Well-posedness and fluctuations of the Dean–Kawasaki equation with coloured noise on bounded domains

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In loving memory of Rahul,
whose strength continues to guide me.

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Abstract

The Dean–Kawasaki equation was discovered independently in the physics literature by Dean [25] and Kawasaki [61] in the nineties, when they studied the evolution of the empirical density of a system of mean field interacting particles. The equation was understood only at a formal level until late 2010s, when Konarovskyi, Lehmann and von Renesse [64, 63] established a rigorous martingale framework for the equation. In the same works, the authors prove a negative "ill-posedness vs triviality" result for the equation and consequently the community shifted its attention to mathematically analysing regularised versions of the Dean–Kawasaki equation that still maintain connections to particle systems.

The aim of this thesis is to study the generalised Dean–Kawasaki stochastic partial differential equation (SPDE) with correlated noise

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho) \circ \dot{\xi}^F + \nu(\rho)) \tag{1}$$

on a C^2-regular bounded domain $U \subset \mathbb{R}^d$ with general Dirichlet boundary conditions. In equation (1), $\circ\dot{\xi}^F$ denotes infinite dimensional Stratonovich noise that is white in time and sufficiently regular in space, and the choice of non-linear functions Φ, σ, ν that we are able to handle include the full range of fast diffusion and porous medium equations $\Phi(\rho) = \rho^m$ for every $m \in (0, \infty)$, noise coefficient $\sigma(\rho) = \Phi^{1/2}(\rho)$ including the degenerate square root $\sigma(\rho) = \sqrt{\rho}$. These choices allow (1) to be interpreted as the fluctuating hydrodynamics of the zero-range process with polynomial jump rates.

As a first result, based on the first paper of the DPhil [82], we present the well-posedness of stochastic kinetic solutions of equations of the type (1) on a bounded domain with Dirichlet boundary data. The results apply to a wide class of non-negative boundary data. In the porus medium regime $\Phi(\xi) = \xi^m$, $m > 1$, the boundary data includes all non-negative constant functions including zero and all smooth functions bounded away from zero. In the classical and fast diffusion cases $\Phi(\xi) = \xi^m$, $m \leq 1$ we can consider any non-negative constant boundary data including zero. This extends the well-posedness theory on the torus of Fehrman and Gess [44].

Subsequently, based on the preprint [81], for $\epsilon \in (0, 1), K = K(\epsilon) \in \mathbb{N}$, under a joint scaling $\epsilon \to 0, K \to \infty$, we prove a central limit theorem and conjecture a large deviations principle for the SPDE

$$\partial_t \rho^\epsilon = \Delta\Phi(\rho^\epsilon) - \sqrt{\epsilon}\nabla \cdot (\sigma(\rho^\epsilon) \circ \dot{\xi}^K) - \nabla \cdot \nu(\rho^\epsilon), \tag{2}$$

where $\dot{\xi}^K$ denotes a finite dimensional truncation of space-time white noise. In the study of the central limit theorem we quantify the rate of convergence of (2) towards the zero noise hydrodynamic limit

$$\partial_t \bar{\rho} = \Delta\Phi(\bar{\rho}) - \nabla \cdot \nu(\bar{\rho}).$$

The large deviations principle allows us to quantify the probability of seeing rare events. That is, whilst we expect $\rho^\epsilon \to \bar{\rho}$ with explicit rate given by the central limit theorem, the large deviation principle quantifies the probability that we see a profile that is very different to $\bar{\rho}$ for small ϵ and large time. We explicitly propose a rate function I such that

$$\mathbb{P}(\rho^\epsilon \in A) \approx \exp(-\epsilon I(A)).$$

The proof of the large deviations is rigorous up to proving the well-posedness of a parabolic-hyperbolic PDE that appears in the rate function. This is the aim of forthcoming work, and in the present work we only give a sketch poof, so we refer to the large deviations result as a conjecture.

Both the central limit theorem and large deviations principle results have connections to the corresponding results for the zero range particle process.

Contents

Chapter 1

Introduction

The aim of this thesis is to study the class of equations known as the generalised Dean–Kawasaki stochastic partial differential equations (SPDEs). The Dean–Kawasaki equation was derived in the physics literature in the nineties by Dean [25] and Kawasaki [61]. In the most simple case, Dean [25] considers the N-particle system $X_t := (X_t^i)_{i=1,\dots,N}$ consisting of independent d-dimensional Brownian motions evolving on the torus $\mathbb{T}^d$. To track the position of the particles he considered the empirical measure, defined by

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}.$$

A simple application of Itô's formula shows that for any smooth test function ψ, the evolution of the empirical measure is governed by

$$\langle \mu_t^N, \psi\rangle = \langle \mu_0^N, \psi\rangle + \frac{1}{2}\int_0^t \langle \mu_s^N, \Delta\psi\rangle\, ds + \frac{1}{N}\sum_{i=1}^N \int_0^t \nabla\psi(X_s^i)\cdot dB_s^i,$$

where the angled brackets represent integration of a function against a measure, defined rigorously below in equation (1.18). The final term in the equation above is the sum of independent martingale noises, whose quadratic variation is

$$\frac{1}{N}\int_0^t \langle \mu_s^N, |\nabla\psi|^2\rangle\, ds.$$

The crucial observation made by Dean [25] was that this is the same quadratic variation we would get if we replaced $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}\, dB^i$ by a space-time white noise of the form $\frac{1}{\sqrt{N}}\sqrt{\mu_t^N(x)}\dot{\xi}(t,x)$ where $\dot{\xi}(t,x)$ is a vector valued space-time cylindrical Wiener process[1]. This motivated that the interacting particle system can be studied by instead considering the Dean–Kawasaki SPDE

$$\partial_t\rho = \frac{1}{2}\Delta\rho - \frac{1}{\sqrt{N}}\nabla\cdot(\sqrt{\rho}\dot{\xi}). \tag{1.1}$$

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[1] The definition and existence of such a white noise is given in the lecture notes by Walsh [94].

This derivation is discussed in more detail and in a more general setting in Sections 1.1 and 1.3 below.

In Section 1.2 we motivate how the Dean–Kawasaki equation arises when studying the fluctuating hydrodynamics of the zero range particle process. Briefly, by studying the rate of convergence of equation (1.1) towards the heat equation as $N \to \infty$, and studying the rare event behaviour of equation (1.1), one obtains the corresponding information about the fluctuations of the zero range particle process. This is important because as we will illustrate in Section 1.4, the zero range particle process is a good model for many real world phenomena, and in these cases it is important to be able to quantify the probability observing rare events.

All this being said, on a mathematical level, a negative result on the well-posedness of martingale solutions of the Dean–Kawasaki equation (1.1) was established in the late 2010s by Konarovskyi, Lehmann and von Renesse [64, 63]. We will outline this result in Section 1.3. Consequently, a primary aim of this thesis is to study a regularised version of the Dean–Kawasaki equation that still maintains the correct fluctuating hydrodynamic behaviour as the particle system. We focus on the equation with correlated noise.

We now present the equation and provide an outline of the main results of the thesis. This thesis is concerned with the generalised[2] Dean–Kawasaki initial boundary value problem on a C^2-regular and bounded domain $U \subset \mathbb{R}^d$,

$$\begin{cases} \partial_t \rho = \Delta \Phi(\rho) - \nabla \cdot (\sigma(\rho) \circ \dot{\xi}^F + \nu(\rho)), & \text{on } U \times (0,T], \\ \Phi(\rho) = \bar{f}, & \text{on } \partial U \times (0,T], \\ \rho(\cdot, t=0) = \rho_0, & \text{on } U \times \{t=0\}. \end{cases} \tag{1.2}$$

Where there is no confusion, throughout the thesis we abuse terminology and refer to equations such as (1.2) as the "generalised Dean–Kawasaki equation", or simply the "Dean–Kawasaki equation", not emphasising the fact that it has correlated noise.

In Chapter 2 we set up the problem by defining the noise and by deriving the kinetic equation corresponding to equation (1.2), which allows us to define stochastic kinetic solutions. For a sequence of continuously differentiable functions $F := \{f_k : U \to \mathbb{R}\}_{k\in\mathbb{N}}$ and a sequence of independent, d-dimensional Brownian motions $\{B^k : [0,T] \to \mathbb{R}^d\}_{k\in\mathbb{N}}$, the Stratonovich noise $\circ\dot{\xi}^F$ in (1.2) is white in time and sufficiently regular in space, defined pointwise by

$$\xi^F : U \times [0,T] \to \mathbb{R}^d, \qquad \xi^F(x,t) := \sum_{k=1}^{\infty} f_k(x) B_t^k. \tag{1.3}$$

The spatial colouring is dictated by an assumption on the spatial component of the noise coefficients F given in Assumption 2.1.2.

We also provide the assumptions on the non-linear coefficients Φ, σ, ν, as well as the assumptions on the boundary data $\bar{f}$ that are needed for well-posedness. The assumptions allow us to consider a wide class of relevant SPDEs such as the full range

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[2] The word "generalised" refers to the fact that we have general non-linear functions Φ, σ, ν present in equation (1.2).

of fast diffusion and porous medium equations, i.e. $\Phi(\rho) = \rho^m$ for every $m \in (0, \infty)$ and the choices $\sigma(\rho) = \Phi^{1/2}(\rho)$ and $\nu(\rho) = \Phi(\rho)$, including the degenerate square root noise coefficient $\sigma(\rho) = \sqrt{\rho}$. These choices allow equation (1.2) to be interpreted as the fluctuating hydrodynamics of the zero-range process with polynomial jump rates, where the ν term models the case of non-symmetric jumps of the particle system.

The irregularity of solutions to (1.2) means that we can only express the boundary condition $\bar{f}$ indirectly via $\Phi(\rho)$. This is done using the trace theorem[3], see Chapter 5.5 of Evans [35]. The boundary data does not depend on time, $\bar{f} = \bar{f}(x)$, for $x \in \partial U$. The well-posedness results apply to a wide class of non-negative boundary data, which is based on certain a priori estimates for the solutions. In the porus medium regime $\Phi(\xi) = \xi^m$, $m > 1$, the boundary data includes all non-negative constant functions including zero and all smooth functions bounded away from zero. In the classical and fast diffusion cases $\Phi(\xi) = \xi^m$, $m \leq 1$ we can consider any non-negative constant boundary data including zero. On the particle level, the Dirichlet boundary condition enables us to model sources and sinks.

Chapter 3 is dedicated to the uniqueness of stochastic kinetic solutions of equation (1.2). The main result is the following theorem.

Theorem 1.0.1 (Uniqueness, Theorem 3.2.2 below)**.** *Suppose that the coefficients of noise ξ^F and the coefficients Φ, σ, ν of equation* (1.2) *satisfy Assumptions 2.1.2 and 2.2.1 respectively. Suppose ρ^1 and ρ^2 are two stochastic kinetic solutions of* (1.2) *in the sense of Definition 2.3.6, with $\mathcal{F}_0$-measurable initial data $\rho_0^1, \rho_0^2 \in L^1(\Omega; L^1(U))$ respectively and with the same boundary data $\bar{f}$. Then almost surely*

$$\sup_{t\in[0,T]} \|\rho^1(\cdot,t) - \rho^2(\cdot,t)\|_{L^1(U)} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(U)}. \tag{1.4}$$

The proof is based on analysis of the kinetic equation corresponding to equation (1.2). The $L^1(U)$-contraction of (1.4) is a stronger statement than just uniqueness and is used to prove the existence of a dynamical system corresponding to solutions of the SPDE, which is important for the proof of the large deviations principle later on.

In Chapter 4 we prove the existence of solutions to equation (1.2). We combine energy estimates for a regularised version of the equation with tightness arguments to obtain the result. The main result can be stated as follows.

Theorem 1.0.2 (Theorem 4.4.8 below)**.** *Suppose that the spatial components of the noise ξ^F, the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in L^1(\Omega; L^1(U))$ be non-negative and $\mathcal{F}_0$-measurable.*

Then there exists a stochastic kinetic solution to the generalised Dean–Kawasaki equation (1.2) *in the sense of Definition 2.3.6.*

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[3] The trace theorem requires that $\Phi(\rho)$ is in the Sobolev space $H^1(U)$. Such a statement is true for $\Phi(\rho)$ via a priori entropy estimate for a regularised version of the equation, but is not true for ρ itself.

With the well-posedness in hand, we study small noise fluctuation theory for the generalised Dean–Kawasaki SPDE. For $\epsilon \in (0,1)$ and $K = K(\epsilon) \in \mathbb{N}$, we study fluctuations the SPDE

$$\begin{cases} \partial_t \rho^\epsilon = \Delta\Phi(\rho^\epsilon) - \sqrt{\epsilon}\nabla\cdot(\sigma(\rho^\epsilon)\circ\dot{\xi}^K) - \nabla\cdot\nu(\rho^\epsilon), & \text{on } U\times(0,T], \\ \Phi(\rho^\epsilon) = \bar{f}, & \text{on } \partial U\times(0,T], \\ \rho^\epsilon(\cdot, t=0) = \rho_0, & \text{on } U\times\{t=0\}, \end{cases} \tag{1.5}$$

where $\dot{\xi}^K$ denotes a finite dimensional truncation of $\mathbb{R}^d$-valued space-time white noise. That is, for $L^2(U)$-orthogonal functions $\{e_k : U \to \mathbb{R}\}_{k\in\mathbb{N}}$ and independent Brownian motions $\{B^k : [0,T] \to \mathbb{R}^d\}_{k\in\mathbb{N}}$, for every $K \in \mathbb{N}$ the noise ξ^K is defined pointwise by

$$\xi^K : U\times[0,T] \to \mathbb{R}^d, \qquad \xi^K(x,t) := \sum_{k=1}^{K} e_k(x)B_t^k. \tag{1.6}$$

We note that the scaling for $K \in \mathbb{N}$ will be chosen entirely a function of ϵ, so we do not make the K-dependence explicit in the superscript of equation (1.5). For every fixed $\epsilon \in (0,1)$ and $K \in \mathbb{N}$, the well-posedness of equation (1.5) follows from the well-posedness results of equation (1.2).

We study the convergence of equation (1.5) as $\epsilon \to 0, K \to \infty$ towards the zero noise, hydrodynamic limit $\bar{\rho}$, which is the unique solution to

$$\partial_t\bar{\rho} = \Delta\Phi(\bar{\rho}) - \nabla\cdot\nu(\bar{\rho}). \tag{1.7}$$

For this chapter, in line with the work of Clini and Fehrman on the torus [20], we will assume that the initial data and boundary data are the same random positive constant. This implies that the solution of the hydrodynamic limit equation (1.7) is given by that constant, and we are looking at convergence of the SPDE towards a constant function.

As a preliminary result we are able to establish a quantitative law of large numbers result for a regularised[4] version of equation (1.5). That is, for the regularised equation $\rho^{n,\epsilon}$ defined in Definition 5.1.9 and the corresponding regularised hydrodynamic limit $\bar{\rho}^n$ defined in equation (5.11), we show in Propositions 5.1.18 and 5.1.20 that under a suitable scaling regime $\epsilon \to 0, K \to \infty$, arbitrary $p \geq 2$ and $n \in \mathbb{N}$, we have that

$$\mathbb{E}\left[\int_0^T \int_U (\rho^{n,\epsilon} - \bar{\rho}^n)^p\right] \to 0. \tag{1.8}$$

The estimate is independent of the regularisation n. The central limit theorem quantifies the rate of this convergence. To this end, define

$$v^\epsilon := \epsilon^{-1/2}(\rho^\epsilon - \bar{\rho}). \tag{1.9}$$

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[4] The regularised equation involves adding a $\frac{1}{n}\Delta\rho$ regularising factor to the equation and smoothing the potentially degenerate non-linearity σ.

We will show that under an appropriate joint scaling regime $\epsilon \to 0, K \to \infty$, solutions v^ϵ converge in probability in a space of distributions to the linearised SPDE

$$\partial_t v = \Delta(\Phi'(\bar{\rho})v)) - \nabla \cdot (\sigma(\bar{\rho})\dot{\xi} + \nu'(\bar{\rho})v), \tag{1.10}$$

with zero initial and boundary data, where $\dot{\xi}$ denotes space-time white noise and $\bar{\rho}$ denotes the solution to the hydrodynamic limit equation (1.7).

The linearisation present in (1.10) can be formally seen by observing that v^ϵ solves

$$\begin{aligned} \partial_t v^\epsilon &= \Delta(\epsilon^{-1/2}(\Phi(\rho^\epsilon) - \Phi(\bar{\rho})) - \nabla \cdot (\epsilon^{-1/2}(\nu(\rho^\epsilon) - \nu(\bar{\rho})) - \nabla \cdot \left(\sigma(\rho^\epsilon) \circ \dot{\xi}^K\right) \\ &= \Delta\left(v^\epsilon \frac{\Phi(\rho^\epsilon) - \Phi(\bar{\rho})}{\rho^\epsilon - \bar{\rho}}\right) - \nabla \cdot \left(v^\epsilon \frac{\nu(\rho^\epsilon) - \nu(\bar{\rho})}{\rho^\epsilon - \bar{\rho}}\right) - \nabla \cdot \left(\sigma(\rho^\epsilon) \circ \dot{\xi}^K\right), \end{aligned} \tag{1.11}$$

with zero initial condition and boundary data. Owing to the law of large numbers (1.8), $\rho^\epsilon \to \bar{\rho}$, and also by definition $\dot{\xi}^K \to \dot{\xi}$, which gives the formal convergence of v^ϵ to v. Solutions ρ^ϵ to the singular equation (1.5) can only be considered in a stochastic kinetic sense. However, the non-linear nature of this solution theory makes it incompatible with convergence in the space of distributions. As a result, the central limit theorem is first proven for the regularised equation.

Theorem 1.0.3 (Theorem 5.2.9 below)**.** *Suppose that the non-linear functions Φ, σ, ν satisfy Assumptions 2.2.1 and 5.1.13, and that the spatial components of the noise ξ^K satisfy assumption 5.1.2. Suppose that ρ_0 and $\bar{f}$ satisfy the simplifying Assumption 5.1.7. Fix an arbitrary regularisation constant $n \in \mathbb{N}$. For $K \in \mathbb{N}$ and $\epsilon \in (0,1)$ let $v^{n,\epsilon}$ denote the weak solution to the regularised equation*[5] (5.36)*, and let v^n be the solution of the regularised, linearised SPDE* (5.38)*.*

There exists a joint scaling regime $\epsilon \to 0, K \to \infty$ such that

$$\mathbb{E}\|v^{n,\epsilon} - v^n\|^2_{L^2([0,T];H^{-s}(U))} \to 0. \tag{1.12}$$

In Theorem 5.2.9 we obtain a quantitative bound for the rate of convergence. The bound is not uniform in the regularisation n, so we can not deduce the desired convergence for the singular equation (1.9). To get around this, we prove an $L^\infty(U \times [0,T])$-estimate, stated in Theorem 5.3.1. We show a quantitative bound that illustrates that for $\Phi^{-1}(M)$ the constant solution to the hydrodynamic limit equation, under a joint scaling regime $\epsilon \to 0$, $K \to \infty$, we have

$$\mathbb{E}\| \left(\rho^\epsilon - \Phi^{-1}(M)\right)_- \|_{L^\infty(U\times[0,T])} \to 0. \tag{1.13}$$

Formally, this estimate says that if our initial condition satisfies $\rho_0 = \Phi^{-1}(M)$ for some $M > 0$, then along appropriate subsequences, the solutions of equation (1.5) satisfy $\rho^\epsilon > \Phi^{-1}(M)/2$ uniformly in space and time with high probability. The estimate is consistent with the intuition that as $\epsilon \to 0$, the solutions of the SPDE ρ^ϵ converge to the constant solution of the hydrodynamic limit $\rho^\epsilon \to \bar{\rho} = \Phi^{-1}(M)$. Using this and the pathwise uniqueness of stochastic kinetic solutions given by Theorem 1.0.1 above,

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[5] $v^{n,\epsilon}$ is like v^ϵ (1.9) but with ρ^ϵ replaced by $\rho^{n,\epsilon}$ and $\bar{\rho}$ replaced by $\bar{\rho}^n$.

in Theorem 5.3.2 we prove the central limit theorem in probability for the singular equation.

In Remark 5.3.4 we identify an appropriate joint scaling regime between $\epsilon \to 0$ and $K \to \infty$ under which the convergences (1.8), (1.12) and (1.13) hold. Formally the scaling depends on how fast the spatial coefficients $\{e_k\}_{k\in\mathbb{N}}$ of the noise (1.6) and their derivatives grow. A canonical example of the noise coefficients are the eigenfunctions of the Laplacian with homogeneous Dirichlet boundary data, see Example 5.1.4. In this case, the scaling limit is explicitly given by

$$\epsilon K^{d+2} \to 0. \tag{1.14}$$

Even though we expect the convergence of ρ^ϵ to $\bar{\rho}$, we are interested in quantifying the (un)likelihood that we see a profile very different to $\bar{\rho}$ after a long time. Whilst unlikely, the occurrence of a rare event can have catastrophic consequences, for instance mechanical failure of a machine or a population going extinct, see Section 1.4 for further examples which are relevant on the bounded domain. The study of rare events is quantified by the large deviations principle, and this is the content of the final chapter, Chapter 6. The main result is the following conjecture.

Conjecture 1.0.4 (Conjecture 6.1.4 below)**.** *Suppose that the non-linear functions Φ, σ, ν, the boundary data $\bar{f}$ and the spatial coefficients of the noise ξ^K satisfy Assumptions 2.2.1, 2.2.9 and 5.1.2 respectively.*

For $\rho_0 \in Ent_\Phi(U)$ and arbitrary $\rho \in L^1(U \times [0,T])$, define

$$I^{\bar{f}}_{\rho_0}(\rho) := \frac{1}{2}\inf_{g\in L^2(U\times[0,T];\mathbb{R}^d)}\Big\{\|g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)} : \partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)g + \nu(\rho)) \\ : \Phi(\rho)|_{\partial U} = \bar{f} \text{ and } \rho(\cdot,0) = \rho_0\Big\}. \tag{1.15}$$

Then under Assumption 6.1.3, for every $\rho \in L^1(U \times [0,T])$, $\rho \mapsto I^{\bar{f}}_{\rho_0}(\rho)$ is a rate function on $L^1(U \times [0,T])$, the family

$$\{I^{\bar{f}}_{\rho_0}(\cdot) : \bar{f} \in H^1(\partial U),\ \rho_0 \in Ent_\Phi(U)\}$$

of rate functions has compact level sets on compacts, and solutions $\{\rho^{\epsilon,K(\epsilon)}\}_{\epsilon\in(0,1)}$ of (1.5) *satisfy the uniform large deviations principle on $L^1(U\times[0,T])$ with rate function $I^{\bar{f}}_{\rho_0}$.*

Since we have for Borel subsets $A \in \mathcal{B}(L^1(U \times [0,T]))$ that

$$\mathbb{P}(\rho^\epsilon \in A) \approx \exp(-\epsilon I(A)) \to \begin{cases} 1 & I(A) = 0 \\ 0, & \text{otherwise,} \end{cases}$$

as $\epsilon \to 0$, it follows that rare events become exponentially unlikely in ϵ unless $I(A) = 0$, in which case the probability of such events converges to 1. Using the form of the rate function (1.15), we have that $I(A) = 0$ if and only if the infimum over the set of g is zero. That is, if the subset A contains the solution to the hydrodynamic limit equation $\bar{\rho}$ as in (1.7).

The large deviations result is based on the variational representation of infinite dimensional Brownian motion introduced by Budhiraja, Dupuis, and Maroulas [11]. As mentioned above, the result is conditional on the well-posedness of the PDE appearing in the rate function (1.15), and is the topic of forthcoming work.

1.1 Application to mean field interacting particle systems

In this section we we motivate how generalised Dean–Kawasaki type equations appear as approximations of weakly interacting particle systems. The discussion in this section follows the recent paper by Djurdjevac, Ji, and Perkowski [31], which in turn is an extension of the earlier work by Djurdjevac, Kremp and Perkowski [32].

In their work, Djurdjevac, Ji, and Perkowski use a regularised version of the Dean–Kawasaki SPDE to approximate a mean field interacting particle system $X_t := (X_t^i)_{i=1,\dots,N}$ on the d-dimensional torus $\mathbb{T}^d$. The dynamics of the particles are given by the distribution dependent stochastic differential equation

$$dX_t^i = b(X_t^i, \mu_t^N) + \sqrt{2}\Sigma(X_t^i, \mu_t^N) \cdot dB_t^i, \tag{1.16}$$

on $\mathbb{T}^d$. Here, $B = (B^i)_{i=1}^N$ denotes independent d-dimensional Brownian motions, for $(x, \mu) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$, we have

$$b(x, \mu) \in \mathbb{R}^d, \quad \Sigma(x, \mu) \in \mathbb{R}^{d \times d},$$

and finally the probability measure μ_t^N is the empirical measure of the particle system

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i}. \tag{1.17}$$

Equation (1.16) conveys that a single particle will evolve with drift and diffusion according to its own position and the state of the whole system through the empirical measure. An important example is the choice

$$b(x, \mu) := (K * \mu)(x), \quad \Sigma(x, \mu) = \sigma((G * \mu)(x))$$

for smooth interaction kernels K, G and appropriate $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$. This choice gives that the evolutions of the particles (1.16) satisfy

$$dX_t^i = \frac{1}{N} \sum_{j=1}^N K(X_t^i - X_t^j)\, dt + \sqrt{2}\sigma \left(\frac{1}{N} \sum_{j=1}^N G(X_t^i - X_t^j) \right) \cdot dB_t^i.$$

Following the computations of Dean [25] and Kawasaki [61], we calculate the formal quadratic variation of the empirical measure. For a smooth test function

$f \in C^\infty(\mathbb{T}^d)$, define

$$\langle f, \mu_t^N \rangle := \int_{\mathbb{T}^d} f(x) \mu_t^N(x)\, dx = \frac{1}{N} \sum_{i=1}^N f(X_t^i). \tag{1.18}$$

Applying Itô's formula to $\langle f, \mu_t^N \rangle$, we have using equations (1.17) and (1.16) that

$$d\langle f, \mu_t^N \rangle = \left\langle [b(\cdot, \rho_{X_t}) \cdot \nabla] f, \mu_t^N \right\rangle dt + \left\langle \left([\Sigma(\cdot, \rho_{X_t}) \cdot \Sigma(\cdot, \rho_{X_t})^t] : \nabla^2 \right) f, \mu_t^N \right\rangle dt + dM_t,$$

where the superscript ^{t} denotes the transpose of a matrix and the semicolon $:$ denotes the Frobenius product of matrices. By the definition of the empirical measure (1.17), the quadratic variation of the martingale term M can be written as

$$\begin{aligned} d[M]_t = d[\langle f, \mu_\cdot^N \rangle]_t &= \frac{2}{N} \left\langle [\Sigma(\cdot, \mu_t^N)^t \cdot \nabla] f, \mu_t^N \right\rangle^2 dt \\ &= \frac{2}{N} \left\langle \left([\Sigma(\cdot, \mu_t^N)^t \cdot \nabla] f \right)^2, \mu_t^N \right\rangle dt \\ &= \frac{2}{N} \sum_{k \in \mathbb{Z}^d} \left| \left\langle \sqrt{\mu_t^N} [\Sigma(\cdot, \mu_t^N)^t \cdot \nabla] f, e_k \right\rangle_{L^2(\mathbb{T}^d)} \right|^2 dt, \end{aligned}$$

where $\{e_k\}_{k \in \mathbb{Z}^d}$ denotes the canonical basis of $L^2(\mathbb{T}^d, \mathbb{C})$ given by Plancherel's identity. The above computation suggests that in order to capture the quadratic variation of the empirical measure (1.17) $\mu_\cdot^N$, with N particles, we can consider the SPDE

$$\begin{aligned} dm_t^N = \nabla^2 : \left[\Sigma(\cdot, m_t^N) \Sigma(\cdot, m_t^N)^t m_t^N \right] dt - \nabla \cdot (b(\cdot, m_t^N) m_t^N)\, dt \\ + \frac{2}{\sqrt{N}} \nabla \cdot \left[\sqrt{m_t^N} \Sigma(\cdot, m_t^N) \cdot dW_t \right], \end{aligned} \tag{1.19}$$

where $(W_t)_{t \geq 0}$ denotes the d-dimensional real cylindrical Wiener process on $L^2(\mathbb{T}^d)$. This is nothing other than the Dean–Kawaski equation with generalised coefficients. However, as mentioned above, equation (1.19) satisfies an "ill-posedness vs triviality" result[6]. Consequently, Djurdjevac, Ji, and Perkowski [31] instead consider an approximation of the SPDE (1.19), given by

$$\begin{aligned} dm_t^N = \nabla^2 : \left[\Sigma(\cdot, m_t^N) \Sigma(\cdot, m_t^N)^t m_t^N \right] dt - \nabla \cdot (b(\cdot, m_t^N) m_t^N)\, dt \\ + \frac{2}{\sqrt{N}} \nabla \cdot \left[f^N(m_t^N) \Sigma(\cdot, m_t^N) \cdot dW_t^N \right], \end{aligned} \tag{1.20}$$

where $f^N : \mathbb{R} \to \mathbb{R}_+$ is a well-chosen approximation of the square root, and $(dW^N)_{N \in \mathbb{N}}$ denotes a family of noises which are white in time, coloured in space and which approximate the genuine space-time white noise. After showing the well-posedness of probabilistically strong $L^2(\mathbb{T}^d)$-solutions to (1.20) under growth conditions on b and Σ, the main result of [31] is to prove a quantitative bound on the weak error of the

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[6] This phenomenon is explained precisely in Section 1.3.1

approximation of the SPDE (1.20) and the particle system (1.17). That is, for every continuous functional $\mathcal{F} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, the smoothed Dean–Kawasaki equation (1.20) approximates the particle system in a weak sense, with a quantitative upper bound

$$\left|\mathbb{E}\mathcal{F}(\mu_t^N) - \mathbb{E}\mathcal{F}(m_t^{N,\epsilon})\right| \leq cN^{-1-\frac{2}{d+2}} \log(N), \tag{1.21}$$

where the the constant $c \in (0, \infty)$ is independent of N. The superscript ϵ in $m_t^{N,\epsilon}$ corresponds to an $H^{-1}(\mathbb{T}^d)$-taming of the equation which is needed to compensate for the equation's quadratic growth, see Section 2.1 of [31].

1.2 Application to hydrodynamic fluctuations of particle systems

In this section we will explain how equations such as (1.2) arise in the study of fluctuating hydrodynamics for particle systems. Consider the concrete example of the symmetric zero range process on the torus $\mathbb{T}_N^d := (\mathbb{Z}^d \setminus N\mathbb{Z}^d)$. That is, the waiting time for a jump when there are k particles at a site is exponentially distributed with parameter k, $Exp(k)$, after which one particle jumps to a neighbour uniformly at random. Jumps at different sites $x, y \in \mathbb{T}_N^d$ occur independently.

For every $N \in \mathbb{N}$, let

$$\eta^N : [0,T] \times \mathbb{T}_N^d \to \mathbb{N}_0$$

denote the configuration of the zero range process on $\mathbb{T}_N^d$. That is, $\eta_t^N(x)$ denotes the number of particles at site $x \in \mathbb{T}_N^d$ at time $t \in [0,T]$. Let

$$\tilde{\mu}_t^N := \frac{1}{N^d} \sum_{x \in (\mathbb{Z}^d/N\mathbb{Z}^d)} \delta_{x/N} \cdot \eta_{N^2 t}^N(x)$$

denote the parabolically rescaled empirical measure for the particle system. The rescaling in space rescales everything back to the unit torus, the rescaling in time allows for more frequent jumps and the factor $\frac{1}{N^d}$ in front rescales the height of the particles to be comparable on every scale.

It was shown in Theorem 1.1 of Chapter 5 of Kipnis and Landim [62] that the discrete measure $\tilde{\mu}^N$ converges to the deterministic, continuous measure $\bar{\rho}\, dx$ where

$$\bar{\rho} : \mathbb{T}^d \times [0,T] \to \mathbb{R} \quad \text{uniquely solves the heat equation} \quad \partial_t \bar{\rho} = \frac{1}{2}\Delta\bar{\rho}.$$

Precisely, it was shown that for every continuous $f : \mathbb{T}^d \to \mathbb{R}$ and $\delta \in (0,1)$, we have for every $t \in [0,T]$,

$$\lim_{N\to\infty} \mathbb{P}\left[|\langle f, \tilde{\mu}^N\rangle - \langle f, \bar{\rho}\rangle| > \delta\right] = 0,$$

where the angled brackets again denote the integral of a function against a measure, just as in (1.18). See Theorem 2.1 of Ferrari, Presutti, Vares [46] for a similar result.

This formally illustrates that we expect particles to uniformly spread out on the torus, which can be understood by the fact that sites with a larger number of particles jump more frequently on average.

Suppose more generally that if there are k particles at a site, we wait an exponential time $Exp\,(g(k))$ before making a jump, for some non-decreasing function function

$$g : \mathbb{N}_0 \to \mathbb{R}_{\geq 0} \quad \text{satisfying} \quad g(0) = 0 \quad \text{and} \quad g(k) > 0 \text{ for } k > 0.$$

In this case $\bar{\rho}$ instead solves the non-linear diffusion equation

$$\partial_t \bar{\rho} = \frac{1}{2} \Delta \Phi(\bar{\rho})$$

for the expected value of the jump rate $\Phi(\alpha) = \mathbb{E}_{\nu_\alpha}[g(\eta^N(0)]$ where ν_α is a parameterisation of the invariant measure of the zero range process $\mathbb{E}_{\nu_\alpha}[\eta^N(0)] = \alpha$, again see Chapter 5 of Kipnis and Landim [62] for further details and a precise statement.

Furthermore, nonequilibrium central limit fluctuation results were obtained by Ferrari, Presutti, and Vares [45], Jankowski [58] and by Gielis, Koukkous, Landim [51]. The measures

$$m^N := N^{d/2}(\mu^N - \bar{\rho}\, dx)$$

are shown to converge in the space of distributions as $N \to \infty$ to a measure m satisfying the linearised SPDE

$$\partial_t m = \Delta(\Phi'(\bar{\rho}) m) - \nabla \cdot (\Phi^{1/2}(\bar{\rho}) \dot{\xi}), \tag{1.22}$$

where $\dot{\xi}$ is d-dimensional space-time white noise and $\bar{\rho}$ is the solution to equation (1.2).

Simulating particle systems to observe central limit convergence and rare event behaviour is expensive when there are a large number of particles. The central limit theorem result described in Theorem 1.0.3 above and in the subsequent paragraph illustrates that if we instead consider the solution ρ^ϵ to the SPDE (1.5), then under a joint scaling $\epsilon \to 0, K \to \infty$, fluctuations of

$$\epsilon^{-1/2}(\rho^\epsilon - \bar{\rho})$$

also converge[7] to the solution m of equation (1.22). That is to say, solutions to SPDEs such as (1.5) correctly describe the non-equilibrium fluctuations of particle systems.

Let us now discuss the large deviations. Based on the central limit theorem $m^N \to m$, a natural object to study the deviations of is the measure arising from the first order expansion

$$\bar{\rho}^N := \bar{\rho}\, dx + N^{-d/2} m.$$

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[7] To align with the particle system scaling (1.22) we could replace the scaling $\epsilon^{-1/2}$ for the SPDE by $\epsilon^{-d/2}$. We would then need to amend the SPDE (1.5) to have a factor $\epsilon^{-d/2}$ in front of the noise term, and we would end up with a different joint scaling regime (1.14), with ϵ replaced by ϵ^d. We use $\epsilon^{-1/2}$ as it is more common in the literature.

Schilder's theorem and the contraction principle prove that $\bar{\rho}^N$ satisfies a large deviations principle with the rate function

$$\bar{I}_{\rho_0}(\rho) := \frac{1}{2}\inf_{g\in L^2(U\times[0,T];\mathbb{R}^d)}\Big\{\|g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)} : \\ \partial_t(\rho-\bar{\rho}) = \Delta\left(\Phi'(\bar{\rho})(\rho-\bar{\rho})\right) - \nabla\cdot(\Phi^{1/2}(\bar{\rho})g) : \rho(\cdot,0)=\rho_0\Big\}.$$

However, it was shown by Benois, Kipnis and Landim [8] that the large deviations of the particle system μ^N are governed by the rate function

$$I_{\rho_0}(\rho) = \frac{1}{2}\inf_{g\in L^2(U\times[0,T];\mathbb{R}^d)}\Big\{\|g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)} : \\ \partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\Phi^{1/2}(\rho)g) : \rho(\cdot,0)=\rho_0\Big\}.$$

This does not coincide with the rate function $\bar{I}_{\rho_0}$ from the first order expansion. On the other hand, as illustrated by Theorem 1.0.4 above, even on a bounded domain the SPDE ρ^ϵ correctly captures the large deviations of the particle system μ^N.

1.3 Background and relevant literature

As we previously mentioned, the Dean–Kawasaki equation was derived independently in the physics literature by Dean [25] and Kawasaki [61] by an analogous derivation as presented in Section 1.1 above, with the choices $\Sigma(x,\mu) = 1/\sqrt{2}$ and pairwise interaction $b(x,\mu) = (\nabla V * \mu)(x)$ in equation (1.16) That is, Dean [25] considered the N-particle system $(X^i)_{i=1,\dots,N}$ following Langevin dynamics

$$dX^i_t = dB^i_t - \frac{1}{N}\sum_{j=1}^N \nabla V(X^i_t - X^j_t)\,dt. \tag{1.23}$$

Choices for the interaction kernel V in equation (1.23) include Coulomb type interactions, Lennard-Jones type interaction or hard sphere repulsion.

By the same arguments as Section 1.1, Dean showed that the empirical measure $\rho^N_t := \frac{1}{N}\sum_{i=1}^N \delta_{X^i_t}$ of the particle system satisfies an equation whose quadratic variation agrees with those of solutions to the SPDE

$$\partial_t\rho = \frac{1}{2}\Delta\rho + \nabla\cdot(\rho(\nabla V * \rho)) + \frac{1}{\sqrt{N}}\nabla\cdot\left(\rho^{1/2}\dot{\xi}\right), \tag{1.24}$$

where $\dot{\xi}$ denotes space-time white noise.

Equation (1.24) is nothing other than the Dean–Kawasaki equation with non-local transport. Hence, Dean [25] noticed that the empirical measure of the particle system and solutions to equation (1.24) can be seen to agree as semi-martingales.

Finally, since it appears in the next subsection, we note that in the same work, Dean formally realised that by considering the coarse grained free energy of the system, denoting by $\rho = \rho(x)$ the time equilibrium density field[8], we can re-write equation (1.24) as

$$\partial_t \rho(x,t) = \nabla \cdot \left(\rho(x,t) \nabla \left. \frac{\delta F}{\delta \rho(x)} \right|_{\rho(x,t)} \right) + \nabla \cdot \left(\eta(x,t) \rho^{1/2}(x,t) \right), \tag{1.25}$$

where $\frac{\delta F}{\delta \rho(x)}$ denotes the functional derivative of F, for the free energy F defined by

$$F[\rho] := \frac{1}{2} \int \int dx\, dy \rho(x) V(x-y) \rho(y) + T \int dx \rho(x) \log \rho(x).$$

That is, formally the deterministic part of Dean's equation can be written as a gradient flow of the free energy functional $F[\rho]$ with respect to the Wasserstein–Otto Riemannian structure on the space of probability densities[9]

1.3.1 Well-posedness results

The irregularity of space-time white noise implies that equations such as (1.1), (1.19), (1.24) or our equation (1.2) if considered with space-time white noise, are not renormalisable using Hairer's regularity structures [55] or Gubinelli, Imkeller and Perkowski's paracontrolled distributions [53], even in dimension $d = 1$. Since the derivative hits space-time white noise in these equations, one can only consider solutions ρ in the space of distributions rather than function valued solutions, in which case the square root $\sqrt{\rho}$ as well as the product $\sqrt{\rho}\dot{\xi}$ have no pathwise classical meaning.

One might hope that there exist non-atomic solutions to these equations. The well-posedness of martingale solutions was studied by Konarovskyi, Lehmann and von Renesse in the non-interacting case in [64] and the smooth interacting case in [63]. In the first work [64], the authors study for $\alpha \in \mathbb{R}$ the Dean–Kawasaki SPDE

$$\partial_t \mu = \frac{\alpha}{2} \Delta \mu + \nabla \cdot (\sqrt{\mu} \dot{\xi}), \quad \text{on} \quad \mathbb{R}^d \times (0,T], \tag{1.26}$$

with initial data $\mu|_{t=0} = \mu_0$, where $\dot{\xi}$ is a vector valued space-time white noise. Theorem 2.2 of Konarovskyi, Lehmann and von Renesse [64] proves that solutions

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[8] ρ denotes the equilibrium configuration of the system. That is, $\rho(x)$ denotes the average number of particles at point x.

[9] That is, the metric tensor at ρ is defined by

$$g_\rho(\xi_1, \xi_2) := \int_U \rho(x) \nabla \phi_1(x) \cdot \nabla \phi_2(x)\, dx, \quad \text{where } \xi_i = -\nabla \cdot (\rho \nabla \psi_i).$$

The corresponding gradient flow reads $\partial_t \rho = -grad_{W_2} F(\rho)$, and the noise term in (1.25) can be seen as a stochastic perturbation of this Wasserstein gradient flow.

of the corresponding martingale problem with parameter α and initial condition μ_0 exist if and only if $\alpha = N \in \mathbb{N}$, and the initial condition is of the form

$$\mu_0 = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i},$$

for points $x_1, \dots x_N \in \mathbb{R}^d$. In this case, the solution is uniquely in law given by the empirical measure

$$\mu_t = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^i_{Nt}},$$

where $\{X^i\}_{i=1}^N$ are N independent diffusion processes each with generator[10] $\frac{1}{2}\Delta$ and starting from point x_i.

Such a result is called an "ill-posedness vs triviality" result. On the other hand, if one adds a non-linear operator Ξ to the equation, formal solutions to the SPDE

$$\partial_t \mu = \frac{\alpha}{2}\Delta\mu + \Xi(\mu) + \nabla\cdot(\sqrt{\mu}\dot{\xi}) \tag{1.27}$$

were constructed by von Renesse and Sturm in [93].

In the second of the works by Konarovskyi, Lehmann and von Renesse [63] obtain a similar ill-posedness vs triviality result for the equation

$$\partial_t \mu = \frac{\alpha}{2}\Delta\mu + \nabla\cdot\left(\mu_t \nabla \frac{\delta F(\mu_t)}{\delta \mu_t}\right) + \nabla\cdot(\sqrt{\mu}\dot{\xi}), \quad \text{on} \quad \mathbb{R}^d \times (0, T],$$

for smooth potentials F. The above equation is analogous to (1.25), and hence the conclusion is that the only martingale solutions of the above equation are given by the measure valued Langevin system studied by Dean (1.23). It is important to remark that such a statement is false for arbitrary drift F, which is illustrated in the case that $\alpha > 0$ by the discussion of equation (1.27) above. The existence of non-trivial solutions in the case $\alpha = 0$ was shown by Marx [78], and the works by Konarovskyi and von Renesse [66, 65].

It is due to this ill-posedness vs triviality result that the Dean–Kawasaki equation is described as a "rigid mathematical object" by Cornalba et. al. [21]. As already illustrated in Section 1.1 above, the rigidity can be overcome when suitable regularisations of the equations are considered, for instance smoothing the square root non-linearity or truncating the noise. Below is a non-exhaustive list of well-posedness results in this direction. We will not repeat the discussion contained in Section 1.1 based on the works by Djurdjevac, Kremp and Perkowski [32] and Djurdjevac, Ji and Perkowski [31].

Firstly we present work that could not handle the square root noise coefficient, and in Section 1.3.3 we outline work that was able to handle the square root using kinetic solution theory.

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[10] That is, the X^i are independent Brownian motions, $dX^i = dB^i$. The authors obtain a more general result, that the same conclusion holds when the Laplacian Δ corresponding to the first term on the right hand side of (1.26) is replaced by a generic symmetric Markov diffusion operator L, and the process takes values on a general polish space E rather than simply $\mathbb{R}^d$.

1. Fehrman and Gess [39, 40] proved the well-posedness of pathwise kinetic solutions, which is a solution theory that combines the rough path techniques of Lyons [76] with kinetic solution theory. In the first of the works [39], the authors consider the stochastic porus-medium equation

   $$\partial_t u = \Delta(|u|^{m-1}u) + \nabla\cdot(A(x,u)\circ dz_t), \quad \text{on} \quad \mathbb{T}^d\times(0,\infty), \tag{1.28}$$

   with $L^2(\mathbb{T}^d)$-integrable initial data. The noise z is an n-dimensional α-Hölder continuous geometric rough path[11] for $\alpha\in(0,1)$, and $A(x,\xi)=(a_{i,j}(x,\xi)):\mathbb{T}^d\times\mathbb{R}\to\mathcal{M}^{d\times n}$ is a matrix valued non-linearity whose regularity is determined by the regularity of the rough path z. In the second of their works [40] they study the well-posedness of same system with linear multiplicative noise. That is, the system

   $$\partial_t u = \Delta(|u|^{m-1}u) + \sum_{k=1}^{n} f_k(x)u\circ dz_t^k, \quad \text{on} \quad U\times(0,\infty),$$

   for a smooth bounded domain $U\subset\mathbb{R}^d$ and homogeneous Dirichlet boundary data. This result on the bounded domain was extended to the case equations of the form (1.28) by Clini [19]. The proofs are motivated by the theory of stochastic viscosity solutions, see the works of Lions and Souganidis [74, 73, 75], and stochastic conservation laws see the works of Lions, Perthame and Souganidis [71, 72], as well as the works by Gess and Souganidis [49, 50].

   Rough path techniques require sufficient regularity on the noise coefficient A to overcome the roughness of the noise z. For instance, in the case of Brownian noise, the coefficient is required to be six times continuously differentiable, so the works fall well outside of the critical square root case.

2. Another approach was by Dareiotis and Gess [24] where they construct probabilistic solutions to equations with non-linear gradient noise of the form

   $$du = (\Delta\Phi(u) + \nabla\cdot G(x,u))\,dt + \sum_{k=1}^{\infty}\left(\nabla\cdot\sigma^k(x,u)\right)\circ dB_t^k \quad \text{on} \quad \mathbb{T}^d\times(0,T].$$

   The proof is via an entropy formulation, based on the earlier work of Dareiotis, Gerencsér and Gess [23]. The entropy approach in [24] requires C_b^3-regularity from the noise coefficient σ, which is an improvement upon the six times differentiability using rough path techniques in the previous point. However, this still remained far from the critical square root case $\sigma(\rho)=\sqrt{\rho}$.

3. In Martini and Mayorcas [77] the authors study local well-posedness of the parabolic-elliptic Keller–Segel-Dean-Kawasaki model in two dimensions with additive space-time noise, which is the coupled system

   $$\begin{cases}\partial_t\rho = \Delta\rho + \nabla\cdot(\rho\nabla\Phi_\rho) + \nabla\cdot(\sigma\dot{\xi}), & \text{on} \quad \mathbb{T}^2\times(0,T],\\ \Delta\Phi_\rho = \rho - \langle\rho,1\rangle_{L^2(\mathbb{T}^d)}, & \text{on} \quad \mathbb{T}^2\times(0,T].\end{cases} \tag{1.29}$$

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[11] Importantly this includes the case of n-dimensional Brownian motion.

Due to the genuine space-time white noise $\dot{\xi}$, the results of Konarovskyi, Lehmann and von Renesse [64] [63] from above apply, so that indeed the equation with smooth drift and $\sigma = \sqrt{\rho}$ only admits solutions that are empirical measures of underlying interacting particle systems.

Hence, motivated by the theory of linear fluctuating hydrodynamics, the authors instead consider the noise coefficient in equation (1.29) given by $\sigma = (\bar{\rho})^{1/2}$ where $\bar{\rho}$ is a solution to the corresponding zero-noise PDE

$$\begin{cases} \partial_t \bar{\rho} = \Delta \bar{\rho} + \nabla \cdot (\bar{\rho} \nabla \Phi_{\bar{\rho}}), & \text{on} \quad \mathbb{T}^2 \times (0,T], \\ \Delta \Phi_{\bar{\rho}} = \bar{\rho} - \langle \bar{\rho}, 1 \rangle_{L^2(\mathbb{T}^d)}, & \text{on} \quad \mathbb{T}^2 \times (0,T]. \end{cases}$$

The well-posedness results of the system (1.29) are via Gubinelli, Imkeller and Perkowski's paracontrolled distribution theory [53].

4. We conclude by noting that there is an extensive literature surrounding the well-posedness of stochastic non-linear diffusion equations on smooth, bounded domains in $\mathbb{R}^d$ with homogeneous Dirichlet boundary conditions. This literature pre-dates that of the previous points. Dating back to the mid 2000s, Barbu et. al [2] show the well-posedness of martingale solutions to stochastic porous media equations with additive noise,

   $$d\rho = \Delta \Phi(\rho)\, dt + \sigma\, dW_t, \quad \text{on} \quad U \times (0,\infty),$$

   where $U \subset \mathbb{R}^d$ is a bounded open set, and the equation has homogeneous Dirichlet boundary data. Here, W_t denotes a cylindrical Wiener process, $\sigma = \sqrt{Q}$ where Q is linear, non-negative, bounded and of finite trace, and Φ' satisfies a polynomial growth condition.

   Well-posedness in the multiplicative noise setting for dimension $1 \leq d \leq 3$ was shown by Barbu, Da Prato and Röckner [3, 4]. Subsequently, Barbu, Da Prato and Röckner [5] proved the well-posedness of strong solutions to the more general equation

   $$d\rho = \Delta \Phi(\rho)\, dt + \sigma(\rho) dW_t, \quad \text{on} \quad U \times (0,T],$$

   in arbitrary dimensions with homogeneous Dirichlet boundary data. Here, Φ satisfies more general monotonicity conditions[12] so that a wider class of porus media type non-linearities can be considered compared to Barbu et. al [2], and σ is Lipschitz.

1.3.2 Kinetic formulation

The kinetic formulation and the notion of kinetic solutions was first introduced in the deterministic partial differential equation (PDE) setting in the mid-nineties by

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[12] In particular no continuity is assumed for Φ, and no growth conditions are assumed at infinity. For instance, $\Phi(\xi) = \exp(a|\xi|^p)$ is permitted, which has exponential growth at $\pm\infty$.

Lions, Perthame and Tadmor [70]. The authors considered the kinetic formulation for multidimensional scalar conservation laws, which can be written in the form

$$\partial_t u + \nabla \cdot A(u) = 0, \quad \text{on} \quad \mathbb{R}^d \times (0,\infty), \tag{1.30}$$

where $A = (A_i)_{i=1}^d$ are $C^{2,\alpha}$-regular functions from $\mathbb{R}$ to $\mathbb{R}$ for some $\alpha > 0$. The kinetic formulation was based on entropy solutions of (1.30). They chose the term "kinetic equation" due to its analogy with the classical kinetic models such as Boltzmann or Vlasov models, see for instance Cercignani [13]. The kinetic formulation helped the authors establish uniqueness of solutions of equation (1.30).

Relevant extensions to degenerate parabolic-hyperbolic PDEs were given by Chen and Pertame [16], Bendahmane and Karlsen [7], and Karlsen and Riseboro [59]. In the most general setting, kinetic solution theory was developed for PDEs of the form

$$\partial_t u + \nabla \cdot (A(x,t,u)) = \nabla \cdot (B(u)\nabla u) + q(x,t,u), \quad \text{on} \quad \mathbb{R}^d \times (0,T],$$

with various assumptions integrability and growth assumptions on A, B and q.

Let us now comment on the use of the kinetic formulation in the stochastic setting. To the author's best knowledge, the first work extending the kinetic solution framework to the stochastic setting was in 2010 by Debussche and Vovelle [28], who extended the PDE results of Lions, Perthame and Tadmor [70] to the case of scalar conservation laws with stochastic forcing

$$du + \nabla \cdot A(u)\, dt = \sigma(u)\, dW_t, \quad \text{on} \quad \mathbb{T}^d \times (0,T],$$

where again W denotes a cylindrical Wiener process. The results hold for the same assumptions on A, and σ sufficiently regular[13]. The results were subsequently extended by Hofmanová [56] to the degenerate parabolic case

$$du + \nabla \cdot (A(u))\, dt = \nabla \cdot (B(x)\nabla u)\, dt + \sigma(u)\, dW_t, \quad \text{on} \quad \mathbb{T}^d \times (0,T] \tag{1.31}$$

and subsequently to the quasilinear case[14] by Debussche, Hofmanová and Vovelle [27].

1.3.3 Kinetic formulation applied to Dean–Kawasaki type equations

The well-posedness results of stochastic kinetic solutions of the generalised Dean–Kawasaki equation (1.2) contained in Chapters 3 and 4 primarily follow the techniques on the torus by Fehrman and Gess [44]. There, the authors also consider the same noise given by (1.3), whose spatial correlation is justified from the particle system point of view due to the natural correlation length given by the grid size of the particles[15].

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[13] The regularity of σ is characterised in terms of the regularity of the functions $\{\sigma(u)e_k\}_{k\in\mathbb{N}}$, where $u \in \mathbb{R}$ and $\{e_k\}_{k\in\mathbb{N}}$ are the spatial components of the cylindrical noise $W = \sum_{k\geq 1} e_k B^k$ for independent Brownian Motions $\{B^k\}_{k\in\mathbb{N}}$.

[14] The same equation as (1.31) where B now depends on the solution $B = B(u)$.

[15] If particles evolve on a lattice with grid size $\epsilon > 0$, then we would expect the noise to be constant on blocks of size $\epsilon/2$, not uncorrelated.

Even though the noise is sufficiently regular in space, the well-posedness of equation (1.2) is tricky in the case of the square root noise coefficient. Indeed, the Itô-to-Stratonovich conversion of (1.2), see equation (2.23) below, introduces a term with factor $(\sigma'(\rho))^2\nabla\rho$ that creates a singularity at zero. This motivates the use of stochastic kinetic solution theory.

We remark that the Stratonovich noise in equation (1.2) is essential to the analysis here, as the mentioned "bad" Itô-to-Stratonovich correction term cancels exactly with an Itô correction term when Itô's formula is applied, for instance see equations (2.28) and (2.29) below in the derivation of the kinetic equation, as well as equations (4.10) and (4.13) in the $L^2(U \times [0,t])$ a priori energy estimates.

It is worth mentioning the relationship between stochastic kinetic solutions and weak solutions in the context of the Dean–Kawasaki equation. When the noise coefficient σ is sufficiently regular so that we can make sense of weak solutions to (1.2), a weak solution is shown to be a stochastic kinetic solution, see Proposition 5.21 of Fehrman and Gess [41]. Conversely, under additional assumptions on σ, stochastic kinetic solutions are also weak solutions, see Corollary 5.31 of Fehrman and Gess [41].

Below we describe a sequence of works relating to kinetic solutions for the Dean–Kawasaki equation which all followed from the result of Fehrman and Gess [41].

1. Motivated by the particle system work of Dean [25], see in particular equation (1.24) above, Wang, Wu and Zhang [95] showed the well-posedness of stochastic kinetic solutions to the SPDE

   $$\partial_t\rho = \Delta\rho - \sqrt{\epsilon}\nabla\cdot(\sqrt{\rho}\circ\dot{\xi}^K) - \nabla\cdot(\rho(V*\rho)), \quad \text{on} \quad \mathbb{T}^d\times(0,T]. \tag{1.32}$$

   The interaction kernel V is assumed to satisfy the Ladyzhenskaya-Prodi-Serrin (LPS) condition, a regularity condition first studied in the context of Navier-Stokes equations by Prodi [84], Serrin [90], and Ladyzhenskaya [68] and applied to SDEs by Krylov and Röckner [67] and distributional dependent SDEs by Röckner and Zhang [87].

   The non-local nature of the convolution extends the well-posedness theory beyond the case of local transport $\nu(\rho)$ which is considered here and by Fehrman and Gess [41].

2. An extension by Fehrman and Gess [43] proves the well-posedness of the generalised Dean–Kawasaki equation with correlated noise (1.2) on the whole space $\mathbb{R}^d\times(0,\infty)$.

3. Recent work of Fehrman [37] extended the well-posedness theory to equations with non-stationary noise with square root non-linearities on a bounded domain with Neumann boundary data. Specifically, Fehrman [37] studies equations of the form

   $$\partial_t\rho = \nabla\cdot(a(x)\nabla\Phi(\rho)) - \nabla\cdot\left(\sqrt{\rho}\circ s(x)\,\dot{\xi}^F\right), \quad \text{on} \quad U\times(0,\infty), \tag{1.33}$$

   with no-flux boundary conditions on a smooth bounded domain $U\subset\mathbb{R}^d$, where the matrix $a := ss^t$ is spatially inhomogeneous and uniformly elliptic. On the level of particles, the no-flux boundary condition corresponds to reflection of the particles at the boundary and ensures that the equation preserves mass.

1.3.4 Central limit theorems

Let us now discuss literature in the direction of central limit theorems. Central limit theorems for stochastic heat equations with Lipschitz continuous noise coefficients were studied by by Huang, Nualart and Viitasaari [57], and Chen, Khoshnevisan, Nualart and Pu [18]. A central limit theorem for the stochastic wave equation with Lipschitz continuous noise coefficients was obtained by Delgado-Vences, Nualart and Zheng [29].

A central limit theorem for the generalised Dean–Kawasaki equation on the torus was proved by Clini and Fehrman [20]. Due to the more technical estimates used in the present work, we are able to handle the full range of fast diffusion and porus medium non-linearities $\Phi(\xi) = \xi^m$ for every $m \in (0, \infty)$. In [20] the authors can only handle the non-linearity for m sufficiently small due to an assumed bound on the growth of $|\Phi''(\xi)|$, see Assumption 2.8(ii) of [20].

A central limit theorem for SPDEs capturing fluctuations of the symmetric simple exclusion process was given by Dirr, Fehrman and Gess [30]. The authors study fluctuations of the SPDE

$$\partial_t \rho^\epsilon = \Delta \rho^\epsilon - \sqrt{\epsilon} \nabla \cdot \left(\sqrt{\rho^\epsilon (1 - \rho^\epsilon)} \circ \dot{\xi}^K \right).$$

The well-posedness of the above equation poses the additional difficulty that there are multiple points of singularity $\{\rho^\epsilon \approx 0\}$ and $\{\rho^\epsilon \approx 1\}$ On the level of kinetic solutions we require a renormalisation of the solution away from both of these sets. On the other hand, due to the fact that solutions are $[0, 1]$-valued, estimates in [30] are often more simple than in the present work.

1.3.5 Large deviations principles

Large deviations for reaction diffusion equations with additive white in time, coloured in space noise was shown by Cerrai and Freidlin [15]. Large deviations for stochastic porus media equations have been shown using exponential estimates and a generalized contraction principle by Röckner, Wang, and Wu [86] and more recently using the weak approach to large deviations by Zhang [100].

The weak approach to large deviations has also developed in the context of Markov chains and random walk models by Dupuis and Ellis [34], and for several systems including infinite dimensional Brownian noise in the more recent book by Budhiraja and Dupuis [10]. Motivated by the same works, Fehrman and Gess [42] prove a large deviations principle for the Dean–Kawasaki equation

$$\partial_t \rho = \Delta \Phi(\rho) - \sqrt{\epsilon} \nabla \cdot (\Phi^{1/2}(\rho) \circ \dot{\xi}^K)$$

on the torus, capturing the large deviation behaviour of the zero range process. This was extended to the whole space by the same authors in [43]. In follow up work, Wu and Zhang [98] prove the large deviations principle for equation (1.32) with non-local transport, again on the torus. For the simple exclusion process the corresponding large deviation result was given by Dirr, Fehrman and Gess [30].

The weak approach to large deviations was also used to study the large deviations of reaction–diffusion equations with an arbitrary polynomial non-linearity by Cerrai and Debussche [14], for stochastic Landau–Lifshitz equations on a bounded interval by Brzeźniak, Goldys, and Jegaraj [9], and for first-order scalar conservation laws perturbed by small multiplicative noise by Dong, Wu, Zhang and Zhang [33].

1.4 Motivation of boundary data

The results contained in this thesis follow several works that were formulated on the torus, for instance the well-posedness results of Fehrman and Gess [44], the central limit results by Clini [20], Fehrman and Gess [42], and Dirr, Fehrman and Gess [30], and the large deviations results of Fehrman and Gess [42].

For mathematicians and physicists, the torus is often a good starting point to prove properties of an equation due to its nice properties like boundedness, connectedness and periodic boundary conditions. In practice, when particle systems are used to model real life phenomena, it is more natural and appropriate to consider the system as evolving on a bounded domain. In this case, homogeneous Neumann boundary conditions are used to model reflection of particles at the boundary, and Dirichlet boundary conditions model absorption or injection of particles. For the Dirichlet boundary data considered here, if the density of particles around the boundary is lower than the boundary value, then the boundary data corresponds to injection of particles. On the other hand, if the density of particles around the boundary is higher than the boundary value, then the boundary condition corresponds to absorption of particles at the boundary.

Particle systems are commonly formulated on bounded domains, such as to model point vortices and biological populations. Below we give several examples motivating physical systems that can be modelled by the zero range particle process with Dirichlet boundary conditions.

1. The most basic example would be to consider the heat equation ($\sigma = \nu = 0$ in (1.2)) which models the heat of a liquid in a container. The Dirichlet boundary condition considered here corresponds to external fixed heat (thermal reservoir) being applied to the system at the boundary of the container.

2. The symmetric zero range process can be used to model vehicle traffic jams or pedestrian movement, see Kaupužs, Mahnke, and Harris [60], where particles represent individuals, moving between different road sections. The boundary data corresponds to individuals leaving or entering the area of interest.

3. In economics the zero range process can be used to model wealth flow, see Pinasco, Cartabia, and Saintier [80] and Cardoso, Iglesias, and Gonçalves [12]. Here the sites represent individuals and the particles represent wealth moving

from one agent to another. The boundary data can model wealth entering or exiting the system, though most wealth models in economics assume conservation of wealth.

1.5 Organisation and main contributions

In Chapter 2 we set up the problem. Firstly, in Section 2.1 we give the definition and assumptions on the nose ξ^F. In Section 2.2 we state the assumptions on the non-linear functions Φ, σ, ν as well as the assumptions on the boundary data $\bar{f}$ that are needed for the well-posedness. The assumptions on the boundary data are required for the a priori estimates that are needed to establish existence of solutions.

In Section 2.3 we subsequently derive the kinetic equation for the generalised Dean–Kawasaki equation, and define what it means to be kinetic solution. The setup and kinetic equation are analogous to Section 2 and 3 of Fehrman and Gess [41], the main difference being that we incorporate the boundary condition in point two of the definition of stochastic kinetic solution, Definition 2.3.6.

Section 2.4 is dedicated to the definitions and properties of convolution kernels and cutoff functions that will be useful in the remainder, in particular in the proof of uniqueness Theorem 3.2.2. Compared to the previous results on the torus, we need to define a new spatial cutoff function, see equation (2.43), that enables us to integrate by parts without picking up additional boundary terms. It is worthwhile to mention that the assumed C^2-regularity of the domain U arises so that we are able to differentiate the distance function that forms part of the definition of the spatial cutoff, as well as to apply Sobolev embedding theorems later on in the a priori estimates.

Chapter 3 is dedicated to the uniqueness of solutions to the generalised Dean–Kawasaki equation. In order to illustrate how to use the kinetic equation, in Section 3.1 we give a formal proof of uniqueness of kinetic solutions for the heat equation. In Section 3.2 the uniqueness of stochastic kinetic solutions of the generalised Dean–Kawaski equation is shown. The main novelty in the proof is in the analysis of the new terms arising when the gradient hits the spatial cutoff function.

Chapter 4 is dedicated to the existence of stochastic kinetic solutions of the generalised Dean–Kawasaki equation. In Section 4.1 we prove various a priori estimates of an appropriately regularised version of the generalised Dean–Kawasaki equation. An important novelty is that we introduce partial differential equations (PDEs) with carefully chosen Dirichlet boundary data, see for example Definition 4.1.1, that ensures that we are able to integrate by parts in the a priori estimates. Bounding the resulting terms leads to the assumptions on the boundary data $\bar{f}$ from Section 2.2. Another novelty compared to the torus is that we have to use techniques from PDE theory such as the Sobolev extension theorem, see Chapter 5.4 of Evans [35], to prove higher order spatial regularity for the SPDE.

In Section 4.2 we prove an entropy estimate for the equation. In Section 4.3 a localised version of the entropy estimate is used to prove a bound for the decay of the kinetic measure at zero, which is a statement that is needed for the uniqueness proof but is proven here for convenience. On the torus the statement is not difficult to prove, but on a bounded domain the arguments do not hold due to a lack of control of boundary terms that appear when integrating by parts. Hence the estimate in Section 4.3 is new and is necessary.

With the a priori estimates in hand, the existence of solutions follows from tightness and compactness arguments, and is presented in Section 4.4. The results in this section are analogous to the torus, and are just provided for completeness.

Section 5 is dedicated to the proof of a central limit theorem for the generalised Dean–Kawasaki SPDE (1.5). In Section 5.1 we define, state the assumptions on, and give an example of the truncated noise ξ^K. Compared to the torus, the noise ξ^K must satisfy homogeneous Dirichlet boundary conditions at the boundary. We prove quantitative law of large number estimates for a regularised version of equation (1.5). We are able to prove a new estimate in Proposition 5.1.20 which allows us to subsequently prove the central limit theorem under more general conditions on the non-linear functions Φ and ν than was known on the torus.

In Section 5.2 we prove a quantitative central limit theorem for the regularised equation. One of the novelties of the section is in the proof of uniqueness of strong solutions to the linearised SPDE (1.10). The extension of the central limit theorem to the singular generalised Dean–Kawasaki equation is given in Section 5.3.

Chapter 6 is devoted to the proof of the large deviations principle. The large deviations is based on the abstract result on the variational representation of infinite dimensional Brownian by Budhiraja, Dupuis, and Maroulas [11]. In Section 6.2 we provide a proof idea of the well-posedness of the parabolic-hyperbolic PDE that appears in the rate function. The lack of $L^p(U \times [0,T])$-estimates for the PDE complicates the proof of entropy estimates on the bounded domain.

Conditions for the large deviations to hold are proven in Section 6.3, 6.5 and 6.6. The uniformity of the large deviations principle is with respect to bounded subsets of the initial data, and in Section 6.4 we prove a negative result that illustrates that the uniformity can not be extended to bounded subsets of the boundary data.

Chapter 2

Setup and Kinetic formulation

The aim of this chapter is to set up the problem by providing a relevant mathematical background for our problem and introducing a lot of the key tools and objects that will be needed in the sequel.

In Section 2.1 we define the noise ξ^F as well as state the relevant assumptions on the noise that are needed for the well-posedness. In Section 2.2 we provide the assumptions on the non-linear functions Φ, σ and ν that are needed for the well-posedness of the equation. Separately we will comment on the necessary assumptions on the boundary data $\bar{f}$. In Section 2.3 we derive the kinetic formulation for the generalised Dean–Kawasaki SPDE. Along the way we define the kinetic function and the kinetic measure. We conclude with the definition of a kinetic solution of the generalised Dean–Kawasaki equation. Finally, in Section 2.4 we define the convolution kernels and cutoff functions that are needed to formalise the uniqueness. In particular, we define a new cutoff function in space that is needed on the bounded domain.

2.1 Definition of the noise

The aim of this section is to define and state the assumptions on the noise ξ^F which appears in equation (1.2). In Definition 2.1.1 we define the noise. The relevant assumptions on the noise are given in Assumption 2.1.2, and relevant comments on the assumptions are provided in Remarks 2.1.3 and 2.1.4. The definition and assumptions on the noise are analogous to those introduced in Section 2 of Fehrman and Gess [44].

Definition 2.1.1 (The noise ξ^F). *Let $F := \{f_k : U \to \mathbb{R}\}_{k\in\mathbb{N}}$ be a sequence of continuously differentiable functions and $\{B^k : [0,T] \to \mathbb{R}^d\}_{k\in\mathbb{N}}$ a sequence of independent, d-dimensional Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. The noise ξ^F, superscripted by F to denote dependence on $\{f_k\}_{k\in\mathbb{N}}$, is defined pointwise by*

$$\xi^F : U \times [0,T] \to \mathbb{R}^d, \qquad \xi^F(x,t) := \sum_{k=1}^{\infty} f_k(x) B_t^k.$$

For ease of notation we define three quantities related to the spatial component of the noise,

$$F_1 : U \to \mathbb{R} \quad \text{defined by} \quad F_1(x) := \sum_{k=1}^{\infty} f_k^2(x), \tag{2.1}$$

$$F_2 : U \to \mathbb{R}^d \quad \text{defined by} \quad F_2(x) := \sum_{k=1}^{\infty} f_k(x)\nabla f_k(x) = \frac{1}{2}\sum_{k=1}^{\infty} \nabla f_k^2(x), \tag{2.2}$$

$$F_3 : U \to \mathbb{R} \quad \text{defined by} \quad F_3(x) := \sum_{k=1}^{\infty} |\nabla f_k(x)|^2. \tag{2.3}$$

We make the following assumptions on the noise.

Assumption 2.1.2 (Assumption on noise). *Suppose that F_1 is continuous on $\bar{U}$, F_2 is continuous and differentiable on $\bar{U}$, and F_3 is continuous on U. That is,*

$$F_1 \in C(\bar{U}), \quad F_2 \in C^1(\bar{U}; \mathbb{R}^d), \quad F_3 \in C(U).$$

Furthermore assume that $\nabla \cdot F_2$ is bounded on U.

By Hölder's inequality, the boundedness of F_1 and F_3 imply the partial sums of F_2 are absolutely convergent. The fact that F_1 is bounded implies that the noise ξ^F is almost surely finite and $L^2(U)$-bounded.

Remark 2.1.3 (Boundedness of F_2 and $\nabla \cdot F_2$). *For every $x \in U$, $F_2(x) \in \mathbb{R}^d$ is a vector. Hence, when we write the supremum norm (or in general any $L^p(U;\mathbb{R}^d)$-norm)*

$$\|F_2\|_{L^\infty(U;\mathbb{R}^d)} := \sup_{x\in U} |F_2(x)|,$$

we will make the choice that $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. However, by the equivalence of norms on $\mathbb{R}^d$, it follows that the growth behaviour and boundedness are the same regardless of the choice of norm on $\mathbb{R}^d$.

Further, by direct computation we have $\nabla \cdot F_2 = F_3 + \sum_{k=1}^{\infty} f_k \Delta f_k$. Therefore, by the triangle inequality and the Cauchy-Schwarz inequality, it holds that

$$\|\nabla \cdot F_2\|_{L^\infty(U)} \leq \|F_3\|_{L^\infty(U)} + \|F_1\|_{L^\infty(U)}^{1/2} \left\| \sum_{k=1}^{\infty} (\Delta f_k)^2 \right\|_{L^\infty(U)}^{1/2}.$$

Hence the boundedness of $\nabla \cdot F_2$ in Assumption 2.1.2 can be translated into an assumption on the uniform boundedness of the sum

$$\sum_{k=1}^{\infty} (\Delta f_k)^2.$$

Remark 2.1.4 (Spatial correlation of the noise ξ^F). *The assumptions on F_1, F_2, F_3 in Assumption 2.1.2 imply that the noise ξ^F is spatially correlated.*

Indeed, for any $x, y \in U$ and $t, s \in [0,T]$, using independence of the Brownian motions, we compute the covariance

$$\mathbb{E}\big[\xi^F(x,t)\cdot\xi^F(y,s)\big] = \sum_{k=1}^{\infty} f_k(x)f_k(y)\,\mathbb{E}[B_t^k \cdot B_s^k] = (t\wedge s)\sum_{k=1}^{\infty} f_k(x)f_k(y).$$

Hence the spatial covariance kernel is given by

$$C(x,y) := \sum_{k=1}^{\infty} f_k(x)f_k(y).$$

In particular, $F_1(x) = C(x,x)$ is the pointwise variance of the noise, while F_2 and F_3 encode first and second order spatial regularity of the kernel. The assumption that F_1 is continuous and F_3 is finite ensures that $C(x,y)$ is well-defined and sufficiently regular, so that $\xi^F(x,t)$ and $\xi^F(y,t)$ remain correlated for nearby points $x \neq y$.

By contrast, space-time white noise would formally correspond to the covariance kernel $C(x,y) = \delta(x-y)$, which is singular on the diagonal and cannot be represented by a sequence $\{f_k\}$ satisfying the above boundedness and regularity assumptions. For example, on the torus $U = \mathbb{T}^d$, the truncated Fourier approximation of white noise,

$$\sum_{k\in\mathbb{Z}^d:|k|\le K} \sqrt{2}\big(\sin(k\cdot x)\,dB_t^k + \cos(k\cdot x)\,dW_t^k\big),$$

satisfies

$$F_1^K(x) = 2\#\{k : |k| \le K\}, \qquad F_3^K(x) = 2\sum_{k:|k|\le K} |k|^2,$$

which both diverge as $K \to \infty$. This reflects the fact that white noise has infinite pointwise variance and no spatial regularity, and is therefore excluded by Assumption 2.1.2.

Thus, the imposed conditions enforce that the noise admits a regular, continuous covariance kernel, which is precisely the signature of spatial correlation.

2.2 Assumptions on the non-linear functions and boundary data

In the paper of the author [82], the assumptions needed to prove existence, entropy estimates and uniqueness are given separately in their respective sections. Whilst this allows the reader to distinguish between which assumptions are needed for which results, many of the assumptions overlap and it is more difficult to ascertain whether there exists any non-linear functions Φ, σ, ν satisfying all of the assumptions. In Assumption 2.2.1 the assumptions on Φ, σ and ν are all gathered. This is followed by relevant remarks in Remarks 2.2.2-2.2.6.

Assumption 2.2.9 gives the relevant assumptions needed on $\bar{f}$ for well-posedness. As mentioned previously, the assumptions allow us to consider a wide class of non-negative boundary data, which is based on certain a priori estimates for the solutions.

We give the definition of the $H^1(\partial U)$-space that is needed for the assumption in Definition 2.2.7 and we conclude with comments on the assumption in Proposition 2.2.11 and Remark 2.2.10.

The assumptions on the non-linear coefficients combines Assumptions 3.1, 4.2, 4.20 and 4.28 of Popat [82].

Assumption 2.2.1 (Assumptions on non-linear functions Φ, σ, ν for well-posedness)**.** *Suppose that the non-linear functions $\Phi, \sigma \in C([0,\infty))$ and $\nu \in C([0,\infty);\mathbb{R}^d)$ satisfy the following assumptions:*

1. *Local regularity: We have*

$$\Phi, \sigma \in C^{1,1}_{loc}([0,\infty)) \quad \textit{and} \quad \nu \in C^1_{loc}([0,\infty);\mathbb{R}^d).$$

2. *The function Φ is strictly increasing and is zero at zero: We have*

$$\Phi(0) = 0 \quad \textit{with} \quad \Phi' > 0 \quad \textit{on} \quad (0,\infty).$$

3. *Growth conditions on Φ and Φ': There exists constants $m \in (0,\infty), c \in (0,\infty)$ such that for every $\xi \in [0,\infty)$*

$$\Phi(\xi) \leq c(1+\xi^m), \quad \textit{and} \quad \Phi'(\xi) \leq c(1+\xi+\Phi(\xi)). \tag{2.4}$$

Let Φ satisfy points 1 and 2 above. Define $\Theta_{\Phi,2} \in C([0,\infty)) \cap C^1_{loc}(0,\infty)$ to be the unique function satisfying

$$\Theta_{\Phi,2}(0) = 0, \quad \Theta'_{\Phi,2}(\xi) = (\Phi'(\xi))^{1/2}. \tag{2.5}$$

Then, either for constants $c \in (0,\infty)$ and $\theta \in [0,1/2]$, we have for every $\xi \in (0,\infty)$ that

$$(\Theta'_{\Phi,2}(\xi))^{-1} = \Phi'(\xi)^{-1/2} \leq c\xi^\theta, \tag{2.6}$$

or there exist constants $c \in (0,\infty)$, $q \in [1,\infty)$ such that for every $\xi, \eta \in [0,\infty)$

$$|\xi - \eta|^q \leq c|\Theta_{\Phi,2}(\xi) - \Theta_{\Phi,2}(\eta)|^2. \tag{2.7}$$

Furthermore, assume that there exists a constants $c_1 \in (0,\infty), c_2 \in [0,\infty)$ such that for every $\xi \in (0,\infty)$ and $m \in (0,\infty)$ as in (2.4), we have

$$\Theta_{\Phi,2}(\xi) \geq c_1\xi^{\frac{m+1}{2}} - c_2. \tag{2.8}$$

4. *Growth conditions on σ, ν and other non-linear functions[1]: There exists a constant $c \in (0,\infty)$ such that for every $\xi \in [0,\infty)$,*

$$|\sigma(\xi)| \leq c\Phi^{1/2}(\xi), \tag{2.9}$$

---

[1] For the well-posedness we just require the more general growth condition on σ, that there exists a constant $c \in (0,\infty)$ such that $\sigma(\xi) \leq c(1+\xi+\Phi(\xi))$. The more strict condition is needed for the entropy estimates, see Section 4.2.

$$|\nu(\xi)| + |\sigma(\xi)\sigma'(\xi)|^2 \leq c(1+\xi+\Phi(\xi)). \tag{2.10}$$

Furthermore, for each $\delta \in (0,1)$ *there exists a constant* $c_\delta \in (0,\infty)$ *such that for every* $\xi \in (\delta,\infty)$*,*

$$\frac{(\sigma'(\xi))^4}{\Phi'(\xi)} \leq c_\delta(1+\xi+\Phi(\xi)).$$

5. *At least linear decay of* σ^2 *at zero: There exists a constant* $c \in (0,\infty)$ *such that*

   $$\limsup_{\xi\to 0^+} \frac{\sigma^2(\xi)}{\xi} \leq c.$$

   In particular this implies that $\sigma(0) = 0$*.*

6. *Behaviour of* $\sigma\sigma'$ *at zero: We have that either* $\sigma\sigma' \in C([0,\infty))$ *with* $(\sigma\sigma')(0) = 0$*, or that* $\nabla \cdot F_2 = 0$ *for* F_2 *defined in* (2.2)*.*

7. *Growth condition on non-linear function of* σ*: At least one of the following three conditions holds:*

   - *The boundary data is constant.*
   - *We have that* $F_2 = 0$*.*
   - *The function* $\Psi_\sigma : [0,\infty) \to \mathbb{R}$ *defined by*

     $$\Psi_\sigma(0) = 0, \quad \Psi_\sigma'(\xi) = [\sigma'(\xi)]^2 \tag{2.11}$$

     is well defined, and satisfies the growth condition that there exists a constant $c \in (0,\infty)$ *such that for every* $\xi \in [0,\infty)$*,*

     $$\Psi_\sigma(\xi) \leq c(1+\xi+\Phi(\xi)). \tag{2.12}$$

8. *Regularity of oscillations of of* σ^2 *and* ν *at infinity: There exists a constant* $c \in [1,\infty)$ *such that for every* $\xi \in [0,\infty)$

   $$\sup_{\xi'\in[0,\xi]} \sigma^2(\xi') \leq c(1+\xi+\sigma^2(\xi)), \quad \text{and} \quad \sup_{\xi'\in[0,\xi]} |\nu(\xi')| \leq c(1+\xi+|\nu(\xi)|). \tag{2.13}$$

9. *Entropy assumption: We have* $\log(\Phi)$ *is locally integrable on* $[0,\infty)$*.*

Remark 2.2.2. *We emphasise that Assumption 2.2.1 includes the cases of interest,* $\Phi(\xi) = \xi^m$ *for the full range* $m \in (0,\infty)$ *and* $\sigma(\xi) = \Phi^{1/2}(\xi)$*, including the critical square root.*

As an important aside, we note that in the paper of the author, Popat [82], the bound on the growth of ν *in* (2.10) *is replaced by the same bound on* ν^2*. That is*

$$|\nu(\xi)|^2 \leq c(1+\xi+\Phi(\xi)). \tag{2.14}$$

The assumption was used in a priori energy estimates to obtain bounds of the form

$$\int_U \nabla h \cdot \nu(\rho)\, dx \leq \|\nabla h\|_{L^2(U;\mathbb{R}^d)} \|\nu(\rho)\|_{L^2(U;\mathbb{R}^d)}, \tag{2.15}$$

for solutions to different harmonic PDEs denoted above by h*. Due to the fact that* h *is harmonic in* U*, first term on the right hand side can be controlled by norms of the boundary data*[2] *of* h*, whereas the bound* (2.14) *was used to control the second term.*

However, we realise that a bound of the form (2.14) *does not allow us to consider* ν *of the form* $\nu(\xi) = \Phi(\xi)$*.*

In this thesis we therefore bound terms such as the left hand side of (2.15) *by requiring higher order integrability of the PDE, namely*

$$\nabla h \in L^\infty(U;\mathbb{R}^d). \tag{2.16}$$

In Proposition 2.2.11 we give criteria on solutions of the harmonic PDE h *that ensures that* (2.16) *holds true.*

Remark 2.2.3. *We refer to Remark 4.2 of Fehrman and Gess [41] for a detailed discussion on the point 8 above. The assumption is satisfied in the case that the functions* σ^2 *and* ν *are increasing, are uniformly continuous, or grow linearly at infinity. The assumption is more general than any of the above three examples and essentially amounts to a restriction on the growth of the magnitude of oscillations, rather than frequency of oscillations at infinity. This is illustrated by the fact that for every* $m, p \in [1, \infty)$*, the following function is permitted*

$$\sigma^2(\xi) = \xi^m + \xi \sin(\xi^p).$$

Remark 2.2.4. *Equations* (2.6) *and* (2.7) *in point 3 enable us to consider* $\Phi(\xi) = \xi^m$ *for every* $m \in (0, \infty)$*.*

- *If* $m < 1$ *then* $\Phi'(\xi)^{-1/2} = m^{-1/2}\xi^{\frac{1-m}{2}}$ *so satisfies* (2.6)*.*
- *If* $m \geq 1$ *then by the following remark, Remark 2.2.5, we have that* $c|\Theta_{\Phi,2}(\xi) - \Theta_{\Phi,2}(\eta)|^2 = cm|\xi^{\frac{m+1}{2}} - \eta^{\frac{m+1}{2}}|^2$ *so satisfies* (2.7) *with* $q = m + 1$*.*

The only additional assumptions compared to the torus is the upper bound of equation (2.8) and point 7, which we comment on now.

Remark 2.2.5 (Growth assumption on $\Theta_{\Phi,2}$)**.** *The lower bound on the growth of* $\Theta_{\Phi,2}$ *in point 3 of the above assumption is essential for obtaining* $L^k(U \times [0,t])$*-estimates of the solution for* $k \in (0, m+1)$ *in Proposition 4.1.6 below. In the model case* $\Phi'(\xi) = m\xi^{m-1}$ *and so the assumption is satisfied since*

$$\Theta_{\Phi,2}(\xi) = m^{1/2} \int_0^\xi \eta^{(m-1)/2}\, d\eta = \frac{2m^{1/2}}{m+1}\xi^{(m+1)/2}.$$

---

[2] Testing the PDE against the solution and integrating by parts allows us to precisely estimate this quantity,

$$\int_U |\nabla h(x)|^2\, dx = \int_{\partial U} h(x) \frac{\partial h}{\partial \hat{\eta}}(x)\, dS(x).$$

Remark 2.2.6 (Point 7 of Assumption 2.2.1)**.** *In the case* $\Phi(\xi) = \xi^m$ *and* $\sigma(\xi) = \Phi^{1/2}(\xi) = \xi^{m/2}$ *for* $m \neq 1$ *we have that*

$$\Psi_\sigma(\xi) = \frac{m^2}{4(m-1)}\xi^{m-1},$$

so satisfies the bound (2.12) *if* $m > 1$.
If $m < 1$*, then the exponent in* ξ^{m-1} *is negative, and so* Ψ_σ *has a singularity at zero. This holds also in the case* $m = 1$*, for which we have that* $\Psi_\sigma(\xi) = \frac{1}{4}\log(\xi)$*. Hence, in the regime when* $m \leq 1$*, the bound* (2.12) *is not satisfied and we require one of the other points in the assumption to hold true.*

In Popat [82] there are also several assumptions on the boundary data, see Assumption 4.2 points 9-11, Definition 4.6 and Assumption 4.9, Assumption 4.20, Assumption 4.23 and Assumption 4.28. The assumptions come from the various energy estimates in Chapter 4. The assumptions encompass all non-negative constant functions including zero and all smooth functions bounded away from zero[3]. All of the assumptions are automatically satisfied in the case of constant boundary data, for example in Chapter 5.

Since we would like to impose as few conditions on the boundary data as possible, based on a priori estimates we a priori only require the boundary data to be $H^1(\partial U)$-regular. We define this space now. Throughout the thesis we denote functions defined on the boundary ∂U with overbars.

Definition 2.2.7 (The space $H^1(\partial U)$)**.** *The space of real valued functions defined on* ∂U *with finite* $H^1(\partial U)$*-Sobolev norm is*

$$H^1(\partial U) := \{\bar{h} : \partial U \to \mathbb{R} : \|\bar{h}\|_{L^2(\partial U)} + \|\nabla_{\partial U}\bar{h}\|_{L^2(\partial U;\mathbb{R}^d)} < \infty\}, \tag{2.17}$$

where $\nabla_{\partial U}\bar{h}$ *denotes the tangential derivative of* $\bar{h}$*. That is, the gradient of* $\bar{h}$ *in directions tangent to the boundary* ∂U*. The* $L^2(\partial U)$*-norm from equation* (2.17) *is defined by*

$$\|\bar{h}\|^2_{L^2(\partial U)} := \int_{\partial U} |\bar{h}(x)|^2 \, dS(x), \tag{2.18}$$

where dS *denotes the* $(d-1)$*-dimensional surface measure on the boundary* ∂U*.*

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[3] This is a subtle point which we briefly explain with an example. To account for the boundary condition, if we want an $L^2(U \times [0,t])$-estimate for the solution ρ, we formally apply Itô's formula to $(\rho - h)^2$, where h is the solution of a PDE with the same boundary data as ρ. Since the Dean–Kawasaki equation is in divergence form, the only contribution of h in the estimate comes through ∇h after an integration by parts. If the boundary data is constant, then the solution to the PDE h is given by a constant, and so has no contribution to the estimate since $\nabla h = 0$. For the entropy estimate in the model case we require regularity of a PDE with logarithm of the solution as the boundary data. This prevents us from considering non-constant boundary data unless it is bounded away from zero.

Remark 2.2.8. *On page 176 of Fabes [36] the $H^1(\partial U)$-norm is rigorously defined on a C^1-regular domain in terms of a local coordinate system of the boundary ∂U. Briefly, the definition involves considering a covering of ∂U by a local co-ordinate chart and measuring the $L^2(\partial U)$-norms of the function and its derivative in these charts. The norm is well-defined and independent of the choice of chart, see Seeley [89].*

The $H^1(\partial U)$-norm is then defined by

$$\|\bar{h}\|^2_{H^1(\partial U)} := \|\bar{h}\|^2_{L^2(\partial U)} + \|\nabla_{\partial U}\bar{h}\|^2_{L^2(\partial U;\mathbb{R}^d)}.$$

Assumption 2.2.9 (Assumptions on boundary data for well-posedness)**.** *Let Φ, σ, ν denote the non-linear functions in equation (1.2), and recall that $\bar{f}$ denotes the boundary data of various equations (via Φ of the solution).*

1. *$L^p(\partial U)$-boundary regularity for non-linear functions: Define $\Theta_\nu : [0,\infty) \to \mathbb{R}^d$ to be the anti-derivative of ν, defined element-wise $\Theta_\nu = (\Theta_{\nu,i})_{i=1,\dots,d}$ by*

   $$\Theta_{\nu,i}(0) = 0, \quad \Theta'_{\nu,i}(\xi) = \nu_i(\xi).$$

   We assume that

   $$\begin{cases} \sigma^2(\Phi^{-1}(\bar{f})) \in L^1(\partial U), \\ \Theta_\nu(\Phi^{-1}(\bar{f})) \in L^1(\partial U;\mathbb{R}^d). \end{cases}$$

   For Ψ_σ as defined in (2.11), we assume that either the boundary data $\bar{f}$ is constant, or

   $$\begin{cases} \bar{f} \in L^2(\partial U), \\ \Phi^{-1}(\bar{f}) \in L^2(\partial U), \\ \Psi_\sigma(\Phi^{-1}(\bar{f})) \in L^2(\partial U). \end{cases}$$

2. *Higher order spatial regularity of non-linear functions: Let the $H^1(\partial U)$-norm be defined as in Definition 2.2.7, and let $\Theta_{\Phi,2}$ be as defined in point 3 of Assumption 2.2.1. Either the boundary data $\bar{f}$ is constant, or we have that*

   $$\Phi^{-1}(\bar{f}) \in H^1(\partial U).$$

   Further assume that either the boundary data is constant or for the unique solution to Laplace's equation $h_{\Phi^{-1}(\bar{f})}$

   $$\begin{cases} -\Delta h_{\Phi^{-1}(\bar{f})} = 0 & \text{on } U, \\ h_{\Phi^{-1}(\bar{f})} = \Phi^{-1}(\bar{f}) & \text{on } \partial U, \end{cases}$$

   we have

   $$\nabla h_{\Phi^{-1}(\bar{f})} \in L^\infty(U;\mathbb{R}^d), \quad \Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})}) \in H^1(U). \tag{2.19}$$

The following assumptions are needed for the entropy estimates in Section 4.2, and are also needed only we don't have constant boundary data.

3. *$L^p(\partial U)$-entropy assumptions: Define the unique function $\Theta_{\Phi,\sigma} : [0,\infty) \to \mathbb{R}$ by*

$$\Theta_{\Phi,\sigma}(0) = 0, \quad \Theta'_{\Phi,\sigma}(\xi) = \frac{\Phi'(\xi)\sigma'(\xi)\sigma(\xi)}{\Phi(\xi)}.$$

   Assume that either $\bar{f}$ is constant, or

$$\Theta_{\Phi,\sigma}(\Phi^{-1}(\bar{f})) \in L^1(\partial U).$$

   For the vector valued functions $\tilde{\Theta}_\nu, \Theta_{\Phi,\nu} : [0,\infty) \to \mathbb{R}^d$ defined element wise $\tilde{\Theta}_\nu = (\tilde{\Theta}_{\nu,i})_{i=1,\ldots,d}$, $\Theta_{\Phi,\nu} = (\Theta_{\Phi,\nu,i})_{i=1,\ldots,d}$ by

$$\begin{cases} \tilde{\Theta}_{\nu,i}(0) = 0, & \tilde{\Theta}'_{\nu,i}(\xi) = \nu_i(\xi)/\xi, \\ \Theta_{\Phi,\nu,i}(0) = 0, & \Theta'_{\Phi,\nu,i}(\xi) = \frac{\Phi'(\xi)\nu_i(\xi)}{\Phi(\xi)}. \end{cases}$$

   We assume that

$$\tilde{\Theta}_\nu(\Phi^{-1}(\bar{f}) \wedge 1) \in L^1(\partial U;\mathbb{R}^d) \quad \Theta_{\Phi,\nu}(\Phi^{-1}(\bar{f})) \in L^1(\partial U;\mathbb{R}^d).$$

   Either $F_2 = 0$, or the unique function Θ_σ defined by

$$\Theta_\sigma(0) = 0, \quad \Theta'_\sigma(\xi) = \frac{\sigma(\xi)\sigma'(\xi)}{\xi}$$

   satisfies

$$\Theta_\sigma(\Phi^{-1}(\bar{f}) \wedge 1) \in L^1(\partial U).$$

4. *Higher order spatial regularity of the logarithm: Either the boundary data $\bar{f}$ is constant or we have that*

$$\log(\bar{f}) \in H^1(\partial U). \tag{2.20}$$

5. *For the unique solution $h_{log(\bar{f})}$ to Laplace's equation with boundary data $\log(\bar{f})$ as defined in Definition 4.2.1, we have that*

$$\nabla h_{log(\bar{f})} \in L^\infty(U;\mathbb{R}^d). \tag{2.21}$$

In the above and throughout the thesis, we will use $h_\cdot$ to denote solutions of Laplace's equation with Dirichlet boundary conditions. We will use a subscript in the notation to denote the boundary data of the equation.

We now make some comments on the above assumption.

We begin with a comment on the assumptions that include the phrase "either the boundary data $\bar{f}$ is constant, or…". To the best of the authors knowledge, a "comparison principle" has not been established for the Dean–Kawasaki equation. That is, if we have two solutions ρ^1, ρ^2 of the Dean–Kawasaki equation (1.2) with the same initial data ρ_0 but different boundary data $\Phi(\rho^1)|_{\partial U} = \bar{f}^1, \Phi(\rho^2)|_{\partial U} = \bar{f}^2$, then does it hold that

$$\bar{f}^1(x) \le \bar{f}^2(x)\ \forall x \in \partial U \implies \rho^1(x,t) \le \rho^2(x,t),\ \forall (x,t) \in U \times [0,T]?$$

Comparison principles in the context of SPDEs have been studied, for example see the works of Yang and Zhou [99], Chen and Huang [17], Wang, Yan and Zhou [96] and references therein. If such a comparison principle were to hold true, then we could replace the assumption of constant boundary data in the above assumption and in point 7 of Assumption 2.2.1 by the assumption of bounded boundary data, which would allow us to handle a larger class of boundary conditions based on the choice of the non-linearities Φ, σ and ν.

Remark 2.2.10. *The integrability assumption on the logarithm* (2.20) *forms the restriction on the boundary data that we are able to handle. If the boundary data is not constant, then we require it to be uniformly bounded away from zero. That is to say, we can not handle boundary data* $\bar{f}$ *that takes the value zero on some part of the boundary, and is positive on other parts.*

We finally comment on the $L^\infty(U;\mathbb{R}^d)$-bounds assumption for the gradients of the PDEs in equations (2.19) and (2.21). These bounds are motivated by the discussion in Remark 2.2.2. The following result shows that in our setup such a bound is expected.

Proposition 2.2.11 (Gradient regularity for solutions to the Laplace equation)**.** *Let* $h_{\bar{h}} : U \to \mathbb{R}$ *be the solution to the following Dirichlet boundary value problem for Laplace's equation,*

$$\begin{cases} -\Delta h_{\bar{h}} = 0 & \text{on } U, \\ h_{\bar{h}} = \bar{h} & \text{on } \partial U. \end{cases} \tag{2.22}$$

If $\bar{h} : \partial U \to \mathbb{R}$ *is smooth, then*

$$\nabla h_{\bar{h}} \in L^\infty(U;\mathbb{R}^d).$$

Proof. Since U is C^2-regular and $\bar{h} \in C^2(\partial U)$ do to the choice of boundary conditions, elliptic regularity theory (see for example Chapter 6 of Gilbarg and Trudinger [52]) implies that the unique solution $h_{\bar{h}}$ to (2.22) satisfies

$$h_{\bar{h}} \in C^2(\bar{U}).$$

In particular, $\nabla h_{\bar{h}} \in C^1(\bar{U};\mathbb{R}^d)$, and hence $\nabla h_{\bar{h}} \in L^\infty(U;\mathbb{R}^d)$.

□

The above gradient estimate is only needed when the boundary data is not constant, in which case Remark 2.2.10 tells us that the data is a smooth function bounded away from zero. It follows that $\Phi^{-1}(\bar{f})$ and $\log(\bar{f})$ are smooth on ∂U, and so (2.19) and (2.21) are satisfied.

2.3 Kinetic formulation of the SPDE

The aim of this section is to define what it means to be a stochastic kinetic solution of the Dean–Kawasaki equation (1.2). After converting equation (1.2) into Itô form and noting the singularity that arises, we derive the kinetic equation. Along the way, we will define the kinetic function in Definition 2.3.4 and the kinetic measure in Definition 2.3.5. The definition of stochastic kinetic solution is then given in Definition 2.3.6. We conclude with Remarks 2.3.7 and 2.3.8 on the definition.

We begin by re-writing equation (1.2) using Itô noise. By Definition 2.1.1 of the noise and the linearity of the divergence operator, we have

$$\partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\circ\dot{\xi}^F + \nu(\rho)) = \Delta\Phi(\rho) - \nabla\cdot\nu(\rho) - \sum_{k=1}^{\infty}\nabla\cdot(\sigma(\rho)f_k\circ dB_t^k).$$

Denoting $F_{\sigma,k}(\xi,x) := \sigma(\xi)f_k(x)$ for fixed $x\in U$, the Itô-to-Stratonovich conversion formula, see Chapter 3.3 of Oksendal [79], the chain rule and product rule give formally that

$$\begin{aligned}\partial_t\rho &= \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\dot{\xi}^F + \nu(\rho)) + \frac{1}{2}\sum_{k=1}^{\infty}\nabla\cdot\left(\frac{\partial F_{\sigma,k}(\rho,x)}{\partial B^k}\right)\\ &= \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\dot{\xi}^F + \nu(\rho)) + \frac{1}{2}\sum_{k=1}^{\infty}\nabla\cdot\left(f_k\sigma'(\rho)\frac{\partial\rho}{\partial B^k}\right)\\ &= \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\dot{\xi}^F + \nu(\rho)) + \frac{1}{2}\sum_{k=1}^{\infty}\nabla\cdot(f_k\sigma'(\rho)\nabla(f_k\sigma(\rho)))\\ &= \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)\dot{\xi}^F + \nu(\rho)) + \frac{1}{2}\nabla\cdot(F_1[\sigma'(\rho)]^2\nabla\rho + \sigma'(\rho)\sigma(\rho)F_2), \end{aligned}\tag{2.23}$$

which we will equivalently sometimes write in the formal SDE notation as

$$d\rho_t = \Delta\Phi(\rho)\,dt - \nabla\cdot(\sigma(\rho)\,d\xi^F + \nu(\rho)\,dt) + \frac{1}{2}\nabla\cdot(F_1[\sigma'(\rho)]^2\nabla\rho + \sigma'(\rho)\sigma(\rho)F_2)\,dt.$$

Remark 2.3.1. *In the model case $\sigma(\rho) = \rho^{1/2}$, the first correction term arising in the Itô equation* (2.23) *is*

$$\frac{1}{8}\nabla\cdot(F_1\rho^{-1}\nabla\rho) = \frac{1}{8}\nabla\cdot(F_1\nabla\log(\rho)).$$

If the solution ρ approaches it's zero set at any time, the above term is a singular. In fact, until recently it was not even clear how we can define the notion of a weak solution since we do not know if $\log(\rho)$ *is locally integrable. An estimate illustrating the integrability of the logarithm on the level of the approximate equation was shown recently by Fehrman in Proposition 2.14 of Fehrman [37].*

The remark below illustrates how to interpret integrals involving the divergence of the Itô noise in (2.23). We use it when interpreting the kinetic equation (2.40) below.

Remark 2.3.2. *For $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $g\in L^2(\Omega\times[0,T];L^2(U))$ and $h\in L^2(\Omega\times[0,T];H^1(U))$ and any $t\in[0,T]$ we define*

$$\int_0^t\int_U g\nabla\cdot(h\,d\xi^F) := \sum_{k=1}^{\infty}\left(\int_0^t\int_U gf_k\nabla h\cdot dB_s^k + \int_0^t\int_U gh\nabla f_k\cdot dB_s^k\right). \tag{2.24}$$

Remark 2.3.3 (Notation, omitting integrand arguments and integrators)**.** *As we did in equation* (2.24)*, for brevity, when clear from the context, we will not always write the arguments of the integrands (e.g.* $g(s,x)$ *might just be written as* g*) and we will sometimes omit the integrators (e.g.* $dx\,ds$*). Furthermore, when integrating against d-dimensional Brownian motions, when we do write the integrators, for arbitrary vector field* $F : U \times [0,T] \to \mathbb{R}^d$*, we write*

$$\int_0^t \int_U F(x,s) \cdot dB_s^k \, dx,$$

to highlight the fact that the noise is multidimensional and so we take the dot product of the noise with F*. This abuses notation, but we consider it clear that the "*dB_s*" term corresponds to the time integral and the "*dx*" corresponds to the spatial integral.*

We are now in a position to derive the kinetic equation for equation (2.23). In Chapter 6 on the large deviations, instead of (2.23), we will need to use kinetic solution theory for the more general equation

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho)\dot{\xi}^F) - \nabla \cdot (\sigma(\rho)g) - \nabla \cdot \nu(\rho) \\ + \frac{1}{2}\nabla \cdot \left(F_1(\sigma'(\rho))^2\nabla\rho + \sigma(\rho)\sigma'(\rho)F_2\right), \quad (2.25)$$

where $g \in L^2(U \times [0,T]; \mathbb{R}^d)$ is an irregular control. Hence we will derive the kinetic equation for this more general equation, but for the purpose of the subsequent two chapters, the reader should just think of the case $g \equiv 0$.

Suppose that we were interested in how functions of the solution behaved. For smooth function $S : \mathbb{R} \to \mathbb{R}$, applying Itô's formula gives that formally[4]

$$\partial_t S(\rho) = S'(\rho)\Delta\Phi(\rho) - S'(\rho)\nabla \cdot (\sigma(\rho)\dot{\xi}^F) - S'(\rho)\nabla \cdot (\sigma(\rho)g) - S'(\rho)\nabla \cdot \nu(\rho) \\ + \frac{1}{2}S'(\rho)\nabla \cdot \left(F_1^K(\sigma'(\rho))^2\nabla\rho + \sigma(\rho)\sigma'(\rho)F_2^K\right) + \frac{1}{2}S''(\rho)\sum_{k=1}^{K} |\nabla(\sigma(\rho)f_k)|^2. \quad (2.26)$$

The final term is the Itô correction term, which is be expanded using the product and chain rule to give

$$\sum_{k=1}^{\infty} |\nabla(\sigma(\rho)f_k)|^2 = F_1|\nabla\sigma(\rho)|^2 + 2\sigma(\rho)\sigma'(\rho)\nabla\rho \cdot F_2 + \sigma^2(\rho)F_3. \quad (2.27)$$

We wish to observe cancellation between the Itô-to-Stratonovich conversion terms in (2.26) and the Itô correction term above (2.27), particularly in the potentially singular term involving $(\sigma'(\rho))^2$. However, we notice that the conversion terms involve $S'(\rho)$, whilst the correction terms involve the second derivative $S''(\rho)$. In order to compare

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[4] The computation is only formal because the equation does not have enough regularity to apply Itô's formula. To get around this, we would need to add a regularisation $\alpha\Delta\rho$ to the right hand side of the equation. So in general the identities would be true with equalities replaced by inequalities.

these terms, we re-write the conversion terms by bringing $S'(\rho)$ into the derivative. Doing this to the first and third terms on the right hand side of equation (2.26) as well as the first term in the second line gives

$$\begin{aligned}
\partial_t S(\rho) = {} & \nabla \cdot (S'(\rho)\Phi'(\rho)\nabla\rho) - S''(\rho)\nabla\rho \cdot \nabla\Phi(\rho) - S'(\rho)\nabla \cdot (\sigma(\rho)\dot{\xi}^F) \\
& - \nabla \cdot (S'(\rho)\sigma(\rho)g) + S''(\rho)\sigma(\rho)\nabla\rho \cdot g - S'(\rho)\nabla \cdot \nu(\rho) \\
& + \frac{1}{2}\nabla \cdot \left(F_1(\sigma'(\rho))^2 S'(\rho)\nabla\rho + \sigma(\rho)\sigma'(\rho)S'(\rho)F_2\right) \\
& - \frac{1}{2}\left(S''(\rho)\nabla\rho \cdot F_1(\sigma'(\rho))^2\nabla\rho + S''(\rho)\nabla\rho \cdot \sigma(\rho)\sigma'(\rho)F_2\right) \\
& + \frac{1}{2}S''(\rho)\left(F_1|\nabla\sigma(\rho)|^2 + 2\sigma(\rho)\sigma'(\rho)\nabla\rho \cdot F_2 + \sigma^2(\rho)F_3\right).
\end{aligned}$$

After the cancellation of terms in the final two lines, the equation simplifies to

$$\begin{aligned}
\partial_t S(\rho) = {} & \nabla \cdot (S'(\rho)\Phi'(\rho)\nabla\rho) - S''(\rho)\nabla\rho \cdot \nabla\Phi(\rho) - S'(\rho)\sqrt{\epsilon}\nabla \cdot (\sigma(\rho)\dot{\xi}^F) \\
& - \nabla \cdot (S'(\rho)\sigma(\rho)g) + S''(\rho)\sigma(\rho)\nabla\rho \cdot g - S'(\rho)\nabla \cdot \nu(\rho) \\
& + \frac{1}{2}\nabla \cdot \left(F_1(\sigma'(\rho))^2 S'(\rho)\nabla\rho + \sigma(\rho)\sigma'(\rho)S'(\rho)F_2\right) \\
& + \frac{1}{2}S''(\rho)\left(\sigma(\rho)\sigma'(\rho)\nabla\rho \cdot F_2 + \sigma^2(\rho)F_3\right). \qquad (2.28)
\end{aligned}$$

Rigorously we interpret the above equation by integrating in space and time against a test function $\psi \in C_c^\infty(U)$ and subsequently integrating by parts. This gives for every fixed $t \in [0,T]$, that

$$\begin{aligned}
\int_U \psi(x)S(\rho)\Big|_{s=0}^{t} = {} & -\int_0^t\int_U \nabla\psi \cdot (S'(\rho)\Phi'(\rho)\nabla\rho) - \int_0^t\int_U \psi S''(\rho)\nabla\rho \cdot \nabla\Phi(\rho) \\
& - \int_0^t\int_U \psi S'(\rho)\nabla \cdot (\sigma(\rho)\, d\xi^F) + \int_0^t\int_U \nabla\psi \cdot (S'(\rho)\sigma(\rho)g) \\
& + \int_0^t\int_U \psi S''(\rho)\sigma(\rho)\nabla\rho \cdot g - \int_0^t\int_U \psi S'(\rho)\nabla \cdot \nu(\rho) \\
& - \frac{1}{2}\int_0^t\int_U \nabla\psi \cdot \left(F_1(\sigma'(\rho))^2 S'(\rho)\nabla\rho + \sigma(\rho)\sigma'(\rho)S'(\rho)F_2\right) \\
& + \frac{1}{2}\int_0^t\int_U \psi S''(\rho)\left(\sigma(\rho)\sigma'(\rho)\nabla\rho \cdot F_2 + \sigma^2(\rho)F_3\right). \qquad (2.29)
\end{aligned}$$

We want to re-write equation (2.29) in terms of the kinetic function.

Definition 2.3.4 (Kinetic function)**.** *Given a non-negative solution ρ of the controlled SPDE* (2.25)*, the kinetic function $\chi : U \times \mathbb{R} \times [0,T] \to \{0,1\}$ of ρ is defined as*

$$\chi(x,\xi,t) := \mathbb{1}_{0<\xi<\rho(x,t)}.$$

The kinetic function of (2.25) can also be viewed as a map from $\chi : [0,\infty) \times \mathbb{R} \to \{0,1\}$, defined by

$$\chi(\rho(x,t),\xi) := \chi(x,\xi,t).$$

In Lions, Perthame and Tadmor [70] the kinetic function is called the velocity distribution or velocity profile since there they view $\xi \in \mathbb{R}$ as a velocity variable. Here we will adopt the same nomenclature and refer to ξ as the velocity variable. By analogy with the theory of gases, χ can be called a pseudo-Maxwellian.

To re-write the equation in terms of the kinetic function, we realise that if $S(0) = 0$, we have the identity

$$S(\rho(x,t)) = \int_{\mathbb{R}} S'(\xi)\chi(x,\xi,t)\, d\xi.$$

This can be substituted into the left hand side of (2.29), and for the right hand side we formally introduce artificial integrals in the velocity variable by adding a Dirac delta term $\delta_0(\xi - \rho)$, which gives

$$\begin{aligned}
&\int_{\mathbb{R}}\int_U \psi(x)S'(\xi)\chi(x,\xi,t)\, dx\, d\xi \bigg|_{s=0}^{t} = -\int_{\mathbb{R}}\int_0^t\int_U \delta_0(\xi-\rho)S'(\xi)\Phi'(\xi)\nabla\psi\cdot\nabla\rho \\
&-\int_{\mathbb{R}}\int_0^t\int_U \delta_0(\xi-\rho)\Phi'(\xi)\psi S''(\xi)|\nabla\rho|^2 - \int_{\mathbb{R}}\int_0^t\int_U \delta_0(\xi-\rho)\psi S'(\xi)\nabla\cdot\left(\sigma(\rho)\, d\xi^F\right) \\
&+\int_{\mathbb{R}}\int_0^t\int_U \delta_0(\xi-\rho)S'(\xi)\sigma(\xi)\nabla\psi\cdot g + \int_{\mathbb{R}}\int_0^t\int_U \delta_0(\xi-\rho)\psi S''(\xi)\sigma(\xi)\nabla\rho\cdot g \\
&-\int_{\mathbb{R}}\int_0^t\int_U \delta_0(\xi-\rho)\psi S'(\xi)\nabla\cdot\nu(\rho) \\
&-\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_U \delta_0(\xi-\rho)\nabla\psi\cdot\left(F_1(\sigma'(\xi))^2S'(\xi)\nabla\rho + \sigma(\xi)\sigma'(\xi)S'(\xi)F_2\right) \\
&+\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_U \delta_0(\xi-\rho)\psi S''(\xi)\left(\sigma(\xi)\sigma'(\xi)\nabla\rho\cdot F_2 + \sigma^2(\xi)F_3\right).
\end{aligned}\tag{2.30}$$

Finally, equation (2.30) can be re-written by factoring terms involving $\psi(x)S'(\xi)$,

$$\begin{aligned}
&\int_{\mathbb{R}}\int_U \psi(x)S'(\xi)\chi(x,\xi,t)\, dx\, d\xi \bigg|_{s=0}^{t} = -\int_{\mathbb{R}}\int_0^t\int_U \nabla\left(\psi S'(\xi)\right)\cdot\delta_0(\xi-\rho)\Phi'(\xi)\nabla\rho \\
&-\int_{\mathbb{R}}\int_0^t\int_U \partial_\xi\left(\psi S'(\xi)\right)\delta_0(\xi-\rho)\Phi'(\xi)|\nabla\rho|^2 - \int_{\mathbb{R}}\int_0^t\int_U \psi S'(\xi)\delta_0(\xi-\rho)\nabla\cdot\left(\sigma(\rho)\, d\xi^F\right) \\
&+\int_{\mathbb{R}}\int_0^t\int_U \nabla\left(\psi S'(\xi)\right)\cdot\delta_0(\xi-\rho)\sigma(\xi)g \\
&+\int_{\mathbb{R}}\int_0^t\int_U \partial_\xi(\psi S'(\xi))\delta_0(\xi-\rho)\sigma(\xi)\nabla\rho\cdot g - \int_{\mathbb{R}}\int_0^t\int_U \psi S'(\xi)\delta_0(\xi-\rho)\nabla\cdot\nu(\rho) \\
&-\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_U \nabla\left(\psi S'(\xi)\right)\cdot\delta_0(\xi-\rho)\left(F_1(\sigma'(\xi))^2\nabla\rho + \sigma(\xi)\sigma'(\xi)F_2\right) \\
&+\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_U \partial_\xi\left(\psi S'(\xi)\right)\delta_0(\xi-\rho)\left(\sigma(\xi)\sigma'(\xi)\nabla\rho\cdot F_2 + \sigma^2(\xi)F_3\right).
\end{aligned}\tag{2.31}$$

Integrating by parts and using the density of functions of the type $\psi(x)S'(\xi)$ in the space $C_c^\infty(U \times (0,\infty))$ allows us to conclude that the kinetic function χ of the SPDE

(2.25) distributionally[5] solves

$$\begin{aligned}\partial_t\chi &= \nabla\cdot(\delta_0(\xi-\rho)\Phi'(\xi)\nabla\rho) + \partial_\xi(\delta_0(\xi-\rho)\Phi'(\xi)|\nabla\rho|^2) - \delta_0(\xi-\rho)\nabla\cdot(\sigma(\rho)\dot{\xi}^F)\\ &\quad - \nabla\cdot(\delta_0(\xi-\rho)\sigma(\xi)g) - \partial_\xi(\delta_0(\xi-\rho)\sigma(\xi)\nabla\rho\cdot g) - \delta_0(\xi-\rho)\nabla\cdot\nu(\rho)\\ &\quad + \frac{1}{2}\nabla(\delta_0(\xi-\rho)\left(F_1(\sigma'(\xi))^2\nabla\rho + \sigma(\xi)\sigma'(\xi)F_2\right))\\ &\quad - \frac{1}{2}\partial_\xi(\delta_0(\xi-\rho)\left(\sigma(\xi)\sigma'(\xi)\nabla\rho\cdot F_2 + \sigma^2(\xi)F_3\right)). \end{aligned} \tag{2.32}$$

However, we mentioned previously that the above computation is formal because the solution ρ did not have enough regularity to enable us to apply Itô's formula to $S(\rho)$. This is resolved by adding a regularisation term to the original equation. For $n \in \mathbb{N}$, instead of (2.25), we consider the regularised equation

$$\begin{aligned}\partial_t\rho_n = \Delta\Phi(\rho_n) + \frac{1}{n}\Delta\rho_n - \nabla\cdot(\sigma(\rho_n)\dot{\xi}^F) - \nabla\cdot(\sigma(\rho_n)g) - \nabla\cdot\nu(\rho_n)&\\ + \frac{1}{2}\nabla\cdot\left(F_1(\sigma'(\rho_n))^2\nabla\rho_n + \sigma(\rho_n)\sigma'(\rho_n)F_2\right).&\end{aligned}$$

Repeating the above analysis would then lead to two additional terms in (2.32),

$$\frac{1}{n}\nabla\cdot(\delta_0(\xi-\rho_n)\nabla\rho) + \frac{1}{n}\partial_\xi(\delta_0(\xi-\rho_n)|\nabla\rho_n|^2), \tag{2.33}$$

which can easily be seen from the first two terms of (2.32) by considering the choice $\Phi(\xi) = \xi$, for which $\Phi'(\xi) = 1$. At the end of the analysis, in the resulting weak formulation we wish to take the limit as $n \to \infty$. We note that $L^2(U \times [0,t])$-energy estimates give the existence of a constant $c \in (0,\infty)$ such that

$$\frac{1}{n}\int_0^t\int_U |\nabla\rho_n|^2 \leq c$$

almost surely along subsequences in n, see for example Proposition 4.1.8 below. This estimate alongside Cauchy–Schwarz inequality gives us that the first term of (2.33) would vanish as $n \to \infty$, since in the weak formulation, we have for test function $\psi \in C_c^\infty(U \times (0,\infty))$ that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{R}}\int_0^t\int_U \psi(x,\xi)\nabla\cdot(\delta_0(\xi-\rho_n)\nabla\rho_n)\,dx\,dt\,d\xi\\ &\qquad = -\lim_{n\to\infty}\frac{1}{n}\int_0^t\int_U(\nabla_x\psi)(x,\rho_n)\cdot\nabla\rho_n\,dx\,dt \leq \lim_{n\to\infty} c\frac{1}{n}\sqrt{n} = 0, \end{aligned} \tag{2.34}$$

where we used the notation $(\nabla_x\psi)(x,\rho) = (\nabla_x\psi)(x,\xi)|_{\xi=\rho}$ to mean that we are only taking the gradient in the first component, rather than the full gradient $\nabla\psi(x,\rho)$.

---

[5] When we write distributionally, we mean an equality that is satisfied when both sides are integrated against test functions.

However, repeating the same computation to the second term in (2.33) results in a non-negative term appearing on the right hand side of (2.32),

$$\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{R}}\int_0^t\int_U \psi(x,\xi)\partial_\xi(\delta_0(\xi-\rho_n)|\nabla\rho_n|^2)\,dx\,dt\,d\xi$$
$$= -\lim_{n\to\infty}\frac{1}{n}\int_0^t\int_U (\partial_\xi\psi)(x,\xi)|_{\xi=\rho_n}|\nabla\rho_n|^2\,dx\,dt\,d\xi.$$

Consequently the equality in (2.32) should actually be replaced with an inequality "$\leq$". On the kinetic level, this entropy inequality is quantified exactly by the kinetic measure, see Section 2 of Chen and Perthame [16], and the derivation of the kinetic measures dq^ϵ in the proof of Theorem 6.6.1. The measure encapsulates contribution of the first term on the right hand side of (2.32) as well as the non-trivial limit described above, and so it is a non-negative measure q on $U\times(0,\infty)\times[0,T]$ satisfying that in the sense of measures

$$\delta_0(\xi-\rho)\Phi'(\xi)|\nabla\rho|^2 \leq q. \tag{2.35}$$

Definition 2.3.5 (Kinetic measure). *Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P})$ be the filtered probability space from Definition 2.1.1. A kinetic measure q is a map from Ω to the space of non-negative, locally finite measures on $U\times(0,\infty)\times[0,T]$ such that for every $\psi\in C_c^\infty(U\times(0,\infty))$ we have*

$$(\omega,t)\in(\Omega,[0,T])\to\int_{\mathbb{R}}\int_0^t\int_U\psi(x,\xi)\,q(dx,d\xi,dt)(\omega) =: \int_{\mathbb{R}}\int_0^t\int_U\psi(x,\xi)\,dq(x,\xi,t)(\omega)$$

is $(\mathcal{F}_t)_{t\geq0}$-predictable.

The resulting weak formulation of (2.32) forms the basis of a stochastic kinetic solution. To capture the boundary data of the solution we introduce the harmonic PDE $h_{\bar f}$, defined by

$$\begin{cases}-\Delta h_{\bar f}=0 & \text{on } U,\\ h_{\bar f}=\bar f & \text{on } \partial U.\end{cases} \tag{2.36}$$

Definition 2.3.6 (Stochastic kinetic solution of the generalised Dean–Kawasaki equation (2.23)). *Let $\rho_0\in L^1(\Omega;L^1(U))$ be a non-negative $\mathcal{F}_0$-measurable initial condition. A stochastic kinetic solution of the generalised Dean–Kawasaki equation* (2.23) *is a non-negative, almost surely continuous $L^1(U)$-valued $(\mathcal{F}_t)_{t\geq0}$-predictable process $\rho\in L^1(\Omega\times[0,T];L^1(U))$ that satisfies*

1. *Integrability of flux: We have*

$$\sigma(\rho)\in L^2(\Omega;L^2(U\times[0,T])) \quad \text{and} \quad \nu(\rho)\in L^1(\Omega;L^1(U\times[0,T];\mathbb{R}^d)).$$

2. *Boundary condition, local regularity of solution: For $h_{\bar f}$ defined in* (2.36), *for each $k\in\mathbb{N}$ we have*

$$((\Phi(\rho)\wedge k)\vee 1/k) - \big((h_{\bar f}\wedge k)\vee 1/k\big)\in L^2(\Omega;L^2([0,T];H_0^1(U))). \tag{2.37}$$

Furthermore, there exists a kinetic measure q that satisfies:

3. *Regularity: Almost surely, in the sense of non-negative measures,*

$$\delta_0(\xi-\rho)\Phi'(\xi)|\nabla\rho|^2 \leq q \text{ on } U\times(0,\infty)\times[0,T]. \tag{2.38}$$

4. *Vanishing at infinity: We have that*

$$\lim_{M\to\infty}\mathbb{E}\left[q(U\times[M,M+1]\times[0,T])\right]=0. \tag{2.39}$$

5. *The kinetic equation: For every $\psi\in C_c^\infty(U\times(0,\infty))$ and every $t\in[0,T]$, almost surely,*

$$\begin{aligned}
&\int_{\mathbb{R}}\int_U \chi(x,\xi,t)\psi(x,\xi)\,dx\,d\xi = \int_{\mathbb{R}}\int_U \chi(x,\xi,t=0)\psi(x,\xi)\,dx\,d\xi \\
&-\int_0^t\int_U \left(\Phi'(\rho)\nabla\rho+\frac{1}{2}F_1[\sigma'(\rho)]^2\nabla\rho+\frac{1}{2}\sigma'(\rho)\sigma(\rho)F_2\right)\cdot(\nabla_x\psi)(x,\xi)|_{\xi=\rho}\,dx\,ds \\
&-\int_{\mathbb{R}}\int_0^t\int_U d_\xi\psi(x,\xi)\,dq+\frac{1}{2}\int_0^t\int_U \left(\sigma'(\rho)\sigma(\rho)\nabla\rho\cdot F_2+\sigma(\rho)^2F_3\right)(\partial_\xi\psi)(x,\xi)|_{\xi=\rho}\,dx\,ds \\
&-\int_0^t\int_U \psi(x,\rho)\nabla\cdot(\sigma(\rho)\,d\xi^F)\,dx-\int_0^t\int_U \psi(x,\rho)\nabla\cdot\nu(\rho)\,dx\,ds.
\end{aligned} \tag{2.40}$$

We conclude the section with a few remarks on the above definition.

Remark 2.3.7 (Point two of Definition 2.3.6)**.** *By standard properties of solutions to the Laplace equation, $h_{\bar{f}}\in L^2([0,T];H^1(U))$, so the second point of Definition 2.3.6 is just a condition on the integrability of $\Phi(\rho)$. Since we assume Φ is strictly increasing, the integrability implies that locally $\rho\in H^1(U)$.*

Furthermore, the fact that the regularity condition holds only when solution is localised away from its zero and infinity set is necessary. The localisation away from the zero set is due to the singularity at zero mentioned in Remark 2.3.1. Additionally, we do not have stable estimates for $\nabla\rho$ when ρ is large when considering initial data that is not $L^2(U)$-integrable, which necessitates the localisation away from infinity.

Remark 2.3.8. *In the kinetic equation it is essential that we integrate against test functions ψ that are compactly supported in $U\times(0,\infty)$. Again noting Remark 2.3.1, the compact support in the velocity variable $\xi\in(0,\infty)$ ensures that equation* (2.40) *needs to hold only away from the zero set of the solution, so avoids potential singularities arising from the logarithm.*

The compact support in space allows us to integrate by parts without worrying about boundary terms.

2.4 Convolution kernels and cutoff functions

Due to the compact support of test functions in the kinetic equation (2.40), we will need to define cutoff functions that will form part of the test functions. To use the kinetic function itself as part of the test functions, we will need to smooth it, so need to define convolution kernels. The definitions are given in Definition 2.4.2. We conclude with Lemma 2.4.3, which is a technical result about the derivatives of the spatial cutoff function.

To define the spatial cutoff, we will first need to define interior regions of the domain U.

Definition 2.4.1 (Interior region of the domain U)**.** *Let $d(x, \partial U)$ denote the usual minimum Euclidean distance from a point to a set. Define the interior regions*

$$U_\gamma := \{x \in U : d(x, \partial U) \geq \gamma\} \subset U, \qquad \partial U_\gamma := \{x \in U : d(x, \partial U) = \gamma\}.$$

Define the real valued positive constant $\gamma_U \in \mathbb{R}_+$ to be the largest distance away from the boundary such that every point in the interior of the domain at most γ_U away from the boundary has a unique closest point on the boundary. That is,

$$\gamma_U := max\{\tilde{\gamma} : \forall x \in \partial U_{\tilde{\gamma}},\ argmin\, d(x, \partial U)\ is\ a\ singleton\}. \tag{2.41}$$

For a non-trivial C^2-regular domain, γ_U is strictly positive, see page 153 of Foote [48].

Definition 2.4.2 (Convolution kernels and cutoff functions)**.**

1. *Convolution kernel in space and velocity: For every $\epsilon, \delta \in (0,1)$ let $\kappa_d^\epsilon : U \to [0,\infty)$ and $\kappa_1^\delta : \mathbb{R} \to [0,\infty)$ be standard convolution kernels/ mollifiers of scale ϵ and δ on U and $\mathbb{R}$ respectively. That is to say, let $\kappa_d \in C_c^\infty(\mathbb{R}^d), \kappa_1 \in C_c^\infty(\mathbb{R})$ be non-negative and integrate to one. For $\epsilon, \delta \in (0,1)$ define*

   $$\kappa_d^\epsilon(x) = \frac{1}{\epsilon^d}\kappa_d\left(\frac{x}{\epsilon}\right),\ \kappa_1^\delta(\xi) = \frac{1}{\delta}\kappa_1\left(\frac{\xi}{\delta}\right). \tag{2.42}$$

   To define the convolution on U, we take any integrable function f and $x \in U_{2\epsilon}$, and define

   $$(f * \kappa_d^\epsilon)(x) := \int_U f(y)\kappa_d^\epsilon(x-y)\, dy.$$

   Let $\kappa^{\epsilon,\delta}$ be defined by the product

   $$\kappa^{\epsilon,\delta}(x,y,\xi,\eta) := \kappa_d^\epsilon(x-y)\kappa_1^\delta(\xi-\eta), \qquad (x-y,\xi,\eta) \in U \times \mathbb{R}^2.$$

2. *Cutoff of small velocity ξ: For every $\beta \in (0,1)$ let $\phi_\beta : \mathbb{R} \to [0,1]$ be the unique non-decreasing piecewise linear function that satisfies*

   $$\phi_\beta(\xi) = 1\ if\ \xi \geq \beta, \quad \phi_\beta(\xi) = 0\ if\ \xi \leq \beta/2, \quad \phi_\beta'(\xi) = \frac{2}{\beta}\mathbb{1}_{\beta/2 \leq \xi \leq \beta}.$$

3. *Cutoff of large velocity ξ: For every $M \in \mathbb{N}$ let $\zeta_M : \mathbb{R} \to [0,1]$ be the unique non-increasing piecewise linear function that satisfies*

   $$\zeta_M(\xi) = 1\ if\ \xi \leq M, \quad \zeta_M(\xi) = 0\ if\ \xi \geq M+1, \quad \zeta_M'(\xi) = -\mathbb{1}_{M \leq \xi \leq M+1}.$$

4. *Spatial cutoff around boundary: The spatial cutoff function is such that it takes the value* 1 *in the interior of the domain,* 0 *along the boundary, and linearly interpolates between the two. Explicitly, for* $\gamma \in (0, \gamma_U)$ *we define the function*

$$\iota_\gamma(x) := \frac{d(x, \partial U) \wedge \gamma}{\gamma} = \begin{cases} 1, & \textit{if}\, d(x, \partial U) > \gamma, \\ \gamma^{-1} d(x, \partial U), & \textit{if}\, 0 \leq d(x, \partial U) \leq \gamma. \end{cases} \tag{2.43}$$

We will use that we can approximate the cutoff functions by smooth approximations, and that we can approximate ι_γ by a compactly supported function. An explicit compactly supported approximation is given in Definition 3.3 of the work the first author and Wu [83], where one considers for $0 < \gamma' < \gamma < \gamma_U$ the function

$$\iota_{\gamma,\gamma'}(x) := \begin{cases} 1, & \text{if}\, d(x, \partial U) > \gamma, \\ (\gamma - \gamma')^{-1} d(x, \partial U_{\gamma'}), & \text{if}\, \gamma' \leq d(x, \partial U) \leq \gamma, \\ 0, & \text{if}\, 0 \leq d(x, \partial U) \leq \gamma'. \end{cases}$$

In this way we may abuse notation and describe ι_γ itself as being compactly supported in U.

We will also need to establish how to define the gradient of the spatial cutoff, stated below as a lemma without proof.

Lemma 2.4.3 (Derivative of spatial cutoff)**.** *To define the spatial derivative of the function* ι_γ*, we will differentiate the distance function appearing in the definition of the spatial cutoff* (2.43)*. Looking at the definition of the cutoff* (2.43)*, we only want to differentiate the distance function for* $x \in U \setminus U_\gamma$*, so it follows that we only need to assume this property for points* x *sufficiently close to the boundary.*

The distance function is differentiable if and only if for every x *we can find a unique closest point* $x^* := \Pi_{\partial U}(x)$ *on the boundary to* x*. The region where this is true is precisely quantified by* γ_U *as in* (2.41)*, which is the reason for defining* ι_γ *on* $\gamma \in (0, \gamma_U)$*. In this range, letting* v_x *denote the inward pointing unit normal at the boundary to point* $x \in U$*, with* x^* *as above, the first derivative of the spatial cutoff is given by*

$$\nabla \iota_\gamma(x) = \gamma^{-1} \frac{x - x^*}{|x - x^*|} \mathbb{1}_{U \setminus U_\gamma}(x) =: \gamma^{-1} v_x \mathbb{1}_{U \setminus U_\gamma}(x).$$

In particular, this implies that the size of the first derivative is of the order γ^{-1}*,*

$$|\nabla \iota_\gamma(x)| = \gamma^{-1} \mathbb{1}_{U \setminus U_\gamma}(x).$$

Chapter 3

Uniqueness

The goal of this chapter is to prove that stochastic kinetic solutions of the generalised Dean–Kawasaki equation are unique. To illustrate the technique, in Section 3.1 we give a formal proof of uniqueness of stochastic kinetic solutions of the heat equation. In Section 3.2 we will rigorously prove the uniqueness of solutions to the generalised Dean–Kawasaki equation.

3.1 Formal uniqueness proof for the heat equation

To give the main idea on how to work with the kinetic function, we give a formal proof of uniqueness of kinetic solutions for the heat equation $\partial_t\rho = \Delta\rho$ on a bounded domain. One of the objectives will be to illustrate how to handle new singular terms that appear due to using the spatial cutoff as part of the test function.

The distributional kinetic equation for the heat equation is obtained by setting $\sigma = \nu = g = 0, \Phi(\xi) = \xi$ in (2.32),

$$\partial_t\chi = \nabla\cdot(\delta_0(\xi-\rho)\nabla\rho) + \partial_\xi(\delta_0(\xi-\rho)|\nabla\rho|^2). \tag{3.1}$$

That is, for every $\psi \in C_c^\infty(U\times(0,\infty))$ and every $t\in[0,T]$, we have that

$$\begin{aligned}\int_{\mathbb{R}}\int_U \chi(x,\xi,t)\psi(x,\xi)\,dx\,d\xi &= \int_{\mathbb{R}}\int_U \chi(x,\xi,t=0)\psi(x,\xi)\,dx\,d\xi\\ -\int_0^t\int_U \nabla\rho(x,s)\cdot\nabla\psi(x,\xi)|_{\xi=\rho}\,dx\,ds &- \int_0^t\int_U \partial_\xi\psi(x,\xi)|_{\xi=\rho}|\nabla\rho(x,s)|^2\,dx\,ds.\end{aligned} \tag{3.2}$$

The uniqueness proof is based on the identity that if ρ^1 and ρ^2 are two stochastic kinetic solutions with kinetic functions χ^1,χ^2 respectively, then

$$|\rho^1(x,t)-\rho^2(x,t)| = \int_{\mathbb{R}}|\chi^1(x,\xi,t)-\chi^2(x,\xi,t)|^2\,d\xi.$$

This just follows from the properties of indicator functions. To control the right hand side, we utilise the identity

$$|\chi^1-\chi^2|^2 = \chi^1+\chi^2-2\chi^1\chi^2. \tag{3.3}$$

It therefore follows that we can quantify the change in the $L^1(U)$-difference of of the two solutions in time by considering

$$\partial_t \int_U |\rho^1 - \rho^2|\, dx = \partial_t \int_{\mathbb{R}} \int_U \chi^1 \, dx\, d\xi + \partial_t \int_{\mathbb{R}} \int_U \chi^2 \, dx\, d\xi - 2\, \partial_t \int_{\mathbb{R}} \int_U \chi^1 \chi^2 \, dx\, d\xi. \tag{3.4}$$

We look to analyse the contribution of each of the terms on the right hand side. On the bounded domain the main novelty comes from the new spatial cutoff, so we aim to highlight the contribution of this term. For the first two terms on the right hand side we pick the test function $\psi = \iota_\gamma$ in the kinetic equation (3.2), which is only formal since it is not compactly supported in the velocity variable. We get from (3.2) and the fact that $\psi = \iota_\gamma$ is independent of ξ, that for $i = 1, 2$,

$$\partial_t \int_{\mathbb{R}} \int_U \chi^i \iota_\gamma \, dx\, d\xi = - \int_0^t \int_U \nabla \rho^i \cdot \nabla \iota_\gamma \, dx\, ds. \tag{3.5}$$

The presence of ι_γ on the left hand side will not be an issue, we will take the $\gamma \to 0$ limit and use the fact that ι_γ converges pointwise to 1. For the final term of (3.4), we have using the product rule that

$$\partial_t \int_{\mathbb{R}} \int_U \chi^1 \chi^2 \, dx\, d\xi = \int_{\mathbb{R}} \int_U (\chi^1\, \partial_t \chi^2 + \chi^2\, \partial_t \chi^1)\, dx\, d\xi. \tag{3.6}$$

For the first term on the right hand side of the above expression, we use the test function $\psi = \chi^1 \iota_\gamma$ in (3.2) to get

$$\begin{aligned}&\int_{\mathbb{R}} \int_U \chi^1 \iota_\gamma \, d\chi^2 \, dx\, d\xi \\ &\qquad = - \int_0^t \int_U \nabla \rho^2 \cdot \nabla (\chi^1 \iota_\gamma)|_{\xi = \rho^2} \, dx\, ds - \int_0^t \int_U \iota_\gamma \partial_\xi \chi^1|_{\xi=\rho^2} |\nabla \rho^2|^2 \, dx\, ds.\end{aligned}$$

Using the product rule for the first term on the right hand side and the distributional identities

$$\partial_\xi \chi^i = \delta_0(\xi) - \delta_0(\xi - \rho^i), \quad \nabla_x \chi^i = \delta_0(\xi - \rho^i) \nabla \rho^i, \tag{3.7}$$

we observe the cancellation

$$\begin{aligned}\int_{\mathbb{R}} \int_U \chi^1 \iota_\gamma \, d\chi^2 \, dx\, d\xi &= - \int_0^t \int_U \chi^1|_{\xi=\rho^2} \nabla \rho^2 \cdot \nabla \iota_\gamma \, dx\, ds \\ - \int_0^t \int_U \iota_\gamma |\nabla \rho^2|^2 \delta_0(\rho^2 - \rho^1)\, dx\, ds &- \int_0^t \int_U \iota_\gamma (\delta_0(\rho^2) - \delta_0(\rho^2 - \rho^1)) |\nabla \rho^2|^2 \, dx\, ds, \\ &= - \int_0^t \int_U \chi^1|_{\xi=\rho^2} \nabla \rho^2 \cdot \nabla \iota_\gamma \, dx\, ds - \int_0^t \int_U \iota_\gamma \delta_0(\rho^2) |\nabla \rho^2|^2 \, dx\, ds.\end{aligned}$$

We get an analogous expression for the final term of (3.6), with ρ^1 and ρ^2 interchanged. Putting everything together, we get the decomposition

$$\partial_t \int_U |\rho^1 - \rho^2|\, dx = I_t^{err} + I_t^{cut},$$

with error term defined by

$$I_t^{err} := 2\int_0^t \int_U \iota_\gamma \delta_0(\rho^2)|\nabla \rho^2|^2 \, dx \, ds + 2\int_0^t \int_U \iota_\gamma \delta_0(\rho^1)|\nabla \rho^1|^2 \, dx \, ds$$

and cutoff term involving terms with the derivative of the spatial cutoff function,

$$\begin{aligned} I_t^{cut} := -\int_0^t \int_U (\nabla \rho^1 + \nabla \rho^2) \cdot \nabla \iota_\gamma + 2\int_0^t \int_U \chi^1|_{\xi=\rho^2} \nabla \rho^2 \cdot \nabla \iota_\gamma \, dx \, ds \\ + 2\int_0^t \int_U \chi^2|_{\xi=\rho^1} \nabla \rho^1 \cdot \nabla \iota_\gamma \, dx \, ds. \end{aligned}$$

For the error term we can directly take the $\gamma \to 0$ limit, and it follows by Stampacchia's lemma (see Chapter 5, Exercises 17,18 of Evans [35]) that $\nabla \rho = 0$ on the set $\{\rho = 0\}$ that

$$\lim_{\gamma \to 0} |I_t^{err}| = 0.$$

For the cutoff term, we regroup the terms and use the identity for $i \neq j \in \{1, 2\}$

$$2\chi^i|_{\xi=\rho^j} - 1 = 2\mathbb{1}_{0<\rho^j<\rho^i} - 1 = \operatorname{sgn}(\rho^i - \rho^j), \tag{3.8}$$

where sgn denotes the sign function. Here and throughout the thesis, we use the convention that the sign function is an odd function[1], so satisfies $\operatorname{sgn}(0) = 0$.

This gives that

$$\begin{aligned} I_t^{cut} = \int_0^t \int_U \nabla \rho^1 \cdot \nabla \iota_\gamma \operatorname{sgn}(\rho^2 - \rho^1) + \nabla \rho^2 \cdot \nabla \iota_\gamma \operatorname{sgn}(\rho^1 - \rho^2) \, dx \, ds \\ = \int_0^t \int_U (\nabla \rho^1 - \nabla \rho^2) \nabla \iota_\gamma \operatorname{sgn}(\rho^2 - \rho^1) \, dx \, ds, \end{aligned}$$

where in the final line we used that sgn is an odd function. Using Lemma 2.4.3, the notation $v_x := \frac{x - x*}{|x - x*|}$ for the inward pointing unit normal, and the fundamental

---

[1] We remark that the expression $2\mathbb{1}_{0<\rho_j<\rho_i} - 1$ is not exactly equivalent to $\operatorname{sgn}(\rho_i - \rho_j)$. However, in the rigorous argument, the left hand side of (3.8) is replaced by the limit of a velocity convolution regularisation. Since the indicator function $\mathbb{1}_{0<\rho_j<\rho_i}$ is discontinuous at $\rho_j = \rho_i$, taking the convolution limit produces an additional term $\frac{1}{2}\mathbb{1}_{\rho_j=\rho_i}$. This extra contribution ensures that (3.8) agrees exactly with the sign function $\operatorname{sgn}(\rho_i - \rho_j)$ satisfying $\operatorname{sgn}(0) = 0$.

theorem of calculus gives

$$
\begin{aligned}
\int_0^t \int_U (\nabla\rho^1 - \nabla\rho^2)\nabla\iota_\gamma \operatorname{sgn}(\rho^2 - \rho^1)\,dx\,ds \\
&= -\int_0^t \int_U \nabla|\rho^1 - \rho^2|\nabla\iota_\gamma\,dx\,ds \\
&= -\gamma^{-1}\int_0^t \int_{U\setminus U_\gamma} \nabla|\rho^1 - \rho^2|\cdot v_x\,dx\,ds \\
= -\gamma^{-1}\int_0^t \int_0^\gamma \int_{\partial U_z} \nabla|\rho^1(x^*+zv_x,s) - \rho^2(x^*+zv_x,s)|\cdot v_x\,dS(x)\,dz\,ds \\
= -\gamma^{-1}\int_0^t \int_0^\gamma \int_{\partial U_z} \frac{\partial}{\partial z}|\rho^1(x^*+zv_x,s) - \rho^2(x^*+zv_x,s)|\,dS(x)\,dz\,ds \\
= \gamma^{-1}\int_0^t \int_{\partial U} |\rho^1 - \rho^2|\,dS(x)\,ds - \gamma^{-1}\int_0^t \int_{\partial U_\gamma} |\rho^1 - \rho^2|\,dS(x)\,ds \le 0.
\end{aligned}
$$

The final line non-positive, the first term vanishes by the fact that the boundary conditions of the solutions coincide, and the second term is non-positive by the fact that the integrand is non-negative for every fixed $\gamma > 0$. Consequently we proved that

$$\lim_{\gamma\to 0} I_t^{cut} \le 0.$$

To conclude, putting all of the above together, we showed that for every fixed $t \in [0, T]$, it holds that

$$
\begin{aligned}
\int_U |\rho^1(x,t) - \rho^2(x,t)|\,dx &= \int_{\mathbb{R}}\int_U |\chi^1(x,\xi,t) - \chi^2(x,\xi,t)|^2\,dx\,d\xi \\
&\le \int_{\mathbb{R}}\int_U |\chi^1(x,\xi,0) - \chi^2(x,\xi,0)|^2\,dx\,d\xi = \int_U |\rho^1(x,0) - \rho^2(x,0)|\,dx.
\end{aligned}
$$

3.2 Uniqueness of solutions to the Dean–Kawasaki equation

In this section we prove the main uniqueness theorem for the generalised Dean–Kawasaki equation by formalising the above steps. Before that, we formalise the distributional equality (3.7) that we used in the formal proof in Lemma 3.2.1. The uniqueness result is then presented in Theorem 3.2.2.

We begin with an integration by parts lemma against the kinetic function. Since we will only deal with test functions that are compactly supported in space, the statement of the first point reads the same as the torus, see Lemma 4.4 of Fehrman and Gess [41].

Lemma 3.2.1 (Integration by parts against kinetic function)**.** *Let $\psi \in C_c^\infty(U \times (0,\infty))$ be a compactly supported test function and χ the kinetic function as defined in Definition 2.3.5. We have that*

1. *Formalising the distributional identity* $\nabla_x\chi(x,\xi,s) = \delta_0(\xi-\rho)\nabla\rho(x,s)$*: For every* $s\in[0,T]$ *we have the equality of vectors*
$$\int_{\mathbb{R}}\int_U (\nabla_x\psi)(x,\xi)\chi(x,\xi,s)\,dx\,d\xi = -\int_U \psi(x,\rho(x,s))\nabla\rho(x,s)\,dx.$$
2. *Formalising the distributional identity* $\partial_\xi\chi(x,\xi,s) = \delta_0(\xi)-\delta_0(\xi-\rho(x,s))$*: For every* $r\in[0,T]$ *we have*
$$\begin{aligned}\int_{\mathbb{R}}\int_U (\partial_\xi\psi)(x,\xi)\chi(x,\xi,s)\,dx\,d\xi &= -\int_U \left(\psi(x,0)-\psi(x,\rho(x,s))\right)\,dx\\ &= \int_U \psi(x,\rho(x,s))\,dx,\end{aligned}$$
where the final equality holds due to the compact support of ψ*.*

Proof. We begin with a proof of the first point. For arbitrary test function $\psi\in C_c^\infty(U\times(0,\infty))$, define $\Psi : U\times(0,\infty)\to\mathbb{R}$ be the unique function satisfying $\Psi(x,0)=0$ and $\partial_\xi\Psi(x,\xi)=\psi(x,\xi)$. By the definition of kinetic function, for any $s\in[0,T]$ we have

$$\begin{aligned}\int_U\int_{\mathbb{R}} (\nabla_x\psi)(x,\xi)\chi(x,\xi,s)\,d\xi\,dx &= \int_U\int_0^{\rho(x,s)} (\nabla_x\psi)(x,\xi)\,d\xi\,dx\\ &= \int_U\int_0^{\rho(x,s)} \partial_\xi(\nabla_x\Psi)(x,\xi)\,d\xi\,dx\\ &= \int_U (\nabla_x\Psi)(x,\rho(x,s))-(\nabla_x\Psi)(x,0)\,dx\\ &= \int_U (\nabla\cdot\Psi)(x,\rho(x,s))-(\partial_\xi\Psi)(x,\rho(x,s))\nabla\rho(x,s)\,dx\\ &= \int_{\partial U}\Psi(x,\rho(x,s))\cdot\hat{\eta}(x)\,dS(x)-\int_U\psi(x,\rho(x,s))\nabla\rho(x,s)\,dx, \quad (3.9)\end{aligned}$$

where the final equality is due to the divergence theorem, and recall that $\hat{\eta}$ denotes the outward pointing unit normal at the boundary. The first term in the final line on the right hand side of (3.9) vanishes due to the compact support of ψ, which proves the first point.

The proof of the second point is a direct consequence of the fundamental theorem of calculus and the fact that ψ is compactly supported in the ξ-variable

$$\begin{aligned}\int_{\mathbb{R}}\int_U (\partial_\xi\psi)(x,\xi)\chi(x,s,\xi)\,dx\,d\xi &= \int_U\int_0^{\rho(x,s)} (\partial_\xi\psi)(x,\xi)\,d\xi\,dx\\ &= \int_U (\psi(x,\rho(x,s))-\psi(x,0))\,dx = \int_U \psi(x,\rho(x,s))\,dx.\end{aligned}$$

□

We are now in a position to prove the uniqueness of stochastic kinetic solutions of (2.23). For the proof we will assume the following decay of the kinetic measure at zero, which is proved in Section 4.3:

$$\liminf_{\beta\to 0}\left(\beta^{-1}q(U\times[\beta/2,\beta]\times[0,T])\right)=0. \tag{3.10}$$

The proof of the uniqueness theorem below follows similar ideas as the torus, see Theorem 4.7 of Fehrman and Gess [41], but is complicated due to the spatial cutoff and having to be more careful when integrating by parts. In the following proof and throughout the thesis, we use $c\in(0,\infty)$ to denote a constant that changes from line to line, and not specify what the constant depends on unless it is important.

Theorem 3.2.2. *Suppose that the coefficients of noise* ξ^F *and the coefficients* Φ,σ,ν *of equation* (2.23) *satisfy Assumptions 2.1.2 and 2.2.1 respectively. Suppose* ρ^1 *and* ρ^2 *are two stochastic kinetic solutions of equation* (2.23) *in the sense of Definition 2.3.6, with kinetic measures* q^1,q^2 *respectively, both satisfying the decay* (3.10), *with* $\mathcal{F}_0$*-measurable initial data* $\rho_0^1,\rho_0^2\in L^1(\Omega;L^1(U))$ *respectively and with the same boundary data* $\Phi(\rho^1)|_{\partial U}=\Phi(\rho^2)|_{\partial U}=\bar{f}$*. Then almost surely*

$$\sup_{t\in[0,T]}\|\rho^1(\cdot,t)-\rho^2(\cdot,t)\|_{L^1(U)}\le\|\rho_0^1-\rho_0^2\|_{L^1(U)}.$$

Proof. Let χ^1,χ^2 be the kinetic functions of ρ^1,ρ^2 respectively. For every $\epsilon,\delta\in(0,1)$, $i\in\{1,2\}$ and $\kappa^{\epsilon,\delta}$ as in Definition 2.4.2, define the smoothed kinetic functions

$$\chi_{t,i}^{\epsilon,\delta}(y,\eta):=(\chi^i(\cdot,\cdot,t)*\kappa^{\epsilon,\delta})(y,\eta),\quad t\in[0,T],y\in U,\eta\in\mathbb{R}.$$

We have by the symmetry of the convolution kernels that for $x,y\in U$ and for $\xi,\eta\in\mathbb{R}$,

$$\nabla_x\kappa_d^\epsilon(y-x)=-\nabla_y\kappa_d^\epsilon(y-x),\quad \partial_\xi\kappa_1^\delta(\eta-\xi)=-\partial_\eta\kappa_1^\delta(\eta-\xi).$$

This implies, as a result of the kinetic equation (2.40), that for every $\epsilon,\delta\in(0,1)$ there is a subset of full probability such that we have for every $i\in\{1,2\}$, $t\in[0,T]$ and $(y,\eta)\in U_{2\epsilon}\times(2\delta,\delta^{-1})$,

$$\begin{aligned}
\chi_{s,i}^{\epsilon,\delta}(y,\eta)\Big|_{s=0}^t&=(\chi^i(\cdot,\cdot,s)*\kappa^{\epsilon,\delta})(y,\eta)\Big|_{s=0}^t=\int_{\mathbb{R}}\int_U\chi^i(x,\xi,s)\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\,dx\,d\xi\Big|_{s=0}^t\\
&=\nabla_y\cdot\left(\int_0^t\int_U\left(\Phi'(\rho^i)\nabla\rho^i+\frac{1}{2}F_1[\sigma'(\rho^i)]^2\nabla\rho^i+\frac{1}{2}\sigma'(\rho^i)\sigma(\rho^i)F_2\right)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\,dx\,ds\right)\\
&\qquad+\partial_\eta\left(\int_0^t\int_{\mathbb{R}}\int_U\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\,dq^i\right)\\
&\qquad-\frac{1}{2}\partial_\eta\left(\int_0^t\int_U\left(\sigma'(\rho^i)\sigma(\rho^i)\nabla\rho^i\cdot F_2+\sigma(\rho^i)^2F_3\right)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\,dx\,ds\right)\\
&\qquad-\int_0^t\int_U\kappa^{\epsilon,\delta}(x,y,\rho,\eta)\nabla\cdot(\sigma(\rho^i)\,d\xi^F)\,dx-\int_0^t\int_U\kappa^{\epsilon,\delta}(x,y,\rho,\eta)\nabla\cdot\nu(\rho^i)\,dx\,ds.
\end{aligned}\tag{3.11}$$

Above we used the standard notation $f(s)|_{s=0}^t:=f(t)-f(0)$. The fact that $(y,\eta)\in U_{2\epsilon}\times(2\delta,\delta^{-1})$ ensures that the convolution kernel $\kappa^{\epsilon,\delta}$ is compactly supported in space and velocity so that it is an admissible test function. To find an expression for the difference in solutions we want to deal with a regularised version of the identity

$$\begin{aligned}
&\int_U |\rho^1(x,s)-\rho^2(x,s)|\,dx\bigg|_{s=0}^{t} \\
&\quad = \int_{\mathbb{R}}\int_U \chi^1(\xi,\rho(x,s)) + \chi^2(\xi,\rho(x,s)) - 2\chi^1(\xi,\rho(x,s))\chi^2(\xi,\rho(x,s))\,dx\,d\xi\bigg|_{s=0}^{t}.
\end{aligned} \tag{3.12}$$

We begin by treating the regularised version of the first two terms on the right hand side of equation (3.12). Testing equation (3.11) against smooth approximations of the product of cutoff functions $\zeta_M\phi_\beta\iota_\gamma$, which are smooth and compactly supported, and subsequently taking the limit of the approximations yields

$$\begin{aligned}
&\int_{\mathbb{R}}\int_U \chi^{\epsilon,\delta}_{s,i}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta\bigg|_{s=0}^{t} = \\
&-\int_{\mathbb{R}}\int_0^t\int_{U^2}\left(\Phi'(\rho^i)\nabla\rho^i + \frac{1}{2}F_1[\sigma'(\rho^i)]^2\nabla\rho^i + \frac{1}{2}\sigma'(\rho^i)\sigma(\rho^i)F_2\right) \\
&\qquad\qquad\qquad\times\,\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\zeta_M(\eta)\phi_\beta(\eta)\cdot\nabla\iota_\gamma\,dy\,dx\,ds\,d\eta \\
&-\int_{\mathbb{R}^2}\int_0^t\int_{U^2}\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)\,dq^i\,dy\,d\eta \\
&+\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_{U^2}\left(\sigma'(\rho^i)\sigma(\rho^i)\nabla\rho^i\cdot F_2 + \sigma(\rho^i)^2F_3\right) \\
&\qquad\qquad\qquad\times\,\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)\,dy\,dx\,ds\,d\eta \\
&-\int_{\mathbb{R}}\int_0^t\int_{U^2}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\nabla\cdot(\sigma(\rho^i)\,d\xi^F)\,dy\,dx\,ds\,d\eta \\
&-\int_{\mathbb{R}}\int_0^t\int_{U^2}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(y,x,\eta,\rho^i)\nabla\cdot\nu(\rho^i)\,dy\,dx\,ds\,d\eta.
\end{aligned} \tag{3.13}$$

For convenience we split this up into three parts,

$$\int_{\mathbb{R}}\int_U \chi^{\epsilon,\delta}_{s,i}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta\bigg|_{s=0}^{t} = I_t^{i,cut} + I_t^{i,mart} + I_t^{i,flux}, \tag{3.14}$$

with the cutoff term being the first three integrals on the right hand side involving derivatives of the cutoff terms, the martingale term being the one that involves the noise term in the penultimate line and the flux term being the integral in the final line. The first term on the right hand side containing the derivative of the spatial cutoff $\nabla\iota_\gamma$ is a new term compared to the torus case, which a priori diverges like γ^{-1}.

To obtain an expression for the final term in equation (3.12), we introduce the notation $(x,\xi)\in U\times\mathbb{R}$ for arguments in χ^1 and related quantities and $(x',\xi')\in U\times\mathbb{R}$ for arguments of χ^2 and related quantities. For brevity we also introduce the notation

$$\bar{k}^{\epsilon,\delta}_{s,1}(x,y,\eta) := \kappa^{\epsilon,\delta}(x,y,\eta,\rho^1(x,s)),\quad \bar{k}^{\epsilon,\delta}_{s,2}(x',y,\eta) := \kappa^{\epsilon,\delta}(x',y,\eta,\rho^2(x',s)). \tag{3.15}$$

The stochastic product rule tells us that almost surely we have, for $\beta \in (0,1)$, $M \in \mathbb{N}$, $\gamma \in (0, \gamma_U)$, $\delta \in (0, \beta/4)$, $\epsilon \in (0, \gamma/4)$,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_U \chi^{\epsilon,\delta}_{s,1}(y,\eta)\chi^{\epsilon,\delta}_{s,2}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta\Big|_{s=0}^{t} \\
&= \int_{\mathbb{R}} \int_0^t \int_U \Big(\chi^{\epsilon,\delta}_{s,1}(y,\eta)d\chi^{\epsilon,\delta}_{s,2}(y,\eta) + \chi^{\epsilon,\delta}_{s,2}(y,\eta)d\chi^{\epsilon,\delta}_{s,1}(y,\eta) \\
&\qquad\qquad + \langle \chi^{\epsilon,\delta}_1, \chi^{\epsilon,\delta}_1 \rangle_s(y,\eta)\Big)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta \\
&= \int_{\mathbb{R}} \int_U \left(\chi^{\epsilon,\delta}_{s,1}(y,\eta)\left[\chi^{\epsilon,\delta}_{s,2}(y,\eta)\Big|_{s=0}^{t}\right] + \chi^{\epsilon,\delta}_{s,2}(y,\eta)\left[\chi^{\epsilon,\delta}_{s,1}(y,\eta)\Big|_{s=0}^{t}\right]\right. \\
&\qquad\qquad \left. + \left[\langle \chi^{\epsilon,\delta}_1, \chi^{\epsilon,\delta}_2 \rangle_s(y,\eta)\Big|_{s=0}^{t}\right]\right)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta,
\end{aligned} \tag{3.16}$$

where we smoothed the kinetic function so are allowed to use it as part of an admissible test function. Compared to the formal computation for the heat equation in Section 3.1, here we have the additional quadratic co-variation term due to the presence of noise in the equation. Using equation (3.11) we can write the first term in the final line of (3.16) as

$$\begin{aligned}
&\int_{\mathbb{R}} \int_U \chi^{\epsilon,\delta}_{s,1}(y,\eta)\left[\chi^{\epsilon,\delta}_{s,2}(y,\eta)\right]\Big|_{s=0}^{t}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta \\
&\quad = \int_{\mathbb{R}} \int_U \zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\chi^{\epsilon,\delta}_{s,1}(y,\eta)\left[\nabla_y \cdot \left(\int_0^t \int_U \left(\Phi'(\rho^2)\nabla\rho^2\right)\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\,ds\right)\right. \\
&\quad + \nabla_y \cdot \left(\int_0^t \int_U \left(\frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2 + \frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\right)\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\,ds\right) \\
&\quad + \partial_\eta\left(\int_{\mathbb{R}} \int_0^t \int_U \kappa^{\epsilon,\delta}(x',y,\xi,\eta)\,dq^2\right) \\
&\quad - \frac{1}{2}\partial_\eta\left(\int_0^t \int_U \left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2 \cdot F_2 + \sigma(\rho^2)^2F_3\right)\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\right)\,ds \\
&\quad \left. - \int_0^t \int_U \bar{k}^{\epsilon,\delta}_{s,2}\nabla \cdot \left(\sigma(\rho^2)\,d\xi^F\right)dx' - \int_0^t \int_U \bar{k}^{\epsilon,\delta}_{s,2}\nabla \cdot \nu(\rho^2)\,dx'\,ds\right]\,dy\,d\eta.
\end{aligned} \tag{3.17}$$

We integrate by parts and move derivatives onto (smooth approximations of) the product $\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\chi^{\epsilon,\delta}_{s,1}(y,\eta)$, which are smooth, compactly supported so can be done using classical integration by parts. We use the product rule when integrating in y then the integration by parts lemma, Lemma 3.2.1 noting the convolution kernel is compactly supported in space since $y, \eta \in U_{2\epsilon} \times (2\delta, \infty)$, which gives

$$\begin{aligned}\nabla_y \chi^{\epsilon,\delta}_{s,1}(y,\eta) &:= \int_{\mathbb{R}} \int_U \chi^i(x,\xi,s) \nabla_y \kappa^{\epsilon,\delta}(y,x,\eta,\xi)\, dx\, d\xi \\ &= -\int_{\mathbb{R}} \int_U \chi^i(x,\xi,s) \nabla_x \kappa^{\epsilon,\delta}(y,x,\eta,\xi)\, dx\, d\xi \\ &= -\int_U \bar{k}^{\epsilon,\delta}_{s,i} \nabla \rho^i(x,r)\, dx. \end{aligned} \tag{3.18}$$

We consequently obtain the decomposition

$$\begin{aligned}&\int_{\mathbb{R}} \int_0^t \int_U \chi^{\epsilon,\delta}_{s,1}(y,\eta) d\chi^{\epsilon,\delta}_{s,2}(y,\eta) \zeta_M(\eta) \phi_\beta(\eta) \iota_\gamma(y)\, dy\, d\eta \\ &\qquad = I^{1,2,err}_t + I^{1,2,meas}_t + I^{1,2,cut}_t + I^{1,2,mart}_t + I^{1,2,flux}_t.\end{aligned}$$

Adding the term

$$\int_{\mathbb{R}} \int_0^t \int_{U^3} [\Phi'(\rho^1)]^{1/2} [\Phi'(\rho^2)]^{1/2} \nabla \rho^1 \cdot \nabla \rho^2 \bar{k}^{\epsilon,\delta}_{s,1} \bar{k}^{\epsilon,\delta}_{s,2} \phi_\beta(\eta) \zeta_M(\eta) \iota_\gamma(y)\, dx\, dx'\, dy\, ds\, d\eta \tag{3.19}$$

to the error term and taking it away from the measure term below (which will allow the error terms below to be nicely factorised) gives for each term separately

$$\begin{aligned}I^{1,2,err}_t &= -\int_{\mathbb{R}} \int_0^t \int_{U^3} \zeta_M(\eta) \phi_\beta(\eta) \iota_\gamma(y) \Phi'(\rho^2) \nabla \rho^2 \cdot \nabla \rho^1 \bar{k}^{\epsilon,\delta}_{s,1} \bar{k}^{\epsilon,\delta}_{s,2}\, dx\, dx'\, dy\, ds\, d\eta \\ &- \int_{\mathbb{R}} \int_0^t \int_{U^3} \zeta_M(\eta) \phi_\beta(\eta) \iota_\gamma(y) \left(\frac{1}{2} F_1 [\sigma'(\rho^2)]^2 \nabla \rho^2 \cdot \nabla \rho^1 + \frac{1}{2} \sigma'(\rho^2) \sigma(\rho^2) F_2 \cdot \nabla \rho^1 \right) \\ &\qquad\qquad \times \bar{k}^{\epsilon,\delta}_{s,1} \bar{k}^{\epsilon,\delta}_{s,2}\, dx\, dx'\, dy\, ds\, d\eta \\ &- \frac{1}{2} \int_{\mathbb{R}} \int_0^t \int_{U^3} \zeta_M(\eta) \phi_\beta(\eta) \iota_\gamma(y) \left(\sigma'(\rho^2) \sigma(\rho^2) \nabla \rho^2 \cdot F_2 + \sigma(\rho^2)^2 F_3 \right) \bar{k}^{\epsilon,\delta}_{s,2} \bar{k}^{\epsilon,\delta}_{s,1}\, dx\, dx'\, dy\, ds\, d\eta \\ &+ \int_{\mathbb{R}} \int_0^t \int_{U^3} [\Phi'(\rho^1)]^{1/2} [\Phi'(\rho^2)]^{1/2} \nabla \rho^1 \cdot \nabla \rho^2 \bar{k}^{\epsilon,\delta}_{s,1} \bar{k}^{\epsilon,\delta}_{s,2} \phi_\beta(\eta) \zeta_M(\eta) \iota_\gamma(y)\, dx\, dx'\, dy\, ds\, d\eta, \end{aligned} \tag{3.20}$$

measure term

$$\begin{aligned}I_t{}^{1,2,meas} &= \int_{\mathbb{R}^2} \int_0^t \int_{U^3} \zeta_M(\eta) \phi_\beta(\eta) \iota_\gamma(y) \kappa^{\epsilon,\delta}(x',y,\xi,\eta) \bar{k}^{\epsilon,\delta}_{s,1}\, dq^2(x',\xi,s)\, dx\, dy\, d\eta \\ &- \int_{\mathbb{R}} \int_0^t \int_{U^3} [\Phi'(\rho^1)]^{1/2} [\Phi'(\rho^2)]^{1/2} \nabla \rho^1 \cdot \nabla \rho^2 \bar{k}^{\epsilon,\delta}_{s,1} \bar{k}^{\epsilon,\delta}_{s,2} \phi_\beta(\eta) \zeta_M(\eta) \iota_\gamma(y)\, dx\, dx'\, dy\, ds\, d\eta, \end{aligned} \tag{3.21}$$

cutoff term defined by

$$\begin{aligned}
I_t^{1,2,cut} &= -\int_{\mathbb{R}^2}\int_0^t\int_{U^2} \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\chi^{\epsilon,\delta}_{s,1}(y,\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(x',y,\xi,\eta)\,dq^2(x',\xi,s)\,dy\,d\eta \\
&+ \frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_{U^2} \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\chi^{\epsilon,\delta}_{s,1}(y,\eta)\iota_\gamma(y)\left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2\cdot F_2+\sigma(\rho^2)^2F_3\right) \\
&\qquad\qquad \times \bar{k}^{\epsilon,\delta}_{s,2}\,dx'\,dy\,ds\,d\eta \\
&- \int_{\mathbb{R}}\int_0^t\int_{U^2} \zeta_M(\eta)\phi_\beta(\eta)\chi^{\epsilon,\delta}_{s,1}(y,\eta)\nabla_y\iota_\gamma(y)\cdot \\
&\qquad \left(\Phi'(\rho^2)\nabla\rho^2+\frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2+\frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\right)\bar{k}^{\epsilon,\delta}_{s,2}\,dx'\,dy\,ds\,d\eta,
\end{aligned}$$

martingale term

$$I_t^{1,2,mart} = -\int_{\mathbb{R}}\int_0^t\int_{U^2} \bar{k}^{\epsilon,\delta}_{s,2}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\chi^{\epsilon,\delta}_{s,1}(y,\eta)\nabla\cdot(\sigma(\rho^2)\,d\xi^F)\,dx'\,dy\,d\eta,$$

and flux term

$$I_t^{1,2,flux} = -\int_{\mathbb{R}}\int_0^t\int_{U^2} \bar{k}^{\epsilon,\delta}_{s,2}\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\chi^{\epsilon,\delta}_{s,1}(y,\eta)\nabla\cdot\nu(\rho^2)\,dx'\,dy\,ds\,d\eta. \tag{3.22}$$

In the equations for the terms above, we have an extra spatial integral for the terms when the derivative falls on the kinetic function due to (3.18). An analogous decomposition holds for the second term on the right hand side of (3.16), and we denote the corresponding error, measure, cutoff, martingale and flux terms of the second term up to time $t \in [0,T]$ by $I_t^{2,1,\cdot}$, where we again artificially add the error term (3.19) and subtract it from the measure term.

Finally we deal with the quadratic co-variation term in equation (3.16). Let us begin by noticing, using Definition 2.1.1 of the noise ξ^F, that formally

$$\begin{aligned}
d\langle\chi^{\epsilon,\delta}_{\cdot,1},\chi^{\epsilon,\delta}_{\cdot,2}\rangle_s(y,\eta) &:= d\langle(\chi^1*\kappa^{\epsilon,\delta}),(\chi^2*\kappa^{\epsilon,\delta})\rangle_s(y,\eta) \\
&= \int_{U^2}\int_{\mathbb{R}^2} d\langle\chi^1,\chi^2\rangle_s\kappa^{\epsilon,\delta}_{s,1}(y,x,\eta,\xi)\kappa^{\epsilon,\delta}_{s,2}(y,x',\eta,\xi')\,d\xi\,d\xi'\,dx\,dx' \\
= \int_{U^2}\int_{\mathbb{R}^2} \delta_0(\xi-\rho^1)\delta_0(\xi'-\rho^2)&\nabla\cdot\left(\sigma(\rho^1)\sum_{k=1}^\infty f_k(x)dB^k_s\right)\nabla\cdot\left(\sigma(\rho^2)\sum_{j=1}^\infty f_j(x')dB^j_s\right) \\
&\times \kappa^{\epsilon,\delta}_{s,1}(x,y,\xi,\eta)\kappa^{\epsilon,\delta}_{s,2}(x',y,\xi',\eta)\,d\xi\,d\xi'\,dx\,dx' \\
= \sum_{j,k=1}^\infty\int_{U^2}&\left(f_k\sigma'(\rho^1)\nabla\rho^1+\sigma(\rho^1)\nabla f_k\right)\left(f_j\sigma'(\rho^2)\nabla\rho^2+\sigma(\rho^2)\nabla f_j\right)d\langle B^k_\cdot,B^j_\cdot\rangle_s \\
&\times \bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}dx\,dx' \\
= \sum_{k=1}^\infty\int_{U^2}&\left(f_k\sigma'(\rho^1)\nabla\rho^1+\sigma(\rho^1)\nabla f_k\right)\left(f_k\sigma'(\rho^2)\nabla\rho^2+\sigma(\rho^2)\nabla f_k\right)\bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\,dx\,dx'\,ds.
\end{aligned}$$

The above can be made rigorous by integrating against smooth approximations of the product $\phi_\beta\zeta_M\iota_\gamma$, and rather than the multiplication of delta functions and using the integration by parts lemma, Lemma 3.2.1. Consequently one obtains that

$$\begin{aligned}\int_{\mathbb{R}}\int_0^t\int_U & d\langle\chi^{\epsilon,\delta}_{\cdot,1},\chi^{\epsilon,\delta}_{\cdot,2}\rangle_s(y,\eta)\,\phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y)\,dy\,d\eta \\ &= \sum_{k=1}^{\infty}\int_{\mathbb{R}}\int_0^t\int_{U^3}\big(f_k\sigma'(\rho^1)\nabla\rho^1+\sigma(\rho^1)\nabla f_k\big)\cdot\big(f_k\sigma'(\rho^2)\nabla\rho^2+\sigma(\rho^2)\nabla f_k\big) \\ &\qquad\qquad \times\,\bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y)\,dx\,dx'\,dy\,ds\,d\eta. \qquad (3.23)\end{aligned}$$

Putting this together, it follows from equation (3.16) and the subsequent computations from (3.17)-(3.23) that we have the decomposition for the cross term as

$$\begin{aligned}\int_{\mathbb{R}}\int_U \chi^{\epsilon,\delta}_{s,1}(y,\eta)\chi^{\epsilon,\delta}_{s,2}(y,\eta)\zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\,dy\,d\eta\Big|_{s=0}^{t} \\ = I^{err}_t+I^{meas}_t+I^{mix,cut}_t+I^{mix,mart}_t+I^{mix,flux}_t.\end{aligned} \qquad (3.24)$$

We put all four terms of the quadratic co-variation term (3.23) into the error term,

$$I^{err}_t = I^{1,2,err}_t + I^{2,1,err}_t + \text{ quadratic co-variation contribution.}$$

The measure term just arises from the first two components of (3.16), that is

$$I^{meas}_t = I^{1,2,meas}_t + I^{2,1,meas}_t. \qquad (3.25)$$

Similarly the contribution of the mixed cutoff, martingale and flux terms just comes from the first two components of (3.16),

$$I^{mix,cut,mart,flux}_t = I^{1,2,cut,mart,flux}_t + I^{2,1,cut,mart,flux}_t.$$

We call these "mixed" terms since we know from (3.12) and the decomposition (3.14) that we still have a contribution from the first order terms to add on. Let us explain this more clearly now. Returning back to the equation of interest that governs the $L^1(U)$-difference of two solutions, equation (3.12), we have the decomposition

$$\int_{\mathbb{R}}\int_U\Big(\chi^{\epsilon,\delta}_{s,1}+\chi^{\epsilon,\delta}_{s,2}-2\chi^{\epsilon,\delta}_{s,1}\chi^{\epsilon,\delta}_{s,2}\Big)\,\phi_\beta\zeta_M\iota_\gamma\Big|_{s=0}^{t} = -2I^{err}_t-2I^{meas}_t+I^{mart}_t+I^{cut}_t+I^{flux}_t. \qquad (3.26)$$

The error and measure term were defined above and arise solely from the mixed term (3.24), the final term on the left hand side of (3.26). The martingale, cutoff and flux terms arise from all three terms in the left hand side of equation (3.26), and are given respectively by

$$I^{mart,cut,flux}_t = I^{1,mart,cut,flux}_t + I^{2,mart,cut,flux}_t - 2I^{mix,mart,cut,flux}_t. \qquad (3.27)$$

We deal with each term on the right hand side of (3.26) separately.

Measure term.

From equations (3.21) and (3.25), the measure term is

$$
\begin{aligned}
I_t^{meas} = & \int_{\mathbb{R}^2}\int_0^t\int_{U^3} \zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(x,y,\xi,\eta)\bar{k}^{\epsilon,\delta}_{s,2}\,dq^1(x,\xi,s)\,dx'\,dy\,d\eta \\
& + \int_{\mathbb{R}^2}\int_0^t\int_{U^3} \zeta_M(\eta)\phi_\beta(\eta)\iota_\gamma(y)\kappa^{\epsilon,\delta}(x',y,\xi,\eta)\bar{k}^{\epsilon,\delta}_{s,1}\,dq^2(x',\xi,s)\,dx\,dy\,d\eta \\
- 2 & \int_{\mathbb{R}}\int_0^t\int_{U^3} [\Phi'(\rho^1)]^{1/2}[\Phi'(\rho^2)]^{1/2}\nabla\rho^1\cdot\nabla\rho^2\bar{k}^{\epsilon,\delta}_{s,1}\bar{k}^{\epsilon,\delta}_{s,2}\phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y)\,dx\,dx'\,dy\,ds\,d\eta.
\end{aligned}
$$

By Hölder's inequality and the bound (2.38) on the kinetic measure from Definition 2.3.6 of stochastic kinetic solutions, we have that

$$
I_t^{meas} \geq 0.
$$

Error term.

Combining equations (3.20) and (3.23), it follows that we have a convenient factorisation for the error term. Specifically, using the dominated convergence theorem and the notation[2] $\bar{k}^\delta_{s,i}(y,\eta) := \kappa^\delta_1(\rho^i(y,s)-\eta)$ for $i = 1,2$, for every fixed $\gamma \in (0,\gamma_U)$, $\delta \in (0,\beta/4)$, after taking the limit as $\epsilon \to 0$ the error term is

$$
\begin{aligned}
\lim_{\epsilon\to 0} I_t^{err} = & -\int_{\mathbb{R}}\int_0^t\int_U \left((\Phi'(\rho^1))^{1/2}-(\Phi'(\rho^2))^{1/2}\right)^2\nabla\rho^1\cdot\nabla\rho^2\bar{k}^\delta_{s,1}\bar{k}^\delta_{s,2}\phi_\beta\zeta_M\iota_\gamma\,dy\,ds\,d\eta \\
-\frac{1}{2} & \int_{\mathbb{R}}\int_0^t\int_U \left(F_1\left(\sigma'(\rho^1)-\sigma'(\rho^2)\right)^2\nabla\rho_1\cdot\nabla\rho_2 + F_3\left(\sigma(\rho^1)-\sigma(\rho^2)\right)^2\right)\bar{k}^\delta_{s,1}\bar{k}^\delta_{s,2}\phi_\beta\zeta_M\iota_\gamma\,dy\,ds\,d\eta \\
-\frac{1}{2} & \int_{\mathbb{R}}\int_0^t\int_U \left(\sigma'(\rho^1)\sigma(\rho^1)+\sigma'(\rho^2)\sigma(\rho^2)-2\sigma'(\rho^1)\sigma(\rho^2)\right)F_2\cdot\nabla\rho^1\bar{k}^\delta_{s,1}\bar{k}^\delta_{s,2}\phi_\beta\zeta_M\iota_\gamma\,dy\,ds\,d\eta \\
-\frac{1}{2} & \int_{\mathbb{R}}\int_0^t\int_U \left(\sigma'(\rho^1)\sigma(\rho^1)+\sigma'(\rho^2)\sigma(\rho^2)-2\sigma'(\rho^2)\sigma(\rho^1)\right)F_2\cdot\nabla\rho^2\bar{k}^\delta_{s,1}\bar{k}^\delta_{s,2}\phi_\beta\zeta_M\iota_\gamma\,dy\,ds\,d\eta.
\end{aligned}
\tag{3.28}
$$

The definition of the convolution kernels imply that whenever we have

$$
\bar{k}^\delta_{s,1}(y,\eta)\bar{k}^\delta_{s,2}(y,\eta)\phi_\beta(\eta)\zeta_M(\eta)\iota_\gamma(y) \neq 0 \tag{3.29}
$$

for $\delta \in (0,\beta/4)$, the difference $|\rho^1(y,s)-\rho^2(y,s)|$ is small. More precisely, owing to Assumption 2.2.1 which gives the local Lipschitz regularity of Φ', the fact that $\Phi' > 0$, the fact that $\delta \in (0,\beta/4)$ which implies that σ is $\frac{1}{2}$-Hölder continuous on the support of the cutoffs (3.29), the triangle inequality, the local boundedness and local Lipschitz regularity of σ and σ', we have for a constant $c \in (0,\infty)$ depending on M and β,

$$
\begin{aligned}
\left((\Phi'(\rho^1))^{1/2}-(\Phi'(\rho^2))^{1/2}\right)^2 + \left(\sigma'(\rho^1)-\sigma'(\rho^2)\right)^2 + \left(\sigma(\rho^1)-\sigma(\rho^2)\right)^2 & \\
+ \left|\sigma'(\rho^1)\sigma(\rho^1)+\sigma'(\rho^2)\sigma(\rho^2)-2\sigma'(\rho^1)\sigma(\rho^2)\right| & \\
& \leq c\delta\mathbb{1}_{0\leq|\rho^1(y,s)-\rho^2(y,s)|\leq c\delta}.
\end{aligned}
$$

---

[2] This is similar to (3.15) without the ϵ-dependence.

Returning to the error term (3.28), the boundedness of F_i for $i = 1, 2, 3$ and Hölder's and Young's inequalities show that there is a constant $c \in (0, \infty)$ depending on M and β such that almost surely, for every $t \in [0, T]$,

$$\begin{aligned}
&\limsup_{\epsilon\to 0} |I_t^{err}| \\
&\quad \le c \int_{\mathbb{R}} \int_0^t \int_U \delta \mathbb{1}_{0\le|\rho^1(y,s)-\rho^2(y,s)|\le c\delta} \left(1 + |\nabla\rho^1|^2 + |\nabla\rho^2|^2\right) \bar{k}^{\delta}_{s,1} \bar{k}^{\delta}_{s,2} \phi_\beta \zeta_M \iota_\gamma \, dy\, ds\, d\eta.
\end{aligned} \tag{3.30}$$

It is then a consequence of the definition of the cutoff functions and convolution kernels, the local $L^2([0, T]; H^1(U))$-regularity property of stochastic kinetic solutions (2.37), the dominated convergence theorem and equation (3.30) that almost surely, for every $t \in [0, T]$,

$$\limsup_{\delta\to 0} \left(\limsup_{\epsilon\to 0} |I_t^{err}| \right) = 0.$$

Cutoff term.

We have for every $\beta \in (0, 1), M \in \mathbb{N}, \gamma \in (0, \gamma_U)$, $\delta \in (0, \beta/4), \epsilon \in (0, \gamma/4)$ that for every $t \in [0, T]$,

$$\begin{aligned}
I_t^{cut} &= \int_{\mathbb{R}^2} \int_0^t \int_{U^2} \kappa^{\epsilon,\delta}(y, x, \eta, \xi) \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)(-1 + 2\chi^{\epsilon,\delta}_{s,2}) \, dq^1(x, \xi, s)\, dy\, d\eta \\
&+ \frac{1}{2} \int_{\mathbb{R}} \int_0^t \int_{U^2} \left(\sigma'(\rho^1)\sigma(\rho^1)\nabla\rho^1 \cdot F_2 + \sigma(\rho^1)^2 F_3\right) \bar{k}^{\epsilon,\delta}_{s,1}(1 - 2\chi^{\epsilon,\delta}_{s,2}) \\
&\qquad\qquad \times \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)\, dy\, dx\, ds\, d\eta \\
&+ \int_{\mathbb{R}^2} \int_0^t \int_{U^2} \kappa^{\epsilon,\delta}(y, x', \eta, \xi) \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)(-1 + 2\chi^{\epsilon,\delta}_{s,1}) \, dq^2(x', \xi, s)\, dy\, d\eta \\
&+ \frac{1}{2} \int_{\mathbb{R}} \int_0^t \int_{U^2} \left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2 \cdot F_2 + \sigma(\rho^2)^2 F_3\right) \bar{k}^{\epsilon,\delta}_{s,2}(1 - 2\chi^{\epsilon,\delta}_{s,1}) \\
&\qquad\qquad \times \partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)\, dy\, dx\, ds\, d\eta \\
&+ \int_{\mathbb{R}} \int_0^t \int_{U^2} \left(\Phi'(\rho^1)\nabla\rho^1 + \frac{1}{2}F_1[\sigma'(\rho^1)]^2\nabla\rho^1 + \frac{1}{2}\sigma'(\rho^1)\sigma(\rho^1)F_2\right) \\
&\qquad\qquad \times \bar{k}^{\epsilon,\delta}_{s,1}\zeta_M(\eta)\phi_\beta(\eta) \cdot \nabla\iota_\gamma(y)(-1 + 2\chi^{\epsilon,\delta}_{s,2})\, dy\, dx\, ds\, d\eta \\
&+ \int_{\mathbb{R}} \int_0^t \int_{U^2} \left(\Phi'(\rho^2)\nabla\rho^2 + \frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2 + \frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\right) \\
&\qquad\qquad \times \bar{k}^{\epsilon,\delta}_{s,2}\zeta_M(\eta)\phi_\beta(\eta) \cdot \nabla\iota_\gamma(y)(-1 + 2\chi^{\epsilon,\delta}_{s,1})\, dy\, dx'\, ds\, d\eta.
\end{aligned} \tag{3.31}$$

Let us begin by bounding the final two lines of I_t^{cut} above, comprising of the new terms involving gradients of the spatial cutoff. The analysis for these terms makes rigorous the formal calculation for the cutoff term in the heat equation in Section 3.1.

We take the $\epsilon, \delta \to 0$ limits first and use the distributional equality for $i, j \in \{1, 2\}$,

$$\lim_{\epsilon,\delta\to 0} \bar{k}^{\epsilon,\delta}_{s,i}(x, y, \eta, \rho)(-1 + 2\chi^{\epsilon,\delta}_{s,j}(y, \eta)) \to \delta_0(x - y)\delta_0(\eta - \rho^i)\mathrm{sgn}(\rho^j - \eta)$$
$$= \delta_0(x - y)\delta_0(\eta - \rho^i)\mathrm{sgn}(\rho^j - \rho^i). \qquad (3.32)$$

Therefore the final two lines of the cutoff term (3.31) can be simplified in the $\epsilon, \delta \to 0$ limit as

$$\begin{aligned}&\int_0^t \int_U \left(\Phi'(\rho^1)\nabla\rho^1 + \frac{1}{2}F_1[\sigma'(\rho^1)]^2\nabla\rho^1 + \frac{1}{2}\sigma'(\rho^1)\sigma(\rho^1)F_2\right)\\ &\qquad\qquad \times \zeta_M(\rho^1)\phi_\beta(\rho^1) \cdot \nabla\iota_\gamma(y)\mathrm{sgn}(\rho^2 - \rho^1)\, dy\, ds\\ &+ \int_0^t \int_{U^2} \left(\Phi'(\rho^2)\nabla\rho^2 + \frac{1}{2}F_1[\sigma'(\rho^2)]^2\nabla\rho^2 + \frac{1}{2}\sigma'(\rho^2)\sigma(\rho^2)F_2\right)\\ &\qquad\qquad \times \zeta_M(\rho^2)\phi_\beta(\rho^2) \cdot \nabla\iota_\gamma(y)\mathrm{sgn}(\rho^1 - \rho^2)\, dy\, ds. \end{aligned} \qquad (3.33)$$

We combine the terms in the two lines using the fact that sign is an odd function and so $\mathrm{sgn}(\rho^1 - \rho^2) = -\mathrm{sgn}(\rho^2 - \rho^1)$. We will deal with the first two terms of each line that have a factor of $\nabla\rho^i$ and the final term separately. First consider the first terms in both the lines of (3.33). Begin by defining the function $\Phi_{M,\beta} : [0, \infty) \to [0, \infty)$ to be the unique function such that $\Phi_{M,\beta}(0) = 0$ and

$$\Phi'_{M,\beta}(\xi) = \zeta_M(\xi)\phi_\beta(\xi)\Phi'(\xi) \geq 0. \qquad (3.34)$$

This says that the function $\Phi_{M,\beta}$ is non-decreasing. To make the below computation rigorous, we define the regularised sgn function $\mathrm{sgn}_\delta := \mathrm{sgn} * \kappa^\delta_1$ for $\delta \in (0, 1)$, where κ^δ_1 is the convolution kernel defined in the first point of Definition 2.4.2, and similarly we define a_δ to be a smooth approximation of the absolute value function satisfying $a_\delta(0) = 0$. Then, using Lemma 2.4.3 to define the spatial derivative of the cutoff, the notation $v_y := \frac{y-y*}{|y-y*|}$ for the inward pointing unit normal and the fundamental theorem of calculus, the contribution coming from the difference of the first terms of

(3.33) is non-negative

$$
\begin{aligned}
&-\int_0^t\int_U \left(\nabla\Phi_{M,\beta}(\rho^2)-\nabla\Phi_{M,\beta}(\rho^1)\right)\cdot\nabla\iota_\gamma(y)\,\mathrm{sgn}(\rho^2-\rho^1)\,dy\,ds\\
&=-\lim_{\delta\to0}\int_0^t\int_U \left(\nabla\Phi_{M,\beta}(\rho^2)-\nabla\Phi_{M,\beta}(\rho^1)\right)\cdot\nabla\iota_\gamma(y)\,\mathrm{sgn}_\delta(\Phi_{M,\beta}(\rho^2)-\Phi_{M,\beta}(\rho^1))\,dy\,ds\\
&=-\lim_{\delta\to0}\int_0^t\int_U \nabla a_\delta\left(\Phi_{M,\beta}(\rho^2)-\Phi_{M,\beta}(\rho^1)\right)\cdot\nabla\iota_\gamma(y)\,dy\,ds\\
&=-\lim_{\delta\to0}\gamma^{-1}\int_0^t\int_{U\setminus U_\gamma} \nabla a_\delta\left(\Phi_{M,\beta}(\rho^2)-\Phi_{M,\beta}(\rho^1)\right)\cdot v_y\,dy\,ds\\
&=-\gamma^{-1}\lim_{\delta\to0}\int_0^t\int_0^\gamma\int_{\partial U_z} \nabla a_\delta\left(\Phi_{M,\beta}(\rho^2(y^*+zv_y,s))-\Phi_{M,\beta}(\rho^1(y^*+zv_y,s))\right)\cdot v_y\,dS(y)\,dz\,ds\\
&=-\gamma^{-1}\lim_{\delta\to0}\int_0^t\int_0^\gamma\int_{\partial U_z} \frac{\partial}{\partial z}a_\delta\left(\Phi_{M,\beta}(\rho^2(y^*+zv_y,s))-\Phi_{M,\beta}(\rho^1(y^*+zv_y,s))\right)\,dS(y)\,dz\,ds\\
&=\gamma^{-1}\lim_{\delta\to0}\int_0^t\int_{\partial U} a_\delta\left(\Phi_{M,\beta}(\rho^2)-\Phi_{M,\beta}(\rho^1)\right)\,dS(y)\,ds\\
&\qquad\qquad-\gamma^{-1}\lim_{\delta\to0}\int_0^t\int_{\partial U_\gamma} a_\delta\left(\Phi_{M,\beta}(\rho^2)-\Phi_{M,\beta}(\rho^1)\right)\,dS(y)\,ds. \qquad (3.35)
\end{aligned}
$$

But by the dominated convergence theorem and the fact that a_δ converges to the absolute value pointwise, we have that the $\delta\to0$ limit can be evaluated to obtain

$$
\begin{aligned}
&=\gamma^{-1}\int_0^t\int_{\partial U}\left|\Phi_{M,\beta}(\rho^2)-\Phi_{M,\beta}(\rho^1)\right|\,dS(y)\,ds\\
&\qquad\qquad-\gamma^{-1}\int_0^t\int_{\partial U_\gamma}\left|\Phi_{M,\beta}(\rho^2)-\Phi_{M,\beta}(\rho^1)\right|\,dS(y)\,ds\le0.
\end{aligned}
$$

This term does not vanish a priori, since the first term is an integral over ∂U and the second is an integral over ∂U_γ. However, it is signed since the first term vanishes due to the solutions coinciding on the boundary, and the second term is non-positive due to the minus sign and the fact that the integrand is non-negative for every fixed $\gamma>0$. By repeating the same arguments, noting that $\frac{1}{2}F_1(\sigma'(\rho^2))^2\ge0$, one can conclude that the combination of second terms of (3.33) are non-positive for every $\gamma>0$. For the final term of (3.33), we have by Lemma 2.4.3 as well as by the boundedness of F_2 and sgn

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_U\left(\sigma'(\rho^1)\sigma(\rho^1)\zeta_M(\rho^1)\phi_\beta(\rho^1)-\sigma'(\rho^2)\sigma(\rho^2)\zeta_M(\rho^2)\phi_\beta(\rho^2)\right)\\
&\qquad\qquad\times\mathrm{sgn}(\rho^2-\rho^1)F_2\cdot\nabla\iota_\gamma(y)\,dy\,ds\\
&\le c\gamma^{-1}\int_0^t\int_U\left|\sigma'(\rho^1)\sigma(\rho^1)\zeta_M(\rho^1)\phi_\beta(\rho^1)-\sigma'(\rho^2)\sigma(\rho^2)\zeta_M(\rho^2)\phi_\beta(\rho^2)\right|\mathbb{1}_{U\setminus U_\gamma}(y)\,dy\,ds. \qquad (3.36)
\end{aligned}
$$

For ease of notation, for every M,β define the function

$$G_{M,\beta}:\mathbb{R}\to\mathbb{R},\quad G_{M,\beta}(\xi)=\sigma'(\xi)\sigma(\xi)\zeta_M(\xi)\phi_\beta(\xi), \qquad (3.37)$$

which is bounded and Lipschitz for every fixed M, β due to the fact that σ and σ' are locally Lipschitz, and is also identically zero outside $[\beta/2, M+1]$ due to the cutoff functions. For every $y \in (0, \gamma_U)$, let $y^* = y^*(y)$ denote the unique closest point on the boundary to y. Due to the boundary condition being implicitly defined via $\Phi(\rho)$, for $i = 1, 2$ we abuse notation and denote by $\rho^i(y^*) := \Phi^{-1}\left(\bar{f}(y^*)\right)$ and $\rho^i_{M,\beta}(y^*) := (\rho^i(y^*) \vee \beta/2) \wedge (M+1)$ for $y^* \in \partial U$. By adding and subtracting this boundary data, the triangle inequality and Lipschitz property of $G_{M,\beta}$, we have

$$
\begin{aligned}
&c\gamma^{-1} \int_0^t \int_{U \setminus U_\gamma} \left|\sigma'(\rho^1)\sigma(\rho^1)\zeta_M(\rho^1)\phi_\beta(\rho^1) - \sigma'(\rho^2)\sigma(\rho^2)\zeta_M(\rho^2)\phi_\beta(\rho^2)\right| \, dy \, ds \\
&\le c\gamma^{-1} \int_0^t \int_{U \setminus U_\gamma} \left(|G_{M,\beta}(\rho^1) - G_{M,\beta}(\rho^1(y^*))| + |G_{M,\beta}(\rho^2(y^*)) - G_{M,\beta}(\rho^2)|\right) \, dy \, ds \\
&\le c\gamma^{-1} \int_0^t \int_{U \setminus U_\gamma} \left(|\rho^1_{M,\beta}(y,s) - \rho^1_{M,\beta}(y^*,s)| + |\rho^2_{M,\beta}(y^*,s) - \rho^2_{M,\beta}(y,s)|\right) \, dy \, ds.
\end{aligned} \tag{3.38}
$$

Both of the above terms are handled in the same way. Write for $i = 1, 2$, fixed distance from the boundary $\gamma' \in (0, \gamma)$, fixed time $s \in [0, t]$, and for constant $c \in (0, \infty)$ that varies from line to line and is independent of the regularisation constants γ', M, β,

$$
\begin{aligned}
\int_{\partial U_{\gamma'}} |\rho^i_{M,\beta}(y,s) - \rho^i_{M,\beta}(y^*,s)| \, dS(y) &= \int_{\partial U_{\gamma'}} \left| \int_0^{\gamma'} \nabla \rho^i_{M,\beta}\left(y^* + z v_y, s\right) \, dz \right| \, dS(y) \\
&\le \int_{U \setminus U_{\gamma'}} |\nabla \rho^i_{M,\beta}(y,s)| \, dy \\
&\le |U \setminus U_{\gamma'}|^{1/2} \, \|\nabla \rho^i_{M,\beta}\|_{L^2(U \setminus U_{\gamma'})} \\
&\le c(\gamma')^{1/2} \, \|\nabla \rho^i_{M,\beta}\|_{L^2(U \setminus U_{\gamma'})} \\
&\le c(\gamma')^{1/2} \, \|\nabla \rho^i_{M,\beta}\|_{L^2(U)},
\end{aligned}
$$

where in the final line we made the norm independent of γ'. We therefore bound the terms of (3.38) by

$$
\begin{aligned}
&\gamma^{-1} \int_0^t \int_{U \setminus U_\gamma} |\rho^i_{M,\beta}(y,s) - \rho^i_{M,\beta}(y^*)| \, dy \, ds \\
&\qquad = c\,\gamma^{-1} \int_0^t \int_0^\gamma \int_{\partial U_{\gamma'}} |\rho^i_{M,\beta}(y,s) - \rho^i_{M,\beta}(y^*)| \, dy \, d\gamma' \, ds \\
&\qquad \le c\,\gamma^{-1} \int_0^t \|\nabla \rho^i_{M,\beta}\|_{L^2(U)} \left(\int_0^\gamma (\gamma')^{1/2} \, d\gamma' \right) ds \\
&\qquad \le c\,\gamma^{1/2} \|\nabla \rho^i_{M,\beta}\|_{L^1([0,t];L^2(U))},
\end{aligned} \tag{3.39}
$$

which converges to 0 as $\gamma \to 0$ for fixed M, β. Therefore we managed to show that the final two lines of the cutoff (3.31) consisting of the new terms are non-positive in the $\gamma \to 0$ limit.

We now make an important comment about the remaining cutoff terms, as well as martingale and flux terms that are yet to be analysed. Following the computations of the uniqueness proof in Fehrman and Gess [44], the terms involving F_2 in the second and fourth lines of the cutoff term (3.31), as well as martingale and flux terms are handled by integration by parts. More precisely the ϵ, δ limits are taken first, then integration by parts is performed before taking M, β limits[3]. On the bounded domain, the presence of a spatial cutoff ι_γ leads to additional terms with factors $\nabla \iota_\gamma$ when integrating by parts. We emphasise that these new terms can be bounded by following the computation from equation (3.36) to (3.39) above, which we illustrate now.

Firstly, to bound the terms involving F_2 in the second and fourth line of (3.31), analogously to above, in the $\epsilon, \delta \to 0$ limit we use the distributional equality (3.32), followed by the fact that sgn is an odd function and finally the product rule to evaluate the derivative of the cutoffs to get

$$\begin{aligned}
\lim_{\epsilon,\delta\to 0}\Bigg(&\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_{U^2}\sigma'(\rho^1)\sigma(\rho^1)\nabla\rho^1\cdot F_2\bar{k}^{\epsilon,\delta}_{s,1}(1-2\chi^{\epsilon,\delta}_{s,2})\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\,\iota_\gamma(y)\,dy\,dx\,ds\,d\eta \\
&+\frac{1}{2}\int_{\mathbb{R}}\int_0^t\int_{U^2}\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2\cdot F_2\bar{k}^{\epsilon,\delta}_{s,2}(1-2\chi^{\epsilon,\delta}_{s,1})\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\,\iota_\gamma(y)\,dy\,dx\,ds\,d\eta\Bigg) \\
&=\frac{1}{2}\int_0^t\int_U\left(\sigma'(\rho^2)\sigma(\rho^2)\nabla\rho^2-\sigma'(\rho^1)\sigma(\rho^1)\nabla\rho^1\right)\cdot F_2\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta)) \\
&\qquad\qquad\operatorname{sgn}(\rho^2-\rho^1)\,\iota_\gamma(x)\,dx\,ds \\
&=\frac{1}{4}\int_0^t\int_U\left(\nabla\sigma^2(\rho^2)-\nabla\sigma^2(\rho^1)\right)\cdot F_2\left(\mathbb{1}_{M<\rho<M+1}+\beta^{-1}\mathbb{1}_{\beta/2<\rho<\beta}\right) \\
&\qquad\qquad\operatorname{sgn}(\rho^2-\rho^1)\,\iota_\gamma(x)\,dx\,ds. \qquad (3.40)
\end{aligned}$$

The indicator functions in the final line allows us to re-write the term in the final line as

$$\begin{aligned}
=\frac{1}{4\beta}\int_0^t\int_U\nabla\Big(&\left(\sigma^2\left((\rho^2\vee\beta)\wedge\beta/2\right)-\sigma^2(\beta/2)\right) \\
&-\left(\sigma^2\left((\rho^1\vee\beta)\wedge\beta/2\right)-\sigma^2(\beta/2)\right)\Big)\cdot F_2\operatorname{sgn}(\rho^2-\rho^1)\,\iota_\gamma(x)\,dx\,ds \\
+\frac{1}{4}\int_0^t\int_U\nabla\Big(&\left(\sigma^2\left((\rho^2\vee(M+1))\wedge M\right)-\sigma^2(M)\right) \\
&-\left(\sigma^2\left((\rho^1\vee(M+1))\wedge M\right)-\sigma^2(M)\right)\Big)\cdot F_2\operatorname{sgn}(\rho^2-\rho^1)\,\iota_\gamma(x)\,dx\,ds.
\end{aligned}$$

Note that σ^2 is not necessarily increasing, so we cannot use that $\operatorname{sgn}(\rho^2-\rho^1) = \operatorname{sgn}(\sigma^2(\rho^2)-\sigma^2(\rho^1))$ as in (3.35). Instead we again first smooth out the sign function and consider $\operatorname{sgn}_\delta := \operatorname{sgn} * \kappa_1^\delta$ before integrating by parts.

The terms involving M in the final line of (3.40) are handled in a similar way to the terms involving β. For convenience introduce the shorthand notation for $i = 1, 2$,

---

[3] Here we also have a spatial cutoff so take the $\gamma \to 0$ limit before the M, β limits.

$(x,t) \in U \times [0,T]$

$$\rho^i_\beta(x,t) := \big(\rho^i(x,t) \vee \beta\big) \wedge \beta/2.$$

The terms involving ρ^1 and ρ^2 are handled in the same way, so we consider the β-terms involving ρ^2 in the final line of (3.40). We have

$$\begin{aligned}
&\frac{1}{4\beta}\int_0^t\int_U \nabla\left(\sigma^2(\rho^2_\beta) - \sigma^2(\beta/2)\right)\cdot F_2 \mathrm{sgn}(\rho^2-\rho^1)\iota_\gamma(x)\,dx\,ds \\
&= \lim_{\delta\to 0}\frac{1}{4\beta}\int_0^t\int_U \nabla\left(\sigma^2(\rho^2_\beta) - \sigma^2(\beta/2)\right)\cdot F_2 \mathrm{sgn}_\delta(\rho^2-\rho^1)\iota_\gamma(x)\,dx\,ds \\
&= -\lim_{\delta\to 0}\frac{1}{4\beta}\int_0^t\int_U \left(\sigma^2(\rho^2_\beta) - \sigma^2(\beta/2)\right)\nabla\cdot F_2 \mathrm{sgn}_\delta(\rho^2-\rho^1)\iota_\gamma(x)\,dx\,ds \\
&\quad -\lim_{\delta\to 0}\frac{1}{4\beta}\int_0^t\int_U \left(\sigma^2(\rho^2_\beta) - \sigma^2(\beta/2)\right)\nabla(\rho^2-\rho^1)\cdot F_2 (\mathrm{sgn}_\delta)'(\rho^2-\rho^1)\iota_\gamma(x)\,dx\,ds \\
&\quad -\lim_{\delta\to 0}\frac{1}{4\beta}\int_0^t\int_U \left(\sigma^2(\rho^2_\beta) - \sigma^2(\beta/2)\right)\nabla\iota_\gamma(x)\cdot F_2\, \mathrm{sgn}_\delta(\rho^2-\rho^1)\,dx\,ds. \qquad (3.41)
\end{aligned}$$

For the first two terms we can directly take the $\gamma \to 0$ limit since the spatial cutoff converges pointwise to 1 and so they can be handled analogously as on the torus, see the proof Theorem 4.6 of Fehrman and Gess [41]. For instance, for the first term, we have using the boundedness of sgn and $\nabla\cdot F_2$, the continuity of $(\sigma^2)' = 2\sigma\sigma'$, the fundamental theorem of calculus, the dominated convergence theorem and the assumption that either $\sigma\sigma'(0) = 0$ or $\nabla \cdot F_2 = 0$, that almost surely

$$\begin{aligned}
-\lim_{\beta\to 0}\lim_{\gamma\to 0}\lim_{\delta\to 0}\frac{1}{4\beta}\int_0^t\int_U \left(\sigma^2(\rho^2_\beta) - \sigma^2(\beta/2)\right)\nabla\cdot F_2 \mathrm{sgn}_\delta(\rho^2-\rho^1)\iota_\gamma(x)\,dx\,ds \\
= -\frac{1}{2}\int_0^t\int_U (\sigma\sigma')(0)\mathbb{1}_{\rho>0}\nabla\cdot F_2 \mathrm{sgn}(\rho^2-\rho^1) = 0.
\end{aligned}$$

The corresponding terms involving M in the final line of (3.40) are treated similarly, but importantly we have to use the assumption on the oscillations of σ^2 at infinity (2.13).

The only fundamentally new term to handle is the final term of equation (3.41) involving the gradient of spatial cutoff. For this term, we can group the ρ^2 and ρ^1 terms, and repeat the same computation from equation (3.36) to (3.39) after realising that $\sigma^2(\cdot \vee \beta \wedge \beta/2)$ is Lipschitz for every fixed $\beta > 0$, F_2 and sgn are bounded, and noting we take $\gamma \to 0$ limit before M and β limits.

We mention also that the terms involving F_3 in the second and fourth lines can be bounded in a similar way as described above, using the $L^2(U)$-integrability of $\sigma(\rho^i)$, Assumption 2.2.1 and the boundedness of F_3.

Returning back to the cutoff term (3.31), we just need to illustrate how to handle the first and third lines of the cutoff term involving the kinetic measures. Taking the derivative of the cutoffs and then taking the limits in ϵ, δ, we have that there exists a constant $c \in (0,\infty)$ such that

$$\limsup_{\epsilon,\delta\to 0}\left|\int_{\mathbb{R}^2}\int_0^t\int_{U^2}\kappa^{\epsilon,\delta}(y,x,\eta,\xi)\partial_\eta(\zeta_M(\eta)\phi_\beta(\eta))\iota_\gamma(y)(-1+2\chi^{\epsilon,\delta}_{s,2})\,dq^1(x,\xi,s)\,dy\,d\eta\right|$$
$$\leq c\left(\beta^{-1}q^1(U\times[\beta/2,\beta]\times[0,T])+q^1(U\times[M,M+1]\times[0,T])\right).$$

An analogous inequality holds for the term involving the kinetic measure q^2. The decays of the kinetic measure at zero (3.10) and infinity (2.39) imply that by Fatou's lemma, there almost surely exists subsequences $\beta\to 0$ and $M\to\infty$ such that

$$\lim_{M\to\infty}\lim_{\beta\to 0}\left(\beta^{-1}q^1(U\times[\beta/2,\beta]\times[0,T])+q^1(U\times[M,M+1]\times[0,T])\right)=0.$$

Putting (3.31) and subsequent computations together, we conclude

$$\lim_{M\to\infty,\beta\to 0}\lim_{\gamma\to 0}\lim_{\epsilon,\delta\to 0}I_t^{cut}\leq 0.$$

Martingale term.

Following the analysis of the martingale term from Fehrman and Gess [44], we have that, for the unique function $\Theta_{M,\beta}:[0,\infty)\to[0,\infty)$ defined by

$$\Theta_{M,\beta}(0)=0,\quad \Theta'_{M,\beta}(\xi)=\phi_\beta(\xi)\zeta_M(\xi)\sigma'(\xi),$$

in the $\epsilon,\delta\to 0$ limits the martingale term can be written as

$$\begin{aligned}\lim_{\epsilon,\delta\to 0}I_t^{mart}&=\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\iota_\gamma(x)\nabla\cdot\left(\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)d\xi^F\right)\,dx\\&+\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\iota_\gamma(x)\left(\phi_\beta(\rho^1)\zeta_M(\rho^1)\sigma(\rho^1)-\Theta_{M,\beta}(\rho^1)\right)\nabla\cdot d\xi^F\,dx\\&-\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\iota_\gamma(x)\left(\phi_\beta(\rho^2)\zeta_M(\rho^2)\sigma(\rho^2)-\Theta_{M,\beta}(\rho^2)\right)\nabla\cdot d\xi^F\,dx.\end{aligned}\tag{3.42}$$

The final two terms of (3.42) can be handled directly by first directly taking the $\gamma\to 0$ limit and using identity

$$\lim_{M\to\infty}\lim_{\beta\to 0}\phi_\beta(\rho^i)\zeta_M(\rho^i)\sigma(\rho^i)=\sigma(\rho^i)\quad\text{strongly in}\quad L^2(U\times[0,T]),\tag{3.43}$$

alongside the fact that there exists a constant $c\in(0,\infty)$ satisfying

$$\left|\sigma(\rho^i)-\Theta_{M,\beta}(\rho^i)\right|\leq c\left(\sup_{\xi\in[0,\beta]}|\sigma(\xi)|+\sigma(\rho^i)\mathbb{1}_{\rho^i>M}+\sup_{\xi\in[M,(M+1)\wedge\rho^i]}|\sigma(\xi)|\mathbb{1}_{\rho^i>M}\right).\tag{3.44}$$

The first term converges to zero as $\beta\to 0$ using $\sigma(0)=0$ and the continuity of σ and the second term converges to zero strongly in $L^2(\Omega\times[0,T];L^2(U))$ as $M\to\infty$ using that $\sigma(\rho^i)$ is $L^2(U)$-integrable and the fact that ρ^i is $L^1(U\times[0,T])$-integrable.

Finally, as a result of the growth of oscillations of σ^2 given in (2.13), the final term of (3.44) converges to zero strongly as $M \to \infty$ in the space $L^2(\Omega \times [0,T]; L^2(U))$.

We are left to deal with the first term of (3.42). Using again the regularisation of the sgn function and subsequently integrating by parts gives

$$\begin{aligned}
&\int_0^t \int_U \operatorname{sgn}(\rho^2-\rho^1)\iota_\gamma(x)\nabla\cdot\left(\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)d\xi^F\right)\,dx \\
&\qquad = -\lim_{\delta\to 0}\int_0^t\int_U \operatorname{sgn}_\delta(\rho^2-\rho^1)\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)\nabla\iota_\gamma(x)\cdot d\xi^F\,dx \\
&\quad -\lim_{\delta\to 0}\int_0^t\int_U (\operatorname{sgn}_\delta)'(\rho^2-\rho^1)\iota_\gamma(x)\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)(\nabla\rho^2-\nabla\rho^1)\cdot d\xi^F\,dx
\end{aligned} \tag{3.45}$$

The bound on the final term of (3.45) follows from the dominated convergence theorem and the local regularity of solutions after observing that, due to the fact that $\Theta_{M,\beta}$ is Lipschitz, there exists a constant $c \in (0,\infty)$ depending on M, β, that for all $\delta \in (0, \beta/4)$,

$$\left|(\operatorname{sgn}_\delta)'(\rho^2-\rho^1)\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)\right| \le c\mathbb{1}_{\{0\le|\rho^1-\rho^2|\le c\delta,\,\text{and}\,\beta/4\le\rho^i\le M+\delta\,\text{for}\,i=1,2\}}.$$

To conclude, we are left to deal with the first term on the right hand side of (3.45), which is a new term. We can take the $\delta \to 0$ limit here owing to the dominated convergence theorem. By using the Burkholder-Davis-Gundy inequality, see Theorem 4.1 of Revuz and Yor [85], as well as the bound on the derivative of the spatial cutoff given in Lemma 2.4.3, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)\nabla\iota_\gamma(x)\cdot d\xi^F\,dx\right|\right] \\
&\le c\mathbb{E}\left[\int_0^T\left(\int_U \operatorname{sgn}(\rho^2-\rho^1)\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)|\nabla\iota_\gamma(x)|\,dx\right)^2 ds\right]^{1/2} \\
&\le c\gamma^{-1}\,\mathbb{E}\left[\int_0^T\left(\int_U \left|(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right|\mathbb{1}_{U\setminus U_\gamma}(x)\,dx\right)^2 ds\right]^{1/2}.
\end{aligned}$$

For every fixed M, β, we have that $\Theta_{M,\beta}$ is Lipschitz, so this term can now just be handled in the same way as the final term of (3.33). The key point is again that we obtain a factor of $\gamma^{3/2}$, which more than compensates the blow up of γ^{-1}. We have for a constant $c \in (0, \infty)$ that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\left(\Theta_{M,\beta}(\rho^1)-\Theta_{M,\beta}(\rho^2)\right)\nabla\iota_\gamma(x)\cdot d\xi^F\,dx\right|\right]$$

$$\leq c\gamma^{-1}\,\mathbb{E}\left[\int_0^T\left(\sum_{i=1}^2\|\nabla\rho^i_{M,\beta}\|_{L^2(U)}\gamma^{3/2}\right)^2 ds\right]^{1/2}$$

$$\leq c\gamma^{1/2}\sum_{i=1}^2\mathbb{E}\|\nabla\rho^i_{M,\beta}\|_{L^2([0,T];L^2(U))}. \quad (3.46)$$

The local regularity property of kinetic solutions (2.37) ensures that the final term in equation (3.46) above is well-defined. Furthermore, it converges to zero in the $\gamma \to 0$ limit for fixed M, β. This implies almost sure convergence of this new term along a subsequence.

Putting everything together, it follows that along appropriate subsequences, we have

$$\lim_{M\to\infty,\beta\to0}\left(\lim_{\gamma\to0}\left(\lim_{\epsilon,\delta\to0}I_t^{mart}\right)\right)=0.$$

Flux term.
The flux term can be handled in the same way as the martingale term above.

After taking the $\epsilon, \delta \to 0$ limits and using the distributional identity (3.32), we obtain

$$\lim_{\epsilon,\delta\to0}I_t^{flux}=\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\iota_\gamma(x)$$
$$\times\left(\nabla\cdot\nu(\rho^1)\zeta_M(\rho^1)\phi_\beta(\rho^1)-\nabla\cdot\nu(\rho^2)\zeta_M(\rho^2)\phi_\beta(\rho^2)\right)dx\,ds.$$

First define the Lipschitz vector valued function $\Psi_{M,\beta,\nu} = (\Psi_{M,\beta,\nu,i})_{i=1}^d$ such that for $\nu = (\nu_i)_{i=1}^d$

$$\Psi_{M,\beta,\nu,i}(0)=0, \quad \Psi'_{M,\beta,\nu,i}(\xi)=\nu_i'(\xi)\phi_\beta(\xi)\zeta_M(\xi).$$

Then with a similar re-writing as the martingale term, we have

$$\lim_{\epsilon,\delta\to0}I_t^{flux}=\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\iota_\gamma(x)\nabla\cdot\left(\Psi_{M,\beta,\nu}(\rho^1)-\Psi_{M,\beta,\nu}(\rho^2)\right)dx\,ds$$
$$+\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\iota_\gamma(x)\left(\nabla\cdot\nu(\rho^1)\zeta_M(\rho^1)\phi_\beta(\rho^1)-\nabla\cdot\Psi_{M,\beta,\nu}(\rho^1)\right)dx\,ds$$
$$-\int_0^t\int_U \operatorname{sgn}(\rho^2-\rho^1)\iota_\gamma(x)\left(\nabla\cdot\nu(\rho^2)\zeta_M(\rho^2)\phi_\beta(\rho^2)-\nabla\cdot\Psi_{M,\beta,\nu}(\rho^2)\right)dx\,ds.$$

This is analogous to the martingale term (3.42), but even more simple due to the lack of stochastic integral. The final two terms can be handled analogously to the final two terms appearing in the martingale term after taking the $\gamma \to 0$ limit, using identities analogous to (3.43) and (3.44).

The first term can be handled using integration by parts, by a simplified version of the steps (3.45)-(3.46) for the corresponding martingale term. Here we use the $L^1(U)$-integrability of $\nu(\rho)$ and the bound on oscillations of ν at infinity (2.13) to apply the dominated convergence theorem. This allows us to conclude that along appropriate subsequences,

$$\lim_{M\to\infty,\beta\to 0}\lim_{\gamma\to 0}\lim_{\epsilon,\delta\to 0} I_t^{flux} = 0.$$

Conclusion.

Putting everything together, we get from equation (3.26) and the subsequent handling of each term, that there are random subsequences $\epsilon, \delta, \beta, \gamma \to 0$, $M \to \infty$ along which

$$\begin{aligned}\int_{\mathbb{R}}\int_U |\chi_s^1 - \chi_s^2|^2\Big|_{s=0}^{t} &= \lim_{\beta\to 0,M\to\infty}\lim_{\gamma\to 0}\lim_{\epsilon,\delta\to 0}\int_{\mathbb{R}}\int_U |\chi_{s,1}^{\epsilon,\delta} - \chi_{s,2}^{\epsilon,\delta}|^2\phi_\beta\zeta_M\iota_\gamma\Big|_{s=0}^{t} \\ &= \lim_{\beta\to 0,M\to\infty}\lim_{\gamma\to 0}\lim_{\epsilon,\delta\to 0}\left(-2I_t^{err} - 2I_t^{meas} + I_t^{mart} + I_t^{cut} + I_t^{flux} + I_t^{bound}\right) \leq 0.\end{aligned}$$

This gives the desired result, since for every $t \in [0,T]$,

$$\int_U |\rho^1(\cdot,t) - \rho^2(\cdot,t)| = \int_{\mathbb{R}}\int_U |\chi_t^1 - \chi_t^2|^2 \leq \int_{\mathbb{R}}\int_U |\chi_0^1 - \chi_0^2|^2 = \int_U |\rho_0^1 - \rho_0^2|.$$

□

Remark 3.2.3. *The pathwise contraction property in the equation above implies the pathwise continuity of solutions with respect to the initial condition. This is a stronger result than the uniqueness of equation (2.23), and will enable us to construct a dynamical system for the SPDE which will be useful in the study of large deviations principles, see Proposition 6.3.5 below.*

Chapter 4

Existence

In this chapter we construct a stochastic kinetic solution of the generalised Dean–Kawasaki equation (2.23) in the sense of Definition 2.3.6. The existence consists of three steps. Firstly, in Section 4.1 we will prove $L^2(U\times[0,t])$-energy estimates for a suitable regularised version of (2.23). We then proceed to prove higher order space-time regularity results for the regularised equation.

As an aside, in Section 4.2, we prove an entropy estimate for the equation and a localised version of this argument helps us to prove the decay of kinetic measure at zero (3.10) that was required for the uniqueness. For all of the estimates we will need to introduce PDEs which are solutions to Laplace's equation, that ensure certain functions vanish along the boundary when applying Itô's formula.

The second and third steps are analogous to Section 5 of Fehrman and Gess [41], and so we are brief in their presentation. For the second step, in the first part of Section 4.4 we show that there exists a stochastic kinetic solution to the regularised equation. The final step, illustrated in the latter half of Section 4.4, we pass to the limit in the regularisation.

4.1 A priori estimates for the regularised equation

In this section we start with some definitions and properties of solutions to Laplace's equation that we will use in the energy estimates. The PDEs are defined in Definition 4.1.1, and in Remark 4.1.2 we state a remark on the specific form of the PDEs.

In Definition 4.1.4 and 4.1.5 we define weak solutions for a regularised version of the Dean–Kawasaki equation. In Proposition 4.1.8 we prove an $L^2(U\times[0,t])$-estimate for the regularised equation for any $t\in[0,T]$. This estimate allows us to prove higher order spatial regularity of the solution in Lemma 4.1.11 and higher order time regularity of the solution cutoff away from zero in Proposition 4.1.14 that will be essential in the tightness arguments.

We begin by introducing two PDEs that allow us to avoid boundary terms in our energy estimate.

Definition 4.1.1 (The PDEs $h_{\Phi^{-1}(\bar{f})}$ and $h_{S'_M(\Phi^{-1}(\bar{f}))}$). *Recall that* $\bar{f}$ *is the boundary condition of the generalised Dean–Kawasaki equation* (1.2)*, defined implicitly through*

$\Phi(\rho)$. Define $h_{\Phi^{-1}(\bar{f})}: U \to \mathbb{R}$ to be the unique weak solution of the unique harmonic PDE

$$\begin{cases} -\Delta h_{\Phi^{-1}(\bar{f})} = 0 & \text{on } U, \\ h_{\Phi^{-1}(\bar{f})} = \Phi^{-1}(\bar{f}) & \text{on } \partial U. \end{cases} \tag{4.1}$$

For fixed $0 < M_1 < M_2$, define $M := (M_1, M_2)$. For the function $S_M : [0,\infty) \to [0,\infty)$ satisfying $S''_M(\xi) = \mathbb{1}_{M_1<\xi<M_2}(\xi)$, define $h_{S'_M(\Phi^{-1}(\bar{f}))} : U \to \mathbb{R}$ to be the unique weak solution to the harmonic PDE

$$\begin{cases} -\Delta h_{S'_M(\Phi^{-1}(\bar{f}))} = 0 & \text{on } U, \\ h_{S'_M(\Phi^{-1}(\bar{f}))} = S'_M(\Phi^{-1}(\bar{f})) & \text{on } \partial U. \end{cases}$$

Since the boundary data $\bar{f}$ and the non-linear function Φ are both non-negative, by the maximum principle, both $h_{\Phi^{-1}(\bar{f})}$ and $h_{S'_M(\Phi^{-1}(\bar{f}))}$ are non-negative. Additionally, since S'_M is $[0, M_2 - M_1]$-valued, the maximum principle also tells us that $h_{S'_M(\Phi^{-1}(\bar{f}))}$ is bounded by $M_2 - M_1$. The assumption on the PDE $h_{\Phi^{-1}(\bar{f})}$ is stated in point 2 of Assumption 2.2.9.

Remark 4.1.2 (Choice of harmonic PDEs.). *The boundary data of the PDEs are chosen to ensure certain functions vanish along the boundary when applying Itô's formula, and consequently allows integration by parts without picking up boundary terms. They are all also chosen to be harmonic in U, and this is for several reasons which we illustrate using the example of PDE $h_{\Phi^{-1}(\bar{f})}$ appearing in equation* (4.13) *in the $L^2(U \times [0,t])$-energy estimate below, which reads*

$$\int_0^t \int_U \nabla h_{\Phi^{-1}(\bar{f})} \cdot \Bigg(\nabla\Phi(\rho^n) + \frac{1}{n}\nabla\rho^n - \sigma_n(\rho^n)\dot{\xi}^F - \nu(\rho^n) + \frac{1}{2}F_1[\sigma'_n(\rho^n)]^2\nabla\rho^n + \frac{1}{2}\sigma'_n(\rho^n)\sigma_n(\rho^n)F_2 \Bigg). \tag{4.2}$$

1. *The first and second terms above involve integrands with a factor of $\nabla h_{\Phi^{-1}(\bar{f})}$ multiplied by another gradient term. To bound these terms we integrate by parts and move the derivative onto the $\nabla h_{\Phi^{-1}(\bar{f})}$ term. Since $h_{\Phi^{-1}(\bar{f})}$ is harmonic we have that they turn into boundary terms. That is, for $f : U \times [0,T] \to \mathbb{R}$ such that the integrals below are well defined, we have for every $t \in [0,T]$,*

$$\begin{aligned} &\int_U \nabla f(x,t) \cdot \nabla h_{\Phi^{-1}(\bar{f})}(x)\, dx \\ &\quad = -\int_U f(x,t)\Delta h_{\Phi^{-1}(\bar{f})}(x)\, dx + \int_{\partial U} f(x,t)\frac{\partial h_{\Phi^{-1}(\bar{f})}}{\partial\hat{\eta}}(x)\, dS(x) \\ &\quad = \int_{\partial U} f(x,t)\frac{\partial h_{\Phi^{-1}(\bar{f})}}{\partial\hat{\eta}}(x)\, dS(x). \end{aligned}$$

   The terms in the final line are then handled using Hölder's and Young's inequality.

2. *For the fourth and final terms of equation* (4.2)*, we have to handle integrands where* $\nabla h_{\Phi^{-1}(\bar{f})}$ *is multiplied by terms without another gradient. Due to the condition* (2.19) *in Assumption 2.2.9 as well as Remark 2.2.2, we use the* $L^\infty(U;\mathbb{R}^d)$*-bound on* $\nabla h_{\Phi^{-1}(\bar{f})}$ *given by Proposition 2.2.11.*

   That is, for an $L^1([0,T];L^1(U;\mathbb{R}^d))$*-integrable vector field* $F : U \times [0,T] \to \mathbb{R}^d$*, we have for every* $t \in [0,T]$*,*

$$\int_0^t \int_U F(x,s) \cdot \nabla h_{\Phi^{-1}(\bar{f})}(x)\, dx\, ds \leq \|\nabla h_{\Phi^{-1}(\bar{f})}\|_{L^\infty(U;\mathbb{R}^d)} \|F\|_{L^1(U\times[0,t];\mathbb{R}^d))},$$

   where we will use assumed growth bounds on F *to control the final term on the right hand side.*

The normal derivative arising in the first point above is handled by the following Theorem, see Theorem 2.4(iii) of Fabes [36] for a more general statement and proof, bounding the $L^p(\partial U)$-norms, for $p \geq 2$, of a whole range of directional derivatives including the normal derivative by the $H^p(\partial U)$-norms of the boundary data. Here we just focus on the case $p = 2$. See also Lemma 4 of Bella, Fehrman and Otto [6] for a similar result on the less regular domain of the cube $U = [0,1]^d$.

Theorem 4.1.3. *Suppose* $U \subset \mathbb{R}^d$ *is a bounded,* C^1*-regular domain satisfying* $\mathbb{R}^d \backslash \bar{U}$ *is connected. Let* $h_{\bar{h}}$ *be the solution to the Laplace equation with boundary data* $\bar{h}$*, defined in* (2.22)*. Then there exists a constant* $c \in (0,\infty)$ *independent of the boundary data* $\bar{h}$ *such that*

$$\left\|\frac{\partial h_{\bar{h}}}{\partial \hat{\eta}}\right\|_{L^2(\partial U)} \leq c\|\bar{h}\|_{H^1(\partial U)}, \tag{4.3}$$

where the $H^1(\partial U)$*-norm is defined in Definition 2.2.7.*

When equation (4.3) is satisfied, say that the Laplace equation $h_{\bar{h}}$ satisfies the "Dirichlet-to-Neumann map" property, since (4.3) illustrates that normal derivatives of $h_{\bar{h}}$ can be bounded up to a constant by the norm of the boundary data $\bar{h}$ and the tangential derivatives of $h_{\bar{h}}$. Importantly, the above theorem allows us to express the subsequent bounds in terms of norms of the boundary data $\bar{f}$ directly rather than in terms of norms the PDEs $h_{\Phi^{-1}(\bar{f})}, h_{S'_M(\Phi^{-1}(\bar{f}))}$.

We once again need to deal with potential singularity from the Itô-to-Stratonovich conversion and ensure the integrals below are well defined, so need the following smoothing of the non-linear function σ. The smoothing will subsequently be dispensed of via an approximation argument in Lemma 4.4.3. Furthermore, the addition of the Laplacian term in the equation below ensures that solutions have the required regularity to apply Itô's formula.

Definition 4.1.4 (Regularised equation)**.** *Let* $\{\sigma_n\}_{n\in\mathbb{N}}$ *be a sequence of functions satisfying for every* $n \in \mathbb{N}$*,*

$$\sigma_n \in C([0,\infty)) \cap C^\infty((0,\infty)), \quad \textit{with} \quad \sigma_n(0) = 0 \quad \textit{and} \quad (\sigma_n)' \in C_c^\infty([0,\infty)). \tag{4.4}$$

For every $n \in \mathbb{N}$, the regularised generalised Dean–Kawasaki equation ρ^n is a solution to the equation

$$\partial_t \rho^n = \Delta\Phi(\rho^n) + \frac{1}{n}\Delta\rho^n - \nabla\cdot\Big(\sigma_n(\rho^n)\dot{\xi}^F + \nu(\rho^n)\Big) \\ + \frac{1}{2}\nabla\cdot\big(F_1[\sigma_n'(\rho^n)]^2\nabla\rho^n + \sigma_n'(\rho^n)\sigma_n(\rho^n)F_2\big)\,, \quad (4.5)$$

with the usual initial condition $\rho^n(\cdot,0) = \rho_0$ and boundary condition $\Phi(\rho^n)|_{\partial U} = \bar{f}$.

For the regularised equation with smoothed σ we can make sense of weak solutions.

Definition 4.1.5 (Weak solution of regularised equation (4.5))**.** *Suppose that the spatial components of the noise ξ^F, the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable, let $h_{\Phi^{-1}(\bar{f})}$ be as in Definition 4.1.1, and let $\Theta_{\Phi,2}$ be defined as in equation* (2.5)*. For fixed regularisation constant $n \in \mathbb{N}$, a weak solution ρ^n of the regularised equation* (4.5) *is a continuous $L^2(U)$-valued, non-negative $(\mathcal{F}_t)_{t\geq 0}$-predictable process satisfying*

1. *Boundary condition, regularity of solution: We have that*

$$\big(\rho^n - h_{\Phi^{-1}(\bar{f})}\big) \in L^2([0,T]; H_0^1(U)).$$

2. *The equation: For every $\psi \in C_c^\infty(U)$, almost surely for every $t \in [0,T]$ we have*

$$\int_U \rho^n(x,t)\psi(x)\,dx = \int_U \rho_0\psi\,dx - \int_0^t\int_U \Phi'(\rho^n)\nabla\rho^n\cdot\nabla\psi\,dx\,ds \\ - \frac{1}{n}\int_0^t\int_U \nabla\rho^n\cdot\nabla\psi\,dx\,ds + \int_0^t\int_U \nu(\rho^n)\cdot\nabla\psi\,dx\,ds + \int_0^t\int_U \sigma_n(\rho^n)\nabla\psi\cdot d\xi^F\,ds \\ - \frac{1}{2}\int_0^t\int_U F_1(\sigma_n'(\rho^n))^2\nabla\rho^n\cdot\nabla\psi\,dx\,ds - \frac{1}{2}\int_0^t\int_U \sigma_n(\rho^n)\sigma_n'(\rho^n)F_2\cdot\nabla\psi\,dx\,ds.$$

As mentioned, the regularisation (4.4) ensures that the first integral in the final line above is well defined. We arrive to the first result of this chapter. We obtain a bound for powers of the solution by comparing it with the non-linear function $\Theta_{\Phi,2}$ defined in equation (2.5). Such an estimate is required because we are not on the torus so do not have preservation of mass, and nor do we have enough regularity of the solution to quantify the flux along the boundary. The proof of this estimate is precisely where we use the new assumption (2.8) on $\Theta_{\Phi,2}$ from Assumption 2.2.1.

Proposition 4.1.6 (Estimate for $L^1([0,T]; L^k(U))$-norm of the regularised equation)**.** *Suppose that Φ satisfies the polynomial growth condition* (2.4) *with constant $m \in (0,\infty)$, let $h_{\Phi^{-1}(\bar{f})}$ be the harmonic PDE defined in Definition 4.1.1 satisfying point 2 of Assumption 2.2.9 and let $\Theta_{\Phi,2}$ be defined as in equation* (2.5)*. For fixed regularisation constant $n \in \mathbb{N}$, let ρ^n denote a weak solution of the regularised equation*

(4.5) *in the sense of Definition 4.1.5. Then one has for every* $\epsilon \in (0,1), t \in [0,T]$ *and* $k \in (0, m+1)$*, the existence of a constant* $c \in (0,\infty)$ *independent of the regularisation* n *such that*

$$\int_0^t \int_U |\rho^n|^k \leq c\left(\frac{t}{\epsilon} + \epsilon t \|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\|^2_{H^1(U)} + \epsilon \int_0^t \int_U |\nabla \Theta_{\Phi,2}(\rho^n)|^2\right).$$

Proof. The bound (2.8) on $\Theta_{\Phi,2}$ given in Assumption 2.2.1 gives for a constant $c \in (0,\infty)$ independent of the regularisation n,

$$\int_0^t \int_U |\rho^n|^k \leq ct + c\int_0^t \int_U |\Theta_{\Phi,2}(\rho^n)|^{\frac{2k}{m+1}}.$$

The exponent in the integrand on the right hand side is strictly less than two. Applying Young's inequality with ϵ and exponent $\frac{m+1}{k} > 1$ and Jensen's inequality then gives for a constant $c \in (0,\infty)$

$$\begin{aligned}\int_0^t \int_U |\Theta_{\Phi,2}(\rho^n)|^{\frac{2k}{m+1}} &\leq \int_0^t \left[\frac{(m+1-k)1^{\frac{m+1}{m+1-k}}}{\epsilon(m+1)} + \frac{\epsilon k}{m+1}\left(\int_U \Theta_{\Phi,2}(\rho^n)^{\frac{2k}{m+1}}\right)^{\frac{m+1}{k}}\right] \\ &\leq \frac{ct}{\epsilon} + c\epsilon \int_0^t \int_U \Theta_{\Phi,2}(\rho^n)^2. \qquad (4.6)\end{aligned}$$

Using the trivial inequality $a^2 \leq 2(a-b)^2 + 2b^2$ with $b = \Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})$, where $h_{\Phi^{-1}(\bar{f})}$ is the weak solution of the harmonic PDE from Definition 4.1.1, using that ρ^n and $h_{\Phi^{-1}(\bar{f})}$ have coincide on ∂U and so applying Poincaré inequality gives the claim,

$$\begin{aligned}&\int_0^t \left(\int_U |\rho^n|\right)^k \\ &\quad\leq \frac{ct}{\epsilon} + c\epsilon \int_0^t \int_U \left(\Theta_{\Phi,2}(\rho^n) - \Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\right)^2 + c\epsilon \int_0^t \int_U \Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})^2 \\ &\quad\leq \frac{ct}{\epsilon} + c\epsilon \int_0^t \int_U \left|\nabla\left(\Theta_{\Phi,2}(\rho^n) - \Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\right)\right|^2 + c\epsilon t \int_U \Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})^2 \\ &\quad\leq \frac{ct}{\epsilon} + c\epsilon \int_0^t \int_U |\nabla \Theta_{\Phi,2}(\rho^n)|^2 + c\epsilon t \left\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\right\|^2_{H^1(U)}. \qquad (4.7)\end{aligned}$$

□

Remark 4.1.7. *Using Jensen's inequality, the above proposition bounding the* $L^k(U \times [0,t])$*-norm also provides a bound the* $L^p([0,t]; L^1(U))$*-norm of the solution, since*

$$\int_0^t \left(\int_U |\rho^n|\right)^k = |U|^k \int_0^t \left(\frac{1}{|U|}\int_U |\rho^n|\right)^k \leq |U|^{k-1} \int_0^t \int_U |\rho^n|^k.$$

Since $k \in (0, m+1)$*, the proposition implies that for the full range* $m \in (0,\infty)$ *we at least have an* $L^1(U \times [0,t])$ *estimate for the solution. Apart from bounding the* $L^1(U)$*-norm, the result is primarily useful when* $m, k > 1$*, since if* $k < 1$ *we can just use interpolation to obtain the integrability*

$$\int_0^t \int_U |\rho^n|^k \leq \int_0^t \int_U (1 + |\rho^n|) < \infty.$$

The proposition below is the first energy estimate that we will prove for weak solutions of the regularised Dean–Kawasaki equation. The estimate is technical as we make explicit the dependence on the boundary data. In Remark 4.1.10 we give a simplified version of the estimate in the case of constant boundary data.

Proposition 4.1.8 (Energy estimate for solution of regularised equation)**.** *Suppose that the spatial components of the noise ξ^F, the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable, let the functions $\Theta_{\Phi,2}$, Θ_ν and Ψ_σ be defined as in Assumption 2.2.9 and let $h_{\Phi^{-1}(\bar{f})}$ be the harmonic PDE defined in Definition 4.1.1. For $n \in \mathbb{N}$, if ρ^n is a weak solution of the regularised equation* (4.5) *in the sense of Definition 4.1.5, then one has for every $t \in [0,T]$, a constant $c \in (0,\infty)$ independent of the regularisation n, the estimate*

$$\begin{aligned}\frac{1}{2}\sup_{s\in[0,t]}\mathbb{E}\left[\int_U\left(\rho^n(x,s)-h_{\Phi^{-1}(\bar{f})}(x)\right)^2\right]+\mathbb{E}\left[\int_0^t\int_U|\nabla\Theta_{\Phi,2}(\rho^n)|^2\right]+\frac{1}{n}\mathbb{E}\left[\int_0^t\int_U|\nabla\rho^n|^2\right]\\ \leq\frac{1}{2}\|\rho_0-h_{\Phi^{-1}(\bar{f})}\|^2_{L^2(U)}+ct\left(1+\|\nabla h_{\Phi^{-1}(\bar{f})}\|_{L^\infty(U;\mathbb{R}^d)}\right)\left(1+\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\|^2_{H^1(U)}\right)\\ +ct\left(\|\sigma_n^2(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)}+\|\Theta_\nu(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U;\mathbb{R}^d)}+\|\bar{f}\|^2_{L^2(\partial U)}+\|\Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\|^2_{L^2(\partial U)}\right)\\ +ct\left(1+\frac{1}{n}\right)\|\Phi^{-1}(\bar{f})\|^2_{H^1(\partial U)}.\end{aligned}\tag{4.8}$$

Let the function S_M and the PDE $h_{S'_M(\Phi^{-1}(\bar{f}))}$ be defined as in Definition 4.1.1, and define the vector-valued function $\Theta_{M,\nu}:[0,\infty)\to\mathbb{R}^d$ by

$$\Theta_{M,\nu}(0)=0,\quad \Theta'_{M,\nu}(\xi)=\mathbb{1}_{M_1<\xi<M_2}\nu(\xi).$$

Then we have for every $M_1 < M_2 \in (0,\infty), t \in [0,T]$ the existence of constant $c \in (0,\infty)$ independent of the regularisation n and both M_1, M_2 such that

$$\begin{aligned}\mathbb{E}\left[\int_0^t\int_U\mathbb{1}_{M_1<\rho^n<M_2}\left(\Phi'(\rho^n)|\nabla\rho^n|^2+\frac{1}{n}|\nabla\rho^n|^2\right)\right]\leq\mathbb{E}\int_U(\rho_0(x)-M_1)_+\\ +\|h_{S'_M(\Phi^{-1}(\bar{f}))}\|_{L^2(U)}\mathbb{E}\|\rho^n(\cdot,t)\|_{L^2(U)}+c\left\|S'_M(\Phi^{-1}(\bar{f}))\right\|_{H^1(\partial U)}\|\nabla h_{S'_M(\Phi^{-1}(\bar{f}))}\|_{L^\infty(U;\mathbb{R}^d)}\\ \times\left(t+t\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\|^2_{H^1(U)}+\mathbb{E}\int_0^t\int_U|\nabla\Theta_{\Phi,2}(\rho^n)|^2\right)+c\mathbb{E}\int_0^t\int_U\mathbb{1}_{\rho^n\geq M_1}\sigma_n^2(\rho^n\wedge M_2)\\ +t\|\Theta_{M,\nu}(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U;\mathbb{R}^d)}+ct\|\left(\sigma_n^2((\Phi^{-1}(\bar{f})\wedge M_2)\vee M_1)-\sigma_n^2(M_1)\right)\|_{L^1(\partial U)}\\ +ct\left\|S'_M(\Phi^{-1}(\bar{f}))\right\|_{H^1(\partial U)}\left(\left\|\bar{f}\right\|_{L^2(\partial U)}+\frac{1}{n}\left\|\Phi^{-1}(\bar{f})\right\|_{L^2(\partial U)}+\left\|\Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\right\|_{L^2(\partial U)}\right).\end{aligned}\tag{4.9}$$

The $L^2(U)$-norm of solution and the $L^2(U\times[0,t])$-norm of $|\nabla\Theta_{\Phi,2}|$ on the right hand side of (4.9) *are both bounded as a consequence of the first estimate. We do not write the bound explicitly for readability.*

Proof. To prove the first claim we apply Itô's formula to the square of the solution minus the solution of the PDE $h_{\Phi^{-1}(\bar{f})}$, which ensures that the difference vanishes along the boundary. This gives for every $t \in [0,T]$ that

$$\int_U \left(\rho^n(x,s) - h_{\Phi^{-1}(\bar f)}(x)\right)^2 \, dx\Big|_{s=0}^{t} = \int_0^t \int_U d\left(\rho^n(x,s) - h_{\Phi^{-1}(\bar f)}(x)\right)^2 \, dx$$
$$= \int_0^t \int_U 2\left(\rho^n - h_{\Phi^{-1}(\bar f)}\right) \, d\left(\rho^n - h_{\Phi^{-1}(\bar f)}\right) + \frac{1}{2} 2\left(\rho^n - h_{\Phi^{-1}(\bar f)}\right)^0 d\left\langle \rho^n - h_{\Phi^{-1}(\bar f)}\right\rangle_s \, dx, \tag{4.10}$$

where $(\langle \cdot \rangle_s)_{s\in[0,T]}$ denotes the quadratic variation process.

For the first term on the right hand side, since $h_{\Phi^{-1}(\bar f)}$ does not depend on time, it follows that $d\left(\rho^n - h_{\Phi^{-1}(\bar f)}\right) = d\rho^n$, and after integrating by parts, we have

$$\int_0^t \int_U 2\left(\rho^n - h_{\Phi^{-1}(\bar f)}\right) \, d\left(\rho^n - h_{\Phi^{-1}(\bar f)}\right) \, dx$$
$$= -2\int_0^t \int_U \nabla\left(\rho^n - h_{\Phi^{-1}(\bar f)}\right) \cdot \Bigg(\nabla\Phi(\rho^n) + \frac{1}{n}\nabla\rho^n - \sigma_n(\rho^n)\dot\xi^F$$
$$-\nu(\rho^n) + \frac{1}{2}F_1[\sigma_n'(\rho^n)]^2\nabla\rho^n + \frac{1}{2}\sigma_n'(\rho^n)\sigma_n(\rho^n)F_2\Bigg) \, dx\, ds. \tag{4.11}$$

Similarly, for the the Itô correction term we have that $d\left\langle \rho^n - h_{\Phi^{-1}(\bar f)}\right\rangle = d\langle\rho^n\rangle$ and using the definition of the noise gives

$$\int_0^t \int_U d\left\langle \rho^n - h_{\Phi^{-1}(\bar f)}\right\rangle_s \, dx$$
$$= \int_0^t \int_U \left(F_1(\sigma_n'(\rho^n))^2|\nabla\rho^n|^2 + 2\sigma_n\sigma_n'(\rho^n)F_2\cdot\nabla\rho^n + F_3\sigma_n^2(\rho^n)\right) \, dx\, ds. \tag{4.12}$$

Putting equation (4.10) together with equations (4.11) and (4.12), we have after cancellation and re-arranging that

$$\frac{1}{2}\int_U \left(\rho^n(x,s) - h_{\Phi^{-1}(\bar f)}(x)\right)^2 \, dx\Big|_{s=0}^{t} + \int_0^t \int_U \Phi'(\rho^n)|\nabla\rho^n|^2 + \frac{1}{n}|\nabla\rho^n|^2$$
$$= \int_0^t \int_U \left(\sigma_n(\rho^n)\nabla\rho^n\cdot\dot\xi^F + \nabla\rho^n\cdot\nu(\rho^n) + \frac{1}{2}\sigma_n'(\rho^n)\sigma_n(\rho^n)\nabla\rho^n\cdot F_2 + \frac{1}{2}F_3\sigma_n^2(\rho^n)\right)$$
$$+ \int_0^t \int_U \nabla h_{\Phi^{-1}(\bar f)} \cdot \Bigg(\nabla\Phi(\rho^n) + \frac{1}{n}\nabla\rho^n - \sigma_n(\rho^n)\dot\xi^F - \nu(\rho^n)$$
$$+\frac{1}{2}F_1[\sigma_n'(\rho^n)]^2\nabla\rho^n + \frac{1}{2}\sigma_n'(\rho^n)\sigma_n(\rho^n)F_2\Bigg). \tag{4.13}$$

We note that the penultimate term in (4.13) is well-defined because of the regularisation σ_n.

We handle each term on the right hand side of (4.13) in turn. The noise term in both lines vanish after taking an expectation. The second term on the right hand side remains as a boundary integral in the following way. As defined in the first point of Assumption 2.2.9, let $\Theta_\nu := (\Theta_{\nu,i})_{i=1,\ldots,d}$ be such that $\Theta_{\nu,i}$ denotes the anti-derivative of ν_i. By the divergence theorem and the trivial fact that $|\hat{\eta}| = 1$, we have the bound

$$\begin{aligned}\int_0^t \int_U \nabla \rho^n \cdot \nu(\rho^n)\,dx\,ds = \int_0^t \int_U \nabla \cdot \Theta_\nu(\rho^n)\,dx\,ds = \int_0^t \int_{\partial U} \Theta_\nu(\Phi^{-1}(\bar{f})) \cdot \hat{\eta}\,dx\,ds \\ \leq \int_0^t \int_{\partial U} \left|\Theta_\nu(\Phi^{-1}(\bar{f})) \cdot \hat{\eta}\right|\,dx\,ds \leq t\|\Theta_\nu(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U;\mathbb{R}^d)},\end{aligned}$$

where we used the fact that the boundary data does not depend on time. The third term can be bounded by integration by parts, noting either $\nabla \cdot F_2 = 0$ or it is bounded, the bound $\sigma^2(\xi) \leq c\Phi(\xi)$ from (2.9) in Assumption 2.2.1, and the fact that $F_2 \cdot \hat{\eta}$ is bounded on ∂U,

$$\begin{aligned}&\frac{1}{4}\int_0^t \int_U \nabla \sigma_n^2(\rho^n) \cdot F_2\,dx\,ds \\ &\quad = -\frac{1}{4}\int_0^t \int_U \sigma_n^2(\rho^n) \nabla \cdot F_2\,dx\,ds + \frac{1}{4}\int_0^t \int_{\partial U} \sigma_n^2(\Phi^{-1}(\bar{f})) F_2 \cdot \hat{\eta}\,dS(x)\,ds \\ &\quad \leq c\int_0^t \int_U \Phi(\rho^n)\,dx\,ds + ct\int_{\partial U} \sigma_n^2(\Phi^{-1}(\bar{f}))\,dS(x). \qquad (4.14)\end{aligned}$$

In the final term above we again used the fact that $\bar{f}$ does not depend on time. We also used, and will repeatedly use below, the fact that we can pick our approximating sequence σ_n to also uniformly satisfy the assumptions imposed on σ from Assumption 2.2.1.

The final term on the right hand side of (4.13) can be bounded by the first term on the final line of (4.14) after noting that F_3 is bounded and once more using the bound $\sigma^2(\xi) \leq c\Phi(\xi)$ from Assumption 2.2.1.

The terms on the right hand side of (4.13) involving $\nabla h_{\Phi^{-1}(\bar{f})}$ would all vanish if the boundary condition was constant. Otherwise they can be bounded precisely as described in points one and two of Remark 4.1.2. The first and second terms involving another gradient are reduced to boundary terms using integration by parts and the fact that $h_{\Phi^{-1}(\bar{f})}$ is harmonic,

$$\begin{aligned}&\int_0^t \int_U \nabla h_{\Phi^{-1}(\bar{f})} \cdot \left(\nabla \Phi(\rho^n) + \frac{1}{n}\nabla \rho^n\right)\,dx\,ds \\ &\qquad = t\int_{\partial U} \frac{\partial h_{\Phi^{-1}(\bar{f})}}{\partial \hat{\eta}} \left(\bar{f} + \frac{1}{n}\Phi^{-1}(\bar{f})\right)\,dS(x).\end{aligned}$$

The fifth term is also handled using integration by parts. Defining Ψ_{σ_n} as in point 7 of Assumption 2.2.1, integration by parts, the product rule and the definition of

$h_{\Phi^{-1}(\bar{f})}$ prove that

$$\begin{aligned}\int_0^t\int_U \nabla h_{\Phi^{-1}(\bar{f})}\cdot\frac{1}{2}F_1[\sigma_n'(\rho^n)]^2\nabla\rho^n\,dx\,ds &= \frac{1}{2}\int_0^t\int_U \nabla h_{\Phi^{-1}(\bar{f})}\cdot F_1\nabla\Psi_{\sigma_n}(\rho^n)\,dx\,ds\\ = -\frac{1}{2}\int_0^t\int_U \nabla\cdot(\nabla h_{\Phi^{-1}(\bar{f})}F_1)\Psi_{\sigma_n}(\rho^n)\,dx\,ds &+ \frac{1}{2}\int_0^t\int_{\partial U}(\nabla h_{\Phi^{-1}(\bar{f})}F_1)\cdot\hat{\eta}\Psi_{\sigma_n}(\rho^n)\,dx\,ds\\ = -\int_0^t\int_U \nabla h_{\Phi^{-1}(\bar{f})}\cdot F_2\Psi_{\sigma_n}(\rho^n)\,dx\,ds &+ \frac{t}{2}\int_{\partial U}\frac{\partial h_{\Phi^{-1}(\bar{f})}}{\partial\hat{\eta}}F_1\Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\,dx\,ds,\end{aligned} \tag{4.15}$$

where in the final line we used the identity $\frac{1}{2}\nabla F_1 = F_2$. The fact that F_1 is continuous on $\bar{U}$ helps to ensure that the final integral is well-defined. We handle the first term in the final line using point 7 of Assumption 2.2.1. The assumption can be used to illustrate that if either the boundary data is constant or $F_2 = 0$, the integral vanishes. If neither of these are true, then the bound (2.12) for Ψ_σ holds, and alongside the facts that $\nabla h_{\Phi^{-1}(\bar{f})}$ and F_2 are $L^\infty(U;\mathbb{R}^d)$-bounded, we have the existence of a constant $c\in(0,\infty)$ such that

$$-\int_0^t\int_U \nabla h_{\Phi^{-1}(\bar{f})}\cdot F_2\Psi_{\sigma_n}(\rho^n)\,dx\,ds \le c\|\nabla h_{\Phi^{-1}(\bar{f})}\|_{L^\infty(U;\mathbb{R}^d)}\int_0^t\int_U(1+\rho^n+\Phi(\rho^n))\,dx\,ds.$$

The remaining two terms in the final integral of (4.13) are also handled using the $L^\infty(U;\mathbb{R}^d)$-bound on $\nabla h_{\Phi^{-1}(\bar{f})}$, the boundedness of F_2 and the bounds (2.10) on ν and $\sigma\sigma'$ from Assumption 2.2.1, which gives for a constant $c\in(0,\infty)$,

$$\begin{aligned}&\int_0^t\int_U \nabla h_{\Phi^{-1}(\bar{f})}\cdot\left(-\nu(\rho^n)+\frac{1}{2}\sigma_n'(\rho^n)\sigma_n(\rho^n)F_2\right)\,dx\,ds\\ &\qquad\le c\|\nabla h_{\Phi^{-1}(\bar{f})}\|_{L^\infty(U;\mathbb{R}^d)}\int_0^t\int_U\left(|\nu(\rho^n)|+|\sigma_n'(\rho^n)\sigma_n(\rho^n)|\right)\,dx\,ds\\ &\qquad\le c\|\nabla h_{\Phi^{-1}(\bar{f})}\|_{L^\infty(U;\mathbb{R}^d)}\int_0^t\int_U(1+\rho^n+\Phi(\rho^n))\,dx\,ds.\end{aligned} \tag{4.16}$$

Putting everything together, (4.13) and the subsequent computations give after taking an expectation

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\left[\int_U\left(\rho^n(x,s)-h_{\Phi^{-1}(\bar{f})}(x)\right)^2dx\Big|_{s=0}^t\right]+\mathbb{E}\left[\int_0^t\int_U|\nabla\Theta_{\Phi,2}(\rho^n)|^2\right]+\frac{1}{n}\mathbb{E}\left[\int_0^t\int_U|\nabla\rho^n|^2\right]\\ &\le c\left(1+\|\nabla h_{\Phi^{-1}(\bar{f})}\|_{L^\infty(U;\mathbb{R}^d)}\right)\mathbb{E}\left[\int_0^t\int_U(1+\rho^n+\Phi(\rho^n))\right]+t\|\Theta_\nu(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U;\mathbb{R}^d)}\\ &\quad+ct\|\sigma_n^2(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)}+t\int_{\partial U}\frac{\partial h_{\Phi^{-1}(\bar{f})}}{\partial\hat{\eta}}\left(\bar{f}+\frac{1}{n}\Phi^{-1}(\bar{f})+\frac{F_1}{2}\Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\right),\end{aligned} \tag{4.17}$$

We look to further simplify the terms on the right hand side of equation (4.17). To bound the first terms on the right hand side, the polynomial growth condition for Φ (2.4) alongside Proposition 4.1.6 gives for a constant $c \in (0,\infty)$ and arbitrary $\epsilon \in (0,1)$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_U \left(|\rho^n| + |\Phi(\rho^n)|\right)\,dx\,ds\right] \\
&\qquad \le ct + c\left(\frac{t}{\epsilon} + \epsilon t \|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar f)})\|^2_{H^1(U)} + \epsilon \mathbb{E}\left[\int_0^t \int_U |\nabla \Theta_{\Phi,2}(\rho^n)|^2\,dx\,ds\right]\right).
\end{aligned}$$

The first three terms remain as part of the estimate, and we choose $\epsilon > 0$ sufficiently small so that the final term can be absorbed into the left hand side of the estimate.

The second and third terms on the right hand side of equation (4.17) appear in the estimate (4.8) so are left unchanged. For the terms in the final line, they are further bounded using Hölder's and Young's inequality, the fact that F_1 is bounded as well as Theorem 4.1.3. Taking the supremum in time gives the first estimate (4.8).

To prove the the second estimate (4.9), for $M := (M_1, M_2)$ and the function $S_M : [0,\infty) \to [0,\infty)$ defined in Definition 4.1.1, define the function $\Psi_{S_M} : U \times [0,\infty) \to \mathbb{R}$ by

$$\Psi_{S_M}(x,0) = 0, \quad \partial_\xi \Psi_{S_M}(x,\xi) = S_M'(\xi) - h_{S_M'(\Phi^{-1}(\bar f))}(x),$$

where $h_{S_M'(\Phi^{-1}(\bar f))}$ satisfies the PDE in Definition 4.1.1 and ensures that $\partial_\xi \Psi_{S_M}(x,\rho^n)$ vanishes along the boundary. Applying Itô's formula to a regularised version of the composition $\Psi_{S_M}(x,\rho^n)$ in a similar way to the first part, integrating by parts and re-arranging gives

$$\begin{aligned}
&\int_U \Psi_{S_M}(x,\rho^n(x,t))\,dx\Big|_{s=0}^{t} \\
&\quad = \int_0^t \int_U (\partial_\xi \Psi_{S_M})(x,\rho^n(x,s))\,d\rho^n\,dx + \frac{1}{2}\int_0^t \int_U (\partial_\xi^2 \Psi_{S_M})(x,\rho^n(x,s))\,d\langle \rho^n\rangle_s\,dx \\
&\quad = -\int_0^t \int_U S''(\rho^n)\nabla\rho^n \cdot \left(\nabla\Phi(\rho^n) + \frac{1}{n}\nabla\rho^n - \sigma_n(\rho^n)\,d\xi^F - \nu(\rho^n) - \sigma_n(\rho^n)\sigma_n'(\rho^n)F_2\right) \\
&\quad + \frac{1}{2}\int_0^t \int_U S''(\rho^n)F_3\sigma_n^2(\rho^n) + \int_0^t \int_U \nabla h_{S_M'(\Phi^{-1}(\bar f))} \cdot \nabla\Phi(\rho^n) + \frac{1}{n}\int_0^t \int_U \nabla h_{S_M'(\Phi^{-1}(\bar f))} \cdot \nabla\rho^n \\
&\quad + \int_0^t \int_U \nabla h_{S_M'(\Phi^{-1}(\bar f))} \cdot \left(-\sigma_n(\rho^n)\,d\xi^F - \nu(\rho^n) + \frac{1}{2}F_1(\sigma_n'(\rho^n))^2\nabla\rho^n + \sigma_n(\rho^n)\sigma_n'(\rho^n)F_2\right).
\end{aligned} \tag{4.18}$$

We deal with the terms in the final equality in a similar way to the first estimate. The first two terms form part of the estimate so are moved to the left hand side, the noise terms vanish in expectation and the fourth term can be re-written as a boundary integral. For the fifth term on the right hand side of the final equality of equation (4.18), we use the identity

$$\begin{aligned}
\mathbb{1}_{M_1<\rho^n<M_2}\sigma_n(\rho^n)\sigma_n'(\rho^n)\nabla\rho^n &= \frac{1}{2}\mathbb{1}_{M_1<\rho^n<M_2}\nabla\sigma_n^2(\rho^n) \\
&= \frac{1}{2}\nabla\left(\sigma_n^2((\rho^n \wedge M_2) \vee M_1) - \sigma_n^2(M_1)\right).
\end{aligned}$$

Then using integration by parts gives

$$\frac{1}{2}\int_0^t\int_U S''(\rho^n)\sigma_n(\rho^n)\sigma_n'(\rho^n)\nabla\rho^n\cdot F_2 =$$
$$-\frac{1}{4}\int_0^t\int_U \left(\sigma_n^2((\rho^n\wedge M_2)\vee M_1)-\sigma_n^2(M_1)\right)\nabla\cdot F_2$$
$$+\frac{1}{4}\int_0^t\int_{\partial U}\left(\sigma_n^2((\Phi^{-1}(\bar{f})\wedge M_2)\vee M_1)-\sigma_n^2(M_1)\right)F_2\cdot\hat{\eta}.$$

By the boundedness of $\nabla\cdot F_2$ and F_3 in U and $F_2\cdot\hat{\eta}$ on ∂U, it follows that there exists a constant $c\in(0,\infty)$ such that

$$\frac{1}{2}\int_0^t\int_U S''(\rho^n)\left(\sigma_n(\rho^n)\sigma_n'(\rho^n)\nabla\rho^n\cdot F_2+F_3\sigma_n^2(\rho^n)\right)$$
$$\leq c\int_0^t\int_U \mathbb{1}_{\rho^n\geq M_1}\sigma_n^2(\rho^n\wedge M_2)+ct\left\|\left(\sigma_n^2((\Phi^{-1}(\bar{f})\wedge M_2)\vee M_1)-\sigma_n^2(M_1)\right)\right\|_{L^1(\partial U)}.$$

Again we mention that the terms in (4.18) involving $\nabla h_{S_M'(\Phi^{-1}(\bar{f}))}$ would vanish if our boundary condition was constant. Otherwise they are dealt with in a similar way to the first estimate, except that we use Hölder's inequality rather than Young's inequality in order to keep the M dependence in these terms through $h_{S_M'(\Phi^{-1}(\bar{f}))}$. Explicitly, we have for the terms that involve another gradient, for Ψ_{σ_n} as in point 7 of Assumption 2.2.1, using point 1 of Remark 4.1.2,

$$\int_0^t\int_U \nabla h_{S_M'(\Phi^{-1}(\bar{f}))}\cdot\left(\nabla\Phi(\rho^n)+\frac{1}{n}\nabla\rho^n+\frac{1}{2}F_1(\sigma_n'(\rho^n))^2\nabla\rho^n\right)$$
$$=t\int_{\partial U}\frac{\partial h_{S_M'(\Phi^{-1}(\bar{f}))}}{\partial\hat{\eta}}\left(\bar{f}+\frac{1}{n}\Phi^{-1}(\bar{f})+\frac{F_1}{2}\Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\right)$$
$$\leq ct\left\|S_M'(\Phi^{-1}(\bar{f}))\right\|_{H^1(\partial U)}\left(\left\|\bar{f}\right\|_{L^2(\partial U)}+\frac{1}{n}\left\|\Phi^{-1}(\bar{f})\right\|_{L^2(\partial U)}+\left\|\Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\right\|_{L^2(\partial U)}\right).$$

For the remaining terms, analogously to (4.16), we have for a constant $c\in(0,\infty)$ that

$$\int_0^t\int_U \nabla h_{S_M'(\Phi^{-1}(\bar{f}))}\cdot\left(-\nu(\rho^n)+\sigma_n(\rho^n)\sigma_n'(\rho^n)F_2-\Psi_{\sigma_n}(\rho^n)\right)\,dx\,ds$$
$$\leq c\|\nabla h_{S_M'(\Phi^{-1}(\bar{f}))}\|_{L^\infty(U;\mathbb{R}^d)}\int_0^t\int_U(1+\rho^n+\Phi(\rho^n))\,dx\,ds. \quad (4.19)$$

The final two terms in (4.19) are dealt with using Proposition 4.1.6, just as in the first estimate. Here we cannot absorb the integral involving $|\nabla\Theta_{\Phi,2}(\rho^n)|^2$ to the left hand side of the estimate. It stays on the right hand side of the estimate, and it is bounded as a consequence of the first estimate.

To complete the estimate, we further note that the definition of Ψ_{S_M} implies that $\Psi_{S_M}(x,\rho^n)=S_M(\rho^n)-h_{S_M'(\Phi^{-1}(\bar{f}))}(x)\rho^n$. Since the product $h_{S_M'(\Phi^{-1}(\bar{f}))}\rho_0$ and

$\int_U S_M(\rho^n(x,t))$ are both non-negative, we can remove these from the left hand side of the estimate. We further use the bound $S_M(\rho_0(x)) \leq (\rho_0(x) - M_1)_+$ to simplify the left hand side of the estimate.

Finally we are left with the additional term $h_{S'_M(\Phi^{-1}(\bar{f}))}\rho^n(\cdot,t)$ on the right hand side of the estimate. Applying Holder's inequality and realising that the $L^2(U)$-norm of $\rho^n(\cdot,t)$ is bounded using the first estimate completes the proof of the second estimate. □

Remark 4.1.9. *The final two terms on the left hand side of equation* (4.8) *are the approximate kinetic measures* q^n*, and the bound in* (4.9) *is a bound on the approximate kinetic measures when* ρ^n *is bounded away from zero and infinity. The right hand side of* (4.9) *is written deliberately to emphasise that it converges to zero as* $M_1, M_2 \to \infty$ *due to the fact that* $h_{S'_M(\Phi^{-1}(\bar{f}))} \equiv 0$ *on* $\bar{U}$ *as* $M_1 \to \infty$.

Remark 4.1.10 (Distinguishing constant and non-constant boundary conditions)**.** *We mentioned several times that the estimates simplify in the case that the boundary condition is a non-negative constant, for instance* $\bar{f} = M \geq 0$*. This type of boundary condition is needed in the classical and fast diffusion cases* $\Phi(\xi) = \xi^m, m \leq 1$*, see Remark 2.2.6. In this case the PDEs* $h_{\Phi^{-1}(\bar{f})}$ *and* $h_{S'_M(\Phi^{-1}(\bar{f}))}$ *as in Definition 4.1.1 are solved by constant functions, for instance*

$$h_{\Phi^{-1}(\bar{f})}(x) = \Phi^{-1}(M), \quad \forall x \in \bar{U}.$$

Hence the gradient $\nabla h_{\Phi^{-1}(\bar{f})}$ *as well as the normal derivative* $\frac{\partial h_{\Phi^{-1}(\bar{f})}}{\partial \hat{\eta}}$ *are zero, and so terms involving either of these factors vanish immediately.*

For example, this means that the first energy estimate (4.8) *reads for constant* $c \in (0,\infty)$ *independent of the boundary data* M*,*

$$\begin{aligned}\frac{1}{2}\sup_{s\in[0,t]} \mathbb{E}\left[\int_U \left(\rho^n(x,s) - h_{\Phi^{-1}(\bar{f})}(x)\right)^2\right] + \mathbb{E}\left[\int_0^t\int_U |\nabla\Theta_{\Phi,2}(\rho^n)|^2\right] + \frac{1}{n}\mathbb{E}\left[\int_0^t\int_U |\nabla\rho^n|^2\right] \\ \leq \|\rho_0 - h_{\Phi^{-1}(\bar{f})}\|^2_{L^2(U)} + c\int_0^t\int_U (1+\rho^n+\Phi(\rho^n)) + ct\|\sigma_n^2(\Phi^{-1}(M))\|_{L^1(\partial U)} \\ \leq \|\rho_0 - h_{\Phi^{-1}(\bar{f})}\|^2_{L^2(U)} + ct\left(1 + \Theta_{\Phi,2}(\Phi^{-1}(M)) + \sigma_n^2(\Phi^{-1}(M))\right). \quad (4.20)\end{aligned}$$

In order to prove tightness of the sequence $\{\rho^n\}_{n\in\mathbb{N}}$, our goal is to use the above energy estimate to prove higher order space and time regularity of the solution. We will need the following lemma that will allow us to prove regularity of (fractional) spatial derivatives of the solution. The proof here is new compared to the torus since we need to use the standard PDE tool of the extension operator, see Chapter 5.4 of Evans [35].

Lemma 4.1.11. *Let* Φ *satisfy Assumption 2.2.1. Let* $z \in H^1(U)$ *be non-negative and such that* $\Theta_{\Phi,2}(z) \in H^1(U)$*. If* Φ *satisfies condition* (2.6)*, then*

$$\|\nabla z\|_{L^1(U;\mathbb{R}^d)} \leq \|z\|^{\theta}_{L^1(U)}\|\nabla\Theta_{\Phi,2}(z)\|_{L^2(U;\mathbb{R}^d)}. \quad (4.21)$$

If Φ satisfies condition (2.7), then for every $\beta \in (0, 1 \wedge 2/q)$, for a constant $c \in (0, \infty)$ depending on β,

$$\|z\|_{W^{\beta,1}(U)} \leq c \left(\|z\|_{L^1(U)} + \|\nabla \Theta_{\Phi,2}(z)\|_{L^2(U;\mathbb{R}^d)}^{\frac{2}{q}} \right). \tag{4.22}$$

Proof. By the chain rule and the fact that $\Phi \in C^1_{loc}((0,\infty))$, we have by the definition of $\Theta_{\Phi,2}$ in equation (2.5) that

$$\nabla z = (\Phi'(z))^{-1/2} \nabla \Theta_{\Phi,2}(z). \tag{4.23}$$

Suppose first that Φ satisfies (2.6). Then using (4.23), (2.6) and Hölder's inequality, it follows that for $\theta \in [0, 1/2]$ as in (2.6), we have that for some constant $c \in (0, \infty)$ that

$$\begin{aligned} \|\nabla z\|_{L^1(U)} \leq c \left\| z^\theta \nabla \Theta_{\Phi,2}(z) \right\|_{L^1(U)} &\leq c \|z^{2\theta}\|_{L^1(U)}^{1/2} \|\nabla \Theta_{\Phi,2}(z)\|_{L^2(U)} \\ &\leq c \|z\|_{L^1(U)}^{\theta} \|\nabla \Theta_{\Phi,2}(z)\|_{L^2(U)}, \end{aligned}$$

which proves the first claim (4.21).

We now look to prove the second claim, (4.22). We will utilise the Sobolev extension theorem from Chapter 5.4 of Evans [35] which we can apply to $\Theta_{\Phi,2}(z)$ since it is $H^1(U)$-regular. The theorem provides the existence of a bounded, linear extension operator

$$E : H^1(U) \to H^1(\mathbb{R}^d),$$

that extends $\Theta_{\Phi,2}$ from U to all of $\mathbb{R}^d$. The extension $E\Theta_{\Phi,2}$ agrees with $\Theta_{\Phi,2}$ on U, and since our domain U is C^2-regular, we have the existence of a constant $c \in (0, \infty)$ depending only on the domain U such that

$$\|E\Theta_{\Phi,2}\|_{H^1(\mathbb{R}^d)} \leq c \|\Theta_{\Phi,2}\|_{H^1(U)}. \tag{4.24}$$

It therefore follows that if Φ satisfies (2.7), then using (2.7), for $q \in [1, \infty)$ as in (2.7), the fractional Sobolev semi norm satisfies for every $\beta \in (0, 2/q \wedge 1)$, for some constant $c \in (0, \infty)$ depending on β,

$$\begin{aligned} \int_{U^2} \frac{|z(x) - z(y)|}{|x-y|^{d+\beta}} \, dx \, dy &\leq c \int_{U^2} \frac{|\Theta_{\Phi,2}(z)(x) - \Theta_{\Phi,2}(z)(y)|^{2/q}}{|x-y|^{d+\beta}} \, dx \, dy \\ &= c \int_{U^2} \frac{|E\Theta_{\Phi,2}(z)(x) - E\Theta_{\Phi,2}(z)(y)|^{2/q}}{|x-y|^{d+\beta}} \, dx \, dy \\ \leq c \int_{U^2} & \left| \int_0^1 \nabla E\Theta_{\Phi,p}(z)(y + s(x-y)) \cdot (x-y) \, ds \right|^{q/2} |x-y|^{-(d+\beta)} \, dx \, dy \\ &\leq c \int_{U^2} \int_0^1 |\nabla E\Theta_{\Phi,p}(z)(y + s(x-y))|^{q/2} \, |x-y|^{-(d+\beta-2/q)} \, ds \, dx \, dy. \end{aligned}$$

An application of Hölder's inequality, the fact that $d+\beta-2/q<d$, the inequality (4.24) and Poincaré's inequality proves that there exists a constant $c\in(0,\infty)$ which depends on β and q such that

$$\int_{U^2}\frac{|z(x)-z(y)|}{|x-y|^{d+\beta}}\,dx\,dy\le c\left(\int_U|\nabla E\Theta_{\Phi,2}(z)|^2\,dx\right)^{1/q}\le c\left(\int_U|\nabla\Theta_{\Phi,2}(z)|^2\,dx\right)^{1/q}.$$

This proves the second claim (4.22) by using the definition of the fractional Sobolev norm. □

As a direct consequence, we have the following higher order spatial regularity estimate.

Corollary 4.1.12 (Higher order spatial regularity of solution to regularised equation)**.** *Suppose that the spatial components of the noise ξ^F, the non-linear functions Φ,σ,ν and the boundary data $\bar f$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let further $\rho_0\in L^2(\Omega;L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable, let the functions $\Theta_{\nu,i}$ and Ψ_σ be defined as in Assumption 2.2.9, and $h_{\Phi^{-1}(\bar f)}$ denote the weak solution of the harmonic PDE defined in Definition 4.1.1. For $n\in\mathbb{N}$, let ρ^n denote a weak solution of the regularised equation* (4.5) *in the sense of Definition 4.1.5.*

1. *If Φ satisfies condition* (2.6)*, then for every $t\in[0,T]$ there is a constant $c\in(0,\infty)$ independent of n and t, satisfying*

$$\begin{aligned}\mathbb{E}\|\rho^n\|_{L^1([0,t];W^{1,1}(U))}&\le c\|\rho_0-h_{\Phi^{-1}(\bar f)}\|^2_{L^2(U)}\\&\quad+ct\left(1+\|\nabla h_{\Phi^{-1}(\bar f)}\|_{L^\infty(U;\mathbb{R}^d)}\right)\left(1+\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar f)})\|^2_{H^1(U)}\right)\\&\quad+ct\left(\|\sigma_n^2(\Phi^{-1}(\bar f))\|_{L^1(\partial U)}+\|\Theta_\nu(\Phi^{-1}(\bar f))\|_{L^1(\partial U;\mathbb{R}^d)}+\|\bar f\|^2_{L^2(\partial U)}\right)\\&\quad+ct\left(\|\Psi_{\sigma_n}(\Phi^{-1}(\bar f))\|^2_{L^2(\partial U)}+\left(1+\frac{1}{n}\right)\|\Phi^{-1}(\bar f)\|^2_{H^1(\partial U)}\right).\end{aligned}\tag{4.25}$$

2. *If Φ satisfies condition* (2.7)*, then for all $\beta\in(0,2/q\wedge1)$ and $t\in[0,T]$ there is a constant $c\in(0,\infty)$ depending on β, but independent of n and t, such that*

$$\begin{aligned}\mathbb{E}\|\rho^n\|_{L^1([0,t];W^{\beta,1}(U))}&\le c\|\rho_0-h_{\Phi^{-1}(\bar f)}\|^2_{L^2(U)}\\&\quad+ct\left(1+\|\nabla h_{\Phi^{-1}(\bar f)}\|_{L^\infty(U;\mathbb{R}^d)}\right)\left(1+\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar f)})\|^2_{H^1(U)}\right)\\&\quad+ct\left(\|\sigma_n^2(\Phi^{-1}(\bar f))\|_{L^1(\partial U)}+\|\Theta_\nu(\Phi^{-1}(\bar f))\|_{L^1(\partial U;\mathbb{R}^d)}+\|\bar f\|^2_{L^2(\partial U)}\right)\\&\quad+ct\left(\|\Psi_{\sigma_n}(\Phi^{-1}(\bar f))\|^2_{L^2(\partial U)}+\left(1+\frac{1}{n}\right)\|\Phi^{-1}(\bar f)\|^2_{H^1(\partial U)}\right).\end{aligned}\tag{4.26}$$

Proof. For both estimates we can use Proposition 4.1.6 to bound the $L^1(U\times[0,t])$ norms with $\epsilon=1$. The first estimate then follows from the first item in Lemma 4.1.11, the fact that $\theta\in[0,1/2]$ and Proposition 4.1.8.

The second estimate follows from the second point in Lemma 4.1.11, the fact that $q\in[1,\infty)$ and Proposition 4.1.8. □

We next aim to show a higher order time regularity result for solutions ρ^n. Since we want the estimate to be stable with respect to the regularisation n, we can only do this for solutions cutoff away from their zero set. The result will subsequently motivate the introduction of a new metric on $L^1(U\times[0,T])$ to prove tightness of the laws of ρ^n, see Definition 4.4.4.

Definition 4.1.13 (Cutoff away from zero)**.** *For $\beta\in(0,1)$ let ϕ_β be the piecewise linear cutoff in Definition 2.4.2. Let $\tilde{\phi}_\beta\in C^\infty([0,\infty))$ be a smooth approximation of ϕ_β. That is to say $0\le\tilde{\phi}_\beta\le 1$ is smooth, non-decreasing and satisfies for a constant $c\in(0,\infty)$ independent of β*

$$\begin{cases}\tilde{\phi}_\beta(\xi)=1, & \xi\ge\beta\\ \tilde{\phi}_\beta(\xi)=0, & \xi\le\beta/2\\ |\tilde{\phi}_\beta'(\xi)|\le c/\beta & \xi\in(0,\infty).\end{cases}$$

For each $\beta\in(0,1)$ we define $\Psi_\beta\in C^\infty([0,\infty))$ by

$$\Psi_\beta(\xi)=\tilde{\phi}_\beta(\xi)\xi. \tag{4.27}$$

Proposition 4.1.14. *Suppose that the spatial components of the noise ξ^F, the nonlinear functions Φ,σ,ν and the boundary data $\bar{f}$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let further $\rho_0\in L^2(\Omega;L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable, let the functions Θ_ν and Ψ_{σ_n} be defined as in Assumption 2.2.9 and $h_{\Phi^{-1}(\bar{f})}$ denote the solution of the harmonic PDE defined in Definition 4.1.1. For $n\in\mathbb{N}$ let ρ^n be a weak solution of the regularised equation (4.5) in the sense of Definition 4.1.5.*

For every $t\in[0,T]$, $\delta\in(0,1/2)$ and $s>\frac{d}{2}+1$ there exists a constant $c\in(0,\infty)$ depending on β,δ and s but independent of the regularisation n such that

$$\begin{aligned}\mathbb{E}\|\Psi_\beta(\rho^n)\|_{W^{\delta,1}([0,t];H^{-s}(U))}&\le ct\|\rho_0-h_{\Phi^{-1}(\bar{f})}\|^2_{L^2(U)}\\ &+ct^2\left(1+\|\nabla h_{\Phi^{-1}(\bar{f})}\|_{L^\infty(U;\mathbb{R}^d)}\right)\left(1+\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\|^2_{H^1(U)}\right)\\ &+ct^2\left(\|\sigma_n^2(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)}+\|\Theta_\nu(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U;\mathbb{R}^d)}+\|\bar{f}\|^2_{L^2(\partial U)}\right)\\ &+ct^2\left(\|\Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\|^2_{L^2(\partial U)}+\left(1+\frac{1}{n}\right)\|\Phi^{-1}(\bar{f})\|^2_{H^1(\partial U)}\right).\end{aligned}$$

The proof is analogous to Proposition 5.14 of Fehrman and Gess [41], we just give the main idea below.

Proof idea. Similarly to the derivation of the kinetic equation, it follows by Itô's formula and then bringing the $\Psi_\beta'(\rho^n)$ factor inside the derivative and using the product rule, that

$$\begin{aligned}
d\Psi_\beta(\rho^n) &= \Psi'_\beta(\rho^n)d\rho^n + \frac{1}{2}\Psi''_\beta(\rho^n)d\langle\rho^n\rangle \\
&= \nabla\cdot\left(\Psi'_\beta(\rho^n)\nabla\Phi(\rho^n) + \frac{1}{n}\Psi'_\beta(\rho^n)\nabla\rho^n - \Psi'_\beta(\rho^n)\sigma_n(\rho^n)d\xi^F - \Psi'_\beta(\rho^n)\nu(\rho^n)\right) \\
&\quad + \nabla\cdot\left(\frac{1}{2}F_1\Psi'_\beta(\rho^n)(\sigma'_n(\rho^n))^2\nabla\rho^n + \frac{1}{2}\Psi'_\beta(\rho^n)\sigma_n(\rho^n)\sigma'_n(\rho^n)F_2\right) \\
&- \Psi''_\beta(\rho^n)\nabla\rho^n\cdot\nabla\Phi(\rho^n) - \frac{1}{n}\Psi''_\beta(\rho^n)|\nabla\rho^n|^2 + \Psi''_\beta(\rho^n)\sigma_n(\rho^n)\nabla\rho^n\cdot d\xi^F + \Psi''_\beta(\rho^n)\nabla\rho^n\cdot\nu(\rho^n) \\
&\quad + \frac{1}{2}\Psi''_\beta(\rho^n)\left(\sigma_n(\rho^n)\sigma'_n(\rho^n)F_2\cdot\nabla\rho^n + F_3\sigma_n^2(\rho^n)\right).
\end{aligned}$$

Doing this allows us to compute the the fractional Sobolev norm more easily. Integrating in time, and writing some derivatives in terms of the function $\Theta_{\Phi,2}$, we have for fixed $x\in U$ the decomposition $\Psi_\beta(\rho^n(x,t)) = \Psi_\beta(\rho_0(x)) + I_t^{f.v.} + I_t^{mart}$ with martingale term

$$I_t^{mart} := -\int_0^t \nabla\cdot\left(\Psi'_\beta(\rho^n)\sigma_n(\rho^n)d\xi^F\right) + \int_0^t \Psi''_\beta(\rho^n)\sigma_n(\rho^n)(\Phi'(\rho^n))^{-1/2}\nabla\Theta_{\Phi,2}(\rho^n)\cdot d\xi^F,$$

and finite variation term

$$\begin{aligned}
I_t^{f.v.} &:= \int_0^t \nabla\cdot\left(\Psi'_\beta(\rho^n)(\Phi'(\rho^n))^{1/2}\nabla\Theta_{\Phi,2}(\rho^n)\right) - \int_0^t \Psi''_\beta(\rho^n)|\nabla\Theta_{\Phi,2}(\rho^n)|^2 \\
&\quad + \frac{1}{n}\int_0^t \nabla\cdot(\Psi'_\beta(\rho^n)\nabla\rho^n) + \frac{1}{n}\int_0^t \nabla\cdot(\Psi'_\beta(\rho^n)\nabla\rho^n) - \frac{1}{n}\int_0^t \Psi''_\beta(\rho^n)|\nabla\rho^n|^2 \\
&+ \frac{1}{2}\int_0^t \nabla\cdot\left(F_1\Psi'_\beta(\rho^n)\frac{(\sigma'_n(\rho^n))^2}{(\Phi'(\rho^n))^{1/2}}\nabla\Theta_{\Phi,2}(\rho^n)\right) + \frac{1}{2}\int_0^t \nabla\cdot\left(\Psi'_\beta(\rho^n)\sigma_n(\rho^n)\sigma'_n(\rho^n)F_2\right) \\
&\quad + \frac{1}{2}\int_0^t \Psi''_\beta(\rho^n)\frac{\sigma_n(\rho^n)\sigma'_n(\rho^n)}{\Phi'(\rho^n)^{1/2}}F_2\cdot\nabla\Theta_{\Phi,2}(\rho^n) + \frac{1}{2}\int_0^t \Psi''_\beta(\rho^n)F_3\sigma_n^2(\rho^n) \\
&\quad - \int_0^t \nabla\cdot(\Psi'_\beta(\rho^n)\nu(\rho^n)) - \int_0^t \nabla\cdot(\Psi'_\beta(\rho^n)\nu(\rho^n)) + \int_0^t \Psi''_\beta(\rho^n)\nabla\rho^n\cdot\nu.
\end{aligned}$$

We begin by showing the martingale term is $W^{\delta,2}([0,t];H^{-s}(U))$-regular. That is

$$\|I_\cdot^{mart}\|^2_{W^{\delta,2}([0,t];H^{-s}(U))} := \int_0^t \|I_u^{mart}\|^2_{H^{-s}(U)}\,du + \int_0^t\int_0^t \frac{\|I_u^{mart} - I_v^{mart}\|^2_{H^{-s}(U)}}{|u-v|^{1+2\delta}}\,dv\,du < \infty. \quad (4.28)$$

Since $s > \frac{d}{2}+1$, the Sobolev embedding theorem tells us that

$$H^s(U) \hookrightarrow L^\infty(U), \quad H^s(U) \hookrightarrow W^{1,\infty}(U), \quad (4.29)$$

so that for test functions ϕ that we use in the definition of negative fractional Sobolev norm, there exists a constant $c\in(0,\infty)$ such that

$$\|\phi\|_{L^\infty(U)} + \|\nabla\phi\|_{L^\infty(U;\mathbb{R}^d)} \le c\|\phi\|_{H^s(U)}. \quad (4.30)$$

Using this and an argument similar to Lemma 2.1 of Flandoli and Gatarek [47] alongside the Burkholder-Davis-Gundy inequality, the fact that Ψ''_β is supported on $[\beta/2,\beta]$ as well as the bounds in Assumption 2.2.1, that the second term in the definition of the norm satisfies

$$\begin{aligned}\mathbb{E}&\left[\int_0^t\int_0^t \frac{\|I_u^{mart}-I_v^{mart}\|^2_{H^{-s}(U)}}{|u-v|^{1+2\delta}}\,dv\,du\right]\\&\leq c\mathbb{E}\left[\int_0^t \|\Theta_{\Phi,2}(\rho^n)\mathbb{1}_{\beta/2\leq\rho^n\leq\beta}\|^2_{L^2(U)}+\|\sigma_n(\rho^n)\|^2_{L^2(U)}\,du\right]\\&\qquad\leq c\mathbb{E}\left[\int_0^t\int_U \left(|\nabla\Theta_{\Phi,2}(\rho^n)|^2+|\Phi(\rho^n)|\right)\,dx\,du\right].\end{aligned}$$

Analogously for the first term in the norm (4.28), we have the same bound

$$\int_0^t \|I_u^{mart}\|^2_{H^{-s}(U)}\,du\leq c\mathbb{E}\left[\int_0^t\int_U\left(|\nabla\Theta_{\Phi,2}(\rho^n)|^2+|\Phi(\rho^n)|\right)\,dx\,du\right].$$

Putting these together, it follows from the bound (2.4) on the growth of Φ and Proposition 4.1.6 that

$$\begin{aligned}&\|I_\cdot^{mart}\|^2_{W^{\delta,2}([0,t];H^{-s}(U))}\\&\qquad\leq c\mathbb{E}\left[t+t\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\|^2_{H^1(U)}+\int_0^t\int_U|\nabla\Theta_{\Phi,2}(\rho^n)|^2\,dx\,du\right].\end{aligned}$$

We next show that the finite variation term is $W^{1,1}([0,t];H^{-1}(U))$-valued. That is

$$\left\|I_\cdot^{f.v.}\right\|^2_{W^{1,1}([0,t];H^{-s}(U))}:=\int_0^t\left\|I_u^{f.v.}\right\|_{H^{-s}(U)}\,du+\int_0^t\left\|\frac{d}{du}I_u^{f.v.}\right\|_{H^{-s}(U)}\,du<\infty. \quad (4.31)$$

It follows from (4.30), the fact that Ψ'_β and Ψ''_β are supported on $[\beta/2,\infty)$ and $[\beta/2,\beta]$ respectively and Young's inequality, that we can bound the first term by

$$\begin{aligned}\int_0^t\left\|I^{f.v.}\right\|_{H^{-s}(U)}\,du\leq c\int_0^t\int_0^v\int_U\Bigg(&\Phi'(\rho^n)\mathbb{1}_{\rho^n>\beta/2}+|\nabla\Theta_{\Phi,2}(\rho^n)|^2\mathbb{1}_{\rho^n>\beta/2}\\&+\frac{1}{n}|\nabla\rho^n|\mathbb{1}_{\rho^n>\beta/2}+\frac{\sigma'_n(\rho^n)^4}{\Phi'(\rho^n)}\mathbb{1}_{\rho^n>\beta/2}+|\sigma_n(\rho^n)\sigma'_n(\rho^n)|\mathbb{1}_{\rho^n>\beta/2}\\&+\frac{(\sigma_n(\rho^n)\sigma'_n(\rho^n))^2}{\Phi'(\rho^n)}\mathbb{1}_{\rho^n>\beta/2}+|\nu(\rho^n)|\mathbb{1}_{\rho^n>\beta/2}+\left(1-\frac{1}{n}\right)|\nabla\rho^n|^2\mathbb{1}_{\beta/2<\rho^n<\beta}\\&+\sigma_n^2(\rho^n)\mathbb{1}_{\beta/2<\rho^n<\beta}+|\nu(\rho^n)|^2\mathbb{1}_{\beta/2<\rho^n<\beta}\Bigg)\,dx\,du\,dv.\end{aligned}$$

Using the fundamental theorem of calculus, the final term of (4.31) consists of the same terms as above but without the inner ds integral. Using that the inner integral is increasing in v, the fact that $n\in\mathbb{N}$ and so $\frac{1}{n}x\leq 1+\frac{1}{n}x^2$, the fact that the final three terms are bounded over the indicator set, Assumptions 2.2.1 and 2.2.9 to bound the various coefficients and using Proposition 4.1.6 analogously to the martingale term, we have

$$\begin{aligned}&\mathbb{E}\left\|I^{f.v.}_{\cdot}\right\|_{W^{1,1}([0,t];H^{-s}(U))}\\&\quad\leq c(1+t)\mathbb{E}\left[t+t\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\|^2_{H^1(U)}+\int_0^t\int_U\left(|\nabla\Theta_{\Phi,2}(\rho^n)|^2+\frac{1}{n}|\nabla\rho^n|^2\right)\right].\end{aligned} \tag{4.32}$$

Using the trivial fact that there is a constant $c\in(0,\infty)$ such that $(1+t)<ct$, the estimate then follows by the first energy estimate in Proposition 4.1.8 alongside the continuous embeddings $W^{\beta,2},W^{1,1}\hookrightarrow W^{\beta,1}$ for every $\beta\in(0,1/2)$. ☐

4.2 Entropy estimate

In this Section we prove an entropy estimate for weak solutions of the regularised Dean–Kawasaki equation. We begin in Definition 4.2.1 by defining a family of solutions of Laplace's equation that are necessary for the estimates. In Definition 4.2.2 we define the space of functions with finite entropy. The entropy estimate is subsequently proven in Proposition 4.2.4 We recall that the additional assumptions required for the entropy estimates are marked clearly in Assumptions 2.2.1 and 2.2.9.

We now introduce the family of PDEs $\{h_{log(\bar{f}+\delta)}\}_{\delta\in[0,1)}$.

Definition 4.2.1. *For every $\delta\in[0,1)$ let $h_{log(\bar{f}+\delta)}$ denote the weak solution of the PDE*

$$\begin{cases}-\Delta h_{log(\bar{f}+\delta)}=0, & \text{in } U,\\ h_{log(\bar{f}+\delta)}=\log(\bar{f}+\delta), & \text{on } \partial U,\end{cases} \tag{4.33}$$

where $\bar{f}$ is the boundary condition in (1.2)*. Let $h_{log(\bar{f})}:U\to\mathbb{R}$ denote the weak solution of the PDE* (4.33) *with the choice $\delta=0$.*

We will need to bound normal derivatives of $h_{log(\bar{f}+\delta)}$ in the $\delta\to 0$ limit, which is guaranteed by equation (2.20) in Assumption 2.2.9.

We now define the entropy space.

Definition 4.2.2 (Entropy space)**.** *Let $\Phi\in C([0,\infty))\cap C^1_{loc}((0,\infty))$ be non-negative, satisfying Assumptions 2.2.1 and $h_{log(\bar{f})}$ be as defined in Definition 4.2.1. Define the function $\Psi_{\Phi,0}:U\times[0,\infty)\to\mathbb{R}$ by*

$$\Psi_{\Phi,0}(x,0)=0,\quad \partial_\xi\Psi_{\Phi,0}(x,\xi)=\log(\Phi(\xi))-h_{log(\bar{f})}(x). \tag{4.34}$$

That is,

$$\Psi_{\Phi,0}(x,\xi)=\int_0^\xi\log(\Phi(\xi'))\,d\xi'-\xi h_{log(\bar{f})}(x).$$

The space of functions with finite entropy (with respect to Φ) is the space

$$Ent_\Phi(U):=\left\{\rho_0\in L^1(U):\rho_0\geq 0\quad\text{and}\quad\int_U\Psi_\Phi(x,\rho_0(x))\,dx<\infty\right\}. \tag{4.35}$$

The boundedness of the $h_{log(\bar{f})}$ term in (4.34) is a consequence of Remark 2.2.10, which tells us that when the boundary data is not zero[1], it is smooth and bounded away from zero, and consequently by the maximum principle $h_{log(\bar{f})}$ is bounded in U.

Hence the boundedness condition in (4.35) is just a condition on the integrability of the first term on the right hand side of (4.34).

Example 4.2.3. *In the case of interest $\Phi(\xi) = \xi^m$, for $m \in (0, \infty)$ we have that*

$$\log(\Phi(\rho)) = m \log(\rho).$$

Since we know that $\rho_0 \in L^1(U)$ and that $h_{log(\bar{f})}$ is bounded, we have

$$\int_U \Psi_\Phi(\rho_0(x))\, dx < \infty \iff \int_U \rho_0 \log(\rho_0) < \infty.$$

We now present the entropy estimate, the proof follows Proposition 5.18 of Fehrman and Gess [44]. We once again repeat the comment in Remark 4.1.10 and emphasise that the analysis below is significantly simplified if the boundary condition is constant.

Proposition 4.2.4 (Entropy estimate)**.** *Suppose that the spatial components of the noise ξ^F, the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in L^1(\Omega; Ent_\Phi(U))$ be $\mathcal{F}_0$-measurable, let the functions $\Theta_{\Phi,\sigma}, \Theta_{\Phi,\nu}, \Psi_\sigma$ be defined as in Assumption 2.2.9, and $h_{\Phi^{-1}(\bar{f})}$ denote the solution of the harmonic PDE defined in Definition 4.1.1 . For $n \in \mathbb{N}$ let ρ^n denote weak solutions of the regularised equation* (4.5)*.*

For the function $\Psi_{\Phi,0}$ defined in (4.34) *and for every $t \in [0, T]$, there exists a constant $c \in (0, \infty)$ independent of the regularisation n such that*

$$\begin{aligned}
\mathbb{E}\left[\int_U \Psi_{\Phi,0}(x, \rho^n(x,t))\, dx\right] + 4\mathbb{E}\left[\int_0^t \int_U |\nabla \Phi^{1/2}(\rho^n)|^2\right] + \frac{1}{n}\mathbb{E}\left[\int_0^t \int_U \frac{\Phi'(\rho^n)}{\Phi(\rho^n)} |\nabla \rho^n|^2\right] \\
\leq \mathbb{E}\left[\int_U \Psi_{\Phi,0}(x, \rho_0(x))\, dx\right] \\
+ c(1 + \|\nabla h_{log(\bar{f})}\|_{L^\infty(U;\mathbb{R}^d)}) \left(t + t\|\Theta_{\Phi,2}(h_{\Phi^{-1}(\bar{f})})\|^2_{H^1(U)} + c\mathbb{E}\left[\int_0^t \int_U |\nabla \Theta_{\Phi,2}(\rho^n)|^2\right]\right) \\
+ ct\|\Theta_{\Phi,\sigma_n}(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)} + t\|\Theta_{\Phi,\nu}(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U;\mathbb{R}^d)} + t\|\bar{f}\|^2_{L^2(\partial U)} \\
+ \frac{1}{n} t\|\Phi^{-1}(\bar{f})\|^2_{L^2(\partial U)} + t\|\Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\|^2_{L^2(\partial U)} + c\left(1 + \frac{1}{n}\right) t\|\log(\Phi^{-1}(\bar{f}))\|^2_{H^1(\partial U)}.
\end{aligned}$$

A bound for the the integral involving $|\nabla \Theta_{\Phi,2}(\rho^n)|^2$ on the right hand side is given by the first energy estimate in Proposition 4.1.8.

Proof. To obtain the bound, apply Itô's formula to the composition $\Psi_{\Phi,\delta}(x, \rho^n(x,s))$, $\delta \in (0,1)$, for the regularised function $\Psi_{\Phi,\delta} : U \times [0,\infty) \to \mathbb{R}$ defined by

$$\Psi_{\Phi,\delta}(x, 0) = 0, \quad (\partial_\xi \Psi_{\Phi,\delta})(x,\xi) = \log(\Phi(\xi) + \delta) - h_{log(\bar{f}+\delta)}(x),$$

---

[1] If the boundary data was the constant zero then we would not need to introduce the PDE in the estimates below at all.

where $h_{log(\bar{f}+\delta)}$ is the unique weak solution of the PDE (4.33) from Definition 4.2.1 and once again is used to ensure that $(\partial_\xi \Psi_{\Phi,\delta})(x,\rho^n(x,s))$ vanishes along the boundary. We get after integrating the first order term by parts and rearranging that

$$\begin{aligned}
&\int_U \Psi_{\Phi,\delta}(x,\rho^n(x,s))\,dx\Big|_{s=0}^{t} + \int_0^t\int_U \left(\frac{|\nabla\Phi(\rho^n)|^2}{\Phi(\rho^n)+\delta} + \frac{1}{n}\frac{\Phi'(\rho^n)|\nabla\rho^n|^2}{\Phi(\rho^n)+\delta}\right)dx\,ds \\
&\quad= \int_0^t\int_U \frac{\sigma_n(\rho^n)\Phi'(\rho^n)\nabla\rho^n\cdot d\xi^F}{\Phi(\rho^n)+\delta}\,dx + \int_0^t\int_U\left(\frac{\Phi'(\rho^n)\nabla\rho^n\cdot\nu(\rho^n)}{\Phi(\rho^n)+\delta}\right. \\
&+\left.\frac{\Phi'(\rho^n)\sigma_n'(\rho^n)\sigma_n(\rho^n)\nabla\rho^n\cdot F_2}{2(\Phi(\rho^n)+\delta)} + \frac{F_3\Phi'(\rho^n)\sigma_n^2(\rho^n)}{2(\Phi(\rho^n)+\delta)}\right)dx\,ds + \int_0^t\int_U \nabla h_{log(\bar{f}+\delta)}\cdot\Big(\nabla\Phi(\rho^n) \\
&\quad+\frac{1}{n}\nabla\rho^n - \sigma_n(\rho^n)d\xi^F - \nu(\rho^n) + \frac{1}{2}F_1[\sigma_n'(\rho^n)]^2\nabla\rho^n + \frac{1}{2}\sigma_n'(\rho^n)\sigma_n(\rho^n)F_2\Big)\,dx\,ds.
\end{aligned} \tag{4.36}$$

The terms are handled in an analogous way to energy estimates already seen thus far in Propositions 4.1.8 and 4.3.2, so we are brief. For the second term on the left hand side, we use the distributional equality

$$\nabla\Phi^{1/2}(\rho^n) = \frac{\Phi'(\rho^n)}{2\Phi^{1/2}(\rho^n)}\nabla\rho^n$$

to re-write it as

$$\int_U \frac{|\nabla\Phi(\rho^n)|^2}{\Phi(\rho^n)+\delta} = \int_U \frac{4\Phi(\rho^n)}{\Phi(\rho^n)+\delta}|\nabla\Phi^{1/2}(\rho^n)|^2.$$

For the terms on the right hand side, taking expectation kills both of the noise terms. The second and third terms on the right hand side of (4.36) can be re-written as boundary integrals.

For the second, we use the functions $\Theta_{\Phi,\nu,\delta,i}$ for $i=1,\ldots,d$ defined by

$$\Theta_{\Phi,\nu,\delta,i}(0)=0, \quad \Theta'_{\Phi,\nu,\delta,i}(\xi) = \frac{\Phi'(\xi)\nu_i(\xi)}{\Phi(\xi)+\delta},$$

and for the third we define the unique function $\Theta_{\Phi,\sigma,\delta}$ satisfying

$$\Theta_{\Phi,\sigma,\delta}(0)=0, \quad \Theta'_{\Phi,\sigma,\delta}(\xi) = \frac{\Phi'(\xi)\sigma'(\xi)\sigma(\xi)}{\Phi(\xi)+\delta},$$

and use integration by parts alongside the boundedness of $\nabla\cdot F_2$ on U and $F_2\cdot\hat{\eta}$ on ∂U. For the fourth term in the first line of (4.36), by the assumption $\sigma\le c\Phi^{1/2}$, the trivial inequality $\frac{x}{x+\delta}<1$ for every $\delta>0$, and the assumption $\Phi'(\xi)\le c(1+\xi+\Phi(\xi))$, we obtain

$$\frac{1}{2}\int_0^t\int_U \frac{F_3\Phi'(\rho^n)\sigma_n^2(\rho^n)}{\Phi(\rho^n)+\delta} \le c\int_0^t\int_U \Phi'(\rho^n) \le c\int_0^t\int_U (1+\rho^n+\Phi(\rho^n)).$$

The terms involving $\nabla h_{log(\bar{f}+\delta)}$ are handled in the way described in points one and two of Remark 4.1.2. The first, second and fifth terms in (4.36) with other gradient terms are handled using integration by parts and turn into boundary terms.

As for the terms without derivatives, we use the final point of Assumption 2.2.9 to deduce that $\nabla h_{log(\bar{f})}$ and $\nabla h_{log(\bar{f}+\delta)}$ are both $L^\infty(U;\mathbb{R}^d)$-valued, which gives alongside the bounds (2.10) from Assumption 2.2.1,

$$\begin{aligned}\int_0^t \int_U \nabla h_{log(\bar{f}+\delta)} \cdot \left(-\nu(\rho^n) + \frac{1}{2}\sigma_n'(\rho^n)\sigma_n(\rho^n)F_2 - \frac{F_2}{2}\Psi_{\sigma_n}(\rho^n)\right) dx\, ds \\ \leq c\|\nabla h_{log(\bar{f}+\delta)}\|_{L^\infty(U;\mathbb{R}^d)} \int_0^t \int_U (1+\rho^n+\Phi(\rho^n))\; dx\, ds.\end{aligned}$$

Putting everything together we get

$$\begin{aligned}&\mathbb{E}\left[\int_U \Psi_{\Phi,\delta}(x,\rho^n(x,t))\, dx\right] + \mathbb{E}\left[\int_0^t\int_U \left(\frac{4\Phi(\rho^n)}{\Phi(\rho^n)+\delta}|\nabla\Phi^{1/2}(\rho^n)|^2 + \frac{1}{n}\frac{\Phi'(\rho^n)}{\Phi(\rho^n)+\delta}|\nabla\rho^n|^2\right)\right] \\ &\leq \mathbb{E}\left[\int_U \Psi_{\Phi,\delta}(x,\rho_0^n(x))\, dx\right] + c\left(1+\|\nabla h_{log(\bar{f}+\delta)}\|_{L^\infty(U;\mathbb{R}^d)}\right)\mathbb{E}\left[\int_0^t\int_U (1+|\rho^n|+|\Phi(\rho^n)|)\right] \\ &\qquad + ct\|\Theta_{\Phi,\sigma_n,\delta}(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U)} + t\|\Theta_{\Phi,\nu,\delta}(\Phi^{-1}(\bar{f}))\|_{L^1(\partial U;\mathbb{R}^d)} \\ &\qquad + t\left(\int_{\partial U}\frac{\partial h_{log(\bar{f}+\delta)}}{\partial\hat{\eta}}\left(\bar{f}+\frac{1}{n}\Phi^{-1}(\bar{f}) + \Psi_{\sigma_n}(\Phi^{-1}(\bar{f}))\right)\right). \qquad (4.37)\end{aligned}$$

Once again we used the fact that the boundary terms in the final two lines are deterministic and do not depend on time. To obtain the desired estimate, we wish to take the $\delta \to 0$ limit in equation (4.37), and therefore we need a handle over the boundary terms which all depend on δ. Alongside Young's inequality, Assumption 2.2.9 precisely allows us to do this. Finally, to bound the integral of ρ^n and $\Phi(\rho^n)$ in the second line, we use Proposition 4.1.6 in the same manner as the energy estimates, noting that we cannot absorb the term involving $|\nabla\Theta_{\Phi,2}(\rho^n)|^2$ into the left hand side, but the term is bounded as a consequence of the first energy estimate (4.8). The estimate is proven. □

Remark 4.2.5 (Comparing entropy estimate with proof of kinetic measure at zero)**.** *Note that the entropy estimate could be used to prove a statement about the kinetic measure at zero if we include the assumption that* $\frac{\Phi(\xi)}{\Phi'(\xi)} \leq c\xi$.

Since we have

$$2\beta^{-1}\mathbb{E}\left[q(U\times[\beta/2,\beta]\times[0,T])\right] \leq \liminf_{n\to\infty}\mathbb{E}\left[\int_{\mathbb{R}}\int_0^T\int_U \frac{1}{\xi}\mathbb{1}_{\beta/2\leq\xi\leq\beta}dq^n\right],$$

that assumption would give

$$\begin{aligned}\frac{1}{\xi}dq^a &= \frac{1}{\xi}\delta_0(\xi-\rho^n)\left(\frac{4\Phi(\rho^n)}{\Phi'(\rho^n)}|\nabla\Phi^{1/2}(\rho^n)|^2 + \frac{1}{n}|\nabla\rho^n|^2\right) \\ &\leq c\delta_0(\xi-\rho^n)\left(4|\nabla\Phi^{1/2}(\rho^n)|^2 + \frac{1}{n}\frac{\Phi'(\rho^n)}{\Phi(\rho^n)}|\nabla\rho^n|^2\right).\end{aligned}$$

One sees that this is the precise quantity which we showed was bounded in the entropy estimate above. Consequently, by the dominated convergence theorem, with the indicator present, the kinetic measures go to zero.

The reason we do not use this estimate is due to the first term on the right hand side of the estimate, which requires $\rho_0 \in L^1(\Omega; Ent_\Phi(U))$. For the definition of stochastic kinetic solution, Definition 2.3.6, we only have $\rho_0 \in L^1(\Omega; L^1(U))$. We circumvent this in the sequel by choosing a test function that cuts off the logarithm at 1.

4.3 Decay of kinetic measure at zero

In this section we will prove the decay of the kinetic measure at zero required in the uniqueness proof. After defining the relevant PDE in Definition 4.3.1 we give the estimate in Proposition 4.3.2.

We consider the following PDE, where the boundary condition is the logarithm cutoff at 1.

Definition 4.3.1 (The PDE $h_{G'(\Phi^{-1}(\bar{f}))}$). *Define the function $G : [0,\infty) \to [0,\infty)$ by*

$$G(0) = 0, \quad G''(\xi) = \frac{1}{\xi}\mathbb{1}_{0<\xi<1}. \tag{4.38}$$

Integrating gives that $G'(\xi) = \log(\xi \wedge 1)$ and $G(\xi) = (\xi \wedge 1)\log(\xi \wedge 1) - (\xi \wedge 1)$.

Define $h_{G'(\Phi^{-1}(\bar{f}))} : U \to \mathbb{R}$ to be the solution of the harmonic PDE

$$\begin{cases} -\Delta h_{G'(\Phi^{-1}(\bar{f}))} = 0 & \text{on } U, \\ h_{G'(\Phi^{-1}(\bar{f}))} = G'(\Phi^{-1}(\bar{f})) & \text{on } \partial U. \end{cases} \tag{4.39}$$

Proposition 4.3.2. *Suppose that the non-linear functions Φ, σ and ν, the boundary data $\bar{f}$ and the spatial components of the noise ξ^F satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively Let further $\rho_0 \in L^1(\Omega; L^1(U))$ be non-negative and $\mathcal{F}_0$-measurable and $h_{G'(\Phi^{-1}(\bar{f}))}$ be defined as in Definition 4.3.1.*

If ρ is a stochastic kinetic solution of (2.23) in the sense of Definition 2.3.6 with corresponding kinetic measure q, then it follows almost surely that

$$\liminf_{\beta \to 0} \left(\beta^{-1} q(U \times [\beta/2, \beta] \times [0,T])\right) = 0.$$

Proof. Let us begin by noting that, whilst we do not know the precise form of the limiting measure q due to the presence of a parabolic defect measure in the limit, see Chen and Perthame [16], for the regularised equation (4.5) we have a precise equation for the kinetic measures, see proof of Proposition 5.21 of Fehrman and Gess [41] as well as the derivation of the kinetic measure from Section 2.3. The kinetic measure for the regularised equation is given by

$$dq^n = \delta_0(\xi - \rho^n)\left(\Phi'(\rho^n)|\nabla\rho^n|^2 + \frac{1}{n}|\nabla\rho^n|^2\right) dx\, dt\, d\xi. \tag{4.40}$$

By Fatou's lemma for measures, we observe that

$$\begin{aligned}\liminf_{\beta\to 0} 2\beta^{-1}\mathbb{E}\left[q(U\times[\beta/2,\beta]\times[0,T])\right] &\leq \liminf_{\beta\to 0}\mathbb{E}\left[\int_{\beta/2}^{\beta}\int_0^T\int_U \frac{1}{\xi}\,dq\right]\\ &= \liminf_{\beta\to 0}\mathbb{E}\left[\int_{\mathbb{R}}\int_0^T\int_U \frac{1}{\xi}\mathbb{1}_{\beta/2\leq\xi\leq\beta}\,dq\right]\\ &\leq \liminf_{n\to\infty}\liminf_{\beta\to 0}\mathbb{E}\left[\int_{\mathbb{R}}\int_0^T\int_U \frac{1}{\xi}\mathbb{1}_{\beta/2\leq\xi\leq\beta}dq^n\right].\end{aligned}$$

Analogous to the energy estimates of Proposition 4.1.8 and to the entropy estimate of Proposition 4.2.4, for fixed regularisation constant $n\in\mathbb{N}$ we apply Itô's formula to the composition $\Psi(x,\rho^n(x,s))$ for a regularised version of the function $\Psi : U\times[0,\infty)\to\mathbb{R}$ defined by

$$\Psi(x,0)=0,\quad (\partial_\xi\Psi)(x,\xi)=G'(\xi)-h_{G'(\Phi^{-1}(\bar f))}(x).$$

Using equation (4.40), one obtains for the functions Θ_σ, Ψ_σ and $\tilde\Theta_\nu$ as in point 3 of Assumption 2.2.9, that there exists a constant $c\in(0,\infty)$ such that

$$\begin{aligned}\mathbb{E}\left[\int_{\mathbb{R}}\int_0^T\int_U\frac{1}{\xi}\mathbb{1}_{0\leq\xi\leq 1}\,dq^n\right] &= \mathbb{E}\left[\int_0^T\int_U\frac{1}{\rho^n}\mathbb{1}_{0<\rho^n\leq 1}\left(\Phi'(\rho^n)|\nabla\rho^n|^2+\frac{1}{n}|\nabla\rho^n|^2\right)dx\,dt\right]\\ &\leq \mathbb{E}\left[\int_U\left(\Psi(x,\rho_0)-\Psi(x,\rho^n(\cdot,t))\right)dx\right]+cT+T\|\tilde\Theta_\nu(\Phi^{-1}(\bar f)\wedge 1)\|_{L^1(\partial U;\mathbb{R}^d)}\\ &\quad+cT\|\Theta_{\sigma_n}(\Phi^{-1}(\bar f)\wedge 1)\|_{L^1(\partial U)}+cT\left(1+\|\nabla h_{G'(\Phi^{-1}(\bar f))}\|_{L^\infty(U;\mathbb{R}^d)}\right)\int_0^T\int_U(1+\rho^n+\Phi(\rho^n))\\ &\quad+T\|\log(\Phi^{-1}(\bar f)\wedge 1)\|_{H^1(\partial U)}\left(\|\bar f\|_{L^2(\partial U)}+\frac{1}{n}\|\Phi^{-1}(\bar f)\|_{L^2(\partial U)}+\|\Psi_{\sigma_n}(\Phi^{-1}(\bar f))\|_{L^2(\partial U)}\right).\end{aligned}\tag{4.41}$$

The terms in the final two lines are handled in the same way as the previous estimates and we do not comment further on these terms.

Furthermore, we have for the first term on the right hand side, that

$$\Psi(x,\rho_0)=G(\rho_0)-\rho_0 h_{G'(\Phi^{-1}(\bar f))}(x)=(\rho_0\wedge 1)\log(\rho_0\wedge 1)-(\rho_0\wedge 1)-\rho_0 h_{G'(\Phi^{-1}(\bar f))}(x).$$

And so by the non-negativity of the solution, the initial condition and $h_{G'(\Phi^{-1}(\bar f))}$, we have the upper bound

$$\begin{aligned}&\mathbb{E}\left[\int_U\left(\Psi(x,\rho_0)-\Psi(x,\rho^n)\right)dx\right]\\ &\qquad\leq\mathbb{E}\left[\int_U\left((\rho_0\wedge 1)\log(\rho_0\wedge 1)+(\rho^n\wedge 1)-\rho^n h_{G'(\Phi^{-1}(\bar f))}(x)\right)dx\right].\end{aligned}$$

Using Hölder's inequality and the $L^2(U\times[0,t])$-energy estimate (4.8), this term is bounded.

Putting everything together, we showed that the right hand side of (4.41) is bounded. By working along the dyadic scale $\beta^{(i)}=2^{-i}$ for $i\in\mathbb{N}_0$, it follows by the boundedness of (4.41) that there exists a constant $c \in (0, \infty)$ that can be made independent of the regularisation n such that

$$\sum_{i=0}^{\infty} \mathbb{E}\left[\int_{\mathbb{R}} \int_0^T \int_U \frac{1}{\beta^{(i)}} \mathbb{1}_{\frac{\beta^{(i)}}{2} \leq \xi \leq \beta^{(i)}} \, dq^n\right] \leq \mathbb{E}\left[\int_{\mathbb{R}} \int_0^T \int_U \frac{1}{\xi} \mathbb{1}_{0 \leq \xi \leq 1} \, dq^n\right] \leq c.$$

The infinite sum being bounded by a constant implies that the individual elements of the sum converge to zero in $L^1(\Omega)$, which implies almost sure convergence along a subsequence. This proves the claim. □

4.4 Existence of solutions to the Dean–Kawasaki equation

In this section we begin in Proposition 4.4.1 by proving the existence of weak solutions of the regularised Dean–Kawasaki equation. In Proposition 4.4.2 we then show that the constructed weak solution is also a stochastic kinetic solution, in the sense that it satisfies a kinetic equation analogous to equation (2.40) but with the additional regularisation terms as in (2.33) above.

The goal will then subsequently be in Lemma 4.4.3 to remove the requirement that σ is smooth and bounded, which will be done with an approximation argument. To take the regularisation n limit, showing the existence of a solution to the generalised Dean–Kawasaki equation (2.23) is done in Theorem 4.4.8. Since the higher order time regularity estimate from Proposition 4.1.14 only applied to solutions cutoff from their zero set, in Definition 4.4.4 we introduce a metric on $L^1(U \times [0,T])$ whose topology coincides with the strong norm topology, and tightness of solutions will be proven with respect to this new topology. The arguments are similar to that on the torus and so follow Section 5 of Fehrman and Gess [41]. We are therefore brief and just provide the main ideas for completeness.

We begin with the proof of the existence of weak solutions to the regularised equation. The proof is more technical than the torus since we need keep track of the boundary data in the analysis.

Proposition 4.4.1 (Existence of weak solution to regularised equation (4.5))**.** *Suppose that the spatial components of the noise ξ^F, the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable.*

For every $n \in \mathbb{N}$ there exists a weak solution to the regularised equation (4.5) *in the sense of Definition 4.1.5. Additionally the solution satisfies the energy estimates of Proposition 4.1.8.*

Proof. We construct weak solutions via a Galerkin approximation with regularised coefficients and pass to the limits in several steps. Throughout, $n \in \mathbb{N}$ is a fixed constant corresponding to a regularisations in the regularised equation (4.5).

Step 1: Regularisation of coefficients and truncation of noise. Let $(\Phi_m)_{m \in \mathbb{N}}$ be a sequence of smooth, bounded, non-decreasing functions with $\Phi_m(0) =$

0, such that $\Phi_m \to \Phi$ and $\Phi'_m \to \Phi'$ locally uniformly, and $\Phi_m^{-1} \to \Phi^{-1}$ uniformly on compact sets. Let $(\nu_m)_{m\in\mathbb{N}}$ be smooth approximations of ν converging locally uniformly to ν.

For $K \in \mathbb{N}$, define the truncated noise

$$\xi^{F,K}(t,x) := \sum_{k=1}^{K} f_k(x) B_t^k,$$

and set

$$F_1^K := \sum_{k=1}^{K} f_k^2, \qquad F_2^K := \sum_{k=1}^{K} f_k \nabla f_k.$$

Step 2: Boundary lifting. For every $m \in \mathbb{N}$, let $h_{\Phi_m^{-1}(\bar{f})}$ denote the harmonic extension of $\Phi_m^{-1}(\bar{f})$. That is, the unique weak solution to

$$\begin{cases} -\Delta h_{\Phi_m^{-1}(\bar{f})} = 0 & \text{in } U, \\ h_{\Phi_m^{-1}(\bar{f})} = \Phi_m^{-1}(\bar{f}) & \text{on } \partial U. \end{cases}$$

Step 3: Galerkin approximation. Let $(e_i)_{i\in\mathbb{N}} \subset H_0^1(U)$ be an orthonormal basis of $L^2(U)$. For $M \in \mathbb{N}$, we define the spaces

$$V_{M,0} := \operatorname{span}\{e_1, \dots, e_M\}, \qquad V_M^m := h_{\Phi_m^{-1}(\bar{f})} + V_{M,0}.$$

We construct an approximate solution by projecting onto the basis vectors, whilst at the same time satisfying the boundary condition. That is, let

$$\rho^{M,m,K}(t,x) = h_{\Phi_m^{-1}(\bar{f})}(x) + \sum_{i=1}^{M} a_i(t) e_i(x),$$

where the coefficients $a_i(t)$ are real-valued stochastic processes. We know that the coefficient $a_i(t)$ should denote the $L^2(U)$-inner product of the unknown solution with e_i. Using the weak formulation of the solution (4.1.5), the coefficients $(a_j(t))_{j=1}^M$ are defined as the unique solutions of

$$\begin{aligned} da_j(t) = &- \int_U \nabla\Phi_m(\rho^{M,m,K}) \cdot \nabla e_j \, dx \, dt - \frac{1}{n} \int_U \nabla\rho^{M,m,K} \cdot \nabla e_j \, dx \, dt \\ &+ \int_U \nu_m(\rho^{M,m,K}) \cdot \nabla e_j \, dx \, dt \\ &- \frac{1}{2} \int_U \Big(F_1^K [\sigma'_n(\rho^{M,m,K})]^2 \nabla\rho^{M,m,K} + \sigma'_n(\rho^{M,m,K}) \sigma_n(\rho^{M,m,K}) F_2^K \Big) \cdot \nabla e_j \, dx \, dt \\ &+ \int_U \sigma_n(\rho^{M,m,K}) \, d\xi^{F,K} \cdot \nabla e_j. \end{aligned}$$

Since Φ_m, ν_m, σ_m are smooth and bounded, the coefficients of this system are globally Lipschitz functions of $(a_1, \dots, a_M)$. Hence, this finite-dimensional system admits a unique strong solution. This allows us to define $\rho^{M,m,K}$.

Step 4: Boundary condition. By construction, $V_{M,0} \subset H^1_0(U)$, and therefore

$$\rho^{M,m,K}(t,\cdot) \in V^m_M \subset h_{\Phi_m^{-1}(\bar{f})} + H^1_0(U).$$

The trace theorem, see Chapter 5.5 of Evans [35], then implies that

$$\rho^{M,m,K}|_{\partial U} = \Phi_m^{-1}(\bar{f}) \quad \text{for all } t \in [0,T].$$

Step 5: Passing to the limits $M, K, m \to \infty$. Applying Itô's formula and using the assumptions on the coefficients, we obtain uniform (in M, K, m) energy estimates as in Proposition 4.1.8. In particular,

$$\rho^{M,m,K} \text{ is bounded in } L^2(\Omega; L^2(0,T; H^1(U))).$$

Using the these bounds and the Aubin–Lions–Simon lemma, we extract a subsequence such that

$$\rho^{M,m,K} \to \rho^{m,K} \quad \text{strongly in } L^2(\Omega; L^2(U \times (0,T))) \text{ as } M \to \infty.$$

Passing to the limit in the weak formulation yields a solution $\rho^{m,K}$.

Using standard stochastic compactness arguments and the convergence of the truncated noise, we can take the $K \to \infty$ limit to obtain a limit ρ^m solving the equation with full noise.

Finally, using the convergence of the non-linearities from Step 1 and compactness, we pass to the limit $m \to \infty$ to obtain a weak solution ρ of the original equation.

Step 6: Boundary condition in the limit. By elliptic regularity, $h_{\Phi_m^{-1}(\bar{f})} \to h_{\Phi^{-1}(\bar{f})}$ in $H^1(U)$ as $m \to \infty$. Since $\rho^{M,m,K} - h_{\Phi_m^{-1}(\bar{f})} \in H^1_0(U)$ and the trace operator is continuous, we obtain

$$\rho|_{\partial U} = \Phi^{-1}(\bar{f}), \quad \text{equivalently } \Phi(\rho)|_{\partial U} = \bar{f}.$$

Step 7: Non-negativity and continuity. Non-negativity follows by applying Itô's formula to suitable approximations of the negative part,similar to what was done in the estimate (4.9). Continuity in $L^2(U)$ follows from the stochastic integral representation.

This completes the proof. □

We now prove that the weak solutions of the Dean–Kawasaki equation are also stochastic kinetic solutions. The proof is analogous to Proposition 5.21 of [41], we provide the idea for completeness.

Proposition 4.4.2 (Stochastic kinetic solution of the regularised Dean–Kawasaki equation (4.5))**.** *Suppose that the spatial components of the noise ξ^F, the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. For m as in equation* (2.4)*, let $\rho_0 \in L^{m+1}(\Omega; L^1(U)) \cap L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable. For $n \in \mathbb{N}$, let ρ^n denote a weak solution of equation* (4.5) *in the sense of Definition 4.1.5, and let*

$$\chi^n(x,\xi,t) = \mathbb{1}_{0<\xi<\rho^n(x,t)}$$

be the corresponding kinetic function on $U \times (0,\infty) \times [0,T]$*. Then* ρ^n *is a stochastic kinetic solution in the sense that for every* $\psi \in C_c^\infty(U \times (0,\infty))$*,* $t \in [0,T]$ *we have almost surely that*

$$\begin{aligned}
\int_{\mathbb{R}}\int_U \chi^n(x,\xi,t)\psi(x,\xi)\,dx\,d\xi &= \int_{\mathbb{R}}\int_U \chi^n(x,\xi,t=0)\psi(x,\xi)\,dx\,d\xi \\
&- \int_0^t\int_U (\partial_\xi\psi)(x,\xi)|_{\xi=\rho^n}\Phi'(\xi)|\nabla\rho^n|^2\,dx\,ds - \frac{1}{n}\int_0^t\int_U \nabla\rho^n\cdot(\nabla_x\psi)(x,\xi)|_{\xi=\rho^n}\,dx\,ds \\
&- \frac{1}{n}\int_0^t\int_U (\partial_\xi\psi)(x,\xi)|_{\xi=\rho^n}|\nabla\rho^n|^2\,dx\,ds - \int_0^t\int_U \psi(x,\rho^n)\nabla\cdot(\sigma_n(\rho^n)\,d\xi^F)\,dx \\
&- \int_0^t\int_U \psi(x,\rho^n)\nabla\cdot\nu(\rho^n)\,dx\,ds \\
&+ \frac{1}{2}\int_0^t\int_U \left(\sigma_n'(\rho^n)\sigma_n(\rho^n)\nabla\rho^n\cdot F_2 + \sigma_n(\rho^n)^2F_3\right)(\partial_\xi\psi)(x,\xi)|_{\xi=\rho^n}\,dx\,ds \\
&- \int_0^t\int_U \left(\Phi'(\rho^n)\nabla\rho^n + \frac{1}{2}F_1[\sigma_n'(\rho^n)]^2\nabla\rho^n + \frac{1}{2}\sigma_n'(\rho^n)\sigma_n(\rho^n)F_2\right)\cdot(\nabla_x\psi)(x,\xi)|_{\xi=\rho^n}\,dx\,dt.
\end{aligned} \tag{4.42}$$

Proof idea. The proof follows precisely the steps for deriving the stochastic kinetic equation (2.40), see Section 2.3. Begin by using Itô's formula to derive an equation for $S(\rho^n)$ for a smooth and bounded function $S : \mathbb{R} \to \mathbb{R}$. Secondly derive a formula for the integral

$$\int_U S(\rho^n)\psi(x)\,dx,$$

for test function $\psi \in C_c^\infty(U)$ using Definition 4.1.5 of a weak solution. Finally the kinetic equation is derived, noting the density of linear combinations of functions of the form $S'(\xi)\psi(x)$ in $C_c^\infty(U \times \mathbb{R})$.

One noteworthy point is that, as mentioned above in the discussion of equation (4.40), the kinetic measure corresponding to the solution ρ^n constructed above is

$$\begin{aligned}
q^n = \delta_0(\xi-\rho^n)\Phi'(\xi)|\nabla\rho^n|^2 &+ \frac{1}{n}\delta_0(\xi-\rho^n)|\nabla\rho^n|^2 \\
&= \delta_0(\xi-\rho^n)\left(|\nabla\Theta_{\Phi,2}(\rho^n)|^2 + \frac{1}{n}|\nabla\rho^n|^2\right).
\end{aligned}$$

The measure is finite due to the estimates of Proposition 4.1.8 and satisfies the other assumptions of a kinetic measure in Definition 2.3.6 due to Assumption 2.2.1. The boundary condition is inherited from the boundary condition of the weak solution, see the first point of Definition 4.1.5. □

Our aim is now to extend the well-posedness to the generalised Dean–Kawasaki equation without regularisation (2.23). The first step is to dispense of the regularisation of σ in Definition 4.1.4.

Lemma 4.4.3 (Approximating σ in C^1_{loc})**.** *Let* σ *satisfy Assumption 2.2.1. Then there exists a sequence* $\{\sigma_n\}_{n\in\mathbb{N}}$ *such that for each* $n \in \mathbb{N}$*, the function* σ_n *satisfies the* *regularity condition* (4.4) *in Definition 4.1.4. Further, the sequence uniformly satisfy Assumption 2.2.1, and*

$$\sigma_n \to \sigma \quad in \quad C^1_{loc}((0,\infty)).$$

Proof. The proof follows from constructing smooth bounded approximations by convolution which can be done due to the regularity of σ from Assumption 2.2.1. $\square$

The difficulty in extending the well-posedness to the singular equation (2.23) is that the weak solution of the regularised equation constructed in Proposition 4.4.1 does not have a stable $W^{\beta,1}([0,t];H^{-s}(U))$-estimate. We only have stable $W^{\beta,1}([0,t];H^{-s}(U))$-estimate for the solution bounded away from its zero set, as in Proposition 4.1.14. We deal with this by defining an equivalent metric on $L^1(U\times[0,T])$ below. Tightness of the cutoff solution $\Psi_\beta(\rho)$ as in Definition (4.1.13) will be proved with respect to this metric.

Definition 4.4.4 (New metric on $L^1(U\times[0,T])$)**.** *For $\beta\in(0,1)$ let Ψ_β be as defined in Definition 4.1.13. Define $D : L^1(U\times[0,T])\times L^1(U\times[0,T])\to[0,\infty)$ by*

$$D(f,g)=\sum_{k=1}^{\infty}2^{-k}\left(\frac{\|\Psi_{1/k}(f)-\Psi_{1/k}(g)\|_{L^1(U\times[0,T])}}{1+\|\Psi_{1/k}(f)-\Psi_{1/k}(g)\|_{L^1(U\times[0,T])}}\right). \tag{4.43}$$

We now state the result that allows us to use the metric D to prove tightness. We expand the ideas presented in Lemma 5.24 of Fehrman and Gess [41] to make their proof rigorous.

Lemma 4.4.5. *The function D defined above is a metric on $L^1(U\times[0,T])$. The metric topology defined by D coincides with the strong norm topology on $L^1(U\times[0,T])$.*

Proof. We split the proof into two steps.

Step 1: Prove that D is a metric. The non-negativity and symmetry of D is clear from the definition. We next aim to show that $D(f,g)=0 \iff f=g$ almost everywhere. If $f=g$ then $D(f,g)=0$ is obvious, so we just need to show the reverse implication. Suppose that $D(f,g)=0$. Then

$$\|\Psi_{1/k}(f)-\Psi_{1/k}(g)\|_{L^1(U\times[0,T])}=0$$

for every $k\in\mathbb{N}$, and so $\Psi_{1/k}(f)=\Psi_{1/k}(g)$ almost everywhere. But for any value of $f>0$, we can find a $k\in\mathbb{N}$ sufficiently large, satisfying $\frac{1}{k}<f$, so that the equality $\Psi_{1/k}(f)=\Psi_{1/k}(g)$ is equivalent to $f=g$ almost everywhere.

We are left to prove the triangle inequality. By the triangle inequality for the $L^1(U\times[0,T])$-norm, it holds that for any $f,g,h\in L^1(U\times[0,T])$ and $k\in\mathbb{N}$

$$\begin{aligned}&\|\Psi_{1/k}(f)-\Psi_{1/k}(g)\|_{L^1(U\times[0,T])}\\&\qquad\le\|\Psi_{1/k}(f)-\Psi_{1/k}(h)\|_{L^1(U\times[0,T])}+\|\Psi_{1/k}(h)-\Psi_{1/k}(g)\|_{L^1(U\times[0,T])}.\end{aligned}$$

The observation that $\zeta(t):=\frac{t}{1+t}$ is concave, so that $\zeta(a+b)\le\zeta(a)+\zeta(b)$, then implies alongside the above equation that

$$\begin{aligned}&\zeta(\|\Psi_{1/k}(f)-\Psi_{1/k}(g)\|_{L^1(U\times[0,T])})\\&\qquad\le\zeta(\|\Psi_{1/k}(f)-\Psi_{1/k}(h)\|_{L^1(U\times[0,T])})+\zeta(\|\Psi_{1/k}(h)-\Psi_{1/k}(g)\|_{L^1(U\times[0,T])}).\end{aligned}$$

Multiplying by 2^{-k} and summing over $k \in \mathbb{N}$ gives the triangle inequality for D. Hence D is a metric.

Step 2: Equivalence of topologies. We wish to show that

$$f_n \to f \text{ in } L^1(U \times [0,T]) \quad \iff \quad D(f_n, f) \to 0.$$

First we prove that $L^1(U \times [0,T])$-convergence implies convergence in the metric D. This follows by the fact that $L^1(U \times [0,T])$-convergence implies that there exists a constant $c \in (0, \infty)$ such that for every $k \in \mathbb{N}$, as $n \to \infty$ we have

$$\|\Psi_{1/k}(f_n) - \Psi_{1/k}(f)\|_{L^1(U\times[0,T])} \leq c\|f_n - f\|_{L^1(U\times[0,T])} \to 0.$$

The inequality above follows by the fact that $\Psi_{1/k}$ is Lipschitz with constant independent of k. Convergence in the metric D then follows by the dominated convergence theorem and the fact that each term in the sum defining $D(f_n, f)$ converges to zero.

We conclude with a proof that convergence in the metric D implies convergence in $L^1(U \times [0,T])$. We first observe by the triangle inequality for real numbers that for every $k \in \mathbb{N}$,

$$|f_n - f| \leq |\Psi_{1/k}(f_n) - \Psi_{1/k}(f)| + |f_n - \Psi_{1/k}(f_n)| + |\Psi_{1/k}(f) - f|. \tag{4.44}$$

Observe for the final two terms that $|f - \Psi_{1/k}(f)| = (1 - \tilde{\phi}_{1/k}(f))|f|$, which is zero when $|f| > 1/k$, so that

$$|f - \Psi_{1/k}(f)| \leq |f|\mathbb{1}_{|f|\leq 1/k} \leq \frac{1}{k}.$$

Integrating (4.44) over $U \times [0,T]$ then gives for every $k \in \mathbb{N}$ that

$$\|f_n - f\|_{L^1(U\times[0,T])} \leq \|\Psi_{1/k}(f_n) - \Psi_{1/k}(f)\|_{L^1(U\times[0,T])} + \frac{2|U|T}{k}. \tag{4.45}$$

Since $D(f_n, f) \to 0$, we have for each $k \in \mathbb{N}$ that

$$2^{-k}\zeta(\|\Psi_{1/k}(f_n) - \Psi_{1/k}(f)\|_{L^1(U\times[0,T])}) \leq D(f_n, f),$$

which implies that for every $k \in \mathbb{N}$, as $n \to \infty$,

$$\zeta(\|\Psi_{1/k}(f_n) - \Psi_{1/k}(f)\|_{L^1(U\times[0,T])}) \leq 2^k D(f_n, f) \to 0.$$

By the fact that ζ is continuous and strictly increasing, this implies for each fixed $k \in \mathbb{N}$ that

$$\|\Psi_{1/k}(f_n) - \Psi_{1/k}(f)\|_{L^1(U\times[0,T])} \to 0 \tag{4.46}$$

Using (4.45) we wish to show that $f_n \to f$ in $L^1(U \times [0,T])$. To that end, fix arbitrary $\epsilon > 0$. Choose $k \in \mathbb{N}$ sufficiently large so that $\frac{2|U|T}{k} < \epsilon/2$. Based on this k, pick $n \in \mathbb{N}$ sufficiently large so that $\|\Psi_{1/k}(f_n) - \Psi_{1/k}(f)\|_{L^1(U\times[0,T])} < \epsilon/2$, which can be done by (4.46). Equation (4.45) then illustrates for sufficiently large n, k, we have

$$\|f_n - f\|_{L^1(U\times[0,T])} < \epsilon.$$

That completes the proof. $\square$

The proposition below is a key element of the existence proof. The statements on the torus can be found in Proposition 5.26 and 5.27 of Fehrman and Gess [41]. We give the details for the proof of the first point and refer the reader to Proposition 5.27 of Fehrman and Gess [41] for a proof of the second point.

Proposition 4.4.6 (Tightness of laws of ρ^n in $L^1(U\times[0,T])$ and of martingale term in $C^\gamma([0,T])$)**.** *Suppose that the spatial components of the noise ξ^F, the non-linear functions Φ,σ,ν and the boundary data $\bar{f}$ satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let $\rho_0\in L^2(\Omega;L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable. For $n\in\mathbb{N}$ let ρ^n denote stochastic kinetic solutions to the regularised generalised Dean–Kawasaki equation* (4.5)*.*

1. *The laws of $\{\rho^n\}_{n\in\mathbb{N}}$ are tight on $L^1(U\times[0,T])$ with respect to the strong norm topology.*
2. *For each test function $\psi\in C_c^\infty(U\times(0,\infty))$, $\gamma\in(0,1/2)$ the laws of the martingales*
$$M_t^{n,\psi}:=\int_0^t\int_U\psi(x,\rho^n)\nabla\cdot(\sigma_n(\rho^n)d\xi^F)$$
are tight on $C^\gamma([0,T])$.

Proof of first point. Owing to Lemma 4.4.5, we just need to show that the laws are tight on $L^1(U\times[0,T])$ with respect to the metric D. We split the proof into several steps.

Step 1: Uniform bound for the cutoff solution $\Psi_{1/k}(\rho^n)$. Fix $k\in\mathbb{N}$. Our aim is to derive uniform in n bounds for $\Psi_{1/k}(\rho^n)$, where $\Psi_\beta,\beta\in(0,1)$ is defined in Definition 4.1.13.

Using Corollary 4.1.12 with the fact that $\Psi'_{1/k}(\xi)\le c$ for a constant $c\in(0,\infty)$ independent of $k\in\mathbb{N}$, we obtain

- If Φ satisfies (2.6), then we can bound
$$\mathbb{E}\|\Psi_{1/k}(\rho^n)\|_{L^1([0,T];W^{1,1}(U))}$$
uniformly in n, by utilising equation (4.25).
- If Φ satisfies (2.7), then for all $\beta\in(0,2/q\wedge 1)$ we can bound
$$\mathbb{E}\|\Psi_{1/k}(\rho^n)\|_{L^1([0,T];W^{\beta,1}(U))}$$
uniformly in n, by utilising equation (4.26).

Furthermore, since $\Psi_{1/k}(\xi)\le\xi$ for every $\xi\in[0,\infty)$, Proposition 4.1.8 yields that
$$\mathbb{E}\|\Psi_{1/k}(\rho^n)\|_{L^\infty([0,T];L^2(U))}$$
is bounded uniformly in n, by utilising equation (4.8).

Step 2: Tightness of cutoff sequence. We want to show that for each $k \in \mathbb{N}$, $\{\Psi_{1/k}(\rho^n)\}_{n\in\mathbb{N}}$ is tight in $L^1(U \times [0,T])$. This is a simple consequence of the Aubin-Lions-Simon lemma, see specifically Corollary 5 of Simon [91], alongside uniform estimates from Step 1 and the compact embeddings $W^{1,1}(U), W^{\beta,1}(U) \hookrightarrow L^1(U)$, the continuous embedding $L^1(U) \hookrightarrow H^{-s}(U)$ for every $s \geq \frac{d}{2} + 1$, alongside the higher order time regularity from Proposition 4.1.14. The claim is proved.

Step 3: Lifting the tightness to ρ^n. Our goal is to transfer the tightness of $\{\Psi_{1/k}(\rho^n)\}_{n\in\mathbb{N}}$ from Step 2 to the regularised solutions $\{\rho^n\}_{n\in\mathbb{N}}$. Define the maps

$$H_k : L^1(U \times [0,T]) \to L^1(U \times [0,T]) \quad \text{defined by} \quad H_k(\rho) = \Psi_{1/k}(\rho).$$

The fact that $\Psi_{1/k}$ is Lipschitz implies that H_k is continuous in the strong norm topology. Fix $\epsilon > 0$. The tightness from Step 2 proves that for every $k \in \mathbb{N}$, there exists a compact set $C_k \subset L^1(U \times [0,T])$ such that for every $n \in \mathbb{N}$ we have

$$\mathbb{P}\left[\Psi_{1/k}(\rho^n) \notin C_k\right] \leq \epsilon/2^k. \tag{4.47}$$

Define the sets

$$D_k := H_k^{-1}(C_k), \quad K = \cap_{k\in\mathbb{N}} D_k.$$

Consequently, by (4.47) and a union bound, it holds that for every $n \in \mathbb{N}$,

$$\mathbb{P}[\rho^n \notin K] \leq \sum_{k=1}^{\infty} \mathbb{P}\left[\Psi_{1/k}(\rho^n) \notin C_k\right] \leq \epsilon.$$

Step 4: Compactness of K in the metric space $(L^1(U \times [0,T]), D)$. To show that the set K is compact, we prove that any sequence $(f_n)_{n\in\mathbb{N}} \subset K$ has a convergent subsequence in K with respect to the metric D.

By construction, it holds that

$$f \in K \implies f \in D_k \quad \forall k \in \mathbb{N} \implies \Psi_{1/k}(f) \in C_k \quad \forall k \in \mathbb{N},$$

which implies $\Psi_{1/k}(K) \subset C_k$. But since for every $k \in \mathbb{N}$, the set C_k is compact in $L^1(U \times [0,T])$, it follows that $\{\Psi_{1/k}(f) : f \in K\}$ is relatively compact in $L^1(U \times [0,T])$.

Using this fact and a diagonalisation argument, for the sequence $(f_n)_{n\in\mathbb{N}}$, there exists a subsequence $(f_{n_j})_{j\in\mathbb{N}}$ such that for every $k \in \mathbb{N}$,

$$\Psi_{1/k}(f_{n_j}) \text{ converges in } L^1(U \times [0,T]) \text{ as } j \to \infty.$$

Consequently, by the definition of the metric D, it follows that $(f_{n_j})_{j\in\mathbb{N}}$ is a Cauchy sequence in $(L^1(U \times [0,T]), D)$, and so converges to some $f \in L^1(U \times [0,T])$ with respect to the metric D. Finally, we realise that K is closed since each of the D_k are closed. Therefore $f \in K$, and K is compact.

Step 5: Conclusion. We showed that for every $\epsilon > 0$ there exists a compact set $K \subset L^1(U \times [0,T])$ such that

$$\sup_n \mathbb{P}\left[\rho^n \notin K\right] \leq \epsilon.$$

Therefore, the laws $\{\rho^n\}_{n\in\mathbb{N}}$ are tight in $(L^1(U \times [0,T]), D)$, and hence by Lemma 4.4.5, the laws are also tight on $L^1(U \times [0,T])$ with respect to the strong norm topology. □

The final ingredient to prove existence will come from the technical lemma below, which is an alternative characterisation for the convergence in probability of a sequence of random variables. See Lemma 1.1 of Gyöngy and Krylov [54] for proof.

Lemma 4.4.7. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $\bar{X}$ *be a complete separable metric space. A sequence* $\{X_n : \Omega \to \bar{X}\}_{n\in\mathbb{N}}$ *of* $\bar{X}$*-valued random variables converges in probability as* $n \to \infty$ *if and only if for every pair of sequences* $(n_k, m_k)_{k\in\mathbb{N}}$ *with* $n_k, m_k \to \infty$ *as* $k \to \infty$*, there is a further subsequence* $(n_{k'}, m_{k'})_{k'\in\mathbb{N}}$ *such that the joint laws* $(X_{n_{k'}}, X_{m_{k'}})$ *converge weakly as* $k' \to \infty$ *to a probability measure* μ *on* $\bar{X} \times \bar{X}$ *satisfying*

$$\mu(\{(x, y) \in \bar{X} \times \bar{X} : x = y\}) = 1.$$

We are ready to prove the main existence result, whose proof follows the proof on the torus, see Theorem 5.29 of Fehrman and Gess [41]. The full proof can be found there, we will just explain the main idea by putting all the previous results from this section together. Similar arguments in a simpler setting can be found in Debussche and Vovelle [26]. See also Theorem 6.2 of Wang, Wu and Zhang [95] for a similar proof.

Theorem 4.4.8 (Existence of solution to the singular equation (2.23))**.** *Suppose that the spatial components of the noise* ξ^F*, the non-linear functions* Φ, σ, ν *and the boundary data* $\bar{f}$ *satisfy Assumptions 2.1.2, 2.2.1 and 2.2.9 respectively. Let further* $\rho_0 \in L^2(\Omega; L^2(U))$ *be non-negative and* $\mathcal{F}_0$*-measurable.*

There exists a stochastic kinetic solution to the generalised Dean–Kawasaki equation (2.23) *in the sense of Definition 2.3.6. Furthermore, the solution satisfies the estimates of Proposition 4.1.8.*

Proof. For clarity, we again split the proof into several steps.

Step 1: Tightness. Recall that the sequence of regularised equations defined in Definition 4.1.4 are denoted by $\{\rho^n\}_{n\in\mathbb{N}}$, the martingales introduced in Proposition 4.4.6 are denoted by $M^{n,\psi}$, and introduce the kinetic measures for the regularised equation

$$q^n := \delta_0(\xi - \rho^n) \left(|\nabla\Theta_{\Phi,2}(\rho^n)|^2 + \frac{1}{n}|\rho^n|^2 \right), n \in \mathbb{N}.$$

Proposition 4.4.2 gives us existence of stochastic kinetic solutions, and the energy estimate of Proposition 4.1.8 allows us to deduce that $\{q^n\}_{n\in\mathbb{N}}$ are a sequence of finite kinetic measures.

Using the kinetic equation given in Proposition 4.4.2, one can write an equation for the kinetic functions χ^n of ρ^n. We analyse each of the random components of the kinetic equation χ^n. Fixing a dense sequence of functions $\{\psi_j\}_{j\in\mathbb{N}}$ of $C_c^\infty(U \times (0,\infty))$ in the strong $H^s(U \times (0,\infty))$ topology (for $s > d/2 + 1$), we consider the random variables,

$$X^n := \left(\rho^n,\, \nabla\Theta_{\Phi,2}(\rho^n),\, \frac{1}{n}\nabla\rho^n,\, q^n, (M^{n,\psi_j})_{j\in\mathbb{N}} \right)$$

on the space

$$\bar{X} := L^1(U \times [0,T]) \times L^2\left(U \times [0,T]; \mathbb{R}^d\right)^2 \times \mathcal{M}(U \times (0,\infty) \times [0,T]) \times C([0,T])^{\mathbb{N}}.$$

We equip the space $\bar{X}$ with the product topology, with the strong topology on $L^1(U \times (0,T))$, the weak topologies on $L^2(U \times (0,T);\mathbb{R}^d)$ and $\mathcal{M}(U \times (0,\infty) \times [0,T])$ and topology of component wise convergence in the strong norm on $C([0,T])^{\mathbb{N}}$, in particular using a similar metric as was constructed in Definition 4.4.4,

$$D_C((f_k),(g_k)) = \sum_{k=1}^{\infty} 2^{-k} \left(\frac{\|f_k - g_k\|_{C([0,T])}}{1 + \|f_k - g_k\|_{C([0,T])}} \right).$$

To show convergence in probability of the random variables X^n, we will use Lemma 4.4.7. To this end, we consider two $\mathbb{N}$-valued subsequences $(n_k)_{k\in\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}}$ such that $n_k, m_k \to \infty$ as $k \to \infty$. Consider the laws of

$$(X^{n_k}, X^{m_k}, B),$$

where $B = (B^j)_{j\in\mathbb{N}}$ are the Brownian motions which form part of the noise ξ^F, defined on the space

$$\bar{Y} := \bar{X} \times \bar{X} \times C([0,T])^{\mathbb{N}}.$$

The energy estimate in Proposition 4.1.8 alongside the two tightness results in Proposition 4.4.6 show that the laws of $(X^n)_{n\in\mathbb{N}}$ are tight on $\bar{X}$.

Step 2: Skorokhod representation theorem. By Prokhrov's theorem, passing to a subsequence still denoted by $k \to \infty$, there is a probability measure μ on $\bar{Y}$ such that as $k \to \infty$, we have the weak convergence of the laws

$$\mathcal{L}(X^{n_k}, X^{m_k}, B) \to \mu \quad \text{on } \bar{Y}.$$

Since the space $\bar{X}$ is separable, this implies that $\bar{Y}$ is also separable, so we can apply the Skorokhod representation theorem. This gives that there is an auxiliary probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that for every $k \in \mathbb{N}$,

$$(\tilde{Y}^k, \tilde{Z}^k, \tilde{\beta}^k) = (X^{n_k}, X^{m_k}, B) \quad \text{in law on } \bar{Y}, \tag{4.48}$$

$$(\tilde{Y}, \tilde{Z}, \tilde{\beta}) = \mu \quad \text{in law on } \bar{Y},$$

and we have the almost sure convergence in the space $\bar{Y}$ as $k \to \infty$:

$$(\tilde{Y}^k, \tilde{Z}^k, \tilde{\beta}^k) \to (\tilde{Y}, \tilde{Z}, \tilde{\beta}). \tag{4.49}$$

We denote expectations with respect to $\tilde{\mathbb{P}}$ by $\tilde{\mathbb{E}}$. To apply Lemma 4.4.7 we would like to show that $\tilde{Y} = \tilde{Z}$.

Step 3: Characterising $\tilde{Y}$. It follows from the equality in law of $\tilde{Y}^k$ and X^{n_k} that there exists

$$\begin{cases} \tilde{\rho}^k \in L^\infty(\Omega \times [0,T]; L^1(U)) \\ \tilde{G}_1^k, \tilde{G}_2^k \in L^2(\Omega; L^2(U \times [0,T]; \mathbb{R}^d)) \\ \tilde{q}^k \in L^1(\Omega; \mathcal{M}(U \times (0,\infty) \times [0,T])) \\ (\tilde{M}^{k,\psi_j})_{j\in\mathbb{N}} \in L^1(\Omega; C([0,T])^{\mathbb{N}}) \end{cases}$$

such that

$$\tilde{Y}^k = (\tilde{\rho}^k, \tilde{G}_1^k, \tilde{G}_2^k, \tilde{q}^k, (\tilde{M}^{k,\psi_j})_{j\in\mathbb{N}}). \tag{4.50}$$

We identify elements of (4.50) by converting various expectations $\tilde{\mathbb{E}}$ into expectations $\mathbb{E}$ using the equality in law (4.48), which gives for every smooth and bounded function ψ that

$$\tilde{\mathbb{E}}\left[\int_0^T \int_U \psi(\tilde{\rho}^k)\,dx\,ds\right] = \mathbb{E}\left[\int_0^T \int_U \psi(\rho^{n_k})\,dx\,ds\right]. \tag{4.51}$$

Let us illustrate how this can be used to identify $\tilde{G}_1^k$. First of all, in analogy with Lemma 4.4.3, by choosing an increasing approximating sequence for $\Theta_{\Phi,2}$, it follows that uniformly in k we have

$$\tilde{\mathbb{E}}\left[\int_0^T \int_U \Theta_{\Phi,2}\left(\tilde{\rho}^k\right)\,dx\,ds\right] < \infty.$$

By (4.48) and (4.51), for every compactly supported $\psi \in C_c^\infty(U)$, $\tilde{A} \in \tilde{\mathcal{F}}$ and $A \in \mathcal{F}$, we have

$$\begin{aligned}\tilde{\mathbb{E}}&\left[\int_0^T \int_U \left(\Theta_{\Phi,2}(\tilde{\rho}^k)\nabla\psi + \psi\tilde{G}_1^k\right)\mathbb{1}_{\tilde{A}}\,dx\,ds\right]\\ &= \mathbb{E}\left[\int_0^T \int_U \left(\Theta_{\Phi,2}(\rho^{n_k})\nabla\psi + \psi\nabla\Theta_{\Phi,2}(\rho^{n_k})\right)\mathbb{1}_A\,dx\,ds\right] = 0.\end{aligned}$$

This allows us to conclude that $\tilde{\mathbb{P}}$-almost surely $\tilde{G}_1^k = \nabla\Theta_{\Phi,2}(\tilde{\rho}^k)$. By virtually an identical argument, it follows that $\tilde{G}_2^k = \frac{1}{n_k}\nabla\tilde{\rho}^k$, that $\tilde{q}^k$ is $\tilde{\mathbb{P}}$-almost surely a kinetic measure for $\tilde{\rho}^k$ in the sense of Definition 2.3.5 satisfying (2.38), and that the continuous paths $(\tilde{M}^{k,\psi_j})_{j\in\mathbb{N}}$ are $\tilde{\mathbb{P}}$-almost surely defined by (4.42) with $\tilde{\rho}^k$ replacing ρ. Furthermore using the energy estimate of Proposition 4.1.8, it holds that as $k \to \infty$,

$$\frac{1}{n_k}\nabla\tilde{\rho}^k \to 0 \quad \text{weakly in} \quad L^2(U \times [0,T]; \mathbb{R}^d).$$

Consequently, in the limit as $k \to \infty$ we have the characterisation

$$\tilde{Y} = (\tilde{\rho}, \nabla\Theta_{\Phi,2}(\tilde{\rho}), 0, \tilde{q}, (\tilde{M}^j)_{j\in\mathbb{N}}),$$

where $\tilde{p} \in L^1(\tilde{\Omega}; L^1(U \times [0,T]))$ and $\tilde{q}$ is the corresponding kinetic measure.

It remains to characterise $\tilde{M}^j$, and to do this we first need to characterise $\tilde{\beta}^k$.

Step 4: The path $\tilde{\beta}$ is a Brownian Motion. Writing for each $k \in \mathbb{N}$, $\tilde{\beta}^k := (\tilde{\beta}^{k,j})_{j\in\mathbb{N}}$, and the limiting process $\tilde{\beta} = (\tilde{\beta}^j)_{j\in\mathbb{N}}$, one obtains using the same trick of interchanging the expectations $\tilde{\mathbb{E}}$ and $\mathbb{E}$ using equalities in law that for any $F : \bar{Y} \to \mathbb{R}$, $s \le t \in [0,T]$, and $k, j \in \mathbb{N}$

$$\begin{aligned}\tilde{\mathbb{E}}&\left[F\left(\tilde{Y}^k|_{[0,s]}, \tilde{Z}^k|_{[0,s]}, \tilde{\beta}^k|_{[0,s]}\right)\left(\tilde{\beta}_t^{k,j} - \tilde{\beta}_s^{k,j}\right)\right]\\ &= \mathbb{E}\left[F\left(X^{n_k}|_{[0,s]}, X^{m_k}|_{[0,s]}, B|_{[0,s]}\right)\left(B_t^j - B_s^j\right)\right] = 0.\end{aligned} \tag{4.52}$$

Passing to the limit as $k \to \infty$ using the uniform integrability of the paths $\tilde{\beta}^{k,j}_t - \tilde{\beta}^{k,j}_s$ implied by the second point of Proposition 4.4.6 and using the equivalence in law (4.48), we have

$$\tilde{\mathbb{E}}\left[F\left(\tilde{Y}|_{[0,s]}, \tilde{Z}|_{[0,s]}, \tilde{\beta}|_{[0,s]}\right)\left(\tilde{\beta}^j_t - \tilde{\beta}^j_s\right)\right] = 0.$$

Identically for $i, j \in \mathbb{N}$, $s \le t \in [0,T]$,

$$\tilde{\mathbb{E}}\left[F\left(\tilde{Y}|_{[0,s]}, \tilde{Z}|_{[0,s]}, \tilde{\beta}|_{[0,s]}\right)\left(\tilde{\beta}^j_t\tilde{\beta}^i_t - \tilde{\beta}^j_s\tilde{\beta}^i_s - \delta_{ij}(t-s)\right)\right] = 0,$$

where $\delta_{ij} := \mathbb{1}_{i=j}$ is the Kronecker delta. Using these and the fact that $\tilde{\beta}^j$ has almost surely continuous paths, we conclude using Lévy's characterisation that $\tilde{\beta}^j$ are independent one dimensional Brownian motions with respect to the filtration

$$\mathcal{G}_t = \sigma\left(\tilde{Y}|_{[0,t]}, \tilde{Z}|_{[0,t]}, \tilde{\beta}|_{[0,t]}\right).$$

By continuity and uniform integrability $\tilde{\beta}$ is a Brownian motion with respect to the augmented filtration $\bar{\mathcal{G}}$ of $\mathcal{G}$.

Step 5: $(\tilde{M}^j)_{j\in\mathbb{N}}$ are $\bar{\mathcal{G}}_t$ martingales. We apply a similar argument as the previous point, see equation (4.52). We obtain that for $s \le t \in [0,T]$ and $j,k \in \mathbb{N}$,

$$\tilde{\mathbb{E}}\left[F\left(\tilde{Y}^k|_{[0,s]}, \tilde{Z}^k|_{[0,s]}, \tilde{\beta}^k|_{[0,s]}\right)\left(\tilde{M}^{k,\psi_j}_t - \tilde{M}^{k,\psi_j}_s\right)\right] = 0.$$

The fact that $(\tilde{M}^j)_{j\in\mathbb{N}}$ satisfy the martingale property with respect to $\bar{\mathcal{G}}_t$ follows by taking the limit as $k \to \infty$ using the uniform integrability of the martingales.

Step 6: $(\tilde{M}^j)_{j\in\mathbb{N}}$ are stochastic integrals with respect to $\tilde{\beta}$. Again this follows from the same techniques as the previous points. First proving the results for the approximations and then taking a limit as $k \to \infty$, one proves that for every $i, j \in \mathbb{N}, s < t \in [0,T]$,

$$\tilde{\mathbb{E}}\Bigg[F\left(\tilde{Y}|_{[0,s]}, \tilde{Z}|_{[0,s]}, \tilde{\beta}|_{[0,s]}\right) \times \left(\tilde{M}^j_t\tilde{\beta}^i_t - \tilde{M}^j_s\tilde{\beta}^i_s - \int_s^t \int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})f_i(x))\right)\,dx\,d\tilde{t}\Bigg] = 0,$$

where recall $(f_i)_{i\in\mathbb{N}}$ are the spatial components of the noise ξ^F introduced in Definition 2.1.1. Hence this shows for each $i \in \mathbb{N}$,

$$\tilde{M}^j_t\tilde{\beta}^i_t - \int_s^t \int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})f_i) \quad \text{is a} \quad \mathcal{G}_t - \text{martingale}.$$

It is easy to see by uniform integrability and time continuity that the process is also a $\bar{\mathcal{G}}_t$ martingale. Identical arguments show that for $j \in \mathbb{N}$

$$(\tilde{M}^j_t)^2 - \int_0^t \sum_{k=1}^{\infty}\left(\int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})f_k)\right)^2$$

is a continuous $\bar{\mathcal{G}}_t$ martingale. Putting everything together, due to an explicit calculation using the quadratic variation of Brownian motion, for every $j \in \mathbb{N}$, $t \in [0,T]$,

$$\tilde{\mathbb{E}}\left(\left(\tilde{M}_t^j - \int_0^t \int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})\tilde{\xi}^F)\right)^2\right) = 0,$$

where $\tilde{\xi}^F$ is defined analogously to ξ^F but with Brownian motion $\tilde{\beta}$ on $\tilde{\Omega}$. It follows that we have the characterisation

$$\tilde{M}_t^j = \int_0^t \int_U \psi_j(x,\tilde{\rho})\nabla\cdot(\sigma(\tilde{\rho})\tilde{\xi}^F).$$

Step 7: Tying up loose ends. One needs to show the following technical steps in order to finish the proof. The results are technical and follow that of Fehrman and Gess [41] so we do not include the details here.

1. Show the limiting kinetic measure $\tilde{q}$ is almost surely a kinetic measure for $\tilde{p}$.
2. Show that $\sigma(\tilde{\rho})$ is $L^2(U\times[0,T])$-integrable.
3. Remove the set $\mathcal{A} := \{t\in[0,T] : \tilde{q}(\{t\}\times U\times\mathbb{R}) \neq 0\}$. Outside $\mathcal{A}$ there is no ambiguity when writing the kinetic equation for the kinetic function $\tilde{\chi}$.
4. Show that $\tilde{\rho}\in L^1(U\times[0,T])$.

   This is quite a technical step, it involves looking at left and right continuous representations of $\tilde{\rho}$. One needs to study properties of the left and right kinetic functions $\chi^{\pm}$. Conclude by showing that the measure $\tilde{q}$ almost surely has no atoms in time.

Step 8: Conclusion. We showed the existence of $\tilde{\rho}$ with representative in $L^1(\tilde{\Omega}\times[0,T];L^1(U))$ that is a stochastic kinetic solution of the Dean–Kawasaki equation (2.23) in the sense of Definition 2.3.6 with respect to Brownian Motion $\tilde{\beta}$ and filtration $(\bar{\mathcal{G}}_t)_{t\in[0,T]}$. That is to say, we showed the existence of a probabilistically weak solution. We conclude by illustrating how to extend this to a probabilistically strong solution.

In Steps 3-7 above, we characterised the process $\tilde{Y}$. We repeat the steps to Characterise $\tilde{Z}$ as in (4.49). Repeating Step 3 above, it follows that we can characterise $\tilde{Z}$ as

$$\tilde{Z} = (\bar{\rho}, \nabla\Theta_{\Phi,2}(\bar{\rho}), 0, \bar{q}, (\bar{M}^j)_{j\in\mathbb{N}}).$$

Continuing, one shows there is an $L^1(U\times[0,T])$-integrable, continuous representation of $\bar{\rho}$ which is a stochastic kinetic solution in the sense of Definition 2.3.6 with respect to Brownian Motion $\tilde{\beta}$ and filtration $(\bar{\mathcal{G}}_t)_{t\in[0,T]}$.

The uniqueness theorem, Theorem 3.2.2 tells us $\tilde{\rho} = \bar{\rho}$ almost surely in the space $L^1(U\times[0,T])$.

By Lemma 4.4.7, it follows that after passing $\{\rho^n\}$ to a subsequence n_k on the original probability space $(\Omega,\mathcal{F},\mathbb{P})$, there is a random variable $\rho\in L^1(\Omega\times[0,T];L^1(U))$ such that ρ^{n_k} converges to ρ in probability.

Working along a further subsequence, it holds that ρ^{n_k} converges almost surely to ρ. Repeating the steps above once more, we can show ρ is a stochastic kinetic solution of the generalised Dean–Kawasaki equation (2.23) in the sense of Definition 2.3.6. Noting that ρ^n are all probabilistically strong solutions to the regularised equation, ρ is also a probabilistically strong solution but for the singular equation. The energy estimates follow from the estimates for the regularised equations and the weak lower semicontinuity of the Sobolev norm. $\square$

Chapter 5

Quantitative law of large numbers and central limit theorem

This chapter is dedicated to the proof of a law of large numbers and central limit theorem result for the Dean–Kawasaki equation. In Section 5.1 we provide a setup of the problem. We begin by defining and stating the assumptions on the noise ξ^K. We then define a regularised version of the equation for which we can define weak solutions, and conclude with a quantitative law of large numbers for the regularised equation. In Section 5.2 we prove a central limit theorem for the approximating equation. The estimate is extended to an estimate for the singular equation via an $L^\infty(U \times [0,T])$-estimate in Section 5.3.

5.1 Setup and quantitative law of large numbers

In this section we begin by defining and stating the assumptions on the noise in Definition 5.1.1 and Assumption 5.1.2 respectively. In Example 5.1.4 we give a concrete example of the noise coefficients that we are able to consider, the eigenfunctions of the Laplacian with homogeneous boundary data. The simplifying assumption of constant boundary data and initial condition is given in Assumption 5.1.7. We then define the regularised equation in Definition 5.9 and give the definition of weak solutions of the regularised equation in Definition 5.1.10. Additional assumptions on the non-linear coefficients Φ, σ and ν required for the central limit theorem are presented in Assumption 5.1.13. Remarks 5.1.14-5.1.16 illustrate that the assumptions are satisfied in the cases of interest. Law of large number estimates are then presented via $L^p(U \times [0,T])$-estimates for the regularised equation in Proposition 5.1.18 and 5.1.20. The estimate in Proposition 5.1.18 is an estimate that is p-independent, and Proposition 5.1.20 presents a novel p-dependent estimate.

Recall from the introduction that are interested in studying fluctuations of equation (1.5), defined for $\epsilon \in (0,1)$ and $K = K(\epsilon) \in \mathbb{N}$ by

$$\begin{cases} \partial_t \rho^\epsilon = \Delta\Phi(\rho^\epsilon) - \sqrt{\epsilon}\nabla \cdot (\sigma(\rho^\epsilon) \circ \dot{\xi}^K) - \nabla \cdot \nu(\rho^\epsilon), & \text{on } U \times (0,T], \\ \Phi(\rho^\epsilon) = \bar{f}, & \text{on } \partial U \times [0,T], \\ \rho^\epsilon(\cdot, t=0) = \rho_0, & \text{on } U \times \{t=0\}, \end{cases} \tag{5.1}$$

with the non-linear functions Φ, σ, ν, the boundary data $\bar{f}$, and the initial data ρ_0 all the same as in the previous chapters. Again we highlight that $K \in \mathbb{N}$ will be chosen as a function of ϵ, so we omit the K-dependence in the notation of the equation[1]. We start by defining the truncated noise ξ^K.

Definition 5.1.1 (Truncated noise)**.** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with adapted, independent d-dimensional Brownian motions $(B^k)_{k\in\mathbb{N}}$. We view the sequence $(B^k)_{k\in\mathbb{N}}$ as being $C([0,\infty);(\mathbb{R}^d)^\infty)$-valued, equipped with the metric topology of co-ordinate wise convergence. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of continuously differentiable real valued functions on the domain $U \subset \mathbb{R}^d$. For every $K \in \mathbb{N}$, define the truncated noise*

$$\xi^K : U \times [0,T] \to \mathbb{R}^d, \qquad \xi^K(x,t) := \sum_{k=1}^{K} f_k(x) B_t^k.$$

In this definition the superscript "K" denotes the level of truncation of the noise. In this section, see Assumption 5.1.2 below, the sequence $\{f_k\}_{k\in\mathbb{N}}$ are orthogonal, so one can view the noise ξ^K as an approximation of space-time white noise.

For every $\epsilon \in (0,1)$ and $K \in \mathbb{N}$, the well-posedness of stochastic kinetic solutions of equation (5.1) is guaranteed by Theorems 3.2.2 and 4.4.8 from the previous chapters.

Analogously to equations (2.1)-(2.3), for every $K \in \mathbb{N}$ we define the truncated sums of the noise coefficients

$$F_1^K : U \to \mathbb{R} \quad \text{defined by} \quad F_1^K(x) := \sum_{k=1}^{K} f_k^2(x), \tag{5.2}$$

$$F_2^K : U \to \mathbb{R}^d \quad \text{defined by} \quad F_2^K(x) := \sum_{k=1}^{K} f_k(x)\nabla f_k(x) = \frac{1}{2}\sum_{k=1}^{K} \nabla f_k^2(x), \tag{5.3}$$

$$F_3^K : U \to \mathbb{R} \quad \text{defined by} \quad F_3^K(x) := \sum_{k=1}^{K} |\nabla f_k(x)|^2. \tag{5.4}$$

We need the following assumption on the noise.

Assumption 5.1.2. *For every $K \in \mathbb{N}$ and $i = 1,2,3$, we assume that*

1. *The sums F_i^K are continuous on $\bar{U}$, and also that $\nabla \cdot F_2^K$ is bounded in U.*

---

[1] Remark 5.3.4 gives the precise scaling between K and ϵ. We can make $K(\epsilon)$ integer valued via the relabelling $K(\epsilon) := \lfloor K(\epsilon) \rfloor$

2. *For every $s > d/2$ we have*

$$\sum_{k=1}^{\infty} \|f_k\|^2_{H^{-s}(U)} < \infty.$$

   Repeatedly, when we want to bound sums of $H^{-s+1}(U)$-norms of the noise coefficients, we will require $s > \frac{d+2}{2}$, which is equivalent to the above assumption.

3. *For every $k \in \mathbb{N}$, f_k satisfies homogeneous Dirichlet boundary conditions. That is, $f_k|_{\partial U} = 0$.*

4. *For $j, k \in \mathbb{N}$, if $j \neq k$, then f_k and f_j are orthogonal in $L^2(U)$.*

A key difference between the noise ξ^F from the previous chapter and the noise ξ^K is that the spatial components of ξ^K are required to be $L^2(U)$-orthogonal, whereas the noise ξ^F was coloured in space. This gives the following important remark.

Remark 5.1.3. *Since the noise coefficients f_k are assumed to be orthogonal in $L^2(U)$, we do not expect the limits $\lim_{K\to\infty} F_i^K$ to exist pointwise for $i = 1, 2, 3$, recall Remark 2.1.4.*

The canonical example of the noise coefficients on a bounded domain that one should have in mind are the eigenfunctions of the Laplacian with homogeneous Dirichlet boundary conditions.

Example 5.1.4 (Eigenfunctions of the Dirichlet Laplacian)**.** *For every $k \in \mathbb{N}$, consider $f_k = e_k$, where $\{e_k\}_{k\in\mathbb{N}}$ are the homogeneous Dirichlet eigenfunctions of the Laplacian on U with corresponding eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. That is, solutions to the equation*

$$\begin{cases} -\Delta e_k = \lambda_k e_k, & x \in U \\ e_k = 0, & x \in \partial U. \end{cases} \tag{5.5}$$

We would then define for $K \in \mathbb{N}$ the spectral truncation

$$\xi^K(x,t) := \sum_{k:\lambda_k \le K} e_k(x) B_t^k.$$

Here, K plays the role of an eigenvalue threshold rather than the number of modes.

Let us illustrate that the eigenfunctions of the Laplacian as above satisfy Assumption 5.1.2, so are a valid choice. We have the following properties, see Chapter 11 of Strauss [92] for further details. We state the properties as a proposition without proof.

Proposition 5.1.5 (Properties of the eigenvalues and eigenfunctions of the Laplacian)**.** *Let $\{e_k, \lambda_k\}_{k\in\mathbb{N}}$ be the eigenfunctions and eigenvalues of the Laplacian respectively as defined in Example 5.1.4. The following properties hold:*

1. *All eigenvalues* $\{\lambda_k\}_{k\in\mathbb{N}}$ *are positive. All eigenfunctions* $\{e_k\}_{k\in\mathbb{N}}$ *can be chosen to be real valued, they are orthogonal in* $L^2(U)$ *and so by Gram-Schmidt they can be chosen to be orthonormal in* $L^2(U)$*. The eigenfunctions are complete in the* $L^2(U)$*-sense.*

2. *Using the local version of Weyl's law, see Weyl [97], we have the bounds*

$$\begin{aligned} E_1^K(x) &:= \sum_{k:\lambda_k\le K} e_k^2(x) \le CK^{\frac{d}{2}}; \quad E_2^K(x) := \sum_{k:\lambda_k\le K} e_k(x)(\nabla e_k)(x) \le CK^{\frac{d+1}{2}}; \\ E_3^K(x) &:= \sum_{k:\lambda_k\le K} |(\nabla e_k)(x)|^2 \le CK^{\frac{d+2}{2}}. \end{aligned} \tag{5.6}$$

Furthermore, by the definition of the eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$*, we also have for every* $K\in\mathbb{N}$*,*

$$\nabla\cdot E_2^K = \nabla\cdot\sum_{k=1}^K e_k\nabla e_k = \sum_{k=1}^K \left(|\nabla e_k|^2 + e_k\Delta e_k\right) = \sum_{k=1}^K \left(|\nabla e_k|^2 - \lambda_k e_k^2\right) \le E_3^K. \tag{5.7}$$

We also have the result below that illustrates that $\{e_k\}_{k\in\mathbb{N}}$ satisfy the second point of Assumption 5.1.2.

Lemma 5.1.6. *Let* $\{e_k\}_{k\in\mathbb{N}}$ *be defined as in Example 5.1.1. Then we have the equivalence*

$$\sum_{k=1}^\infty \|e_k\|^2_{H^{-s}(U)} < \infty \iff s > \frac{d}{2}.$$

Proof. The $H^{-s}(U)$-norm of the eigenfunctions of the Laplacian is computed using the definition

$$\|e_k\|_{H^{-s}(U)} := \sup_{f\in H^s(U):\,\|f\|_{H^s(U)}\le 1} |\langle e_k, f\rangle_s|,$$

where the inner product $\langle\cdot,\cdot\rangle_s$ denotes the dual pairing of $H^{-s}(U)$ and $H^s(U)$, which can be interpreted as an $L^2(U)$ inner product by using $e_k\in L^2(U)\subset H^{-s}(U)$. Since $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal basis in $L^2(U)$, we can write a general element $f\in L^2(U)$ as

$$f(x) = \sum_{k=1}^\infty f_k e_k(x), \quad f_k = \langle f, e_k\rangle_{L^2(U)},$$

by which it follows from spectral calculus that the $H^s(U)$-norm of f is given by

$$\|f\|^2_{H^s(U)} = \sum_{j=1}^\infty (1+\lambda_j)^s|f_j|^2.$$

The maximisation problem becomes

$$\|e_k\|_{H^{-s}(U)} = \sup_{f\in H^s(U):\,\sum_{j\in\mathbb{N}}(1+\lambda_j)^s|f_j|^2\le 1} |f_k|.$$

The supremum is attained when $f_j = 0$ for every $j \neq k$, and for the k'th element we have $(1+\lambda_k)^s |f_k|^2 = 1$. Rearranging gives

$$\|e_k\|_{H^{-s}(U)} = (1+\lambda_k)^{-s/2}.$$

By a corollary of Weyl's law [97], we have that on a C^2-regular bounded domain $U \subset \mathbb{R}^d$, the eigenvalues grow like $\lambda_k \asymp k^{2/d}$, where we used the notation $\asymp$ to mean that there exists constants $c_1, c_2 \in (0,\infty)$ such that for all large k, $c_1 k^{2/d} \leq \lambda_k \leq c_2 k^{2/d}$. Hence we have

$$\sum_{k=1}^{\infty} \|e_k\|^2_{H^{-s}(U)} \asymp \sum_{k=1}^{\infty} \frac{1}{(1+k^{2/d})^s} < \infty \iff 2s/d > 1 \iff s > \frac{d}{2}.$$

□

For the subsequent results, we will have to make the following simplifying assumption. The assumption has the same consequence as the assumption required on the torus, see Assumption 2.1 of Clini and Fehrman [20].

Assumption 5.1.7 (Simplifying Assumption). *Suppose that the $\mathcal{F}_0$-measurable initial data and boundary data are the same random positive constant. That is, there exists an $M > 0$ such that*

$$\Phi(\rho^\epsilon)|_{\partial U} = M, \quad \rho_0|_U = \Phi^{-1}(M).$$

Remark 5.1.8. *The consequence of Assumption 5.1.7 is that the solution $\bar{\rho}$ of the hydrodynamic limit equation*

$$\partial_t \bar{\rho} = \Delta\Phi(\bar{\rho}) - \nabla \cdot \nu(\bar{\rho}), \tag{5.8}$$

with initial data ρ_0 and boundary data $\bar{f}$ is given uniquely[2] by the constant $\bar{\rho} = \Phi^{-1}(M)$, and therefore we are essentially looking at fluctuations around a constant function.

A remark illustrating the necessity of this assumption is given after the proof of the central limit theorem, see Remark 5.1.7.

Just as we did in Definition 4.1.4 in the previous chapter, in order to consider weak solutions we need to smooth the potential singularity coming from σ, and to obtain energy estimates we need to add a regularisation.

Definition 5.1.9 (Regularised equation). *Let $\{\sigma_n\}_{n\in\mathbb{N}}$ be the sequence of smooth approximations of σ as in (4.4) from Definition 4.1.4. For $n \in \mathbb{N}$, let $\rho^{n,\epsilon}$ denote the solution to the regularised equation*

$$\partial_t \rho^{n,\epsilon} = \Delta\Phi(\rho^{n,\epsilon}) + \frac{1}{n}\Delta\rho^{n,\epsilon} - \sqrt{\epsilon}\nabla \cdot (\sigma_n(\rho^{n,\epsilon}) \circ \dot{\xi}^K) - \nabla \cdot \nu(\rho^{n,\epsilon}), \tag{5.9}$$

with boundary data $\Phi(\rho^{n,\epsilon})|_{\partial U} = \bar{f}$ and initial condition $\rho^{n,\epsilon}(\cdot, t=0) = \rho_0(\cdot)$.

---

[2] The uniqueness follows by the same arguments as Theorem 3.2.2 in the special case $\sigma = 0$ (no noise), and the equivalence of stochastic kinetic and weak solutions.

By repeating the same computation as equation (2.23), the Itô-to-Stratonovich conversion of (5.9) is

$$\begin{aligned}\partial_t\rho^{n,\epsilon} = \Delta\Phi(\rho^{n,\epsilon}) + \frac{1}{n}\Delta\rho^{n,\epsilon} - \sqrt{\epsilon}\nabla\cdot(\sigma_n(\rho^{n,\epsilon})\dot{\xi}^K) - \nabla\cdot\nu(\rho^{n,\epsilon})&\\ + \frac{\epsilon}{2}\nabla\cdot(F_1^K(\sigma_n'(\rho^{n,\epsilon})^2\nabla\rho^{n,\epsilon} + \sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})F_2^K).&\quad (5.10)\end{aligned}$$

Since we added a $\frac{1}{n}\Delta\rho^{n,\epsilon}$-term to the equation, our goal is to study convergence of the regularised equation (5.10) as $\epsilon\to 0$ to the regularised zero noise hydrodynamic limit equation

$$\partial_t\bar{\rho}^n = \Delta\Phi(\bar{\rho}^n) + \frac{1}{n}\Delta\bar{\rho}^n - \nabla\cdot\nu(\bar{\rho}^n). \quad (5.11)$$

However, owing to Assumption 5.1.7, the unique weak solution of (5.11) is still the constant function $\bar{\rho}^n = \Phi^{-1}(M)$, for every $n\in\mathbb{N}$.

With the regularisations mentioned in Definition 5.1.9, we can make sense of weak solutions to (5.9) in an analogous way as Definition 4.1.5.

Definition 5.1.10 (Weak solution of regularised equation)**.** *Suppose that the non-linear functions Φ,σ,ν, the boundary data $\bar{f}$ and spatial components of the noise ξ^K satisfy Assumptions 2.2.1, 2.2.9 and 5.1.2 respectively. Let further $\rho_0\in L^2(\Omega;L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable, let $h_{\Phi^{-1}(\bar{f})}$ be as in Definition 4.1.1. A weak solution $\rho^{n,\epsilon}$ of the regularised equation* (5.9) *with initial condition ρ_0 is a continuous $L^2(U)$-valued, non-negative, $(\mathcal{F}_t)_{t\geq 0}$-predictable process satisfying*

1. *Boundary condition, regularity of solution: For every $k\in\mathbb{N}$,*

$$((\rho^{n,\epsilon}\wedge k)\vee 1/k) - \big((h_{\Phi^{-1}(\bar{f})}\wedge k)\vee 1/k\big)\in L^2(\Omega\times[0,T];H_0^1(U)).$$

2. *The equation: For all test functions $\psi\in C_c^\infty(U)$ and every $t\in[0,T]$,*

$$\begin{aligned}\int_U \rho^{n,\epsilon}(x,t)\psi(x) &= \int_U \rho_0(x)\psi(x) - \int_0^t\int_U \Phi'(\rho^{n,\epsilon})\nabla\psi\cdot\nabla\rho^{n,\epsilon}\\ &- \frac{1}{n}\int_0^t\int_U \nabla\psi\cdot\nabla\rho^{n,\epsilon} + \sqrt{\epsilon}\int_0^t\int_U \sigma_n(\rho^{n,\epsilon})\nabla\psi\cdot d\xi^K + \int_0^t\int_U\nabla\psi\cdot\nu(\rho^{n,\epsilon})\\ &+ \frac{\epsilon}{2}\int_0^t\int_U F_1^K[\sigma_n'(\rho^{n,\epsilon})]^2\nabla\psi\cdot\nabla\rho^{n,\epsilon} + \frac{\epsilon}{2}\int_0^t\int_U \sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})\nabla\psi\cdot F_2^K.\end{aligned}$$

Proposition 5.1.11 (Well-posedness of weak solutions to regularised equation)**.** *Suppose that the non-linear functions Φ,σ,ν and the spatial components of the noise ξ^K satisfy Assumptions 2.2.1 and 5.1.2 respectively. Suppose that ρ_0 and $\bar{f}$ satisfy the simplifying Assumption 5.1.7. For every $n\in\mathbb{N}$, there exists a unique weak solution $\rho^{n,\epsilon}$ of equation* (5.10)*. Furthermore, weak solutions and stochastic kinetic solutions coincide.*

Proof. The argument is a simplified version of Proposition 4.4.1 above, since our noise is already finite dimensional we skip the approximation of the noise step and everything else is the same. □

For the results below, we need the auxiliary functions, which can only defined for general $p > 2$ due to Assumption 5.1.7 which tells us $\bar{\rho}$ is a fixed constant, independent of space or time.

Definition 5.1.12 (Auxiliary functions for Φ and σ)**.** *Under Assumption 5.1.7, let $\bar{\rho}$ denote the constant solution to the hydrodynamic limit equation* (5.8)*. For every $p > 2$, define $\Theta_{\Phi,p}$ to be the unique function satisfying*

$$\Theta_{\Phi,p}(\bar{\rho}) = 0, \quad \textit{and} \quad \Theta'_{\Phi,p}(\xi) = (\xi - \bar{\rho})^{\frac{p-2}{2}}(\Phi'(\xi))^{1/2}, \tag{5.12}$$

and define $\Theta_{\sigma,p}$ to be the unique function satisfying

$$\Theta_{\sigma,p}(\bar{\rho}) = 0, \quad \textit{and} \quad \Theta'_{\sigma,p}(\xi) = (\xi - \bar{\rho})^{p-2}\sigma(\xi)\sigma'(\xi). \tag{5.13}$$

For $p = 2$, the functions $\Theta_{\Phi,2}$ and $\Theta_{\sigma,2}$ are defined with the same antiderivatives as (5.12) *and* (5.13) *respectively, but with initial conditions $\Theta_{\Phi,2}(0) = \Theta_{\sigma,2}(0) = 0$.*

For the central limit theorem results we need the following additional assumptions.

Assumption 5.1.13.

1. *Assume that $\Phi, \nu \in C^2(0,\infty)$ are such that there exists constants $c, \beta \in (0,\infty)$ such that for every $\xi \in (0,\infty)$,*

$$|\Phi''(\xi)| + |\nu''(\xi)| \leq c(1 + \xi^{\beta}). \tag{5.14}$$

2. *For $\bar{\rho}$ the solution to the regularised hydrodynamic limit equation* (5.8) *satisfying Assumption 5.1.13, and for the functions $\Theta_{\Phi,p}$ and $\Theta_{\sigma,p}$ as in equations* (5.12) *and* (5.13) *respectively, for every $p \geq 2$, suppose that there exists constants $q \in (0,2)$ and $c \in (0,\infty)$ such that for every $\xi \in (0,\infty)$,*

$$(\xi - \bar{\rho})^{p-2}\sigma^2(\xi) + \Theta_{\sigma,p}(\xi) \leq c(1 + \Theta^q_{\Phi,p}(\xi)). \tag{5.15}$$

3. *For $\Theta_{\sigma,p}$ as in* (5.13)*, suppose that for every $p \geq 2$ there exists constants $\gamma, c \in (0,\infty)$ such that for every $\xi \in (0,\infty)$,*

$$\sigma^p(\xi) + \left(\frac{\Theta_{\sigma,p}(\xi)}{(\xi - \bar{\rho})^{p-2}}\right)^{p/2} \leq c(1 + \xi^{\gamma}). \tag{5.16}$$

Remark 5.1.14. *The first point is used in the proof of the central limit theorem for the regularised equation, when we make rigorous the formal computation from* (1.11)*, in particular see the the computation between equations* (5.44) *and* (5.46) *below. We note that due to the presence of the constant on the right hand side of* (5.14) *and Assumption 5.1.7, the bound can be alternatively stated as*

$$|\Phi''(\xi)| + |\nu''(\xi)| \leq c(1 + (\xi - \bar{\rho})^{\beta}).$$

This can be seen rigorously via the inequality $a^{\beta} \leq 2^{\beta}((a-b)^{\beta} + b^{\beta})$.

The second and third points in Assumption 5.1.13 are used to prove two different energy estimates in Propositions 5.1.18 and 5.1.20 respectively. We give a comment below that justifies why they are reasonable.

Remark 5.1.15 (Verifying point two of Assumption 5.1.13)**.** *Since for small values of the argument $\xi \in (0,\infty)$, the functions on the left hand side of* (5.15) *are bounded, the presence of the constant term on the right hand side gives that the inequality is always true.*

Hence, we just need to verify the inequality (5.15) *for large $\xi \in (0,\infty)$, where by "large", we mean large relative to $\bar{\rho}$ so that we have $(\xi - \bar{\rho}) \asymp \xi$.*

In this case, for the model case $\Phi(\xi) = \xi^m$ we have that $\Theta'_{\Phi,p}(\xi) \asymp \xi^{\frac{p+m-3}{2}}$, and consequently it follows

$$\Theta_{\Phi,p}(\xi) \asymp \xi^{\frac{p+m-1}{2}}. \tag{5.17}$$

Furthermore, when $\sigma(\xi) = \Phi^{1/2}(\xi)$ as in the particle system example (1.22)*, for large ξ we have that $(\xi - \bar{\rho})^{p-2}\sigma^2(\xi) \asymp \xi^{p+m-2}$ and since $\sigma(\xi)\sigma'(\xi) \asymp \xi^{m-1}$, we also have $\Theta_{\sigma,p}(\xi) \asymp \xi^{m+p-2}$. Hence point 2 of Assumption 5.1.13 is satisfied with the choice*

$$q = \frac{2(m+p-2)}{m+p-1} < 2. \tag{5.18}$$

We note that this also holds in the case of the classical Dean–Kawasaki equation, $\Phi(\xi) = \xi, \sigma(\xi) = \sqrt{\xi}$, with the choice $m = 1$ in (5.18)*.*

Remark 5.1.16 (Verifying point 3 of Assumption 5.1.13)**.** *Recall from Remark 5.1.15, that in the case of interest from the particle system perspective $\Phi(\xi) = \xi^m$ and $\sigma(\xi) = \Phi^{1/2}(\xi)$, we had $\Theta_{\sigma,p}(\xi) \asymp \xi^{m+p-2}$. Hence the quotient in* (5.16) *is well defined as $\Theta_{\sigma,p}(\xi)$ decays to zero faster than $(\xi - \bar{\rho})^{p-2}$ as $\xi \to \bar{\rho}$.*

Again by the same reasoning as Remark 5.1.15, we just need to check the inequality (5.16) *for large values of the argument $\xi \in (0,\infty)$. In this case we have that*

$$\sigma^p(\xi) + \left(\frac{\Theta_{\sigma,p}(\xi)}{(\xi - \bar{\rho})^{p-2}}\right)^{p/2} \asymp \xi^{mp/2},$$

so the choice $\gamma = \frac{mp}{2}$ suffices. By the same reasoning as Remark 5.1.14, we can replace the bound (5.16) *with the more convenient*

$$\sigma^p(\xi) + \left(\frac{\Theta_{\sigma,p}(\xi)}{(\xi - \bar{\rho})^{p-2}}\right)^{p/2} \leq c(1 + (\xi - \bar{\rho})^{\gamma}). \tag{5.19}$$

To bound the right hand side of equation (5.15) which will appear in the energy estimate, we need the following technical lemma. The proof is a simplified version of Proposition 4.1.6, following the methods of equation (4.6) and (4.7).

Lemma 5.1.17 (Interpolation estimate)**.** *Let Ψ be any function, and $z : U \times [0,T] \to \mathbb{R}$ be measurable with constant boundary $z|_{\partial U} = \Phi^{-1}(M)$, $M \geq 0$, and $\Psi(z) \in L^2([0,T]; H^1(U))$. Then for every $q \in (0,2), \delta \in (0,1)$ and $t \in [0,T]$, there exists a constant $c \in (0,\infty)$ depending on M such that*

$$\int_0^t \int_U \Psi^q(z) \leq c\left(\frac{t}{\delta} + \delta \int_0^t \int_U |\nabla \Psi(z)|^2\right).$$

Proof. Since q is strictly less than 2, multiplying the integrand by 1 and using Cauchy-Schwarz and Young's inequality with exponent $\frac{2}{q} > 1$ then gives for every $\delta \in (0,1)$ and $t \in [0,T]$

$$\int_0^t \int_U |\Psi(z)|^q \leq c\delta^{-1}t + c\delta \int_0^t \int_U \Psi(z)^2.$$

Using the trivial inequality $a^2 \leq 2(a-b)^2 + 2b^2$ with the constant $b = \Psi(\Phi^{-1}(M))$, and subsequently applying Poincaré inequality gives the claim,

$$\begin{aligned}\int_0^t \int_U |\Psi(z)|^q &\leq c\delta^{-1}t + c\delta \int_0^t \int_U (\Psi(z) - \Psi(M))^2 + c\delta \int_0^t \int_U \Psi(M)^2 \\ &\leq c\delta^{-1}t + c\delta \int_0^t \int_U |\nabla(\Psi(z) - \Psi(M))|^2 \\ &\leq c\delta^{-1}t + c\delta \int_0^t \int_U |\nabla \Psi(z)|^2.\end{aligned}$$

□

The first result of this chapter will be proving two $L^p(U \times [0,t])$-energy estimates for the regularised equation, both of which will be useful in the sequel. The main difference of the estimate below to the above Proposition 4.1.8 is that we are able to prove $L^p(U \times [0,t])$-estimates for arbitrary $p \geq 2$, rather than just $L^2([0,t] \times U)$-estimates due to the fact that the boundary data is assumed to be constant.

Proposition 5.1.18 (p-independent energy estimate). *Suppose that the non-linear functions Φ, σ, ν satisfy Assumptions 2.2.1 and 5.1.13, and that the spatial components of the noise ξ^K satisfy assumption 5.1.2. Suppose that ρ_0 and $\bar{f}$ satisfy the simplifying Assumption 5.1.7. Let $\Theta_{\Phi,p}$ be the unique function defined in Definition 5.1.12. For $n \in \mathbb{N}$, let $\rho^{n,\epsilon}$ be the weak solution of the regularised equation* (5.9) *and let $\bar{\rho}^n$ be the solution to the regularised hydrodynamic limit equation* (5.11).

For every $\epsilon \in (0,1), t \in (0,T]$ there exists a constant $c \in (0,\infty)$ independent of t, ϵ, n and K such that for every $p \geq 2$,

$$\begin{aligned}\frac{1}{tp(p-1)}\mathbb{E}\left[\int_0^t \int_U (\rho^{n,\epsilon} - \bar{\rho}^n)^p\right] + \mathbb{E}\left[\int_0^t \int_U |\nabla \Theta_{\Phi,p}(\rho^{n,\epsilon})|^2\right] \\ + \frac{4}{np^2}\mathbb{E}\left[\int_0^t \int_U |\nabla(\rho^{n,\epsilon} - \bar{\rho}^n)^{p/2}|^2\right] \\ \leq c\epsilon t \left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)}\right), \quad (5.20)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{p(p-1)} \sup_{s \in [0,t]} \mathbb{E}\left[\int_U (\rho^{n,\epsilon} - \bar{\rho}^n)^p\right] + \mathbb{E}\left[\int_0^t \int_U |\nabla \Theta_{\Phi,p}(\rho^{n,\epsilon})|^2\right] \\ + \frac{4}{np^2}\mathbb{E}\left[\int_0^t \int_U |\nabla(\rho^{n,\epsilon} - \bar{\rho}^n)^{p/2}|^2\right] \\ \leq c\epsilon t \left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)}\right). \quad (5.21)\end{aligned}$$

Proof. Applying Itô's formula, Assumption 5.1.7, integrating by parts, using that $\nabla(\rho^{n,\epsilon}-\bar{\rho}^n)=\nabla\rho^{n,\epsilon}$ and the cancellation of the Itô correction and Itô-to-Stratonovich conversion terms (similar to the $L^2(U\times[0,t])$-estimate, see discussion proceeding equation (4.13) in Proposition 4.1.8 in the previous chapter), gives after re-arranging for every $p\geq 2$ for the function $\Theta_{\Phi,p}$ defined in Definition 5.1.12 that

$$\frac{1}{p(p-1)}\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^p+\int_0^t\int_U|\nabla\Theta_{\Phi,p}(\rho^{n,\epsilon})|^2+\frac{4}{np^2}\int_0^t\int_U|\nabla(\rho^{n,\epsilon}-\bar{\rho}^n)^{p/2}|^2=$$
$$\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}\nabla\rho^{n,\epsilon}\cdot\nu(\rho^{n,\epsilon})-\frac{\sqrt{\epsilon}}{(p-1)}\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-1}\nabla\cdot(\sigma_n(\rho^{n,\epsilon})\dot{\xi}^K)$$
$$+\frac{\epsilon}{2}\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}\left(\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})F_2^K\cdot\nabla\rho^{n,\epsilon}+F_3^K\sigma_n^2(\rho^{n,\epsilon})\right). \quad (5.22)$$

We deal with each of the terms in (5.22) in turn. The first term on the right hand side vanishes due to the fact that the boundary condition is constant,

$$\begin{aligned}\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}\nabla\rho^{n,\epsilon}\cdot\nu(\rho^{n,\epsilon,K})&=\sum_{i=1}^d\int_0^t\int_U\partial_i\left(\Theta_{\nu,p,i}(\rho^{n,\epsilon})\right)\\&=t\sum_{i=1}^d\Theta_{\nu,p,i}(\Phi^{-1}(\bar{f}))\int_{\partial U}\hat{\eta}_i=0,\end{aligned} \quad (5.23)$$

where we defined for $i=1,\ldots,d$ the unique functions $\Theta_{\nu,p,i}$ by

$$\Theta_{\nu,p,i}(0)=0,\quad\Theta_{\nu,p,i}'(\xi)=(\xi-\bar{\rho}^n)^{p-2}\nu_i(\xi),$$

and recall that $\hat{\eta}=(\hat{\eta}_1,\ldots,\hat{\eta}_d)$ denotes the outward pointing unit normal at the boundary. The final equality in (5.23) follows from the divergence theorem, letting e_i denote the standard basis vector in $\mathbb{R}^d$ in the i'th direction, we have

$$\int_{\partial U}\hat{\eta}_i=\int_{\partial U}e_i\cdot\hat{\eta}=\int_U\nabla\cdot e_i=\int_U\frac{\partial}{\partial_{x_i}}(1)=0.$$

The noise term in (5.22) is a martingale, so vanishes under expectation. We are left with handling the two terms in the final line of (5.22). For the first, we have using integration by parts that

$$\frac{\epsilon}{2}\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})F_2^K\cdot\nabla\rho^{n,\epsilon}=-\frac{\epsilon}{2}\int_0^t\int_U\Theta_{\sigma_n,p}(\rho^{n,\epsilon})\nabla\cdot F_2^K. \quad (5.24)$$

for the function $\Theta_{\sigma_n,p}$ defined in Definition 5.1.12. In the case $p=2$ we do not pick up a boundary term when integrating by parts due to the Dirichlet boundary data of the noise coefficients, see point 3 of Definition 5.1.2, that gives

$$\frac{\epsilon}{2}\int_0^t\int_{\partial U}\sigma_n^2(M)F_2^K\cdot\hat{\eta}=\frac{\epsilon\sigma_n^2(M)}{2}\sum_{k=1}^K\int_0^t\int_{\partial U}f_k\nabla f_k\cdot\hat{\eta}=0. \quad (5.25)$$

Using an $L^\infty(U)$-estimate for $\nabla \cdot F_2^K$, and using point 2 of Assumption 5.1.13, we have that there exists constants $q \in (0,2)$ and $c \in (0,\infty)$ such that

$$\frac{\epsilon}{2}\int_0^t \int_U (\rho^{n,\epsilon} - \bar{\rho}^n)^{p-2}\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})F_2^K \cdot \nabla \rho^{n,\epsilon} \leq c\epsilon \|\nabla \cdot F_2^K\|_{L^\infty(U)} \int_0^t \int_U (1 + \Theta^q_{\Phi,p}(\rho^{n,\epsilon})).$$

For the final term of (5.22), using an $L^\infty(U)$-estimate for F_3^K and point 2 of Assumption 5.1.13 gives the existence of constants $q \in (0,2)$ and $c \in (0,\infty)$ such that

$$\frac{\epsilon}{2}\int_0^t \int_U (\rho^{n,\epsilon} - \bar{\rho}^n)^{p-2} F_3^K \sigma_n^2(\rho^{n,\epsilon}) \leq c\epsilon \|F_3^K\|_{L^\infty(U)} \int_0^t \int_U (1 + \Theta^q_{\Phi,p}(\rho^{n,\epsilon})).$$

The integrals of $\Theta^q_{\Phi,p}$ are controlled using Lemma 5.1.17, again picking $\delta > 0$ small enough to absorb the resulting gradient term to the left hand side of the estimate. Putting (5.22) and the subsequent computations together, relabelling t as s, after taking an expectation we are left with

$$\frac{1}{p(p-1)}\mathbb{E}\left[\int_U (\rho^{n,\epsilon} - \bar{\rho}^n)^p\right] + \mathbb{E}\left[\int_0^s \int_U |\nabla \Theta_{\Phi,p}(\rho^{n,\epsilon})|^2\right] + \frac{4}{np^2}\mathbb{E}\left[\int_0^s \int_U |\nabla (\rho^{n,\epsilon} - \bar{\rho}^n)^{p/2}|^2\right] \leq c\epsilon s \left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)}\right). \quad (5.26)$$

The first estimate (5.20) then follows from integrating the inequality (5.26) over $[0,t]$ with respect to s, changing the order of integration, and noticing that when f is non-negative,

$$\int_0^t \int_0^s f(u)\,du\,ds = \int_0^t (t-u)f(u)\,du \leq t\int_0^t f(u)\,du.$$

The second estimate (5.21) follows by taking supremum in time on both sides of (5.26). □

Remark 5.1.19. *The above estimate is stable for every fixed $K \in \mathbb{N}$. However in order to take the $K \to \infty$ limit and ensure the right hand side of the estimates converge to zero, we observe that one needs a joint scaling regime $\epsilon \to 0, K \to \infty$ such that*

$$\epsilon\left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)}\right) \to 0.$$

Based on other energy estimates below, we will deduce a final joint scaling regime later in Remark 5.3.4.

Using the third point of Assumption 5.1.13, we can also obtain a bound depending on p on the right hand side.

Proposition 5.1.20 (*p*-dependent estimate)**.** *Suppose that the non-linear functions* Φ, σ, ν *satisfy Assumptions 2.2.1 and 5.1.13, and that the spatial components of the noise* ξ^K *satisfy assumption 5.1.2. Suppose that* ρ_0 *and* $\bar{f}$ *satisfy the simplifying Assumption 5.1.7. Let* $\Theta_{\Phi,p}$ *be the unique function defined in Definition 5.1.12. For* $n \in \mathbb{N}$*, let* $\rho^{n,\epsilon}$ *be the weak solution of the regularised equation* (5.9) *and let* $\bar{\rho}^n$ *be the solution to the regularised hydrodynamic limit equation* (5.11)*.*

For every $\epsilon \in (0,1), t \in (0,T]$ *there exists a constant* $c \in (0,\infty)$ *independent of* t, ϵ*,* n *and* K *such that for every* $p \geq 2$*,*

$$\begin{aligned}\frac{1}{tp(p-1)}\mathbb{E}\left[\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^p\right]+\mathbb{E}\left[\int_0^t\int_U|\nabla\Theta_{\Phi,p}(\rho^{n,\epsilon})|^2\right]&\\ +\frac{4}{np^2}\mathbb{E}\left[\int_0^t\int_U|\nabla(\rho^{n,\epsilon}-\bar{\rho}^n)^{p/2}|^2\right]&\\ \leq c\epsilon^{p/2}t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}^{p/2}+\|F_3^K\|_{L^\infty(U)}^{p/2}\right)\left(1+\epsilon t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}+\|F_3^K\|_{L^\infty(U)}\right)\right),&\end{aligned}$$

and

$$\begin{aligned}\frac{1}{p(p-1)}\sup_{s\in[0,t]}\mathbb{E}\left[\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^p\right]+\mathbb{E}\left[\int_0^t\int_U|\nabla\Theta_{\Phi,p}(\rho^{n,\epsilon})|^2\right]&\\ +\frac{4}{np^2}\mathbb{E}\left[\int_0^t\int_U|\nabla(\rho^{n,\epsilon}-\bar{\rho}^n)^{p/2}|^2\right]&\\ \leq c\epsilon^{p/2}t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}^{p/2}+\|F_3^K\|_{L^\infty(U)}^{p/2}\right)\left(1+\epsilon t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}+\|F_3^K\|_{L^\infty(U)}\right)\right).&\end{aligned}$$

Proof. The proof follows via the same estimates as the proof of Proposition 5.1.18. It is just a matter of how the terms in the final line of equation (5.22) are handled. For the first term, after integrating by parts as in (5.24) and using an $L^\infty(U)$-estimate for $\nabla\cdot F_2^K$, Hölder's inequality and Young's inequality with exponent $\frac{p}{p-2}>1$, and point 3 of Assumption 5.1.13 (specifically (5.19)) gives that there exists constants $\beta, c \in (0,\infty)$ such that for every $\delta \in (0,1)$,

$$\begin{aligned}\frac{\epsilon}{2}\int_0^t\int_U\Theta_{\sigma_n,p}(\rho^{n,\epsilon})\nabla\cdot F_2^K &\leq c\epsilon\|\nabla\cdot F_2^K\|_{L^\infty(U)}\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}\frac{\Theta_{\sigma_n,p}(\rho^{n,\epsilon})}{(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}}\\ &\leq \delta\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^p+c\delta^{-1}\epsilon^{p/2}\|\nabla\cdot F_2^K\|_{L^\infty(U)}^{p/2}\int_0^t\int_U\left(\frac{\Theta_{\sigma_n,p}(\rho^{n,\epsilon})}{(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}}\right)^{p/2}\\ &\leq \delta\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^p+c\delta^{-1}\epsilon^{p/2}\|\nabla\cdot F_2^K\|_{L^\infty(U)}^{p/2}\int_0^t\int_U(1+(\rho^{n,\epsilon}-\bar{\rho}^n)^\gamma).\end{aligned}\tag{5.27}$$

In precisely the same way, by taking the $L^\infty(U)$-norm of F_3^K, Hölder's inequality and Young's inequality with exponent $\frac{p}{p-2}>1$, and point 3 of Assumption 5.1.13 (specifically (5.19)), we get for the final term of (5.22) that there exists constants

$\beta, c \in (0,\infty)$ such that for every $\delta \in (0,1)$,

$$\begin{aligned}\frac{\epsilon}{2}\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}F_3^K\sigma_n^2(\rho^{n,\epsilon}) &\le \frac{\epsilon}{2}\|F_3^K\|_{L^\infty(U)}\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^{p-2}\sigma_n^2(\rho^{n,\epsilon})\\ &\le \delta\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^p + c\delta^{-1}\epsilon^{p/2}\|F_3^K\|_{L^\infty(U)}^{p/2}\int_0^t\int_U\sigma_n^p(\rho^{n,\epsilon})\\ &\le \delta\int_0^t\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^p + c\delta^{-1}\epsilon^{p/2}\|F_3^K\|_{L^\infty(U)}^{p/2}\int_0^t\int_U(1+(\rho^{n,\epsilon}-\bar{\rho}^n)^\gamma).\end{aligned}$$

For both the above term and the final line of equation (5.27), we pick $\delta > 0$ small enough so that the first term can be absorbed onto the left hand side. For the final term above, without loss of generality, by adjusting the constant term we can pick $\gamma \ge 2$, for which we can use the first energy estimate Proposition 5.1.18. This gives the bound

$$\begin{aligned}&\frac{1}{p(p-1)}\mathbb{E}\left[\int_U(\rho^{n,\epsilon}-\bar{\rho}^n)^p\right]+\mathbb{E}\left[\int_0^t\int_U|\nabla\Theta_{\Phi,p}(\rho^{n,\epsilon})|^2\right]\\ &\qquad\qquad+\frac{4}{np^2}\mathbb{E}\left[\int_0^t\int_U|\nabla(\rho^{n,\epsilon}-\bar{\rho}^n)^{p/2}|^2\right]\\ &\le c\epsilon^{p/2}t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}^{p/2}+\|F_3^K\|_{L^\infty(U)}^{p/2}\right)\left(1+\epsilon t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}+\|F_3^K\|_{L^\infty(U)}\right)\right).\end{aligned}$$

The two estimates are then obtained in the same way as Proposition 5.1.18. □

Remark 5.1.21. *Based on the joint scaling from Remark 5.1.19, whenever we have that*

$$\epsilon\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}+\|F_3^K\|_{L^\infty(U)}\right)\le 1,$$

the rate of convergence to zero of the right hand of Proposition 5.1.20 is governed by

$$\epsilon^{p/2}\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}^{p/2}+\|F_3^K\|_{L^\infty(U)}^{p/2}\right).$$

Whenever $p > 2$, this is an improved rate of convergence compared to the p-independent estimate of Proposition 5.1.18.

5.2 Central limit theorem for the SPDE with regularised diffusion coefficient

On the singular level, the central limit theorem involves showing that the random variables $v^\epsilon := \epsilon^{-1/2}(\rho^\epsilon - \bar{\rho})$ converge in the space of distributions to the linearised SPDE

$$\partial_t v = \Delta(\Phi'(\bar{\rho})v)) - \nabla\cdot(\sigma(\bar{\rho})\dot{\xi} + \nu'(\bar{\rho})v), \tag{5.28}$$

where the equation has zero initial condition and boundary data, and where $\dot{\xi}$ denotes an $\mathbb{R}^d$-valued space-time white noise[3], and $\bar{\rho}$ denotes the weak solution to the hydrodynamic limit equation (5.8) . In this section we will prove a central limit theorem for the regularised equation (5.9).

We begin in Definition 5.2.1 by defining what it means to be a strong solution to equation (5.28). After proving two technical results in Lemma 5.2.4 and Corollary 5.2.5, we prove the well-posedness of strong solutions of the linearised SPDE in Proposition 5.2.6. Finally the central limit theorem for the regularised equation is proven in Theorem 5.2.9.

Throughout this section, $\langle \cdot, \cdot \rangle_s$ denotes the dual pairing between $H^{-s}(U)$ and $H^s(U)$.

Definition 5.2.1 (Strong solution to linearised SPDE (5.28))**.** *Suppose that $\bar{\rho}$ satisfies Assumption 5.1.7. A $H^{-s}(U)$-strong solution to the linearised SPDE* (5.28) *is an $(\mathcal{F}_t)_{t\geq 0}$-adapted and almost surely continuous $H^{-s}(U)$-valued process v satisfying that for every $\psi \in H_0^s(U)$,*

$$\langle v(t), \psi \rangle_s = \int_0^t \langle v(r), \Phi'(\bar{\rho})\Delta\psi(r) + \nu'(\bar{\rho}) \cdot \nabla\psi \rangle_s \, dr + \int_0^t \int_U \sigma(\bar{\rho}) \nabla\psi \cdot \dot{\xi} \, dr.$$

Remark 5.2.2. *The boundary condition is implicit in the above expression. In particular, it can be seen in the first term when we applied integration by parts and did not pick up any boundary terms.*

Remark 5.2.3. *In the central limit theorem results we always assume that Assumption 5.1.7 holds. This implies that $\sigma(\bar{\rho})$ is either identically zero (in which case there is no noise in the system), or is a constant, uniformly bounded away from zero. In both cases we do not need to smooth σ.*

Furthermore, owing to point 2 of Assumption 2.2.1, $\Phi'(\bar{\rho})$ is a strictly positive constant and so the first term on the right hand side of (5.28) *behaves like a Laplacian and has the corresponding regularising affect. Hence, regularisation in way of equations* (5.9) *and* (5.11) *is not required for energy estimates of the equation.*

To prove the well-posedness of strong solutions to the linearised SPDE, we will need to use the following standard technical result.

Lemma 5.2.4. *Let $z \in L^2(U \times [0,T])$. Then for every $s \geq 1$, $(-\Delta)^{-s/2}\nabla z \in L^2(U \times [0,T]; \mathbb{R}^d)$ and there exists a constant $c \in (0,\infty)$ such that for every $t \in [0,T]$*

$$\|(-\Delta)^{-s/2}\nabla z\|_{L^2(U\times[0,t];\mathbb{R}^d)} \leq c\|z\|_{L^2(U\times[0,t])}.$$

Proof. Let $(e_k, \lambda_k)_{k\in\mathbb{N}}$ be the eigenfunctions and eigenvalues of the Dirichlet Laplacian. Since $z \in L^2(U \times [0,T])$, we can write

$$z(t,x) = \sum_{k=1}^{\infty} z_k(t)e_k(x), \quad z_k(t) = \langle z(t,\cdot), e_k \rangle_{L^2(U)}.$$

---

[3] The definition and existence of space-time white noise is given in the lecture notes by Walsh [94].

Using the spectral definition of fractional powers of $(-\Delta)$, we have

$$f := (-\Delta)^{-s/2} z = \sum_{k=1}^{\infty} z_k(t) \lambda_k^{-s/2} e_k(x).$$

By the spectral definition,

$$(-\Delta)^{-s/2} \nabla z = \nabla f,$$

it follows that

$$\|(-\Delta)^{-s/2} \nabla z\|^2_{L^2(U \times [0,t]; \mathbb{R}^d)} = \int_0^t \|\nabla f(s, \cdot)\|^2_{L^2(U;\mathbb{R}^d)} \, ds.$$

Now, using the spectral characterization of the Dirichlet Laplacian and orthogonality of $\{e_k\}_{k \in \mathbb{N}}$, we have

$$\|\nabla f(t, \cdot)\|^2_{L^2(U)} = \langle -\Delta f, f \rangle_{L^2(U)} = \left\langle \sum_{k=1}^{\infty} \lambda_k f_k e_k, \sum_{j=1}^{\infty} f_j e_j \right\rangle_{L^2(U)} = \sum_{k=1}^{\infty} \lambda_k |f_k(t)|^2,$$

where $f_k(t) = \langle f, e_k \rangle_{L^2(U)} = z_k(t) \lambda_k^{-s/2}$. Therefore

$$\|\nabla f(t, \cdot)\|^2_{L^2(U)} = \sum_{k=1}^{\infty} \lambda_k |z_k(t)|^2 \lambda_k^{-s} = \sum_{k=1}^{\infty} |z_k(t)|^2 \lambda_k^{-s+1}.$$

Integrating in time yields

$$\|(-\Delta)^{-s/2} \nabla z\|^2_{L^2(U \times [0,t])} = \int_0^t \sum_{k=1}^{\infty} |z_k(u)|^2 \lambda_k^{-s+1} \, du.$$

Since $s > 1$, we have $\lambda_k^{-s+1} \leq \lambda_1^{-s+1}$ for all k, where λ_1 is the smallest eigenvalue, and hence

$$\sum_{k=1}^{\infty} |z_k(u)|^2 \lambda_k^{-s+1} \leq \lambda_1^{-s+1} \sum_{k=1}^{\infty} |z_k(u)|^2.$$

Integrating again in time gives for any $t \in [0, T]$

$$\|(-\Delta)^{-s/2} \nabla z\|^2_{L^2(U \times [0,t])} \leq \lambda_1^{-s+1} \|z\|^2_{L^2(U \times [0,t])}.$$

Taking square roots completes the proof. □

We will also make use of following result, which is another instance of the same spectral mapping property for the operator $\nabla(-\Delta)^{-s}$.

Corollary 5.2.5. *Let* $s > \frac{1}{2}$ *and* $w \in H^{-s+\frac{1}{2}}(U)$. *Then* $\nabla(-\Delta)^{-s} w \in H^s(U; \mathbb{R}^d)$, *and there exists a constant* $c \in (0, \infty)$ *such that*

$$\|\nabla(-\Delta)^{-s} w\|_{H^s(U)} \leq c \|w\|_{H^{-s+\frac{1}{2}}(U)}.$$

Proof. Following the proof of Lemma 5.2.4, for $(e_k, \lambda_k)_{k\in\mathbb{N}}$ defined there, since $w \in H^{-s+\frac{1}{2}}(U)$, we can write $w = \sum_{k=1}^{\infty} w_k e_k$ with the coefficients $w_k := \langle w, e_k\rangle_{s-\frac{1}{2}}$, and

$$\nabla(-\Delta)^{-s} w = \sum_{k=1}^{\infty} w_k \lambda_k^{-s} \nabla e_k.$$

We now estimate the $H^s(U)$-norm. By definition and the the re-writing $(-\Delta)^{s/2}\nabla(-\Delta)^{-s} = \nabla(-\Delta)^{-1/2}(-\Delta)^{-(s-\frac{1}{2})}$, we obtain that

$$\|\nabla(-\Delta)^{-s}w\|_{H^s(U)} = \|(-\Delta)^{s/2}\nabla(-\Delta)^{-s}w\|_{L^2(U)} = \|\nabla(-\Delta)^{-1/2}(-\Delta)^{-(s-\frac{1}{2})}w\|_{L^2(U)}.$$

By Lemma 5.2.4, applied with $z = (-\Delta)^{-(s-\frac{1}{2})}w \in L^2(U)$, we obtain

$$\|\nabla(-\Delta)^{-1/2}z\|_{L^2(U)} \leq c\|z\|_{L^2(U)}.$$

Translating this into w, we obtain the result

$$\|\nabla(-\Delta)^{-s}w\|_{H^s(U)} \leq c\|(-\Delta)^{-(s-\frac{1}{2})}w\|_{L^2(U)} = \|w\|_{H^{-s+\frac{1}{2}}(U)}.$$

□

Proposition 5.2.6 (Existence and uniqueness of strong solutions to the linearised SPDE)**.** *Suppose that the non-linear functions Φ, σ, ν and the spatial components of the noise ξ^K satisfy assumption Assumptions 2.2.1 and 5.1.2 respectively. Suppose that ρ_0 and $\bar{f}$ satisfy the simplifying Assumption 5.1.7. There exists a unique strong solution of the linearised SPDE* (5.28) *as defined in Definition 5.2.1.*

The existence in the proof below follows Proposition 3.7 of Dirr, Fehrman and Gess [30], but the uniqueness is new.

Proof. Let ξ^K be the finite dimensional noise from Definition 5.1.1, satisfying Assumption 5.1.2 and $\bar{\rho}$ the solution of hydrodynamic limit equation (5.8) satisfying Assumption 5.1.7. Let v_K denote the solution of the linearised SPDE with the finite dimensional noise, that is, the solution to

$$\partial_t v_K = \Delta(\Phi'(\bar{\rho})v_K)) - \nabla\cdot(\sigma(\bar{\rho})\dot{\xi}^K + \nu'(\bar{\rho})v_K), \tag{5.29}$$

with zero initial data and boundary condition. Simplified versions of Theorems 3.2.2 and 4.4.8 prove uniqueness and existence of stochastic kinetic solutions and weak solutions of (5.29) for every $K \in \mathbb{N}$.

For $s > \frac{d+2}{2}$ consider $z_K := (-\Delta)^{-s/2}v_K$, which is function valued.

Applying Itô's formula to z_K^2 and using Assumption 5.1.7 gives for the first order term

$$\begin{aligned}
&\int_0^t \int_U z_K dz_K \\
&\quad = \int_0^t \int_U z_K \Big((-\Delta)^{-s/2} \Delta(\Phi'(\bar{\rho}) v_K)) - (-\Delta)^{-s/2} \nabla \cdot (\sigma(\bar{\rho}) \dot{\xi}^K + \nu'(\bar{\rho}) v_K) \Big) \\
&\quad = -\int_0^t \int_U \nabla z_K \cdot \Big((-\Delta)^{-s/2} \nabla(\Phi'(\bar{\rho}) v_K)) - (-\Delta)^{-s/2} (\sigma(\bar{\rho}) \dot{\xi}^K + \nu'(\bar{\rho}) v_K) \Big) \\
&= -\int_0^t \int_U \Phi'(\bar{\rho}) |\nabla z_K|^2 + \int_0^t \int_U \sigma(\bar{\rho}) (-\Delta)^{-s/2} \nabla z_K \cdot d\xi^K + \int_0^t \int_U \nabla z_K \cdot \nu'(\bar{\rho}) z_K.
\end{aligned} \tag{5.30}$$

The final term on the right hand side above vanishes by analogous reasoning to the term involving ν in Proposition 5.1.18, see equation (5.23) there.

The Itô correction can be bounded by Assumption 5.1.2 since $s > \frac{d+2}{2}$ to give for a constant $c \in (0, \infty)$ independent of K,

$$\frac{1}{2} \int_0^t \int_U d\langle z_K \rangle_t = \frac{1}{2} \sum_{k=1}^K \int_0^t \int_U \sigma^2(\bar{\rho}) |(-\Delta)^{-s/2} \nabla f_k|^2 = \frac{t|\sigma^2(\bar{\rho})|}{2} \sum_{k=1}^K \|f_k\|_{H^{-s+1}(U)}^2 \leq ct.$$

The final inequality follows from the fact that we can upper bound the partial sum by the infinite sum, which is bounded and independent of K. Putting everything together and taking supremum over time and an expectation, we have

$$\begin{aligned}
&\mathbb{E} \left[\sup_{t \in [0,T]} \|z_K\|_{L^2(U)}^2 + \Phi'(\bar{\rho}) \int_0^T \int_U |\nabla z_K|^2 \right] \\
&\qquad\qquad \leq \mathbb{E} \left[\sup_{t \in [0,T]} \left| \int_0^t \int_U \sigma(\bar{\rho}) \left((-\Delta)^{-s/2} \nabla z_K \right) \cdot d\xi^K \right| \right] + cT.
\end{aligned}$$

For the noise term in the above equation, Assumption 5.1.7 alongside Burkholder-Davis-Gundy inequality, Hölder and Young inequalities, the fact that $s > \frac{3}{2}$ so we can apply Lemma 5.2.4, shows that for a constant $c \in (0, \infty)$ depending on $\bar{\rho}$, but independent of T and K,

$$\begin{aligned}
&\mathbb{E} \left[\sup_{t \in [0,T]} \left| \int_0^t \int_U \sigma(\bar{\rho}) \left((-\Delta)^{-s/2} \nabla z_K \right) \cdot d\xi^K \right| \right] \\
&\qquad\qquad \leq c|\sigma(\bar{\rho})| \mathbb{E} \|(-\Delta)^{-s/2} \nabla z_K\|_{L^2(U \times [0,T]); \mathbb{R}^d} \leq c \mathbb{E} \|z_K\|_{L^2(U \times [0,T])}.
\end{aligned}$$

Putting (5.30) and subsequent computations together, we obtain the existence of a constant $c \in (0, \infty)$ independent of T and K such that

$$\mathbb{E} \left[\sup_{t \in [0,T]} \|z_K\|_{L^2(U)}^2 + \Phi'(\bar{\rho}) \int_0^T \int_U |\nabla z_K|^2 \right] \leq c(T + \mathbb{E} \|z_K\|_{L^2(U \times [0,T])}).$$

Transferring the estimate of z_K to v_K and using Grönwall's inequality, it follows that there exists a constant $c \in (0,\infty)$ independent of K such that

$$\mathbb{E}\left[\|v_K\|^2_{L^\infty([0,T];H^{-s}(U))} + \|v_K\|^2_{L^2([0,T];H^{-s+1}(U))}\right] \leq cT. \tag{5.31}$$

Furthermore, to estimate higher order time regularity, observe that distributionally we have due to Assumption 5.1.7 that

$$v_K(\cdot,t) = \int_0^t \Phi'(\bar\rho)\Delta v_K - \nabla\cdot(\nu'(\bar\rho)v_K)\,dt - \int_0^t \nabla\cdot(\sigma(\bar\rho)d\xi^K) = I_t^{f.v.}(\cdot) + I_t^{mart}(\cdot).$$

By equation (5.31) we get that the finite variation term satisfies for constant $c \in (0,\infty)$ independent of K,

$$\|I_\cdot^{f.v.}\|_{W^{1,2}([0,T];H^{-s+1}(U))} \leq c\|v_K\|_{L^2([0,T];H^{-s}(U))}.$$

From Assumption 5.1.7, the Burkholder-Davis-Gundy inequality and Lemma 5.1.6, we have for a constant c depending on $T,\beta,\bar\rho$, but independent of K, such that

$$\begin{aligned}\mathbb{E}\|I_\cdot^{mart}\|^2_{W^{\beta,2}([0,T];H^{-(s+1)}(U))} &= \mathbb{E}\int_0^T\int_0^T |s-t|^{-(1+2\beta)}\left\|\sum_{k=1}^K\int_s^t \sigma(\bar\rho)f_k\,dB_t^k\right\|^2_{H^{-s}(U)}\\ &\leq c\sigma^2(\bar\rho)\mathbb{E}\int_0^T\int_0^T|s-t|^{-(1+2\beta)}\sum_{k=1}^K\int_s^t\|f_k\|^2_{H^{-s}(U)}\\ &\leq c\int_0^T\int_0^T|s-t|^{-2\beta}\leq c,\end{aligned} \tag{5.32}$$

where again in the penultimate inequality we upper bounded the partial sum up to K by the infinite sum and used point 2 of Assumption 5.1.2. By the above estimates and the compact embedding of $H^{-s}(U)$ into $H^{-s'}(U)$ whenever $s < s' \in (0,\infty)$, the Aubin-Lions-Simon lemma [1, 69, 91] tells us that due to estimates (5.31) and (5.32), the laws of $\{v_K\}_{K\in\mathbb{N}}$ are tight on the space $L^2([0,T];H^{-s}(U))$ for every $s > d/2$. Following arguments similar to Theorem 4.4.8, we get that in the $K\to\infty$ limit, $\{v_K\}_{k\in\mathbb{N}}$ converge in law in $L^2([0,T];H^{-s}(U))$ to an element $v \in L^2([0,T]\times\Omega;H^{-s}(U))$ for every $s > d/2$, satisfying for every $\psi \in C_c^\infty(U)$,

$$\langle v(t),\psi\rangle_s = \int_0^t \langle v(r), \Phi'(\bar\rho)\Delta\psi(r) + \nu'(\bar\rho)\cdot\nabla\psi\rangle_s\,dr + \int_0^t\int_U \sigma(\bar\rho)\nabla\psi\cdot d\xi\,dr.$$

Both terms on the right hand side are continuous in time, so it follows that for every $k\in\mathbb{N}$, and $\{f_k\}_{k\in\mathbb{N}}$ as in Definition 5.1.1, $t\mapsto\langle v(t),f_k\rangle_s$ has a continuous modification in $L^2([0,T])$. By the $\mathbb{P}$-a.s. boundedness of v in $L^2([0,T];H^{-s}(U))$, we know that there exists a $H^{-s}(U)$-continuous modification, denoted again by v, satisfying the above equation for every $\psi\in C_c^\infty(U)$ and $t\in[0,T]$. This completes the proof of existence.

For uniqueness, let $v, \tilde{v}$ be two strong solutions of (5.28) with the same initial condition and boundary data. Define $w := v - \tilde{v} \in L^2([0,T]; H^{-s}(U))$. Then w satisfies

$$\partial_t w = \Phi'(\bar{\rho})\Delta w - \nabla \cdot (\nu'(\bar{\rho})w), \quad w|_{t=0} = w|_{\partial U} = 0,$$

in the sense of distributions.

Let $s > \frac{d}{2}$ and define the energy

$$E(t) := \|w(t)\|^2_{H^{-s}(U)} = \langle w(t), (-\Delta)^{-s} w(t)\rangle_s.$$

Since $w \in L^2([0,T]; H^{-s}(U))$ and the equation holds in the distributional sense, we can differentiate $E(t)$ to obtain

$$\frac{1}{2}\frac{d}{dt}E(t) = \langle \partial_t w, (-\Delta)^{-s} w\rangle_s.$$

Substituting the equation for w, we obtain

$$\frac{1}{2}\frac{d}{dt}E(t) = \Phi'(\bar{\rho})\langle \Delta w, (-\Delta)^{-s} w\rangle_{s+1} - \langle \nabla \cdot (\nu'(\bar{\rho})w), (-\Delta)^{-s} w\rangle_{s+\frac{1}{2}}. \tag{5.33}$$

We treat the two terms on the right hand side separately.

For the first, the self-adjointness of $-\Delta$ in the duality sense gives

$$\langle \Delta w, (-\Delta)^{-s} w\rangle_{s+1} = -\langle w, (-\Delta)^{-s+1} w\rangle_s = -\|w\|^2_{H^{-s+1}(U)}. \tag{5.34}$$

For the final term of (5.33), integrating by parts in the weak sense gives

$$\langle \nabla \cdot (\nu'(\bar{\rho})w), (-\Delta)^{-s} w\rangle_{s+\frac{1}{2}} = -\langle \nu'(\bar{\rho})w, \nabla(-\Delta)^{-s} w\rangle_s.$$

Applying Cauchy–Schwarz inequality, Corollary 5.2.5, the inerpolation inequality $\|w\|_{H^{-s+\frac{1}{2}}(U)} \leq c\|w\|^{1/2}_{H^{-s}(U)}\|w\|^{1/2}_{H^{-s+1}(U)}$ and Young's inequality, there exists a constant $c \in (0,\infty)$ such that for every $\epsilon \geq 0$

$$\begin{aligned}
\left|\langle \nu'(\bar{\rho})w, \nabla(-\Delta)^{-s} w\rangle_s\right| &\leq |\nu'(\bar{\rho})|\|w\|_{H^{-s}(U)}\|\nabla(-\Delta)^{-s} w\|_{H^s(U)} \\
&\leq |\nu'(\bar{\rho})|\|w\|_{H^{-s}(U)}\|\nabla(-\Delta)^{-s} w\|_{H^s(U)} \\
&\leq c|\nu'(\bar{\rho})|\|w\|^{3/2}_{H^{-s}(U)}\|w\|^{1/2}_{H^{-s+1}(U)} \\
&\leq \frac{\epsilon}{4}\|w\|^2_{H^{-s+1}(U)} + \frac{3(c|\nu'(\bar{\rho})|)^{4/3}}{4}\epsilon^{-1}\|w\|^2_{H^{-s}(U)}.
\end{aligned} \tag{5.35}$$

Putting (5.33) and the estimates (5.34)-(5.35) together, denoting $\tilde{c} := \frac{3(c|\nu'(\bar{\rho})|)^{4/3}}{2}$ we obtain

$$\frac{d}{dt}E(t) \leq -2\Phi'(\bar{\rho})\|w\|^2_{H^{-s+1}(U)} + \tilde{c}\epsilon^{-1}\|w\|^2_{H^{-s}(U)} + \frac{\epsilon}{2}\|w\|^2_{H^{-s+1}(U)}.$$

Choosing ϵ sufficiently small so that the overall contribution of the $H^{-s+1}(U)$-norm on the right hand side is negative, so can be removed from the estimate, we obtain that there exists a constant $c \in (0,\infty)$ such that

$$\frac{d}{dt}E(t) \leq cE(t).$$

Since $E(0) = 0$, Grönwall's inequality implies that

$$E(t) = 0 \quad \text{for all } t \in [0, T].$$

Hence $w = 0$ in $L^2([0,T]; H^{-s}(U))$, and therefore $v = \tilde{v}$.

□

Now that we have the well-posedness of the regularised equation $\rho^{n,\epsilon}$ and the well-posedness of strong solutions of the limiting equation v, let us turn our focus on formulating the central limit theorem result for the regularised equation.

Analogous to v^ϵ defined at the beginning of this section, we introduce the regularised version

$$v^{n,\epsilon} := \epsilon^{-1/2}(\rho^{n,\epsilon} - \bar{\rho}^n) \tag{5.36}$$

for $\rho^{n,\epsilon}$ the weak solution to the regularised equation (5.9) in the sense of Proposition 5.1.11, and $\bar{\rho}^n$ the solution of the regularised hydrodynamic limit equation (5.11).

Just as the formal computation in equation (1.11) shows, we expect that $v^{n,\epsilon}$ converges to v as $\epsilon \to 0$ and $n \to \infty$, and $v \in H^{-s}(U)$ is only distribution valued. So we would expect all $L^p(U \times [0,T])$-norms of $v^{n,\epsilon}$ to blow up as $\epsilon \to 0, K \to \infty$. The rate of blow up is quantified by the following estimate.

Proposition 5.2.7. *Suppose that the non-linear functions Φ, σ, ν satisfy Assumptions 2.2.1 and 5.1.13, and that the spatial components of the noise ξ^K satisfy assumption 5.1.2. Suppose that ρ_0 and $\bar{f}$ satisfy the simplifying Assumption 5.1.7. For fixed $n, K \in \mathbb{N}$ and $\epsilon \in (0,1)$, Let $v^{n,\epsilon}$ denote the weak solution to equation* (5.36) *above.*

For every $p \geq 2, t \in [0,T]$, we have the bound

$$\begin{aligned}
&\mathbb{E}\|v^{n,\epsilon}\|^p_{L^p(U\times[0,t])} \\
&\quad \leq ct^2 \left(\|\nabla \cdot F_2^K\|^{p/2}_{L^\infty(U)} + \|F_3^K\|^{p/2}_{L^\infty(U)} \right) \left(1 + \epsilon t \left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)} \right)\right).
\end{aligned} \tag{5.37}$$

Proof. The proof is a direct consequence of the p-dependent energy estimate, Proposition 5.1.20, by which it follows that

$$\begin{aligned}
\int_0^t \int_U (v^{n,\epsilon})^p &= \epsilon^{-p/2} \int_0^t \int_U (\rho^{n,\epsilon} - \bar{\rho}^n)^p \\
&\leq ct^2 \left(\|\nabla \cdot F_2^K\|^{p/2}_{L^\infty(U)} + \|F_3^K\|^{p/2}_{L^\infty(U)} \right) \left(1 + \epsilon t \left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)} \right)\right).
\end{aligned}$$

□

Remark 5.2.8 (Divergence of $L^p(U \times [0,t])$-norms of $v^{n,\epsilon}$)**.** *In contrast to Remark 5.1.19 where we can find a joint scaling $\epsilon \to 0, K(\epsilon) \to \infty$ such that the $L^p(U \times [0,t])$-norms of $(\rho^{n,\epsilon} - \bar{\rho}^n)$ converge to zero, here clearly the first term in the product* (5.37) *is independent of ϵ. Consequently we have divergence of the $L^p(U \times [0,t])$-norms of $v^{n,\epsilon}$ under any scaling regime where the $L^\infty(U)$-norms of $\nabla \cdot F_2^K$ and F_3^K diverge as*

$K \to \infty$. Furthermore, based on the joint scaling from Remark 5.1.19, whenever ϵ, K satisfy $\epsilon \left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)} \right) \leq 1$, the norms of $v^{n,\epsilon}$ diverge with rate

$$\left(\|\nabla \cdot F_2^K\|_{L^\infty(U)}^{p/2} + \|F_3^K\|_{L^\infty(U)}^{p/2} \right).$$

For the choice of noise as in Example 5.1.4, equations (5.6) *and* (5.7) *show that the rate of divergence is solely governed by*

$$\|F_3^K\|_{L^\infty(U)}^{p/2} \sim K^{p(d+2)}.$$

We conclude this section by proving a central limit theorem for the approximating SPDE. Just as in equations (5.9) and (5.11), for $n \in \mathbb{N}$, we define the regularised equation v^n as the strong solution to

$$\partial_t v^n = \Delta(\Phi'(\bar{\rho}^n)v^n)) + \frac{1}{n}\Delta v^n - \nabla \cdot (\sigma(\bar{\rho}^n)\dot{\xi} + \nu'(\bar{\rho}^n)v^n). \tag{5.38}$$

We mentioned in Remark 5.2.3 that the regularisation in (5.38) is not necessary to prove well-posedness in light of Assumption 5.1.7. However, it is necessary in the proof of the central limit theorem because the regularisation term $\frac{1}{n}\Delta v^n$ groups exactly with the regularisation term in the equation for $v^{n,\epsilon}$, see equation (5.41) below. Furthermore, replacing $\bar{\rho}$ by $\bar{\rho}^n$ is not necessary and just plays a technical role rather than a conceptual one. It is used in the computation of equation (5.46) below and we write it since we only stated the above $L^p(U \times [0,t])$-estimates for $(\rho^{n,\epsilon} - \bar{\rho}^n)$ rather than for $(\rho^{n,\epsilon} - \bar{\rho})$. Existence of strong solutions to the regularised equation (5.38) follows in the same way as Proposition 5.2.6.

Fundamentally, the estimate below is different from that of Propositions 5.1.18, 5.1.20 and 5.2.7, since neither quantity in the difference $v^{n,\epsilon} - v$ is solved by a constant, so we will see in the proof below that one really needs to handle the difference of the corresponding non-linear terms.

Theorem 5.2.9 (CLT for approximating equation)**.** *Suppose that the non-linear functions Φ, σ, ν satisfy Assumptions 2.2.1 and 5.1.13, and that the spatial components of the noise ξ^K satisfy assumption 5.1.2. Suppose that ρ_0 and $\bar{f}$ satisfy the simplifying Assumption 5.1.7. For fixed $s > \frac{d+2}{2}$, for every $K \in \mathbb{N}$, denote by $\mathcal{T}_s(K)$ the tail sum*

$$\mathcal{T}_s(K) := \sum_{k=K}^{\infty} \|f_k\|_{H^{-s+1}(U)}. \tag{5.39}$$

Fix $\epsilon \in (0,1)$ and $n, K \in \mathbb{N}$. Let $v^{n,\epsilon}$ denote the weak solution to the regularised equation (5.36)*, and let v^n be the solution of the regularised, linearised SPDE* (5.38)*. For $\beta \in (0,\infty)$ as in point 1 of Assumption 5.1.13 and any $t \in [0,T]$, there exists a* *constant $c \in (0,\infty)$ independent of n, ϵ and K such that*

$$\begin{aligned}\mathbb{E}\|v^{n,\epsilon} - v^n\|^2_{L^2([0,t];H^{-s}(U))} &\leq c\left(1 + \epsilon t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)}\right)\right) \\ \times &\left[\epsilon t^2\left(\|\nabla\cdot F_2^K\|^2_{L^\infty(U)} + \|F_3^K\|^2_{L^\infty(U)}\right) + \epsilon^{\beta+1}t^2\left(\|\nabla\cdot F_2^K\|^{\beta+2}_{L^\infty(U)} + \|F_3^K\|^{\beta+2}_{L^\infty(U)}\right)\right. \\ &\left. + \epsilon t\left(\|F_1^K\|^2_{L^\infty(U)}\|\sigma_n'\|^2_{L^\infty((0,\infty))} + \|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)}\right)\right] \\ &+ c\mathbb{E}\|\sigma_n(\rho^{n,\epsilon}) - \sigma(\bar\rho^n)\|^2_{L^2(U\times[0,t])} + ct\,\mathcal{T}_s(K). \quad (5.40)\end{aligned}$$

Proof. For $s > \frac{d+2}{2}$ define $z^{n,\epsilon} := (-\Delta)^{-s/2}(v^{n,\epsilon} - v^n)$. Since $s > \frac{d+2}{2}$, we have that $\mathbb{P}$-a.s. $z^{n,\epsilon} \in L^2([0,T];H^1(U))$ and satisfies

$$\begin{aligned}\partial_t z^{n,\epsilon} &= \epsilon^{-1/2}(-\Delta)^{-s/2}(\Delta\Phi(\rho^{n,\epsilon}) - \Delta\Phi(\bar\rho^n)) - \Phi'(\bar\rho^n)(-\Delta)^{-s/2}\Delta v \\ &+ \frac{1}{n}\epsilon^{-1/2}(-\Delta)^{-s/2}(\Delta\rho^{n,\epsilon} - \Delta\bar\rho^n) - \frac{1}{n}(-\Delta)^{-s/2}\Delta v \\ &+ \epsilon^{-1/2}(-\Delta)^{-s/2}\left(\nabla\cdot\nu(\rho^{n,\epsilon}) - \nabla\cdot\nu(\bar\rho^n)\right) - (-\Delta)^{-s/2}\nabla\cdot(\nu'(\bar\rho^n)v) \\ &- (-\Delta)^{-s/2}\nabla\cdot(\sigma_n(\rho^{n,\epsilon})\dot\xi^K) + (-\Delta)^{-s/2}\nabla\cdot(\sigma(\bar\rho^n)\dot\xi) \\ &+ \frac{\epsilon^{1/2}}{2}(-\Delta)^{-s/2}\nabla\cdot(F_1^K(\sigma_n'(\rho^{n,\epsilon}))^2\nabla\rho^{n,\epsilon} + \sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})F_2^K). \quad (5.41)\end{aligned}$$

By Itô's formula, we have

$$\int_0^t\int_U d(z^{n,\epsilon})^2 = 2\int_0^t\int_U z^{n,\epsilon}\,dz^{n,\epsilon} + \int_0^t\int_U d\langle z^{n,\epsilon}\rangle. \quad (5.42)$$

First we compute the quadratic variation term in (5.42) and bound it right away. By Assumption 5.1.2 we have that $\dot\xi = \sum_{k\in\mathbb{N}} f_k dB_t^k$, so we can re-write the noise terms in (5.41) as

$$\begin{aligned}&-(-\Delta)^{-s/2}\nabla\cdot(\sigma_n(\rho^{n,\epsilon})\dot\xi^K) + (-\Delta)^{-s/2}\nabla\cdot(\sigma(\bar\rho^n)\dot\xi) \\ &= -\sum_{k=1}^K(-\Delta)^{-s/2}\nabla((\sigma_n(\rho^{n,\epsilon}) - \sigma(\bar\rho^n))f_k)\cdot dB_t^k + \sum_{k=K+1}^\infty(-\Delta)^{-s/2}\nabla(\sigma(\bar\rho^n))f_k)\cdot dB_t^k.\end{aligned}$$

It follows by the $L^2(U)$-orthonormality of $\{f_k\}_{k\in\mathbb{N}}$ and the fact that $s > 3/2$ that we can apply Lemma 5.2.4 which tells us that the first of the two terms can be bounded by

$$\sum_{k=1}^K\int_0^t\int_U\left((-\Delta)^{-s/2}\nabla((\sigma_n(\rho^{n,\epsilon}) - \sigma(\bar\rho^n))f_k)\right)^2 \leq c\|\sigma_n(\rho^{n,\epsilon}) - \sigma(\bar\rho^n)\|^2_{L^2(U\times[0,t])},$$

and for the second, using Assumption 5.1.7 alongside the definition of the $H^{-s}(U)$-norm gives that there exists a constant $c \in (0,\infty)$ such that

$$\begin{aligned}\sum_{k=K+1}^\infty\int_0^t\int_U\left((-\Delta)^{-s/2}\nabla(\sigma(\bar\rho^n))f_k)\right)^2 &\leq \sigma^2(\bar\rho^n)t\sum_{k=K+1}^\infty\int_U\left((-\Delta)^{-s/2}\nabla f_k\right)^2 \\ &\leq ct\mathcal{T}_s(K),\end{aligned}$$

where $\mathcal{T}_s(K)$ is defined in (5.39) and in the final inequality we used $\mathcal{T}_s(K+1) \leq \mathcal{T}_s(K)$. By point 2 of Assumption 5.1.2, $\mathcal{T}_s(K)$ decays to zero as $K \to \infty$ with an explicit rate that depends on the specific choice of $\{f_k\}_{k\in\mathbb{N}}$, see Example 5.2.13 below.

Putting everything together, Itô's formula (5.42) and the equation for z (5.41) alongside the simplification of the quadratic variation term above gives that

$$\begin{aligned}\frac{1}{2}\int_0^t\int_U (z^{n,\epsilon})^2 \\ \leq \int_0^t\int_U z^{n,\epsilon}\Bigg[&\epsilon^{-1/2}(-\Delta)^{-s/2}(\Delta\Phi(\rho^{n,\epsilon})-\Delta\Phi(\bar{\rho}^n)) - \Phi'(\bar{\rho}^n)(-\Delta)^{-s/2}\Delta v \\ &+\frac{1}{n}\epsilon^{-1/2}(-\Delta)^{-s/2}(\Delta\rho^{n,\epsilon}-\Delta\bar{\rho}^n) - \frac{1}{n}(-\Delta)^{-s/2}\Delta v \\ &+\epsilon^{-1/2}(-\Delta)^{-s/2}\left(\nabla\cdot\nu(\rho^{n,\epsilon})-\nabla\cdot\nu(\bar{\rho}^n)\right) - (-\Delta)^{-s/2}\nabla\cdot(\nu'(\bar{\rho}^n)v) \\ &-(-\Delta)^{-s/2}\nabla\cdot(\sigma_n(\rho^{n,\epsilon})\dot{\xi}^K) + (-\Delta)^{-s/2}\nabla\cdot(\sigma(\bar{\rho}^n)\dot{\xi}) \\ &+\frac{\epsilon^{1/2}}{2}(-\Delta)^{-s/2}\nabla\cdot(F_1^K(\sigma_n'(\rho^{n,\epsilon}))^2\nabla\rho^{n,\epsilon} + \sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})F_2^K)\Bigg] \\ &+c\|\sigma_n(\rho^{n,\epsilon})-\sigma(\bar{\rho}^n)\|^2_{L^2(U\times[0,t])} + \sigma^2(\bar{\rho}^n)t\,\mathcal{T}_s(K). \quad (5.43)\end{aligned}$$

We deal with each term of (5.43) in turn.

First of all, for the terms in the second line involving the $\frac{1}{n}$-regularisation, they can be re-written using integration by parts. They become

$$2\frac{1}{n}\int_0^t\int_U z^{n,\epsilon}\Delta z^{n,\epsilon} = -2\frac{1}{n}\int_0^t\int_U |\nabla z^{n,\epsilon}|^2,$$

which can be moved over to the left hand side of the estimate.

Next, the terms in the first line on the right hand side of equation (5.43) can be re-written as

$$\int_0^t\int_U z^{n,\epsilon}\left((-\Delta)^{-s/2}\Delta\left(v^{n,\epsilon}\frac{\Phi(\rho^{n,\epsilon})-\Phi(\bar{\rho}^n)}{\rho^{n,\epsilon}-\bar{\rho}^n} - \Phi'(\bar{\rho}^n)v^n\right)\right).$$

We write the above term in the inner bracket as "what we want" plus a correction,

$$\begin{aligned}&\left(v^{n,\epsilon}\frac{\Phi(\rho^{n,\epsilon})-\Phi(\bar{\rho}^n)}{\rho^{n,\epsilon}-\bar{\rho}^n} - \Phi'(\bar{\rho}^n)v^n\right) \\ &\qquad = \Phi'(\bar{\rho}^n)(v^{n,\epsilon}-v^n) + v^{n,\epsilon}\left(\frac{\Phi(\rho^{n,\epsilon})-\Phi(\bar{\rho}^n)}{\rho^{n,\epsilon}-\bar{\rho}^n} - \Phi'(\bar{\rho}^n)\right). \quad (5.44)\end{aligned}$$

Using integration by parts and Assumption 5.1.7, we get that the first term in (5.44) can be re-written as

$$\int_0^t\int_U z^{n,\epsilon}(-\Delta)^{-s/2}\Delta\Phi'(\bar{\rho}^n)(v^{n,\epsilon}-v^n) = -\Phi'(\bar{\rho}^n)\int_0^t\int_U |\nabla z^{n,\epsilon}|^2, \quad (5.45)$$

which can also be moved onto the left hand side of the estimate. By Assumption 5.1.7 and point 2 of Assumption 2.2.1 we have $\Phi'(\bar{\rho}^n)$ is strictly positive. This will allow us to absorb space-time integrals of $|\nabla z^{n,\epsilon}|^2$ to the left hand side in a way that does not depend on the regularisation n.

We now aim to bound the final term corresponding to the correction in (5.44). Fix $(x,t) \in U \times [0,T]$. By the mean value theorem, we have that there exists a $\xi = \xi(x,t) \in (\bar{\rho}^n, \rho^{n,\epsilon})$ (if $\rho^{n,\epsilon}(x,t) > \bar{\rho}^n$), otherwise $\xi \in (\rho^{n,\epsilon}, \bar{\rho}^n)$ such that

$$\Phi(\rho^{n,\epsilon}) - \Phi(\bar{\rho}^n) = \Phi'(\xi)(\rho^{n,\epsilon} - \bar{\rho}^n).$$

Consequently, substituting this into the final term of (5.44) and applying the mean value theorem again gives for $\xi' = \xi'(x,t) \in (\bar{\rho}^n, \xi)$ (if $\xi(x,t) > \bar{\rho}^n$) otherwise for $\xi' \in (\xi, \bar{\rho}^n)$, that

$$\left(\frac{\Phi(\rho^{n,\epsilon}) - \Phi(\bar{\rho}^n)}{\rho^{n,\epsilon} - \bar{\rho}^n} - \Phi'(\bar{\rho}^n) \right) = \Phi'(\xi) - \Phi'(\bar{\rho}^n) = \Phi''(\xi')(\xi - \bar{\rho}^n).$$

Note that the space-time integral of the final term is guaranteed due to point 1 of Assumption 5.1.13. Then, using Cauchy-Schwarz and Young's inequalities, the fact that $s > \frac{d+2}{2}$, the bound $|\xi - \bar{\rho}^n| \le |\rho^{n,\epsilon} - \bar{\rho}^n|$ and the bound on Φ'' given in point 1 of Assumption 5.1.13 proves that there exists constants $c, \beta \in (0,\infty)$ such that for every $\delta \in (0,1)$,

$$\begin{aligned}
&\int_0^t \int_U z^{n,\epsilon} (-\Delta)^{-s/2} \Delta \left(v^{n,\epsilon} \Phi''(\xi')(\xi - \bar{\rho}^n) \right) \\
&\qquad \le \frac{\delta}{2} \int_0^t \int_U (z^{n,\epsilon})^2 + \frac{1}{2\delta} \int_0^t \int_U \left((-\Delta)^{-s/2} \Delta \left(v^{n,\epsilon} \Phi''(\xi')(\xi - \bar{\rho}^n) \right) \right)^2 \\
&\qquad \le \frac{\delta}{2} \int_0^t \int_U (z^{n,\epsilon})^2 + \frac{c}{\delta} \int_0^t \int_U \left(v^{n,\epsilon} (\rho^{n,\epsilon} - \bar{\rho}^n)(1 + (\rho^{n,\epsilon} - \bar{\rho})^{\beta}) \right)^2 \\
&\qquad \le \frac{\delta}{2} \int_0^t \int_U (z^{n,\epsilon})^2 + \frac{c}{\delta} \int_0^t \int_U \left(\epsilon^{1/2} (v^{n,\epsilon})^2 + \epsilon^{\frac{\beta+1}{2}} (v^{n,\epsilon})^{\beta+2} \right)^2 \\
&\qquad \le \frac{\delta}{2} \int_0^t \int_U (z^{n,\epsilon})^2 + \frac{c\epsilon}{\delta} \int_0^t \int_U (v^{n,\epsilon})^4 + \frac{c\epsilon^{\beta+1}}{\delta} \int_0^t \int_U (v^{n,\epsilon})^{2\beta+4}. \qquad (5.46)
\end{aligned}$$

We can pick $\delta > 0$ sufficiently small so that the first term in the final line can be absorbed onto the left hand side of (5.43). We will show how to bound the second and third terms on the right hand side of (5.46) later.

The terms involving ν in the third line of (5.43) can be handled in the same way as the above terms, again using point 1 of Assumption 5.1.13. We have that there exists constants $c, \beta \in (0,\infty)$ such that for every $\delta > 0$, ξ, ξ' in the same intervals as above,

$$\begin{aligned}
\int_0^t \int_U z^{n,\epsilon} \left[\epsilon^{-1/2}(-\Delta)^{-s/2}\left(\nabla\cdot\nu(\rho^{n,\epsilon}) - \nabla\cdot\nu(\bar{\rho}^n)\right) - (-\Delta)^{-s/2}\nabla\cdot(\nu'(\bar{\rho}^n)v^n)\right] \\
= \int_0^t \int_U \nabla z^{n,\epsilon}\cdot\nu'(\bar{\rho}^n)z^{n,\epsilon} \\
+ \int_0^t \int_U z^{n,\epsilon}(-\Delta)^{-s/2}\nabla\cdot\left(v^{n,\epsilon}\left(\frac{\nu(\rho^{n,\epsilon})-\nu(\bar{\rho}^n)}{\rho^{n,\epsilon}-\bar{\rho}^n} - \nu'(\bar{\rho}^n)\right)\right) \\
= \int_0^t \int_U \nabla z^{n,\epsilon}\cdot\nu'(\bar{\rho}^n)z^{n,\epsilon} + \int_0^t \int_U z^{n,\epsilon}(-\Delta)^{-s/2}\nabla\cdot\left(v^{n,\epsilon}\nu''(\xi')(\xi-\bar{\rho}^n)\right) \\
\leq \int_0^t \int_U \nabla z^{n,\epsilon}\cdot\nu'(\bar{\rho}^n)z^{n,\epsilon} + c\delta\int_0^t \int_U |\nabla z^{n,\epsilon}|^2 \\
+ \frac{c\epsilon}{\delta}\int_0^t \int_U (v^{n,\epsilon})^4 + \frac{c\epsilon^{\beta+1}}{\delta}\int_0^t \int_U (v^{n,\epsilon})^{2\beta+4}.
\end{aligned}$$

The first term on the right hand side in the final line vanishes due to Assumption 5.1.7 in analogy with terms involving ν in previous estimates, see for instance equation (5.23). The second term can be absorbed onto the left hand side of the estimate by picking $\delta > 0$ sufficiently small, independently of n. We will deal with the remaining two terms involving powers of $v^{n,\epsilon}$ using Proposition 5.2.7 below, but for now let's return to the equation (5.43) for $z^{n,\epsilon}$.

The noise terms in the fourth line on the right hand side of (5.43) vanish under expectation.

For the first term in the penultimate line of (5.43), we have using integration by parts, Cauchy-Schwarz inequality, Lemma 5.2.4 and Young's inequality, that there exists a constant $c \in (0,\infty)$ such that for every $\delta \in (0,1)$,

$$\begin{aligned}
&\frac{\epsilon^{1/2}}{2}\int_0^t \int_U z^{n,\epsilon}(-\Delta)^{-s/2}\nabla\cdot\left(F_1^K(\sigma_n'(\rho^{n,\epsilon}))^2\nabla\rho^{n,\epsilon}\right) \\
&= -\frac{\epsilon^{1/2}}{2}\int_0^t \int_U \nabla z^{n,\epsilon}\cdot(-\Delta)^{-s/2}\left(F_1^K(\sigma_n'(\rho^{n,\epsilon}))^2\nabla\rho^{n,\epsilon}\right) \\
&\leq \frac{\epsilon^{1/2}\|F_1^K\|_{L^\infty(U)}\|\sigma_n'\|_{L^\infty((0,\infty))}}{2}\int_0^t \int_U |\nabla z^{n,\epsilon}|\,|(-\Delta)^{-s/2}\nabla\sigma_n(\rho^{n,\epsilon})| \\
&\leq \frac{\delta}{4}\int_0^t \int_U |\nabla z^{n,\epsilon}|^2 + \frac{\epsilon\|F_1^K\|^2_{L^\infty(U)}\|\sigma_n'\|^2_{L^\infty((0,\infty))}}{\delta}\int_0^t \int_U |\sigma_n(\rho^{n,\epsilon})|^2. \qquad (5.47)
\end{aligned}$$

Similarly, for the second term in the penultimate line, integration by parts, Cauchy-Schwarz and Young's inequalities imply that there exists a constant $c \in (0,\infty)$ such that for every $\delta \in (0,1)$,

$$\begin{aligned}\frac{\epsilon^{1/2}}{2}\int_0^t\int_U z^{n,\epsilon}(-\Delta)^{-s/2}\nabla\cdot(\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})F_2^K)&\\ =-\frac{\epsilon^{1/2}}{2}\int_0^t\int_U \nabla z^{n,\epsilon}(-\Delta)^{-s/2}(\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})F_2^K)&\\ \leq \frac{\delta}{4}\int_0^t\int_U|\nabla z^{n,\epsilon}|^2+\frac{\epsilon\|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)}}{\delta}\int_0^t\int_U(\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon}))^2&\end{aligned}$$

Again pick δ sufficiently small, independent of n, so that the first term of the above equation and equation (5.47) can be moved to the left hand side. Putting everything together, it follows from (5.43) and subsequent computations that we have the bound

$$\begin{aligned}\mathbb{E}\left[\int_0^t\int_U\frac{1}{2}(z^{n,\epsilon})^2+\left(\Phi'(\bar{\rho}^n)+\frac{1}{n}\right)|\nabla z^{n,\epsilon}|^2\right]&\\ \leq c\epsilon\mathbb{E}\left[\int_0^t\int_U(v^{n,\epsilon})^4\right]+c\epsilon^{\beta+1}\mathbb{E}\left[\int_0^t\int_U(v^{n,\epsilon})^{2\beta+4}\right]&\\ +\epsilon\|F_1^K\|^2_{L^\infty(U)}\|\sigma_n'\|^2_{L^\infty((0,\infty))}\mathbb{E}\left[\int_0^t\int_U|\sigma(\rho^{n,\epsilon})|^2\right]+c\mathbb{E}\|\sigma_n(\rho^{n,\epsilon})-\sigma(\bar{\rho})\|^2_{L^2(U\times[0,t])}&\\ +\epsilon\|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)}\mathbb{E}\left[\int_0^t\int_U(\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon}))^2\right]+\sigma^2(\bar{\rho})t\,\mathcal{T}_s(K).&\quad(5.48)\end{aligned}$$

For the first two terms on the right hand side, we use Proposition 5.2.7 and the resulting terms appear on the right hand side of the estimate. Importantly, these terms have factors of ϵ in front, implying the existence of a joint scaling in ϵ, K under which the terms vanish.

To estimate the first terms in the penultimate and final line of (5.48), we use the bounds on σ^2 and $(\sigma\sigma')^2$ given in equations (2.9) and (2.10) in Assumption 2.2.1 alongside the assumed growth on Φ in equation (2.4) which we alter in the same way as in Remark 5.1.14 to give for constant $c\in(0,\infty)$

$$\begin{aligned}\epsilon\|F_1^K\|^2_{L^\infty(U)}\|\sigma_n'\|^2_{L^\infty((0,\infty))}\mathbb{E}\left[\int_0^t\int_U|\sigma(\rho^{n,\epsilon})|^2\right]&\\ +\epsilon\|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)}\mathbb{E}\left[\int_0^t\int_U(\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon}))^2\right]&\\ \leq c\epsilon\left(\|F_1^K\|^2_{L^\infty(U)}\|\sigma_n'\|^2_{L^\infty((0,\infty))}+\|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)}\right)\mathbb{E}\left[\int_0^t\int_U(1+(\rho^{n,\epsilon}-\bar{\rho})^m)\right].&\end{aligned}$$

Using the p-independent energy estimate of Proposition 5.1.18 and Assumption 5.1.7 gives that the above term is bounded above by

$$\begin{aligned}c\epsilon t\left(\|F_1^K\|^2_{L^\infty(U)}\|\sigma_n'\|^2_{L^\infty((0,\infty))}+\|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)}\right)&\\ \times\left(1+\epsilon t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}+\|F_3^K\|_{L^\infty(U)}\right)\right).&\end{aligned}$$

Putting everything together then gives the estimate. □

Remark 5.2.10 (Simplifying the estimate)**.** *The estimate in Theorem 5.2.9 contains a delicate balance of coefficients involving powers of* ϵ *and powers of the* $L^\infty(U)$*-norms of sums of noise coefficients, making it impossible to simplify further in a straightforward way. It comes down to how we bound the two norms of* $v^{n,\epsilon}$ *in equation* (5.48)*. Using Proposition 5.2.7 followed by the facts that that we can upper bound the* $L^\infty(U)$*-norms of the noise coefficients by their squares and* $\epsilon^{\beta+1} \leq \epsilon^{(\beta+2)/2}$ *for every* $\epsilon \in (0,1), \beta > 1$*, and the trivial inequality* $(1+x)(x+x^{\frac{\beta+2}{2}}) \leq 8(x+x^{\frac{\beta+4}{2}})$ *we could have alternatively stated the bound for these two terms as*

$$\begin{aligned}\epsilon\mathbb{E}\left[\int_0^t\int_U (v^{n,\epsilon})^4\right] &+ \epsilon^{\beta+1}\mathbb{E}\left[\int_0^t\int_U (v^{n,\epsilon})^{2\beta+4}\right]\\ &\leq ct^2\left(1+\epsilon t\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)}\right)\right)\\ \times\Big(\epsilon\left(\|\nabla\cdot F_2^K\|^2_{L^\infty(U)} + \|F_3^K\|^2_{L^\infty(U)}\right) &+ \epsilon^{\beta+1}\Big(\|\nabla\cdot F_2^K\|^{\beta+2}_{L^\infty(U)} + \|F_3^K\|^{\beta+2}_{L^\infty(U)}\Big)\Big)\\ \leq ct^3\Big(\epsilon\left(\|\nabla\cdot F_2^K\|^2_{L^\infty(U)} + \|F_3^K\|^2_{L^\infty(U)}\right) &+ \epsilon^{\frac{\beta+4}{2}}\Big(\|\nabla\cdot F_2^K\|^{\beta+4}_{L^\infty(U)} + \|F_3^K\|^{\beta+4}_{L^\infty(U)}\Big)\Big).\end{aligned}$$

This makes the scaling between ϵ *and* K *more explicit.*

Remark 5.2.11. *It is not possible to directly take the limit of the regularisation* $n \to \infty$ *in equation* (5.40) *and get a statement for the singular equation. This is due to the presence of the term* $\|\sigma_n'\|_{L^\infty((0,\infty))}$ *which would diverge in the classical case when* σ_n *are approximating the square root.*

Remark 5.2.12 (Necessity of Assumption 5.1.7)**.** *It was illustrated in the* $L^2(U\times[0,t])$ *estimate of Proposition 4.1.8 above, that if the boundary data is non-constant, then one can only estimate* $L^2(U\times[0,t])$*-norms of the solutions* $\rho^{n,\epsilon}$ *of* (5.1) *rather than* $L^p(U\times[0,t])$*-norms for general* $p \geq 2$ *that is presented in Proposition 5.1.18.*

Briefly, if we tried to repeat the computations in Proposition 5.1.18 if $\bar\rho$ *was not constant, we have that the first order term involving the diffusion* Φ *can be written up to a constant as*

$$\begin{aligned}&\int_0^t\int_U (\rho^{n,\epsilon}-\bar\rho)^{p-1}\Delta(\Phi(\rho^{n,\epsilon})-\Phi(\bar\rho))\\ &\qquad\qquad = -(p-1)\int_0^t\int_U (\rho^{n,\epsilon}-\bar\rho)^{p-2}\nabla(\rho^{n,\epsilon}-\bar\rho)\cdot\nabla(\Phi(\rho^{n,\epsilon})-\Phi(\bar\rho)).\end{aligned}$$

It is not clear to the author how to bound such a term, even in the one dimensional interval $U = [0,1]$ *and even if the solution to the hydrodynamic limit* $\bar\rho$ *was a linear function. This is because we don't have a quantitative control over the difference* $\rho^{n,\epsilon} - \bar\rho$*.*

Consequently we could only hope to obtain $L^2(U\times[0,t])$*-estimates for* $v^{n,\epsilon}$ *rather than the result of Proposition 5.2.7. However, in the proof of the central limit theorem above, we saw in equation* (5.46) *that we needed to bound the* $L^4(U\times[0,t])$*-norm and* $L^{2\beta+4}(U\times[0,t])$*-norm of* $v^{n,\epsilon}$ *for* $\beta \in (0,\infty)$*. This is one example which highlights the necessity of Assumption 5.1.7.*

Example 5.2.13 (Explicit bound for $\mathcal{T}_s(K)$ for eigenfunctions of Laplacian)**.** *For $k \in \mathbb{N}$, if $f_k = e_k$ are the eigenfunctions of the Laplacian as in Example 5.1.4, the computation in Lemma 5.1.6 above shows that by Weyl's law, see Weyl [97], we have for every $s > \frac{d+2}{2}$ the approximation*

$$\mathcal{T}_s(K) = \sum_{k=K}^{\infty} \|f_k\|^2_{H^{-s+1}(U)} \sim \int_K^{\infty} k^{-2(s-1)/d}\, dk \sim K^{1-2(s-1)/d}, \tag{5.49}$$

which is an explicit rate of decay for the tail sum in terms of K, since

$$s > \frac{d+2}{2} \iff 2(s-1)/d > 1 \iff 1 - 2(s-1)/d < 0,$$

so the right hand side of (5.49) *does indeed converge to zero as $K \to \infty$.*

In general, if the noise coefficients have the growth

$$\|f_k\|^2_{H^{-s}(U)} \sim k^{-\alpha} \quad \text{for some} \quad \alpha > 1,$$

then the tail sum $\mathcal{T}_s(K)$ grows like $K^{1-\alpha}$.

5.3 Central limit theorem for the SPDE with singular coefficients

In this section we first establish an $L^\infty(U \times [0,T])$-estimate for the solutions of the singular equation (5.1) in Theorem 5.3.1. Combining this estimate with the central limit theorem for the regularised equation gives us the quantitative central limit theorem in probability for the singular equation, which we prove in Theorem 5.3.2.

Let us begin with the $L^\infty(U \times [0,T])$-estimate. Formally the estimate says that if the initial condition starts away from zero, then with high probability the solution stays away from zero uniformly in space and time. The computations following equation (5.54) in the proof below follow closely to Theorem 3.9 of Dirr, Fehrman and Gess [30], so we just present the main idea.

Theorem 5.3.1 ($L^\infty(U \times [0,T])$-estimate for equation with singular coefficients)**.** *Suppose that the non-linear functions Φ, σ, ν satisfy Assumptions 2.2.1 and 5.1.13, and that the spatial components of the noise ξ^K satisfy Assumption 5.1.2. Suppose that ρ_0 and $\bar{f}$ satisfy the simplifying Assumption 5.1.7.*

Then, for stochastic kinetic solutions ρ^ϵ of (5.1)*, for M as in Assumption 5.1.7, there exists constants $c, \gamma \in (0,\infty)$ independent of ϵ, K such that*

$$\mathbb{E}\| \left(\rho^\epsilon - \Phi^{-1}(M)\right)_- \|_{L^\infty(U\times[0,T])} \leq c \left(\epsilon \left(\|F_3^K\|_{L^\infty(U)} + \|\nabla \cdot F_2^K\|_{L^\infty(U)}\right)\right)^\gamma.$$

Proof. In order to apply Itô's formula, we will need to work with the regularised equation (5.9). However, the estimates are completely uniform with respect to the regularisation parameter $n \in \mathbb{N}$, so in the end we can just apply dominated convergence and take the limit as $n \to \infty$. Under Assumption 5.1.7 the solution of the PDE

$h_{\Phi^{-1}(\bar{f})}$ is given by the constant function $\Phi^{-1}(M)$, so below we abuse notation and write $\Phi^{-1}(M)$ to denote the function defined on $\bar{U}$ taking the constant value $\Phi^{-1}(M)$ everywhere. For fixed $p \geq 2, n, K \in \mathbb{N}, \epsilon \in (0,1)$ and $\rho^{n,\epsilon}$ as in (5.9), Itô's formula tells us

$$d\left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_-^p = \frac{1}{n}\left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_-^{p-1} d\left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)$$
$$+ \frac{1}{2}p(p-1)\left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_-^{p-2} d\langle\rho^{n,\epsilon} - \Phi^{-1}(M)\rangle.$$

This gives after integrating by parts and rearranging, that

$$\frac{1}{p(p-1)}\int_U (\rho^{n,\epsilon}(x,t) - \Phi^{-1}(M))_-^p + \int_0^t\int_U \left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_-^{p-2} \Phi'(\rho^{n,\epsilon})|\nabla\rho^{n,\epsilon}|^2$$
$$+ \frac{1}{n}\int_0^t\int_U \left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_-^{p-2} |\nabla\rho^{n,\epsilon}|^2$$
$$= \int_0^t\int_U \left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_-^{p-2}\left(\sqrt{\epsilon}\sigma_n(\rho^{n,\epsilon})\nabla\rho^{n,\epsilon}\cdot d\xi^K + \nabla\rho^{n,\epsilon}\cdot\nu(\rho^{n,\epsilon})\right)$$
$$+ \int_0^t\int_U \left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_-^{p-2}\left(\frac{\epsilon}{2}F_3^K\sigma_n^2(\rho^{n,\epsilon}) + \frac{\epsilon}{2}\sigma_n(\rho^{n,\epsilon})\sigma_n'(\rho^{n,\epsilon})\nabla\rho^{n,\epsilon}\cdot F_2^K\right). \quad (5.50)$$

The key observation to make is that since $\rho^{n,\epsilon}$ is non-negative, we have that

$$\left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_- \neq 0 \iff \rho^{n,\epsilon} \in [0, \Phi^{-1}(M)),$$

which is a bounded set. On this set, since $\sigma \in C_{loc}([0,\infty))$, we have that uniformly in the n regularisation,

$$\left|\sigma_n(\rho^{n,\epsilon})\mathbb{1}_{(\rho^{n,\epsilon}-\Phi^{-1}(M))_-\neq 0}\right| \leq c. \quad (5.51)$$

We want to use this to handle the first term on the right hand side of (5.50). For every $n \in \mathbb{N}$, there exists a constant $c \in (0,\infty)$ independent of n such that we have the following bound for the unique function $\Psi_{\sigma_n,p}(\xi)$ defined by,

$$\Psi_{\sigma_n,p}(\xi) := \int_0^\xi (\xi' - \Phi^{-1}(M))_-^{p-2}\sigma_n(\xi')\,d\xi' \leq c\int_0^\xi (\xi' - \Phi^{-1}(M))_-^{p-2}$$
$$= \frac{c}{p-1}(\xi - \Phi^{-1}(M))_-^{p-1}. \quad (5.52)$$

Consequently, integration by parts gives that

$$\int_0^t\int_U \left(\rho^{n,\epsilon} - \Phi^{-1}(M)\right)_-^{p-2}\sigma_n(\rho^{n,\epsilon})\nabla\rho^{n,\epsilon}\cdot d\xi^K = -\int_0^t\int_U \Psi_{\sigma_n,p}(\rho^{n,\epsilon})\nabla\cdot d\xi^K.$$

Analogous reasoning as equation (5.25) illustrates that we do not pick up additional boundary terms when integrating by parts in the case $p = 2$. The Burkholder-Davis-Gundy inequality, Hölder's inequality and Young's inequality prove that there exists a constant $c \in (0,\infty)$ such that for every $\delta \in (0,1)$,

$$\begin{aligned}
&\mathbb{E}\left[\sqrt{\epsilon}\sup_{t\in[0,T]}\left|\int_0^t\int_U \left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}\sigma_n(\rho^{n,\epsilon})\nabla\rho^{n,\epsilon}\cdot d\xi^K\right|\right] \\
&\qquad\qquad \le \frac{c\sqrt{\epsilon}}{p-1}\mathbb{E}\left[\left(\int_0^T\sum_{i=1}^d\sum_{k=1}^K\left(\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-1}\partial_i f_k\right)^2\right)^{1/2}\right] \\
&\le c\sqrt{\epsilon}\mathbb{E}\left[\left(\sup_{t\in[0,T]}\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p}\right)^{1/2}\left(\int_0^T\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}F_3^K\right)^{1/2}\right] \\
&\le \frac{\delta}{2}\mathbb{E}\left[\sup_{t\in[0,T]}\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p}\right]+\frac{c\epsilon\|F_3^K\|_{L^\infty(U)}}{2\delta}\mathbb{E}\left[\int_0^T\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}\right].
\end{aligned} \tag{5.53}$$

We take δ sufficiently small so that the first term in the final line can be absorbed onto the left hand side (after taking supremum in time). Let us illustrate how to deal with the other terms on the right hand side of equation (5.50). The term involving ν vanishes due to Assumption 5.1.7 by the same reasoning as usual, see for example equation (5.23). For the first term in the final line of (5.50), we again use equation (5.51) which allows us to obtain for constant $c \in (0,\infty)$ independent of n,

$$\frac{\epsilon}{2}\int_0^t\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}F_3^K\sigma_n^2(\rho^{n,\epsilon}) \le c\epsilon\|F_3^K\|_{L^\infty(U)}\int_0^t\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}.$$

Due to the assumption $(\sigma(\xi)\sigma'(\xi))^2 \le c(1+\xi^m)$ which can be deduced from equations (2.4) and (2.10) from Assumption 2.2.1, and the trivial observation that if $\xi \in [0,\Phi^{-1}(M))$ then $\xi^m \le (\Phi^{-1}(M))^m$ is bounded, we have using integration by parts that for the final term on the right hand side of (5.50), there exists a constant $c \in (0,\infty)$ such that

$$\begin{aligned}
&\frac{\epsilon}{2}\int_0^t\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}\sigma(\rho^{n,\epsilon})\sigma'(\rho^{n,\epsilon})\nabla\rho^{n,\epsilon}\cdot F_2^K \\
&\qquad\qquad \le c\epsilon\int_0^t\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}\nabla\rho^{n,\epsilon}\cdot F_2^K \\
&\qquad\qquad\qquad \le c\epsilon\|\nabla\cdot F_2^K\|_{L^\infty(U)}\int_0^t\int_U\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-1}.
\end{aligned}$$

As above, by an analogous computation to (5.25), in the case $p=2$ we don't pick up boundary terms when integrating by parts. This term can then handled in the same way as the noise term (5.53), which can be easily seen by multiplying the integrand by 1 and applying Hölder's inequality.

Putting equation (5.50) together with the subsequent computations and taking supremum in time and expectation gives

$$\begin{aligned}
&\frac{1}{p(p-1)}\mathbb{E}\left[\sup_{t\in[0,T]}\int_U (\rho^{n,\epsilon}(x,t)-\Phi^{-1}(M))_-^p\right] \\
&+\mathbb{E}\left[\int_0^T\int_U \left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}\Phi'(\rho^{n,\epsilon})|\nabla\rho^\epsilon|^2\right]+\frac{1}{n}\mathbb{E}\left[\int_0^T\int_U \left|\nabla\left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p/2}\right|^2\right] \\
&\qquad\le c\epsilon\left(\|F_3^K\|_{L^\infty(U)}+\|\nabla\cdot F_2^K\|_{L^\infty(U)}\right)\mathbb{E}\left[\int_0^T\int_U \left(\rho^{n,\epsilon}-\Phi^{-1}(M)\right)_-^{p-2}\right]. \quad (5.54)
\end{aligned}$$

The right hand side of the above estimate is independent of n in the sense that it does not depend on the regularised σ_n. Hence, by dominated convergence, we take the limit as $n\to\infty$, which is represented below by replacing $\rho^{n,\epsilon}$ with ρ^ϵ. Furthermore, it follows by Chapter 4, Proposition 4.7 and Exercise 4.30 of Revuz and Yor [85] and Hölder's inequality, that for every $p\in(1,\infty)$ and $n_p := p^{-1}$ that we can raise both sides to the power n_p at the cost of picking up an additional constant,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{t\in[0,T]}\int_U (\rho^\epsilon(x,t)-\Phi^{-1}(M))_-^p + p(p-1)\int_0^T\int_U \left(\rho^\epsilon-\Phi^{-1}(M)\right)_-^{p-2}\Phi'(\rho^\epsilon)|\nabla\rho^\epsilon|^2\right.\right. \\
&\qquad\left.\left.+\frac{1}{n}p(p-1)\int_0^T\int_U\left|\nabla\left(\rho^\epsilon-\Phi^{-1}(M)\right)_-^{p/2}\right|^2\right)^{1/p}\right] \\
&\qquad\le \frac{n_p^{-n_p}}{1-n_p}\left(cp^2\epsilon\left(\|F_3^K\|_{L^\infty(U)}+\|\nabla\cdot F_2^K\|_{L^\infty(U)}\right)\right)^{1/p} \\
&\qquad\qquad\times\mathbb{E}\left[\|\left(\rho^\epsilon-\Phi^{-1}(M)\right)_-\|_{L^{p-2}(U\times[0,T])}\right]^{\frac{p-2}{p}}, \quad (5.55)
\end{aligned}$$

where on the right hand side we also used the trivial bound $p(p-1)<p^2$.

We now want to use a Moser iteration to conclude. The above bound, bounding the $L^p(U)$-norm by the $L^{p-2}(U)$-norm will not be sufficient for us, because if we used that to iterate, the constants that we accumulate would diverge. Instead, we we will need the following bound. By the same computation as in Theorem 3.9 of Dirr Fehrman and Gess [30], using the log-convex inequality, the Sobolev inequality and Hölder's inequality, we have the improved estimate

$$\begin{aligned}
&\mathbb{E}\|\left(\rho^\epsilon-\Phi^{-1}(M)\right)_-\|_{L^{\frac{(2+d)p}{d}}(U\times[0,T])} \\
&\qquad\le \frac{n_p^{-n_p}}{1-n_p}\left(cp^2\epsilon\left(\|F_3^K\|_{L^\infty(U)}+\|\nabla\cdot F_2^K\|_{L^\infty(U)}\right)\right)^{1/p} \\
&\qquad\qquad\times\mathbb{E}\left[\|\left(\rho^\epsilon-\Phi^{-1}(M)\right)_-\|_{L^{p-2}(U\times[0,T])}\right]^{\frac{p-2}{p}}. \quad (5.56)
\end{aligned}$$

Now, compared to (5.55), when we perform the Mozer iteration below, at each step our iterates decrease by a multiplicative factor of $\frac{2+d}{d}$ which will ensure the constants that we accumulate on the right hand side remain bounded in the limit. Proceed iteratively, define $p_0 = 0$ and for $k \in \mathbb{N}$ define

$$p_k = \frac{2+d}{d} p_{k-1} + 2.$$

Trivially the growth bound $p_k > \left(\frac{d+2}{d}\right)^k$ is satisfied. Inequality (5.56) gives that for $q_k := p_{k-1} + 2$,

$$\begin{aligned}
\mathbb{E}\| \left(\rho^\epsilon - \Phi^{-1}(M)\right)_- \|_{L^{p_k}(U\times[0,T])} &\le \frac{n_{q_k}^{-n_{q_k}}}{1-n_{q_k}} \left(c\epsilon q_k^2 \left(\|F_3^K\|_{L^\infty(U)} + \|\nabla\cdot F_2^K\|_{L^\infty(U)}\right)\right)^{n_{q_k}} \\
&\quad \times \mathbb{E}\left[\| \left(\rho^\epsilon - \Phi^{-1}(M)\right)_- \|_{L^{p_{k-1}}(U\times[0,T])}\right]^{\frac{q_k}{q_k+2}} \\
&\le \prod_{r=1}^{k} \left(\frac{n_{q_r}^{-n_{q_r}}}{1-n_{q_r}} (cq_r)^{2n_{q_r}} \right)^{\prod_{s=r+1}^{k} \frac{q_s}{q_s+2}} \\
&\quad \times \left(\epsilon \left(\|F_3^K\|_{L^\infty(U)} + \|\nabla\cdot F_2^K\|_{L^\infty(U)}\right)\right)^{\sum_{r=1}^k n_{q_r} \prod_{s=r+1}^{k} \frac{q_s}{q_s+2}}. \qquad (5.57)
\end{aligned}$$

We conclude by showing that the constant and exponent in the final inequality do not diverge in the $k \to \infty$ limit. We begin analysing what happens to the exponent of the final term of (5.57). First of all, since we have $q_k > \left(\frac{d+2}{d}\right)^{k-1} + 2$, and the fact that $\log(1-x) \ge -\frac{x}{1-x}$ for every $x \in (0,1)$, we have

$$\begin{aligned}
\liminf_{N\to\infty} \log\left(\prod_{s=2}^{N} \frac{q_s}{q_s+2}\right) &= \liminf_{N\to\infty} \sum_{s=2}^{N} \log\left(1 - \frac{2}{q_s+2}\right) \\
&\ge \liminf_{N\to\infty} \sum_{s=2}^{N} -\frac{2}{q_s} > \liminf_{N\to\infty} \sum_{s=2}^{N} -2\left(\frac{d}{d+2}\right)^{s-1}.
\end{aligned}$$

We note that the right hand side is a geometric series with ratio $r = \frac{d}{d+2} < 1$ and so the series converges, and hence the left hand side is finite.

Furthermore, since $1/q_r$ decays exponentially due to the bound on q_r, we have that there is a constant $\gamma \in (0,\infty)$ that bounds the exponent on the final term of (5.57),

$$\lim_{k\to\infty} \left(\sum_{r=1}^{k} n_{q_r} \prod_{s=r+1}^{k} \frac{q_s}{q_s+2} \right) = \gamma.$$

Now we need to analyse the product of constants on the right hand side of (5.57). The definition of n_{q_r} gives the bound

$$\limsup_{k\to\infty} \prod_{r=1}^{k} \left((cq_r)^{2n_{q_r}} \frac{n_{q_r}^{-n_{q_r}}}{1-n_{q_r}} \right)^{\prod_{s=r}^{k-1} \frac{q_s}{q_s+2}} \le \limsup_{k\to\infty} \prod_{r=1}^{k} \left((cq_r)^{2n_{q_r}} \frac{q_r^{1+1/q_r}}{q_r - 1} \right),$$

and so for every $k \in \mathbb{N}$, it holds that

$$\begin{aligned}\log\left(\prod_{r=1}^{k}(cq_r)^{2n_{q_r}}\,\frac{q_r^{1+1/q_r}}{q_r-1}\right) &= \sum_{r=1}^{k} 2/q_r\log(cq_r)+1/q_r\log(q_r)+\log\left(\frac{q_r}{q_r-1}\right)\\ &= \sum_{r=1}^{k} 2/q_r\log(cq_r)+1/q_r\log(q_r)-\log(1+(q_r-1)^{-1}).\end{aligned}$$

Since $q_r > \left(\frac{d+2}{d}\right)^{r-1}$, we get that

$$\limsup_{k\to\infty}\sum_{r=1}^{k} 2n_{q_r}\log(cq_r)+1/q_r\log(q_r)-\log(1+(q_r-1)^{-1})<\infty.$$

This implies that there is a constant $\tilde{c}\in(0,\infty)$ such that

$$\limsup_{k\to\infty}\prod_{r=1}^{k}\left(c\frac{n_{q_r}^{-n_{q_r}}}{1-n_{q_r}}\right)^{\prod_{s=r+1}^{k}\frac{q_s}{q_s+2}}\le\tilde{c}.$$

Therefore, putting everything together and passing to the limit as $k\to\infty$ in (5.57), the statement is proved. □

We are now in a position to prove the central limit theorem for the equation with singular coefficients. The proof combines the central limit theorem result for the regularised equation, Theorem 5.2.9, with the uniform estimate above. We note that the methods are follow the proof of Theorem 3.10 of Dirr Fehrman and Gess [30]. We provide the proof for completeness.

Theorem 5.3.2 (Central limit theorem for the singular equation)**.** *Suppose that the non-linear functions Φ, σ, ν satisfy Assumptions 2.2.1 and 5.1.13, and that the spatial components of the noise ξ^K satisfy assumption 5.1.2. Suppose that ρ_0 and $\bar{f}$ satisfy the simplifying Assumption 5.1.7. Let ρ^ϵ be the stochastic kinetic solution of equation* (5.1) *and $\bar{\rho}$ be the solution of the hydrodynamic limit equation* (5.8)*. Let v^ϵ be a weak solution to* (1.9)*, and v a strong solution to the linearised SPDE* (5.28) *in the sense of Definition 5.2.1.*

For $\beta\in(0,\infty)$ as in point 1 of Assumption 5.1.13 and every $s>\frac{d+2}{2}$, there exists constants $c,\gamma\in(0,\infty)$ such that for every $\eta\in(0,1)$,

$$\begin{aligned}\mathbb{P}\left[\|v^\epsilon-v\|_{L^2([0,T];H^{-s}(U))}\ge\eta\right] &\le c\eta^{-2}\left(1+\epsilon T\left(\|\nabla\cdot F_2^K\|_{L^\infty(U)}+\|F_3^K\|_{L^\infty(U)}\right)\right)\\ &\times\Bigg[\epsilon T^2\left(\|\nabla\cdot F_2^K\|^2_{L^\infty(U)}+\|F_3^K\|^2_{L^\infty(U)}\right)+\epsilon^{\beta+1}T^2\left(\|\nabla\cdot F_2^K\|^{\beta+2}_{L^\infty(U)}+\|F_3^K\|^{\beta+2}_{L^\infty(U)}\right)\\ &\qquad+\epsilon T\left(\|F_1^K\|^2_{L^\infty(U)}+\|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)}\right)\Bigg]\\ &\quad+c\eta^{-2}T\,\mathcal{T}_s(K)+c\left(\epsilon\left(\|F_3^K\|_{L^\infty(U)}+\|\nabla\cdot F_2^K\|_{L^\infty(U)}\right)\right)^\gamma.\end{aligned}\tag{5.58}$$

Proof. By Assumption 5.1.7, the initial condition of ρ^ϵ is strictly positive $\rho_0 = \Phi^{-1}(M) > 0$. Furthermore, let $\tilde{\sigma} \in C([0,\infty)) \cap C^\infty((0,\infty))$ be an arbitrary function satisfying for a constant $c \in (0,\infty)$ potentially depending on M[4],

$$\tilde{\sigma}(\xi) = \sigma(\xi) \text{ for every } \xi \in [\Phi^{-1}(M)/2, \infty), \text{ and } |\tilde{\sigma}'(\xi)| \leq c \text{ for every } \xi \in (0,\infty). \tag{5.59}$$

Let $\tilde{\rho}^\epsilon$ be the unique stochastic kinetic solution to (5.1) with σ replaced by $\tilde{\sigma}$. Due to Theorem 3.2.2 that gives pathwise uniqueness of solutions, the two solutions ρ^ϵ, and $\tilde{\rho}^\epsilon$ coincide on the event when σ and $\tilde{\sigma}$ coincide. That is,

$$\rho^\epsilon = \tilde{\rho}^\epsilon \quad \text{in} \quad L^1(U \times [0,T]),$$

on the event $(\mathcal{S} \cap \tilde{\mathcal{S}}) \subseteq \Omega$ defined by

$$\mathcal{S} = \{\|\rho^\epsilon\|_{L^\infty(U\times[0,T])} \geq \Phi^{-1}(M)/2\} \quad \text{and} \quad \tilde{\mathcal{S}} = \{\|\tilde{\rho}^\epsilon\|_{L^\infty(U\times[0,T])} \geq \Phi^{-1}(M)/2\}.$$

Furthermore, define $\tilde{v}^\epsilon := \epsilon^{-1/2}(\tilde{\rho}^\epsilon - \bar{\rho})$, and let v denote the solution to the linearised SPDE (5.28). We have the upper bound

$$\mathbb{P}\left[\|v^\epsilon - v\|_{L^2([0,T];H^{-s}(U))} \geq \eta\right] \leq \mathbb{P}\left[\|\tilde{v}^\epsilon - v\|_{L^2([0,T];H^{-s}(U))} \geq \eta\right] + \mathbb{P}[\mathcal{S}^c] + \mathbb{P}[\tilde{\mathcal{S}}^c]. \tag{5.60}$$

For the first term, by Chebyshev's inequality and the central limit theorem for the approximate equation Theorem 5.2.9, replacing the computation in (5.47) by incorporating the uniform bound (5.59) on σ', we get

$$\begin{aligned}\mathbb{P}\left[\|\tilde{v}^\epsilon - v\|_{L^2([0,T];H^{-s}(U))} > \eta\right] &\leq c\eta^{-2}\left(1 + \epsilon T\left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)}\right)\right) \\ &\times \Bigg[\epsilon T^2\left(\|\nabla \cdot F_2^K\|^2_{L^\infty(U)} + \|F_3^K\|^2_{L^\infty(U)}\right) + \epsilon^{\beta+1}T^2\left(\|\nabla \cdot F_2^K\|^{\beta+2}_{L^\infty(U)} + \|F_3^K\|^{\beta+2}_{L^\infty(U)}\right) \\ &\quad + \epsilon T\left(\|F_1^K\|^2_{L^\infty(U)} + \|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)}\right)\Bigg] + c\eta^{-2}T\,\mathcal{T}_s(K). \end{aligned}\tag{5.61}$$

We also simplified the above estimate using the uniform bound on σ from equation (5.59), which allowed us to obtain using Proposition 5.1.18 with $p = 2$ that

$$\begin{aligned}\mathbb{E}\|\sigma(\tilde{\rho}^\epsilon) - \sigma(\bar{\rho})\|^2_{L^2(U\times[0,T])} &\leq c\mathbb{E}\|\tilde{\rho}^\epsilon - \bar{\rho}\|^2_{L^2(U\times[0,T])} \\ &\leq c\epsilon T^2\left(\|\nabla \cdot F_2^K\|_{L^\infty(U)} + \|F_3^K\|_{L^\infty(U)}\right),\end{aligned}$$

where the right hand side is already present on the right hand side of the estimate (5.61). For the final two terms of (5.60), it follows from the $L^\infty(U \times [0,T])$-estimate of Theorem 5.3.1, which equally applies to $\tilde{\rho}^\epsilon$ due to (5.59), that

$$\mathbb{P}[\mathcal{S}^c] + \mathbb{P}[\tilde{\mathcal{S}}^c] \leq c\left(\epsilon\left(\|F_3^K\|_{L^\infty(U)} + \|\nabla \cdot F_2^K\|_{L^\infty(U)}\right)\right)^\gamma.$$

Putting both parts of estimate (5.60) together completes the proof. □

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[4] In the square root diffusion case we would have $|\tilde{\sigma}'(\xi)| \leq \sqrt{2}(\Phi^{-1}(M))^{-1/2}$.

Remark 5.3.3. *We point out that the convergence in probability in Theorem 5.3.2 above is weaker than the* $L^1(\Omega; L^2([0,T]; H^{-s}(U)))$*-convergence for the regularised equation stated in Theorem 5.2.9. We can not hope to achieve the convergence in expectation for the singular equation due to a lack of control of the expectation*

$$\mathbb{E}\left[\|v^\epsilon - v\|^2_{L^2([0,T];H^{-s}(U))}\mathbb{1}_{\mathcal{S}^c}\right].$$

This is due to the lack of uniform energy estimate for the approximate equation (5.40)*.*

Remark 5.3.4 (Choice of joint scaling)**.** *Based on the previous estimates (Propositions 5.1.18 and 5.1.20, Theorems 5.2.9, 5.3.1 and 5.3.2), we choose a joint scaling such that the right hand side of the estimates converge to zero as* $\epsilon \to 0$ *and* $K \to \infty$*. A sufficient joint scaling is one which ensures that as* $\epsilon \to 0, K \to \infty$ *we have*

$$\epsilon\left(\|F_1^K\|^2_{L^\infty(U)} + \|F_2^K\|^2_{L^\infty(U;\mathbb{R}^d)} + \|F_3^K\|^2_{L^\infty(U)} + \|\nabla \cdot F_2^K\|^2_{L^\infty(U)}\right) \to 0. \tag{5.62}$$

In the case of the eigenfunctions of Laplacian from Example 5.1.4, equation (5.6) *gives explicitly the scaling relation needed between* ϵ *and* K *as*

$$\epsilon K^{d+2} \to 0.$$

Chapter 6

The large deviations principle

The goal of this chapter is to set up the structure to prove a large deviations principle for solutions of the generalised Dean–Kawasaki equation. We aim to apply abstract theorem from Budhiraja, Dupuis, and Maroulas [11] which is based on the variational representation of infinite dimensional Brownian Motion.

In Section 6.1 we begin by defining what it means for a sequence of functions to satisfy the large deviations principle and make a comment on different notions of uniform large deviations principles. We then state the large deviations result that we will use. Section 6.2 is dedicated to the analysis of a degenerate parabolic hyperbolic PDE that appears in the statement of the large deviations principle. We outline the proof of the well-posedness and $L^1(U)$-contraction for solutions to the skeleton equation. This result is proven rigorously in forthcoming work.

The abstract theorem of Budhiraja, Dupuis, and Maroulas [11] relies on the existence of solution maps. We prove the existence of such maps in Section 6.3. Whilst the large deviations principle is stated uniformly with respect to bounded subsets of the initial data, in Section 6.4 we show that the uniformity does not extend to bounded subsets of the boundary data. After proving energy estimates for the regularised SPDE with control in Section 6.5, we conclude by proving the remaining conditions for the large deviations in Section 6.6.

6.1 Setup and statement of the large deviations principle

In this section we begin in Definitions 6.1.1 and 6.1.2 with the definition of large deviations principle. We subsequently discuss various notions of uniform large deviations principles. After setting up the notation, in Assumption 6.1.3 we give the assumptions for the large deviations principle to hold true and then in Theorem 6.1.4 we conclude by stating the large deviations result from Budhiraja, Dupuis, and Maroulas [11] that we aim to apply.

Concretely, our aim is to prove that under the joint scaling regime $\epsilon \to 0, K(\epsilon) \to \infty$ from Remark 5.3.4, solutions ρ^ϵ of the generalised Dean–Kawasaki equation with truncated noise (5.1),

$$\partial_t \rho^\epsilon = \Delta\Phi(\rho^\epsilon) - \sqrt{\epsilon}\nabla\cdot(\sigma(\rho^\epsilon)\circ\dot{\xi}^K) - \nabla\cdot\nu(\rho^\epsilon) \tag{6.1}$$

satisfy a large deviations principle, and also to identify the corresponding rate function. As mentioned in the final part of Section 1.2, the motivation is that one can study the large deviations principle for the zero range particle system by looking at the large deviations of the SPDE, in the sense that the rate functions coincide.

We begin by stating what it means for the solutions of the SPDE ρ^ϵ to satisfy the large deviations principle. From the well-posedness results above, we know that solutions ρ^ϵ live in the space $L^1(U\times[0,T])$. The theory of large deviations is concerned with events A in the Borel set $\mathcal{B}(L^1(U\times[0,T]))$ such that the probability $\mathbb{P}(\rho^\epsilon\in A)$ decays exponentially under the joint scaling of Remark 5.3.4. The exponential rate is quantified in terms of a "rate function" $I : L^1(U\times[0,T])\to\mathbb{R}$. The following definitions are standard, for example see Definitions 1 and 2 of Budhiraja, Dupuis, and Maroulas [11].

Definition 6.1.1 (Rate function)**.** *A function $I : L^1(U\times[0,T])\to\mathbb{R}$ is called a rate function on $L^1(U\times[0,T])$ if for every constant $M<\infty$, the level set*

$$\{\rho\in L^1(U\times[0,T]) : I(\rho)\le M\}$$

is a compact subset of $L^1(U\times[0,T])$. For $A\in\mathcal{B}(L^1(U\times[0,T]))$, define

$$I(A) := \inf_{\rho\in A} I(f).$$

Definition 6.1.2 (Large deviations principle)**.** *Let I be a rate function on $L^1(U\times[0,T])$ as defined in Definition 6.1.1. The sequence $\{\rho^\epsilon\}_{\epsilon\in(0,1)}$ satisfies the large deviation principle on $L^1(U\times[0,T])$ if the following two conditions hold*

1. *Large deviation upper bound: For each closed subset F of $L^1(U\times[0,T])$,*

$$\limsup_{\epsilon\to0}\epsilon\log\mathbb{P}(\rho^\epsilon\in F)\le -I(F). \tag{6.2}$$

2. *Large deviation lower bound: For each open subset G of $L^1(U\times[0,T])$,*

$$\liminf_{\epsilon\to0}\epsilon\log\mathbb{P}(\rho^\epsilon\in G)\ge -I(G). \tag{6.3}$$

It is well known that if a sequence of random variables satisfies the large deviation principle with some rate function, then the rate function is unique, see for example Theorem 1.3.1 of Dupuis and Ellis [34]. Combining the upper and lower bounds (6.2)-(6.3) gives that approximately we have for every Borel set $A\in\mathcal{B}(L^1(U\times[0,T]))$ and small ϵ,

$$\mathbb{P}(\rho^\epsilon\in A)\approx e^{-\epsilon I(A)}.$$

If the set A contains the solution to the hydrodynamic limit equation $\bar{\rho}$, then we will see that $I(A)=0$, which implies that

$$\mathbb{P}(\rho^\epsilon\in A)\to 1 \quad \text{as} \quad \epsilon\to 0.$$

Otherwise $I(A) \neq 0$ and we have a quantitative exponential decay

$$\mathbb{P}(\rho^\epsilon \in A) \to 0 \quad \text{as} \quad \epsilon \to 0$$

governed by the rate function $I(A)$.

As mentioned above, we will rely on the abstract result of Budhiraja, Dupuis, and Maroulas [11]. In their work, they prove the existence of a uniform Laplace principle. For us, the uniformity of the large deviations will be proven with respect to bounded subsets of the initial data, and the boundary data will be fixed, but arbitrary. On the level of large deviations, it has been shown in Section 1.2 of Dupuis and Ellis [34] that the large deviations principle and the Laplace principle are equivalent. However, it was shown by Salins [88] that the same is not true on the level of "uniform" large deviations, where there are three distinct concepts. Firstly the uniform Laplace principle (ULP), secondly Freidlin-Wentzell uniform large deviations principle (FWULDP), and finally the Dembo-Zeitouni uniform large deviations principle (DZULDP), see Definitions 2.1-2.3 of Salins [88] and references therein.

In the present work, we will prove that our family of rate functions $I^{\bar f}_{\rho_0}$, indexed by the initial data, has compact level sets on compacts (in the sense of Definition 4 of Budhiraja, Dupuis, and Maroulas [11], which is a consequence of Theorem 6.1.4 below), under which Theorem 2.5 of Salins [88] gives that ULP and FWULP are equivalent. To motivate what the FWULDP is, we show how to modify the large deviations lower bound (6.3). The random variables $\{\rho^\epsilon(\bar f, \rho_0)\}_{\epsilon\in(0,1)}$ (solutions to (6.1) with initial data ρ_0 and boundary data $\Phi(\rho^\epsilon(\bar f,\rho_0))|_{\partial U} = \bar f$) satisfy the FWULDP lower bound with rate functions $I^{\bar f}_{\rho_0}$ uniformly over bounded subsets of the initial data if for every $\delta > 0$, every Borel set $A \in \mathcal{B}(Ent_{\Phi,R}(U))$ defined in (6.7), every $\bar f \in H^1(\partial U)$ satisfying Assumption 2.2.9 and every $s > 0$,

$$\liminf_{\epsilon\to 0} \inf_{\rho_0 \in A} \inf_{\tilde\rho \in \Gamma_{\rho_0}(s)} \Big(\big(\epsilon \log \mathbb{P}\left(\|\rho^\epsilon(\bar f, \rho_0) - \tilde\rho\| < \delta\right) + I^{\bar f}_{\rho_0}(\tilde\rho)\big) \Big) \geq 0, \tag{6.4}$$

where

$$\Gamma_{\rho_0}(s) := \{\tilde\rho \in L^1(U \times [0,T]) : I^{\bar f}_{\rho_0}(\tilde\rho) \leq s\}.$$

The key point is the infimum over $\rho_0 \in A$, which tells us that the large deviations lower bound holds uniformly over all initial data in the Borel set A.

To set up the problem and to state the relevant result, we first need to recall the spaces where various objects live. First of all, based on the assumptions on the boundary data stated in Assumption 2.2.9, we require that the boundary data $\bar f$ lives in the space $H^1(\partial U)$.

Due to the definition of stochastic kinetic solutions, see Definition 2.3.6, a priori one considers that the non-negative initial data lives in the space $L^1(U)$. However, the formal a priori estimate in Section 6.2.2 will prove that the correct space for the initla data is the entropy space $Ent_\Phi(U)$ as defined in Definition 4.2.2.

As in Definition 5.1.1, we view the infinite sequence of Brownian motions

$$B := (B^k)_{k\in\mathbb{N}}$$

used to define the noise ξ^K in Definition 5.1.1 as living in the space $C([0,T];(\mathbb{R}^d)^\infty)$ equipped with the metric topology of co-ordinate wise convergence. In this chapter we again assume that the noise satisfies Assumption 5.1.2.

In the notation of Budhiraja, Dupuis, and Maroulas [11], for $\epsilon \in [0,1)$ and fixed boundary data $\bar{f} \in H^1(\partial U)$ satisfying Assumption 2.2.9, we will denote the solution map for equation (6.1) by

$$\mathcal{G}^{\bar{f},\epsilon} : Ent_\Phi(U) \times C([0,T];(\mathbb{R}^d)^\infty) \to L^1(U \times [0,T]).$$

That is, if we let $\rho^\epsilon(\bar{f},\rho_0)$ denote the stochastic kinetic solution of (6.1) with boundary data $\Phi(\rho^\epsilon)|_{\partial U} = \bar{f} \in H^1(\partial U)$ and initial condition $\rho_0 \in Ent_\Phi(U)$, then distributionally we have

$$\rho^\epsilon(\bar{f},\rho_0) = \mathcal{G}^{\bar{f},\epsilon}(\rho_0, \sqrt{\epsilon}B). \tag{6.5}$$

The existence of such a map $\mathcal{G}^{\bar{f},\epsilon}$ is shown as part of the proof of Proposition 6.3.5 below. We state an assumption under which the ULP holds for equation (6.1), see Assumption 2 of Budhiraja, Dupuis, and Maroulas [11].

Assumption 6.1.3 (Assumption for large deviations principle)**.** *Fix $\bar{f} \in H^1(\partial U)$ satisfying Assumption 2.2.9. Suppose that there exists a measurable map $\mathcal{G}^{\bar{f},0} : Ent_\Phi(U) \times C([0,T];(\mathbb{R}^d)^\infty) \to L^1(U \times [0,T])$ such that*

1. *For every $R < \infty$ and compact subset $K \subset Ent_\Phi(U)$, the set*

   $$\Gamma_{R,K} := \left\{ \mathcal{G}^{\bar{f},0}\left(\rho_0, \int_0^\cdot g(s)\,ds\right) : g \in L^2_R(U \times [0,T]), \rho_0 \in K \right\}$$

   is a compact subset of $L^1(U \times [0,T])$, where we defined the bounded $L^2(U \times [0,T])$ space by

   $$L^2_R(U \times [0,T]) := \{g \in L^2(U \times [0,T];\mathbb{R}^d) : \|g\|_{L^2(U\times[0,T])} \le R\}.$$

2. *For an arbitrary family of initial conditions $\{\rho_0^\epsilon\}_{\epsilon\in(0,1)} \subset Ent_\Phi(U)$ and controls $\{g^\epsilon\}_{\epsilon\in(0,1)} \subset L^\infty(\Omega; L^2(U\times[0,T];\mathbb{R}^d))$, whenever we have the weak convergences*[1] *$\rho_0^\epsilon \to \rho_0$ in $L^1(U)$ and $g^\epsilon \to g$ in $L^2(U \times [0,T])$ as $\epsilon \to 0$, then we have that distributionally in $L^1(U \times [0,T])$, as $\epsilon \to 0$,*

   $$\mathcal{G}^{\bar{f},\epsilon}\left(\rho_0^\epsilon, \sqrt{\epsilon}B + \left(\int_0^\cdot g_k^\epsilon(s)\,ds\right)_{k\in\mathbb{N}}\right) \to \mathcal{G}^{\bar{f},0}\left(\rho_0, \left(\int_0^\cdot g_k(s)\,ds\right)_{k\in\mathbb{N}}\right),$$

   where for $k \in \mathbb{N}$, $g_k^\epsilon(s) := \langle g^\epsilon(\cdot,s), f_k\rangle_{L^2(U)}$ and $g_k(s) := \langle g(\cdot,s), f_k\rangle_{L^2(U)}$ denote projections onto the spatial components of the noise.

The below follows from a direct application of Theorem 6 of Budhiraja, Dupuis, and Maroulas [11].

---

[1] We ask for weak convergence of the initial data in the space $L^1(U)$ since it is more intuitive to understand what it means to converge weakly in $L^1(U)$. That is to say, the dual space of $L^1(U)$, the space of bounded linear functionals on $L^1(U)$, is better understood than the dual space of $Ent_\Phi(U)$.

Theorem 6.1.4 (Sufficient condition for ULP to hold)**.** *For $\bar{f} \in H^1(\partial U)$ satisfying Assumption 2.2.9, $\rho_0 \in Ent_\Phi(U)$ and $\rho \in L^1(U \times [0,T])$, define*

$$
\begin{aligned}
I^{\bar{f}}_{\rho_0}(\rho) &:= \inf_{g \in L^2(U \times [0,T];\mathbb{R}^d) : \rho = \mathcal{G}^{\bar{f},0}(\rho_0, \int_0^\cdot g(s)\,ds)} \left\{ \frac{1}{2} \int_0^T \|g(\cdot,s)\|^2_{L^2(U;\mathbb{R}^d)}\, ds \right\} \\
&= \frac{1}{2} \inf_{g \in L^2(U \times [0,T];\mathbb{R}^d)} \Big\{ \|g\|^2_{L^2(U \times [0,T];\mathbb{R}^d)} : \partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho) g + \nu(\rho)) \\
&\qquad\qquad : \Phi(\rho)|_{\partial U} = \bar{f} \text{ and } \rho(\cdot,0) = \rho_0 \Big\}. \qquad (6.6)
\end{aligned}
$$

Suppose that Assumption 6.1.3 holds, and also that for every $\rho \in L^1(U \times [0,T])$, the map $\rho_0 \mapsto I^{\bar{f}}_{\rho_0}(\rho)$ is lower semi-continuous from $Ent_\Phi(U)$ to $[0,\infty]$.

Then, for arbitrary fixed fixed $\bar{f} \in H^1(\partial U)$ and every $\rho_0 \in Ent_\Phi(U)$, the map $\rho \mapsto I^{\bar{f}}_{\rho_0}(\rho)$ is a rate function on $L^1(U \times [0,T])$, the family $\{I^{\bar{f}}_{\rho_0}(\cdot) : \bar{f} \in H^1(\partial U),\ \rho_0 \in Ent_\Phi(U)\}$ of rate functions has compact level sets on compacts, and $\{\rho^\epsilon(\bar{f},\rho_0)\}_{\epsilon \in (0,1)}$ satisfies the uniform Laplace principle on $L^1(U \times [0,T])$ with rate function $I^{\bar{f}}_{\rho_0}$ in the sense of Definition 5 of Budhiraja, Dupuis, and Maroulas [11]. The "uniformity" is with is with respect to compact subsets of the initial condition space $Ent_{\Phi,R}(U)$ defined by

$$
Ent_{\Phi,R}(U) := \left\{ \rho_0 \in Ent_\Phi(U) : \int_U \Psi_\Phi(\rho_0)\, dx < R \right\}. \qquad (6.7)
$$

6.2 Analysis of the skeleton equation

This section is split into three sub-sections. The goal will be to prove various properties of the parabolic-hyperbolic PDE that appears in the rate function (6.6), which we refer to as the skeleton equation. That is, the equation

$$
\begin{cases}
\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot (\sigma(\rho) g + \nu(\rho)), & \text{on } U \times (0,T], \\
\Phi(\rho) = \bar{f}, & \text{on } \partial U \times [0,T], \\
\rho(\cdot, t=0) = \rho_0, & \text{on } U \times \{t=0\}.
\end{cases} \qquad (6.8)
$$

In Section 6.2.1 we prove via a scaling argument that for controls $g \in L^2(U \times [0,T];\mathbb{R}^d)$, the skeleton equation is energy critical for initial data in $L^1(U)$. In Section 6.2.2 we prove that for the skeleton equation, due to the irregularity of the control $g \in L^2(U \times [0,T];\mathbb{R}^d)$ we only have an entropy type estimate for the equation. This is then used in Section 6.2.3 to justify the well-posedness of the skeleton equation.

As mentioned previously, the results in this subsection are part of ongoing work and may be subject to change as the analysis is further developed.

6.2.1 Energy criticality

In this section we recall the formal computation of Section 2.1 of Fehrman and Gess [42] which illustrates that for controls $g \in L^2(U \times [0,T];\mathbb{R}^d)$, the skeleton equation (6.8) is energy critical for initial data $\rho_0 \in L^1(U)$.

To illustrate the scaling argument, we consider the specific case of the porus medium equation $\Phi(\xi) = \xi^m$ for some $m \in [1, \infty)$, $\sigma(\xi) = \Phi^{1/2}(\xi)$ and $\nu = 0$, and we look at the equation on the whole space[2] $\mathbb{R}^d \times [0, \infty)$ in which case there is no boundary data.

That is, we analyse at the equation

$$\partial_t \rho = \Delta \rho^m - \nabla \cdot (\rho^{m/2} g)$$

with initial data $\rho_0 \in L^r(\mathbb{R}^d)$ for arbitrary $r \geq 1$. We "zoom in" to the equation and consider the behaviour around the origin $(x, t) = (0, 0)$ in the sense that for $\lambda, \eta, \tau \in (0, 1)$, we consider the rescaling

$$\tilde{\rho}(x, t) := \lambda \rho(\eta x, \tau t).$$

We want to consider the evolution of the rescaled equation $\tilde{\rho}$. Using the chain rule gives the identities

$$\Delta(\tilde{\rho}(x, t))^m = \lambda^m \Delta(\rho(\eta x, \tau t)^m) = \lambda^m \eta^2 (\Delta \rho^m)(\eta x, \tau t),$$

and

$$\begin{aligned} \nabla \cdot (\tilde{\rho}(x, t)^{m/2} g(\eta x, \tau t) = \lambda^{m/2} \nabla \cdot (\rho(\eta x, \tau t)^{m/2} g(\eta x, \tau t)) \\ = \lambda^{m/2} \eta (\nabla \cdot (\rho g))(\eta x, \tau t). \end{aligned}$$

There is some subtlety in the notation above. The terms in the middle are functions that are evaluated at the points, and then the derivatives are taken, whereas the final terms are compositions of the derivatives and the functions which are then evaluated at the points. Therefore, writing

$$\tilde{g}(x, t) := \frac{\tau}{\eta \lambda^{m/2-1}} g(\eta x, \tau t),$$

we have that

$$\begin{aligned} \partial_t \tilde{\rho}(x, t) = \lambda \partial_t (\rho(\eta x, \tau t)) = \lambda \tau (\partial_t \rho)(\eta x, \tau t) = \lambda \tau \left(\Delta \rho^m - \nabla \cdot (\rho^{m/2} g) \right) (\eta x, \tau t) \\ = \left(\frac{\tau}{\eta^2 \lambda^{m-1}} \Delta \tilde{\rho}^m - \nabla \cdot (\tilde{\rho}^{m/2} \tilde{g}) \right) (x, t), \end{aligned} \tag{6.9}$$

with initial data

$$\tilde{\rho}(x, 0) = \lambda \rho_0(\eta x). \tag{6.10}$$

We want to see how the scaling affects the balance between the parabolic and hyperbolic terms. Preserving the $L^r(\mathbb{R}^d)$-norm of the initial data (6.10) under the rescaling gives the condition $\lambda = \eta^{d/r}$.

---

[2] The equation needs to be considered on the whole space so that we can "zoom in" around the origin and everything still makes sense.

Preserving the diffusion (6.9) under the rescaling gives the condition

$$\frac{\tau}{\eta^2\lambda^{m-1}} = 1 \iff \tau = \eta^2\eta^{d/r(m-1)}.$$

In this case, it holds that

$$\begin{aligned}
\|\tilde{g}\|_{L^2(\mathbb{R}^d\times(0,\infty);\mathbb{R}^d)} &= \left(\int_0^\infty\int_{\mathbb{R}^d}\left(\frac{\tau}{\eta\lambda^{m/2-1}}g(\eta x,\tau t)\right)^2 dx\,dt\right)^{1/2}\\
&= \frac{\tau}{\eta\lambda^{m/2-1}}\left(\int_0^\infty\int_{\mathbb{R}^d}\left(g(\eta x,\tau t)\right)^2 dx\,dt\right)^{1/2}\\
&= \eta^{1+d/r(m/2)}\left(\int_0^\infty\int_{\mathbb{R}^d}\left(g(\eta x,\tau t)\right)^2 dx\,dt\right)^{1/2}\\
&= \eta^{1+d/r(m/2)}\left(\frac{1}{\eta^d\tau}\right)^{1/2}\|g\|_{L^2(\mathbb{R}^d\times(0,\infty))}\\
&= \eta^{-d/2+d/r(1/2)}\|g\|_{L^2(\mathbb{R}^d\times(0,\infty))}.
\end{aligned}$$

To ensure that the norm does not diverge as $\eta \to 0$, we obtain our condition on the integrability of the initial data

$$\frac{d}{2r} \geq d/2 \iff r = 1.$$

Hence, via a scaling argument, preserving the norm of the initial data and the diffusion under the scaling, we see that if we want the equation to remain well-posed under the rescaling, we require the initial data to be $L^1(U)$-integrable.

6.2.2 A priori estimate for skeleton equation

In this section we produce a formal estimate for the skeleton equation (6.8) under the assumption that the initial data is non-negative which will suggest that the entropy space $Ent_\Phi(U) \subset L^1(U)$ is the correct space for the initial data. The estimate follows similar arguments to the entropy estimate Proposition 4.2.4 in Section 4.2.

Let $\psi : [0,\infty) \to \mathbb{R}$ be an arbitrary function. In this section we will not assume that the boundary data is a fixed constant (but only that it satisfies Assumption 2.2.9), and so we consider the function $\Psi : U \times [0,\infty) \to \mathbb{R}$ defined by

$$\Psi(x,\xi) := \int_0^\xi \left(\psi(\xi') - h_{\psi(\Phi^{-1}(\bar{f}))}(x)\right) d\xi', \tag{6.11}$$

where for each fixed test function ψ, $h_{\psi(\Phi^{-1}(\bar{f}))}$ denotes the weak of the harmonic PDE

$$\begin{cases} -\Delta h_{\psi(\Phi^{-1}(\bar{f}))} = 0, & \text{on } U,\\ h_{\psi(\Phi^{-1}(\bar{f}))} = \psi\left(\Phi^{-1}(\bar{f})\right), & \text{on } \partial U, \end{cases} \tag{6.12}$$

which is present to ensure that $\partial_\xi \Psi$ is compactly supported in space. Suppose that ψ is chosen in such a way to ensure by Proposition 2.2.11 that $\nabla h_{\psi(\Phi^{-1}(\bar{f}))} \in L^\infty(U;\mathbb{R}^d)$.

Denoting by ρ the solution of the skeleton equation (6.8), it follows using integration by parts that the composition $\Psi(x,\rho(x,t))$ satisfies

$$\begin{aligned}\partial_t \int_U \Psi(x,\rho)\,dx &= \int_U \partial_\xi \Psi(x,\rho)\partial_t \rho\,dx\\ &= \int_U \left(\psi(\rho) - h_{\psi(\Phi^{-1}(\bar{f}))}(x)\right)\left(\Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)g + \nu(\rho))\right)\,dx\\ &= -\int_U \psi'(\rho)\Phi'(\rho)|\nabla\rho|^2\,dx + \int_U \psi'(\rho)\sigma(\rho)\nabla\rho\cdot g + \psi'(\rho)\nabla\rho\cdot\nu(\rho)\,dx\\ &+\int_U \nabla h_{\psi(\Phi^{-1}(\bar{f}))}(x)\cdot\nabla\Phi(\rho)\,dx - \int_U \sigma(\rho)\nabla h_{\psi(\Phi^{-1}(\bar{f}))}(x)\cdot g + \nabla h_{\psi(\Phi^{-1}(\bar{f}))}(x)\cdot\nu(\rho)\,dx.\end{aligned} \tag{6.13}$$

Our goal is to pick the test function ψ in such a way that we are able close the energy estimate. The terms in (6.13) are handled similarly to the energy estimates of Section 4.1.

The first term in the final equality can be moved over to the left hand side of the estimate.

For the second term, since we only have that the control is $L^2(U\times[0,T];\mathbb{R}^d)$-integrable, we use Cauchy-Schwartz and Young's inequalities to give

$$\int_U \psi'(\rho)\sigma(\rho)\nabla\rho\cdot g\,dx \leq \frac{1}{2}\|g\|_{L^2(U;\mathbb{R}^d)} + \frac{1}{2}\int_U (\psi'(\rho))^2\sigma^2(\rho)|\nabla\rho|^2\,dx.$$

The final term in the first line involving ν shows up as boundary terms in the estimate. For $i = 1,\dots,d$, define the unique vector valued function $\Psi_{\psi,\nu} := (\Psi_{i,\psi,\nu})_{i=1,\dots,d}$ element-wise by

$$\Psi_{i,\psi,\nu}(0) = 0, \quad \Psi'_{i,\psi,\nu}(\xi) = \psi'(\xi)\nu_i(\xi).$$

Then we have by the divergence theorem

$$\int_U \psi'(\rho)\nabla\rho\cdot\nu(\rho)\,dx = \int_U \nabla\cdot\Psi_{\psi,\nu}\,dx = \int_{\partial U}\Psi_{\psi,\nu}\cdot\hat{\eta}\,dS(x) \leq \|\Psi_{\psi,\nu}\|_{L^1(\partial U;\mathbb{R}^d)}.$$

Let us now deal with the boundary terms, the terms in the final line of (6.13). For the first term, using integration by parts, the formula (6.12) satisfied by $h_{\psi(\Phi^{-1}(\bar{f}))}$, Cauchy Schwartz and Young's inequalities, we have a constant $c \in (0,\infty)$ such that

$$\begin{aligned}\int_U \nabla h_{\psi(\Phi^{-1}(\bar{f}))}(x)\cdot\nabla\Phi(\rho)\,dx &= \int_{\partial U}\nabla h_{\psi(\Phi^{-1}(\bar{f}))}\cdot\hat{\eta}\bar{f}\,dS(x)\\ &\leq \frac{1}{2}\|\bar{f}\|^2_{L^2(\partial U)} + \frac{1}{2}\left\|\frac{\partial h_{\psi(\Phi^{-1}(\bar{f}))}}{\partial\hat{\eta}}\right\|^2_{L^2(\partial U)}\\ &\leq \frac{1}{2}\|\bar{f}\|^2_{L^2(\partial U)} + c\left\|\psi\left(\Phi^{-1}(\bar{f})\right)\right\|^2_{H^1(\partial U)}\end{aligned} \tag{6.14}$$

where the final inequality comes from Theorem 4.1.3. Handling the middle term in the final line of (6.13) is where we need the assumption that $\nabla h_{\psi(\Phi^{-1}(\bar{f}))} \in L^{\infty}(U;\mathbb{R}^d)$ since we only have the integrability $\sigma(\rho) \in L^2(U)$. Applying this alongside Cauchy-Schwartz and Young's inequalities, we have

$$\int_U \sigma(\rho)\nabla h_{\psi(\Phi^{-1}(\bar{f}))}(x)\cdot g\,dx \leq \frac{1}{2}\|g\|^2_{L^2(U;\mathbb{R}^d)} + \frac{\|\nabla h_{\psi(\Phi^{-1}(\bar{f}))}\|^2_{L^{\infty}(U;\mathbb{R}^d)}}{2}\int_U \sigma^2(\rho)\,dx. \tag{6.15}$$

Similarly, since we only have the integrability $\nu \in L^1(U;\mathbb{R}^d)$, for the final term of (6.13) we use the gradient bound $\nabla h_{\psi(\Phi^{-1}(\bar{f}))} \in L^{\infty}(U;\mathbb{R}^d)$ to give

$$\int_U \nabla h_{\psi(\Phi^{-1}(\bar{f}))}(x)\cdot\nu(\rho)\,dx \leq \|\nabla h_{\psi(\Phi^{-1}(\bar{f}))}\|_{L^{\infty}(U;\mathbb{R}^d)}\int_U |\nu(\rho)|\,dx. \tag{6.16}$$

Putting everything together gives the existence of a constant $c \in (0,\infty)$ such that

$$\begin{aligned}
&\partial_t \int_U \Psi(x,\rho)\,dx + \int_U \psi'(\rho)\Phi'(\rho)|\nabla\rho|^2\,dx \\
&\quad\leq \frac{1}{2}\int_U \left(\psi'(\rho)\right)^2\sigma^2(\rho)|\nabla\rho|^2\,dx + c\|\nabla h_{\psi(\Phi^{-1}(\bar{f}))}\|_{L^{\infty}(U;\mathbb{R}^d)}\int_U \left(\sigma^2(\rho)+|\nu(\rho)|\right)\,dx \\
&\qquad + \|g\|_{L^2(U;\mathbb{R}^d)} + \frac{1}{2}\|\bar{f}\|_{L^2(\partial U)} + \|\Psi_{\psi,\nu}\|_{L^1(\partial U;\mathbb{R}^d)} + c\left\|\psi\left(\Phi^{-1}(\bar{f})\right)\right\|_{H^1(\partial U)}.
\end{aligned} \tag{6.17}$$

The key challenge of this estimate is in how to bound the first term on the right hand side of (6.17). This term can be absorbed into the left hand side of the estimate only if for every $\xi \in (0,\infty)$ it holds that

$$((\psi'(\xi))^2\sigma^2(\xi) \leq \psi'(\xi)\Phi'(\xi).$$

Using Assumption (2.9) that $\sigma \leq c\Phi^{1/2}$ it follows that a sufficient condition for the above inequality to hold is that there exists a constant $c \in (0,\infty)$ such that

$$\psi'(\xi) \leq c\frac{\Phi'(\xi)}{\Phi(\xi)}. \tag{6.18}$$

By inspection, a candidate choice for the test function is

$$\psi(\xi) = \log(\Phi(\xi)).$$

To close the estimate, we just need to bound the terms involving integrals of σ^2 and ν in (6.17). We have by (2.9) and (2.10) from Assumption 2.2.1 and interpolation that there exists a constant $c \in (0,\infty)$ such that

$$\int_U \left(\sigma^2(\rho)+|\nu(\rho)|\right)\,dx \leq c\int_U (1+\Phi(\rho))\,dx.$$

The constant remains part of the estimate. To bound the integral of Φ, we combine the Gagliardo-Sobolev-Nirenberg inequality with Young's inequality to obtain for a constant $c \in (0,\infty)$

$$\int_U \Phi(\rho) \leq \epsilon \int_U |\nabla \Phi^{1/2}(\rho)| + c\left(\int_U \Phi^{1/2}(\rho)\right)^2.$$

The first term can be absorbed into the left hand side of the estimate. We observe that the final can be written as equal to the first term on the right hand side up to a boundary term using the same methods as Proposition 4.1.6, in particular using Poincaré inequality in the computation (4.7).

To make the above estimate rigorous we would follow the techniques of Proposition 4.2.4 and apply Itô's formula to a regularised version of the logarithm.

Remark 6.2.1. *Note $\psi(x) = x$ does not satisfy the inequality* (6.18)*, hence $L^2(U)$-estimates can not be obtained for the skeleton equation.*

This can be seen by the if we attempt to find an $L^2(U \times [0,T])$-estimate, we need to absorb the integral

$$\epsilon \int_0^T \int_U \sigma^2(\rho)|\nabla\rho|^2 \, dx \, ds$$

into the term on the left hand side

$$\int_0^T \int_U \Phi'(\rho)|\nabla\rho|^2 \, dx \, ds.$$

This can not be done since we do not have that $\sigma^2(\xi) \leq \Phi'(\xi)$ for every $\xi \in (0,\infty)$.

6.2.3 Well-posedness of the skeleton equation

In this section we use the entropy estimate above to outline how to prove the well-posedness of solutions to the skeleton equation. In Definition 6.2.2 we define weak solutions of the skeleton equation and in Definition 6.2.3 we define kinetic solutions of the skeleton equation. Via a tightness criterion based on regularity of $\Phi^{1/2}(\rho)$ in Lemma 6.2.4, we postulate the well-posedness and the $L^1(U)$-contraction of solutions in Corollary 6.2.5.

Let us begin by defining what it means to be a weak and kinetic solution of the skeleton equation. Similarly to Definition 2.3.6, we need to introduce the following harmonic PDE to capture the boundary condition.

$$\begin{cases} -\Delta h_{\Phi^{1/2}(\Phi^{-1}(\bar{f}))} = 0 & \text{on } U, \\ h_{\Phi^{1/2}(\Phi^{-1}(\bar{f}))} = \Phi^{1/2}\left(\Phi^{-1}(\bar{f})\right) & \text{on } \partial U. \end{cases} \tag{6.19}$$

Definition 6.2.2 (Weak solution of the skeleton equation)**.** *Suppose that the nonlinear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in Ent_\Phi(U)$ be $\mathcal{F}_0$-measurable, $g \in L^2(U \times [0,T]; \mathbb{R}^d)$ be* *an $\mathcal{F}_t$-predictable control, and let $h_{\Phi^{1/2}(\Phi^{-1}(\bar{f}))}$ denote the solution to the PDE defined in* (6.19)*.*

A weak solution of the skeleton equation (6.8) *with initial data ρ_0 and boundary data $\bar{f}$ is a continuous $L^1(U)$-valued function $\rho \in L^\infty([0,T];L^1(U))$ satisfying*

1. *Boundary condition and regularity: We have that*
$$(\Phi^{1/2}(\rho) - h_{\Phi^{1/2}(\Phi^{-1}(\bar{f}))}) \in L^2([0,T];H_0^1(U)).$$
2. *The equation: For every $t \in [0,T]$ and $\psi \in C_c^\infty(U)$ we have*
$$\int_U \rho(x,t)\psi(x)\,dx = \int_U \rho_0(x)\psi(x)\,dx - \int_0^t \int_U \Phi'(\rho)\nabla\rho \cdot \nabla\psi\,dx\,dt \\ + \int_0^t \int_U \sigma(\rho)g \cdot \nabla\psi\,dx\,dt + \int_0^t \int_U \nu(\rho) \cdot \nabla\psi\,dx\,dt.$$

Based on the derivation of kinetic solutions in Section 2.3 with no noise ($\sigma = 0$), we have the following definition of kinetic solutions for the skeleton equation (6.8).

Definition 6.2.3 (Kinetic solution of the Skeleton equation)**.** *Suppose that the nonlinear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in Ent_\Phi(U)$ be $\mathcal{F}_0$-measurable, $g \in L^2(U \times [0,T];\mathbb{R}^d)$ be an $\mathcal{F}_t$-predictable control, and let $h_{\Phi^{1/2}(\Phi^{-1}(\bar{f}))}$ denote the weak solution to the PDE defined in* (6.19)*.*

A kinetic solution of (6.8) *is a non-negative function $\rho \in L^\infty([0,T];L^1(U))$ satisfying*

1. *Integrability of flux: We have*
$$\sigma(\rho) \in L^2(U \times [0,T]) \quad \text{and} \quad \nu(\rho) \in L^1(U \times [0,T];\mathbb{R}^d).$$
2. *Boundary condition and regularity of the solution: We have that*
$$(\Phi^{1/2}(\rho) - h_{\Phi^{1/2}(\Phi^{-1}(\bar{f}))}) \in L^2([0,T];H_0^1(U)).$$
3. *The kinetic equation: Define the deterministic kinetic measure q, a non-negative measure on $U \times \mathbb{R} \times [0,T]$ satisfying that in the sense of measures,*
$$q := \delta_0(\xi - \rho)\Phi'(\xi)|\nabla\rho|^2 = \delta_0(\xi-\rho)\frac{4\Phi(\xi)}{\Phi'(\xi)}|\nabla\Phi^{1/2}(\rho)|^2. \tag{6.20}$$
The kinetic equation satisfies for every $\psi \in C_c^\infty(U \times (0,\infty))$ and for every $t \in [0,T]$, that
$$\int_{\mathbb{R}}\int_U \chi(x,\xi,t)\psi(x,\xi) = \int_{\mathbb{R}}\int_U \chi(x,\xi,0)\psi(x,\xi) - \int_0^t\int_U \Phi'(\rho)\nabla\rho \cdot (\nabla_x\psi)(x,\rho) \\ - \int_{\mathbb{R}}\int_0^t\int_U \partial_\xi\psi(x,\xi)\,dq + \int_0^t\int_U \sigma(\rho)(\nabla_x\psi)(x,\rho)\cdot g \\ + \int_0^t\int_U (\partial_\xi\psi)(x,\rho)\sigma(\rho)\nabla\rho\cdot g - \int_0^t\int_U \psi(x,\rho)\nabla\cdot\nu(\rho). \tag{6.21}$$

The existence of weak solutions for the skeleton equation (6.8) is based on the following tightness criteria.

Lemma 6.2.4 (Sufficient condition for tightness)**.** *Let* $\{\rho^\epsilon\}_{\epsilon\in(0,1)}$ *be a sequence that satisfies for every* $s \geq \frac{d+2}{2}$, *that there exists a constant* $c \in (0,\infty)$ *independent of* ϵ *such that*

$$\|\rho^\epsilon\|_{L^\infty([0,T];L^2(U))} + \|\Phi^{1/2}(\rho^\epsilon)\|_{L^2([0,T];H^1(U))} + \|\partial_t\rho^\epsilon\|_{L^1([0,T];H^{-s}(U))} \leq c.$$

Then $\{\rho^\epsilon\}_{\epsilon\in(0,1)}$ *is relatively pre-compact in* $L^1(U\times[0,T])$, *and* $\{\Phi^{1/2}(\rho^\epsilon)\}_{\epsilon\in(0,1)}$ *is relatively pre-compact in* $L^2(U\times[0,T])$.

Proof. The proof a simplified version of Proposition 4.4.6 and is a consequence of the Aubin-Lions-Simon lemma, see Corollary 5 of Simon [91]. □

We now state the key result for the well-posedness of the skeleton equation. The result is the content of upcoming work, so is stated here as a conjecture. We nonetheless give an outline of the proof.

Conjecture 6.2.5 (Well-posedness of the skeleton equation)**.** *Suppose that the non-linear functions* Φ, σ, ν *and the boundary data* $\bar{f}$ *satisfy Assumptions 2.2.1 and 2.2.9 respectively. Suppose that the non-linearity* Φ *additionally satisfies Assumptions 6 and 10 of Fehrman and Gess*[3] *[42]. Let further* $\rho_0 \in Ent_\Phi(U)$ *be* $\mathcal{F}_0$*-measurable and* $g \in L^2(U\times[0,T];\mathbb{R}^d)$ *be an* $\mathcal{F}_t$*-predictable control.*

Then there exists a unique weak solution to the skeleton equation (6.8)*. Furthermore, we have the following equivalence of kinetic solutions and weak solutions,*

$$\rho \text{ is a kinetic solution of } (6.8) \text{ in the sense of Definition 6.2.3}$$
$$\iff \rho \text{ is a weak solution of } (6.8) \text{ in the sense of Definition 6.2.2.}$$

Finally, if ρ^1, ρ^2 *are two weak solutions to* (6.8) *with initial data* $\rho_0^1, \rho_0^2 \in Ent_\Phi(U)$ *and the same control* $g \in L^2(U\times[0,T];\mathbb{R}^d)$, *then we have the contraction*

$$\sup_{t\in[0,T]} \|\rho^1(\cdot,t) - \rho_2(\cdot,t)\|_{L^1(U)} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(U)}. \tag{6.22}$$

Proof idea. The proof on the torus for the choice of coefficients $\nu = 0, \sigma = \Phi^{1/2}$ is precisely the content of Theorems 8, 14 and Proposition 20 of Fehrman and Gess [42].

Theorem 8 proves the uniqueness of kinetic solutions and the $L^1(U)$-contraction (6.22). The result is a simplified version of Theorem 3.2.2 above so we don't repeat it here.

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[3] Assumption 6 of [42] assumes that Φ' is locally 1/2-Hölder continuous, and assumes the growth condition

$$\sup_{\xi\in[0,M]} |\Phi(\xi)/\Phi'(\xi)| \leq cM.$$

Assumption 10 assumed a different growth condition on Φ depending on whether $\Phi^{1/2}$ is convex or concave. The assumptions are satisfied by $\Phi(\xi) = \xi^m$ for every $m \in (1,\infty)$.

Proposition 20 subsequently proves existence of weak solutions, following an argument analogous to Proposition 4.4.1 above. We approximate the skeleton equation (6.8) by the regularised equation

$$\partial_t\rho = \Delta\left[(\Phi^{1/2,\eta_1}(\rho))^2\right] + \eta_2\Delta\rho - \nabla\cdot(\sigma^{\eta_3}(\rho)g - \nu^{\eta_4}(\rho)), \tag{6.23}$$

with $\eta_1, \eta_2, \eta_3, \eta_4 \in (0,1)$, where $\Phi^{1/2,\eta_1}, \sigma^{\eta_3}, \nu^{\eta_4}$ are smooth approximations of $\Phi^{1/2}, \sigma$ and ν respectively. Proving the existence and $L^1(U)$-contraction of solutions to this regularised equation (6.23) is quite straightforward. We can also prove entropy estimates similar to that of Section 6.2.2, where the estimates are uniform in the regularisation coefficients. The existence of weak solutions of the skeleton equation (6.8) then follows from applying Lemma 6.2.4.

The existence and uniqueness are tied together by the equivalence of weak and kinetic solutions, which is the content of Theorem 14 of [42]. The fact that kinetic solutions are weak solutions is straightforward, and can be formally seen by choosing the test function in (6.21) to be independent of the ξ variable. Proving that weak solutions are stochastic kinetic solutions is more tricky. The method follows the derivation of the kinetic equation, see for example Section 2.3, with the additional technical difficulty that we only have regularity for $\Phi^{1/2}(\rho)$ rather than having regularity of the solution itself. $\square$

6.3 Existence of solution maps

Our goal is now to unpack and prove the various conditions of Assumption 6.1.3 and Theorem 6.1.4, which will allow us to conclude the large deviations principle. We begin in this section with Proposition 6.3.1 which proves that there exists a solution map for the skeleton equation. In Remark 6.3.2 the first condition in Assumption 6.1.3 is verified. We conclude by proving the existence of a solution map for the controlled SPDE in Proposition 6.3.5

Let us begin with a result on the existence of a solution map for the skeleton equation (6.8).

Proposition 6.3.1 (Existence of solution map for skeleton equation)**.** *Suppose that the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in Ent_\Phi(U)$ be $\mathcal{F}_0$-measurable and $g \in L^2(U \times [0,T]; \mathbb{R}^d)$ be an $\mathcal{F}_t$-predictable control.*

Assuming that Conjecture 6.2.5 holds, there exists a measurable map

$$\mathcal{G}^{\bar{f},0} : Ent_\Phi(U) \times C([0,T]; (\mathbb{R}^d)^\infty) \to L^1(U \times [0,T])$$

such that for $g_k(s) := \langle g(\cdot, s), f_k\rangle_{L^2(U)}$,

$$\rho(\bar{f}, \rho_0) = \mathcal{G}^{\bar{f},0}\left(\rho_0, \left(\int_0^{\cdot} g_k(s)\,ds\right)_{k\in\mathbb{N}}\right),$$

where $\rho(\bar{f}, \rho_0)$ denotes the unique weak solution of (6.8) with initial data ρ_0 and boundary data $\Phi(\rho)|_{\partial U} = \bar{f}$ in the sense of Definition 6.2.2 above.

Proof. The proof is a simplified version of the proof for Proposition 6.3.5 below, which proves the existence of a solution map for the controlled SPDE (6.25). The proof relies on two main ingredients. The first is the well-posedness of the equation and the second is the $L^1(U)$-contraction, both conditions were assumed above in Conjecture 6.2.5. □

Remark 6.3.2 (Point 1 of Assumption 6.1.3)**.** *The first point of Assumption 6.1.3 requires us to verify that for every $R < \infty$ and compact subset $K \subset Ent_\Phi(U)$, the set*

$$\Gamma_{R,K} := \left\{ \mathcal{G}^{\bar{f},0}\left(\rho_0, \int_0^{\cdot} g(s)\, ds\right) : g \in L^2_R(U \times [0,T]), \rho_0 \in K \right\}$$

is a compact subset of $L^1(U \times [0,T])$. However, this is a simple consequence of the entropy estimate for the skeleton equation given in Section 6.2.2 combined with the tightness criteria of Lemma 6.2.4.

Let us turn our attention to proving the existence of a solution map

$$\mathcal{G}^{\bar{f},\epsilon}\left(\rho_0^\epsilon, \sqrt{\epsilon}B + \left(\int_0^{\cdot} g_k^\epsilon(s)\, ds\right)_{k\in\mathbb{N}}\right) \tag{6.24}$$

appearing in point 2 of Assumption 6.1.3. That is, for fixed $\epsilon \in (0,1)$ and boundary data $\bar{f} \in H^1(U)$ satisfying Assumption 2.2.9, we want to prove the existence of a solution map for the controlled SPDE

$$\partial_t \rho^{\epsilon,g^\epsilon} = \Delta\Phi(\rho^{\epsilon,g^\epsilon}) - \sqrt{\epsilon}\nabla\cdot(\sigma(\rho^{\epsilon,g^\epsilon}) \circ \dot{\xi}^{K(\epsilon)}) - \nabla\cdot(\sigma(\rho^{\epsilon,g^\epsilon})P_{K(\epsilon)}g^\epsilon) - \nabla\cdot\nu(\rho^{\epsilon,g^\epsilon}), \tag{6.25}$$

with initial data ρ_0^ϵ, boundary data $\Phi(\rho^{\epsilon,g^\epsilon})|_{\partial U} = \bar{f}$, and where $P_{K(\epsilon)}g^\epsilon$ denotes the projection of the control $g^\epsilon \in L^2(U \times [0,T]; \mathbb{R}^d)$ onto the span of $\{f_k\}_{k=1}^{K(\epsilon)}$.

Remark 6.3.3 (Notation in equation (6.25))**.** *Initially it may appear as if we are abusing notation and using ϵ to denote both the scaling in front of the noise term in* (6.25) *and also to denote the approximating sequences $\{g^\epsilon\}_{\epsilon\in(0,1)}, \{\rho_0^\epsilon\}_{\epsilon\in(0,1)}$. Put differently, ϵ appears in the solution map* (6.24) *both when writing the approximating solution $\mathcal{G}^\epsilon$ which corresponds to the scaling in front of the noise term and the arguments inside the function corresponding to the approximating sequence of coefficients. This is the convention in Budhiraja, Dupuis, and Maroulas [11] and there is no issue with choosing the same scaling in ϵ for both objects.*

We will need the following result, see Proposition 27 of Fehrman and Gess [42] or Proposition 7.12 of Fehrman and Gess [38] for proof.

Proposition 6.3.4 (Separability of $Ent_\Phi(U)$)**.** *Let $\Phi \in C([0,\infty)) \cap C^1_{loc}((0,\infty))$ be non-negative and satisfy Assumption 2.2.1. The space $Ent_\Phi(U)$ equipped with the $L^1(U)$-topology is a complete, separable metric space.*

The existence of a solution map for the controlled SPDE (6.25) below is similar to Theorem 23 and Proposition 24 of Fehrman and Gess [42]. In the proof below we fix the boundary data $\bar{f} \in H^1(\partial U)$ and prove that the solution map exists for fixed (but arbitrary) boundary data.

Proposition 6.3.5 (Existence of solution map for controlled SPDE)**.** *Suppose that the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in Ent_\Phi(U)$ be $\mathcal{F}_0$-measurable and $g \in L^2(U \times [0,T]; \mathbb{R}^d)$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable control.*

For every $\epsilon \in (0,1)$ and $g^\epsilon_k(s) := \langle g^\epsilon(\cdot, s), f_k\rangle_{L^2(U)}$, there exists a measurable map

$$\mathcal{G}^{\bar{f},\epsilon} : Ent_\Phi(U) \times C([0,T]; (\mathbb{R}^d)^\infty) \to L^1(U \times [0,T])$$

such that

$$\rho^{\epsilon,g^\epsilon}(\bar{f}, \rho_0) = \mathcal{G}^{\bar{f},\epsilon}\left(\rho_0, \sqrt{\epsilon}B + \left(\int_0^{\cdot} g^\epsilon_k(s)\, ds\right)_{k\in\mathbb{N}}\right),$$

where $\rho^{\epsilon,g^\epsilon}(\bar{f}, \rho_0)$ is the unique stochastic kinetic solution of (6.25) *with initial data ρ_0, boundary data $\Phi(\rho^{\epsilon,g^\epsilon})|_{\partial U} = \bar{f}$ and control g^ϵ.*

Proof. The proof consists of two steps. Firstly we will we will show the existence of a map $\mathcal{G}^{\bar{f},\epsilon}$ such that for the uncontrolled system (6.1), we have

$$\rho^\epsilon(\bar{f}, \rho_0) = \mathcal{G}^{\bar{f},\epsilon}\left(\rho_0, \sqrt{\epsilon}B\right).$$

The second step will be to extend the result to the controlled SPDE $\rho^{\epsilon,g^\epsilon}$ (6.25) via Girsanov theorem.

For every fixed $\epsilon \in (0,1)$ and $K \in \mathbb{N}$, the well-posedness of stochastic kinetic solutions of equation (6.1) is a consequence Chapters 3 and 4. Due to the well-posedness, we know that for fixed boundary data $\bar{f} \in H^1(\partial U)$ satisfying Assumption 2.2.9 and initial condition $\rho_0 \in Ent_\Phi(U)$, there exists a measurable function $\mathcal{G}^\epsilon_{\bar{f},\rho_0} : C([0,T]; (\mathbb{R}^d)^\infty) \to L^1(U \times [0,T])$ that takes the noise to the solution of (6.1). That is

$$\mathcal{G}^\epsilon_{\bar{f},\rho_0}(\sqrt{\epsilon}B) = \rho^\epsilon(\bar{f}, \rho_0).$$

Since it follows from Proposition 6.3.4 above that $Ent_\Phi(U)$ is separable with the $L^1(U)$-topology, by looking at a countable dense subset $\{\rho_n\}_{n\in\mathbb{N}}$ of $Ent_\Phi(U)$, the uniqueness result of Theorem 3.2.2 implies that for every $n, m \in \mathbb{N}$, we have

$$\sup_{t\in[0,T]} \|\mathcal{G}^\epsilon_{\bar{f},\rho_n}\left(\sqrt{\epsilon}B\right) - \mathcal{G}^\epsilon_{\bar{f},\rho_m}\left(\sqrt{\epsilon}B\right)\|_{L^1(U)} \leq \|\rho_n - \rho_m\|_{L^1(U)}. \tag{6.26}$$

As a consequence of the density of $\{\rho_n\}_{n\in\mathbb{N}}$ with respect to the $L^1(U)$-norm, it follows that there exists a measurable function $\mathcal{G}^\epsilon_{\bar{f},\sqrt{\epsilon}B} : Ent_\Phi(U) \to L^1(U \times [0,T])$ that maps for every $n \in \mathbb{N}$ the initial condition ρ_n to the solution of (6.1),

$$\mathcal{G}^\epsilon_{\bar{f},\sqrt{\epsilon}B}(\rho_n) = \rho^\epsilon(\bar{f}, \rho_n) = \mathcal{G}^\epsilon_{\bar{f},\rho_n}(\sqrt{\epsilon}B).$$

The fact that $\{\rho_n\}_{n\in\mathbb{N}}$ is dense then implies that for arbitrary $\rho_0 \in Ent_\Phi(U)$, on a subset of full probability depending on ρ_0,

$$\mathcal{G}^\epsilon_{\bar{f},\sqrt{\epsilon}B}(\rho_0) = \rho^\epsilon(\bar{f}, \rho_0) = \mathcal{G}^\epsilon_{\bar{f},\rho_0}(\sqrt{\epsilon}B).$$

Finally, the desired solution map $\mathcal{G}^{\bar{f},\epsilon}$ is defined by

$$\mathcal{G}^{\bar{f},\epsilon}\left(\rho_0, \sqrt{\epsilon}B\right) := \mathcal{G}^{\epsilon}_{\bar{f},\sqrt{\epsilon}B}\left(\rho_0\right).$$

The measurability of $\mathcal{G}^{\bar{f},\epsilon}$ follows from the measurability of the map $\mathcal{G}^{\bar{f},\epsilon}\left(\rho_0, \cdot\right)$ viewed as a function of the noise, and the strong continuity of the map $\mathcal{G}^{\bar{f},\epsilon}\left(\cdot, \sqrt{\epsilon}B\right)$ viewed as a function of the initial datum.

Let us now move onto the second step, which we note follows from Theorem 10 of Budhiraja, Dupuis and Maroulas [11]. Fix $\epsilon \in (0,1)$ and measurable control $g^\epsilon \in L^2(U \times [0,T]; \mathbb{R}^d)$. The measure

$$d\gamma^{\epsilon,g} := \exp\left\{-\frac{1}{\sqrt{\epsilon}}\sum_{k=1}^{K}\int_0^T\int_U g^\epsilon(x,s)\,dx\,dB_s^k - \frac{1}{2\epsilon}\sum_{k=1}^{K}\int_0^T\int_U (g^\epsilon)^2(x,s)\,dx\,ds\right\}d\mathbb{P}$$

is a probability measure on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined in Definition 5.1.1. The measure $\gamma^{\epsilon,g}$ is absolutely continuous with respect to $\mathbb{P}$, and by Girsanov's theorem, see Theorem 10.14 of Da Prato and Zabczyk [22], the process

$$\{\tilde{B}^k_\cdot\}_{k\in\mathbb{N}} := \left\{B^k_\cdot + \epsilon^{-1/2}\int_0^\cdot g^\epsilon_k\right\}_{k\in\mathbb{N}}$$

on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \gamma^{\epsilon,g})$ is an infinite sequence of independent d-dimensional Brownian motions. Hence, by the first step and uniqueness of solutions given by Theorem 3.2.2, we have that the quantity of interest

$$\mathcal{G}^{\bar{f},\epsilon}\left(\rho_0, \sqrt{\epsilon}B + \left(\int_0^\cdot g^\epsilon_k(s)\,ds\right)_{k\in\mathbb{N}}\right)$$

is the unique stochastic kinetic solution to (6.1) with B replaced by $\tilde{B}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \gamma^{\epsilon,g})$. But (6.1) with $\tilde{B}$ is precisely the same as $\rho^{\epsilon,g^\epsilon}$ as in (6.25). The result then follows on the original probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ using the fact that the measures $\mathbb{P}$ and $\gamma^{\epsilon,g}$ are absolutely continuous. □

6.4 Lack of uniformity of solution map with respect to boundary data

It is natural to wonder whether it is possible to extend the uniformity of the Laplace principle to include uniformly bounded subsets of boundary data as well. That is, whether we can take an infimum over bounded subsets of $H^1(\partial U)$ in equation (6.4). This section is dedicated to illustrating why this is not possible.

Rather than looking at the solution map $\mathcal{G}^{\bar{f},\epsilon}$ as in (6.5), we would need the existence of a solution map that also takes the boundary data as input,

$$\mathcal{G}^\epsilon : H^1(\partial U) \times Ent_\Phi(U) \times C([0,T]; (\mathbb{R}^d)^\infty) \to L^1(U \times [0,T]).$$

Recall that a key step in the proof of Proposition 6.3.5 was the $L^1(U)$-contraction

$$\sup_{t\in[0,T]} \|\rho^{\epsilon,1}(\cdot,t) - \rho^{\epsilon,2}(\cdot,t)\|_{L^1(U)} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(U)}$$

for two stochastic kinetic solutions $\rho^{\epsilon,1}, \rho^{\epsilon,2}$ of (6.1) with the same boundary data $\bar{f}$ and different initial conditions ρ_0^1, ρ_0^2. To have uniformity with respect to the boundary data we would need to prove that if the two stochastic kinetic solutions had different boundary data, say $\bar{f}^1, \bar{f}^2 \in H^1(\partial U)$ respectively, both satisfying Assumption 2.2.9, then for some constant $c \in (0,\infty)$ we have an inequality of the form

$$\sup_{t\in[0,T]} \|\rho^{\epsilon,1}(\cdot,t) - \rho^{\epsilon,2}(\cdot,t)\|_{L^1(U)} \leq \|\rho_0^1 - \rho_0^2\|_{L^1(U)} + cT\|\bar{f}^1 - \bar{f}^2\|_{H^1(\partial U)}. \tag{6.27}$$

In this section we briefly discuss how the uniqueness proof of Theorem 3.2.2, would be amended and subsequently conclude that it is not possible to obtain an inequality of the form (6.27).

Recall that a key observation in the uniqueness proof was that the $L^1(U)$-difference of solutions can be written in terms of the corresponding kinetic functions χ^1, χ^2 of the two solutions. That is, for every $t \in [0,T]$ we have the identity

$$\begin{aligned}&\int_U |\rho^{\epsilon,1}(x,s) - \rho^{\epsilon,2}(x,s)|\,dx\Big|_{s=0}^{t}\\ &= \int_{\mathbb{R}}\int_U \chi^1(\xi,\rho^{\epsilon,1}(x,s)) + \chi^2(\xi,\rho^{\epsilon,1}(x,s)) - 2\chi^1(\xi,\rho^{\epsilon,1}(x,s))\chi^2(\xi,\rho^{\epsilon,1}(x,s))\Big|_{s=0}^{t}.\end{aligned}$$

The terms on the right hand side were handled using the kinetic equation (2.40). Since test functions in the kinetic equation need to be compactly supported in the spatial and velocity variables, for the first two terms above we need to test against (regularised versions) of relevant cutoff functions. For the final term we need to include the kinetic function as part of the test function, which required us to smooth the kinetic function via convolution.

The presence of the cutoff function in space ι_γ with parameter $\gamma \in (0,1)$, see equation (2.43) of Definition 2.4.2, enabled us to integrate by parts and perform analysis without the presence of additional boundary dependent terms. It is precisely when we take the limit in the cutoff parameter $\gamma \to 0$ that boundary terms arise. Specifically, for the cutoff term (3.31), we notice from (3.33) that before taking limits in the cutoff functions, we have the term

$$\gamma^{-1}\left(\int_0^t\int_{\partial U} |\Phi_{M,\beta}(\rho^{\epsilon,2}) - \Phi_{M,\beta}(\rho^{\epsilon,1})| - \int_0^t\int_{\partial U_\gamma} |\Phi_{M,\beta}(\rho^{\epsilon,2}) - \Phi_{M,\beta}(\rho^{\epsilon,1})|\right), \tag{6.28}$$

where $\Phi_{M,\beta}$ was defined in equation (3.34).

The technical lemma below illustrates why one can not expect a bound such as (6.27) to hold for a difference such as (6.28). The main point is that the $H^1(\partial U)$-norm as in Definition 2.2.7 looks at tangential derivatives of f along the boundary, whereas the lemma below illustrates that a bound can only be obtained in terms of normal derivatives at the boundary.

Lemma 6.4.1. *Let $U \subset \mathbb{R}^2$ be a bounded C^2-regular domain, and let $f \in C^2(\bar{U})$. Then we have the estimate*

$$\lim_{\gamma \to 0} \left(\gamma^{-1} \left(\int_{\partial U} f(x)\, dS(x) - \int_{\partial U_\gamma} f(x)\, dS_\gamma(x) \right) \right) \\ = \int_{\partial U} H(x) f(x)\, dS(x) + \int_{\partial U} \nabla f(x) \cdot \hat{\eta}(x)\, dS(x), \quad (6.29)$$

where $H(x)$ denotes the mean curvature of the boundary ∂U at the point x, recall that $\hat{\eta}(x)$ denotes the outward pointing normal vector field at point x, and finally we used the notation that dS and dS_γ denote the surface measures on $\partial U, \partial U_\gamma$ respectively.

Proof. We first observe that for γ sufficiently small, one can write points on ∂U_γ as a smooth perturbation of points on ∂U by moving along the inward pointing normal vector. For every $x \in \partial U_\gamma$, let $x^* = x^*(x) := \Pi_{\partial U}(x)$ denote the closest point on the boundary to x. Since U is a C^2-domain, we know that for $\gamma \in (0, \gamma_U)$ where γ_U is as in (2.41), such a point exists and is unique. If $\hat{\eta}(x^*)$ denotes the outward pointing unit normal vector field at point $x^* \in \partial U$, then

$$x = x^* - \gamma \hat{\eta}(x^*) + O(\gamma^2),$$

where the final term denotes dependence on terms depending on γ of order γ^2 or higher. Similarly, we have for the surface measure dS_γ on ∂U_γ the first order expansion[4]

$$dS_\gamma(x) = \big(1 - \gamma H(x^*) + O(\gamma^2)\big)\, dS(x^*).$$

Putting these two facts together gives

$$\int_{\partial U_\gamma} f(x) dS_\gamma(x) = \int_{\partial U} f\left(x^* - \gamma \hat{\eta}(x^*) + O(\gamma^2)\right) \big(1 - \gamma H(x^*) + O(\gamma^2)\big)\, dS(x^*). \quad (6.30)$$

Since $f \in C^2(\bar{U})$, the first order Taylor expansion

$$f\left(x^* - \gamma \hat{\eta}(x^*) + O(\gamma^2)\right) = f(x^*) - \gamma \nabla f(x^*) \cdot \hat{\eta}(x^*) + O(\gamma^2)$$

implies that we can write (6.30) as

$$\begin{aligned} &\int_{\partial U_\gamma} f(x) dS_\gamma(x) \\ &\quad = \int_{\partial U} \big(f(x^*) - \gamma \nabla f(x^*) \cdot \hat{\eta}(x^*) + O(\gamma^2)\big) \big(1 - \gamma H(x^*) + O(\gamma^2)\big)\, dS(x^*) \\ &\qquad = \int_{\partial U} f(x^*) - \gamma \int_{\partial U} \nabla f(x^*) \cdot \hat{\eta}(x^*) - \gamma \int_{\partial U} f(x^*) H(x^*) + O(\gamma^2), \end{aligned}$$

---

[4] The formula is derived form the first order expansion of the Jacobian determinant associated with the normal flow at the boundary.

where we observed in the final line that several of the terms in the product were absorbed into $O(\gamma^2)$. Finally, the desired difference on the left hand side of (6.29) is

$$\gamma^{-1}\left(\int_{\partial U} f(x^*) - \int_{\partial U_\gamma} f(x)\right) = \int_{\partial U} \nabla f(x^*)\cdot\hat{\eta}(x^*) + \int_{\partial U} f(x^*)H(x^*) + O(\gamma),$$

which proves the statement after taking the limit $\gamma \to 0$ on both sides. $\square$

The difficulty for the difference (6.28) is that stochastic kinetic solutions $\rho^{\epsilon,1}, \rho^{\epsilon,1}$ of the Dirichlet problem (6.1) do not have sufficient regularity to evaluate the normal derivative

$$\int_{\partial U} \frac{\partial}{\partial\hat{\eta}}|\Phi_{M,\beta}(\rho^{\epsilon,2}) - \Phi_{M,\beta}(\rho^{\epsilon,1})|\, dS(x),$$

let alone control it by the $H^1(\partial U)$-norm of the difference of the corresponding boundary datum. We recall that in general, normal derivatives can only be controlled by the tangential derivative whenever a "Dirichlet-to-Neumann map" exists. Such a map only is only known for simple evolution equations such as harmonic PDEs, see Theorem 4.1.3 above. Hence, even if the normal derivative could be evaluated, there is no hope to obtain an estimate of the form (6.27).

6.5 Energy estimates for the regularised controlled SPDE

In order to prove the weak convergence of solutions required in point 2 of Assumption 6.1.3, we have to first establish tightness of the laws of the controlled SPDE $\rho^{\epsilon,g}$ (6.25). In this section we will aim to adapt the estimates of Sections 4.1 and 4.2, which requires us to add a regularisation into the controlled SPDE. To do this we will need to handle the additional contribution from the control g. To just emphasise the impact of these terms, in Assumption 6.5.1 we assume that the boundary data is constant. Proposition 6.5.2 then adapts the $L^2(U\times[0,T])$-estimate, Proposition 6.5.3 adapts the higher order spatial regularity and Proposition 6.5.4 adapts the higher order time regularity for solutions cutoff from zero. We finish with an entropy estimate in Proposition 6.5.5 which will be useful in the sequel. The estimates allow us to prove the tightness of the laws of the regularised equations in Corollary 6.5.7.

In order to prove relevant energy estimates that allow us to prove tightness, for $\epsilon \in (0,1)$ and $n \in \mathbb{N}$, for $\{\sigma_n\}_{n\in\mathbb{N}}$ as in Lemma 4.4.3, we need to introduce the regularised controlled SPDE,

$$\begin{aligned}\partial_t\rho^{n,\epsilon,g} = \Delta\Phi(\rho^{n,\epsilon,g}) + \frac{1}{n}\Delta\rho^{n,\epsilon,g} - \sqrt{\epsilon}\nabla\cdot(\sigma_n(\rho^{n,\epsilon,g})\circ\dot{\xi}^{K(\epsilon)})\\ - \nabla\cdot(\sigma_n(\rho^{n,\epsilon,g})P_{K(\epsilon)}g) - \nabla\cdot\nu(\rho^{n,\epsilon,g}),\end{aligned} \quad (6.31)$$

with initial data $\rho^{n,\epsilon,g}(\cdot,0) = \rho_0$ and boundary data $\Phi(\rho^{n,\epsilon,g})|_{\partial U} = \bar{f}$.

We recall that the regularisation of the square root enables us to consider weak solutions, and the regularisation of adding the Laplacian provides sufficient regularity of solutions to enable us to apply Itô's formula. The definition of weak solutions for (6.31) is similar to the definition for the uncontrolled equation, see Definitions 4.1.5 and 5.1.10 above. With the regularisations as in (6.31) the existence of weak solutions follows the same argument as the uncontrolled case, see Proposition 4.4.1 above.

In Chapters 3 and 4 the infinite dimensional noise is correlated in space, and we needed to assume we have the pointwise boundedness for F_i for $i = 1, 2, 3$. Due to the $L^2(U)$-orthogonality of the coefficients in $\xi^{K(\epsilon)}$, this is no longer true, but the estimates still hold under the scaling regime of Remark 5.3.4 due to the fact that in equation (5.10), the noise coefficients are multiplied by a factor of ϵ. Owing to Remark 5.3.4, picking $\epsilon \in (0, 1)$ sufficiently small so that

$$\sqrt{\epsilon}\left(\|F_1^{K(\epsilon)}\|_{L^\infty(U)} + \|F_2^{K(\epsilon)}\|_{L^\infty(U)} + \|F_3^{K(\epsilon)}\|_{L^\infty(U)} + \|\nabla\cdot F_2^{K(\epsilon)}\|_{L^\infty(U)}\right) < 1 \quad (6.32)$$

ensures that the bounds from Section 4.1 can be directly used.

The new part of estimates we want to capture is the dependence on the control g. Hence, for simplicity of presentation, we make the assumption below which just enables us to ignore the boundary dependency of the estimates in Section 4.1.

Assumption 6.5.1 (Constant boundary data)**.** *Assume that the non-negative boundary data of the regularised controlled SPDE $\rho^{n,\epsilon,g}$ as in equation* (6.31) *is a constant,*

$$\Phi(\rho^{n,\epsilon,g})|_{\partial U} = M \geq 0.$$

We emphasise that the above assumption is not necessary. We are able to consider the same boundary data as Chapter 4, namely all non-negative constant data including zero and all smooth functions bounded away from zero.

Proposition 6.5.2 ($L^2(U \times [0, T])$-estimate for the regularised controlled equation)**.** *Suppose that the non-linear functions Φ, σ, ν, the spatial components of the noise ξ^K and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1, 5.1.2 and 6.5.1 respectively. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable and $g \in L^2(U \times [0, T]; \mathbb{R}^d)$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable control. Suppose that $\epsilon \in (0, 1)$ is sufficiently small so that the inequality* (6.32) *is satisfied.*

For fixed regularisation constant $n \in \mathbb{N}$, let $\rho^{n,\epsilon,g}$ denote weak solutions of the regularised controlled SPDE (6.31)*. Then there exists a constant $c \in (0, \infty)$ independent of n, ϵ such that*

$$\frac{1}{2}\sup_{t\in[0,T]}\mathbb{E}\left[\int_U (\rho^{n,\epsilon,g} - \Phi^{-1}(M))^2\right] + \mathbb{E}\left[\int_0^T\int_U |\nabla\Theta_{\Phi,2}(\rho^{n,\epsilon,g})|^2\right] + \frac{1}{n}\mathbb{E}\left[\int_0^T\int_U |\nabla\rho^{n,\epsilon,g}|^2\right]$$
$$\leq \frac{1}{2}\|\rho_0 - \Phi^{-1}(M)\|_{L^2(U)}^2 + cT + c\int_0^T\int_U |P_{K(\epsilon)}g|^2, \quad (6.33)$$

where $\Theta_{\Phi,2}$ is defined in point 3 of Assumption 2.2.1.

Proof. The only novelty compared to Proposition 4.1.8 is the presence of the control term in (6.31). This leads to the additional term

$$\int_0^t\int_U \sigma_n(\rho^{n,\epsilon,g})\nabla\rho^{n,\epsilon,g}\cdot P_{K(\epsilon)}g$$

on the right hand side of the estimate. Using from Assumption 2.2.1 that $\sigma \leq c\Phi^{1/2}$, Cauchy-Schwarz and Young's inequalities, for every $\delta \in (0,1)$ we have the bound

$$\begin{aligned}\int_0^t \int_U \sigma_n(\rho^{n,\epsilon,g}) \nabla \rho^{n,\epsilon,g} \cdot g &\leq c\delta \int_0^t \int_U \Phi(\rho^{n,\epsilon,g}) |\nabla \rho^{n,\epsilon,g}|^2 + \delta^{-1} \int_0^t \int_U |P_{K(\epsilon)} g|^2 \\ &= c\delta \int_0^t \int_U |\nabla \Theta_{\Phi,2}(\rho^{n,\epsilon,g})|^2 + \delta^{-1} \int_0^t \int_U |P_{K(\epsilon)} g|^2. \quad (6.34)\end{aligned}$$

Picking $\delta \in (0,1)$ sufficiently small so that the first term can be absorbed onto the left hand side of the estimate completes the proof. □

We recall that the presence of the $L^2(U)$-norm of the initial data in the above estimate (6.33) will necessitate the higher order regularity of the initial data for the subsequent estimates.

The proof of tightness for laws of the regularised controlled SPDE (6.31) is based on Proposition 4.4.6 above. Our goal will therefore be to prove the two higher order space and time regularity estimates for the regularised controlled SPDE. We begin with the higher order spatial regularity estimate, which we recall is split into two cases considering whether we have fast diffusion type equations $\Phi(\rho) = \rho^m$ for $m \in (0,1)$ or porus medium type equations $m > 1$.

Proposition 6.5.3 (Higher order spatial regularity for the controlled SPDE)**.** *Suppose that the non-linear functions Φ, σ, ν, the spatial components of the noise ξ^K and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1, 5.1.2 and 6.5.1 respectively. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable and $g \in L^2(U \times [0,T]; \mathbb{R}^d)$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable control. Suppose that $\epsilon \in (0,1)$ is sufficiently small so that the inequality* (6.32) *is satisfied.*

For fixed regularisation constant $n \in \mathbb{N}$, let $\rho^{n,\epsilon,g}$ denote weak solutions of the regularised controlled SPDE (6.31)*.*

1. *If Φ satisfies equation* (2.6) *in point 3 of Assumption 2.2.1, then there exists a constant $c \in (0,\infty)$ independent of n, ϵ such that*

$$\mathbb{E}\|\rho^{n,\epsilon,g}\|_{L^1([0,T];W^{1,1}(U))} \leq c\left(T + \|\rho_0\|^2_{L^2(U)} + \|P_{K(\epsilon)} g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)}\right).$$

2. *If Φ satisfies equation* (2.7) *in point 3 of Assumption 2.2.1, then for every $\gamma \in (0,1)$ there exists a constant $c \in (0,\infty)$ independent of n, ϵ such that*

$$\mathbb{E}\|\rho^{n,\epsilon,g}\|_{L^1([0,T];W^{\gamma,1}(U))} \leq c\left(T + \|\rho_0\|^2_{L^2(U)} + \|P_{K(\epsilon)} g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)}\right).$$

Proof. For the regularised Dean–Kawasaki equation without a control term $\rho^{n,\epsilon}$ as in (5.10), the bound for the $L^1(U \times [0,T])$-norm of the solutions follows from the technical Proposition 4.1.6, and the bound for the higher order spatial regularity is precisely contained in Lemma 4.1.11. The estimate for the controlled equation $\rho^{n,\epsilon,g}$ follows from these estimates combined with the $L^2(U \times [0,T])$-estimate of Proposition 6.5.2. □

We will now state the additional time regularity result.

Proposition 6.5.4 (Higher order time regularity for the regularised controlled SPDE (6.31))**.** *Suppose that the non-linear functions Φ, σ, ν, the spatial components of the noise ξ^K and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1, 5.1.2 and 6.5.1 respectively. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable and $g \in L^2(U \times [0,T]; \mathbb{R}^d)$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable control. Suppose that $\epsilon \in (0,1)$ is sufficiently small so that the inequality* (6.32) *is satisfied.*

For fixed regularisation constant $n \in \mathbb{N}$, let $\rho^{n,\epsilon,g}$ denote weak solutions of the regularised controlled SPDE (6.31)*.*

For every $\beta \in (0,1)$, let Ψ_β be as defined in Definition 4.1.13 above. For every $\beta \in (0,1)$, $\delta \in (0,1/2)$ and $s > \frac{d+1}{2}$ there exists a constant $c \in (0,\infty)$ independent of n and ϵ such that

$$\mathbb{E}\|\Psi_\beta(\rho^{n,\epsilon,g})\|_{W^{\delta,1}([0,T];H^{-s}(U))} \leq c\left(T + \|\rho_0\|^2_{L^2(U)} + \|P_{K(\epsilon)}g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)}\right).$$

Proof. See Proposition 4.1.14 for a proof for the uncontrolled equation. The presence of the additional control term here does not cause additional difficulty, and we just need to handle the extra finite variation term

$$\int_0^t \Psi'_\beta(\rho^{n,\epsilon,g})\nabla\cdot(\sigma_n(\rho^{n,\epsilon,g})P_{K(\epsilon)}g)$$
$$= \int_0^t \nabla\cdot(\Psi'_\beta(\rho^{n,\epsilon,g})\sigma_n(\rho^{n,\epsilon,g})P_{K(\epsilon)}g) - \int_0^t \Psi''_\beta(\rho^{n,\epsilon,g})\sigma_n(\rho^{n,\epsilon,g})\nabla\rho^{n,\epsilon,g}\cdot P_{K(\epsilon)}g.$$

The $W^{1,1}([0,T];H^{-s}(U))$-norm of the above terms can be bounded exactly in the same way as the other finite variations terms of Proposition 4.1.14, see equations (4.31)-(4.32), combining it with the estimate (6.34), which results in the additional factor of the $L^2(U\times[0,T])$-norm of $P_{K(\epsilon)}(g)$ on the right hand side of the estimate. □

The final estimate we will show is the entropy estimate. The estimate is used to control gradient terms when we show convergence of the controlled SPDE (6.25) to the skeleton equation (6.8) in Theorem 6.6.1 below.

Proposition 6.5.5 (Entropy estimate)**.** *Suppose that the non-linear functions Φ, σ, ν, the spatial components of the noise ξ^K and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1, 5.1.2 and 6.5.1 respectively. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable and $g \in L^2(U \times [0,T]; \mathbb{R}^d)$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable control. Suppose that $\epsilon \in (0,1)$ is sufficiently small so that the inequality* (6.32) *is satisfied.*

For fixed regularisation constant $n \in \mathbb{N}$, let $\rho^{n,\epsilon,g}$ denote weak solutions of the regularised controlled SPDE (6.31)*, and let $\Psi_{\Phi,0} : U \times [0,\infty) \to \mathbb{R}$ be defined as in* (4.34)*. For every $t \in [0,T]$ there exists a constant $c \in (0,\infty)$ independent of n and ϵ such that*

$$\mathbb{E}\left[\int_U \Psi_{\Phi,0}(x,\rho^{n,\epsilon,g})\,dx\right] + 4\mathbb{E}\left[\int_0^t\int_U |\nabla\Phi^{1/2}(\rho^{n,\epsilon,g})|^2\right] + \frac{1}{n}\mathbb{E}\left[\int_0^t\int_U \frac{\Phi'(\rho)}{\Phi(\rho)}|\nabla\rho^{n,\epsilon,g}|^2\right]$$
$$\leq \mathbb{E}\left[\int_U \Psi_{\Phi,0}(x,\rho_0(x))\,dx\right] + c\left(t + \|\rho_0\|^2_{L^2(U)} + \|P_{K(\epsilon)}g\|^2_{L^2(U\times[0,t];\mathbb{R}^d)}\right).$$

Proof. The estimate is a consequence of the entropy estimate in Proposition 4.2.4, which gives the existence of a constant $c \in (0,\infty)$ such that

$$\begin{aligned}\mathbb{E}\left[\int_U \Psi_{\Phi,0}(x,\rho^{n,\epsilon,g})\,dx\right] + 4\mathbb{E}\left[\int_0^t\int_U |\nabla\Phi^{1/2}(\rho^{n,\epsilon,g})|^2\right] + \frac{1}{n}\mathbb{E}\left[\int_0^t\int_U \frac{\Phi'(\rho)}{\Phi(\rho)}|\nabla\rho^{n,\epsilon,g}|^2\right] \\ \leq ct + \mathbb{E}\left[\int_U \Psi_{\Phi,0}(x,\rho_0(x))\,dx\right] + \mathbb{E}\left[\int_0^t\int_U |\nabla\Theta_{\Phi,2}(\rho^{n,\epsilon,g})|^2\right] \\ + \mathbb{E}\left[\lim_{\delta\to 0}\int_0^t\int_U \frac{\Phi'(\rho^{n,\epsilon,g})}{\Phi(\rho^{n,\epsilon,g})+\delta}\sigma_n(\rho^{n,\epsilon,g})\nabla\rho^{n,\epsilon,g}\cdot P_{K(\epsilon)}g\right], \quad (6.35)\end{aligned}$$

where $\Theta_{\Phi,2}$ is defined in (2.5). A bound for the third term on the right hand side of (6.35) is obtained using Proposition 6.5.2 above. Furthermore, due to the presence of the control, we have the final term of (6.35) that needs to be handled. An argument similar to (6.34) using the assumption that $\sigma \leq c\Phi^{1/2}$, Cauchy-Schwarz and Young's inequalities and the fact that $\frac{x}{x+\delta} \leq 1$ for every $\delta \in (0,1)$ gives for every $\gamma \in (0,1)$

$$\begin{aligned}\int_0^t\int_U \frac{\Phi'(\rho^{n,\epsilon,g})}{\Phi(\rho^{n,\epsilon,g})+\delta}\sigma_n(\rho^{n,\epsilon,g})\nabla\rho^{n,\epsilon,g}\cdot P_{K(\epsilon)}g \\ \leq c\gamma\int_0^t\int_U \frac{(\Phi'(\rho^{n,\epsilon,g}))^2}{(\Phi(\rho^{n,\epsilon,g})+\delta)}|\nabla\rho^{n,\epsilon,g}|^2 + c\gamma^{-1}\int_0^t\int_U |P_{K(\epsilon)}g|^2.\end{aligned}$$

The first term can be absorbed onto the left hand side of (6.35) after taking the limit $\delta \to 0$ by picking $\gamma > 0$ sufficiently small. Putting everything together then gives the estimate. ☐

Remark 6.5.6. *If we want to remove the dependence of the projection $P_{K(\epsilon)}$ in the estimates of Propositions 6.5.2, 6.5.3, 6.5.4 and 6.5.5, we can use the trivial fact that by the definition of the projection map, it holds that*

$$\int_0^t\int_U |P_{K(\epsilon)}g|^2\,dx\,ds \leq \int_0^t\int_U |g|^2\,dx\,ds.$$

Using the above estimates, we conclude the section with tightness of laws of the regularised controlled SPDE $\rho^{n,\epsilon,g}$.

Corollary 6.5.7 (Tightness of laws of controlled SPDE)**.** *Suppose that the non-linear functions Φ, σ, ν, the boundary data $\bar{f}$ and the spatial components of the noise ξ^K and satisfy Assumptions 2.2.1, 2.2.9 5.1.2 respectively*[5]*. Let further $\rho_0 \in L^2(\Omega; L^2(U))$ be non-negative and $\mathcal{F}_0$-measurable and $g \in L^2(U \times [0,T]; \mathbb{R}^d)$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable control.*

If $\epsilon \in (0,1), K(\epsilon) \in \mathbb{N}$ satisfy the joint scaling (5.62) *from Remark 5.3.4, then the laws of the solutions $\{\rho^{n,\epsilon,g}\}_{\epsilon\in(0,1),n\in\mathbb{N}}$ of the regularised controlled SPDE* (6.31) *with initial condition ρ_0, control g and boundary data $\bar{f}$ are tight*[6] *on $L^1(U \times [0,T])$ with respect to the strong norm topology.*

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[5] Note that we are back to the setting of more general initial data.

[6] Recall that the tightness is along appropriate subsequences of $n \to \infty, \epsilon \to 0$, which follows from the fact that the right hand side of the energy estimates above are stable in all three $, n, \epsilon$.

Proof. The proof is identical to Proposition 4.4.6 and follows by combining the estimates in Propositions 6.5.3 and 6.5.4 with the Aubin-Lions-Simon lemma. □

6.6 The uniform large deviations principle

We are in a position to prove the the remaining conditions of Theorem 6.1.4. In Theorem 6.6.1 we prove the weak convergence of solutions of the controlled SPDE towards solutions of the skeleton equation as $\epsilon \to 0$. The chapter is concluded with Proposition 6.6.2 where we prove that the rate function is lower semi-continuous. We will then have shown all the required conditions for Theorem 6.1.4. This gives us the large deviations principle.

Theorem 6.6.1. *Suppose that the non-linear functions Φ, σ, ν, the boundary data $\bar{f}$ and the spatial components of the noise ξ^K and satisfy Assumptions 2.2.1, 2.2.9 5.1.2 respectively. Furthermore assume that the non-linearity Φ satisfies for constant $c \in (0,\infty)$ and every $M \in (0,\infty)$*

$$\sup_{0\leq\xi\leq M} \frac{\Phi(\xi)}{\Phi'(\xi)} \leq cM. \tag{6.36}$$

Finally assume that Conjecture 6.2.5 holds. Let $\{g^\epsilon\}_{\epsilon\in(0,1)}$ and g be $L^2(\Omega; L^2(U \times [0,T]; \mathbb{R}^d))$-valued, $(\mathcal{F}_t)_{t\geq 0}$-predictable processes satisfying

$$\sup_{\epsilon\in(0,1)} \|g^\epsilon\|_{L^\infty(\Omega;L^2(U\times[0,T];\mathbb{R}^d))} < \infty, \quad g^\epsilon \to g \quad \text{weakly in } L^2(U \times [0,T]; \mathbb{R}^d) \text{ as } \epsilon \to 0,$$

and let $\{\rho_0^\epsilon\}_{\epsilon\in(0,1)}, \rho_0 \in Ent_\Phi(U)$ be such that

$$\sup_{\epsilon\in(0,1)} \left(\int_U \Psi_\Phi(\rho_0^\epsilon) \right) < \infty, \quad \rho_0^\epsilon \to \rho_0 \quad \text{weakly in } L^1(U) \text{ as } \epsilon \to 0. \tag{6.37}$$

Let $\{\epsilon, K(\epsilon)\}_{\epsilon\in(0,1)}$ be such that the joint scaling (5.62) *in Remark 5.3.4 is satisfied. For every $\epsilon \in (0,1)$ and fixed $\bar{f} \in H^1(\partial U)$, denote by $\rho^{\epsilon,g^\epsilon}(\bar{f}, \rho_0^\epsilon)$ the unique stochastic kinetic solutions of the controlled SPDE* (6.25) *with control g^ϵ, boundary data $\bar{f}$, and initial data ρ_0^ϵ. Then we have that*

$$\rho^{\epsilon,g^\epsilon}(\bar{f}, \rho_0^\epsilon) \to \rho(\bar{f}, \rho_0) \quad \text{weakly in } L^1(U \times [0,T]) \text{ as } \epsilon \to 0,$$

where as above ρ denotes the solution of the skeleton equation (6.8) *in the sense of Definition 6.2.2 and Conjecture 6.2.5.*

Proof. Let us begin by considering non-negative initial data $\{\rho_0^\epsilon\}_{\epsilon\in(0,1)}$ and ρ_0 that are uniformly bounded in $L^2(U)$. In this case we can use Corollary 6.5.7 that gives us tightness of laws of the regularised controlled SPDEs $\{\rho^{n,\epsilon,g^\epsilon}(\bar{f}, \rho_0^\epsilon)\}_{\epsilon\in(0,1),n\in\mathbb{N}}$.

We recall that the kinetic formulation for the equation is presented in Section 2.3. The sequence of non-negative kinetic measures $\{q^\epsilon\}_{\epsilon\in(0,1)}$ corresponding to the regularised equation (6.31) are defined for fixed regularisation constant $n \in \mathbb{N}$ by

$$dq^\epsilon := \delta_0(\xi - \rho^{n,\epsilon,g^\epsilon}) \left(|\nabla \Theta_{\Phi,2}(\rho^{n,\epsilon,g^\epsilon})|^2 + \frac{1}{n} |\nabla \rho^{n,\epsilon,g^\epsilon}|^2 \right) dx\, dt\, d\xi.$$

Due to Proposition 6.5.2 and the assumed the $L^2(U)$-integrability of the initial data, the non-negative kinetic measures are uniformly bounded in expectation, in the sense that

$$\sup_{\epsilon\in(0,1)} \mathbb{E}\left[q^\epsilon(U\times(0,\infty)\times[0,T])\right]<\infty.$$

Let the kinetic function χ^ϵ corresponding to the regularised controlled equation $\rho^{n,\epsilon,g^\epsilon}$ be as defined in Definition 2.3.4. In Section 2.3. we deduced that for every $t\in[0,T]$, $\psi\in C_c^\infty(U\times(0,\infty))$, $\mathbb{P}-a.s.$ the kinetic equation for fixed regularised coefficients $n\in\mathbb{N}$ is the equation

$$\begin{aligned}
\int_{\mathbb{R}}\int_U \chi^\epsilon(x,\xi,t)\psi(x,\xi) &= \int_{\mathbb{R}}\int_U \chi^\epsilon(x,\xi,0)\psi(x,\xi)\\
&- \int_0^t\int_U \Phi'(\rho^{n,\epsilon,g^\epsilon})\nabla\rho^{n,\epsilon,g^\epsilon}\cdot(\nabla_x\psi)(x,\rho^{n,\epsilon,g^\epsilon}) - \int_0^t\int_{\mathbb{R}}\int_U \partial_\xi\psi(x,\xi)\,dq^\epsilon\\
&-\sqrt{\epsilon}\int_0^t\int_U \psi(x,\rho^{n,\epsilon,g^\epsilon})\nabla\cdot(\sigma_n(\rho^{n,\epsilon,g^\epsilon})\,d\xi^{K(\epsilon)}) + \int_0^t\int_U \sigma_n(\rho^{n,\epsilon,g^\epsilon})(\nabla_x\psi)(x,\rho^{n,\epsilon,g^\epsilon})\cdot P_{K(\epsilon)}g^\epsilon\\
&+\int_0^t\int_U (\partial_\xi\psi)(x,\rho^{n,\epsilon,g^\epsilon})\sigma_n(\rho^{n,\epsilon,g^\epsilon})\nabla\rho^{n,\epsilon,g^\epsilon}\cdot P_{K(\epsilon)}g - \int_0^t\int_U \psi(x,\rho^{n,\epsilon,g^\epsilon})\nabla\cdot\nu(\rho^{n,\epsilon,g^\epsilon})\\
&-\frac{\epsilon}{2}\int_0^t\int_U (\nabla_x\psi)(x,\rho^{n,\epsilon,g^\epsilon})\cdot\Big(F_1^{K(\epsilon)}(\sigma_n'(\rho^{n,\epsilon,g^\epsilon}))^2\nabla\rho^{n,\epsilon,g^\epsilon} + \sigma_n(\rho^{n,\epsilon,g^\epsilon})\sigma_n'(\rho^{n,\epsilon,g^\epsilon})F_2^{K(\epsilon)}\Big)\\
&+\frac{\epsilon}{2}\int_0^t\int_U (\partial_\xi\psi)(x,\rho^{n,\epsilon,g^\epsilon})\Big(\sigma_n(\rho^{n,\epsilon,g^\epsilon})\sigma_n'(\rho^{n,\epsilon,g^\epsilon})\nabla\rho^{n,\epsilon,g^\epsilon}\cdot F_2^{K(\epsilon)} + \sigma_n^2(\rho^{n,\epsilon,g^\epsilon})F_3^{K(\epsilon)}\Big).
\end{aligned}\tag{6.38}$$

The terms in the final two lines involving a products of factors of ϵ and noise coefficients $\{F_i^{K(\epsilon)}\}_{i=1,2,3}$ dictate the joint scaling needed for the large deviations principle, and we note that it is the first term in the penultimate line involving a factor of $(\sigma_n'(\rho^{n,\epsilon,g^\epsilon}))^2$ that necessitates the use of compactly supported test functions in the velocity variable ξ.

We note that the energy estimate of Proposition 6.5.5 only gives us weak convergence of the random variables $\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})$ in the space $L^2(U\times[0,T];\mathbb{R}^d)$ rather weak convergence of the gradients $\nabla\rho^{n,\epsilon,g^\epsilon}$. Hence using the distributional equality

$$\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon}) = \frac{1}{2}\Phi^{-1/2}(\rho^{n,\epsilon,g^\epsilon})\Phi'(\rho^{n,\epsilon,g^\epsilon})\nabla\rho^{n,\epsilon,g^\epsilon}$$

we re-write the kinetic equation (6.38) as

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_U \chi^\epsilon(x,\xi,t)\psi(x,\xi) = \int_{\mathbb{R}}\int_U \chi^\epsilon(x,\xi,0)\psi(x,\xi) \\
&- 2\int_0^t\int_U \Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})\cdot(\nabla_x\psi)(x,\rho^{n,\epsilon,g^\epsilon}) - \int_0^t\int_{\mathbb{R}}\int_U \partial_\xi\psi(x,\xi)\,dq^\epsilon \\
&- \sqrt{\epsilon}\int_0^t\int_U \psi(x,\rho^{n,\epsilon,g^\epsilon})\nabla\cdot(\sigma_n(\rho^{n,\epsilon,g^\epsilon})\,d\xi^{K(\epsilon)}) \\
&+ \int_0^t\int_U \sigma_n(\rho^{n,\epsilon,g^\epsilon})(\nabla_x\psi)(x,\rho^{n,\epsilon,g^\epsilon})\cdot P_{K(\epsilon)}g^\epsilon - \int_0^t\int_U \psi(x,\rho^{n,\epsilon,g^\epsilon})\nabla\cdot\nu(\rho^{n,\epsilon,g^\epsilon}) \\
&+ \int_0^t\int_U (\partial_\xi\psi)(x,\rho^{n,\epsilon,g^\epsilon})\frac{2\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})\sigma_n(\rho^{n,\epsilon,g^\epsilon})}{\Phi'(\rho^{n,\epsilon,g^\epsilon})}\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})\cdot P_{K(\epsilon)}g \\
&- \frac{\epsilon}{2}\int_0^t\int_U (\nabla_x\psi)(x,\rho^{n,\epsilon,g^\epsilon}) \\
&\qquad\times\left(F_1^{K(\epsilon)}\frac{2\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})(\sigma_n'(\rho^{n,\epsilon,g^\epsilon}))^2}{\Phi'(\rho^{n,\epsilon,g^\epsilon})}\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon}) + \sigma_n(\rho^{n,\epsilon,g^\epsilon})\sigma_n'(\rho^{n,\epsilon,g^\epsilon})F_2^{K(\epsilon)}\right) \\
&+ \frac{\epsilon}{2}\int_0^t\int_U (\partial_\xi\psi)(x,\rho^{n,\epsilon,g^\epsilon}) \\
&\qquad\times\left(\frac{2\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})\sigma_n(\rho^{n,\epsilon,g^\epsilon})\sigma_n'(\rho^{n,\epsilon,g^\epsilon})}{\Phi'(\rho^{n,\epsilon,g^\epsilon})}\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})\cdot F_2^{K(\epsilon)} + \sigma_n^2(\rho^{n,\epsilon,g^\epsilon})F_3^{K(\epsilon)}\right),
\end{aligned}
\tag{6.39}
$$

where the quotient terms are integrable due to the compact support of the test function ψ in the velocity variable. Due to the tightness of laws of $\rho^{n,\epsilon,g^\epsilon}(\bar{f},\rho_0^\epsilon)$ given by Corollary 6.5.7, we want to use Skorokhod representation theorem to pass almost surely to the to the $\epsilon\to 0, n\to\infty$ limits on an auxiliary probability space. However, the reason that we can not do this directly is due to the term in the fifth line of (6.39), where we have the product of two random variables $\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})$ and $P_{K(\epsilon)}g$ which are both only weakly convergent, so it is not possible to characterise convergence of their product. The key observation is that such a product does not appear in the weak formulation of the skeleton equation, see Definition 6.2.2. Let us upper bound the contribution of this term by introducing the measures p^ϵ. For $\epsilon\in(0,1)$ we define p^ϵ to be the non-negative, almost surely finite measure on $U\times(0,\infty)\times[0,T]$ given by

$$dp^\epsilon := \delta_0(\xi-\rho^{n,\epsilon,g^\epsilon})|\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})|\,|P_{K(\epsilon)}g^\epsilon|\,dx\,dt\,d\xi.$$

The finiteness of the measures follows from the $L^2(U\times[0,T])$-boundedness of the two terms $|\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})|$ and $|P_{K(\epsilon)}g^\epsilon|$. By the assumption $\sigma\le c\Phi^{1/2}$ for some constant $c\in(0,\infty)$, and by construction, we have the bound

$$
\begin{aligned}
&\left|\int_0^t\int_U (\partial_\xi\psi)(x,\rho^{n,\epsilon,g^\epsilon})\frac{2\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})\sigma_n(\rho^{n,\epsilon,g^\epsilon})}{\Phi'(\rho^{n,\epsilon,g^\epsilon})}\nabla\Phi^{1/2}(\rho^{n,\epsilon,g^\epsilon})\cdot P_{K(\epsilon)}g\right| \\
&\qquad\qquad\le c\int_{\mathbb{R}_+}\int_0^t\int_U |\partial_\xi\psi(x,\xi)|\frac{2\Phi(\xi)}{\Phi'(\xi)}\,dp^\epsilon.
\end{aligned}
\tag{6.40}
$$

We return back to the kinetic equation (6.39). To understand the limiting behaviour of the stochastic integral, again omitting the dependence on the regularisation coefficients, define for fixed $n \in \mathbb{N}$ and test function $\psi \in C_c^\infty(U \times (0,\infty))$

$$M_t^\psi := \int_0^t \int_U \psi(x, \rho^{n,\epsilon,g^\epsilon}) \nabla \cdot (\sigma_n(\rho^{n,\epsilon,g^\epsilon}) \dot{\xi}^{K(\epsilon)}).$$

Analogous to Proposition 4.4.6 above, Proposition 5.27 of Fehrman and Gess [41] proves that for $\gamma \in (0, 1/2)$, the laws of the martingales are tight on $C^\gamma([0,T])$.

We now follow the methods of the uniqueness proof, Theorem 3.2.2 above, to characterise the limiting behaviour of the solutions $\rho^{n,\epsilon,g^\epsilon}$. To do so, we need to characterise the limiting behaviour of each of the components in the kinetic equation (6.39) above. For $s > \frac{d+2}{2}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ a countable sequence of dense functions in $C_c^\infty(U \times (0,\infty))$ with respect to the $H^s(U \times (0,\infty))$-topology, we want to establish the tightness of the random variables

$$X^{n,\epsilon} := \left(\rho^{n,\epsilon,g^\epsilon}, \rho_0^\epsilon, \nabla\Theta_{\Phi,2}(\rho^{n,\epsilon,g^\epsilon}), \frac{1}{n}\nabla\rho^{n,\epsilon,g^\epsilon}, g^\epsilon, \frac{q^\epsilon}{|q^\epsilon|}, \frac{p^\epsilon}{|p^\epsilon|}, |q^\epsilon|, |p^\epsilon|, \sqrt{\epsilon}(M_t^{\psi_j})_{j\in\mathbb{N}} \right), \tag{6.41}$$

where $|q^\epsilon| := q^\epsilon(U \times (0,\infty) \times [0,T])$ and analogously for $|p^\epsilon|$, on the product metric topology of the state space

$$\begin{aligned} \bar{X} := L^1(U \times [0,T]) \times L^2(U) \times L^2(U \times [0,T]; \mathbb{R}^d)^3 \times \mathcal{P}(U \times (0,\infty) \times [0,T])^2 \\ \times \mathbb{R}^2 \times C^\gamma([0,T])^{\mathbb{N}}, \end{aligned}$$

where $\mathcal{P}(U \times (0,\infty) \times [0,T])$ denotes the space of non-negative probability measures on $U \times (0,\infty) \times [0,T]$. The space $L^1(U \times [0,T])$ is equipped with the strong topology, the spaces $L^2(U)$ and $L^2(U \times [0,T]; \mathbb{R}^d)$ are equipped with the weak topology, and the space $C^\gamma([0,T])^{\mathbb{N}}$ is equipped with the topology of component-wise convergence in the strong norm induced by the metric defined in Definition 4.4.4.

In much the same way Theorem 3.2.2 above, by using Prokhorov's theorem and the Skorokhod representation theorem, owing to Corollary 6.5.7 and tightness of the martingale term, we have convergence of $X^{n,\epsilon}$ along a subsequence $n_k \to \infty, \epsilon_k \to 0$ in an auxiliary probability space.

To prove the desired result it therefore suffices to prove that if along this subsequence, if $X^{n,\epsilon}$ converges $\mathbb{P}$-a.s. to a random variable

$$X := (\rho, \rho_0, \nabla\Theta_{\Phi,2}(\rho), 0, g, \tilde{q}, \tilde{p}, a, b, 0), \tag{6.42}$$

for probability measures $\tilde{q}, \tilde{p} \in \mathcal{P}(U \times (0,\infty) \times [0,T])$ and random constants $a, b \in \mathbb{R}$, then ρ is a $\mathbb{P}$-a.s. weak solution to the skeleton equation with control g, initial data ρ_0 and boundary data $\bar{f}$.

The regularisations in n were used to obtain the energy estimates that led to tightness, so do not play any further role in the proof and below we denote by $\rho^{\epsilon,g^\epsilon}$ the solution to the limiting singular controlled SPDE (6.25). The sufficient condition is proved by passing from the kinetic equation (6.39) to the weak formulation of the equation along the subsequence $\epsilon_k \to 0$. The kinetic equation has the additional velocity component that is not seen by the weak formulation, so to remove this dependency, for $\delta \in (0, 1/2)$, let $\tau_\delta : [0,\infty) \to [0,1]$ be a smooth function satisfying for some constant $c \in (0,\infty)$

$$\begin{cases} \tau_\delta = 0 & \text{on } [0,\delta] \cup [2\delta^{-1},\infty) \\ \tau_\delta = 1 & \text{on } [2\delta,\delta^{-1}] \\ |\tau_\delta'| \leq c\delta^{-1}\mathbb{1}_{\xi\in[\delta,2\delta]} + c\delta\mathbb{1}_{\xi\in[\delta^{-1},2\delta^{-1}]}. \end{cases} \tag{6.43}$$

The function τ_δ behaves like the product of the velocity cutoff functions $\phi_\beta\zeta_M$ from Definition 2.4.2. For an arbitrary function $\psi \in C_c^\infty(U)$, we will choose test functions in the kinetic equation of the form $\tilde{\Psi}_\delta(x,\xi) = \tau_\delta(\xi)\psi(x)$. We first pass to the limit $\epsilon \to 0$ and subsequently pass to the velocity limit $\delta \to 0$ to recover the weak formulation of the skeleton equation. We proceed term by term.

For the initial data ρ_0^ϵ, we only have that it converges weakly to ρ_0, which is not compatible with the non-linear convergence of the kinetic function, so using the definition of the kinetic function, we write the relevant term as "what we want" plus a correction,

$$\int_{\mathbb{R}}\int_U \chi^\epsilon(x,\xi,0)\tilde{\Psi}_\delta(x,\xi) = \int_U \rho_0^\epsilon(x)\psi(x) + \int_{\mathbb{R}}\int_U \chi^\epsilon(x,\xi,0)\psi(x)\left(\tau_\delta(\xi) - 1\right).$$

By re-arranging, it follows from the support of τ_δ, the boundedness of ψ and the weak convergence of ρ_0^ϵ to ρ_0, that there exists a constant $c \in (0,\infty)$ such that

$$\limsup_{\epsilon\to 0}\left|\int_{\mathbb{R}}\int_U \chi^\epsilon(x,\xi,0)\tilde{\Psi}_\delta(x,\xi) - \int_U \rho_0(x)\psi(x)\right| \leq c\delta. \tag{6.44}$$

The choice of scaling (5.62), Proposition 6.5.2 and the compact support in spatial component of $\tilde{\Psi}_\delta$ then implies that the terms in the penultimate line of (6.39) satisfy $\mathbb{P} - a.s.$ for every fixed $\delta \in (0,1)$,

$$\begin{aligned} \lim_{\epsilon\to 0}\sup_{t\in[0,T]}\Bigg|&\frac{\epsilon}{2}\int_0^t\int_U (\nabla_x\tilde{\Psi}_\delta)(x,\rho^{\epsilon,g^\epsilon}) \\ &\times\left(F_1^{K(\epsilon)}\frac{2\Phi^{1/2}(\rho^{\epsilon,g^\epsilon})(\sigma'(\rho^{\epsilon,g^\epsilon}))^2}{\Phi'(\rho^{\epsilon,g^\epsilon})}\nabla\Phi^{1/2}(\rho^{\epsilon,g^\epsilon}) + \sigma(\rho^{\epsilon,g^\epsilon})\sigma'(\rho^{\epsilon,g^\epsilon})F_2^{K(\epsilon)}\right)\Bigg| = 0, \end{aligned} \tag{6.45}$$

and analogously for terms in the final line of (6.39)

$$\begin{aligned} \lim_{\epsilon\to 0}\sup_{t\in[0,T]}\Bigg|&\frac{\epsilon}{2}\int_0^t\int_U (\partial_\xi\tilde{\Psi}_\delta)(x,\rho^{\epsilon,g^\epsilon}) \\ &\times\left(\frac{2\Phi^{1/2}(\rho^{\epsilon,g^\epsilon})\sigma(\rho^{\epsilon,g^\epsilon})\sigma'(\rho^{\epsilon,g^\epsilon})}{\Phi'(\rho^{\epsilon,g^\epsilon})}\nabla\Phi^{1/2}(\rho^{\epsilon,g^\epsilon})\cdot F_2^{K(\epsilon)} + \sigma^2(\rho^{\epsilon,g^\epsilon})F_3^{K(\epsilon)}\right)\Bigg| = 0. \end{aligned} \tag{6.46}$$

For the martingale term, we have by Burkholder-Davis-Gundy inequality and the compact support of $\tilde{\Psi}_\delta$ that there exists a constant $c \in (0, \infty)$ such that for every $\epsilon, \delta \in (0,1)$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\sqrt{\epsilon}\int_0^t\int_U \tilde{\Psi}_\delta(x,\rho^{\epsilon,g^\epsilon})\nabla\cdot(\sigma(\rho^{\epsilon,g^\epsilon})\,d\xi^{K(\epsilon)})\right|\right] \\
&\leq c\epsilon\int_0^T\int_U \tilde{\Psi}_\delta^2(x,\rho^{\epsilon,g^\epsilon})\left(\sigma^2(\rho^{\epsilon,g^\epsilon})F_1^{K(\epsilon)} + 2\sigma(\rho^{\epsilon,g^\epsilon})\nabla\sigma(\rho^{\epsilon,g^\epsilon})F_2^{K(\epsilon)} + |\nabla\sigma(\rho^{\epsilon,g^\epsilon})|^2F_3^{K(\epsilon)}\right) \\
&\qquad\qquad \leq c\epsilon\left(\|F_1^{K(\epsilon)}\|_{L^\infty(U)} + \|F_2^{K(\epsilon)}\|_{L^\infty(U;\mathbb{R}^d)} + \|F_3^{K(\epsilon)}\|_{L^\infty(U)}\right) \\
&\qquad\qquad\qquad\qquad \times\int_0^T\int_U \tilde{\Psi}_\delta^2(x,\rho^{\epsilon,g^\epsilon})\left(\sigma^2(\rho^{\epsilon,g^\epsilon}) + (\sigma'(\rho^{\epsilon,g^\epsilon}))^2|\nabla\rho^{\epsilon,g^\epsilon}|^2\right).
\end{aligned}
$$

It therefore follows by the choice of scaling (5.62) that along a further subsequence, for every fixed $\delta \in (0,1)$, as $\epsilon \to 0$, $\mathbb{P}$-almost surely,

$$
\lim_{\epsilon\to 0}\sup_{t\in[0,T]}\left|\sqrt{\epsilon}\int_0^t\int_U \tilde{\Psi}_\delta(x,\rho^{\epsilon,g^\epsilon})\nabla\cdot(\sigma(\rho^{\epsilon,g^\epsilon})\,d\xi^{K(\epsilon)})\right| = 0. \tag{6.47}
$$

The remaining terms are handled using the strong convergence of $\rho^{\epsilon,g^\epsilon}$ to ρ, the weak convergence of $P_{K(\epsilon)}g$ to g, the weak convergence of $\nabla\Phi^{1/2}(\rho^{\epsilon,g^\epsilon})$ to $\nabla\Phi^{1/2}(\rho)$, the weak convergence of q^ϵ to $a\tilde{q}$ and the weak convergence of p^ϵ to $b\tilde{p}$, where recall that a, b are the constants in (6.42), representing the mass of the limiting measures from (6.41). Using this, equation (6.39) and the subsequent analysis (6.40), (6.44), (6.45), (6.46), and (6.47) illustrates that after passing to a subsequence $\epsilon \to 0$, $\mathbb{P}$-a.s. for almost every $t \in [0,T]$, if χ denotes the kinetic function of the skeleton equation ρ, then the difference between the kinetic function and the weak formulation satisfies for a constant $c \in (0,\infty)$ independent of $\delta \in (0,1)$,

$$
\begin{aligned}
&\bigg|\int_{\mathbb{R}}\int_U \chi(x,\xi,t)\tilde{\Psi}_\delta(x,\xi) - \int_U \rho_0(x)\psi(x) + 2\int_0^t\int_U \Phi^{1/2}(\rho)\nabla\Phi^{1/2}(\rho)\cdot(\nabla_x\tilde{\Psi}_\delta)(x,\rho) \\
&\qquad\qquad - \int_0^t\int_U \sigma(\rho)(\nabla_x\tilde{\Psi}_\delta)(x,\rho)\cdot g + \int_0^t\int_U \tilde{\Psi}_\delta(x,\rho)\nabla\cdot\nu(\rho)\bigg| \\
&\qquad\qquad\qquad \leq c\delta + c\left|\int_{\mathbb{R}}\int_0^t\int_U |\tau_\delta'(\xi)|\psi(x)\frac{2\Phi(\xi)}{\Phi'(\xi)}b\,d\tilde{p} + |\tau_\delta'(\xi)|\psi(x)a\,d\tilde{q}\right|.
\end{aligned} \tag{6.48}
$$

We now want to pass to the $\delta \to 0$ limit. It follows from the support of τ_δ', Proposition 4.3.2 which bounds the kinetic measure $\tilde{q}$ when the velocity argument approaches zero, the boundedness of ψ and the finiteness of $b\tilde{q}$ that

$$
\liminf_{\delta\to 0}\max_{t\in[0,T]}\mathbb{E}\left[\int_{\mathbb{R}}\int_0^t\int_U |\tau_\delta'(\xi)|\psi(x)a\,d\tilde{q}\right] = 0.
$$

From the support of τ_δ', the boundedness of ψ, the finiteness of $\tilde{p}$ and assumption

(6.36) in the statement of the theorem, we have

$$\begin{aligned}\liminf_{\delta\to 0}\max_{t\in[0,T]}\mathbb{E}&\left[\int_{\mathbb{R}}\int_0^t\int_U|\tau_\delta'(\xi)|\psi(x)\frac{2\Phi(\xi)}{\Phi'(\xi)}b\,d\tilde{p}\right]\\ &\le c\liminf_{\delta\to 0}\mathbb{E}\left[\int_{[\delta,2\delta]}\int_0^T\int_U\delta^{-1}\frac{2\Phi(\xi)}{\Phi'(\xi)}\,d\tilde{p}+\int_{[\delta^{-1},2\delta^{-1}]}\int_0^T\int_U\delta\frac{2\Phi(\xi)}{\Phi'(\xi)}\,d\tilde{p}\right]\\ &\le c\liminf_{\delta\to 0}\mathbb{E}\left[\int_{[\delta,2\delta]\cup[\delta^{-1},2\delta^{-1}]}\int_0^T\int_U d\tilde{p}\right]=0.\end{aligned}$$

We are left to deal with the terms on the left hand side of (6.48) in the $\delta\to 0$ limit. For the first, we have from the support of τ_δ and the definition of the kinetic function that $\mathbb{P}$-a.s. for every $t\in[0,T]$,

$$\lim_{\delta\to 0}\int_{\mathbb{R}}\int_U\chi^\epsilon(x,\xi,t)\tilde{\Psi}_\delta(x,\xi)=\int_{\mathbb{R}}\int_U\chi^\epsilon(x,\xi,t)\psi(x)=\int_U\rho(x,t)\psi(x),$$

and analogously for the transport term

$$\lim_{\delta\to 0}\int_0^t\int_U\tilde{\Psi}_\delta(x,\rho)\nabla\cdot\nu(\rho)=\int_0^t\int_U\psi(x)\nabla\cdot\nu(\rho).$$

For the remaining two terms on the left hand side of (6.48), using the identity $(\nabla_x\tilde{\Psi}_\delta)(x,\xi)|_{\xi=\rho}=\nabla\psi(x)\tau_\delta(\rho)$, it similarly follows that

$$\lim_{\delta\to 0}\int_0^t\int_U\Phi'(\rho)\nabla\rho\cdot(\nabla_x\tilde{\Psi}_\delta)(x,\rho)=\int_0^t\int_U\Phi'(\rho)\nabla\rho\cdot\nabla\psi(x)$$

and

$$\lim_{\delta\to 0}\int_0^t\int_U\sigma(\rho)(\nabla_x\tilde{\Psi}_\delta)(x,\rho)\cdot g=\int_0^t\int_U\sigma(\rho)\nabla\psi(x)\cdot g.$$

Putting equation (6.48) and subsequent computations together, after passing to a subsequence $\delta\to 0$, $\mathbb{P}$-a.s. for almost every $t\in[0,T]$ we have

$$\begin{aligned}\int_U\rho(x,t)\psi(x)=\int_U\rho_0(x)\psi(x)-\int_0^t\int_U 2\Phi^{1/2}(\rho)\nabla\Phi^{1/2}(\rho)\cdot\nabla\psi(x)\\ +\int_0^t\int_U\sigma(\rho)\nabla\psi(x)\cdot g-\int_0^t\int_U\psi(x)\nabla\cdot\nu(\rho).\end{aligned}$$

This is a re-writing of the weak formulation of the skeleton equation, see Definition 6.2.2. Furthermore, by applying arguments similar to the uniqueness proof in Theorem 3.2.2 we can obtain that ρ has an $L^1(U\times[0,T])$-continuous representative. The additional regularity of $\Phi^{1/2}(\rho)$ is inherited by the entropy estimate Proposition 6.5.5, and the boundary data is constant throughout the analysis, so is inherited by the skeleton equation.

Hence, ρ is a $\mathbb{P}$-a.s. weak solution of the skeleton equation (6.8) with control g, initial data ρ_0 and boundary data $\Phi(\rho)|_{\partial U}=\bar{f}$. Uniqueness then follows by combining the equivalence of weak and kinetic solutions of Conjecture 6.2.5 with the uniqueness of kinetic solutions which follows from a simplified version of Theorem 3.2.2.

To extend the result to less regular initial data $\rho_0^\epsilon, \rho_0 \in Ent_\Phi(U)$, we just apply an approximation argument. Approximate $\rho_0^\epsilon \in Ent_\Phi(U)$ by $(\rho_0^\epsilon \wedge n) \in L^2(U)$ for each $n \in \mathbb{N}$ for which the above result holds, and then the $L^1(U)$-contraction of kinetic solutions and triangle inequality gives the result. □

We now check the final remaining condition in Theorem 6.1.4.

Proposition 6.6.2. *Suppose that the non-linear functions Φ, σ, ν and the boundary data $\bar{f}$ satisfy Assumptions 2.2.1 and 2.2.9 respectively. Let further $\rho_0 \in Ent_\Phi(U)$ be $\mathcal{F}_0$-measurable and $g \in L^2(U \times [0,T];\mathbb{R}^d)$ be an $(\mathcal{F}_t)_{t\geq 0}$-predictable control.*

Recall from equation (6.6) that the rate function is given for arbitrary $\rho \in L^1(U \times [0,T])$ by

$$I^{\bar{f}}_{\rho_0}(\rho) = \frac{1}{2}\inf_{g\in L^2(U\times[0,T];\mathbb{R}^d)}\Big\{\|g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)} : \partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot(\sigma(\rho)g + \nu(\rho)) \\ : \Phi(\rho)|_{\partial U} = \bar{f}, \rho(\cdot,0) = \rho_0\Big\}.$$

For every $\bar{f} \in H^1(\partial U)$ and $\rho \in L^1(U \times [0,T])$, we have that $\rho_0 \mapsto I^{\bar{f}}_{\rho_0}(\rho)$ is a lower semi-continuous map from $Ent_\Phi(U)$ to $[0,\infty]$.

Proof. Fix $\rho \in L^1(U \times [0,T])$ and let $\{\rho_0^\epsilon\}_{\epsilon\in(0,1)} \subset Ent_\Phi(U)$ be a sequence converging to ρ_0 in the sense of (6.37). To prove lower semicontinuity, we aim to show that

$$I^{\bar{f}}_{\rho_0}(\rho) \leq \liminf_{\epsilon\to 0} I^{\bar{f}}_{\rho_0^\epsilon}(\rho). \tag{6.49}$$

If $\liminf_{\epsilon\to 0} I^{\bar{f}}_{\rho_0^\epsilon}(\rho) = \infty$, there is nothing to prove. Hence, up to extracting a subsequence, we may assume that $\sup_{\epsilon\in(0,1)} I^{\bar{f}}_{\rho_0^\epsilon}(\rho) < \infty$.

By definition of the rate function, for each $\epsilon \in (0,1)$ there exists a control $g^\epsilon \in L^2(U \times [0,T];\mathbb{R}^d)$ such that ρ solves

$$\partial_t\rho = \Delta\Phi(\rho) - \nabla\cdot\big(\sigma(\rho)g^\epsilon + \nu(\rho)\big) \tag{6.50}$$

with boundary condition $\Phi(\rho)|_{\partial U} = \bar{f}$ and initial condition $\rho(\cdot,0) = \rho_0^\epsilon$, and

$$\frac{1}{2}\|g^\epsilon\|^2_{L^2(U\times[0,T];\mathbb{R}^d)} \leq I^{\bar{f}}_{\rho_0^\epsilon}(\rho) + \epsilon. \tag{6.51}$$

In particular, $\{g^\epsilon\}_{\epsilon\in(0,1)}$ is bounded in $L^2(U \times [0,T];\mathbb{R}^d)$, and therefore, up to extracting a further subsequence, there exists $g \in L^2(U \times [0,T];\mathbb{R}^d)$ such that

$$g^\epsilon \to g \quad \text{weakly in } L^2(U \times [0,T];\mathbb{R}^d).$$

We now pass to the limit in (6.50). Writing the equation in weak form and using the convergence of ρ_0^ϵ together with the weak convergence of g^ϵ, we may invoke the weak-strong continuity of solutions to the skeleton equation, see Theorem 21 of Fehrman and Gess [42], to conclude that ρ satisfies

$$\partial_t \rho = \Delta\Phi(\rho) - \nabla \cdot \big(\sigma(\rho) g + \nu(\rho)\big)$$

with boundary condition $\Phi(\rho)|_{\partial U} = \bar{f}$ and initial condition $\rho(\cdot, 0) = \rho_0$. In particular, g is an admissible control in the definition of $I^{\bar{f}}_{\rho_0}(\rho)$, and therefore

$$I^{\bar{f}}_{\rho_0}(\rho) \leq \frac{1}{2}\|g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)}.$$

Finally, by weak lower semicontinuity of the $L^2(U \times [0,T]; \mathbb{R}^d)$-norm and equation (6.51), we obtain

$$\frac{1}{2}\|g\|^2_{L^2(U\times[0,T];\mathbb{R}^d)} \leq \liminf_{\epsilon\to 0} \frac{1}{2}\|g^\epsilon\|^2_{L^2(U\times[0,T];\mathbb{R}^d)} \leq \liminf_{\epsilon\to 0} I^{\bar{f}}_{\rho_0^\epsilon}(\rho).$$

Combining the previous two inequalities yields the desired result (6.49). $\square$

Remark 6.3.2, Propositions 6.3.1, 6.3.5, Theorem 6.6.1 and Proposition 6.6.2 verify the conditions to prove the large deviations principle for equation (6.1) from Theorem 6.1.4, under the presumption that Corollary 6.2.5 holds.

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