

ISOTHERMAL LIMIT OF ENTROPY SOLUTIONS OF THE EULER EQUATIONS FOR ISENTROPIC GAS DYNAMICS*

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Abstract. We are concerned with the isothermal limit of entropy solutions in L^∞ , containing the vacuum states, of the Euler equations for isentropic gas dynamics. We prove that the entropy solutions in L^∞ of the isentropic Euler equations converge strongly to the corresponding entropy solutions of the isothermal Euler equations, when the adiabatic exponent $\gamma \rightarrow 1$. This is achieved by combining careful entropy analysis and refined kinetic formulation with a compensated compactness argument to obtain the required uniform estimates for the limit. The entropy analysis involves careful estimates for the relation between the corresponding entropy pairs for the isentropic and isothermal Euler equations when the adiabatic exponent $\gamma \rightarrow 1$. The kinetic formulation for the entropy solutions of the isentropic Euler equations with the uniformly bounded initial data is refined, so that the total variation of the dissipation measures in the formulation is locally uniformly bounded with respect to $\gamma > 1$. The explicit asymptotic analysis of the Riemann solutions containing the vacuum states is also presented.

Key words. isothermal limit, adiabatic exponent, entropy solutions, vacuum, singular limit, isentropic Euler equations, isothermal Euler equations, compactness framework, entropy analysis, kinetic formulation, strong convergence

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1. Introduction. We are concerned with the isothermal limit of entropy solutions in L^∞ , containing the vacuum states, of the Euler equations for isentropic gas dynamics—a system that is the oldest, but still most prominent, paradigm for the analysis of hyperbolic systems of conservation laws. The one-dimensional Euler equations for barotropic gas dynamics take the form

$$(1.1) \quad \begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\frac{m^2}{\rho} + p \right) = 0, \end{cases}$$

where ρ denotes the fluid density, $p = p(\rho)$ is the pressure, and $m = \rho u$ is the momentum. When $\rho > 0$, $u := \frac{m}{\rho}$ represents the fluid velocity.

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The pressure-density relation under consideration can be written by scaling as

$$(1.2) \quad p = \frac{\rho^\gamma}{\gamma} \quad \text{for } \gamma > 1$$

for ideal isentropic gases, while

$$(1.3) \quad p = \rho$$

for the isothermal gas.

Consider the Cauchy problem for system (1.1) with large Cauchy data:

$$(1.4) \quad (\rho, m)|_{t=0} = (\rho_0, m_0)(x)$$

and $\rho_0(x) \geq 0$.

The global existence of solutions with large initial data in L^∞ was first established in DiPerna [14] for $\gamma = 1 + \frac{2}{2k+1}$ with $k \geq 2$ integer. For the general interval $1 < \gamma \leq \frac{5}{3}$, the global existence problem was solved in Ding, Chen, and Luo [11, 12] and Chen [2]. The adiabatic exponent range $\gamma \geq 3$ was solved in Lions, Perthame, and Tadmor [19], and the remaining case $\frac{5}{3} < \gamma < 3$ was closed in Lions, Perthame, and Souganidis [18]. Later, the existence problem for the general pressure $p(\rho)$ with (1.2) as its leading asymptotic term as $\rho \rightarrow 0$ was solved in Chen and LeFloch [6, 7], whose approach further simplifies the proofs for the γ -law case for all $\gamma > 1$; see also [25].

For the isothermal case (1.3), the first existence result was obtained in Nishida [21] for BV solutions without vacuum for large initial data. For general L^∞ -solutions containing the vacuum states, it was first solved in Huang and Wang [16]; see also [17] for a different approach.

For the isothermal limit when the adiabatic exponent $\gamma \rightarrow 1$, the previous analysis is in the framework of BV solutions away from the vacuum. The first result was established by Nishida and Smoller [22] in the Lagrangian coordinates. Chen, Christoforou, and Zhang [4] established the L^1 -dependence estimate of the BV solutions with respect to $\gamma > 1$ in Eulerian coordinates. The isothermal limit for the BV solutions from the full Euler equations away from the vacuum was solved by Chen, Christoforou, and Zhang in [5]. It is well known that the vacuum states occur generically in the entropy solutions of the Euler equations, even starting with the initial data without vacuum states. A prototypical example is the Riemann problem with Riemann initial data away from the vacuum for which there exists a global Riemann solution consisting of two rarefaction waves with an intermediate vacuum state for the isentropic Euler equations; see [3] and section 7.1 below. For the Riemann solution containing the vacuum states, the fluid velocity u is bounded for $\gamma > 1$ near the vacuum, while $|u|$ goes to ∞ as the density goes to 0 for $\gamma = 1$; see also section 7.2. Another difference is the sound speed as the speed of the pressure disturbance traveling through the medium vanishes as the density goes to 0 for $\gamma > 1$, while the sound speed c is a fixed constant independent of the density for $\gamma = 1$; see section 7.2. Therefore, the isothermal limit of entropy solutions in L^∞ , containing the vacuum states, with large initial data has been a longstanding open problem in the analysis of nonlinear partial differential equations and mathematical fluid mechanics.

The main objective of this paper is to provide an affirmative answer to this problem by providing a rigorous convergence proof of the isothermal limit. More precisely, we prove that the entropy solutions in L^∞ , containing the vacuum states, of the isentropic Euler equations converge strongly to the corresponding entropy solutions of the isothermal Euler equations when the adiabatic exponent $\gamma \rightarrow 1$. This is achieved

by combining careful entropy analysis and refined kinetic formulation with a compensated compactness argument to obtain the required uniform estimates for the isothermal limit. These effective approaches for the analysis of the isentropic Euler equations are based on the subtle analysis for the Euler–Poisson–Darboux equation, while the entropy equation of the isothermal Euler equations is not governed by the Euler–Poisson–Darboux equation. The entropy analysis involves careful estimates for the relation between the corresponding entropy pairs for the isentropic and isothermal Euler equations when the adiabatic exponent $\gamma \rightarrow 1$. The kinetic formulation for the entropy solutions of the isentropic Euler equations with the uniformly bounded initial data is refined, so that the total variation of the dissipation measures in the formulation is locally uniformly bounded with respect to $\gamma > 1$. The compensated compactness argument is based on the H^{-1} -compactness of entropy dissipation measures, the div-curl lemma in [20, 26], and the Young’s measure presentation theorem (cf. [1, 26]); see also [10, 15].

The rest of this paper is organized as follows: In section 2, we state some existence results and properties of entropy solutions of the compressible Euler equations and present the main theorem of this paper, Theorem 2.3, regarding the isothermal limit of the entropy solutions. The uniform L^∞ -estimate of the entropy solutions with respect to the adiabatic exponents $\gamma > 1$ is shown in section 3, while the refined kinetic formulation is obtained in section 4. The uniform relation between the corresponding entropy pairs for the isentropic and isothermal Euler equations is established in section 5. In section 6, we complete the proof of the main theorem, Theorem 2.3. Finally, in section 7, the explicit asymptotic analysis of the Riemann solutions containing the vacuum states is presented. In particular, the phenomenon of decavitation is shown as $\gamma \rightarrow 1$, which is different from the formation of cavitation and concentration in the vanishing pressure limit (equivalently, the high Mach limit) presented in Chen and Liu [8].

2. Entropy solutions of the compressible Euler equations. In this section, we first present some existence results and properties of entropy solutions of the compressible Euler equations (1.1), which can be rewritten as a hyperbolic system of conservation laws of the form

$$(2.1) \quad \partial_t U + \partial_x F^{(\theta)}(U) = 0$$

with $U = (\rho, m)$, $F^{(\theta)}(U) = (m, \frac{m^2}{\rho} + p^{(\theta)}(\rho))$, and $\theta := \frac{\gamma-1}{2} \geq 0$. The pressure-density relation (1.2) can be rewritten as

$$(2.2) \quad p^{(\theta)}(\rho) = \frac{\rho^{2\theta+1}}{2\theta+1}.$$

System (2.1) represents the isentropic Euler equations when $\theta > 0$ and the isothermal Euler equations when $\theta = 0$. Then the entropy pair $(\eta_*^{(\theta)}, q_*^{(\theta)})$ of the mechanical energy and energy flux is of the form

$$(2.3) \quad (\eta_*^{(\theta)}, q_*^{(\theta)})(\rho, m) = \left(\frac{1}{2} \frac{m^2}{\rho} + \rho e^{(\theta)}(\rho), \frac{1}{2} \frac{m^3}{\rho^2} + m \frac{d(\rho e^{(\theta)}(\rho))}{d\rho} \right),$$

where $e^{(\theta)}(\rho) = \int_1^\rho \frac{p^{(\theta)}(\tau)}{\tau^2} d\tau$ is the specific internal energy. Notice that $c^{(\theta)}(\rho) = \rho^\theta$, which implies that $\lim_{\rho \rightarrow 0} c^{(\theta)}(\rho) = 0$, while $c^{(0)}(\rho) \equiv 1$.

We now introduce the following broad class of entropy solutions.

DEFINITION 2.1. Let $\theta \geq 0$. A vector function $U(t, x) = (\rho, m)(t, x) \in L^\infty(\mathbb{R}_+^2)$ is called an entropy solution of the Cauchy problem (1.1)–(1.4) with initial data $(\rho_0, m_0)(x) \in L^\infty(\mathbb{R})$ if, for any test function $\phi \in C_0^\infty(\mathbb{R}_+^2)$,

$$(2.4) \quad \int_{\mathbb{R}_+^2} (U \partial_t \phi + F^{(\theta)}(U) \partial_x \phi) \, dx dt + \int_{\mathbb{R}} U_0(x) \phi(0, x) \, dx = 0,$$

and, for any test function $\phi \in C_0^\infty(\mathbb{R}_+^2)$ with $\phi \geq 0$,

$$(2.5) \quad \int_{\mathbb{R}_+^2} (\eta_*^{(\theta)}(U) \partial_t \phi + q_*^{(\theta)}(U) \partial_x \phi) \, dx dt + \int_{\mathbb{R}} \eta_*^{(\theta)}(U_0)(x) \phi(0, x) \, dx \geq 0.$$

For system (2.1), a general entropy pair (η, q) obeys the following linear hyperbolic system:

$$(2.6) \quad \nabla q(\rho, m) = \nabla \eta(\rho, m) \nabla (F^{(\theta)}(\rho, m))^\top.$$

A weak entropy is an entropy $\eta(\rho, m)$ that vanishes at $\rho = 0$ (vacuum), and then the corresponding pair (η, q) is called a weak entropy pair. Notice that $u(t, x) := \frac{m(t, x)}{\rho(t, x)}$ is not well-defined on the vacuum set $\{\rho(t, x) = 0\}$, which is one of the main reasons why the weak entropy is used, while the momentum function $m(t, x)$ itself is well-defined to be 0 on $\{\rho(t, x) = 0\}$ almost everywhere. We will often use m and $u = \frac{m}{\rho}$ alternatively when $\rho > 0$ without ambiguity.

Any entropy solution $(\rho, m)(t, x)$ is also required (whenever available) to satisfy the additional entropy inequality

$$(2.7) \quad \partial_t \eta(\rho, m) + \partial_x q(\rho, m) \leq 0$$

in the sense of distributions for any weak entropy pair $(\eta, q)(\rho, m)$ satisfying that $\eta(\rho, m)$ is a convex function with respect to (ρ, m) .

As in [2, 6, 7, 11, 12, 14, 16, 18, 19], the weak entropy $\eta(\rho, u)$ satisfies

$$(2.8) \quad \begin{cases} \partial_{\rho\rho} \eta - \frac{p'(\rho)}{\rho^2} \partial_{uu} \eta = 0 & \text{for } \rho > 0, \\ \eta|_{\rho=0} = 0, \end{cases}$$

and the corresponding entropy flux $q(\rho, u)$ satisfies

$$\partial_\rho q(\rho, u) = u \partial_\rho \eta + \frac{p'(\rho)}{\rho} \partial_u \eta, \quad \partial_u q(\rho, u) = \rho \partial_\rho \eta + u \partial_u \eta$$

for a general pressure function $p(\rho)$, including $p^{(\theta)}(\rho)$ for $\theta \geq 0$.

2.1. Isentropic case. When $\theta > 0$, the Riemann invariants of system (2.1) are

$$w_j^{(\theta)} = \frac{m}{\rho} + (-1)^{j+1} \frac{\rho^\theta}{\theta} \quad \text{for } j = 1, 2.$$

The weak entropy kernel of (2.1) is

$$(2.9) \quad \chi^{(\theta)}(\rho; s - u) = a_\theta \left[\left(\frac{\rho^\theta}{\theta} \right)^2 - (s - u)^2 \right]_+^{\frac{1-\theta}{2\theta}} = a_\theta \left[(w_1^{(\theta)} - s)(s - w_2^{(\theta)}) \right]_+^{\frac{1-\theta}{2\theta}}$$

with $a_\theta = \theta^{\frac{1}{\theta}} \left(\int_{-1}^1 [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau \right)^{-1}$, which is a fundamental solution of (2.8), determined by

$$\begin{cases} \partial_{\rho\rho}\chi^{(\theta)} - \rho^{\gamma-3}\partial_{uu}\chi^{(\theta)} = 0 & \text{for } \rho > 0, \\ \chi^{(\theta)}|_{\rho=0} = 0, \quad \chi_\rho^{(\theta)}|_{\rho=0} = \delta_{u=s}, \end{cases}$$

where we have used the notation: $[x]_+ = \max(0, x)$. Then we have the following representation of weak entropy pairs.

LEMMA 2.2. *For $\theta > 0$, the weak entropy pair of (2.1) can be represented as*

$$\begin{aligned} \eta^{(\theta)}(\rho, m; \psi) &= \int_{\mathbb{R}} \chi^{(\theta)}(\rho; s - \frac{m}{\rho}) \psi(s) ds, \\ q^{(\theta)}(\rho, m; \psi) &= \int_{\mathbb{R}} (\theta s + (1 - \theta) \frac{m}{\rho}) \chi^{(\theta)}(\rho; s - \frac{m}{\rho}) \psi(s) ds \end{aligned}$$

for any $\psi \in C^2(\mathbb{R})$. Furthermore, $\eta^{(\theta)}(\rho, m; \psi)$ is convex if and only if ψ is convex (cf. [19]).

Using the standard change of variable $s = u + \frac{\rho^\theta}{\theta} \tau$, we have

$$\begin{aligned} (2.10) \quad \eta^{(\theta)}(\rho, m; \psi) &= a_\theta \int_{\mathbb{R}} [(\frac{\rho^\theta}{\theta})^2 - (u - s)^2]_+^{\frac{1-\theta}{2\theta}} \psi(s) ds \\ &= \frac{\rho \int_{\mathbb{R}} \psi(u + \frac{\rho^\theta}{\theta} \tau) [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}{\int_{\mathbb{R}} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}, \end{aligned}$$

$$\begin{aligned} (2.11) \quad q^{(\theta)}(\rho, m; \psi) &= a_\theta \int_{\mathbb{R}} (\theta s + (1 - \theta) u) [(\frac{\rho^\theta}{\theta})^2 - (u - s)^2]_+^{\frac{1-\theta}{2\theta}} \psi(s) ds \\ &= \frac{\rho \int_{\mathbb{R}} (u + \rho^\theta \tau) \psi(u + \frac{\rho^\theta}{\theta} \tau) [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}{\int_{\mathbb{R}} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}. \end{aligned}$$

In particular, the energy-energy flux pair $(\eta_*^{(\theta)}, q_*^{(\theta)})$ may also be obtained by choosing $\psi = \frac{1}{2}s^2 - \frac{1}{2\theta(2\theta+1)}$.

THEOREM 2.3 (Existence Theorem for $\theta > 0$). *For any $\theta > 0$, there exist both a bounded entropy solution $(\rho, m)(t, x)$ of the Cauchy problem (1.1) and (1.4) on \mathbb{R}_+^2 and a corresponding bounded dissipation measure $D^{(\theta)} \leq 0$ on $\mathbb{R}_+^2 \times \mathbb{R}$ such that the weak entropy kernel $\chi^{(\theta)}(\rho; s - \frac{m}{\rho})$ satisfies the kinetic formulation:*

$$(2.12) \quad \partial_t \chi^{(\theta)} + \partial_x \left((\theta s + (1 - \theta) \frac{m}{\rho}) \chi^{(\theta)} \right) = \partial_{ss} D^{(\theta)}(t, x; s)$$

with initial data $(\rho_0, m_0)(x)$ such that $\rho_0(x) \geq 0$ a.e. $x \in \mathbb{R}$ and $(\rho_0, \frac{m_0}{\rho_0})(x) \in L^\infty(\mathbb{R})$. Moreover, $(\rho, m)(t, x)$ satisfies that

$$(2.13) \quad \frac{|m(t, x)|}{\rho(t, x)} + \frac{\rho(t, x)^\theta - 1}{\theta} \leq \left\| \frac{|m_0|}{\rho_0} + \frac{\rho_0^\theta - 1}{\theta} \right\|_{L^\infty(\mathbb{R})} =: w_0^{(\theta)}$$

and $\text{supp } D^{(\theta)}(t, x; \cdot) \subset [-w_0^{(\theta)}, w_0^{(\theta)}]$.

See [2, 6, 7, 11, 12, 14, 18, 19, 23] for the details.

Remark 2.4. The kinetic formulation (2.12) is equivalent to the weak entropy inequality (2.7) for L^∞ solutions. See Chen and Perepelitsa [9], Perthame [23], and Perthame and Tzavaras [24].

2.2. Isothermal case. When $\theta = 0$, the Riemann invariants are

$$w_j^{(0)} = \rho e^{(-1)^{j+1} \frac{m}{\rho}} \quad \text{for } j = 1, 2.$$

Then typical weak entropy pairs are

$$(2.14) \quad \eta_\xi(\rho, m) = \rho^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2} \frac{m}{\rho}}, \quad q_\xi(\rho, m) = \left(\frac{m}{\rho} + \xi\right) \eta_\xi \quad \text{for } \xi \in (-1, 1).$$

This generates the following family of weak entropy pairs.

LEMMA 2.5. When $\theta = 0$, the entropy pair $(\eta^{(0)}, q^{(0)})$,

$$(2.15) \quad \begin{cases} \eta^{(0)}(\rho, m; \psi) = \int_{-1}^1 \eta_\xi(\rho, m) \psi(\xi) d\xi, \\ q^{(0)}(\rho, m; \psi) = \int_{-1}^1 \left(\frac{m}{\rho} + \xi\right) \eta_\xi(\rho, m) \psi(\xi) d\xi, \end{cases}$$

is a weak entropy pair of (2.1) for any $\psi \in L^\infty(\mathbb{R})$ with $\text{supp } \psi(\cdot) \Subset (-1, 1)$. Furthermore, $\eta^{(0)}(\rho, m; \psi)$ is convex for any $\psi \geq 0$.

The representation formula (2.15) is the result in [16, Lemma 2.1]. It has also been shown in [16, Lemma 3.1] that, for any $\xi \in (-1, 1)$, $\eta_\xi(\rho, m)$ is strictly convex, which implies that $\eta^{(0)}(\rho, m; \psi)$ is convex for any $\psi \geq 0$.

THEOREM 2.6 (Compactness Framework for $\theta = 0$). Let $(\rho^\varepsilon, m^\varepsilon)(t, x)$ be a sequence of approximate solutions of (2.1) with $\theta = 0$ satisfying

$$(2.16) \quad 0 \leq \rho^\varepsilon(t, x) \leq C, \quad |m^\varepsilon(t, x)| \leq \rho^\varepsilon(t, x)(|\ln \rho^\varepsilon(t, x)| + C) \quad \text{a.e. } (t, x) \in \mathbb{R}_+^2,$$

where $C > 0$ is a constant independent of ε . Assume that there exists a small constant $\delta > 0$ such that, for any $\xi \in (-\delta, \delta)$,

$$(2.17) \quad \partial_t \eta_\xi(\rho^\varepsilon, m^\varepsilon) + \partial_x q_\xi(\rho^\varepsilon, m^\varepsilon) \quad \text{is compact in } H_{loc}^{-1}$$

for the weak entropy pairs (η_ξ, q_ξ) defined in (2.14). Then there exist both a subsequence (still denoted) $(\rho^\varepsilon, m^\varepsilon)(t, x)$ and a vector function $(\rho, m)(t, x)$ such that

$$(\rho^\varepsilon, m^\varepsilon)(t, x) \rightarrow (\rho, m)(t, x) \quad \text{in } L_{loc}^p(\mathbb{R}_+^2) \text{ for all } p \in [1, \infty) \text{ as } \varepsilon \rightarrow 0.$$

Remark 2.7. In [16], Theorem 2.6 is stated when condition (2.17) holds for the case that $\delta = 1$. It can be directly checked that Theorem 2.6 is still valid as stated above for some small constant $\delta > 0$, by following the same argument as in [16].

Remark 2.8. Condition (2.16) is equivalent to

$$0 \leq w_j^{(0)}(\rho^\varepsilon(t, x), m^\varepsilon(t, x)) \leq C \quad \text{for } j = 1, 2,$$

where $C > 0$ is a constant independent of ε .

2.3. Main theorem. We now state the main theorem of this paper. Since our main focus is on the isothermal limit $\theta \rightarrow 0$, we always assume $\theta \in (0, \theta_0]$ for some fixed $\theta_0 \in (0, \infty)$ without loss of generality; for simplicity, we take any fixed $\theta_0 \in (0, 1)$ in our analysis from now on throughout this paper.

THEOREM 2.9 (Main Theorem). For $\theta > 0$, let $(\rho^{(\theta)}, m^{(\theta)})(t, x)$ be entropy solutions of (2.1) satisfying (2.12)–(2.13) with initial data $(\rho_0^{(\theta)}, m_0^{(\theta)})(x)$, as constructed in Theorem 2.3. Assume that there exists a constant $w_0 > 0$ (e.g., $w_0 = \sup_{0 < \theta \leq \theta_0} w_0^{(\theta)}$) independent of $\theta \in (0, \theta_0]$ such that the initial data $(\rho_0^{(\theta)}, m_0^{(\theta)})(x)$ satisfy that

$$(2.18) \quad \frac{|m_0^{(\theta)}(x)|}{\rho_0^{(\theta)}(x)} + \frac{(\rho_0^{(\theta)}(x))^\theta - 1}{\theta} \leq w_0 \quad \text{a.e. } x \in \mathbb{R}.$$

Then there exist both a subsequence (still denoted) $(\rho^{(\theta)}, m^{(\theta)})$ and a vector function (ρ, m) such that

$$(\rho^{(\theta)}, m^{(\theta)})(t, x) \rightarrow (\rho, m)(t, x) \quad \text{in } L_{loc}^p(\mathbb{R}_+^2) \text{ for all } p \in [1, \infty) \text{ as } \theta \rightarrow 0,$$

and $(\rho, m)(t, x)$ is an entropy solution of (2.1) with $\theta = 0$ satisfying

$$0 \leq \rho(t, x) \leq e^{w_0}, \quad |m(t, x)| \leq \rho(t, x)(|\ln \rho(t, x)| + w_0) \quad \text{a.e. } (t, x) \in \mathbb{R}_+^2,$$

and the entropy inequality (2.7) in the sense of distributions for any weak entropy pair (η_ξ, q_ξ) for $\xi \in (-\sqrt{2} + 1, \sqrt{2} - 1)$. In particular, for any nonnegative function $\psi(\xi) \in L^\infty(\mathbb{R})$ with $\text{supp } \psi(\xi) \Subset (-\sqrt{2} + 1, \sqrt{2} - 1)$, $(\rho, m)(t, x)$ satisfies

$$\partial_t \eta^{(0)}(\rho, m; \psi) + \partial_x q^{(0)}(\rho, m; \psi) \leq 0$$

in the sense of distributions.

3. Uniform L^∞ -estimate. In this section, we establish the uniform L^∞ -estimate of the entropy solutions in Theorem 2.3 with respect to the adiabatic exponents $\gamma > 1$, i.e., $\theta > 0$.

LEMMA 3.1 (Uniform L^∞ -Bound with respect to $\theta > 0$). Let $(\rho^{(\theta)}, m^{(\theta)})(t, x)$ be the entropy solutions of (2.1) with initial data $(\rho_0^{(\theta)}, m_0^{(\theta)})(x)$ satisfying (2.18), as constructed in Theorem 2.3. Then, for almost everywhere $(t, x) \in \mathbb{R}_+^2$,

$$(3.1) \quad 0 \leq \rho^{(\theta)}(t, x) \leq e^{w_0}, \quad |m^{(\theta)}(t, x)| \leq \rho^{(\theta)}(t, x)(|\ln \rho^{(\theta)}(t, x)| + w_0),$$

and $(\eta_*^{(\theta)}, q_*^{(\theta)})(\rho^{(\theta)}(t, x), m^{(\theta)}(t, x))$ are uniformly bounded with respect to $\theta \in (0, \theta_0]$.

Proof. Using Theorem 2.3 and condition (2.18), for each fixed $\theta > 0$, the entropy solution $(\rho^{(\theta)}, m^{(\theta)})(t, x)$ of (2.1) satisfies that, for a.e. $(t, x) \in \mathbb{R}_+^2$,

$$(3.2) \quad 0 \leq \frac{(\rho^{(\theta)}(t, x))^\theta}{\theta} \leq w_0 + \frac{1}{\theta}, \quad \frac{|m^{(\theta)}(t, x)|}{\rho^{(\theta)}(t, x)} \leq w_0 + \frac{1 - (\rho^{(\theta)}(t, x))^\theta}{\theta},$$

which implies

$$(3.3) \quad 0 \leq \rho^{(\theta)}(t, x) \leq (1 + \theta w_0)^{\frac{1}{\theta}} \leq e^{w_0} \quad \text{for a.e. } (t, x) \in \mathbb{R}_+^2.$$

On the other hand,

$$\rho^{(\theta)}(t, x) e^{\frac{|m^{(\theta)}(t, x)|}{\rho^{(\theta)}(t, x)}} \leq \rho^{(\theta)}(t, x) e^{-\frac{(\rho^{(\theta)}(t, x))^\theta - 1}{\theta}} e^{w_0} \quad \text{for a.e. } (t, x) \in \mathbb{R}_+^2.$$

Using

$$(3.4) \quad \rho^\theta - 1 - \theta \ln \rho \geq 0 \quad \text{for any } \rho \geq 0 \text{ and } \theta > 0,$$

we see that $\rho^{(\theta)} e^{-\frac{(\rho^{(\theta)})^\theta - 1}{\theta}} = e^{-\frac{1}{\theta}((\rho^{(\theta)})^\theta - 1 - \theta \ln \rho^{(\theta)})} \leq 1$ so that

$$\rho^{(\theta)}(t, x) e^{\frac{|m^{(\theta)}(t, x)|}{\rho^{(\theta)}(t, x)}} \leq e^{w_0} \quad \text{for a.e. } (t, x) \in \mathbb{R}_+^2,$$

which is uniformly bounded with respect to $\theta \in (0, \theta_0]$.

We now show the uniform boundedness of the energy-energy flux pair $(\eta_*^{(\theta)}, q_*^{(\theta)})$. From (2.3), for $\theta > 0$, we have

$$(3.5) \quad (\eta_*^{(\theta)}, q_*^{(\theta)})(\rho, m) = \left(\frac{1}{2} \frac{m^2}{\rho} + \rho \frac{\rho^{2\theta} - 1}{2\theta(2\theta + 1)}, \frac{1}{2} \frac{m^3}{\rho^2} + m \left(\frac{\rho^{2\theta} - 1}{2\theta} + \frac{1}{2\theta + 1} \right) \right).$$

Based on estimates (3.2)–(3.3) and formula (3.5), it suffices to estimate $\frac{m^{(\theta)}}{(\rho^{(\theta)})^{\frac{2}{3}}}$. We first obtain from (3.3)–(3.4) that

$$\frac{|m^{(\theta)}|}{(\rho^{(\theta)})^{\frac{2}{3}}} \leq w_0 (\rho^{(\theta)})^{\frac{1}{3}} + (\rho^{(\theta)})^{\frac{1}{3}} \frac{1 - (\rho^{(\theta)})^\theta}{\theta} \leq w_0 e^{\frac{w_0}{3}} - (\rho^{(\theta)})^{\frac{1}{3}} \ln \rho^{(\theta)} \leq w_0 e^{\frac{w_0}{3}} + 3e^{-1}.$$

Then there exists a universal constant $C := C(w_0, \theta_0) > 0$ independent of $\theta \in (0, \theta_0]$ such that

$$|m^{(\theta)}| + \frac{|m^{(\theta)}|^2}{\rho^{(\theta)}} \leq C(\rho^{(\theta)})^{\frac{1}{3}} \leq C, \quad \left| m^{(\theta)} \frac{(\rho^{(\theta)})^{2\theta} - 1}{2\theta} \right| \leq C(\rho^{(\theta)})^{\frac{1}{3}} \frac{|(\rho^{(\theta)})^\theta - 1|}{\theta} \leq C.$$

Thus, the uniform boundedness of $(\eta_*^{(\theta)}, q_*^{(\theta)})(\rho^{(\theta)}(t, x), m^{(\theta)}(t, x))$ with respect to $\theta \in (0, \theta_0]$ follows. This completes the proof. \square

4. Refined kinetic formulation for $\theta > 0$. In this section, we show that the total variation of the dissipation measures in the kinetic formulation of the entropy solutions of the isentropic Euler equations (2.1) with initial data satisfying (2.18) are locally uniformly bounded with respect to $\theta \in (0, \theta_0]$. As indicated in Theorem 2.3, for fixed $\theta > 0$, $(\rho^{(\theta)}, m^{(\theta)})$ is an entropy solution of (2.1) satisfying the kinetic equation (2.12) with kernel $\chi^{(\theta)}$ defined in (2.9) and the dissipation measure $D^{(\theta)}$ on $\mathbb{R}_+^2 \times \mathbb{R}$.

LEMMA 4.1 (Uniformly Bounded Total Variation of the Dissipation Measures). *Let the initial data $(\rho_0^{(\theta)}, m_0^{(\theta)})(x)$ satisfy (2.18). Assume that the entropy solutions $(\rho^{(\theta)}, m^{(\theta)})(t, x)$ and the dissipation measures $D^{(\theta)}(t, x; s)$ are as constructed in Theorem 2.3. Then the total variation of the dissipation measures $D^{(\theta)}(t, x; s)$ is locally uniformly bounded with respect to $\theta \in (0, \theta_0]$: For any compact set $K \Subset \mathbb{R}_+^2$, there exists $C(w_0; K) > 0$ such that*

$$(4.1) \quad \int_{\mathbb{R}} |D^{(\theta)}(\cdot, \cdot; s)|(K) ds \leq C(w_0; K),$$

and $\text{supp } D^{(\theta)}(t, x; \cdot) \subset [-w_0, w_0]$.

Proof. For any convex function $\psi(s)$, i.e., $\psi''(s) \geq 0$, it follows from Theorem 2.3 that the entropy solutions $(\rho^{(\theta)}, m^{(\theta)})$ satisfy

$$(4.2) \quad \partial_t \eta^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}; \psi) + \partial_x q^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}; \psi) = \int_{\mathbb{R}} D^{(\theta)}(t, x; s) \psi''(s) ds \leq 0$$

in the sense of distributions, since $D^{(\theta)}(t, x; s) \leq 0$ on $\mathbb{R}_+^2 \times \mathbb{R}$.

In (4.2), we choose $\psi(s) = \psi_*(s) := \frac{1}{2}s^2 - \frac{1}{2\theta(2\theta+1)}$ with $\psi_*''(s) = 1$, which leads to

$$(4.3) \quad \begin{aligned} & \partial_t \eta_*^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}; \psi) + \partial_x q_*^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}; \psi) \\ &= \int_{\mathbb{R}} D^{(\theta)}(t, x; s) \, ds = - \int_{\mathbb{R}} |D^{(\theta)}(t, x; s)| \, ds \end{aligned}$$

in the sense of distributions. We take the smooth cutoff function $\varphi(t, x) \geq 0$ as a test function in (4.3) such that $\varphi \equiv 1$ on K , and $\hat{K} := \text{supp } \varphi$ is compact in \mathbb{R}_+^2 so that $K \subset \hat{K} \Subset \mathbb{R}_+^2$. Then we have

$$\begin{aligned} \int_{\mathbb{R}} |D^{(\theta)}(\cdot, \cdot; s)|(K) \, ds &\leq - \int_{\mathbb{R}} \langle D^{(\theta)}(\cdot, \cdot; s), \varphi \rangle \, ds \\ &= \int_{\hat{K}} \left(\eta_*^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}) \partial_t \varphi + q_*^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}) \partial_x \varphi \right) \, dx \, dt \\ &\leq C(w_0; K), \end{aligned}$$

where we have used Lemma 3.1, $D^{(\theta)}(t, x; s) \leq 0$, and (4.2) by taking φ as the test function. Therefore, we conclude that the total variation of $D^{(\theta)}(t, x; s)$ is locally uniformly bounded with respect to $\theta \in (0, \theta_0]$ as measures, as indicated in (4.1).

Moreover, we know that $\text{supp } D^{(\theta)}(t, x; \cdot) \subset [-w_0^{(\theta)}, w_0^{(\theta)}]$. Since $0 \leq w_0^{(\theta)} \leq w_0$, we conclude that $\text{supp } D^{(\theta)}(t, x; \cdot) \subset [-w_0, w_0]$. \square

5. Relation between $(\eta_\xi^{(\theta)}, q_\xi^{(\theta)})$ and (η_ξ, q_ξ) . In this section, we show a uniform relation between a family of weak entropy pairs $(\eta_\xi^{(\theta)}, q_\xi^{(\theta)})$ and the corresponding entropy pair (η_ξ, q_ξ) on any function $(\rho^{(\theta)}, m^{(\theta)})$ satisfying (3.1) with respect to $\theta \in (0, \theta_0]$.

The weak entropy pairs $(\eta_\xi^{(\theta)}, q_\xi^{(\theta)})$ of system (2.1) with $\theta > 0$ under consideration are

$$(5.1) \quad \eta_\xi^{(\theta)}(\rho, m) := \frac{\rho \int_{\mathbb{R}} e^{\frac{\xi}{1-\xi^2}(\frac{m}{\rho} + \frac{\rho^\theta}{\theta} \tau)} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} \, d\tau}{\int_{\mathbb{R}} e^{\frac{\xi \tau}{\theta(1-\xi^2)}} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} \, d\tau} \geq 0,$$

$$(5.2) \quad q_\xi^{(\theta)}(\rho, m) := \frac{\rho \int_{\mathbb{R}} (\frac{m}{\rho} + \rho^\theta \tau) e^{\frac{\xi}{1-\xi^2}(\frac{m}{\rho} + \frac{\rho^\theta}{\theta} \tau)} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} \, d\tau}{\int_{\mathbb{R}} e^{\frac{\xi \tau}{\theta(1-\xi^2)}} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} \, d\tau}.$$

These entropy pairs are obtained by choosing $\psi(s)$ in (2.10)–(2.11) as

$$(5.3) \quad \psi_\xi^{(\theta)}(s) = \frac{\int_{\mathbb{R}} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} \, d\tau}{\int_{\mathbb{R}} e^{\frac{\xi \tau}{\theta(1-\xi^2)}} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} \, d\tau} e^{\frac{\xi}{1-\xi^2} s}.$$

We first consider the uniform boundedness of $(\eta_\xi^{(\theta)}, q_\xi^{(\theta)})$.

LEMMA 5.1. *For any function $(\rho^{(\theta)}, m^{(\theta)})$ satisfying (3.1), $(\eta_\xi^{(\theta)}, q_\xi^{(\theta)})(\rho^{(\theta)}, m^{(\theta)})$ defined in (5.1)–(5.2) are uniformly bounded with respect to $\theta \in (0, \theta_0]$ when $\xi \in [-\sqrt{2} + 1, \sqrt{2} - 1]$.*

Proof. For simplicity of notation, we drop the superscript in $(\rho^{(\theta)}, m^{(\theta)})$ in this proof.

First, $\xi \in [-\sqrt{2}+1, \sqrt{2}-1]$ is equivalent to the inequality: $\frac{2|\xi|}{1-\xi^2} \leq 1$. By definition, we have

$$\begin{aligned} |\eta_\xi^{(\theta)}(\rho, m)| &= \frac{\int_{-1}^1 (\rho e^{\frac{\xi}{1-\xi^2} \frac{m}{\rho}} e^{\frac{\xi\tau}{1-\xi^2} \frac{\rho^\theta-1}{\theta}}) e^{\frac{\xi\tau}{\theta(1-\xi^2)}} [1-\tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}{\int_{-1}^1 e^{\frac{\xi\tau}{\theta(1-\xi^2)}} [1-\tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau} \\ &\leq \sup_{\tau \in [-1,1]} \left(\rho e^{\frac{\xi}{1-\xi^2} \frac{m}{\rho}} e^{\frac{\xi\tau}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} \right) \\ &\leq \left(\rho e^{\frac{|m|}{\rho}} \right)^{\frac{|\xi|}{1-\xi^2}} \sup_{\tau \in [-1,1]} \left(\rho^{1-\frac{|\xi|}{1-\xi^2}} e^{\frac{\xi\tau}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} \right) \\ &\leq e^{\frac{|\xi|}{1-\xi^2} w_0} \sup_{\tau \in [-1,1]} \left(\rho^{1-\frac{|\xi|}{1-\xi^2}} e^{\frac{\xi\tau}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} \right). \end{aligned}$$

When $\rho \geq 1$,

$$\sup_{\tau \in [-1,1]} \left(\rho^{1-\frac{|\xi|}{1-\xi^2}} e^{\frac{\xi\tau}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} \right) = \rho^{1-\frac{|\xi|}{1-\xi^2}} e^{\frac{|\xi|}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} \leq e^{\left(1-\frac{|\xi|}{1-\xi^2}\right) w_0 + \frac{|\xi|}{1-\xi^2} w_0} \leq e^{w_0};$$

when $\rho \in (0, 1)$,

$$\sup_{\tau \in [-1,1]} \left(\rho^{1-\frac{|\xi|}{1-\xi^2}} e^{\frac{\xi\tau}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} \right) = \rho^{1-\frac{|\xi|}{1-\xi^2}} e^{-\frac{|\xi|}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} \leq \rho^{1-\frac{2|\xi|}{1-\xi^2}} \leq 1 \leq e^{w_0},$$

where we have used (3.4) and $\frac{\rho^\theta-1}{\theta} \leq w_0$. The argument for the uniform boundedness of $q_\xi^{(\theta)}(\rho^{(\theta)}, m^{(\theta)})$ is similar. \square

The main estimate of this section is the following:

PROPOSITION 5.2. *Given any $\theta > 0$ and any function $(\rho^{(\theta)}, m^{(\theta)})$ satisfying (3.1), then, for any interval $[a, b] \Subset (-\sqrt{2}+1, \sqrt{2}-1)$, there exists $C := C(w_0; a, b) > 0$ independent of $\theta \in (0, \theta_0]$ such that*

$$|\eta_\xi^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}) - \eta_\xi(\rho^{(\theta)}, m^{(\theta)})| + |q_\xi^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}) - q_\xi(\rho^{(\theta)}, m^{(\theta)})| \leq C\sqrt{\theta}$$

for any $\xi \in [a, b]$.

Proof. For simplicity, we drop the superscript in $(\rho^{(\theta)}, m^{(\theta)})$ in the proof. We divide the proof into four steps.

1. The difference between $\eta_\xi^{(\theta)}$ and η_ξ is

$$\begin{aligned} (5.4) \quad &\eta_\xi^{(\theta)}(\rho, m) - \eta_\xi(\rho, m) \\ &= \frac{\int_{-1}^1 \rho e^{\frac{\xi}{1-\xi^2} \frac{m}{\rho}} \left(e^{\frac{\xi}{1-\xi^2} \frac{\rho^\theta-1}{\theta} \tau} - \rho^{\frac{\xi^2}{1-\xi^2}} \right) e^{\frac{\xi\tau}{\theta(1-\xi^2)}} [1-\tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}{\int_{-1}^1 e^{\frac{\xi\tau}{\theta(1-\xi^2)}} [1-\tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau} \\ &=: \langle \nu_\xi^{(\theta)}(\cdot), f_\xi(\cdot) \rangle \\ &= \langle \nu_\xi^{(\theta)}(\cdot), f_\xi(\cdot) - f_\xi(\xi) \rangle + f_\xi(\xi), \end{aligned}$$

where

$$(5.5) \quad f_\xi(\tau) = \rho e^{\frac{\xi}{1-\xi^2} \frac{m}{\rho}} \left(e^{\frac{\xi}{1-\xi^2} \frac{\rho^\theta-1}{\theta} \tau} - \rho^{\frac{\xi^2}{1-\xi^2}} \right)$$

is a C^1 -function in $\tau \in [-1, 1]$, and

$$\nu_\xi^{(\theta)}(\tau) = \frac{e^{\frac{\xi\tau}{\theta(1-\xi^2)}} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}}}{\int_{-1}^1 e^{\frac{\xi\tau}{\theta(1-\xi^2)}} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}$$

may be regarded as a probability measure over $C([-1, 1])$ for each fixed $\theta > 0$.

2. *Claim 1:* For any C^1 -function $f(\tau)$ and $\theta \in (0, \theta_0]$,

$$|\langle \nu_\xi^{(\theta)}, f(\cdot) - f(\xi) \rangle| \leq C \|f'\|_{L^\infty} \sqrt{\theta} \quad \text{for any } \xi \in (-\sqrt{2} + 1, \sqrt{2} - 1),$$

where $C > 0$ is a constant independent of $\theta \in (0, \theta_0]$, $f \in C^1([-1, 1])$, and $\xi \in (-\sqrt{2} + 1, \sqrt{2} - 1)$.

The claim can be shown as follows: Notice that

$$\begin{aligned} (5.6) \quad & |\langle \nu_\xi^{(\theta)}, f(\cdot) - f(\xi) \rangle| \\ &= \left| \frac{\int_{\mathbb{R}} (f(\tau) - f(\xi)) e^{\frac{\xi}{\theta(1-\xi^2)}\tau} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}{\int_{\mathbb{R}} e^{\frac{\xi}{\theta(1-\xi^2)}\tau} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau} \right| \\ &\leq \frac{\int_{\mathbb{R}} |f(\tau) - f(\xi)| e^{\frac{\xi}{\theta(1-\xi^2)}\tau} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau}{\int_{\mathbb{R}} e^{\frac{\xi}{\theta(1-\xi^2)}\tau} [1 - \tau^2]_+^{\frac{1-\theta}{2\theta}} d\tau} \\ &= \frac{\int_{\mathbb{R}} |f(\tau) - f(\xi)| e^{\frac{\xi}{1-\xi^2}\tau} (e^{\frac{\xi}{1-\xi^2}\tau} [1 - \tau^2]_+^{\frac{1}{2}})^{\frac{1-\theta}{\theta}} d\tau}{\int_{\mathbb{R}} e^{\frac{\xi}{1-\xi^2}\tau} (e^{\frac{\xi}{1-\xi^2}\tau} [1 - \tau^2]_+^{\frac{1}{2}})^{\frac{1-\theta}{\theta}} d\tau} \\ &= \frac{\int_{\mathbb{R}} |f(\tau) - f(\xi)| e^{\frac{\xi}{1-\xi^2}\tau} (e^{\frac{\xi}{1-\xi^2}\tau} [1 - \tau^2]_+^{\frac{1}{2}})^{\frac{1-\theta}{\theta}} (e^{\frac{\xi}{1-\xi^2}\xi} (1 - \xi^2)^{\frac{1}{2}})^{-\frac{1-\theta}{\theta}} d\tau}{\int_{\mathbb{R}} e^{\frac{\xi}{1-\xi^2}\tau} (e^{\frac{\xi}{1-\xi^2}\tau} [1 - \tau^2]_+^{\frac{1}{2}})^{\frac{1-\theta}{\theta}} (e^{\frac{\xi}{1-\xi^2}\xi} (1 - \xi^2)^{\frac{1}{2}})^{-\frac{1-\theta}{\theta}} d\tau} \\ &= \frac{\int_{-1}^1 |f(\tau) - f(\xi)| e^{\frac{\xi}{1-\xi^2}\tau} e^{\frac{g(\tau)-g(\xi)}{h(\theta)}} d\tau}{\int_{-1}^1 e^{\frac{\xi}{1-\xi^2}\tau} e^{\frac{g(\tau)-g(\xi)}{h(\theta)}} d\tau}, \end{aligned}$$

where $h(\theta) = \frac{\theta}{1-\theta} > 0$, and $g(\tau) = \frac{\xi}{1-\xi^2}\tau + \frac{1}{2}\ln(1 - \tau^2)$ with $g'(\xi) = 0$ and $g''(\tau) = -\frac{1+\tau^2}{(1-\tau^2)^2}$ which is singular near $\tau = \pm 1$. Again, using the Taylor expansion, we have

$$g(\tau) - g(\xi) = \frac{1}{2}g''(\tilde{\tau})(\tau - \xi)^2 = -\frac{1 + \tilde{\tau}^2}{2(1 - \tilde{\tau}^2)^2}(\tau - \xi)^2 \leq -\frac{1}{2}(\tau - \xi)^2$$

for some $\tilde{\tau}$ between ξ and τ .

On the other hand, when $\tau \in (-\frac{1}{2}, \frac{1}{2})$, $\tilde{\tau}$ must be in $(-\frac{1}{2}, \frac{1}{2})$ for any $\xi \in (-\sqrt{2} + 1, \sqrt{2} - 1)$. Then we have

$$(5.7) \quad g(\tau) - g(\xi) \geq -\frac{1 + \frac{1}{4}}{2(1 - \frac{1}{4})^2}(\tau - \xi)^2 = -\frac{10}{9}(\tau - \xi)^2 \quad \text{if } \tau \in (-\frac{1}{2}, \frac{1}{2}).$$

Thus, we see from (5.6)–(5.7) that

$$\begin{aligned}
 |\langle \nu_\xi^{(\theta)}, f(\cdot) - f(\xi) \rangle| &\leq \frac{e^{\frac{2|\xi|}{1-\xi^2}} \int_{-1}^1 |f(\tau) - f(\xi)| e^{-\frac{(\tau-\xi)^2}{2h(\theta)}} d\tau}{\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\frac{10(\tau-\xi)^2}{9h(\theta)}} d\tau} \\
 &\leq \frac{e^{\frac{1-\xi}{\sqrt{h(\theta)}}} |f(\xi + \sqrt{h(\theta)}s) - f(\xi)| e^{-\frac{1}{2}s^2} ds}{\int_{-\frac{1-2\xi}{2\sqrt{h(\theta)}}}^{\frac{1-2\xi}{2\sqrt{h(\theta)}}} e^{-\frac{10}{9}s^2} ds} \\
 &\leq C \|f'\|_{L^\infty} \sqrt{h(\theta)} \frac{\int_{-\infty}^{\infty} |s| e^{-\frac{1}{2}s^2} ds}{\int_{-\frac{3}{2}+\sqrt{2}}^{\frac{3}{2}+\sqrt{2}} e^{-\frac{10}{9}s^2} ds} \leq C \|f'\|_{L^\infty} \sqrt{\theta},
 \end{aligned}$$

where $C > 0$ is a generic constant independent of $\theta > 0$. We conclude the proof of Claim 1.

3. *Claim 2:* For any interval $[a, b] \Subset (-\sqrt{2}+1, \sqrt{2}-1)$, there exists $C = C(w_0; a, b) > 0$ independent of $\theta \in (0, \theta_0]$ such that, for any (ρ, m) satisfying (3.1), $f_\xi(\tau)$ defined in (5.5) satisfies that

$$|f'_\xi(\tau)| \leq C(w_0; a, b) \quad \text{for any } \xi \in [a, b] \text{ and } \tau \in [-1, 1].$$

This can be seen as follows: First, we have

$$f'_\xi(\tau) = \frac{\xi}{1-\xi^2} \frac{\rho^\theta - 1}{\theta} \rho e^{\frac{-\xi}{1-\xi^2}(\frac{m}{\rho} + \frac{\rho^\theta - 1}{\theta}\tau)}.$$

We consider two cases.

Case 1: $\rho \geq 1$. Then

$$(5.8) \quad |f'_\xi(\tau)| \leq \frac{|\xi|}{1-\xi^2} \frac{\rho^\theta - 1}{\theta} \rho e^{\frac{|\xi|}{1-\xi^2}(|\frac{m}{\rho}| + \frac{\rho^\theta - 1}{\theta})} \leq \frac{|\xi|}{1-\xi^2} w_0 e^{w_0} e^{\frac{|\xi|}{1-\xi^2} w_0} \leq \frac{w_0}{2} e^{\frac{3w_0}{2}},$$

where we have used that

$$(5.9) \quad \frac{|\xi|}{1-\xi^2} \leq \frac{1}{2}, \quad \rho^\theta - 1 \leq \theta w_0, \quad \rho \leq e^{w_0}, \quad \frac{|m|}{\rho} + \frac{\rho^\theta - 1}{\theta} \leq w_0.$$

Case 2: $\rho \in [0, 1)$. Then, for any $\xi \in [a, b] \Subset (-\sqrt{2}+1, \sqrt{2}-1)$,

$$\begin{aligned}
 (5.10) \quad |f'_\xi(\tau)| &\leq \frac{1-\rho^\theta}{2\theta} \rho e^{\frac{|\xi|}{1-\xi^2}(|\frac{m}{\rho}| + \frac{1-\rho^\theta}{\theta})} \leq \frac{1}{2} |\ln \rho| \rho e^{\frac{|\xi|}{1-\xi^2}(w_0 - 2 \ln \rho)} \\
 &\leq \frac{1}{2} e^{\frac{w_0}{2}} \rho^{1-\frac{2|\xi|}{1-\xi^2}} |\ln \rho| \leq C(w_0; a, b)
 \end{aligned}$$

for $C(w_0; a, b) > 0$ depending only on w_0 and (a, b) , but independent of $\theta \in (0, \theta_0]$, where we have used (5.9) and $1 - \frac{2|\xi|}{1-\xi^2} \geq \delta(a, b) > 0$ for some $\delta(a, b)$ depending on (a, b) .

Therefore, Claim 2 follows by combining estimates (5.8)–(5.10) together.

4. *Claim 3:* For any (ρ, m) satisfying (3.1),

$$(5.11) \quad |f_\xi(\xi)| \leq C\theta \quad \text{for any } \xi \in (-\sqrt{2}+1, \sqrt{2}-1),$$

where $C := C(w_0) > 0$ is a constant depending only on w_0 , but independent of $\theta \in (0, \theta_0]$ and $\xi \in (-\sqrt{2}+1, \sqrt{2}-1)$.

This can be seen as follows: Similarly to Lemma 5.1, we have

$$\begin{aligned} |f_\xi(\xi)| &\leq (\rho e^{\frac{|m|}{\rho}})^{\frac{|\xi|}{1-\xi^2}} \rho^{1-\frac{|\xi|}{1-\xi^2}} \left| e^{\frac{\xi^2}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} - \rho^{\frac{\xi^2}{1-\xi^2}} \right| \\ &\leq e^{\frac{|\xi|}{1-\xi^2} w_0} \rho^{1-\frac{|\xi|}{1-\xi^2}} \left| e^{\frac{\xi^2}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} - \rho^{\frac{\xi^2}{1-\xi^2}} \right|. \end{aligned}$$

Then statement (5.11) is clear when $\rho = 0$. When $\rho > 0$, we use that $\lim_{\theta \rightarrow 0} e^{\beta \frac{\rho^\theta-1}{\theta}} = \rho^\beta$ for any β to obtain

$$e^{\frac{\xi^2}{1-\xi^2} \frac{\rho^\theta-1}{\theta}} - \rho^{\frac{\xi^2}{1-\xi^2}} = \int_0^\theta \frac{d}{ds} \left(e^{\frac{\xi^2}{1-\xi^2} \frac{\rho^s-1}{s}} \right) ds = \frac{\xi^2}{1-\xi^2} \int_0^\theta e^{\frac{\xi^2}{1-\xi^2} \frac{\rho^s-1}{s}} \frac{d}{ds} \left(\frac{\rho^s-1}{s} \right) ds.$$

Then

$$\begin{aligned} |f_\xi(\xi)| &\leq C \frac{|\xi|^2}{1-\xi^2} \rho^{1-\frac{|\xi|}{1-\xi^2}} \left| \int_0^\theta e^{\frac{\xi^2}{1-\xi^2} \frac{\rho^s-1}{s}} \frac{d}{ds} \left(\frac{\rho^s-1}{s} \right) ds \right| \\ &\leq C \rho^{1-\frac{|\xi|}{1-\xi^2}} \max_{s \in (0, \theta)} \left(e^{\frac{\xi^2}{1-\xi^2} \frac{\rho^s-1}{s}} \right) \int_0^\theta \left| \frac{d}{ds} \left(\frac{\rho^s-1}{s} \right) \right| ds \\ &\leq C e^{\frac{\xi^2}{1-\xi^2} w_0} \rho^{1-\frac{|\xi|}{1-\xi^2}} \frac{\rho^\theta - 1 - \theta \ln \rho}{\theta} \\ &\leq C \rho^{1-\frac{|\xi|}{1-\xi^2}} \frac{\rho^\theta - 1 - \theta \ln \rho}{\theta}, \end{aligned}$$

since $\rho^\theta - 1 - \theta \ln \rho \geq 0$, where we have used the fact that

$$\frac{d}{ds} \left(\frac{\rho^s-1}{s} \right) = \frac{1}{s^2} (s \rho^s \ln \rho - \rho^s + 1) = \frac{1}{s^2} (\rho^s \ln \rho^s - \rho^s + 1) \geq 0 \quad \text{for } s \in (0, \theta].$$

On the other hand, the Taylor expansion with respect to θ gives

$$(5.12) \quad \rho^{1-\frac{|\xi|}{1-\xi^2}} (\rho^\theta - 1 - \theta \ln \rho) = \frac{1}{2} \rho^{1-\frac{|\xi|}{1-\xi^2}} (\ln \rho)^2 \tilde{\theta}^2 \leq C \theta^2 \quad \text{for some } \tilde{\theta} \in (0, \theta).$$

Then we conclude Claim 3.

5. Combining Claims 1–3 above together, we see from (5.4) that

$$\begin{aligned} |\eta_\xi^{(\theta)}(\rho, m) - \eta_\xi(\rho, m)| &\leq |\langle \nu_\xi^{(\theta)}(\tau), f_\xi(\tau) - f_\xi(\xi) \rangle| + |f_\xi(\xi)| \\ &\leq C \|f'\|_{L^\infty} \sqrt{\theta} + C \theta \leq C \sqrt{\theta}. \end{aligned}$$

The proof for the estimate that $|q_\xi^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}) - q_\xi(\rho^{(\theta)}, m^{(\theta)})| \leq C \sqrt{\theta}$ follows the same argument. This completes the proof. \square

By a similar computation to (5.12), we obtain

$$(5.13) \quad |\eta_*^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}) - \eta_*^{(0)}(\rho^{(\theta)}, m^{(\theta)})| + |q_*^{(\theta)}(\rho^{(\theta)}, m^{(\theta)}) - q_*^{(0)}(\rho^{(\theta)}, m^{(\theta)})| \leq C \theta.$$

Remark 5.3. From this lemma, we can see that the restriction that $\xi \in (-\sqrt{2} + 1, \sqrt{2} - 1)$ is natural for Claims 1–2 in steps 2–3 in the proof above.

6. Proof of the main theorem. We divide the proof of the main theorem, Theorem 2.9, into three steps.

1. For any $\theta > 0$, assume that $(\rho^{(\theta)}, m^{(\theta)})$ is the corresponding entropy solution of (2.1), constructed in Theorem 2.3. It follows from (3.1) that the solution sequence $(\rho^{(\theta)}, m^{(\theta)})$ is uniformly bounded with respect to $\theta \in (0, \theta_0]$, which satisfies condition (2.16) in Theorem 2.6.

We now show that the solution sequence $(\rho^{(\theta)}, m^{(\theta)})$ satisfies condition (2.17) in Theorem 2.6 for any $\delta \in (0, \sqrt{2} - 1)$. Notice that

$$(6.1) \quad \partial_t \eta_\xi(\rho^{(\theta)}, m^{(\theta)}) + \partial_x q_\xi(\rho^{(\theta)}, m^{(\theta)}) =: J_1^{(\theta)} + J_2^{(\theta)}$$

with

$$\begin{aligned} J_1^{(\theta)} &= \partial_t (\eta_\xi(\rho^{(\theta)}, m^{(\theta)}) - \eta_\xi^{(\theta)}(\rho^{(\theta)}, m^{(\theta)})) + \partial_x (q_\xi(\rho^{(\theta)}, m^{(\theta)}) - q_\xi^{(\theta)}(\rho^{(\theta)}, m^{(\theta)})), \\ J_2^{(\theta)} &= \int_{-w_0}^{w_0} (\psi_\xi^{(\theta)})''(s) D^{(\theta)}(t, x; s) ds. \end{aligned}$$

By Proposition 5.2, we have

$$\|J_1^{(\theta)}\|_{H^{-1}(K)} \leq C_K \sqrt{\theta} \rightarrow 0 \quad \text{as } \theta \rightarrow 0$$

for any subset $K \Subset \mathbb{R}_+^2$. This implies that $J_1^{(\theta)}$ is compact in H_{loc}^{-1} .

Notice that $\text{supp } D^{(\theta)}(t, x; \cdot) \subset [-w_0, w_0]$. By (5.3), we see that, for $s \in [-w_0, w_0]$,

$$(\psi_\xi^{(\theta)})''(s) = \frac{\xi^2}{(1 - \xi^2)^2} \frac{e^{\frac{\xi}{1-\xi^2}s} \int_{-1}^1 (1 - \tau^2)^{\frac{1-\theta}{2\theta}} d\tau}{\int_{-1}^1 e^{\frac{\xi\tau}{\theta(1-\xi^2)}} (1 - \tau^2)^{\frac{1-\theta}{2\theta}} d\tau} \leq \frac{2\xi^2}{(1 - \xi^2)^2} e^{\frac{|\xi|w_0}{1-\xi^2}} \leq C,$$

where we have used the fact that $e^{\frac{\xi\tau}{\theta(1-\xi^2)}} \geq 1$ when $\xi\tau \geq 0$. Thus, $J_2^{(\theta)}$ is uniformly bounded as a Radon measure sequence, which implies that $J_2^{(\theta)}$ is compact in $W_{\text{loc}}^{-1,q}$ for $q \in (1, 2)$.

Combining the compactness of both $J_1^{(\theta)}$ and $J_2^{(\theta)}$ above together, we conclude that

$$J_1^{(\theta)} + J_2^{(\theta)} \quad \text{is compact in } W_{\text{loc}}^{-1,q} \text{ for } q \in (1, 2).$$

On the other hand,

$$J_1^{(\theta)} + J_2^{(\theta)} = \partial_t \eta_\xi(\rho^{(\theta)}, m^{(\theta)}) + \partial_x q_\xi(\rho^{(\theta)}, m^{(\theta)}) \quad \text{is uniformly bounded in } W_{\text{loc}}^{-1,\infty}.$$

Then the interpolation embedding compactness (cf. [11, 13]) implies that $J_1^{(\theta)} + J_2^{(\theta)}$ is H_{loc}^{-1} -compact. Therefore, the solution sequence $(\rho^{(\theta)}, m^{(\theta)})$ satisfies condition (2.17).

We now employ Theorem 2.6 to conclude that $(\rho^{(\theta)}, m^{(\theta)})(t, x)$ (extracting a subsequence if necessary) strongly converges to a function $(\rho, m)(t, x) \in L^\infty$ a.e. $(t, x) \in \mathbb{R}_+^2$. From the uniform estimates in section 3, we conclude that

$$0 \leq \rho(t, x) \leq e^{w_0}, \quad |m(t, x)| \leq \rho(t, x)(|\ln \rho(t, x)| + w_0) \quad \text{a.e. } (t, x) \in \mathbb{R}_+^2.$$

2. We next prove that $(\rho, m)(t, x)$ satisfies the energy inequality. For any fixed $\theta > 0$, $(\rho^{(\theta)}, m^{(\theta)})(t, x)$ satisfies

$$(6.2) \quad \partial_t \eta_*(\rho^{(\theta)}, m^{(\theta)}) + \partial_x q_*(\rho^{(\theta)}, m^{(\theta)}) \leq 0$$

in the sense of distributions. Notice that (5.13) holds for any (ρ, m) satisfying (3.1). Taking $\theta \rightarrow 0$ in (6.2), we have

$$\partial_t \eta_*^{(0)}(\rho, m) + \partial_x q_*^{(0)}(\rho, m) \leq 0$$

in the sense of distributions. This shows that (ρ, m) is an entropy solution of system (2.1) with $\theta = 0$ in the sense of Definition 2.1.

3. For $\xi \in (-\sqrt{2} + 1, \sqrt{2} - 1)$, there exists a nonpositive measure $D(t, x; \xi)$ such that

$$\int_{\mathbb{R}} (\psi_\xi^{(\theta)})''(s) D^{(\theta)}(t, x; s) ds \rightharpoonup D(t, x; \xi) \quad \text{in Radon measures as } \theta \rightarrow 0.$$

Taking the weak limit in (6.1), we conclude that, for each $\xi \in (-\sqrt{2} + 1, \sqrt{2} - 1)$,

$$\partial_t \eta_\xi(\rho, m) + \partial_x q_\xi(\rho, m) = D(t, x; \xi) \leq 0$$

in the sense of distributions. This implies that (ρ, m) satisfies the entropy inequality (2.7) with entropy pairs (η_ξ, q_ξ) in the sense of distributions.

In particular, for any nonnegative $\psi(\xi) \in L^\infty(\mathbb{R})$ with $\text{supp } \psi(\xi) \subseteq [-\sqrt{2} + 1, \sqrt{2} - 1]$, (ρ, m) satisfies

$$\partial_t \eta^{(0)}(\rho, m; \psi) + \partial_x q^{(0)}(\rho, m; \psi) \leq 0$$

in the sense of distributions.

This completes the proof of Theorem 2.9.

7. Convergence of Riemann solutions containing the vacuum states. In this section, we show the pointwise convergence of Riemann solutions that include the vanishing vacuum states between the two rarefaction waves and the limit of one-sided vacuum states, which are discussed in sections 7.1 and 7.2, respectively. Naturally, all the Riemann solutions in this section satisfy the conditions of Theorem 2.9, especially (2.18).

Consider the Riemann solutions of system (2.1) for $\theta \geq 0$ with initial data

$$(\rho_0, \frac{m_0}{\rho_0})(x) = \begin{cases} (\rho_L, u_L) & \text{for } x < 0, \\ (\rho_R, u_R) & \text{for } x > 0. \end{cases}$$

LEMMA 7.1. *For $\theta > 0$, the shock curves are*

$$(7.1) \quad S_1^{(\theta)} : u_R = u_L - \sqrt{\left(\frac{1}{\rho_L} - \frac{1}{\rho_R}\right) \frac{\rho_R^{2\theta+1} - \rho_L^{2\theta+1}}{2\theta+1}} \quad \text{for } u_L > u_R \text{ and } \rho_L < \rho_R,$$

$$(7.2) \quad S_2^{(\theta)} : u_R = u_L - \sqrt{\left(\frac{1}{\rho_L} - \frac{1}{\rho_R}\right) \frac{\rho_R^{2\theta+1} - \rho_L^{2\theta+1}}{2\theta+1}} \quad \text{for } u_L > u_R \text{ and } \rho_L > \rho_R,$$

and the respective solutions of shocks $S_j^{(\theta)}$, $j = 1, 2$, are

$$(\rho^{(\theta)}, u^{(\theta)}) = \begin{cases} (\rho_L, u_L) & \text{for } x < \sigma t, \\ (\rho_R, u_R) & \text{for } x > \sigma t \end{cases}$$

with shock speed $\sigma := \frac{[\rho u]}{[\rho]} = \frac{\rho_R u_R - \rho_L u_L}{\rho_R - \rho_L}$. Correspondingly, the rarefaction curves are

$$(7.3) \quad R_1^{(\theta)} : w_1^{(\theta)}(\rho_R, u_R) = w_1^{(\theta)}(\rho_L, u_L) =: w_1^{(\theta)} \quad \text{for } u_L < u_R \text{ and } \rho_L > \rho_R,$$

$$(7.4) \quad R_2^{(\theta)} : w_2^{(\theta)}(\rho_R, u_R) = w_2^{(\theta)}(\rho_L, u_L) =: w_2^{(\theta)} \quad \text{for } u_L < u_R \text{ and } \rho_L < \rho_R,$$

where the respective solutions of rarefaction waves $R_j^{(\theta)}$, $j = 1, 2$, are

$$(\rho^{(\theta)}, u^{(\theta)}) = \begin{cases} (\rho_L, u_L) & \text{for } \frac{x}{t} \leq \lambda_j^{(\theta)}(\rho_L, u_L), \\ ((\frac{(-1)^j \theta (\frac{x}{t} - w_j^{(\theta)})}{\theta + 1})^{\frac{1}{\theta}}, \frac{x}{t} + \theta w_j^{(\theta)}) & \text{for } \lambda_j^{(\theta)}(\rho_L, u_L) < \frac{x}{t} < \lambda_j^{(\theta)}(\rho_R, u_R), \\ (\rho_R, u_R) & \text{for } \frac{x}{t} \geq \lambda_j^{(\theta)}(\rho_R, u_R), \end{cases}$$

where $\lambda_j^{(\theta)}(\rho, u) = u + (-1)^j \rho^\theta$, $j = 1, 2$, are two eigenvalues.

For $\theta = 0$, the shock curves are

$$(7.5) \quad S_1^{(0)} : u_R = u_L - \sqrt{\left(\frac{1}{\rho_L} - \frac{1}{\rho_R}\right)(\rho_R - \rho_L)} \quad \text{for } u_L > u_R \text{ and } \rho_L < \rho_R,$$

$$(7.6) \quad S_2^{(0)} : u_R = u_L - \sqrt{\left(\frac{1}{\rho_L} - \frac{1}{\rho_R}\right)(\rho_R - \rho_L)} \quad \text{for } u_L > u_R \text{ and } \rho_L > \rho_R,$$

and the respective solutions of shocks $S_j^{(0)}$, $j = 1, 2$, are

$$(\rho^{(0)}, u^{(0)}) = \begin{cases} (\rho_L, u_L) & \text{for } x < \sigma t, \\ (\rho_R, u_R) & \text{for } x > \sigma t \end{cases}$$

with shock speed $\sigma := \frac{[\rho u]}{[\rho]} = \frac{\rho_R u_R - \rho_L u_L}{\rho_R - \rho_L}$. Correspondingly, the rarefaction curves are

$$(7.7) \quad R_1^{(0)} : w_1^{(0)}(\rho_R, u_R) = w_1^{(0)}(\rho_L, u_L) =: w_1^{(0)} \quad \text{for } u_L < u_R \text{ and } \rho_L > \rho_R,$$

$$(7.8) \quad R_2^{(0)} : w_2^{(0)}(\rho_R, u_R) = w_2^{(0)}(\rho_L, u_L) =: w_2^{(0)} \quad \text{for } u_L < u_R \text{ and } \rho_L < \rho_R,$$

where the respective solutions of rarefaction waves $R_j^{(0)}$ are

$$(\rho^{(0)}, u^{(0)}) = \begin{cases} (\rho_L, u_L) & \text{for } \frac{x}{t} \leq \lambda_j^{(0)}(\rho_L, u_L), \\ (w_j^{(0)} e^{(-1)^j \frac{x}{t} - 1}, \frac{x}{t} - (-1)^j) & \text{for } \lambda_j^{(0)}(\rho_L, u_L) < \frac{x}{t} < \lambda_j^{(0)}(\rho_R, u_R), \\ (\rho_R, u_R) & \text{for } \frac{x}{t} \geq \lambda_j^{(0)}(\rho_R, u_R), \end{cases}$$

where $\lambda_j^{(0)}(\rho, u) = u + (-1)^j$, $j = 1, 2$, are two eigenvalues.

From (7.1)–(7.2) and (7.5)–(7.6), if $\rho_L = 0$ or $\rho_R = 0$, then $\rho_L = \rho_R = 0$, which indicates that, for $\theta \geq 0$, shock waves cannot connect with a vacuum state. However, for $\theta \geq 0$, (7.3) and (7.7) show that the right-states of $R_1^{(\theta)}$ and $R_1^{(0)}$ may be a vacuum state $\rho_R = 0$, while (7.4) and (7.8) show the left-states of $R_2^{(\theta)}$ and $R_2^{(0)}$ may also be a vacuum state $\rho_L = 0$. Based on the above analysis, we now separate our discussion into the two-sided nonvacuum case and the one-sided vacuum case.

7.1. Two-sided nonvacuum case. For this case, $\rho_L > 0$ and $\rho_R > 0$, and the possible vacuum occurs in the middle state. The general statement is the following:

PROPOSITION 7.2. *For fixed (ρ_L, u_L) and (ρ_R, u_R) with $\rho_L > 0$ and $\rho_R > 0$, there exists a unique Riemann solution $(\rho^{(\theta)}, u^{(\theta)})$ for $\theta \geq 0$, and $(\rho^{(\theta)}, u^{(\theta)}) \rightarrow (\rho^{(0)}, u^{(0)})$ as $\theta \rightarrow 0$.*

Proof. We divide the proof into four steps.

1. For $\theta > 0$, by [10], we have the following phase plane for fixed (ρ_L, u_L) as shown in Figure 7.1 on which $(0, u_{\text{cr}}^{(\theta)})$ with $u_{\text{cr}}^{(\theta)} = u_L + \frac{\rho_L^{(\theta)}}{\theta}$ is the intersect point of $R_1^{(\theta)}$ and $\{\rho = 0\}$, and $R_{2,\text{cr}}^{(\theta)}$ is the 2-rarefaction curve starting from $(0, u_{\text{cr}}^{(\theta)})$ separating region $IV_2^{(\theta)}$ from $IV_1^{(\theta)}$.

- If $(\rho_R, u_R) \in I^{(\theta)}$, then the solution $(\rho^{(\theta)}, u^{(\theta)})$ is $S_1^{(\theta)} + R_2^{(\theta)}$.
- If $(\rho_R, u_R) \in II^{(\theta)}$, then the solution $(\rho^{(\theta)}, u^{(\theta)})$ is $S_1^{(\theta)} + S_2^{(\theta)}$.
- If $(\rho_R, u_R) \in III^{(\theta)}$, then the solution $(\rho^{(\theta)}, u^{(\theta)})$ is $R_1^{(\theta)} + S_2^{(\theta)}$.
- If $(\rho_R, u_R) \in IV_1^{(\theta)}$, then the solution $(\rho^{(\theta)}, u^{(\theta)})$ is $R_1^{(\theta)} + R_2^{(\theta)}$ with the middle state $\rho_M^{(\theta)} > 0$.
- If $(\rho_R, u_R) \in IV_2^{(\theta)}$, then the solution $(\rho^{(\theta)}, u^{(\theta)})$ is $R_1^{(\theta)} + R_2^{(\theta)}$ with the middle state $\rho_M^{(\theta)} = 0$.

The criterion for the last case is $\rho_M^{(\theta)} = 0$ if and only if $\xi_1^{(\theta)} := u_L + \frac{\rho_L^{(\theta)}}{\theta} \leq u_R - \frac{\rho_R^{(\theta)}}{\theta} =: \xi_2^{(\theta)}$.

2. For $\theta = 0$, we have the following phase plane for fixed (ρ_L, u_L) as shown in Figure 7.2 similarly, on which $R_1^{(0)}$ and $\{\rho = 0\}$ do not intersect.

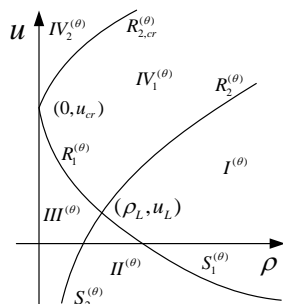


FIG. 7.1.

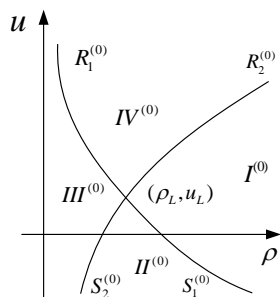


FIG. 7.2.

- If $(\rho_R, u_R) \in I^{(0)}$, then the solution $(\rho^{(0)}, u^{(0)})$ is $S_1^{(0)} + R_2^{(0)}$.
- If $(\rho_R, u_R) \in II^{(0)}$, then the solution $(\rho^{(0)}, u^{(0)})$ is $S_1^{(0)} + S_2^{(0)}$.
- If $(\rho_R, u_R) \in III^{(0)}$, then the solution $(\rho^{(0)}, u^{(0)})$ is $R_1^{(0)} + S_2^{(0)}$.
- If $(\rho_R, u_R) \in IV^{(0)}$, then the solution $(\rho^{(0)}, u^{(0)})$ is $R_1^{(0)} + R_2^{(0)}$.

Then we have the existence and uniqueness of the Riemann solution $(\rho^{(\theta)}, u^{(\theta)})$ for all $\theta \geq 0$.

3. For the shock curves, the direct computation leads to $(S_1^{(\theta)}, S_2^{(\theta)}) \rightarrow (S_1^{(0)}, S_2^{(0)})$ and the shock speed $\sigma^{(\theta)} \rightarrow \sigma^{(0)}$ uniformly as $\theta \rightarrow 0$, since ρ_L and $\rho_R > 0$. Similarly to the computation of (5.12), for the case that $\rho_L > 0$ and $\rho_R > 0$, $(R_1^{(\theta)}, R_2^{(\theta)}) \rightarrow (R_1^{(0)}, R_2^{(0)})$ and

$$(\lambda_1^{(\theta)}, \lambda_2^{(\theta)}) = (u - \rho^\theta, u + \rho^\theta) \rightarrow (\lambda_1^{(0)}, \lambda_2^{(0)}) = (u - 1, u + 1) \quad \text{uniformly as } \theta \rightarrow 0.$$

Thus, we conclude that the shock and rarefaction wave curves converge strongly, for the two-sided nonvacuum case.

4. With the convergence of each wave curve, we turn to the convergence of the corresponding Riemann solutions $(\rho^{(\theta)}, u^{(\theta)})$. The argument is divided into two cases:

Case 1: $\rho_M^{(\theta)} > 0$ for all $\theta > 0$. In this case, for the fixed left-state (ρ_L, u_L) , the right-state (ρ_R, u_R) belongs to $\bigcap_{\theta \in (0, \theta_0]} (I^{(\theta)} \cup II^{(\theta)} \cup III^{(\theta)} \cup IV^{(\theta)})$. The conclusion of step 3 leads to the uniform convergence of $(\rho^{(\theta)}, u^{(\theta)}) \rightarrow (\rho^{(0)}, u^{(0)})$ as $\theta \rightarrow 0$.

Case 2: There exists θ_* such that $\rho_M^{(\theta_*)} = 0$. For this case, $(\rho_R, u_R) \in IV^{(\theta_*)}$ so that the solution $(\rho^{(\theta_*)}, u^{(\theta_*)})$ is $R_1^{(\theta_*)} + R_2^{(\theta_*)}$ with $\xi_1^{(\theta_*)} \leq \xi_2^{(\theta_*)}$. Notice that

$$\xi_2^{(\theta)} - \xi_1^{(\theta)} = (u_R - \frac{\rho_R^\theta}{\theta}) - (u_L + \frac{\rho_L^\theta}{\theta}) = (u_R - u_L) - \frac{\rho_L^\theta + \rho_R^\theta}{\theta}.$$

There exists $\underline{\theta}_* < \theta_*$ such that $\xi_2^{(\theta)} < \xi_1^{(\theta)}$ for $\theta < \underline{\theta}_*$, which leads to $\rho_M^{(\theta)} > 0$. The rest of the argument is the same as in Case 1. \square

Remark 7.3. Since $\rho_L > 0$ and $\rho_R > 0$ in Proposition 7.2, condition (2.18) is satisfied.

Remark 7.4. In Case 2 of step 4 in the proof above, we observe that the centered vacuum state with $\rho_M^{(\theta)} = 0$ vanishes when θ is small enough. This represents the phenomenon of decavitation as $\gamma \rightarrow 1$, which is different from the formation of cavitation and concentration in the vanishing pressure limit (equivalently, the high Mach limit) is presented in Chen and Liu [8].

7.2. One-sided vacuum case. Next, we consider the one-sided vacuum case. Without loss of generality, we assume that $\rho_L = 0$, which connects $R_2^{(\theta)}$ with $R_2^{(0)}$. Notice that the fluid velocity u_L is not well-defined in the vacuum with $\rho_L = 0$ in general. In the following analysis, we fix the right-state (ρ_R, u_R) .

For $\theta > 0$, the Riemann solution $(\rho^{(\theta)}, u^{(\theta)})$ as shown in Figure 7.3 is

$$(\rho^{(\theta)}, u^{(\theta)}) = \begin{cases} ((\frac{\theta \frac{x}{t} - \theta u_R + \rho_R^\theta}{\theta + 1})^{\frac{1}{\theta}}, \frac{\frac{x}{t} + \theta u_R - \rho_R^\theta}{\theta + 1}) & \text{for } u_R - \frac{\rho_R^\theta}{\theta} < \frac{x}{t} < u_R + \rho_R^\theta, \\ (\rho_R, u_R) & \text{for } \frac{x}{t} \geq u_R + \rho_R^\theta, \end{cases}$$

and $\rho^{(\theta)} = 0$ when $\frac{x}{t} \leq u_R - \frac{\rho_R^\theta}{\theta}$. Moreover, $u^{(\theta)}(t, x) = u_R - \frac{\rho_R^\theta}{\theta}$ at $\frac{x}{t} = u_R - \frac{\rho_R^\theta}{\theta} + 0$.

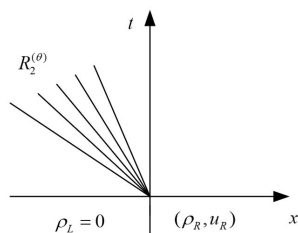


FIG. 7.3.

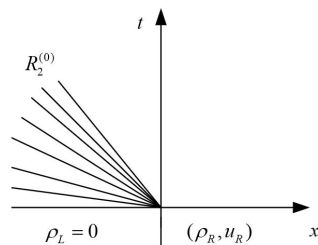


FIG. 7.4.

For $\theta = 0$, the Riemann solution $(\rho^{(0)}, u^{(0)})$ as shown in Figure 7.4 is

$$(\rho^{(0)}, u^{(0)}) = \begin{cases} (\rho_R e^{\frac{x}{t} - u_R - 1}, \frac{x}{t} - 1) & \text{for } \frac{x}{t} < u_R + 1, \\ (\rho_R, u_R) & \text{for } \frac{x}{t} \geq u_R + 1. \end{cases}$$

Moreover, $u^{(0)}(t, x) \rightarrow -\infty$ and $m^{(0)}(t, x) \rightarrow 0$ as $\frac{x}{t} \rightarrow -\infty$.

For $\theta \rightarrow 0$, it is direct to see that $u_R - \frac{\rho_R^\theta}{\theta} \rightarrow -\infty$ and $u_R + \rho_R^\theta \rightarrow u_R + 1$, which leads to $(\rho^{(\theta)}, u^{(\theta)}) \rightarrow (\rho^{(0)}, u^{(0)})$.

Then we conclude the following:

PROPOSITION 7.5. *For fixed (ρ_R, u_R) with $\rho_R > 0$ and $\rho_L = 0$, there exists a unique Riemann solution $(\rho^{(\theta)}, u^{(\theta)})$ for $\theta \geq 0$, and $(\rho^{(\theta)}, u^{(\theta)}) \rightarrow (\rho^{(0)}, u^{(0)})$ as $\theta \rightarrow 0$.*

Remark 7.6. For the case: $\rho_R = 0$ and $\rho_L > 0$, the same convergence result can be obtained similarly.

Notice that

$$\lim_{\theta \rightarrow 0} \lim_{\frac{x}{t} \rightarrow u_R - \frac{\rho_R^\theta}{\theta} + 0} c^{(\theta)}(\rho^{(\theta)}(t, x)) = 0, \quad \lim_{\theta \rightarrow 0} c^{(\theta)}(\rho^{(\theta)}(t, x)) \equiv 1,$$

which indicate the order of the limits on the sound speed $c^{(\theta)}$ close to the vacuum cannot be exchanged in general.

Here, u_L does not have a direct definition on the vacuum and may be defined only as the limit from the nonvacuum region. In this case, we may regard $u_L^{(\theta)}$ as $u^{(\theta)}(t, x) = u_R - \frac{\rho_R^\theta}{\theta}$ at $\frac{x}{t} = u_R - \frac{\rho_R^\theta}{\theta} + 0$, which varies with θ . When θ is small enough, it is direct to see that $u_L^{(\theta)} = u_R - \frac{\rho_R^\theta}{\theta} < 0$. Then

$$|u_L^{(\theta)}| + \frac{\rho_L^\theta - 1}{\theta} = -u_R + \frac{\rho_R^\theta - 1}{\theta} \leq w_0,$$

which indicates (2.18).

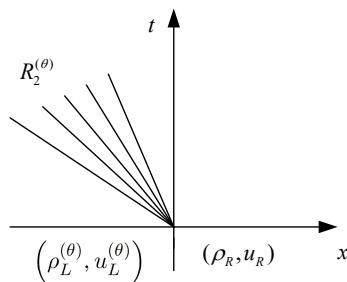


FIG. 7.5.

For the direct definition of $(\rho_L^{(\theta)}, u_L^{(\theta)})$ close to the vacuum states and (2.18), we construct a family of Riemann solutions $(\rho_A^{(\theta)}, u_A^{(\theta)})$ approaching $(\rho^{(0)}, u^{(0)})$ with the initial data satisfying (2.18). Choose the initial data

$$(\rho_A^{(\theta)}, u_A^{(\theta)})(0, x) = \begin{cases} (\rho_L^{(\theta)}, u_L^{(\theta)}) & \text{for } x < 0, \\ (\rho_R, u_R) & \text{for } x > 0 \end{cases}$$

with $\rho_L^{(\theta)} = O(\theta^{\frac{1}{\theta}})$ and $u_L^{(\theta)} = u_R - \frac{\rho_R^{(\theta)} - (\rho_L^{(\theta)})^\theta}{\theta}$. It is direct to check that $(\rho_A^{(\theta)}, u_A^{(\theta)})(0, x)$ satisfy (2.18).

Then the Riemann solutions $(\rho_A^{(\theta)}, u_A^{(\theta)})$ as shown in Figure 7.5 are

$$(\rho_A^{(\theta)}, u_A^{(\theta)}) = \begin{cases} (\rho_L^{(\theta)}, u_L^{(\theta)}) & \text{for } \frac{x}{t} \leq u_L^{(\theta)} + (\rho_L^{(\theta)})^\theta, \\ ((\frac{\theta \frac{x}{t} - \theta u_R + \rho_R^\theta}{\theta + 1})^{\frac{1}{\theta}}, \frac{\frac{x}{t} + \theta u_R - \rho_R^\theta}{\theta + 1}) & \text{for } u_L^{(\theta)} + (\rho_L^{(\theta)})^\theta < \frac{x}{t} < u_R + \rho_R^\theta, \\ (\rho_R, u_R) & \text{for } \frac{x}{t} \geq u_R + \rho_R^\theta. \end{cases}$$

When $\theta \rightarrow 0$, we see that $\rho_L^{(\theta)} \rightarrow 0$, $u_L^{(\theta)} \rightarrow -\infty$, $u_L^{(\theta)} + (\rho_L^{(\theta)})^\theta \rightarrow -\infty$, and $u_R + \rho_R^\theta \rightarrow u_R + 1$. Then we conclude that $(\rho_A^{(\theta)}, u_A^{(\theta)}) \rightarrow (\rho^{(0)}, u^{(0)})$.

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