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Kripke on Gödel Incompleteness

Daniel Isaacson

Philosophy Faculty, Oxford University, Oxford, UK

Correspondence: Daniel Isaacson (daniel.isaacson@philosophy.ox.ac.uk)

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ABSTRACT

This paper surveys six of Saul Kripke's highly creative ideas and results on Gödel incompleteness, from when he was an undergraduate to last publications. These include his extension of incompleteness from sentences to predicates, his model-theoretic proof of incompleteness of arithmetic, his compelling analysis of incompleteness in terms of the heterological paradox rather than the liar paradox (cited by Gödel) for a heuristic account of incompleteness, his ingenious demonstration that Hilbert's programme bore within itself the seeds of its collapse independently of Gödel incompleteness, his revisionist point that there was incompleteness in mathematics well before the Paris-Harrington Theorem, and his demonstration, contrary to the common view, that direct self-reference need not be contradictory and can be used to obtain a Gödel sentence containing a numeral for its own Gödel number. Kripke published the work surveyed here with the exception of his model-theoretic proof of incompleteness, which he presented in a lecture in 1978. These ideas constitute mathematical results of great ingenuity and highly illuminating philosophical insights.

1 | Introduction

Saul Kripke's second published paper, after his epoch making "Completeness Theorem in Modal Logic", was on Gödel incompleteness with the title "'Flexible' predicates of formal number theory" Kripke (1962). It was written and published while he was an undergraduate. His next major work on incompleteness was a model-theoretic proof of incompleteness, presented in a lecture in Oxford in 1978 under the title, "A model-theoretic proof of Gödel's theorem" Kripke (1978). Over the next 40 years, Kripke continued to develop insights into the phenomenon of incompleteness, and in the last decade of his life four papers of his on incompleteness were prepared and accepted for publication Kripke (2014, 2022a, 2022b, 2023) (thanks to the industry and expertise of the Kripke Center in CUNY Graduate Center). In this paper, I survey these six contributions to understanding the incompleteness of formal systems.[1]

2 | "'Flexible' Predicates of Formal Number Theory" (1962)

Kripke was 19 when he submitted "'Flexible' predicates of formal number theory" for publication, and it was published in *Proceedings of the American Mathematical Society* in 1962 Kripke (1962). In this short paper (four pages), Kripke extended Gödel incompleteness from sentences to predicates. It displays extraordinary mastery of Kleene's *Introduction to Metamathematics*, the bible of the subject at that time (a comparable accomplishment to Kreisel's mastery of volume 2 of Hilbert and Bernays, *Grundlagen der Mathematik* as an undergraduate). In a note written much later Kripke (2018), Kripke remarks that while an undergraduate at Harvard, he was being discouraged from publishing his many further results on modal and intuitionistic logic (by Quine and Dreben, I take it he meant), and says that he published this paper, "almost to show that I could do something else".

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For a given a base theory **F**, Kripke calls an m-place predicate $P_n(x_1, \ldots, x_m)$ in the language of **F** *flexible* for m-place Σ_n-formulas iff for every Σ_n-formula $Q(x_1, \ldots, x_m)$ with m-free variables, the sentence $\forall x_1 \ldots \forall x_m (P_n(x_1, \ldots, x_m) \leftrightarrow Q(x_1, \ldots, x_m))$ is consistent with **F**, i.e., $P_n(x_1, \ldots, x_m)$ can be consistently interpreted to have any Σ_n-definable extension. In this paper, Kripke proved the existence of flexible predicates. Mostowski had obtained similar results at that time. In Kripke (2018), Kripke reports that "When I met Andrzej Mostowski in 1962 at a conference on modal and many-valued logics held in Helsinki and told him of my proof, he said that I should have remarked that it was an 'essential improvement' over his result, because he could not get his result for arbitrary systems containing the theory R, while my version does so." Kripke also notes that, "Mostowski's argument was much longer than mine."

This paper remains relevant to current research. Joel Hamkins makes use of this result of Kripke's in his paper, "The modal logic of arithmetic potentialism and the universal algorithm" Hamkins (2018), where he generalizes it to a uniform version: There is a computable sequence of formulas $\sigma_n(x)$ for $n \geq 2$, with σ_n having complexity Σ_n, such that for any model of arithmetic M and any sequence of formulas ϕ_n coded in M, with ϕ_n of complexity Σ_n, there is an end-extension M^* of M with $M^* \models \forall x(\sigma_n(x) \leftrightarrow \phi_n(x)$ for all $n \geq 2$ (Theorem 22(2), p. 16). Hamkins obtains this result by use of Woodin's universal algorithm that "there is a Turing machine program that can in principle enumerate any desired finite sequence of numbers, if only it is run in the right universe; and furthermore, in any model of arithmetic, one can realize any desired further extension of the enumerated sequence by moving to a taller model of arithmetic end-extending the previous one" (p. 10).[2]

3 | Model-Theoretic Proof of the Incompleteness of Arithmetic (1978)

Kripke spent the academic year 1977–1978 in Oxford as a Visiting Fellow at All Souls College. On 2 February 1978, he gave a lecture at the Mathematical Institute, "A model-theoretic proof of Gödel's theorem" Kripke (1978). The lecture began at 5.00 pm and finished at 7.30 pm, by which time much of the original audience had left. One of those who stayed to the end was Joseph Quinsey, a graduate student in the Mathematical Institute Logic Group, who was inspired by that lecture to write his D.Phil. thesis on *Applications of Kripke's Notion of Fulfilment* (Quinsey 1980), supervised by Dana Scott and Robin Gandy. Another who stayed to the end and talked with Kripke afterwards was Jeff Paris, who Kripke had asked to attend.

The context in which Kripke carried out this work was the then-recent model-theoretic proof of the incompleteness of Peano Arithmetic obtained by Jeff Paris and Leo Harrington "A mathematical incompleteness in Peano Arithmetic" (PA) published in the *Handbook of Mathematical Logic* in 1977 Paris and Harrington (1977) in which they showed by a model-theoretic proof that a variant of finite Ramsey's Theorem cannot be proved in PA. For his own result, Kripke introduced his powerful technique of fulfillability, which applies to systems weaker and stronger than PA, as well as to PA itself. Kripke in a 1982 joint publication with Simon Kochen, "Non-standard models of Peano Arithmetic" Kochen and Kripke (1982), referred to this work briefly (in Section VII (d)), but otherwise never published these results. Joseph Quinsey's doctoral thesis was not easily available until 2019, when it was transcribed into LaTeX and posted online Quinsey (1980). In 1990, Warren Goldfarb published an extremely nice paper, "Herbrand's theorem and the incompleteness of arithmetic" Goldfarb (1990), which took account of Kochen and Kripke's paper and Quinsey's thesis and developed these ideas in the context of Herbrand's theorem and Hilbert's program. In 2000, Putnam (2000) published a brief and, in my view, unsatisfactory account of these ideas of Kripke. In 2017 Mattias Granberg Olsson wrote a Master's thesis, *A Model-Theoretic Proof of Godel's Theorem: Kripke's Notion of Fulfilment* (2017) in which fulfilment is formalised in weak arithmetic. In 2022, Roy Cook published a paper "Fulfillability, instability, and incompleteness" Cook (2022) in order "to publicize, and to a more limited extent, to further develop, an alternative proof of Gödel's incompleteness theorem due to Saul Kripke, based on a notion called *fulfillability*." Cook's paper has come to my attention too late to offer a view on it here.

In the following, I give an account of Kripke's notion of *fulfillability* and his use of it to give a model-theoretic proof of the Π_2-incompleteness of formal arithmetic.[3]

Kripke's notion of fulfillability is defined in terms of a game for two players played with a strictly increasing sequence of natural numbers σ, which may be either finite or infinite, and a sentence A in the language of arithmetic which is either Π_0 (no unbounded quantifiers) or of the form $\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k B(x_1, y_1. \ldots, x_k, y_k)$ for $k \geq 1$ and $B(x_1, y_1. \ldots, x_k, y_k)$ a Π_0-formula.

Notation: For σ a finite sequence, $l(\sigma)$ denotes the length of σ.

Notation: For σ a sequence and i a natural number, $\sigma(i)$ denotes the ith element of σ.

Definition 1. Game G Let σ be a strictly increasing sequence of natural numbers, finite or infinite, and let A be a sentence in the language of arithmetic which is either Π_0 or of the form $\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k B(x_1, y_1. \ldots, x_k, y_k)$ where $B(x_1, y_1. \ldots, x_k, y_k)$ is Π_0 for $k \geq 1$. If A is Π_0, Player II wins if A is true, otherwise Player I wins. If A is not Π_0, Player I picks an element $\sigma(i)$ of σ, with the constraint that if σ is finite, $i < l(\sigma)$, and then chooses a number $m_1 < \sigma(i)$ by which to instantiate $\forall x_1$. Player II then chooses a number $n_1 < \sigma(i+1)$ by which to instantiate $\exists y_1$. If there are further quantifiers in the prefix, Player I then picks an element $\sigma(j)$ of σ such that $j < l(\sigma)$ if σ is finite and chooses a number $m_2 < \sigma(j)$ by which to instantiate $\forall x_2$. Player II then chooses a number $n_2 < \sigma(max(i,j)+1)$ by which to instantiate $\exists y_2$. The game continues in this way for each further pair of quantifiers $\forall x_i \exists y_i$, and ends when numbers have been picked for all $2k$ quantifiers in the prefix of A. At each round of the game, for $\sigma(i_1), \ldots \sigma(i_r)$, the elements of σ chosen as bounds by Player I in this and previous rounds, Player II picks a number $< \sigma(max(i_1, \ldots i_r)+1)$. Player II wins if the Π_0-matrix of A instantiated with the chosen numbers, $B(m_1, n_1, \ldots, m_k, n_k)$, is true. Otherwise, Player I wins.

Definition 2. Fulfillability of a Π_{2k} sentence by a sequence For a sentence A in the language of arithmetic that is Π_{2k} for $k \geq 0$ and σ a strictly increasing sequence, we say that σ fulfills A if and only if Player II has a winning strategy in the game G played with A and σ.

Note that if A is Π_0, A is fulfilled for any sequence σ if and only if A is true, since Player II has a winning strategy, namely do nothing.

Lemma 1. *Every sentence A in the language of arithmetic is provably equivalent in PA to a Π_{2k}-sentence for $k \geq 0$.*

Proof. Every sentence A is provably equivalent by pure logic to a sentence in prenex normal form. In Peano Arithmetic, every sentence in prenex normal form is provably equivalent to a sentence with no adjacent like quantifiers. If the result of prenexing A and combining like adjacent quantifiers begins with an existential quantifier, prefix a vacuous universal quantifier, and if it ends with a universal quantifier, add a vacuous existential quantifier at the end of the prefix. □

Convention and notation: In general, there is no unique Π_{2k}-sentence provably equivalent to a given sentence A, but we can stipulate a procedure for generating a Π_{2k}-sentence provably equivalent to a given sentence A, which will be unique, which we denote by A^*.

Lemma 2. *There is a Δ_1-formula $\Pi i(x,y)$ in the language of arithmetic such that $\Pi i(x,y)$ if and only if $y = \ulcorner E^*_x \urcorner$, i.e., y is the Gödel number of the Π_{2k}-sentence E^*_x.*

Proof. Prenexing a formula can be carried out effectively. Transformation of a formula with adjacent like unbounded quantifiers in the quantifier prefix to obtain a formula with just one unbounded quantifier and new bounded quantifiers can also be carried out effectively. □

Lemma 3. *Fulfillability of a Π_{2k}-sentence A for particular k by a finite strictly increasing sequence σ is expressible by a Π_0 -sentence.*

Proof. A is the form of the form $\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k B(x_1, y_1. \ldots, x_k, y_k)$, where $B(x_1, y_1, \ldots, x_k, y_k)$ is Π_0 for $k \geq 1$.

Then a finite sequence σ fulfills A if and only if the Π_0-sentence

$(\forall i_1 < l(\sigma))(\forall x_1 < \sigma(i_1))(\exists y_1 < \sigma(i_1+1))\ldots(\forall i_k < l(\sigma))(\forall x_k < \sigma(i_k))$
$(\exists y_k < \sigma(max(i_1, \ldots i_k)+1)B(x_1, y_1, \ldots, x_k, y_k)$ is true. □

To express fulfillability uniformly for arbitrary k, i.e., with free variable k (which in the following lemma is signified by use of Feferman's dot notation), we use the Δ_1 truth definition for Π_0-sentences, $Tr_{\Pi_0}(x)$.

Lemma 4. *Sentences of the form $\forall x_1 \exists y_1 \ldots \forall x_{\dot{k}} \exists y_{\dot{k}} B(x_1, y_1. \ldots, x_{\dot{k}}, y_{\dot{k}})$, where $B(x_1, y_1, \ldots, x_{\dot{k}}, y_{\dot{k}})$ is Π_0 are fulfillable by a strictly increasing sequence σ if and only if $Tr_{\Pi_0}(\ulcorner(\forall i_1 < l(\sigma))(\forall x_1 < \sigma(i_1))(\exists y_1 < \sigma(i_1+1))\ldots(\forall i_{\dot{k}} < l(\sigma))$
$(\forall x_{\dot{k}} < \sigma(i_{\dot{k}}))(\exists y_{\dot{k}} < \sigma(max(i_1, \ldots i_{\dot{k}})+1)B(x_1, y_1, \ldots, x_{\dot{k}}, y_{\dot{k}})\urcorner)$*

Proof. By arithmetization of syntax in PA. □

Definition 3. Fulfillability of a sentence by a sequence For a sentence A in the language of arithmetic and A^* a Π_{2k} sentence for $k \geq 0$ such that $PA \vdash (A \leftrightarrow A^*)$ and σ a strictly increasing sequence, we say that σ fulfills A if and only σ fulfills A^* by Definition 2.

Lemma 5. *Fulfillability of a Π_{2k}-sentence A by an infinite strictly increasing sequence is expressible by a Π_1-sentence in the language of arithmetic plus a symbol for the infinite sequence σ.*

Proof. Fulfillability by an infinite sequence corresponds to fulfillability by a finite sequence except without a natural number as the length of the sequence. Deleting the bounds by the length of σ in the formula expressing fulfillability by a finite sequence, given in the proof of Lemma 3, yields the sentence

$\forall i_1(\forall x_1 < \sigma(i_1))(\exists y_1 < \sigma(i_1+1))\ldots\forall i_k(\forall x_k < \sigma(i_k))$
$(\exists y_k < \sigma(max(i_1, \ldots i_k)+1)B(x_1, y_1, \ldots, x_k, y_k)$. □

Lemma 6. *A Π_{2k}-sentence $\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k B(x_1, y_1. \ldots, x_k, y_k)$ in the language of arithmetic is true iff some infinite sequence fulfills A.*

Proof.

i If A is fulfillable by an infinite sequence σ, then the sentence in Lemma 5 is true, and it is immediate that A, without those bounds on the quantifiers, is true.

ii To show that if A is true, then some infinite sequence fulfills A:
  1 Assume that A is true. Then there exist Skolem functions $f_1(x_1)$, $f_2(x_1, x_2)$, ..., $f_k(x_1, \ldots, x_k)$ such that $\forall x_1 \ldots \forall x_k B(x_1, f_1(x_1), \ldots, x_k, f_k(x_1, \ldots, x_k))$ is true.
  2 The following recursive definition generates an infinite strictly increasing sequence σ:
  $\sigma(1) = k$ for some number $k > 0$.
  $\sigma(i+1) = max\{f_1(x_1)+1, \ldots, f_i(x_1, \ldots, x_i)+1,$
  $x_u + x_v, x_u \cdot x_v : x_1, \ldots, x_i, x_u, x_v < \sigma(i)\}$.
  3 Note that by the condition that $\sigma(i+1)$ is closed under addition and multiplication of numbers $< \sigma(i), \sigma$ is strictly increasing, even if, for example, there is only one $y_1 = \ldots = y_k$ such that $\forall x_1 \exists y_1 \ldots \forall x_k \exists y_k B(x_1, y_1, \ldots, x_k, y_k)$.
  4 The sequence σ defined in (2) fulfills A since for each existential quantifier, $\sigma(i)$ bounds the Skolem function for that quantifier. □

Definition 4. A finite sequence is nice We call a finite sequence σ *nice* iff $\sigma(1) > l(\sigma)$, i.e., its initial element exceeds its length.

Definition 5. A sentence is nicely n-fulfillable A sentence A is nicely n-fulfillable by a finite sequence σ (or a finite sequence σ nicely n-fulfills A) iff some nice sequence σ of length n fulfils A.

Lemma 7. *If A is true, then for every n, there is a nice sequence σ of length n such that A is nicely n-fulfilled by σ.*

Proof. If A is true, then the infinite sequence σ defined in Lemma 6 fulfills A. For any n, choose a finite portion of σ

beginning with an element of σ bigger than n and include the next $n-1$ elements. The resulting finite monotone increasing sequence of length n nicely fulfills A. ☐

Notation: P_n is the nth axiom of PA in the enumeration by Gödel numbering.

$Seq(x)$ is a Π_0-formula such that $Seq(x)$ iff x is the Gödel number of a strictly increasing sequence of natural numbers.

$l(x)$ is such that if $Seq(x)$, $l(x)$ is the length of the sequence whose code is x.

x_i is such that if $Seq(x)$. x_i is the ith element of the sequence whose code is x.

Lemma 8. *There is a Π_0-formula $Ful(x,y)$ in the language of PA, which expresses that a finite strictly increasing sequence σ fulfills a sentence A, i.e., $Ful(m,n)$ if and only if $m = \ulcorner\sigma\urcorner$ and σ is strictly increasing and $m = \ulcorner A\urcorner$ and σ fulfills A.*

Proof. Finite sequences can be coded by natural numbers, and being strictly increasing is expressible in the language of arithmetic. There is an effective procedure for obtaining from a sentence A a Π_{2k} sentence A^* such that $PA \vdash (A \leftrightarrow A^*)$. The condition that σ fulfills A^* is expressed by the Π_0-sentence in Lemma 3. ☐

By the arithmetised syntax of Lemma 8, Lemma 7 is provable in PA:

Lemma 9. *For every sentence A in the language of arithmetic, $PA \vdash (A \rightarrow \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner A\urcorner)))$.*

Proof. The proof of Lemma 7 depended on the construction of an infinite increasing sequence using Skolem functions for A^*. This construction can be formalized in PA because PA has definable Skolem functions by the least number principle. ☐

Lemma 10. *There is a Π_0-formula $S(x,y)$ such that $S(x,y)$ if and only if y is the Gödel number of a finite sequence σ and x is the Gödel number of a subsequence of σ.*

Lemma 11. *$PA \vdash (S(x,y) \wedge Ful(y,z) \rightarrow Ful(x,z)$*

For a given Gödel numbering of the language of arithmetic, there is a one-place Π_0 formula $P(x)$ which expresses the property of being the Gödel number of an axiom of PA. This numbering gives an enumeration of the axioms of PA.

Kripke constructed a true Π_2-sentence in the language of arithmetic which, we will see by use of non-standard models of PA, is unprovable in PA on the assumption that PA is consistent.

Definition 6. Kripke's sentence that is undecidable in PA

$$K =_{df} \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner(P_1 \wedge \ldots \wedge P_{\dot{x}})\urcorner))$$

Note that x is a free variable in the enumeration of the first x axioms of PA in K (signified by use of Feferman's dot notation).

Theorem 12. *If PA is Σ_1-sound, Kripke's sentence K is true.*

Proof. By Lemma 9, for each k,

$PA \vdash ((P_1 \wedge \cdots \wedge P_k) \rightarrow \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner(P_1 \wedge \cdots \wedge P_k)\urcorner)))$. Then, since $PA \vdash (P_1 \wedge \cdots \wedge P_k)$, $PA \vdash \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner(P_1 \wedge \cdots \wedge P_k)\urcorner))$. Then, by $\forall$-elimination, $PA \vdash \exists y(Seq(y) \wedge l(y) = k \wedge y_1 > k \wedge Ful(y, \ulcorner(P_1 \wedge \cdots \wedge P_k)\urcorner))$. Then, on the assumption that PA is Σ_1-sound, for each k, $\exists y(Seq(y) \wedge l(y) = k \wedge y_1 > k \wedge Ful(y, \ulcorner(P_1 \wedge \cdots \wedge P_k)\urcorner))$ is true, so $\forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner(P_1 \wedge \ldots \wedge P_{\dot{x}}))\urcorner)$ is true. ☐

Notation: We denote by $\mathfrak{N}$ the standard model of PA; we denote by $\mathfrak{M}$ a countable non-standard model of PA.

Theorem 13. Model-theoretic proof that PA is incomplete *On the assumption that PA is consistent, there is a non-standard model of PA in which K is false, so $PA \nvdash K$, and on the assumption that PA is Σ_2-sound, $PA \nvdash \neg K$.*

Proof.

i Proof that $PA \nvdash K$.
  1. Assume $PA \vdash K$.
  2. Let $\mathfrak{M}$ be a countable non-standard model of PA, which exists by the assumption that PA is consistent and the compactness theorem for first-order logic. This is the one place in the proof where we require the hypothesis that PA is consistent. Countability, which follows from the Löwenheim–Skolem Theorem, is not strictly necessary but helps in envisaging the construction. By assumption (1), $\mathfrak{M} \models K$, which is to say that
  3. $\mathfrak{M} \models \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner(P_1 \wedge \ldots \wedge P_{\dot{x}}))\urcorner)$.
  4. Let z be a non-standard number in $\mathfrak{M}$. By $\forall$ -elimination in $\mathfrak{M}$, we have that $\mathfrak{M} \models \exists y(Seq(y) \wedge l(y) = z \wedge y_1 > z \wedge Ful(y, \ulcorner(P_1 \wedge \ldots \wedge P_z))\urcorner)$andlet$y \in \mathfrak{M}$besuchthat $\mathfrak{M} \models (Seq(y) \wedge l(y) = z \wedge y_1 > z \wedge Ful(y, \ulcorner(P_1 \wedge \ldots \wedge P_z))\urcorner)$. Let τ be a bounded, strictly increasing sequence in $\mathfrak{M}$ such that $\ulcorner\tau\urcorner = y$ and $l(\tau) = z$, $\tau(1) > z$, and $\mathfrak{M} \models Ful(\ulcorner\tau\urcorner, \ulcorner(P_1 \wedge \ldots \wedge P_z))\urcorner)$. (Because $\tau(1) > z$ and τ is strictly increasing, all members of τ are non-standard since z is non-standard.)
  5. By the least number principle in $\mathfrak{M}$, we may take $\tau(z)$ to be minimal among all $\sigma(z)$ for nice sequences σ such that $\mathfrak{M} \models Ful(\ulcorner\sigma\urcorner, \ulcorner(P_1 \wedge \ldots \wedge P_z))\urcorner)$.
  6. We specify a submodel $\mathfrak{M}^*$ of $\mathfrak{M}$ in terms of τ as follows: $\mathfrak{M}^* =_{df} \{a \in \mathfrak{M} : \text{for some n} \in \mathfrak{N}, a \leq \tau(n)\}$. Since by niceness of τ, $\tau(1) > l(\tau) = z$, $z \in \mathfrak{M}^*$. By the condition on $\tau(i+1)$ from the proof of the second half of Lemma 6, $\mathfrak{M}^*$ is closed under $+$ and $\cdot$. We will show that $\mathfrak{M}^*$ is a model of $(P_1 \wedge \ldots \wedge P_z)$.
  7. Let τ^* be the subsequence of τ determined by $x \in \tau * \equiv_{df} (\exists i \in \omega) x = \tau(i)$, i.e., τ^* is an ω-sequence and an initial subsequence of τ. Then by (4) and Lemma 11, $\mathfrak{M} \models Ful(\ulcorner\tau^*\urcorner, \ulcorner(P_1 \wedge \ldots \wedge P_z))\urcorner)$.
  8. Then by Definition 3 and Lemma 4, $\mathfrak{M} \models Tr_{\Pi_0}(\ulcorner(\forall i_1 < \omega)(\forall x_1 < \tau^*(i_1))(\exists y_1 < \tau^*(i_1+1))\ldots(\forall i_k < \omega)(\forall x_k < \tau^*(i_k))(\exists y_k < \tau^*(max(i_1, \ldots i_k)+1))B(x_1, y_1, \ldots, x_k, y_k)\urcorner)$, where k is computable from z.

9 Then since $PA \vdash (Tr_{\Pi_0}(\ulcorner \phi \urcorner) \rightarrow \phi)$, $\mathfrak{M} \vDash (\forall i_1 < \omega)(\forall x_1 < \tau^*(i_1))(\exists y_1 < \tau^*(i_1 + 1)) \ldots (\forall i_k < \omega)(\forall x_k < \tau^*(i_k))(\exists y_k < \tau^*(max(i_1, \ldots i_k) + 1))B(x_1, y_1, \ldots, x_k, y_k)$.

10 Since $\mathfrak{M}^*$ is an initial segment of $\mathfrak{M}$ and all the elements of τ^* are in $\mathfrak{M}^*$, (9) implies that $\mathfrak{M}^* \vDash (\forall i_1 < \omega)(\forall x_1 < \tau^*(i_1))(\exists y_1 < \tau^*(i_1 + 1)) \ldots (\forall i_k < \omega)(\forall x_k < \tau^*(i_k))(\exists y_k < \tau^*(max(i_1, \ldots i_k) + 1)B(x_1, y_1, \ldots, x_k, y_k)$

11 Since τ^* is unbounded in $\mathfrak{M}^*$, then by Lemmas 5 and 6 and Definition 3, $\mathfrak{M}^* \vDash (P_1 \wedge \ldots \wedge P_z)$.

12 Since $(P_1 \wedge \ldots \wedge P_z)$ is the conjunction of the axioms P_i of PA such that $i \leq z$ and z is a non-standard number in $\mathfrak{M}^*$, for every $n \in \mathfrak{N}$, $n < z$. Hence the infinitely many axioms of PA are all included in this conjunction. So $\mathfrak{M}^*$ is a model of PA.

13 We now show that $\mathfrak{M}^* \nvDash K$. Suppose $\mathfrak{M}^* \vDash \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner (P_1 \wedge \ldots \wedge P_{\dot{x}}) \urcorner))$. The quantifier $\forall x$ can be instantiated by non-standard z in $\mathfrak{M}^*$, so $\mathfrak{M}^* \vDash \exists y(Seq(y) \wedge l(y) = z \wedge y_1 > z \wedge Ful(y, \ulcorner (P_1 \wedge \ldots \wedge P_z) \urcorner))$.

14 Let ρ be a nice sequence of length z that nicely z-fulfils $(P_1 \wedge \ldots \wedge P_z)$ in $\mathfrak{M}^*$. The statement that ρ nicely z-fulfills $(P_1 \wedge \ldots \wedge P_z)$ is Π_0. Hence, since $\mathfrak{M}^*$ is an initial segment of $\mathfrak{M}$, ρ nicely z-fulfills $(P_1 \wedge \ldots \wedge P_z)$ in $\mathfrak{M}$.

15 Since ρ instantiates an existential quantifier in $\mathfrak{M}^*$, its elements are in $\mathfrak{M}^*$, so in particular, $\rho(z) \in \mathfrak{M}^*$, which means that for some $k \in \mathfrak{N}$, $\rho(z) < \tau(k)$. Since k is a standard number and z is a non-standard number, $k < z$ and since τ in $\mathfrak{M}$ is strictly increasing, $\tau(k) < \tau(z)$, so in $\mathfrak{M}$, $\rho(z) < \tau(z)$. But as stipulated at (5), $\tau(z)$ is minimal. This contradiction has been derived from the supposition that $\mathfrak{M}^* \vDash K$, so $\mathfrak{M}^* \nvDash K$. Since we have at (12) that $\mathfrak{M}^*$ is a model of PA, $PA \nvdash K$.

ii Proof that $PA \nvdash \neg K$ on the assumption that the PA is Σ_2 -sound.

16 The assumption that PA is Σ_2-sound implies that PA is Σ_1 -sound, which by Theorem 12 implies that K is true. $\neg K$ is false and Σ_2, so on the assumption that PA is Σ_2-sound, $PA \nvdash \neg K$. □

Corollary 14. PA not finitely axiomatizable

Proof. It is a corollary of Lemma 9 and Theorem 13 that PA is not finitely axiomatizable.

PA. Suppose PA is axiomatizable by the sentence B. By Lemma 9,

1 $PA \vdash (B \rightarrow \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner B \urcorner)))$, so

2 $PA \vdash \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner B \urcorner))))$.

3 If B axiomatizes PA, then $PA \vdash Pr_{PA}(\ulcorner (B \rightarrow (P_1 \wedge \ldots \wedge P_{\dot{x}})) \urcorner)$.

4 From (2) and (3) we have $PA \vdash \forall x \exists y(Seq(y) \wedge l(y) = x \wedge y_1 > x \wedge Ful(y, \ulcorner (P_1 \wedge \ldots \wedge P_{\dot{x}}) \urcorner))$

5 This contradicts Theorem 13, so PA is not finitely axiomatizable. □

The first proof of non-finite axiomatizability of Peano Arithmetic was by Ryll-Nardzewski (1952), published in 1952, using non-standard models of Peano Arithmetic. I find Kripke's proof of this result, as a corollary of his model-theoretic proof of Π_2-incompleteness, more perspicuous than Ryll-Nardzewski's, although Alex Wilkie has said in correspondence that he does not think that Kripke's proof is fundamentally different from Ryll-Nardzewski's. Also in 1952, Andrzej Mostowski published a quite different proof Mostowski (1952) by showing that Peano Arithmetic proves the consistency of each of its proper sub-theories, so by Gödel's second incompleteness theorem, it cannot be finitely axiomatizable. Mostowski says in his paper: "The impossibility of a finite axiomatization of the arithmetic of positive integers was first shown by Ryll-Nardzewski (1952)" (fn 26, p. 150). He notes that Ryll-Nardzewski obtained his proof "by a different method" (fn 27, p. 154), and also claims that his result is "slightly stronger" (fn 26, p. 150).

4 | "The Road to Gödel" (2014)

This paper was published in 2014 in a volume edited by Jonathan Berg that was based on the proceedings of a conference, "Naming, Necessity, and More," held at the University of Haifa in 1999, in honor of Saul Kripke on the occasion of his being awarded an honorary doctorate. At that conference, Kripke gave a lecture "The Road to Gödel," and he lectured under this title a number of other times, including in Utrecht and in Oxford in February 2001.

Kripke sets out two things he wants to do in this paper: "first, to present the Gödel theorem as almost the inevitable result of a historic line of thought. I don't mean that it *did* happen that way; I mean that it *could* have, and perhaps *should* have [...]. Second, I want to show that the Gödel statement, the one Gödel proves to be undecidable in the first incompleteness theorem, makes a fairly intelligible assertion that can actually be stated" (p. 223).

Kripke argues that the Gödel incompleteness theorem is an inevitable outgrowth of the inconsistency of the naive (unrestricted) comprehension principle, and indeed a special case of it, and that this is the best way to understand Gödel incompleteness, rather than as an analogue of the Liar Paradox (which Gödel suggested in his publication of the result).

Kripke notes that "It is a matter of pure first-order logic that the unrestricted comprehension axiom schema is inconsistent. Russell already realized this in his example of the barber who shaves all and only those who do not shave themselves ... the interpretation of the epsilon relation is irrelevant" (p. 228).

Kripke then derives from this fact a non-constructive proof of the incompleteness of formal systems of arithmetic containing plus and times, i.e., there exists is a true unprovable sentence in the language of arithmetic, without finding such a sentence (pp. 232–235). He then goes on to show how consideration in terms of paradox rather than pure logic can yield a constructive proof of incompleteness. Rather than using Russell's paradox, or the Liar, Kripke focuses on Kurt Grelling's heterological paradox, which he quotes from Quine, "The ways of paradox":

> The adjective 'short' is short; the adjective 'English' is English; the adjective 'adjectival' is adjectival; the adjective 'polysyllabic' is polysyllabic. Each of these adjectives is, in Grelling's terminology, autological: each is true of itself. Other adjectives are heterological;

> thus ‘long’, which is not a long adjective; ‘German’, which is not a German adjective; ‘monosyllabic’, which is not a monosyllabic one. Grelling's paradox arises from the query: Is the adjective ‘heterological’ an autological or a heterological one? (Quine 1966, 6)

Kripke gives the following account of how to construct the Gödel sentence from Grelling's paradox:

> Suppose now we try to imitate the ‘heterological’ paradox, only replacing satisfaction (or ‘true of’) by ‘provability of’. Replacing adjectives, or adjectival phrases, by formulae with one free variable (Gödel's ‘class signs’—remember that we could even fix the free variable as x_1), a class sign $A(x_1)$ is naturally called ‘provable of’ a number n if $A(0^n)$ is provable, or alternatively, as we have seen, if $(\exists x_1)(x_1 = 0^n \wedge A(x_1))$ is provable. Suppose that we identify formulae with their Gödel numbers. Then a particular formula $Pr(x, y)$ with two free variables says that x is provable of y. $\neg Pr(x_1, x_1)$ is a class sign (in Gödel's sense) that says that a formula is unprovable of itself. It itself has a particular Gödel number n, and $\neg Pr(0^n, 0^n)$ is simply a way of saying:

> ‘Unprovable of itself’ is unprovable of itself.

> This is precisely the statement G constructed by Gödel. (pp. 238–239)

Kripke's concludes that

> Thus the basic statement G can be called Gödel's form of the ‘heterological’ paradox, and to the present writer its content is clearer than if it is regarded in terms of the Liar paradox. (*loc. cit.*)

5 | “The Collapse of the Hilbert Program: A Variation on the Gödelian Theme” (2022)

This paper was published in *The Bulletin of Symbolic Logic* in 2022 Kripke (2022a). Kripke had earlier published an extended abstract of an Invited Special Talk on this topic given at the 2008 winter meeting of the ASL Kripke (2009). The usual view is that the Hilbert program was shown to be unrealizable by Gödel incompleteness. In this paper, Kripke shows that the impossibility of Hilbert's program was internal to the program itself. This new insight comes from Kripke's demonstration that realization of the Hilbert program for Π_2-sentences is expressible by a Π_2-sentence which is about all Π_2-sentences, and that this self-application is contradictory.

The key idea of Hilbert's program is the ε-substitution method, by which provable formulas of the form $\exists x A(x)$ are replaced by $A(\varepsilon x A(x))$, where $\varepsilon x A(x)$ denotes any true instance of $A(x)$ and an arbitrary object if there are none, axiomatized by the schema $A(t) \rightarrow A(\varepsilon x A(x))$ for all terms t in the language. If the Hilbert program were realized for a given formal system S, it would establish not only the consistency of S but also its Σ_1-soundness (equivalent to 1-consistency): by the ε-substitution method, if $S \vdash \exists x A(x)$, for $A(x)$ a Σ_0-formula, then $A(\varepsilon x A(x))$ is true. But it would show more than this. It would establish Π_2-soundness, since if $S \vdash \forall x \exists y B(x, y)$, then for all n, $S \vdash \exists y B(\overline{n}, y)$, so by Σ_1-soundness of S, $\forall x \exists y B(x, y)$ is true. Kripke's key insight here is that the realization of Hilbert's program in (a weak, i.e., finitary subsystem of) a system S for Π_2-sentences in the language of T can itself be expressed by a Π_2-sentence in the language of S, call it K, and that the realizability of the Hilbert program for S is tantamount to $S \vdash K$. Kripke derives a contradiction from the supposition that $S \vdash K$, and establishes thereby that Hilbert's program is not realizable.

By the preceding considerations, realization of the Hilbert program in a system S implies that

1 Given a Σ_0-formula $A(x, y)$, for any Π_2-sentence $\forall x \exists y A(x, y)$, if $S \vdash \forall x \exists y A(x, y)$, then for every x there is some number n such that $S \vdash A(x, n)$. We can formalize this statement in arithmetized syntax by expressing the following predicate and relation:

2 $B(x)$ to express “x is the Gödel number of a proof in S of the Π_2-sentence $\forall x \exists y A(x, y)$”

3 $C(x, y)$ to express “there is a number $n < y$ such that y is the Gödel number of a proof of $A(x, n)$”

4 $B(x)$ and $C(x, y)$ are expressible by Σ_0-formulas.

5 A restricted form of statement (1) can be expressed in arithmetized syntax of S by

$\forall x \exists y (B(x) \rightarrow C(x, y))$, which by (4) is a Π_2-sentence.

6 Note that (5) does not fully express (1) since the x in $C(x, y)$ is not an arbitrary number but the Gödel number of a proof of a given Π_2-sentence. However, there is no loss of generality in this formalization since the point of the argument is to show that (1) is not provable, and if a literal formalization of (1) were provable, (5) would be provable by universal instantiation.

7 Note also that the condition in (3) that $n < y$ (which is needed so that $C(x, y)$ can be expressed by a Σ_0-formula) reflects the fact that the numeral for n occurs in the proof whose Gödel number is y (so it is an attribute of arithmetization).

8 Suppose that a is the Gödel number of a proof in S of $\forall x \exists y (B(x) \rightarrow C(x, y))$.

9 Application of the ε-substitution method and (8) yields a number b and a proof in S of $(B(\overline{a}) \rightarrow C(\overline{a}, \overline{b}))$

10 By the least number principle, we may take b to be the least y such that $(B(\overline{a}) \rightarrow C(\overline{a}, \overline{y}))$ has a proof in S.

11 By (5) and (8), a is the Gödel number of a proof of a Π_2-sentence, so by (2), (4), and Σ_0-completeness of any theory in which arithmetization of syntax can be carried out, $S \vdash B(\overline{a})$.

12 Hence by (9), $S \vdash C(\overline{a}, \overline{b})$.

13 Then by (3), there is a number $n < b$ such that b is a proof of $(B(\overline{a}) \rightarrow C(\overline{a}, \overline{n}))$.

14 This contradicts and therefore refutes supposition (10), which was on the basis of supposition (8), so there is no proof in S of $\forall x \exists y (B(x) \rightarrow C(x, y))$, which establishes that the Hilbert program cannot be realized (pp. 423–424). This result transforms our understanding of Hilbert's program.

Two remarks about the failure of Hilbert's program:

1. What failed was the Hilbert program as originally conceived, i.e., looking for proofs of the consistency of infinitary branches of mathematics by finitary means, where finitary mathematics is a subpart of infinitary mathematics. Gentzen, working on Hilbert's program under supervision from Bernays and Hilbert, gave a consistency proof for Peano Arithmetic using ε_0-transfinite induction applied to finitary mathematics, which is not provable in PA, but is constructive though not finitary (as Gödel later noted), which made the development of Hilbert's program tenable, and it continued as proof theory, one of the four main branches of mathematical logic, to the present day (as shown by the recently published textbook on proof theory Mancosu et al. 2021).
2. While Kripke showed, without question, that the impossibility of realizing Hilbert's program as originally conceived by Hilbert does not depend on Gödel's incompleteness theorems—in particular the second incompleteness theorem—but is intrinsic to the program itself, we should remark, as Kripke does (p. 424) that his proof of this fact relies on the arithmetization of syntax by which Gödel proved his incompleteness theorems. At the same time, one can say that arithmetization of syntax itself was implicit in Hilbert's program, as when he said in his 1925 lecture "On the infinite": "A formalized proof, like a numeral, is a concrete and surveyable object" (Hilbert 1926, 383), but it was only Gödel who realized the game-changing implications of this insight.

The structure of Kripke's proof is reminiscent of his model-theoretic proof of incompleteness of arithmetic, which Kripke remarks about in footnote 7 of this paper: "I myself arrived at the present result through a circuitous route. I had already found a purely model-theoretic version of the Gödel theorem and realized that it could also be carried out syntactically, using appropriate finite approximations and semantic tableaux. But then I saw that the ladder could be kicked away and that, formulated in detail, the result the Hilbertians were attempting to obtain in fact implies its own impossibility."

If I may be permitted an entirely personal remark, I am very struck by an acknowledgment by Kripke related to this last remark: "I am indebted to Burton Dreben for his insistence that the Hilbert program or approach (Ansatz) was not merely to prove the consistency of mathematics by finitary means, but was a specific program for interpreting proofs. Thus, as Dreben emphasized, it is a kind of constructive model theory." (p. 424). From time to time over many years, I heard Saul express resentment at Dreben as having been unsupportive or indeed discouraging of his work on modal logic when he arrived at Harvard as an undergraduate. That Saul writes with warmth and appreciation at having learned something important from an idea that was central to Dreben's thinking is very striking and—given my affection for Dreben, who was my undergraduate tutor, and for Saul—gratifying to me.

6 | "Mathematical Incompleteness Results in First-Order Peano Arithmetic: A Revisionist View of the Early History" (2022)

When Jeff Paris and Leo Harrington published their paper "A mathematical incompleteness in Peano Arithmetic" in the *Handbook of Mathematical Logic* Paris and Harrington (1977), the editor of the Handbook, Jon Barwise, declared that "Since 1931, the year Gödel's Incompleteness Theorems were published, mathematicians have been looking for a strictly mathematical example of an incompleteness in first-order Peano arithmetic, one which is mathematically simple and interesting and does not require the numerical coding of notions from logic," and claimed that the Paris–Harrington sentence is the first such. In Kripke (2022b), Kripke takes exception to this claim, incontrovertibly, it seems to me, citing Gentzen's proof that transfinite induction of order type ε_0 is not derivable in Peano Arithmetic. This does not quite strictly meet Barwise's criterion, since the Gentzen theorem requires coding of ordinals $< \varepsilon_0$. However, Goodstein's theorem obviates that objection by giving a purely number-theoretic formulation of Gentzen's result with no coding, although, of course, to see the truth of the Goodstein sentence does require coding of base ω notations of ordinals less than ε_0. But equally, the formulation of the finite Ramsey's Theorem and its variant in the language of PA requires coding of finite sets of numbers. Kripke also cites the work of Matiyasevich (1970), building on earlier work by Davis, Putnam, and Robinson, as having already by the time of the Paris-Harrington paper "shown, in effect, that in any consistent recursively axiomatized system (in which some weak theory such as R is interpretable), we can effectively find a Diophantine equation that has no solution, but where this fact cannot be proved in the system." (p. 177).

7 | "Gödel's Theorem and Direct Self-Reference" (2023)

In this short paper Kripke (2023) (with publication date June 2023, though published online on 2 December 2021), Kripke demonstrated that contrary to the usual view, stemming from Gödel (1931), direct self-reference need not be contradictory, and it is possible to prove the first incompleteness theorem with a non-standard Gödel numbering "allowing a statement to contain a numeral designating its own Gödel number." (p. 651). The trick is the following:

> Let the 'original' Gödel numbering be Gödel's own prime power product numbering, except that the smallest prime used is 3, so that Gödel numbers are always odd. In the 'new' numbering, all Gödel

> numbers coincide with the 'original', except that for each n, the formula
>
> $$\exists x_1(x_1 = 0^{(2k_n)} \wedge A_n(x_1)),$$
>
> gets the Gödel number $2k_n$, where k_n is the original Gödel number of $\exists x_1(x_1 = 0^{(n)} \wedge A_n(x_1))$. The 'new' numbering allows a formula to contain a numeral designating its Gödel number, and in that sense it is a self-referential Gödel numbering.

> In this self-referential Gödel numbering, every formula $A_n(x_1)$ has an 'instance' $\exists x_1(x_1 = 0^{(2k_n)} \wedge A_n(x_1))$ asserting that its own Gödel number satisfies $A_n(x_1)$. The Gödel incompleteness theorem is the special case where $A_n(x_1)$ is unprovability in the system. (pp. 651-652)

Kripke noted in this paper that he had already in his "Outline of a theory of truth" (1975) stated that the Gödel theorem could be proved using 'direct' self-reference (pp. 650-651). He also cited papers on self-reference by Grabmayr, Halbach, Heck, Picollo, and Visser.

8 | Conclusion

These six contributions by Saul Kripke to understanding incompleteness of formal systems of arithmetic and other systems are very rich. They are also very diverse, and do not, as such, constitute a research program. Rather, each one gives us a gem of new understanding of the phenomenon of incompleteness, although that said, there is a very striking connection between the model-theoretic proof of incompleteness and the proof-theoretic internal refutation of Hilbert's program, as Kripke notes. Both establish Π_2-incompleteness, and both make central use of the least number principle.

These contributions differ one from another in the extent to which they are mathematical or philosophical. I would classify as primarily mathematical the paper on flexible predicates, the model-theoretic proof of incompleteness, and the paper on Gödel's theorem and direct self-reference. The paper on the collapse of the Hilbert program obtains a result that is highly significant both mathematically and philosophically. The "revisionist" view of the early history of incompleteness results is essentially philosophical. The mathematical results in these contributions display tremendous ingenuity. What all these contributions have in common is the uniqueness of their viewpoint, inviting us to think differently about each of these topics.

Endnotes

[1] I presented an earlier version of this paper in the Saul Kripke Memorial Conference in May 2023. I am grateful to Romina Birman, Yale Weiss, and Anandi Hattiangadi for inviting me to remember and honor Saul in this way. I am also grateful to Joseph Almog for inviting me to contribute to this special issue of *Theoria* in memory of Saul Kripke. I knew Saul from my senior year as an undergraduate at Harvard, 1966–1967, when I took his courses Phil 151 and Phil 143 in which he lectured on the topics that became *Naming and Necessity*. Later, I was a junior colleague of his for 2 years at the Rockefeller University, and then in Oxford where Kripke twice held a college visiting fellowship for an academic year and visited from time to time for shorter periods. I am extremely grateful to have known Saul Kripke for those 56 years. He was not only a genius but a wonderful person.

[2] I am grateful to Joel Hamkins for information on his use of Kripke's flexible predicates.

[3] I am grateful to Jeff Paris and Joe Quinsey for discussing with me their perceptions of the ideas and results Kripke presented in his 1978 lecture in Oxford, and to Joe Quinsey for answering my questions about this result. I am especially grateful to Alex Wilkie for his generous help in understanding this topic.

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