

SUPERCritical ELLIPTIC EQUATIONS

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ABSTRACT. For an elliptic model equation with supercritical power non-linearity we give a complete description of radial solutions and discuss self-similar blow-up solutions.

1. INTRODUCTION

Not much is known about non-trivial solutions u to the equation

$$(1.1) \quad -\Delta u = |u|^{p-2}u \text{ in } \Omega$$

on a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ when $n \geq 3$ and when the exponent p in the nonlinear term is in the “super-critical” range $p > 2^* = \frac{2n}{n-2}$.

Here, in a first attempt to understand this equation more thoroughly, we focus on solutions of (1.1) on $\mathbb{R}^n \setminus \{0\}$ which either are radially symmetric or self-similar in the sense that $u = u_R$ for all $R > 0$, where

$$(1.2) \quad u_R(x) = R^a u(Rx), \quad R > 0,$$

with $a := \frac{2}{p-2}$. Note that this scaling of a solution u of equation (1.1) on $\mathbb{R}^n \setminus \{0\}$ again produces a solution u_R to (1.1), and $u = u_R$ for all $R > 0$ if and only if

$$(1.3) \quad u(x) = |x|^{-a} v\left(\frac{x}{|x|}\right), \quad x \neq 0,$$

where v solves the equation

$$(1.4) \quad -\Delta_{S^{n-1}} v + b \cdot v = |v|^{p-2} v \text{ on } S^{n-1}$$

with $b := a(n-2-a)$. Such solutions also might arise from blow-up of singular solutions of (1.1) on a domain $\Omega \subset \mathbb{R}^n$ which are stationary in the sense of Pacard [6].

Conversely, if $p > 2^*$ any solution $v \neq 0$ to (1.4) of class $H^1 \cap L^p$ via (1.3) induces a solution $u \in H_{loc}^1 \cap L_{loc}^p(\mathbb{R}^n)$ of (1.1) blowing up at the origin. Equation (1.4) is similar to equation (1.1), but – unless $p = 2^+ := \frac{2(n-1)}{n-3}$, the critical Sobolev exponent in $n-1$ dimensions – no longer has any of the invariance properties of the latter. When $p \leq 2^+$ any solution v to (1.4) of class H^1 is smooth, and solutions $v > 0$ have been classified completely. In contrast, there is a vast abundance of sign-changing self-similar solutions; see Section 3. When $p > 2^+$ we expect equation (1.4) to admit also singular solutions $0 < v \in H^1 \cap L^p$, inducing solutions $u \in H^1$ of (1.1) blowing up on a higher-dimensional singular set. In a sequel [8] to this paper we plan to study this latter case more closely.

Our results thus are most conclusive for solutions $u > 0$ and may be summarized, as follows. On the one hand, in Theorem 2.1 we characterize the self-similar solution

$$(1.5) \quad u^*(r) = b^{\frac{1}{p-2}} r^{-a} \in H_{loc}^1 \cap L_{loc}^p(\mathbb{R}^n)$$

as the strong limit $u^* = \lim_{R \rightarrow \infty} u_R$ of rescalings of any smooth, radial entire solution u of (1.1) with $u(0) > 0$. On the other hand, as a consequence of a result of Gidas and Spruck [4], for mildly super-critical exponents $2^* < p < 2^+$ the solution u^* also is the unique positive, self-similar solution of (1.1) of class $H_{loc}^1 \cap L_{loc}^p$ on \mathbb{R}^n .

Moreover, for arbitrary $p > 2^*$ the function u^* is the only radially symmetric solution of (1.1) with a mild singularity at $r = 0$; indeed, by Theorems 2.1 and 2.3 any radial solution $u \neq \pm u^*$ which is singular at $r = 0$ changes sign infinitely often and blows up at the origin with a predetermined rate which, in particular, implies that $u \notin H_{loc}^1$.

2. RADIAL SOLUTIONS

For any dimension $n \geq 3$ and for any given $p > 2^* = \frac{2n}{n-2}$ we consider solutions $u = u(r)$ of the equation

$$(2.1) \quad u_{rr} + \frac{n-1}{r} u_r + |u|^{p-2} u = 0 \text{ on }]0, \infty[,$$

corresponding to (1.1). In our first result we study entire solutions $u \in C^2([0, \infty[)$ of (2.1) with a well-defined limit

$$(2.2) \quad u(0) = u_0, u_r(0) = 0 \text{ at } r = 0,$$

giving rise to classical solutions of (1.1) on \mathbb{R}^n . Moreover, we prove that the function u^* defined in (1.5) is the only radial solution of (1.1) with a mild singularity at the origin.

Theorem 2.1. *i) Let $u \in C^2([0, \infty[)$ solve (2.1), (2.2) for some $u_0 > 0$. Then $u(r) > 0$, and $u'(r) < 0$ for $r > 0$. Finally, there holds*

$$r^a u(r) \rightarrow b^{\frac{1}{p-2}} \text{ as } r \rightarrow \infty,$$

for $a = \frac{2}{p-2}$ and $b = a(n-a-2)$ as above.

ii) Suppose that $u \in C^2(]0, \infty[)$ solves (2.1) with

$$(2.3) \quad \liminf_{r \rightarrow 0} r^{\frac{2p}{p-2}} (|u_r|^2 + |u|^p) < \infty.$$

Then either $u \in C^2([0, \infty[)$, and u is as in i), or $u = \pm u^$.*

From Theorem 2.1.i) we immediately obtain the characterization $u^* = \lim_{R \rightarrow \infty} u_R$ for any solution $0 < u \in C^2([0, \infty[)$ of (1.1).

Note that the regular solutions of (2.1) are uniquely determined by the choice of u_0 in (2.2). Using (1.2) and invariance of (1.1) under reflection $u \rightarrow -u$ they can be obtained by rescaling the solution satisfying (2.2) with initial data $u_0 = 1$.

Before giving the proof of Theorem 2.1, we first remark that for equation (2.1) the classical Pohozaev identity [7] takes the following form.

Theorem 2.2. *Let $u \in C^2(]0, R[)$ be a solution of equation (2.1). Then we have*

$$(2.4) \quad \frac{d}{dr} \left[r^n \left(\frac{1}{2} |u_r|^2 + \frac{1}{p} |u|^p + \frac{n-2}{2r} u_r u \right) \right] + \left(\frac{n-2}{2} - \frac{n}{p} \right) r^{n-1} |u|^p = 0 \text{ in }]0, R[.$$

A similarly useful identity results if we consider equation (2.1) in scaled coordinates. Introducing $s = \log r$, we define $v(s) = r^a u(r)$ so that $u(r) = e^{-as} v(s)$. Compute

$$\begin{aligned} r^{a+1} u_r(r) &= v'(s) - av(s), \\ r^{a+2} u_{rr}(r) &= v''(s) - (2a+1)v'(s) + a(a+1)v(s). \end{aligned}$$

Hence (2.1) translates into the equation

$$(2.5) \quad v''(s) + \mu v'(s) - b \cdot v(s) + |v|^{p-2} v(s) = r^{a+2} (u_{rr} + \frac{n-1}{r} u_r + |u|^{p-2} u) = 0$$

for v , with $\mu = n - \frac{2p}{p-2} = n - 2 - 2a$ and $b = a(n - 2 - a) = a(\mu + a)$ as defined above. Note that $\mu > 0$ if (and only if) $p > 2^*$.

Multiplying (2.5) with $2v'$ we find the identity

$$(2.6) \quad \frac{d}{ds} (|v'|^2 - bv^2 + \frac{2}{p} |v|^p) + 2\mu |v'|^2 = 0.$$

For later reference, for a solution v to (2.5) we define

$$(2.7) \quad g(s) = (|v'|^2 - bv^2 + \frac{2}{p} |v|^p)(s).$$

By (2.6) then for any such v the function $s \mapsto g(s)$ is non-increasing. Moreover, $g \geq -\frac{p-2}{p} b \frac{p}{p-2}$ is uniformly bounded from below; we remark that this lower bound is achieved for the constant solution $v \equiv b^{\frac{1}{p-2}}$ of (2.5) which represents the selfsimilar solution u^* of (2.1). We furthermore remark that solutions of (2.1) satisfying the assumption (2.3) of Theorem 2.1.ii) correspond exactly to those solutions for which g is bounded.

Proof of Theorem 2.1. i) As observed earlier, it suffices to consider the case $u_0 = 1$. Integrating (2.4) from $\rho = 0$ to r , we find that $u_r u < 0$ on $]0, \infty[$. Since $u(0) = 1 > 0$ it follows that $u(r) > 0$, $u_r(r) < 0$ for $r > 0$.

Next, from (2.2) we see that the corresponding function $v(s) = r^a u(r) \rightarrow 0$ and $v'(s) \rightarrow 0$ as $s \rightarrow -\infty$ so

$$(2.8) \quad C_{-\infty} := \lim_{s \rightarrow -\infty} g(s) = 0.$$

Recalling that the quantity g defined in (2.7) is non-increasing and bounded from below we conclude that also the limit

$$(2.9) \quad C_{\infty} := \lim_{s \rightarrow \infty} g(s) = \lim_{s \rightarrow \infty} (|v'|^2 - bv^2 + \frac{2}{p} v^p) \leq 0$$

exists; moreover, (2.6) yields that $v' \in L^2(\mathbb{R})$ and

$$(2.10) \quad C_{\infty} + 2\mu \int_{-\infty}^{\infty} |v'|^2 ds = C_{-\infty}.$$

It then follows that

$$(2.11) \quad \lim_{s \rightarrow \infty} v'(s) = 0, \quad \lim_{s \rightarrow \infty} \left(\frac{2}{p} v^p - bv^2 \right) = C_{\infty},$$

because for any $\epsilon > 0$ by (2.10) we have $\inf_{[s, s+\epsilon]} |v'|^2 \rightarrow 0$ ($s \rightarrow \infty$), and (2.5), (2.9) limit the rate of change of v' . But (2.11) implies convergence $v(s) \rightarrow v_{\infty}$

as $s \rightarrow \infty$; moreover, together with (2.5) the relation (2.11) yields that also $v''(s)$ converges to a limit which must be zero as v' converges. Therefore

$$-bv + v^{p-1} \rightarrow 0 \text{ as } s \rightarrow \infty,$$

and we conclude that either $v_\infty = 0$ and $C_\infty = 0$, or $v_\infty = b^{\frac{1}{p-2}}$ and $C_\infty = -\frac{p-2}{p}b^{\frac{p}{p-2}} = \min g$.

But if $C_\infty = 0$, from (2.8) and (2.10) we conclude that $v' \equiv 0$ and hence $v \equiv \lim_{s \rightarrow \infty} v(s) = 0$, which implies that also $u \equiv 0$ and thus contradicts our assumption that $u(0) = 1$. Therefore we must have $v(s) \rightarrow b^{\frac{1}{p-2}}$ as $s \rightarrow \infty$, that is, $r^a u(r) \rightarrow b^{\frac{1}{p-2}}$ ($r \rightarrow \infty$), as claimed.

ii) Observe the relation $b - a^2 = a(n - 2 - 2a) = a\mu > 0$. Thus, by Young's inequality we can estimate

$$|v'|^2 - bv^2 = (r^{a+1}u_r + a \cdot r^a u)^2 - br^{2a}u^2 \leq Cr^{2(a+1)}u_r^2 = Cr^{\frac{2p}{p-2}}u_r^2$$

with a constant $C = C(p, n) > 0$. If now we assume that

$$(2.12) \quad C_{-\infty} := \lim_{s \rightarrow -\infty} (|v'|^2 - bv^2 + \frac{2}{p}|v|^p) \leq C \liminf_{r \rightarrow 0} r^{\frac{2p}{p-2}}(|u_r|^2 + |u|^p) < \infty,$$

similar to the above we conclude that $v'(s), v''(s) \rightarrow 0$ as $s \rightarrow -\infty$, and either $v(s) \rightarrow \pm b^{\frac{1}{p-2}}$ as $s \rightarrow -\infty$ and $C_{-\infty} = \min g$, or $v(s) \rightarrow 0$ as $s \rightarrow -\infty$. In the latter case it follows that $r^a u(r), r^{a+1}u_r(r) \rightarrow 0$ as $r \rightarrow 0$; in particular,

$$\frac{1}{r^\mu} \int_{B_r(0)} (|u_r|^2 + |u|^p) dx \rightarrow 0 \text{ as } r \rightarrow 0,$$

since $\mu = n - 2 - 2a = n - \frac{2p}{p-2} = n - ap$. From Pacard's [6] Proposition 1 and Lemma 8 then we obtain regularity for u at $x = 0$, so u must be as in i). (As observed in [1], any $p > 2^*$ is admissible for Pacard's result.)

Thus we may assume that $C_{-\infty} = \min g$ and thus that $s \mapsto g(s)$ is constant. It follows that $v \equiv \lim_{s \rightarrow -\infty} v(s) = \pm b^{\frac{1}{p-2}}$, and $u = \pm u^*$. \square

From Theorem 2.1 we know that any solution $u \in C^2([0, \infty[)$ of (2.1) that vanishes at some point $R > 0$ must be strongly singular at $x = 0$. In fact, we show in the next result that also the converse is true. Furthermore we obtain that statement ii) of the previous result remains valid under much weaker regularity assumptions.

Theorem 2.3. *Suppose that $u \in C^2([0, \infty[)$ solves equation (2.1) with*

$$(2.13) \quad \liminf_{r \rightarrow 0} r^{\frac{2p}{p-2}}(|u_r|^2 + |u|^p) = \infty.$$

Then u changes sign infinitely often on any interval $]0, R[$, $R > 0$, and there exists a number $c_0 > 0$ such that

$$(2.14) \quad |u_r|^2 + \frac{2}{p}|u|^p = c_0 r^{-(n+d)} + O(r^{-n}) \text{ as } r \rightarrow 0,$$

where $d = \frac{2p}{p+2}(n-1) - n > 0$.

Since $n + d > n > \frac{2p}{p-2}$ for $p > 2^*$, we thus see that the only radially symmetric solutions of (1.1) that are locally in H^1 are the selfsimilar solutions $\pm u^*$ and the regular solutions $u \in C^2([0, \infty[)$ described in Theorem 2.1.

Proof. i) Let u be as above. For brevity in the following we use the shorthand notation $u' = u_r$ also with reference to r . Consider the quantity

$$\Phi_0(r) := r^n(|u'|^2 + \frac{2}{p}|u|^p)$$

from Theorem 2.2 which satisfies

$$\Phi_0'(r) = nr^{-1}\Phi_0(r) + 2r^n(u'' + |u|^{p-2}u) \cdot u' = nr^{-1}\Phi_0(r) - 2(n-1)|u'|^2 \cdot r^{n-1}.$$

While not monotone itself, a slight modification of Φ_0 similar to (2.4) will give us a monotone quantity that can be used to study the asymptotic behaviour of u near zero.

Writing equation (2.1) in divergence form

$$(2.15) \quad (r^{n-1}u')' = -r^{n-1}|u|^{p-2}u$$

and letting

$$\Phi(r) := \Phi_0(r) + c \cdot r^{n-1}u' \cdot u,$$

with a constant $c \in \mathbb{R}$ to be determined, we find

$$\begin{aligned} \Phi'(r) &= \Phi_0'(r) + cr^{n-1}(|u'|^2 - |u|^p) \\ &= nr^{-1}\Phi_0 - [(2n-2-c)|u'|^2 + \frac{cp}{2} \cdot \frac{2}{p}|u|^p]r^{n-1}. \end{aligned}$$

For $c = \frac{4}{2+p}(n-1)$ chosen so that $(2n-2-c) = \frac{cp}{2}$ we then obtain the identity

$$(2.16) \quad \Phi'(r) = -dr^{-1}\Phi_0(r)$$

for the positive number d defined in the statement of Theorem 2.3.

Remark that for solutions of (2.1) satisfying (2.13) the difference $\Phi - \Phi_0$ is small compared with Φ . Indeed, for any number $\varepsilon \in]0, 1[$ we may estimate

$$(2.17) \quad |\Phi - \Phi_0| \leq C\Phi_0^{\frac{2+p}{2p}}r^{n-1-\frac{2+p}{2p}n} = C(r^d\Phi_0)^{\frac{2+p}{2p}} \leq \frac{1}{8}\varepsilon\Phi_0 + C_\varepsilon r^\mu,$$

so that by assumption (2.13)

$$|\Phi - \Phi_0| \leq \frac{1}{4}\varepsilon\Phi_0 \leq \frac{\varepsilon}{4-\varepsilon}\Phi$$

for sufficiently small $r > 0$. Hence we also have

$$(2.18) \quad |\Phi - \Phi_0| \leq \frac{\varepsilon}{d}\Phi$$

for $0 < r < r_\varepsilon$ small enough. Equation (2.16) thus gives the differential inequality

$$-(d+\varepsilon)r^{-1}\Phi(r) \leq \Phi'(r) \leq -(d-\varepsilon)r^{-1}\Phi(r)$$

which results in estimates of the form

$$(2.19) \quad C_1 \cdot r^{-d+\varepsilon} \leq \Phi(r), \Phi_0(r) \leq C_2 \cdot r^{-d-\varepsilon},$$

on $]0, r_\varepsilon[$, with numbers $C_{1,2} > 0$ depending on ε .

Inserting the bound (2.19) in (2.17) we can now improve (2.18) to obtain

$$(2.20) \quad |\Phi - \Phi_0| \leq C(r^d\Phi_0)^{\frac{2+p}{2p}} \leq Cr^{-\frac{2+p}{2p}\varepsilon} \leq Cr^{-\varepsilon}.$$

This bound in turn gives the improved differential inequality

$$|(r^d\Phi(r))'| = r^d|\Phi' + dr^{-1}\Phi| = dr^{d-1}|\Phi - \Phi_0| \leq Cr^{d-1-\varepsilon} \text{ on }]0, r_\varepsilon[.$$

Integrating this inequality for a fixed number $\varepsilon \in]0, d[$ we conclude that there exists

$$c_0 = \lim_{r \rightarrow 0} r^d \Phi(r) = \lim_{r \rightarrow 0} r^d \Phi_0(r).$$

By (2.17) then $|\Phi - \Phi_0| \leq C$, and we find the improved bound $|(r^d \Phi(r))'| \leq Cr^{d-1}$ from which we obtain that

$$|r^d \Phi_0(r) - c_0| \leq Cr^d + |r^d \Phi(r) - c_0| \leq Cr^d,$$

as claimed. Observe that the limit c_0 must be positive; otherwise estimate (2.19) would be violated.

ii) In a second step we now show that any solution u of (2.1) satisfying (2.13), and therefore also satisfying (2.14), changes sign infinitely often as $r \downarrow 0$. Assume by contradiction that some such u does not change sign on an interval $]0, r_1[$ for some $r_1 > 0$, say $u(r) \geq 0$ for $0 < r < r_1$.

We first claim that u must be decreasing in $]0, r_1[$. Indeed, since the maximum principle implies that solutions of (2.1) cannot achieve non-negative local minima, the only alternative would be that u is increasing on an interval $]0, r_2[$ for some $0 < r_2 < r_1$. However the resulting bound $0 \leq u \leq u(r_2)$ and the asymptotics of Φ_0 shown above then imply that $u'(r) \geq c \cdot r^{-\frac{n+d}{2}}$ as $r \rightarrow 0$, which contradicts our assumption that $u \geq 0$.

Now recall that $c_0 = \lim_{r \rightarrow 0} \Phi_0(r)r^d > 0$ and consider the sets

$$(2.21) \quad \begin{aligned} U_1(R) &:= \{r \in [R, 2R]; |u'(r)|^2 \geq \frac{1}{4}c_0 r^{-n-d}\}, \\ U_2(R) &:= \{r \in [R, 2R]; \frac{2}{p}u(r)^p \geq \frac{1}{4}c_0 r^{-n-d}\}, \end{aligned}$$

whose union contains the whole interval $[R, 2R]$, provided $R > 0$ is sufficiently small.

Since $u' \leq 0$ and $u > 0$ near zero, for the 1-dimensional Lebesgue measure $\mu(U_1(R))$ we obtain the inequality

$$u(R) \geq - \int_R^{2R} u'(r) dr \geq c_1 R^{-(n+d)/2} \cdot \mu(U_1(R)),$$

with a constant $c_1 > 0$ depending on c_0 but independent of R . In view of the estimate

$$u(R) \leq C(R^{-n}\Phi_0(R))^{1/p} \leq CR^{-(n+d)/p}$$

we then find the upper bound

$$\mu(U_1(R)) \leq CR^\gamma, \text{ with } \gamma = (n+d) \cdot \frac{p-2}{2p} > 1.$$

On the other hand, upon integrating equation (2.15) we find that

$$- \int_0^{2R} (u' r^{n-1})' dr = \int_0^{2R} u^{p-1} r^{n-1} dr \geq c R^{n-1-(n+d) \cdot \frac{p-1}{p}} \mu(U_2(R)).$$

Since $u'(r)r^{n-1} \rightarrow 0$ as $r \rightarrow 0$ the left hand side is equal to $-u'(2R)(2R)^{n-1}$ and thus bounded from above by $C \cdot R^{n-1-(n+d)/2}$. We conclude that also the measure of the second set is bounded by $\mu(U_2(R)) \leq CR^\gamma$ and thus in total

$$\mu(U_1(R)) + \mu(U_2(R)) \leq CR^\gamma \ll R = \mu([R, 2R])$$

for small $R > 0$, which contradicts the fact that $[R, 2R] \subset U_1(R) \cup U_2(R)$. \square

Thus, sign-changing solutions are necessarily oscillatory near the origin. On the other hand, we now show that the sign of any solution to (2.1) stabilizes near infinity.

Theorem 2.4. *Let $u \in C^2([0, \infty[)$ solve (2.1). Then there exists $R > 0$ such that $u(r) \neq 0$ for $r > R$, and either $\frac{u(r)}{u^*(r)} \rightarrow \pm 1$ as $r \rightarrow \infty$, or there exists a number $c_1 \neq 0$ such that*

$$(2.22) \quad u(r) = r^{-(n-2)}(c_1 + O(r^{-(2+\mu(p-2))})).$$

Proof. Let $v(s) = r^a u(r)$ be defined as above, satisfying (2.5). As in the proof of Theorem 2.2, from (2.6) we conclude that the limit

$$C_\infty := \lim_{s \rightarrow \infty} g(s) = \lim_{s \rightarrow \infty} (|v'|^2 - bv^2 + \frac{2}{p}|v|^p) < \infty$$

exists and that $v' \in L^2([0, \infty[)$ with

$$C_\infty + 2\mu \int_0^\infty |v'|^2 ds = g(0) < \infty.$$

Still following the proof of Theorem 2.2, we then conclude that

$$v'(s) \rightarrow 0 \text{ as } s \rightarrow \infty,$$

and either $v(s) \rightarrow 0$ as $s \rightarrow \infty$, or $v(s) \rightarrow \pm b^{\frac{1}{p-2}}$. In the latter case, $\frac{u(r)}{u^*(r)} \rightarrow \pm 1$ and clearly our claim is true. In the following we therefore may assume that $\lim_{s \rightarrow \infty} v(s) = 0$.

Let now $s_1 \in \mathbb{R}$ be such that $|v|^{p-2} < b$ for $s > s_1$. Then on $[s_1, \infty)$ the expression $v''(s) + \mu v'(s) = (b - |v|^{p-2})v$ has the same sign as $v(s)$, so v can neither achieve a positive maximum nor a negative minimum on $[s_1, \infty)$. So if there is any point $s_0 > s_1$ with $v(s_0) = 0$ we thus find that v is monotone on $[s_0, \infty)$; in particular, $v(s) \neq 0$ for $s > s_0$. If no such s_0 exists we define $s_0 := s_1$ so that again $v(s) \neq 0$ for $s > s_0$.

To see also the claimed asymptotic behavior, for $s > s_0$ we divide (2.5) by v to find the identity

$$(2.23) \quad \left(\frac{v'}{v}\right)' + \left(\frac{v'}{v}\right)^2 + \mu \frac{v'}{v} = \frac{v''}{v} + \mu \frac{v'}{v} = b - |v|^{p-2}$$

for $b = a(\mu + a)$ as above.

For $f := v'/v$ this gives the equation

$$(2.24) \quad f' + f^2 + \mu f - b = f' + (f - a)(f + \mu + a) = -|v|^{p-2}.$$

We claim that $\lim_{s \rightarrow \infty} f(s) = -(\mu + a)$. First of all, if $f(s_2) < -(\mu + a)$ for some s_2 then f would be decreasing on $[s_2, \infty)$ and thus satisfy $f'(s) \leq -\varepsilon f^2(s)$ for some $\varepsilon > 0$ so that $f(s) \rightarrow -\infty$ as $s \rightarrow s_-$ for some $s_- < \infty$ which is impossible as $v(s) \neq 0$ for $s > s_0$.

On the other hand, inspection of the sign of the nonlinearity combined with the assumption that $v(s) \rightarrow 0$ shows that if $\limsup_{s \rightarrow \infty} f(s) > -(\mu + a)$ then f tends to $a = \frac{2}{p-2}$ as $s \rightarrow \infty$. But then $v' \geq v/(p-2)$ for large s , and $v(s) \rightarrow \infty$ as $s \rightarrow \infty$, contrary to our assumption. We thus conclude that $f(s) \rightarrow -(\mu + a)$ as $s \rightarrow \infty$.

Given any $\varepsilon > 0$ we then find that for sufficiently large $s > s_0(\varepsilon)$

$$(2.25) \quad -(\mu + a) \leq \frac{v'}{v} \leq (-\mu + a + \varepsilon)$$

which gives

$$(2.26) \quad v(s_0)e^{-(\mu+a)(s-s_0)} \leq v(s) \leq v(s_0)e^{-(\mu+a-\varepsilon)(s-s_0)}.$$

as a first estimate on the asymptotics of v . Since $\mu + 2a = n - 2$ this translates into the bound

$$(2.27) \quad u(r_0)(r/r_0)^{-(n-2)} \leq u(r) \leq u(r_0)(r/r_0)^{-(n-2-\varepsilon)}$$

for sufficiently large $r > r_0(\varepsilon)$.

To obtain the more precise asymptotic expansion claimed in Theorem 2.4 we now consider the function $h(s) := f(s) + \mu + a = f(s) + n - 2 > 0$ which satisfies

$$(2.28) \quad h' - (n-2)h = -h^2 - |v|^{p-2} \geq -\varepsilon h - |v|^{p-2}$$

for $r \geq r_0(\varepsilon)$ sufficiently large. Inserting the bound on v derived in (2.26) we find that

$$(2.29) \quad e^{(n-2-\varepsilon)s} \cdot (h(s) \cdot e^{-(n-2-\varepsilon)s})' \geq -ce^{-(p-2)(\mu+a-\varepsilon)s}$$

for s large. Since $h(s) \rightarrow 0$ as $s \rightarrow \infty$ this gives the estimate

$$(2.30) \quad h(s) \leq Ce^{-(p-2)(\mu+a-\varepsilon)s}$$

which in turn allows us to improve the differential inequality (2.25) for v to

$$0 \leq v' + (\mu + a)v \leq vh \leq Ce^{-(p-1)(\mu+a-\varepsilon)s}$$

so that

$$(e^{(\mu+a)s}v(s))' \leq Ce^{-(p-2)(\mu+a-\varepsilon)s}.$$

Since $p > 2$ we conclude that the limit

$$c_1 = \lim_{s \rightarrow \infty} e^{(\mu+a)s}v(s) = \lim_{r \rightarrow \infty} r^{n-2}u(r)$$

exists, and thus

$$(2.31) \quad v(s) \leq Ce^{-(\mu+a)s} \text{ for large } s.$$

Repeating the above argument with the bound

$$\begin{aligned} h' - (n-2)h &= -h^2 - |v|^{p-2} \geq -Ce^{-2(p-2)(\mu+a-\varepsilon)s} - Ce^{-(p-2)(\mu+a)s} \\ &\geq -Ce^{-(p-2)(\mu+a)s}, \quad s \gg 1, \end{aligned}$$

resulting from (2.30) and (2.31) we obtain the estimate claimed in (2.22). \square

3. SELF-SIMILAR SOLUTIONS

Recall that a solution u of (1.1) on $\mathbb{R}^n \setminus \{0\}$ is self-similar if

$$(3.1) \quad u(x) = u_R(x) = R^a u(Rx) \text{ for all } x \neq 0 \text{ and any } R > 0.$$

As we have seen in Section 2, equation (1.1) admits the self-similar radial solution u^* , and $u^* \in H_{loc}^1 \cap L_{loc}^p(\mathbb{R}^n)$ when $p > 2^*$. By Pacard's work [6] we also expect to find not necessarily radial self-similar weak solutions of (1.1) on \mathbb{R}^n as limits of blow-ups of singular “stationary” weak solutions u of equation (1.1) on a domain, and the question of classifying these singularity profiles is of interest. Self-similar solutions to (1.1) also may arise as asymptotic profiles of stationary solutions at infinity, as described in our paper [8], or as singularity profiles of solutions of the parabolic evolution problem associated with (1.1) at blow-up points of Type I, as defined in the forthcoming work [1] of Simon Blatt and the second author, which extends the work of Giga-Kohn [3] to the critical and super-critical range and also

removes the restriction on the range of admissible exponents p in the results of Pacard [6].

Differentiating relation (3.1) in R for fixed $x \in S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$, we find the relations

$$(3.2) \quad u_r + au/r = 0, u_{rr} = a(a+1)u/r^2 \text{ for } r = |x| > 0.$$

Moreover choosing $R = |x|^{-1}$ for each $x \neq 0$, we obtain the representation (1.3) with a function $v: S^{n-1} \rightarrow \mathbb{R}$.

Thus, evaluating (1.1) at a point $x \in S^{n-1}$, from (3.2) we obtain the equation

$$\begin{aligned} 0 &= \Delta u + |u|^{p-2}u = u_{rr} + (n-1)u_r + \Delta_{S^{n-1}}u + |u|^{p-2}u \\ &= \Delta_{S^{n-1}}v - a(n-2-a)v + |v|^{p-2}v; \end{aligned}$$

that is,

$$(3.3) \quad -\Delta_{S^{n-1}}v + b \cdot v = |v|^{p-2}v \text{ on } S^{n-1},$$

or (1.4). We conclude this paper with the following remarks characterizing the solutions of (3.3) in the range $2^* < p \leq 2^+ := \frac{2(n-1)}{n-3}$, the critical exponent in $n-1$ dimensions (which almost equals the range covered by Pacard's [6] results). Observe that by standard regularity results for $p \leq 2^+$ any solution $v \in H^1$ of (3.3) is smooth.

By using the method of Ding [2] one can show that for any $p < \frac{2m}{m-2}$, where $n = m + k$ with $2 \leq k \leq m$, equation (3.3) possesses infinitely many distinct solutions of changing sign, giving rise to infinitely many distinct sign-changing solutions of (1.1) which blow up at the origin at the rate $|x|^{-a}$. Choosing the minimal number m with the above property, for even dimensions n we have $m = \frac{n}{2}$ and $\frac{2m}{m-2} = \frac{2n}{n-4} > 2^+$; likewise $\frac{2m}{m-2} = \frac{2(n+1)}{n-3} > 2^+$ when $n = 2m-1$ is odd. Thus, in general there is a huge multitude of self-similar solutions of (1.1).

However, the picture changes if we restrict our attention to positive solutions. In fact, by a result of Gidas and Spruck [4] for $p < 2^+$ any positive solution of (3.3) is constant. Indeed, even though for $p < 2^+$ we have $b > \frac{(n-1)(n-3)}{4}$ while there holds $R_{ij} = (n-2)g_{ij}$ for the Ricci curvature of S^{n-1} in the standard metric $g = (g_{ij})$, the condition (B.10) in the Remark following Theorem B.2 in [4] is satisfied for the number $\gamma > 0$ given by (B.11), and the conclusion follows from [4], Theorem B.2.

Finally, in the critical case $p = 2^+ = \frac{n-1}{n-3}$ we have

$$b = \frac{(n-1)(n-3)}{4} = \frac{1}{c(n-1)} R_{g_{S^{n-1}}}$$

with R_h the scalar curvature and $c(k) = 4 \frac{k-1}{k-2}$ the constant appearing in the conformal Laplace operator

$$L_h = -c(k)\Delta_h + R_h$$

on a k -dimensional manifold (M, h) , and (3.3) is the Yamabe equation on S^{n-1} , with infinitely many positive solutions u exactly arising as the coefficient in the conformal factor $u^{\frac{4}{n-3}} g_{S^{n-1}}$ of the metric obtained by pulling back the standard spherical metric $g_{S^{n-1}}$ with a Möbius transformation; see Obata [5].

In a sequel [8] to this paper we plan to investigate (1.4) also in the super-critical range $p > 2^+$ and, in particular, construct weak solutions $0 < v \in H^1$ depending only on the distance $\rho(q) = \text{dist}(q, N)$ of the point $q \in S^{n-1}$ from the North pole N and blowing up at the rate $v(\rho) \sim \rho^{-a}$ as $\rho \rightarrow 0$, similar to u^* .

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