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Graded Differential Vector Spaces, Cartan–Eilenberg Systems and Conjectures in Conley Index Theory

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Abstract

Cartan–Eilenberg systems play a prominent role in the homological algebra of filtered and graded differential vector spaces and (co)chain complexes in particular. We define the concept of Cartan–Eilenberg systems of vector spaces over a poset. Our main result states that a filtered chain isomorphism between P-graded differential vector spaces is equivalent to an isomorphism between associated Cartan–Eilenberg systems. An application of this result to the theory of dynamical systems addresses two open conjectures posed by J. Robbin and D. Salamon regarding uniqueness type questions for connection matrices, as detailed in Robbin and Salamon (Ergod Theory Dyn Syst 12(1):153–183, 1992. https://doi.org/10.1017/S0143385700006647) and Franzosa and Mischaikow (J Differ Equ 71(2):270–287, 1988. https://doi.org/10.1016/0022-0396(88)90028-9). The main result of this paper also proves that the connection matrix theories in Franzosa (Trans Am Math Soc 311(2):561–592, 1989. https://doi.org/10.2307/2001142), Robbin and Salamon (1992) and Harker et al. (J Appl Comput Topol 5(3):459–529, 2021. https://doi.org/10.1007/s41468-021-00073-3) are equivalent in the setting of vector spaces, as well as uniqueness of connection matrices for Morse–Smale gradient systems, cf. Reineck (Trans Am Math Soc 322(2):523–545, 1990. https://doi.org/10.2307/2001713).

Keywords Graded differential vector space · Cartan–Eilenberg system · Chain isomorphism · Connection matrix.

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1 Introduction

In the realm of algebraic topology and homological algebra, differential vector spaces graded by a poset P play a crucial role in the theory of spectral sequences and spectral systems and their applications in various fields, in particular dynamical systems. This paper studies the intricate relationship between P-graded differential vector spaces and Cartan–Eilenberg systems, cf. [1]. Our investigation revisits the traditional concept of Cartan–Eilenberg systems by considering vector spaces over an arbitrary poset. In particular, an application of our main result to dynamical systems theory confirms two open conjectures posited by J. Robbin and D. Salamon, referenced in [4, 11], which concern the uniqueness of connection matrices-a pivotal concept in the analysis of the global behavior of dynamical systems.

In order to state the main results in this paper we start with a brief, intuitive definition of Cartan–Eilenberg systems over a lattice. Let P be a finite poset and let $\mathsf{O}(\mathsf{P})$ be the lattice of down-sets in P. A Cartan–Eilenberg system $\mathbf{E}$ over $\mathsf{O}(\mathsf{P})$ consists of a functor E from the arrow category of $\mathsf{O}(\mathsf{P})$, which consists of ordered pairs (α, β), with $\alpha \subseteq \beta$, to the category $\mathbb{K}$-**Vect** of $\mathbb{K}$-vector spaces and an additional structure that accounts for exact triangles in homology, cf. Sect. 4.1 for a detailed definition. If we unpack the above informal definition we obtain the application $(\alpha, \beta) \mapsto \mathsf{E}(\alpha, \beta) = E_\alpha^\beta \in \mathrm{Ob}(\mathbb{K}\text{-}\mathbf{Vect})$. The functor E yields the homomorphisms $\ell\colon E_\alpha^\beta \to E_{\alpha'}^{\beta'}$ for all $(\alpha, \beta) \le (\alpha', \beta')$,[1] and the composition $E_\alpha^\beta \xrightarrow{\ell} E_{\alpha'}^{\beta'} \xrightarrow{\ell} E_{\alpha''}^{\beta''}$ is given by $\ell\colon E_\alpha^\beta \to E_{\alpha''}^{\beta''}$ due to the transitivity in $\mathsf{O}(\mathsf{P})$. Moreover there exist connecting homomorphisms $k\colon E_\beta^\gamma \to E_\alpha^\beta$ and $k\colon E_{\beta'}^{\gamma'} \to E_{\alpha'}^{\beta'}$ such that the diagrams[2]

$$\begin{tikzcd}
E_\alpha^\beta \arrow[rr, "i"] & & E_\alpha^\gamma \arrow[dl, "j"] \\
& E_\beta^\gamma \arrow[ul, "k"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
E_\beta^\gamma \arrow[r, "k"] \arrow[d, "\ell"'] & E_\alpha^\beta \arrow[d, "\ell"] \\
E_{\beta'}^{\gamma'} \arrow[r, "k"] & E_{\alpha'}^{\beta'}
\end{tikzcd}
\tag{1.1}$$

are exact and commutative respectively for all pairs of ordered triples $(\alpha, \beta, \gamma) \le (\alpha', \beta', \gamma')$.[3] By construction $\ell\colon E_\alpha^\beta \xrightarrow[\mathrm{id}]{\ell} E_\alpha^\beta$ is the identity homomorphism and the exactness of (1.1)[left] shows that $E_\alpha^\alpha = 0$ for all $\alpha \in \mathsf{O}(\mathsf{P})$, cf. [6]. A Cartan–Eilenberg system is *excisive* if $E_\alpha^\beta \cong E_{\alpha'}^{\beta'}$ for all $\beta \smallsetminus \alpha = \beta' \smallsetminus \alpha'$, cf. Lemma 4.3. A morphism between Cartan–Eilenberg systems $\mathbf{E}$ and $\mathbf{E}'$ is given by homomorphisms $h_\alpha^\beta\colon E_\alpha^\beta \to E_\alpha'^\beta$ which satisfy the commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & E_\alpha^\beta \arrow[r, "i"] \arrow[d, "h_\alpha^\beta"] & E_\alpha^\gamma \arrow[r, "j"] \arrow[d, "h_\alpha^\gamma"] & E_\beta^\gamma \arrow[r, "k"] \arrow[d, "h_\beta^\gamma"] & E_\alpha^\beta \arrow[r] \arrow[d, "h_\alpha^\beta"] & {} \\
{} \arrow[r] & E_\alpha'^\beta \arrow[r, "i'"] & E_\alpha'^\gamma \arrow[r, "j'"] & E_\beta'^\gamma \arrow[r, "k'"] & E_\alpha'^\beta \arrow[r] & {}
\end{tikzcd}
\tag{1.2}$$

for any $\alpha \subseteq \beta \subseteq \gamma$. In this case we write $h\colon \mathbf{E} \to \mathbf{E}'$ as a morphism of Cartan–Eilenberg systems.

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[1] The order on ordered pairs is point wise, i.e. $(\alpha, \beta) \le (\alpha', \beta')$ if and only $\alpha \subseteq \alpha'$ and $\beta \subseteq \beta'$.

[2] In the special cases $(\alpha, \beta) \le (\alpha, \gamma)$ and $(\alpha, \gamma) \le (\beta, \gamma)$ the morphisms ℓ are denoted by i and j respectively.

[3] An ordered triple (α, β, γ) is defined by $\alpha \subseteq \beta \subseteq \gamma$. The order on ordered triples is again point wise.

A P-graded differential vector space is a vector space C with (i) a P-grading given by the decomposition

$$C = \bigoplus_{p \in \mathsf{P}} G_p C, \tag{1.3}$$

where $G_p C \subseteq C$ are subspaces, and (ii) an endomorphism $\mathrm{d}_C \colon C \to C$, with $\mathrm{d}_C^2 = 0$, that is $\mathsf{O}(\mathsf{P})$-filtered, i.e. d_C is upper-triangular with respect to P, cf. Sect. 2. A P-graded differential vector space is strict if the restrictions $\mathrm{d}_C \colon G_p C \to G_p C$ are trivial for all $p \in \mathsf{P}$. A homomorphism of differential vector spaces is called a chain homomorphism. A chain homomorphism is $\mathsf{O}(\mathsf{P})$-filtered if it preserves the filterings. Precise definitions are provided in Sects. 2 and 2.1. As detailed in Sect. 2.2, a P-graded differential vector space (C, d_C) gives rise to a Cartan–Eilenberg system $\mathbf{E}(C, \mathrm{d}_C)$ with E-terms $E_\alpha^\beta = H(G_{\beta \smallsetminus \alpha} C)$, where $(G_{\beta \smallsetminus \alpha} C, \mathrm{d}_C)$ is a well-defined differential vector space with $G_{\beta \smallsetminus \alpha} C = \bigoplus_{p \in \beta \smallsetminus \alpha} G_p C$ a subspace and E_α^β is its homology. The main result of this paper can be phrased as follows:

Theorem A (cf. Theorem 5.1) *Two finite dimensional, strict,* P*-graded differential vector spaces are* $\mathsf{O}(\mathsf{P})$*-filtered chain isomorphic if and only if their respective Cartan–Eilenberg systems are isomorphic.*

An important application of Theorem A is to the theory of connection matrices developed by R. Franzosa, who introduces connection matrices in the algebraic topological theory of Morse representations for dynamical systems, cf. [3] and Sect. 6. Connection matrices contain information about the connecting orbit structure between the Morse sets in a Morse representation and generate the central algebraic structure referred to as a *module braid* in [2, 3], which serves as an invariant for the Morse representation in question. In Sect. 7 we prove that Franzosa's module braids in the setting of Morse representations are examples of Cartan–Eilenberg systems. The main result in [3] states that these module braids are always generated, under some mild conditions, by an appropriately constructed graded differential vector space—a *connection matrix*. Theorem A shows that two connection matrices assigned to a Morse representation are conjugate via filtered chain isomorphisms and thus unique up to filtered conjugacy. A first partial result in this direction was obtained in [4, 5] in the context of vector spaces, cf. Remark 5.4. Theorem A settles the two central conjectures formulated in [11] concerning the similarity of connection matrices, and completes the result in [4]. In Sect. 7 we give an extensive account of these conjectures and their proofs. Another consequence of Theorem A concerns connection matrix theories in the setting of vector spaces.

Theorem B (cf. Theorem 7.3) *Let* (D, d_D) *be an* $\mathsf{O}(\mathsf{P})$*-filtered differential vector space. Then, every strict,* P*-graded differential vector space* (A, d_A) *whose Cartan–Eilenberg systems is isomorphic to* $\boldsymbol{E}(D, \mathrm{d}_D)$*, is* $\mathsf{O}(\mathsf{P})$*-filtered chain equivalent to* (D, d_D).

As a consequence of Theorem B the connection matrix theories of [3, 5, 11] in the setting of finite dimensional vector spaces are equivalent. Without the restriction of finite dimensionality the notions in [3, 11] are equivalent.

Another application of Theorem A (Theorem 5.1) involves the uniqueness of connection matrices for Morse–Smale gradient flows. A P-graded chain complex (C, d_C) of $\mathbb{Z}_2$-vector spaces is Morse–Smale graded if there exists an order-preserving map $\mu \colon \mathsf{P} \to \mathbb{Z}$ such that $\mu(p) = \mu(q)$, $p \neq q$, then $p \parallel q$, and $G_p C_k = \mathbb{Z}_2$. The pair (P, μ) is called a Morse–Smale grading. For such chain complexes we have the following uniqueness result.

Theorem C (cf. Theorem 7.5) *Let* (C, d_C) *and* (C, d'_C) *be algebraic Morse–Smale graded chain complexes with Morse–Smale grading* (P, μ) *for both chain complexes. Then,* $\mathrm{d}_C = \mathrm{d}'_C$ *if and only if* $\boldsymbol{E}(C, \mathrm{d}_C) \cong \boldsymbol{E}(C, \mathrm{d}'_C)$.

Theorem C applies to Morse–Smale gradient flows and provides an algebraic proof of a result by J. Reineck, cf. [10], that connection matrices for such systems are unique, cf. Sect. 7.3.

Remark 1.1 In this paper we choose to present the results in the category $\mathbb{K}$-**Vect** of $\mathbb{K}$-vector spaces. The main result (Theorem A) is expected to remain valid if we replace vector spaces by free, finitely generated R-modules where the ring R is a principal ideal domain, cf. Remark 5.3.

2 Graded and Filtered Differential Vector Spaces

An *ordered decomposition* of a vector space C is a finite poset $\Pi = (\Pi, \le)$ consisting of non-trivial projections $\pi : C \to C$ such that

(i) $\pi \circ \pi' = 0$ for all distinct pairs $\pi, \pi' \in \Pi$;
(ii) $\sum_{\pi \in \Pi} \pi = \mathrm{id}$.

Let $(\mathsf{P}, \le)$ be a finite poset and let $\mathsf{O}(\mathsf{P})$ be the bounded, distributive lattice of down-sets in P. A P-*grading* on C is an order-embedding

$$\mathrm{grd} \colon \Pi \hookrightarrow \mathsf{P},$$

and the pair (C, grd) is called a P-*graded vector space*. A (finite) filtering on a vector space C is a homomorphism

$$\mathrm{flt} \colon \mathsf{O}(\mathsf{P}) \to \mathsf{Sub}\, C,$$

where $\mathsf{Sub}\, C$ is the lattice of subspaces of C. The image of flt is a sublattice consisting of subspaces $F_\alpha C \subseteq C$ ordered by inclusion. Note that $\alpha \mapsto F_\alpha C$ being a lattice homomorphism implies that $F_{\alpha \cup \beta} C = F_\alpha C + F_\beta C$ and $F_{\alpha \cap \beta} C = F_\alpha C \cap F_\beta C$. The pair (C, flt) is called an $\mathsf{O}(\mathsf{P})$-*filtered vector space*. A graded vector space yields a filtering $\mathrm{flt} \colon \mathsf{O}(\mathsf{P}) \to \mathsf{Sub}\, C$ of C given by $\alpha \mapsto F_\alpha C := \bigoplus_{\pi \in \mathrm{grd}^{-1}(\alpha)} \pi C$. We use the convention that the empty sum is 0. The homomorphism flt extends to a meet-semilattice homomorphism $\beta \smallsetminus \alpha \mapsto G_{\beta \smallsetminus \alpha} C := \bigoplus_{\pi \in \mathrm{grd}^{-1}(\beta \smallsetminus \alpha)} \pi C$. In particular, this assigns the 'atoms' $\{p\} \mapsto G_p C$ and associated decomposition: $G_{\beta \smallsetminus \alpha} C = \bigoplus_{p \in \beta \smallsetminus \alpha} G_p C$, with $G_p C = \pi C$, if $\mathrm{grd}^{-1}(\{p\}) = \{\pi\}$ and $G_p C = 0$ if $\mathrm{grd}^{-1}(\{p\}) = \varnothing$. We refer to the subspaces $G_p C$ as the subspaces of homogeneous elements of degree p. Since grd need not be onto, some of the subspaces $G_p C$ may be the trivial subspace 0 for some p and therefore the subspaces $G_p C$ coincide with the components $\mathrm{im}\, \pi$ whenever $G_p C$ is non-trivial. It is customary to denote the P-graded vector space by its decomposition $C = \bigoplus_{p \in \mathsf{P}} G_p C$. Conversely, an $\mathsf{O}(\mathsf{P})$ filtered vector space C also yields the P-graded vector space $\widetilde{C} = \bigoplus_{p \in \mathsf{P}} F_\beta C / F_\alpha C$, where $\beta \smallsetminus \alpha = \{p\}$. There exist subspaces $G_p C \subseteq C$, with $C_p C \cong F_\beta C / F_\alpha C$, such that $F_\alpha C = \bigoplus_{p \in \alpha} G_p C$.[4]

A homomorphism $f \colon C \to C'$ of P-graded vector spaces is called *homogeneous*, or P-*graded* if it satisfies the property: for all $p \in \mathsf{P}$ there exists a $q \in \mathsf{P}$ such that $f\big(G_p C\big) \subseteq$

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[4] For R-modules $\widetilde{C}$ is not isomorphic to C in general. If the subquotients $F_\beta C / F_\alpha C$ are free there exist submodules $G_p C \subseteq C$, with $C_p C \cong F_\beta C / F_\alpha C$, such that $F_\alpha C = \bigoplus_{p \in \alpha} G_p C$, cf. [11].

$G_q C'$. A homomorphism $f\colon C \to C'$ of P-graded, or $\mathsf{O}(\mathsf{P})$-filtered vector spaces that satisfies $f\big(F_\alpha C\big) \subseteq F_\alpha C'$ is called $\mathsf{O}(\mathsf{P})$-*filtered*. Note that homomorphisms of graded vector spaces may be filtered when the graded vector spaces are regarded as filtered vector spaces, with $F_\alpha C = \bigoplus_{p\in\alpha} G_p C$ being the filtering induced by the P-grading on C. An isomorphism $f\colon C \to C'$ is an $\mathsf{O}(\mathsf{P})$-filtered isomorphism if

$$f(F_\alpha C) = F_\alpha C', \quad \forall \alpha \in \mathsf{O}(\mathsf{P}). \tag{2.1}$$

If the order structure of the grading, or filtering is clear from context we refer to (homogeneous) graded and filtered homomorphisms.

Remark 2.1 An isomorphism that preserves the filterings is not necessarily a filtered isomorphism in the sense of Condition (2.1) as the following example shows. Let $C = \mathbb{R}^3$ and $C' = \mathbb{R}^3$ and $f\colon C \to C'$ is the identity homomorphism. Suppose C is filtered as $0 \subseteq \mathbb{R} \subseteq \mathbb{R}^3$ and C' is filtered as $0 \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$. Clearly f is an isomorphism that is $\mathsf{O}(\mathsf{P})$-filtered, but $f(\mathbb{R}) \subsetneq \mathbb{R}^2$. If both maps f and f^{-1} are $\mathsf{O}(\mathsf{P})$-filtered, then $F_\alpha C = f^{-1} f(F_\alpha C) \subseteq f^{-1}(F_\alpha C') \subseteq F_\alpha C$ which yields (2.1). Thus a filtered isomorphism is an isomorphism for which both f and f^{-1} are filtered.

2.1 Differential Vector Spaces

A *differential vector space* (C, d_C) a vector space C endowed with an endomorphism $\mathrm{d}_C : C \to C$, with $\mathrm{d}_C^2 = 0$, cf. [8, II.1]. The elements $c \in C$ are referred to as *chains* and the endomorphism d_C is referred to as the *differential*. We fix the following notation: $Z(C) := \ker \mathrm{d}_C$ are the *cycles* in C and $B(C) := \operatorname{im} \mathrm{d}_C$ are the *boundaries* in C. The *homology* of (C, d_C) is defined as $H(C) := Z(C)/B(C)$. A homomorphism $f\colon (C, \mathrm{d}_C) \to (A, \mathrm{d}_A)$ of differential vector spaces is a homomorphism which satisfies $\mathrm{d}_A f = f \mathrm{d}_C$ and is called a *chain homomorphism*, or *chain isomorphism* if f is an isomorphism of vector spaces. Differential vector spaces and chain homomorphisms form the category $\mathbf{C}(\mathbb{K}\text{-}\mathbf{Vect})$. Homology for a differential vector space defines a covariant functor $\mathsf{H}\colon \mathbf{C}(\mathbb{K}\text{-}\mathbf{Vect}) \to \mathbb{K}\text{-}\mathbf{Vect}$ given by $(C, \mathrm{d}_C) \mapsto H(C)$, called the *homology functor*. The homology may be regarded as a differential vector space with $\mathrm{d} = 0$. Next we consider grading and filtering in combination with the differental d_C.

Definition 2.2 A triple $\big(C, \mathrm{d}_C, \mathrm{grd}\big)$ is called a P-*graded differential vector space* if

(i) (C, d_C) is a differential vector space;
(ii) (C, grd) is a P-graded vector space with grading $\mathrm{grd}\colon \Pi \hookrightarrow \mathsf{P}$;
(iii) d_C is $\mathsf{O}(\mathsf{P})$-filtered.

A P-graded differential vector space is *strict* if

(iv) $\mathrm{d}_C F_\alpha C \subseteq F_{\alpha \smallsetminus \{p\}} C$, for all p maximal in α.

A chain homomorphism, or chain isomorphism between graded or filtered differential vector spaces preserving the filtering is called an $\mathsf{O}(\mathsf{P})$-*filtered chain homomorphism*, or $\mathsf{O}(\mathsf{P})$-*filtered chain isomorphism*. For a P-graded differential vector space the condition on d_C to be $\mathsf{O}(\mathsf{P})$-filtered implies that the restriction $\mathrm{d}_C\colon G_q C \to G_p C$ is well-defined. The latter may be non-trivial only if $p \le q$. If the differential is strict then the latter inequality is strict, i.e. $p < q$. For any convex set $\beta \smallsetminus \alpha$ the pair $(G_{\beta\smallsetminus\alpha} C, \mathrm{d}_C)$ is a differential vector space with $G_{\beta\smallsetminus\alpha} C \subseteq C$ a subspace. More generally a triple $(C, \mathrm{d}_C, \mathrm{flt})$ is called an $\mathsf{O}(\mathsf{P})$-*filtered differential vector space* if (C, flt) is $\mathsf{O}(\mathsf{P})$-filtered and the differential satisfies $\mathrm{d}_C F_\alpha C \subseteq F_\alpha C$ for all $\alpha \in \mathsf{O}(\mathsf{P})$.

The $\mathsf{O}(\mathsf{P})$-filtering induced by a P-graded differential vector space automatically satisfies the conditions of an $\mathsf{O}(\mathsf{P})$-filtered differential vector space via $F_\alpha C := \bigoplus_{p\in\alpha} G_p C$.

Example 2.3 A chain complex (C, d_C) is an example of a graded differential vector space, traditionally denoted by $C = \bigoplus_k C_k$, where the differential d_C is homogeneous of degree -1, and thus filtered. This is equivalent to $\mathrm{d}_C = \bigoplus_k \mathrm{d}_k$, with $\mathrm{d}_k : C_k \to C_{k-1}$ and $\mathrm{d}_k\mathrm{d}_{k+1} = 0$. The chain maps are taken be to homogeneous, i.e. $f : C \to C'$, $f = \bigoplus_k f_k$, with $f_k : C_k \to C'_k$ and $\mathrm{d}'_k f_k = f_{k-1}\mathrm{d}_k$, for all k. The same holds for cochain complexes.

Remark 2.4 For graded and filtered differential vector spaces we will omit grd and flt from the notation if there is no ambiguity about the grading or filtering, and simply write (C, d_C).

2.2 Cartan–Eilenberg Systems

In Sect. 1 a basic definition of Cartan–Eilenberg systems in the category of vector spaces is given. Filtered differential vector spaces are a prime source for generating excisive Cartan–Eilenberg systems. Let (C, d_C) be a filtered differential vector space. Since the differential d_C preserves the filtering we have the short exact sequence of differential vector spaces:

$$0 \longrightarrow \frac{F_\beta C}{F_\alpha C} \xrightarrow{i} \frac{F_\gamma C}{F_\alpha C} \xrightarrow{j} \frac{F_\gamma C}{F_\beta C} \longrightarrow 0\ , \quad \alpha \subseteq \beta \subseteq \gamma. \tag{2.2}$$

The associated homology is given by $E_\alpha^\beta := H(F_\beta C/F_\alpha C)$, $E_\alpha^\gamma := H(F_\gamma C/F_\beta C)$ and $E_\beta^\gamma := H\big(F_\gamma C/F_\beta C\big)$. This yields the exact triangles in (1.1) where the connecting homomorphism $k\colon H\big(F_\beta C/F_\alpha C\big) \to H(F_\alpha C)$ is constructed in the usual way. All other axioms are readily verified which yields a Cartan–Eilenberg system denoted $\mathbf{E}(C, \mathrm{d}_C)$. The excisive property follows from the fact that $F_\beta C/F_{\alpha\cap\beta} C = F_\beta C/(F_\alpha C \cap F_\beta C) \cong (F_\alpha C + F_\beta C)/F_\alpha C = F_{\alpha\cup\beta} C/F_\alpha C$, cf. Lemma 4.3. The (excisive) Cartan–Eilenberg system $\mathbf{E}(C, \mathrm{d}_C)$ is the Cartan–Eilenberg system of the $\mathsf{O}(\mathsf{P})$-filtered differential vector space (C, d_C). Since graded differential vector spaces are filtered differential vector spaces via $F_\alpha C := \bigoplus_{p\in\alpha} G_p C$, we define the Cartan–Eilenberg system of a P-graded differential vector space as outlined above. The induced Cartan–Eilenberg system is again denoted by $\mathbf{E}(C, \mathrm{d}_C)$.

Remark 2.5 For finite dimensional, strict P-graded differential vector spaces the homology satisfies $H(G_p C, \mathrm{d}_C) = G_p C \cong E_\alpha^\beta$, with $\beta\smallsetminus\alpha = \{p\}$ for all $p \in \mathsf{P}$.

3 Induced Chain Homomorphisms

In this section we prove two results, Theorems 3.12 and 3.14, concerning chain homomorphisms that lift homomorphisms on homology. We start with a number of preliminary lemmas.

Lemma 3.1 *Let D be a vector space, (A, d_A) a differential vector space, and let $h\colon D \to A$ be a homomorphism such that $\mathrm{d}_A h = 0$ and $H(h) = 0$.*[5] *Then, $h\colon D \to B(A)$ and there exists a lift $l\colon D \to A$ such that the diagram*

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[5] If D is regarded as differential vector space with $\mathrm{d} = 0$, then the condition $\mathrm{d}_A h = 0$ is equivalent to h being a chain homomorphism. The induced homology map is denoted by $H(h)\colon D \to H(A)$. We say that D is a *boundaryless* differential vector space.

$$\begin{tikzcd}
 & A \arrow[d, two heads, "\mathrm{d}_A"] \\
D \arrow[ru, dashed, "l"] \arrow[r, "h"'] & B(A)
\end{tikzcd} \tag{3.1}$$

commutes, i.e. $h = \mathrm{d}_A l$.

Proof By the assumption that $\mathrm{d}_A h = 0$ we have that $h\colon D \to Z(A)$. Observe the short exact sequence for homology

$$0 \longrightarrow B(A) \xrightarrow{e} Z(A) \xrightarrow{\rho} H(A) \longrightarrow 0, \tag{3.2}$$

where ρ is the projection $x \mapsto [x] \in H(A)$ and e is inclusion. From the assumption that $H(h) = \rho h = 0$ and the *exactness* of (3.2) we conclude that h maps into $B(A)$ and thus $h\colon D \to B(A) \subseteq Z(A)$. Consider the diagram:

$$\begin{tikzcd}
 & A \arrow[d, two heads, "\mathrm{d}_A"'] & D \arrow[l, "l"'] \arrow[ld, dashed, "h"'] \arrow[d, "h"] \arrow[rd, "H(h)"] & & \\
0 \arrow[r] & B(A) \arrow[r, "e"] & Z(A) \arrow[r, "\rho"] & H(A) \arrow[r] & 0
\end{tikzcd} \tag{3.3}$$

Since D is a vector space it is projective and therefore there exists a lift $l\colon D \to A$ such that the diagram in (3.1) commutes. □

Remark 3.2 For the action of differentials we omit parentheses and for other homomorphisms we adopt the usual functional notation, i.e. $\mathrm{d}c$, λc and $h(c)$. Parentheses are also omitted in compositions of maps.

Remark 3.3 In the remainder of the paper $C \oplus C'$ (and $A \oplus A'$) denotes the Cartesian product as direct sum with specifically constructed differentials to emphasize that this is not the product of differential vector spaces with the product differential but a product vector space with a chosen differential.

Remark 3.4 In the special case that (A, d_A) is a chain complex and $h_k\colon D \to A_k$, with $\mathrm{d}_k h_k = 0$, then $h_k\colon D \to B_k(A)$ and there exists a lift $l_k\colon D \to A_{k+1}$ such that $h_k = \mathrm{d}_{k+1} l_k$, which yields the following diagram

$$\begin{tikzcd}
 & A_{k+1} \arrow[d, two heads, "\mathrm{d}_{k+1}"'] & D \arrow[l, "l_k"'] \arrow[ld, dashed, "h_k"'] \arrow[d, "h_k"] \arrow[rd, "H_k(h_k)"] & & \\
0 \arrow[r] & B_k(A) \arrow[r, "e_k"] & Z_k(A) \arrow[r, "\rho_k"] & H_k(A) \arrow[r] & 0
\end{tikzcd}$$

where $B_k(A) = \operatorname{im} \mathrm{d}_{k+1}$, $Z_k(A) = \ker \mathrm{d}_k$ and $H_k(A) = Z_k(A)/B_k(A)$.

Lemma 3.5 *Let* C, C' *be vector spaces and consider the short exact sequence*

$$0 \longrightarrow C \xrightarrow{i} C \oplus C' \xrightarrow{j} C' \longrightarrow 0, \tag{3.4}$$

where $i(c) = (c, 0)$, $j(c, c') = c'$. *Let* d_C *be a differential for* C *and let* C' *be boundaryless. Then, for* i *and* j *to be chain homomorphisms the differential on* $C \oplus C'$ *is of the form*

$$\mathrm{d}_{C\oplus C'} = \begin{pmatrix} \mathrm{d}_C & \lambda \\ 0 & 0 \end{pmatrix}, \tag{3.5}$$

where $\lambda\colon C' \to C$ *is a chain homomorphisms, i.e.* $\mathrm{d}_C\lambda = 0$. *Moreover, for the long exact sequence (exact triangle)*

$$\longrightarrow H(C) \xrightarrow{i} H(C \oplus C') \xrightarrow{j} C' \xrightarrow{H(\lambda)} H(C) \longrightarrow \tag{3.6}$$

the canonical connecting homomorphism is given by $H(\lambda)$.

Proof Let us start with the differential in (3.5). In order for i and j to be chain maps we need: $i\mathrm{d}_C = \mathrm{d}_{C\oplus C'} i$ and $j\mathrm{d}_{C\oplus C'} = \mathrm{d}_{C'} j$. By assumption $\mathrm{d}_{C\oplus C'}$ is given by

$$\mathrm{d}_{C\oplus C'} = \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix},$$

for some homomorphisms $\kappa\colon C \to C$, $\lambda\colon C' \to C$, $\mu\colon C \to C'$ and $\nu\colon C' \to C'$. From the conditions on i and j, and the fact that C' is boundaryless ($\mathrm{d}_{C'} = 0$), we derive that $\big(\mathrm{d}_C c, 0\big) = i\mathrm{d}_C c = \mathrm{d}_{C\oplus C'} i(c) = \big(\kappa c, \mu c\big)$, which yields that $\kappa = \mathrm{d}_C$ and $\mu = 0$. Moreover, $\nu c' = j\mathrm{d}_{C\oplus C'}(c, c') = 0 \cdot j(c, c') = 0$, which implies that $\nu = 0$ and therefore $\mathrm{d}_{C\oplus C'}$ is given by (3.5) for some homomorphism $\lambda\colon C' \to C$. Since $\mathrm{d}_{C\oplus C'}$ is a differential we obtain the condition $\mathrm{d}_C\lambda = 0$, which makes λ a chain homomorphism.

As for the long exact sequence we follow the standard construction for the connecting homomorphism. Let $c' \in C' = Z(C')$. By exactness, $c' = j(\tilde{c}, c')$ for some $\tilde{c} \in C$. Since j is a chain homomorphism we have that $j\mathrm{d}_{C\oplus C'}(\tilde{c}, c') = 0 \cdot j(\tilde{c}, c')$ and thus $\mathrm{d}_{C\oplus C'}(\tilde{c}, c') \in \ker j = \operatorname{im} i$, which implies that $\mathrm{d}_C\tilde{c} + \lambda c' = c$ for some $c \in C$. Observe that $\mathrm{d}_C c = \mathrm{d}_C\big(\mathrm{d}_C\tilde{c} + \lambda c'\big) = \mathrm{d}_C^2\tilde{c} + \mathrm{d}_C\lambda c' = 0$. Define the connecting homomorphism $k[c'] := [c] = [\mathrm{d}_C\tilde{c} + \lambda c'] = [\lambda c'] = H(\lambda)[c']$, which establishes the connecting homomorphism. □

Remark 3.6 If (C, d_C) is a chain complex the short exact sequence in (3.4) holds degree wise and yields:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_k \arrow[r, "i_k"] \arrow[d, "\mathrm{d}_k"] & C_k \oplus C'_k \arrow[r, "j_k"] \arrow[d, "\bar{\mathrm{d}}_k"] & C'_k \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & C_{k-1} \arrow[r, "i_{k-1}"] \arrow[d] & C_{k-1} \oplus C'_{k-1} \arrow[r, "j_{k-1}"] \arrow[d] & C'_{k-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd} \tag{3.7}$$

In this case $\bar{\mathrm{d}}_k(c, c') = \mathrm{d}_k c + \lambda_k c'$, where $\lambda_k\colon C'_k \to C_{k-1}$ satisfies $\mathrm{d}_{k-1}\lambda_k = 0$, and for the long exact sequence $H_k(\lambda_k)\colon C'_k \to H_{k-1}(C)$ is the connecting homomorphism. The homology in degree k is given by the functor H_k.

Lemma 3.7 *Consider the following commutative diagram of differential vector spaces with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "f'"'] & C \oplus C' \arrow[r, "j"] & C' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & A \arrow[r, "i'"] & A \oplus A' \arrow[r, "j'"] & A' \arrow[r] & 0
\end{tikzcd} \tag{3.8}$$

where $i(c) = (c, 0)$, $j(c, c') = c'$, $i'(a) = (a, 0)$, $j'(a, a') = a'$, *and* f' *and* f'' *are chain homomorphisms and where* C' *and* A' *are boundaryless. Then, the differentials on* $C \oplus C'$ *and* $A \oplus A'$ *are given by Lemma* 3.5 *and are of the form*

$$\mathrm{d}_{C\oplus C'} = \begin{pmatrix} \mathrm{d}_C & \lambda \\ 0 & 0 \end{pmatrix}, \quad \mathrm{d}_{A\oplus A'} = \begin{pmatrix} \mathrm{d}_A & \lambda' \\ 0 & 0 \end{pmatrix}, \tag{3.9}$$

where $\lambda\colon C' \to C$ *and* $\lambda'\colon A' \to A$ *are chain homomorphisms, i.e.* $\mathrm{d}_C\lambda = 0$ *and* $\mathrm{d}_A\lambda' = 0$. *Let* $K := jZ(C \oplus C') \subseteq C'$ *and* $K' := jZ(A \oplus A') \subseteq A'$. *If,*

$$H(f')H(\lambda) = H(\lambda')f'', \tag{3.10}$$

then the map f'' *induces a chain homomorphism* $f''\colon K \to K'$ *by restriction, such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & Z(C) \arrow[r, "i"] \arrow[d, "f'"'] & Z(C \oplus C') \arrow[r, "j"] & K \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & Z(A) \arrow[r, "i'"] & Z(A \oplus A') \arrow[r, "j'"] & K' \arrow[r] & 0
\end{tikzcd} \tag{3.11}$$

commutes with exact rows.

Proof By definition $Z(C \oplus C') = \big\{(z, z') \mid \mathrm{d}_C z + \lambda z' = 0\big\}$ and the projection K is characterized by

$$K = \big\{z' \in C' \mid \mathrm{d}_C z + \lambda z' = 0 \text{ for some } z \in C\big\}. \tag{3.12}$$

The same holds for $Z(A \oplus A')$ and K'. With the chain maps i and j the rows in (3.11) are exact. Since f' is a chain homomorphism the restriction of f' to $Z(C)$ defines the homomorphism $f'\colon Z(C) \to Z(A)$. Moreover, the diagram in (3.11) commutes provided that f'' is a well-defined homomorphism from K to K'. Consider the compositions $f'\lambda$ and $\lambda' f''$. Then,

$$\mathrm{d}_A f'\lambda = f' d_C \lambda = 0, \quad \text{and} \quad \mathrm{d}_A \lambda' f'' = 0, \tag{3.13}$$

which follows from the fact that $d_C\lambda = 0$ and $d_A\lambda' = 0$. We conclude that $f'\lambda\colon C' \to Z(A)$ and $\lambda' f''\colon C' \to Z(A)$. Define the homomorphism $h = f'\lambda - \lambda' f''$ acting from C' to A. Then, $\mathrm{d}_A h = 0$ and thus $h\colon C' \to Z(A)$. Consider the diagram:

$$\begin{tikzcd}
 & & C' \arrow[d, "h"] \arrow[dr, "H(h)"] & & \\
0 \arrow[r] & B(A) \arrow[r, "e"] & Z(A) \arrow[r, "\rho"] & H(A) \arrow[r] & 0
\end{tikzcd} \tag{3.14}$$

Observe that $H(h) = 0$. Indeed, since both $f'(\lambda c') \in Z(A)$ and $\lambda' f''(c') \in Z(A)$ the homology class satisfies:

$$\begin{aligned}
H(h)[c'] &= [h(c')] = \rho h(c') = \rho f'(\lambda c') - \rho\lambda' f''(c') = [f'(\lambda c')] - [\lambda' f''(c')] \\
&= H(f')[\lambda c'] - H(\lambda') f''(c') = H(f')H(\lambda)[c'] - H(\lambda') f''(c') = 0,
\end{aligned}$$

where $[f''(c')] = f''(c')$ and $[c'] = c'$, and which uses (3.10).

Let $z' \in K$. We show that $f''(z') \in K'$. Under restriction of h to $K \subseteq C'$ we have again $\mathrm{d}_A h|_K = 0$ and $H(h|_K) = 0$. We can apply Lemma 3.1 with respect to the vector space K, the differential vector space (A, d_A) and the homomorphism $h|_K\colon K \to A$. Consequently, $h|_K\colon K \to B(A)$ and there exists a lift $l\colon K \to A$ such that $h|_K = \mathrm{d}_A l$. For $z' \in K$ define an element $a \in A$, given by $a = l(z')$, so that $\mathrm{d}_A a = \mathrm{d}_A l(z') = h|_K(z')$. This implies that $\mathrm{d}_A a = f'(\lambda z' \mathrm{d}_A a = f'\lambda(z') - \lambda' f''(z')$ and thus $f'\lambda(z') = \lambda' f''(z') + \mathrm{d}_A a$. For $z' \in K$ the definition of the latter yields a $z \in C$ such that $(z, z') \in Z(C \oplus C')$, i.e. $\mathrm{d}_C z + \lambda z' = 0$, cf. (3.12). From a we define $\zeta = \big(f'(z) + a, f''(z')\big) \in A \oplus A'$. Then, by the choice of a and the fact that f' is a chain homomorphism, we have that:

$$\begin{aligned}
\mathrm{d}_{A\oplus A'}\zeta &= \mathrm{d}_A(f'(z) + a) + \lambda' f''(z') = f'\mathrm{d}_C z + \lambda' f''(z') + \mathrm{d}_A a \\
&= f'\mathrm{d}_C z + f'\lambda(z') = f'(\mathrm{d}_C z + \lambda z') = 0,
\end{aligned}$$

which shows that $\zeta \in Z(A \oplus A')$. By definition $f''(z') = j(\zeta) \in K'$ proving our assertion. The restriction of f'' to K is obviously a chain homomorphism since C' and A' are boundaryless. □

Remark 3.8 Equation (3.10) yields a commutative diagram in homology and a cube of homomorphisms of which the front and back faces are not commutative in general. The right cube is commutative in the homotopy category $\mathbf{K}(\mathbb{K}\text{-}\mathbf{Vect})$:

$$\begin{tikzcd}
\boldsymbol{C}' \arrow[r, "H(\lambda)"] \arrow[d, "f''"'] & H(C) \arrow[d, "H(f')"] \\
\boldsymbol{A}' \arrow[r, "H(\lambda')"] & H(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
& C' \arrow[rr, "\lambda"] \arrow[dl, "0"'] \arrow[dd, "f''" near start] & & C \arrow[dl, "\mathrm{d}_C"] \arrow[dd, "f'"] \\
C' \arrow[rr, "\lambda" near end, crossing over] \arrow[dd, "f''"'] & & C & \\
& A' \arrow[rr, "\lambda'" near start] \arrow[dl, "0"'] & & A \arrow[dl, "\mathrm{d}_A"] \\
A' \arrow[rr, "\lambda'"] & & A \arrow[from=uu, "f'" near start, crossing over] &
\end{tikzcd}$$

where $\boldsymbol{C}' = H(C')$, $\boldsymbol{A}' = H(A')$, $f'' = H(f'')$ and λ and λ' are chain homomorphisms with $f'\lambda \sim \lambda' f''$, cf. (3.27).

Remark 3.9 If (C, d_C) and (A, d_A) are chain complexes we consider the chain homomorphisms f' and f'' degree wise, i.e. $f'_k\colon C_k \to A_k$ and $f''_k\colon C'_k \to A'_k$ for all k. In this case (3.10) becomes $H_{k-1}(f'_{k-1})H_k(\lambda_k) = H_k(\lambda'_k)f''_k$, with $\lambda_k\colon C'_k \to C_{k-1}$ and $\lambda'_k\colon A'_k \to A_{k-1}$. Lemma 3.7(iii) holds degree wise with complemented subspaces K_k and K'_k for all k.

Lemma 3.10 *Consider the commutative diagram of vector spaces:*

$$\begin{tikzcd}
& P' \arrow[rr] \arrow[dl, "f'"'] \arrow[dd, "\epsilon'" near start] & & P' \oplus P'' \arrow[rr] \arrow[dd, "\epsilon" near start] & & P'' \arrow[dl, "f''"'] \arrow[dd, "\epsilon''"] \\
Q' \arrow[rr, crossing over] \arrow[dd, "\eta'"] & & Q' \oplus Q'' \arrow[rr, crossing over] & & Q'' & \\
& D' \arrow[rr, "i_D" near start] \arrow[dl, "g'"] & & D \arrow[rr, "j_D" near start] \arrow[dll, "g"] & & D'' \arrow[dl, "g''"] \\
B' \arrow[rr, "i_B"'] & & B \arrow[rr, "j_B"'] \arrow[from=uu, "\eta" near start, crossing over] & & B'' \arrow[from=uu, "\eta''" near start, crossing over] &
\end{tikzcd} \tag{3.15}$$

where η' is surjective, the sequence $0 \longrightarrow B' \xrightarrow{i_B} B \xrightarrow{j_B} B''$ is exact. The maps on the top part of the diagram are the usual inclusion and projection homomorphisms. Then, there exists a homomorphism $\gamma\colon P'' \to Q'$ such that for the choice

$$f = \begin{pmatrix} f' & \gamma \\ 0 & f'' \end{pmatrix} : P' \oplus P'' \to Q' \oplus Q'', \tag{3.16}$$

the diagram

$$\begin{tikzcd}
& P' \arrow[rrr] \arrow[dl, "f'"'] \arrow[dd, "\epsilon'"] & & & P' \oplus P'' \arrow[rrr] \arrow[dl, "f"'] \arrow[dd, "\epsilon"] & & & P'' \arrow[dl, "f''"'] \arrow[dd, "\epsilon''"] \\
Q' \arrow[rrr] \arrow[dd, "\eta'"] & & & Q' \oplus Q'' \arrow[rrr] \arrow[dd, "\eta"] & & & Q'' \arrow[dd, "\eta''"] & \\
& D' \arrow[rrr, "i_D"] \arrow[dl, "g'"] & & & D \arrow[rrr, "j_D"] \arrow[dl, "g"] & & & D'' \arrow[dl, "g''"] \\
B' \arrow[rrr, "i_B"'] & & & B \arrow[rrr, "j_B"'] & & & B'' &
\end{tikzcd} \tag{3.17}$$

commutes and f *is a lift of* g *in the sense that* $g\epsilon = \eta f$.

Proof We define a homomorphism $\gamma\colon P'' \to Q'$ so that f given in (3.16) is a lift of g in the sense that $g\epsilon = \eta f$. Regardless of the definition of γ, we already have commutativity on $P' \oplus 0$. Let $(x, 0) \in P' \oplus P''$. Then, $f(x, 0) = \big(f'(x), 0\big)$ and a diagram chase yields

$$\eta f(x, 0) = \eta(f'(x), 0) = i_B \eta' f'(x) = i_B g' \epsilon'(x) = g i_D \epsilon'(x) = g\epsilon(x, 0). \tag{3.18}$$

It remains to define γ on P''. Let $(x, y) \in P' \oplus P''$. In order for f to be a lift of g we need $g\epsilon(x, y) = \eta f(x, y)$ and therefore

$$\begin{aligned} g\epsilon(x, y) &= \eta f(x, y) = \eta\big(f'(x) + \gamma(y), f''(y)\big) \\ &= \eta\big(f'(x), f''(y)\big) + \eta\big(\gamma(y), 0\big) = \eta\big(f'(x), f''(y)\big) + i_B \eta' \gamma(y), \end{aligned}$$

which is equivalent to

$$i_B \eta' \gamma(y) = g\epsilon(x, y) - \eta\big(f'(x), f''(y)\big). \tag{3.19}$$

The remainder of the proof is devoted to establishing (3.19). Using the commutativity with respect to elements $(x, 0) \in P' \oplus P''$ we have that $g\epsilon(x, 0) - \eta\big(f'(x), 0\big) = 0$, cf. (3.18), and thus

$$\begin{aligned} g\epsilon(x, y) - \eta\big(f'(x), f''(y)\big) &= g\epsilon(x, 0) - \eta\big(f'(x), 0\big) + g\epsilon(0, y) - \eta\big(0, f''(y)\big) \\ &= g\epsilon(0, y) - \eta\big(0, f''(y)\big), \end{aligned}$$

showing that the expression $g\epsilon(x, y) - \eta\big(f'(x), f''(y)\big)$ is independent of x. Observe that

$$\begin{aligned} j_B\Big[g\epsilon(x, y) - \eta\big(f'(x), f''(y)\big)\Big] &= j_B g\epsilon(x, y) - j_B \eta\big(f'(x), f''(y)\big) \\ &= g'' j_D \epsilon(x, y) - \eta'' f''(y) \\ &= g'' \epsilon''(y) - \eta'' f''(y) = 0, \end{aligned}$$

using the fact that f'' is a lift of g''. Since $0 \longrightarrow B' \xrightarrow{i_B} B \xrightarrow{j_B} B''$ is *exact* we have that $\ker j_B = \operatorname{im} i_B$ and i_B is injective. Consequently,

$$\begin{aligned} b &= g\epsilon(x, y) - \eta\big(f'(x), f''(y)\big) \\ &= g\epsilon(0, y) - \eta\big(0, f''(y)\big) \in \operatorname{im} i_B, \end{aligned}$$

and is independent of x. Due to the latter and the injectivity of i_B this defines a homomorphism $\beta\colon P'' \to B'$ given by $\beta(y) = i_B^{-1}\big(g\epsilon(0, y) - \eta\big(0, f''(y)\big)\big)$. By assumption we have the *surjective* homomorphism $\eta'\colon Q' \twoheadrightarrow B'$. Since P'' is a vector space and thus projective, there exists a lift $\gamma\colon P'' \to Q'$ such that the diagram

$$\begin{tikzcd}
& Q' \arrow[d, two heads, "\eta'"] \\
P'' \arrow[ur, dashed, "\gamma"] \arrow[r, "\beta"'] & B'
\end{tikzcd}$$

commutes. Therefore, $\eta'\gamma(y) = \beta(y)$ and thus $i_B\eta'\gamma(y) = i_B\beta(y) = b$, which establishes (3.19) and concludes the construction of f. ◻

Remark 3.11 In the case of chain complexes Lemma 3.10 can be applied degree wise which yields a homomorphism f in every degree, cf. Theorem 3.12.

Theorem 3.12 *Consider the following commutative diagram of differential vector spaces with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "f'"'] & C \oplus C' \arrow[r, "j"] & C' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & A \arrow[r, "i'"] & A \oplus A' \arrow[r, "j'"] & A' \arrow[r] & 0
\end{tikzcd} \tag{3.20}$$

where C' and A' are finite dimensional and boundaryless differential vector spaces, i' and j' are given in Lemma 3.7 and f' and f'' are chain homomorphisms. By Lemma 3.5 the differentials $\mathrm{d}_{C\oplus C'}$ and $\mathrm{d}_{A\oplus A'}$ are given by (3.9). Assume that the long exact sequences in homology make the commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & H(C) \arrow[r, "H(i)"] \arrow[d, "H(f')"] & H(C \oplus C') \arrow[r, "H(j)"] \arrow[d, "g"] & C' \arrow[r, "H(\lambda)"] \arrow[d, "f''"] & H(C) \arrow[r] \arrow[d, "H(f')"] & {} \\
{} \arrow[r] & H(A) \arrow[r, "H(i')"] & H(A \oplus A') \arrow[r, "H(j')"] & A' \arrow[r, "H(\lambda')"] & H(A) \arrow[r] & {}
\end{tikzcd} \tag{3.21}$$

where $g\colon H(C \oplus C') \to H(A \oplus A')$ is a homomorphism, and $H(f')$ and $f'' = H(f'')$ are the induced homomorphisms on homology and the rows are exact (exact triangles). Then, there exists a chain homomorphism f that is a lift for g such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "f'"] & C \oplus C' \arrow[r, "j"] \arrow[d, "f"] & C' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & A \arrow[r, "i'"] & A \oplus A' \arrow[r, "j'"] & A' \arrow[r] & 0
\end{tikzcd} \tag{3.22}$$

is a morphism of short exact sequences of differential vector spaces, i.e. the diagram commutes with short exact rows.

Proof We construct a homomorphism $f\colon C \oplus C' \to A \oplus A'$ which satisfies two requirements: (i) f is a chain homomorphism which fits into Diagram (3.22), and (ii) f induces g, i.e. $g = H(f)$. By Lemma 3.7 we have the subspaces $K = jZ(C \oplus C')$ and $K' = j'Z(A \oplus A')$. Given the diagram in (3.20) and by the commutativity in (3.21), which implies (3.10), Lemma 3.7 yields the commutative diagram in (3.11). Since the rows in Diagram (3.11) are exact the short exact sequences split as:

$$\begin{tikzcd}
0 \arrow[r] & Z(C) \arrow[r] \arrow[d, "f'"] & Z(C) \oplus K \arrow[r] & K \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & Z(A) \arrow[r] & Z(A) \oplus K' \arrow[r] & K' \arrow[r] & 0
\end{tikzcd} \tag{3.23}$$

i.e. $Z(C \oplus C') \cong Z(C) \oplus K$ and $Z(A \oplus A') \cong Z(A) \oplus K'$. To be more precise, since K is a subspace it is projective and thus there exists a section map $s\colon K \to Z(C \oplus C')$ and the diagram

$$\begin{tikzcd}
 & & & K \arrow[ld, "s"'] \arrow[d, "\mathrm{id}"] & \\
0 \arrow[r] & Z(C) \arrow[r, tail, "i"] & Z(C \oplus C') \arrow[r, two heads, "j"] & K \arrow[r] & 0
\end{tikzcd}$$

such that $js = \mathrm{id}$. Moreover, s is an isomorphism from K to $s(K)$. By construction $Z(C \oplus C') = \ker j \oplus \operatorname{im} s = Z(C) \oplus s(K) \cong Z(C) \oplus K$. The homomorphism s is given by the formula $z' \mapsto \big(\sigma(z'), z'\big)$, for some $\sigma : K \to C$ such that $\mathrm{d}_C\sigma(z') + \lambda z' = 0$, i.e.

$$\mathrm{d}_C\sigma = -\lambda, \quad \text{on } K. \tag{3.24}$$

Summarizing we obtain:

$$\begin{tikzcd}
 & & Z(C) \oplus K \arrow[d, "r", "\cong"'] \arrow[rd, two heads, "j"] & & \\
0 \arrow[r] & Z(C) \arrow[ru, tail, "i"] \arrow[r, tail, "i"] & Z(C) \oplus s(K) \arrow[r, two heads, "j"] \arrow[d, equal] & K \arrow[r] & 0 \\
 & & Z(C \oplus C') & &
\end{tikzcd}$$

where both rows are exact and the isomorphism r is given by

$$r = \begin{pmatrix} \mathrm{id} & \sigma \\ 0 & \mathrm{id} \end{pmatrix} : Z(C) \oplus K \to Z(C \oplus C'). \tag{3.25}$$

Indeed, for $c \in Z(C)$ and $r(c, z') = \big(c+\sigma(z'), z'\big)$ and $\mathrm{d}_C\big(c+\sigma(z')\big)+\lambda z' = \mathrm{d}_C\sigma(z')+\lambda z' = 0$. The map r extends to an automorphism on $C \oplus C'$ by defining $\sigma : C' \to C$. By assumption C' is finite dimensional and therefore K is complemented with $C' = K \oplus K^c$. We define $\sigma|_{K^c} = 0$. The same analysis applies to $Z(A \oplus A') \cong Z(A) \oplus K'$, with automorphism r' and $\sigma'\colon A' \to A$ and $\sigma'|_{K'^c} = 0$ (also K' is complemented since A' is finite dimensional). In order to apply Lemma 3.10 we conjugate the differentials $\mathrm{d}_{C\oplus C'}$ and $\mathrm{d}_{A\oplus A'}$:

$$\bar{\mathrm{d}}_{C\oplus C'} := r^{-1}\mathrm{d}_{C\oplus C'}r, \quad \text{and} \quad \bar{\mathrm{d}}_{A\oplus A'} := r'^{-1}\mathrm{d}_{A\oplus A'}r'.$$

A direct calculation shows that

$$\bar{\mathrm{d}}_{C\oplus C'} = \begin{pmatrix} \mathrm{d}_C & \mathrm{d}_C\sigma + \lambda \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{d}_C & \bar{\lambda} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{d}_C & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } C \oplus K,$$

where $\bar{\lambda} = \mathrm{d}_C\sigma + \lambda = 0$ on K. Moreover, $\bar{\mathrm{d}}_{C\oplus C'} = \mathrm{d}_{C\oplus C'}$ on $C \oplus K^c$, i.e. $\bar{\lambda}|_K = 0$ and $\bar{\lambda}|_{K^c} = \lambda$. The same characterization applies to $\bar{\mathrm{d}}_{A\oplus A'}$, with $\bar{\lambda}' = \mathrm{d}_A\sigma' + \lambda'$. Under the automorphism r the homology of $\big(C \oplus C', \bar{\mathrm{d}}_{C\oplus C'}\big)$ is given by $\bar{H}(C) \oplus K \cong H(C \oplus C')$, with $\bar{H}(A) = Z(A)/\big(B(A) + \bar{\lambda}'K'^c\big)$.[6] Similarly, $\bar{H}(A) \oplus K' \cong H(A \oplus A')$, with $\bar{H}(A) =$

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[6] Clearly $\bar{\lambda}K^c \subseteq Z(C)$ since $\mathrm{d}_C\bar{\lambda} = 0$, and $\bar{\lambda}K^c \subseteq B(C)$ exactly when $H(\bar{\lambda}) = 0$, cf. Lemma 3.5. Since $\bar{\lambda}|_K = 0$ we can also write $\bar{H}(C) = Z(C)/\big(B(C) + \bar{\lambda}C'\big)$.

$Z(A)/\big(B(A)+\bar\lambda' K'^c\big)$. Homology classes in $\overline{H}(C)$ and $\overline{H}(A)$ are denoted by $[\overline{c}]$ and $[\overline{a}]$ respectively. The homomorphism $g\colon H(C\oplus C')\to H(A\oplus A')$ yields the homomorphism $\bar g\colon \overline{H}(C)\oplus K\to \overline{H}(A)\oplus K'$. The homology map $g\colon H(C\oplus C')\to H(A\oplus A')$ lifts to a chain map $\psi\colon C\oplus C'\to A\oplus A'$ such that $g=H(\psi)$. For $\bar\psi = r'^{-1}\psi r$ we obtain

$$\bar g = H(\bar\psi) = H(r')^{-1}H(\psi)H(r) = H(r')^{-1}gH(r). \tag{3.26}$$

For the long exact sequence $\to A' \xrightarrow{H(\lambda')} H(A) \xrightarrow{H(i')} \bar H(A)\oplus K' \xrightarrow{j'} A' \to$ we have that $H(i')[a]=[i(a)]=[(a,0)]=\big([\overline{a}],0\big)\in \overline{H}(A)\oplus K'$. This yields the commutative diagram:

$$\begin{tikzcd}
H(A) \arrow[r, "H(i')"] \arrow[d, "\pi'"'] & \overline{H}(A)\oplus K' \arrow[d, "\widehat{\pi}'"] \\
\overline{H}(A) \arrow[ur, "i'"] \arrow[r, leftrightarrow, "\mathrm{id}"] & \overline{H}(A)
\end{tikzcd}$$

where $\widehat{\pi}'$ is the projection onto the first factor and $\overline{\pi}'=\widehat{\pi}'\circ H(i')$. The map i' is given by $[\overline{a}]\mapsto \big([\overline{a}],0\big)$. By construction the sequence

$$0\to \overline{H}(A) \xrightarrow{i'} \overline{H}(A)\oplus K' \xrightarrow{j'} A'$$

is exact, where j' are the standard projection onto the second factor. Now define $[f']\colon H(C)\to \overline{H}(A)$ by $[f'][c]:=\overline{\pi}'H(f')[c]$. With Diagrams (3.21) and (3.23) as input we obtain the commutative diagram:

$$\begin{tikzcd}
& Z(C) \arrow[rr, "i"] \arrow[dl, "f'"'] \arrow[dd, "\pi"] & & Z(C)\oplus K \arrow[rr, "j"] \arrow[dl, "\bar f"'] \arrow[dd, "\pi\oplus\mathrm{id}"] & & K \arrow[dl, "f''"'] \arrow[dd, "\mathrm{id}"] \\
Z(A) \arrow[rr, "i'"] \arrow[dd, "\pi'"] & & Z(A)\oplus K' \arrow[rr, "j'"] \arrow[dd, "\pi'\oplus\mathrm{id}"] & & K' \arrow[dd, "\mathrm{id}"] & \\
& H(C) \arrow[rr, "H(i)"'] \arrow[dl, "{[f']}"] & & \overline{H}(C)\oplus K \arrow[rr, "j"'] \arrow[dl, "\bar g"] & & C' \arrow[dl, "f''"] \\
\overline{H}(A) \arrow[rr, "i'"'] & & \overline{H}(A)\oplus K' \arrow[rr, "j'"'] & & A' &
\end{tikzcd}$$

where the projection $Z(A)\xrightarrow{\pi'} H(A) \xrightarrow{\overline{\pi}'} \overline{H}(A)$ is again denoted by π'. The same applies to the projections $\pi\oplus\mathrm{id}$ and $\pi'\oplus\mathrm{id}$. The commutativity of the bottom left square follows from Diagram (3.21) and the fact that $[f']=\bar\pi' H(f')$ and $H(i')=i'\circ\pi'$. Indeed,

$$i'[f'] = i'\bar\pi' H(f') = H(i')H(f') = \bar g H(i).$$

Now apply Lemma 3.10 to obtain the homomorphism $\bar f\colon Z(C)\oplus K\to Z(A)\oplus K'$ is given by

$$\bar f|_{Z(C)\oplus K} = \begin{pmatrix} f' & \bar\gamma|_K \\ 0 & f'' \end{pmatrix} : Z(C)\oplus K\to Z(A)\oplus K',$$

for some homomorphism $\bar\gamma|_K\colon K\to Z(A)$, i.e. $\mathrm{d}_A\bar\gamma|_K=0$. In addition, $\bar f$ trivially satisfies the chain homomorphism property since the differentials on $Z(C)\oplus K$ and $Z(A)\oplus K'$ vanish. Moreover, $\bar f$ is a lift for the homology map $\bar g : \bar H(C)\oplus K\to \bar H(A)\oplus K'$.

Because f' and f'' are defined on C and C' respectively it remains to extend $\bar{\gamma}|_K$ as a map $\bar{\gamma}\colon C' \to A$ in order to construct a chain homomorphism $\bar{f}\colon C \oplus C' \to A \oplus A'$. For $\bar{f}$ to be a chain homomorphism we need $\bar{\mathrm{d}}_{A\oplus A'}\bar{f} = \bar{f}\bar{\mathrm{d}}_{C\oplus C'}$. The compositions are given by

$$\bar{\mathrm{d}}_{A\oplus A'}\bar{f} = \begin{pmatrix} \mathrm{d}_A & \bar{\lambda}' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f' & \bar{\gamma} \\ 0 & f'' \end{pmatrix} = \begin{pmatrix} \mathrm{d}_A f' & \mathrm{d}_A\bar{\gamma} + \bar{\lambda}' f'' \\ 0 & 0 \end{pmatrix}$$

$$\bar{f}\bar{\mathrm{d}}_{C\oplus C'} = \begin{pmatrix} f' & \bar{\gamma} \\ 0 & f'' \end{pmatrix}\begin{pmatrix} \mathrm{d}_C & \bar{\lambda} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f'\mathrm{d}_C & f'\bar{\lambda} \\ 0 & 0 \end{pmatrix}$$

This yields the condition:

$$\mathrm{d}_A\bar{\gamma} = f'\bar{\lambda} - \bar{\lambda}' f'' = \bar{h}, \tag{3.27}$$

where $\bar{h} = f'\bar{\lambda} - \bar{\lambda}' f'' = h + f'\mathrm{d}_C\sigma - \mathrm{d}_A\sigma' f''$. We have, since $\bar{\lambda}|_K = 0$ and $\bar{\lambda}'|_{K'} = 0$, that

$$\bar{h} = \begin{cases} 0, & \text{on } K \\ h, & \text{on } K^c, \end{cases} \quad \text{and} \quad \mathrm{d}_A\bar{h} = 0 \text{ on } C',$$

where the latter follows from the fact that on the subspace K^c, $\mathrm{d}_A\bar{h} = \mathrm{d}_A h = \mathrm{d}_A f'\lambda - \mathrm{d}_A\lambda' f'' = f'\mathrm{d}_C\lambda - \mathrm{d}_A\lambda' f'' = 0$ by (3.13). Similarly, on K^c, $H(\bar{h}) = H(h) = H(f')H(\lambda) - H(\lambda')f'' = 0$ by (3.10) and thus $H(\bar{h}) = 0$ as homology map $H(\bar{h})\colon C' \to H(A)$. Since K^c is a vector space we can apply Lemma 3.1 (using the fact that the differential on C' and thus K^c is zero) which yields the homomorphism $\bar{h}|_{K^c}\colon K^c \to B(A)$ and a lift $\bar{\gamma}|_{K^c}\colon K^c \to A$, such that $\mathrm{d}_A\bar{\gamma}|_{K^c} = \bar{h}|_{K^c} = h$. Since $C' = K \oplus K^c$ we can combine the latter with the construction of $\bar{\gamma}$ on K. Define,

$$\bar{\gamma}(z' + z'') := \bar{\gamma}|_K(z') + \bar{\gamma}|_{K^c}(z''), \quad c' = z' + z'' \in C', \ \ z' \in K, \ z'' \in K^c,$$

which concludes that homomorphism $\bar{\gamma}$ is defined on all of C' and $\mathrm{d}_A\bar{\gamma} = \bar{h}$. This completes the construction of $\bar{f}\colon C \oplus C' \to A \oplus A'$. The restriction to $Z(C) \oplus K$ is a lift for $\bar{g}$ and therefore $\bar{f}$ is a lift for $\bar{g}$.

The chain homomorphism f is obtained from $\bar{f}$ via $f = r'\bar{f}r^{-1}$ and is given by

$$f = r'\bar{f}r^{-1} = \begin{pmatrix} f' & -f'\sigma + \sigma' f'' + \bar{\gamma} \\ 0 & f'' \end{pmatrix} = \begin{pmatrix} f' & \gamma \\ 0 & f'' \end{pmatrix} : C \oplus C' \to A \oplus A', \tag{3.28}$$

where $\gamma = \bar{\gamma} - f'\sigma + \sigma' f''$. By construction the chain homomorphism $\bar{f}$ is a lift for $\bar{g} = H(\bar{f}) = H(r')^{-1}gH(r)$, cf. (3.26). From the latter we obtain

$$g = H(r')H(\bar{f})H(r)^{-1} = H(r'\bar{f}r^{-1}) = H(f),$$

which completes the proof. □

Remark 3.13 As for Lemma 3.10, in the case of chain complexes Theorem 3.12 can be applied degree wise which yields a homomorphism f in every degree.

Consider subspaces $C, C' \subseteq \overline{C}$ and $A, A' \subseteq \overline{A}$. The subspaces C, C' and A, A' are equipped with differentials $\mathrm{d}_C, \mathrm{d}_{C'}$ and $\mathrm{d}_A, \mathrm{d}_{A'}$ respectively. The next result concerns the sums $C + C'$ and $A + A'$ as differential vector spaces.

Theorem 3.14 *Let $C, C' \subseteq \overline{C}$ and $A, A' \subseteq \overline{A}$ be subspaces. Consider the following diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & C \cap C' \arrow[r, "\varrho"] \arrow[d, "f^{\dagger}"'] & C \times C' \arrow[r, "\varpi"] \arrow[d, "f' \times f''"'] & C + C' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A \cap A' \arrow[r, "\varrho"] & A \times A' \arrow[r, "\varpi"] & A + A' \arrow[r] & 0
\end{tikzcd} \tag{3.29}$$

where $\varrho(c) = (c, -c)$ and $\varpi(c, c') = c + c'$. We equip C, C' and A, A' with differentials $\mathrm{d}_C, \mathrm{d}_{C'}$ and $\mathrm{d}_A, \mathrm{d}_{A'}$ respectively. The product differentials are given by

$$\mathrm{d}_{C \times C'} = \mathrm{d}_C \times \mathrm{d}_{C'} = \begin{pmatrix} \mathrm{d}_C & 0 \\ 0 & \mathrm{d}_{C'} \end{pmatrix}, \quad \text{and} \quad \mathrm{d}_{A \times A'} = \mathrm{d}_A \times \mathrm{d}_{A'} = \begin{pmatrix} \mathrm{d}_A & 0 \\ 0 & \mathrm{d}_{A'} \end{pmatrix}.$$

Suppose the homomorphisms $f' \colon C \to A$ and $f'' \colon C' \to A'$ are chain homomorphisms with respect to d_C and d_A, and $\mathrm{d}_{C'}$ and $\mathrm{d}_{A'}$ respectively. Then,

(i) *the maps ϱ and ϖ are chain homomorphisms if and only if $\mathrm{d}_{C \cap C'} c = \mathrm{d}_C c = \mathrm{d}_{C'} c$ for all $c \in C \cap C'$ and $\mathrm{d}_{C+C'}(c + c') := \mathrm{d}_C c + \mathrm{d}_{C'} c'$ for all $c \in C$ and all $c' \in C'$, and the same for $\mathrm{d}_{A \cap A'}$ and $\mathrm{d}_{A+A'}$;*

(ii) *Diagram (3.29) is commutative and the maps $f^{\dagger}$ and f are chain homomorphisms if and only if $f^{\dagger}(c) = f'(c) = f''(c)$ for all $c \in C \cap C'$ and $f(c + c') := f'(c) + f''(c')$ for all $c \in C$ and all $c' \in C'$.*

Given the commutative diagrams in homology:

$$\begin{tikzcd}
H(C) \arrow[r, "i"] \arrow[d, "H(f')"'] & H(C + C') \arrow[d, "g"] & H(C') \arrow[r, "i"] \arrow[d, "H(f'')"'] & H(C + C') \arrow[d, "g"] \\
H(A) \arrow[r, "i"] & H(A + A') & H(A') \arrow[r, "i"] & H(A + A')
\end{tikzcd} \tag{3.30}$$

where $i([c]_C) = [c]_{C+C'}$ and $i([c']_{C'}) = [c']_{C+C'}$, and analogously for A, A' and $A + A'$. Then,

(iii) *the chain homomorphism f lifts g, i.e. $g = H(f)$.*

Proof Let us start with the observation that the sum of f' and f'' is well-defined under the assumption that $f' = f''$ on $C \cap C'$. The same applies to the sum of differentials d_C and $\mathrm{d}_{C'}$, and d_A and $\mathrm{d}_{A'}$. Consider $c + c' = \bar{c} + \bar{c}'$, $c, \bar{c} \in C$ and $c', \bar{c}' \in C'$. This implies that $c - \bar{c} = \bar{c}' - c' \in C \cap C'$. Then,

$$f'(c) - f'(\bar{c}) + f''(c') - f''(\bar{c}') = f'(c - \bar{c}) - f''(\bar{c}' - c') = 0,$$

since $c - \bar{c} = \bar{c}' - c' \in C \cap C'$ and thus $f' = f''$ on $C \cap C'$. This shows that $f'(c) + f''(c') = f'(\bar{c}) + f''(\bar{c}')$ and $f \colon C + C' \to A + A'$ is well-defined. Similarly, $\mathrm{d}_{C+C'} \colon C + C' \to C + C'$ and $\mathrm{d}_{A+A'} \colon A + A' \to A + A'$ are well-defined.

For the diagrams:

$$\begin{tikzcd}
C \cap C' \arrow[r, "\varrho"] \arrow[d, "\mathrm{d}_{C \cap C'}"'] & C \times C' \arrow[d, "\mathrm{d}_{C \times C'}"] & C \times C' \arrow[r, "\varpi"] \arrow[d, "\mathrm{d}_{C \times C'}"'] & C + C' \arrow[d, "\mathrm{d}_{C+C'}"] \\
C \cap C' \arrow[r, "\varrho"] & C \times C' & C \times C' \arrow[r, "\varpi"] & C + C'
\end{tikzcd}$$

to commute we obtain $\mathrm{d}_{C \times C'}(c, -c) = \big(\mathrm{d}_C c, -\mathrm{d}_{C'} c\big) = \big(\mathrm{d}_{C \cap C'} c, -\mathrm{d}_{C \cap C'} c\big)$ for all $c \in C \cap C'$, which is equivalent to $\mathrm{d}_{C \cap C'} = \mathrm{d}_C = \mathrm{d}_{C'}$. By the same token the second diagram yields $\mathrm{d}_{C+C'}(c+c') = \mathrm{d}_C c + \mathrm{d}_{C'}c'$, which is well-defined by our above observation. The same arguments apply to d_A, $\mathrm{d}_{A'}$, $\mathrm{d}_{A\cap A'}$ and $\mathrm{d}_{A+A'}$ concluding the proof of (i).

Consider the diagrams:

$$\begin{tikzcd}
C \cap C' \arrow[r, "\varrho"] \arrow[d, "f^\dagger"'] & C \times C' \arrow[d, "f' \times f''"] \\
A \cap A' \arrow[r, "\varrho"] & A \times A'
\end{tikzcd}
\qquad
\begin{tikzcd}
C \times C' \arrow[r, "\varpi"] \arrow[d, "f' \times f''"'] & C + C' \arrow[d, "f"] \\
A \times A' \arrow[r, "\varpi"] & A + A'
\end{tikzcd}$$

In order for the first diagram to commute we chase the diagram which yields the condition $(f' \times f'')(c, -c) = \big(f'(c), -f''(c')\big) = \big(f^\dagger(c), -f^\dagger(c)\big)$ for all $c \in C \cap C'$. This condition is equivalent to $f^\dagger(c) = f'(c) = f''(c)$ for all $c \in C \cap C'$. As for the second diagram we obtain $f(c+c') = f'(c) + f''(c')$ for all $c \in C$ and $c' \in C'$, and f is well-defined by the above observation. It remains to show that $f^\dagger$ and f satisfy the chain homomorphism property. Since f' and f'' are chain homomorphisms, and the differentials coincide on the intersections, $f^\dagger$ is a chain homomorphism. For f we have

$$\begin{aligned}
f\big(\mathrm{d}_{C+C'}(c+c')\big) &= f\big(\mathrm{d}_C c + \mathrm{d}_{C'}c'\big) = f'(\mathrm{d}_C c) + f''(\mathrm{d}_{C'}c') \\
&= \mathrm{d}_A f'(c) + \mathrm{d}_{A'} f''(c') = \mathrm{d}_{A+A'}\big(f'(c) + f''(c')\big) \\
&= \mathrm{d}_{A+A'} f(c+c'),
\end{aligned}$$

proving (ii).

As for (iii) we argue as follows. From (3.30) we have that $g\big([c]_{C+C'}\big) = [f'(c)]_{A+A'}$, $c \in C$ and $g\big([c']_{C+C'}\big) = [f''(c')]_{A+A'}$, $c' \in C'$. Consequently,

$$\begin{aligned}
g\big([c+c']_{C+C'}\big) &= g\big([c]_{C+C'}\big) + g\big([c']_{C+C'}\big) = [f'(c)]_{A+A'} + [f''(c')]_{A+A'} \\
&= [f'(c) + f''(c')]_{A+A'} = [f(c+c')]_{A+A'} \\
&= H(f)\big([c+c']_{C+C'}\big),
\end{aligned}$$

which completes the proof. □

Remark 3.15 As before Theorem 3.14 can be carried out degree wise which is of particular interest for chain complexes.

Remark 3.16 Chain complexes are the primary sources for differential vector spaces, especially in their applications to dynamical systems. As noted in Example 2.3, a chain complex is a graded differential vector space with an off-diagonal differential. This characteristic ensures that the homology is also graded: $H(C) = \bigoplus_k H_k(C)$, where $H_k(C) = \ker \mathrm{d}_k / \operatorname{im} \mathrm{d}_{k+1}$. The chain maps for chain complexes are homogeneous and induce homogeneous homomorphisms on homology: $H(f) = \bigoplus H_k(f_k)$ with $H_k(f_k) : H_k(C) \to H_k(A)$. An $\mathsf{O}(\mathsf{P})$-filtering is defined by $\alpha \mapsto F_\alpha C_k$ for all k—degree wise filtering—which is a filtering on C, but not vice versa. Chain homomorphisms are assumed to be filtered degree wise: $f_k(F_\alpha C_k) \subseteq F_\alpha A_k$ for all k. The morphisms for the associated Cartan–Eilenberg systems are homogeneous with respect to the integer grading induced by the chain complexes. The assertions of Theorems 3.12 and 3.14 remain valid if the differential vector spaces are chain complexes. The techniques used in the proofs of Theorems 3.12 and 3.14 allow us to construct the chain maps separately in each degree. In the previous paragraphs we indicated the restriction to chain complexes in Remarks 3.4, 3.6, 3.9, 3.11, 3.13 and 3.15. For simplicity of exposition, we do not use the specific grading of chain complexes.

4 Properties of Excisive Cartan–Eilenberg Systems

In Sect. 1 we introduced the notion of Cartan–Eilenberg system over a general poset. In this section we will provide a formal definition and derive some essential properties needed to prove the main theorem.

4.1 A Categorical Definition

Let $(\mathsf{Q}, \leq)$ be a poset, or more generally a pre-order. We may regard Q as a small (thin) category, denoted $\mathbf{Q_1}$, where the objects in the category are the elements in Q and the order relations $\alpha \leq \beta$ account for the morphisms, or arrows, i.e. $\alpha \leq \beta$ yields the arrow $\alpha \to \beta$. The *arrow category* of $\mathbf{Q_1}$ consists of pairs (α, β), with $\alpha \leq \beta$, and unique morphisms $(\alpha, \beta) \to (\gamma, \delta)$ for $\alpha \leq \gamma$ and $\beta \leq \delta$, and is denoted by $\mathbf{Q_2}$. The latter corresponds to commutative diagrams in $\mathbf{Q_1}$. The (functor) categories $\mathbf{Q_n}$ consist of concatenations of $n-1$ arrows and are given by n-tuples[7] $(\alpha_1, \ldots, \alpha_n) \in \mathsf{Q} \times \cdots \times \mathsf{Q}$, $\alpha_i \in \mathsf{Q}$, with the additional requirement that $\alpha_1 \leq \cdots \leq \alpha_n$.[8] An object in $\mathbf{Q_n}$ may also be regarded as a path of length n in the directed graph induced by the poset Q. Following [6] we consider the covariant functors p_0, p_1 and p_2 acting from $\mathbf{Q_3}$ to $\mathbf{Q_2}$, given by

$$(\alpha, \beta, \gamma) \overset{\mathsf{p}_0}{\longmapsto} (\alpha, \beta), \quad (\alpha, \beta, \gamma) \overset{\mathsf{p}_1}{\longmapsto} (\alpha, \gamma), \quad \text{and} \quad (\alpha, \beta, \gamma) \overset{\mathsf{p}_2}{\longmapsto} (\beta, \gamma),$$

respectively, and the natural transformations $\mathsf{i} \colon \mathsf{p}_0 \Rightarrow \mathsf{p}_1$ and $\mathsf{j} \colon \mathsf{p}_1 \Rightarrow \mathsf{p}_2$ whose components are given by

$$(\alpha, \beta) \xmapsto{\mathsf{i}_{(\alpha,\beta,\gamma)}} (\alpha, \gamma) \quad \text{and} \quad (\alpha, \gamma) \xmapsto{\mathsf{j}_{(\alpha,\beta,\gamma)}} (\beta, \gamma),$$

respectively.

Definition 4.1 A *Cartan–Eilenberg system*[9] over a poset Q consists of a covariant functor $\mathsf{E} \colon \mathbf{Q_2} \to \mathbb{K}\text{-}\mathbf{Vect}$ and a natural transformation $\mathsf{k} \colon \mathsf{E}\mathsf{p}_2 \Rightarrow \mathsf{E}\mathsf{p}_0$ between the composite functors $\mathsf{E}\mathsf{p}_2$ and $\mathsf{E}\mathsf{p}_0$, called the *connecting homomorphism*, such that

$$\begin{tikzcd}
\mathsf{E}\mathsf{p}_0 \arrow[rr, Rightarrow, "\mathsf{E}\mathsf{i}"] & & \mathsf{E}\mathsf{p}_1 \arrow[dl, Rightarrow, "\mathsf{E}\mathsf{j}"] \\
& \mathsf{E}\mathsf{p}_2 \arrow[ul, Rightarrow, "\mathsf{k}"] &
\end{tikzcd}$$

is an exact triangle, where the natural transformations $\mathsf{E}\mathsf{i}$ and $\mathsf{E}\mathsf{j}$ are the right whiskerings of E and i, and E and j respectively. A Cartan–Eilenberg system over Q is denoted by $\mathbf{E} = \big(\mathbf{Q_2}, \mathsf{E}, \mathsf{k}\big)$.

Remark 4.2 In [1] the poset Q are the extended integers. The above definition is based on the extension in [6] for linearly ordered posets.

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[7] The entries α_i in $(\alpha_1, \ldots, \alpha_n)$ satisfy $\alpha_1 \leq \cdots \leq \alpha_n$ and are not necessarily distinct elements of Q.

[8] The notation $\mathbf{Q_n}$ indicates the functor category of functors $F \colon \mathbf{n} \to \mathbf{Q_1}$, where $\mathbf{n}$ is the category $\bullet \to \bullet \to \cdots \to \bullet \to \bullet$ consisting of $n+1$ objects and n arrows. The objects are identified with nested n-tuples in Q and the morphisms are given by the induced order on n-tuples.

[9] Cartan–Eilenberg systems can be formulated in any abelian category such as R-modules, abelian groups and $\mathbb{K}$-vector spaces.

Morphisms between Cartan–Eilenberg systems $\mathbf{E}$ and $\mathbf{E}'$ are given as natural transformations $\mathsf{h}\colon \mathsf{E} \Rightarrow \mathsf{E}'$ between the functors E and E' such that

$$\begin{tikzcd}
\mathsf{Ep}_2 \arrow[r, Rightarrow, "\mathsf{k}"] \arrow[d, Rightarrow, "\mathsf{hp}_2"'] & \mathsf{Ep}_0 \arrow[d, Rightarrow, "\mathsf{hp}_0"] \\
\mathsf{E}'\mathsf{p}_2 \arrow[r, Rightarrow, "\mathsf{k}'"] & \mathsf{E}'\mathsf{p}_0
\end{tikzcd} \tag{4.1}$$

is commutative, where hp_2 and hp_1 are the left whiskerings of h and p_2 and p_0 respectively. The components are given in (1.2).

4.2 Incomparable Convex Sets

We restrict here to excisive Cartan–Eilenberg systems over a finite distributive lattice $\mathsf{Q} = \mathsf{O}(\mathsf{P})$, where the latter is the lattice of down-sets of a finite poset P. Recall that a Cartan–Eilenberg system is excisive if $\ell : E^{\beta}_{\alpha\cap\beta} \xrightarrow{\cong} E^{\alpha\cup\beta}_{\alpha}$ is an isomorphism for all $\alpha, \beta \in \mathsf{O}(\mathsf{P})$.

Lemma 4.3 *Let* $(\alpha, \beta), (\alpha', \beta') \in \boldsymbol{Q_2}$ *with* $\beta\smallsetminus\alpha = \beta'\smallsetminus\alpha'$. *Then,* $E^{\beta}_{\alpha} \cong E^{\beta'}_{\alpha'}$.

Proof Define $\tilde\alpha = \alpha \vee \alpha'$ and $\tilde\beta = \beta \vee \beta'$. Then, $\alpha, \alpha' \le \tilde\alpha$, $\beta, \beta' \le \tilde\beta$ and $\beta\smallsetminus\alpha = \beta'\smallsetminus\alpha' = \tilde\beta\smallsetminus\tilde\alpha$. Observe that $(\beta \cap \tilde\beta)\smallsetminus(\alpha \cup \tilde\alpha) = \beta\smallsetminus\tilde\alpha = \tilde\beta\smallsetminus\tilde\alpha$. Consequently, $(\beta \cup \tilde\alpha)\smallsetminus\tilde\alpha = \tilde\beta\smallsetminus\tilde\alpha$ and therefore $\tilde\beta = \beta \cup \tilde\alpha$. Similarly, $\beta\smallsetminus(\beta \cap \tilde\alpha) = \beta\smallsetminus\alpha$ which implies that $\alpha = \beta \cap \tilde\alpha$. By the excisive property we obtain $E^{\beta}_{\alpha} = E^{\beta}_{\tilde\alpha\cap\beta} \xrightarrow{\ell} E^{\tilde\alpha\cup\beta}_{\tilde\alpha} = E^{\tilde\beta}_{\tilde\alpha}$. By the same token one proves that $E^{\beta'}_{\alpha'} = E^{\beta'}_{\tilde\alpha\cap\beta'} \xrightarrow{\ell'} E^{\tilde\alpha\cup\beta'}_{\tilde\alpha} = E^{\tilde\beta}_{\tilde\alpha}$, and thus $(\ell')^{-1}\ell\colon E^{\beta}_{\alpha} \xrightarrow{\cong} E^{\beta'}_{\alpha'}$. □.

In an excisive Cartan–Eilenberg system the isomorphisms between E-terms yield additional relations on the maps i and j.

Lemma 4.4 *Consider non-trivial triples* $(\alpha, \beta, \gamma), (\alpha, \beta', \gamma)$*, i.e.* $\alpha \subsetneq \gamma$*, such that* $\beta\smallsetminus\alpha = \gamma\smallsetminus\beta'$ *and* $\beta'\smallsetminus\alpha = \gamma\smallsetminus\beta$. *Then, in the exact triangles*

$$\begin{tikzcd}
E^{\beta}_{\alpha} \arrow[rr, "i"] & & E^{\gamma}_{\alpha} \arrow[dl, "j"] & E^{\beta'}_{\alpha} \arrow[rr, "i'"] & & E^{\gamma}_{\alpha} \arrow[dl, "j'"] \\
& E^{\gamma}_{\beta} \arrow[ul, "k"] & & & E^{\gamma}_{\beta'} \arrow[ul, "k'"] &
\end{tikzcd} \tag{4.2}$$

the homomorphisms i, i' *and* j, j' *satisfy:* $\ell^{-1}j'i = \mathrm{id}$ *on* E^{β}_{α} *and* $ji'\ell'^{-1} = \mathrm{id}$ *on* E^{γ}_{β}*, where* $\ell\colon E^{\beta}_{\alpha} \to E^{\gamma}_{\beta'}$ *and* $\ell'\colon E^{\beta'}_{\alpha} \to E^{\gamma}_{\beta}$ *are the isomorphisms by the excisive property.*

Proof An excisive Cartan–Eilenberg system yields the isomorphisms $\ell\colon E^{\beta}_{\alpha} \to E^{\gamma}_{\beta'}$ and $\ell'\colon E^{\gamma}_{\beta} \to E^{\beta'}_{\alpha}$. By the definition of the functor E and the categorical interpretation of ordered pairs, $j'i$ is the homomorphism $\ell\colon E^{\beta}_{\alpha} \to E^{\gamma}_{\beta'}$, which implies that $j'i$ is an isomorphism and $\ell^{-1}j'i = \mathrm{id}$ on E^{β}_{α}. The same applies to isomorphism $\ell'\colon E^{\beta'}_{\alpha} \to E^{\gamma}_{\beta}$ and $ji'\ell'^{-1} = \mathrm{id}$ on E^{γ}_{β}. □

Remark 4.5 For non-trivial triples $(\alpha, \beta, \gamma), (\alpha, \beta', \gamma)$, such that $\beta\smallsetminus\alpha = \gamma\smallsetminus\beta'$ and $\beta'\smallsetminus\alpha = \gamma\smallsetminus\beta$, the convex sets $\xi = \beta\smallsetminus\alpha$ and $\eta = \gamma\smallsetminus\beta$ are *incomparable*, or *parallel*, i.e. ξ and η

are incomparable with respect induced order by P. They satisfy the additional property that $\xi \cap \eta = \varnothing$ and $\xi \cup \eta = \zeta = \gamma \smallsetminus \alpha$, as well as $\beta \cap \beta' = \alpha$ and $\beta \cup \beta' = \gamma$. In pariticular, $\beta \neq \beta'$.

For the E-terms of the triples (α, β, γ), (α, β', γ) in Lemma 4.4 the homomorphisms i, i' are injective and the homomorphisms j, j' are surjective. In particular, using Lemma 4.4,

$$\begin{tikzcd}
0 \arrow[r] & E_\alpha^\beta \arrow[r, tail, "i"] & E_\alpha^\gamma \arrow[r, two heads, "j"] & E_\beta^\gamma \arrow[r] \arrow[l, bend left, "i'\ell'^{-1}"] & 0,
\end{tikzcd}$$

which shows that the exact sequence is right split and thus a split exact sequence with $E_\alpha^\gamma \cong E_\alpha^\beta \oplus E_\beta^\gamma$. The same construction follows by using the other triple. The connecting homomorphisms k and k' are trivial proving that the exact triangle in (4.2) are in fact split exact sequences.

If we consider Cartan–Eilenberg systems generated by a P-graded differential vector space we can derive properties of the differential for the above case of triples describing incomparable convex sets. As before consider triples (α, β, γ), (α, β', γ), such that $\beta \smallsetminus \alpha = \gamma \smallsetminus \beta'$ and $\beta' \smallsetminus \alpha = \gamma \smallsetminus \beta$. Let $C = \bigoplus_{p \in \mathsf{P}} G_p C$, then (2.2) yields the exact sequences

$$\begin{tikzcd}
 & G_{\gamma \smallsetminus \beta'} C \arrow[d, equal] & G_{\beta \smallsetminus \alpha} C \oplus G_{\gamma \smallsetminus \beta} C \arrow[d, equal] & & \\
0 \arrow[r] & G_{\beta \smallsetminus \alpha} C \arrow[r, "i"] & G_{\gamma \smallsetminus \alpha} C \arrow[r, "j"] \arrow[d, equal] & G_{\gamma \smallsetminus \beta} C \arrow[r] & 0 \\
0 \arrow[r] & G_{\beta' \smallsetminus \alpha} C \arrow[r, "i'"] \arrow[d, equal] & G_{\gamma \smallsetminus \alpha} C \arrow[r, "j'"] \arrow[d, equal] & G_{\gamma \smallsetminus \beta'} C \arrow[r] & 0 \\
 & G_{\gamma \smallsetminus \beta} C & G_{\beta' \smallsetminus \alpha} C \oplus G_{\gamma \smallsetminus \beta'} C & &
\end{tikzcd} \tag{4.3}$$

Since i, i' are chain homomorphisms by the assumption that the differential d_C is $\mathsf{O}(\mathsf{P})$-filtered we obtain the following splitting lemma for the differential d_C.

Lemma 4.6 *Consider the triples* (α, β, γ), (α, β', γ) *with* $\beta \smallsetminus \alpha = \gamma \smallsetminus \beta'$ *and* $\beta' \smallsetminus \alpha = \gamma \smallsetminus \beta$. *Then, the differential* $\mathrm{d}_C \colon G_{\gamma \smallsetminus \alpha} C \to G_{\gamma \smallsetminus \alpha} C$ *is given by:*

$$\mathrm{d}_{G_{\gamma \smallsetminus \alpha} C} = \begin{pmatrix} \mathrm{d}_{G_{\beta \smallsetminus \alpha} C} & 0 \\ 0 & \mathrm{d}_{G_{\gamma \smallsetminus \beta} C} \end{pmatrix} : G_{\beta \smallsetminus \alpha} C \oplus G_{\gamma \smallsetminus \beta} \to G_{\beta \smallsetminus \alpha} C \oplus G_{\gamma \smallsetminus \beta}, \tag{4.4}$$

where $\mathrm{d}_{G_{\beta \smallsetminus \alpha} C}$ *and* $\mathrm{d}_{G_{\gamma \smallsetminus \beta} C}$ *are the restrictions of* d_C *to* $G_{\beta \smallsetminus \alpha} C$ *and* $G_{\gamma \smallsetminus \beta}$ *respectively.*

Proof If we apply the first part of the proof of Lemma 3.5 to both exact sequences in (4.3) we obtain that the off diagonal terms are zero completing the proof. □

Remark 4.7 For Cartan–Eilenberg systems generated by P-graded differential vector spaces the conclusion of Lemma 4.4 follows immediately since $ji' = \mathrm{id}$ and $j'i = \mathrm{id}$ as indicated in Diagram (4.3). The direct sum decomposition is immediate from the form of the differential.

Remark 4.8 The considerations in this section hold for excisive Cartan–Eilenberg systems in any abelian category.

5 Strict Representations

Two $\mathsf{O}(\mathsf{P})$-filtered chain homomorphisms $f, \widetilde{f}\colon (C, \mathrm{d}_C) \to (A, \mathrm{d}_A)$ are said to be $\mathsf{O}(\mathsf{P})$-filtered chain homotopic if there is an $\mathsf{O}(\mathsf{P})$-filtered homomorphism $h\colon C \to A$ such that $f - \widetilde{f} = h\mathrm{d}_C + \mathrm{d}_A h$; such a map h is called an $\mathsf{O}(\mathsf{P})$-filtered chain homotopy from f to $\widetilde{f}$. Notation: $f \sim \widetilde{f}$. An $\mathsf{O}(\mathsf{P})$-filtered chain homomorphism $f\colon (C, \mathrm{d}_C) \to (A, \mathrm{d}_A)$ is an $\mathsf{O}(\mathsf{P})$-filtered chain (homotopy) equivalence if there is an $\mathsf{O}(\mathsf{P})$-filtered chain homomorphism $f^\dagger\colon (A, \mathrm{d}_A) \to (C, \mathrm{d}_C)$ such that $f^\dagger f \sim \mathrm{id}_C$ and $ff^\dagger \sim \mathrm{id}_A$. In this case (C, d_C) and (A, d_A) are $\mathsf{O}(\mathsf{P})$-*filtered chain equivalent*. Notation: $(C, \mathrm{d}_C) \simeq (A, \mathrm{d}_A)$. Two $\mathsf{O}(\mathsf{P})$-filtered, or P-graded differential vector spaces (C, d_C) and (A, d_A) are $\mathsf{O}(\mathsf{P})$-*filtered chain isomorphic* if there exists an $\mathsf{O}(\mathsf{P})$-filtered isomorphism $f\colon (C, \mathrm{d}_C) \to (A, \mathrm{d}_A)$, cf. (2.1), that is also a chain isomorphism.[10] In this case f is called an $\mathsf{O}(\mathsf{P})$-filtered chain isomorphism. Notation: $(C, \mathrm{d}_C) \cong (A, \mathrm{d}_A)$.

In this section we establish the fundamental equivalence result in the setting of strict, P-graded differential vector spaces.

Theorem 5.1 *Two finite dimensional, strict,* P*-graded differential vector spaces* (C, d_C) *and* (A, d_A) *are* $\mathsf{O}(\mathsf{P})$*-filtered chain isomorphic if and only if* $\boldsymbol{E}(C, \mathrm{d}_C) \cong \boldsymbol{E}(A, \mathrm{d}_A)$.

Proof If (C, d_C) and (A, d_A) are $\mathsf{O}(\mathsf{P})$-filtered chain isomorphic then the associated Cartan–Eilenberg systems are isomorphic which follows directly from the definition of Cartan–Eilenberg system for a graded differential vector space. For the converse we argue by induction using the lattice structure of $\mathsf{O}(\mathsf{P})$.

Let $q \in \mathsf{P}$ be a minimal element and let $\alpha = \{q\}$. Since $\mathbf{E}(C, \mathrm{d}_C) \cong \mathbf{E}(A, \mathrm{d}_A)$ we have $F_\alpha C = G_q C$, $F_\alpha A = G_q A$, and a chain isomorphism $f\colon G_q C \xrightarrow{\cong} G_q A$, with $f(F_\alpha C) = F_\alpha A$, which uses the fact that (C, d_C) and (A, d_A) are finite dimensional and strict, cf. Remark 2.5. Since the differentials on $G_q C$ and $G_q A$ vanish, $H(f) = f$. This holds for all minimal elements $q \in \mathsf{P}$, which concludes the first step in the induction.

Let $\alpha \in \mathsf{O}(\mathsf{P})$ and suppose $f'\colon F_\alpha C \to F_\alpha A$ is an $\mathsf{O}(\mathsf{P})$-filtered chain isomorphism with

$$f'(F_{\alpha'} C) = F_{\alpha'} A, \quad \forall \alpha' \subseteq \alpha. \tag{5.1}$$

Let $q \in \mathsf{P} \smallsetminus \alpha$ be a minimal element. Define $\beta = \downarrow q$, which is a join-irreducible down-set in $\mathsf{O}(\mathsf{P})$ with unique immediate predecessor $\beta^\dagger \subseteq \alpha$. This implies that $\beta = \beta^\dagger \cup \{q\}$. The next step is to construct a chain isomorphism $f\colon F_\beta C \xrightarrow{\cong} F_\beta A$, where $F_\beta C = F_{\beta^\dagger} C \oplus G_q C$ and $F_\beta A = F_{\beta^\dagger} A \oplus G_q A$, and which satisfies $f(F_\beta C) = F_\beta A$. Since $\mathbf{E}(C, \mathrm{d}_C) \cong \mathbf{E}(A, \mathrm{d}_A)$ we have a chain isomorphism $f''\colon G_q C \xrightarrow{\cong} G_q A$ and an isomorphism $g\colon H(F_{\beta^\dagger} C \oplus G_q C) \to H(F_{\beta^\dagger} A \oplus G_q A)$ given by the exact triangles:

---

[10] By Remark 2.1 this equivalent to having $\mathsf{O}(\mathsf{P})$-filtered chain homomorphisms $f\colon (C, \mathrm{d}_C) \to (A, \mathrm{d}_A)$ and $f^\dagger\colon (A, \mathrm{d}_A) \to (C, \mathrm{d}_C)$ such that $f^\dagger f = \mathrm{id}_C$ and $ff^\dagger = \mathrm{id}_A$.

$$\begin{tikzcd}
& H(F_{\beta^\dagger}C) \arrow[d, equal] & H(F_{\beta^\dagger}C \oplus G_qC) \arrow[d, equal] & G_qC \arrow[d, equal] & & \\
{} \arrow[r] & E^{\beta^\dagger}_{\emptyset} \arrow[r] \arrow[d, "H(f')"', "\cong"] & E^{\beta}_{\emptyset} \arrow[r] \arrow[d, "\cong"', "g"] & E^{\beta}_{\beta^\dagger} \arrow[r] \arrow[d, "\cong"', "f''"] & E^{\beta^\dagger}_{\emptyset} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & E'^{\beta^\dagger}_{\emptyset} \arrow[r] & E'^{\beta}_{\emptyset} \arrow[r] & E'^{\beta}_{\beta^\dagger} \arrow[r] & E'^{\beta^\dagger}_{\emptyset} \arrow[r] & {} \\
& H(F_{\beta^\dagger}A) \arrow[u, equal] & H(F_{\beta^\dagger}A \oplus G_qA) \arrow[u, equal] & G_qA \arrow[u, equal] & &
\end{tikzcd}$$

Since (C, d_C) and (A, d_A) are finite dimensional and strict we have $H(G_qC) = G_qC$ and $H(G_qA) = G_qA$. On the chain level this yields the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_{\beta^\dagger}C \arrow[r] \arrow[d, "f'"'] & F_{\beta^\dagger}C \oplus G_qC \arrow[r] \arrow[d, dashed, "f"] & G_qC \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & F_{\beta^\dagger}A \arrow[r] & F_{\beta^\dagger}A \oplus G_qA \arrow[r] & G_qA \arrow[r] & 0
\end{tikzcd} \tag{5.2}$$

As before, since both (C, d_C) and (A, d_A) are finite dimensional and strict it follows that G_qC and G_qA are finite dimensional, boundaryless vector spaces (and thus projective). The components $F_{\beta^\dagger}C$ and $F_{\beta^\dagger}A$ are also finite dimensional differential vector spaces. By Theorem 3.12 there exists a chain homomorphism f that induces g. The chain homomorphism f is a chain isomorphism via the five lemma since both f' (by induction) and f'' are chain isomorphisms. The isomorphism f is filtered by construction, i.e. $f(F_\beta C) \subseteq F_\beta A$. The above procedure yields a more precise statement, i.e. $\beta' \subsetneq \beta$ if and only if $\beta' \subseteq \beta^\dagger$, which implies, by the induction hypotheses, that $f(F_{\beta'}C) = f'(F_{\beta'}C) = F_{\beta'}A$ for all $\beta' \subseteq \beta^\dagger \subseteq \alpha$. By the form of f given in (3.28), and the fact that $f''(G_qC) = G_qA$, we have

$$\begin{aligned}
f(F_\beta C) = f\big(F_{\beta^\dagger}C \oplus G_qC\big) &= \big[f'(F_{\beta^\dagger}C) + \gamma(G_qC)\big] \oplus f''(G_qC) \\
&= F_{\beta^\dagger}A \oplus G_qA = F_\beta A,
\end{aligned}$$

since $\gamma(G_qC) \subseteq F_{\beta^\dagger}A$.

It remains to construct f on $F_{\beta'}C$ for down-sets $\beta' \subseteq \alpha \cup \{q\}$, where $\alpha \cup \{q\} \in \mathsf{O}(\mathsf{P})$. Since every down-set can be uniquely represented as an irredundant union of join-irreducible down-sets, the down-set β' is represented as

$$\beta' = \downarrow q \cup \downarrow p_{i_1} \cup \cdots \cup \downarrow p_{i_k} = \beta \cup \gamma, \quad p_{i_1}, \ldots, p_{i_k} \in \alpha, \quad \beta = \downarrow q, \tag{5.3}$$

and $\gamma = \downarrow p_{i_1} \cup \cdots \cup \downarrow p_{i_k} \subseteq \alpha$. For $\beta \cup \gamma$ and $\beta \cap \gamma$ we have the following diagram:

$$\begin{tikzcd}
& (\beta \cap \gamma) \cup (\beta \smallsetminus \gamma) \cup (\gamma \smallsetminus \beta) \arrow[dl] \arrow[dr] & \\
(\beta \cap \gamma) \cup (\beta \smallsetminus \gamma) \arrow[dr] & & (\beta \cap \gamma) \cup (\gamma \smallsetminus \beta) \arrow[dl] \\
& \beta \cap \gamma &
\end{tikzcd}$$

where $(\beta \cap \gamma) \cup (\beta \smallsetminus \gamma) \cup (\gamma \smallsetminus \beta) = \beta \cup \gamma = \beta'$. Consider the decompositions:

$$F_\beta C = F_{\beta \cap \gamma}C \oplus G_{\beta \smallsetminus \gamma}C, \quad F_\gamma C = F_{\beta \cap \gamma}C \oplus G_{\gamma \smallsetminus \beta}C$$

and

$$F_{\beta'}C = F_{\beta\cap\gamma}C \oplus \Big[G_{\beta\setminus\gamma}C \oplus G_{\gamma\setminus\beta}C\Big] = F_\beta C + F_\gamma C.$$

The same decomposition holds for $F_\beta A$, $F_\gamma A$ and $F_{\beta'}A$. Since $\beta\setminus\gamma$ and $\gamma\setminus\beta$ are uncomparable convex sets, we can apply Lemma 4.6, with respect to the triples $(\beta\cap\gamma, \beta, \beta\cup\gamma)$ and $(\beta\cap\gamma, \gamma, \beta\cup\gamma)$. This implies that the differential on $G_{\beta\setminus\gamma}C \oplus G_{\gamma\setminus\beta}C$ is a direct sum and thus the differentials are of the form:

$$\mathrm{d}_{F_\beta C} = \begin{pmatrix} \mathrm{d}_{F_{\beta\cap\gamma}C} & \Lambda \\ 0 & \mathrm{d}_{G_{\beta\setminus\gamma}C} \end{pmatrix}, \quad \mathrm{d}_{F_\gamma C} = \begin{pmatrix} \mathrm{d}_{F_{\beta\cap\gamma}C} & \Lambda' \\ 0 & \mathrm{d}_{G_{\gamma\setminus\beta}C} \end{pmatrix}, \tag{5.4}$$

and

$$\mathrm{d}_{F_{\beta'}C} = \begin{pmatrix} \mathrm{d}_{F_{\beta\cap\gamma}C} & \Lambda & \Lambda' \\ 0 & \mathrm{d}_{G_{\beta\setminus\gamma}C} & 0 \\ 0 & 0 & \mathrm{d}_{G_{\gamma\setminus\beta}C} \end{pmatrix}. \tag{5.5}$$

It is readily verified that with $C = F_\beta C$, $C' = F_\gamma C$, $A = F_\beta A$ and $A' = F_\gamma A$, and $C + C' = F_{\beta'}C$, $A + A' = F_{\beta'}A$ and $C \cap C' = F_{\beta\cap\gamma}C$, $A \cap A' = F_{\beta\cap\gamma}A$, Theorem 3.14(i) is satisfied for the differentials with $\mathrm{d}_{F_\beta C} = \mathrm{d}_{F_\gamma C} = \mathrm{d}_{F_{\beta\cap\gamma}C}$ on $F_{\beta\cap\gamma}C$ and $\mathrm{d}_{F_\beta A} = \mathrm{d}_{F_\gamma A} = \mathrm{d}_{F_{\beta\cap\gamma}A}$ on $F_{\beta\cap\gamma}A$. By construction the chain isomorphisms $f' = f|_{F_\beta C}$ and $f'' = f|_{F_\gamma C}$ satisfy Theorem 3.14(ii) and therefore $f := f|_{F_\beta C + F_\gamma C}$ is a well-defined chain homomorphism on $F_{\beta'}C$ and $f(F_{\beta'}C) = F_{\beta'}A$. Indeed, $f'(F_\beta C) = F_\beta A$ and $f''(F_\gamma C) = F_\gamma A$, and $f(c + c') = f'(c) + f''(c')$ for $c \in F_\beta C$ and $c' \in F_\gamma C$. This implies that

$$f(F_{\beta'}C) = f(F_\beta C + F_\gamma C) = F_\beta A + F_\gamma A = F_{\beta'}A.$$

and therefore the chain homomorphism is well-defined on $F_{\alpha\cup\downarrow q}C$ and is filtered, i.e. $f(F_{\beta'}C) = F_{\beta'}A$ for all $\beta' \subseteq \alpha \cup \downarrow q$.

Because $f(F_{\beta\cap\gamma}C) = F_{\beta\cap\gamma}A$, the homomorphism f is of the form

$$f' = f|_{F_\beta C} = \begin{pmatrix} f|_{F_{\beta\cap\gamma}C} & \chi \\ 0 & f|_{G_{\beta\setminus\gamma}C} \end{pmatrix}, \quad f'' = f|_{F_\gamma C} = \begin{pmatrix} f|_{F_{\beta\cap\gamma}C} & \chi' \\ 0 & f|_{G_{\gamma\setminus\beta}C} \end{pmatrix}$$

Since $f|_{F_\beta C}$ and $f|_{F_\gamma C}$ are isomorphisms so are $f|_{G_{\beta\setminus\gamma}C}$ and $f|_{G_{\gamma\setminus\beta}C}$, and by construction $f(G_{\beta\setminus\gamma}C) \subseteq G_{\beta\setminus\gamma}A$ and $f(G_{\gamma\setminus\beta}C) \subseteq G_{\gamma\setminus\beta}A$. Since $\beta'\setminus\beta = \gamma\setminus\beta$ and $\beta'\setminus\gamma = \beta\setminus\gamma$ we have the exact triangles:

$$\begin{tikzcd} E^{\beta}_{\varnothing} \arrow[rr, "i"] & & E^{\beta'}_{\varnothing} \arrow[dl, "j"] \\ & E^{\beta'}_{\beta} \cong E^{\gamma}_{\beta} \arrow[ul, "k"] & \end{tikzcd} \qquad \begin{tikzcd} E^{\gamma}_{\varnothing} \arrow[rr, "i"] & & E^{\beta'}_{\varnothing} \arrow[dl, "j"] \\ & E^{\beta'}_{\gamma} \cong E^{\beta}_{\gamma} \arrow[ul, "k"] & \end{tikzcd} \tag{5.6}$$

From the isomorphism between Cartan–Eilenberg systems we obtain the commutative squares:

$$\begin{tikzcd} E^{\beta}_{\varnothing} \arrow[r, "i"] \arrow[d, "H(f')"'] & E^{\beta'}_{\varnothing} \arrow[d, "g"] \\ E'^{\beta}_{\varnothing} \arrow[r, "i"] & E'^{\beta'}_{\varnothing} \end{tikzcd} \qquad \begin{tikzcd} E^{\gamma}_{\varnothing} \arrow[r, "i"] \arrow[d, "H(f'')"'] & E^{\beta'}_{\varnothing} \arrow[d, "g"] \\ E'^{\gamma}_{\varnothing} \arrow[r, "i"] & E'^{\beta'}_{\varnothing} \end{tikzcd}$$

By Theorem 3.14(iii), f lifts g, i.e. $g = H(f)$. Since f' and f'' are chain isomorphisms by construction, the five lemma applied to the short exact sequences associated to the exact triangles in (5.6),

$$\begin{tikzcd}
0 \arrow[r] & F_\beta C \arrow[r] \arrow[d, "f'"', "\cong"] & F_\beta C + F_\gamma C \arrow[r] \arrow[d, dashed, "f"] & G_{\gamma\smallsetminus\beta} C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & F_\beta A \arrow[r] & F_\beta A + F_\gamma A \arrow[r] & G_{\gamma\smallsetminus\beta} A \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & F_\gamma C \arrow[r] \arrow[d, "f''"', "\cong"] & F_\beta C + F_\gamma C \arrow[r] \arrow[d, dashed, "f"] & G_{\beta\smallsetminus\gamma} C \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & F_\gamma A \arrow[r] & F_\beta A + F_\gamma A \arrow[r] & G_{\beta\smallsetminus\gamma} A \arrow[r] & 0
\end{tikzcd}$$

imply that also f is a chain isomorphism, which completes the construction of f on $F_{\beta'}C$ for down-sets $\beta' \subseteq \alpha \cup \{q\}$. This process terminates after finitely many steps. □

Remark 5.2 If h is a morphism between $\mathbf{E}$ and $\mathbf{E}'$ such that $h|_{E_\alpha^\beta}$ is an isomorphism for all $\beta\smallsetminus\alpha = \{p\}$, $p \in \mathsf{P}$, then h is a isomorphism on all E-terms E_α^β, for all $\alpha \subseteq \beta$. This is a direct consequence of the five lemma.

Remark 5.3 Theorem 3.12 is an important ingredient from homological algebra, crucial to the proof of Theorem 5.1, which we prove in the category of $\mathbb{K}$-vector spaces. An extension of Theorem 3.12 in the category of R-modules for appropriate classes of rings R, such as principal ideal domains, is subject of future study

Remark 5.4 In [4] a special version of Theorem 5.1 was proved in the theory of algebraic transition matrices. Although stated in more general terms the result applies in the setting of vector spaces under specific restrictions on the poset P: N-free posets. The latter is not required for Theorem 5.1.

6 Equivalence to Franzosa's Module Braids

In [3] a data structure is developed to relate Conley indices of Morse sets in a Morse representation, cf. [7]. In this section we show that Franzosa's data structure is equivalent to an excisive Cartan–Eilenberg system.

6.1 Module Braids

The starting point in [3] is a finite poset $(\mathsf{P}, \leq)$. We recall some notation from [3] to relate elements in $\mathsf{Co}(\mathsf{P})$, the meet-semilattice of convex sets in P, and formulate these concepts in terms of lattice theory. Two elements $\xi, \eta \in \mathsf{Co}(\mathsf{P})$ are *adjacent* if and only if there exists an ordered triple (α, β, γ) such that $\xi = \beta\smallsetminus\alpha$ and $\eta = \gamma\smallsetminus\beta$. Then, $\xi \cap \eta = 0$ and the union satisfies $\xi\eta := \xi \cup \eta = \gamma\smallsetminus\alpha$, which is a *decomposition* of convex sets. For every convex set $\xi \in \mathsf{Co}(\mathsf{P})$ assign a differential vector space (D_ξ, d_ξ) such that

$$0 \longrightarrow D_\xi \xrightarrow{i} D_{\xi\eta} \xrightarrow{j} D_\eta \longrightarrow 0 \ ,$$

is exact.[11] If ξ and η are incomparable then $j'i = \mathrm{id}$ on D_ξ where $i\colon D_\xi \to D_{\xi\eta}$ and $j'\colon D_{\eta\xi} \to D_\xi$. Franzosa refers to the above structure as a ‘chain complex braid’. The prime example of a chain complex braid is given by an $\mathsf{O(P)}$-filtered differential vector space (D, d_D) as displayed in (2.2). As in Sect. 4.2 one proves that $G_{\beta\smallsetminus\alpha}D \cong G_{\beta'\smallsetminus\alpha'}D$ for $\beta\smallsetminus\alpha = \beta'\smallsetminus\alpha'$ along with the other properties. The homology is denoted by $E_\xi := H(D_\xi, \mathrm{d}_\xi)$ and satisfies the following properties:

Definition 6.1 A *module braid* $\mathbf{B}$ over a meet-semilattice $\mathsf{Co(P)}$ is a collection of vector spaces ($\mathbb{K}$-modules) $E_\xi \in \mathrm{Ob}(\mathbb{K}\text{-}\mathbf{Vect})$, indexed by $\xi \in \mathsf{Co(P)}$, such that

(i) for every pair of *adjacent* $\xi, \eta \in \mathsf{Co(P)}$ the triangles

$$\begin{tikzcd}
E_\xi \arrow[rr, "i"] & & E_{\xi\eta} \arrow[dl, "j"] \\
& E_\eta \arrow[ul, "k"] &
\end{tikzcd} \tag{6.1}$$

are exact;

(ii) for *incomparable* elements $\xi, \eta \in \mathsf{Co(P)}$, cf. Remark 4.5, both triangles

$$\begin{tikzcd}
E_\xi \arrow[rr, "i"] & & E_{\xi\eta} \arrow[dl, "j"] & & E_\eta \arrow[rr, "i'"] & & E_{\eta\xi} \arrow[dl, "j'"] \\
& E_\eta \arrow[ul, "k"] & & & & E_\xi \arrow[ul, "k'"] &
\end{tikzcd} \tag{6.2}$$

are exact and $j'i = \mathrm{id}$ on E_ξ and $ji' = \mathrm{id}$ on E_η.

(iii) for every triple of adjacent elements[12] $\xi, \eta, \zeta \in \mathsf{Co(P)}$ the ‘braid diagram’

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[dl] \arrow[dr, dashed] & {} \arrow[d] \\
E_\xi \arrow[dr, "i"] \arrow[dd, bend right, "i"'] & & E_\zeta \arrow[dl, "k"'] \arrow[dd, bend left, dashed, "k"] \\
& E_{\xi\eta} \arrow[dl, "i"'] \arrow[dr, "j"] & \\
E_{\xi\eta\zeta} \arrow[dr, "j"] \arrow[dd, bend right, "j"'] & & E_\eta \arrow[dl, dashed, "i"'] \arrow[dd, bend left, "k"] \\
& E_{\eta\zeta} \arrow[dl, dashed, "j"'] \arrow[dr, "k"] & \\
E_\zeta \arrow[dr, "k"] \arrow[dd, bend right, dashed, "k"'] & & E_\xi \arrow[dl, "i"'] \arrow[dd, bend left, "i"] \\
& E_{\xi\eta} \arrow[dl, "j"'] \arrow[dr, "i"] & \\
E_\eta \arrow[dr, dashed] \arrow[d] & & E_{\xi\eta\zeta} \arrow[dl] \arrow[d] \\
{} & {} & {}
\end{tikzcd} \tag{6.3}$$

commutes with exact ‘strands’.

Remark 6.2 The terminology module braid is best explained since the exact triangles in (6.1) correspond to the strands in the braid diagram in (6.3). In the spirit of triangulated categories the braid diagram can also be reshaped as a commutative diagram of exact triangles:

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[11] In [3] the slightly more general assumption of weak exactness is assumed. The results in this section remain true under the assumption of weak exactness. Weak exactness in [3] can be avoided by using Borel-Moore homology for example.

[12] A triple ξ, η, ζ of adjacent convex sets is equivalent to a ordered quadruple $(\alpha, \beta, \gamma, \delta)$ such that $\xi = \beta\smallsetminus\alpha$, $\eta = \gamma\smallsetminus\beta$ and $\zeta = \delta\smallsetminus\gamma$. These convex sets are mutually disjoint and their union is $\delta\smallsetminus\alpha$.

$$\begin{tikzcd}
& & E_\xi \arrow[lldd, bend right, "i"'] \arrow[rrdd, bend left, "i"] & & \\
& E_\eta \arrow[ur, "k"] \arrow[rr, dashed, "i"'] & & E_{\eta\zeta} \arrow[ul, "k"'] \arrow[dl, dashed, "j"'] & \\
E_{\xi\eta} \arrow[ur, "j"] \arrow[rrrr, bend right, "i"'] & & E_\zeta \arrow[ll, "k"] \arrow[ul, dashed, "k"'] & & E_{\xi\eta\zeta} \arrow[ll, "j"] \arrow[ul, "j"']
\end{tikzcd} \tag{6.4}$$

The dashed 'strand' corresponds to the middle exact triangle in (6.4), which makes the diagram commutative.

Lemma 6.3 *Let* $\mathbf{E}$ *be a Cartan–Eilenberg system over* $\mathsf{O}(\mathsf{P})$*, with* P *a finite poset. Then, the diagram*

$$\begin{tikzcd}
& & E^\beta_\alpha \arrow[lldd, bend right, "i"'] \arrow[rrdd, bend left, "i"] & & \\
& E^\gamma_\beta \arrow[ur, "k"] \arrow[rr, dashed, "i"'] & & E^\delta_\beta \arrow[ul, "k"'] \arrow[dl, dashed, "j"'] & \\
E^\gamma_\alpha \arrow[ur, "j"] \arrow[rrrr, bend right, "i"'] & & E^\delta_\gamma \arrow[ll, "k"] \arrow[ul, dashed, "k"'] & & E^\delta_\alpha \arrow[ll, "j"] \arrow[ul, "j"']
\end{tikzcd} \tag{6.5}$$

commutes for every ordered quadruple $(\alpha, \beta, \gamma, \delta)$ *in* $\mathsf{O}(\mathsf{P})$.

Proof The commutative diagram follows by defining the dashed triangle by considering the compositions: $E^\gamma_\beta \xrightarrow{k} E^\beta_\alpha \xrightarrow{i} E^\delta_\alpha \xrightarrow{j} E^\delta_\beta$, $E^\delta_\beta \xrightarrow{k} E^\beta_\alpha \xrightarrow{i} E^\delta_\alpha \xrightarrow{j} E^\delta_\gamma$, and $E^\delta_\gamma \xrightarrow{k} E^\gamma_\alpha \xrightarrow{j} E^\gamma_\beta$. The first two compositions are equal to the homomorphisms $i\colon E^\gamma_\beta \to E^\delta_\beta$ and $j\colon E^\delta_\beta \to E^\delta_\gamma$, since nested ordered pairs correspond to unique homomorphisms. The latter composition is the connecting homomorphism $k\colon E^\delta_\gamma \to E^\gamma_\beta$ due to the commutative squares for k in (1.1)[right]. This implies that the dashed triangle is the exact triangle for (β, γ, δ). $\square$

The diagram in (6.5) is a reshaped version of the octahedral diagram in triangulated categories and is a structure present in Cartan–Eilenberg systems, cf. [9, Lem. 4.8]. The octahedral diagram can also be used in the definition of a Cartan–Eilenberg system by removing the commutative squares in (1.1)[right] and adding some additional conditions. This idea lies at the heart of proving that Franzosa's module braids correspond to excisive Cartan–Eilenberg systems.

Theorem 6.4 *A module braid* $\mathbf{B}$ *over a meet-semilattice* $\mathsf{Co}(\mathsf{P})$ *for some finite poset* P *is equivalent to an excisive Cartan–Eilenberg system* $\mathbf{E}$ *over* $\mathsf{O}(\mathsf{P})$.

Proof We start with the observation that for an excisive Cartan–Eilenberg system we choose E-terms over $\mathsf{Co}(\mathsf{P})$ by setting $E_\xi = E^\beta_\alpha \cong E^{\beta'}_{\alpha'}$, with $\xi = \beta \smallsetminus \alpha = \beta' \smallsetminus \alpha'$. The exact triangles for $\mathbf{E}$ yield the exact triangle for $\mathbf{B}$ in (i). Property (ii) follows from Lemma 4.4 and the octahedral diagram (module braid diagram) is given by Lemma 6.3, which establishes Property (iii). We conclude that an excisive Cartan–Eilenberg system $\mathsf{O}(\mathsf{P})$ defines a module braid over $\mathsf{Co}(\mathsf{P})$.

For the converse we want to show that (i)-(iii) in the definition of module braid define an excisive Cartan–Eilenberg system in the sense that the commutative squares in (1.1)[right] exist. Define $E_\alpha^\beta := E_\xi$ for all ordered pairs (α, β) such that $\beta \smallsetminus \alpha = \xi$. In (6.1) choose $\xi = \alpha$ and $\eta = \beta \smallsetminus \alpha$ for any the ordered pair (α, β), which establishes the associated exact triangles in (1.1)[left], together with the maps i, j and k. The next step is to define the maps $\ell \colon E_\alpha^\beta \to E_{\alpha'}^{\beta'}$ and $\ell \colon E_\beta^\gamma \to E_{\beta'}^{\gamma'}$ in (1.1)[right] for ordered triples $(\alpha, \beta, \gamma) \le (\alpha', \beta', \gamma')$. Consider the octahedral diagrams in (6.4) in the following two configurations:[13]

$$\begin{aligned} &\alpha \underbrace{\subseteq}_{\xi} \beta \underbrace{\subseteq}_{\eta} \gamma \underbrace{\subseteq}_{\zeta} \gamma' \\ &\alpha \underbrace{\subseteq}_{\xi} \beta \underbrace{\subseteq}_{\eta'} \beta' \underbrace{\subseteq}_{\zeta'} \gamma' \end{aligned} \qquad \begin{tikzcd} E_\eta \arrow[r, "k_1"] \arrow[d, "i"'] & E_\xi \arrow[dd, "i_1'"] \\ E_{\eta\zeta} = E_{\eta'\zeta'} \arrow[ur, "k_2", "k_2'"'] \arrow[d, "j'"'] & \\ E_{\zeta'} \arrow[r, "k_3'"] & E_{\xi\eta'} \end{tikzcd} \tag{6.6}$$

Diagram (6.7) below gives the octahedral diagrams for both configurations. By using the first configuration Diagram (6.7) yields the upper part of Diagram (6.6)[right] and by using the second configuration in (6.7) we obtain the lower part of Diagram (6.6)[right].

$$\begin{tikzcd} & & E_\xi \arrow[ddll, bend right, "i_1"'] \arrow[ddrr, bend left, "i_2"] & & \\ & E_\eta \arrow[ur, "k_1"] \arrow[rr, dashed, "i"] & & E_{\eta\zeta} \arrow[ul, "k_2"'] \arrow[dl, dashed, "j"] & \\ E_{\xi\eta} \arrow[ur, "j_1"] \arrow[rrrr, bend right, "i_3"'] & & E_\zeta \arrow[ul, dashed, "k"] \arrow[ll, "k_3"] & & E_{\xi\eta\zeta} \arrow[ul, "j_2"'] \arrow[ll, "j_3"] \end{tikzcd} \quad \begin{tikzcd} & & E_\xi \arrow[ddll, bend right, "i_1'"'] \arrow[ddrr, bend left, "i_2'"] & & \\ & E_{\eta'} \arrow[ur, "k_1'"] \arrow[rr, dashed, "i'"] & & E_{\eta'\zeta'} \arrow[ul, "k_2'"'] \arrow[dl, dashed, "j'"] & \\ E_{\xi\eta'} \arrow[ur, "j_1'"] \arrow[rrrr, bend right, "i_3'"'] & & E_{\zeta'} \arrow[ul, dashed, "k'"] \arrow[ll, "k_3'"] & & E_{\xi\eta'\zeta'} \arrow[ul, "j_2'"'] \arrow[ll, "j_3'"] \end{tikzcd} \tag{6.7}$$

Since the upper and lower diagrams in (6.6)[right] commute the composition is also a commutative diagram. This construction defines a homomorphism $\ell \colon E_\beta^\gamma \to E_{\beta'}^{\gamma'}$ for every ordered pair $(\beta, \gamma) \le (\beta', \gamma')$. The composition properties for the maps ℓ follows by building adjacent diagrams from (6.6)[right], which yields $E_\alpha^\beta \xrightarrow{\ell} E_{\alpha'}^{\beta'} \xrightarrow{\ell'} E_{\alpha''}^{\beta''}$. Diagram (6.6)[right] together with composition yields the following commutative diagrams:

$$\begin{tikzcd} E_\beta^\gamma \arrow[r, "k_1"] \arrow[d, "j'i"'] & E_\alpha^\beta \arrow[r, "\mathrm{id}"] \arrow[d, "i_1'"] & E_\alpha^\beta \arrow[d, "\ell"] \\ E_{\beta'}^{\gamma'} \arrow[r, "k_3'"] & E_\alpha^{\beta'} \arrow[r, "\ell'"] & E_{\alpha'}^{\beta'} \end{tikzcd} \qquad \begin{tikzcd} E_{\beta'}^{\gamma'} \arrow[r, "k_3'"] \arrow[d, "\mathrm{id}"'] \arrow[dr, "k_3'"] & E_\alpha^{\beta'} \arrow[d, "\ell'"] \\ E_{\beta'}^{\gamma'} \arrow[r, "k_3' j"] & E_{\alpha'}^{\beta'} \end{tikzcd}$$

which establishes Diagram (1.1)[right]. It remains to show that $\ell \colon E_\alpha^\beta \to E_{\alpha'}^{\beta'}$ is the identity map whenever $\beta \smallsetminus \alpha = \beta' \smallsetminus \alpha'$ and $(\alpha, \beta) \le (\alpha', \beta')$. Then, the ordered triples (α, β, β') and $(\alpha, \alpha', \beta')$ satisfy the conditions in Lemma 4.4 which provides the exact triangles (ii) of the definition of module braid

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[13] The notation $\alpha \underbrace{\subseteq}_{\eta} \beta$ indicates the convex set $\eta = \beta \smallsetminus \alpha$.

$$\begin{tikzcd}
E_\xi \arrow[rr, "i"] & & E_{\xi\eta} \arrow[dl, "j"] & & E_\eta \arrow[rr, "i'"] & & E_{\eta\xi} \arrow[dl, "j'"] \\
& E_\eta \arrow[ul, "k"] & & & & E_\xi \arrow[ul, "k'"] &
\end{tikzcd} \tag{6.8}$$

with $\xi = \beta \smallsetminus \alpha = \beta' \smallsetminus \alpha'$ and $\eta = \alpha' \smallsetminus \alpha = \beta' \smallsetminus \beta$ incomparable convex sets, and $\xi\eta = \beta' \smallsetminus \alpha$. The homomorphism $\ell \colon E_\alpha^\beta \to E_{\alpha'}^{\beta'}$ is given by the composition $E_\alpha^\beta \xrightarrow{i} E_\alpha^{\beta'} \xrightarrow{j'} E_{\alpha'}^{\beta'}$. By Definition 6.1(ii) we have that $j'i = \mathrm{id}$ which proves that $\ell = \mathrm{id}$. The E-terms constructed in this proof satisfy the hypotheses of a Cartan–Eilenberg system with the isomorphisms of excisive pairs being the identity isomorphism, i.e. $E_{\alpha\cap\beta}^\beta = E_\alpha^{\alpha\cup\beta}$ for all $\alpha, \beta \in \mathsf{O}(\mathsf{P})$. □

6.2 Connection Matrices

In [3] a powerful representation result is established for module braids that are generated by a chain complex braid. Due to Theorem 6.4 we can reformulate this result for Cartan–Eilenberg systems generated by an $\mathsf{O}(\mathsf{P})$-filtered differential vector space. Let (D, d_D) be an $\mathsf{O}(\mathsf{P})$-filtered differential vector space and let $\mathbf{E}(D, \mathrm{d}_D)$ be the associated Cartan–Eilenberg system. We say that the latter is finite dimensional if the vector spaces E_α^β are finite dimensional for each pair (α, β) with $\beta \smallsetminus \alpha = \{p\}$ for all $p \in \mathsf{P}$. Define a special finite dimensional P-graded vector space A with the decomposition

$$A = \bigoplus_{p \in \mathsf{P}} G_p A, \quad G_p A \cong E_\alpha^\beta, \quad \beta \smallsetminus \alpha = \{p\}. \tag{6.9}$$

An appropriate differential is constructed by the following theorem:

Theorem 6.5 [cf. [3], Thm. 4.8] *Let (D, d_D) be an $\mathsf{O}(\mathsf{P})$-filtered differential vector space with the property that the associated Cartan–Eilenberg system $\boldsymbol{E}(D, \mathrm{d}_D)$ is finite dimensional. Let A be defined as in (6.9). Then, there exists a strict $\mathsf{O}(\mathsf{P})$-filtered differential $\mathrm{d}_A \colon A \to A$ such that*

$$\boldsymbol{E}(A, \mathrm{d}_A) \cong \boldsymbol{E}(D, \mathrm{d}_D).$$

In particular, there exists an $\mathsf{O}(\mathsf{P})$-filtered quasi-isomorphism[14] *$f \colon (A, \mathrm{d}_A) \to (D, \mathrm{d}_D)$, such that the isomorphisms $H(f)_\alpha^\beta \colon E_\alpha^\beta(A, \mathrm{d}_A) \to E_\alpha^\beta(D, \mathrm{d}_D)$, for all $\alpha \subseteq \beta$, define an isomorphism of Cartan–Eilenberg systems.*

In [3] the strict, P-graded differential vector space (A, d_A) is called a *connection matrix* for $\mathbf{E}(D, \mathrm{d}_D)$. Note that d_A restricted to the vector spaces $G_p A$ is the zero map and therefore $H(G_p A) = G_p A$. The differential may be regarded as a strict, upper-triangular matrix on A. In dynamical systems theory the non-trivial entries of d_A contain information about heterclinic orbits between Morse sets. In [3] the notion of connection matrix is defined in a slightly more general context. If we translate this to Cartan–Eilenberg systems we obtain:

Definition 6.6 [3, Defn. 3.6.B] Let $\mathbf{E}$ be a Cartan–Eilenberg system over $\mathsf{O}(\mathsf{P})$. A strict, P-graded differential vector space (A, d_A) as defined in (6.9), such that $\mathbf{E}(A, \mathrm{d}_A) \cong \mathbf{E}$, is called a connection matrix for $\mathbf{E}$.

Remark 6.7 The above theorem is a translation of the more general result in [3, Thm. 4.8] to the case of Cartan–Eilenberg systems generated by filtered differential vector spaces.

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[14] A quasi-isomorphism is a chain homomorphism that is an isomorphism on homology.

Existence of connection matrices is only established in the setting of [3, Thm. 4.8], i.e. for Cartan–Eilenberg systems $\mathbf{E}(D, \mathrm{d}_D)$ in which case a connection matrix is quasi-isomorphic to (D, d_D).

7 Discussion

The main result of this paper in Theorem 5.1 has various implications for the theory of connection matrices as used in dynamical systems theory. In this section we discuss the applications and resolutions of the conjectures by Robbin and Salamon, as well as the implications for the transition matrix theory by Franzosa and Mischaikow, cf. [4].

7.1 The Conjectures by J. Robbin and D. Salamon

In their seminal paper on dynamical systems and connection matrices, cf. [11], Robbin and Salamon formulate a number of conjectures concerning non-uniqueness of connection matrices. Theorem 5.1 is posed as Conjecture 8.5 in [11] in the setting of flows. Theorem 5.1 resolves non-uniqueness issues in Franzosa's connection matrix theory (see [3, Section 6.3]): all connection matrices are $\mathsf{O}(\mathsf{P})$-filtered chain isomorphic, i.e. all connection matrices according to Definition 6.6 are conjugated via $\mathsf{O}(\mathsf{P})$-filtered chain isomorphisms, which generalizes the main result in [4, Theorem 3.5].

In [11, Thm. 8.1] it is proven that an $\mathsf{O}(\mathsf{P})$-filtered differential vector space (D, d_D) is $\mathsf{O}(\mathsf{P})$-filtered chain equivalent to a strict, P-graded differential vector space (A, d_A), such that $\mathbf{E}(D, \mathrm{d}_D) \cong \mathbf{E}(A, \mathrm{d}_A)$. In this setting our main result solves the following conjecture:

Corollary 7.1 (cf. [11, Conj. 7.4]) *Let* (D, d_D) *and* $(D', \mathrm{d}_{D'})$ *be* $\mathsf{O}(\mathsf{P})$*-filtered differential vector spaces over a field* $\mathbb{K}$*. Then,* $\mathbf{E}(D, \mathrm{d}_D) \cong \mathbf{E}(D', \mathrm{d}_{D'})$ *if and only if* (D, d_D) *and* $(D', \mathrm{d}_{D'})$ *are* $\mathsf{O}(\mathsf{P})$*-filtered chain equivalent, i.e.* $(D, \mathrm{d}_D) \simeq (D', \mathrm{d}_{D'})$.

Proof If (D, d_D) and $(D', \mathrm{d}_{D'})$ are $\mathsf{O}(\mathsf{P})$-chain equivalent then the equivalence $\mathbf{E}(D, \mathrm{d}_D) \cong \mathbf{E}(D', \mathrm{d}_{D'})$ is immediate. For vector spaces the existence theorems in [5, 11] show that there exist strict, P-graded differential vector spaces (A, d_A) and (C, d_C), and chain equivalences $(A, \mathrm{d}_A) \simeq (D, \mathrm{d}_D)$ and $(C, \mathrm{d}_C) \simeq (D', \mathrm{d}_{D'})$, which implies that $\mathbf{E}(D, \mathrm{d}_D) \cong \mathbf{E}(A, \mathrm{d}_A)$ and $\mathbf{E}(D', \mathrm{d}_{D'}) \cong \mathbf{E}(C, \mathrm{d}_C)$. By assumption, $\mathbf{E}(D, \mathrm{d}_D) \cong \mathbf{E}(D', \mathrm{d}_{D'})$ and thus $\mathbf{E}(A, \mathrm{d}_A) \cong \mathbf{E}(C, \mathrm{d}_C)$. Consequently, $(A, \mathrm{d}_A) \cong (C, \mathrm{d}_C)$ are $\mathsf{O}(\mathsf{P})$-chain isomorphic via Theorem 5.1, and therefore, $(D, \mathrm{d}_D) \simeq (A, \mathrm{d}_A) \cong (C, \mathrm{d}_C) \simeq (D', \mathrm{d}_{D'})$, which proves $(D, \mathrm{d}_D) \simeq (D', \mathrm{d}_{D'})$, i.e. they are $\mathsf{O}(\mathsf{P})$-chain equivalent. □

Remark 7.2 Conjecture 8.4 in [11] is false in the setting of connection matrices since in the case of ring coefficients the homologies E_α^β, $\beta \smallsetminus \alpha = \{p\}$ are not necessarily free, or projective, cf. [3, Exam. 6.3]. Therefore a connection matrix is not an option. This problem is a subject of further study.

7.2 Connection Matrix Theories for Vector Spaces

In [5] the theory of connection matrices for field coefficients is treated in the setting of homotopy categories without using homology. For an $\mathsf{O}(\mathsf{P})$-filtered differential vector space (D, d_D), or equivalently, a P-graded differential vector space consider the decomposition

$$D = \bigoplus_{p\in \mathsf{P}} G_p D, \quad \alpha \mapsto F_\alpha D = \bigoplus_{p\in\alpha} G_p D, \quad \alpha \in \mathsf{O}(\mathsf{P}).$$

In [5, Defn. 4.23], a strict, P-graded differential vector space (A, d_A) is a connection matrix for (D, d_D) if (A, d_A) is $\mathsf{O}(\mathsf{P})$-filtered chain equivalent to (D, d_D). An algorithm is proven in [5] to construct a strict P-graded differential vector space (A, d_A) given a finite dimensional $\mathsf{O}(\mathsf{P})$-filtered differential vector space (D, d_D). This procedure is reminiscent of the inductive existence result in [11, Thm. 8.1]. In [5, Prop. 4.27] it is proven that connection matrices defined in the sense of [5, Defn. 4.23] are $\mathsf{O}(\mathsf{P})$-filtered chain isomorphic, which covers a special case of Theorem 5.1.

In Sect. 6 we have summarized the connection matrix theory by Franzosa. Consider the $\mathsf{O}(\mathsf{P})$-filtered differential vector space (D, d_D). In Franzosa's theory a strict, P-graded differential vector space (A, d_A) is a connection matrix for an $\mathsf{O}(\mathsf{P})$-filtered differential vector space (D, d_D) if $\mathbf{E}(A, \mathrm{d}_A) \cong \mathbf{E}(D, \mathrm{d}_D)$, cf. Definition 6.6. Existence of a connection matrix is given by [3, Thm. 4.8]. The latter is not assumed to be a chain equivalence. Note that Franzosa's notion of connection matrix does not even require a map at the chain level. The following result concerns all possible connection matrices according to Definition 6.6.

Theorem 7.3 *Let (D, d_D) be an $\mathsf{O}(\mathsf{P})$-filtered differential vector space. Then, every strict, P-graded differential vector space (A, d_A) which satisfies*

$$\boldsymbol{E}(A, \mathrm{d}_A) \cong \boldsymbol{E}(D, \mathrm{d}_D),$$

is $\mathsf{O}(\mathsf{P})$-filtered chain equivalent to (D, d_D), i.e. $(A, \mathrm{d}_A) \simeq (D, \mathrm{d}_D)$.

Proof The P-graded differential vector space (A, d_A) is also $\mathsf{O}(\mathsf{P})$-filtered. The $\mathsf{O}(\mathsf{P})$-filtered differential vector spaces (D, d_D) and (A, d_A) satisfy $\mathbf{E}(D, \mathrm{d}_D) \cong \mathbf{E}(A, \mathrm{d}_A)$. By Corollary 7.1 this implies that (D, d_D) and (A, d_A) are $\mathsf{O}(\mathsf{P})$-filtered chain equivalent, which proves the theorem. □

As a consequence of Theorem 7.3 the connection matrix theories of [3, 5, 11] in the setting of (finite dimensional) differential vector spaces are equivalent. The advantage of the latter theory is that the existence theory is based on an effective algorithm for producing connection matrices.

7.3 Morse–Smale Gradings and Unique Connection Matrices

Suppose (C, d_C) is a chain complex, in which case the chain homomorphisms are given by $f_k\colon C_k \to C_k$, i.e. $f_k(F_\alpha C_k) \subseteq F_\alpha C_k$, for all $\alpha \in \mathsf{O}(\mathsf{P})$ and for all k, cf. Example 2.3 and Remark 3.16. For simplicity of exposition we consider P-graded chain complexes in the category of $\mathbb{Z}_2$-vector spaces, where both gradings satisfy a natural order-preserving property.

Definition 7.4 A strict, P-graded chain complex (C, d_C) is referred to as an *algebraic Morse–Smale graded chain complex* if there exists a map $\mu\colon \mathsf{P} \to \mathbb{Z}$ such that

(i) $p < q$ implies $\mu(p) < \mu(q)$;

(ii) $G_p C_k = \begin{cases} \mathbb{Z}_2 & \text{if } k = \mu(p); \\ 0 & \text{otherwise.} \end{cases}$

The pair (P, μ) is called a *Morse–Smale grading* for (C, d_C).

By definition algebraic Morse–Smale graded chain complexes are finite dimensional and $C = \mathbb{Z}_2^{|\mathsf{P}|}$, where $|\mathsf{P}|$ is the number of elements in P. As a consequence of Condition (i) we have that

$$\mu(p) = \mu(q), \quad p \neq q \quad \Longrightarrow \quad p \parallel q. \tag{7.1}$$

The Morse–Smale grading restricts the differentials on C.

Theorem 7.5 *Let* (C, d_C) *and* (C, d'_C) *be algebraic Morse–Smale graded chain complexes with Morse–Smale grading* (P, μ) *for both chain complexes. Then,* $\mathrm{d}_C = \mathrm{d}'_C$ *if and only if* $\boldsymbol{E}(C, \mathrm{d}_C) \cong \boldsymbol{E}(C, \mathrm{d}'_C)$.

Proof By assumption both chain complexes have the same Morse–Smale grading with associated choice of basis. Theorem 5.1 yields the existence of an $\mathsf{O}(\mathsf{P})$-filtered chain automorphism $f : C \to C$ given by $f_k : C_k \to C_k$, which are $\mathsf{O}(\mathsf{P})$-filtered for all k. By the P-grading each f_k may be regarded as matrix with entries $f_k^{pq} : G_q C_k \to G_p C_k$, for all $\mu(p) = \mu(q) = k$. The filtering condition on f_k translates to: $f_k^{pq} \neq 0$ implies $p \leq q$. Consequently, by (7.1), and thus Definition 7.4(i), $f_k^{pq} = 0$ for all $p \neq q$ and f_k^{pp} are automorphisms. By Definition 7.4(ii) the factors are given by $G_p C_k = \mathbb{Z}_2$ and therefore $f_k^{pp} = \mathrm{id}$ for all p with $\mu(p) = k$ since the only $\mathbb{Z}_2$-vector space automorphism on $\mathbb{Z}_2$ is the identify map. We conclude that $f = \mathrm{id}$ and thus $\mathrm{d}_C = \mathrm{d}'_C$. □

The term Morse–Smale graded chain complex is motivated by Morse–Smale flows in dynamical systems theory. In [10], J. Reineck proves that connection matrices for Morse–Smale flows without periodic orbits are unique. Via Conley index theory such flows yield a Morse–Smale graded chain complex where the vector spaces $G_p C_k$ record the critical points of index k, cf. [2, 12]. Theorem 7.5 provides an algebraic proof of Reineck's result.

Remark 7.6 The proof of Theorem 7.5 can be repeated for chain complexes of $\mathbb{K}$-vector spaces. For chain complexes of $\mathbb{K}$-vector spaces the result stays the same except for the $\mathbb{K}$-vector space automorphisms of $\mathbb{K}$ given by $\mathrm{Aut}(\mathbb{K}) = \mathbb{K}^\times$ (the multiplicative group $\mathbb{K} \smallsetminus \{0\}$).

We conclude that the entries of the differentials d_C and d'_C are the same up to conjugacy, which is a statement of uniqueness.

Remark 7.7 Theorem 7.5 can be proved in a more general context when (C, d_C) is an algebraic Morse–Smale graded chain complex and (A, d_A) is a finite dimensional, strict P-graded chain complex (both of $\mathbb{K}$-vector spaces). Under the condition that $\mathbf{E}(C, \mathrm{d}_C) \cong \mathbf{E}(A, \mathrm{d}_A)$, then $G_p C_k$ and $G_p A_k$ are isomorphic and the entries in d_C and d_A are related by conjugation. This is the general statement that there exists only one strict differential to represent $\mathbf{E}$ in the Morse–Smale setting.

Remark 7.8 The condition that the $\mathsf{O}(\mathsf{P})$-filtered chain isomorphisms are also homogeneous with respect to the chain complex grading is crucial for Theorem 7.5. If an $\mathsf{O}(\mathsf{P})$-filtered chain isomorphism is not necessarily homogeneous with respect to the chain complex grading, then there may be multiple *similar* differentials.

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