

Automorphic Compatible Systems of Galois Representations



Federico Amadio Guidi
Somerville College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy

Trinity 2019

Acknowledgements

First, I would like to express my deepest gratitude to my supervisor Andrew Wiles for introducing me to this area of research, for the many inspiring ideas he shared with me, and for his constant guidance and support.

A special thank goes to my examiners Minhyong Kim and Jack Thorne for their very stimulating comments on a preliminary version of this thesis.

Over the last four years, I have benefited from many interesting conversations with several people. Among them, I would especially like to thank Francis Brown, Laura Capuano, Toby Gee, Giacomo Micheli, and Damian Rössler.

I would finally like to thank everyone who made these years in Oxford very enjoyable, both in and out of the Mathematical Institute.

Abstract

This thesis investigates properties of compatible systems of Galois representations, mainly focusing on the compatible systems which are attached to certain classes of automorphic representations of GL_n .

We develop a general method to prove independence results for algebraic monodromy groups in abstract compatible systems of representations, and give applications both in characteristic zero and in positive characteristic settings. In the case of automorphic compatible systems (and actually for a slightly larger class of geometric compatible systems), we apply our method to deduce an independence result, assuming a classical irreducibility conjecture. In addition, we also deduce an independence result in the case of compatible systems of lisse sheaves on normal varieties over finite fields.

We then focus on the study of the geometry of (pseudo)deformation spaces of Galois representations and definite unitary groups eigenvarieties at points corresponding to certain classical automorphic representations. In this context, we present smoothness results known in the literature, and suggest possible implications for automorphic compatible systems.

Contents

1	Background material	1
1.1	Galois representations	2
1.1.1	Generalities on ℓ -adic Galois representations	2
1.1.2	Weil-Deligne representations	4
1.1.3	Local Galois representations when $\ell \neq p$	7
1.2	A survey of basic p -adic Hodge theory	7
1.2.1	Fontaine's p -adic period rings	8
1.2.2	Local Galois representations when $\ell = p$	13
1.2.3	Applications to p -adic comparison isomorphisms	19
1.3	Geometric compatible systems of Galois representations	20
1.4	The local Langlands correspondence	24
1.4.1	Representation theory over characteristic zero local fields	24
1.4.2	The L -group and the conjectural correspondence	27
1.4.3	The case of GL_n	29
1.5	Algebraic automorphic representations	30
1.5.1	Generalities on automorphic forms and representations	30
1.5.2	The Satake isomorphism	36
1.5.3	Local definitions	38
1.5.4	Global definitions	39
1.6	Compatible systems attached to algebraic automorphic representations	42
1.7	Potential automorphy and irreducibility of compatible systems	44
2	Independence of algebraic monodromy groups in compatible systems	51
2.1	Abstract compatible systems and algebraic monodromy groups	51
2.2	The formal character and the variety of characteristic polynomials	53
2.3	Frobenius tori	54
2.4	Abstract independence results	56

2.5	The case of geometric compatible systems of Galois representations	61
2.6	Compatible systems in the positive characteristic case	64
3	Deformations of automorphic Galois representations	69
3.1	Deformation theory of pseudorepresentations	69
3.1.1	Deformations of pseudorepresentations	70
3.1.2	Cayley-Hamilton algebras, GMA structures, and reducibility ideals	74
3.1.3	Deformation conditions	77
3.2	Eigenvarieties for definite unitary groups	78
3.2.1	Triangulations and refinements of crystalline representation	79
3.2.2	Definition and basic properties of eigenvarieties	83
3.2.3	Galois pseudorepresentations on eigenvarieties	87
3.3	Automorphic points in (pseudo)deformation spaces	89
3.3.1	Smoothness results	89
3.3.2	Plan of future research	94
	Bibliography	97

Introduction

This thesis investigates properties of compatible systems of Galois representations. We mainly focus on the compatible systems which are attached to certain classes of automorphic representations of GL_n .

Compatible systems of Galois representations have been introduced in arithmetic algebraic geometry by Serre in [89], generalising the role played by the family of ℓ -adic Tate modules of elliptic curves or abelian varieties defined over a number field. Let us sketch a definition of compatible system (refer to §1.3 for the definition of the so called *geometric* compatible systems of Galois representations, and to §2.1 for the definition of *abstract* compatible systems). For simplicity, let us focus on the number field case (positive characteristic counterparts will be studied only in §2.6). Let E and F be number fields, and let $n \geq 1$ be an integer. A *compatible system* of rank n representations of the absolute Galois group of F defined over E is a family $\{\rho_\lambda\}_\lambda$ of continuous semisimple representations

$$\rho_\lambda : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_n(\overline{E}_\lambda)$$

parametrised by the finite places λ of E , and satisfying some “compatibility” or “ λ -independence” property. More precisely, we require that there exists a finite set S of “bad places” of F such that if $v \notin S$ and λ does not divide the residue characteristic of v , then ρ_λ is unramified at v , and the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_v)$ has coefficients in E , and is independent of λ .

Further λ -independence properties are expected to hold for compatible systems. A classical question to ask in this context is whether the *algebraic monodromy groups* are independent of λ . Questions on independence of algebraic monodromy groups in compatible systems originate in the literature from a classical result by Serre, see [89], which states that if X is an elliptic curve without complex multiplication over a number field, then the image of the Galois representation on the ℓ -adic Tate module of X is an open subgroup of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for every prime ℓ , and is equal to $\mathrm{GL}_2(\mathbb{Z}_\ell)$ for all but finitely many primes ℓ . The problem of extending this result to more general

geometric settings led, for instance, to the formulation of the Mumford-Tate conjecture for abelian varieties, see [81]. Let us consider the problem of independence of algebraic monodromy groups for general compatible systems. Let then $\{\rho_\lambda\}_\lambda$ be a compatible system of rank n representations of the absolute Galois group of F defined over E . For each λ , we define the algebraic monodromy group of ρ_λ to be the Zariski closure G_λ of the image of ρ_λ in $\mathrm{GL}_n(\overline{E}_\lambda)$. This is a reductive algebraic subgroup of $\mathrm{GL}_{n,\overline{E}_\lambda}$ (actually defined over a finite extension of E_λ). For simplicity, let us assume that each ρ_λ takes values in $\mathrm{GL}_n(E_\lambda)$ (so that G_λ is defined over E_λ). The prototype of the independence questions that we study in this thesis is the following.

Question 0.0.1. Does there exist an algebraic subgroup G of $\mathrm{GL}_{n,E}$ such that $G \times_E E_\lambda$ is conjugate to G_λ over E_λ for each λ ?

Giving an answer to this question is in general way too difficult. For this reason, we focus on a specific class of compatible systems, that is the class of those compatible systems which are attached to certain *algebraic* automorphic representations of GL_n via instances of the *global Langlands conjecture* for GL_n . We elaborate briefly on this.

Let us fix a prime ℓ , and an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$. Then, the global Langlands (-Clozel-Fontaine-Mazur) conjecture for GL_n over F predicts the existence of a unique bijection

$$\mathcal{A}_F^0(n) \xleftrightarrow{\sim} \mathcal{G}_F^0(n)$$

between the set $\mathcal{A}_F^0(n)$ of (isomorphism classes of) algebraic, cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$, and the set $\mathcal{G}_F^0(n)$ of (isomorphism classes of) irreducible continuous Galois representations $\rho : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, which are unramified outside a finite set of places of F , and de Rham at places of F above ℓ . This bijection is conjecturally characterised by local-global compatibilities at the finite places of F . More precisely, for each finite place v of F we require the bijection to be compatible with the *local Langlands correspondence* for GL_n over F_v , which is a theorem of Harris and Taylor [60] and (independently) of Henniart [62]. Note that these local-global compatibilities imply the matching of the *L-functions* of the objects appearing on the two sides of the correspondence.

Since the analytic continuation and functional equation is known for *L-functions* on the “automorphic side” by results of Godement and Jacquet [55], then if X is a smooth projective variety over F and we know that the Galois representation on the ℓ -adic étale cohomology of $X_{\overline{F}} = X \times_F \overline{F}$ is automorphic, then we obtain the Hasse-Weil conjecture for X , see for instance [90].

Let us also remark that the set $\mathcal{A}_F^0(n)$ is countable by results of Harish-Chandra [58], and that all the elements of $\mathcal{G}_F^0(n)$ should arise from the ℓ -adic étale cohomology of smooth proper varieties over F by the conjecture of Fontaine and Mazur [51]. The above bijection should therefore be a bijection of (conjecturally) countable sets. Another consequence of the conjecture of Fontaine and Mazur is that all the representations on the “Galois side” should arise in compatible systems.

In this generality, the global Langlands conjecture is a highly open problem. Nevertheless, when the field F is totally real or CM, the arrow from “automorphic” to “Galois” has been mostly established (see §1.6 for a more detailed exposition). In particular, (semisimple) Galois representations ρ_π can be constructed from regular algebraic, cuspidal automorphic representations π , and are proved to satisfy (most cases of) compatibility with the local Langlands correspondence. When the automorphic representation π is also (conjugate) self-dual, all the requested properties of ρ_π have been proved, except for the irreducibility, which remains open in general, and for which only partial progress has been obtained so far (see §1.7 for an overview of known results). Even less is known for the arrow from “Galois” to “automorphic”. In this direction, the available techniques are generalisations of the method of [108] and [99], see for instance [34], [98], and the more recent [25]. These have been applied to prove “potential automorphy” results, as for instance those of [6], and the very recent ones of [2].

For a regular algebraic, (conjugate) self-dual, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, when the field F is totally real or CM, varying ℓ over the rational primes it is possible to show that the representations ρ_π live in a (geometric) compatible system $\{\rho_{\pi,\lambda}\}_\lambda$ of rank n representations, and that there exists a number field E_π such that each $\rho_{\pi,\lambda}$ takes values in $\mathrm{GL}_n(E_{\pi,\lambda})$. In this thesis, we study Question 0.0.1 for this class (and actually for a slightly larger class) of compatible systems.

Our approach to this problem consists in developing a general method to prove independence of algebraic monodromy groups in abstract compatible systems. This method is meant to be applicable in those contexts where some instances of the “global Langlands correspondence” are known. This turns out to be not only the case of automorphic compatible systems, but also the case of some compatible systems arising in positive characteristic settings (note that for GL_n over a global field of positive characteristic the global Langlands correspondence is actually a theorem of L. Lafforgue [71]).

Let us briefly explain how our strategy works. Recall that Question 0.0.1 has been studied for abstract rational compatible systems through algebraic methods in the

foundational work of Larsen and Pink [75]. The main results of [75] turn out to be extendible verbatim to compatible systems having coefficients in any number field. Using these results, we develop a general method to prove λ -independence of the neutral components of the algebraic monodromy groups of compatible systems over a finite extension of the field of coefficients, and over a set of primes of Dirichlet density 1. The main idea is that one can prove a λ -independence result of this form whenever a compatible system admits what we call a *Lie-irreducible decomposition* over a finite extension of its field of coefficients (see §2.4 for the definition of Lie-irreducible decomposition). The proof of the existence of a Lie-irreducible decomposition essentially reduces to the problem of extending a single λ -adic representation to a compatible system.

Let us see how this method works in the case of automorphic compatible systems (and actually for a slightly larger class of geometric compatible systems). First of all, let us remark that if π is a regular algebraic, conjugate self-dual, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, when F is a CM field, then it is natural to ask whether it is possible (at least) to show the irreducibility of $\rho_{\pi,\lambda}$ for $\lambda \mid \ell$ and ℓ in a set of rational primes of Dirichlet density 1 (see Conjecture 1.7.3). We prove the following result.

Theorem 0.0.2 (see Corollary 2.5.2). *Let F be a CM field, and let $\{\rho_\lambda\}_\lambda$ be a pure, regular, totally odd, conjugate self-dual geometric compatible system of rank n representations of Γ_F with coefficients in E . Assume Conjecture 1.7.3. Then, there exist a set \mathcal{L} of rational primes of Dirichlet density 1, a finite extension E' of E , and a split connected reductive algebraic subgroup G of $\mathrm{GL}_{n,E'}$ such that for each $\ell \in \mathcal{L}$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ and $G \times_{E'} E'_{\lambda'}$ are conjugate over $E'_{\lambda'}$.*

As explained above, the key point consists in showing that certain λ -adic representations of this kind extend to compatible systems. This is achieved by the potential automorphy techniques of [6]. We remark that our result is conditional on Conjecture 1.7.3. However, before spending more words on this problem, let us make a digression about some positive characteristic counterparts.

In the positive characteristic setting, we use the global Langlands correspondence of [71] and arguments of Drinfeld [42] (actually taken from [15]) to prove a λ -independence result for compatible systems of lisse sheaves on normal varieties over finite fields (see §2.6 for the relevant definitions in this context). For a (geometrically connected) variety X over a finite field \mathbb{F}_q , and a compatible system $\{\mathcal{F}_\lambda\}_\lambda$ of semisimple rank n

lisse E_λ -sheaves on X , for each λ we denote by G_λ the Zariski closure of the image of the representation of $\pi_1^{\text{ét}}(X)$ corresponding to \mathcal{F}_λ in $\text{GL}_n(E_\lambda)$.

Theorem 0.0.3 (see Corollary 2.6.5). *Let X be a normal geometrically connected irreducible variety over \mathbb{F}_q , and let $\{\mathcal{F}_\lambda\}_\lambda$ be a compatible system of semisimple lisse E_λ -sheaves on X . Then, there exist a set \mathcal{L} of rational primes of Dirichlet density 1, a finite extension E' of E , and a split connected reductive algebraic subgroup G of $\text{GL}_{n,E'}$ such that for each $\ell \in \mathcal{L}$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ and $G \times_{E'} E'_{\lambda'}$ are conjugate over $E'_{\lambda'}$.*

Let us now focus on Conjecture 1.7.3 for automorphic compatible systems. A natural approach to this problem is by means of an L -function argument, which relies on results in the direction “Galois” to “automorphic” in the bijection (recalled above) predicted by the global Langlands conjecture for GL_n . We sketch here a rough and simplified version of this argument (we refer to §1.7 for a precise version of it). Let p be a prime, let F be a totally real or CM field, and let π be a regular algebraic, (conjugate) self-dual, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. We want to show that the corresponding Galois representation $\rho_\pi : \text{Gal}(\overline{F}/F) \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$ is irreducible. Since local-global compatibility with the local Langlands correspondence holds, then outside a finite set S of places of F we have an identity of (partial) L -functions $L^S(\rho_\pi \otimes \rho_\pi^\vee, s) = L^S(\pi \times \pi^\vee, s)$. By results of Shahidi [92] and of Jacquet and Shalika [65] we have that $\text{ord}_{s=1} L^S(\pi \times \pi^\vee, s) = -1$. Let $\rho_\pi = \bigoplus_{i=1}^k \rho_i$ be an irreducible decomposition into pairwise non isomorphic subrepresentations, so that now each ρ_i is in $\mathcal{G}_F^0(n_i)$. Assume that each ρ_i is automorphic, that is there exist π_i in $\mathcal{A}_F^0(n_i)$ such that $\rho_i = \rho_{\pi_i}$. We then have

$$\begin{aligned} L^S(\rho_\pi \otimes \rho_\pi^\vee, s) &= \prod_{i,j=1}^k L^S(\rho_i \times \rho_j^\vee, s) \\ &= \prod_{i,j=1}^k L^S(\pi_i \times \pi_j^\vee, s), \end{aligned}$$

so that

$$\begin{aligned} \text{ord}_{s=1} L^S(\rho_\pi \otimes \rho_\pi^\vee, s) &= \sum_{i,j=1}^k \text{ord}_{s=1} L^S(\pi_i \times \pi_j^\vee, s) \\ &= \sum_{i=1}^k \text{ord}_{s=1} L^S(\pi_i \times \pi_i^\vee, s) \\ &= -k, \end{aligned}$$

again by results of [92] and [65]. It follows that $k = 1$, and hence ρ_π is irreducible.

The known instances of Conjecture 1.7.3 rely on an argument of this kind. The key point is always the use of some “Galois” to “automorphic” result, i.e. some arrow $\mathcal{G}_F^0(n') \rightarrow \mathcal{A}_F^0(n')$. So far, the results which have been applied in this context are the potential automorphy techniques of [6]. Refined potential automorphy techniques should eventually imply Conjecture 1.7.3 (see the discussion in §1.7), thus also making Theorem 0.0.2 unconditional.

Even though we don’t know Conjecture 1.7.3 at this stage, we highlight how the irreducibility of automorphic Galois representations has interesting geometric consequences for the p -adic rigid analytic spaces parametrising p -adic families of Galois representations and automorphic representations. Let us in fact recall that the set $\mathcal{A}_F^0(n)$ is countable, and that the set $\mathcal{G}_F^0(n)$ is conjecturally countable. Nevertheless, these sets can be embedded (roughly speaking) in certain p -adic rigid analytic spaces $\mathcal{A}_F^0(n) \subset \mathcal{E}$ and $\mathcal{G}_F^0(n) \subset \mathcal{X}$. The space \mathcal{X} is easier to describe, and is just (essentially) a moduli space of representations (see §3.1.1 and §3.3.1 for the details). The space \mathcal{E} is a so-called *eigenvariety* (see §3.2.2 for the details).

Let us fix a prime p appropriately, and an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_p}$. Let π be a regular algebraic, conjugate self-dual, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, for F a CM field, and let $\rho_\pi : \mathrm{Gal}(\overline{F}/F) \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p})$. Then ρ_π defines a point of \mathcal{X} , and π (together with the choice of an additional datum, see §3.2.2) defines a point of \mathcal{E} . Under certain circumstances, it is known that the irreducibility of ρ_π implies the smoothness of \mathcal{X} and \mathcal{E} at these points (see §3.3.1 for the details). Motivated by explicit examples constructed by Bellaïche [7] and [9], we suggest that converse results should hold as well, i.e. if the representation ρ_π is reducible then \mathcal{X} (resp. \mathcal{E}) cannot be smooth at ρ_π (resp. π), and we explain how it is reasonable to expect some regularity of these spaces in general. Nevertheless, we don’t formulate precise conjectures, nor prove any precise result in this context, which we then leave as a topic of future research.

The structure of this thesis is the following. In §1 we introduce the preliminary material on Galois representations and automorphic representations, we present the construction of automorphic compatible systems, and we review known potential automorphy results, relating them to the problem of irreducibility of automorphic compatible systems. Then, in §2 we present our results on λ -independence of algebraic monodromy groups, with applications to automorphic compatible systems and to compatible systems arising in positive characteristic settings. Note that the material presented in §2 essentially coincides with the one presented in the author’s preprint

[3]. Finally, in §3 we study p -adic families of Galois representations and automorphic representations, and we focus on smoothness results for the corresponding p -adic rigid analytic spaces.

Notation

Given a ring with unity R , we let R^\times denote the (multiplicative) group of invertible elements in R . Given integers $m, n \geq 1$, we let $M_{m \times n}(R)$ denote the ring of $m \times n$ matrices with coefficients in R . If R is commutative, we let $W(R)$ denote the ring of Witt vectors of R , and for any $x \in R$, we denote by $[x]$ its Teichmüller representative $(x, 0, \dots)$ in $W(R)$. If R is a domain, we let $\text{Frac } R$ denote its field of fractions.

For a representation $\rho : \Gamma \rightarrow \text{GL}(V)$ of a group Γ on a vector space V over a field \mathbb{k} , we denote by ρ^\vee the dual representation to ρ .

Given a variety X over a field \mathbb{k} , and an inclusion of fields $\mathbb{k} \subset \mathbb{K}$, we let $X_{\mathbb{K}} = X \times_{\mathbb{k}} \mathbb{K} = X \times_{\text{Spec } \mathbb{k}} \text{Spec } \mathbb{K}$.

Given a field M , we let \overline{M} be an algebraic closure of M , and M^s be a separable closure of M inside \overline{M} , and we denote by $\Gamma_M = \text{Gal}(M^s/M)$ the absolute Galois group of M . For every integer $n \geq 1$, we denote by ζ_n a primitive n^{th} -root of unity in M^s , and by $\mu_n(M^s)$ the group of n^{th} -root of unity in M^s . Given a finite extension M' of M , we denote by $N_{M'/M} : M' \rightarrow M$ its norm.

Given a discrete valuation field K , we denote by v_K the discrete valuation on K , by \mathcal{O}_K the ring of integers (i.e. the valuation ring) of K , by \mathfrak{m}_K the (unique) maximal ideal of \mathcal{O}_K , by ϖ_K a uniformiser of \mathcal{O}_K , and by $k_K = k(v_K)$ the residue field of v_K . For a prime p , we simply write v_p for the discrete valuation on $\overline{\mathbb{Q}}_p$ normalised such that $v_p(p) = 1$.

For a non Archimedean local field F , we denote by F^{un} the maximal unramified extension of F , we let I_F be the inertia subgroup of Γ_F , so that $I_F = \Gamma_{F^{\text{un}}} = \Gamma_{\widehat{F}^{\text{un}}}$, where \widehat{F}^{un} is the completion of F^{un} , and we let $\text{Frob}_F \in \Gamma_F/I_F$ be the geometric Frobenius. We denote by W_F the Weil group of F , and we let W_F^{ab} denote the group $W_F/\overline{[W_F, W_F]}$.

When F is a global field, we denote by $|F|$ the set of finite places of F . If Σ is a set of places of F , and ℓ is a prime, we set $\Sigma_\ell = \{v \in \Sigma : v \mid \ell\}$.

If S is a finite set of places of F , we let F_S be the maximal extension of F inside a (fixed) separable closure F^s of F which is unramified outside S , and we set $\Gamma_{F,S} = \text{Gal}(F_S/F)$. For a place v of F , we let F_v be the completion of F at v , which is a local field. We let $|\cdot|_v$ be the absolute value on F_v corresponding to v , normalised

to give uniformisers absolute value $1/\#k(v)$ when v is finite. In this case, F_v is a non Archimedean local field, and we write $\text{Frob}_v = \text{Frob}_{F_v}$.

For a global field F , we let \mathcal{O}_F be its ring of integers. We let $\mathbb{A}_F = \prod'_v F_v$ denote the ring of adèles over F (where the infinite product is restricted with respect to the rings of integers \mathcal{O}_{F_v} of F_v). We also let $|\cdot|_F = \prod_v |\cdot|_v$. When F is a number field, we write $\mathbb{A}_F = F_\infty \times \mathbb{A}_{F,f}$, where $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v|\infty} F_v$ is the ring of infinite adèles over F , and $\mathbb{A}_{F,f} = \prod'_{v \nmid \infty} F_v$ is the ring of finite adèles over F . If S is a finite subset of finite places of F , we write $\mathbb{A}_{F,f}^S = \prod'_{v \notin S, v \nmid \infty} F_v$. If p is a prime, we also write $\mathbb{A}_{F,f}^p = \mathbb{A}_{F,f}^{|\mathbb{F}_p|}$ and $\mathbb{A}_{F,f}^{S,p} = \mathbb{A}_{F,f}^{S \cup |\mathbb{F}_p|}$.

Given a variety X over a finite field \mathbb{F}_q , where q is a power of a prime p , we denote by $|X|$ its set of closed points. For each $x \in |X|$, let \bar{x} be an algebraic geometric point of X localised at x , and let $\text{Frob}_x \in \Gamma_{k(x)} = \text{Gal}(k(\bar{x})/k(x))$ be the geometric Frobenius. Given a lisse $\overline{\mathbb{Q}}_\ell$ -sheaf \mathcal{F} on X , with $\ell \neq p$, we write $\det(1 - \text{Frob}_x t, \mathcal{F})$ for the characteristic polynomial of the image of Frob_x under the monodromy representation of $\Gamma_{k(x)}$ on $\mathcal{F}_{\bar{x}}$.

Chapter 1

Background material

In this chapter we introduce the basic terminology, and recall the preliminary results and some open questions which will be the background of this work. The purpose of this overview is to give a precise statement of the construction of geometric compatible systems of Galois representations attached to regular algebraic, conjugate self-dual, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, for F a CM field, and to present the relevant potential automorphy results, as well as some open questions, in this context. In §1.1 we start by recalling some generalities on ℓ -adic Galois representations, mostly focusing on the local ones. A more precise study of local Galois representations is then accomplished in §1.2 via p -adic Hodge theory. In §1.3 we move to global Galois representations, where we introduce the notion of geometric compatible system, for which we present a list of basic definitions, largely following [6], and a fundamental (motivating) geometric example. In the two subsequent sections we focus on the representation theory side of the picture. In §1.4 we sketch some representation theory of reductive algebraic groups over (characteristic zero) local fields, and introduce the (conjectural) local Langlands correspondence, giving then a precise statement of it in the (known) case of GL_n . In §1.5 we move to the global setting, where we give an overview of the definitions of automorphic forms and representations for reductive algebraic groups over number fields, before focusing on the class of algebraic automorphic representations, to which (in certain cases) geometric compatible systems of Galois representations can be attached. The announced construction of automorphic compatible systems is finally stated in §1.6. After this, in §1.7 we focus on the “inverse” problem of determining when a geometric compatible system is automorphic. In this context, we state the main results of [6], and relate them to the classical conjecture predicting irreducibility of the Galois representations attached to algebraic, cuspidal automorphic representations, of which we present the current status, and the natural approach (so far) to its solution.

1.1 Galois representations

In this section, we introduce the background material on ℓ -adic Galois representations. First of all, in §1.1.1 we give some generalities, fix some of the notation and conventions adopted throughout this work, and present some basic examples. Then, we introduce the category of Weil-Deligne representations in §1.1.2, which in some cases correspond to certain classes of local ℓ -adic Galois representations. For local ℓ -adic Galois representations when $\ell \neq p$ this correspondence is quite elementary, and is presented in §1.1.3.

1.1.1 Generalities on ℓ -adic Galois representations

A *Galois representation* is a continuous representation $\rho : \Gamma \rightarrow \mathrm{GL}(V)$ of the Galois group $\Gamma = \mathrm{Gal}(M'/M)$ of some Galois extension M'/M on a finite dimensional vector space V over a topological field \mathbb{k} .

The theory of Galois representations depends strongly on the topology of the field \mathbb{k} . For instance, if \mathbb{k} is the field \mathbb{C} of complex numbers with the usual topology, or if \mathbb{k} has the discrete topology, e.g. if \mathbb{k} is a finite field, then the image of ρ is finite, and ρ factors through the Galois group $\mathrm{Gal}(M''/M)$ of a finite extension M'' of M inside M' .

On the other hand, there are contexts in which Galois representations have, in general, infinite image. In this work, we mostly focus on Galois representations of this kind.

Let ℓ be a prime. An *ℓ -adic Galois representation* is a Galois representation $\rho : \Gamma \rightarrow \mathrm{GL}(V)$ on a finite dimensional vector space V over an algebraic extension \mathbb{k} of \mathbb{Q}_ℓ . If V has dimension n over \mathbb{k} , by fixing a basis of V we can see ρ as a continuous representation $\rho : \Gamma \rightarrow \mathrm{GL}_n(\mathbb{k})$. As a consequence of the Baire category theorem, see for instance the proof of [41, Corollary 5], we have that the image of ρ is contained in $\mathrm{GL}_n(E)$, for some finite extension E of \mathbb{Q}_ℓ inside \mathbb{k} . It follows that any ℓ -adic representation can be realised on a finite dimensional \mathbb{Q}_ℓ -vector space.

Remark 1.1.1. In some contexts, however, it would be useful to see ℓ -adic representations as representations on finite dimensional $\overline{\mathbb{Q}_\ell}$ -vector spaces, as, for instance, in the case of geometric compatible systems, see §1.3. Also, we sometimes call *λ -adic Galois representation* a Galois representation on a finite dimensional vector space over an algebraic extension of the completion of a number field at a finite place λ .

We first consider the following example.

Example 1.1.2. Let M be a field, and let Γ_M be the absolute Galois group of M . Let ℓ be a prime different from the characteristic of M . The groups $\mu_{\ell^n}(M^s)$ of ℓ^n -th roots of unity in M^s , together with the ℓ -powering maps $\mu_{\ell^{n+1}}(M^s) \rightarrow \mu_{\ell^n}(M^s)$, form an inverse system, whose limit is the Tate module $T_\ell(\mathbb{G}_{m,M}) = \varprojlim \mu_{\ell^n}(M^s)$ of the multiplicative group scheme $\mathbb{G}_{m,M}$ over M . This is a free \mathbb{Z}_ℓ -module of rank 1, and it carries a natural continuous action of Γ_M , induced by the actions on $\mu_{\ell^n}(M^s)$. Then, $V_\ell(\mathbb{G}_{m,M}) = T_\ell(\mathbb{G}_{m,M}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a 1-dimensional \mathbb{Q}_ℓ -vector space, endowed with a continuous action of Γ_M . We call ℓ -adic cyclotomic character of Γ_M the corresponding ℓ -adic representation $\epsilon : \Gamma_M \rightarrow \mathbb{Q}_\ell^\times$. Usually, we also write $T_\ell(\mathbb{G}_{m,M}) = \mathbb{Z}_\ell(1)$ and $V_\ell(\mathbb{G}_{m,M}) = \mathbb{Q}_\ell(1)$. Also, for any ℓ -adic representation $\rho : \Gamma_M \rightarrow \mathrm{GL}(V)$ and $i \in \mathbb{Z}$, we define the i -th Tate twist of V to be $V(i) = V \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)^{\otimes i}$ (where for a negative i we mean tensoring with the $-i$ -th tensor power of the dual $\mathbb{Q}_\ell(-1) = \mathbb{Q}_\ell(1)^\vee$ of $\mathbb{Q}_\ell(1)$).

A relevant class of ℓ -adic representations arises in geometric contexts, as explained in the following example.

Example 1.1.3. Let M be a field, and let Γ_M be the absolute Galois group of M . Let X be a smooth proper variety over M , and let $X_{M^s} = X \times_M M^s$. Let ℓ be a prime different from the characteristic of M . For every $i \in \mathbb{N}$, the étale cohomology groups $H_{\text{ét}}^i(X_{M^s}, \mathbb{Z}/\ell^n \mathbb{Z})$, together with the maps $H_{\text{ét}}^i(X_{M^s}, \mathbb{Z}/\ell^{n+1} \mathbb{Z}) \rightarrow H_{\text{ét}}^i(X_{M^s}, \mathbb{Z}/\ell^n \mathbb{Z})$ induced by the reduction maps $\mathbb{Z}/\ell^{n+1} \mathbb{Z} \rightarrow \mathbb{Z}/\ell^n \mathbb{Z}$, form an inverse system. We let $H_{\text{ét}}^i(X_{M^s}, \mathbb{Z}_\ell) = \varprojlim H_{\text{ét}}^i(X_{M^s}, \mathbb{Z}/\ell^n \mathbb{Z})$, and $H_{\text{ét}}^i(X_{M^s}, \mathbb{Q}_\ell) = H_{\text{ét}}^i(X_{M^s}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, which is a finite dimensional \mathbb{Q}_ℓ -vector space, say of dimension n over \mathbb{Q}_ℓ . The natural action of Γ_M on X_{M^s} induces an action on $H_{\text{ét}}^i(X_{M^s}, \mathbb{Z}/\ell^n \mathbb{Z})$, which in turn induces an action on $H_{\text{ét}}^i(X_{M^s}, \mathbb{Q}_\ell)$. This therefore defines an ℓ -adic representation $\rho_X : \Gamma_M \rightarrow \mathrm{GL}_n(\mathbb{Q}_\ell)$.

Given an ℓ -adic representation $\rho : \Gamma \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$, if E is a finite extension of \mathbb{Q}_ℓ such that the image of ρ is contained in $\mathrm{GL}_n(E)$, it is easy to show that we can conjugate ρ to be valued in $\mathrm{GL}_n(\mathcal{O}_E)$. Then, we can compose ρ with the map $\mathrm{GL}_n(\mathcal{O}_E) \rightarrow \mathrm{GL}_n(k_E)$ induced by the reduction modulo the maximal ideal \mathfrak{m}_E of \mathcal{O}_E , and semisimplify to get a continuous representation $\bar{\rho} : \Gamma \rightarrow \mathrm{GL}_n(k_E)$. We often denote it by $\bar{\rho} : \Gamma \rightarrow \mathrm{GL}_n(\overline{\mathbb{F}_\ell})$, if the field E is not specified.

The two main classes of ℓ -adic representations appearing in this work are the following.

- *Local representations*, i.e. ℓ -adic representations of the absolute Galois group Γ_F of a finite extension F of \mathbb{Q}_p , for a prime p .

- *Global representations*, i.e. ℓ -adic representations of the absolute Galois group Γ_F of a number field F (or of $\Gamma_{F,S}$, for S a finite set of places of F).

If $\rho : \Gamma_F \rightarrow \mathrm{GL}(V)$ is a global ℓ -adic representation of Γ_F , where F is a number field, and v is a finite place of F , then $\rho|_{\Gamma_{F_v}}$ is obviously a local ℓ -adic representation. Given any property of local ℓ -adic representations, we say that ρ satisfies that property at v if $\rho|_{\Gamma_{F_v}}$ satisfies it.

We say that a local ℓ -adic representation $\rho : \Gamma_F \rightarrow \mathrm{GL}(V)$, for F a finite extension of \mathbb{Q}_p , is *unramified* if $\rho(I_F) = \{1\}$. If S is a finite set of places of a number field F , then a global ℓ -adic representation $\rho : \Gamma_F \rightarrow \mathrm{GL}(V)$ is unramified outside S if and only if it factors through $\Gamma_{F,S}$. In this case, if S contains all the places above ℓ , if ρ has coefficients in a finite extension E of \mathbb{Q}_ℓ , and for any $v \notin S$ we let $Q_v(t) = \det(1 - t\rho(\mathrm{Frob}_v)) \in E[t]$ be the characteristic polynomial of $\rho(\mathrm{Frob}_v)$, then given an embedding $\iota : E \rightarrow \mathbb{C}$ we define the *partial L-function* of ρ as

$$L^S(\iota\rho, s) = \prod_{v \notin S} ((\#k(v))^{ns} / \iota Q_v((\#k(v))^s)), \quad s \in \mathbb{C},$$

where the infinite product may or may not converge.

Let us first focus on the case of local ℓ -adic Galois representations of Γ_F , for F a finite extension of \mathbb{Q}_p . These can be often described in terms of representations of the Weil group W_F of F . In this context, the theory varies significantly if $\ell \neq p$ or $\ell = p$. The case $\ell \neq p$ is easier, and is treated in §1.1.3. On the other hand, the case $\ell = p$ is far more complicated, and requires tools from *p-adic Hodge theory*; we treat it in §1.2.2.

Global ℓ -adic Galois representations, and more precisely the so called *geometric* ones, are treated in §1.3, where we study *geometric compatible systems*, in which they naturally arise.

1.1.2 Weil-Deligne representations

Let p be a prime, and let F be a finite extension of \mathbb{Q}_p . Local class field theory provides a canonical isomorphism $\mathrm{Art}_F : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$, normalised to send uniformisers to lifts of Frob_F . It follows that the characters of W_F are completely classified by the representations of $F^\times = \mathrm{GL}_1(F)$. In general, for $n \geq 1$ the local Langlands correspondence allows us to describe the n -dimensional representations of W_F in terms of the representations of $\mathrm{GL}_n(F)$, see §1.4.3.

Let us now focus on the representations of W_F . Let \mathbb{k} be a field of characteristic zero. We give the following definition.

Definition 1.1.4. A *Weil-Deligne representation* of W_F over \mathbb{k} is a pair (r, N) consisting of a representation $r : W_F \rightarrow \mathrm{GL}(V)$ on a finite dimensional \mathbb{k} -vector space V , and an endomorphism $N \in \mathrm{End}(V)$ such that

$$r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_F^{-1}(\sigma)|_F N$$

for all $\sigma \in W_F$. When $N = 0$, we simply call it a *Weil representation* of W_F over \mathbb{k} .

Note that for any Weil-Deligne representation (r, N) of W_F over \mathbb{k} , since I_F is compact and open in W_F , then $r(I_F)$ is necessarily finite. Also, the endomorphism N is necessarily nilpotent. We say that a Weil-Deligne representation (r, N) of W_F over \mathbb{k} is *unramified* if $r(I_F) = \{1\}$ and $N = 0$.

A morphism between two Weil-Deligne representations (r, N) and (r', N') of W_F over \mathbb{k} , where $r : W_F \rightarrow \mathrm{GL}(V)$ and $r' : W_F \rightarrow \mathrm{GL}(V')$, is a W_F -equivariant \mathbb{k} -linear map $f : V \rightarrow V'$ such that $f \circ N = N' \circ f$.

Contrary to the case of Galois representations, in the definition of Weil-Deligne representations there is no reference to a topology on \mathbb{k} . We then have that the category of Weil-Deligne representations of W_F over \mathbb{k} is independent of the topology of \mathbb{k} (this would allow us, for instance, to identify the categories of Weil-Deligne representations over \mathbb{C} and over $\overline{\mathbb{Q}}_\ell$, up to the choice of an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$).

Let (r, N) be Weil-Deligne representations of W_F over \mathbb{k} , and let F' be a finite extension of F . Then, $(r, N)|_{W_{F'}} = (r|_{W_{F'}}, N)$ is a Weil-Deligne representation of $W_{F'}$ over \mathbb{k} .

We say that a Weil-Deligne representation (r, N) of W_F over \mathbb{k} is *Frobenius semisimple* if r is semisimple. Given any (r, N) , we choose a lift ϕ of Frob_F to W_F , and we let $r(\phi) = su = us$, where $s \in \mathrm{GL}(V)$ is semisimple, and $u \in \mathrm{GL}(V)$ is unipotent. For $n \in \mathbb{Z}$ and $\sigma \in I_F$, we set $r^{\mathrm{ss}}(\phi^n \sigma) = s^n r(\sigma)$. Then, $(r, N)^{\mathrm{F-ss}} = (r^{\mathrm{ss}}, N)$ is a Frobenius semisimple Weil-Deligne representation of W_F over \mathbb{k} , which is called the *Frobenius semisimplification* of (r, N) , and is independent of the above choices.

Restriction to the inertia subgroup I_F of W_F defines an equivalence relation on Weil-Deligne representations of W_F over \mathbb{k} . We say that two Weil-Deligne representations (r, N) and (r', N') of W_F over \mathbb{k} are *inertially equivalent*, and we write $(r, N) \sim_{I_F} (r', N')$, if $r|_{I_F} \cong r'|_{I_F}$. We can also define an order relation \prec_{I_F} on each equivalence class for \sim_{I_F} in the following way. Assume that $(r, N) \sim_{I_F} (r', N')$, and fix bases so that r and r' can be seen as representations $r, r' : W_F \rightarrow \mathrm{GL}_n(\mathbb{k})$. We write $(r, N) \prec_{I_F} (r', N')$ if N is in the Zariski closure of the set of matrices $PN'P^{-1}$ in $M_{n \times n}(\mathbb{k})$, where P runs among the matrices in $\mathrm{GL}_n(\mathbb{k})$ such that $Pr'|_{I_F} P^{-1} = r|_{I_F}$.

Let now (r, N) be a Weil-Deligne representation of W_F over \mathbb{C} , where $r : W_F \rightarrow \mathrm{GL}(V)$. Set $V_N = \ker(N)$, and choose a lift ϕ of Frob_F to W_F . Define the *L-function* of (r, N) as

$$L((r, N), s) = \det(1 - (\#k_F)^{-s} \phi|_{V_N^{I_F}})^{-1}, \quad s \in \mathbb{C}.$$

The definition of the ϵ -function of a Weil-Deligne representation over \mathbb{C} is highly more technical, and not explicit, refer to [96, §3.4, 3.6] for a deeper discussion. A result of Deligne, see [37, Théorème 4.1], previously proved by Langlands, shows the existence of a unique function which attaches to any representation $r : W_F \rightarrow \mathrm{GL}(V)$ over \mathbb{C} , any nontrivial additive character $\psi : F \rightarrow \mathbb{C}^\times$, and any additive Haar measure dx on F , a number $\epsilon(r, \psi, dx) \in \mathbb{C}^\times$, satisfying certain properties, see for instance [96, §3.4].

For a Weil-Deligne representation (r, N) of W_F over \mathbb{C} , and a nontrivial additive character $\psi : F \rightarrow \mathbb{C}^\times$, let us fix an additive Haar measure dx_ψ on F which is self-dual with respect to ψ , and let us define the ϵ -function of (r, N) as

$$\epsilon((r, N), \psi, s) = \epsilon(r \otimes |\mathrm{Art}_F^{-1}|_F^s, \psi, dx_\psi), \quad s \in \mathbb{C}.$$

Let $q \in \mathbb{R}_{>0}$. Then, an algebraic number $\alpha \in \overline{\mathbb{Q}}$ is called a *Weil q -number* if for each embedding $\iota : \overline{\mathbb{Q}} \rightarrow \mathbb{C}$ we have $|\iota\alpha|^2 = q$.

Let ℓ be a prime. Given an integer $w \in \mathbb{Z}$, we say that a Weil-Deligne representation (r, N) of W_F over \mathbb{Q}_ℓ is *strictly pure* of weight w if for some (and hence every) lift ϕ of Frob_F to W_F , every eigenvalue α of $r(\phi)$ is a Weil $(\#k_F)^w$ -number. In this case, we must have $N = 0$.

Let (r, N) be a Weil-Deligne representation of W_F over \mathbb{Q}_ℓ , where $r : W_F \rightarrow \mathrm{GL}(V)$. We say that (r, N) is *mixed* if there is a filtration $(F_i^W V)_{i \in \mathbb{Z}}$ of V with $F_i^W V = V$ for $i \gg 0$ and $F_i^W V = \{0\}$ for $i \ll 0$, such that each i -th graded piece $\mathrm{gr}_i^W V$ is strictly pure of weight i . If (r, N) is mixed, then there is a unique choice of filtration $(F_i^W V)_{i \in \mathbb{Z}}$, and we have $N(F_i^W V) \subset F_{i-2}^W V$ for all $i \in \mathbb{Z}$.

Finally, given $w \in \mathbb{Z}$, we say that (r, N) is *pure* of weight w if it is mixed with all weights in $w + \mathbb{Z}$, and if for all $i \in \mathbb{Z}_{>0}$ we have

$$N^i : \mathrm{gr}_{w+i}^W V \xrightarrow{\sim} \mathrm{gr}_{w-i}^W V.$$

We say that a Weil-Deligne representation (r, N) of W_F over \mathbb{Q}_ℓ is *bounded* if for some (and hence all) $\sigma \in W_F \setminus I_F$, all the eigenvalues of $r(\sigma)$ in $\overline{\mathbb{Q}_\ell}$ are ℓ -adic units.

1.1.3 Local Galois representations when $\ell \neq p$

Grothendieck's ℓ -adic monodromy theorem allows us to attach to any representation $\rho : \Gamma_F \rightarrow \mathrm{GL}(V)$ on a finite dimensional \mathbb{Q}_ℓ -vector space, where $\ell \neq p$, a bounded Weil-Deligne representation (r, N) of W_F over \mathbb{Q}_ℓ in a functorial way. Let us recall this result.

Theorem 1.1.5. *If $\rho : \Gamma_F \rightarrow \mathrm{GL}(V)$ is a representation of Γ_F over \mathbb{Q}_ℓ , with $\ell \neq p$, then there exists a finite extension F' of F such that $\rho(I_{F'})$ is unipotent.*

Choose a lift ϕ of Frob_F to W_F , and a compatible system $(\zeta_{\ell^n})_{n=0}^\infty$ of primitive ℓ -power roots of unity $\zeta_{\ell^n} \in \overline{\mathbb{Q}_\ell}$, and let $t : I_F \rightarrow \mathbb{Z}_\ell$ be the continuous surjective homomorphism determined by $\sigma(\varpi_F^{1/\ell^n}) = \zeta_{\ell^n}^{t(\sigma)} \varpi_F^{1/\ell^n}$.

Let then $\rho : \Gamma_F \rightarrow \mathrm{GL}(V)$ be as above, and let F' be a finite extension of F such that $\rho(I_{F'})$ is unipotent. For any $\sigma' \in I_{F'}$, the map

$$N = \frac{\log(\rho(\sigma'))}{t(\sigma')} \in \mathrm{End}(V)$$

is well defined, and independent of σ' . If $\sigma \in W_F$ lies above Frob_F^n for some $n \in \mathbb{Z}$, set

$$r(\sigma) = \rho(\sigma) \exp(-t(\phi^{-n}\sigma)N) \in \mathrm{GL}(V).$$

Then,

$$\mathrm{WD}(\rho) = (r, N)$$

is a bounded Weil-Deligne representation of W_F over \mathbb{Q}_ℓ , which up to isomorphism is independent of the choices of ϕ and $(\zeta_{\ell^n})_{n=0}^\infty$. In fact, WD defines an equivalence between the category of representation of Γ_F over \mathbb{Q}_ℓ and the category of bounded Weil-Deligne representations of W_F over \mathbb{Q}_ℓ . The functor WD commutes with restriction to open subgroups and tensor operations. Furthermore, we have that ρ is unramified if and only if $\mathrm{WD}(\rho)$ is unramified.

When $\ell = p$, it is still possible to attach Weil-Deligne representations of W_F over \mathbb{Q}_p to certain classes of representations of Γ_F over \mathbb{Q}_p . This construction is highly more technical, and is presented at the end of §1.2.2.

1.2 A survey of basic p -adic Hodge theory

In this section, we introduce some basic notions of p -adic Hodge theory. Our exposition consists of three parts. First of all, we define Fontaine's p -adic period rings B_{HT} , B_{dR} , B_{cris} , and B_{st} , and recall their main properties, see §1.2.1. Secondly, we

focus on the classes of Galois representations that those rings define, allowing us to deal with the case of local Galois representations when $\ell = p$, see §1.2.2. Finally, we consider local Galois representations arising from arithmetic geometry, and state p -adic comparison isomorphisms, see §1.2.3. For additional details, we refer mainly to [48] for the first part, and to [50] for the second and third parts.

Let us first of all fix some notation. Let p be a prime. By a *p -adic field* we mean a field K of characteristic 0 that is complete with respect to a fixed discrete valuation that has a perfect residue field k of characteristic p . We let W be the ring of Witt vectors of k , and we let $K_0 = W[1/p]$. Also, we let C be the completion of an algebraic closure of K .

1.2.1 Fontaine's p -adic period rings

The first period ring we define is the Hodge-Tate period ring. Denote by t a generator of $\mathbb{Z}_p(1)$. The *Hodge-Tate period ring* is defined as

$$\begin{aligned} B_{\text{HT}} &= \bigoplus_{i \in \mathbb{Z}} C(i) \\ &= C[t, 1/t]. \end{aligned}$$

We equip B_{HT} with the multiplicative structure given by

$$ct^i \cdot c't^j = cc't^{i+j}, \quad c, c' \in C, i, j \in \mathbb{Z},$$

where we denote the element $c \otimes t^i \in C(i) = C \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$ by ct^i , and with the obvious action of Γ_K . Clearly, the definition of B_{HT} only depends on C , and not on K .

The definition of the de Rham period ring is slightly more technical, and requires some additional work; we sketch it here briefly, and refer to [48, §1] for the details.

First of all, consider the perfect ring

$$\begin{aligned} R = R(\mathcal{O}_C) &= \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p \\ &= \left\{ (x_n)_{n=0}^\infty \in \prod_{n=0}^\infty \mathcal{O}_C/p : x_{n+1}^p = x_n, \forall n \geq 0 \right\}, \end{aligned}$$

and let $\bar{\theta} : R \rightarrow \mathcal{O}_C/p$ be the natural homomorphism given by mapping $(x_n)_{n=0}^\infty$ to x_0 .

Given $x = (x_n)_{n=0}^\infty \in R$, for every n choose a lifting \tilde{x}_n of x_n to \mathcal{O}_C . Then, for every n the sequence $\tilde{x}_{m+n}^{p^m}$ converges for $m \rightarrow \infty$ to an element $x^{(n)} \in \mathcal{O}_C$, which does not depend on the choices of liftings. The map $x \mapsto (x^{(n)})_{n=0}^\infty$ defines a bijection between R and the set of tuples $(x^{(n)})_{n=0}^\infty$ of elements of \mathcal{O}_C such that $(x^{(n+1)})^p = x^{(n)}$ for every n .

Define the W -algebra $A_{\text{inf}} = W(R)$. The choice of a compatible system $(\zeta_{p^n})_{n=0}^{\infty}$ of primitive p -power roots of unity $\zeta_{p^n} \in \mathcal{O}_C$ defines an element $\epsilon \in R$ by the above bijection. Let then

$$\xi = \frac{[\epsilon] - 1}{[\epsilon]^{1/p} - 1} = \sum_{i=0}^{p-1} [\epsilon]^{i/p} \in A_{\text{inf}}.$$

The homomorphism $\bar{\theta}$ induces a surjective homomorphism $\theta : A_{\text{inf}} \rightarrow \mathcal{O}_C$ of W -algebras, whose kernel is the principal ideal generated by ξ . This naturally extends to a surjective homomorphism $\theta : A_{\text{inf}}[1/p] \rightarrow C$ of K_0 -algebras, whose kernel is again the principal ideal generated by ξ .

Define

$$B_{\text{dR}}^+ = \varprojlim_{n \in \mathbb{N}} A_{\text{inf}}[1/p]/(\xi)^n.$$

This is a complete discrete valuation ring with residue field C , and uniformiser ξ , and has a natural action of Γ_K . If we equip each $A_{\text{inf}}[1/p]/(\xi)^n$ with the topology induced by the p -adic topology on $A_{\text{inf}}[1/p]$, the corresponding topology of the inverse limit on B_{dR}^+ turns out to be (strictly) less fine than its topology as a discrete valuation ring. We call this first topology the *natural topology* on B_{dR}^+ . The *de Rham period ring* is then defined as $B_{\text{dR}} = \text{Frac } B_{\text{dR}}^+ = B_{\text{dR}}^+[1/\xi]$. It inherits a natural action of Γ_K from B_{dR}^+ .

Note that, as for B_{HT} , the definition of the ring B_{dR}^+ , and therefore that of B_{dR} , only depends on C , and not on K . Namely, if $K' \subset C$ is a complete discretely-valued subfield, we get the same ring B_{dR}^+ whether we use K or K' .

It can be seen that the natural projection $\theta : B_{\text{dR}}^+ \rightarrow C$ has a section $s : C \rightarrow B_{\text{dR}}^+$, that is a ring homomorphism satisfying $\theta(s(x)) = x$ for all $x \in C$, so that it is possible to see B_{dR}^+ , and therefore B_{dR} , as C -algebras. However, the section s is not unique, and there is no such s which is continuous with respect to the natural topology, and which commutes with the action of Γ_K . Nevertheless, there exists a unique continuous homomorphism $s : \bar{K} \rightarrow B_{\text{dR}}^+$ commuting with the action of Γ_K , and such that $\theta(s(x)) = x$ for all $x \in \bar{K}$. This allows us to see B_{dR}^+ , and therefore B_{dR} , canonically as \bar{K} -algebras, and then $\theta : B_{\text{dR}}^+ \rightarrow C$ as a homomorphism of \bar{K} -algebras. A result of Colmez, see [48, §A1], claims that the image of \bar{K} in B_{dR}^+ is dense with respect to the subspace topology induced by the natural topology of B_{dR}^+ .

The series

$$\log([\epsilon]) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\epsilon] - 1)^n}{n}$$

converges in B_{dR}^+ with respect to the natural topology, and so we set $t = \log([\epsilon]) \in B_{\text{dR}}^+$. This enables us to identify $\mathbb{Z}_p(1)$ to the sub- \mathbb{Z}_p -module of B_{dR}^+ generated by t . The

element t is a uniformiser for B_{dR}^+ , and we have $B_{\text{dR}} = B_{\text{dR}}^+[1/t]$. Also, we can define a canonical filtration on B_{dR} given by

$$F^i B_{\text{dR}} = t^i B_{\text{dR}}^+, \quad i \in \mathbb{Z},$$

with respect to which we have that $\text{gr } B_{\text{dR}} = B_{\text{HT}}$. The choice of a section $s : C \rightarrow B_{\text{dR}}^+$ induces an isomorphism $B_{\text{dR}} \cong C((t))$ of C -algebras. Note that this isomorphism is not continuous, and does not commute with the action of Γ_K .

We define now the crystalline and the semistable period rings and state some of their properties. Since this material is quite technical as well, we just give here a rough exposition, and refer to [48, §2 - §4] for the details.

Let A_{cris}^0 be the A_{inf} -algebra generated by the elements $\frac{\xi^n}{n!}$, for $n \geq 1$, inside $A_{\text{inf}}[1/p]$, let A_{cris} be its p -adic completion, and let $B_{\text{cris}}^+ = A_{\text{cris}}[1/p] \subset B_{\text{dR}}^+$. Consider the element $t = \log([\epsilon]) \in B_{\text{dR}}^+$. We have $t \in A_{\text{cris}}$. The *crystalline period ring* is then defined as $B_{\text{cris}} = B_{\text{cris}}^+[1/t]$. Since we also have $t^{p-1} \in pA_{\text{cris}}$, we see that $B_{\text{cris}} = A_{\text{cris}}^+[1/t]$. There is a natural action of Γ_K on B_{cris}^+ , which induces a natural action on B_{cris} .

Consider the Frobenius map

$$\phi : A_{\text{inf}} \rightarrow A_{\text{inf}}, \quad \phi((x_n)_{n=0}^\infty) = (x_n^p)_{n=0}^\infty.$$

Since $\phi(\xi) = \xi^p + p\eta$ for some $\eta \in A_{\text{inf}}$, we deduce that $\phi(\frac{\xi^n}{n!}) = \frac{p^n}{n!}(\eta + (p-1)\frac{\xi^p}{p!})^n$, for $n \geq 1$, and so $\phi(A_{\text{cris}}^0) \subset A_{\text{cris}}^0$. We can then extend ϕ by continuity to A_{cris} and B_{cris}^+ . Also, we have $\phi(t) = pt$, and so ϕ can be extended to B_{cris} by setting $\phi(1/t) = 1/pt$. The action of ϕ on B_{cris} commutes with the action of Γ_K .

Note that the definitions of B_{cris}^+ and B_{cris} only depend on C and not on K , as for B_{dR}^+ and B_{dR} . Also, we can show that the natural Γ_K -equivariant map $K \otimes_{K_0} B_{\text{cris}} \rightarrow B_{\text{dR}}$ is injective, and so we get a Γ_K -equivariant embedding of B_{cris} in B_{dR} . This allows us to give B_{cris} the subspace filtration

$$F^i B_{\text{cris}} = B_{\text{cris}} \cap F^i B_{\text{dR}}, \quad i \in \mathbb{Z},$$

from B_{dR} . Note that this filtration is not ϕ -stable. More precisely, it can be proved, see [48, Théorème 5.3.7], that for any $i \in \mathbb{Z}$ there is an exact sequence

$$0 \rightarrow \mathbb{Q}_p(i) \rightarrow F^i B_{\text{cris}} \xrightarrow{p^{-i}\phi-1} B_{\text{cris}} \rightarrow 0.$$

Denote by \mathfrak{m}_R the maximal ideal of R . Then, we have $R^\times = \bar{k}^\times \times (1 + \mathfrak{m}_R)$. For any $x \in 1 + \mathfrak{m}_R$, we have $\frac{([x]-1)^n}{n} \in A_{\text{cris}}$ for $n \gg 0$, and the sequence $\frac{([x]-1)^n}{n}$ converges

p -adically to 0. In particular, the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{([x] - 1)^n}{n}$$

converges to an element $\lambda(x) \in B_{\text{cris}}^+$. By setting $\lambda(x) = 0$ for any $x \in \bar{k}^\times$, we then get a homomorphism $\lambda : R^\times \rightarrow B_{\text{cris}}^+$, which is Γ_K -equivariant, and satisfies $\phi(\lambda(x)) = p\lambda(x)$ for any $x \in R^\times$.

Let us now consider the category consisting of the couples (S, λ_S) , where S is a B_{cris}^+ -algebra, and $\lambda_S : (\text{Frac } R)^\times \rightarrow S$ is a homomorphism extending λ , and with the obvious morphisms. This category has an initial object, which is unique up to isomorphism. We let B_{st}^+ be the B_{cris}^+ -algebra so obtained, and we let $\lambda : (\text{Frac } R)^\times \rightarrow B_{\text{st}}^+$ be the corresponding homomorphism. We then define the *semistable period ring* to be $B_{\text{st}} = B_{\text{st}}^+[1/t]$. By the universal property defining B_{st}^+ , the action of Γ_K extends uniquely to B_{st}^+ and to B_{st} .

Note that we have

$$B_{\text{st}}^+ = \text{Sym}((\text{Frac } R)^\times) \otimes_{\text{Sym}(R^\times)} B_{\text{cris}}^+,$$

so that, non-canonically, $B_{\text{st}}^+ \cong B_{\text{cris}}^+[X]$ and $B_{\text{st}} \cong B_{\text{cris}}[X]$, upon choosing $y \in (\text{Frac } R)^\times$ with $y \neq R^\times$, and setting $X = \lambda(y)$.

Since the Frobenius ϕ on B_{cris}^+ satisfies $\phi(\lambda(x)) = p\lambda(x)$ for any $x \in R^\times$, we can canonically extend it to B_{st}^+ via the requirement $\phi(\lambda(x)) = p\lambda(x)$ for any $x \in (\text{Frac } R)^\times$. Also, since $\phi(t) = pt$, we can further extended it to B_{st} . It would clearly commute with the action of Γ_K .

Note that, as for all the rings introduced so far, the rings B_{st}^+ and B_{st} only depend on C , and not on K .

Besides the Frobenius ϕ , the semistable period ring B_{st} admits an additional operator: a monodromy operator N which commutes with the action of Γ_K , and whose interaction with ϕ satisfies $N\phi = p\phi N$. Loosely speaking, the idea is to consider $N = \frac{d}{dX}$ on $B_{\text{st}}^+ \cong B_{\text{cris}}^+[X]$ and extend it to B_{st} . Nevertheless, to give a precise definition, let us choose a valuation v on \bar{K} with values in \mathbb{Q} . Note that we have isomorphisms $(\text{Frac } R)^\times / R^\times \cong C^\times / \mathcal{O}_C^\times \cong \bar{K}^\times / \mathcal{O}_{\bar{K}}^\times \cong \mathbb{Q}$, and the map

$$\kappa : B_{\text{st}} \otimes_{\mathbb{Q}} (\bar{K} / \mathcal{O}_{\bar{K}}^\times) \rightarrow \Omega_{B_{\text{st}}/B_{\text{cris}}}^1, \quad \kappa(b \otimes \tilde{x}) = bd(\lambda(x)),$$

where \tilde{x} denotes the image of $x \in (\text{Frac } R)^\times$ in \bar{K}^\times via the above isomorphisms, is an isomorphism. We can think of v as the choice of an isomorphism $\bar{K}^\times / \mathcal{O}_{\bar{K}}^\times \cong \mathbb{Q}$. Therefore, by extending scalars, v gives an isomorphism $\tilde{v} : B_{\text{st}} \otimes_{\mathbb{Q}} (\bar{K}^\times / \mathcal{O}_{\bar{K}}^\times) \xrightarrow{\sim} B_{\text{st}}$.

Define then the *monodromy operator* N associated to v to be the derivation

$$N = \tilde{v} \circ \kappa^{-1} \circ d : B_{\text{st}} \rightarrow B_{\text{st}}.$$

This is the unique B_{cris} -derivation of B_{st} with values in B_{st} such that $N(\lambda(x)) = v_R(x)$ for any $x \in (\text{Frac } R)^\times$, where we denote by v_R the extension of v to $\text{Frac } R$. Furthermore, we have that N commutes with the action of Γ_K , it satisfies $N\phi = p\phi N$, and we have an exact sequence of B_{cris} -modules

$$0 \rightarrow B_{\text{cris}} \rightarrow B_{\text{st}} \xrightarrow{N} B_{\text{st}} \rightarrow 0.$$

When v is the valuation v_0 normalised by $v_0(p) = 1$ we call the corresponding monodromy operator the *canonical* monodromy operator on B_{st} .

We conclude this survey on period rings by defining a Γ_K -equivariant embedding of B_{st} in B_{dR} . We first need to define a Γ_K -equivariant homomorphism $\iota^+ : B_{\text{st}}^+ \rightarrow B_{\text{dR}}^+$ of B_{cris}^+ -algebras. By the universal property defining B_{st}^+ , it is enough to define a homomorphism $\lambda_{\text{dR}} : (\text{Frac } R)^\times \rightarrow B_{\text{dR}}^+$ extending λ . In order to do so, we choose an extension of the usual p -adic logarithm $\log : \mathcal{O}_{\overline{K}}^\times \rightarrow \overline{K}$ to a homomorphism $\log : \overline{K}^\times \rightarrow \overline{K}$ commuting with the action of Γ_K . For $x \in (\text{Frac } R)^\times$ such that $x^{(0)} \in \overline{K}^\times$, the series

$$\log \left(\frac{[x]}{x^{(0)}} \right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\frac{[x]}{x^{(0)}} - 1 \right)^n}{n}$$

converges in B_{dR}^+ , and we set

$$\lambda_{\text{dR}}(x) = \log \left(\frac{[x]}{x^{(0)}} \right) + \log(x^{(0)}).$$

In general, for any $x \in (\text{Frac } R)^\times$, we write $x = ab$, with $a \in R^\times$ and $b^{(0)} \in \overline{K}^\times$, and we set

$$\lambda_{\text{dR}}(x) = \lambda(a) + \lambda_{\text{dR}}(b).$$

We then get the required ι^+ , which induces a Γ_K -equivariant homomorphism of B_{cris} -algebras $\iota : B_{\text{st}} \rightarrow B_{\text{dR}}$.

By extension of scalars, ι^+ and ι induce Γ_K -equivariant homomorphisms $K \otimes_{K_0} B_{\text{st}}^+ \rightarrow B_{\text{dR}}^+$ and $K \otimes_{K_0} B_{\text{st}} \rightarrow B_{\text{dR}}$, which can be proved to be injective. It follows that there is a Γ_K -equivariant embedding of B_{st} in B_{dR} .

We remark that the embedding of $K \otimes_{K_0} B_{\text{st}}$ in B_{dR} , and therefore that of B_{st} in B_{dR} , depends on the choice of the extension of the p -adic logarithm to \overline{K}^\times . Nevertheless, we can see that the image of $K \otimes_{K_0} B_{\text{st}}$ in B_{dR} is independent on this choice. When the extension of the logarithm is chosen so that $\log(p) = 0$, we call the corresponding embedding the *natural* embedding of B_{st} in B_{dR} .

1.2.2 Local Galois representations when $\ell = p$

Let us now move to the second part of this exposition, where we present a glossary of p -adic Galois representations in terms of Fontaine's period rings. Let K be a p -adic field, and let Γ_K be the absolute Galois group of K . A p -adic representation¹ of Γ_K is a continuous representation $\rho : \Gamma_K \rightarrow \mathrm{GL}(V)$ of Γ_K on a finite dimensional vector space V over \mathbb{Q}_p . Fontaine's period rings allow us to define some remarkable classes of p -adic representations. In order to do so, we first of all need to recall the formalism of admissible representations, for which we refer to [50, §1].

Let Γ be a topological group, and let B be a topological commutative ring equipped with a continuous action of Γ . A finitely generated B -module V is said to be a B -representation of Γ if it is equipped with a semilinear continuous action of Γ , that is such that

$$\begin{aligned} g(x + x') &= g(x) + g(x'), \\ g(bx) &= g(b)g(x), \quad \forall x, x' \in V, \forall b \in B, \forall g \in \Gamma. \end{aligned}$$

Note that a p -adic representation of Γ_K is then just a \mathbb{Q}_p -representation of Γ_K in this sense. A B -representation V of Γ is said to be *free* if it is free as a B -module, *trivial* if it is free and if one of the following equivalent conditions holds:

- (1) there exists a basis of V consisting of elements of V^Γ ;
- (2) there exists an isomorphism $V \cong B^d$ of B -representations of Γ .

Assume that $\mathbb{K} = B^\Gamma$ is a field, and let \mathbb{k} be a closed subfield of \mathbb{K} . If B is a domain, the continuous action of Γ on B extends to a continuous action of Γ on $\mathrm{Frac} B$, in the obvious way. We say that B is (\mathbb{k}, Γ) -regular if the following conditions hold:

- (1) B is a domain.
- (2) $B^\Gamma = (\mathrm{Frac} B)^\Gamma$.
- (3) If $b \in B$, $b \neq 0$, is such that the line $\mathbb{k}b$ is stable by Γ , then it is invertible in B .

¹We consider here a class of representations which is larger than the class of ℓ -adic representations, as in §1.1, with $\ell = p$. In fact, we allow K to be a p -adic field, and not just a finite extension F of \mathbb{Q}_p , since it is useful to include the case of the completion $\widehat{F}^{\mathrm{un}}$ of the maximal unramified extension of a finite extension F of \mathbb{Q}_p , for which we have $\Gamma_{\widehat{F}^{\mathrm{un}}} = I_F$. Relevant properties of the representations of Γ_F can in fact be detected on I_F .

Let V be a \mathbb{k} -representation of Γ . Then, $B \otimes_{\mathbb{k}} V$ is a free B -representation of Γ . We say that V is *B-admissible* if $B \otimes_{\mathbb{k}} V$ is a trivial B -representation of Γ .

For any \mathbb{k} -representation V of Γ , set

$$D_B(V) = (B \otimes_{\mathbb{k}} V)^{\Gamma}.$$

We get a map

$$\alpha_V : B \otimes_{\mathbb{k}} D_B(V) \rightarrow B \otimes_{\mathbb{k}} V, \quad b \otimes x \mapsto bx,$$

which is B -linear and commutes with the action of Γ .

Assume that B is (\mathbb{k}, Γ) -regular, and let V be a \mathbb{k} -representation. Then:

- (1) The map α_V is injective, and $\dim_{\mathbb{k}} D_B(V) \leq \dim_{\mathbb{k}} V$.
- (2) The following conditions are equivalent:
 - (a) V is B -admissible.
 - (b) α_V is an isomorphism.
 - (c) $\dim_{\mathbb{k}} D_B(V) = \dim_{\mathbb{k}} V$.

Denote by $\mathbf{Rep}_{\mathbb{k}}(\Gamma)$ the category of \mathbb{k} -representations of Γ . It is well known that this is a neutral Tannakian category over \mathbb{k} , in the sense of [40]. The map D_B can be seen as an additive functor

$$D_B : \mathbf{Rep}_{\mathbb{k}}(\Gamma) \rightarrow \mathbf{Vect}_{\mathbb{k}}$$

to the category of finite dimensional \mathbb{k} -vector spaces. Denote by $\mathbf{Rep}_{\mathbb{k}}^B(\Gamma)$ the category of \mathbb{k} -representations of Γ which are B -admissible. Then, $\mathbf{Rep}_{\mathbb{k}}^B(\Gamma)$ is a Tannakian subcategory of $\mathbf{Rep}_{\mathbb{k}}(\Gamma)$, and the restriction of D_B to $\mathbf{Rep}_{\mathbb{k}}^B(\Gamma)$ is a fibre functor.

Admissibility with respect to Fontaine's period rings allows us to define some classes of p -adic Galois representations. We refer to [50, §3 - §5] for the details.

Let us first focus on the Hodge-Tate and the de Rham period rings. We have that B_{HT} and B_{dR} are (\mathbb{Q}_p, Γ_K) -regular, and we can give the following definitions.

Definition 1.2.1. We say that a p -adic representation of Γ_K is *Hodge-Tate* (resp. *de Rham*) if it is B_{HT} -admissible (resp. B_{dR} -admissible).

Let us denote by $\mathbf{Rep}_{\mathbb{Q}_p}^{\text{HT}}(\Gamma_K)$ (resp. $\mathbf{Rep}_{\mathbb{Q}_p}^{\text{dR}}(\Gamma_K)$) the category of p -adic representations of Γ_K which are Hodge-Tate (resp. de Rham).

We let \mathbf{Grad}_K be the category of finite dimensional K -vector spaces equipped with a grading indexed by \mathbb{Z} , and we let \mathbf{Fil}_K denote the category of finite dimensional K -vector spaces D equipped with a decreasing filtration $(F^i D)_{i \in \mathbb{Z}}$ which is exhaustive and separated, that is

- (1) $F^i D$ are K -vector subspaces of D ;
- (2) $F^{i+1} D \subset F^i D$ for all $i \in \mathbb{Z}$;
- (3) $\cup_{i \in \mathbb{Z}} F^i D = D$, and $\cap_{i \in \mathbb{Z}} F^i D = \{0\}$.

A morphism between two objects D, D' in \mathbf{Fil}_K is a K -linear map $\eta : D \rightarrow D'$ such that

$$\eta(F^i D) \subset F^i D', \quad \text{for all } i \in \mathbb{Z}.$$

The category \mathbf{Grad}_K is abelian, while \mathbf{Fil}_K is just additive, and there is an obvious additive functor $\text{gr} : \mathbf{Fil}_K \rightarrow \mathbf{Grad}_K$ given by mapping each filtered K -vector space to the associated graded K -vector space.

Let us set $D_{\text{HT}} = D_{B_{\text{HT}}}$ and $D_{\text{dR}} = D_{B_{\text{dR}}}$. For each $\rho : \Gamma_K \rightarrow \text{GL}(V)$, we have an obvious structure of graded K -vector space on $D_{\text{HT}}(V)$, and a structure of filtered K -vector space on $D_{\text{dR}}(V)$ given by

$$F^i D_{\text{dR}}(V) = (F^i B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{\Gamma_K}, \quad i \in \mathbb{Z}.$$

We can then see D_{HT} and D_{dR} as additive functors

$$\begin{aligned} D_{\text{HT}} &: \mathbf{Rep}_{\mathbb{Q}_p}(\Gamma_K) \rightarrow \mathbf{Grad}_K, \\ D_{\text{dR}} &: \mathbf{Rep}_{\mathbb{Q}_p}(\Gamma_K) \rightarrow \mathbf{Fil}_K. \end{aligned}$$

The identity $\text{gr } B_{\text{dR}} = B_{\text{HT}}$ induces a natural inclusion of $\text{gr } D_{\text{dR}}(V)$ in $D_{\text{HT}}(V)$ for each $\rho : \Gamma_K \rightarrow \text{GL}(V)$. The following result holds.

Theorem 1.2.2.

- (1) *If $\rho : \Gamma_K \rightarrow \text{GL}(V)$ is a de Rham representation, then it is Hodge-Tate. Furthermore, the isomorphism $B_{\text{dR}} \otimes_K D_{\text{dR}}(V) \rightarrow B_{\text{dR}} \otimes_{\mathbb{Q}_p} V$ is strictly compatible with the filtrations, and $\text{gr } D_{\text{dR}}(V) = D_{\text{HT}}(V)$.*
- (2) *We have inclusions*

$$\mathbf{Rep}_{\mathbb{Q}_p}^{\text{dR}}(\Gamma_K) \subset \mathbf{Rep}_{\mathbb{Q}_p}^{\text{HT}}(\Gamma_K) \subset \mathbf{Rep}_{\mathbb{Q}_p}(\Gamma_K),$$

where each of these categories is a Tannakian subcategory of the following. Also, the restriction of D_{HT} (resp. D_{dR}) to $\mathbf{Rep}_{\mathbb{Q}_p}^{\text{HT}}(\Gamma_K)$ (resp. $\mathbf{Rep}_{\mathbb{Q}_p}^{\text{dR}}(\Gamma_K)$) is an exact, faithful, tensor functor.

Note that there are explicit examples of Hodge-Tate representations which are not de Rham.

Let $\rho : \Gamma_K \rightarrow \mathrm{GL}(V)$ be a Hodge-Tate representation. We define the multiset $\mathrm{HT}(\rho)$ of *Hodge-Tate weights* of ρ to be the multiset of $\dim_{\mathbb{Q}_p} V$ integers containing i with multiplicity

$$h_i = \dim_{\mathbb{Q}_p} (V \otimes_{\mathbb{Q}_p} C(i))^{\Gamma_K}.$$

Remark 1.2.3. With the convention of [6], if F is a finite extension of \mathbb{Q}_p , and $\rho : \Gamma_F \rightarrow \mathrm{GL}(V)$ is a representation on a finite dimensional $\overline{\mathbb{Q}_p}$ -vector space which is Hodge-Tate, for any embedding $\tau : F \rightarrow \overline{\mathbb{Q}_p}$ we define the multiset $\mathrm{HT}_\tau(\rho)$ of τ -*Hodge-Tate weights* of ρ to be the multiset of $\dim_{\overline{\mathbb{Q}_p}} V$ integers containing i with multiplicity $h_i = \dim_{\overline{\mathbb{Q}_p}} (V \otimes_{\tau, F} C(i))^{\Gamma_F}$. When F is a number field, and $\rho : \Gamma_F \rightarrow \mathrm{GL}(V)$ is a representation on a finite dimensional $\overline{\mathbb{Q}_p}$ -vector space, for any embedding $\tau : F \rightarrow \overline{\mathbb{Q}_p}$, if ρ is Hodge-Tate at the place $v(\tau)$ of F induced by τ , we let $\mathrm{HT}_\tau(\rho) = \mathrm{HT}(\rho|_{\Gamma_{F_{v(\tau)}}})$. In this case, we say that ρ is *regular* if for any embedding $\tau : F \rightarrow \overline{\mathbb{Q}_p}$, every element of the multiset $\mathrm{HT}_\tau(\rho)$ has multiplicity 1.

Before moving to crystalline and semistable representations, let us remark that, since B_{HT} and B_{dR} are \overline{K} -algebras, if $\rho : \Gamma_K \rightarrow \mathrm{GL}(V)$ is a p -adic representation such that $\rho|_{\Gamma_{K'}}$ is Hodge-Tate (resp. de Rham) for some finite extension K' of K , then ρ itself is Hodge-Tate (resp. de Rham). This is not true in general for crystalline and semistable representations.

Let us now consider the crystalline and the semistable period rings. The rings B_{cris} and B_{st} are (\mathbb{Q}_p, Γ_K) -regular, and we can give the following definition.

Definition 1.2.4. We say that a p -adic representation of Γ_K is *crystalline* (resp. *semistable*) if it is B_{cris} -admissible (resp. B_{st} -admissible).

We denote by $\mathbf{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ (resp. $\mathbf{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(\Gamma_K)$) the category of p -adic representations of Γ_K which are crystalline (resp. semistable). It can be proved that a p -adic representation $\rho : \Gamma_K \rightarrow \mathrm{GL}(V)$ is crystalline (resp. semistable) if and only if $\rho|_{I_K}$ is crystalline (resp. semistable).

Let us now introduce categories of K_0 -vector spaces corresponding to crystalline and semistable representations. A (ϕ, N) -*module* over K_0 is a finite dimensional K_0 -vector space D equipped with two maps

$$\phi, N : D \rightarrow D$$

satisfying the following properties:

- (1) ϕ is injective, and semilinear with respect to the absolute Frobenius of K_0 ;
- (2) N is K_0 -linear;
- (3) $N\phi = p\phi N$.

Note that the action of N on D is necessarily nilpotent. A morphism of (ϕ, N) -modules is just a K_0 -linear map between them, which commutes with ϕ and N .

A (ϕ, N) -module D is said to be *filtered* if the K -vector space $D_K = K \otimes_{K_0} D$ is equipped with a decreasing filtration $(F^i D_K)_{i \in \mathbb{Z}}$, which is exhausted and separated. A morphism between two filtered (ϕ, N) -modules D and D' is a morphism of (ϕ, N) -modules $\eta : D \rightarrow D'$ such that the induced K -linear map $\eta_K : D_K \rightarrow D'_K$ is a morphism of filtered K -vector space. We denote by $\mathbf{MF}_K(\phi, N)$ the category of filtered (ϕ, N) -modules. Note that $\mathbf{MF}_K(\phi, N)$ is an additive category, but not abelian. We call (filtered) ϕ -module a (filtered) $(\phi, 0)$ -module, and we write $\mathbf{MF}_K(\phi) = \mathbf{MF}_K(\phi, 0)$.

Let us set $D_{\text{cris}} = D_{B_{\text{cris}}}$ and $D_{\text{st}} = D_{B_{\text{st}}}$. For each $\rho : \Gamma_K \rightarrow \text{GL}(V)$, the action of the Frobenius ϕ on B_{cris} gives $D_{\text{cris}}(V)$ a structure of ϕ -module, and the action of ϕ and the monodromy operator N on B_{st} gives $D_{\text{st}}(V)$ a structure of (ϕ, N) -module. Also, the inclusion of B_{cris} in B_{st} identifies $D_{\text{cris}}(V)$ as the kernel of N in $D_{\text{st}}(V)$. The injective homomorphism $K \otimes_{K_0} B_{\text{st}} \rightarrow B_{\text{dR}}$ allows us to see $D_{\text{st}}(V)_K = K \otimes_{K_0} D_{\text{st}}(V)$ as a subspace of $D_{\text{dR}}(V)$, which can then be equipped with the induced filtration. It follows that we can see D_{cris} and D_{st} as additive functors

$$\begin{aligned} D_{\text{cris}} &: \mathbf{Rep}_{\mathbb{Q}_p}(\Gamma_K) \rightarrow \mathbf{MF}_K(\phi), \\ D_{\text{st}} &: \mathbf{Rep}_{\mathbb{Q}_p}(\Gamma_K) \rightarrow \mathbf{MF}_K(\phi, N). \end{aligned}$$

Let $\rho : \Gamma_K \rightarrow \text{GL}(V)$ be a p -adic representation. It can be proved that if ρ is semistable, then $D_{\text{st}}(V)_K = D_{\text{dR}}(V)$, and ρ is de Rham. Analogously, it can be proved that if ρ is crystalline, then it is semistable, and we have $D_{\text{cris}}(V) = D_{\text{st}}(V)$, with $N = 0$, and $D_{\text{cris}}(V)_K = D_{\text{dR}}(V)$.

We say that a filtered (ϕ, N) -module D is *admissible* if there exist a semistable representation $\rho : \Gamma_K \rightarrow \text{GL}(V)$ and an isomorphism between D and $D_{\text{st}}(V)$. We denote by $\mathbf{MF}_K^{\text{ad}}(\phi, N)$ the full subcategory of $\mathbf{MF}_K(\phi, N)$ whose objects are the (ϕ, N) -modules which are admissible. We write $\mathbf{MF}_K^{\text{ad}}(\phi) = \mathbf{MF}_K^{\text{ad}}(\phi, 0)$. In conclusion, the following result holds.

Theorem 1.2.5. *We have inclusions*

$$\mathbf{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K) \subset \mathbf{Rep}_{\mathbb{Q}_p}^{\text{st}}(\Gamma_K) \subset \mathbf{Rep}_{\mathbb{Q}_p}^{\text{dR}}(\Gamma_K),$$

where each of these categories is a Tannakian subcategory of the following. Also, the restriction of D_{st} to $\mathbf{Rep}_{\mathbb{Q}_p}^{\text{st}}(\Gamma_K)$ is a fully faithful, tensor functor, which induces an equivalence of categories between $\mathbf{Rep}_{\mathbb{Q}_p}^{\text{st}}(\Gamma_K)$ and $\mathbf{MF}_K^{\text{ad}}(\phi, N)$.

Notice that for $N = 0$, we have that the restriction of D_{cris} to $\mathbf{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ is a fully faithful, tensor functor, which induces an equivalence of categories between $\mathbf{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ and $\mathbf{MF}_K^{\text{ad}}(\phi)$.

Let us now see a p -adic analogue of the ℓ -adic monodromy theorem of Grothendieck presented in §1.1. We first need an additional definition.

Definition 1.2.6. We say that a p -adic representation $\rho : \Gamma_K \rightarrow \text{GL}(V)$ is *potentially semistable* if there exists a finite extension K' of K such that $\rho|_{\Gamma_{K'}}$ is semistable.

Clearly, any potentially semistable representation is de Rham. The converse result, conjectured by Fontaine in [50, §6.2], and known as the *p -adic monodromy conjecture*, holds true as well, and has been first proved by Berger, see [13, Corollaire 5.22].

Theorem 1.2.7. *Any p -adic de Rham representation of Γ_K is potentially semistable.*

By the above theorem, given a finite extension F of \mathbb{Q}_p , we can attach to any ℓ -adic (where $\ell = p$) de Rham representation $\rho : \Gamma_F \rightarrow \text{GL}(V)$ of Γ_F a Weil-Deligne representation of W_F in a functorial way. This construction is due to Fontaine, and we refer to [49, §2.3] for the details.

Let $\rho : \Gamma_F \rightarrow \text{GL}(V)$ be de Rham. Then, ρ is potentially semistable, that is there exists a finite extension F' of F such that $\rho|_{\Gamma_{F'}}$ is semistable. Let k' be the residue field of the discrete valuation on F' , let W' be the ring of Witt vectors of k' , and let $F'_0 = W'[1/p]$. Since V is B_{st} -admissible as a representation of $\Gamma_{F'}$, we have that $(B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^{\Gamma_{F'}}$ is a finite dimensional F'_0 -vector space of dimension $\dim_{\mathbb{Q}_p} V$.

Choose an extension F' such that $\rho|_{\Gamma_{F'}}$ is semistable, and set

$$W = (B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^{\Gamma_{F'}}.$$

If $\sigma \in W_F$ lies above Frob_F^n for some $n \in \mathbb{Z}$, set

$$r(\sigma) = (\sigma \phi^{n[k:\mathbb{F}_p]}) \otimes \rho(\sigma) \in \text{GL}(W).$$

Then,

$$\text{WD}(\rho) = (r, N \otimes 1)$$

is a Weil-Deligne representation of W_F over \mathbb{Q}_p , which up to isomorphism is independent of the choice of F' . In fact, we can see that WD is a functor from the category of

ℓ -adic (where $\ell = p$) de Rham representations of Γ_F to the category of Weil-Deligne representations of W_F over \mathbb{Q}_p . Also in this case, the functor WD commutes with restriction to open subgroups and tensor operations. However, contrary to the case $\ell \neq p$, one cannot recover a p -adic de Rham representation from the corresponding Weil-Deligne representation, since this does not take into account the filtration. Furthermore, we have that ρ is crystalline if and only if $\text{WD}(\rho)$ is unramified.

1.2.3 Applications to p -adic comparison isomorphisms

We conclude this section by recalling p -adic comparison theorems with étale cohomology. Let us fix some notation. Let K be a p -adic field, and let X be a proper smooth variety over K . For every $i \in \mathbb{N}$, the étale cohomology group $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p)$ is a finite dimensional \mathbb{Q}_p -vector space, equipped with a natural action of Γ_K , and so an object of $\mathbf{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, see Example 1.1.3. We see now some p -adic Hodge theoretic properties of these p -adic representations.

For every $i \in \mathbb{N}$, the de Rham cohomology group $H_{\text{dR}}^i(X/K)$ is the hypercohomology group $\mathbb{H}^i(\Omega_{X/K}^\bullet)$ of the de Rham complex

$$\Omega_{X/K}^\bullet : \mathcal{O}_{X/K} \rightarrow \Omega_{X/K}^1 \rightarrow \Omega_{X/K}^2 \rightarrow \cdots$$

Each $H_{\text{dR}}^i(X/K)$ is a finite dimensional K -vector space, equipped with the Hodge filtration, and so it is an object of the category \mathbf{Fil}_K . The following result, conjectured by Fontaine in [47, §A.6], has been proved by Faltings in [44, §VIII].

Theorem 1.2.8. *For every $i \in \mathbb{N}$, there exists an isomorphism*

$$B_{\text{dR}} \otimes_{\mathbb{Q}_p} H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p) \cong B_{\text{dR}} \otimes_K H_{\text{dR}}^i(X/K)$$

of B_{dR} -vector spaces, which is compatible with the filtration and the action of Γ_K . In other words, the p -adic representation $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p)$ is de Rham, and there exists an isomorphism

$$D_{\text{dR}}(H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p)) \cong H_{\text{dR}}^i(X/K)$$

of filtered K -vector spaces.

Let us now assume that X has good reduction, that is there exists a smooth proper scheme \mathcal{X} over \mathcal{O}_K whose generic fibre $\mathcal{X} \times_{\mathcal{O}_K} K$ is isomorphic to X . For every $i \in \mathbb{N}$, we let $H_{\text{cris}}^i(\mathcal{X}_k/W)$ be the crystalline cohomology group of the special fibre $\mathcal{X}_k = \mathcal{X} \times_{\mathcal{O}_K} k$. Each $H_{\text{cris}}^i(\mathcal{X}_k/W)$ is a finitely generated W -module. Let $H_{\text{cris}}^i(\mathcal{X}_k) = K_0 \otimes_W H_{\text{cris}}^i(\mathcal{X}_k/W)$. By the comparison isomorphism [14, Théorème

V.2.3.2] we can give $H_{\text{cris}}^i(\mathcal{X}_k)$ a structure of filtered ϕ -module, which is independent of the choice of the smooth model \mathcal{X} of X over \mathcal{O}_K , see [54, §B]. We then denote by $H_{\text{cris}}^i(X)$ the corresponding element of $\mathbf{MF}_K(\phi)$. The following result has also been conjectured by Fontaine in [47, §A.11], and proved by Faltings in [44, §V].

Theorem 1.2.9. *For every $i \in \mathbb{N}$, there exists an isomorphism*

$$B_{\text{cris}} \otimes_{\mathbb{Q}_p} H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p) \cong B_{\text{cris}} \otimes_{K_0} H_{\text{cris}}^i(X)$$

of B_{cris} -modules, which is compatible with the action of Γ_K , the action of ϕ , and (after extending scalars to B_{dR}) with the filtration. In other words, the p -adic representation $H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, and there exists an isomorphism

$$D_{\text{cris}}(H_{\text{ét}}^i(X_{\overline{K}}, \mathbb{Q}_p)) \cong H_{\text{cris}}^i(X)$$

of filtered ϕ -modules.

Note that, by extending scalars to B_{dR} , this implies Theorem 1.2.8 for varieties with good reduction.

1.3 Geometric compatible systems of Galois representations

Let F be a number field, let Γ_F denote the absolute Galois group of F . Let ℓ be a prime. A continuous representation $\rho : \Gamma_F \rightarrow \text{GL}(V)$ of Γ_F over \mathbb{Q}_ℓ is said to be *geometric* if it is unramified outside a finite set of places of F , and if it is de Rham at each place v of F above ℓ .

In practice (and conjecturally always, by the conjecture of Fontaine-Mazur [51]), geometric Galois representations arise as part of *geometric compatible systems*. Among the various possible (and conjecturally equivalent) definitions of compatible systems, we follow the one given in [6, §5.1], with some small variations in the terminology.

Let again F be a number field, and let S be a finite set of places of F . Let E be a number field, and let Λ be a set of places of E . Let $n \geq 1$ be an integer. We recall the following definition.

Definition 1.3.1. A *geometric² compatible system* of rank n representations of Γ_F defined over E , unramified outside S , and indexed by Λ is a family $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of

²We add “geometric” to the terminology of [6, §5.1] in order to distinguish these compatible systems from the “abstract” compatible systems of Definition 2.1.1.

continuous semisimple representations

$$\rho_\lambda : \Gamma_F \rightarrow \mathrm{GL}_n(\overline{E}_\lambda)$$

such that:

- (1) If $v \notin S$ is a finite place of F , then for all λ not dividing the residue characteristic of v , the representation ρ_λ is unramified at v , and the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_v)$ has coefficients in E and is independent of λ .
- (2) Each representation ρ_λ is de Rham at all places above the residue characteristic of λ , and in fact crystalline at any place $v \notin S$ above the residue characteristic of λ .
- (3) For each embedding $\tau : F \rightarrow \overline{E}$ the τ -Hodge-Tate weights of ρ_λ are independent of λ .

From now on, we will assume all the geometric compatible systems to be indexed by set of places of the form $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$, for \mathcal{L} a set of rational primes of Dirichlet density 1.

Given a geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$, for $v \notin S$ and $\lambda \in \Lambda$ not dividing the residue characteristic of v we let $Q_v(t) = \det(1 - t\rho_\lambda(\mathrm{Frob}_v)) \in E[t]$ be the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_v)$, and for an embedding $\tau : F \rightarrow \overline{E}$ and $\lambda \in \Lambda$ we denote by H_τ the multiset of τ -Hodge-Tate weights of ρ_λ .

A geometric compatible system \mathcal{R} is called *regular* if for each embedding $\tau : F \rightarrow \overline{E}$ every element of the multiset H_τ has multiplicity 1. It is called *extremely regular* if it is regular, and for some embedding $\tau : F \rightarrow \overline{E}$ the set H_τ has the property that if H and H' are subsets of H_τ of the same cardinality, and if $\sum_{h \in H} h = \sum_{h' \in H'} h'$, then $H = H'$.

We say that a geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ defined over a number field E has *coefficients* in a finite extension E' of E if for any $\lambda \in \Lambda$ there exists a place λ' of E' above λ such that the image of ρ_λ is contained in $\mathrm{GL}_n(E'_{\lambda'})$.

Given two sets \mathcal{L} and \mathcal{L}' of rational primes of Dirichlet density 1, we say that two geometric compatible systems $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ and $\mathcal{R}' = \{\rho'_\lambda\}_{\lambda \in \Lambda'}$ defined over a number field E and indexed by $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$ and $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E|_\ell$ are *isomorphic* if there exists a subset $\mathcal{L}'' \subset \mathcal{L} \cap \mathcal{L}'$ of Dirichlet density 1 such that $\rho_\lambda \cong \rho'_\lambda$ for all $\ell \in \mathcal{L}''$ and $\lambda \mid \ell$. In this case, we write $\mathcal{R} \cong_{(\mathcal{L}'')} \mathcal{R}'$, or for simplicity just $\mathcal{R} \cong \mathcal{R}'$. When \mathcal{R} and \mathcal{R}' have coefficients in E , we say that they are *isomorphic over E* if the isomorphisms $\rho_\lambda \cong \rho'_\lambda$ are defined over E .

A geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ defined over a number field E and indexed by $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$, for \mathcal{L} a set of rational primes of Dirichlet density 1, is said to be *irreducible* if there exists a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1 such that ρ_λ is irreducible for all $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$.

Given a geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of representations of Γ_F , for any finite extension F' of F the family $\mathcal{R}|_{\Gamma_{F'}} = \{\rho_\lambda|_{\Gamma_{F'}}\}_{\lambda \in \Lambda}$ is a geometric compatible system of representations of $\Gamma_{F'}$.

A geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ is said to be *strictly compatible* if for each finite place v of F there is a Weil-Deligne representation $\text{WD}_v(\mathcal{R})$ of W_{F_v} over \overline{E} such that for each $\lambda \in \Lambda$ not dividing the residue characteristic of v and every E -linear embedding $\varsigma : \overline{E} \rightarrow \overline{E}_\lambda$ there is an isomorphism

$$\varsigma \text{WD}_v(\mathcal{R}) \cong \text{WD}(\rho_\lambda|_{\Gamma_{F_v}})^{\text{F-ss}}.$$

Given an integer $w \in \mathbb{Z}$, we say that a geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ is *pure* of weight w if:

- for each $v \notin S$, each root α of $Q_v(t)$ in \overline{E} is a Weil $(\#k(v))^w$ -number;
- for each embedding $\tau : F \rightarrow \overline{E}$ and each complex conjugation c in $\text{Gal}(\overline{E}/\mathbb{Q})$ we have that

$$H_{c \circ \tau} = \{w - h : h \in H_\tau\}.$$

We say that \mathcal{R} is *strictly pure* of weight w if:

- \mathcal{R} is strictly compatible, and for each place v of F the Weil-Deligne representation $\text{WD}_v(\mathcal{R})$ is pure of weight w .
- for each embedding $\tau : F \rightarrow \overline{E}$ and each complex conjugation c in $\text{Gal}(\overline{E}/\mathbb{Q})$ we have that

$$H_{c \circ \tau} = \{w - h : h \in H_\tau\}.$$

Given a geometric compatible system \mathcal{R} defined over E and unramified outside S , and an embedding $\iota : E \rightarrow \mathbb{C}$, we define the *partial L-function* of \mathcal{R} as

$$L^S(\iota\mathcal{R}, s) = \prod_{v \notin S} ((\#k(v))^{ns} / \iota Q_v((\#k(v))^s)), \quad s \in \mathbb{C}.$$

Again, this infinite product may or may not converge. If \mathcal{R} is pure of weight w , then $L^S(\iota\mathcal{R}, s)$ converges to an analytic function in $\text{Re } s > 1 + w/2$.

Assume now that F is a CM field³, with maximal totally real subfield F^+ , and let c be a generator of $\text{Gal}(F/F^+)$. A *polarised* λ -adic representation of Γ_F is a pair (ρ, μ) , where $\rho : \Gamma_F \rightarrow \text{GL}_n(\overline{E}_\lambda)$ is a λ -adic representation, and $\mu : \Gamma_{F^+} \rightarrow \overline{E}_\lambda^\times$ is a continuous character, such that for some (and hence all) $v \mid \infty$ of F^+ , there exists a non-degenerate pairing $\langle \cdot, \cdot \rangle_v$ on \overline{E}_λ^n such that

$$\langle x, y \rangle_v = -\mu(c_v) \langle y, x \rangle_v$$

and

$$\langle \rho(\sigma)x, \rho(c_v \sigma c_v)y \rangle_v = \mu(\sigma) \langle x, y \rangle_v$$

for all $x, y \in \overline{E}_\lambda$ and $\sigma \in \Gamma_F$. We say that (ρ, μ) is *totally odd* if $\mu(c_v) = -1$ for all $v \mid \infty$.

A *polarised* (resp. *totally odd*, *polarised*) geometric compatible system of representations of Γ_F is a pair $(\mathcal{R}, \mathcal{M})$, where $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ is a geometric compatible system of representations of Γ_F , and $\mathcal{M} = \{\mu_\lambda\}_{\lambda \in \Lambda}$ is a geometric compatible system of characters of Γ_{F^+} , such that for all $\lambda \in \Lambda$ the pair $(\rho_\lambda, \mu_\lambda)$ is a polarised (resp. totally odd, polarised) λ -adic representation of Γ_F .

We say that a λ -adic representation $\rho : \Gamma_F \rightarrow \text{GL}_n(\overline{E}_\lambda)$ is *conjugate self-dual* (resp. *totally odd*, *conjugate self-dual*) if $(\rho, \epsilon^{1-n} \delta_{F/F^+}^n)$, where ϵ denotes the cyclotomic character of Γ_{F^+} and δ_{F/F^+} denotes the quadratic character of Γ_{F^+} attached to the extension F/F^+ , is a polarised (resp. totally odd, polarised) λ -adic representation of Γ_F . We say that a geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of representations of Γ_F is *conjugate self-dual* (resp. *totally odd*, *conjugate self-dual*) if each ρ_λ is a conjugate self-dual (resp. totally odd, conjugate self-dual) λ -adic representation of Γ_F .

Geometric compatible systems naturally arise from geometric contexts, as shown in the following example.

Example 1.3.2. Let F be a number field, let X be a smooth projective variety over F , and let $i \in \mathbb{N}$. For every prime ℓ , the ℓ -adic étale cohomology group $V_\ell = H_{\text{ét}}^i(X_{\overline{F}}, \mathbb{Q}_\ell)$ is a finite-dimensional vector space over \mathbb{Q}_ℓ , with a natural continuous action of Γ_F . As a consequence of the comparison isomorphism [5, Exposé XI, Théorème 4.4] we get that each V_ℓ has dimension n over \mathbb{Q}_ℓ , where n is the dimension of $H^i(X(\mathbb{C}), \mathbb{Q})$ over \mathbb{Q} , for a fixed embedding of F into \mathbb{C} . Let V_ℓ^{ss} denote the semisimplification of V_ℓ . Then, the action of Γ_F on V_ℓ^{ss} defines a family $\mathcal{R}_X = \{\rho_{X,\ell}\}_{\ell \in |\mathbb{Q}|}$ of semisimple

³By a *CM field* we mean an imaginary CM field. Most of the results presented in this work can be adapted also to the totally real case. For the sake of simplicity, however, we only restrict to the CM case.

continuous rank n representations $\rho_{X,\ell} : \Gamma_F \rightarrow \mathrm{GL}_n(\mathbb{Q}_\ell)$. Let S be a finite set of places of F such that X has good reduction outside S , and containing all the ramified places for the extension F/\mathbb{Q} . We have:

- for each $v \notin S$, with $v \nmid \ell$, the representation $\rho_{X,\ell}$ is unramified at v , and the characteristic polynomial of $\rho_{X,\ell}(\mathrm{Frob}_v)$ is independent of ℓ , and its roots in $\overline{\mathbb{Q}}$ are Weil $(\#k(v))^i$ -numbers, by results of Deligne, see [38, Théorème 1.6];
- for each $v \mid \ell$, the representation $\rho_{X,\ell}|_{\Gamma_{F_v}}$ is de Rham, with Hodge-Tate weights given by the jumps of the Hodge filtration on the i -th de Rham cohomology group of X_{F_v} , and crystalline if $v \notin S$, by the results of Faltings presented in §1.2.3.

The family \mathcal{R}_X is then a geometric compatible system, pure of weight i . Conjecturally, it is strictly compatible, and there is no need to semisimplify the representations. Note that both these properties are known when X is an abelian variety.

In §1.6, we will see how geometric compatible systems also arise in the context of automorphic representations.

1.4 The local Langlands correspondence

In this section we present the (conjectural) local Langlands correspondence for reductive algebraic groups over (characteristic zero) local fields. We start by recalling the representation theory background in §1.4.1, and we give an introduction to the expected correspondence in §1.4.2. In the case of GL_n , the local Langlands correspondence has been fully proved. We present the main result in this context in §1.4.3, which will allow us to make precise the local-global compatibility statements in the main theorem on the construction of automorphic compatible systems in §1.6.

1.4.1 Representation theory over characteristic zero local fields

Let F be a characteristic zero local field, and let G be a connected reductive algebraic group over F . The group $G(F)$ of F -rational points of G is a locally compact, totally disconnected, Hausdorff topological group. Fix a (left invariant) Haar measure “meas” on $G(F)$.

Assume that F is Archimedean, and fix a maximal compact subgroup K of $G(F)$. Let also \mathfrak{g} be the Lie algebra of $G(F)$, viewed as a real Lie group, and let \mathfrak{k} be the Lie

algebra of K . Given a representation σ of $G(F)$ on a Hilbert space V such that $\sigma|_K$ is unitary, for each equivalence class of irreducible representation γ of K we write

$$V^\gamma = \{f \in V : \langle \sigma|_K f \rangle \cong \gamma\}$$

for the γ -isotypic subspace of V . A vector $f \in V$ is said to have K -type γ if $f \in V^\gamma$.

A (\mathfrak{g}, K) -module is a \mathbb{C} -vector space V , together with a representation π of \mathfrak{g} and K , satisfying the following conditions.

- (1) The space V is a countable algebraic direct sum $V = \bigoplus_i V_i$, where each V_i is a finite dimensional K -invariant vector space.
- (2) For $X \in \mathfrak{k}$ and $f \in V$, we have

$$\begin{aligned} \pi(X)f &= Xf = \frac{d}{dt} \exp(tX)f|_{t=0} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (\pi(\exp(tX))f - f). \end{aligned}$$

- (3) For $k \in K$, $X \in \mathfrak{g}$, and $f \in V$, we have $\pi(k)\pi(X)\pi(k^{-1})f = \pi(\text{Ad}(k)X)f$.

We say that a (\mathfrak{g}, K) -module V is *admissible* if we can choose the V_i 's to have distinct K -types.

Define the *Hecke algebra* of $(G(F), K)$ to be the convolution algebra $\mathcal{H}_{\mathbb{C}}(G(F), K)$ of distributions on $G(F)$ with support in K , which are K -finite under both left and right translation. For simplicity of notation, set $\mathcal{H} = \mathcal{H}_{\mathbb{C}}(G(F), K)$.

Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ (resp. $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$) be the complexified Lie algebra of $G(F)$ (resp. K), and let $U(\mathfrak{g}_{\mathbb{C}})$ (resp. $U(\mathfrak{k}_{\mathbb{C}})$) be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{k}_{\mathbb{C}}$). The algebra \mathcal{H} contains the algebra $\mathcal{H}_{\mathbb{C}}(K)$ of K -finite measures on K , viewed as distributions on $G(F)$. It can be proved that the map $U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{k}_{\mathbb{C}})} \mathcal{H}_{\mathbb{C}}(K) \rightarrow \mathcal{H}$ given by $(X, \mu) \mapsto X * \mu$ is a vector space isomorphism. Let dK denote a Haar measure on K , normalised to give unit volume to K . For any finite dimensional irreducible representation γ of K , we let d_γ be its rank, and χ_γ be its character. The measure

$$e_\gamma = \frac{1}{d_\gamma \text{meas}(K)} \chi_\gamma dK$$

on K is an idempotent of the algebra $\mathcal{H}_{\mathbb{C}}(K)$. By the above isomorphism, the idempotents e_γ of $\mathcal{H}_{\mathbb{C}}(K)$ define idempotents of the Hecke algebra \mathcal{H} , which becomes then an idempotent algebra.

A \mathcal{H} -module V is said to be *nondegenerate* if $\mathcal{H}V = V$, *admissible* if it is nondegenerate, and if $e_\gamma V$ is finite dimensional for all finite dimensional irreducible representations γ of K .

Given a (\mathfrak{g}, K) -module V , endowed with a representation π of \mathfrak{g} and K , for $X \in U(\mathfrak{g}_{\mathbb{C}})$, $\mu \in \mathcal{H}(K)$, and $f \in V$, the formula $(X \otimes \mu)f = \pi(X)\pi(\mu)f$ defines on V a structure of nondegenerate \mathcal{H} -module. We have that V is admissible as a (\mathfrak{g}, K) -module if and only if it is admissible as a \mathcal{H} -module. It is not difficult to show that this construction gives an isomorphism between the category of (\mathfrak{g}, K) -modules and the category of nondegenerate \mathcal{H} -modules.

Assume now that F is non Archimedean. We say that a representation π of $G(F)$ on a \mathbb{C} -vector space V is *smooth* if the stabiliser of every vector in V is open; equivalently, if $V = \cup_K V^K$, where K runs over the compact open subgroups of $G(F)$. We say that π is *admissible* if it is smooth, and for all compact open subgroups K of $G(F)$ the subspace V^K of K -fixed vectors of V is finite dimensional. Any irreducible smooth representation of $G(F)$ is admissible.

Let Z be the centre of G , and let π be an irreducible admissible representation of $G(F)$ on a \mathbb{C} -vector space V . By Schur's lemma, $Z(F)$ acts on V via a character χ_π , that is $\pi(z) = \chi_\pi(z)\text{Id}$ for all $z \in Z(F)$. We call χ_π the *central character* of π .

Given a commutative ring R with unit, define the *Hecke algebra* of $G(F)$ over R to be the convolution algebra $\mathcal{H}_R(G(F))$ of locally constant and compactly supported R -valued functions on $G(F)$. This is an associative algebra, which has no unit element unless G is discrete, and is commutative if and only if $G(F)$ is commutative.

For any compact open subgroup K of $G(F)$, the function

$$e_K = \frac{1}{\text{meas}(K)} \mathbf{1}_K,$$

where $\mathbf{1}_K$ denotes the characteristic function of K , is an idempotent of $\mathcal{H}_R(G(F))$. The algebra $\mathcal{H}_R(G(F), K) = e_K \mathcal{H}_R(G(F)) e_K$ can be identified with the space of locally constant and compactly supported R -valued functions on $G(F)$ which are K -bi-invariant. For simplicity of notation, set $\mathcal{H} = \mathcal{H}_{\mathbb{C}}(G(F))$, and set $\mathcal{H}(G(F), K) = \mathcal{H}_{\mathbb{C}}(G(F), K)$.

A \mathcal{H} -module V is said to be *nondegenerate* if $\mathcal{H}V = V$. If π is a smooth representation of $G(F)$ on a \mathbb{C} -vector space V , then V has a natural structure of \mathcal{H} -module, and for every compact open subgroup K of $G(F)$, the space $e_K V = V^K$ has a natural structure of $\mathcal{H}(G(F), K)$ -module. Viceversa, any nondegenerate \mathcal{H} -module is associated to a unique smooth representation of $G(F)$. In other words, there is an

isomorphism between the category of smooth representations of $G(F)$ and the category of nondegenerate \mathcal{H} -modules.

We say that a \mathcal{H} -module V is *admissible* if it is nondegenerate and if for all compact open subgroups K of $G(F)$, the space $\mathbb{1}_K V$ is finite dimensional. Clearly, a smooth representation of $G(F)$ is admissible if and only if the corresponding \mathcal{H} -module is admissible.

Recall that G is said to be unramified if G is quasi-split, and split over an unramified extension of F . If G is unramified, and K is a hyperspecial compact open subgroup of $G(F)$, in the sense of [102, §3], then $(G(F), K)$ is a Gelfand pair, and the algebra $\mathcal{H}(G(F), K)$, which in this case is known as the *spherical Hecke algebra* of $(G(F), K)$, is commutative. This result can be proved directly via Gelfand's lemma, see [56, §2] for a sketch of the argument, or alternatively via the *Satake isomorphism*, whose study is postponed to §1.5.2. When $\mathcal{H}(G(F), K)$ is commutative, a result of Flath, see [45, §1], implies that if π is an irreducible admissible representation of $G(F)$ on a \mathbb{C} -vector space V , then the space V^K of K -fixed vectors in V (which is then finite dimensional) has dimension at most 1.

Let π be a smooth representation π of $G(F)$ on a \mathbb{C} -vector space V . For any compact open subgroup K of $G(F)$, let $V^\vee(K)$ be the space of linear functionals v^\vee on V such that $\langle v^\vee, \pi(e_K)v \rangle = \langle v^\vee, v \rangle$ for every $v \in V$. The space $V^\vee = \cup_K V^\vee(K)$ is called the *smooth dual* of V . There is a smooth representation π^\vee of $G(F)$ on V^\vee , called the *contragredient* to π , characterised by the relation $\langle \pi^\vee(g)v^\vee, v \rangle = \langle v^\vee, \pi(g^{-1})v \rangle$ for $g \in G(F)$, $v \in V$, and $v^\vee \in V^\vee$. We have that π is admissible if and only if π^\vee is admissible.

1.4.2 The L -group and the conjectural correspondence

Let F be a local field. For the purposes of this work, let us assume that F has characteristic zero⁴. Let G be a connected reductive algebraic group over F . First of all, we review here briefly the construction of the *L -group* of G , for which we refer to [17, §I]. Also refer to [95] for the background on reductive groups.

If we let $T \subset B$ be a maximal torus, contained in a Borel subgroup of $G_{\overline{F}}$, we can associate the based root datum $\Psi_0(G, B, T) = (X^*(T), \Delta^*(B), X_*(T), \Delta_*(B))$ consisting of the character and cocharacter groups of T , and the roots and coroots

⁴A conjectural local Langlands correspondence can be formulated also in the positive characteristic setting, where it has been proved for GL_n by Laumon, Rapoport, and Stuhler in [76]. Quite recently, an arrow of the correspondence for a general reductive group has been established by Genestier and V. Lafforgue in [53].

which are simple and positive with respect to the ordering defined by B . Let Z be the centre of G . If $T' \subset B'$ is a different choice of a maximal torus, contained in a Borel subgroup of $G_{\overline{F}}$, and $\Psi_0(G, B', T')$ is the corresponding based root datum, then there is an element of $(G/Z)(\overline{F})$ mapping B' to B and T' to T , and all these elements induce the same isomorphism of based root data $\Psi_0(G, B, T) \xrightarrow{\sim} \Psi_0(G, B', T')$. Following Kottwitz, we define then $\Psi_0(G) = (X^*, \Delta^*, X_*, \Delta_*)$ to be the projective limit of the $\Psi_0(G, B, T)$ via the above isomorphisms. There is a natural group homomorphism $\mu_G : \Gamma_F \rightarrow \text{Aut}(\Psi_0(G))$, which factors through $\text{Gal}(F'/F)$ if F' is a Galois extension of F inside \overline{F} that splits G .

Let \widehat{G} be a connected reductive algebraic group over \mathbb{C} , the complex *dual* of G , equipped with a given isomorphism of root data $\Psi_0(\widehat{G}) \xrightarrow{\sim} \Psi_0(G)^\vee$, where $\Psi_0(G)^\vee = (X_*, \Delta_*, X^*, \Delta^*)$ is the dual root datum to $\Psi_0(G)$. There is a canonical group isomorphism $\text{Aut}(\Psi_0(G)) \cong \text{Aut}(\Psi_0(G)^\vee)$ given by mapping an automorphism of X^* to its inverse, and hence a canonical action of Γ_F on $\Psi_0(G)^\vee$, which then induces an action of Γ_F on $\Psi_0(\widehat{G})$.

Fix a maximal torus $\widehat{T} \subset \widehat{B}$, contained in a Borel subgroup of \widehat{G} , and a splitting (also called a pinning in the literature, see [95, §2.14] for the details) of the exact sequence

$$1 \rightarrow \text{Int}(\widehat{G}) \rightarrow \text{Aut}(\widehat{G}) \rightarrow \text{Aut}(\Psi_0(\widehat{G})) \rightarrow 1.$$

Then, this data induces an action of Γ_F on \widehat{G} . Define then the L -group of G to be the resulting semidirect product

$${}^L G = \widehat{G} \rtimes \Gamma_F,$$

with identity component \widehat{G} and component group Γ_F . By construction, there is a natural projection ${}^L G \rightarrow \Gamma_F$. Note that if G splits over F , then the action of Γ_F on \widehat{G} is trivial, and so ${}^L G = \widehat{G} \times \Gamma_F$.

Define the *Langlands group* of F to be

$$L_F = \begin{cases} W_F, & \text{if } F \text{ is Archimedean,} \\ W_F \times \text{SU}(2), & \text{if } F \text{ is non Archimedean.} \end{cases}$$

Recall that W_F is equipped with a continuous homomorphism $W_F \rightarrow \Gamma_F$ with dense image, so that we have a projection $L_F \rightarrow \Gamma_F$.

We say that a continuous homomorphism $\phi : L_F \rightarrow {}^L G(\mathbb{C})$ is *admissible* if it commutes with the projections to Γ_F . Two admissible homomorphisms $\phi, \phi' : L_F \rightarrow {}^L G(\mathbb{C})$ are said to be *equivalent* if they differ by conjugation by an element of $\widehat{G}(\mathbb{C})$.

The conjectural *local Langlands correspondence* for G over F predicts the existence of a partition of the set of isomorphism classes of irreducible admissible (\mathfrak{g}, K) -modules when F is Archimedean, and of irreducible admissible representations of $G(F)$ when F is non Archimedean, into disjoint finite subset Π_ϕ , called *L-packets*, parametrised by equivalence classes of admissible continuous homomorphisms $\phi : L_F \rightarrow {}^L G(\mathbb{C})$, called *L-parameters*. In addition, this conjectural parametrisation should satisfy some desiderata, i.e. the matching of certain (conjectural) L -functions and ϵ -functions, see [72].

In the Archimedean case, such a correspondence has been proved by Langlands in [73]. We omit here the details, for which we refer to Langlands' paper.

1.4.3 The case of GL_n

In the case of GL_n , the local Langlands correspondence has been proved also over non Archimedean local fields.

Fix a finite extension F of \mathbb{Q}_p , for some prime p , and an integer $n \geq 1$. For GL_n over F , the L -packets are singletons, i.e. equivalence classes of single irreducible admissible representations of $GL_n(F)$ over \mathbb{C} . We then let $\mathcal{A}_F^0(n)$ denote the set of isomorphism classes of irreducible admissible representations of $GL_n(F)$ over \mathbb{C} .

Let us remark that since GL_n is split over F , and $\widehat{GL}_n = GL_n$, then ${}^L GL_n = GL_n \times \Gamma_F$, so that in this case L -parameters corresponds to (equivalence classes of) continuous representations $\phi : W_F \times SU(2) \rightarrow GL_n(\mathbb{C})$. The group $W_F \times SU(2)$ is often called the *Weil-Deligne group* of F , and its representations over \mathbb{C} are in bijection with Weil-Deligne representations of W_F over \mathbb{C} , see for instance [96, §4.1]. We let $\mathcal{G}_F^0(n)$ denote the set of isomorphism classes of irreducible rank n Weil-Deligne representations of W_F over \mathbb{C} .

Given irreducible admissible representations π and π' respectively of $GL_n(F)$ and $GL_{n'}(F)$ over \mathbb{C} , and an additive character $\psi : F \rightarrow \mathbb{C}^\times$, it is possible to define an L -function $L(\pi \times \pi', s)$ and an ϵ -function $\epsilon(\pi \times \pi', \psi, s)$, for $s \in \mathbb{C}$. We refer, for instance, to [63] and [64] for these constructions.

The following result has been proved independently by Harris and Taylor, see [60, Theorem VII.2.20], and by Henniart, see [62, Théorème 4.2]. Later, an alternative proof has been given by Scholze, see [87, Theorem 1.2].

Theorem 1.4.1. *For any finite extension F of \mathbb{Q}_p there exist a collection of bijections*

$$\text{rec}_F : \mathcal{A}_F^0(n) \xrightarrow{\sim} \mathcal{G}_F^0(n)$$

for every integer $n \geq 1$ satisfying the following properties.

(1) If $\chi \in \mathcal{A}_F^0(1)$, then $\text{rec}_F(\chi) = \chi \circ \text{Art}_F^{-1}$.

(2) If $\pi \in \mathcal{A}_F^0(n)$, and $\pi' \in \mathcal{A}_F^0(n')$, then

$$\begin{aligned} L(\pi \times \pi', s) &= L(\text{rec}_F(\pi) \otimes \text{rec}_F(\pi'), s), \\ \epsilon(\pi \times \pi', \psi, s) &= \epsilon(\text{rec}_F(\pi) \otimes \text{rec}_F(\pi'), \psi, s). \end{aligned}$$

(3) If $\pi \in \mathcal{A}_F^0(n)$, and $\chi \in \mathcal{A}_F^0(1)$, then $\text{rec}_F(\pi \otimes (\chi \circ \det)) = \text{rec}_F(\pi) \otimes \text{rec}_F(\chi)$.

(4) If $\pi \in \mathcal{A}_F^0(n)$ has central character χ , then $\det \text{rec}_F(\pi) = \text{rec}_F(\chi)$.

(5) If $\pi \in \mathcal{A}_F^0(n)$, then $\text{rec}_F(\pi^\vee) = \text{rec}_F(\pi)^\vee$.

A collection of bijections rec_F satisfying properties (1)-(5) of the above theorem is called a *local Langlands correspondence* for GL_n over F . It is unique by results of Henniart, see [61, Théorème 4.1].

1.5 Algebraic automorphic representations

In this section, we introduce the class of algebraic automorphic representations to which geometric Galois representations are conjecturally (and in some cases actually) attached. We start by recalling the general definitions of automorphic forms and representations for reductive algebraic groups over number fields in §1.5.1. Afterwards, we quickly move again to the local setting, where we present the Satake isomorphism for spherical Hecke algebras in §1.5.2, and some natural algebraicity conditions over Archimedean local fields in §1.5.3. These local preliminaries will allow us to define algebraicity conditions for automorphic representations in §1.5.4.

1.5.1 Generalities on automorphic forms and representations

Let F be a number field, and let G be a connected reductive algebraic group over F . Fix maximal compact subgroups K_v of $G(F_v)$ for each infinite place v of F .

Let $G(F_\infty) = \prod_{v|\infty} G(F_v)$ and $G(\mathbb{A}_{F,f}) = \prod'_{v|\infty} G(F_v)$, where the second product is restricted with respect to a choice of compact open subgroups K_v of $G(F_v)$ for each finite place v of F , and let $G(\mathbb{A}_F) = G(F_\infty) \times G(\mathbb{A}_{F,f})$ be the group of adelic points of G over F . We review here some representation theory of $G(F_\infty)$ and $G(\mathbb{A}_{F,f})$, roughly following the exposition of [45]. Throughout this exposition, we fix Haar measures, which we all denote (if not otherwise specified) by “meas”, on each of the groups $G(F_v)$, $G(F_\infty)$, and $G(\mathbb{A}_{F,f})$.

Let us first of all consider the Archimedean case. For each infinite place v of F , let \mathfrak{g}_v be the Lie algebra of $G(F_v)$, viewed as a real Lie group. The product $K_\infty = \prod_{v|\infty} K_v$ is a maximal compact subgroup of $G(F_\infty)$. We let $\mathfrak{g} = \bigoplus_{v|\infty} \mathfrak{g}_v$ be the Lie algebra of $G(F_\infty)$, viewed as a real Lie group. It is then possible to define a notion of (\mathfrak{g}, K_∞) -module (resp. *admissible* (\mathfrak{g}, K_∞) -module) exactly as in §1.4.1.

For each infinite place v of F , let $\mathcal{H}_v = \mathcal{H}_{\mathbb{C}}(G(F_v), K_v)$ be the Hecke algebra of $(G(F_v), K_v)$. Define the *Hecke algebra* of $(G(F_\infty), K_\infty)$ to be the convolution algebra $\mathcal{H}_\infty = \mathcal{H}_{\mathbb{C}}(G(F_\infty), K_\infty)$ of distributions on $G(F_\infty)$ with support in K_∞ , which are K_∞ -finite under both left and right translation. We have a decomposition $\mathcal{H}_\infty = \bigotimes_{v|\infty} \mathcal{H}_v$, and we have that \mathcal{H}_∞ is naturally an idempotent algebra, with idempotents of the form

$$e_\gamma = \frac{1}{d_\gamma \text{meas}(K_\infty)} \chi_\gamma dK_\infty,$$

for γ finite dimensional irreducible representations of K_∞ , of rank d_γ and character χ_γ , and where dK_∞ denotes a Haar measure on K_∞ , normalised to give unit volume to K_∞ .

A \mathcal{H}_∞ -module V is said to be *nondegenerate* if $\mathcal{H}_\infty V = V$, *admissible* if $e_\gamma V$ is finite dimensional for all finite dimensional irreducible representations γ of K_∞ . There is an isomorphism between the category of (\mathfrak{g}, K_∞) -modules and the category of nondegenerate \mathcal{H}_∞ -modules. Furthermore, a (\mathfrak{g}, K_∞) -module is admissible if and only if the corresponding \mathcal{H}_∞ -module is admissible.

Let us now move to the non Archimedean setting. We say that a $G(\mathbb{A}_{F,f})$ -module V *smooth* if the stabiliser of every vector in V is open; equivalently, if $V = \bigcup_{K_f} V^{K_f}$, where K_f runs over the compact open subgroups of $G(\mathbb{A}_{F,f})$. We say that V is *admissible* if it is smooth, and for all compact open subgroups K_f of $G(\mathbb{A}_{F,f})$ the space V^{K_f} of K_f -fixed vectors in V is finite dimensional.

Given a commutative ring R with unit, for any finite place v of F , let $\mathcal{H}_R(G(F_v))$ be the Hecke algebra of $G(F_v)$ over R , and for any compact open subgroup K_v of $G(F_v)$, let e_{K_v} be the corresponding idempotent of $\mathcal{H}_R(G(F_v))$. Define the *Hecke algebra* of $G(\mathbb{A}_{F,f})$ over R to be the convolution algebra $\mathcal{H}_R(G(\mathbb{A}_{F,f}))$ of locally constant R -valued functions on $G(\mathbb{A}_{F,f})$ with compact support. We have a decomposition $\mathcal{H}_R(G(\mathbb{A}_{F,f})) = \bigotimes'_{v|\infty} \mathcal{H}_R(G(F_v))$, where the tensor product is restricted with respect to the idempotents e_{K_v} . The algebra $\mathcal{H}_R(G(\mathbb{A}_{F,f}))$ is then an idempotent algebra. For simplicity of notation, set $\mathcal{H}_v = \mathcal{H}_{\mathbb{C}}(G(F_v))$ for any v , and $\mathcal{H}_f = \mathcal{H}_{\mathbb{C}}(G(\mathbb{A}_{F,f}))$.

A \mathcal{H}_f -module V is said to be *nondegenerate* if $\mathcal{H}_f V = V$, *admissible* if it is nondegenerate and if for all compact open subgroups K_f of $G(\mathbb{A}_{F,f})$ the space $\mathbb{1}_{K_f} V$, where $\mathbb{1}_{K_f}$ denotes the characteristic function of K_f , is finite dimensional. There is an

isomorphism between the category of smooth $G(\mathbb{A}_{F,f})$ -modules and the category of nondegenerate \mathcal{H}_f -modules. Furthermore, a $G(\mathbb{A}_{F,f})$ -module is admissible if and only if the corresponding \mathcal{H}_f -module is admissible.

An *admissible $G(\mathbb{A}_F)$ -module* is a vector space V which is both an admissible (\mathfrak{g}, K_∞) -module, and an admissible $G(\mathbb{A}_{F,f})$ -module, and such that the action of $G(\mathbb{A}_{F,f})$ commutes with the action of \mathfrak{g} and K_∞ .

Let us conclude this abstract representation theory overview with the statement of Flath's tensor product decomposition theorem. For each finite place v of F , fix compact open subgroups K_v of $G(F_v)$. There exists a finite set S of places of F , containing all the infinite places, such that G extends to a reductive group scheme \mathcal{G} over the ring $\mathcal{O}_{F,S}$ of S -integers of F . For each $v \notin S$, the compact open subgroup $\mathcal{G}(\mathcal{O}_{F_v})$ of $G(F_v)$ is hyperspecial by [102, §3.9.1]. At these places, we can then choose $K_v = \mathcal{G}(\mathcal{O}_{F_v})$. Recall that in this case the algebra $\mathcal{H}(G(F_v), K_v) = \mathcal{H}_{\mathbb{C}}(G(F_v), K_v)$ is commutative, and hence for any irreducible admissible $G(F_v)$ -module V_v the space $V_v^{K_v}$ of K_v -fixed vectors in V_v has dimension at most 1. The following decomposition theorem has been proved by Flath in [45, §3].

Theorem 1.5.1. *Let V be an irreducible admissible $G(\mathbb{A}_F)$ -module. Then, for each infinite place v of F there exists an irreducible admissible (\mathfrak{g}, K_v) -module V_v , and for each finite place v of F there exists an irreducible admissible $G(F_v)$ -module V_v such that, for almost all finite v , the space $V_v^{K_v}$ contains a nonzero vector f_v^0 , such that*

$$V \cong \otimes'_v V_v,$$

where the tensor product is restricted with respect to the vectors f_v^0 .

We can now define automorphic forms and representations of $G(\mathbb{A}_F)$, following roughly the exposition of [19]. Let us start by recalling the notion of classical automorphic form on $G(F_\infty)$.

A *norm* $\|\cdot\|$ on $G(F_\infty)$ is a function on $G(F_\infty)$ of the form $\|x\| = (\mathrm{Tr}(\sigma(x)^* \sigma(x)))^{1/2}$, where σ is a representation of $G(F_\infty)$ on a finite dimensional Hilbert space V , which has finite kernel, and such that $\sigma|_{K_\infty}$ is unitary. (Here $*$ denotes the adjoint with respect to the inner product on V). We say that a complex valued function $f : G(F_\infty) \rightarrow \mathbb{C}$ is *slowly increasing* if there exists a norm $\|\cdot\|$ on $G(F_\infty)$, a constant $C > 0$, and an integer $r > 0$ such that

$$|f(x)| \leq C \|x\|^r, \quad \forall x \in G(F_\infty).$$

Two subgroups Γ and Γ' of the same group are said to be *commensurable* if $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ' . Fix a faithful representation $G \rightarrow \mathrm{GL}_{n,F}$, and let \mathcal{G} be the schematic closure of G in $\mathrm{GL}_{n,\mathcal{O}_F}$. We say that a subgroup Γ of $G(F)$ is *arithmetic* if Γ and $\mathcal{G}(\mathcal{O}_F)$ are commensurable. It is possible to show that the notion of arithmeticity does not depend on the choice of the faithful representation $G \rightarrow \mathrm{GL}_{n,F}$.

Let us fix an arithmetic subgroup Γ of $G(F)$, and let $Z(\mathfrak{g}_{\mathbb{C}})$ denote the centre of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We give the following definition.

Definition 1.5.2. A \mathcal{C}^{∞} complex valued function

$$f : G(F_{\infty}) \rightarrow \mathbb{C}$$

is said to be an *automorphic form* on $G(F_{\infty})$ with respect to Γ if it satisfies the following conditions.

- (1) It is left Γ -invariant.
- (2) There is an idempotent $\xi_{\infty} \in \mathcal{H}_{\infty}$ such that $f * \xi_{\infty} = f$.
- (3) There is an ideal J of $Z(\mathfrak{g}_{\mathbb{C}})$ of finite codimension which annihilates f .
- (4) It is slowly increasing.

In this case, we sometimes say that f is of *type* $(\Gamma, \xi_{\infty}, J, K_{\infty})$

We denote by $\mathcal{A}(\Gamma, \xi_{\infty}, J, K_{\infty})$ the space of automorphic forms of type $(\Gamma, \xi_{\infty}, J, K_{\infty})$ on $G(F_{\infty})$, and we let $\mathcal{A}(\Gamma, J, K_{\infty}) = \cup_{\xi} \mathcal{A}(\Gamma, \xi_{\infty}, J, K_{\infty})$, which is naturally a $(\mathfrak{g}, K_{\infty})$ -module. A result of Harish-Chandra gives that $\mathcal{A}(\Gamma, J, K_{\infty})$ is admissible as a $(\mathfrak{g}, K_{\infty})$ -module, and hence each $\mathcal{A}(\Gamma, \xi_{\infty}, J, K_{\infty})$ is finite dimensional, see [58, Theorem 1].

Let us now move to the adelic setting. A complex valued function $f : G(\mathbb{A}_F) \rightarrow \mathbb{C}$ is said to be *smooth* if it is continuous, and viewed as a function of two arguments $x \in G(F_{\infty})$ and $y \in G(\mathbb{A}_{F,f})$ it is \mathcal{C}^{∞} in x (resp. locally constant in y) for fixed y (resp. fixed x).

Define the *Hecke algebra* of $G(\mathbb{A}_F)$ to be $\mathcal{H} = \mathcal{H}_{\infty} \otimes \mathcal{H}_f$. We say that an element $\xi \in \mathcal{H}$ is *simple* if it is of the form $\xi = \xi_{\infty} \otimes \xi_f$, where $\xi_{\infty} \in \mathcal{H}_{\infty}$, and $\xi_f \in \mathcal{H}_f$. We give the following definition.

Definition 1.5.3. A smooth complex valued function

$$f : G(\mathbb{A}_F) \rightarrow \mathbb{C}$$

is said to be an *automorphic form* on $G(\mathbb{A}_F)$ if it satisfies the following conditions.

- (1) It is left $G(F)$ -invariant.
- (2) There is a simple element $\xi \in \mathcal{H}$ such that $f * \xi = f$.
- (3) There is an ideal J of $Z(\mathfrak{g}_{\mathbb{C}})$ of finite codimension which annihilates f .
- (4) For each $y \in G(\mathbb{A}_{F,f})$ the function $x \mapsto f(xy)$ on $G(F_{\infty})$ is slowly increasing.

In this case, we sometimes say that f is of *type* (ξ, J, K_{∞}) .

We denote by $\mathcal{A}(\xi, J, K_{\infty})$ the space of automorphic forms of type (ξ, J, K_{∞}) on $G(\mathbb{A}_F)$. Given a compact open subgroup K_f of $G(\mathbb{A}_{F,f})$, we define the idempotent

$$e_{K_f} = \frac{1}{\text{meas}(K_f)} \mathbf{1}_{K_f}$$

of \mathcal{H}_f , where, as above, $\mathbf{1}_{K_f}$ denotes the characteristic function of K_f . Note that there exists a compact open subgroup K_f of $G(\mathbb{A}_{F,f})$ such that $\xi_f * e_{K_f} = \xi_f$. In this case, we see that $\mathcal{A}(\xi, J, K_{\infty}) \subset \mathcal{A}(\xi_{\infty} * e_{K_f}, J, K_{\infty})$. Without loss of generality, let us then assume that $\xi = \xi_{\infty} \otimes e_{K_f}$ for some K_f . The double quotient $G(F) \backslash G(\mathbb{A}) / G(F_{\infty}) K_f$ is finite, see [16], and so there is a finite set T of double coset representatives, which we can assume to be contained in $G(\mathbb{A}_{F,f})$. For $t \in T$, we let

$$\Gamma_t = G(F) \cap (G(F_{\infty}) \times t \cdot K_f \cdot t^{-1}),$$

which is an arithmetic subgroup of $G(F)$. Given a complex valued function f on $G(\mathbb{A}_F)$, and $t \in T$, we define a function f_t on $G(F_{\infty})$ by $f_t(x) = f(tx)$, for $x \in G(F_{\infty})$. It is immediate to check that f is left $G(F)$ -invariant if and only if f_t is left Γ_t -invariant for every $t \in T$. The map $f \mapsto (f_t)_{t \in T}$ induces then an isomorphism

$$\mathcal{A}(\xi, J, K_{\infty}) \xrightarrow{\sim} \bigoplus_{t \in T} \mathcal{A}(\Gamma_t, \xi_{\infty}, J, K_{\infty}),$$

which immediately implies that $\mathcal{A}(\xi, J, K_{\infty})$ is finite dimensional.

Let N be the unipotent radical of any proper parabolic subgroup P of G defined over F . A continuous (resp. measurable) complex valued function $f : G(\mathbb{A}_F) \rightarrow \mathbb{C}$ is said to be *cuspidal* if

$$\int_{N(F) \backslash N(\mathbb{A}_F)} f(nx) dn = 0$$

for all (resp. almost all) $x \in G(\mathbb{A}_F)$.

An automorphic form f on $G(\mathbb{A}_F)$ is said to be a *cuspidal form* if it is cuspidal. We denote by $\mathcal{A}^0(\xi, J, K_{\infty})$ the space of cuspidal forms of type (ξ, J, K_{∞}) on $G(\mathbb{A}_F)$.

The global Hecke algebra \mathcal{H} naturally acts by right convolution on the spaces $\mathcal{A}(\xi, J, K_{\infty})$ of automorphic forms on $G(\mathbb{A}_F)$. We can finally give the following definition.

Definition 1.5.4. An *automorphic representation* (resp. *cuspidal automorphic representation*) of $G(\mathbb{A}_F)$ is an irreducible \mathcal{H} -module which is isomorphic to a subquotient of a representation of \mathcal{H} on a space of automorphic forms (resp. cusp forms) on $G(\mathbb{A}_F)$.

Let π be an automorphic representation of $G(\mathbb{A}_F)$. Then, π decomposes as a tensor product $\pi \cong \pi_\infty \otimes \pi_f$, where π_∞ is an irreducible \mathcal{H}_∞ -module, and π_f is an irreducible \mathcal{H}_f -module, each of which is admissible by [19, Proposition 4.5]. It follows that π is naturally an irreducible admissible $G(\mathbb{A}_F)$ -module, and so by Theorem 1.5.1 it has a decomposition

$$\pi \cong \otimes'_v \pi_v.$$

For any finite set S of places of F , we let $\pi^S = \otimes'_{v \notin S} \pi_v$, and we define the *contragredient* of π to be $\pi^\vee = \otimes'_v \pi_v^\vee$.

Let Z be the greatest torus, split over F , of the centre of F , and let π be an automorphic representation of $G(\mathbb{A}_F)$ on a \mathbb{C} -vector space V . Then, by Schur's lemma, the group $Z(\mathbb{A}_F)$ acts on V via a character χ_π , that is $\pi(z) = \chi_\pi(z)\text{Id}$ for all $z \in Z(\mathbb{A}_F)$. We call χ_π the *central character* of π . Note that since automorphic forms are left $G(F)$ -invariant, then χ_π factors through the quotient $Z(F)\backslash Z(\mathbb{A}_F)$.

Let χ be a complex character of $Z(F)\backslash Z(\mathbb{A}_F)$. Define the space

$$\begin{aligned} & L^2(G(F)\backslash G(\mathbb{A}_F))_\chi \\ &= \{f \in L^2(G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)) : f(zx) = \chi(z)f(x), \forall z \in Z(\mathbb{A}_F), \forall x \in G(\mathbb{A}_F)\}. \end{aligned}$$

The group $G(\mathbb{A}_F)$ acts on $L^2(G(F)\backslash G(\mathbb{A}_F))_\chi$ by right translation. A function $f \in L^2(G(F)\backslash G(\mathbb{A}_F))_\chi$ is said to be *K-finite*, for some maximal compact subgroup K of $G(\mathbb{A}_F)$, if its right translates by elements of K span a finite dimensional vector space. By [45, Theorem 4], for any $G(\mathbb{A}_F)$ -invariant irreducible closed subspace V of $L^2(G(F)\backslash G(\mathbb{A}_F))_\chi$, the subspace V_{fin} of K -finite vectors in V is an automorphic representation of $G(\mathbb{A}_F)$. We call *discrete* an automorphic representation of $G(\mathbb{A}_F)$ arising in this way. A result of Gelfand and Piatetski-Shapiro, see [52], gives that the subspace $L^2_0(G(F)\backslash G(\mathbb{A}_F))_\chi$ of cuspidal functions in $L^2(G(F)\backslash G(\mathbb{A}_F))_\chi$ decomposes as a discrete sum with finite multiplicities of closed irreducible $G(\mathbb{A}_F)$ -invariant subspaces. The subspaces of K -finite vectors in these subspaces give, up to isomorphism, all the cuspidal automorphic representations of $G(\mathbb{A}_F)$ with central character χ . In other terms, all cuspidal automorphic representations of $G(\mathbb{A}_F)$ are discrete.

1.5.2 The Satake isomorphism

Let F be a finite extension of \mathbb{Q}_p , for some prime p , and let G be a connected reductive algebraic group over F . Assume that G is unramified. Recall that in this case, if K is an hyperspecial subgroup of $G(F)$, and π is an irreducible admissible representation of $G(F)$ on a \mathbb{C} -vector space, then V^K has dimension at most 1. We say that π is *unramified*, or *spherical*, if $V^K \neq \{0\}$.

Let now π be an automorphic representation. By Flath's tensor product decomposition theorem we have an isomorphism $\pi \cong \otimes'_v \pi_v$. Part of this result actually consists in showing that the representations π_v are unramified for all but finitely many finite places v . We show here how unramified representations can be characterised by means of their (so called) *Satake parameters*.

For simplicity, let us assume that G is split, refer to [28, §4.2] for the general unramified case. Let $T \subset B$ be a maximal torus, contained in a Borel subgroup of G , let \widehat{G} be the complex dual group of G , and let $\widehat{T} \subset \widehat{B}$ be a maximal torus, contained in a Borel subgroup of \widehat{G} , so that there is an isomorphism $X^*(\widehat{T}) \cong X_*(T)$.

Fix a hyperspecial subgroup K of $G(F)$, and let $\mathcal{H}(G(F), K)$ be the Hecke algebra of $(G(F), K)$. As anticipated in §1.4.1, in this case we have that $\mathcal{H}(G(F), K)$ is commutative. This can be proved straightforwardly via the *Satake isomorphism*.

Let N be the unipotent radical of any proper parabolic subgroup P of G defined over F , and let dn be the unique Haar measure on $N(F)$ which gives $N(F) \cap K$ volume 1. Let $\delta : B(F) \rightarrow \mathbb{R}_{>0}$ be the modular function on $B(F)$ defined by the formula $d(bnb^{-1}) = \delta(b)dn$, for all $b \in B(F)$. Since δ is trivial on $N(F)$, it defines a character $\delta : T(F) \rightarrow \mathbb{R}_{>0}$. For any $f \in \mathcal{H}(G(F), K)$, define its *Satake transform* to be the function $\mathcal{S}f$ on $T(F)$ given by the integral

$$\mathcal{S}f(t) = \delta(t)^{1/2} \int_{N(F)} f(tn)dn, \quad t \in T(F).$$

The function $\mathcal{S}f$ is actually a function on $T(F)/(T(F) \cap K) = X_*(T)$, with values in \mathbb{C} . This gives that the Satake transform \mathcal{S} induces a ring homomorphism from $\mathcal{H}(G(F), K)$ to the ring $\mathbb{C}[X_*(T)]$ of \mathbb{C} -valued functions on $X_*(T)$. Recall that $X^*(\widehat{T}) \cong X_*(T)$, so that \mathcal{S} can be considered to take values in $\mathbb{C}[X^*(\widehat{T})]$. Let $W(\widehat{G}, \widehat{T})$ be the Weyl group of \widehat{G} with respect to \widehat{T} . The main result is that the image of \mathcal{S} lies in the subring $\mathbb{C}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})}$ of $W(\widehat{G}, \widehat{T})$ -invariants in $\mathbb{C}[X^*(\widehat{T})]$. More precisely, the following result holds, see [28, Theorem 4.1].

Proposition 1.5.5. *The Satake transform gives a ring isomorphism*

$$\mathcal{S} : \mathcal{H}(G(F), K) \xrightarrow{\sim} \mathbb{C}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})}.$$

The isomorphism of Proposition 1.5.5 is often called the *Satake isomorphism*. Note that, this immediately implies that the spherical Hecke algebra $\mathcal{H}(G(F), K)$ is commutative.

Let $\text{Rep}_{\mathbb{C}}(\widehat{G}(\mathbb{C}))$ be the representation ring of $\widehat{G}(\mathbb{C})$ over \mathbb{C} , that is the Grothendieck ring of the category $\mathbf{Rep}_{\mathbb{C}}(\widehat{G}(\mathbb{C}))$ of finite dimensional representations of $\widehat{G}(\mathbb{C})$ over \mathbb{C} . We have $\text{Rep}_{\mathbb{C}}(\widehat{G}(\mathbb{C})) \cong \mathbb{C}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})}$, and so $\mathcal{H}(G(F), K) \cong \text{Rep}_{\mathbb{C}}(\widehat{G}(\mathbb{C}))$ via \mathcal{S} . A further consequence of Proposition 1.5.5 is that we can associate (uniquely) to any (isomorphism class of) unramified irreducible admissible representations of $G(F)$ over \mathbb{C} a semisimple conjugacy class in $\widehat{G}(\mathbb{C})$. This relies on the fact that the complex characters of $\text{Rep}_{\mathbb{C}}(\widehat{G}(\mathbb{C}))$ are indexed by the semisimple conjugacy classes in $\widehat{G}(\mathbb{C})$.

Let π be an unramified irreducible admissible representation of $G(F)$ on a \mathbb{C} -vector space V . Let $f \in \mathcal{H}(G(F), K)$. Then, $\pi(f)$ acts via a scalar on V^K , and hence on all V . We denote this scalar by $\text{Tr } \pi(f)$. The map

$$\mathcal{H}(G(F), K) \rightarrow \mathbb{C}, \quad f \mapsto \text{Tr } \pi(f),$$

is then a character of $\mathcal{H}(G(F), K)$. Via \mathcal{S} , this induces a character of $\text{Rep}_{\mathbb{C}}(\widehat{G}(\mathbb{C}))$. We denote by $c(\pi)$ the semisimple conjugacy class in $\widehat{G}(\mathbb{C})$ attached to it. We refer to [56, Proposition 6.4] for the following result.

Proposition 1.5.6. *The map $\pi \mapsto c(\pi)$ gives a bijection between the set of isomorphism classes of unramified irreducible admissible representations of $G(F)$ over \mathbb{C} and the set of semisimple conjugacy classes in $\widehat{G}(\mathbb{C})$.*

Given π as above, we call *Satake parameters* on π the eigenvalues of a representative of the conjugacy class $c(\pi)$.

For the purposes of this section, see in particular §1.5.4, it would be useful to consider, following [23, §2.2], the natural \mathbb{Q} -structures on $\mathcal{H}(G(F), K)$ coming from the \mathbb{Q} -valued functions $\mathcal{H}_{\mathbb{Q}}(G(F), K)$ and from $\mathbb{Q}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})}$ via the Satake isomorphism⁵. If π is an unramified irreducible admissible representation of $G(F)$ on a \mathbb{C} -vector space V , then the action of $\mathcal{H}(G(F), K)$ on V^K induces maps $\mathcal{H}_{\mathbb{Q}}(G(F), K) \rightarrow \mathbb{C}$ and $\mathbb{Q}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})} \rightarrow \mathbb{C}$. Given a subfield E of \mathbb{C} , we say that

- (1) π is defined over E if $\mathcal{H}_{\mathbb{Q}}(G(F), K) \rightarrow \mathbb{C}$ takes values in E ;
- (2) the Satake parameter of π is defined over E if $\mathbb{Q}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})} \rightarrow \mathbb{C}$ takes values in E .

⁵Note that the Satake transform on $\mathcal{H}_{\mathbb{Q}}(G(F), K)$ does not induce in general an isomorphism $\mathcal{H}_{\mathbb{Q}}(G(F), K) \cong \mathbb{Q}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})}$, and so $\mathcal{H}_{\mathbb{Q}}(G(F), K)$ and $\mathbb{Q}[X^*(\widehat{T})]^{W(\widehat{G}, \widehat{T})}$ give in general distinct \mathbb{Q} -structures on $\mathcal{H}(G(F), K)$.

The Satake parameters can be used to give a quick definition of global partial L -functions of automorphic representations, for which we refer, for instance, to [17]. For the purposes of this work, we restrict to the case of GL_n , with its standard representation.

Let F be a number field, and let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Decompose π as a tensor product $\pi \cong \otimes'_v \pi_v$. Then, there is a finite set S of places of F such that π_v is unramified for all $v \notin S$. For each $v \notin S$, let $c(\pi_v)$ be the corresponding semisimple conjugacy class in $\widehat{\mathrm{GL}}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$ given by Proposition 1.5.6. Define then the *partial L -function* of π as

$$L^S(\pi, s) = \prod_{v \notin S} \det(1 - (\#k(v))^{-s} c(\pi_v))^{-1},$$

where the infinite product converges for $\mathrm{Re} s$ large enough. Also, as a consequence of a classical result of Godement and Jacquet, see [55, Corollary 13.8], the partial L -function $L^S(\pi, s)$ is known to have meromorphic continuation to the whole complex plane.

1.5.3 Local definitions

Let F be an Archimedean local field, and let G be a connected reductive algebraic group over F . Let \mathfrak{g} be the Lie algebra of $G(F)$, viewed as a real Lie group, and let K be a maximal compact subgroup of $G(F)$. Recall that the local Langlands correspondence for G over F , proved by Langlands in [73], establishes a bijection between the set of irreducible admissible (\mathfrak{g}, K) -modules and the set of $\widehat{G}(\mathbb{C})$ -conjugacy classes of continuous admissible homomorphisms $\phi : W_F \rightarrow {}^L G(\mathbb{C})$, see §1.4.2. For any of such π , we let ϕ_π be the corresponding admissible homomorphism.

Fix an algebraic closure \overline{F} of F , and let $T \subset B$ be a maximal torus, contained in a Borel subgroup of $G_{\overline{F}}$. Let us also choose a maximal torus $\widehat{T} \subset \widehat{B}$, contained in a Borel subgroup of \widehat{G} . Let now $\phi : W_F \rightarrow {}^L G(\mathbb{C})$ be a continuous admissible homomorphism. The group W_F contains a finite index subgroup canonically isomorphic to \overline{F}^\times . After possibly conjugating ϕ by an element of $\widehat{G}(\mathbb{C})$, let us assume that $\phi(\overline{F}^\times) \subset \widehat{T}(\mathbb{C})$. Let $\sigma, \tau : \overline{F} \xrightarrow{\sim} \mathbb{C}$ be the two \mathbb{R} -isomorphisms. Then, for $z \in \overline{F}^\times$ we have $\phi(z) = \sigma(z)^{\lambda_\sigma} \tau(z)^{\lambda_\tau}$, for $\lambda_\sigma, \lambda_\tau \in X_*(\widehat{T}) \otimes_{\mathbb{Q}} \mathbb{C}$ such that $\lambda_\sigma - \lambda_\tau \in X_*(\widehat{T})$. Since there is usually more than one way of conjugating $\phi(\overline{F}^\times)$ into $\widehat{T}(\mathbb{C})$, we have that the pair $(\lambda_\sigma, \lambda_\tau) \in (X_*(\widehat{T}) \otimes_{\mathbb{Q}} \mathbb{C})^2$ is a well-defined invariant of ϕ only up to the (diagonal) action of the Weyl group $W(G, T)$.

A natural notion of “algebraicity” is then the following.

Definition 1.5.7. We say that $\phi : W_F \rightarrow {}^L G(\mathbb{C})$ is *L-algebraic* if $\lambda_\sigma \in X^*(T)$. We say that an irreducible admissible (\mathfrak{g}, K) -module π is *L-algebraic* if ϕ_π is *L-algebraic*.

Note that the notion of *L-algebraicity* is well-defined, as λ_σ is well-defined up to an element of $W(G, T)$, but $W(G, T)$ preserves $X_*(\widehat{T}) = X^*(T)$. Also, since $\lambda_\sigma - \lambda_\tau \in X_*(\widehat{T}) = X^*(T)$, we have that $\lambda_\sigma \in X^*(T)$ if and only if $\lambda_\tau \in X^*(T)$, and so the notion of *L-algebraicity* does not depend on the choice of the isomorphism $\sigma : \overline{F} \xrightarrow{\sim} \mathbb{C}$. Let us also remark that the notion of *L-algebraicity* for an *L-parameter* depends only on its restriction to \overline{F}^\times , and the notion of *L-algebraicity* for an admissible representation of $G(F)$ depends only on its central character.

Let us now introduce a second notion of algebraicity, which generalises to any reductive group Clozel's original definition for GL_n , see [33, §1.2.3]. We denote by $\delta \in X^*(T) \otimes_{\mathbb{Q}} \mathbb{C}$ half the sum of the positive roots. We give the following definition.

Definition 1.5.8. We say that $\phi : W_F \rightarrow {}^L G(\mathbb{C})$ is *C-algebraic* if $\lambda_\sigma - \delta \in X^*(T)$. We say that an irreducible admissible (\mathfrak{g}, K) -module π is *C-algebraic* if ϕ_π is *C-algebraic*.

Again, the notion of *C-algebraicity* is independent of the $W(G, T)$ -orbit of λ_σ , as if $\lambda_\sigma - \delta \in X^*(T)$ and $w \in W(G, T)$, then $w.\lambda_\sigma - \delta = w(\lambda_\sigma - \delta) - (\delta - w.\delta) \in X^*(T)$, and of the choice of the isomorphism $\sigma : \overline{F} \xrightarrow{\sim} \mathbb{C}$, as if $\lambda_\sigma - \delta \in X^*(T)$, then $\lambda_\tau - \delta = (\lambda_\sigma - \delta) + (\lambda_\tau - \lambda_\sigma) \in X^*(T)$. Also, *C-algebraicity* for an *L-parameter* depends only on its restriction to \overline{F}^\times , and *C-algebraicity* for an admissible representation of $G(F)$ depends only on its central character.

We say that a *C-algebraic* continuous admissible homomorphism $\phi : W_F \rightarrow {}^L G(\mathbb{C})$ is *regular C-algebraic* if $\phi|_{\overline{F}^\times}$ is not invariant under any nontrivial element of the Weyl group $W(\widehat{G}, \widehat{T})$. We say that an irreducible admissible representation π of $G(F)$ is *regular C-algebraic* if ϕ_π is *regular C-algebraic*.

1.5.4 Global definitions

Let F be a number field, and let G be a connected reductive algebraic group defined over F . For any automorphic representation π of $G(\mathbb{A}_F)$, let $\pi \cong \otimes'_v \pi_v$ be its restricted tensor product decomposition. We now give globalised versions of the definition of §1.5.3.

Definition 1.5.9. We say that π is *L-algebraic* if π_v is *L-algebraic* for all infinite places v of F .

Definition 1.5.10. We say that π is *C-algebraic* if π_v is *C-algebraic* for all infinite places v of F .

A C -algebraic automorphic representation π of $G(\mathbb{A}_F)$ is said to be *regular C -algebraic* if π_v is regular C -algebraic for all infinite places v of F .

In the case of GL_n , the notion of C -algebraic (resp. regular C -algebraic) automorphic representation coincides (in the isobaric case) with Clozel's notion of *algebraic* (resp. *regular algebraic*) automorphic representation introduced in [33, §1.2.3].

Definition 1.5.11. We say that π is *L -arithmetic* if there is a finite set S of places of F , containing all the infinite places and all places where π is ramified, and a number field $E \subset \mathbb{C}$, such that the Satake parameter of π_v is defined over E for all $v \notin S$.

Definition 1.5.12. We say that π is *C -arithmetic* if there is a finite set S of places of F , containing all the infinite places and all places where π is ramified, and a number field $E \subset \mathbb{C}$, such that π_v is defined over E for all $v \notin S$.

When there is a number field $E \subset \mathbb{C}$ such that π_v is defined over E for all finite places v of F , we say that π is *strongly C -arithmetic*.

For any finite set S of places of F , the *field of rationality* $\mathbb{Q}(\pi^S)$ is defined as the fixed field of \mathbb{C} under the group $\{\sigma \in \mathrm{Aut}(\mathbb{C}) : (\pi^S)^\sigma \cong \pi^S\}$. When S is just the set of all infinite places of F , we simply denote it by $\mathbb{Q}(\pi)$. If π is C -algebraic, then $\mathbb{Q}(\pi^S)$ is finite over \mathbb{Q} for some finite set S of places of F , containing all the infinite places and all places where π is ramified. If π is strongly C -algebraic, then $\mathbb{Q}(\pi)$ is finite over \mathbb{Q} .

It is conjectured, see [23, Conjecture 5.14, 5.15], that the notions of L -algebraic (resp. C -algebraic) and L -arithmetic (resp. C -arithmetic) automorphic representation should coincide. However, not much is known in this generality.

A class of algebraic automorphic representations which naturally arise in many contexts considered in this work is that of *cohomological* representations.

Let \mathfrak{g} be the Lie algebra of $G(F_\infty)$, let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and let K_∞ be a subgroup of $G(F_\infty)$, whose image in $G^{\mathrm{ad}}(F_\infty)$ is a maximal compact subgroup. Let K_∞° be the connected component of the identity in K_∞ . Let π be an automorphic representation of $G(\mathbb{A}_F)$. We give the following definition.

Definition 1.5.13. We say that π is *ξ -cohomological*, for some irreducible algebraic representation ξ of $\mathrm{Res}_{F/\mathbb{Q}}G$ over \mathbb{C} , if $H^i(\mathfrak{g}_{\mathbb{C}}, K_\infty^\circ, \pi_\infty \otimes \xi) \neq \{0\}$ for some $i \geq 0$.

For any sufficiently small open compact subgroup K_f of $G(\mathbb{A}_{F,f})$, consider the locally symmetric space

$$Y_{G,K_f} = G(F) \backslash G(\mathbb{A}_F) / K_\infty^\circ K_f.$$

Let ξ be an irreducible algebraic representation of $\text{Res}_{F/\mathbb{Q}}G$ over \mathbb{C} , and denote by \mathcal{L}_ξ the associated local system of \mathbb{C} -vector spaces on Y_{G,K_f} . The following result holds.

Proposition 1.5.14. *Let π be a ξ -cohomological automorphic representation of $G(\mathbb{A}_F)$. Then, π is C -algebraic. Moreover, if π is also cuspidal, then*

- (1) π_f is a $G(\mathbb{A}_{F,f})$ -module direct summand of $\varinjlim_{K_f} H^i(Y_{G,K_f}, \mathcal{L}_\xi)$ for some $i \geq 0$.
- (2) π is strongly C -arithmetic.

Refer to [23, Lemma 5.52], or [94, Lemma 2.14], for a proof of the first part of this proposition, and to [94, Proposition 2.15] for the second part, which for general linear groups has been shown by Clozel in [33, Théorème 3.13, Lemme 3.14, 3.15].

In the case of GL_n , the picture is way more clear. We have the following result.

Proposition 1.5.15. *Let π be an automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. Then*

- (1) π is L -algebraic if and only if $\pi \otimes |\det|_F^{(n-1)/2}$ is C -algebraic.
- (2) π is L -arithmetic if and only if $\pi \otimes |\det|_F^{(n-1)/2}$ is C -arithmetic.
- (3) If π is cuspidal, and regular C -algebraic, then it is ξ -cohomological for some ξ .

Refer to [23, Proposition 5.35, 5.36] respectively for the first and the second part, and to [33, Lemme 3.14] for the third part. Note in particular that for any regular C -algebraic, cuspidal automorphic representation π of $\text{GL}_n(\mathbb{A}_F)$, the field of rationality $\mathbb{Q}(\pi)$ is a number field, as π is strongly C -arithmetic.

Following Clozel's original terminology, in this work we simply say that an automorphic representation π of $\text{GL}_n(\mathbb{A}_F)$ is *regular algebraic* if it is regular C -algebraic. In this case, if π is also cuspidal, then it is ξ -cohomological, for some irreducible algebraic representation ξ of $\text{Res}_{F/\mathbb{Q}}\text{GL}_n$ over \mathbb{C} , and so π_∞ has the same infinitesimal character as ξ .

Denote by $(\mathbb{Z}^n)^{\text{Hom}(F,\mathbb{C}),+}$ the set of elements $a = (a_{\tau,i}) \in (\mathbb{Z}^n)^{\text{Hom}(F,\mathbb{C})}$ satisfying

$$a_{\tau,1} \geq \dots \geq a_{\tau,n}.$$

For $a \in (\mathbb{Z}^n)^{\text{Hom}(F,\mathbb{C}),+}$, denote by ξ_a the irreducible algebraic representation of $\text{Res}_{F/\mathbb{Q}}\text{GL}_n$ over \mathbb{C} which is the tensor product over $\tau \in \text{Hom}(F,\mathbb{C})$ of the irreducible representations of GL_n with highest weights a_τ . We say that a regular algebraic

cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ has *weight* a if π_∞ has the same infinitesimal character as ξ_a^\vee .

We say that the weight $a = (a_{\tau,i})$ of a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ is *extremely regular* if there exists an embedding $\tau : F \rightarrow \mathbb{C}$ such that, if H and H' are subsets of $\{a_{\tau,i} + n - i : i = 1, \dots, n\}$ of the same cardinality, and if $\sum_{h \in H} h = \sum_{h' \in H'} h'$, then $H = H'$.

If π is a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $a = (a_{\tau,i})$, by Clozel's *purity lemma*, see [33, Lemme 4.9], there exists an integer $w \in \mathbb{Z}$ such that $a_{\tau,i} + a_{\bar{\tau},n+1-i} = w$ for each $\tau \in \mathrm{Hom}(F, \mathbb{C})$ and $i = 1, \dots, n$, where we denoted by $\bar{\tau}$ the complex conjugate of τ .

1.6 Compatible systems attached to algebraic automorphic representations

Assume now that F is a CM field, denote by F^+ its maximal totally real subfield, and by c a generator of $\mathrm{Gal}(F/F^+)$. Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let $\chi : \mathbb{A}_{F^+}^\times / (F^+)^\times \rightarrow \mathbb{C}$ be a continuous character such that $\chi_v(-1) = (-1)^n$ for all $v \mid \infty$. We say that the pair (π, χ) is a *polarised* automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ if $\pi^c \cong \pi^\vee \otimes (\chi \circ N_{F/F^+} \circ \det)$. (Here π^c denotes the composition of π with the complex conjugation c on $\mathrm{GL}_n(\mathbb{A}_F)$). When $\pi^c \cong \pi^\vee$, we just say that π is *conjugate self-dual*⁶.

Thanks to the work of many people, e.g. [33], [69], [60], [93], [32], geometric compatible systems can be attached to regular algebraic, conjugate self-dual, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, for F a CM field⁷. We recall here the general statement of this construction, which also involves results of [100], [26] and [27]. We have:

Theorem 1.6.1. *Let π be a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, unramified outside a finite set S of places of F . Then, there exist a number field E_π , a compatible system $\mathcal{R}_\pi = \{\rho_{\pi,\lambda}\}_{\lambda \in |E_\pi|}$ of rank n semisimple representations*

$$\rho_{\pi,\lambda} : \Gamma_F \rightarrow \mathrm{GL}_n(E_{\pi,\lambda})$$

with coefficients in E_π , and an integer w such that:

⁶For simplicity, in this work we will just consider conjugate self-dual automorphic representations, instead of polarised ones as in [6].

⁷An analogous construction holds also when F is a totally real field.

- (1) $\rho_{\pi,\lambda}$ is totally odd, conjugate self-dual.
- (2) If v is a place of F not dividing the residue characteristic of λ , then, given an isomorphism $\iota : \overline{E}_{\pi,\lambda} \xrightarrow{\sim} \mathbb{C}$, we have the local-global compatibility

$$\iota \mathrm{WD}(\rho_{\pi,\lambda}|_{\Gamma_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}_v(\pi_v \otimes |\det|_v^{(1-n)/2}),$$

and these Weil-Deligne representations are pure of weight w .

- (3) Each representation $\rho_{\pi,\lambda}$ is de Rham at all places above the residue characteristic of λ , and for each embedding $\tau : F \rightarrow \overline{E}_\pi$ the τ -Hodge-Tate weights of $\rho_{\pi,\lambda}$ have multiplicity at most one, and are given by

$$\mathrm{HT}_\tau(\rho_{\pi,\lambda}) = \{a_{\tau,1} + n - 1, a_{\tau,2} + n - 2, \dots, a_{\tau,n}\},$$

where $a = (a_{\tau,i})$ is the weight of π . Moreover

$$\mathrm{HT}_{\tau \circ c}(\rho_{\pi,\lambda}) = \{w - h : h \in \mathrm{HT}_\tau(\rho_{\pi,\lambda})\}.$$

- (4) If v is a place of F dividing the residue characteristic of λ , then, given an isomorphism $\iota : \overline{E}_{\pi,\lambda} \xrightarrow{\sim} \mathbb{C}$, we have the local-global compatibility

$$\iota \mathrm{WD}(\rho_{\pi,\lambda}|_{\Gamma_{F_v}})^{\mathrm{F-ss}} \cong \mathrm{rec}_v(\pi_v \otimes |\det|_v^{(1-n)/2}).$$

In particular, $\rho_{\pi,\lambda}$ is semistable at v , and if $v \notin S$ then it is crystalline.

The compatible system \mathcal{R}_π is then a geometric compatible system of rank n representations of Γ_F with coefficients in E_π and unramified outside S , in the sense of Definition 1.3.1. Moreover, it is strictly pure of weight w , regular, and conjugate self-dual.

Irreducibility of the compatible system \mathcal{R}_π is conjectural, and only partial progress has been obtained in this direction so far⁸.

Given an embedding $\iota : E_\pi \rightarrow \mathbb{C}$, we have $L^S(\iota \mathcal{R}_\pi, s) = L^S(\pi, s)$ for $\mathrm{Re} s$ large enough. Since $L^S(\pi, s)$ has meromorphic continuation to the whole complex plane by the above mentioned result of Godement and Jacquet, it makes sense to continue $L^S(\iota \mathcal{R}_\pi, s)$ as a meromorphic function on the whole complex plane.

We say that a geometric compatible system \mathcal{R} of rank n representations of Γ_F defined over a number field E and unramified outside a finite set S of places of F is *automorphic* if there exists a regular algebraic, cuspidal automorphic representation π

⁸We discuss this topic more precisely in §1.7.

of $\mathrm{GL}_n(\mathbb{A}_F)$ and an embedding $\iota : E \rightarrow \mathbb{C}$ such that if $v \notin S$, then π_v is unramified, and $\mathrm{rec}_v(\pi_v \otimes |\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\iota Q_v(t)$. Note that if \mathcal{R} is conjugate self-dual, so is π . Also, it follows from [33, Théorème 3.13] that if \mathcal{R} is automorphic for some π and ι , then for any embedding $\iota' : E \rightarrow \mathbb{C}$ there is a regular algebraic, cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbb{A}_F)$ such that if $v \notin S$, then π'_v is unramified, and $\mathrm{rec}_v(\pi'_v \otimes |\det|_v^{(1-n)/2})(\mathrm{Frob}_v)$ has characteristic polynomial $\iota' Q_v(t)$.

When a conjugate self-dual geometric compatible system \mathcal{R} is automorphic for some regular algebraic, conjugate self-dual, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, then $\mathcal{R} \cong \check{\mathcal{R}}_\pi$ by the Čebotarev density theorem and [20, §12.1, Proposition 3].

Remark 1.6.2. A construction of ℓ -adic Galois representations for regular algebraic cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, for F totally real or CM, without a self-duality assumption has been recently achieved by Harris, Lan, Taylor and Thorne in [59], and by Scholze in [88]. However, the proof that these representations are de Rham at places $v \mid \ell$ and the proof of a local-global compatibility and crystallinity statement at the places $v \mid \ell$ where the automorphic representation is unramified is still (to the extent of the author's knowledge) work in progress by Varma [104]. General constructions of geometric Galois representations from algebraic automorphic representations of $G(\mathbb{A}_F)$, for any connected reductive group G over a number field F , are conjectured by Buzzard and Gee in [23, §3.2, 5.3].

1.7 Potential automorphy and irreducibility of compatible systems

Let F be a number field, let ℓ be a prime, and let $n \geq 1$ be an integer. The global Langlands conjecture for GL_n over F predicts the existence of a unique bijection between the set of isomorphism classes of L -algebraic (resp. C -algebraic) cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$ and the set of isomorphism classes of irreducible geometric representations $\rho_\pi : \Gamma_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Furthermore, this bijection is conjectured to be compatible with the local Langlands correspondence for GL_n at the finite places of F .

In §1.6, we have seen how geometric Galois representations can be attached to regular algebraic, conjugate self-dual, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, when F is a CM field. Also, we have seen how these automorphic Galois

representations naturally arise as part of geometric compatible systems. On the basis of the construction of §1.6, we then expect the existence of a unique bijection between the set of regular algebraic, conjugate self-dual, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, and the set of (isomorphism classes of) regular, totally odd, conjugate self-dual, irreducible geometric compatible systems of rank n representations of Γ_F . Also in this case, the bijection is required to be compatible with the local Langlands correspondence for GL_n at the finite places of F .

The only remaining issue with the arrow from “automorphic” to “Galois” in this conjectural bijection is that of the irreducibility of automorphic compatible systems. As for the arrow from “Galois” to “automorphic”, on the other hand, significant progress has been obtained by the work of Barnet-Lamb, Gee, Geraghty and Taylor in [6].

The main result of [6] in this direction is the following potential automorphy theorem for (finite collections of) λ -adic Galois representations⁹.

Theorem 1.7.1 ([6, Theorem 4.5.1]). *Let F/F_0 be a finite Galois extension of CM fields, and let F^{avoid}/F be a finite Galois extension. Let $\{\rho_i\}_{i=1}^k$ be a family of regular, totally odd, conjugate self-dual representations $\rho_i : \Gamma_F \rightarrow \mathrm{GL}_{n_i}(\overline{E}_{\lambda_i})$. For each i , let ℓ_i be the residue characteristic of λ_i , and let d_i be the maximum dimension of an irreducible constituent of the restriction of $\overline{\rho}_i$ to the closed subgroup of Γ_F generated by all the Sylow pro- ℓ_i -subgroups. Assume that for each i :*

- $\ell_i \geq 2(d_i + 1)$ and $\zeta_{\ell_i} \notin F$.
- ρ_i is potentially diagonalisable at each place of F^+ above ℓ_i .
- $\overline{\rho}_i|_{\Gamma_{F(\zeta_{\ell_i})}}$ is irreducible.

Then, there are a finite CM extension F' of F , linearly disjoint from F^{avoid} over F and with F'/F_0 Galois, and for each i a regular algebraic, conjugate self-dual, cuspidal automorphic representation π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_{F'})$ which is unramified at each place of F' above ℓ_i , and such that, if λ'_i is a place of E_{π_i} above ℓ_i , then $\rho_{\pi_i, \lambda'_i} \otimes_{E_{\pi_i, \lambda'_i}} \overline{E}_{\pi_i, \lambda'_i} \cong \rho_i|_{\Gamma_{F'}}$.

When the conclusion of this theorem holds for a collection $\{\rho_i\}_{i=1}^k$ of (non necessarily totally odd, conjugate self-dual¹⁰) regular representations ρ_i , we say that

⁹An analogous result holds also in the totally real case.

¹⁰Recall that a construction of automorphic Galois representations is available also in the non conjugate self-dual case, see Remark 1.6.2.

the representations ρ_i are *simultaneously potentially automorphic*. At the level of geometric compatible systems, Theorem 1.7.1 implies the following¹¹.

Corollary 1.7.2 ([6, Corollary 5.4.2]). *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a regular, totally odd, conjugate self-dual, irreducible geometric compatible system of rank n representations of Γ_F . Then, there is a finite CM Galois extension F' of F such that $\mathcal{R}|_{\Gamma_{F'}}$ is automorphic, for some regular algebraic, conjugate self-dual, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_{F'})$.*

When the conclusion of this theorem holds, we say that the compatible system \mathcal{R} is *potentially automorphic*. Another relevant result in this direction has been obtained by Patrikis and Taylor, see [84, Theorem 2.1], where irreducibility is replaced by a purity assumption, and the given geometric compatible system is proved to be isomorphic to a direct sum of potentially automorphic ones.

Note that if a geometric compatible system \mathcal{R} of representations of Γ_F , defined over a number field E and unramified outside a finite set S of places of F , is potentially automorphic (or it is a direct sum of potentially automorphic ones) then it is strictly pure, and given an embedding $\iota : E \rightarrow \mathbb{C}$, the partial L -function $L^S(\iota\mathcal{R}, s)$ has meromorphic continuation to the whole complex plane, see [6, Corollary 5.4.3].

It is then clear how a proof of the irreducibility of automorphic compatible systems would be of fundamental importance in establishing the expected global correspondence for geometric compatible systems (at least up to a finite extension of the base field).

Recall that for a regular algebraic, conjugate self-dual, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ the global Langlands conjecture predicts all the representations $\rho_{\pi, \lambda}$ in the corresponding geometric compatible system $\mathcal{R}_\pi = \{\rho_{\pi, \lambda}\}_{\lambda \in |E_\pi|}$ to be absolutely irreducible. Nevertheless, for the purposes of this work, see §2.5, it is enough to formulate the following weaker conjecture.

Conjecture 1.7.3. *Let π be a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then, there exists a set of rational primes \mathcal{L} of Dirichlet density 1 such that for all $\ell \in \mathcal{L}$ and $\lambda \mid \ell$ the representation*

$$\rho_{\pi, \lambda} : \Gamma_F \rightarrow \mathrm{GL}_n(E_{\pi, \lambda})$$

is absolutely irreducible.

¹¹Also in this case, we have a version for finite collections of geometric compatible systems (which we decided to omit to simplify our exposition). Again, an analogous result holds in the totally real case as well.

So far, Conjecture 1.7.3 has been only proved either:

- under local assumptions on the automorphic representation¹², see [100];
- in lower dimensional cases, see [24]¹³ (over totally real fields) and [109];
- when the automorphic representation has extremely regular weight, see [6];
- at a set of primes of positive Dirichlet density, see [84].

All these irreducibility results (apart for the first one) rely on an L -function argument, which we are now going to summarise in a single proposition. The proof of it is essentially the same as the proof of [6, Theorem 5.5.2], but we decided to include it here for the sake of completeness.

Before presenting the L -function argument let us recall that, if F is a number field, and π and π' are cuspidal automorphic representations respectively of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_{n'}(\mathbb{A}_F)$, with tensor product decompositions respectively $\pi \cong \otimes' \pi_v$ and $\pi' \cong \otimes' \pi'_v$, and S is a finite set of places of F , one can define a (Jacquet-Shalika) partial L -function

$$L^S(\pi \times \pi', s) = \prod_{v \notin S} L(\pi_v \times \pi'_v, s), \quad s \in \mathbb{C},$$

where the product converges for $\mathrm{Re} s$ large enough, and for $\mathrm{Re} s > 1$ if π and π' are unitary. In addition, when π and π' are unitary, the partial L -function $L^S(\pi \times \pi', s)$ has meromorphic continuation to the whole \mathbb{C} , and is holomorphic and nonzero at $s = 1$, unless $\pi \cong (\pi')^\vee$, in which case it has a simple pole, with $\mathrm{ord}_{s=1} L^S(\pi \times \pi^\vee) = -1$, by results of Shahidi, see [92], and of Jacquet and Shalika, see [65].

If π and π' are regular algebraic and cuspidal, then there exist integers w and w' such that $\pi \otimes |\det|_F^{(w+1-n)/2}$ and $\pi' \otimes |\det|_F^{(w'+1-n')/2}$ are unitary, and so

$$\begin{aligned} & L^S(\pi \times (\pi')^\vee, s + (w - w' + n' - n)/2) \\ &= L^S((\pi \otimes |\det|_F^{(w+1-n)/2}) \times ((\pi' \otimes |\det|_F^{(w'+1-n')/2})^\vee), s) \end{aligned}$$

has meromorphic continuation to the whole \mathbb{C} , and is holomorphic and nonzero at $s = 1$, unless $\pi \cong \pi' \otimes |\det|_F^{(w'-w+n-n')/2}$, in which case it has a simple pole, of order -1 .

¹²In this case, one can actually get (easily) absolute irreducibility of $\rho_{\pi,\lambda}$ for all λ . In fact, if we assume for instance π to be square-integrable at some finite place v of F , then $\mathrm{rec}_v(\pi_v \otimes |\det|_v^{(1-n)/2})$ would be indecomposable. Now, since local-global compatibility holds at v , and $\rho_{\pi,\lambda}$ is semisimple, this implies that $\rho_{\pi,\lambda}$ is absolutely irreducible.

¹³More precisely, we refer here to [24, Theorem 3.2]. This result is not affected by the published erratum of [24].

The announced proposition summarising the L -function argument is the following.

Proposition 1.7.4. *Let π be a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let \mathcal{L} be a set of rational primes such that, if $\ell \in \mathcal{L}$ and $\lambda \mid \ell$, the irreducible constituents of $\rho_{\pi,\lambda} \otimes_{E_{\pi,\lambda}} \overline{E}_{\pi,\lambda}$ are simultaneously potentially automorphic. Then, for all $\ell \in \mathcal{L}$ and $\lambda \mid \ell$ the representation $\rho_{\pi,\lambda}$ is absolutely irreducible.*

Proof. Let $\ell \in \mathcal{L}$, let $\lambda \mid \ell$, let $\rho = \rho_{\pi,\lambda} \otimes_{E_{\pi,\lambda}} \overline{E}_{\pi,\lambda}$, and let

$$\rho \cong \bigoplus_{i=1}^k \rho_i$$

be an irreducible decomposition. Let n_i be the rank of ρ_i . Since the representations ρ_i are simultaneously potentially automorphic by assumption, that is the conclusion of Theorem 1.7.1 holds for the collection $\{\rho_i\}_{i=1}^k$, with F^{avoid} being the compositum of the $\overline{F}^{\ker(\overline{\rho}_i)}$, for $i = 1, \dots, k$, there exist a finite extension F'/F , linearly disjoint from F^{avoid} over F , and for each i a regular algebraic, cuspidal automorphic representation π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_{F'})$ which is unramified at each place of F' above ℓ , and such that, if λ'_i is a place of E_{π_i} above ℓ , then $\rho_{\pi_i,\lambda'_i} \otimes_{E_{\pi_i,\lambda'_i}} \overline{E}_{\pi_i,\lambda'_i} \cong \rho_i|_{\Gamma_{F'}}$, and $\rho_i|_{\Gamma_{F'}}$ is irreducible, since $\overline{\rho}_i|_{\Gamma_{F'}}$ is irreducible.

Let S be the finite set of places of F which divide ℓ , or above which π ramifies, or above which F' ramifies. Then, given an isomorphism $\iota : \overline{E}_{\pi,\lambda} \xrightarrow{\sim} \mathbb{C}$, we have

$$L^S(\iota(\mathcal{R}_\pi \otimes \mathcal{R}_\pi^\vee), s) = L^S(\pi \times \pi^\vee, s)$$

as meromorphic functions on the whole \mathbb{C} . The partial L -function $L^S(\pi \times \pi^\vee, s)$ has a simple pole at $s = 1$, with $\mathrm{ord}_{s=1} L^S(\pi \times \pi^\vee, s) = -1$, and so $\mathrm{ord}_{s=1} L^S(\iota(\mathcal{R}_\pi \otimes \mathcal{R}_\pi^\vee), s) = -1$. We show now that $\mathrm{ord}_{s=1} L^S(\iota(\mathcal{R}_\pi \otimes \mathcal{R}_\pi^\vee), s) = -k$, so that we obtain $k = 1$, as desired.

By Brauer's theorem for the (finite) group $\mathrm{Gal}(F'/F)$, there exist intermediate fields $F \subset F'_\alpha \subset F'$, with F'/F'_α soluble, characters $\psi_\alpha : \mathrm{Gal}(F'/F'_\alpha) \rightarrow \mathbb{C}^\times$, and integers n'_α , for $\alpha = 1, \dots, N$, such that

$$1 = \sum_{\alpha=1}^N n'_\alpha [\mathrm{Ind}_{\Gamma_{F'_\alpha}}^{\Gamma_F} \psi_\alpha]$$

in the Grothendieck group of finite dimensional representations of $\mathrm{Gal}(F'/F)$ over \mathbb{C} , so that

$$\rho \otimes \rho^\vee \cong \bigoplus_{\alpha=1}^N (\mathrm{Ind}_{\Gamma_{F'_\alpha}}^{\Gamma_F} (\rho|_{\Gamma_{F'_\alpha}} \otimes \rho^\vee|_{\Gamma_{F'_\alpha}}) \otimes \iota^{-1} \psi_\alpha)^{n'_\alpha}.$$

We can then write

$$\begin{aligned} L^S(\iota(\mathcal{R}_\pi \otimes \mathcal{R}_\pi^\vee), s) &= L^S(\iota(\rho \otimes \rho^\vee), s) \\ &= \prod_{\alpha=1}^N L^S(\iota(\rho|_{\Gamma_{F'_\alpha}} \otimes \rho^\vee|_{\Gamma_{F'_\alpha}}) \otimes \psi_\alpha, s)^{n'_\alpha}. \end{aligned}$$

By applying the weak lifting result of Arthur and Clozel, see [4, §3, Theorem 4.2], to any π_i with respect to the extensions F'/F'_α as in the proof of [34, Lemma 4.2.2], we get regular algebraic, cuspidal automorphic representations $\pi_i^{(F'_\alpha)}$ of $\mathrm{GL}_{n_i}(\mathbb{A}_{F'_\alpha})$ whose attached λ'_i -adic representations are isomorphic (after extension of scalars) to $\rho_i|_{\Gamma_{F'_\alpha}}$. Moreover, we have that each $\rho_i|_{\Gamma_{F'_\alpha}}$ is irreducible.

Therefore, for any α, i, j we get

$$\begin{aligned} &\mathrm{ord}_{s=1} L^S(\iota(\rho_i|_{\Gamma_{F'_\alpha}} \otimes \rho_j^\vee|_{\Gamma_{F'_\alpha}}) \otimes \psi_\alpha, s) \\ &= \mathrm{ord}_{s=1} L^S(\pi_i^{(F'_\alpha)} \times (\pi_j^{(F'_\alpha)})^\vee \times (\psi_\alpha \circ \mathrm{Art}_{F'_\alpha}), s + (n_j - n_i)/2) \\ &= -\delta_{i,j} \delta_{\psi_\alpha, 1} \\ &= -\dim_{\overline{E}_{\pi, \lambda}} \mathrm{Hom}_{\overline{E}_{\pi, \lambda}[\Gamma_{F'_\alpha}]}(\rho_i|_{\Gamma_{F'_\alpha}} \otimes \psi_\alpha, \rho_j|_{\Gamma_{F'_\alpha}}). \end{aligned}$$

Summing over i, j , it follows that

$$\begin{aligned} &\mathrm{ord}_{s=1} L^S(\iota(\rho|_{\Gamma_{F'_\alpha}} \otimes \rho^\vee|_{\Gamma_{F'_\alpha}}) \otimes \psi_\alpha, s) \\ &= -\dim_{\overline{E}_{\pi, \lambda}} \mathrm{Hom}_{\overline{E}_{\pi, \lambda}[\Gamma_{F'_\alpha}]}(\rho|_{\Gamma_{F'_\alpha}} \otimes \psi_\alpha, \rho|_{\Gamma_{F'_\alpha}}). \end{aligned}$$

Therefore

$$\begin{aligned} &\mathrm{ord}_{s=1} L^S(\iota(\mathcal{R}_\pi \otimes \mathcal{R}_\pi^\vee), s) \\ &= \sum_{\alpha=1}^N n'_\alpha \mathrm{ord}_{s=1} L^S(\iota(\rho|_{\Gamma_{F'_\alpha}} \otimes \rho^\vee|_{\Gamma_{F'_\alpha}}) \otimes \psi_\alpha, s) \\ &= -\sum_{\alpha=1}^N n'_\alpha \dim_{\overline{E}_{\pi, \lambda}} \mathrm{Hom}_{\overline{E}_{\pi, \lambda}[\Gamma_{F'_\alpha}]}(\rho|_{\Gamma_{F'_\alpha}} \otimes \psi_\alpha, \rho|_{\Gamma_{F'_\alpha}}) \\ &= -\sum_{\alpha=1}^N n'_\alpha \dim_{\overline{E}_{\pi, \lambda}} \mathrm{Hom}_{\overline{E}_{\pi, \lambda}[\Gamma_F]}(\mathrm{Ind}_{\Gamma_{F'_\alpha}}^{\Gamma_F}(\rho|_{\Gamma_{F'_\alpha}} \otimes \psi_\alpha), \rho) \\ &= -\dim_{\overline{E}_{\pi, \lambda}} \mathrm{Hom}_{\overline{E}_{\pi, \lambda}[\Gamma_F]}(\rho, \rho) \\ &= -k. \end{aligned}$$

By the above identity, we then get that $k = 1$, and so $\rho_{\pi, \lambda}$ is absolutely irreducible. \square

Remark 1.7.5. If π is a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and $\mathcal{R}_\pi = \{\rho_{\pi,\lambda}\}_{\lambda \in |E_\pi|}$ is the attached geometric compatible system of rank n representations of Γ_F , then there is a set \mathcal{L} of rational primes of Dirichlet density 1 such that, if $\ell \in \mathcal{L}$, $\lambda \mid \ell$, and ρ_i is an irreducible constituent of $\rho_{\pi,\lambda} \otimes_{E_{\pi,\lambda}} \overline{E}_{\pi,\lambda}$, then

- ρ_i is regular (obviously as it is a subrepresentation of a regular representation).
- $\ell \geq 2(n+1)$ and $\zeta_\ell \notin F$.
- ℓ is unramified in F , ρ_i is unramified (and hence crystalline) at any place above ℓ , and its Hodge-Tate weights lie in a range of the form $[a_i, a_i + \ell - 2]$, so that it is potentially diagonalisable by [6, Lemma 1.4.3.(2)].
- $\overline{\rho}_i|_{\Gamma_{F(\zeta_\ell)}}$ is irreducible¹⁴.

In order to apply Theorem 1.7.1 to get simultaneous potential automorphy of the representations ρ_i , and hence prove Conjecture 1.7.3 by the L -function argument of Proposition 1.7.4, it only remains to show that each ρ_i is totally odd, conjugate self-dual. Note that if a ρ_i is conjugate self-dual, then it is automatically totally odd by [11, Theorem 1.2]. The key point is then to show that each ρ_i is conjugate self-dual. When π has extremely regular weight, this holds true for any λ , see [6, Lemma 5.4.5], and so Conjecture 1.7.3 follows in this case, see [6, Theorem 5.5.2]. In full generality, however, this can be only achieved at places λ lying above a set of primes of positive Dirichlet density, as a consequence of the CM nature of the field of rationality $\mathbb{Q}(\pi)$ of π , see [84, Lemma 1.6], and so Conjecture 1.7.3 is proved at least at a set of positive Dirichlet density, see [84, Theorem 1.7]. When $n \leq 5$, it can be shown directly that (if the field F is totally real) each ρ_i is totally odd (essentially) self-dual, so that Conjecture 1.7.3 follows in this case, see [24, Theorem 3.2]. When $n \leq 6$, Conjecture 1.7.3 is proved by combining the results of [84] with some independence of λ and classification results for the Lie algebra of the Zariski closure of the image of $\rho_{\pi,\lambda}$, see [109, §9].

It is obvious to remark how finer potential automorphy results, possibly removing the self-duality assumption, would imply Conjecture 1.7.3 in higher generality. However, the only available potential automorphy result in the non self-dual case applies so far (to the extent of the author's knowledge) to certain 2-dimensional representations and to their symmetric powers, see [2, Theorem 7.1.10].

¹⁴The set of rational primes ℓ such that this property holds is known to have Dirichlet density 1 by [6, 5.3.2].

Chapter 2

Independence of algebraic monodromy groups in compatible systems

This chapter focuses on the study of problems of λ -independence of algebraic monodromy groups in compatible systems. Our approach can be presented in a general setting. In §2.1 we introduce the basic terminology, while in §2.2 and §2.3 we recall and extend to our setting the main tools from [75], which will be useful in §2.4. Our general method to prove λ -independence results for abstract compatible systems is presented in §2.4. Finally, we see applications of the results of §2.4 to the case of geometric compatible systems in §2.5, and to compatible systems in the positive characteristic case in §2.6.

2.1 Abstract compatible systems and algebraic monodromy groups

A *group with Frobenii*, or simply an *F-group*, is a pair $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ where Γ is a profinite group, and $\{F_\alpha\}_{\alpha \in A}$ is a collection of elements of Γ , called *Frobenius elements*, which are dense in Γ . Motivating examples for the definition of group with Frobenii are clearly the absolute Galois group of a global field, together with the collection of geometric Frobenii indexed by the finite places of its separable closure, and the étale fundamental group of a variety over a finite field, together with the collection of geometric Frobenii indexed by its closed points.

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, let E be a number field, and let Λ be a set of finite places of E . We give the following definition¹.

¹Compare [75, Definition 6.5].

Definition 2.1.1. A *compatible system* of rank n representations of Γ defined over E and indexed by Λ is a family $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of continuous representations

$$\rho_\lambda : \Gamma \rightarrow \mathrm{GL}_n(\overline{E}_\lambda)$$

such that there is a subset $\mathcal{X} \subset A \times \Lambda$ satisfying the following conditions.

- (1) For every $\alpha \in A$, $(\alpha, \lambda) \in \mathcal{X}$ for all but finitely many $\lambda \in \Lambda$.
- (2) For all $(\alpha, \lambda) \in \mathcal{X}$, the characteristic polynomial of $\rho_\lambda(F_\alpha)$ has coefficients in E and is independent of λ .
- (3) For any places $\lambda_1, \dots, \lambda_m \in \Lambda$, the set $\{F_\alpha : (\alpha, \lambda_i) \in \mathcal{X} \text{ for all } i = 1, \dots, m\}$ is dense in Γ .

Remark 2.1.2. Clearly, any geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of representations of Γ_F , for F a number field, as in Definition 1.3.1 is a compatible system in the sense of Definition 2.1.1, with \mathcal{X} being the set of pairs (\bar{v}, λ) , where \bar{v} is a finite place of \overline{F} lying above no places in S , and λ is an element of Λ not dividing the residue characteristic of \bar{v} . In §2.6, we will consider another fundamental class of examples of compatible systems, i.e. compatible systems of lisse sheaves on varieties over finite fields.

As in the case of geometric compatible systems, we will assume from now on all the compatible systems to be indexed by set of places of the form $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$, for \mathcal{L} a set of rational primes of Dirichlet density 1. Also, we say that $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ has *coefficients* in a finite extension E' of E if for any $\lambda \in \Lambda$ the image of ρ_λ is contained in $\mathrm{GL}_n(E'_{\lambda'})$ for some place λ' of E' above λ . Given two sets \mathcal{L} and \mathcal{L}' of rational primes of Dirichlet density 1, we say that two compatible systems $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ and $\mathcal{R}' = \{\rho'_\lambda\}_{\lambda \in \Lambda'}$ defined over a number field E and indexed by $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$ and $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E|_\ell$ are *isomorphic* if there exists a subset $\mathcal{L}'' \subset \mathcal{L} \cap \mathcal{L}'$ of Dirichlet density 1 such that $\rho_\lambda \cong \rho'_\lambda$ for all $\ell \in \mathcal{L}''$ and $\lambda \mid \ell$, and we write $\mathcal{R} \cong_{(\mathcal{L}'')} \mathcal{R}'$, or for simplicity just $\mathcal{R} \cong \mathcal{R}'$. If they both have coefficients in E , we say that they are *isomorphic over E* if the isomorphisms $\rho_\lambda \cong \rho'_\lambda$ are defined over E .

Given an open subgroup Γ' of Γ of finite index, if we let $A' = \{\alpha \in A : F_\alpha \in \Gamma'\}$, then the pair $(\Gamma', \{F_\alpha\}_{\alpha \in A'})$ is also a group with Frobenii, and $\mathcal{R}|_{\Gamma'} = \{\rho_\lambda|_{\Gamma'}\}_{\lambda \in \Lambda}$ forms a compatible system, with $\mathcal{X}' = \mathcal{X} \cap (A' \times \Lambda)$.

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, and let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n representations of Γ with coefficients in a number field E . For any

$\lambda \in \Lambda$, denote by G_λ the Zariski closure of $\rho_\lambda(\Gamma)$ in $\mathrm{GL}_n(E_\lambda)$. This is an algebraic subgroup of $\mathrm{GL}_{n,E_\lambda}$, which we call the *algebraic monodromy group* of ρ_λ . We recall the following lemma.

Lemma 2.1.3 ([75, Lemma 6.9]²). *If ρ_λ is semisimple, then G_λ is reductive. If moreover ρ_λ is absolutely irreducible, then the natural representation $G_\lambda \rightarrow \mathrm{GL}_{n,E_\lambda}$ is absolutely irreducible.*

Denote by G_λ° the connected component of the identity of G_λ . It is immediate to notice that if Γ' is an open subgroup of Γ , and G'_λ denotes the algebraic monodromy group of $\rho_\lambda|_{\Gamma'}$, then the connected component of the identity G'°_λ of G'_λ coincides with G_λ° .

In the next sections, we consider the problem of studying the algebraic monodromy groups G_λ of $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ varying λ . The hope in this context is to find a global algebraic subgroup G of $\mathrm{GL}_{n,E}$ defined over E such that G_λ is conjugate to $G \times_E E_\lambda$ over E_λ for any $\lambda \in \Lambda$. This is the prototype of a *λ -independence* problem for the compatible system \mathcal{R} . Unfortunately, proving a λ -independence result of this form in the abstract setting is actually too difficult. For this reason, we will focus on weaker λ -independence statements.

2.2 The formal character and the variety of characteristic polynomials

If G is a reductive group over a field \mathbb{k} of characteristic zero, $T \subset G$ is a maximal torus of G over \mathbb{k} , and $\rho : G \rightarrow \mathrm{GL}_{n,\mathbb{k}}$ is a representation of G over \mathbb{k} , we define the *formal character* of ρ to be the restriction of ρ to T .

Let $\mathrm{ch} : \mathrm{GL}_{n,\mathbb{k}} \rightarrow \mathbb{G}_{m,\mathbb{k}} \times \mathbb{A}_{\mathbb{k}}^{n-1}$ be the morphism which associates to a matrix the coefficients of its characteristic polynomial. Given a connected reductive algebraic subgroup G of $\mathrm{GL}_{n,\mathbb{k}}$, and a maximal torus $T \subset G$ over \mathbb{k} , we have that $\mathrm{ch}(G) = \mathrm{ch}(T)$ is a variety defined over \mathbb{Q} , which determines uniquely the formal character of the natural representation $G \rightarrow \mathrm{GL}_{n,\mathbb{k}}$ up to isomorphism, see [75, §4].

Let $\mathbb{G}_{m,\mathbb{k}}^n$ be the split maximal torus of $\mathrm{GL}_{n,\mathbb{k}}$. The Weyl group of $\mathrm{GL}_{n,\mathbb{k}}$ with respect to $\mathbb{G}_{m,\mathbb{k}}^n$ is the symmetric group \mathfrak{S}_n , which acts by permutation of factors. Let $T_0 \subset \mathbb{G}_{m,\mathbb{k}}^n$ be a subtorus such that $T_0 \times_{\mathbb{k}} \bar{\mathbb{k}}$ is conjugate to $T \times_{\mathbb{k}} \bar{\mathbb{k}}$ over $\bar{\mathbb{k}}$. Since

²This result is actually proved in [75] in the case $E = \mathbb{Q}$. The adaptation to the general case is immediate. In the rest of this chapter, whenever we will cite results from [75] without any further clarification, it will mean that the adaptation to our setting is minor.

the semisimple part of any point in G can be conjugated into T_0 over $\bar{\mathbb{k}}$, we have that $\text{ch}(G) = \text{ch}(T_0)$ pointwise. For every $\sigma \in \mathfrak{S}_n \setminus \text{Cent}_{\mathfrak{S}_n}(T_0)$, we define a proper subgroup $H_\sigma \subset T_0$ in the following way. We let $H_\sigma = \{t \in T : \sigma(t) = t\}$ if $\sigma(T_0) = T_0$, and $H_\sigma = T_0 \cap \sigma(T_0)$ otherwise. We define now Y to be the union of all $\text{ch}(H_\sigma)$. This is a Zariski-closed proper subset of $\text{ch}(T_0) = \text{ch}(G)$. We say that a point $g \in G$ is Γ -regular if $\text{ch}(g) \notin Y$. We have the following result.

Proposition 2.2.1 ([75, Proposition 4.7]).

- (1) For every $x \in (\text{ch}(G) \setminus Y)(\mathbb{k})$, there exists a torus $T \subset \text{GL}_{n,\mathbb{k}}$, and an element $t \in T(\mathbb{k})$, such that $\text{ch}(T) = \text{ch}(G)$ and $\text{ch}(t) = x$. The pair (t, T) is unique up to conjugation by $\text{GL}_n(\mathbb{k})$.
- (2) Let $g \in G(\mathbb{k})$ be Γ -regular. Then g lies in a unique maximal torus T of G , and the $\text{GL}_n(\mathbb{k})$ -conjugacy class of (g, T) is uniquely determined by $\text{ch}(g)$ and $\text{ch}(G)$.

Let us move back to the setting of §2.1. Let then $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, let E be a number field, let \mathcal{L} be a set of rational primes of Dirichlet density 1, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n semisimple representations of Γ with coefficients in E and indexed by Λ . Keep notations as in §2.1. We have:

Proposition 2.2.2 ([75, Proposition 6.12]). *The formal character of the natural representation $G_\lambda^\circ \times_{E_\lambda} \bar{E}_\lambda \rightarrow \text{GL}_{n,\bar{E}_\lambda}$ is independent of λ . More precisely, there exist a torus T_0 over E and a faithful representation $\rho_0 : T_0 \rightarrow \text{GL}_{n,E}$ such that $T_0 \times_E \bar{E}_\lambda$ is isomorphic to a maximal torus in $G_\lambda^\circ \times_{E_\lambda} \bar{E}_\lambda$, and $\rho_0 \otimes_E \bar{E}_\lambda$ is equivalent to the formal character of $G_\lambda^\circ \times_{E_\lambda} \bar{E}_\lambda \rightarrow \text{GL}_{n,\bar{E}_\lambda}$ for all $\lambda \in \Lambda$.*

Proposition 2.2.3 ([75, Proposition 6.14]). *There is an open normal subgroup $\Gamma' \subset \Gamma$ such that ρ_λ induces an isomorphism $\Gamma/\Gamma' \xrightarrow{\sim} G_\lambda/G_\lambda^\circ$ for all $\lambda \in \Lambda$.*

Remark 2.2.4. Note that Proposition 2.2.3 implies that the group $G_\lambda/G_\lambda^\circ$ of connected components of G_λ is independent of λ . Also, $\mathcal{R}' = \mathcal{R}|_{\Gamma'}$ is a compatible system, and the algebraic monodromy group of $\rho_\lambda|_{\Gamma'}$ is G_λ° for every $\lambda \in \Lambda$.

2.3 Frobenius tori

In this section, we extend the theory of Frobenius tori of [75, §7] to compatible systems with coefficients in any number field E . From this, we clarify how the main results of [75, §8, 9] can be deduced in this setting.

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, let E be a number field, let \mathcal{L} be a set of rational primes of Dirichlet density 1, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n representations of Γ with coefficients in E and indexed by Λ . Throughout this section, we assume that each ρ_λ is semisimple, and that the algebraic monodromy group G_λ of ρ_λ is connected for each $\lambda \in \Lambda$.

The following result holds.

Proposition 2.3.1 ([75, Proposition 7.2]). *For any $\lambda \in \Lambda$, the set of all $\gamma \in \Gamma$ such that $\rho_\lambda(\gamma)$ is Γ -regular is open and dense in Γ .*

Since for every $(\alpha, \lambda) \in \mathcal{X}$ the condition on $\rho_\lambda(F_\alpha)$ being Γ -regular does depend only on $\text{ch}(\rho_\lambda(F_\alpha)) \in (\mathbb{G}_{m,E} \times \mathbb{A}_E^{n-1})(E)$, which in turn depends only on α , we will simply say that α is Γ -regular whenever this condition is satisfied. By Proposition 2.3.1, if we replace A with the set of Γ -regular $\alpha \in A$, the corresponding set $\mathcal{X} \subset A \times \Lambda$ still satisfies the conditions (1)-(3) in Definition 2.1.1. From now on, let us assume that every $\alpha \in A$ is actually Γ -regular.

We prove the following result, which generalises [75, §7.5] to our context.

Proposition 2.3.2. *For every $\alpha \in A$, there exists a torus $T_\alpha \subset \text{GL}_{n,E}$, unique up to conjugation by $\text{GL}_n(E)$, such that:*

- (1) *For all λ with $(\alpha, \lambda) \in \mathcal{X}$, $T_\alpha \times_E E_\lambda$ is conjugate to a maximal torus of G_λ .*
- (2) *For all pairwise distinct places $\lambda_1, \dots, \lambda_k \in \Lambda$, and all maximal tori T_i of G_{λ_i} , there exists an $\alpha \in A$ such that, for every i , $(\alpha, \lambda_i) \in \mathcal{X}$ and $T_\alpha \times_E E_{\lambda_i}$ is conjugate to T_i .*

Proof. Fix $\alpha \in A$. Then, for all but finitely many places $\lambda \in \Lambda$, $(\lambda, \alpha) \in \mathcal{X}$ and the characteristic polynomial of $\rho_\lambda(F_\alpha)$ has coefficients in E and is independent of λ . Recall also that $\rho_\lambda(F_\alpha)$ is Γ -regular by assumption. Therefore, by Proposition 2.2.1, there exists a torus $T_\alpha \subset \text{GL}_{n,E}$, unique up to conjugation by $\text{GL}_n(E)$, such that $T_\alpha \times_E E_\lambda$ is conjugate to the unique maximal torus of G_λ containing $\rho_\lambda(F_\alpha)$. This proves the first part of the proposition.

Let us now move to the second part. Let then $\lambda_1, \dots, \lambda_k \in \Lambda$ be pairwise distinct places, and let T_i be maximal tori of G_{λ_i} . For each i , let ℓ_i be the residue characteristic of λ_i , and let $\rho_{\ell_i} = \oplus_{\lambda|\ell_i} \rho_\lambda$, which we regard as a representation of Γ over \mathbb{Q}_{ℓ_i} of dimension $n[E : \mathbb{Q}]$. The algebraic monodromy group G_{ℓ_i} of ρ_{ℓ_i} is a closed subgroup of $\prod_{\lambda|\ell_i} G_\lambda$. For each $\lambda \mid \ell_i$, the composition p_λ of the inclusion of G_{ℓ_i} into $\prod_{\lambda|\ell_i} G_\lambda$ with the canonical projection onto the λ -th factor is a morphism of algebraic groups,

and hence $p_\lambda(G_{\ell_i})$ is a closed subgroup of G_λ , see [18, Corollary 1.4]. By the definition of G_{ℓ_i} and G_λ , we then have $p_\lambda(G_{\ell_i}) = G_\lambda$, and so p_λ is a surjective morphism of algebraic groups. By [18, Proposition 11.14], there exists a maximal torus T_{ℓ_i} of G_{ℓ_i} such that $p_{\lambda_i}(T_{\ell_i}) = T_i$. Combining now property (3) of \mathcal{X} , see Definition 2.1.1, with Proposition 2.3.1 and [75, Proposition 7.3], we have that there exists $\alpha \in A$ such that $(\alpha, \lambda_i) \in \mathcal{X}$ and $\rho_{\ell_i}(F_\alpha)$ is conjugate to an element of $T_{\ell_i}(\mathbb{Q}_{\ell_i})$ over \mathbb{Q}_{ℓ_i} for each i . But then $\rho_\lambda(F_\alpha) = p_{\lambda_i}(\rho_{\ell_i}(F_\alpha))$ is conjugate to an element of $T_i(E_{\lambda_i})$ over E_{λ_i} for each i . It follows that $T_\alpha \times_E E_{\lambda_i}$ is conjugate to T_i . \square

The tori T_α over E will be called *Frobenius tori*. For every $\alpha \in A$, let L_α be the splitting field of T_α over E , that is the smallest degree extension L_α/E such that $T_\alpha \times_E L_\alpha$ is split. Let L be the intersection of all L_α . We call this field the *splitting field* of the compatible system \mathcal{R} . The following result holds.

Proposition 2.3.3 (compare [75, Proposition 8.9]³). *There exists a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1 such that for every $\ell \in \mathcal{L}'$ and every $\lambda \mid \ell$ the connected reductive group G_λ is unramified, and split over LE_λ .*

Let us denote by c_λ the formal character of the natural representation $G_\lambda \rightarrow \mathrm{GL}_{n, E_\lambda}$. Let us also denote by Ψ_λ the root datum of G_λ , and by $W(\Psi_\lambda)$ the Weyl group of Ψ_λ . We have:

Theorem 2.3.4 (compare [75, Theorem 9.4]⁴). *Assume that each ρ_λ is absolutely irreducible, and that $L = E$. Let \mathcal{L}' be a subset of \mathcal{L} such that for each $\ell \in \mathcal{L}'$ and each $\lambda \mid \ell$ the connected reductive group G_λ splits over E_λ . Then, for all $\ell \in \mathcal{L}'$ and all $\lambda \mid \ell$ the triple $(\Psi_\lambda, c_\lambda, W(\Psi_\lambda))$ is independent of λ up to isomorphism.*

2.4 Abstract independence results

Let $(\Gamma, \{F_\alpha\}_{\alpha \in A})$ be a group with Frobenii, let E be a number field, let \mathcal{L} be a set of rational primes of Dirichlet density 1, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. From the results of §2.3 we deduce the following:

³Again, this result is actually proved in [75] in the case $E = \mathbb{Q}$. Notice however that the results of [75, §8] only rely on the results of [75, §3], which are proved over any perfect field, and on [75, §7.5], which we extended to our setting in Proposition 2.3.2.

⁴The result in [75], again proved in the case $E = \mathbb{Q}$, only relies on the results of [75, §2] on abstract root data, on [75, Proposition 6.12], on the results of [75, §5], which are applied after base change to \mathbb{C} , and on [75, §7.5]. In our setting [75, Proposition 6.12] and [75, §7.5] are extended respectively in Proposition 2.2.2 and Proposition 2.3.2.

Proposition 2.4.1. *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n absolutely Lie-irreducible⁵ representations of Γ with coefficients in E and indexed by Λ . Let \mathcal{L}' be a subset of \mathcal{L} of Dirichlet density 1 such that for each $\ell \in \mathcal{L}'$ and each $\lambda \mid \ell$ the connected reductive group G_λ° is unramified. Then, there exist a finite extension E' of E and a split connected reductive algebraic subgroup G of $\mathrm{GL}_{n,E'}$ such that for each $\ell \in \mathcal{L}'$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ and $G \times_{E'} E'_{\lambda'}$ are conjugate over $E'_{\lambda'}$.*

Proof. Let Γ' be the open subgroup of Γ such that $\Gamma/\Gamma' \xrightarrow{\sim} G_\lambda/G_\lambda^\circ$ for all λ as in Proposition 2.2.3. As each ρ_λ is absolutely Lie-irreducible, the restrictions $\rho_\lambda|_{\Gamma'}$ are still absolutely irreducible. Up to restrict to Γ' , we can assume each G_λ to be connected. We can therefore apply the results of §2.3. Let E' be the splitting field of \mathcal{R} . By Proposition 2.3.3, for each $\ell \in \mathcal{L}'$ and each $\lambda \mid \ell$ the group G_λ splits over $E'E_\lambda$. Let $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E'|_\ell$, and for each $\lambda' \in \Lambda'$, denoting by λ its restriction to E , let $\rho'_{\lambda'} = \rho_\lambda \otimes_{E_\lambda} E'_{\lambda'}$. Consider now the compatible system $\{\rho'_{\lambda'}\}_{\lambda' \in \Lambda'}$. Clearly, the splitting field of $\{\rho'_{\lambda'}\}_{\lambda' \in \Lambda'}$ is equal to E' , and the algebraic monodromy group $G'_{\lambda'}$ of $\rho'_{\lambda'}$ is equal to $G_\lambda \times_{E_\lambda} E'_{\lambda'}$ by construction. By Theorem 2.3.4, we then have that for all $\lambda' \in \Lambda'$ the triple $(\Psi'_{\lambda'}, c'_{\lambda'}, W(\Psi'_{\lambda'}))$, where $\Psi'_{\lambda'}$ denotes the root datum of $G'_{\lambda'}$, $c'_{\lambda'}$ denotes the formal character of the natural representation of $G'_{\lambda'}$ into $\mathrm{GL}_{n,E'_{\lambda'}}$, and $W(\Psi'_{\lambda'})$ denotes the Weyl group of $\Psi'_{\lambda'}$, is independent of λ' . Fix then such a triple $(\Psi, c', W(\Psi'))$, and let G be the (unique up to conjugation, by [74, Theorem 4]⁶) split connected reductive group over E' with root datum Ψ and formal character c' . \square

Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n representations of Γ with coefficients in E and indexed by Λ . We introduce the following definition.

Definition 2.4.2. We say that \mathcal{R} has a *Lie-irreducible decomposition* over E and at a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1 if there exist open subgroups Γ_i of Γ , compatible systems $\mathcal{S}_i = \{\sigma_{i,\lambda}\}_{\lambda \in \Lambda'}$ of rank m_i absolutely Lie-irreducible representations of Γ_i with coefficients in E and indexed by $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E|_\ell$, and rank d_i Artin representations⁷ ω_i

⁵We say that a rank n representation ρ of Γ with coefficients in a field \mathbb{k} of characteristic zero is *absolutely Lie-irreducible* if $\rho|_{\Gamma'}$ is absolutely irreducible for any open subgroup Γ' of Γ .

⁶This result shows that the formal character uniquely determines the conjugacy class of an irreducible semisimple subgroup of the general linear group over \mathbb{C} , up to certain (completely classified) root system ambiguities. It is immediate to extend it to any split semisimple subgroup of the general linear group over any field \mathbb{k} of characteristic zero. We are going to apply it in this form also in the proof of Proposition 2.4.4. The extension to the case of reductive groups follows from [75, §5.2].

⁷In this setting, we say that a continuous representation $\omega : \Gamma \rightarrow \mathrm{GL}_d(\overline{E}_\lambda)$ is *Artin* if it factors through a finite quotient of Γ . Any Artin representation ω can be actually realised over $\overline{\mathbb{Q}}$.

of Γ_i , for $i = 1, \dots, k$, such that

$$\rho_\lambda \cong \bigoplus_{i=1}^k \text{Ind}_{\Gamma_i}^\Gamma(\sigma_{i,\lambda} \otimes \omega_i)$$

over E for all $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$. In this case, we write

$$\mathcal{R} \cong_{(\mathcal{L}')} \bigoplus_{i=1}^k \text{Ind}_{\Gamma_i}^\Gamma(\mathcal{S}_i \otimes \omega_i).$$

Remark 2.4.3. For some choices of the group Γ , one can prove that any λ -adic representation $\rho : \Gamma \rightarrow \text{GL}_n(\overline{E}_\lambda)$ has a Lie-irreducible decomposition $\rho \cong \bigoplus_{i=1}^k \text{Ind}_{\Gamma_i}^\Gamma(\sigma_i \otimes \omega_i)$, where Γ_i is an open subgroup of Γ , $\sigma_i : \Gamma_i \rightarrow \text{GL}_{m_i}(\overline{E}_\lambda)$ is Lie-irreducible, and $\omega_i : \Gamma_i \rightarrow \text{GL}_{d_i}(\overline{E}_\lambda)$ is Artin, for $i = 1, \dots, k$. This has been proved by Katz, see [67, Proposition 1], when Γ is the étale fundamental group of a smooth connected affine curve over an algebraically closed field of positive characteristic, and by Patrikis, see [82, Proposition 3.4.1], by adapting Katz's argument, when Γ is the absolute Galois group of a number field. To prove a decomposition result for a compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ of representations of Γ , one would then start from a single λ -adic representation ρ_λ in \mathcal{R} , decompose it as $\rho_\lambda \cong \bigoplus_{i=1}^k \text{Ind}_{\Gamma_i}^\Gamma(\sigma_{i,\lambda} \otimes \omega_i)$, and extend each $\sigma_{i,\lambda}$ to a compatible system \mathcal{S}_i of Lie-irreducible representations. The isomorphism $\mathcal{R} \cong_{(\mathcal{L}')} \bigoplus_{i=1}^k \text{Ind}_{\Gamma_i}^\Gamma(\mathcal{S}_i \otimes \omega_i)$ would then follow by considerations on the traces of Frobenius elements, see the proofs of Theorem 2.5.1 and Theorem 2.6.2. However, extending a λ -adic representation to a compatible system is a highly non trivial problem, and can be achieved, for instance, under suitable assumptions in the number field case by means of potential automorphy techniques, see §2.5, and in the positive characteristic case by means of the global Langlands correspondence for GL_n of L. Lafforgue, see §2.6.

We prove the following result.

Proposition 2.4.4. *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of rank n representations of Γ with coefficients in E and indexed by Λ which has a Lie-irreducible decomposition over E and at a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1. Let \mathcal{L}'' be a subset of \mathcal{L}' of Dirichlet density 1 such that for each $\ell \in \mathcal{L}''$ and each $\lambda \mid \ell$ the connected reductive group G_λ° is unramified. Then, there exist a finite extension E' of E and a split connected reductive algebraic subgroup G of $\text{GL}_{n,E'}$ such that for each $\ell \in \mathcal{L}''$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ and $G \times_{E'} E'_{\lambda'}$ are conjugate over $E'_{\lambda'}$.*

Proof. As \mathcal{R} has a Lie-irreducible decomposition over E at \mathcal{L}' , we have that there exist open subgroups Γ_i of Γ , compatible systems $\mathcal{S}_i = \{\sigma_{i,\lambda}\}_{\lambda \in \Lambda'}$ of rank m_i absolutely Lie-irreducible representations of Γ_i with coefficients in E and indexed by $\Lambda' = \cup_{\ell \in \mathcal{L}'} |E|_\ell$, and rank d_i Artin representations ω_i of Γ_i , for $i = 1, \dots, k$, such that for each $\ell \in \mathcal{L}'$ and each $\lambda \mid \ell$ we have $\rho_\lambda \cong \bigoplus_{i=1}^k \text{Ind}_{\Gamma_i}^\Gamma(\sigma_{i,\lambda} \otimes \omega_i)$.

For each i and $\lambda \in \Lambda'$, let us denote by $G_{i,\lambda}$ the algebraic monodromy group of $\rho_{i,\lambda}$, and by $G_{i,\lambda}^\circ$ its connected component of the identity. The group G_λ° is then a closed subgroup of $\prod_{i=1}^k G_{i,\lambda}^\circ$, and each of the natural projections map G_λ° surjectively onto $G_{i,\lambda}^\circ$. Let $\Lambda'' = \cup_{\ell \in \mathcal{L}''} |E|_\ell$. Since for each $\lambda \in \Lambda''$ the group G_λ° is unramified, it follows that each $G_{i,\lambda}^\circ$ is also unramified by [18, Theorem 22.6]. For each $\lambda \in \Lambda'$, let us denote by $H_{i,\lambda}$ the algebraic monodromy group of $\sigma_{i,\lambda}$, and by $H_{i,\lambda}^\circ$ its connected component of the identity. Let Γ'_i be an open normal subgroup of Γ contained in Γ_i and such that $\rho_{i,\lambda}|_{\Gamma'_i} \cong \sigma_{i,\lambda}^{\oplus d_i}|_{\Gamma'_i}$. Since restriction to an open subgroup leaves the connected component of the identity of the algebraic monodromy group of a λ -adic representation invariant, we have that $G_{i,\lambda}^\circ$ coincides with the connected component of the identity of the algebraic monodromy group of $\rho_{i,\lambda}|_{\Gamma'_i}$, which in turns coincides with the connected component of the identity of the algebraic monodromy group of $\sigma_{i,\lambda}^{\oplus d_i}|_{\Gamma'_i}$, and hence with that of $\sigma_{i,\lambda}^{\oplus d_i}$. Now, the connected component of the identity of the algebraic monodromy group of $\sigma_{i,\lambda}^{\oplus d_i}$ coincides with $H_{i,\lambda}^\circ$, diagonally embedded into the product of d_i copies of $H_{i,\lambda}^\circ$. It follows that for each $\lambda \in \Lambda''$ the group H_λ° is unramified. As each $\sigma_{i,\lambda}$ for $\lambda \in \Lambda$ is absolutely Lie-irreducible, Proposition 2.4.1 applies to \mathcal{S}_i , and so there exist a finite extension E'_i of E , and a split connected reductive algebraic subgroup H_i of $\text{GL}_{m_i, E'}$, such that for each $\ell \in \mathcal{L}''$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $H_{i,\lambda}^\circ \times_{E_\lambda} E'_{i,\lambda'}$ and $H_i \times_{E'_i} E'_{i,\lambda'}$ are conjugate over $E'_{i,\lambda'}$. Define then G_i to be the group H_i , diagonally embedded into $H_i^{d_i}$, and hence a connected reductive algebraic subgroup of $\text{GL}_{n_i, E'}$, where $n_i = m_i d_i$. We have that for each $\ell \in \mathcal{L}''$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_{i,\lambda}^\circ \times_{E_\lambda} E'_{i,\lambda'}$ and $G_i \times_{E'_i} E'_{i,\lambda'}$ are conjugate over $E'_{i,\lambda'}$.

Recall that for each $\lambda \in \Lambda'$, the group G_λ° is a closed subgroup of $\prod_{i=1}^k G_{i,\lambda}^\circ$, and each $G_{i,\lambda}^\circ$ coincides with $H_{i,\lambda}^\circ$, diagonally embedded into $(H_{i,\lambda}^\circ)^{d_i}$. Let $\text{GL}_{\underline{n}, E_\lambda} = \prod_{i=1}^k \text{GL}_{m_i, E_\lambda}^{d_i}$. The group G_λ° can then be seen as a closed subgroup of $\text{GL}_{\underline{n}, E_\lambda}$, where each standard representation $G_\lambda^\circ \rightarrow \text{GL}_{m_i, E_\lambda}$ is absolutely irreducible, as each $\sigma_{i,\lambda}$ is absolutely Lie-irreducible. Let G_λ^{der} be the derived subgroup of G_λ° . We want to show that it is enough to prove λ -independence of G_λ^{der} . Let then $\det_{m_i} : \text{GL}_{m_i, E_\lambda} \rightarrow \mathbb{G}_{m, E_\lambda}$ be the usual determinant map, and let $\det_{\underline{n}} = \prod_{i=1}^k \det_{m_i}^{d_i} : \text{GL}_{\underline{n}, E_\lambda} \rightarrow \mathbb{G}_{m, E_\lambda}^d$, where $d = \sum_{i=1}^k d_i$. Then, $\det_{\underline{n}}$ maps G_λ° onto a subtorus T_λ of $\mathbb{G}_{m, E_\lambda}^d$. The centre of $\text{GL}_{\underline{n}, E_\lambda}$

maps onto \mathbb{G}_{m,E_λ} , and the preimage of T_λ is the connected component of the identity of the centre of G_λ° . Since $\{\det_{\underline{n}} \circ \rho_\lambda\}_{\lambda \in \Lambda'}$ is a compatible system whose algebraic monodromy groups are tori, we have that there exists a subtorus T of $\mathbb{G}_{m,E}^d$ such that $T_\lambda = T \times_E E_\lambda$ for each $\lambda \in \Lambda'$. It follows that the connected components of the identity of the centres of G_λ° come from a fixed torus in the centre of $\mathrm{GL}_{\underline{n},E}$ for each $\lambda \in \Lambda'$. For this reason, for λ -independence questions it is enough to deal with G_λ^{der} .

Let us first assume that $k = 2$. Let then E' be the compositum of E'_1 and E'_2 inside a fixed algebraic closure \overline{E} of E , let $\ell \in \mathcal{L}''$, let λ be a place of E above ℓ , and let λ' be a place of E' above ℓ . Let $\mathfrak{g}_{\lambda'}$ be the Lie algebra of $G_\lambda^{\mathrm{der}} \times_{E_\lambda} E'_{\lambda'}$, let G_i^{der} denote the derived subgroup of G_i , let \mathfrak{g}_i be the Lie algebra of $G_i^{\mathrm{der}} \times_{E_i} E'$, and let $\mathfrak{g}_{i,E'_{\lambda'}} = \mathfrak{g}_i \otimes_{E'} E'_{\lambda'}$, for $i = 1, 2$. We have that $\mathfrak{g}_{\lambda'}$ is a subalgebra of $\mathfrak{g}_{1,E'_{\lambda'}} \oplus \mathfrak{g}_{2,E'_{\lambda'}}$, such that $\pi_i(\mathfrak{g}_{\lambda'}) = \mathfrak{g}_{i,E'_{\lambda'}}$, where $\pi_i : \mathfrak{g}_{1,E'_{\lambda'}} \oplus \mathfrak{g}_{2,E'_{\lambda'}} \rightarrow \mathfrak{g}_{i,E'_{\lambda'}}$ denotes the canonical projection, for $i = 1, 2$. Since \mathfrak{g}_1 and \mathfrak{g}_2 are semisimple, they are direct sums of simple factors, and so there exists a semisimple Lie algebra \mathfrak{s} over E' such that $\mathfrak{g}_i \cong \mathfrak{s} \oplus \tilde{\mathfrak{g}}_i$, for $i = 1, 2$, and such that $\tilde{\mathfrak{g}}_1$ and $\tilde{\mathfrak{g}}_2$ have no common simple factors. Since $\mathfrak{g}_{\lambda'}$ is semisimple, and $\pi_i(\mathfrak{g}_{\lambda'}) = \mathfrak{g}_{i,E'_{\lambda'}}$, for $i = 1, 2$, we have $\mathfrak{g}_{\lambda'} \cong \mathfrak{s}_{E'_{\lambda'}} \oplus \tilde{\mathfrak{g}}_{\lambda'}$, where $\mathfrak{s}_{E'_{\lambda'}} = \mathfrak{s} \otimes_{E'} E'_{\lambda'}$, and $\pi_i(\tilde{\mathfrak{g}}_{\lambda'}) = \tilde{\mathfrak{g}}_{i,E'_{\lambda'}}$, where $\tilde{\mathfrak{g}}_{i,E'_{\lambda'}} = \tilde{\mathfrak{g}}_i \otimes_{E'} E'_{\lambda'}$, for $i = 1, 2$. By Goursat's lemma, since there exist no nonzero ideals $\mathfrak{n}_{i,\lambda}$ of $\tilde{\mathfrak{g}}_{i,E'_{\lambda'}}$, for $i = 1, 2$, such that $\tilde{\mathfrak{g}}_{1,E'_{\lambda'}}/\mathfrak{n}_{1,\lambda} \cong \tilde{\mathfrak{g}}_{2,E'_{\lambda'}}/\mathfrak{n}_{2,\lambda}$, as $\tilde{\mathfrak{g}}_{1,E'_{\lambda'}}$ and $\tilde{\mathfrak{g}}_{2,E'_{\lambda'}}$ have no common simple factors, we deduce that $\tilde{\mathfrak{g}}_{\lambda'} \cong \tilde{\mathfrak{g}}_{1,E'_{\lambda'}} \oplus \tilde{\mathfrak{g}}_{2,E'_{\lambda'}}$, and so $\mathfrak{g}_{\lambda'} \cong \mathfrak{s}_{E'_{\lambda'}} \oplus \tilde{\mathfrak{g}}_{1,E'_{\lambda'}} \oplus \tilde{\mathfrak{g}}_{2,E'_{\lambda'}}$, which gives that $\mathfrak{g}_{\lambda'}$ is independent of λ' . This implies that the root system $\Phi_{\lambda'}$ of $G_\lambda^{\mathrm{der}} \times_{E_\lambda} E'_{\lambda'}$ is independent of λ' .

When $k > 2$, by working iteratively with the compatible systems $\{\rho_\lambda^{(j)} \oplus \rho_{j+1,\lambda}\}_{\lambda \in \Lambda'}$, where $\rho_\lambda^{(j)} = \bigoplus_{i=1}^j \rho_{i,\lambda}$, for $j = 1, \dots, k-1$, by the above argument (notice that we didn't use the fact that $\rho_{1,\lambda}$ and $\rho_{2,\lambda}$ are absolutely irreducible) we get that the root system $\Phi_{\lambda'}$ of $G_\lambda^{\mathrm{der}} \times_{E_\lambda} E'_{\lambda'}$ is independent of λ' , where now E' denotes the compositum of the extensions E'_i of E inside \overline{E} .

Since by Proposition 2.2.2 the formal character of each standard representation $G_\lambda^{\mathrm{der}} \times_{E_\lambda} E'_{\lambda'} \rightarrow \mathrm{GL}_{m_i,E'_{\lambda'}}$ is also independent of λ' , by [74, Theorem 4] we have that the conjugacy class of $G_\lambda^{\mathrm{der}} \times_{E_\lambda} E'_{\lambda'}$ inside each $\mathrm{GL}_{m_i,E'_{\lambda'}}$ is independent of λ' . It then follows that the conjugacy class of $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ inside $\mathrm{GL}_{n,E'_{\lambda'}}$ is independent of λ' , as required. \square

2.5 The case of geometric compatible systems of Galois representations

Let F be a CM field, and let $n \geq 1$ be an integer. In this section, we prove a decomposition theorem, and hence an independence result, for a class of geometric compatible systems of rank n representations of Γ_F , assuming that Conjecture 1.7.3 holds true. Let E be a number field, let \mathcal{L} be a set of rational primes of Dirichlet density 1, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. The decomposition theorem is the following.

Theorem 2.5.1. *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a pure, regular, totally odd, conjugate self-dual geometric compatible system of rank n representations of Γ_F with coefficients in E and indexed by Λ . Assume Conjecture 1.7.3. Then, \mathcal{R} has a Lie-irreducible decomposition over a finite extension E' of E and at a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1.*

Proof. Assuming Conjecture 1.7.3, by [84, Theorem 2.1] we have that there exists a set \mathcal{L}^0 of rational primes of Dirichlet density 1, and strictly pure, regular, totally odd, conjugate self-dual compatible systems $\mathcal{R}_i = \{\rho_{i,\lambda}\}_{\lambda \in \Lambda}$ of irreducible representations $\rho_{i,\lambda} : \Gamma_F \rightarrow \mathrm{GL}_{n_i}(\overline{E}_\lambda)$ indexed by $\Lambda^0 = \cup_{\ell \in \mathcal{L}^0} |E|_\ell$, for $i = 1, \dots, k$, such that

$$\rho_\lambda \otimes_{E_\lambda} \overline{E}_\lambda \cong \bigoplus_{i=1}^k \rho_{i,\lambda},$$

for each $\ell \in \mathcal{L}^0$ and $\lambda \mid \ell$. Furthermore, after possibly removing finitely many primes from \mathcal{L}^0 , we can assume that for each $\ell \in \mathcal{L}^0$ we have $\ell \geq 2(n+1)$.

Fix an i , and consider the compatible system \mathcal{R}_i . By [84, Lemma 1.2], we can assume that \mathcal{R}_i is defined over a CM field E_i . Let $\Lambda_i^0 = \cup_{\ell \in \mathcal{L}^0} |E_i|_\ell$. For all $\lambda \in \Lambda_i^0$, let $G_{i,\lambda}$ be the algebraic monodromy group⁸ of $\rho_{i,\lambda}$, and let $G_{i,\lambda}^\circ$ be the connected component of the identity of $G_{i,\lambda}$. By Lemma 2.2.3 there exists a finite Galois extension F'_i/F such that the representation $\rho_{i,\lambda}$ induces an isomorphism $\mathrm{Gal}(F'_i/F) \xrightarrow{\sim} G_{i,\lambda}/G_{i,\lambda}^\circ$ for all $\lambda \in \Lambda_i^0$. By applying [6, Proposition 5.3.2], we get that there exists a subset \mathcal{L}_i of \mathcal{L}^0 of Dirichlet density 1 such that if $\ell \in \mathcal{L}_i$, then $\overline{\rho}_{i,\lambda} \mid_{\Gamma_{F(\zeta_\ell)}}$ is irreducible for every $\lambda \mid \ell$. Let $\Lambda_i = \cup_{\ell \in \mathcal{L}_i} |E_i|_\ell$. Up to removing finitely many places from \mathcal{L}_i , we may further assume that if $\ell \in \mathcal{L}_i$, and $\lambda \mid \ell$, then

- $\zeta_\ell \notin F'_i$,
- ℓ is unramified in F'_i and lies below no element of the set of bad places for \mathcal{R}_i ,
- the Hodge-Tate weights of $\rho_{i,\lambda}$ lie in a range of the form $[a_i, a_i + \ell - 2]$.

⁸This would be a reductive algebraic subgroup of $\mathrm{GL}_{n,E'_i,\lambda'}$, for E'_i a finite extension of E_i where \mathcal{R}_i takes coefficients, and λ' a place of E'_i above λ .

Let $F_i^{\prime, \text{cm}}$ denote the maximal CM subfield of F_i' . Then, assuming Conjecture 1.7.3, by [84, Theorem 2.1] we have that there exist subsets \mathcal{L}'_i of \mathcal{L}_i of Dirichlet density 1 such that each irreducible component of $\rho_{i, \lambda}|_{\Gamma_{F_i^{\prime, \text{cm}}}}$ is totally odd, conjugate self-dual for each $\ell \in \mathcal{L}'_i$ and $\lambda \mid \ell$. Let $\Lambda'_i = \cup_{\ell \in \mathcal{L}'_i} |E_i|_\ell$.

By [6, Lemma 5.3.1(2)] each $\rho_{i, \lambda}$ is Lie-multiplicity free⁹. Therefore, for each $\lambda \in \Lambda'_i$, by [82, Lemma 3.4.6(1)] there exist an intermediate field $F \subset F_i^\lambda \subset F_i'$, and a rank m_i^λ Lie-irreducible representation σ_i^λ of $\Gamma_{F_i^\lambda}$ such that we can write $\rho_{i, \lambda} \cong \text{Ind}_{\Gamma_{F_i^\lambda}}^{\Gamma_F} \sigma_i^\lambda$. We now prove that there exists $\lambda_0 \in \Lambda'_i$ such that $F_i^{\lambda_0}$ is a CM field¹⁰. Denote by $F_i^{\lambda, \text{cm}}$ the maximal CM subfield of F_i^λ . For simplicity, let us enlarge E_i to contain the maximal CM subfield of F_i' , so that $F_i^{\lambda, \text{cm}} \subset E_i$ for all $\lambda \in \Lambda'_i$, and let us take its Galois closure. With a slight abuse of notation, let us still denote its corresponding set of places by Λ'_i . If $F_i^\lambda \neq F_i^{\lambda, \text{cm}}$ for all $\lambda \in \Lambda'_i$, then we can find a place $\lambda_0 \in \Lambda'_i$ with residue characteristic ℓ_0 such that ℓ_0 splits in E_i but not in $F_i^{\lambda_0}$. Let now w_0 be a non-split place of $F_i^{\lambda_0}$ of residue characteristic ℓ_0 . Since $F_{i, w_0}^{\lambda_0}$ does not embed in $E_{i, \lambda_0} = \mathbb{Q}_{\ell_0}$ by assumption, we can deduce by an argument analogous to that of [82, Lemma 3.4.13] that ρ_{i, λ_0} is not regular, which is a contradiction. Therefore, we get that $F_i^{\lambda_0} = F_i^{\lambda_0, \text{cm}}$. Also, $\ell_0 \geq 2(m_i^{\lambda_0} + 1)$ and $\zeta_{\ell_0} \notin F_i^{\lambda_0}$ by the previous assumptions. For simplicity of notation, we set $F_i = F_i^{\lambda_0}$.

The representation $\sigma_i^{\lambda_0}$ is an irreducible component of $\rho_{i, \lambda_0}|_{\Gamma_{F_i}}$, and so it is totally odd, conjugate self-dual, as $F_i \subset F_i^{\prime, \text{cm}}$, and its set of Hodge-Tate weights is a subset of the set of Hodge-Tate weights of ρ_{i, λ_0} . This gives that $\sigma_i^{\lambda_0}$ has Hodge-Tate weights lying in $[a_i, a_i + \ell_0 - 2]$. Since ℓ_0 is unramified in F_i' , and so it is in F_i as well, we have that $\sigma_i^{\lambda_0}$ is crystalline at each place w_0 of F_i lying above ℓ_0 by [82, Lemma 2.2.9]. Therefore, $\sigma_i^{\lambda_0}$ is potentially diagonalizable at each place w_0 of F_i lying above ℓ_0 .

By applying Mackey restriction formula to $\bar{\rho}_{i, \lambda_0}|_{\Gamma_{F(\zeta_{\ell_0})}} \cong \text{Res}_{\Gamma_{F(\zeta_{\ell_0})}}^{\Gamma_F} \text{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \bar{\sigma}_i^{\lambda_0}$, which is irreducible by assumption, we deduce that $\bar{\sigma}_i^{\lambda_0}|_{\Gamma_{F_i(\zeta_{\ell_0})}}$ is irreducible. Therefore, we can apply [6, Theorem 5.5.1] to get that $\sigma_i^{\lambda_0}$ is part of a strictly pure compatible system $\mathcal{S}_i = \{\sigma_{i, \lambda}\}_{\lambda \in \Lambda'_i}$ of representations of Γ_{F_i} defined over E_i .

Since \mathcal{S}_i is regular, each representation $\sigma_{i, \lambda}$ is Lie-multiplicity free, again by [6, Lemma 5.3.1(2)]. Also, again by assuming Conjecture 1.7.3, we have that there exists a subset \mathcal{L}''_i of \mathcal{L}'_i of Dirichlet density 1, with $\ell_0 \in \mathcal{L}''_i$, such that $\sigma_{i, \lambda}$ is irreducible for each $\ell \in \mathcal{L}''_i$ and $\lambda \mid \ell$. Since $\sigma_{i, \lambda_0} \cong \sigma_i^{\lambda_0}$ is Lie-irreducible, and Lie-irreducibility in a

⁹We say that a rank n representation ρ of a profinite group Γ with coefficients in an algebraically closed field \mathbb{k} of characteristic zero is *Lie-multiplicity free* if for any open subgroup Γ' of Γ any irreducible Γ' -subrepresentation of ρ has multiplicity 1.

¹⁰The following argument is largely adapted on that of [82, Lemma 3.4.13].

compatible system of irreducible, Lie-multiplicity free representations is independent of λ by [82, Corollary 3.4.11], we get that $\sigma_{i,\lambda}$ is Lie-irreducible for each $\ell \in \mathcal{L}_i''$ and $\lambda \mid \ell$.

For every finite place v of F outside the set of bad places for \mathcal{R}_i and every $\lambda \in \Lambda'_i$ not lying above the residue characteristic of v we have that

$$\begin{aligned} \mathrm{Tr} \rho_{i,\lambda}(\mathrm{Frob}_v) &= \mathrm{Tr} \rho_{i,\lambda_0}(\mathrm{Frob}_v) \\ &= \mathrm{Tr} \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \sigma_i^{\lambda_0}(\mathrm{Frob}_v) \\ &= \mathrm{Tr} \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \sigma_{i,\lambda}(\mathrm{Frob}_v), \end{aligned}$$

where the first equality follows from the independence of λ of the characteristic polynomials at the Frobenius elements in the compatible system \mathcal{R}_i , and the last equality follows from the usual formula for the trace of an induced representation, see [6, §5.5] for instance, and the independence of λ of the characteristic polynomials at the Frobenius elements in the compatible system \mathcal{S}_i . Combining the Čebotarev density theorem with [20, §12.1, Proposition 3] we deduce that $\rho_{i,\lambda} \cong \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \sigma_{i,\lambda}$ for every $\lambda \in \Lambda'_i$.

Up to enlarging E_i , we can assume that each $\sigma_{i,\lambda}$ is actually defined over E_i , so that \mathcal{S}_i has coefficients in E_i . Let then $\mathcal{L}' = \cap_{i=1}^k \mathcal{L}_i''$, which also has Dirichlet density 1, and let E' be an extension of E containing all the fields E_i . Since $\sigma_{i,\lambda}$ is absolutely Lie-irreducible for each $\ell \in \mathcal{L}'$ and $\lambda \mid \ell$, we have that

$$\mathcal{R} \otimes_E E' \cong_{(\mathcal{L}')} \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} \mathcal{S}_i \otimes_{E_i} E'$$

is a Lie-irreducible decomposition over E' and at \mathcal{L}' . \square

Combining this with Proposition 2.4.4, we get the following result.

Corollary 2.5.2. *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a pure, regular, totally odd, conjugate self-dual geometric compatible system of rank n representations of Γ_F with coefficients in E and indexed by Λ . Assume Conjecture 1.7.3. Then, there exist a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1, a finite extension E' of E , and a split connected reductive algebraic subgroup G of $\mathrm{GL}_{n,E'}$ such that for each $\ell \in \mathcal{L}'$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ and $G \times_{E'} E'_{\lambda'}$ are conjugate over $E'_{\lambda'}$.*

Remark 2.5.3. In Theorem 2.5.1, assuming Conjecture 1.7.3 is required in order to apply [6, Theorem 5.5.1] to get potential automorphy of certain λ -adic representations. This in turn guarantees that such representations can be “extended” to a compatible system. Refined potential automorphy results would then eventually imply more

general versions of Theorem 2.5.1 and Corollary 2.5.2. Notice also that the conclusions of Theorem 2.5.1 and Corollary 2.5.2 hold unconditionally when the compatible system has extremely regular weights, in the sense of [6, §5.1].

Remark 2.5.4. Let π be a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and let $\mathcal{R}_\pi = \{\rho_{\pi,\lambda}\}_{\lambda \in |E_\pi|}$ be the corresponding compatible system of representations of Γ_F . Assuming Conjecture 1.7.3, the conclusion of Corollary 2.5.2 clearly holds for \mathcal{R}_π . There exists then a split connected reductive group G_π defined over a finite extension of E_π which interpolates the groups of connected components of the algebraic monodromy groups of $\rho_{\pi,\lambda}$, for λ above a set of rational primes of Dirichlet density 1. In analogy with the Mumford-Tate conjecture, the group G_π should be related to a notion of “Mumford-Tate group” for π .

2.6 Compatible systems in the positive characteristic case

In this last section we give applications of the abstract independence results of §2.4 to positive characteristic settings. We start with the case of compatible systems of representations of the absolute Galois group of a global function field, and we then move to compatible systems of lisse sheaves on a variety over a finite field.

Let p be a prime, let q be a power of p , and let C be a smooth projective curve, geometrically connected over \mathbb{F}_q . Let F be the field $\mathbb{F}_q(C)$ of rational functions on C , and let Γ_F denote the absolute Galois group of F . The following preliminary result is essentially a reformulation in the characteristic p setting of [82, Proposition 3.4.1], which in turn adapts the arguments of [67, Proposition 1]. We include here the proof for the sake of completeness.

Lemma 2.6.1. *Let $\rho : \Gamma_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ be an irreducible representation. Then either ρ is induced from a representation of an open proper subgroup of Γ_F , or ρ is Lie-irreducible, or there exist an integer $d \geq 2$ dividing n , a Lie-irreducible representation σ of Γ_F of dimension n/d , and an Artin representation ω of Γ_F of dimension d such that $\rho \cong \sigma \otimes \omega$.*

Proof. Let G denote the Zariski closure of $\rho(\Gamma_F)$ in $\mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$, and let G° denote the connected component of the identity of G . Then, either ρ as a representation of G is induced from a proper subgroup H of G containing G° , or its restriction to G° is isotypical. If ρ is induced from such an H as a representation of G , then as a

representation of Γ_F it is induced from the open proper subgroup $\rho^{-1}(H)$ of Γ_F , and we are done.

Let us then assume that $\rho|_{G^\circ}$ is isotypical, that is $\rho|_{G^\circ} \cong \sigma_0^{\oplus d}$ for some irreducible representation σ_0 of G° and some $d \geq 1$. If $d = 1$, then ρ is Lie-irreducible.

So suppose $d \geq 2$. Let F' be any sufficiently large extension of F such that $\rho(\Gamma_{F'}) \subset G^\circ$. Then, $\rho(\Gamma_{F'})$ is Zariski dense in G° , and so σ_0 is irreducible as a representation of $\Gamma_{F'}$, and we have $\rho|_{\Gamma_{F'}} \cong \sigma_0^{\oplus d}$. Since σ_0 is Γ_F -invariant, it extends to a projective representation of Γ_F . Now, we have $H^2(\Gamma_F, \mathbb{Q}/\mathbb{Z}) = 0$ by Tate's theorem, see [91, Theorem 4], and so this projective representation lifts to a representation σ_1 of Γ_F , so that for some character $\chi : \Gamma_{F'} \rightarrow \overline{\mathbb{Q}}_\ell^\times$ we have $\sigma_1^{\oplus d}|_{\Gamma_{F'}} \cong \rho|_{\Gamma_{F'}} \otimes \chi$. Let $\mu = \det(\rho)/\det(\sigma_1^{\oplus d})$. Then $\mu|_{\Gamma_{F'}} = \chi^{-n}$, and we can find characters $\mu_0, \mu_1 : \Gamma_F \rightarrow \overline{\mathbb{Q}}_\ell^\times$, with μ_0 of finite order, such that $\mu = \mu_1^n \mu_0$. Then $(\chi \mu_1|_{\Gamma_{F'}})^n = (\mu^{-1} \mu_1^n)|_{\Gamma_{F'}} = \mu_0^{-1}|_{\Gamma_{F'}}$, so that setting $\sigma = \sigma_1 \otimes \mu_1$, and replacing F' by a finite extension trivialising μ_0 , we find a Lie-irreducible representation σ of Γ_F and a finite extension F' of F such that $\rho|_{\Gamma_{F'}} \cong \sigma^{\oplus d}|_{\Gamma_{F'}}$. Let then $\omega = \text{Hom}_{\Gamma_{F'}}(\sigma, \rho)$. We have that ω is an Artin representation of Γ_F of dimension d , and the natural map $\sigma \otimes \omega \rightarrow \rho$ is an isomorphism. \square

Let E be a number field, let \mathcal{L} be a set of rational primes of Dirichlet density 1, with $p \notin \mathcal{L}$, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. We prove the following result.

Theorem 2.6.2. *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of semisimple representations of Γ_F with coefficients in E . Then, \mathcal{R} has a Lie-irreducible decomposition over a finite extension E' of E and at \mathcal{L} .*

Proof. Fix $\lambda_0 \in \Lambda$. By Lemma 2.6.1 there exist finite extensions F_i of F , Lie-irreducible representations $\sigma_i : \Gamma_{F_i} \rightarrow \text{GL}_{m_i}(\overline{E}_{\lambda_0})$, Artin representations $\omega_i : \Gamma_{F_i} \rightarrow \text{GL}_{d_i}(\overline{E}_{\lambda_0})$, for $i = 1, \dots, k$, such that

$$\rho_{\lambda_0} \otimes_{E_{\lambda_0}} \overline{E}_{\lambda_0} \cong \bigoplus_{i=1}^k \text{Ind}_{\Gamma_{F_i}}^{\Gamma_F} (\sigma_i \otimes \omega_i).$$

For each i , choose $\chi_i : \Gamma_{F_i} \rightarrow \overline{E}_{\lambda_0}^\times$ such that the determinant of $\tau_i = \sigma_i \otimes \chi_i$ has finite order. Then, τ_i extends to a compatible system $\{\tau_{i,\lambda}\}_{\lambda \in \Lambda}$ by the global Langlands correspondence [71, Théorème VI.9]. Also, by global class field theory we can extend χ_i to a compatible system $\{\chi_{i,\lambda}\}_{\lambda \in \Lambda}$. For each $\lambda \in \Lambda$, let $\sigma_{i,\lambda} = \tau_{i,\lambda} \otimes \chi_{i,\lambda}^{-1}$. We then have that $\mathcal{S}_i = \{\sigma_{i,\lambda}\}_{\lambda \in \Lambda}$ is a compatible system extending σ_i .

Let E' be a finite extension of E such that each \mathcal{S}_i has coefficients in E' , and each ω_i can be realised over E' . Then, for any place v of F and any $\lambda \in \Lambda$ not lying above

the residue characteristic of v we have

$$\begin{aligned} \mathrm{Tr} \rho_\lambda(\mathrm{Frob}_v) &= \mathrm{Tr} \rho_{\lambda_0}(\mathrm{Frob}_v) \\ &= \mathrm{Tr} \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} (\sigma_i \otimes \omega_i)(\mathrm{Frob}_v) \\ &= \mathrm{Tr} \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} (\sigma_{i,\lambda} \otimes \omega_i)(\mathrm{Frob}_v), \end{aligned}$$

by independence of λ of the characteristic polynomials at the Frobenius elements, and by the usual formulas for the traces of direct sums, tensor products, and induced representations. Combining the Čebotarev density theorem with [20, §12.1, Proposition 3], it then follows that

$$\mathcal{R} \otimes_E E' \cong_{(\mathcal{L})} \bigoplus_{i=1}^k \mathrm{Ind}_{\Gamma_{F_i}}^{\Gamma_F} (\mathcal{S}_i \otimes \omega_i) \otimes_E E'$$

is a Lie-irreducible decomposition over E' and at \mathcal{L} . \square

By Proposition 2.4.4 we then have the following immediate consequence.

Corollary 2.6.3. *Let $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of semisimple representations of Γ_F with coefficients in E and indexed by Λ . Then, there exist a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1, a finite extension E' of E , and a split connected reductive algebraic subgroup G of $\mathrm{GL}_{n,E'}$ such that for each $\ell \in \mathcal{L}'$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ and $G \times_{E'} E'_{\lambda'}$ are conjugate over $E'_{\lambda'}$.*

Let us now consider the case of compatible systems of lisse sheaves on a variety over \mathbb{F}_q . Refer to [39, §1.1 - §1.3] for the basic definitions and results in this setting.

Let X be a geometrically connected variety over \mathbb{F}_q . Let $\bar{\eta}$ be a geometric point of X , and let $\pi_1^{\text{ét}}(X, \bar{\eta})$ be the arithmetic fundamental group of X . Let $\ell \neq p$ be a prime. The functor which assigns to each lisse $\bar{\mathbb{Q}}_\ell$ -sheaf \mathcal{L} on X its fibre $\mathcal{L}_{\bar{\eta}}$ over $\bar{\eta}$ induces an equivalence between the category of lisse $\bar{\mathbb{Q}}_\ell$ -sheaves on X and the category of finite dimensional continuous representations of $\pi_1^{\text{ét}}(X, \bar{\eta})$ with coefficients in $\bar{\mathbb{Q}}_\ell$. Via this equivalence, standard notions about representations (e.g rank, semisimplicity, irreducibility, or Lie-irreducibility) can be translated to the context of lisse sheaves.

Let E be a number field, let \mathcal{L} be a set of rational primes of Dirichlet density 1, with $p \notin \mathcal{L}$, and let $\Lambda = \cup_{\ell \in \mathcal{L}} |E|_\ell$. We recall the following definition.

Definition 2.6.4. *A compatible system of lisse E_λ -sheaves on X indexed by Λ is a family $\mathcal{R} = \{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ of lisse E_λ -sheaves \mathcal{F}_λ on X such that, for any closed point $x \in |X|$ of X and any $\lambda \in \Lambda$, the polynomial $\det(1 - \mathrm{Frob}_x t, \mathcal{F}_\lambda)$ has coefficients in E and is independent of λ .*

Any compatible system of lisse E_λ -sheaves $\mathcal{R} = \{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ on X indexed by Λ defines a compatible system $\{\rho_\lambda\}_{\lambda \in \Lambda}$ of representations of $\pi_1^{\text{ét}}(X, \bar{\eta})$ with coefficients in E and indexed by Λ , in the sense of Definition 2.1.1, where ρ_λ is the representation $\mathcal{F}_{\lambda, \bar{\eta}}$, and $\mathcal{X} = |X| \times \Lambda$. The algebraic monodromy group G_λ of ρ_λ is called the *arithmetic monodromy group* of \mathcal{F}_λ . As usual, we denote by G_λ° the connected component of the identity of G_λ . Also, for simplicity of notation we omit the base point $\bar{\eta}$ and write just $\pi_1^{\text{ét}}(X)$ for the étale fundamental group of X .

From Corollary 2.6.3 we deduce the following λ -independence result.

Corollary 2.6.5. *Let X be a normal geometrically connected irreducible variety over \mathbb{F}_q , and let $\mathcal{R} = \{\mathcal{F}_\lambda\}_{\lambda \in \Lambda}$ be a compatible system of semisimple lisse E_λ -sheaves on X indexed by Λ . Then, there exist a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1, a finite extension E' of E , and a split connected reductive algebraic subgroup G of $\text{GL}_{n, E'}$ such that for each $\ell \in \mathcal{L}'$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ and $G \times_{E'} E'_{\lambda'}$ are conjugate over $E'_{\lambda'}$.*

Proof. Let $\{\rho_\lambda\}_{\lambda \in \Lambda}$ be the compatible system of representations of $\pi_1^{\text{ét}}(X)$ corresponding to \mathcal{R} . By [15, Corollary 3.33] there exist a finite étale cover $\phi : X' \rightarrow X$, and a curve C embedding into X' via some ι such that $\rho_\lambda(\phi_* \pi_1^{\text{ét}}(X')) = \rho_\lambda(\phi_* \iota_* \pi_1^{\text{ét}}(C))$, for all $\lambda \in \Lambda$.

For any $\lambda \in \Lambda$, let $\rho'_\lambda = \rho_\lambda \circ \phi_*$, and let G'_λ be its algebraic monodromy group. Since the morphism ϕ is finite étale, then $\phi_* \pi_1^{\text{ét}}(X')$ is an open subgroup of $\pi_1^{\text{ét}}(X)$, and so $G'^{\circ}_\lambda = G^\circ_\lambda$, since restriction to an open subgroup leaves the connected component of the identity of the algebraic monodromy group of a λ -adic representation invariant.

Up to birational equivalence, we can assume that C is smooth and projective. Let F be the field of rational functions on C , and let $\pi_C : \Gamma_F \rightarrow \pi_1^{\text{ét}}(X')$ be the composition of the natural surjection of Γ_F onto $\pi_1^{\text{ét}}(C)$ with ι_* .

For any $\lambda \in \Lambda$, let $\rho_{C, \lambda} = \rho'_\lambda \circ \pi_C$. Note that $\rho_{C, \lambda}(\Gamma_F) = \rho'_\lambda(\pi_1^{\text{ét}}(X'))$, so that the algebraic monodromy groups of $\rho_{C, \lambda}$ and ρ'_λ coincide. The family $\{\rho_{C, \lambda}\}_{\lambda \in \Lambda}$ is a compatible system of semisimple representations of Γ_F , and then, by Corollary 2.6.3 there exist a subset \mathcal{L}' of \mathcal{L} of Dirichlet density 1, a finite extension E' of E , and a split connected reductive algebraic subgroup G of $\text{GL}_{n, E'}$ such that for each $\ell \in \mathcal{L}'$, each place λ of E above ℓ , and each place λ' of E' above λ , the groups $G_\lambda^\circ \times_{E_\lambda} E'_{\lambda'}$ and $G \times_{E'} E'_{\lambda'}$ are conjugate over $E'_{\lambda'}$. \square

Remark 2.6.6. A crucial point in the proof of this result is the reduction to the case of curves, which relies on the work of Böckle, Gajda, and Petersen in [15, §3.3]. A similar λ -independence result, for all $\lambda \in \Lambda$, has been obtained with a different

approach by D'Addezio, see [36, Theorem 4.3.2], assuming the variety X to be also smooth.

Chapter 3

Deformations of automorphic Galois representations

In this chapter, we study p -adic families of Galois representations and automorphic representations, and the geometry of the corresponding p -adic rigid analytic spaces at points arising from (classical) automorphic representations. In §3.1 we focus on the deformation theory of representations and pseudorepresentations, and introduce the corresponding rigid analytic p -adic (pseudo)deformation spaces. Then, in §3.2 we move to the automorphic setting, where we introduce eigenvarieties for definite unitary groups, i.e. p -adic rigid analytic spaces parametrising p -adic systems of Hecke eigenvalues appearing in spaces of p -adic automorphic forms. We finally fix a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, for F a CM field, and consider the automorphic Galois representation attached to it by the results presented in §1. The automorphic Galois representation defines an “automorphic point” in a suitable p -adic rigid analytic (pseudo)deformation space, and the automorphic representation (together with an additional datum, see §3.2.2) defines a “classical point” in a suitable eigenvariety. In §3.3 we study the smoothness of these points, and suggest possible converse results.

3.1 Deformation theory of pseudorepresentations

We study here the deformation theory of representations and pseudorepresentations. In §3.1.1 we introduce the background material on deformation functors, pseudorepresentations, and pseudodeformation functors, and we define p -adic rigid analytic pseudodeformation spaces. In §3.1.2, we present deeper algebraic results on the structure of pseudorepresentations, while in §3.1.3 we quickly focus on the theory of deformation conditions for pseudorepresentations.

3.1.1 Deformations of pseudorepresentations

Let p be a prime, and let E be a finite extension of \mathbb{Q}_p . Let \mathcal{O} denote the ring of integers of E , let \mathfrak{m} be the maximal ideal of \mathcal{O} , and let $\mathbb{F} = \mathcal{O}/\mathfrak{m}$. Let Γ be a profinite group, let $n \geq 1$ be an integer, and let

$$\bar{\rho} : \Gamma \rightarrow \mathrm{GL}_n(\mathbb{F})$$

be a continuous representation. Also, let $V_{\bar{\rho}} = \mathbb{F}^n$, equipped with the $\mathbb{F}[\Gamma]$ -module structure given by $\bar{\rho}$.

Let $\widehat{\mathcal{C}}_{\mathcal{O}}$ denote the category of complete Noetherian local commutative \mathcal{O} -algebras with residue field \mathbb{F} . Given an object A in $\widehat{\mathcal{C}}_{\mathcal{O}}$, we say that a continuous representation $\rho : \Gamma \rightarrow \mathrm{GL}_n(A)$ is a *lifting* of $\bar{\rho}$ to A if ρ reduces to $\bar{\rho}$ modulo the maximal ideal \mathfrak{m}_A of A . A *deformation* of $\bar{\rho}$ to A is an equivalence class of liftings of $\bar{\rho}$ to A modulo the action by conjugation of $1 + M_{n \times n}(\mathfrak{m}_A)$.

Let

$$\mathrm{Def}_{\bar{\rho}}^{\square} : \widehat{\mathcal{C}}_{\mathcal{O}} \rightarrow \mathbf{Sets}$$

be the functor which attaches to each object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$ the set of liftings of $\bar{\rho}$ to A , and let

$$\mathrm{Def}_{\bar{\rho}} : \widehat{\mathcal{C}}_{\mathcal{O}} \rightarrow \mathbf{Sets}$$

be the functor which attaches to each object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$ the set of deformations of $\bar{\rho}$ to A .

Assume from now on that the group Γ satisfies Mazur's Φ_p condition, that is for all finite index subgroups $\Gamma' \subset \Gamma$, the maximal pro- p quotient of Γ' is topologically finitely generated. Then, by a classical result due to Mazur, see [78], the functor $\mathrm{Def}_{\bar{\rho}}^{\square}$ is representable by a complete Noetherian local commutative \mathcal{O} -algebra $R_{\bar{\rho}}^{\square}$, called the *universal framed deformation ring* of $\bar{\rho}$, together with a *universal lifting* (or *framed deformation*) $\rho^{\square} : \Gamma \rightarrow \mathrm{GL}_n(R_{\bar{\rho}}^{\square})$ of $\bar{\rho}$ to $R_{\bar{\rho}}^{\square}$. Furthermore, if $\mathrm{End}_{\mathbb{F}[\Gamma]} V_{\bar{\rho}} \cong \mathbb{F}$, then the functor $\mathrm{Def}_{\bar{\rho}}$ is representable by a complete Noetherian local commutative \mathcal{O} -algebra $R_{\bar{\rho}}$, called the *universal deformation ring* of $\bar{\rho}$, together with a *universal deformation* $\rho^{\mathrm{univ}} : \Gamma \rightarrow \mathrm{GL}_n(R_{\bar{\rho}})$ of $\bar{\rho}$ to $R_{\bar{\rho}}$.

Assume that $\mathrm{Def}_{\bar{\rho}}$ is representable by $R_{\bar{\rho}}$. Then the closed points x of the affine scheme $\mathrm{Spec} R_{\bar{\rho}}[1/p]$ correspond to E -algebras homomorphisms $x : R_{\bar{\rho}}[1/p] \rightarrow E'$ for finite extensions E' of E . Composing with the universal deformation ρ^{univ} , these induce representations $\rho_x : \Gamma \rightarrow \mathrm{GL}_n(E')$. Since each x actually maps $R_{\bar{\rho}} \subset R_{\bar{\rho}}[1/p]$ into $\mathcal{O}_{E'} \subset E'$, then each ρ_x can be conjugate to a representation $\rho_x : \Gamma \rightarrow \mathrm{GL}_n(\mathcal{O}_{E'})$, which is a deformation of $\bar{\rho}$. For this reason, with a slight abuse of language, we say that

the closed points of $\text{Spec } R_{\bar{\rho}}[1/p]$ parametrise deformations of $\bar{\rho}$ to finite extensions of E . In order to deal with the case when $\text{Def}_{\bar{\rho}}$ is not necessarily representable, e.g. when $\bar{\rho}$ is not absolutely irreducible, we should introduce the notion of *pseudorepresentation*, and consider a corresponding deformation problem. This will allow us to define (in the general setting) a rigid analytic space \mathcal{X} over E whose points are in bijection with deformations of $\bar{\rho}$ to finite extensions of E .

The notion of pseudorepresentation of a group has been firstly introduced in number theory by Wiles [107] in dimension 2, and by Taylor [97] in general. Later, Taylor's definition has been also considered by Rouquier [86] in the context of A -algebras, where pseudorepresentations are called *pseudocharacters*. In this work, we consider the definition given by Chenevier [31], where pseudorepresentations are instead called *determinants*. Under certain assumptions, Taylor-Rouquier's and Chenevier's notions turn out to be equivalent.

Let A be a commutative ring with unit. An A -polynomial law between two A -modules M and M' is a natural transformation P between the functors $M \otimes_A -$ and $M' \otimes_A -$ from the category of A -algebras to the category of sets. An A -polynomial law P is said to be *homogeneous* of degree n , for an integer $n \geq 1$, if $P_B(bx) = b^n P_B(x)$ for any A -algebra B , any $b \in B$, and any $x \in M \otimes_A B$. If M and M' are A -algebras, then an homogeneous A -polynomial law P of degree n between M and M' is said to be *multiplicative* if $P_B(1) = 1$ and if $P_B(xy) = P_B(x)P_B(y)$ for any A -algebra B , and any $x, y \in M \otimes_A B$.

Let R be an A -algebra. We give the following definition.

Definition 3.1.1. An n -dimensional A -valued *pseudorepresentation* of R is a multiplicative A -polynomial law

$$D : R \rightarrow A$$

which is homogeneous of degree n . When $R = A[\Gamma]$ for some profinite group Γ , we simply say that D is a pseudorepresentation of Γ .

A natural source of examples of pseudorepresentations is the following.

Example 3.1.2. The natural transformation $D = \det \circ \rho$, where $\rho : R \rightarrow M_{n \times n}(A)$ is a given representation, and \det is the natural transformation defined by the determinant map in the usual way¹, is an n -dimensional A -valued pseudorepresentation of R .

¹For any commutative A -algebra B , we let $\det_B : M_{n \times n}(B) \rightarrow B$ be the usual determinant map.

Under suitable assumptions on A or D it holds true that any pseudorepresentation of R arises from a representation of R as in Example 3.1.2. A result in this direction is the following.

Theorem 3.1.3 ([31, Theorem 2.12]). *Let \mathbb{k} be an algebraically closed field, and let R be a \mathbb{k} -algebra. Then, for any n -dimensional pseudorepresentation $D : R \rightarrow \mathbb{k}$, there exists a semisimple representation $\rho : R \rightarrow M_{n \times n}(\mathbb{k})$ such that $D = \det \circ \rho$. Moreover, such a ρ is unique up to isomorphism.*

Given a field \mathbb{k} (not necessarily algebraically closed), we say that an n -dimensional pseudorepresentation $D : R \rightarrow \mathbb{k}$ is *split* if there exists a semisimple representation $\rho : R \rightarrow M_{n \times n}(\mathbb{k})$ such that $D = \det \circ \rho$. We say that $D : R \rightarrow \mathbb{k}$ is *absolutely multiplicity free* (resp. *absolutely irreducible*) if the corresponding representation $\rho : R \rightarrow M_{n \times n}(\bar{\mathbb{k}})$ over $\bar{\mathbb{k}}$ given by Theorem 3.1.3 is multiplicity free (resp. irreducible).

Let now D be an homogeneous of degree n multiplicative polynomial law between R and A . For each $r \in R$, define the *characteristic polynomial* $\chi(r, t) \in A[t]$ of r with respect to D by the formula

$$\begin{aligned} \chi(r, t) &= D_{A[t]}(1 - tr) \\ &= \sum_{i=0}^n (-1)^i \Lambda_{n-i}(r) t^i, \end{aligned}$$

where each Λ_i is thus an A -polynomial law between R and A which is homogeneous of degree i . In particular, we have $\Lambda_0 = 1$, $\Lambda_n = D$, and Λ_1 is an A -linear map which is denoted by Tr_D , and called the *trace* associated to D .

If D is an A -valued n -dimensional pseudorepresentation of R , then the A -polynomial laws Λ_i define an A -polynomial law

$$\chi : R \rightarrow R, \quad \chi(r) = \sum_{i=0}^n (-1)^i \Lambda_{n-i}(r) r^i,$$

which is homogeneous of degree n .

Note that, given an A -valued pseudorepresentation D of R , its trace Tr_D is an A -valued pseudocharacter on R in the sense of Rouquier. Furthermore, the map $D \mapsto \text{Tr}_D$ induces a bijection between the set of n -dimensional A -valued pseudorepresentations of R and the set of n -dimensional A -valued pseudocharacters on R if either $2 \in R^\times$ and $n = 2$, or $(2n)! \in R^\times$, see [31, Proposition 1.29].

When $R = A[\Gamma]$ for some profinite group Γ , any n -dimensional A -valued pseudorepresentation $D : A[\Gamma] \rightarrow A$ is uniquely determined by the (finite) set of functions

$$D^{[\alpha]} : \Gamma^n \rightarrow A,$$

with $\alpha \in I_n = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n : \alpha_1 + \dots + \alpha_n = n\}$, defined by

$$D_{A[t_1, \dots, t_n]} \left(\sum_{i=1}^n t_i \sigma_i \right) = \sum_{\alpha \in I_n} (\sigma_1, \dots, \sigma_n) t^\alpha,$$

where $t^\alpha = \prod_{i=1}^n t_i^{\alpha_i}$, for each $(\sigma_1, \dots, \sigma_n) \in \Gamma^n$. When A is a topological ring, we say that $D : A[\Gamma] \rightarrow A$ is *continuous* if for each $\alpha \in I_n$ the function $D^{[\alpha]} : \Gamma^n \rightarrow A$ is continuous.

Let us now introduce the deformation problem for pseudorepresentations. Fix an n -dimensional continuous pseudorepresentation $\bar{D} : \mathbb{F}[\Gamma] \rightarrow \mathbb{F}$. By Theorem 3.1.3, it is equivalent to give such a pseudorepresentation and (the isomorphism class of) a continuous representation $\bar{\rho} : \Gamma \rightarrow \mathrm{GL}_n(\bar{\mathbb{F}})$ such that $\det(1 + t\bar{\rho}(\sigma)) \in \mathbb{F}[t]$ for all $\sigma \in \Gamma$, and $\bar{D} = \det \circ \bar{\rho}$.

Given an object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$, we say that an n -dimensional continuous pseudorepresentation $D : A[\Gamma] \rightarrow A$ is a *deformation* of \bar{D} to A , or a *pseudodeformation* of \bar{D} to A , if $D \otimes_A \mathbb{F} = \bar{D}$. For each object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$, let then $\mathrm{Def}_{\bar{D}}(A)$ be the set of deformations of \bar{D} to A . This defines a functor

$$\mathrm{Def}_{\bar{D}} : \widehat{\mathcal{C}}_{\mathcal{O}} \rightarrow \mathbf{Sets}.$$

The following result is due to Chenevier.

Theorem 3.1.4 ([31, Propositions 3.3 and 3.7]). *Assume that Γ satisfies Φ_p . Then, the functor $\mathrm{Def}_{\bar{D}}$ is representable by a complete Noetherian local commutative \mathcal{O} -algebra $R_{\bar{D}}$, together with a universal deformation $D^{\mathrm{univ}} : R_{\bar{D}}[\Gamma] \rightarrow R_{\bar{D}}$ of \bar{D} to $R_{\bar{D}}$.*

Remark 3.1.5. When $\bar{\rho}$ is absolutely irreducible, [31, Theorem 2.22] implies that there is an isomorphism of functors on $\widehat{\mathcal{C}}_{\mathcal{O}}$ between $\mathrm{Def}_{\bar{\rho}}$ and $\mathrm{Def}_{\bar{D}}$. Nevertheless, even when this condition is not necessarily satisfied, by Theorem 3.1.3 there is a bijection between the set of closed points of the affine scheme $\mathrm{Spec} R_{\bar{D}}[1/p]$ and the set of deformations of $\bar{\rho}$ to finite extensions of E . For this reason, the affine scheme $\mathrm{Spec} R_{\bar{D}}[1/p]$ can be thought as a ‘‘coarse moduli space’’ for deformations of $\bar{\rho}$. A deeper moduli theoretic interpretation of the ring $R_{\bar{D}}$ has been given by Wang-Erickson in [106], where it is proved that if \bar{D} is absolutely multiplicity free, then $R_{\bar{D}}$ is essentially the GIT quotient ring of the framed deformation ring $R_{\bar{\rho}}^{\square}$ of $\bar{\rho}$ for the action of PGL_n , see [106, Theorem 3.8].

Consider the affine formal scheme $\mathfrak{X} = \mathrm{Spf} R_{\bar{D}}$ over \mathcal{O} , and let $\mathcal{X} = \mathfrak{X}^{\mathrm{rig}}$ be the rigid analytic space over E attached to \mathfrak{X} via Berthelot’s functor. It can be proved,

see [31, Theorem 3.17], that \mathcal{X} represents the functor which attaches to each rigid analytic space X over E the set of deformations of \bar{D} to $\mathcal{O}(X)$, and is isomorphic to a closed subspace of some rigid analytic open unit ball. The closed points of \mathcal{X} are in bijection with the closed points of $\text{Spec } R_{\bar{D}}[1/p]$, which in turn parametrise deformations of $\bar{\rho}$ to finite extensions of E .

3.1.2 Cayley-Hamilton algebras, GMA structures, and reducibility ideals

Let us first of all introduce the notion of *kernel* of a polynomial law. Let A be a commutative ring with unit, and let M and M' be A -modules. Given an A -polynomial law P between M and M' , define $\ker(P)$ to be the subset of elements $m \in M$ such that $P_B(m \otimes b + x) = P_B(x)$ for any A -algebra B , any $b \in B$, and any $x \in M \otimes_A B$. Equivalently, $m \in \ker(P)$ if and only if for any integer N and any $m_1, \dots, m_N \in M$, the element $P_{A[t_1, \dots, t_N]}(tm + t_1m_1 + \dots + t_Nm_N) \in M'[t_1, \dots, t_N]$ is independent of t (i.e. lies in $M'[t_1, \dots, t_N]$). By definition, $\ker(P)$ is an A -submodule of M . We say that P is *faithful* if $\ker(P) = 0$.

Let now R be an A -module, let $n \geq 1$ be an integer, and let $D : R \rightarrow A$ be an n -dimensional A -valued pseudorepresentation of R . We denote by $\text{CH}(D)$ the (two-sided) ideal of R generated by the coefficients of

$$\chi(t_1r_1 + \dots + t_Nr_N) \in R[t_1, \dots, t_N],$$

with $r_1, \dots, r_N \in R$, and $N \geq 1$. We say that D is *Cayley-Hamilton* if $\text{CH}(D) = \{0\}$. Equivalently, D is Cayley-Hamilton if and only if the A -polynomial law $\chi : R \rightarrow R$ is identically zero. When D is Cayley-Hamilton, we also say that (R, D) is a *Cayley-Hamilton algebra* for D . It can be proved that $\ker(D)$ contains $\text{CH}(D)$, so that if D is faithful then D is Cayley-Hamilton.

Let $D : R \rightarrow A$ be an n -dimensional A -valued pseudorepresentation of R . A quotient S of R by a two-sided ideal of R contained in $\ker(D)$, and such that the induced pseudorepresentation $D : S \rightarrow A$ is Cayley-Hamilton, is called a *Cayley-Hamilton quotient* of (R, D) .

Given an n -dimensional A -valued pseudorepresentation D of Γ , a *Cayley-Hamilton representation* of D is the datum of a Cayley-Hamilton A -algebra (R', D') and a group homomorphism $\rho : \Gamma \rightarrow R'^{\times}$ such that, if we let $f : A[\Gamma] \rightarrow R'$ be the A -algebra homomorphism induced by ρ , we have $D_B(x) = D'_B((f \otimes_A B)(x))$ for any A -algebra B , and any $x \in A[\Gamma] \otimes_A B$.

In [31, §2.21], building on work of [10, §1.4.3], Chenevier proves a structure theorem for Cayley-Hamilton algebras (R, D) such that D is valued in an Henselian local ring A , and such that the residual pseudorepresentation $\overline{D} = D \otimes_A A/\mathfrak{m}_A$ of D , where \mathfrak{m}_A denotes the maximal ideal of A , is split and absolutely multiplicity free. In order to introduce this result, we need some preliminaries about *generalised matrix algebras* from [10, §1.3].

We fix positive integers k and n_1, \dots, n_k , and we let $n = n_1 + \dots + n_k$. We say that an A -algebra R is a *generalised matrix algebra* (GMA) of type (n_1, \dots, n_k) if R is equipped with:

1. a family of orthogonal idempotents e_1, \dots, e_k of sum 1,
2. for each i , an A -algebras isomorphism $\psi_i : e_i R e_i \xrightarrow{\sim} M_{n_i \times n_i}(A)$,

such that the *trace map* $T : R \rightarrow A$ of R , defined by $T(x) = \sum_{i=1}^k \text{Tr} \psi_i(e_i x e_i)$ for $x \in R$, satisfies $T(xy) = T(yx)$ for all $x, y \in R$.

Every GMA turns out to be isomorphic to a so called *standard* GMA in the following way. For each i , we let E_i be the unique element of $e_i R e_i$ such that $\psi_i(E_i)$ is the elementary matrix of $M_{n_i \times n_i}(A)$ with 1 at the entry $(1, 1)$, and 0 elsewhere, and for each i, j we let $A_{i,j} = E_i R E_j$. Each $A_{i,j}$ is an A -submodule of finite type of R . Moreover we have $A_{h,i} A_{i,j} \subset A_{h,j}$ for all h, i, j , and $A_{i,i} \cong A$ via T for all i . As a consequence of Morita equivalence, there is a canonical isomorphism of A -algebras

$$R \cong \begin{pmatrix} M_{n_1 \times n_1}(A_{1,1}) & \cdots & M_{n_1 \times n_k}(A_{1,k}) \\ \vdots & \ddots & \vdots \\ M_{n_k \times n_1}(A_{k,1}) & \cdots & M_{n_k \times n_k}(A_{k,k}) \end{pmatrix}.$$

Let us go back to the context of Cayley-Hamilton algebras. We let A be an Henselian local ring, with maximal ideal \mathfrak{m}_A , we let R be an A -algebra, and we let $D : R \rightarrow A$ be Cayley-Hamilton. Assume that the residual pseudorepresentation $\overline{D} = D \otimes_A A/\mathfrak{m}_A$ of D is split, and absolutely multiplicity free. Let k be the number of (pairwise non isomorphic) subrepresentations in an irreducible decomposition of the representation corresponding to \overline{D} , and let n_1, \dots, n_k be their ranks. Then, the following result holds.

Theorem 3.1.6 (part of [31, Theorem 2.22]). *The A -algebra R is a generalised matrix algebra of type (n_1, \dots, n_k) , with trace map $T = \text{Tr}_D$.*

Let us now see an application of this structure theorem to the deformation theory of pseudorepresentations. Fix notations as in §3.1.1. Let us fix $\bar{\rho} : \Gamma \rightarrow \mathrm{GL}_n(\mathbb{F})$, and let $\bar{D} = \det \circ \bar{\rho}$, so that \bar{D} is split by construction. Let us assume that

$$\bar{\rho} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \cong \bigoplus_{i=1}^k \bar{\rho}_i$$

is an irreducible decomposition into pairwise non isomorphic subrepresentations, so that \bar{D} is also absolutely multiplicity free. For each i , let n_i be the rank of $\bar{\rho}_i$.

Let A be an object in $\widehat{\mathcal{C}}_{\mathcal{O}}$, and let D be a deformation of \bar{D} to A . Let S be a Cayley-Hamilton quotient of $(A[\Gamma], D)$. Then, by Theorem 3.1.6, the A -algebra S is a generalised matrix algebra of type (n_1, \dots, n_k) . Let then $A_{i,j}$ be the corresponding A -submodules of S . A notion of *complexity* for D has been introduced by Bellaïche in [8] using these data². Let us now recall the definition.

For $i \geq 1$ an integer, an application $\gamma : \{0, \dots, i\} \rightarrow \{1, \dots, k\}$ is called a *path* of length i . We say that γ is *closed* if $\gamma(0) = \gamma(i)$. We say that a closed γ is *simple* if it is injective on $\{0, \dots, i-1\}$. Given any path γ of length i , we set

$$A_{\gamma} = A_{\gamma(0), \gamma(1)} \cdots A_{\gamma(n-1), \gamma(n)}.$$

By [8, Lemma 1], the A -modules A_{γ} only depend on A and Tr_D , and not on the Cayley-Hamilton quotient S . We say that D has *complexity* less than or equal to i if $A_{\gamma} = 0$ for any simple closed path γ of length greater than i .

For $i = 1, \dots, k$, denote by $\mathrm{Def}_{\bar{D}, i}$ the subfunctor of $\mathrm{Def}_{\bar{D}}$ which attaches to each object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$ the set of deformations of \bar{D} to A which have complexity less than or equal to i . Notice that $\mathrm{Def}_{\bar{D}, k} = \mathrm{Def}_{\bar{D}}$. We have the following result.

Proposition 3.1.7 ([8, Proposition 1]). *For every $i = 1, \dots, k$, the functor $\mathrm{Def}_{\bar{D}, i}$ is representable by a local quotient $R_{\bar{D}, i}$ of $R_{\bar{D}}$.*

The ideals J_i of $R_{\bar{D}}$ such that $R_{\bar{D}, i} = R_{\bar{D}}/J_i$, which can be defined as

$$J_i = \sum_{\substack{\gamma \text{ simple closed path} \\ \text{of length } > i}} (R_{\bar{D}})_{\gamma}$$

are examples of the so called *reducibility ideals* of D , in the sense of [10, §1.5]. We have $J_k = \{0\}$, and we call $J_{\mathrm{tot}} = J_1$ the *total reducibility ideal* of D .

²Bellaïche's original definition, see [8, §2], was actually formulated in the language of pseudocharacters. With a slight abuse of terminology, we will refer Bellaïche's definition to the context of pseudorepresentations.

3.1.3 Deformation conditions

Ramakrishna's theory of deformation conditions [85] has been recently extended to the context of pseudodeformations by Wake and Wang-Erickson [105].

Let us keep notations as in §3.1.1 and §3.1.2. We start by recalling the notion of *stability* for a (pseudo)deformation condition. Let us denote by $\mathbf{Mod}_{\mathcal{O}[\Gamma]}$ the category of $\mathcal{O}[\Gamma]$ -modules, and by $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$ its full subcategory whose objects have finite cardinality. A *condition* on $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$ is a full subcategory \mathcal{C} of $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$, and we say that an object of $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$ *satisfies* condition \mathcal{C} if it belongs to \mathcal{C} . A condition \mathcal{C} is said to be *stable* if it is closed under isomorphisms, subquotients, and direct sums in $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$, that is:

1. for every object M of \mathcal{C} and all isomorphisms $f : M \rightarrow M'$ in $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$, the object M' is in \mathcal{C} ;
2. for every object M of \mathcal{C} and all morphisms $f : M \rightarrow M'$ and $g : M'' \rightarrow M$ in $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$, the kernel of f and the cokernel of g are in \mathcal{C} ;
3. for every finite collection of objects $\{M_i\}_{i=1}^n$ of \mathcal{C} , the direct sum $\bigoplus_{i=1}^n M_i$ in $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$ is in \mathcal{C} .

Let us fix a stable deformation condition \mathcal{C} on $\mathbf{Mod}_{\mathcal{O}[\Gamma]}^{\text{fin}}$. Given an object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$ with maximal ideal \mathfrak{m}_A , we say that an $A[\Gamma]$ -module that is finitely generated as an A -module *satisfies* condition \mathcal{C} if $M/\mathfrak{m}_A^i M$ satisfies \mathcal{C} for all $i \geq 1$.

Let $\bar{\rho} : \Gamma \rightarrow \text{GL}_n(\mathbb{F})$, and let us assume that $V_{\bar{\rho}} = \mathbb{F}^n$, equipped with the $\mathbb{F}[\Gamma]$ -module structure given by $\bar{\rho}$, satisfies \mathcal{C} . Given an object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$, we say that a deformation ρ of $\bar{\rho}$ to A satisfies \mathcal{C} if $V_{\rho} = A^n$, equipped with the $A[\Gamma]$ -module structure given by ρ , satisfies \mathcal{C} . We define then a functor

$$\text{Def}_{\bar{\rho}, \mathcal{C}} : \widehat{\mathcal{C}}_{\mathcal{O}} \rightarrow \mathbf{Sets}$$

by attaching to each object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$ the set of deformations of $\bar{\rho}$ to A which satisfy condition \mathcal{C} .

The main result of [85] implies that if $\text{End}_{\mathbb{F}[\Gamma]} V_{\bar{\rho}} \cong \mathbb{F}$, then functor $\text{Def}_{\bar{\rho}, \mathcal{C}}$ is representable by a local quotient $R_{\bar{\rho}, \mathcal{C}}$ of $R_{\bar{\rho}}$. In particular, we have that a deformation ρ of $\bar{\rho}$ to A satisfies condition \mathcal{C} if and only if $R_{\bar{\rho}} \rightarrow A$ factors through $R_{\bar{\rho}} \rightarrow R_{\bar{\rho}, \mathcal{C}}$.

In order to define a functor parametrising pseudodeformations satisfying a stable condition, we should firstly focus on Cayley-Hamilton representations. Let $\bar{D} = \det \circ \bar{\rho}$, let A be an object of $\widehat{\mathcal{C}}_{\mathcal{O}}$, and let D be a deformation of \bar{D} to A . We say that a

Cayley-Hamilton representation (R', D') of D , with $\rho : \Gamma \rightarrow R'^{\times}$, satisfies \mathcal{C} if R' , equipped with the $A[\Gamma]$ -module structure given by ρ , satisfies \mathcal{C} .

Finally, we say that a deformation D of \overline{D} to A satisfies condition \mathcal{C} if there exists a Cayley-Hamilton representation of D satisfying condition \mathcal{C} . We let

$$\mathrm{Def}_{\overline{D}, \mathcal{C}} : \widehat{\mathcal{C}}_{\mathcal{O}} \rightarrow \mathbf{Sets}$$

be the functor which attaches to each object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$ the set of deformations of \overline{D} to A which satisfy condition \mathcal{C} . The following result has been proved by Wake and Wang-Erickson.

Theorem 3.1.8 ([105, Theorem 2.2.5]). *The functor $\mathrm{Def}_{\overline{D}, \mathcal{C}}$ is representable by a local quotient $R_{\overline{D}, \mathcal{C}}$ of $R_{\overline{D}}$.*

In particular, we have that a deformation D of \overline{D} to A satisfies condition \mathcal{C} if and only if $R_{\overline{D}} \rightarrow A$ factors through $R_{\overline{D}} \rightarrow R_{\overline{D}, \mathcal{C}}$. It then follows that the closed points of the affine scheme $\mathrm{Spec} R_{\overline{D}, \mathcal{C}}[1/p]$ parametrise deformations of $\overline{\rho}$ to finite extensions of E satisfying condition \mathcal{C} . Furthermore, when $\overline{\rho}$ is absolutely irreducible, it can be proved that there is a canonical isomorphism between $R_{\overline{D}, \mathcal{C}}$ and $R_{\overline{\rho}, \mathcal{C}}$, see [105, Corollary 3.3.5].

3.2 Eigenvarieties for definite unitary groups

We review here the construction of eigenvarieties for definite unitary groups with respect to a purely imaginary quadratic extension of a totally real field, and we focus on the p -adic families of Galois representations attached to them. General constructions of eigenvarieties have been provided for instance in [43], [77], [103], and [57]. For the purposes of this work, however, it is enough to focus on the case of definite unitary groups, following the original construction of [29], and its more recent versions presented for instance in [10] and [30]. At a preliminary stage, we study in §3.2.1 triangulations and refinements attached to crystalline representations, and their deformation theory. The definition and basic properties of definite unitary groups eigenvarieties are instead presented in §3.2.2. Finally, in §3.2.3 we construct Galois pseudorepresentations on definite unitary groups eigenvarieties interpolating classical automorphic Galois representations.

3.2.1 Triangulations and refinements of crystalline representations

This is essentially an addendum to §1.2, where we introduced some background on p -adic Hodge theory. We see here some further notions, which will be useful in the study of p -adic families of automorphic representations. Our exposition roughly follows the one of [10, §2] and [12, §2].

We fix a prime p , and a p -adic field K , and we let $K_\infty = K(\zeta_{p^\infty})$ be the field obtained by adjoining all the p -power roots of unity to K . We define the *Robba ring* over K to be the ring \mathcal{R}_K of power series $f = \sum_{i \in \mathbb{Z}} a_i T^i$ defined over the maximal absolutely unramified extension of K_∞ and converging on an annulus of the form $r(f) < |T| < 1$. When $K = \mathbb{Q}_p$ we simply write \mathcal{R} for the Robba ring over \mathbb{Q}_p . Let A be a finite dimensional local commutative \mathbb{Q}_p -algebra, equipped with its unique Banach \mathbb{Q}_p -algebra topology, with maximal ideal \mathfrak{m}_A , and with residue field L (which is a finite extension of \mathbb{Q}_p). We write $\mathcal{R}_{K,A} = \mathcal{R}_K \otimes_{\mathbb{Q}_p} A$.

The ring $\mathcal{R}_{K,A}$ can be equipped with commuting, A -linear, continuous actions of the Frobenius operator ϕ and the Galois group $\Gamma = \text{Gal}(K_\infty/K)$ given by

$$\begin{aligned}\phi(f)(T) &= f(T^p), \\ \gamma(f)(T) &= f(T^\gamma), \quad f \in \mathcal{R}_{K,A}, \quad \gamma \in \Gamma.\end{aligned}$$

A (ϕ, Γ) -module over $\mathcal{R}_{K,A}$ is a finite free $\mathcal{R}_{K,A}$ -module D equipped with commuting, $\mathcal{R}_{K,A}$ -semilinear, continuous actions of ϕ and Γ , such that $\phi(D)$ generates D as $\mathcal{R}_{K,A}$ -module. The *rank* of D is its rank as $\mathcal{R}_{K,A}$ -module. A *morphism* between two (ϕ, Γ) -modules D and D' over $\mathcal{R}_{K,A}$ is just a $\mathcal{R}_{K,A}$ -linear map $\eta : D \rightarrow D'$ which commutes with the actions of ϕ and Γ .

By work of Kedlaya, see [68, Theorem 6.10], it is possible to attach to each (ϕ, Γ) -module D of rank n over $\mathcal{R}_{K,L}$ a sequence of rational numbers $s_1 \leq \dots \leq s_n$, called the *slopes* of D . The (ϕ, Γ) -module D is then said to be *étale* if $s_1 = \dots = s_n = 0$.

The functor $V \mapsto D_{\text{rig}}^\dagger(V)$ introduced by Berger in [13] defines a tensor equivalence between the category of continuous rank n representations of Γ_K over L and the category of étale (ϕ, Γ) -modules of rank n over $\mathcal{R}_{K,L}$, see for instance [35, Proposition 1.7]. Recall that Berger's construction of [13, §3.4] actually allows to define a (ϕ, Γ) -module $D_{\text{rig}}^\dagger(V)$ over $\mathcal{R}_{K,A}$ for any A -module V of finite type, equipped with a continuous action of Γ_K , in a functorial way.

Rank one (ϕ, Γ) -modules over $\mathcal{R}_{K,A}$ are classified by continuous characters $\delta : K^\times \rightarrow A^\times$ in the following way. Let us fix a uniformiser ϖ_K . (The construction

below can be easily proved to be independent of this choice). Given a continuous character $\delta : K^\times \rightarrow A^\times$, we let $\mathcal{R}_{K,A}(\delta)$ be the (ϕ, Γ) -module $\mathcal{R}_{K,A}$ equipped with the $\mathcal{R}_{K,A}$ -semilinear continuous actions of ϕ and Γ given by

$$\begin{aligned}\phi(1) &= \delta(\varpi_K), \\ \gamma(1) &= \delta(1), \quad \gamma \in \Gamma.\end{aligned}$$

If K is a finite extension of \mathbb{Q}_p , then to any δ as above it is possible to associate a continuous homomorphism $\delta \circ \text{Art}_K^{-1} : W_K \rightarrow A^\times$. This extends to a continuous character of Γ_K if and only if $v_L(\delta(\varpi_K) \bmod \mathfrak{m}_A) = 0$. In this case, we can see that $\mathcal{R}_{K,A}(\delta) = D_{\text{rig}}^\dagger(\delta \circ \text{Art}_K^{-1})$.

In general, it is possible to show, see for instance [10, Proposition 2.3.1], that any (ϕ, Γ) -module of rank one over $\mathcal{R}_{K,A}$ is isomorphic to $\mathcal{R}_{K,A}(\delta)$ for a unique δ .

As in §1.2, we let $K_0 = W(k_K)[1/p]$. Given a continuous representation $\rho : \Gamma_K \rightarrow \text{GL}(V)$ of Γ_K on a finite dimensional L -vector space V , the K_0 -vector spaces $D_{\text{cris}}(V)$ and $D_{\text{st}}(V)$ introduced by Fontaine, see §1.2.2, can be recovered from $D_{\text{rig}}^\dagger(V)$. For the purposes of this work, we only focus on $D_{\text{cris}}(V)$, and refer to [13] for the case of $D_{\text{st}}(V)$.

Let D be a (ϕ, Γ) -module over $\mathcal{R}_{K,L}$, and let $t \in \mathcal{R}$ be as in §1.2.1. Set then $D_{\text{cris}}(D) = D[1/t]^\Gamma$. By [13, Théorème 3.6], given a continuous representation $\rho : \Gamma_K \rightarrow \text{GL}(V)$ of Γ_K on a finite dimensional L -vector space V , there is a canonical isomorphism $D_{\text{cris}}(V) \cong D_{\text{cris}}(D_{\text{rig}}^\dagger(V))$. We say that a (ϕ, Γ) -module D of rank n over $\mathcal{R}_{K,L}$ is *crystalline* if $D_{\text{cris}}(D)$ has dimension n over L . Clearly, an étale (ϕ, Γ) -module D over $\mathcal{R}_{K,L}$ is crystalline if and only if it comes from a crystalline representation $\rho : \Gamma_K \rightarrow \text{GL}(V)$ of Γ_K via Berger's functor D_{rig}^\dagger .

Again, let A be a finite dimensional commutative \mathbb{Q}_p -algebra with residue field L . A *triangulation* of a (ϕ, Γ) -module D of rank n over $\mathcal{R}_{K,A}$ is a strictly increasing filtration $\Delta = (\Delta_i)_{i=0}^n$ of (ϕ, Γ) -submodules

$$\Delta_0 = \{0\} \subsetneq \Delta_1 \subsetneq \cdots \subsetneq \Delta_{n-1} \subsetneq \Delta_n = D$$

of D which are free and direct summand as $\mathcal{R}_{K,A}$ -modules. The *parameter* of $\Delta = (\Delta_i)_{i=0}^n$ is the ordered n -tuple $(\delta_1, \dots, \delta_n)$ of continuous characters $\delta_i : K^\times \rightarrow A^\times$ such that $\Delta_i/\Delta_{i-1} = \mathcal{R}_{K,A}(\delta_i)$. Following Colmez, we say that a (ϕ, Γ) -module D of rank n over $\mathcal{R}_{K,A}$ is *triangulable* if it can be equipped with a triangulation $\Delta = (\Delta_i)_{i=0}^n$.

Let us now study the triangulations of crystalline (ϕ, Γ) -modules and their deformation theory. For the purposes of this work, we just focus on crystalline étale (ϕ, Γ) -modules, coming then from crystalline representations of Γ_K via D_{rig}^\dagger . Note that

the theory still makes sense without this assumption, see for instance [10]. Also, let us restrict to the case where K is a finite extension of \mathbb{Q}_p .

Let then $\rho : \Gamma_K \rightarrow \mathrm{GL}(V)$ be a crystalline representation of Γ_K on an n -dimensional L -vector space V , and let $D = D_{\mathrm{rig}}^\dagger(V)$. Recall from §1.2.2 that $D_{\mathrm{cris}}(D)$ is a finite free $K_0 \otimes_{\mathbb{Q}_p} L$ -module, equipped with a K_0 -semilinear (but L -linear) Frobenius operator ϕ , and $D_{\mathrm{cris}}(D)_K = K \otimes_{K_0} D_{\mathrm{cris}}(D)$ is equipped with an exhausted and separated decreasing filtration $(F^i D_{\mathrm{cris}}(D)_K)_{i \in \mathbb{Z}}$ by $K \otimes_{\mathbb{Q}_p} L$ -submodules. Given an embedding $\tau : K \rightarrow L$, the filtration $(F^i D_{\mathrm{cris}}(D)_K)_{i \in \mathbb{Z}}$ of $D_{\mathrm{cris}}(D)_K$ induces a filtration of $D_{\mathrm{cris}}(D)_K \otimes_{K, \tau} L$, whose jumps are precisely the τ -Hodge-Tate weights $h_{\tau,1} \leq \dots \leq h_{\tau,n}$ of V . The operator $\phi^{[k_K : \mathbb{F}_p]}$ acts K_0 -linearly on $D_{\mathrm{cris}}(D)$, and we refer to its eigenvalues as the *crystalline eigenvalues* of $D_{\mathrm{cris}}(D)$. Up to enlarging L , let us assume that every crystalline eigenvalue of $D_{\mathrm{cris}}(D)$ is contained in L^\times .

A *refinement* of V is a full strictly increasing filtration $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ of ϕ -submodules

$$\mathcal{F}_0 = \{0\} \subsetneq \mathcal{F}_1 \subsetneq \dots \subsetneq \mathcal{F}_{n-1} \subsetneq \mathcal{F}_n = D_{\mathrm{cris}}(V)$$

of $D_{\mathrm{cris}}(V)$. A refinement $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ of V naturally induces an ordering $(h'_{\tau,1}, \dots, h'_{\tau,n})$ of the τ -Hodge-Tate weights of V , for a given embedding $\tau : K \rightarrow L$, defined by the property that the jumps of the filtration of $\mathcal{F}_i \otimes_{K_0, \tau} L$ induced by the filtration of $D_{\mathrm{cris}}(D)_K \otimes_{K, \tau} L$ are $(h'_{\tau,1}, \dots, h'_{\tau,i})$ for each i , and an ordering (ϕ_1, \dots, ϕ_n) of the crystalline eigenvalues of $D_{\mathrm{cris}}(V)$ defined by the identity $\det(t - \phi|_{\mathcal{F}_i}) = \prod_{j=1}^i (t - \phi_j)$ in $L[t]$ for each i . If the crystalline eigenvalues are distinct, then an ordering (ϕ_1, \dots, ϕ_n) induces a refinement of V by setting $\mathcal{F}_i = \sum_{j=1}^i D_{\mathrm{cris}}(D)^{\phi^{[k_K : \mathbb{F}_p]} = \phi_j}$. In this case, there is a bijection between the set of refinements of V and the set of orderings of the crystalline eigenvalues of $D_{\mathrm{cris}}(D)$.

The (ϕ, Γ) -module $D = D_{\mathrm{rig}}^\dagger(V)$ has rank n over $\mathcal{R}_{K,L}$. If $\Delta = (\Delta_i)_{i=0}^n$ is a triangulation of D , then $D_{\mathrm{cris}}(\Delta_i) = \Delta_i[1/t]^\Gamma$ is a filtered ϕ -submodule of rank i of $D_{\mathrm{cris}}(D)$. It is possible to show, see [10, Proposition 2.4.1], that the assignment $\Delta_i \mapsto D_{\mathrm{cris}}(\Delta_i)$ induces a bijection between the set of triangulations of D and the set of refinements of V , whose inverse is given by $\mathcal{F}_i \mapsto (\mathcal{R}_{K,L}[1/t]\mathcal{F}_i) \cap D$. Note that if the crystalline eigenvalues of $D_{\mathrm{cris}}(V)$ are distinct, then this gives a bijection between the set of triangulations of D and the set of orderings of the crystalline eigenvalues of $D_{\mathrm{cris}}(D)$.

Let us briefly see how the Hodge-Tate weights match under this bijection. Let $\tau : K \rightarrow L$ be an embedding, and let $h_{\tau,1} \leq \dots \leq h_{\tau,n}$ be the τ -Hodge-Tate weights of V . Let $\Delta = (\Delta_i)_{i=0}^n$ be a triangulation of D , and let $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ be the corresponding refinement of V . Let $(h'_{\tau,1}, \dots, h'_{\tau,n})$ be the ordering of the τ -Hodge-Tate weights of V

induced by $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$. If $(\delta_1, \dots, \delta_n)$ is the parameter of $\Delta = (\Delta_i)_{i=0}^n$, then $h'_{\tau,i}$ is precisely the τ -Hodge-Tate weight of δ_i .

Assume that for each embedding $\tau : K \rightarrow L$ the τ -Hodge-Tate weights of V are distinct. We call the *critical type* of a triangulation $\Delta = (\Delta_i)_{i=0}^n$ of D (resp. of a refinement $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ of V) the collection $(\sigma_\tau)_\tau$ of permutations of $\{1, \dots, n\}$ such that $h'_{\tau,i} = h_{\tau, \sigma_\tau(i)}$ for all τ and i . If $\sigma_\tau = \text{Id}$ for each τ , we say that $\Delta = (\Delta_i)_{i=0}^n$ (resp. $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$) is *non critical*.

Let us now study the deformation theory of the objects introduced above. Before considering the case of a crystalline representation of Γ_K , let us give some abstract definitions. Let us then assume again that K is a p -adic field, and that L is a finite extension of \mathbb{Q}_p . Let $\widehat{\mathcal{C}}_L$ be the category of complete Noetherian local commutative L -algebras with residue field L , and let \mathcal{C}_L be its full subcategory of finite Artinian L -algebras. Also, let us fix a (ϕ, Γ) module D of rank n over $\mathcal{R}_{K,L}$. Given an object A of \mathcal{C}_L , a *deformation* of D to A is a pair (D_A, π_A) consisting of a (ϕ, Γ) -module D_A over $\mathcal{R}_{K,A}$ and a $\mathcal{R}_{K,A}$ -linear morphism $\pi_A : D_A \rightarrow D$ of (ϕ, Γ) -modules inducing an isomorphism $D_A \otimes_A L \cong D$. We define a functor

$$\text{Def}_D : \mathcal{C}_L \rightarrow \mathbf{Sets}$$

by mapping each object A of \mathcal{C}_L to the set of isomorphism classes of deformations of D to A .

Let us assume that D is triangulable, and let $\Delta = (\Delta_i)_{i=0}^n$ be a triangulation of D . We define the *trianguline deformation functor* of the pair (D, Δ) to be the functor

$$\text{Def}_{D,\Delta} : \mathcal{C}_L \rightarrow \mathbf{Sets}$$

given by mapping each object A of \mathcal{C}_L to the set of isomorphism classes of triples (D_A, π_A, Δ_A) where (D_A, π_A) is a deformation of D to A , and $\Delta_A = (\Delta_{A,i})_{i=0}^n$ is a triangulation of D_A such that $\pi_A(\Delta_{A,i}) = \Delta_i$ for each i .

Let us now move back to the context of crystalline representations. Let $\rho : \Gamma_K \rightarrow \text{GL}(V)$ be a crystalline representation of Γ_K on an n -dimensional L -vector space V , and let $D = D_{\text{rig}}^\dagger(V)$. The choice of a triangulation Δ of D is equivalent to the choice of a refinement \mathcal{F} of V . We let $\text{Def}_\rho = \text{Def}_D$ and $\text{Def}_{\rho,\mathcal{F}} = \text{Def}_{D,\Delta}$. Note that Def_ρ is just the functor which maps each object A of \mathcal{C}_L , with maximal ideal \mathfrak{m}_A , to the set of isomorphism classes of continuous representations $\rho_A : \Gamma_K \rightarrow \text{GL}(V_A)$ of Γ_K on A -modules V_A of finite type together with homomorphisms $\pi_A : V_A \rightarrow V$ of $A[\Gamma_K]$ -modules inducing isomorphisms $V_A \otimes_A L \cong V$ (this definition makes sense for any continuous representation $\rho : \Gamma_K \rightarrow \text{GL}(V)$, not just for a crystalline one). By

construction, we have a natural transformation $\text{Def}_{\rho, \mathcal{F}} \rightarrow \text{Def}_{\rho}$. The following result holds.

Proposition 3.2.1 ([10, Proposition 2.5.8]). *If the crystalline eigenvalues of $D_{\text{cris}}(V)$ are distinct, then $\text{Def}_{\rho, \mathcal{F}}$ is a subfunctor of Def_{ρ} , and $\text{Def}_{\rho, \mathcal{F}} \rightarrow \text{Def}_{\rho}$ is relatively representable. Moreover, if \mathcal{F} is non critical, then the subfunctor $\text{Def}_{\rho, \text{cris}}$ of Def_{ρ} parametrising crystalline deformations factors through $\text{Def}_{\rho, \mathcal{F}}$.*

3.2.2 Definition and basic properties of eigenvarieties

First of all, let us define the unitary groups for which we are going to construct eigenvarieties. In general, let \mathbb{k} be a field, let E be an étale \mathbb{k} -algebra of degree 2, equipped with a nontrivial \mathbb{k} -automorphism c , let $n \geq 1$ be an integer, and let Δ be a simple central E -algebra of rank n^2 , equipped with a \mathbb{k} -algebra anti-involution $x \mapsto x^*$ of the second kind, i.e. which coincides with c on E . We can attach to the datum $(\Delta, *)$ a linear algebraic group G over \mathbb{k} , whose points on every \mathbb{k} -algebra R are given by

$$G(R) = \{x \in (\Delta \otimes_{\mathbb{k}} R)^{\times} : xx^* = 1\}.$$

Note that the base change $G \times_{\mathbb{k}} E$ of G to E is isomorphic to the group Δ^{\times} over E of invertible elements of Δ , and hence G is a twisted \mathbb{k} -form of $\text{GL}_{n, E}$. Actually, every twisted \mathbb{k} -form of $\text{GL}_{n, E}$ is isomorphic to a group of this form.

If E is a field, then we say that G is a *unitary group* attached to E/\mathbb{k} . When moreover $\Delta = M_{n \times n}(E)$, and $*$ is the adjunction with respect to a c -Hermitian form h on E^n , we refer to G as a *n -variables unitary group* attached to E/\mathbb{k} .

Let now p be a fixed prime, and let $\iota_p : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ and $\iota_{\infty} : \overline{\mathbb{Q}} \rightarrow \mathbb{C}$ be fixed embeddings. Let F be a CM field field, with maximal totally real subfield F^+ , let c be a generator of $\text{Gal}(F/F^+)$, and assume that all the places of F^+ above p split in F . Let $n \geq 1$ be an integer, which is either odd, or if it is even then $n[F^+ : \mathbb{Q}] \equiv 0 \pmod{4}$. By the Hasse principle, there exists a (unique) n -variables unitary group U attached to F/F^+ which is split over F , quasi-split at each finite place of F^+ , and definite, i.e. such that $U(F_v^+)$ is compact for every real place v of F^+ . If v is a place of F^+ above p , then the choice of a place \tilde{v} of F above v defines an isomorphism $U(F_v^+) \cong \text{GL}_n(F_{\tilde{v}}) = \text{GL}_n(F_v^+)$. From now on, let us choose \tilde{v} above v for all v above p .

Let us fix a finite set S of finite places v of F^+ such that $v \nmid p$, and containing all the places of F^+ at which U is ramified.

Fix a compact open subgroup $K^S = \prod_{v \notin S} K_v$ of $U(\mathbb{A}_{F^+,f}^S)$ such that $K_v = U(\mathcal{O}_{F_v^+})$ for almost all $v \notin S$, and such that K_v is maximal hyperspecial if $v \nmid p$, and is an Iwahori subgroup of $U(F_v^+) \cong \mathrm{GL}_n(F_v^+)$ if $v \mid p$. Let $E \subset \overline{\mathbb{Q}}$ be a Galois number field, and let \mathcal{H} be a commutative E -subalgebra of the Hecke algebra of $U(\mathbb{A}_{F^+,f}^p)$ over E , containing the algebra \mathcal{H}^S of locally constant and compactly supported \mathbb{Z} -valued functions on $U(\mathbb{A}_{F^+,f}^{S,p})$ which are bi-invariant under the action of $K^{S,p} = \prod_{v \notin S, v \nmid p} K_v$. Finally, for each $v \in S$, fix idempotents e_v of the Hecke algebra of $U(F_v^+)$ over E , and set $e = (\otimes_{v \in S} e_v) \otimes \mathbf{1}_{K^{S,p}}$, which is an idempotent of the Hecke algebra of $U(\mathbb{A}_{F^+,f}^p)$ over E . Finally, let L be the Galois closure of $\iota_p(E)$ and $\iota_p(F)$ inside $\overline{\mathbb{Q}_p}$.

For every $v \mid p$, let T_v be the diagonal \mathbb{Q}_p -torus of $U(F_v^+) \cong \mathrm{GL}_n(F_v^+)$, let T_v^0 be a maximal compact subgroup of T_v , and let B_v be the Borel \mathbb{Q}_p -subgroup of upper triangular matrices of $U(F_v^+) \cong \mathrm{GL}_n(F_v^+)$. Let then $T = \prod_{v \mid p} T_v$, $T^0 = \prod_{v \mid p} T_v^0$, and $B = \prod_{v \mid p} B_v$. Let δ_v be the modular function on B_v , and let $\delta = \otimes_{v \mid p} \delta_v$ be the modular function on B , as in §1.5.2. The choice of B allows us to define a notion of dominant character (with respect to B) in the group $X^*(T)$ of algebraic characters of T . Note that every element κ of $X^*(T)$ is entirely determined by a tuple $(\kappa_{\tau,i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^+, \overline{\mathbb{Q}_p})}$ via the relation

$$\kappa(x_1, \dots, x_n) = \prod \tau((x_i)_{v(\tau)})^{\kappa_{\tau,i}},$$

where for each embedding $\tau : F^+ \rightarrow \overline{\mathbb{Q}_p}$ we denote by $v(\tau)$ is the place of F^+ induced by τ . The element κ is then dominant if $\kappa_{\tau,1} \geq \dots \geq \kappa_{\tau,n}$ for each τ , i.e. $(\kappa_{\tau,i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^+, \overline{\mathbb{Q}_p}),+}$ with the notation of §1.5.4.

We let $\mathcal{T} = \mathrm{Hom}_{\mathrm{cts}}(T, \mathbb{G}_m^{\mathrm{rig}})$ (resp. $\mathcal{W} = \mathrm{Hom}_{\mathrm{cts}}(T^0, \mathbb{G}_m^{\mathrm{rig}})$) be the rigid analytic space over \mathbb{Q}_p whose points over any affinoid \mathbb{Q}_p -algebra R parametrise the continuous characters $T \rightarrow R^\times$ (resp. $T^0 \rightarrow R^\times$). The group $X^*(T)$ of algebraic characters of T is then a subgroup of $\mathcal{T}(L)$.

Since U is a definite unitary group over F^+ , all the automorphic representations π of $U(\mathbb{A}_{F^+})$ are discrete, and cohomological in degree 0. Also, the infinite part π_∞ of π is finite dimensional over \mathbb{C} , and the finite part π_f of π is defined over $\overline{\mathbb{Q}}$. The highest weight of π_∞ naturally defines an element κ_π of $X^*(T) \subset \mathcal{T}(L)$.

Assume that $e(\pi_f)^{K_p} \neq \{0\}$, where $K_p = \prod_{v \mid p} K_v$. Then, $e(\pi_f)^{K_p}$ is finite dimensional over \mathbb{C} , and so it can be triangularised as \mathcal{H} -module. After semisimplification, this gives a finite set of ring homomorphisms $\mathcal{H} \rightarrow \mathbb{C}$. Since π_f is defined over $\overline{\mathbb{Q}}$, these homomorphisms can be seen as taking values in $\overline{\mathbb{Q}}$ via ι_∞^{-1} . After composing with ι_p , we then get ring homomorphisms $\psi : \mathcal{H} \rightarrow \overline{\mathbb{Q}_p}$. Such a ring homomorphism ψ is called a $\overline{\mathbb{Q}_p}$ -system of Hecke eigenvalues attached to π . When $\mathcal{H} = \mathcal{H}^S$, there is a unique $\psi = \psi_\pi$, which is completely determined by the Satake parameter of π .

Let $v \mid p$. An *accessible refinement* of $\pi_v \otimes |\det|_v^{(1-n)/2}$ is a character $\vartheta_v : T_v/T_v^0 \rightarrow \mathbb{C}^\times$ such that $\pi_v \otimes |\det|_v^{(1-n)/2}$ occurs as a subrepresentation of the normalised smooth parabolic induction

$$\begin{aligned} & \text{Ind}_{B_v}^{\text{GL}_n(F_v^+)} \vartheta_v \\ &= \{f : \text{GL}_n(F_v^+) \rightarrow \mathbb{C} \text{ smooth} : f(bg) = (\vartheta_v \delta_v^{1/2})(b)f(g), \forall b \in B_v, \forall g \in \text{GL}_n(F_v^+)\} \end{aligned}$$

of ϑ_v to $U(F_v^+) \cong \text{GL}_n(F_v^+)$. When π_v is unramified, an accessible refinement of $\pi_v \otimes |\det|_v^{(1-n)/2}$ is equivalently an ordering of its Satake parameters. Explicitly, if ϑ_v is an accessible refinement, then an ordering $(\phi_{v,1}, \dots, \phi_{v,n})$ of the Satake parameters of $\pi_v \otimes |\det|_v^{(1-n)/2}$ is given by setting $\phi_{v,i} = \vartheta_v(t_{v,i})$, for $t_{v,i} = (1, \dots, 1, \varpi_v, 1, \dots, 1) \in T_v$, where the uniformiser ϖ_v of $\mathcal{O}_{F_v^+}$ is in i -th position, for each i .

Let ϑ_v be an accessible refinement of $\pi_v \otimes |\det|_v^{(1-n)/2}$. Up to composing with $\iota_p \iota_\infty^{-1}$, we can see ϑ_v as a character $\vartheta_v : T_v/T_v^0 \rightarrow \overline{\mathbb{Q}}_p^\times$. Let $\vartheta = \{\vartheta_v\}_{v \mid p}$. Then, we can attach to the pair (π, ϑ) an element $\nu_{(\pi, \vartheta)} = \kappa_\pi(\otimes_{v \mid p} \vartheta_v)(\delta^{-1/2} \prod_{v \mid p} |\det|_v^{(n-1)/2})$ of $\mathcal{T}(\overline{\mathbb{Q}}_p)$. Note that $\nu_{(\pi, \vartheta)}$ determines exactly κ_π and ϑ .

We call *p-refined* automorphic representation of $U(\mathbb{A}_{F^+, f})$ a pair (π, ϑ) consisting of an automorphic representation π of $U(\mathbb{A}_{F^+, f}^+)$ and a collection $\vartheta = \{\vartheta_v\}_{v \mid p}$ of accessible refinements of $\pi_v \otimes |\det|_v^{(1-n)/2}$ for $v \mid p$ as above. Let $Z^{\text{cl}} \subset \text{Hom}_{\text{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p) \times \mathcal{T}(\overline{\mathbb{Q}}_p)$ be the subset consisting of the pairs (ψ, ν) such that there exists a p -refined automorphic representation (π, ϑ) of $U(\mathbb{A}_{F^+})$ with $e(\pi_f)^{K_p} \neq \{0\}$, and such that ψ is a $\overline{\mathbb{Q}}_p$ -system of Hecke eigenvalues attached to π , and $\nu = \nu_{(\pi, \vartheta)}$. The following theorem is [30, Théorème 1.6]. We refer to [10, §7] for a more detailed discussion of this result.

Theorem 3.2.2. *There exists a unique 4-tuple $(\mathcal{E}, \psi, \nu, Z^{\text{cl}})$ where:*

- \mathcal{E} is a reduced rigid analytic space over L ;
- $\psi : \mathcal{H} \rightarrow \mathcal{O}(\mathcal{E})$ is a ring homomorphism;
- $\nu : \mathcal{E} \rightarrow \mathcal{T}$ is a finite analytic morphism;
- $Z^{\text{cl}} \subset \mathcal{E}(\overline{\mathbb{Q}}_p)$ is an accumulation and Zariski-dense subset³;

satisfying the following properties:

- (1) For every open affinoid $V \subset \mathcal{T}$, the map $\psi \otimes \nu^\# : \mathcal{H} \otimes_{\mathbb{Z}} \mathcal{O}(V) \rightarrow \mathcal{O}(\nu^{-1}(V))$ is surjective.

³A subset $Z \subset \mathcal{E}(\overline{\mathbb{Q}}_p)$ is said to be *accumulation and Zariski-dense* if it meets any irreducible component of \mathcal{E} , and if for all $z \in Z$ there is a basis of affinoid neighbourhoods U of z such that $Z \cap U$ is Zariski-dense in U .

- (2) The evaluation map $\mathcal{E}(\overline{\mathbb{Q}}_p) \rightarrow \mathrm{Hom}_{\mathrm{ring}}(\mathcal{H}, \overline{\mathbb{Q}}_p)$, $x \mapsto \psi_x = (h \mapsto \psi(h)(x))$, induces a bijection $Z^{\mathrm{cl}} \xrightarrow{\sim} \mathcal{Z}^{\mathrm{cl}}$, $z \mapsto (\psi_z, \nu(z))$.
- (3) The space \mathcal{E} is equidimensional of dimension $\dim \mathcal{W} = n[F^+ : \mathbb{Q}]$.
- (4) Let $\kappa : \mathcal{E} \rightarrow \mathcal{W}$ be the composition of ν with the natural projection $\mathcal{T} \rightarrow \mathcal{W}$. Then, the space \mathcal{E} has an admissible covering by open affinoids $\Omega \subset \mathcal{E}$ such that $\kappa(\Omega) \subset \mathcal{W}$ is an open affinoid, and such that $\kappa|_{\Omega} : \Omega \rightarrow \kappa(\Omega)$ is finite, and surjective when restricted to any irreducible component of Ω . Moreover, the image via κ of any irreducible component of \mathcal{E} is Zariski-open in \mathcal{W} .
- (5) We have $\psi(\mathcal{H}) \subset \mathcal{O}(\mathcal{E})^0$.
- (6) Let $x \in \mathcal{E}(\overline{\mathbb{Q}}_p)$ be such that $\kappa(x) \in \mathcal{W}(\overline{\mathbb{Q}}_p)$ is algebraic and dominant, let $(\kappa_{\tau,i}) \in (\mathbb{Z}^n)^{\mathrm{Hom}(F^+, \overline{\mathbb{Q}}_p), +}$ be the corresponding tuple. Assume that

$$\frac{v_p(\phi_{v(\tau),1} \cdots \phi_{v(\tau),i})}{v_p(\varpi_{v(\tau)})} < \kappa_{\tau,i} - \kappa_{\tau,i+1} + 1$$

for each $\tau : F^+ \rightarrow \overline{\mathbb{Q}}_p$ and $i = 1, \dots, n$. Then $x \in Z^{\mathrm{cl}}$.

The rigid analytic space \mathcal{E} over L is called a reduced *eigenvariety* for U of idempotent type e , and the closed points in Z^{cl} are called the *classical points* of \mathcal{E} .

Given a compact open subgroup $K^p = \prod_{v \nmid p} K_v$ of $U(\mathbb{A}_{F^+,f}^p)$, usually referred to as a *tame level* away from p , we let S be the (finite) set of places $v \nmid p$ of F^+ such that K_v is not maximal hyperspecial, we let $\mathcal{H} = \mathcal{H}^S$, and we let $e = \mathbf{1}_{K^p}$. We call the corresponding eigenvariety \mathcal{E}_{K^p} an eigenvariety for U of tame level K^p . The eigenvariety \mathcal{E}_{K^p} is equidimensional of dimension $n[F^+ : \mathbb{Q}]$. We say that an automorphic representation π of $U(\mathbb{A}_{F^+})$ has tame level K^p if $(\pi_f)^{K^p} \neq \{0\}$. Any p -refined automorphic representation (π, ϑ) of $U(\mathbb{A}_{F^+})$ such that π has tame level K^p naturally defines a classical point of \mathcal{E}_{K^p} . We also say that \mathcal{E}_{K^p} contains (π, ϑ) .

Given a p -refined automorphic representation (π, ϑ) of $U(\mathbb{A}_{F^+})$, instead of considering a whole tame level eigenvariety containing (π, ϑ) , it would be useful, for the purposes of this work, to consider a *minimal level* eigenvariety containing (π, ϑ) . In order to define such an object, we need to fix suitable data. We refer to [30, §3.6] for the details of this construction.

Fix a tame level $K^p = \prod_{v \nmid p} K_v$ of $U(\mathbb{A}_{F^+,f}^p)$ for π , and let S be the set of places $v \nmid p$ of F^+ such that π_v is ramified. For each $v \in S$, fix a Bernstein connected component \mathcal{B}_v of the category of smooth representations of $U(F_v^+) \cong \mathrm{GL}_n(F_v^+)$ over \mathbb{C} . Let us fix a Galois number field $E \subset \overline{\mathbb{Q}}$ such that \mathcal{B}_v and its Bernstein centre \mathfrak{z}_v

are defined over E . We can choose an idempotent e_v of the Hecke algebra of $U(F_v^+)$ over E such that:

- $b_v e_v = e_v$, where b_v is the Bernstein central idempotent associated to \mathcal{B}_v ;
- $e_v(\pi_v) \neq 0$;
- $\text{rec}_v(\sigma) \prec_{I_{F_v^+}} \text{rec}_v(\pi_v)$ for each irreducible σ in \mathcal{B}_v such that $e_v(\sigma) \neq 0$.

Let us finally fix the E -algebra

$$\mathcal{H} = (\otimes_{v \in S} \mathfrak{z}_v) \otimes_{\mathbb{Z}} \mathcal{H}^S,$$

(where the tensor product $\otimes_{v \in S} \mathfrak{z}_v$ is taken over E). The *minimal level* eigenvariety containing (π, ϑ) is then the eigenvariety \mathcal{E} of idempotent type $e = (\otimes_{v \in S} e_v) \otimes \mathbf{1}_{K^S, p}$ attached to these data. This is a closed rigid analytic subvariety of the eigenvariety \mathcal{E}_{K^p} of tame level K^p , which is equidimensional of dimension $n[F^+ : \mathbb{Q}]$ as well. The pair (π, ϑ) naturally defines a classical point of \mathcal{E} .

3.2.3 Galois pseudorepresentations on eigenvarieties

Let us fix notations as in §3.2.2. Let us consider a tame level $K^p = \prod_{v \nmid p} K_v$ of $U(\mathbb{A}_{F^+, f}^p)$, and let \mathcal{E}_{K^p} be the corresponding eigenvariety. Let \mathcal{Z}^{reg} be the subset of \mathcal{Z}^{cl} of pairs (ψ, ν) corresponding to p -refined automorphic representations (π, ϑ) of $U(\mathbb{A}_{F, f}^+)$ such that π_∞ has regular weight. Let Z^{reg} be the corresponding subset of Z^{cl} via $Z^{\text{cl}} \xrightarrow{\sim} \mathcal{Z}^{\text{cl}}$. By [10, Lemma 7.5.3], the set Z^{reg} is a Zariski dense subset of $\mathcal{E}_{K^p}(\overline{\mathbb{Q}}_p)$, accumulating at each point of Z^{cl} .

Let $z \in Z^{\text{reg}}$, and let π be the corresponding automorphic representation of $U(\mathbb{A}_{F, f}^+)$. By results of Labesse, see [70], and of Mœglin and Waldspurger, see [79], the (weak) base change $\Pi = \text{BC}(\pi)$ of π to $\text{GL}_n(\mathbb{A}_F)$ is a well-defined automorphic representation, which decomposes as an isobaric sum⁴ $\Pi = \boxplus_{i=1}^k \Pi_i$ of discrete conjugate self-dual automorphic representations Π_i of $\text{GL}_{n_i}(\mathbb{A}_F)$ of the form $\Pi_i = \boxplus_{j=1}^{d_i} (\Pi_i^0 \otimes |\det|_F^{j-(d_i+1)/2})$, for conjugate self-dual cuspidal automorphic representations Π_i^0 of $\text{GL}_{m_i}(\mathbb{A}_F)$, where $n_i = m_i d_i$. By Theorem 1.6.1, we can then attach to Π a Galois representation $\rho_\Pi : \Gamma_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$ satisfying

$$\iota_\infty \iota_p^{-1} \text{WD}(\rho_\Pi |_{\Gamma_{F_v}})^{\text{F-ss}} \cong \text{rec}_v(\pi_v \otimes |\det|_v^{(1-n)/2})$$

⁴If Π and Π' are automorphic representations of $\text{GL}_n(\mathbb{A}_F)$ and $\text{GL}_{n'}(\mathbb{A}_F)$ respectively, then the *isobaric sum* of Π and Π' is the automorphic representation $\Pi \boxplus \Pi'$ of $\text{GL}_{n+n'}(\mathbb{A}_F)$ satisfying $(\Pi \boxplus \Pi')_v = \Pi_v \boxplus \Pi'_v$ for each place v of F , where $\Pi_v \boxplus \Pi'_v$ is defined by $\phi_{\Pi_v \boxplus \Pi'_v} = \phi_{\Pi_v} \oplus \phi_{\Pi'_v}$ if v is infinite, and by $\text{rec}_v(\Pi_v \boxplus \Pi'_v) = \text{rec}_v(\Pi_v) \oplus \text{rec}_v(\Pi'_v)$ if v is finite.

for each $v \mid p$ (recall that \tilde{v} is a fixed place of F above v). We set $\rho_z = \rho_\Pi$. Note that if π_v is unramified, then ρ_z is crystalline at \tilde{v} , and the crystalline eigenvalues of $D_{\text{cris}}(\rho_z |_{\Gamma_{F_{\tilde{v}}}})$ match with the Satake parameters of $\pi_v \otimes |\det|_v^{(1-n)/2}$. It follows that the accessible refinement ϑ_v defines (uniquely) an ordering of the crystalline eigenvalues of $D_{\text{cris}}(\rho_z |_{\Gamma_{F_{\tilde{v}}}})$, and therefore a refinement of $\rho_z |_{\Gamma_{F_{\tilde{v}}}}$, and a triangulation of $D_{\text{rig}}^\dagger(\rho_z |_{\Gamma_{F_{\tilde{v}}}})$, by the results presented in §3.2.1.

Let S be the set of places v of F^+ such that K_v is not maximal hyperspecial, and let \tilde{S} be the set of places of S which are either above the places of S or above p . For each $w \notin \tilde{S}$ and $i = 1, \dots, n$, if v is the place of F^+ below w , we let $h_{w,i}$ be the element of $\mathcal{H}_{\mathbb{Z}}(U(F_v^+), K_v)$ corresponding to the double coset of the matrices of the form

$$\text{diag}(\underbrace{\varpi_v, \dots, \varpi_v}_i, 1, \dots, 1).$$

Combining the Zariski-density of Z^{reg} in $\mathcal{E}_{K^p}(\overline{\mathbb{Q}}_p)$, the existence of Galois representations $\rho_z : \Gamma_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$ for each $z \in Z^{\text{reg}}$, and an interpolation argument due to Chenevier, see [29, Proposition 7.1.1], we get the following result⁵.

Proposition 3.2.3. *There exists a unique continuous n -dimensional pseudorepresentation $D : \mathcal{O}(\mathcal{E}_{K^p})[\Gamma_{F, \tilde{S}}] \rightarrow \mathcal{O}(\mathcal{E}_{K^p})$ such that $\overline{D}_z = \det \circ \rho_z$ for each $z \in Z^{\text{reg}}$, and*

$$D_{\mathcal{O}(\mathcal{E}_{K^p})[t]}(1 - t \text{Frob}_w) = 1 + \sum_{i=1}^n (-1)^i (\#k(w))^{i(i-1)/2} \psi(h_{w,i}) t^i, \quad \text{in } \mathcal{O}(\mathcal{E}_{K^p})[t],$$

for each $w \notin \tilde{S}$.

Let us fix a mod p representation $\bar{\rho} : \Gamma_{F, \tilde{S}} \rightarrow \text{GL}_n(\overline{\mathbb{F}}_p)$, and let $\mathcal{E}_{K^p, \bar{\rho}}$ be the (possibly empty) rigid analytic subspace of \mathcal{E}_{K^p} whose closed points z have attached Galois representations ρ_z reducing to $\bar{\rho}$ modulo p . Let \mathcal{X} be the universal pseudodeformation rigid analytic space for $\overline{D} = \det \circ \bar{\rho}$ introduced in §3.1.1, and let \mathcal{X}_L be its base change to L . By Proposition 3.2.3 and the universal property of \mathcal{X} , there exists then a morphism $\mathcal{E}_{K^p, \bar{\rho}} \rightarrow \mathcal{X}_L$ of rigid analytic spaces. We will look more precisely at this morphism in 3.3.1.

⁵Note that Chenevier's argument in [29, Proposition 7.1.1] is presented in the context of pseudocharacters. However, it turns out to be easily extendible to the context of pseudorepresentations.

3.3 Automorphic points in (pseudo)deformation spaces

This final section studies the geometry of p -adic rigid analytic (pseudo)deformation spaces and eigenvarieties at points arising from (classical) automorphic representations. We fix a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, for F a CM field, and consider the attached automorphic Galois representation. In §3.3.1 we see how the irreducibility of the automorphic Galois representation implies smoothness results for deformation spaces and (under additional “generic” assumptions) for eigenvarieties at the corresponding points. In §3.3.2 we sketch a plan of future research on possible converses to the smoothness results of §3.3.1.

3.3.1 Smoothness results

Let F be a CM field, let F^+ be its maximal totally real subfield, let c be a generator of $\mathrm{Gal}(F/F^+)$, and let $n \geq 1$ be an integer. Let π be a regular algebraic, conjugate self-dual, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Let p be a prime (which we assume to be odd), and let

$$\rho_\pi : \Gamma_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

be the Galois representation attached to π via Theorem 1.6.1. For simplicity of notation, set $\rho = \rho_\pi$, and let $\bar{\rho}$ be (as usual) the semisimplification of the reduction of ρ modulo p . As predicted by the Langlands conjecture for GL_n , the representation ρ should be absolutely irreducible. We have seen in §1.7 how this is still an open question in general. Nevertheless, let us assume here that ρ is absolutely irreducible (actually, assume that $\bar{\rho}$ is absolutely irreducible⁶). In this section, we see smoothness results for (pseudo-)deformation spaces and eigenvarieties attached to $\bar{\rho}$ and π .

Let U be a quasi-split definite unitary group in n variables attached to F/F^+ , as in §3.2.2. Then, by results of Mok, see [80], the representation π descends to an automorphic representation of $U(\mathbb{A}_{F^+})$ via standard base change. With a slight abuse of notation, let us still denote this representation by π . Let S be the (finite) set of places of F^+ where π ramifies, and let \tilde{S} be a finite set of places of F such that every $\tilde{w} \in \tilde{S}$ is split over some $w \in S$, and \tilde{S} contains at most one place above any $w \in S$.

⁶It is known, see [83, Corollary 1.3], that if a geometric compatible system $\mathcal{R} = \{\rho_\lambda\}_{\lambda \in \Lambda}$ with coefficients in a number field E is absolutely irreducible, then there exists a subset Λ' of Λ of residual Dirichlet density 1 such that $\bar{\rho}_\lambda$ is absolutely irreducible for every $\lambda \in \Lambda'$. Therefore, if Conjecture 1.7.3 holds true, then this result applies to the automorphic compatible system $\mathcal{R}_\pi = \{\rho_{\pi, \lambda}\}_{\lambda \in |E_\pi|}$

If we denote by F_S the maximal extension of F inside a fixed algebraic closure \overline{F} of F which is unramified outside of any of the places in F above those in S , then F_S/F^+ and F_S/F are Galois extensions, with $\text{Gal}(F_S/F^+)/\text{Gal}(F_S/F) \cong \text{Gal}(F/F^+)$, and ρ (and therefore $\overline{\rho}$) factors through $\text{Gal}(F_S/F)$. Let E be a finite extension of \mathbb{Q}_p inside $\overline{\mathbb{Q}_p}$ such that the image of ρ is contained in $\text{GL}_n(E)$, and let \mathcal{X} be the rigid analytic pseudodeformation space for $\overline{D} = \det \circ \overline{\rho}$ over E . For $\overline{\rho}$ absolutely irreducible, this coincides with the rigid analytic space attached to the universal deformation ring of $\overline{\rho}$. The representation ρ naturally defines a closed point x of \mathcal{X} .

Let us now introduce a different deformation space for $\overline{\rho}$. First of all, we recall the definition of the Clozel-Harris-Taylor group scheme \mathcal{G}_n , see [34, §2.1], which is the group scheme over \mathbb{Z} defined as the semidirect product

$$\mathcal{G}_n = (\text{GL}_n \times \text{GL}_1) \rtimes \{1, j\},$$

where the group $\{1, j\}$ acts on $\text{GL}_n \times \text{GL}_1$ by $j(g, a)j^{-1} = (a^t g^{-1}, a)$. There is a homomorphism $\nu : \mathcal{G}_n \rightarrow \text{GL}_1$ which sends (g, a) to a , and j to -1 . Let \mathcal{G}_n° denote the connected component of the identity of \mathcal{G}_n , and let “ad” denote the adjoint action of \mathcal{G}_n on \mathfrak{gl}_n , given by

$$\begin{aligned} \text{ad}(g, a)(X) &= gXg^{-1} \\ \text{ad}(j)(X) &= -{}^t X, \quad X \in \mathfrak{gl}_n. \end{aligned}$$

Given a group Γ , a ring A , and a group homomorphism $r : \Gamma \rightarrow \mathcal{G}_n(A)$, we write $\text{ad}(r)$ for $\mathfrak{gl}_n(A)$ with the adjoint action $\text{ad} \circ r$ of Γ .

Let now Γ be a topological group, and let Δ be an open subgroup of Γ of index 2. If A is a topological ring, and $\gamma_0 \in \Gamma \setminus \Delta$ is fixed, then by [34, Lemma 2.1.1] there is a natural bijection between the following two sets.

- (1) Continuous homomorphisms $r : \Gamma \rightarrow \mathcal{G}_n(A)$ inducing an isomorphism $\Gamma/\Delta \xrightarrow{\sim} \mathcal{G}_n(A)/\mathcal{G}_n^\circ(A)$.
- (2) Triples $(\rho, \mu, \langle \cdot, \cdot \rangle)$, where $\rho : \Delta \rightarrow \text{GL}_n(A)$ and $\mu : \Gamma \rightarrow A^\times$ are continuous homomorphisms, and $\langle \cdot, \cdot \rangle$ is a perfect A -linear pairing on A^n satisfying

$$\begin{aligned} \langle \rho(\delta)x, \rho(\gamma_0 \delta \gamma_0^{-1})y \rangle &= \mu(\delta) \langle x, y \rangle, \\ \langle x, \rho(\gamma_0^2)y \rangle &= -\mu(\gamma_0) \langle y, x \rangle, \quad \forall \delta \in \Delta, \forall x, y \in A^n. \end{aligned}$$

Under this bijection, we have $\mu(\gamma) = (\nu \circ r)(\gamma)$ for all $\gamma \in \Gamma$, and $\langle x, y \rangle = {}^t x P^{-1} y$ for $r(\gamma_0) = (P, -\mu(\gamma_0))j$.

Let \mathcal{O} be the ring of integers of E , let \mathfrak{m} be the maximal ideal of \mathcal{O} , and let $\mathbb{F} = \mathcal{O}/\mathfrak{m}$. Let $\bar{r} : \text{Gal}(F_S/F^+) \rightarrow \mathcal{G}_n(\mathbb{F})$ be the continuous homomorphism corresponding to $\bar{\rho} : \text{Gal}(F_S/F) \rightarrow \text{GL}_n(\mathbb{F})$ via the above bijection, and let $\mu : \text{Gal}(F_S/F^+) \rightarrow \mathcal{O}^\times$ be a continuous character such that $\mu \pmod{\mathfrak{m}} = \nu \circ \bar{r}$.

For each $\tilde{w} \in \tilde{S}$, let $\bar{r}_{\tilde{w}} = \bar{r}|_{\Gamma_{F_{\tilde{w}}}}$, and let $R_{\tilde{w}}$ be a quotient of the universal framed deformation ring $R_{\tilde{w}}^\square = R_{\bar{r}_{\tilde{w}}}^\square$ of $\bar{r}_{\tilde{w}}$ such that, if $r : \Gamma_{F_{\tilde{w}}} \rightarrow \text{GL}_n$ is a lifting of $\bar{r}_{\tilde{w}}$ to an object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$ with maximal ideal \mathfrak{m}_A and $g \in 1 + M_{n \times n}(\mathfrak{m}_A)$, then the map $R_{\tilde{w}}^\square \rightarrow A$ induced by r factors through $R_{\tilde{w}}$ if and only if the map $R_{\tilde{w}}^\square \rightarrow A$ induced by grg^{-1} factors through $R_{\tilde{w}}$.

The tuple $\mathcal{S} = (F/F^+, S, \tilde{S}, \mathcal{O}, \bar{r}, \mu, \{R_{\tilde{w}}\}_{\tilde{w} \in \tilde{S}})$ is then a *global deformation datum* in the sense of [1, §1.3] (a slight variation of the data considered in [34, §2.3]). Given an object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$, we say that a continuous homomorphism $r : \Gamma_{F^+} \rightarrow \mathcal{G}_n(A)$ is a *lifting* of \bar{r} to A if it reduces to \bar{r} modulo the maximal ideal \mathfrak{m}_A of A . A *deformation* of \bar{r} to A is an equivalence class of liftings of \bar{r} to A modulo the action by conjugation of $1 + M_{n \times n}(\mathfrak{m}_A)$. We say that a lifting $r : \Gamma_{F^+} \rightarrow \mathcal{G}_n(A)$ of \bar{r} to A is of *type \mathcal{S}* if r factors through $\text{Gal}(F_S/F^+)$, if $\nu \circ r = \mu$, and for each $\tilde{w} \in \tilde{S}$, the map $R_{\tilde{w}}^\square \rightarrow A$ induced by the lifting $r|_{\Gamma_{F_{\tilde{w}}}}$ of $\bar{r}_{\tilde{w}}$ factors through $R_{\tilde{w}}$. We say that a deformation of \bar{r} to A is of *type \mathcal{S}* if one (equivalently any) lifting in its deformation class is of type \mathcal{S} . Let

$$\text{Def}_{\mathcal{S}} : \widehat{\mathcal{C}}_{\mathcal{O}} \rightarrow \mathbf{Sets}$$

be the functor which attaches to each object A of $\widehat{\mathcal{C}}_{\mathcal{O}}$ the set of deformations of \bar{r} to A of type \mathcal{S} . The following result holds.

Proposition 3.3.1 (part of [1, Proposition 1.3.9]). *Assume that $\bar{\rho}$ is absolutely irreducible. Then, the functor $\text{Def}_{\mathcal{S}}$ is representable by a complete Noetherian local commutative \mathcal{O} -algebra $R_{\mathcal{S}}$, together with a universal deformation $r_{\mathcal{S}}^{\text{univ}} : \text{Gal}(F_S/F^+) \rightarrow \mathcal{G}_n(R_{\mathcal{S}})$ of \bar{r} to $R_{\mathcal{S}}$.*

Under the assumptions of Proposition 3.3.1, there is a bijection between the set of closed points of the affine scheme $\text{Spec } R_{\mathcal{S}}[1/p]$ and the set of deformations of type \mathcal{S} of \bar{r} to finite extensions of E . Let $\mathcal{X}_{\mathcal{S}} = (\text{Spf } R_{\mathcal{S}})^{\text{rig}}$ be the rigid analytic space over E attached to the formal scheme $\text{Spf } R_{\mathcal{S}}$ via Berthelot's functor. Then, the closed points of $\mathcal{X}_{\mathcal{S}}$ are in bijection with the set of deformations of type \mathcal{S} of \bar{r} to finite extensions of E .

Let us assume as above that $\bar{\rho}$ is absolutely irreducible. Assume also that $R_{\tilde{w}} = R_{\tilde{w}}^\square$ for all $\tilde{w} \in \tilde{S}$. Then, the homomorphism $r : \text{Gal}(F_S/F^+) \rightarrow \mathcal{G}_n(E)$ induced by ρ (the

Galois representation attached to π) defines a closed point of \mathcal{X}_S , which we still denote by x . The main result of [1] implies the following.

Theorem 3.3.2 (consequence of [1, Theorem 3.2.3]). *Assume that $\zeta_p \notin F$, and that $\bar{\rho}(\Gamma_{F(\zeta_p)})$ is adequate⁷. Then \mathcal{X}_S is smooth at x of dimension $\frac{n(n+1)}{2}[F^+ : \mathbb{Q}]$.*

This smoothness result relies on the proof of the vanishing of the geometric Bloch-Kato Selmer group $H_g^1(\text{Gal}(F_S/F^+), \text{ad}(r))$, predicted by the Bloch-Kato conjecture for $\text{ad}(r)$, see [1, §3.2.4]. Note that if p is sufficiently large, then $\zeta_p \notin F$, and if $\bar{\rho}$ is irreducible then $\bar{\rho}(\Gamma_{F(\zeta_p)})$ is adequate.

Let us now move to the context of eigenvarieties. Let us assume that all the places of F^+ above p split in F . As in §3.2.2, let us fix embeddings $\iota_p : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \rightarrow \mathbb{C}$, let us choose a place \tilde{v} of F above each place v of F above p , and let L be the Galois closure of E and $\iota_p(F)$ inside $\overline{\mathbb{Q}}_p$. Let ϵ be the cyclotomic character of Γ_{F^+} , and let δ_{F/F^+} be the quadratic character of Γ_{F^+} attached to the extension F/F^+ . For each $\tilde{w} \in \tilde{S}$, let $R_{\tilde{w}}$ be the maximal reduced p -torsion-free quotient of $R_{\tilde{w}}^\square$. Let us consider the global deformation datum $\mathcal{S} = (F/F^+, S, \tilde{S}, \mathcal{O}_L, \bar{r}, \epsilon^{1-n} \delta_{F/F^+}^n, \{R_{\tilde{w}}\}_{\tilde{w} \in \tilde{S}})$, let \mathcal{X}_S be the corresponding rigid analytic deformation space over E , and let $\mathcal{X}_{S,L}$ be the base change of \mathcal{X}_S to L .

Assume that π_v is an unramified principal series for each finite place v of F^+ above p . For all such v , let $\rho_{\tilde{v}} = \rho|_{\Gamma_{F_{\tilde{v}}}}$. Note that the representation $\rho_{\tilde{v}}$ is crystalline for all \tilde{v} . Let us choose a collection $\vartheta = \{\vartheta_v\}_{v|p}$ of accessible refinements of $\pi_v \otimes |\det|_v^{(1-n)/2}$ for $v | p$. Equivalently, this corresponds to the choice of an ordering of the crystalline eigenvalues of $D_{\text{cris}}(\rho_{\tilde{v}})$, or equivalently to the choice of a refinement $\mathcal{F}_{\tilde{v}}$ of $\rho_{\tilde{v}}$, or of a triangulation of $D_{\text{rig}}^\dagger(\rho_{\tilde{v}})$. Let us write $\mathcal{F} = \{\mathcal{F}_{\tilde{v}}\}_{v|p}$.

Given a tame level $K^p = \prod_{v \nmid p} K_v$ of $U(\mathbb{A}_{F^+,f}^p)$ for π , the pair (π, ϑ) defines a (classical) closed point y of the eigenvariety \mathcal{E}_{K^p} of tame level K^p , as in §3.2.2. Note that y is actually a closed point of the $\bar{\rho}$ -component $\mathcal{E}_{K^p, \bar{\rho}}$ of \mathcal{E}_{K^p} . Using Emerton's construction of eigenvarieties [43], then $\mathcal{E}_{K^p, \bar{\rho}}$ can be identified with a rigid analytic subspace of $\mathcal{X}_{S,L} \times \mathcal{W}_L$, where \mathcal{W}_L is the base change of \mathcal{W} to L , see [22, §4.1].

Define a functor

$$\text{Def}_\rho : \mathcal{C}_L \rightarrow \mathbf{Sets}$$

by mapping each object A of \mathcal{C}_L , with maximal ideal \mathfrak{m}_A , to the set of equivalence classes modulo the action by conjugation of $1 + M_{n \times n}(\mathfrak{m}_A)$ of continuous representations $\rho_A : \text{Gal}(F_S/F) \rightarrow \text{GL}_n(A)$ which reduce to $\bar{\rho}$ modulo \mathfrak{m}_A . It is a well-known result,

⁷Refer to the appendix to [101] for the notion of *adequate subgroup* of $\text{GL}_n(\mathbb{F})$ and for the relevant results concerning these objects.

see for instance [31, §4.1], that the completed rigid analytic stalk $\widehat{\mathcal{O}}_{\mathcal{X}_L, x}$ (which is an object of $\widehat{\mathcal{C}}_L$) pro-represents Def_ρ . By [22, Lemme 4.10], we have an identification $\widehat{\mathcal{O}}_{\mathcal{X}_L, x} = \widehat{\mathcal{O}}_{\mathcal{X}_{S, L}, x}$, and so obviously $\widehat{\mathcal{O}}_{\mathcal{X}_{S, L}, x}$ pro-represents Def_ρ as well.

Let us now define a subfunctor

$$\text{Def}_{\rho, \mathcal{F}} : \mathcal{C}_L \rightarrow \mathbf{Sets}$$

of Def_ρ by mapping each object A of \mathcal{C}_L to those ρ_A such that $\rho_A|_{\Gamma_{F_{\tilde{v}}}}$ is in $\text{Def}_{\rho_{\tilde{v}}, \mathcal{F}_{\tilde{v}}}(A)$ for each finite place v of F^+ above p , and whose restriction to an inertia subgroup at each finite place \tilde{w} of F not above p is constant. By Proposition 3.2.1, we have that $\text{Def}_{\rho, \mathcal{F}}$ is pro-representable by an object of $\widehat{\mathcal{C}}_L$. The following result of Breuil, Hellmann, and Shraen proves an $R = T$ conjecture of Bellaïche and Chenevier, see [10, Conjecture 7.6.12].

Theorem 3.3.3 (consequence of [22, Théorème 4.8]). *The completed rigid analytic stalk $\widehat{\mathcal{O}}_{\mathcal{E}_{K^p, \bar{\rho}, y}}$ pro-represents $\text{Def}_{\rho, \mathcal{F}}$.*

Let now \mathcal{E} be the minimal eigenvariety containing (π, ϑ) . Let $(\sigma_{\tilde{v}, \tau})_\tau$ be the critical type of the refinement $\mathcal{F}_{\tilde{v}}$ of $\rho_{\tilde{v}}$ for each finite place v of F^+ above p . We introduce additional deformation functors. Firstly, let $\text{Def}_\rho^{\text{csd}}$ be the representable subfunctor of Def_ρ parametrising conjugate self-dual deformations. For each finite place \tilde{w} of F not above p , let $\rho_{\tilde{w}} = \rho|_{\Gamma_{F_{\tilde{w}}}}$, and let $\text{Def}_{\rho_{\tilde{w}}}^{\text{min}}$ be the subfunctor of $\text{Def}_{\rho_{\tilde{w}}}$ parametrising minimally ramified deformations. That is, the functor which maps each object A of \mathcal{C}_L to the set of deformations ρ_A of $\rho_{\tilde{w}}$ such that $\rho_A|_{I_{\tilde{w}}} \cong \rho_{\tilde{w}}|_{I_{\tilde{w}}} \otimes_L A$. It is not difficult to check that the natural transformation $\text{Def}_{\rho_{\tilde{w}}}^{\text{min}} \rightarrow \text{Def}_{\rho_{\tilde{w}}}$ is relatively representable. We let then $\text{Def}_{\rho, \mathcal{F}}^{\text{min}}$ be the fibre product defined by the diagram

$$\begin{array}{ccc} \text{Def}_{\rho, \mathcal{F}}^{\text{min}} & \longrightarrow & \text{Def}_\rho^{\text{csd}} \\ \downarrow & & \downarrow \\ \prod_{\tilde{w} \nmid p} \text{Def}_{\rho_{\tilde{w}}}^{\text{min}} \times \prod_{v|p} \text{Def}_{\rho_{\tilde{v}}, \mathcal{F}_{\tilde{v}}} & \longrightarrow & \prod_{\tilde{w} \nmid p} \text{Def}_{\rho_{\tilde{w}}} \times \prod_{v|p} \text{Def}_{\rho_{\tilde{v}}} \end{array}$$

Since the bottom arrow is relatively representable, and $\text{Def}_\rho^{\text{csd}}$ is pro-representable by an object of $\widehat{\mathcal{C}}_L$, we have that $\text{Def}_{\rho, \mathcal{F}}^{\text{min}}$ is itself pro-representable by an object of $\widehat{\mathcal{C}}_L$. The main results of [1] and [12] then imply the following.

Theorem 3.3.4 (consequence of [1, Theorem 3.2.3] and [12, Theorem 4.7]). *Assume that each $\sigma_{\tilde{v}, \tau}$ is a product of distinct simple transpositions, that $\zeta_p \notin F$, and that $\bar{\rho}(\Gamma_{F(\zeta_p)})$ is adequate. Then \mathcal{E} is smooth at y of dimension $n[F^+ : \mathbb{Q}]$, and the completed rigid analytic stalk $\widehat{\mathcal{O}}_{\mathcal{E}, y}$ pro-represents $\text{Def}_{\rho, \mathcal{F}}^{\text{min}}$.*

3.3.2 Plan of future research

We have seen how the irreducibility of ρ allows one to prove smoothness results for deformation spaces and eigenvarieties. Motivated by explicit examples constructed by Bellaïche in [7] and [9], we suggest that some converse results should hold true as well.

More precisely, Bellaïche constructs in [7] a p -refined automorphic representation (π, ϑ) of $U(\mathbb{A}_{\mathbb{Q}})$, for U a definite unitary group in 3 variables attached to an imaginary quadratic extension F/\mathbb{Q} , such that the Galois representation ρ_{π} attached to π decomposes as $\rho_{\pi} = \rho_0 \oplus 1$, for ρ_0 a 2-dimensional representation of Γ_F , and 1 the trivial representation, and shows that the minimal eigenvariety for U containing (π, ϑ) is singular at the classical point defined by (π, ϑ) , see [7, Theorem 6]. The key point of the proof consists in showing that since $\rho_{\pi} = \rho_0 \oplus 1$ is reducible, then the corresponding completed rigid analytic stalk of the eigenvariety is not factorial.

An analogous method has been used by Bellaïche in [9] to construct a class of higher dimensional examples, see [9, Theorem 1].

Let us fix notations as in §3.3.1. However, let us assume that

$$\rho \otimes_E \bar{E} \cong \bigoplus_{i=1}^k \rho_i$$

is an irreducible decomposition, where each ρ_i has rank n_i . Up to choosing p in a set of rational primes of Dirichlet density 1, we can assume that ρ is multiplicity free, i.e. the representations ρ_i are pairwise non isomorphic, and that the representations $\bar{\rho}_i$ are absolutely irreducible and pairwise non isomorphic as well. Moreover, up to choosing p sufficiently large, we can assume that $\zeta_p \notin F$, and that $\bar{\rho}_i(\Gamma_{F(\zeta_p)})$ is an adequate subgroup of $\mathrm{GL}_{n_i}(\bar{\mathbb{F}}_p)$. Let \mathcal{X} be the pseudodeformation rigid analytic space over E for $\bar{D} = \det \circ \bar{\rho}$. The representation ρ naturally defines a closed point x of \mathcal{X} . By fixing a suitable global deformation datum \mathcal{S} , we should be able to construct a ring $R_{\mathcal{S}}$ parametrising pseudodeformations of type \mathcal{S} of \bar{r} , using the theory of §3.1.3. We would then let $\mathcal{X}_{\mathcal{S}}$ be the corresponding rigid analytic space over E . The representation ρ would then define a closed point of $\mathcal{X}_{\mathcal{S}}$, which we still denote by x . We suggest that if $k > 1$, then $\mathcal{X}_{\mathcal{S}}$ should be singular at x .

If we also assume π_v to be unramified, and with distinct Satake parameters for each $v \mid p$, then we can choose a refinement ϑ_v of $\pi_v \otimes |\det|_v^{(1-n)/2}$ for each $v \mid p$, and let $\vartheta = \{\vartheta_v\}_{v \mid p}$. Let \mathcal{E} be the minimal eigenvariety over L containing the p -refined automorphic representation (π, ϑ) . Then, (π, ϑ) defines a closed point y of \mathcal{E} . For each $v \mid p$, let $(\sigma_{\bar{v}, \tau})_{\tau}$ be the critical type of the refinement of $\rho|_{\Gamma_{F_{\bar{v}}}}$ induced by ϑ_v . It is known that if there exists a $\sigma_{\bar{v}, \tau}$ which is not a product of distinct simple transpositions,

then \mathcal{E} is singular at y , see [21, Theorem 5.4.2]. Let us then assume that each $\sigma_{\bar{v},\tau}$ is a product of distinct simple transpositions. We suggest that also if $k > 1$, then \mathcal{E} should be singular at y .

The key algebraic-combinatorial result of [9] linking the reducibility of the Galois representation at a point of the eigenvariety with the “regularity” of the completed rigid analytic stalk at that point is the following.

Proposition 3.3.5 ([9, Proposition 3.1]). *Let A be a local ring with maximal ideal \mathfrak{m}_A . Let $A_{i,j}$ be fractional ideals of A , for $1 \leq i, j \leq k$, such that*

- (1) $A_{i,j}A_{j,i} \subset \mathfrak{m}_A$ for every distinct $1 \leq i, j \leq k$;
- (2) $A_{h,i}A_{i,j} \subset A_{h,j}$ for every distinct $1 \leq h, i, j \leq k$;
- (3) $\sum_{i \neq j} A_{i,j}A_{j,i} = \mathfrak{m}_A$;
- (4) $A_{i,j}$ and $A_{j,i}$ are isomorphic as A -modules for every distinct $1 \leq i, j \leq k$.

Then, if A is factorial we have $k > \dim A$.

Let us make some comments on this proposition. First of all, let us clarify how it is applied to the context of eigenvarieties.

Remark 3.3.6. In the proof of [9, Theorem 1], the ring A is the completed rigid analytic stalk $\widehat{\mathcal{O}}_{\mathcal{E},y}$. If \mathcal{E} is smooth at y , then A is regular, and hence factorial by the Auslander-Buchsbaum theorem. Let $D_y : A[\Gamma_{F,\bar{\mathcal{S}}}] \rightarrow A$ be the n -dimensional pseudorepresentation given by specialising the Galois pseudorepresentation on \mathcal{E}_{K^p} of §3.2.3 to y . If S is a Cayley-Hamilton quotient of $(A[\Gamma_{F,\bar{\mathcal{S}}}], D_y)$, then S is a generalised matrix algebra of type (n_1, \dots, n_k) by Theorem 3.1.6. Let $A_{i,j}$ be the corresponding fractional ideals of A . Then, they obviously satisfy properties (1) and (2) of Proposition 3.3.5. In order to ensure that the remaining properties hold true as well, Bellaïche considers automorphic representations π such that the geometric Bloch-Kato Selmer groups $H_g^1(\text{Gal}(F_S/F), \text{ad}(\rho_i))$ are trivial for each i , so that the total reducibility ideal J_{tot} of D_y is the maximal ideal \mathfrak{m}_A , and hence (3) holds, and such that each ρ_i is conjugate self-dual, so that (4) holds, see [9, §3.1]. Then, the conclusion of Proposition 3.3.5 holds for A , and so $k > \dim A$. However, this violates the fact that $\dim A = n[F^+ : \mathbb{Q}]$, and so \mathcal{E} cannot be smooth at y . We suggest that an analogous result should hold true even when the hypotheses (3) and (4) are relaxed, so that we can at least avoid to require each ρ_i to be conjugate self-dual. Note that fractional ideals $A_{i,j}$ satisfying (1) and (2) are well defined for any pseudodeformation

of $D = \det \circ \rho$. We can therefore replace the completed rigid analytic stalk of the minimal eigenvariety \mathcal{E} with the completed rigid analytic stalk of any suitable rigid analytic pseudodeformation space for \overline{D} , e.g. \mathcal{X}_S . In this context, note that by choosing suitable quotients $R_{\tilde{w}}$ of $R_{\tilde{w}}^\square$ for $\tilde{w} \in \tilde{S}$, it should be possible to deduce property (3) from some Bloch-Kato statement for the representations $\text{ad}(\rho_i)$.

Let us explain now how it is reasonable to expect some (pseudo)deformation rings of automorphic Galois representations to be at least factorial.

Remark 3.3.7. If \mathcal{X}_S is constructed so that the identification $\widehat{\mathcal{O}}_{\mathcal{X}_L, x} = \widehat{\mathcal{O}}_{\mathcal{X}_{S, L}, x}$ holds, then $\widehat{\mathcal{O}}_{\mathcal{X}_{S, L}, x}$ is conjecturally (at least) factorial. A classical conjecture of Jannsen, see [66, Conjecture 1], predicts in fact the vanishing of the cohomology group $H^2(\text{Gal}(F_S/F), \text{ad}(\rho))$. If this holds true, then with an argument analogous to that of [78, Proposition 2] we deduce that the framed deformation ring R_ρ^\square is a formal power series ring over E , and hence a regular ring. Now, since ρ is multiplicity free, by results of [106] we have that $\widehat{\mathcal{O}}_{\mathcal{X}, x}$ is isomorphic to the GIT quotient ring of R_ρ^\square for the action of PGL_n , and so it is factorial by [46, §V, Exercise 6], since R_ρ^\square is regular. It follows that $\widehat{\mathcal{O}}_{\mathcal{X}_L, x}$ is factorial, and so is $\widehat{\mathcal{O}}_{\mathcal{X}_{S, L}, x}$. We remark that Jannsen's conjecture for $\text{ad}(\rho)$ is closely related to the Bloch-Kato conjecture for $\text{ad}(\rho)$.

The above results and remarks suggest that smoothness properties for completed rigid analytic stalks of pseudodeformation spaces and eigenvarieties, irreducibility of automorphic Galois representations, and suitable Bloch-Kato statements are closely related. We plan to further investigate their connections in the future.

Bibliography

- [1] Patrick Allen. Deformations of polarized automorphic Galois representations and adjoint Selmer groups. *Duke Math. J.*, 165(13):2407–2460, 2016.
- [2] Patrick B. Allen, Frank Calegari, Ana Caraiani, Toby Gee, David Helm, Bao V. Le Hung, James Newton, Peter Scholze, Richard Taylor, and Jack A. Thorne. Potential automorphy over CM fields. *ArXiv preprint arXiv:1812.09999*, 2018.
- [3] Federico Amadio Guidi. Independence of algebraic monodromy groups in compatible systems. *ArXiv preprint arXiv:1905.05028*, 2019.
- [4] James Arthur and Laurent Clozel. *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, volume 120 of *Ann. of Math. Stud.* Princeton University Press, Princeton, 1989.
- [5] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Théorie des topos et cohomologie étale des schemas (SGA 4) - Tome 3*, volume 305 of *Lecture Notes in Math.* Springer-Verlag, Berlin New York, 1972. Séminaire de Géométrie Algébrique du Bois Marie - 1963-64. Directed by A. Grothendieck.
- [6] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor. Potential automorphy and change of weight. *Ann. of Math.*, 179:501–609, 2014.
- [7] Joël Bellaïche. Nonsmooth classical points on eigenvarieties. *Duke Math. J.*, 145(14):71–90, 2008.
- [8] Joël Bellaïche. Pseudodeformations. *Math. Z.*, 270(3-4):1163–1180, 2012.
- [9] Joël Bellaïche. Unitary Eigenvarieties at Isobaric Points. *Canad. J. Math.*, 67(2):315–329, 2015.
- [10] Joël Bellaïche and Gaëtan Chenevier. Families of Galois representations and Selmer groups. *Astérisque*, 324:1–314, 2009.

- [11] Joël Bellaïche and Gaëtan Chenevier. The sign of Galois representations attached to automorphic forms for unitary groups. *Compos. Math.*, 147(5):1337–1352, 2011.
- [12] John Bergdall. Smoothness of definite unitary eigenvarieties at critical points. *J. Reine Angew. Math.*, 2017.
- [13] Laurent Berger. Représentations p -adiques et équations différentielles. *Invent. Math.*, 148(2):219–284, 2002.
- [14] Pierre Berthelot. *Cohomologie Cristalline des Schémas de Caractéristique $p > 0$* , volume 407 of *Lecture Notes in Math.* Springer-Verlag, Berlin Heidelberg New York, 1974.
- [15] Gebhard Böckle, Wojciech Gajda, and Sebastian Petersen. On the semisimplicity of reductions and adelic openness for E -rational compatible systems over global function fields. *Trans. Amer. Math. Soc.*, 2019.
- [16] Armand Borel. Some finiteness properties of adèle groups over number fields. *Publ. Math. Inst. Hautes Études Sci.*, 16:5–30, 1963.
- [17] Armand Borel. Automorphic L -functions. In *Automorphic Forms, Representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, volume 33, Part 2, pages 27–61, 1979.
- [18] Armand Borel. *Linear Algebraic Groups - Second Enlarged Edition*, volume 126 of *Grad. Texts in Math.* Springer-Verlag, New York, 1991.
- [19] Armand Borel and Hervé Jacquet. Automorphic forms and automorphic representations. In *Automorphic Forms, Representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, volume 33, Part 1, pages 189–207, 1979.
- [20] Nicolas Bourbaki. *Algèbre. Chapitre 8. Modules et anneaux semi-simples.* Hermann, Paris, 1958.
- [21] Christophe Breuil, Eugen Hellmann, and Benjamin Schraen. A local model for the trianguline variety and applications. *ArXiv preprint arXiv:1702.02192*, 2017.
- [22] Christophe Breuil, Eugen Hellmann, and Benjamin Schraen. Une interprétation modulaire de la variété trianguline. *Math. Ann.*, 367(3-4):1587–1645, 2017.

- [23] Kevin Buzzard and Toby Gee. The conjectural connections between automorphic representations and Galois representations. In Fred Diamond, Payman L. Kassei, and Minhyong Kim, editors, *Automorphic Forms and Galois Representations - Volume 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, chapter 5, pages 135–187. Cambridge University Press, Cambridge, 2014.
- [24] Frank Calegari and Toby Gee. Irreducibility of automorphic Galois representations of $\mathrm{GL}(n)$, n at most 5. *Ann. Inst. Fourier*, 63(5):1881–1912, 2013.
- [25] Frank Calegari and David Geraghty. Modularity lifting beyond the Taylor-Wiles method. *Invent. Math.*, 211(1):297–433, 2018.
- [26] Ana Caraiani. Local-global compatibility and the action of monodromy on nearby cycles. *Duke Math. J.*, 161(12):2311–2413, 2012.
- [27] Ana Caraiani. Monodromy and local-global compatibility for $l = p$. *Algebra Number Theory*, 8(7):1597–1646, 2014.
- [28] Pierre Cartier. Representations of p -adic groups: a survey. In *Automorphic Forms, Representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, volume 33, Part 1, pages 111–155, 1979.
- [29] Gaëtan Chenevier. Familles p -adiques de formes automorphes pour GL_n . *J. Reine Angew. Math.*, 570:143–217, 2004.
- [30] Gaëtan Chenevier. Une application des variétés de Hecke des groupes unitaires. *preprint*, 2009.
- [31] Gaëtan Chenevier. The p -adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings. In Fred Diamond, Payman L. Kassei, and Minhyong Kim, editors, *Automorphic Forms and Galois Representations - Volume 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, chapter 7, pages 221–285. Cambridge University Press, Cambridge, 2014.
- [32] Gaëtan Chenevier and Michael Harris. Construction of automorphic Galois representations, II. *Camb. J. Math.*, 1:53–73, 2013.
- [33] Laurent Clozel. Motifs et Formes Automorphes: Applications du Principe de Fonctorialité. In Laurent Clozel and James S. Milne, editors, *Automorphic Forms, Shimura Varieties, and L-functions - Volume I*, volume 10 of *Perspect. Math.*, chapter 2, pages 77–159. Academic Press, San Diego, 1990.

- [34] Laurent Clozel, Michael Harris, and Richard Taylor. Automorphy for some l -adic lifts of automorphic mod l Galois representations. *Publ. Math. Inst. Hautes Études Sci.*, 108:1–181, 2008.
- [35] Pierre Colmez. Représentations triangulines de dimension 2. *Astérisque*, 319:213–258, 2008.
- [36] Marco D’Addezio. The monodromy groups of lisse sheaves and overconvergent F -isocrystals. *ArXiv preprint arXiv:1711.06669*, 2017.
- [37] Pierre Deligne. Les constantes des équations fonctionnelles des fonctions L . In Pierre Deligne and Willem Kuyk, editors, *Modular functions of one variable II*, volume 349 of *Lecture Notes in Math.*, pages 501–595. Springer-Verlag, Berlin Heidelberg, 1973.
- [38] Pierre Deligne. La Conjecture de Weil, I. *Publ. Math. Inst. Hautes Études Sci.*, 43:273–307, 1974.
- [39] Pierre Deligne. La Conjecture de Weil, II. *Publ. Math. Inst. Hautes Études Sci.*, 52:137–252, 1980.
- [40] Pierre Deligne. Catégories Tannakiennes. In Pierre Cartier, Luc Illusie, Nicholas M. Katz, Gérard Laumon, Yuri I. Manin, and Kenneth A. Ribet, editors, *The Grothendieck Festschrift, Volume II*, chapter 3, pages 111–195. Birkhäuser, Basel, 2007.
- [41] Mark Dickinson. On the modularity of certain 2-adic Galois representations. *Duke Math. J.*, 109(2):319–382, 2001.
- [42] Vladimir Gershonovich Drinfeld. On a conjecture of Deligne. *Mosc. Math. J.*, 12(3):512–542, 2012.
- [43] Matthew Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.*, 164(1):1–84, 2006.
- [44] Gerd Faltings. Crystalline cohomology and p -adic Galois representations. In Jun-Ichi Igusa, editor, *Algebraic Analysis, Geometry, and Number Theory: Proceedings of the JAMI Inaugural Conference*, pages 25–81. Johns Hopkins University Press, 1989.

- [45] Dan Flath. Decomposition of representations into tensor products. In *Automorphic Forms, Representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, volume 33, Part 1, pages 179–183, 1979.
- [46] John Fogarty. *Invariant Theory*. W. A. Benjamin, INC., New York Amsterdam, 1969.
- [47] Jean-Marc Fontaine. Sur certains types de représentations p -adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate. *Ann. of Math.*, 115(2):529–577, 1982.
- [48] Jean-Marc Fontaine. Les corps des périodes p -adiques. *Astérisque*, 223:59–111, 1994.
- [49] Jean-Marc Fontaine. Représentations ℓ -adiques potentiellement semi-stables. *Astérisque*, 223:321–347, 1994.
- [50] Jean-Marc Fontaine. Représentations p -adiques semi-stables. *Astérisque*, 223:113–184, 1994.
- [51] Jean-Marc Fontaine and Barry Mazur. Geometric Galois representations. *Elliptic curves, modular forms, & Fermat's last theorem (Hong Kong, 1993)*, Ser. Number Theory, I, pages 41–78, 1995.
- [52] Israel M. Gelfand and Ilya I. Piatetski-Shapiro. Automorphic functions and representation theory. *Trudy Moscov. Mat. Obšč.*, 12:389–412, 1963.
- [53] Alain Genestier and Vincent Lafforgue. Chtoucas restreints pour les groupes réductifs et paramétrisation de Langlands locale. *ArXiv preprint arXiv:1709.00978*, 2019.
- [54] Henri Gillet and William Messing. Cycle classes and Riemann-Roch for crystalline cohomology. *Duke Math. J.*, 55(3):501–538, 1987.
- [55] Roger Godement and Hervé Jacquet. *Zeta Functions of Simple Algebras*, volume 260 of *Lecture Notes in Math*. Springer-Verlag, Berlin Heidelberg New York, 1972.
- [56] Benedict H. Gross. On the Satake isomorphism. In Anthony J. Scholl and Richard L. Taylor, editors, *Galois Representations in Arithmetic Algebraic Geometry*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 223–238. Cambridge University Press, Cambridge, 1998.

- [57] David Hansen. Universal eigenvarieties, trianguline Galois representations, and p -adic Langlands functoriality. *J. Reine Angew. Math.*, 730:1–64, 2017.
- [58] Harish-Chandra. *Automorphic Forms on Semisimple Lie Groups*, volume 62 of *Notes by J. G. M. Mars. Lecture Notes in Math.* Springer-Verlag, Berlin Heidelberg New York, 1968.
- [59] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne. On the rigid cohomology of certain Shimura varieties. *Res. Math. Sci.*, 3(37), 2016.
- [60] Michael Harris and Richard Taylor. *The Geometry and Cohomology of Some Simple Shimura Varieties*, volume 151 of *Ann. of Math. Stud.* Princeton University Press, Princeton and Oxford, 2001.
- [61] Guy Henniart. Caractérisation de la correspondance de Langlands locale par les facteurs ε de paires. *Invent. Math.*, 113(1):339–350, 1993.
- [62] Guy Henniart. Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p -adique. *Invent. Math.*, 139(2):439–455, 2000.
- [63] Hervé Jacquet, Ilya I. Piatetski-Shapiro, and Joseph A. Shalika. Facteurs L et ε du groupe linéaire. *C. R. Acad. Sci. Paris*, 209:59–61, 1979.
- [64] Hervé Jacquet, Ilya I. Piatetski-Shapiro, and Joseph A. Shalika. Rankin-Selberg convolutions. *Amer. J. Math.*, 105(2):367–464, 1983.
- [65] Hervé Jacquet and Joseph A. Shalika. On Euler products and the classification of automorphic forms II. *Amer. J. Math.*, 103(4):777–815, 1981.
- [66] Uwe Jannsen. On the ℓ -adic cohomology of varieties over number fields and its Galois cohomology. In Kenneth Ribet, Yasutaka Ihara, and Jean-Pierre Serre, editors, *Galois groups over \mathbb{Q}* , volume 16 of *Math. Sci. Res. Inst. Publ.*, chapter 5, pages 315–360. Springer, New York, 1989.
- [67] Nicholas M. Katz. On the monodromy groups attached to certain families of exponential sums. *Duke Math. J.*, 54(1):41–56, 1987.
- [68] Kiran S. Kedlaya. A p -adic local monodromy theorem. *Ann. of Math.*, 160:93–184, 2004.
- [69] Robert E. Kottwitz. On the λ -adic representations associated to some simple Shimura varieties. *Invent. Math.*, 108:653–665, 1992.

- [70] Jean-Pierre Labesse. Changement de base CM et séries discrètes. In Laurent Clozel, Michael Harris, Jean-Pierre Labesse, and Bao-Châu Ngô, editors, *Stabilization of the trace formula, Shimura varieties, and arithmetic applications - Volume 1: On the Stabilization of the Trace Formula*, pages 429–470. International Press, Somerville, 2011.
- [71] Laurent Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. *Invent. Math.*, 147(1):1–241, 2002.
- [72] Robert P. Langlands. Problems in the theory of automorphic forms. In *Lectures in Modern Analysis and Applications III*, volume 170 of *Lecture Notes in Math.*, pages 18–61. Springer, New York, 1970.
- [73] Robert P. Langlands. On the irreducible representations of real algebraic groups. In Paul J. Sally Jr. and David A. Vogan Jr., editors, *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, volume 31 of *Math. Surveys Monogr.*, pages 101–170. American Mathematical Society, Providence, 1989.
- [74] Michael Larsen and Richard Pink. Determining representations from invariant dimensions. *Invent. Math.*, 102(1):377–398, 1990.
- [75] Michael Larsen and Richard Pink. On ℓ -independence of Algebraic Monodromy Groups in Compatible Systems of Representations. *Invent. Math.*, 107(3):603–636, 1992.
- [76] Gérard Laumon, Michael Rapoport, and Ulrich Stuhler. \mathcal{D} -elliptic sheaves and the Langlands correspondence. *Invent. Math.*, 113(1):217–338, 1993.
- [77] David Loeffler. Overconvergent algebraic automorphic forms. *Proc. Lond. Math. Soc.*, 102(2):193–228, 2011.
- [78] Barry Mazur. Deforming Galois representations. In Kenneth Ribet, Yasutaka Ihara, and Jean-Pierre Serre, editors, *Galois groups over \mathbb{Q}* , volume 16 of *Math. Sci. Res. Inst. Publ.*, chapter 6, pages 385–437. Springer, New York, 1989.
- [79] Colette Mœglin and Jean-Loup Waldspurger. Le spectre résiduel de $GL(n)$. *Ann. Sci. Éc. Norm. Supér.*, 22(4):605–674, 1989.
- [80] Chung Pang Mok. *Endoscopic classification of representations of quasi-split unitary groups*, volume 235. American Mathematical Society, Providence, 2015.

- [81] David Mumford. Families of Abelian Varieties. *Proc. Sympos. Pure Math.*, 9:347–351, 1966.
- [82] Stefan Patrikis. Variations on a theorem of Tate. *ArXiv preprint arXiv:1207.6724v4*, 2012.
- [83] Stefan Patrikis, Andrew Snowden, and Andrew Wiles. Residual irreducibility of compatible systems. *Int. Math. Res. Not. IMRN*, 2018(2):571–587, 2016.
- [84] Stefan Patrikis and Richard Taylor. Automorphy and irreducibility of some l -adic representations. *Compos. Math.*, 151(2):207–229, 2015.
- [85] Ravi Ramakrishna. On a variation of Mazur’s deformation functor. *Compos. Math.*, 87(3):269–286, 1993.
- [86] Raphaël Rouquier. Caractérisation des caracteres et pseudo-caracteres. *J. Algebra*, 180(2):571–586, 1996.
- [87] Peter Scholze. The Local Langlands Correspondence for GL_n over p -adic fields. *Invent. Math.*, 192:663–715, 2013.
- [88] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math.*, 182(3):945–1066, 2015.
- [89] Jean-Pierre Serre. *Abelian l -Adic Representations and Elliptic Curves*. W. A. Benjamin, New York, 1968.
- [90] Jean-Pierre Serre. Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures. *Séminaire Delange-Pisot-Poitou. Théorie des nombres*, 11(2):1–15, 1969.
- [91] Jean-Pierre Serre. Modular forms of weight one and Galois representations. In *Algebraic number fields: L -functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975)*, pages 193–268, London, 1977. Academic Press.
- [92] Freydoon Shahidi. On Certain L -Functions. *Amer. J. Math.*, 103(2):297–355, 1981.
- [93] Sug Woo Shin. Galois representations arising from some compact Shimura varieties. *Ann. of Math. (2)*, 173(3):1645–1741, 2011.

- [94] Sug Woo Shin and Nicolas Templier. On fields of rationality for automorphic representations. *Compos. Math.*, 150(12):2003–2053, 2014.
- [95] Tonny A. Springer. Reductive groups. In *Automorphic Forms, Representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, volume 33, Part 1, pages 3–27, 1979.
- [96] John Tate. Number theoretic background. In *Automorphic Forms, Representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, volume 33, Part 2, pages 3–26, 1979.
- [97] Richard Taylor. Galois representations associated to Siegel modular forms of low weight. *Duke Math. J.*, 63(2):281–332, 1991.
- [98] Richard Taylor. Automorphy for some l -adic lifts of automorphic mod l Galois representations. II. *Publ. Math. Inst. Hautes Études Sci.*, 108:183–239, 2008.
- [99] Richard Taylor and Andrew Wiles. Ring-Theoretic Properties of Certain Hecke Algebras. *Ann. of Math.*, 141(3):553–572, 1995.
- [100] Richard Taylor and Teruyoshi Yoshida. Compatibility of local and global Langlands correspondences. *J. Amer. Math. Soc.*, 20:467–493, 2007.
- [101] Jack Thorne. On the automorphy of l -adic Galois representations with small residual image, with an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Jack Thorne. *J. Inst. Math. Jussieu.*, 11(4):855–920, 2012.
- [102] Jacques Tits. Reductive groups over local fields. In *Automorphic Forms, Representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, volume 33, Part 1, pages 29–69, 1979.
- [103] Eric Urban. Eigenvarieties for reductive groups. *Ann. of Math.*, 174(3):1685–1784, 2011.
- [104] Ila Varma. Crystallinity of Galois representations associated to regular algebraic cuspidal automorphic representations of GL_n . *In preparation*.
- [105] Preston Wake and Carl Wang-Erickson. Deformation conditions for pseudorepresentations. *ArXiv preprint arXiv:1707.01896*, 2017.
- [106] Carl Wang-Erickson. Algebraic families of Galois representations and potentially semi-stable pseudodeformation rings. *Math. Ann.*, pages 1–67, 2017.

- [107] Andrew Wiles. On ordinary λ -adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.
- [108] Andrew Wiles. Modular Elliptic Curves and Fermat’s Last Theorem. *Ann. of Math.*, 141(3):443–551, 1995.
- [109] Yuhou Xia. Irreducibility of Automorphic Galois Representations of Low Dimensions. *Math. Ann.*, 374(3-4):1953–1986, 2019.