Application of Linear Momentum Actuator Disc Theory to Open Channel Flow

by

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Report No. OUEL 2296/08

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January 2008

ABSTRACT

This report presents a step by step extension of the simple one-dimensional linear momentum actuator disc theory (LMADT), which results in the well known Betz-Lanchester limit for wind turbines, to a general cross sectional array of tidal turbines in an open channel tidal flow. Unlike previous models no restriction is placed on the geometry of the turbine array or the Froude number of the flow.

One of the key findings from applying LMADT to open channel flow is that the efficiency of an arbitrary array of turbines can be determined relative to the total power extracted from the channel flow, including the effects of downstream mixing. A general form of this dimensionless efficiency may be more important for open channel flow, given the possibility of downstream constraints, than the typical dimensionless power co-efficient used for wind turbines.

1. INTRODUCTION

The growing worldwide demand for renewable energy, coupled with the apparent pool of energy within the world's tidal currents, has led to considerable interest in tidal power development over the last 25 years. Current developments in the U.K. leading towards tidal power generation include initiatives such as the Marine Energy Challenge (Carbon Trust, 2007) and the commissioning of an Atlas of Marine Renewable Energy Resources (DTI, 2007). However, despite this activity, very little has been established about the actual limit to power extraction from a flow in an open channel. Without this limit, it becomes hard to benchmark the efficiency of a given tidal power device or scheme, and subsequently to optimize a design for full scale generation.

A method to determine the limit of power extraction in a fluid is the simple LMADT, first introduced by Betz in the 1920's (Burton *et al.*, 2001). The application of the model in an infinite volume of air is used in the analysis and design of wind turbines. However, it is well known that the flow of air in the atmosphere is different to that of a liquid constrained in an open channel (Bryden *et al.* 2007). For example, the atmospheric flow of air is substantially less constrained, due to the air's negligible density, and therefore a given stream tube will expand relatively freely when the flow slows. By contrast an open channel flow is constrained by the fluid's density, and the resulting free surface that forms. For these reasons it is acknowledged (Bryden *et al.* 2007) that the adoption of a Betz-Lanchester calculation, in its standard form, is irrelevant for tidal streams.

The purpose of this report is to demonstrate how the simple LMADT can be extended to a flow that characterises an open channel more precisely. To illustrate how the extension can be achieved a series of four flows is analysed. The first of these flows is the simple LMADT in an infinite medium, which we can call the Betz analysis for convenience. The second and third flows then introduce the concept of a finite medium, through the addition of a constant pressure boundary and a constant volume boundary respectively. Finally, the fourth flow will introduce a finite medium for a dense fluid - a condition that represents an open channel.

The application of actuator disc models to open channel flow has been attempted before by Whelan *et al.* (2007) and by Garrett & Cummins (2004, 2007). These attempts have included restrictions on the flow conditions that do not permit general analysis of a cross sectional array of turbines in an open channel. For instance, Whelan *et al*. (2007) restrict their model to an infinite row of turbines in an open channel. Alternatively Garrett & Cummins (2007) require a sufficiently low Froude number for their model to be accurate. (This requirement stems from Garrett & Cummins' assumption of no change in height along the channel – a somewhat contradictory assumption given that at the same time they allow the pressure within the fluid to vary.) The model presented in this report has no restrictions on the geometry of the turbine array or the Froude number of the flow.

The analysis of the finite flows allows for a thorough examination of the downstream mixing that results in the far wake of the actuator disc. The mixing process involves a loss of energy that will, in most cases, be an unavoidable by-product of the extraction of power at the actuator disc. If the total power that is removed from a channel is restricted, (for example by environmental constraints (Bryden *et al*. 2007)), the ability to extract as much power as possible, while minimising mixing losses will be desirable. This gives rise to the need to understand the efficiency of a tidal turbine device in an open channel flow. A measure of efficiency that can be used to characterise a turbine is presented in this report.

2. LAYOUT OF THIS REPORT

The four sections §6-§9 present the four flow conditions used to illustrate the extension of LMADT to an open channel flow. For each flow a standard framework is adopted, which consists of

1. A diagram illustrating the flow conditions
2. A table that develops the continuity relations between the different regions of the flow
3. Commentary and manipulation of the relevant integral equations
4. A proposed calculation sequence that can be used to 'solve' the integral equations (for the open channel flow an additional section on the solution space of the model is included)

A summary table is presented at the end to allow comparison of the models.

3. ACKNOWLEDGEMENTS

The second author gratefully acknowledges the support of the Rhodes Trust.

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5. NOMENCLATURE

Symbol	**Definition**
u	Stream velocity (uniform)
ρ	Fluid density
g	Gravity
p	Pressure (gauge)
h	Stream height/hydrostatic head
α_2	Turbine flow velocity coefficient
α_4	Turbine wake flow velocity coefficient
β_4	Bypass flow velocity coefficient
A	Area of the turbine defined as an actuator disc
R	Area ratio
b	Width of flow (open channel)
B	§8: Blockage ratio ($1/R$) §9: Blockage ratio (A/bh)
T	Thrust from the actuator disc to the fluid
X	Reaction between the turbine flow and bypass flow
P	Power extracted by the turbine
P_W	Power dissipate in downstream mixing
C_P	Dimensionless power coefficient, normalised by upstream kinetic flux
C_{P*}	Dimensionless power coefficient, normalised by upstream kinetic flux and the pressure drop across the actuator disc
C_{PW}	Dimensionless power dissipation in downstream mixing, normalised by upstream kinetic flux
C_T	Dimensionless thrust coefficient, normalised by upstream kinetic pressure
C_{TL}	Dimensionless thrust coefficient, normalised by turbine kinetic pressure
η	Efficiency of a turbine in a finite flow
F_r	Froude number = $u/\sqrt{gh}$

Subscripts	**Definition**
t	Turbine flow
b	Bypass flow
1,2,3...	Station of the flow

6. THE STANDARD 'BETZ' LINEAR MOMENTUM ACTUATOR DISC THEORY

6.1 Geometry of the flow

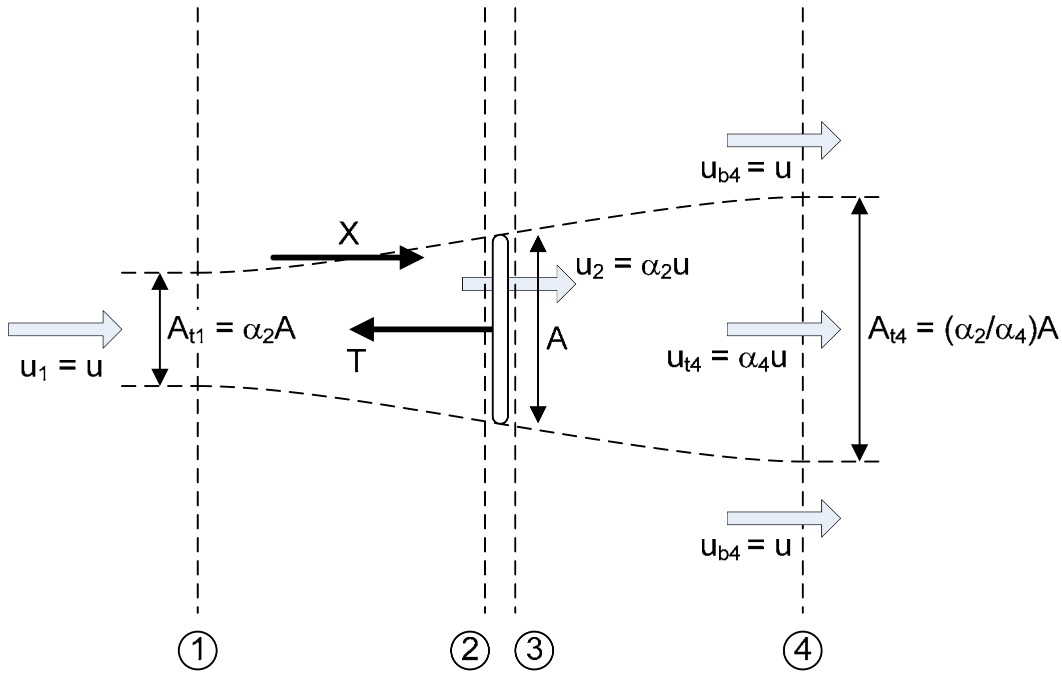

Figure 1: One dimensional linear momentum actuator disc theory in an infinite medium.

6.2 Continuity relations

Region		Station 1	Station 2	Station 3	Station 4
Turbine	Area	$A_{1t} = A\alpha_2$	$A_{2t} = A_{3t} = A$		$A_{4t} = A\dfrac{\alpha_2}{\alpha_4}$
	Velocity	$u_{1t} = u$	$u_{2t} = u_{3t} = u\alpha_2$		$u_{4t} = u\alpha_4$
	Volumetric flow	$q_{1t} = q_t = uA\alpha_2$	$q_{2t} = q_{3t} = uA\alpha_2$		$q_{4t} = uA\alpha_2$
	Pressure	$p_{1t} = p$	p_{2t}	p_{3t}	$p_{4t} = p$
By-pass	Velocity	$u_{1b} = u$			$u_{4b} = u$
	Pressure	$p_{1b} = p$			$p_{4b} = p$

Table 1

6.3 Commentary and derivation

The basic Betz calculation, as applied to the power generation problem, relates to flow through a turbine in a medium of infinite extent. The analysis addresses just the flow through the turbine itself. We use the terminology defined in Figure 1 in which four stations are identified (1) far upstream of the turbine, (2) immediately upstream of the turbine, (3) immediately downstream of the turbine and (4) sufficiently far downstream from the turbine that the pressure can again be treated as uniform (although the velocity is not). Variables relating to each station will be identified by appropriate subscripts, and in addition the subscript "*t*" is used for that part of the flow passing through the turbine and "*b*" for the remainder of the flow (the by-pass flow, in this case infinite in extent).

In this simple case the analysis can be confined to the flow passing through the turbine. It is assumed that at station 1 the pressure and velocity are uniform so that $p_{1t} = p_{1b} = p$ and $u_{1t} = u_{1b} = u$. At the turbine it is assumed that the velocity has been reduced to $u_{2t} = u_{3t} = u\alpha_2$, and at station 4 that it is further reduced to $u_{4t} = u\alpha_4$. At station (4) it is assumed that the pressure is once more uniform across the flow so that $p_{4t} = p_{4b} = p$ (in the by-pass region it is assumed that the pressure is p throughout). The volumetric flux through the turbine is $q_t = uA\alpha_2$.

Applying Bernoulli from station 1 to station 4 in the by-pass flow simply gives the result that, since the pressures are the same at these two stations, so are the velocities, so $u_{4b} = u$. Applying Bernoulli from stations 1 to 2 and from 3 to 4 in the turbine flow gives:

$$p + \frac{1}{2}\rho u^2 = p_{2t} + \frac{1}{2}\rho u^2 \alpha_2^2 \qquad \ldots(1)$$

$$p_{3t} + \frac{1}{2}\rho u^2 \alpha_2^2 = p + \frac{1}{2}\rho u^2 \alpha_4^2 \qquad \ldots(2)$$

and equilibrium across the turbine gives

$$p_{2t} - p_{3t} = \frac{T}{A} \qquad \ldots(3)$$

Combining the above three equations gives:

$$\frac{1}{2}\rho u^2 \left(1 - \alpha_4^2\right) = \frac{T}{A} \qquad \ldots(4)$$

We now consider the momentum equation. If the net axial force on the surface of the stream tube between stations 1 and 4 is X above the force due to ambient pressure, the momentum equation is simply:

$$X - T = \rho q_t \left(u_{4t} - u_{1t}\right) = \rho u^2 A \alpha_2 \left(\alpha_4 - 1\right) \qquad \ldots(5)$$

If we assume that there is no net change of momentum in the by-pass flow, then we can deduce that $X = 0$. (Strictly we cannot make this deduction as an infinitesimal momentum change of an infinite volume could occur. However, the finite flow case considered below confirms that, in the limit of the infinite flow, the $X = 0$ assumption is justifiable.) We can therefore obtain:

$$\frac{T}{A} = \rho u^2 \alpha_2 \left(1 - \alpha_4\right) \qquad \ldots(6)$$

Equating (4) and (6) we immediately obtain $\alpha_2 = \frac{1 + \alpha_4}{2}$ and we can express the flow in terms of a single parameter family as a function of α_4. In particular we can write an expression for the power absorbed by the turbine (and possibly output as useful power) as:

$$P = T\alpha_2 u = \frac{1}{2}\rho u^3 A \alpha_2 \left(1 - \alpha_4^2\right) = \frac{1}{2}\rho u^3 A \frac{\left(1 + \alpha_4\right)}{2}\left(1 - \alpha_4^2\right) = \frac{1}{2}\rho u^3 A C_P \qquad \ldots(7)$$

The maximum power is extracted when the power coefficient C_P is maximised as a function of α_4. Simple differentiation reveals that this occurs when $\alpha_4 = \frac{1}{3}$, $\alpha_2 = \frac{2}{3}$ and $C_P = \frac{16}{27}$. Furthermore we can write:

$$T = \frac{1}{2}\rho u^2 A\left(1-\alpha_4^2\right) = \frac{1}{2}\rho u^2 A C_T \qquad \ldots(8)$$

And note that at the optimum conditions $C_T = \frac{8}{9}$. For some purposes it might be more useful to define the thrust in terms of a local velocity, therefore:

$$T = \frac{1}{2}\rho u_2^2 A C_{TL} = \frac{1}{2}\rho u^2 A \alpha_2^2 C_{TL} \qquad \ldots(9)$$

and we note that at optimal conditions $C_{TL} = 2$.

It is worth noting some of the approximations and anomalies in the Betz analysis. Firstly it assumes that only axial components of velocity are significant – so that radial velocities and tangential (swirl) velocities are ignored.

Secondly there is clearly an anomaly in the calculation of pressure in the turbine flow region and the by-pass region. Upstream of the turbine the Bernoulli calculation implies that the pressure in the turbine region is higher than in the by-pass, and downstream of the turbine the pressure in the turbine region is lower than in the by-pass. The ambiguity of the pressure along this boundary means that the unknown force X cannot be derived from the pressure calculations.

Finally note that far downstream from the turbine (even further than station 4) the wake will eventually mix with the by-pass flow. The infinite boundary condition means that the pressure and velocity far downstream will be the same as far upstream. At first sight this implies that no energy has been extracted, but this is of course erroneous – in this case it is clear that the integral of an infinitesimal change over an infinite area will lead to a finite energy loss. In fact there is an additional energy loss in the wake mixing process. This issue can only be resolved by considering a finite flow, as is addressed in section 7 below.

6.4 Calculation sequence

The calculation sequence below includes calculations of the wake energy loss and overall efficiency, as addressed in section 7.

1. Specify principal dimensioning parameters ρ, u and A
2. (Optionally specify upstream pressure p, which acts as purely additive term to all pressures)
3. Specify dimensionless velocity factor $0 \le \alpha_4 \le 1$
4. Calculate dimensionless quantities:
    a. $\alpha_2 = \frac{1+\alpha_4}{2}$
    b. $C_T = \left(1-\alpha_4^2\right)$
    c. $C_{TL} = \frac{C_T}{\alpha_2^2}$
    d. $C_P = \alpha_2 C_T$
    e. $C_{PW} = \alpha_2 \left(1-\alpha_4\right)^2$
    f. $\eta = \frac{C_P}{C_P + C_{PW}} = \frac{P}{P + P_W}$

5. Calculate dimensioned quantities:
   g. $T = \frac{1}{2}\rho u^2 A C_T$
   h. $P = \frac{1}{2}\rho u^3 A C_P$
   i. $P_W = \frac{1}{2}\rho u^3 A C_{PW}$
   j. Pressure drop across turbine $\Delta p_T = \frac{T}{A}$

7. LINEAR MOMENTUM ACTUATOR DISC THEORY IN A FINITE FLOW (PRESSURE CONSTRAINED)

7.1 Geometry of the flow

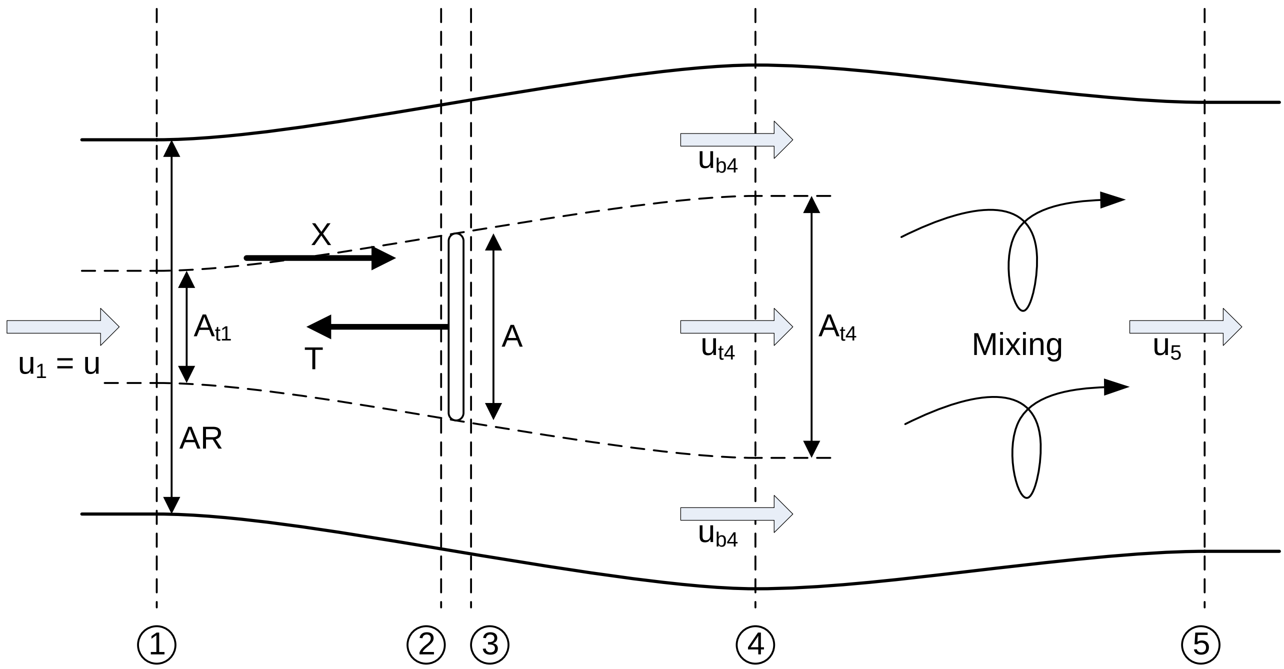

Figure 2: One dimensional linear momentum actuator disc theory in a finite medium bounded by a constant pressure boundary

7.2 Continuity Relations

Region		Station 1	Station 2	Station 3	Station 4	Station 5
Turbine	Area	$A_{1t} = A\alpha_2$	$A_{2t} = A_{3t} = A$		$A_{4t} = A\dfrac{\alpha_2}{\alpha_4}$	
	Velocity	$u_{1t} = u$	$u_{2t} = u_{3t} = u\alpha_2$		$u_{4t} = u\alpha_4$	
	Volumetric Flow	$q_{1t} = q_t = uA\alpha_2$	$q_{2t} = q_{3t} = uA\alpha_2$		$q_{4t} = uA\alpha_2$	
	Pressure	$p_{1t} = p$	p_{2t}	p_{3t}	$p_{4t} = p$	
By-pass	Area	$A_{1b} = A(R-\alpha_2)$			$A_{4b} = A(R-\alpha_2)$	
	Velocity	$u_{1b} = u$			$u_{4b} = u$	
	Volumetric Flow	$q_{1b} = q_b = uA(R-\alpha_2)$			$q_{4b} = uA(R-\alpha_2)$	
	Pressure	$p_{1b} = p$			$p_{4b} = p$	
Total	Area	$A_1 = AR$			$A_4 = A\left(R-\alpha_2+\dfrac{\alpha_2}{\alpha_4}\right)$	A_5
	Velocity	$u_1 = u$	Varies		Varies	u_5
	Volumetric Flow	$q_1 = q = uAR$			$q_4 = uAR$	$q_5 = uAR$
	Pressure	$p_1 = p$	Varies	Varies	$p_4 = p$	$p_5 = p$

Table 2

7.3 Commentary and derivation

We now consider an equivalent calculation to §6 for a flow of finite dimensions. We assume that the flow occurs within a region that has a constant pressure boundary condition around the outside (quite how this could be realised in practical terms is uncertain, but theoretically it is of value). The main variables are set out in Table 2, and Figure 2 shows the main features of the flow. We now add consideration of Station 5, which is sufficiently far downstream that mixing has occurred and the flow is of uniform velocity. The dimension of the flow far upstream is taken as $A_1 = AR$, where R is a dimensionless ratio. For some applications the blockage factor $B = 1/R$ may be more convenient.

The analysis proceeds much as before. Again the constant pressure condition leads to $u_{1t} = u_{1b} = u$. Equations (1), (2), (3) and (4) are unchanged. Consideration of the momentum change (in fact zero) of the by-pass flow between stations 1 and 4 leads to the conclusion $X = 0$, so that equation (6) still applies, and again we deduce $\alpha_2 = \frac{1+\alpha_4}{2}$. Once more the flow is a function of a single parameter α_4. The conditions for optimal power extraction are unchanged. All of these are useful results which indicate that the standard result is an appropriate limiting condition as $R \to \infty$.

The additional analysis that is now possible is consideration of the mixing zone. The momentum equation between stations 4 and 5 gives (given the fact that there is no net force on this zone):

$$u^2 A\alpha_2\alpha_4 + u^2 A(R-\alpha_2) = uARu_5 \qquad \text{…(10)}$$

From which we deduce that $u_5 = u\frac{(R-\alpha_2+\alpha_2\alpha_4)}{R} = u\left(1-\frac{(1-\alpha_4^2)}{2R}\right)$. The additional energy loss in the wake mixing process is:

$$\begin{aligned} P_W &= \frac{1}{2}\rho u^3 A\alpha_2\alpha_4^2 + \frac{1}{2}\rho u^3 A(R-\alpha_2) - \frac{1}{2}\rho uARu_5^2 \\ &= \frac{1}{2}\rho u^3 A\left(\alpha_2\alpha_4^2 + R - \alpha_2 - \frac{(R-\alpha_2+\alpha_2\alpha_4)^2}{R}\right) \\ &= \frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4)^2\left(1-\frac{\alpha_2}{R}\right) = \frac{1}{2}\rho u^3 A\frac{(1+\alpha_4)}{2}(1-\alpha_4)^2\left(1-\frac{(1+\alpha_4)}{2R}\right) \end{aligned} \qquad \text{…(11)}$$

Note we can also write $P_W = P\frac{(1-\alpha_4)}{(1+\alpha_4)}\left(1-\frac{\alpha_2}{R}\right)$.

As $R \Rightarrow \infty$, $P_W = P\frac{(1-\alpha_4)}{(1+\alpha_4)}$, so that at optimal conditions ($\alpha_4 = 1/3$), $P_W = P/2$. The important conclusion is that there is an inevitable further power loss in the flow due to wake mixing, over and above any useful power extracted at the turbine.

As a result we can define an efficiency factor $\eta = \frac{P}{P+P_W}$, which is the proportion of the total energy extracted from the flow that can usefully be extracted, the remainder being lost in the wake. It is straightforward to show that:

$$\eta = \frac{P}{P+P_W} = \frac{(1+\alpha_4)}{(1+\alpha_4)+(1-\alpha_4)(1-B\alpha_2)} \qquad \ldots(12)$$

7.4 Calculation sequence

1. Specify principal dimensioning parameters ρ, u and A
2. (Optionally specify upstream pressure p, which acts as purely additive term to all pressures)
3. Specify area ratio $1 \le R \le \infty$ and dimensionless velocity factor $0 \le \alpha_4 \le 1$
4. Calculate dimensionless quantities:
   a. $\alpha_2 = \dfrac{1+\alpha_4}{2}$
   b. $C_T = \left(1-\alpha_4^2\right)$
   c. $C_{TL} = \dfrac{C_T}{\alpha_2^2}$
   d. $C_P = \alpha_2 C_T$
   e. $C_{PW} = \alpha_2 (1-\alpha_4)^2 \left(1-\dfrac{\alpha_2}{R}\right)$
   f. $\eta = \dfrac{C_P}{C_P + C_{PW}} = \dfrac{P}{P+P_W}$
5. Calculate dimensioned quantities:
   g. $T = \dfrac{1}{2}\rho u^2 A C_T$
   h. $P = \dfrac{1}{2}\rho u^3 A C_P$
   i. $P_W = \dfrac{1}{2}\rho u^3 A C_{PW}$
   j. Pressure drop across turbine $\Delta p_T = \dfrac{T}{A}$
   k. Downstream velocity $u_5 = u\left(1-\dfrac{\alpha_2(1-\alpha_4)}{R}\right)$

8. LINEAR MOMENTUM ACTUATOR DISC THEORY IN A PARALLEL-SIDED TUBE

8.1 Geometry of the flow

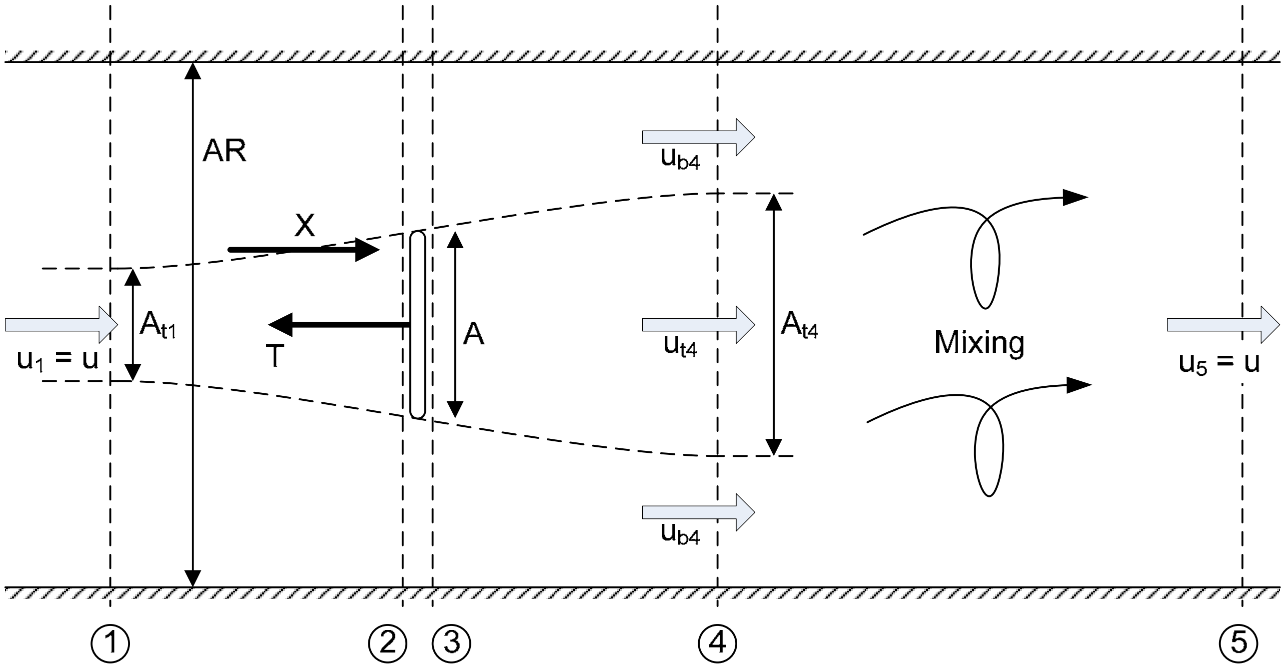

Figure 3: One dimensional linear momentum actuator disc theory in a finite medium bounded by a parallel-sided tube

8.2 Continuity Relations

Region		Station 1	Station 2	Station 3	Station 4	Station 5
Turbine	Area	$A_{1t} = A\alpha_2$	$A_{2t} = A_{3t} = A$		$A_{4t} = A\dfrac{\alpha_2}{\alpha_4}$	
	Velocity	$u_{1t} = u$	$u_{2t} = u_{3t} = u\alpha_2$		$u_{4t} = u\alpha_4$	
	Volumetric flow	$q_{1t} = q_t = uA\alpha_2$	$q_{2t} = q_{3t} = uA\alpha_2$		$q_{4t} = uA\alpha_2$	
	Pressure	$p_{1t} = p$	p_{2t}	p_{3t}	$p_{4t} = p_4$	
By-pass	Area	$A_{1b} = A(R-\alpha_2)$			$A_{4b} = A\left(R - \dfrac{\alpha_2}{\alpha_4}\right)$	
	Velocity	$u_{1b} = u$			$u_{4b} = u\dfrac{(R-\alpha_2)}{(R-\alpha_2/\alpha_4)}$	
	Volumetric flow	$q_{1b} = q_b = uA(R-\alpha_2)$			$q_{4b} = uA(R-\alpha_2)$	
	Pressure	$p_{1b} = p$			$p_{4b} = p_4$	
Total	Area	$A_1 = AR$			$A_4 = AR$	$A_5 = AR$
	Velocity	$u_1 = u$	Varies		Varies	$u_5 = u$
	Volumetric flow	$q_1 = q = uAR$			$q_4 = uAR$	$q_5 = uAR$
	Pressure	$p_1 = p$	Varies	Varies	p_4	$p_5 = p - \dfrac{T}{AR} = p - \Delta p$

Table 3

8.3 Commentary and derivation

We now consider another finite flow, but this time a flow in a confined tube. The main parameters are defined in Table 3 and shown in Figure 3.
The key difference is that the by-pass flow is no longer at constant velocity and one can deduce $u_{4b} = u\dfrac{(R-\alpha_2)}{(R-\alpha_2/\alpha_4)} = u\beta_4$. It follows that Bernoulli in the by-pass flow gives:

$$p_4 - p = \frac{1}{2}\rho u^2\left(1-\beta_4^2\right) = \frac{1}{2}\rho u^2\left(1 - \frac{(R-\alpha_2)^2}{(R-\alpha_2/\alpha_4)^2}\right) \qquad \ldots(13)$$

Equations (1) and (3) are unchanged, but (2) becomes

$$p_{3t} + \frac{1}{2}\rho u^2\alpha_2^2 = p_4 + \frac{1}{2}\rho u^2\alpha_4^2 \qquad \ldots(14)$$

On combining (1), (3), (13) and (14) one obtains:

$$\frac{1}{2}\rho u^2\left(\beta_4^2 - \alpha_4^2\right) = \frac{1}{2}\rho u^2\left(\frac{(R-\alpha_2)^2}{(R-\alpha_2/\alpha_4)^2} - \alpha_4^2\right) = \frac{T}{A} \qquad (15)$$

The momentum equation for the entire flow between stations 1 and 4 is written:

$$pAR - p_4 AR - T = u^2 A\rho\alpha_2(\alpha_4 - 1) + u^2 A\rho(R - \alpha_2)\left(\frac{(R-\alpha_2)}{(R-\alpha_2/\alpha_4)} - 1\right) \quad \ldots(16)$$

which can be simplified to

$$p - p_4 = \frac{T}{RA} + \rho u^2 \frac{\alpha_2}{\alpha_4}\frac{(1-\alpha_4)^2}{(R-\alpha_2/\alpha_4)} \quad \ldots (17)$$

Combining (17) with (13) and (15) gives

$$-\frac{1}{2}\rho u^2\left(1-\beta_4^2\right) = \frac{1}{2}\rho u^2 \frac{\left(\beta_4^2 - \alpha_4^2\right)}{R} + \rho u^2 \frac{\alpha_2}{\alpha_4}\frac{(1-\alpha_4)^2}{(R-\alpha_2/\alpha_4)} \quad \ldots(18)$$

After some manipulation this leads to:

$$R\alpha_4^2(2\alpha_2 - 1 - \alpha_4) + \alpha_2\left(2\alpha_4^2 + \alpha_2 - 3\alpha_4\alpha_2\right) = 0 \quad \ldots(19)$$

Now we can consider two limits:

As $R \to \infty$, $\alpha_2 = \dfrac{1+\alpha_4}{2}$

As $R \to 1$, $-\alpha_4^2 - \alpha_4^3 + \alpha_2\left(4\alpha_4^2 + \alpha_2 - 3\alpha_4\alpha_2\right) = 0$, which is consistent with $\alpha_2 \to 1$, $\alpha_4 \to 1$, and in fact leads to $\alpha_2 = \alpha_4$.

For general R, solve the quadratic:

$$(1-3\alpha_4)\alpha_2^2 + 2(R+1)\alpha_4^2\alpha_2 - R\alpha_4^2(1+\alpha_4) = 0$$

The most convenient form of the solution is

$$\alpha_2 = \frac{R(1+\alpha_4)}{(R+1) + \sqrt{(R-1)^2 + R(1-1/\alpha_4)^2}} = \frac{(1+\alpha_4)}{(1+B) + \sqrt{(1-B)^2 + B(1-1/\alpha_4)^2}} \quad \ldots(20)$$

The power is then given by:

$$\begin{aligned} P = Tu_{2t} = Tu\alpha_2 &= \frac{1}{2}\rho A u^3 \alpha_2\left(\frac{(R-\alpha_2)^2}{(R-\alpha_2/\alpha_4)^2} - \alpha_4^2\right) \\ &= \frac{1}{2}\rho A u^3 R\alpha_2\alpha_4^2(1-\alpha_4)\left(\frac{R(1+\alpha_4) - 2\alpha_2}{(R\alpha_4 - \alpha_2)^2}\right) \quad \ldots(21) \\ &= \frac{1}{2}\rho A u^3 \alpha_2(1-\alpha_4)\left(\frac{(1+\alpha_4) - 2B\alpha_2}{(1-B\alpha_2/\alpha_4)^2}\right) = \frac{1}{2}\rho A u^3 C_P \end{aligned}$$

where the solution for α_2 from equation (20) should be substituted.

It is found numerically that this is always maximised by $\alpha_4 = \dfrac{1}{3}$ for which $\alpha_2 = \dfrac{2R}{3(R+1)}$ and the power is:

$$P = \frac{1}{2}\rho A u^3 \frac{16}{27}\left(\frac{R}{(R-1)}\right)^2 = \frac{1}{2}\rho A u^3 C_P \quad \ldots(22)$$

Note, however, that as $R \to 1$ the power extracted becomes infinite. This is because of the drop of pressure in the tube. Since from simple statics:

$$\Delta p = \frac{T}{AR} = \frac{1}{2}\rho u^2 B(1-\alpha_4)\left(\frac{(1+\alpha_4)-2B\alpha_2}{(1-B\alpha_2/\alpha_4)^2}\right)$$

A more rational measure of the performance might be $C_{P*} = \dfrac{P}{\frac{1}{2}\rho u^3 A + uA\Delta p} = \dfrac{C_P}{1+BC_T}$, where $C_T = \dfrac{T}{\frac{1}{2}\rho u^2 A} = (1-\alpha_4)\left(\dfrac{(1+\alpha_4)-2B\alpha_2}{(1-B\alpha_2/\alpha_4)^2}\right)$. Thus:

$$C_{P*} = \frac{\alpha_2(1-\alpha_4)\left(\frac{(1+\alpha_4)-2B\alpha_2}{(1-B\alpha_2/\alpha_4)^2}\right)}{1+B(1-\alpha_4)\left(\frac{(1+\alpha_4)-2B\alpha_2}{(1-B\alpha_2/\alpha_4)^2}\right)} = \frac{\alpha_2(1-\alpha_4)}{\left(\frac{(1-B\alpha_2/\alpha_4)^2}{(1+\alpha_4)-2B\alpha_2}\right)+B(1-\alpha_4)} \quad \ldots(23)$$

After substituting (20), this may be optimised as a function of α_4 value for each value of B. Note that all the above solutions are entirely compatible with the original Betz solution as $R \to \infty$.

At this stage we can also reconsider the force X that is acting between the turbine flow and the bypass flow. Previously this force has been zero, but now with the inclusion of the volume boundary we can expect that it is finite and positive. Considering momentum across the bypass flow we can write:

$$pA(R-\alpha_2) - p_4 A\left(R - \frac{\alpha_2}{\alpha_4}\right) - X = u^2 A\rho(R-\alpha_2)\left(\frac{(R-\alpha_2)}{(R-\alpha_2/\alpha_4)} - 1\right) \quad \ldots(24)$$

If we concern ourselves only with pressures above atmospheric we can take $p = 0$, also substituting for p_4 from (13), (24) can be rewritten as

$$-\frac{1}{2}\rho u^2 A\left(1-\beta_4^2\right)\left(R - \frac{\alpha_2}{\alpha_4}\right) - X = u^2 A\rho(R-\alpha_2)(\beta_4 - 1)$$

so that

$$\begin{aligned}
X &= \frac{1}{2}\rho u^2 A\left(2(R-\alpha_2)(\beta_4-1) + \left(1-\beta_4^2\right)\left(R-\frac{\alpha_2}{\alpha_4}\right)\right) \\
&= \frac{1}{2}\rho u^2 A(1-\beta_4)\left(-2(R-\alpha_2) + (1+\beta_4)\left(R-\frac{\alpha_2}{\alpha_4}\right)\right) \\
&= \frac{1}{2}\rho u^2 A(1-\beta_4)\left(-2(R-\alpha_2) + \left(2R - \frac{\alpha_2}{\alpha_4} - \alpha_2\right)\right) \\
&= \frac{1}{2}\rho u^2 A(1-\beta_4)\left(\alpha_2 - \frac{\alpha_2}{\alpha_4}\right)
\end{aligned}$$

Since $\alpha_2 > \alpha_4$ and $\beta_4 > 1$ for all values of R, it follows that X must also be greater than 0 for all values of R.

The power lost in the wake mixing process may also be determined. First of all it is necessary to determine the change of pressure from stations 4 to 5:

$$(p_4 - p_5)RA = \rho u^2 A\left(R - \alpha_2\alpha_4 - (R-\alpha_2)\frac{(R-\alpha_2)}{(R-\alpha_2/\alpha_4)}\right) \qquad \ldots(25)$$

$$p_4 - p_5 = -\rho u^2 \frac{\alpha_2(1-\alpha_4)^2}{\alpha_4(R-\alpha_2/\alpha_4)} \qquad \ldots(26)$$

$$\begin{aligned}
P_W &= \frac{1}{2}\rho u^3 A\alpha_2\alpha_4^2 + \frac{1}{2}\rho u^3 A(R-\alpha_2)\beta_4^2 - \frac{1}{2}\rho u^3 AR + ARu(p_4 - p_5) \\
&= \frac{1}{2}\rho u^3 A\left(\alpha_2\alpha_4^2 + \frac{(R-\alpha_2)^3}{(R-\alpha_2/\alpha_4)^2} - R\right) - \rho u^3 AR\frac{\alpha_2(1-\alpha_4)^2}{\alpha_4(R-\alpha_2/\alpha_4)} \\
&= \frac{1}{2}\rho u^3 A\alpha_2 R\left(\frac{R\left(\alpha_4^2 + 2/\alpha_4 - 3\right) + \alpha_2\left(3 - 2\alpha_4 - 1/\alpha_4^2\right) - 2\left(1/\alpha_4 - 2 + \alpha_4\right)\left(R - \alpha_2/\alpha_4\right)}{(R-\alpha_2/\alpha_4)^2}\right) \\
&= \frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4)^2 R\left(\frac{R + \alpha_2(1-2\alpha_4)/\alpha_4^2}{(R-\alpha_2/\alpha_4)^2}\right) \\
&= \frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4)^2\left(1 + \frac{B\alpha_2(1-B\alpha_2)}{\alpha_4^2(1-B\alpha_2/\alpha_4)^2}\right)
\end{aligned} \qquad \ldots(27)$$

Note that as $R \to \infty$ this gives the same asymptotic solution as for the constant pressure case. We can also calculate:

$$\frac{P_W}{P} = (1-\alpha_4)\left(\frac{1 + B\alpha_2(1-2\alpha_4)/\alpha_4^2}{(1+\alpha_4) - 2B\alpha_2}\right) \qquad \ldots(28)$$

8.4 Calculation sequence

1. Specify principal dimensioning parameters ρ, u and A
2. (Optionally specify upstream pressure p, which acts as purely additive term to all pressures)
3. Specify blockage ratio $0 \le B \le 1$ and dimensionless velocity factor $0 \le \alpha_4 \le 1$
4. Calculate dimensionless quantities:
   a. $\alpha_2 = \dfrac{(1+\alpha_4)}{(1+B) + \sqrt{(1-B)^2 + B(1-1/\alpha_4)^2}}$
   b. $\beta_4 = \dfrac{(1-B\alpha_2)}{(1-B\alpha_2/\alpha_4)}$
   c. $C_T = \left(\beta_4^2 - \alpha_4^2\right)$
   d. $C_{TL} = \dfrac{C_T}{\alpha_2^2}$

e. $C_P = \alpha_2 C_T$

f. $C_{P*} = \dfrac{C_P}{1+BC_T}$

g. $C_{PW} = \alpha_2(1-\alpha_4)^2\left(1+\dfrac{B\alpha_2(1-B\alpha_2)}{\alpha_4^2(1-B\alpha_2/\alpha_4)^2}\right)$

h. $\eta = \dfrac{C_P}{C_P + C_{PW}} = \dfrac{P}{P+P_W}$

5. Calculate dimensioned quantities:

i. $T = \dfrac{1}{2}\rho u^2 AC_T$

j. $P = \dfrac{1}{2}\rho u^3 AC_P$

k. $P_W = \dfrac{1}{2}\rho u^3 AC_{PW}$

l. Pressure drop across turbine $\Delta p_T = \dfrac{T}{A}$

m. Downstream pressure $p_5 = p - \dfrac{BT}{A}$

9. LINEAR MOMENTUM ACTUATOR DISC THEORY IN AN OPEN CHANNEL FLOW

9.1 Geometry of the flow

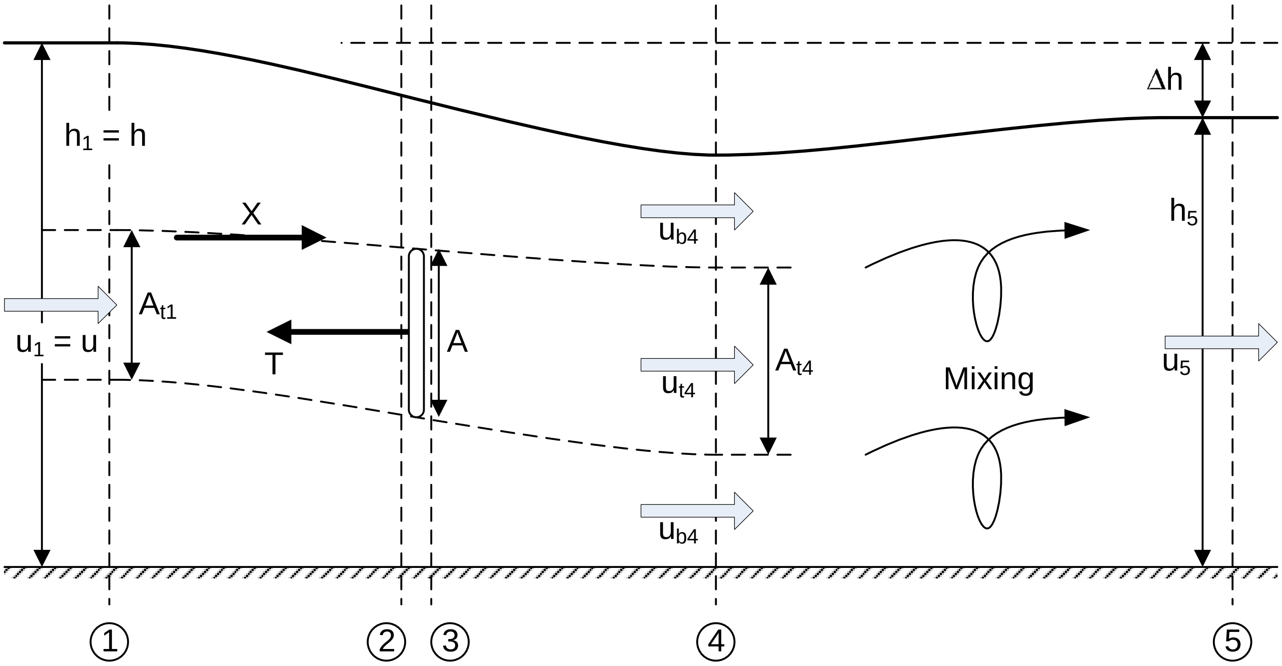

Figure 4: One dimensional linear momentum actuator disc theory in an open channel flow

9.2 Continuity relations

Region		Station 1	Station 2	Station 3	Station 4	Station 5
Turbine	Area	$A_{1t} = bhB\alpha_2$	$A_{2t} = A_{3t} = bhB$		$A_{4t} = bhB\dfrac{\alpha_2}{\alpha_4}$	
	Velocity	$u_{1t} = u$	$u_{2t} = u_{3t} = u\alpha_2$		$u_{4t} = u\alpha_4$	
	Volumetric flow	$q_{1t} = q_t = ubhB\alpha_2$	$q_{2t} = q_{3t} = ubhB\alpha_2$		$q_{4t} = ubhB\alpha_2$	
	Elevation head	$h_{1t} = h$	h_{2t}	h_{3t}	$h_{4t} = h_4$	
By-pass	Area	$A_{1b} = bh(1 - B\alpha_2)$			$A_{4b} = bh\dfrac{(1 - B\alpha_2)}{\beta_4}$	
	Velocity	$u_{1b} = u$			$u_{4b} = u\beta_4$	
	Volumetric flow	$q_{1b} = q_b = uhb(1 - B\alpha_2)$			$q_{4b} = uhb(1 - B\alpha_2)$	
	Elevation head	$h_{1b} = h$			$h_{4b} = h_4$	
Total	Depth	$h_1 = h$			h_4	$h_5 = h - \Delta h$
	Velocity	$u_1 = u$	Varies		Varies	$u_5 = \dfrac{uh}{(h - \Delta h)}$
	Volumetric flow	$q_1 = q = ubh$			$q_4 = ubh$	$q_5 = ubh$
	Pressure force	$p_1 = \frac{1}{2}\rho g h^2$	Varies	Varies	$p_4 = \frac{1}{2}\rho g h_4^2$	$p_5 = \frac{1}{2}\rho g(h - \Delta h)^2$

Table 4

9.3 Commentary and derivation

The open channel flow calculation follows a similar pattern to before, except that in the Bernoulli calculation the total head is now employed. We assume that at stations 1, 4, and 5 the pressure can be treated as hydrostatic. In some senses the calculation is a hybrid between the calculation at constant pressure and the one in a fixed tube: the downstream dimensions of the flow are not fixed, but there are relationships between dimension and velocity and between dimension and pressure force.

We start by noting that in the by-pass flow:

$$h + \frac{u^2}{2g} = h_4 + \frac{u^2\beta_4^2}{2g} \quad \text{…(29)}$$

As before, Bernoulli in the turbine flow upstream and downstream of the turbine gives:

$$h + \frac{u^2}{2g} = h_{2t} + \frac{u^2\alpha_2^2}{2g} \quad \text{…(30)}$$

$$h_{3t} + \frac{u^2\alpha_2^2}{2g} = h_4 + \frac{u^2\alpha_4^2}{2g} \quad \text{…(31)}$$

And the equilibrium of the turbine gives:

$$\rho g\left(h_{2t} - h_{3t}\right)Bbh = T \qquad \ldots(32)$$

Combining equations (29), (30), (31) and (32) gives:

$$h_{2t} - h_{3t} = \frac{T}{\rho gBbh} = \frac{u^2}{2g}\left(\beta_4^2 - \alpha_4^2\right) \qquad \ldots(33)$$

$$T = \frac{\rho u^2 Bbh}{2}\left(\beta_4^2 - \alpha_4^2\right) \qquad \ldots(34)$$

Now consider the momentum equation between stations 1 and 4, which gives:

$$\frac{1}{2}\rho gb\left(h^2 - h_4^2\right) - T = \rho u^2 bhB\alpha_2\left(\alpha_4 - 1\right) + \rho u^2 hb\left(1 - B\alpha_2\right)\left(\beta_4 - 1\right) \qquad \ldots(35)$$

Eliminating T between (34) and (35) gives

$$\frac{1}{2}g\left(h^2 - h_4^2\right) - Bh\frac{u^2}{2}\left(\beta_4^2 - \alpha_4^2\right) = u^2 hB\alpha_2\left(\alpha_4 - 1\right) + u^2 h\left(1 - B\alpha_2\right)\left(\beta_4 - 1\right) \qquad \ldots(36)$$

And we can make use of the continuity relationship

$$h_4 = Bh\frac{\alpha_2}{\alpha_4} + h\frac{\left(1 - B\alpha_2\right)}{\beta_4} \qquad \ldots(37)$$

Note also the following forms

$$\beta_4 = h\left(1 - B\alpha_2\right)/\left(h_4 - Bh\,\alpha_2/\alpha_4\right) \qquad \ldots(38)$$

$$\alpha_2 = \frac{\alpha_4}{Bh}\frac{\left(h\left(1 - \beta_4\right) + \beta_4.\left(h - h_4\right)\right)}{\left(\alpha_4 - \beta_4\right)} \qquad \ldots(39)$$

To eliminate (in principle) h_4 and β_4 between (29), (36) and (37), leaving, as previously, a relationship between α_2 and α_4. First eliminate h_4 to give:

$$\left(1 - \left(\frac{B\alpha_2}{\alpha_4} + \frac{\left(1 - B\alpha_2\right)}{\beta_4}\right)\right) = \frac{u^2}{2gh}\left(\beta_4^2 - 1\right) \qquad \ldots(40)$$

and

$$\left(1 - \left(\frac{B\alpha_2}{\alpha_4} + \frac{\left(1 - B\alpha_2\right)}{\beta_4}\right)^2\right) = \frac{u^2}{gh}\left(2B\alpha_2\left(\alpha_4 - 1\right) + 2\left(1 - B\alpha_2\right)\left(\beta_4 - 1\right) + B\left(\beta_4^2 - \alpha_4^2\right)\right)$$

$$= \frac{u^2}{gh}\left(2B\alpha_2\left(\alpha_4 - \beta_4\right) + 2\left(\beta_4 - 1\right) + B\left(\beta_4^2 - \alpha_4^2\right)\right) \qquad \ldots(41)$$

It is convenient later to write the results in terms of the upstream Froude number $F_r = u/\sqrt{gh}$. Dividing (41) by (40) we obtain

$$\left(1 + \frac{B\alpha_2}{\alpha_4} + \frac{\left(1 - B\alpha_2\right)}{\beta_4}\right) = \frac{2}{\left(\beta_4^2 - 1\right)}\left(2B\alpha_2\left(\alpha_4 - \beta_4\right) + 2\left(\beta_4 - 1\right) + B\left(\beta_4^2 - \alpha_4^2\right)\right) \qquad \ldots(42)$$

which re-arranges to

$$B\alpha_2\left(\beta_4-\alpha_4\right)\left(4+\frac{\left(\beta_4^2-1\right)}{\alpha_4\beta_4}\right)=2B\left(\beta_4^2-\alpha_4^2\right)+\frac{\left(1-\beta_4\right)^3}{\beta_4} \qquad \ldots(43)$$

leading to the solution:

$$\alpha_2=\frac{2\left(\beta_4+\alpha_4\right)-\dfrac{\left(\beta_4-1\right)^3}{B\beta_4\left(\beta_4-\alpha_4\right)}}{4+\dfrac{\left(\beta_4^2-1\right)}{\alpha_4\beta_4}} \qquad \ldots(44)$$

Rewriting (40) as

$$B\alpha_2\frac{\left(\beta_4-\alpha_4\right)}{\alpha_4\beta_4}=\frac{\left(\beta_4-1\right)}{\beta_4}-\frac{u^2}{2gh}\left(\beta_4^2-1\right) \qquad \ldots(45)$$

And dividing (44) and (45) to eliminate α_2 we obtain after some manipulation:

$$\left(4\alpha_4\beta_4+\left(\beta_4^2-1\right)\right)\left(\left(\beta_4-1\right)-\frac{F_r^2}{2}\left(\beta_4^2-1\right)\beta_4\right)=2B\left(\beta_4^2-\alpha_4^2\right)\beta_4-\left(\beta_4-1\right)^3 \qquad \ldots(46)$$

Which is a quartic in β_4

$$\frac{F_r^2}{2}\beta_4^4+2\alpha_4F_r^2\beta_4^3-\left(2-2B+F_r^2\right)\beta_4^2-\left(4\alpha_4+2\alpha_4F_r^2-4\right)\beta_4+\left(\frac{F_r^2}{2}+4\alpha_4-2B\alpha_4^2-2\right)=0 \qquad \ldots(47)$$

As $B\to 0$ and $\beta_4\to 1$ note the limit

$$\frac{B}{\left(\beta_4-1\right)}=\frac{2\alpha_4}{\left(1-\alpha_4^2\right)}\left(1-\frac{u^2}{gh}\right) \qquad \ldots(48)$$

The downstream head drop can be calculated from overall momentum:

$$\frac{1}{2}\rho gb\left(h^2-\left(h-\Delta h\right)^2\right)-T=\rho bhu\left(\frac{uh}{h-\Delta h}-u\right) \qquad \ldots(49)$$

$$\frac{1}{2}\left(2\frac{\Delta h}{h}-\left(\frac{\Delta h}{h}\right)^2\right)-\frac{T}{\rho bgh^2}=\frac{u^2}{gh}\left(\frac{\Delta h}{h-\Delta h}\right) \qquad \ldots(50)$$

$$\frac{1}{2}\left(2\frac{\Delta h}{h}-\left(\frac{\Delta h}{h}\right)^2\right)-\frac{T}{\rho bgh^2}=F_r^2\left(\frac{\Delta h/h}{1-\Delta h/h}\right) \qquad \ldots(51)$$

Where $C_T=\dfrac{T}{\frac{1}{2}\rho Bbhu^2}$ so that $\dfrac{T}{\rho bgh^2}=\dfrac{C_TBF_r^2}{2}$. This is a cubic in $\Delta h/h$:

$$\frac{1}{2}\left(\frac{\Delta h}{h}\right)^3-\frac{3}{2}\left(\frac{\Delta h}{h}\right)^2+\left(1-F_r^2+\frac{C_TBF_r^2}{2}\right)\frac{\Delta h}{h}-\frac{C_TBF_r^2}{2}=0 \qquad \ldots(52)$$

The power lost in the mixing is calculated as:

$$P_W = \frac{1}{2}\rho u^3 bhB\alpha_2\alpha_4^2 + \frac{1}{2}\rho u^3 bh(1-B\alpha_2)\beta_4^2 - \frac{1}{2}\rho u^3 bh\left(\frac{h}{h-\Delta h}\right)^2 + hbu(h_4 - h_5)\rho g$$
$$= \frac{1}{2}\rho u^3 Bbh\left(\alpha_2\alpha_4^2 + \frac{(1-B\alpha_2)}{B}\beta_4^2 - \frac{1}{B}\left(\frac{1}{1-\Delta h/h}\right)^2 + \frac{2(h_4-h_5)g}{u^2 B}\right) \quad \ldots(53)$$

Alternatively it can be useful simply to calculate the total power taken out of the flow:

$$P + P_W = \frac{1}{2}\rho u^3 bh - \frac{1}{2}\rho u^3 bh\left(\frac{h}{h-\Delta h}\right)^2 + hbu(h - h_5)\rho g$$
$$= \frac{1}{2}\rho u^3 bh\left(1 - \left(\frac{1}{1-\Delta h/h}\right)^2 + \frac{2\Delta h/h}{F_r^2}\right) = \rho gubh\Delta h\left(1 - F_r^2\,\frac{1-\Delta h/2h}{(1-\Delta h/h)^2}\right) \quad \ldots(54)$$

Therefore the efficiency of the turbine is simply:

$$\eta = \frac{P}{P+P_W} = \frac{P}{\rho gubh\Delta h}\left(1 - F_r^2\,\frac{1-\Delta h/2h}{(1-\Delta h/h)^2}\right)^{-1} \quad \ldots(55)$$

For small Froude number flows this may be approximated by $\eta \approx \dfrac{P}{\rho gubh\Delta h}$.

9.4 Calculation sequence

1. Specify principal dimensioning parameters ρ, g and h
2. (Optionally specify width b, which acts as purely scaling term on power and force)
3. Specify upstream Froude number $F_r = u/\sqrt{gh}$, blockage ratio $0 \le B \le 1$ and dimensionless velocity factor $0 \le \alpha_4 \le 1$
4. Calculate dimensionless quantities:
   a. Solve for β_4 from:
   $$\frac{F_r^2}{2}\beta_4^4 + 2\alpha_4 F_r^2\beta_4^3 - \left(2 - 2B + F_r^2\right)\beta_4^2 - \left(4\alpha_4 + 2\alpha_4 F_r^2 - 4\right)\beta_4 + \left(\frac{F_r^2}{2} + 4\alpha_4 - 2B\alpha_4^2 - 2\right) = 0$$
   such that $\beta_4 > 1 \text{ and } 1 > \alpha_2 > \alpha_4$.
   b. $$\alpha_2 = \frac{2(\beta_4 + \alpha_4) - \dfrac{(\beta_4 - 1)^3}{B\beta_4(\beta_4 - \alpha_4)}}{4 + \dfrac{(\beta_4^2 - 1)}{\alpha_4\beta_4}}$$
   c. $C_T = \left(\beta_4^2 - \alpha_4^2\right)$
   d. $C_{TL} = \dfrac{C_T}{\alpha_2^2}$
   e. Solve for $\Delta h/h$ from:
   $$\frac{1}{2}\left(\frac{\Delta h}{h}\right)^3 - \frac{3}{2}\left(\frac{\Delta h}{h}\right)^2 + \left(1 - F_r^2 + \frac{C_T BF_r^2}{2}\right)\frac{\Delta h}{h} - \frac{C_T BF_r^2}{2} = 0$$
   f. $C_P = \alpha_2 C_T$

g. $C_P + C_{PW} = \frac{1}{B}\left(1-\left(\frac{1}{1-\Delta h/h}\right)^2 + \frac{2\Delta h/h}{F_r^2}\right)$

h. $\eta = \frac{C_P}{C_P + C_{PW}} = \frac{P}{P + P_W}$

5. Calculate dimensioned quantities:

i. $T = \frac{1}{2}\rho u^2 BbhC_T$

j. $P = \frac{1}{2}\rho u^3 BbhC_P$

k. $P_W = \frac{1}{2}\rho u^3 BbhC_{PW}$

l. $\Delta h = h\frac{\Delta h}{h}$

m. Pressure drop across turbine $\Delta p_T = \frac{T}{Bbh}$

9.5 Solution space of the model

The quartic defined by equation (47) will yield real solutions for β_4 only for a subset of input variables F_r, B, α_4. To determine the range of this subset we can reconsider the equations derived in §9.3. It is clear that both equation (29) and equation (35) express quantities that will have a minimum value when plotted against h_4. These minimum values indicate that the flow within the bypass and the far wake, respectively, will be exactly critical. If h_4 is specified as less than this critical point no real solutions will exist for a given upstream discharge rate. More specifically the turbine will 'block' the flow and a hydraulic jump will result.

To determine the critical point consider equation (29). Mathematically the condition of critical flow can be expressed as

$$\frac{dE}{dh_4} = \frac{d}{dh_4}\left(\frac{\beta_4^2 V^2}{2g} + h_4\right) = \frac{hFr^2}{2}\frac{d}{dh_4}\beta_4^2 + 1 = 0 \qquad \ldots(56)$$

Giving the condition

$$\frac{d(\beta_4^2)}{dh_4} = -\frac{2}{hFr^2} \qquad \ldots(57)$$

A similar exercise can be done for equation (35) to determine the minimum momentum. However it can be shown numerically that in all cases the bypass condition given by (57) is reached at the point when solutions to the quartic (47) become complex. The far wake will never reach critical conditions before the bypass flow.

Therefore the solution space of this open channel model is bounded by the requirement that the bypass flow remains sub-critical, or mathematically $\frac{d(\beta_4^2)}{dh_4} < -\frac{2}{hFr^2}$.

10. SUMMARY

Variable	Unbounded	Finite: pressure controlled	Finite: tube	Open channel
Upstream pressure or depth	p	p	p	h
Upstream velocity	u	u	u	u
Variable β_4	1	1	$\frac{(1-B\alpha_2)}{(1-B\alpha_2/\alpha_4)}$	From solution of quartic
Variable α_2	$\frac{1+\alpha_4}{2}$	$\frac{1+\alpha_4}{2}$	$\frac{(1+\alpha_4)}{(1+B)+\sqrt{(1-B)^2+B(1-1/\alpha_4)^2}}$	$\frac{2(\beta_4+\alpha_4)-\frac{(\beta_4-1)^3}{B\beta_4(\beta_4-\alpha_4)}}{4+\frac{(\beta_4^2-1)}{\alpha_4\beta_4}}$
Power P	$\frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4^2)$	$\frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4^2)$	$\frac{1}{2}\rho u^3 A\alpha_2(\beta_4^2-\alpha_4^2)$	$\frac{1}{2}\rho u^3 Bbh\alpha_2(\beta_4^2-\alpha_4^2)$
Wake loss P_W	$\frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4)^2$	$\frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4)^2\left(1-\frac{\alpha_2}{R}\right)$	$\frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4)^2\frac{\left(1-B\alpha_2\frac{(2\alpha_4-1)}{\alpha_4^2}\right)}{(1-B\alpha_2/\alpha_4)^2}$	See equation 52.
Total power removed $P+P_w$	$\frac{1}{2}\rho u^3 A\,2\alpha_2(1-\alpha_4)$	$\frac{1}{2}\rho u^3 A\alpha_2(1-\alpha_4)\left(2-\frac{(1-\alpha_4^2)}{2R}\right)$		$\rho gubh\Delta h\left(1-F_r^2\frac{1-\Delta h/2h}{(1-\Delta h/h)^2}\right)$
Thrust T	$\frac{1}{2}\rho u^2 A(1-\alpha_4^2)$	$\frac{1}{2}\rho u^2 A(1-\alpha_4^2)$	$\frac{1}{2}\rho u^2 A(\beta_4^2-\alpha_4^2)$	$\frac{1}{2}\rho u^2 Bbh(\beta_4^2-\alpha_4^2)$
Downstream pressure or depth	$\sim p$	p	$p-\frac{1}{2}\rho u^2 B(1-\alpha_4^2)\frac{\left(1-\frac{2B\alpha_2}{(1+\alpha_4)}\right)}{(1-B\alpha_2/\alpha_4)^2}$	$h-\Delta h$, from solution of cubic for $\Delta h/h$
Downstream velocity	$\sim u$	$u\left(1-\frac{(1-\alpha_4^2)}{2R}\right)$	u	$\frac{uh}{h-\Delta h}$