

On Certain Degenerate Whittaker Models for Cuspidal Representations of $\mathrm{GL}_{k \cdot n}(\mathbb{F}_q)$

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Abstract

Let π be an irreducible cuspidal representation of $\mathrm{GL}_{kn}(\mathbb{F}_q)$. Assume that $\pi = \pi_\theta$, corresponds to a regular character θ of $\mathbb{F}_{q^{kn}}^*$. We consider the twisted Jacquet module of π with respect to a non-degenerate character of the unipotent radical corresponding to the partition $(n, n, \dots, n)$ of kn . We show that, as a $\mathrm{GL}_n(\mathbb{F}_q)$ -representation, this Jacquet module is isomorphic to $\pi_{\theta|_{\mathbb{F}_n^*}} \otimes \mathrm{St}^{\otimes(k-1)}$, where St is the Steinberg representation of $\mathrm{GL}_n(\mathbb{F}_q)$. This generalizes a theorem of D. Prasad, who considered the case $k = 2$. We prove and rely heavily on a formidable identity involving q -hypergeometric series and linear algebra.

1 Introduction

Let $\mathbb{F} := \mathbb{F}_q$ be the finite field of size q . We fix a nontrivial character ψ_0 of $\mathbb{F}$. Denote by $\mathbb{F}_m := \mathbb{F}_{q^m}$ the unique degree m field extension of $\mathbb{F}$. For a positive integer r , we denote the diagonal subgroup of $(\mathrm{GL}_\ell(\mathbb{F}))^r$ by

$$\Delta^r(\mathrm{GL}_\ell(\mathbb{F})) := \{(g, \dots, g) \in (\mathrm{GL}_\ell(\mathbb{F}))^r \mid g \in \mathrm{GL}_\ell(\mathbb{F})\}.$$

For a partition $\rho = (k_1, k_2, \dots, k_s)$ of ℓ , denote by P_ρ the corresponding standard parabolic subgroup of $\mathrm{GL}_\ell(\mathbb{F})$. Let M_ρ and N_ρ be the corresponding standard Levi subgroup and unipotent radical.

Fix $k \geq 1$. Let $\rho = (n, n, \dots, n)$ be the partition of kn consisting of k parts of size n . In this paper we denote $G := \mathrm{GL}_{kn}(\mathbb{F})$, $P := P_\rho$, $M := M_\rho$ and $N := N_\rho$. We have the Levi decomposition $P = M \ltimes N$. We write $U \in N$ in the form

$$U = \begin{pmatrix} I_n & X_{1,1} & X_{1,2} & \cdots & X_{1,k-2} & X_{1,k-1} \\ 0 & I_n & X_{2,2} & \cdots & X_{2,k-2} & X_{2,k-1} \\ 0 & 0 & I_n & \cdots & X_{3,k-2} & X_{3,k-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_n & X_{k-1,k-1} \\ 0 & 0 & 0 & \cdots & 0 & I_n \end{pmatrix}, \quad (1.1)$$

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where the matrices $X_{i,j}$ ($1 \leq i \leq j \leq k-1$) are elements of $M_n(\mathbb{F})$.

Definition 1.1. A character $\psi : N \rightarrow \mathbb{C}^*$ is said to be non-degenerate if it is of the form

$$\psi(U) := \psi_0 \left(\text{tr} \left(\sum_{i=1}^{k-1} A_i X_{i,i} \right) \right) = \prod_{i=1}^{k-1} \psi_0(\text{tr}(A_i X_{i,i})),$$

where the matrices A_i are invertible.

Let $\psi : N \rightarrow \mathbb{C}^*$ be a non-degenerate character. Let π be an irreducible representation of G , acting on a space V_π . We denote by $V_{\pi_{k,N,\psi}}$ the largest subspace of V_π , on which N operates through ψ , i.e.

$$V_{\pi_{k,N,\psi}} = \{v \in V_\pi \mid \pi(U)v = \psi(U)v, \forall U \in N\}.$$

This is the (N, ψ) -isotypic subspace of V_π and it is the image of the canonical projection of V_π on $V_{\pi_{k,N,\psi}}$ given by

$$P_{k,N,\psi}(v) = \frac{1}{|N|} \sum_{U \in N} \overline{\psi}(U) \pi(U)v. \quad (1.2)$$

Since M normalizes N , it acts on the characters of N as follows. If $m \in M$, then for all $U \in N$

$$(m \cdot \psi)(U) = \psi(m^{-1}Um).$$

We have, for $m \in M$,

$$\pi(m)V_{\pi_{k,N,\psi}} = V_{\pi_{k,N,m \cdot \psi}}.$$

Let us compute the stabilizer of ψ in M . If

$$m = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_k \end{pmatrix},$$

where $B_i \in \text{GL}_n(\mathbb{F})$ for all $1 \leq i \leq k$, then

$$(m \cdot \psi)(U) = \psi_0 \left(\text{tr} \left(\sum_{i=1}^{k-1} A_i B_i^{-1} X_{i,i} B_{i+1} \right) \right).$$

Thus, $m \cdot \psi = \psi$ if and only if $B_i = B_{i+1}$ for all $1 \leq i \leq k-1$. In other words,

$$\text{stab}_M \psi = \Delta^k(\text{GL}_n(\mathbb{F})) \cong \text{GL}_n(\mathbb{F}).$$

Therefore, $V_{\pi_{k,N,\psi}}$ is a $\text{GL}_n(\mathbb{F})$ -module. We denote by $\pi_{k,N,\psi}$ the resulting representation of $\text{GL}_n(\mathbb{F})$ on $V_{\pi_{k,N,\psi}}$. It is easy to see that by conjugation with an element in the standard Levi subgroup, we may simply take all the A_i to be the identity matrix. The corresponding twisted Jacquet modules are isomorphic. In the rest of the paper we assume $A_i = I_n$ and fix

$$\psi(U) := \psi_0 \left(\text{tr} \left(\sum_{i=1}^{k-1} X_{i,i} \right) \right).$$

The goal of this paper is to calculate the character of $\pi_{k,N,\psi}$, and to describe it in terms of more familiar representations, for an irreducible, cuspidal representation $\pi = \pi_\theta$ of $\mathrm{GL}_{kn}(\mathbb{F})$, associated to a regular character θ of $\mathbb{F}_{kn}^*$. The paper generalizes Prasad's result for the case $k = 2$ stated below.

Theorem. [*Pra00, Thm. 1*] *Let π be an irreducible cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{F})$ obtained from a character θ of $\mathbb{F}_{2n}^*$. Then*

$$\pi_{2,N,\psi} \cong \mathrm{Ind}_{\mathbb{F}_n^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \downarrow_{\mathbb{F}_n^*}. \quad (1.3)$$

Prasad proved this theorem by an explicit calculation of the characters of $\pi_{2,N,\psi}$ and of the induced representation $\mathrm{Ind}_{\mathbb{F}_n^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \downarrow_{\mathbb{F}_n^*}$. At any element of $\mathrm{GL}_n(\mathbb{F})$ the characters are the same. Therefore, the two representations are equivalent.

The methods used in this paper are generalizations of the methods used by the second author in his thesis [Haz16] for the case $k = 3$. From the character calculation, done in Theorem 3 below, we are able to describe in Theorem 4 $\pi_{k,N,\psi}$ in terms of the representations $\mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \downarrow_{\mathbb{F}_\ell^*}$, where $\ell \mid n$. This reduces immediately to Prasad's result when $k = 2$. Furthermore, we give a compact description of $\pi_{k,N,\psi}$ in terms of the Steinberg representation in the following theorem.

Theorem 1. *Let $k \geq 1$. Let π_θ be an irreducible cuspidal representation of $\mathrm{GL}_{kn}(\mathbb{F})$ obtained from a character θ of $\mathbb{F}_{kn}^*$. Then*

$$\pi_{k,N,\psi} \cong \pi_{\theta \downarrow_{\mathbb{F}_n^*}} \otimes \mathrm{St}^{\otimes(k-1)},$$

where $\pi_{\theta \downarrow_{\mathbb{F}_n^*}}$ is the irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$ obtained from $\theta \downarrow_{\mathbb{F}_n^*}$, and $\mathrm{St}^{\otimes(k-1)}$ is the $(k-1)$ -fold tensor product of the Steinberg representation of $\mathrm{GL}_n(\mathbb{F})$ with itself.

Note that for $n = 1$, Theorem 1 gives $\pi_{k,N,\psi} \cong \theta \downarrow_{\mathbb{F}^*}$, which also follows from Gel'fand-Graev [GG62] in case of $\mathrm{GL}_k(\mathbb{F})$ (cf. [Car93, Ch. 8.1]).

We are currently investigating an analogous construction for a non-Archimedean local field.

1.1 Structure of the paper

In §2 we set the background material from several topics that are needed in the paper: linear algebra, representation theory, q -hypergeometric identities and arithmetic identities.

In §3 we calculate the dimension of $\pi_{k,N,\psi}$. Green's formula allows us to express the dimension as rather complicated sum. We use q -hypergeometric identities and linear algebra to show that this sum admits the following compact form.

Theorem 2. *Let $k \geq 2$. We have*

$$\dim(\pi_{k,N,\psi}) = q^{(k-2)\frac{n(n-1)}{2}} \frac{|\mathrm{GL}_n(\mathbb{F})|}{q^n - 1}.$$

In §4 we compute the character of $\pi_{k,N,\psi}$, denoted by $\Theta_{k,N,\psi}$. Apart from the tools used in Theorem 2 this requires understanding of some conjugacy classes of $\mathrm{GL}_n(\mathbb{F})$. When $d \mid m$, we have an embedding $\mathbb{F}_d^* \hookrightarrow \mathrm{GL}_m(\mathbb{F})$ (see §2.1). The elements in $\mathrm{GL}_m(\mathbb{F})$ conjugate to an element in the image of this embedding are said to come from $\mathbb{F}_d$.

Theorem 3. *Let $k \geq 2$. Let $g = s \cdot u$ be the Jordan decomposition of an element g in $\mathrm{GL}_n(\mathbb{F})$, where s and u are the semisimple part and unipotent part, respectively.*

(I) *If s does not come from $\mathbb{F}_n$, then*

$$\Theta_{k,N,\psi}(g) = 0.$$

(II) *If the $u \neq I_n$, then*

$$\Theta_{k,N,\psi}(g) = 0.$$

(III) *Assume that $u = I_n$ and s comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ and $d \mid n$ is minimal. Let λ be an eigenvalue of s which generates $\mathbb{F}_d$ over $\mathbb{F}$. Then,*

$$\Theta_{k,N,\psi}(s) = (-1)^{k(n-d')} q^{(k-2)\frac{n(d'-1)}{2}} \cdot \left[\sum_{i=0}^{d'-1} \theta(\lambda^{q^i}) \right] \cdot \frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{q^n - 1},$$

where $d' = n/d$.

In §5 we obtain from Theorem 3 and Lemma 2.10 an isomorphism of representation relating between $\pi_{k,N,\psi}$ and $\mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \upharpoonright_{\mathbb{F}_\ell^*}$ for all $\ell \mid n$. We write $a|b|c$ for $a|b$ and $b|c$. For any ℓ dividing n and any $k \geq 2$, let

$$a_{k;n,\ell}(q) = \frac{q^\ell - 1}{q^n - 1} \sum_{m: \ell|m|n} \mu\left(\frac{m}{\ell}\right) (-1)^{k(n-\frac{n}{m})} q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}, \quad (1.4)$$

where μ is the Möbius function.

Theorem 4. *Let $k \geq 2$.*

(I) *If k is even or n is odd, we have*

$$\pi_{k,N,\psi} \cong \bigoplus_{\ell|n} a_{k;n,\ell}(q) \cdot \mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \upharpoonright_{\mathbb{F}_\ell^*}. \quad (1.5)$$

(II) *If k is odd and n is even, we have*

$$\left(\pi_{k,N,\psi} \oplus \bigoplus_{\ell: \ell|n, 2 \nmid \frac{n}{\ell}} (-a_{k;n,\ell}(q)) \cdot \mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \upharpoonright_{\mathbb{F}_\ell^*} \right) \cong \bigoplus_{\ell: \ell|n, 2 \mid \frac{n}{\ell}} a_{k;n,\ell}(q) \cdot \mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})} \theta \upharpoonright_{\mathbb{F}_\ell^*}. \quad (1.6)$$

We note that the coefficients in Theorem 4 are non-negative integers. Indeed, when $k = 2$, it is easily shown (see Lemma 2.10) that $a_{2;n,\ell}(q) = \delta_{\ell,n}$, which gives (1.3). If $k > 2$ we show in Lemma 2.10 that $a_{k;n,\ell}(q)$ is a positive integer, except when k is odd, n is even and $2 \nmid \frac{n}{\ell}$, in which case $-a_{k;n,\ell}(q)$ is a positive integer.

In §6 we deduce Theorem 1 from Theorem 3.

2 Preliminaries

2.1 Cuspidal representations

We review the irreducible cuspidal representations of $\mathrm{GL}_m(\mathbb{F})$ as in S.I. Gel'fand [Gel75, §6] (originally in J. A. Green [Gre55]). Irreducible cuspidal representations of $\mathrm{GL}_m(\mathbb{F})$, from which all the other irreducible representations of $\mathrm{GL}_m(\mathbb{F})$ are obtained via the process of parabolic induction, are associated to regular characters of $\mathbb{F}_m^*$. A multiplicative character θ of $\mathbb{F}_m^*$ is called *regular* if, under the action of the Galois group of $\mathbb{F}_m$ over $\mathbb{F}$, the orbit of θ consists of m distinct characters of $\mathbb{F}_m^*$.

We denote the irreducible cuspidal representation of $\mathrm{GL}_m(\mathbb{F})$ associated to a regular character θ of $\mathbb{F}_m^*$ by π_θ and the character of the representation π_θ by Θ_θ .

Given $a \in \mathbb{F}_m$, consider the map $m_a : \mathbb{F}_m \rightarrow \mathbb{F}_m$, defined by $m_a(x) = ax$. The map $a \mapsto m_a$ is an injective homomorphism of algebras $\mathbb{F}_m \hookrightarrow \mathrm{End}_{\mathbb{F}}(\mathbb{F}_m)$. This way, every element of $\mathbb{F}_m^*$ gives rise to a well-defined conjugacy class in $\mathrm{GL}_m(\mathbb{F})$. The elements in the conjugacy classes in $\mathrm{GL}_m(\mathbb{F})$, which are so obtained from elements of $\mathbb{F}_m^*$, are said to come from $\mathbb{F}_m^*$.

We summarize the information about the character Θ_θ in the following theorem. We refer to the paper [Pra00, Thm. 2] for the statement of this theorem in this explicit form, which is originally due to Green [Gre55, Thm. 14] (cf. [Gel75, SZ84]).

Theorem 2.1 (Green [Gre55]). *Let Θ_θ be the character of a cuspidal representation π_θ of $\mathrm{GL}_m(\mathbb{F})$ associated to a regular character θ of $\mathbb{F}_m^*$. Let $g = s \cdot u$ be the Jordan decomposition of an element g in $\mathrm{GL}_m(\mathbb{F})$ (s is a semisimple element, u is unipotent and s, u commute). If $\Theta_\theta(g) \neq 0$, then the semisimple element s must come from $\mathbb{F}_m^*$. Suppose that s comes from $\mathbb{F}_m^*$. Let λ be an eigenvalue of s in $\mathbb{F}_m^*$, and let $t = \dim_{\mathbb{F}_m} \ker(g - \lambda I)$. Then*

$$\Theta_\theta(s \cdot u) = (-1)^{m-1} \left[\sum_{\alpha=0}^{d-1} \theta(\lambda^{q^\alpha}) \right] (1 - q^d)(1 - (q^d)^2) \cdots (1 - (q^d)^{t-1}) \quad (2.1)$$

where q^d is the cardinality of the field generated by λ over $\mathbb{F}$, and the summation is over the various distinct Galois conjugates of λ .

Corollary 2.2. *The value $\Theta_\theta(g)$ is determined by the eigenvalue of g and the number of Jordan blocks of g , which, in turn, is determined by $\dim_{\mathbb{F}_m} \ker(g - \lambda I)$.*

2.2 Characters induced from subfields

The following lemma summarizes the information about the character of $\mathrm{Ind}_{\mathbb{F}_\ell^*}^{\mathrm{GL}_n(\mathbb{F})}(\theta \upharpoonright_{\mathbb{F}_\ell^*})$, where $\ell \mid n$ and θ is a character of $\mathbb{F}_n^*$.

Lemma 2.3. [Haz16, Lem. 2.4] *Let θ be a character of $\mathbb{F}_n^*$. Suppose that $s \in \mathrm{GL}_n(\mathbb{F})$ comes from $\mathbb{F}_d \subseteq \mathbb{F}_\ell$ ($d \mid \ell$ is minimal). Let λ be an eigenvalue of s*

in $\mathbb{F}_d^*$. Then, the character Θ_{Ind_ℓ} of $\text{Ind}_{\mathbb{F}_\ell^*}^{\text{GL}_n(\mathbb{F})}(\theta \upharpoonright_{\mathbb{F}_\ell^*})$ at s is given by

$$\Theta_{\text{Ind}_\ell}(s) = \frac{1}{q^\ell - 1} \sum_{\substack{g \in \text{GL}_n(\mathbb{F}) \\ g^{-1}sg \in \mathbb{F}_\ell^*}} \theta(g^{-1}sg) \quad (2.2)$$

$$= \frac{|\text{GL}_{d'}(\mathbb{F}_d)|}{q^\ell - 1} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right], \quad (2.3)$$

where $d' = n/d$, and the last sum is over the various distinct Galois conjugates of λ . The value of the character Θ_{Ind_ℓ} at an element of $\text{GL}_n(\mathbb{F})$ which does not come from $\mathbb{F}_\ell$ is zero.

Remark 2.4. Recall that in (2.2) $\mathbb{F}_\ell^*$ is considered a subgroup of $\text{GL}_n(\mathbb{F})$ by the injective map $a \mapsto [m_a]$, where $[m_a]$ is the representing matrix of m_a with respect to a fixed basis of $\mathbb{F}_n$ over $\mathbb{F}$. Note that the choice of basis for $[m_a]$ does not affect the values of Θ_{Ind_ℓ} .

2.3 On some conjugacy classes of $\text{GL}_n(\mathbb{F})$

2.3.1 Analogue of Jordan form

Let $g \in \text{GL}_n(\mathbb{F})$ and $g = s \cdot u$ be its Jordan decomposition. Assume that s comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ ($d \mid n$ is minimal). Let $\lambda \in \mathbb{F}_d^*$ be an eigenvalue of s , which generates the field $\mathbb{F}_d$ over $\mathbb{F}$. Denote by f the characteristic polynomial of λ (of degree d), and by $L_f \in \text{GL}_d(\mathbb{F})$ the companion matrix of f . For $\ell \geq 1$ we denote

$$L_{f,\ell} = \begin{pmatrix} L_f & I_d & & \\ & L_f & & \\ & & \ddots & I_d \\ & & & L_f \end{pmatrix} \in \text{GL}_{\ell \cdot d}(\mathbb{F}).$$

This is an analogue of a Jordan block. As in [Gel75, Gre55], there exists $\rho = (\ell_1, \dots, \ell_r)$, a partition of $\frac{n}{d}$, $\ell_1 \geq \ell_2 \geq \dots \geq \ell_r$, such that g is conjugate to

$$L_\rho(f) := \begin{pmatrix} L_{f,\ell_1} & & & \\ & L_{f,\ell_2} & & \\ & & \ddots & \\ & & & L_{f,\ell_r} \end{pmatrix},$$

i.e. there exists $R \in \text{GL}_n(\mathbb{F})$ such that

$$R^{-1}gR = L_\rho(f). \quad (2.4)$$

Notice that in case $u = I_n$ (g is semisimple), we have $\rho = (1^{n/d})$ and there exists $R \in \text{GL}_n(\mathbb{F})$ such that $R^{-1}gR$ is a block diagonal matrix with $d' = n/d$ times L_f on the diagonal. Otherwise, $\ell_1 > 1$ and, in particular, there exists $R \in \text{GL}_n(\mathbb{F})$ such that the upper $2d \times 2d$ left corner of $R^{-1}gR$ is

$$\begin{pmatrix} L_f & I_d \\ & L_f \end{pmatrix}.$$

Now, s (and so g) has d different eigenvalues obtained by applying the Frobenius automorphism σ , which generates the Galois group $\text{Gal}(\mathbb{F}_d/\mathbb{F})$, namely

$$\{\lambda, \sigma(\lambda), \dots, \sigma^{d-1}(\lambda)\} = \{\lambda, \lambda^q, \dots, \lambda^{q^{d-1}}\},$$

all of multiplicity $d' = n/d$ in the characteristic polynomial of s . Let $0 \neq v_0 \in \mathbb{F}_d^d$ satisfy $L_f \cdot v_0 = \lambda v_0$. So $L_f \cdot \sigma^i(v_0) = \lambda^{q^i} \sigma^i(v_0)$, for $0 \leq i \leq d-1$. Hence, $B = \{v_0, \sigma(v_0), \dots, \sigma^{d-1}(v_0)\} \subseteq \mathbb{F}_d^d$ is linearly independent over $\mathbb{F}_d$, since its elements are eigenvectors of L_f for different eigenvalues. Let $T \in \text{GL}_d(\mathbb{F}_d)$ be the diagonalizing matrix of L_f obtained by B , i.e.

$$T^{-1} L_f T = D, \quad (2.5)$$

where

$$D := \text{diag}(\lambda, \dots, \lambda^{q^{d-1}}).$$

Denote by $\Delta^{d'}(T)$ the block diagonal matrix with d' times T on the diagonal. Explicitly, the columns of $\Delta^{d'}(T)$ are the vectors of the basis

$$C = \{v_0(i, j)\}_{0 \leq j \leq d'-1, 0 \leq i \leq d-1}, \quad (2.6)$$

whose $(j \cdot d + i)$ -th vector is given by

$$v_0(i, j) = \begin{pmatrix} 0_{j \cdot d} \\ \sigma^i(v_0) \\ 0_{n-(j+1) \cdot d} \end{pmatrix} \in \mathbb{F}_d^n,$$

where $0 \leq i \leq d-1$ and $0 \leq j \leq d'-1$. Thus, in case $u = I_n$

$$\Delta^{d'}(T^{-1}) R^{-1} g R \Delta^{d'}(T) = \begin{pmatrix} D & & \\ & \ddots & \\ & & D \end{pmatrix}.$$

Otherwise

$$\Delta^{d'}(T^{-1}) R^{-1} g R \Delta^{d'}(T) = \begin{pmatrix} D & I_d & & \\ & D & & \\ & & D & * \\ & & & \ddots & * \\ & & & & D \end{pmatrix},$$

where $*$ means either I_d or 0_d above the diagonal. We denote

$$g_\rho := g_{\rho, R} = \Delta^{d'}(T^{-1}) R^{-1} g R \Delta^{d'}(T). \quad (2.7)$$

The matrix g_ρ is sometimes referred to as an analogue of the Jordan form of g [Gel75, §0].

2.3.2 Conjugating an arbitrary matrix

We use the notation of §2.3.1. In particular, we have a fixed $g \in \text{GL}_n(\mathbb{F})$ and corresponding R and T as defined in (2.4) and (2.5). Let $A \in M_n(\mathbb{F})$. We study the following conjugation

$$A_\rho := A_{\rho,R} = \Delta^{d'} (T^{-1}) R^{-1} A R \Delta^{d'} (T) \in M_n(\mathbb{F}_d).$$

Define A_R by $A_R = R^{-1} A R$, and so $A_\rho = \Delta^{d'} (T^{-1}) A_R \Delta^{d'} (T)$.

Let $B \in M_n(\mathbb{F}_d)$. Let us represent the vectors $B \cdot v_0(0, m)$, for any $0 \leq m \leq d' - 1$, as a linear combination of the basis C given in (2.6):

$$B \cdot v_0(0, m) = \sum_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} a_{m,i;j} \cdot v_0(i, j), \quad a_{m,i;j} \in \mathbb{F}_d.$$

A necessary and sufficient condition for $B \in M_n(\mathbb{F})$ is that for all $0 \leq m \leq d' - 1$, $0 \leq r \leq d - 1$,

$$B \cdot v_0(r, m) = \sum_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} \sigma^r(a_{m,i;j}) \cdot v_0(i + r \pmod{d}, j). \quad (2.8)$$

By taking $B = A_R \in M_n(\mathbb{F})$, we get that (2.8) holds for A_R . Therefore, $[A_R]_C = A_\rho$ is a $d' \times d'$ matrix with entries from $M_d(\mathbb{F}_d)$. For $0 \leq m, j \leq d' - 1$, the m -th row and j -th column of A_ρ , denoted by $A_{m,j}$, is given by

$$A_{m,j} = (\sigma^r(a_{m,i-r \pmod{d};j}))_{0 \leq i, r \leq d-1}, \quad (2.9)$$

i.e. $A_{m,j} \in M_d(\mathbb{F}_d)$ and for $0 \leq i, r \leq d - 1$, the i -th row and r -th column of $A_{m,j}$ is $\sigma^r(a_{m,i-r \pmod{d};j})$. The above discussion can be summarized in the following lemma.

Lemma 2.5. *In the above notations, the map $A \mapsto A_\rho$ induces an $\mathbb{F}$ -linear isomorphism $M_n(\mathbb{F}) \rightarrow M_{n \times d'}(\mathbb{F}_d) \cong [M_{d \times d'}(\mathbb{F}_d)]^{d'}$. It is given by*

$$A \mapsto \begin{pmatrix} (a_{0,i;j})_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} \\ \vdots \\ (a_{d'-1,i;j})_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} \end{pmatrix},$$

where the $(m \cdot d + i)$ -th row and j -th column of the image of A is $a_{m,i;j} \in \mathbb{F}_d$, for $0 \leq m, j \leq d' - 1$ and $0 \leq i \leq d - 1$.

2.3.3 Trace under conjugation

For $g \in \text{GL}_n(\mathbb{F})$ and $A \in M_n(\mathbb{F})$ we shall be interested in $\text{tr}(g^{-1}A)$. We use the notation of §2.3.1 and §2.3.2. By (2.7), we have

$$\text{tr}(g^{-1}A) = \text{tr}(g_\rho^{-1}A_\rho).$$

The inverse of an analogue of a Jordan block of order $d \cdot \ell$ is given by

$$\left(\left(\begin{pmatrix} D & I_d \\ & \ddots & I_d \\ & & D \end{pmatrix} \right)^{-1} \right)_{i,j} = \begin{cases} (-1)^{j-i} D^{-j+i-1}, & i \leq j \\ 0, & i > j, \end{cases} \quad (2.10)$$

for $0 \leq i, j \leq \ell$, where the LHS of (2.10) denotes the block matrix in the i -th row and j -th column. We have

$$\begin{aligned} \text{tr}(g_\rho^{-1} A_\rho) &= \sum_{m=0}^{d'-1} \text{tr}(D^{-1} A_{m,m} + D^{-2} \alpha_m(g, D^{-1}, A_\rho)) \\ &= \text{tr} \left(\sum_{m=0}^{d'-1} D^{-1} A_{m,m} \right) + \sum_{m=0}^{d'-1} \text{tr}(D^{-2} \alpha_m(g, D^{-1}, A_\rho)), \end{aligned} \quad (2.11)$$

where $\alpha_m(g, D^{-1}, A_\rho)$, for $0 \leq m \leq d' - 1$ are determined by the analogous Jordan form of g . Notice, that in case g is semisimple, then $\alpha_m(g, D^{-1}, A_\rho) = 0$ for all $0 \leq m \leq d' - 1$. Otherwise, for $0 \leq m \leq d' - 1$, $D^{-2} \alpha_m(g, D^{-1}, A_\rho)$ equals to a sum of terms of the form $(-1)^\ell D^{-\ell-1} A_{\ell,m}$, where $m < \ell \leq d' - 1$.

By (2.9) we have

$$D^{-1} A_{m,m} = \left((\lambda^{-1})^{q^r} \sigma^r(a_{m,i-r \pmod{d};m}) \right)_{1 \leq i, r \leq d-1}.$$

So the first sum in the RHS of (2.11) becomes

$$\sum_{m=0}^{d'-1} \sum_{r=0}^{d-1} (\lambda^{-1})^{q^r} \sigma^r(a_{m,0;m}) = \sum_{r=0}^{d-1} \sigma^r \left(\lambda^{-1} \sum_{m=0}^{d'-1} a_{m,0;m} \right) = \text{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot \sum_{m=0}^{d'-1} a_{m,0;m} \right).$$

On the other hand, for each $0 \leq m \leq d' - 1$, the term $\text{tr}(D^{-2} \alpha_m(g, D^{-1}, A_\rho))$ in (2.11) does not depend on the elements $a_{\ell,0;m}$, where $\ell = m$. Each such term depends only on λ and on $a_{\ell,i,m}$ where $\ell > m$. We summarize the above results in the following lemma.

Lemma 2.6. *In the above notations,*

$$\text{tr}(g^{-1} A) = \text{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot \sum_{m=0}^{d'-1} a_{m,0;m} \right) + \sum_{m=0}^{d'-1} \text{tr}(D^{-2} \alpha_m(g, D^{-1}, A_\rho)),$$

and each summand $\text{tr}(D^{-2} \alpha_m(g, D^{-1}, A_\rho))$ is independent of $a_{m,0;m}$ appearing in the first summand, for all $0 \leq m \leq d' - 1$.

In case $g = s$ is semisimple we have

$$\text{tr}(g^{-1} A) = \text{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot \sum_{m=0}^{d'-1} a_{m,0;m} \right).$$

2.4 q -hypergeometric identity

In order to calculate the dimension of $\pi_{k,N,\psi}$, we need a combinatorial identity related to ranks of triangular block matrices. We first prove a lemma that is a special case of a q -analogue of the Chu-Vandermonde identity, phrased in a manner that we use in the proof of the combinatorial identity. We recall the definition of the q -Pochhammer symbol:

$$(a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i).$$

Lemma 2.7. *Let $R_q(n, m, r)$ be the number of $n \times m$ matrices of rank r over the finite field of size q (n, m may be 0, with the convention that the empty matrix has rank 0). Let a be an integer greater or equal to $n + m$. Then*

$$\sum_{r \geq 0} R_q(n, m, r) (q; q)_{a-r} = q^{nm} \frac{(q; q)_{a-n} (q; q)_{a-m}}{(q; q)_{a-n-m}}.$$

Proof. We start by stating a q -analogue of the Chu-Vandermonde identity [GR04, Eq. (1.5.2)]:

$$\sum_{r=0}^i \frac{(q^{-i}; q)_r (b; q)_r}{(c; q)_r (q; q)_r} \left(\frac{cq^i}{b} \right)^r = \frac{(c/b; q)_i}{(c; q)_i},$$

where i is a non-negative integer, and b, c are complex numbers that satisfy $b \neq 0$ and $c \notin \{q^{-1}, \dots, q^{-(i-1)}\}$. Choosing $i = n$, $b = q^{-m}$, $c = q^{-a}$, we obtain

$$\sum_{r=0}^n \frac{(q^{-n}; q)_r (q^{-m}; q)_r}{(q^{-a}; q)_r (q; q)_r} q^{(n+m-a)r} = \frac{(q^{m-a}; q)_n}{(q^{-a}; q)_n}. \quad (2.12)$$

We have the following formula for $R_q(n, m, r)$ by Landsberg [Lan93]:

$$R_q(n, m, r) = \frac{(-1)^r (q^{-n}; q)_r (q^{-m}; q)_r q^{(n+m)r - \binom{r}{2}}}{(q; q)_r}.$$

By expressing the r -th summand of (2.12) as

$$\begin{aligned} & \frac{(-1)^r (q^{-n}; q)_r (q^{-m}; q)_r q^{(n+m)r - \binom{r}{2}}}{(q; q)_r} \cdot \frac{q^{-ar + \binom{r}{2}}}{(-1)^r (q^{-a}; q)_r} \\ &= R_q(n, m, r) \cdot \frac{q^{-ar + \binom{r}{2}}}{(-1)^r (q^{-a}; q)_r}, \end{aligned}$$

we obtain that

$$\sum_{r=0}^n R_q(n, m, r) \frac{q^{-ar + \binom{r}{2}} (-1)^r}{(q^{-a}; q)_r} = \frac{(q^{m-a}; q)_n}{(q^{-a}; q)_n}. \quad (2.13)$$

The proof is concluded by applying to (2.13) the simple identity

$$(q^{-x}; q)_y = (-1)^y q^{\binom{y}{2} - xy} \frac{(q; q)_x}{(q; q)_{x-y}}$$

with $(x, y) \in \{(a, n), (a - m, n), (a, r)\}$. □

We now state our main combinatorial identity needed for computing the dimension. Let k be a positive integer. We define the following family of functions.

$$f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right) = \sum_A (q; q)_{a-\text{rk}A}, \quad (2.14)$$

where $\{n_i\}_{i=1}^k, \{m_j\}_{j=1}^k$ are sequences of non-negative integers, a is an integer such that

$$a \geq \max\left\{\sum_{j=1}^i n_j + \sum_{j=i}^k m_j \mid 1 \leq i \leq k\right\} \quad (2.15)$$

and the sum is over all matrices $A \in M_{(\sum_{i=1}^k n_i) \times (\sum_{j=1}^k m_j)}(\mathbb{F})$ of the form

$$A = \begin{pmatrix} Y_{1,1} & Y_{1,2} & \cdots & Y_{1,k} \\ 0 & Y_{2,2} & \cdots & Y_{2,k} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Y_{k,k} \end{pmatrix}, \quad (2.16)$$

where $Y_{i,j} \in M_{n_i \times m_j}(\mathbb{F})$ for all $1 \leq i \leq j \leq k$.

Proposition 2.8. *Let $k \geq 1$. For any sequences of non-negative integers, $\{n_i\}_{i=1}^k$ and $\{m_j\}_{j=1}^k$, and for any integer a satisfying (2.15), we have*

$$f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right) = q^{1 \leq i \leq j \leq k} n_i m_j \cdot \frac{\prod_{i=0}^k (q; q)_{a - \sum_{j=1}^{k-i} n_j - \sum_{j=k-i+1}^k m_j}}{\prod_{i=1}^k (q; q)_{a - \sum_{j=1}^{k-i+1} n_j - \sum_{j=k-i+1}^k m_j}}. \quad (2.17)$$

Proof. We use the following notation:

$$I_{r,n,m} = \begin{pmatrix} I_r & 0_{m-r} \\ 0_{n-r} & 0 \end{pmatrix}, \quad (r \leq \min\{n, m\}). \quad (2.18)$$

We prove the proposition by induction on k . Let $k = 1$. Then

$$f_{1,q}\left(a; \begin{smallmatrix} n \\ m \end{smallmatrix}\right) = \sum_{A \in M_{n \times m}(\mathbb{F})} (q; q)_{a-\text{rk}A} = \sum_{r \geq 0} R_q(n, m, r) (q; q)_{a-r}.$$

By Lemma 2.7 we find that

$$f_1\left(a; \begin{smallmatrix} n \\ m \end{smallmatrix}\right) = q^{nm} \frac{(q; q)_{a-n} (q; q)_{a-m}}{(q; q)_{a-n-m}},$$

as needed. We now perform the induction step, i.e. assume that (2.17) holds for $k-1$ in place of k , and prove it for k . We split the sum defining $f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right)$ as follows:

$$f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right) = \sum_{\substack{Y_{i,i} \in M_{n_i \times m_i}(\mathbb{F}) \\ 1 \leq i \leq k}} \sum_{\substack{Y_{i,j} \in M_{n_i \times m_j}(\mathbb{F}) \\ 1 \leq i < j \leq k}} (q; q)_{a-\text{rk}A}. \quad (2.19)$$

In the inner sum of (2.19) the ranks of $Y_{i,i}$ are fixed for all $1 \leq i \leq k$, so we set $r_i = \text{rk}(Y_{i,i})$. There exist invertible matrices E_i, C_i such that $Y_{i,i} = E_i I_{r_i, n_i, m_i} C_i$, for all $1 \leq i \leq k$. So, one can write A in the inner sum of (2.19) as $\text{diag}(E_1, \dots, E_k) \cdot \tilde{A} \cdot \text{diag}(C_1, \dots, C_k)$, where

$$\tilde{A} = \begin{pmatrix} I_{r_1, n_1, m_1} & \tilde{Y}_{1,2} & \cdots & \tilde{Y}_{1,k} \\ 0 & I_{r_2, n_2, m_2} & \cdots & \tilde{Y}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{r_k, n_k, m_k} \end{pmatrix} \quad (2.20)$$

and $\tilde{Y}_{i,j} = E_i^{-1} Y_{i,j} C_j^{-1}$ for all $1 \leq i < j \leq k$. Together with the fact that rank is invariant under elementary operations, (2.19) becomes

$$f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right) = \sum_{\substack{\forall 1 \leq i \leq k: i=1 \\ r_i \geq 0}} \prod_{i=1}^k R_q(n_i, m_i, r_i) \sum_{\tilde{A}} (q; q)_{a - \text{rk} \tilde{A}}, \quad (2.21)$$

where the inner sum is over matrices $\tilde{A}$ of the form (2.20). We can use Gaussian elimination operations on $\tilde{Y}_{i,j}$ for all $1 \leq i < j \leq k$ (which do not affect the rank of $\tilde{A}$) as follows: the first r_i rows of each $\tilde{Y}_{i,j}$ are being canceled by the pivot elements in I_{r_i, n_i} (using elementary row operations) and the first r_j columns of each $\tilde{Y}_{i,j}$ are being canceled by the pivot elements in I_{r_j, n_j} (using elementary column operations). Formally, the composition of these elementary operations maps the sequence of matrices $\{\tilde{Y}_{i,j}\}_{1 \leq i < j \leq k}$ $\mathbb{F}$ -linearly to a sequence of matrices

$$\{\hat{Y}_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & Z_{i,j} \end{pmatrix}\}_{1 \leq i < j \leq k}, \quad (2.22)$$

where $Z_{i,j} \in M_{(n_i - r_i) \times (m_j - r_j)}(\mathbb{F})$. This linear map is a projection by construction. Its kernel is of size $q^{\sum_{t=1}^{k-1} r_t \sum_{\ell=t+1}^k m_\ell + \sum_{t=2}^k r_t \sum_{\ell=1}^{t-1} (n_\ell - r_\ell)}$. The dimension of the kernel corresponds to the number of elements which we canceled. Equation (2.21) becomes

$$f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right) = \sum_{\substack{\forall 1 \leq i \leq k: i=1 \\ r_i \geq 0}} \prod_{i=1}^k R_q(n_i, m_i, r_i) q^{\sum_{t=1}^{k-1} r_t \sum_{\ell=t+1}^k m_\ell + \sum_{t=2}^k r_t \sum_{\ell=1}^{t-1} (n_\ell - r_\ell)} \\ \cdot \sum_{\hat{A}} (q; q)_{a - \text{rk} \hat{A}}, \quad (2.23)$$

where the inner sum is over matrices of the form

$$\hat{A} = \begin{pmatrix} I_{r_1, n_1, m_1} & \hat{Y}_{1,2} & \cdots & \hat{Y}_{1,k} \\ 0 & I_{r_2, n_2, m_2} & \cdots & \hat{Y}_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{r_k, n_k, m_k} \end{pmatrix},$$

and $\widehat{Y}_{i,j}$ are as defined in (2.22). Note that $\text{rk}\widehat{A} = \sum_{j=1}^k r_j + \text{rk}Z$, where

$$Z = \begin{pmatrix} Z_{1,2} & \cdots & Z_{1,k} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Z_{k-1,k} \end{pmatrix}.$$

Hence, from (2.23) we obtain the following recursive relation:

$$\begin{aligned} f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right) &= \sum_{\substack{\forall 1 \leq i \leq k: \\ r_i \geq 0}} \prod_{i=1}^k R_q(n_i, m_i, r_i) q^{\sum_{t=1}^{k-1} r_t \sum_{\ell=t+1}^k m_\ell + \sum_{t=2}^k r_t \sum_{\ell=1}^{t-1} (n_\ell - r_\ell)} \\ &\quad \cdot f_{k-1,q}\left(a - \sum_{j=1}^k r_j; \begin{smallmatrix} n_1 - r_1, \dots, n_{k-1} - r_{k-1} \\ m_2 - r_2, \dots, m_k - r_k \end{smallmatrix}\right). \end{aligned} \quad (2.24)$$

Plugging the induction assumption in (2.24) we get that $f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right)$ equals

$$\begin{aligned} &\sum_{\substack{\forall 1 \leq i \leq k: \\ r_i \geq 0}} \prod_{i=1}^k R_q(n_i, m_i, r_i) q^{\sum_{t=1}^{k-1} r_t \sum_{\ell=t+1}^k m_\ell + \sum_{t=2}^k r_t \sum_{\ell=1}^{t-1} (n_\ell - r_\ell)} \\ &\quad \cdot q^{\sum_{1 \leq i \leq j \leq k-1} (n_i - r_i) \cdot (m_{j+1} - r_{j+1})} \cdot \frac{\prod_{i=0}^{k-1} (q; q)_{a - \sum_{j=1}^k r_j - \sum_{j=1}^{k-1-i} (n_j - r_j) - \sum_{j=k-i}^{k-1} (m_{j+1} - r_{j+1})}}{\prod_{i=1}^{k-1} (q; q)_{a - \sum_{j=1}^k r_j - \sum_{j=1}^{k-i} (n_j - r_j) - \sum_{j=k-i}^{k-1} (m_{j+1} - r_{j+1})}}. \end{aligned} \quad (2.25)$$

Rearranging (2.25), we see that the sum over $r_1, \dots, r_k$ may be written as a product over k sums, where the i -th sum is over r_i :

$$\begin{aligned} f_{k,q}\left(a; \begin{smallmatrix} n_1, \dots, n_k \\ m_1, \dots, m_k \end{smallmatrix}\right) &= \frac{q^{\sum_{1 \leq i \leq j \leq k-1} n_i m_{j+1}}}{\prod_{i=1}^{k-1} (q; q)_{a - \sum_{j=1}^{k-i} n_j - \sum_{j=k-i}^{k-1} m_{j+1}}} \\ &\quad \cdot \prod_{i=1}^k \left(\sum_{r_i \geq 0} R_q(n_i, m_i, r_i) (q; q)_{a - r_i - \sum_{j=1}^{i-1} n_j - \sum_{j=i}^{k-1} m_{j+1}} \right). \end{aligned} \quad (2.26)$$

Using Lemma 2.7 we substitute each inner sum of (2.26) with

$$q^{n_i \cdot m_i} \frac{(q; q)_{a - \sum_{j=1}^i n_j - \sum_{j=i}^{k-1} m_{j+1}} (q; q)_{s - \sum_{j=1}^{i-1} n_j - \sum_{j=i-1}^{k-1} m_{j+1}}}{(q; q)_{a - \sum_{j=1}^i n_j - \sum_{j=i-1}^{k-1} m_{j+1}}},$$

and by simplifying we complete the induction step and obtain the desired identity. $\square$

Remark 2.9. Solomon [Sol90] proved a relation between the following two quantities: the number of placements of k non-attacking rooks on a $n \times n$ chessboard, counted with certain weights depending on q , and the number of matrices in $M_{n \times n}(\mathbb{F})$ of rank k . Haglund generalized Solomon's result to any "Ferrers board" [Hag98, Thm. 1], which means that the number of matrices of the form (2.16) over $\mathbb{F}$ of rank k is related to the q -rook polynomial $R_k(B, q)$, where B is a certain Ferrers board associated with (2.16). For the definition of a Ferrers board and $R_k(B, q)$, see the introduction to the paper by Garsia and Remmel [GR86]. In particular, Proposition 2.8 may be deduced from a result of Garcia and Remmel on q -rook polynomials, see [Hag98, Cor. 2]. Our proof of Proposition 2.8 is direct and so we believe it is more accessible. More importantly, the ideas used in the proof reappear in the proofs of Theorem 2 and Theorem 3.

2.5 Arithmetic properties of certain polynomials

For any d dividing n and any $k \geq 2$, let

$$a_{k;n,d}(x) = \frac{x^d - 1}{x^n - 1} \sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) (-1)^{k(n-\frac{n}{m})} x^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} \in \mathbb{Q}(x), \quad (2.27)$$

where $\mu : \mathbb{N} \rightarrow \mathbb{C}$ is the Möbius function, defined by $\mu(1) = 1$ and

$$\mu(n) = \begin{cases} 0 & \text{if } p^2 \mid n \text{ for some prime } p, \\ (-1)^m & \text{if } n = p_1 p_2 \dots p_m, \text{ where } p_i \text{ are distinct primes.} \end{cases}$$

We recall the following properties of μ [IR90, Ch. 2].

- The divisor sum $\sum_{d|n} \mu(d)$ is given by

$$\sum_{d|n} \mu(d) = \delta_{1,n}. \quad (2.28)$$

- The Möbius function is multiplicative.

Lemma 2.10. *Let $k \geq 2$. The following hold.*

- (I) *For any $d \mid n$, $a_{k;n,d}(x)$ is a polynomial in $\mathbb{Z}[x]$. Furthermore, in case $d \notin \{n, \frac{n}{2}\}$, $a_{k;n,d}(x)$ is divisible by $x^d - 1$. In the remaining cases we have*

$$a_{k;n,d}(x) = \begin{cases} (-1)^{k(n-1)} & \text{if } d = n, \\ \frac{x^{\frac{(k-2)n}{2}} + (-1)^{k+1}}{x^{\frac{n}{2}} + 1} & \text{if } d = \frac{n}{2}. \end{cases} \quad (2.29)$$

- (II) *If $k > 2$ we have $\deg(a_{k;n,d}) = \frac{(n(k-2)-2d)(n-d)}{2d}$, and $a_{k;n,d}$ has leading coefficient $(-1)^{k(n-\frac{n}{d})}$. If $k = 2$, we have $a_{k;n,d} = \delta_{n,d}$.*

- (III) *Assume $k > 2$. For any prime power q , $a_{k;n,d}(q)$ is a non-zero integer. Its sign equals the sign of $(-1)^{k(n-\frac{n}{d})}$, i.e. it is a positive integer unless k is odd, n is even and $2 \nmid \frac{n}{d}$.*

Proof. We begin by proving the first part of the lemma. If $d \in \{n, \frac{n}{2}\}$, a short calculation reveals that (2.29) holds. From now on we assume that $d \notin \{n, \frac{n}{2}\}$. We shall show that

$$x^n - 1 \mid \sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) (-1)^{k(n-\frac{n}{m})} x^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} \quad (2.30)$$

in $\mathbb{Q}[x]$, which implies that $a_{k;n,d}(x)$ is a polynomial divisible by $x^d - 1$. Gauss's lemma, applied to (2.30), implies that $a_{k;n,d}(x) \in \mathbb{Z}[x]$. We now prove (2.30).

Let z be a root of unity of order dividing n . Assume first that n is odd or that k is even. Then for all $m \mid n$ we have

$$z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = (z^n)^{(k-2)\frac{n-1}{2}} = 1.$$

Hence, using (2.28),

$$\sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) (-1)^{k(n-\frac{n}{m})} z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = \sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) = \sum_{a: a|\frac{n}{d}} \mu(a) = \delta_{d,n} = 0. \quad (2.31)$$

Now we assume instead that n is even and k is odd. We are led to consider two cases.

- If $z^{\frac{n}{2}} = -1$ then for all $m \mid n$ we have,

$$z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = (-1)^{\frac{n}{m}-1}.$$

Hence, using (2.28),

$$\sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) (-1)^{k(n-\frac{n}{m})} z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = - \sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) = - \sum_{a|\frac{n}{d}} \mu(a) = -\delta_{d,n} = 0. \quad (2.32)$$

- If $z^{\frac{n}{2}} = 1$ then for all $m \mid n$ we have,

$$z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} = 1.$$

Hence,

$$\begin{aligned} \sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) (-1)^{k(n-\frac{n}{m})} z^{(k-2)\frac{n}{2}(\frac{n}{m}-1)} &= \sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) (-1)^{\frac{n}{m}} = \sum_{a|\frac{n}{d}} \mu(a) (-1)^{\frac{n}{ad}} \\ &= \sum_{\substack{a|\frac{n}{d} \\ 2|\frac{n}{ad}}} \mu(a) - \sum_{\substack{a|\frac{n}{d} \\ 2 \nmid \frac{n}{ad}}} \mu(a) \\ &= \begin{cases} 0 - \sum_{a|\frac{n}{d}} \mu(a) & \text{if } 2 \nmid \frac{n}{d} \\ \sum_{a|\frac{n}{2d}} \mu(a) - \sum_{\substack{a|\frac{n}{d} \\ 2|a}} \mu(2 \cdot \frac{a}{2}) & \text{if } 2 \mid \frac{n}{d}, 4 \nmid \frac{n}{d} \\ \sum_{a|\frac{n}{2d}} \mu(a) - \sum_{\substack{a|\frac{n}{d} \\ 2 \nmid \frac{n}{ad}}} \mu(4 \cdot \frac{a}{4}) & \text{if } 4 \mid \frac{n}{d} \end{cases} \\ &= \begin{cases} -\delta_{d,n} & \text{if } 2 \nmid \frac{n}{d} \\ \delta_{2d,n} - \mu(2)\delta_{2d,n} & \text{if } 2 \mid \frac{n}{d}, 4 \nmid \frac{n}{d} \\ \delta_{2d,n} & \text{if } 4 \mid \frac{n}{d} \end{cases} \\ &= 0. \end{aligned} \quad (2.33)$$

Equations (2.31), (2.32) and (2.33) show that the RHS of (2.30) vanishes on each root of the separable polynomial $x^n - 1$, which establishes (2.30). This concludes the proof of the first part of the lemma.

The second part of the lemma for $k > 2$ follows from the observation that the numerator of $a_{k;n,d}(x)$ has degree $d + (k-2)\frac{n}{2}(\frac{n}{d} - 1)$ (arising from the term corresponding to $m = d$) and leading coefficient equal to $(-1)^{k(n-\frac{n}{d})}$, while the denominator of $a_{k;n,d}(x)$ has degree n and leading coefficient equal to 1.

When $k = 2$, all terms in the sum in (2.27) are constants, and we have

$$a_{2;n,d}(x) = \frac{x^d - 1}{x^n - 1} \sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) = \frac{x^d - 1}{x^n - 1} \delta_{n,d} = \delta_{n,d}.$$

We now turn to the third part of the lemma. Since $a_{k;n,d}(x)$ has integer coefficients, $a_{k;n,d}(q)$ is an integer. We now determine its sign when $k > 2$, and in particular show that it is non-zero.

Since $q^d - 1$, $q^n - 1$, $q^{\frac{n}{2}}$ are positive, we deal with the expression

$$\begin{aligned} \tilde{a}_{k;n,d}(q) &:= \frac{q^n - 1}{q^d - 1} q^{(k-2)\frac{n}{2}} \cdot a_{k;n,d}(q) \\ &= \sum_{m: d|m|n} \mu\left(\frac{m}{d}\right) (-1)^{k(n-\frac{n}{m})} (q^{(k-2)\frac{n}{2}})^{\frac{n}{m}} \\ &= \sum_{a|\frac{n}{d}} \mu(a) (-1)^{k(n-\frac{n}{ad})} (q^{(k-2)\frac{n}{2}})^{\frac{n}{ad}}, \end{aligned}$$

whose sign is the same as the sign of $a_{k;n,d}(q)$. If $d = n$ then

$$(-1)^{k(n-\frac{n}{d})} \tilde{a}_{k;n,d}(q) = q^{(k-2)\frac{n}{2}} > 0.$$

If $d = \frac{n}{2}$ then

$$(-1)^{k(n-\frac{n}{d})} \tilde{a}_{k;n,d}(q) = (q^{(k-2)\frac{n}{2}})^2 + (-1)^{k+1} q^{(k-2)\frac{n}{2}} > 0.$$

If $\frac{n}{d} \geq 3$, we set $t = q^{(k-2)\frac{n}{2}}$. Then, $t \geq 2^{\frac{3}{2}} > 2$ and

$$\begin{aligned} (-1)^{k(n-\frac{n}{d})} \tilde{a}_{k;n,d}(q) &\geq (q^{(k-2)\frac{n}{2}})^{\frac{n}{d}} - \sum_{1 \leq i \leq \frac{n}{2d}} (q^{(k-2)\frac{n}{2}})^i \geq (q^{(k-2)\frac{n}{2}})^{\frac{n}{d}} - \frac{(q^{(k-2)\frac{n}{2}})^{\frac{n}{d}}}{1 - q^{-(k-2)\frac{n}{2}}} \\ &= (q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}} \left((q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}} - \frac{1}{1 - q^{-(k-2)\frac{n}{2}}} \right) \\ &\geq (q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}} \left((q^{(k-2)\frac{n}{2}})^{\frac{3}{2}} - \frac{1}{1 - q^{-(k-2)\frac{n}{2}}} \right) \\ &= \frac{(q^{(k-2)\frac{n}{2}})^{\frac{n}{2d}}}{1 - q^{-(k-2)\frac{n}{2}}} \left(t^{\frac{1}{2}}(t - 1) - 1 \right) > 0. \end{aligned}$$

□

Remark 2.11. The polynomials $a_{k;n,d}(x)$ may be expressed using the necklace polynomials (see Moreau [Mor72]), defined by

$$M_n(x) = \frac{1}{n} \sum_{d|n} \mu(d) x^{\frac{n}{d}}.$$

Indeed,

$$a_{k;n,d}(x) = \frac{x^d - 1}{x^n - 1} \cdot \left(\frac{(-1)^n}{x^{\frac{n}{2}}} \right)^{k-2} \cdot M_{\frac{n}{d}} \left((-x^{\frac{n}{2}})^{k-2} \right).$$

3 Calculation of the Dimension of $\pi_{k,N,\psi}$

Here we prove Theorem 2. Recall that Θ_θ is the character of the irreducible cuspidal representation π_θ associated to a regular character θ of $\mathbb{F}_n^*$. Given $U \in N$, we write it in the notation of (1.1). From (1.2),

$$\begin{aligned} \dim(\pi_{k,N,\psi}) &= \frac{1}{|N|} \sum_{U \in N} \Theta_\theta(U) \bar{\psi}(U) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta(U) \bar{\psi}(U) \\ &= \frac{1}{q^{\binom{k}{2}n^2}} \sum_{\substack{X_{i,i} \in M_n(\mathbb{F}) \\ 1 \leq i \leq k-1}} \sum_{\substack{X_{i,j} \in M_n(\mathbb{F}) \\ 1 \leq i < j \leq k-1}} \Theta_\theta(U) \bar{\psi}(U). \end{aligned}$$

The character $\psi(U) = \psi(X_{1,1}, \dots, X_{k-1,k-1})$ is determined by the traces of $X_{i,i}$, $1 \leq i \leq k-1$. Hence,

$$\dim(\pi_{k,N,\psi}) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{\substack{X_{i,i} \in M_n(\mathbb{F}) \\ 1 \leq i \leq k-1}} \bar{\psi}(U) \sum_{\substack{X_{i,j} \in M_n(\mathbb{F}) \\ 1 \leq i < j \leq k-1}} \Theta_\theta(U). \quad (3.1)$$

By Corollary 2.2, the value $\Theta_\theta(U)$ is determined by $\dim_{\mathbb{F}_{kn}} \ker(U - I)$ which is in turn determined by $\text{rank}_{\mathbb{F}_{kn}}(U - I)$. In the inner sum of (3.1) set $r_i = \text{rk}(X_{i,i})$ for $1 \leq i \leq k-1$. We write $I_{r,n} := I_{r,n,n}$ as defined in (2.18). There exist invertible matrices E_i, C_{i+1} such that $X_{i,i} = E_i I_{r_i,n} C_{i+1}$. So, one can write U in the inner sum of (3.1) as I_{kn} plus

$$\text{diag}(E_1, \dots, E_{k-1}, I_n) \begin{pmatrix} 0 & I_{r_1,n} & \cdots & \tilde{X}_{1,k-2} & \tilde{X}_{1,k-1} \\ 0 & 0 & \cdots & \tilde{X}_{2,k-2} & \tilde{X}_{2,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{r_{k-1},n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \text{diag}(I_n, C_2, \dots, C_k),$$

where $\tilde{X}_{i,j} = E_i^{-1} X_{i,j} C_{j+1}^{-1}$ for all $1 \leq i < j \leq k-1$. Together with the fact that rank is invariant under elementary operations, we now have

$$\begin{aligned} \dim(\pi_{k,N,\psi}) &= \frac{1}{q^{\binom{k}{2}n^2}} \sum_{\substack{X_{i,i} \in M_n(\mathbb{F}) \\ 1 \leq i \leq k-1}} \bar{\psi}(U) \\ &\quad \cdot \sum_{\substack{\tilde{X}_{i,j} \in M_n(\mathbb{F}) \\ 1 \leq i < j \leq k-1}} \Theta_\theta \left(I_{kn} + \begin{pmatrix} 0 & I_{r_1,n} & \cdots & \tilde{X}_{1,k-2} & \tilde{X}_{1,k-1} \\ 0 & 0 & \cdots & \tilde{X}_{2,k-2} & \tilde{X}_{2,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{r_{k-1},n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \right). \end{aligned} \quad (3.2)$$

As in the proof of Proposition 2.8, we can use Gaussian elimination operations on $\tilde{X}_{i,j}$ for all $1 \leq i < j \leq k-1$ (which do not affect the rank nor dimension

of the kernel of the matrix minus I_{kn} , and the number of Jordan blocks is not affected as well) in such a way that the sequence of matrices $\{\tilde{X}_{i,j}\}_{1 \leq i < j \leq k-1}$ is mapped $\mathbb{F}$ -linearly to a sequence of matrices

$$\{\hat{\tilde{X}}_{i,j} = \begin{pmatrix} 0 & 0 \\ 0 & Y_{i,j} \end{pmatrix}\}_{1 \leq i < j \leq k-1},$$

where $Y_{i,j} \in M_{(n-r_i) \times (n-r_j)}(\mathbb{F})$. The kernel of this mapping is of size $q^{\sum_{i=1}^{k-2} r_i(k-i-1)n + \sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1} (n-r_j)}$. The dimension of the kernel corresponds to the number of elements which we cancel. Equation (3.2) becomes

$$\dim(\pi_{k,N,\psi}) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{\substack{X_{i,i} \in M_n(\mathbb{F}) \\ 1 \leq i \leq k-1}} \bar{\psi}(U) q^{\sum_{i=1}^{k-2} r_i(k-i-1)n + \sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1} (n-r_j)} \cdot \sum_{\substack{Y_{i,j} \in M_{(n-r_i) \times (n-r_j)}(\mathbb{F}) \\ 1 \leq i < j \leq k-1}} \Theta_\theta(g), \quad (3.3)$$

where

$$g = I_{kn} + \begin{pmatrix} 0 & I_{r_1,n} & \cdots & \hat{\tilde{X}}_{1,k-2} & \hat{\tilde{X}}_{1,k-1} \\ 0 & 0 & \cdots & \hat{\tilde{X}}_{2,k-2} & \hat{\tilde{X}}_{2,k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_{r_{k-1},n} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Using the character formula (2.1), we can calculate $\Theta_\theta(g)$. In this case $m = kn$, $g = s \cdot u$ where $s = I_{kn}$, so $\lambda = 1$ and

$$t = \dim \ker(g - I) = kn - \text{rk}(g - I) = kn - \sum_{i=1}^{k-1} r_i - \text{rk} A,$$

where

$$A = \begin{pmatrix} Y_{1,2} & \cdots & Y_{1,k-1} \\ \vdots & & \vdots \\ 0 & \cdots & Y_{k-2,k-1} \end{pmatrix}, \quad 1 \leq i < j \leq k-1. \quad (3.4)$$

So,

$$\begin{aligned} \Theta_\theta(g) &= (-1)^{kn-1} (1-q)(1-q^2) \cdots (1-q^{kn - \sum_{i=1}^{k-1} r_i - \text{rk} A - 1}) \\ &= (-1)^{kn-1} (q; q)_{kn - \sum_{i=1}^{k-1} r_i - \text{rk} A - 1}. \end{aligned}$$

Equation (3.3) can now be written as

$$\dim(\pi_{k,N,\psi}) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{\substack{X_{i,i} \in M_n(\mathbb{F}) \\ 1 \leq i \leq k-1}} \bar{\psi}(U) q^{\sum_{i=1}^{k-2} r_i(k-i-1)n + \sum_{i=2}^{k-1} r_i \sum_{j=1}^{i-1} (n-r_j)} \cdot (-1)^{kn-1} \sum_A (q; q)_{kn - \sum_{i=1}^{k-1} r_i - \text{rk} A - 1}, \quad (3.5)$$

where the inner sum is over all matrices of the form (3.4) and by the definition (2.14) it is equal to

$$f_{k-2,q}\left(kn - \sum_{i=1}^{k-1} r_i - 1; \begin{smallmatrix} n-r_1, \dots, n-r_{k-2} \\ n-r_2, \dots, n-r_{k-1} \end{smallmatrix}\right).$$

By applying Proposition 2.8 we replace the inner sum in (3.5) by

$$q^{\sum_{1 \leq i \leq j \leq k-2} (n-r_i) \cdot (n-r_{j+1})} \cdot \frac{\prod_{i=0}^{k-2} (q; q)_{kn - \sum_{j=1}^{k-1} r_j - 1 - \sum_{j=1}^{k-2-i} (n-r_j) - \sum_{j=k-i-1}^{k-2} (n-r_{j+1})}}{\prod_{i=1}^{k-2} (q; q)_{kn - \sum_{j=1}^{k-1} r_j - 1 - \sum_{j=1}^{k-i-1} (n-r_j) - \sum_{j=k-i-1}^{k-2} (n-r_{j+1})}},$$

which equals

$$q^{\sum_{1 \leq i \leq j \leq k-2} (n-r_i) \cdot (n-r_{j+1})} \cdot \frac{\prod_{i=1}^{k-1} (q; q)_{2n-1-r_i}}{((q; q)_{n-1})^{(k-2)}}.$$

Now (3.5) becomes

$$\dim(\pi_{k,N,\psi}) = \frac{(-1)^{kn-1}}{((q; q)_{n-1})^{(k-2)} q^{(k-1)n^2}} \sum_{\substack{X_{i,i} \in M_n(\mathbb{F}) \\ 1 \leq i \leq k-1}} \prod_{i=1}^{k-1} \overline{\psi_0}(\text{tr}(X_{i,i})) (q; q)_{2n-1-r_i}. \quad (3.6)$$

Changing the order of sum and product in (3.6) we get that

$$\dim(\pi_{k,N,\psi}) = \frac{(-1)^{kn-1}}{((q; q)_{n-1})^{(k-2)} q^{(k-1)n^2}} \prod_{i=1}^{k-1} \sum_{X_{i,i} \in M_n(\mathbb{F})} \overline{\psi_0}(\text{tr}(X_{i,i})) (q; q)_{2n-1-r_i}. \quad (3.7)$$

From Section 5 of [Pra00], each inner sum in (3.7) is equal to

$$\sum_{X_{i,i} \in M_n(\mathbb{F})} \overline{\psi_0}(\text{tr}(X_{i,i})) (q; q)_{2n-1-r_i} = (-1)^n \cdot q^{n^2} \cdot q^{\binom{n}{2}} (q; q)_{n-1}. \quad (3.8)$$

Plugging (3.8) in (3.7), we obtain

$$\dim(\pi_{k,N,\psi}) = q^{(k-1)\binom{n}{2}} (-1)^{n-1} (q; q)_{n-1} = q^{(k-2)\binom{n}{2}} \frac{|\text{GL}_n(\mathbb{F})|}{q^n - 1},$$

as needed. $\square$

4 Calculation of the Character $\Theta_{k,N,\psi}$

In this section we prove Theorem 3. Namely, we calculate $\Theta_{k,N,\psi}$. From now on we use the following notations:

$$h_{g;U} = \begin{pmatrix} g & X_{1,1} & X_{1,2} & \cdots & X_{1,k-2} & X_{1,k-1} \\ 0 & g & X_{2,2} & \cdots & X_{2,k-2} & X_{2,k-1} \\ 0 & 0 & g & \cdots & X_{3,k-2} & X_{3,k-1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & g & X_{k-1,k-1} \\ 0 & 0 & 0 & \cdots & 0 & g \end{pmatrix},$$

where U (and so $X_{i,j}$) is as in (1.1). Note that $h_{I_n, U} = U$. We also define

$$\Delta^r(g) = \text{diag}(g, \dots, g) \in \Delta^r(\text{GL}_n(\mathbb{F})), \quad g \in \text{GL}_n(\mathbb{F}).$$

By definition,

$$\begin{aligned} \Theta_{k,N,\psi}(g) &= \text{tr}(\pi_{k,N,\psi}(g)) = \text{tr}(\pi(\Delta^k(g)) \upharpoonright_{V_{k,N,\psi}}) \\ &= \text{tr}(\pi(\Delta^k(g)) \circ P_{k,N,\psi}). \end{aligned} \quad (4.1)$$

Substituting (1.2) into (4.1) we have

$$\begin{aligned} \Theta_{k,N,\psi}(g) &= \text{tr} \left(\frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \pi[\Delta^k(g) \cdot U] \bar{\psi}(U) \right) \\ &= \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \text{tr}(\pi[\Delta^k(g) \cdot U]) \bar{\psi}(U). \end{aligned} \quad (4.2)$$

Now we perform the change of variables

$$X_{i,j} \mapsto g^{-1}X_{i,j}, \quad 1 \leq i \leq j \leq k-1$$

in (4.2) and obtain

$$\Theta_{k,N,\psi}(g) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta(h_{g;U}) \bar{\psi}(g^{-1}X_{1,1}, \dots, g^{-1}X_{k-1,k-1}). \quad (4.3)$$

In parts §4.1, §4.2 and §4.3 we prove parts (I), (II) and (III) of Theorem 3, respectively.

4.1 Character at $g = s \cdot u$ such that the semisimple part s does not come from $\mathbb{F}_n$

Let $g = s \cdot u$. Assume that the semisimple part s does not come from $\mathbb{F}_n$. The semisimple part of $h_{g;U}$ is $\Delta^k(s)$, which also does not come from $\mathbb{F}_n$. By Theorem 2.1, we have $\Theta_\theta(h_{g;U}) = 0$. Hence, by (4.3) $\Theta_{k,N,\psi}(g) = 0$. $\square$

4.2 Character calculation at a non-semisimple element

Assume that s comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ and $d \mid n$ is minimal. In addition, $d < n$ since g is not semisimple. Let $\lambda \in \mathbb{F}_d^*$ be an eigenvalue of s which generates the field $\mathbb{F}_d$ over $\mathbb{F}$. We use the notation of §2.3. Thus, there exist $R \in \text{GL}_n(\mathbb{F})$ and ρ a partition of $d' = n/d$ such that $R^{-1}gR = L_\rho(f)$. There exists $\Delta^{d'}(T) \in \text{GL}_n(\mathbb{F}_d)$ such that

$$g_\rho = \Delta^{d'}(T^{-1}) R^{-1}gR \Delta^{d'}(T),$$

the analogue of the Jordan form of g . Recall that by Lemma 2.5, the map

$$A \mapsto A_\rho := A_{\rho,R} = \Delta^{d'}(T^{-1}) R^{-1}AR \Delta^{d'}(T)$$

induces an isomorphism. By the notation of §2.3.2 we have for each

$$X_{a,b}, \quad \forall 1 \leq a \leq b \leq k-1,$$

the corresponding isomorphism of Lemma 2.5

$$X_{a,b} \mapsto (X_{a,b})_\rho = \begin{pmatrix} \left(x_{0,i;j}^{(a,b)} \right)_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} \\ \vdots \\ \left(x_{d'-1,i;j}^{(a,b)} \right)_{\substack{0 \leq i \leq d-1 \\ 0 \leq j \leq d'-1}} \end{pmatrix}.$$

Note that

$$\Delta^k \left(\Delta^{d'} (T^{-1}) \right) \Delta^k (R^{-1}) h_{g;U} \Delta^k (R) \Delta^k \left(\Delta^{d'} (T) \right) = h_{g_\rho;U_\rho}, \quad (4.4)$$

where U_ρ is the element of N with $(X_{a,b})_\rho$ instead of $X_{a,b}$. From (4.4) we obtain

$$\text{rk} (h_{g-\lambda I_n;U}) = \text{rk} (h_{g_\rho-\lambda I_n;U_\rho}).$$

We prove that $\text{rk} (h_{g-\lambda I_n;U})$ (which by Corollary 2.2 determines the value of $\Theta_\theta (h_{g;U})$) is independent of $x_{1,0;1}^{(k-1,k-1)} \in \mathbb{F}_d$. The matrix $h_{g_\rho-\lambda I_n;U_\rho}$ is of the form

$$h_{g_\rho-\lambda I_n;U_\rho} = \begin{pmatrix} g_\rho - \lambda I_n & (X_{1,1})_\rho & \cdots & (X_{1,k-2})_\rho & (X_{1,k-1})_\rho \\ 0 & g_\rho - \lambda I_n & \cdots & (X_{2,k-2})_\rho & (X_{2,k-1})_\rho \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & g_\rho - \lambda I_n & (X_{k-1,k-1})_\rho \\ 0 & 0 & \cdots & 0 & \boxed{g_\rho - \lambda I_n} \end{pmatrix}. \quad (4.5)$$

Consider the boxed block in (4.5). The $2d \times 2d$ upper left block of the boxed matrix $g_\rho - \lambda I_n$ is of the form

$$\begin{pmatrix} 0 & & & \boxed{1} & & \\ & \lambda^q - \lambda & & & 1 & \\ & & \ddots & & & 1 \\ & & & \lambda^{q^{d-1}} - \lambda & & 1 \\ & & & 0 & & \\ & & & & \lambda^q - \lambda & \\ & & & & & \ddots \\ & & & & & & \lambda^{q^{d-1}} - \lambda \end{pmatrix} \quad (4.6)$$

Let $Z := X_{k-1,k-1}$, $Z_\rho := (X_{k-1,k-1})_\rho$ and $z_{m,i;j} := x_{m,i;j}^{(k-1,k-1)}$. One can eliminate the $(d+1)$ -th column in Z_ρ by the boxed 1 from (4.6), i.e. all the elements $\{z_{m,i;1}\}_{\substack{0 \leq i \leq d-1 \\ 0 \leq m \leq d'-1}}$. In particular, $z_{1,0;1} = x_{1,0;1}^{(k-1,k-1)}$ is eliminated. Now, by Lemma 2.6, (4.3) can be written as

$$\begin{aligned} \Theta_{k,N,\psi}(g) &= \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta (h_{g;U}) \cdot \prod_{i=1}^{k-2} \overline{\psi}_0 (g^{-1} X_{i,i}) \\ &\quad \cdot \overline{\psi}_0 \left(\text{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot \sum_{m=0}^{d'-1} z_{m,0;m} \right) + \text{tr} (D^{-2} \alpha (g, D^{-1}, Z_\rho)) \right). \end{aligned} \quad (4.7)$$

By Lemma 2.5, going over $Z \in M_n(\mathbb{F})$ is equivalent to going over $(z_{m,i;j})_{\substack{0 \leq i \leq d-1 \\ 0 \leq j, m \leq d'-1}}$, $z_{m,i;j} \in \mathbb{F}_d$. We have just shown that $\Theta_\theta(h_{g;U})$ is independent of $z_{1,0;1}$, and by Lemma 2.6 $\text{tr}(D^{-2}\alpha(g, D^{-1}, Z_\rho))$ in (4.7) is also independent of $z_{1,0;1}$. Thus, we may write (4.7) as the following double sum, where the inner sum is over $z_{1,0;1}$ and the outer sum is over the rest of the coordinates of U :

$$\begin{aligned} \Theta_{k,N,\psi}(g) &= \frac{1}{q^{\binom{k}{2}n^2}} \sum_{\substack{X_{i,j} \in N, (i,j) \neq (k-1, k-1) \\ z_{m,i;j} \in \mathbb{F}_d, (m,i,j) \neq (1,0,1)}} \Theta_\theta(h_{g;U}) \cdot \prod_{i=1}^{k-2} \overline{\psi}_0(g^{-1}X_{i,i}) \\ &\quad \cdot \overline{\psi}_0(\text{tr}(D^{-2}\alpha(g, D^{-1}, Z_\rho))) \cdot \overline{\psi}_0 \left(\text{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot \sum_{\substack{0 \leq m \leq d'-1 \\ m \neq 1}} z_{m,0;m} \right) \right) \\ &\quad \cdot \sum_{z_{1,0;1} \in \mathbb{F}_d} \overline{\psi}_0(\text{Tr}_{\mathbb{F}_d/\mathbb{F}}(\lambda^{-1} \cdot z_{1,0;1})). \end{aligned}$$

Since $\overline{\psi}_0 \circ \text{Tr}_{\mathbb{F}_d/\mathbb{F}}$ is a nontrivial character, we have

$$\sum_{z_{1,0;1} \in \mathbb{F}_d} \overline{\psi}_0(\text{Tr}_{\mathbb{F}_d/\mathbb{F}}(\lambda^{-1} \cdot z_{1,0;1})) = 0.$$

Thus, $\Theta_{k,N,\psi}(g) = 0$. □

4.3 Character calculation at a semisimple element

Here we use (4.3) to calculate the value of $\Theta_{k,N,\psi}(g)$ for $g = s$ where s is semisimple element which comes from a subfield of $\mathbb{F}_n$ ($u = I_n$). Again, we use the notation of §2.3. Thus, there exist $R \in \text{GL}_n(\mathbb{F})$, ρ a partition of n/d and $\Delta^{d'}(T) \in \text{GL}_n(\mathbb{F}_d)$ such that

$$s_\rho = \Delta^{d'}(T^{-1}) R^{-1} s R \Delta^{d'}(T), \quad (4.8)$$

the analogue of the Jordan form of s . We also use the notation of §2.3.2, and in particular define $(X_{a,b})_\rho$ as in §4.2.

Let $\lambda \in \mathbb{F}_n^*$ be an eigenvalue of s . If $\lambda \in \mathbb{F}^*$ then $s = \lambda I$, and we have by (4.3)

$$\Theta_{k,N,\psi}(\lambda I) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta(h_{\lambda I;U}) \overline{\psi}(\lambda^{-1}X_{1,1}, \dots, \lambda^{-1}X_{k-1,k-1}).$$

By the change of variables

$$X_{i,j} \mapsto \lambda X_{i,j},$$

we get

$$\Theta_{k,N,\psi}(\lambda I) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta(\lambda h_{I;U}) \overline{\psi}(X_{1,1}, \dots, X_{k-1,k-1}).$$

By Theorem 2.1, we have $\Theta_\theta(\lambda \cdot h_{I;U}) = \theta(\lambda) \Theta_\theta(h_{I;U})$, and so

$$\Theta_{k,N,\psi}(\lambda I) = \theta(\lambda) \Theta_{k,N,\psi}(I) = \theta(\lambda) \dim(\pi_{k,N,\psi}).$$

By Theorem 2, this proves the case $\lambda \in \mathbb{F}^*$.

If $\lambda \in \mathbb{F}_d^* \subseteq \mathbb{F}_n^*$ is an eigenvalue of s and $1 < d \mid n$ is such that $\mathbb{F}_d$ is generated by λ over $\mathbb{F}$, we have by (4.3)

$$\Theta_{k,N,\psi}(s) = \frac{1}{q^{\binom{k}{2}n^2}} \sum_{U \in N} \Theta_\theta(h_{s;U}) \bar{\psi}(s^{-1}X_{1,1}, \dots, s^{-1}X_{k-1,k-1}). \quad (4.9)$$

In order to compute $\Theta_\theta(h_{s;U})$, we need to find conditions for $X_{i,j}$, such that $h_{s;U}$ will have a fixed number of Jordan blocks. This is equivalent to saying that $h_{s;U} - \lambda I_{kn}$ will have a given kernel dimension, or a given rank. Rank and trace are invariant under conjugation, so let us denote by h_{s_ρ, U_ρ} , the matrix $h_{s;U}$ conjugated by $\Delta^k(R)\Delta^k(\Delta^{d'}(T))$, where R and T are defined by s in (4.8):

$$h_{s_\rho, U_\rho} := \Delta^k(\Delta^{d'}(T^{-1})) \Delta^k(R^{-1}) h_{s;U} \Delta^k(R) \Delta^k(\Delta^{d'}(T)).$$

We have a matrix in $\text{GL}_{kn}(\mathbb{F}_d)$ and our goal is to find out how many matrices of the form

$$h_{s_\rho, U_\rho} - \lambda I_{kn} = h_{s_\rho - \lambda I_n, U_\rho},$$

where U varies, have a given rank ℓ .

First, notice that by the invariance of rank under elementary row and column operations on $h_{s_\rho - \lambda I_n, U_\rho}$, we can use the nonzero elements on the diagonal of $s_\rho - \lambda I_n$ to cancel the corresponding elements of $(X_{a,b})_\rho$. These elementary operations map the sequence of matrices $\{(X_{a,b})_\rho\}_{1 \leq a \leq b \leq k-1}$ $\mathbb{F}_d$ -linearly to the sequence

$$\left\{ (\hat{X}_{a,b})_\rho = \begin{pmatrix} x_{0,0;0}^{(a,b)} & \cdots & x_{d'-1,0;0}^{(a,b)} \\ \vdots & \ddots & \vdots \\ x_{0,0;d'-1}^{(a,b)} & \cdots & x_{d'-1,0;d'-1}^{(a,b)} \end{pmatrix} \in M_{d'}(\mathbb{F}_d) \right\}_{1 \leq a \leq b \leq k-1}.$$

The dimension of the kernel of this map is $\binom{k}{2}(n-d')d'$, corresponding to the number of elements we canceled. Hence, the number of matrices $h_{s_\rho - \lambda I_n, U_\rho}$ of rank ℓ is $(q^d)^{\binom{k}{2}(n-d')d'}$ times the number of matrices of the form

$$A := \begin{pmatrix} (\hat{X}_{1,1})_\rho & \cdots & (\hat{X}_{1,k-2})_\rho & (\hat{X}_{1,k-1})_\rho \\ 0 & \cdots & (\hat{X}_{2,k-2})_\rho & (\hat{X}_{2,k-1})_\rho \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & (\hat{X}_{k-1,k-1})_\rho \end{pmatrix} \in M_{(k-1)d'}(\mathbb{F}_d) \quad (4.10)$$

of rank $\ell - k(n-d')$. Using the character formula (2.1), we can calculate $\Theta_\theta(h_{s;U})$. In this case $m = kn$, $g = h_{s;U}$ and

$$t = \dim \ker(h_{s;U} - I) = kn - \text{rk}(h_{s;U} - I) = kn - k(n-d') - \text{rk}A = kd' - \text{rk}A.$$

Thus

$$\begin{aligned} \Theta_\theta(h_{s;U}) &= (-1)^{kn-1} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right] (1-q^d)(1-(q^d)^2) \cdots (1-(q^d)^{kd'-\text{rk}A-1}) \\ &= (-1)^{kn-1} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right] (q^d; q^d)_{kd'-\text{rk}A-1}. \end{aligned} \quad (4.11)$$

Now, by (4.11) and Lemma 2.6, (4.9) can be written as

$$\Theta_{k,N,\psi}(s) = \frac{(-1)^{kn-1} (q^d)^{\binom{k}{2}(n-d')d'}}{q^{\binom{k}{2}n^2}} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right] \sum_A (q^d; q^d)_{k d' - \text{rk} A - 1} \cdot \prod_{i=1}^{k-1} \bar{\psi}_0 \left(\text{Tr}_{\mathbb{F}_d/\mathbb{F}} \left(\lambda^{-1} \cdot \sum_{m=0}^{d'-1} x_{m,0;m}^{(i,i)} \right) \right), \quad (4.12)$$

where the sum is over matrices A as in (4.10). By the character formula (2.1), the RHS of (4.12) is $(-1)^{k(n-d')} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right]$ times the RHS of (3.1), when one replaces n with d' , q with q^d and ψ_0 with

$$\psi'_0 : \mathbb{F}_d \rightarrow \mathbb{C}^*, \quad \psi'_0(x) = \psi_0 \left(\text{Tr}_{\mathbb{F}_d/\mathbb{F}}(\lambda^{-1}x) \right).$$

Thus, the RHS of (4.12) is equal to $\dim(\pi_{k,N,\psi})$ (which is calculated in Theorem 2) after the substitution of n, q, ψ_0 with the relevant values. Hence,

$$\Theta_{k,N,\psi}(s) = (-1)^{k(n-d')} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right] (q^d)^{(k-2)\frac{d'(d'-1)}{2}} \frac{|\text{GL}_{d'}(\mathbb{F}_d)|}{q^n - 1},$$

as desired. $\square$

5 Proof of Theorem 4

Notice first that by part (III) of Lemma 2.10, the coefficients in both (1.5) and (1.6) are positive integers, unless $k = 2$ in which case they may also be zero.

Representations of a finite group are equivalent if the corresponding characters coincide. Hence, both parts of the theorem are equivalent to

$$\forall g \in \text{GL}_n(\mathbb{F}) : \Theta_{k;N,\psi}(g) = \sum_{\ell|n} a_{k;n,\ell}(q) \cdot \Theta_{\text{Ind}_\ell}(g), \quad (5.1)$$

where Θ_{Ind_ℓ} is the character of $\text{Ind}_{\mathbb{F}_\ell^*}^{\text{GL}_n(\mathbb{F})}(\theta \downarrow_{\mathbb{F}_\ell^*})$. We prove now (5.1) for any $g \in \text{GL}_n(\mathbb{F})$. If g is not semisimple or does not come from $\mathbb{F}_n$ then the LHS of (5.1) is zero by parts (I) and (II) of Theorem 3. The RHS of (5.1) is also zero on such elements by Lemma 2.3.

Let g be a semisimple element, which comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ and $d \mid n$ is minimal. Let λ be an eigenvalue of s , which generates $\mathbb{F}_d$ over $\mathbb{F}$. For such g , part (III) of Theorem 3 and Lemma 2.3 imply that (5.1) is equivalent to

$$(-1)^{k(n-d')} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right] q^{(k-2)\frac{n(d'-1)}{2}} \frac{|\text{GL}_{d'}(\mathbb{F}_d)|}{q^n - 1} = \sum_{\ell: d|\ell|n} a_{k;n,\ell}(q) \frac{|\text{GL}_{d'}(\mathbb{F}_d)|}{q^\ell - 1} \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right], \quad (5.2)$$

where $d' = n/d$. The following identity, which we now prove, establishes (5.2):

$$\frac{(-1)^{k(n-d')} q^{(k-2)\frac{n(d'-1)}{2}}}{q^n - 1} = \sum_{\ell: d|\ell|n} \frac{a_{k;n,\ell}(q)}{q^\ell - 1}. \quad (5.3)$$

Using (1.4), the RHS of (5.3) is

$$\sum_{\ell: d|\ell|n} \sum_{m: \ell|m|n} \frac{\mu\left(\frac{m}{\ell}\right) (-1)^{k(n-\frac{n}{m})} q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}}{q^n - 1}. \quad (5.4)$$

We simplify (5.4) using (2.28) as follows:

$$\begin{aligned} & \sum_{\ell: d|\ell|n} \sum_{m: \ell|m|n} \frac{\mu\left(\frac{m}{\ell}\right) (-1)^{k(n-\frac{n}{m})} q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}}{q^n - 1} \\ &= \sum_{m: d|m|n} \frac{(-1)^{k(n-\frac{n}{m})} q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}}{q^n - 1} \sum_{\ell: d|\ell|m} \mu\left(\frac{m}{\ell}\right) \\ &= \sum_{m: d|m|n} (-1)^{k(n-\frac{n}{m})} \frac{q^{(k-2)\frac{n}{2}(\frac{n}{m}-1)}}{q^n - 1} \delta_{d,m} \\ &= (-1)^{k(n-\frac{n}{d})} \frac{q^{(k-2)\frac{n}{2}(\frac{n}{d}-1)}}{q^n - 1}, \end{aligned}$$

which is the LHS of (5.3). Hence the proof is complete. $\square$

6 Proof of Theorem 1

Representations of a finite group are equivalent if the corresponding characters coincide. Hence, the theorem is equivalent to

$$\forall g \in \mathrm{GL}_n(\mathbb{F}) : \quad \Theta_{k,N,\psi}(g) = \Theta_{\theta|_{\mathbb{F}_n^*}}(g) \cdot (\mathrm{St}(g))^{k-1}, \quad (6.1)$$

where we use the notation St also for the character of the Steinberg representation. We prove now (6.1) for any $g \in \mathrm{GL}_n(\mathbb{F})$.

We first prove (6.1) for $k = 1$. Note that $N = \{I_n\}$ and so

$$V_{\pi_{1,N,\psi}} = \{v \in V_{\pi_\theta} \mid \pi(I_n)v = v\} = V_{\pi_\theta}.$$

Hence $\pi_{1,N,\psi}(g) = \pi_\theta(g)$ as needed.

Now assume $k \geq 2$. If the semisimple part s of g does not come from $\mathbb{F}_n$, or g is not semisimple, then $\Theta_{k,N,\psi}(g) = 0$ by Theorem 3. From Theorem 2.1, we have $\Theta_{\theta|_{\mathbb{F}_n^*}}(g) = 0$. Hence, (6.1) is proved in that case.

Otherwise, $g = s$ is a semisimple element which comes from $\mathbb{F}_d \subseteq \mathbb{F}_n$ and $d \mid n$ is minimal. We begin by calculating the character value $\mathrm{St}(g)$. For any prime p , let m_p be the p -part of m . By [Car93, Thm. 6.5.9],

$$\mathrm{St}(g) = \varepsilon_{\mathrm{GL}_n} \varepsilon_{C(g)^\circ} \left| C(g)^\mathbb{F} \right|_{\mathrm{char}(\mathbb{F})},$$

where ε_G is (-1) to the power of the $\mathbb{F}$ -rank of G , $C(g)$ is the centralizer of g in $\mathrm{GL}_n(\overline{\mathbb{F}})$, $C(g)^\circ$ is its identity component and $C(g)^\mathbb{F}$ is the subgroup of $\mathbb{F}$ -rational points in $C(g)$. The $\mathbb{F}$ -rank of GL_n is n . Let $\rho = (1, 1, \dots, 1)$, a partition of $d' = \frac{n}{d}$ and let f be the characteristic polynomial of s . By §2.3.1, the centralizer $C(g)^\mathbb{F}$ is isomorphic to $C(L_{f,\rho})^\mathbb{F}$, which in turn is isomorphic

to $\mathrm{GL}_{d'}(\mathbb{F}_d)$ (cf. [Gre55, Lem. 2.4] and the discussion preceding it). Thus, $\varepsilon_{C(g)^\circ} = \varepsilon_{\mathrm{GL}_{d'}} = (-1)^{d'}$ and

$$|C(g)^\mathbb{F}| = q^{\sum_{i=1}^{d'} d(d'-i)} \prod_{k=1}^{d'} (q^{dk} - 1), \quad |C(g)^\mathbb{F}|_{\mathrm{char}(\mathbb{F})} = q^{\frac{n(d'-1)}{2}}.$$

The discussion shows that

$$\mathrm{St}(g) = (-1)^{n-d'} q^{\frac{n(d'-1)}{2}}. \quad (6.2)$$

By Theorem 2.1,

$$\begin{aligned} \Theta_{\theta|_{\mathbb{F}_n^*}}(g) &= (-1)^{n-1} \left[\sum_{\alpha=0}^{d-1} \theta(\lambda^{q^\alpha}) \right] (1 - q^d)(1 - (q^d)^2) \cdots (1 - (q^d)^{d'-1}) \\ &= (-1)^{n-d'} \left[\sum_{\alpha=0}^{d-1} \theta(\lambda^{q^\alpha}) \right] (q^d - 1)(q^{2d} - 1) \cdots (q^{n-d} - 1) \frac{q^n - 1}{q^n - 1} \\ &= (-1)^{n-d'} \left[\sum_{\alpha=0}^{d-1} \theta(\lambda^{q^\alpha}) \right] \frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{(q^n - 1) q^{\frac{n(d'-1)}{2}}}, \end{aligned} \quad (6.3)$$

where λ is an eigenvalue of g . By Theorem 3

$$\Theta_{k,N,\psi}(g) = (-1)^{k(n-d')} q^{(k-2)\frac{n(d'-1)}{2}} \cdot \left[\sum_{i=0}^{d-1} \theta(\lambda^{q^i}) \right] \cdot \frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{q^n - 1}. \quad (6.4)$$

Multiplying (6.3) by (6.2) raised to the $(k-1)$ -th power, we get (6.4) as needed. $\square$

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References

- [Car93] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons, Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.
- [Gel75] S. I. Gel'fand. Representations of the general linear group over a finite field. *Lie groups and their representations (Proc. Summer School on Group Representations of the Bolya: János Math. Soc., Budapest, 1971)*, pages 119–132, 1975.

- [GG62] I. M. Gel'fand and M. I. Graev. Construction of irreducible representations of simple algebraic groups over a finite field. *Dokl. Akad. Nauk SSSR*, 147:529–532, 1962.
- [GR86] A. M. Garsia and J. B. Remmel. Q -counting rook configurations and a formula of Frobenius. *J. Combin. Theory Ser. A*, 41(2):246–275, 1986.
- [GR04] George Gasper and Mizan Rahman. *Basic hypergeometric series*, volume 96 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Richard Askey.
- [Gre55] J. A. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.
- [Hag98] James Haglund. q -rook polynomials and matrices over finite fields. *Adv. in Appl. Math.*, 20(4):450–487, 1998.
- [Haz16] Zahi Hazan. On certain degenerate whittaker models for $GL(n)$ over finite fields. Master's thesis, Tel Aviv University, October 2016. <http://primage.tau.ac.il/libraries/theses/exeng/free/2980107.pdf>.
- [IR90] Kenneth Ireland and Michael Rosen. *A classical introduction to modern number theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
- [Lan93] Georg Landsberg. Ueber eine Anzahlbestimmung und eine damit zusammenhängende Reihe. *J. Reine Angew. Math.*, 111:87–88, 1893.
- [Mor72] C. Moreau. Sur les permutations circulaires distinctes. *Nouvelles annales de mathématiques, journal des candidats aux écoles polytechnique et normale*, 11:309–314, 1872.
- [Pra00] Dipendra Prasad. The space of degenerate Whittaker models for general linear groups over a finite field. *Internat. Math. Res. Notices*, 2000(11):579–595, 2000.
- [Sol90] Louis Solomon. The Bruhat decomposition, Tits system and Iwahori ring for the monoid of matrices over a finite field. *Geom. Dedicata*, 36(1):15–49, 1990.
- [SZ84] T. A. Springer and A. V. Zelevinsky. Characters of $GL(n, \mathbf{F}_q)$ and Hopf algebras. *J. London Math. Soc. (2)*, 30(1):27–43, 1984.