

On global solutions to the Navier-Stokes system with large $L^{3,\infty}$ initial data

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March 22, 2018

Abstract

This paper addresses a question concerning the behaviour of a sequence of global solutions to the Navier-Stokes equations, with the corresponding sequence of smooth initial data being bounded in the (non-energy class) weak Lebesgue space $L^{3,\infty}$. It is closely related to the question of what would be a reasonable definition of global weak solutions with a non-energy class of initial data, including the aforementioned Lorentz space. This paper can be regarded as an extension of a similar problem regarding the Lebesgue space L_3 to the weak Lebesgue space $L^{3,\infty}$, whose norms are both scale invariant with the respect to the Navier-Stokes scaling.

1 Introduction

In our paper we consider the Cauchy problem for the Navier-Stokes system in the space-time domain $Q_\infty = \mathbb{R}^3 \times]0, \infty[$ for vector-valued function $v = (v_1, v_2, v_3) = (v_i)$ and scalar function q , satisfying the equations

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q, \quad \operatorname{div} v = 0 \quad (1.1)$$

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in Q_∞ , the boundary conditions

$$v(x, t) \rightarrow 0 \quad (1.2)$$

as $|x| \rightarrow \infty$ for all $t \in [0, \infty[$, and the initial conditions

$$v(\cdot, 0) = u_0(\cdot) \quad (1.3)$$

with divergence free function u_0 belonging to a weak $L_3(\mathbb{R}^3)$ space denoted in the paper as $L^{3,\infty}(\mathbb{R}^3)$.

Let us recall the definition of the Lorentz spaces. For a measurable function $f : \Omega \rightarrow \mathbb{R}^m$ define:

$$d_{f,\Omega}(\alpha) := |\{x \in \Omega : |f(x)| > \alpha\}|. \quad (1.4)$$

Let $s \in]0, \infty[$ and $l \in]0, \infty]$. Given a measurable $\Omega \subseteq \mathbb{R}^n$, the Lorentz space $L^{s,l}(\Omega)$ is the set of all measurable functions g on Ω such that the quasinorm $\|g\|_{L^{s,l}(\Omega)}$ is finite. Here:

$$\|g\|_{L^{s,l}(\Omega)} := \left(s \int_0^\infty \alpha^l d_{g,\Omega}(\alpha)^{\frac{l}{s}} \frac{d\alpha}{\alpha} \right)^{\frac{1}{l}}, \quad (1.5)$$

$$\|g\|_{L^{s,\infty}(\Omega)} := \sup_{\alpha>0} \alpha d_{g,\Omega}(\alpha)^{\frac{1}{s}}. \quad (1.6)$$

As is the case for the $L_3(\mathbb{R}^3)$ norm, the Lorentz norm $L^{3,\infty}(\mathbb{R}^3)$ is scale invariant with respect to the Navier-Stokes scaling

$$v^\lambda(x, t) = \lambda v(\lambda x, \lambda^2 t), \quad q^\lambda(x, t) = \lambda^2 q(\lambda x, \lambda^2 t).$$

The important difference between the above spaces is that the norm in the space $L_3(\mathbb{R}^3)$ possesses a shrinking property, i.e., the norm over a ball vanishes as the radius of this ball goes to zero, while the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$ does not meet such a property for its norm. The difference can also be expressed in terms of the density of smooth compactly supported functions. The special interest of the space $L^{3,\infty}(\mathbb{R}^3)$ as a phase space for the Navier-Stokes equations is due to the fact that, in contrast to the space $L_3(\mathbb{R}^3)$, it contains minus one homogeneous divergence free functions.

The local in time existence of strong solutions to the above Cauchy problem is a relatively well known fact proved in a number of papers, see for

example, [5], [7], [9], [17], [19], [28], and [34], with the help of Kato's arguments [13]. The typical outcome is local in time existence of the so-called mild solutions under certain assumptions on the initial data¹. This technique has a perturbative character as it does not take into account the skew symmetry of nonlinear term in full generality. Consequently, there is an absence of results about global solvability for large initial data².

A breakthrough result in this direction has been established by Lemarie-Rieusset, see [24]. He showed that, for a very wide class of initial data³, there exists a certain global solution to the initial value problem (1.1)-(1.3) that in addition satisfies the local energy inequality. Such a solution exists globally in time if $u_0 \in L^{3,\infty}(\mathbb{R}^3)$. However, the class of Lemarie-Rieusset's solutions seems to be too wide and one can expect that additionally some global norms are bounded if more restrictive classes of initial data with unbounded energy⁴ are considered.

Moreover, there are some additional requirements for the class of weak global solutions. First of them is some kind of stability with respect to weak or weak-(*) convergence of initial data. To be precise, in our case, this would mean the following. Assuming that a sequence $u_0^{(k)}$ converges weakly-(*) to u_0 in $L^{3,\infty}(\mathbb{R}^3)$, we need to show that the corresponding solutions $u^{(k)}$ with initial data $u_0^{(k)}$ converges in a sense to a solution u with initial data u_0 . This issue appears if one wants to show that scale invariant norms blow up as time approaches potential blowup time, see [29] and [2]. The second important point is that the conception of Lemarie-Rieusset solutions has not been developed yet for unbounded domains different to the whole space \mathbb{R}^3 . This makes it desirable to have a notion of weak global solutions that can be extended to other unbounded domains.

In the paper [31], the notion of global weak L_3 -solutions has been introduced in the case of initial data belonging to the Lebesgue space $L_3(\mathbb{R}^3)$, which respects the above two requirements. In addition, in [31], regularity of weak L_3 -solutions has been proven on a finite time interval, the length of which depends on the initial data, that in turn implies uniqueness of weak

¹We are not discussing global existence of mild solutions for small initial data which is a very interesting topic itself but outside of our scope.

²We restrict our considerations to three dimensional case only.

³The completion of smooth compactly supported divergence free functions in the space $L_{s,\text{unif}}$ with the finite norm $\|u\|_{s,\text{unif}} := \sup_{x \in \mathbb{R}^3} \|u\|_{L_s(B(x,1))}$ for $s = 2$.

⁴ $u_0 \notin L_2(\mathbb{R}^3)$.

L_3 -solutions on this finite time interval. The aim of the paper is to implement this program in the case of initial data belonging to the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$.

To define our weak solution, we need to introduce additional notation:

$$S(t)u_0(x) = \int_{\mathbb{R}^3} \Gamma(x-y, t)u_0(y)dy,$$

where Γ is a known heat kernel, $V(x, t) := S(t)u_0(x)$;

$L_s(\Omega)$ is a Lebesgue space in $\Omega \subseteq \mathbb{R}^3$ so that $L_s(\Omega) = L^{s,s}(\Omega)$ and abbreviations $L_s := L_s(\mathbb{R}^3)$ and $L^{s,l} := L^{s,l}(\mathbb{R}^3)$ are used;

J and $\overset{\circ}{J}^{\frac{1}{2}}$ are the completion of the space

$$C_{0,0}^\infty(\mathbb{R}^3) := \{v \in C_0^\infty(\mathbb{R}^3) : \operatorname{div} v = 0\}$$

with respect to L_2 -norm and the Dirichlet integral

$$\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^{\frac{1}{2}},$$

correspondingly. Additionally, we define the space-time domains $Q_T := \mathbb{R}^3 \times]0, T[$ and $Q_\infty := \mathbb{R}^3 \times]0, \infty[$.

Definition 1.1. *We say that v is a weak $L^{3,\infty}$ -solution to Navier-Stokes IBVP in Q_T if*

$$v = V + u, \tag{1.7}$$

with $u \in L_\infty(0, T; J) \cap L_2(0, T; \overset{\circ}{J}^{\frac{1}{2}})$ and there exists $q \in L^{\frac{3}{2}, \text{loc}}_2(Q_T)$ such that u and q satisfy the perturbed Navier Stokes system in the sense of distributions:

$$\partial_t u + v \cdot \nabla v - \Delta u = -\nabla q, \quad \operatorname{div} u = 0 \tag{1.8}$$

in Q_T . Additionally, it is required that for any $w \in L_2$:

$$t \rightarrow \int_{\mathbb{R}^3} w(x) \cdot u(x, t) dx \tag{1.9}$$

is a continuous function on $[0, T]$. Moreover, u satisfies the energy inequality:

$$\|u(\cdot, t)\|_{L_2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla u(x, t')|^2 dx dt' \leq$$

$$\leq 2 \int_0^t \int_{\mathbb{R}^3} (V \otimes u + V \otimes V) : \nabla u dx dt' \quad (1.10)$$

for all $t \in [0, T]$.

Finally, it is required that v and q satisfy the local energy inequality. Namely, for a.a. $t \in]0, T[$,

$$\begin{aligned} & \int_{\mathbb{R}^3} \phi(x, t) |v(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \phi |\nabla v|^2 dx dt' \leq \\ & \leq \int_0^t \int_{\mathbb{R}^3} [|v|^2 (\partial_t \phi + \Delta \phi) + v \cdot \nabla \phi (|v|^2 + 2q)] dx dt' \end{aligned} \quad (1.11)$$

for all non negative functions $\phi \in C_0^\infty(Q_T)$.

v is called a global weak $L^{3,\infty}$ -weak solution if it is a weak solution in Q_T for any $T > 0$.

Remark 1.2. One can see that the right hand side in the energy inequality (1.1) is finite and thus the function u satisfies the initial condition in the strong L_2 -sense, i.e., $u(\cdot, t) \rightarrow 0$ in L_2 .

With regards to V , we can show that $\|V(\cdot, t) - u_0\|_{L_{s,\text{unif}}} \rightarrow 0$ as $t \rightarrow 0$ for any $s < 3$. In general, $V(\cdot, t)$ does not tends to u_0 in $L^{3,\infty}$ which can be easily seen for minus one homogeneous initial data, see [5].

The main result of the paper reads the following.

Theorem 1.3. Let $u_0^{(k)} \xrightarrow{*} u_0$ in $L^{3,\infty}$ and let $v^{(k)}$ be a sequence of a global weak $L^{3,\infty}$ -solutions to the Cauchy problem for the Navier-Stokes system with initial data $u_0^{(k)}$. Then there exists a subsequence still denoted $v^{(k)}$ that converges to a global weak $L^{3,\infty}$ -solution v to the Cauchy problem for the Navier-Stokes system with initial data u_0 , in the sense of distributions.

Corollary 1.4. There exists at least one global weak $L^{3,\infty}$ -solution to the Cauchy problem (1.1)-(1.3).

It is worth noticing that the smooth forward self-similar solution, the existence of which has been proved recently in [12], is a global weak $L^{3,\infty}$ -solution.

Certain uniqueness and regularity statements, regarding weak $L^{3,\infty}$ -solutions, provide further justification of the definition of weak $L^{3,\infty}$ -solutions. We start with conditional uniqueness results.

Theorem 1.5. *Let v be a global weak $L^{3,\infty}$ -solution to the Cauchy problem for the Navier-Stokes equations with the initial data $u_0 \in L^{3,\infty}$. There is a universal constant $\varepsilon_0 > 0$ with the following property. If*

$$\limsup_{R \rightarrow 0} \|u_0\|_{L^{3,\infty}(B(x_0, R))} < \varepsilon_0 \quad (1.12)$$

for any $x_0 \in \mathbb{R}^3$ and

$$\|v(\cdot, t) - u_0(\cdot)\|_{L^{3,\infty}(\mathbb{R}^3)} < \varepsilon_0 \quad (1.13)$$

holds for all $t \in]0, T[$, then v is of class C^∞ in Q_T .

Moreover, if \tilde{v} is another global weak $L^{3,\infty}$ -solution to the Cauchy problem for the Navier-Stokes equations with the the same initial data u_0 , then $\tilde{v} = v$ in Q_T .

Corollary 1.6. *Let v and \tilde{v} be two global weak $L^{3,\infty}$ -solution to the Cauchy problem for the Navier-Stokes equations with the same initial data u_0 . Suppose that $v \in C([0, T]; L^{3,\infty})$. Then $\tilde{v} = v$ in Q_T .*

As to regularity, we can state the following.

Theorem 1.7. *Suppose that $u_0 \in L^{3,\infty}$. There exists a universal constant $\varepsilon > 0$ such that if*

$$\langle V \rangle_{Q_T} := \sup_{0 < t < T} t^{\frac{1}{5}} \|V(\cdot, t)\|_{L^5} \leq \varepsilon, \quad (1.14)$$

where $V(\cdot, t) = S(t)u_0(\cdot)$, then there exists a v that is a weak $L^{3,\infty}$ -solution to the Cauchy problem for the Navier-Stokes system in Q_T and satisfies the property

$$\langle v \rangle_{Q_T} < 2\langle V \rangle_{Q_T}. \quad (1.15)$$

Moreover, the following estimate is valid

$$\|v - V\|_{L^\infty(0, T; L^3)} < \langle V \rangle_{Q_T} \quad (1.16)$$

It is easy to verify that a solution of Theorem 1.7 is infinitely smooth in Q_T .

Although the main condition (1.14) holds for a wide class initial data, it does not work for large minus one homogeneous initial data, see details in [5].

Finally, there will be shown that under Kozono-Yamazaki condition, see [20], any global weak $L^{3,\infty}$ -solution is unique and smooth on a short time interval.

Proposition 1.8. *Let $u_0 \in L^{3,\infty}$. There exists an $\varepsilon_3 > 0$ such that if*

$$\limsup_{\alpha \rightarrow \infty} \alpha d_{u_0, \mathbb{R}^3}(\alpha)^{\frac{1}{3}} < \varepsilon_3 \quad (1.17)$$

then there exists a $T = T(u_0) > 0$ such that all global weak $L^{3,\infty}$ solutions, with initial data $u_0 \in L^{3,\infty}$, coincide on Q_T .

2 Preliminaries

Now we state a fact about Lorentz spaces concerning a decomposition. The proof can be found in [2]. This will be formulated as a Lemma. Analogous statement is Lemma II.I proven by Calderon in [4].

Lemma 2.1. *Take $1 < t < r < s \leq \infty$, and suppose that $g \in L^{r,\infty}(\Omega)$. For any $N > 0$, we let $g_-^N := g\chi_{|g| \leq N}$ and $g_+^N := g - g_-^N$. Then*

$$\|g_-^N\|_{L^s(\Omega)}^s \leq \frac{s}{s-r} N^{s-r} \|g\|_{L^{r,\infty}(\Omega)}^r - N^s d_g(N) \quad (2.1)$$

if $s < \infty$, and

$$\|g_+^N\|_{L^t(\Omega)}^t \leq \frac{r}{r-t} N^{t-r} \|g\|_{L^{r,\infty}(\Omega)}^r. \quad (2.2)$$

Moreover, for $\Omega = \mathbb{R}^3$, if $g \in L^{r,l}$ with $1 \leq l \leq \infty$ and $\operatorname{div} g = 0$, then $g = \bar{g}^N + \tilde{g}^N$ where $\bar{g}^N \in [C_{0,0}^\infty(\mathbb{R}^3)]^{L^s(\mathbb{R}^3)}$ with

$$\|\bar{g}^N\|_{L^s}^s \leq \frac{Cs}{s-r} N^{s-r} \|g\|_{L^{r,\infty}}^r \quad (2.3)$$

and $\tilde{g}^N \in [C_{0,0}^\infty(\mathbb{R}^3)]^{L_t(\mathbb{R}^3)}$ with

$$\|\tilde{g}^N\|_{L^t}^t \leq \frac{Cr}{r-t} N^{t-r} \|g\|_{L^{r,\infty}}^r. \quad (2.4)$$

Remark 2.2. *Looking at the proof of the second part of of Lemma 2, we can easily see that*

$$\|\bar{g}^N\|_{L^{r,\infty}} + \|\tilde{g}^N\|_{L^{r,\infty}} \leq c(r) \|g\|_{L^{r,\infty}}. \quad (2.5)$$

Let us recall the well known properties of $L^{s,1}$, for $1 < s < \infty$, such as separability and density of smooth compactly supported functions. Also, recall that

$$(L^{s,1})' = L^{s',\infty}, \quad s' = \frac{s}{s-1}.$$

The identification is as follows, if $f \in L^{s',\infty}$ and $g \in L^{s,1}$:

$$T_f(g) = \int_{\mathbb{R}^3} f g dx.$$

The following proposition concerns weak-star approximation of $L^{3,\infty}$ functions.

Proposition 2.3. *Let $u_0 \in L^{3,\infty}$ be divergence free, in the sense of distributions. Then there exists a sequence $u_0^{(k)} \in C_{0,0}^\infty(\mathbb{R}^3)$ such that*

$$u_0^{(k)} \xrightarrow{*} u_0$$

in $L^{3,\infty}$.

The proof is based on the estimates of solutions to the Neumann boundary problem in the terms of the Lorentz space $L^{\frac{3}{2},1}$.

Now, consider the following Cauchy problem for the heat equation

$$\partial_t u - \Delta u = 0 \tag{2.6}$$

in Q_∞ ,

$$u(\cdot, 0) = u_0(\cdot) \in L^{3,\infty} \tag{2.7}$$

in \mathbb{R}^3 .

Let us recall some known facts about solution operators of $S(t)$ for the corresponding semi-group. Indeed, $u(\cdot, t) = V(\cdot, t) = S(t)u_0(\cdot)$.

Proposition 2.4. *We have*

$$\|S(t)u_0\|_{L^{3,\infty}} \leq C\|u_0\|_{L^{3,\infty}}. \tag{2.8}$$

Moreover for $3 < r < \infty$, $m, k \in \mathbb{N}$:

$$\|\partial_t^m \nabla^k S(t)u_0\|_{L^r} \leq \frac{C\|u_0\|_{L^{3,\infty}}}{t^{m+\frac{k}{2}+\frac{3}{2}(\frac{1}{3}-\frac{1}{r})}}. \tag{2.9}$$

Furthermore for $1 \leq q < 3$ the following limits exist as $t \rightarrow 0$:

$$\|S(t)u_0 - u_0\|_{L_{q,unif}} \rightarrow 0, \tag{2.10}$$

$$S(t)u_0 \xrightarrow{*} u_0 \tag{2.11}$$

in $L^{3,\infty}$. Under the additional constraint that $u_0 \in \mathbb{L}^{3,\infty} := [C_{0,0}^\infty]^{L^{3,\infty}}$ Then we have that $S(t)u_0 \in \mathbb{L}^{3,\infty}$ and

$$\lim_{t \rightarrow 0} \|S(t)u_0 - u_0\|_{L^{3,\infty}} = 0. \tag{2.12}$$

Proof. The first two estimates are follows from convolution structure of the heat potential and the corresponding inequalities.

Recall the definition

$$\|f\|_{L_{p,unif}} := \sup_{x_0 \in \mathbb{R}^3} \|f\|_{L_p(B(x_0,1))}.$$

Now, let us focus only on proving (2.10), as all other statements follow from this and (2.8). From Lemma 2.1 we can write

$$u_0 := \bar{u}_0^1 + \tilde{u}_0^1, \quad (2.13)$$

so that

$$\bar{u}_0^1 \in [C_{0,0}^\infty]^{L_s} \cap L^{3,\infty}, \quad \tilde{u}_0^1 \in [C_{0,0}^\infty]^{L_q} \cap L^{3,\infty}$$

with $1 < q < 3 < s < \infty$. It is clear that

$$\lim_{t \rightarrow 0} \|S(t)\bar{u}_0^1 - \bar{u}_0^1\|_{L_s} = 0,$$

$$\lim_{t \rightarrow 0} \|S(t)\tilde{u}_0^1 - \tilde{u}_0^1\|_{L_q} = 0.$$

From here, (2.10) is obtained without difficulty. \square

Proposition 2.5. *Let*

$$u_0^{(k)} \xrightarrow{*} u_0$$

in $L^{3,\infty}$. Then, for any $\phi \in C_0^\infty(Q_\infty)$:

$$\int_0^\infty \int_{\mathbb{R}^3} S(t)u_0^{(k)}(x)\phi(x,t)dxdt \rightarrow \int_0^\infty \int_{\mathbb{R}^3} S(t)u_0(x)\phi(x,t)dxdt. \quad (2.14)$$

Proof. By Lemma 2.1, we have

$$u_0^{(k)} := \bar{u}_0^{(k)1} + \tilde{u}_0^{(k)1}$$

and

$$\sup_k \|\bar{u}_0^{(k)1}\|_{L_s} + \sup_k \|\tilde{u}_0^{(k)1}\|_{L_q} \leq C(s, q) \sup_k \|u_0^{(k)}\|_{L^{3,\infty}}.$$

It is clear that $\bar{u}_0^{(k)1} \rightharpoonup \bar{u}_0$, $S(t)\bar{u}_0^{(k)1} \rightharpoonup S(t)\bar{u}_0$ in L_s and $\tilde{u}_0^{(k)1} \rightharpoonup \tilde{u}_0$, $S(t)\tilde{u}_0^{(k)1} \rightharpoonup S(t)\tilde{u}_0$ in L_q . Obviously, $u_0 = \bar{u}_0 + \tilde{u}_0$. From here the conclusion is easily reached. \square

3 Existence of global weak $L^{3,\infty}(\mathbb{R}^3)$ -solutions

3.1 Apriori estimates

Let $L_{s,l}(Q_T)$, $W_{s,l}^{1,0}(Q_T)$, $W_{s,l}^{2,1}(Q_T)$ be anisotropic (or parabolic) Lebesgues and Sobolev spaces with norms

$$\|u\|_{L_{s,l}(Q_T)} = \left(\int_0^T \|u(\cdot, t)\|_{L_s}^l dt \right)^{\frac{1}{l}}, \quad \|u\|_{W_{s,l}^{1,0}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},$$

$$\|u\|_{W_{s,l}^{2,1}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)}.$$

Lemma 3.1. *Assume that $u \in L_\infty(0, T; J) \cap L_2(0, T; \overset{\circ}{J} \frac{1}{2})$ and let $u_0 \in L^{3,\infty}$ be divergence free. Then*

$$V \cdot \nabla V \in L_{\frac{11}{7}}(Q_T), \quad (3.1)$$

$$V \cdot \nabla u + u \cdot \nabla V \in L_{\frac{5}{4}, \frac{3}{2}}(Q_T), \quad (3.2)$$

$$V \otimes u : \nabla u \in L_1(Q_T). \quad (3.3)$$

Proof. By Holder inequality and Proposition 2.4:

$$\begin{aligned} \int_{\mathbb{R}^3} |V \cdot \nabla V|^{\frac{11}{7}} dx &\leq \|V\|_{L^{\frac{22}{7}}}^{\frac{11}{7}} \|\nabla V\|_{L^{\frac{22}{7}}}^{\frac{11}{7}} \leq \\ &\leq c \frac{\|u_0\|_{L^{3,\infty}}^{\frac{22}{7}}}{t^{\frac{6}{7}}}. \end{aligned}$$

From here, (3.1) is easily established. Again, by Holder inequality and Proposition 2.4:

$$\|u \cdot \nabla V\|_{L_{\frac{5}{4}}} \leq \|\nabla V\|_{L_{\frac{10}{3}}} \|u\|_{L_2} \leq c \frac{\|u_0\|_{L^{3,\infty}} \|u\|_{L_{2,\infty}(Q_T)}}{t^{\frac{11}{20}}}.$$

From this it is immediate that $u \cdot \nabla V \in L_{\frac{5}{4}, \frac{3}{2}}(Q_T)$. Again by Holder inequality, it is not difficult to verify

$$\int_0^T \|V \cdot \nabla u\|_{L_{\frac{5}{4}}}^{\frac{3}{2}} dt \leq \left(\int_0^T \|\nabla u\|_{L_2}^2 dt \right)^{\frac{3}{4}} \left(\int_0^T \|V\|_{L_{\frac{10}{3}}}^6 dt \right)^{\frac{1}{4}}.$$

The conclusion is easily reached by noting that Proposition 2.4 gives:

$$\|V\|_{L^{\frac{10}{3}}}^6 \leq c \frac{\|u_0\|_{L^{3,\infty}}}{t^{\frac{6}{20}}}.$$

The last estimate is known and shows why there are difficulties to prove energy estimate for u . By O'Neil's inequality and Proposition 2.4:

$$\begin{aligned} \int_{\mathbb{R}^3} |V \otimes u : \nabla u| dx &\leq \|V\|_{L^{3,\infty}} \|u\|_{L^{6,2}} \|\nabla u\|_{L_2} \\ &\leq c \|u_0\|_{L^{3,\infty}} \|\nabla u\|_{L_2}^2. \end{aligned}$$

We have used fact that $L^{6,2}(\Omega) \hookrightarrow W_2^1(\Omega)$. See [1], for example. \square

The next statement is a direct consequence of Lemma 3.1 and coercive estimates of solutions to the Stokes problem.

Lemma 3.2. *Let v be a global weak $L^{3,\infty}$ -solution with functions u and q as in Definition 1.1. Then*

$$(u, q) = \sum_{i=1}^3 (u^i, p_i) \quad (3.4)$$

such that for any finite T :

$$(u^i, \nabla p_i) \in W_{s_i, l_i}^{2,1}(Q_T) \times L_{s_i, l_i}(Q_T) \quad (3.5)$$

and

$$(s_1, l_1) = (9/8, 3/2), s_2 = l_2 = 11/7, (s_3, l_3) = (5/4, 3/2). \quad (3.6)$$

In addition (u^i, p_i) satisfy the following:

$$\partial_t u^1 - \Delta u^1 + \nabla p_1 = -u \cdot \nabla u, \quad (3.7)$$

$$\partial_t u^2 - \Delta u^2 + \nabla p_2 = -V \cdot \nabla V, \quad (3.8)$$

$$\partial_t u^3 - \Delta u^3 + \nabla p_3 = -V \cdot \nabla u - u \cdot \nabla V \quad (3.9)$$

in Q_∞ , and

$$\operatorname{div} u^i = 0 \quad (3.10)$$

in Q_∞ for $i = 1, 2, 3$,

$$u^i(\cdot, 0) = 0 \quad (3.11)$$

for all $x \in \mathbb{R}^3$ and $i = 1, 2, 3$.

Before the next Lemma let us introduce some notation. Let u , v and u_0 be as in Definition 1.1. Let $u_0 = \bar{u}_0^N + \tilde{u}_0^N$ denote the splitting from Lemma 2.1. Let us define the following:

$$\bar{V}^N(\cdot, t) := S(t)\bar{u}_0^N(\cdot, t), \quad (3.12)$$

$$\tilde{V}^N(\cdot, t) := S(t)\tilde{u}_0^N(\cdot, t) \quad (3.13)$$

and

$$w^N(x, t) := u(x, t) + \tilde{V}^N(x, t). \quad (3.14)$$

Lemma 3.3. *In the above notation, we have the following global energy inequality*

$$\begin{aligned} & \|w^N(\cdot, t)\|_{L_2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla w^N(x, t')|^2 dx dt' \leq \\ & \leq \|\tilde{u}_0^N\|_{L_2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} (\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N dx dt' \end{aligned} \quad (3.15)$$

that is valid for positive N and t .

Proof. The first stage is showing that w^N satisfies the local energy inequality. Let us briefly sketch how this can be done. Let $\phi \in C_0^\infty(Q_\infty)$ be a positive function. Observe that the assumptions in Definition 1.1 imply that the following function

$$t \rightarrow \int_{\Omega} w^N(x, t) \cdot \bar{V}^N(x, t) \phi(x, t) dx \quad (3.16)$$

is continuous for all $t \geq 0$. It is not so difficult to show that this term has the following expression:

$$\begin{aligned} & \int_{\mathbb{R}^3} w^N(x, t) \cdot \bar{V}^N(x, t) \phi(x, t) dx = \int_0^t \int_{\mathbb{R}^3} (w^N \cdot \bar{V}^N) (\Delta \phi + \partial_t \phi) dx dt' - \\ & - 2 \int_0^t \int_{\mathbb{R}^3} \nabla w^N : \nabla \bar{V}^N \phi dx dt' + \int_0^t \int_{\mathbb{R}^3} \bar{V}^N \cdot \nabla \phi q dx dt' + \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} (|v|^2 - |w^N|^2) v \cdot \nabla \phi dx dt' - \\
& - \int_0^t \int_{\mathbb{R}^3} (\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N \phi dx dt' - \\
& - \int_0^t \int_{\mathbb{R}^3} (\bar{V}^N \otimes \bar{V}^N + \bar{V}^N \otimes w^N) : (w^N \otimes \nabla \phi) dx dt'. \tag{3.17}
\end{aligned}$$

It is also readily shown that

$$\begin{aligned}
\int_{\mathbb{R}^3} |\bar{V}^N(x, t)|^2 \phi(x, t) dx &= \int_0^t \int_{\mathbb{R}^3} |\bar{V}^N(x, t')|^2 (\Delta \phi(x, t') + \partial_t \phi(x, t')) dx dt' - \\
& - 2 \int_0^t \int_{\mathbb{R}^3} |\nabla \bar{V}^N|^2 \phi dx dt'. \tag{3.18}
\end{aligned}$$

Using (1.1), together with (3.1)-(3.1), we obtain that for all $t \in]0, \infty[$ and for all non negative functions $\phi \in C_0^\infty(Q_\infty)$:

$$\begin{aligned}
& \int_{\mathbb{R}^3} \phi(x, t) |w^N(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \phi |\nabla w^N|^2 dx dt' \leq \\
& \leq \int_0^t \int_{\mathbb{R}^3} |w^N|^2 (\partial_t \phi + \Delta \phi) + 2 q w^N \cdot \nabla \phi dx dt' + \\
& + \int_0^t \int_{\mathbb{R}^3} |w^N|^2 v \cdot \nabla \phi dx dt' + \\
& + 2 \int_0^t \int_{\mathbb{R}^3} (\bar{V}^N \otimes \bar{V}^N + \bar{V}^N \otimes w^N) : (\nabla w^N \phi + w^N \otimes \nabla \phi) dx dt' \tag{3.19}
\end{aligned}$$

In the next part of the proof, let $\phi(x, t) = \phi_1(t)\phi_R(x)$. Here, $\phi_1 \in C_0^\infty(0, \infty)$ and $\phi_R \in C_0^\infty(B(2R))$ are positive functions. Moreover, $\phi_R = 1$ on $B(R)$, $0 \leq \phi_R \leq 1$,

$$|\nabla \phi_R| \leq c/R,$$

$$|\nabla^2 \phi_R| \leq c/R^2.$$

Since $\tilde{u}_0^N \in [C_{0,0}^\infty(\mathbb{R}^3)]^{L_2(\mathbb{R}^3)}$, it is obvious that for $\tilde{V}^N(\cdot, t) := S(t)\tilde{u}_0^N(\cdot, t)$ we the energy equality:

$$\|\tilde{V}^N(\cdot, t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \tilde{V}^N|^2 dx dt' = \|\tilde{u}_0^N\|_{L_2}^2. \quad (3.20)$$

By semigroup estimates, we have for $2 \leq p \leq \infty$, $10/3 \leq q \leq \infty$:

$$\|\tilde{V}^N(\cdot, t)\|_{L_p} \leq \frac{C(p)}{t^{\frac{3}{2}(\frac{1}{2}-\frac{1}{p})}} \|\tilde{u}_0^N\|_{L_2}, \quad (3.21)$$

$$\|\bar{V}^N(\cdot, t)\|_{L_q} \leq \frac{C(q)}{t^{\frac{3}{2}(\frac{3}{10}-\frac{1}{q})}} \|\bar{u}_0^N\|_{L_{\frac{10}{3}}}. \quad (3.22)$$

Hence, we have $w^N \in C_w([0, T]; J) \cap L_2(0, T; \overset{\circ}{J}^{\frac{1}{2}})$. Here, T is finite and $C_w([0, T]; J)$ denotes continuity with respect to the weak topology. Using these facts, and usual multiplicative inequalities, it is obvious that the following limits hold:

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\mathbb{R}^3} \phi_R(x) \phi_1(t) |w^N(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \phi_R \phi_1 |\nabla w^N|^2 dx dt' = \\ & = \int_{\mathbb{R}^3} \phi_1(t) |w^N(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \phi_1 |\nabla w^N|^2 dx dt', \\ & \lim_{R \rightarrow \infty} \int_0^t \int_{\mathbb{R}^3} (|w^N|^2 \partial_t \phi_1 \phi_R + 2(\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N \phi_1 \phi_R) dx dt' = \\ & = \int_0^t \int_{\mathbb{R}^3} (|w^N|^2 \partial_t \phi_1 + 2(\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N \phi_1) dx dt', \end{aligned}$$

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_0^t \int_{\mathbb{R}^3} (|w^N|^2 \phi_1 \Delta \phi_R + \phi_1 |w^N|^2 v \cdot \nabla \phi_R + \\ & + 2\phi_1 (\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : (w^N \otimes \nabla \phi_R)) dx dt' = 0. \end{aligned}$$

Let us focus on the term containing the pressure, namely

$$\int_0^t \int_{\mathbb{R}^3} q w^N \cdot \nabla \phi_R \phi_1 dx dt'.$$

Define $T(R) := B(2R) \setminus B(R)$. We can instead treat

$$\int_0^t \int_{T_+(R)} (q - [q]_{B(2R)}) w^N \cdot \nabla \phi_R \phi_1 dx dt'.$$

Using Poincare inequality, it is not so difficult to show:

$$\begin{aligned} & \left| \int_0^t \int_{T(R)} (p_1 - [p_1]_{B(2R)}) w^N \cdot \nabla \phi_R \phi_1 dx dt' \right| \leq \\ & \leq \frac{C \|\phi_1\|_{L_\infty(0,t)}}{R^{\frac{2}{3}}} \|w^N\|_{L_3(T(R) \times]0,t])} \|\nabla p_1\|_{L_{\frac{9}{8}, \frac{3}{2}}(Q_t)}, \end{aligned} \quad (3.23)$$

$$\begin{aligned} & \left| \int_0^t \int_{T(R)} (p_2 - [p_2]_{B(2R)}) w^N \cdot \nabla \phi_R \phi_1 dx dt' \right| \leq \\ & \leq C \|\phi_1\|_{L_\infty(0,t)} \|w^N\|_{L_{\frac{11}{4}}(T(R) \times]0,t])} \|\nabla p_2\|_{L_{\frac{11}{7}}(Q_t)}, \end{aligned} \quad (3.24)$$

$$\begin{aligned} & \left| \int_0^t \int_{T_+(R)} (p_3 - [p_3]_{B(2R)}) (w^N \cdot \nabla \phi_R) \phi_1 dx dt' \right| \leq \\ & \leq \frac{C \|\phi_1\|_{L_\infty(0,t)}}{R^{\frac{2}{5}}} \|w^N\|_{L_3(T(R) \times]0,t])} \|\nabla p_3\|_{L_{\frac{5}{4}, \frac{3}{2}}(Q_t)}. \end{aligned} \quad (3.25)$$

Using (3.1)-(3.1), multiplicative inequalities and properties of the pressure decomposition in Definition 1.1 we infer that

$$\lim_{R \rightarrow \infty} \int_0^t \int_{T(R)} q w^N \cdot \nabla \phi_R \phi_1 dx dt' = 0.$$

Thus, putting everything together, we get for arbitrary positive function $\phi_1 \in C_0^\infty(0, \infty)$:

$$\begin{aligned} & \int_{\mathbb{R}^3} \phi_1(t) |w^N(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \phi_1(t) |\nabla w^N|^2 dx dt' \leq \\ & \leq \int_0^t \int_{\mathbb{R}^3} |w^N|^2 \partial_t \phi_1 + 2(\bar{V}^N \otimes w^N + \bar{V}^N \otimes \bar{V}^N) : \nabla w^N \phi_1 dx dt' \end{aligned} \quad (3.26)$$

From Remark 1.2, we see that

$$\lim_{t \rightarrow 0} \|w^N(\cdot, t) - \tilde{u}_0^N(\cdot)\|_{L_2} = 0. \quad (3.27)$$

Using known arguments from [2], we have the following estimates:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3} |\bar{V}^N \otimes w^N : \nabla w^N| dx dt' \leq \\ & \leq C N^{\frac{1}{10}} \|u_0\|_{L^{3,\infty}}^{\frac{9}{10}} \left(\int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 dx dt' \right)^{\frac{4}{5}} \left(\int_0^t \frac{\|w^N(\cdot, \tau)\|_{L_2}^2}{\tau^{\frac{3}{4}}} d\tau \right)^{\frac{1}{5}}, \end{aligned} \quad (3.28)$$

$$\int_0^t \int_{\mathbb{R}^3} |\bar{V}^N \otimes \bar{V}^N : \nabla w^N| dx dt' \leq C t^{\frac{7}{20}} N^{\frac{1}{5}} \|u_0\|_{L^{3,\infty}}^{\frac{9}{5}} \|\nabla w^N\|_{L_2(Q_t)}. \quad (3.29)$$

Using (3.1), (3.27) and (3.1)-(3.29), we infer (3.3) by standard arguments involving an appropriate choices of $\phi_1(t) = \phi_\epsilon(t)$ and letting ϵ tend to zero. \square

Lemma 3.4. *Let u , v and u_0 be as in Definition 1.1. Then the following estimate is valid for all $N, t > 0$:*

$$\begin{aligned} \|u(\cdot, t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx dt' &\leq C(N^{-1} \|u_0\|_{L^{3,\infty}}^3 + t^{\frac{7}{10}} N^{\frac{2}{5}} \|u_0\|_{L^{3,\infty}}^{\frac{18}{5}}) + \\ &+ C \exp(C t^{\frac{1}{4}} N^{\frac{1}{2}} \|u_0\|_{L^{3,\infty}}^{\frac{9}{2}}) (N^{-\frac{1}{2}} t^{\frac{1}{4}} \|u_0\|_{L^{3,\infty}}^{\frac{33}{8}} + t^{\frac{19}{20}} N^{\frac{9}{10}} \|u_0\|_{L^{3,\infty}}^{\frac{199}{40}}). \end{aligned} \quad (3.30)$$

Hence, taking $N = t^{-\frac{1}{2}}$ gives the following scale invariant estimate:

$$\begin{aligned} \|u(\cdot, t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx dt' &\leq \\ &\leq C t^{\frac{1}{2}} \exp(C \|u_0\|_{L^{3,\infty}}^{\frac{9}{2}}) (\|u_0\|_{L^{3,\infty}}^{\frac{9}{8}} + 1) (\|u_0\|_{L^{3,\infty}}^3 + \|u_0\|_{L^{3,\infty}}^{\frac{18}{5}}). \end{aligned} \quad (3.31)$$

Proof. First observe that $u = w^N - \tilde{v}^N$. Thus, using (3.20) we see that

$$\begin{aligned} \|u(\cdot, t)\|_{L_2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx dt' &\leq \\ &\leq 2 \|\tilde{u}_0^N\|_{L_2}^2 + 2 \|w^N(\cdot, t)\|_{L_2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 dx dt'. \end{aligned}$$

By (2.4):

$$\|\tilde{u}_0^N\|_{L_2}^2 \leq C N^{-1} \|u_0\|_{L^{3,\infty}}^3. \quad (3.32)$$

Thus, it is sufficient to prove (3.4) for w^N in place of u . From now on, denote

$$y_N(t) := \|w^N(\cdot, t)\|_{L_2}^2.$$

Using (3.3), estimates (3.1)-(3.29), (3.32) and the Young's inequality obtain that

$$\begin{aligned} y_N(t) + \int_0^t \int_{\mathbb{R}^3} |\nabla w^N|^2 dx dt' &\leq C N^{\frac{1}{2}} \|u_0\|_{L^{3,\infty}}^{\frac{9}{2}} \int_0^t \frac{y_N(\tau)}{\tau^{\frac{3}{4}}} d\tau + \\ &+ C(N^{-1} \|u_0\|_{L^{3,\infty}}^3 + t^{\frac{7}{10}} N^{\frac{2}{5}} \|u_0\|_{L^{3,\infty}}^{\frac{18}{5}}). \end{aligned}$$

The conclusion is then easily reached using a Gronwall type Lemma. \square

3.2 Existence of global weak $L^{3,\infty}(\mathbb{R}^3)$ -solutions

Proof of Theorem 1.3 We have

$$u_0^{(k)} \xrightarrow{*} u_0$$

in $L^{3,\infty}$ and may assume that

$$M := \sup_k \|u_0^{(k)}\|_{L^{3,\infty}} < \infty.$$

Firstly, define

$$V^{(k)}(\cdot, t) := S(t)u_0^{(k)}(\cdot, t), \quad V(\cdot, t) := S(t)u_0(\cdot, t).$$

By Proposition 2.5, we see that $V^{(k)}$ converges to V on Q_∞ in the sense of distributions. By Proposition 2.4, we see that

$$\|V^{(k)}(\cdot, t)\|_{L^{3,\infty}} \leq CM, \quad (3.33)$$

$$\|\partial_t^m \nabla^l V^{(k)}(\cdot, t)\|_{L^r} \leq \frac{CM}{t^{m+\frac{l}{2}+\frac{3}{2}(\frac{1}{3}-\frac{1}{r})}}. \quad (3.34)$$

Here $r \in [3, \infty]$. For $T < \infty$ and $l \in]1, \infty[$, we have the compact embedding

$$W_l^{2,1}(B(n) \times]0, T]) \hookrightarrow C([0, T]; L_l(B(n))).$$

From this and (3.34) one immediately infers that for every $n \in \mathbb{N}$ and $l \in]1, \infty[$:

$$\partial_t^m \nabla^l V^{(k)} \rightarrow \partial_t^m \nabla^l V \text{ in } C([1/n, n]; L_l(B(n))). \quad (3.35)$$

Fixing $N = 1$ in Lemma 3.4 we have:

$$\|u^{(k)}(\cdot, t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla u^{(k)}|^2 dx dt' \leq f_0(M, t). \quad (3.36)$$

By means of a Cantor diagonalisation argument, we can abstract a subsequence such that for any finite $T > 0$:

$$u^{(k)} \xrightarrow{*} u \text{ in } L_{2,\infty}(Q_T), \quad (3.37)$$

$$\nabla u^{(k)} \rightharpoonup \nabla u \text{ in } L_2(Q_T). \quad (3.38)$$

Using (3.37), together with (3.4), we also get that:

$$\|u\|_{L_{2,\infty}(Q_t)} \leq C(M)t^{\frac{1}{2}}. \quad (3.39)$$

From (3.36) it is easily inferred that

$$\|u^{(k)} \cdot \nabla u^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_t)} \leq f_1(M, t). \quad (3.40)$$

By the same reasoning as in Lemma 3.1, we obtain:

$$\|V^{(k)} \cdot \nabla V^{(k)}\|_{L_{\frac{11}{7}}(Q_t)} \leq f_2(M, t), \quad (3.41)$$

$$\|V^{(k)} \cdot \nabla u^{(k)} + u^{(k)} \cdot \nabla V^{(k)}\|_{L_{\frac{5}{4},\frac{3}{2}}(Q_t)} \leq f_3(M, t). \quad (3.42)$$

Split $u^{(k)} = \sum_{i=1}^3 u^{i(k)}$ according to Definition 1.1, namely (3.4). By coercive estimates for the Stokes system, along with (3.40) obtain:

$$\|u^{1(k)}\|_{W_{\frac{9}{8},\frac{3}{2}}^{2,1}(Q_t)} + \|\nabla p_1^{(k)}\|_{L_{\frac{9}{8},\frac{3}{2}}(Q_t)} \leq C f_1(M, t), \quad (3.43)$$

$$\|u^{2(k)}\|_{W_{\frac{11}{7}}^{2,1}(Q_t)} + \|\nabla p_2^{(k)}\|_{L_{\frac{11}{7}}(Q_t)} \leq C f_2(M, t), \quad (3.44)$$

$$\|u^{3(k)}\|_{W_{\frac{5}{4},\frac{3}{2}}^{2,1}(Q_t)} + \|\nabla p_3^{(k)}\|_{L_{\frac{5}{4},\frac{3}{2}}(Q_t)} \leq C f_3(M, t). \quad (3.45)$$

By the previously mentioned embeddings, we infer from (3.43)-(3.45) that for any $n \in \mathbb{N}$ we have the following convergence for a certain subsequence:

$$u^{(k)} \rightarrow u \text{ in } C([0, n]; L_{\frac{9}{8}}(B(n))). \quad (3.46)$$

Hence, using (3.36), it is standard to infer that for any $s \in]1, 10/3[$

$$u^{(k)} \rightarrow u \text{ in } L_s(B(n) \times]0, n[). \quad (3.47)$$

It is also not so difficult to show that for any $f \in L_2$ and for any $n \in \mathbb{N}$:

$$\int_{\mathbb{R}^3} u^{(k)}(x, t) \cdot f(x) dx \rightarrow \int_{\mathbb{R}^3} u(x, t) \cdot f(x) dx \text{ in } C([0, n]). \quad (3.48)$$

Using (3.39) with (3.48), we establish that

$$\lim_{t \rightarrow 0} \|u(\cdot, t)\|_{L_2} = 0. \quad (3.49)$$

All that remains to show is establishing the local energy inequality (1.1) for the limit and establishing the energy inequality (3.3) for u . Verifying the local energy inequality is not so difficult and hence omitted. Let us focus on verifying (3.3) for u . By identical reasoning to Lemma 3.3, we have that for an arbitrary positive function $\phi_1(t) \in C_0^\infty(0, \infty)$:

$$\begin{aligned} & \int_{\mathbb{R}^3} \phi_1(t) |u(x, t)|^2 dx + 2 \int_0^t \int_{\mathbb{R}^3} \phi_1(t) |\nabla u|^2 dx dt' \leq \\ & \leq \int_0^t \int_{\mathbb{R}^3} |u|^2 \partial_t \phi_1 + 2(V \otimes u + V \otimes V) : \nabla u \phi_1 dx dt'. \end{aligned} \quad (3.50)$$

From Lemma 3.1 and semigroup estimates, we have that

$$(V \otimes u + V \otimes V) : \nabla u \in L_1(Q_T)$$

for any positive finite T . Using these facts and (3.49), the conclusion is reached by choosing appropriate $\phi_\epsilon = \phi_1$ and taking a limit. \square

Let us comment on Corollary 1.4. Recall that by Proposition 2.3, there exists a sequence $u_0^{(k)} \in C_{0,0}^\infty(\mathbb{R}^3)$ such that

$$u_0^{(k)} \xrightarrow{*} u_0$$

in $L^{3,\infty}$. It was shown in [31] that for any k there exists a global L_3 -weak solution $v^{(k)}$. Now, Corollary 1.4 follows from Theorem 1.3.

4 Uniqueness

First we introduce the notation $Q(z_0, R) = B(x_0, R) \times]t - R^2, t[$. Here, $z_0 = (x_0, t) \in Q_\infty$.

Proof of Theorem 1.5 Step I. Regularity. Our first remark is that, given $\varepsilon > 0$ and $R > 0$, there exists a number $R_*(T, R, \varepsilon) > 0$ such that if $B(x_0, R) \subset \mathbb{R}^3 \setminus B(R_*)$ and $t_0 - R^2 > 0$ then

$$\frac{1}{R^2} \int_{Q(z_0, R)} (|v|^3 + |q - [q]_{B(x_0, R)}|^{\frac{3}{2}}) dx dt \leq \varepsilon.$$

For v it is certainly true. For q , we can use Lemma 3.3. Indeed, if $q = p_1 + p_2 + p_3$, then, for example, we have

$$\begin{aligned}
& \frac{1}{R^2} \int_{Q(z_0, R)} |p_1 - [p_1]_{B(x_0, R)}|^{\frac{3}{2}} dx ds \leq \\
& \leq \frac{1}{R^2} \int_0^T \int_{B(x_0, R)} |p_1 - [p_1]_{B(x_0, R)}|^{\frac{3}{2}} dx ds \leq \frac{1}{R^{\frac{3}{2}}} \int_0^T \left(\int_{B(x_0, R)} |\nabla p_1|^{\frac{9}{8}} dx \right)^{\frac{4}{3}} dt \leq \\
& \leq \frac{1}{R^{\frac{3}{2}}} \int_0^T \left(\int_{\mathbb{R}^3 \setminus B(R_*)} |\nabla p_1|^{\frac{9}{8}} dx \right)^{\frac{4}{3}} dt \rightarrow 0
\end{aligned}$$

as $R_* \rightarrow \infty$ for any fixed $R > 0$. Since the pair v and q satisfies the local energy inequality, by ε -regularity theory developed in [3], we can claim that

$$|v(z_0)| \leq \frac{c}{R}$$

as long as z_0 and R satisfy the conditions above.

Now, our aim is to show that v is locally bounded. To this end, we can use condition (1.12) and state that there exists $R_0(x_0, \varepsilon_0) > 0$ such that

$$\|u_0\|_{L^{3,\infty}(B(x_0, R))} < \varepsilon_0$$

for all $0 < R < R_0(x_0, \varepsilon_0)$. Then

$$\|v(\cdot, t)\|_{L^{3,\infty}(B(x_0, R))} \leq \|u_0\|_{L^{3,\infty}(B(x_0, R))} + \varepsilon_0 < 2\varepsilon_0$$

for all $0 < R < R_0(x_0, \varepsilon_0)$ and for all $t \in]0, T[$.

Using Hölder inequality for Lorentz spaces, we have

$$\begin{aligned}
& \frac{1}{r} \left(\int_{t_0-r^2}^{t_0} \left(\int_{B(x_0, r)} |v|^2 dx \right)^2 dt \right)^{\frac{1}{4}} \leq \\
& \leq c \sup_{t_0-r^2 < t < t_0} \|v(\cdot, t)\|_{L^{3,\infty}(B(x_0, r))} \leq c\varepsilon_0
\end{aligned}$$

for all $t_0 \in]0, T]$, for all $0 < r < R_0(x_0, \varepsilon_0)$ satisfying $t_0 - r^2 > 0$, and c is a positive universal constant. Then the local boundedness follows from

ε -regularity conditions derived in [35] with a suitable choice of the constant ε_0 .

So, we can ensure that $v \in L_\infty(Q_{\delta,T})$ for any $\delta > 0$. Here, $Q_{\delta,T} = \mathbb{R}^3 \times]\delta, T[$. Then, we can easily deduce that, for any $\delta > 0$, $u \in W_2^{2,1}(Q_{\delta,T})$, $\nabla u \in L_{2,\infty}(Q_{\delta,T})$, and $\nabla q \in L_2(Q_{\delta,T})$. By iterative arguments, we complete the proof of the theorem.

Step II. Uniqueness. Regularity results proved above allow us to state that the energy identity

$$\frac{1}{2} \int_{\mathbb{R}^3} |u(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 dx ds = \int_0^t \int_{\mathbb{R}^3} V \otimes v : \nabla u dx ds$$

holds for any $t > 0$ and, moreover,

$$\int_{\mathbb{R}^3} \left(\partial_t u(x, t) \cdot w(x) + (v(x, t) \cdot \nabla v(x, t)) \cdot w(x) + \nabla u(x, t) : \nabla w(x) \right) dx = 0$$

for any $w \in C_{0,0}^\infty(\mathbb{R}^3)$ and for all $t \in]0, T[$.

Letting $\tilde{u} = \tilde{v} - V$ and $w = \tilde{u} - u$, we can repeat the same arguments as in [31] to obtain

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} |w(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \leq \\ & \leq \int_0^{t_0} \int_{\mathbb{R}^3} \left(\tilde{v} \otimes \tilde{v} : \nabla w - v \otimes v : \nabla w \right) dx dt = \int_0^{t_0} \int_{\mathbb{R}^3} (w \otimes v + v \otimes w) : \nabla w dx dt. \end{aligned}$$

So, finally,

$$\begin{aligned} I &:= \int_{\mathbb{R}^3} |w(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \leq \\ & \leq c \int_0^{t_0} \int_{\mathbb{R}^3} |v|^2 |w|^2 dx dt. \end{aligned}$$

Let us fix $s \in]0, T[$, then

$$I \leq cI_1 + cI_2 + cI_3,$$

where

$$\begin{aligned}
I_1 &= \int_0^{t_0} \int_{\mathbb{R}^3} |v(x, t) - u_0(x)|^2 |w(x, t)|^2 dx dt, \\
I_2 &= \int_0^{t_0} \int_{\mathbb{R}^3} |v(x, s) - u_0(x)|^2 |w(x, t)|^2 dx dt, \\
I_3 &= \int_0^{t_0} \int_{\mathbb{R}^3} |v(x, s)|^2 |w(x, t)|^2 dx dt.
\end{aligned}$$

The first two integrals are evaluated in the same way with the help of the Hölder inequality for Lorentz spaces:

$$\begin{aligned}
c(I_1 + I_2) &\leq c \int_0^{t_0} (\|v(\cdot, t) - u_0(\cdot)\|_{L^{3,\infty}}^2 + \\
&\quad + \|v(\cdot, s) - u_0(\cdot)\|_{L^{3,\infty}}^2) \|w(\cdot, t)\|_{L^{6,2}}^2 dt.
\end{aligned}$$

By assumptions of the theorem,

$$c(I_1 + I_2) \leq c\varepsilon_0 \int_0^{t_0} \|w(\cdot, t)\|_{L^{6,2}}^2 dt.$$

It remains to apply the Sobolev inequality and conclude that

$$c(I_1 + I_2) \leq c\varepsilon_0 \int_0^{t_0} \|\nabla w(\cdot, t)\|_{L_2}^2 dt.$$

To estimate I_3 , we are going to use the fact that $v(\cdot, s)$ is bounded for positive $s \leq T$, i.e.,

$$\|v(\cdot, s)\|_{L^\infty} \leq g(s).$$

Here, it might happen that $g(s) \rightarrow \infty$ if $s \rightarrow 0$. So,

$$I_3 \leq g^2(s) \int_0^{t_0} \int_{\mathbb{R}^3} |w(x, t)|^2 dx dt.$$

Then reducing ε_0 if necessary, we find

$$\int_{\mathbb{R}^3} |w(x, t_0)|^2 dx + \int_0^{t_0} \int_{\mathbb{R}^3} |\nabla w|^2 dx dt \leq cg^2(s) \int_0^{t_0} \int_{\mathbb{R}^3} |w(x, t)|^2 dx dt$$

for all $t_0 \in]0, T[$, which implies that $w(\cdot, t) = 0$ for the same t . \square

To justify Corollary 1.6, we can argue as follows. First, it can be shown that

$$\|u_0\|_{L^{3,\infty}(B(x_0, R))} \rightarrow 0$$

as $R \rightarrow 0$. Indeed, if v is a weak $L^{3,\infty}$ -solution in Q_T , then for a.a. $t \in]0, T[$ we have $v(\cdot, t) \in L^{3,\infty}$ along with the following property. Namely, for all $x_0 \in \mathbb{R}^3$:

$$\|v(\cdot, t)\|_{L^{3,\infty}(B(x_0, R))} \rightarrow 0$$

as $R \rightarrow 0$. Since it is assumed that $v \in C([0, T]; L^{3,\infty})$, the above property in fact holds for all $t \in [0, T]$.

Now, one should split the interval $[0, T]$ into sufficiently small pieces by points $t_k = kT/N$ with $k = 1, 2, \dots, N$ so that

$$\|v(\cdot, t) - v(\cdot, t_{k-1})\|_{L^{3,\infty}(\mathbb{R}^3)} < \varepsilon_0$$

for any $t \in [t_{k-1}, t_k]$ and for all $k = 1, 2, \dots, N$. It remains to apply Theorem 1.5 successively for $k = 1, 2, \dots, N$.

5 Regularity

Proof of Theorem 1.7 We use the Kato iteration scheme. Let us define the following, for $k = 1, 2, \dots$,

$$v^{(1)} = V, \quad V^{(k+1)} = V + u^{(k+1)},$$

where $u^{(k+1)}$ solves the following problem

$$\partial_t u^{(k+1)} - \Delta u^{(k+1)} + \nabla q^{(k+1)} = -\operatorname{div} v^{(k)} \otimes v^{(k)}, \quad \operatorname{div} u^{(k+1)} = 0$$

in Q_T ,

$$u^{(k+1)}(\cdot, 0) = 0$$

in \mathbb{R}^3 . It is easy to check that for solutions to the above linear problem the following estimates are true

$$\langle u^{(k+1)} \rangle_{Q_T} \leq c \langle v^{(k)} \rangle_{Q_T}^2,$$

$$\|u^{(k+1)}\|_{L_\infty(0,T;L_3)} \leq c \langle v^{(k)} \rangle_{Q_T}^2$$

and thus we have

$$\langle v^{(k+1)} \rangle_{Q_T} \leq \langle V \rangle_{Q_T} + c \langle v^{(k)} \rangle_{Q_T}^2,$$

$$\|v^{(k+1)}\|_{L_\infty(0,T;L^{3,\infty})} \leq \|V\|_{L_\infty(0,T;L^{3,\infty})} + c \langle v^{(k)} \rangle_{Q_T}^2,$$

and

$$\|v^{(k+1)} - V\|_{L_\infty(0,T;L_3)} \leq c \langle v^{(k)} \rangle_{Q_T}^2$$

for all $k = 1, 2, \dots$. Using Kato's arguments, one easily show that for $\varepsilon < \frac{1}{4c}$ we shall have

$$\langle v^{(k)} \rangle_{Q_T} < 2 \langle V \rangle_{Q_T} \quad (5.1)$$

for all $k = 1, 2, \dots$. We get, in addition, that

$$\|v^{(k)}\|_{L_\infty(0,T;L^{3,\infty})} \leq \|V\|_{L_\infty(0,T;L^{3,\infty})} + \langle V \rangle_{Q_T}, \quad (5.2)$$

$$\|v^{(k+1)} - V\|_{L_\infty(0,T;L_3)} \leq \langle V \rangle_{Q_T} \quad (5.3)$$

for all $k = 1, 2, \dots$. Furthermore, Kato's arguments also give that there is a $v = V + u$ such that

$$\langle v^{(k)} - v \rangle_{Q_T}, \langle u^{(k)} - u \rangle_{Q_T} \rightarrow 0, \quad (5.4)$$

$$\|v^{(k)} - v\|_{L_\infty(0,T;L^{3,\infty})}, \|u^{(k)} - u\|_{L_\infty(0,T;L_3)} \rightarrow 0. \quad (5.5)$$

Next we note that by interpolation:

$$t^{\frac{1}{8}} \|g(\cdot, t)\|_{L_4} \leq C (\|g(\cdot, t)\|_{L^{3,\infty}})^{\frac{3}{8}} (t^{\frac{1}{5}} \|g(\cdot, t)\|_{L_5})^{\frac{5}{8}}. \quad (5.6)$$

Using this and (5.4)-(5.5), we immediately see that

$$\|v^{(k)} - v\|_{L_4(Q_T)}, \|u^{(k)} - u\|_{L_4(Q_T)} \rightarrow 0. \quad (5.7)$$

We also can exploit our equation, together with the pressure equation, to derive the following estimate for the energy and pressure:

$$\|u^{(k)} - u^{(m)}\|_{2,\infty,Q_T}^2 + \|\nabla u^{(k)} - \nabla u^{(m)}\|_{2,Q_T}^2 + \|q^{(k)} - q^{(m)}\|_{2,Q_T}^2 \leq$$

$$\leq c \int_0^T \int_{\mathbb{R}^3} |v^{(k)} \otimes v^{(k)} - v^{(m)} \otimes v^{(m)}|^2 dx dt. \quad (5.8)$$

Using (5.7), we immediately see the following

$$u^{(k)} \rightarrow u \text{ in } W_2^{1,0}(Q_T) \cap C([0, T]; L_2(\mathbb{R}^3)) \cap L_4(Q_T), \quad (5.9)$$

$$u(\cdot, 0) = 0, \quad (5.10)$$

$$q^{(k)} \rightarrow q \text{ in } L_2(Q_T). \quad (5.11)$$

Clearly, the pair v and q satisfies the Navier-Stokes equations, in a distributional sense. It is easily verified that

$$S(t)u_0 \in L_4(Q_T) \cap L_{2,\infty}(B(R) \times]0, T[) \cap W_2^{1,0}(B(R) \times]\epsilon, T[) \quad (5.12)$$

for any $0 < R$, $0 < \epsilon < T$. By (5.9)-(5.11), v has the same property. It is known that this, along with $q \in L_2(Q_T)$, is sufficient to infer that the pair v and q satisfies the local energy equality. This can be shown by a mollification argument. Showing that u satisfies the energy inequality (on Q_T) present in our definition of global weak $L^{3,\infty}$ solution (in fact, in this case it is an equality), can now be carried out in a similar way to Lemma 3.3. Here, certain decay properties of u, q from (5.9)-(5.11) are needed, as well as the fact that $\lim_{t \rightarrow 0^+} \|u(\cdot, t)\|_{L_2(\mathbb{R}^3)} = 0$. \square

Proof of Theorem 1.8 Condition (1.17) ensures that there exists an $N > 0$ such that

$$\|(u_0)_+^N\|_{L^{3,\infty}} < \varepsilon_3.$$

Thus, by the convolution inequality,

$$\langle S(t)(u_0)_+^N \rangle_{Q_T}, \|S(t)(u_0)_+^N\|_{L_\infty(0,T;L^{3,\infty})} < C\varepsilon_3.$$

By Lemma 2.1, we have that

$$\|(u_0)_-^N\|_{L^5} \leq CN^{\frac{2}{5}} \|u_0\|_{L^{3,\infty}}^{\frac{3}{5}}.$$

Thus,

$$\langle V \rangle_{Q_T} < C\varepsilon_3 + T^{\frac{1}{5}} CN^{\frac{2}{5}} \|u_0\|_{L^{3,\infty}}^{\frac{3}{5}}.$$

Taking $T := T(u_0)$ and ε_3 sufficiently small gives, by Theorem 1.7, the existence of weak $L^{3,\infty}$ solution on Q_T such that

$$\|v - S_1(t)(u_0)_-^N\|_{L_\infty(0,T;L^{3,\infty})} \leq$$

$$\begin{aligned} &\leq \|v - V\|_{L_\infty(0,T;L^{3,\infty})} + \|S_1(t)(u_0)_+^N\|_{L_\infty(0,T;L^{3,\infty})} < \\ &< V >_{Q_T} + C\varepsilon_3 < \varepsilon_0. \end{aligned} \quad (5.13)$$

Next we notice that $S_1(t)(u_0)_-^N$ is bounded in (Q_T) and moreover

$$\|S_1(t)(u_0)_-^N\|_{L^{3,\infty}(B(x_0,R))} \leq CRN. \quad (5.14)$$

These facts, along with (5), are enough to conclude by using minor adaptations to the proof of Theorem 1.5. \square

Remark 5.1. *Furthermore, there is the lower bound for T :*

$$T \geq \frac{\min(\varepsilon^5, \varepsilon_0^5)}{CN^2\|u_0\|_{L^{3,\infty}}^3}. \quad (5.15)$$

Here, C is a universal constant. Moreover, ε and ε_0 are from Theorems 1.5 and 1.7 respectively.

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