

A penalty scheme and policy iteration for nonlocal HJB variational inequalities with monotone nonlinearities

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Abstract. We propose a class of numerical schemes for nonlocal HJB variational inequalities (HJBVIs) with monotone nonlinearities arising from mixed optimal stopping and control of processes with infinite activity jumps, where the objective is specified by a monotone recursive preference. The solution and free boundary of the HJBVI are constructed from a sequence of penalized equations, for which the penalization error is estimated. The penalized equation is then discretized by a class of semi-implicit monotone approximations. We present a novel analysis technique for the well-posedness of the discrete equation, and demonstrate the convergence of the scheme, which subsequently gives a constructive proof for the existence of a solution to the penalized equation and variational inequality. We further propose an efficient iterative algorithm with local superlinear convergence for solving the discrete equation. Numerical experiments are presented for an optimal investment problem under ambiguity and a two-dimensional recursive consumption-portfolio allocation problem.

Key words. HJB variational inequalities, non-Lipschitz Epstein-Zin preferences, monotone nonlinearity, penalization, semi-smooth Newton methods, optimal investment

AMS subject classifications. 65M06, 65M12, 62L15, 93E20, 91G80

1 Introduction

In this article, we consider the following nonlocal Hamilton-Jacobi-Bellman variational inequality (HJBVI) on $\mathcal{Q}_T = (0, T] \times \mathbb{R}^d$ with $T > 0$:

$$\begin{aligned} 0 &= F(t, x, u, Du, D^2u, \{K^\alpha u\}_{\alpha \in \mathbf{A}}, \{B^\alpha u\}_{\alpha \in \mathbf{A}}) \\ &= \begin{cases} \min \{u - \zeta, u_t + \inf_{\alpha \in \mathbf{A}} (-L^\alpha u - f^\alpha(t, x, u, (\sigma^\alpha)^T Du, B^\alpha u))\}, & (t, x) \in \mathcal{Q}_T, \\ u(0, x) - g(x), & x \in \mathbb{R}^d, \end{cases} \end{aligned} \quad (1.1)$$

with the operators $L^\alpha := A^\alpha + K^\alpha$ and B^α satisfying for $\phi \in C^{1,2}(\bar{\mathcal{Q}}_T)$ that

$$A^\alpha \phi(t, x) = \frac{1}{2} \text{tr}(\sigma^\alpha(t, x)(\sigma^\alpha(t, x))^T D^2 \phi(t, x)) + b^\alpha(t, x) \cdot D \phi(t, x), \quad (1.2)$$

$$K^\alpha \phi(t, x) = \int_E (\phi(t, x + \eta^\alpha(t, x, e)) - \phi(t, x) - \eta^\alpha(t, x, e) \cdot D \phi(t, x)) \nu(de), \quad (1.3)$$

$$B^\alpha \phi(t, x) = \int_E m(\phi(t, x + \eta^\alpha(t, x, e)) - \phi(t, x)) \gamma(t, x, e) \nu(de), \quad (1.4)$$

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where $u : \bar{Q}_T \rightarrow \mathbb{R}$ is an unknown function, \mathbf{A} is a given nonempty compact metric space, ν is a given σ -finite measure on $E = \mathbb{R}^n \setminus \{0\}$ satisfying $\int_E (1 \wedge |e|^2) \nu(de) < \infty$, $m : \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R}^d \rightarrow \mathbb{R}$, $\zeta : \bar{Q}_T \rightarrow \mathbb{R}$, $\gamma : \bar{Q}_T \times E \rightarrow \mathbb{R}$ are given continuous functions, for each $\alpha \in \mathbf{A}$, $b^\alpha : \bar{Q}_T \rightarrow \mathbb{R}^d$, $\sigma^\alpha : \bar{Q}_T \rightarrow \mathbb{R}^{d \times d}$, $\eta^\alpha : \bar{Q}_T \times E \rightarrow \mathbb{R}^d$, $f^\alpha : \bar{Q}_T \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions, and for each $\phi \in C^{1,2}(\bar{Q}_T)$ and $(t, x) \in \bar{Q}_T$, $D\phi(t, x)$ and $D^2\phi(t, x)$ are the gradient and Hessian of ϕ at (t, x) , respectively. We shall specify the precise conditions on the coefficients of (1.1) in Assumption 1.

Note that the integral operators K^α and B^α create a non-locality of (1.1), more precisely, the value of the solution u at each point $(t, x) \in \bar{Q}_T$ evolves based on the weighted average of values of u at other points with respect to the measure ν . These integral operators arise naturally as the generators of pure jump Lévy processes with Lévy measure ν (see e.g. [10]). Here we allow ν to be a singular measure such that $\nu(\{e \in E \mid |e| < 1\}) = \infty$, which is associated with infinite activity Lévy processes, i.e., Lévy processes that admit an infinite number of jumps in each time interval.

We shall allow the function f^α , called the driver of (1.1), to be monotone, possibly non-Fréchet-differentiable, and of arbitrary growth in its third component. Equation (1.1) with monotone nonlinearities extends the classical HJBVIs with linear drivers, i.e., $f^\alpha(t, x, y, z, k) \equiv \ell^\alpha(t, x) - r^\alpha(t, x)y$ for some functions $\ell^\alpha, r^\alpha : \bar{Q}_T \rightarrow \mathbb{R}$, and play an important role in modern finance, including the following: models for American options in a market with constrained portfolios [18, 17, 2], recursive utility optimization problems [20], and robust pricing and risk measures under probability model uncertainty [3, 38, 35]. We remark that imposing merely monotonicity assumptions on the drivers allows us to consider several important non-smooth Lipschitz drivers stemming from robust pricing [38, 19, 35] and non-Lipschitz drivers arising in stochastic recursive control problems (e.g. the Epstein-Zin preference in [20, 30, 34]), while including an extra nonlinearity in the operator B^α enables us to incorporate ambiguity in the jump processes [38]. In particular, we present two worked-out examples in Section 6, where the first one describes the situation that an investor chooses the optimal wealth allocation and liquidation time of an asset, whose price process has infinite activity jumps, by taking potential model misspecification into consideration, while the second one describes the situation that an agent chooses their optimal consumption and investment strategies based on the (nonlinear) Epstein-Zin utility. These problems lead naturally to HJBVIs of the type (1.1) with the key features present (i.e., singular non-local term, non-smooth/non-Lipschitz driver, control optimization, obstacle term). As the solution to (1.1) is in general not known analytically, it is important to construct effective and robust numerical schemes for solving these fully nonlinear equations.

To the best of our knowledge, there is no published numerical scheme covering the generality of (1.1). However, there is a vast literature on monotone approximations for local HJB equations (e.g., [11, 4, 31, 12]) and on monotone finite-difference quadrature schemes for nonlocal HJB equations (e.g., [6, 5]). A penalty approximation to parabolic variational inequalities (with applications to American option pricing problems) is first given in [16], where convergence of the penalized, discretized solutions and global monotone convergence of a nonlinear iteration for the discretized system is shown, with a finite termination property. We refer the reader to [25, 23] for further analyses of these parabolic penalty equations, and to [22, 24, 29] for penalty approximations to elliptic variational inequalities.

Works covering specific extensions to the aforementioned references include [7, 40] for an application of policy iteration together with penalization to solve HJB obstacle problems with linear drivers, to [15] for schemes to HJB obstacle problems with Lipschitz drivers based on piecewise constant policy time stepping, and to [41] for policy iteration for (finite-dimensional) static HJB equations with Fréchet-differentiable concave drivers and finite control sets.

In this paper, we shall construct a class of monotone schemes for solving (1.1) with a monotone (possibly non-Fréchet-differentiable) driver and a compact set of controls. Note that monotonicity of the scheme is crucial, since it is well-known that non-monotone schemes may fail to converge or even converge to false “solutions” [12]. By Godunov’s Theorem [21], in general, one can expect a monotone scheme to be at most first-order accurate.

Recently, a class of non-monotone “filtered” schemes has been proposed and analyzed in [8] for parabolic HJB equations (without the integral operators K^α and B^α), which combines monotone schemes with high-order non-monotone discretizations, and exhibits an overall high order convergence behaviour for smooth enough solutions. However, as pointed out in [8, Remark 2.8], it is more challenging to establish the well-posedness and convergence of such schemes in a general setting due to the loss of monotonicity. The presence of nonlocal operators K^α and B^α and the non-smooth non-Lipschitz driver f^α further complicates the construction of high-order time and space discretizations of (1.1). Hence, we focus on monotone discretizations of (1.1) in this paper.

We emphasize that the non-Lipschitz setting of the drivers prevents us from adopting the standard Banach fixed-point arguments (see e.g. [12, 15]) to establish the well-posedness and stability of the discrete approximations of (1.1), and hence new analysis techniques are required. Moreover, the non-differentiability and non-convexity of the driver in y and its nonlinear dependence on z and k also introduce substantial difficulties in designing efficient iterative algorithms for solving the discrete equations.

The main contributions of this work are as follows:

- We formulate a penalty approximation to (1.1) with a monotone driver, and demonstrate that, as the penalty parameter tends to infinity, the solution of the penalized equation converges to the solution of (1.1) monotonically from below, at a rate depending explicitly on the regularity of the obstacle, which extends the results in [29] to nonlocal equations with monotone drivers and time-dependent obstacles. These convergence results further lead us to a convergent approximation of the free boundary of (1.1), which to our best knowledge is new, even in the classical cases with linear drivers.
- We propose a class of semi-implicit monotone approximations to the penalized equations, which enjoy a stability condition independent of the penalty parameter. We further present a novel analysis technique for the well-posedness of the resulting (infinite-dimensional) discrete equation by constructing Lipschitz approximations of the monotone driver via smoothing and truncation. The convergence of the scheme is demonstrated, which subsequently gives a constructive proof for the existence of a bounded viscosity solution to the penalized equations and the HJBVI (1.1).
- For practical implementations, we propose an efficient iterative algorithm for a localized discrete equation with a slantly differentiable driver, and demonstrate the local superlinear convergence for the value functions, which extends the results obtained in the cases with linear drivers (see [7, 40]) or with Fréchet-differentiable convex drivers and finite control sets [41].
- Numerical examples for an optimal investment problem for jump-diffusion models under ambiguity and a two-dimensional recursive consumption-portfolio allocation problem with stochastic volatility models are included to investigate the convergence order of the scheme with respect to different discretization parameters.

We organize this paper as follows. Section 2 gives standard definitions and assumptions on the HJBVI (1.1). We shall propose a penalty approximation to the HJBVI in Section 3 and study

its convergence properties. Then we derive a class of fully discrete monotone schemes for the penalized equations in Section 4.1, and establish their convergence in Section 4.2. A Newton-type iterative method with local superlinear convergence is constructed in Section 5 to solve the resulting discrete equations. Numerical examples for an optimal investment problem under ambiguity and a recursive consumption-portfolio allocation problem are presented in Section 6 to illustrate the effectiveness of our algorithms.

2 Main assumptions and preliminaries

In this section, we state our main assumptions on the coefficients of (1.1) and introduce related concepts of solutions. We start by collecting some useful notation which is needed frequently throughout this work.

For any given function $g : \mathbb{R}^{d_1} \mapsto \mathbb{R}^{d_2}$, we define by $g^+ := \max(g, 0)$ and $g^- := \max(-g, 0)$ the (component-wise) positive and negative part of g , respectively. Also for a function $f : \bar{Q}_T := [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d_1 \times d_2}$ we define the following (semi-)norms:

$$|f|_0 = \sup_{(t,x) \in \bar{Q}_T} |f(t,x)|, \quad |f|_1 = \sup_{t \in [0,T], x, x' \in \mathbb{R}^d, x \neq x'} \frac{|f(t,x) - f(t,x')|}{|x - x'|}, \quad \|f\|_1 = |f|_0 + |f|_1,$$

which extend naturally to time-independent functions or vectors.

Now we turn to the standing assumptions on the coefficients of the HJBVI (1.1):

Assumption 1. *Let \mathbf{A} be a non-empty compact metric space. Moreover, there exists $C > 0$ and $\mu \in \mathbb{R}$ such that it holds for any $(\alpha, e, t, x, z, k) \in \mathbf{A} \times E \times \bar{Q}_T \times \mathbb{R}^d \times \mathbb{R}$ and $y, y' \in \mathbb{R}$ that:*

- (1) *The mappings $\mathbf{A} \times [0, T] \ni (\alpha, t) \mapsto (b^\alpha(t, x), \sigma^\alpha(t, x), \eta^\alpha(t, x, e)) \in \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R}^d$ and $[0, T] \ni t \mapsto (\gamma(t, x, e), \zeta(t, x)) \in \mathbb{R} \times \mathbb{R}$ are continuous, and the following estimates are satisfied: $g(x) \geq \zeta(0, x)$, $\gamma(t, x, e) \geq 0$, $\|b^\alpha\|_1 + \|\sigma^\alpha\|_1 + \|\zeta\|_1 + \|g\|_1 \leq C$,*

$$\|\eta^\alpha(\cdot, \cdot, e)\|_1 + |\gamma(\cdot, \cdot, e)|_0 \leq C(1 \wedge |e|), \quad |\gamma(\cdot, \cdot, e)|_1 \leq C(1 \wedge |e|^2).$$

- (2) (a) (Boundedness.) $|f^\alpha(t, x, 0, 0, 0)| \leq C$.

- (b) (Monotonicity.) *The mapping $\mathbb{R} \ni y \mapsto f^\alpha(t, x, y, z, k) \in \mathbb{R}$ is monotone in the sense that there exists a continuous increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that*

$$(y - y')(f^\alpha(t, x, y, z, k) - f^\alpha(t, x, y', z, k)) \leq \mu|y - y'|^2, \quad (2.1)$$

$$|f^\alpha(t, x, y, z, k)| \leq |f^\alpha(t, x, 0, 0, 0)| + \varphi(|y|) + C(|z| + |k|), \quad (2.2)$$

and $\mathbb{R} \ni k \mapsto f^\alpha(t, x, y, z, k) \in \mathbb{R}$ is non-decreasing.

- (c) (Continuity.) *The mapping $\mathbf{A} \times [0, T] \times \mathbb{R} \ni (\alpha, t, y) \mapsto f^\alpha(t, x, y, z, k) \in \mathbb{R}$ is continuous, and the mapping $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \ni (x, z, k) \mapsto f^\alpha(t, x, y, z, k) \in \mathbb{R}$ is Lipschitz continuous with the constant C .*

- (3) $m : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous and non-decreasing with $m(0) = 0$.

- (4) ν is a σ -finite Borel measure on $E = \mathbb{R}^n \setminus \{0\}$ and satisfies $\int_E (1 \wedge |e|^2) \nu(de) < \infty$.

Remark 1. *Although our discussions focus on the cases with coefficients $b^\alpha, \sigma^\alpha, \zeta, g$ bounded in x , similar results and analysis are valid for coefficients with polynomial growth as well. Moreover,*

as pointed out in [28], there is no loss of generality by assuming f^α is strictly monotone in y with $\mu < 0$ in (2.1) (this can be seen by carrying out an exponential time scaling of the solution).

Finally, we remark that for any Lipschitz continuous and non-decreasing function $\psi : \mathbb{R} \rightarrow \mathbb{R}$, in particular the functions $m : \mathbb{R} \rightarrow \mathbb{R}$ and $\mathbb{R} \ni k \mapsto f^\alpha(t, x, u, p, k) \in \mathbb{R}$, one has $\psi(x) - \psi(y) \leq C(x - y)^+$ for any $x, y \in \mathbb{R}$, which will be used frequently in our subsequent analysis.

We emphasize that Assumption 1 only requires f^α to be monotone in y (up to an additive linear function) and allows the coefficients σ^α and η^α to vanish at certain points. Therefore, due to lack of regularization from a Laplacian or fractional Laplacian operator, the solutions of (1.1) are typically not smooth, and we shall understand the equation in the viscosity sense [28]:

Definition 2.1 (Viscosity solution). *An upper (resp. lower) semicontinuous function u is said to be a viscosity subsolution (resp. supersolution) of (1.1) if and only if for any point $\mathbf{x}_0 = (t_0, x_0)$ and for any $\phi \in C^{1,2}(\bar{Q}_T)$ such that $\phi(\mathbf{x}_0) = u(\mathbf{x}_0)$ and $u - \phi$ attains its global maximum (resp. minimum) at \mathbf{x}_0 , one has*

$$\begin{aligned} & F_*(\mathbf{x}_0, u(\mathbf{x}_0), D\phi(\mathbf{x}_0), D^2\phi(\mathbf{x}_0), \{K^\alpha\phi(\mathbf{x}_0)\}_{\alpha \in \mathbf{A}}, \{B^\alpha\phi(\mathbf{x}_0)\}_{\alpha \in \mathbf{A}}) \leq 0 \\ & (\text{resp. } F^*(\mathbf{x}_0, u(\mathbf{x}_0), D\phi(\mathbf{x}_0), D^2\phi(\mathbf{x}_0), \{K^\alpha\phi(\mathbf{x}_0)\}_{\alpha \in \mathbf{A}}, \{B^\alpha\phi(\mathbf{x}_0)\}_{\alpha \in \mathbf{A}}) \geq 0). \end{aligned}$$

A continuous function is a viscosity solution of the HJBVI (1.1) if it is both a viscosity sub- and supersolution.

We will demonstrate that (1.1) admits a unique bounded solution under Assumption 1. The uniqueness follows directly from the comparison principle which we establish below (see Proposition 3.1), while a continuous bounded solution can be explicitly constructed through discrete approximations (see Remark 2).

3 Penalty approximations for the HJBVI

In this section, we shall propose a penalty approximation for the HJBVI (1.1), which is an extension of the ideas used for local HJB obstacle problems (with linear drivers) in [29, 40] and for American options in [23].

For any given parameter $\rho \geq 0$, we shall consider the following penalized problem:

$$\begin{aligned} 0 &= F^\rho(\mathbf{x}, u, Du, D^2u, \{K^\alpha u\}_{\alpha \in \mathbf{A}}, \{B^\alpha u\}_{\alpha \in \mathbf{A}}) \\ &= \begin{cases} u_t^\rho + \inf_{\alpha \in \mathbf{A}} (-L^\alpha u^\rho - f^\alpha(\mathbf{x}, u^\rho, (\sigma^\alpha)^T Du^\rho, B^\alpha u^\rho)) - \rho(\zeta - u^\rho)^+, & \mathbf{x} \in Q_T, \\ u^\rho(0, x) - g(x), & x \in \mathbb{R}^d, \end{cases} \end{aligned} \quad (3.1)$$

which will be interpreted in the viscosity sense similar to Definition 2.1 by virtue of the possible degeneracy of the equation.

In the following, we shall focus on the uniqueness of viscosity solutions. The proof for the existence of bounded solutions will be deferred to Section 4, where we will construct continuous solutions of (3.1) through numerical schemes and demonstrate they are bounded independent of the penalty parameter ρ (see Remark 2).

The next proposition presents a continuous dependence estimate for the solutions of the penalized equation, which quantifies the stability properties of solutions with respect to the obstacle. As the reader will see immediately, this estimate not only implies the comparison principle and uniqueness of viscosity solutions, but also enables us to derive the convergence rate of the penalty approximation to the HJBVI (1.1) and construct convergent approximations for the free boundary.

The proof of this estimate follows essentially along the standard doubling of variables argument, with extra technicalities arising from the nonlinearities of f^α and B^α . We include a detailed proof in Appendix A for the convenience of the reader.

Proposition 3.1. *Consider (1.1) and (3.1) with obstacles ζ_1 and ζ_2 (instead of ζ), such that in each case Assumption 1 is satisfied (with ζ in Item (1) replaced by ζ_1 and ζ_2 , respectively). Let u_1 (resp. u_2) be a bounded subsolution (resp. supersolution) to (1.1) with ζ_1 (resp. ζ_2) and u_1^ρ (resp. u_2^ρ) be a bounded subsolution (resp. supersolution) to (3.1) with ζ_1 (resp. ζ_2), then it holds for any $(t, x) \in \bar{Q}_T$ that*

$$u_1(t, x) - u_2(t, x) \leq |(u_1(0, \cdot) - u_2(0, \cdot))^+|_0 + |\zeta_1 - \zeta_2|_0, \quad (3.2)$$

$$u_1^\rho(t, x) - u_2^\rho(t, x) \leq |(u_1^\rho(0, \cdot) - u_2^\rho(0, \cdot))^+|_0 + |\zeta_1 - \zeta_2|_0. \quad (3.3)$$

We now proceed to study the convergence of the penalized equation (3.1) to the HJBVI (1.1). The following theorem illustrates the monotone convergence of $\{u^\rho\}_{\rho>0}$ in terms of the penalty parameter ρ .

Theorem 3.2. *Suppose Assumption 1 holds. Let u and u^ρ be the bounded viscosity solutions to, respectively, (1.1) and (3.1) with parameter $\rho \geq 0$. Then $\{u^\rho\}_{\rho>0}$ converges monotonically from below to u as $\rho \rightarrow \infty$, uniformly on compact sets.*

Proof. We start with the following two important observations, which can be established directly from the definitions: (1) If u^ρ is a subsolution to (3.1) with any $\rho \geq 0$, then u^ρ is a subsolution to (1.1); (2) If $\rho_1 \leq \rho_2$ and u^{ρ_2} is a supersolution to (3.1) with the parameter ρ_2 , then u^{ρ_2} is a supersolution to (3.1) with the parameter ρ_1 . Then the comparison principles for (1.1) and (3.1) enable us to conclude that $u^{\rho_1} \leq u^{\rho_2} \leq u$ for any $\rho_1 \leq \rho_2$. Finally, the monotonicity of $\{u^\rho\}_{\rho>0}$ and the boundedness of u imply that $\{u^\rho\}_{\rho>0}$ converge monotonically to some function \bar{u} , which is the unique solution of (1.1) due to the stability of viscosity solutions (see e.g. [2]). \square

The next result asserts the convergence rate of the penalized equation to the HJBVI, which depends on the regularity of the obstacle as observed in [29, 23].

Theorem 3.3. *Let u and u^ρ be the bounded viscosity solution to (1.1) and (3.1). Suppose Assumption 1 holds and the obstacle ζ is continuously differentiable in t with a bounded derivative. Then there exists a constant $C_0 > 0$, independent of the penalty parameter ρ , such that*

$$0 \leq u(\mathbf{x}) - u^\rho(\mathbf{x}) \leq C_0 \rho^{-1/2}, \quad \mathbf{x} \in \bar{Q}_T. \quad (3.4)$$

If we further assume ζ is semiconvex in x , i.e., there exists some $C \in \mathbb{R}$ such that $\zeta(t, x) - C|x|^2$ is convex in x , then we have

$$0 \leq u(\mathbf{x}) - u^\rho(\mathbf{x}) \leq C_0/\rho, \quad \mathbf{x} \in \bar{Q}_T. \quad (3.5)$$

Proof. The proof follows essentially the steps in the arguments for [29, Theorem 2.1], together with the continuous dependence estimate as ascertained by Proposition 3.1. The main step is first establishing (3.5) for (1.1) with regular obstacle $\zeta \in C_b^{1,2}(Q_T)$. Then, for (1.1) with a general obstacle, we shall consider a sequence of regularized HJBVIs and regularized penalized equations with the following initial conditions and obstacles: for all $(t, x) \in \bar{Q}_T$,

$$g_\varepsilon(x) = \int_{\mathbb{R}^d} g(y) \varepsilon^{-d} \varrho(\varepsilon^{-1}(x - y)) dy, \quad \zeta_\varepsilon(t, x) = \int_{\mathbb{R}^d} \zeta(t, y) \varepsilon^{-d} \varrho(\varepsilon^{-1}(x - y)) dy, \quad \varepsilon > 0,$$

where ϱ is a positive smooth function supported in $\{|x| < 1\}$ with mass one. Note that $\zeta_\varepsilon \in C^{1,2}(\mathcal{Q}_T)$ and the assumption $g(\cdot) \geq \zeta(0, \cdot)$ implies $g_\varepsilon(x) \geq \zeta_\varepsilon(0, x)$ for all $\varepsilon > 0$ and $x \in \mathbb{R}^d$. Moreover, the Lipschitz continuity of g and ζ in x shows that $|g_\varepsilon - g|_0 + |\zeta_\varepsilon - \zeta|_0 \leq C\varepsilon$, for some constant C independent of ε . Then, we can estimate the penalization error for the regularized HJBVI and the regularization error by using (3.5) and Proposition 3.1, respectively, and balance the approximation error in the two cases to obtain the desired conclusions. \square

We end this section with a convergent approximation of the free boundary, $\Gamma = \{\mathbf{x} \in \bar{\mathcal{Q}}_T \mid u(\mathbf{x}) = \zeta(\mathbf{x})\}$, of the HJBVI (1.1) using the solution of penalized equations. Suppose the estimate $0 \leq u(\mathbf{x}) - u^\rho(\mathbf{x}) \leq C_0\rho^{-\mu}$ holds for some constants $C_0 > 0$ and $\mu \in (0, 1]$, we then define for each $\rho > 0$ the set

$$\Gamma_\rho = \{\mathbf{x} \in \bar{\mathcal{Q}}_T \mid \zeta(\mathbf{x}) - C_0\rho^{-\mu} \leq u^\rho(\mathbf{x}) \leq \zeta(\mathbf{x})\}. \quad (3.6)$$

It follows directly from the estimates for u^ρ and u that $\Gamma \subset \Gamma_\rho$ for all $\rho > 0$. The next result demonstrates that Γ_ρ in fact converges to Γ in terms of the Hausdorff metric.

Theorem 3.4. *It holds for any given compact set $K \subset \bar{\mathcal{Q}}_T$ that*

$$\lim_{\rho \rightarrow \infty} d_{\mathcal{H}}(\Gamma_\rho \cap K, \Gamma \cap K) = \lim_{\rho \rightarrow \infty} \sup_{\mathbf{y} \in \Gamma_\rho \cap K} \inf_{\mathbf{x} \in \Gamma \cap K} |\mathbf{x} - \mathbf{y}| = 0.$$

Proof. Suppose the statement does not hold for a given compact set K , then there exist a constant $\varepsilon > 0$ and sequences $\{\rho_n\}$ and $\{\mathbf{y}_n\} = \{t_n, y_n\}$ such that $\rho_n \rightarrow \infty$, $\mathbf{y}_n \in \Gamma_{\rho_n} \cap K$ and $\inf_{\mathbf{x} \in \Gamma \cap K} |\mathbf{x} - \mathbf{y}_n| \geq \varepsilon$. By passing to a subsequence, we can assume that $\mathbf{y}_n \rightarrow \mathbf{y}^* \in K$ and $\inf_{\mathbf{x} \in \Gamma \cap K} |\mathbf{x} - \mathbf{y}^*| \geq \varepsilon$, which implies that $\mathbf{y}^* \notin \Gamma$. However, the definition of Γ_{ρ_n} gives

$$\begin{aligned} u(\mathbf{y}^*) - \zeta(\mathbf{y}^*) &= u(\mathbf{y}^*) - u(\mathbf{y}_n) + u(\mathbf{y}_n) - u^{\rho_n}(\mathbf{y}_n) + u^{\rho_n}(\mathbf{y}_n) - \zeta(\mathbf{y}_n) + \zeta(\mathbf{y}_n) - \zeta(\mathbf{y}^*) \\ &\leq u(\mathbf{y}^*) - u(\mathbf{y}_n) + C_0\rho_n^{-\mu} + 0 + \zeta(\mathbf{y}_n) - \zeta(\mathbf{y}^*) \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$, which together with the fact that $u \geq \zeta$ implies $\mathbf{y}^* \in \Gamma$, and hence a contradiction. \square

4 Discrete approximations for penalized equations

In this section, we propose a class of semi-implicit monotone approximations for solving the penalized equation (3.1) with a fixed penalty parameter $\rho \geq 0$. We shall construct the schemes in Section 4.1 and perform their analysis in Section 4.2. In order to derive more accurate estimates for the truncation error and the stability condition of the schemes, throughout this section we shall impose the following condition on the Lévy measure:

Assumption 2. *The Lévy measure ν admits a density $k(e)$ with the following estimate: it holds for some constants $C > 0$ and $\kappa \in [0, 2)$ that*

$$0 \leq k(e) \leq C|e|^{-n-\kappa}, \quad |e| < 1, \quad e \in E = \mathbb{R}^n \setminus \{0\}. \quad (4.1)$$

Typical examples of ν are the tempered κ -stable Lévy measure $\nu(de) = c_\kappa \exp(-K|e|)|e|^{-n-\kappa}de$ or the truncated κ -stable Lévy measures $\nu(de) = 1_{|e| < 1}c_\kappa|e|^{-n-\kappa}de$ for some constants $c_\kappa, K > 0$. Near zero these measures have a similar behaviour as the Lévy measure associated to the fractional Laplacian $(-\Delta)^{\kappa/2}$ with a (pseudo-differential) order κ .

We emphasize that Assumption 2 is imposed for the sake of preciseness, and will only be used in Section 4. Without the above estimate, one can still establish the consistency and stability of the schemes, but with more pessimistic results (see for example [15]).

4.1 Semi-implicit numerical methods

In this section, we shall derive semi-implicit monotone approximations for (3.1). We recall that it is crucial to construct a monotone discretization, since in general non-monotone schemes may fail to converge or even converge to false “solutions” (see [12]). For simplicity, we focus on the uniform spatial grid $\{x_i\}_i = h\mathbb{Z}^d$ on \mathbb{R}^d and a time partition $\{t_n\}_{n=0}^N$ with $\max_n |t_{n+1} - t_n| = \Delta t$, but similar results are valid for unstructured nondegenerate grids as well.

We start with the approximation of the nonlocal operators by truncating the small jumps of the Lévy measure and compensating it with an additional diffusion term as suggested in [15]. More precisely, for any $r \in (0, 1)$, we introduce the truncated Lévy measure $\nu_r(de) = 1_{|e|>r}\nu(de)$ and the modified diffusion coefficient σ_r^α such that for $i \neq j$, $\sigma_{r,ij}^\alpha = \sigma_{ij}^\alpha$, and for $i = 1, \dots, d$,

$$\sigma_{r,ii}^\alpha(t, x) = \text{sgn}(\sigma_{ii}^\alpha(t, x)) \left((\sigma_{ii}^\alpha(t, x))^2 + \int_{|e|<r} |\eta_i^\alpha(t, x, e)|^2 \nu(de) \right)^{1/2}, \quad (t, x) \in \bar{\mathcal{Q}}_T. \quad (4.2)$$

We shall approximate the linear operators $\{L^\alpha\}_{\alpha \in \mathbf{A}}$ in (3.1) by replacing the coefficient σ^α and the singular measure ν with the modified coefficient σ_r^α and the truncated nonsingular measure ν_r , respectively: for any given $\alpha \in \mathbf{A}$, $\phi \in C^{1,2}(\bar{\mathcal{Q}}_T)$ and $(t, x) \in \bar{\mathcal{Q}}_T$,

$$\begin{aligned} L^\alpha \phi(t, x) &= A^\alpha \phi(t, x) + K^\alpha \phi(t, x) \\ &\approx \frac{1}{2} \text{tr}(\sigma_r^\alpha(t, x)(\sigma_r^\alpha(t, x))^T D^2 \phi(t, x)) + b^\alpha(t, x) \cdot D\phi(t, x) \\ &\quad + \int_E (\phi(t, x + \eta^\alpha(t, x, e)) - \phi(t, x) - \eta^\alpha(t, x, e) \cdot D\phi(t, x)) 1_{|e|>r} \nu(de) \\ &= A_r^\alpha \phi(t, x) + K_r^\alpha \phi(t, x), \end{aligned}$$

where we have the modified operators:

$$A_r^\alpha \phi(t, x) := \frac{1}{2} \text{tr}(\sigma_r^\alpha(t, x)(\sigma_r^\alpha(t, x))^T D^2 \phi(t, x)) + \left(b^\alpha(t, x) - \int_{|e|>r} \eta^\alpha(t, x, e) \nu(de) \right) \cdot D\phi(t, x), \quad (4.3)$$

$$K_r^\alpha \phi(t, x) := \int_{|e|>r} (\phi(t, x + \eta^\alpha(t, x, e)) - \phi(t, x)) \nu(de). \quad (4.4)$$

Similarly, we replace the measure ν in the operators $\{B^\alpha\}_{\alpha \in \mathbf{A}}$ of (3.1) by the truncated measure ν_r , which leads to the following approximation:

$$B^\alpha \phi(t, x) \approx B_r^\alpha \phi(t, x) := \int_{|e|>r} m(\phi(t, x + \eta^\alpha(t, x, e)) - \phi(t, x)) \gamma(t, x, e) \nu(de). \quad (4.5)$$

By using Taylor’s theorem, Assumption 1 Items (1)(3) and Assumption 2, one can easily quantify the truncation errors of these approximations for any given $\phi \in C^{1,2}(\bar{\mathcal{Q}}_T)$ and $(t, x) \in \bar{\mathcal{Q}}_T$,

$$\begin{aligned} &|L^\alpha \phi(t, x) - A_r^\alpha \phi(t, x) - K_r^\alpha \phi(t, x)| + |B^\alpha \phi(t, x) - B_r^\alpha \phi(t, x)| \\ &\leq C \max(1, |D\phi|_0, |D^2\phi|_0) \int_{|e|<r} |e|^2 \nu(de) \\ &\leq C \max(1, |D\phi|_0, |D^2\phi|_0) \int_{|e|<r} |e|^{2-n-\kappa} de \leq C \max(1, |D\phi|_0, |D^2\phi|_0) r^{2-\kappa}, \end{aligned} \quad (4.6)$$

where the constant $\kappa \in [0, 2)$ is the same as that in Assumption 2, and $r \in (0, 1)$ is the truncation size of the singular measure ν near zero. Note that such a truncation removes the singularity of ν , which is essential for the numerical stability of the scheme (see the CFL condition (4.22)).

The nonlocal operators K_r^α and B_r^α are then approximated by a combination of interpolation and quadrature rules as in [6, 12, 15, 5]. Let \mathcal{I}_h be a second-order positive interpolation operator on the spatial grid $\{x_i\}_{i \in \mathbb{Z}^d}$ such that for all $x \in \mathbb{R}^d$,

$$\mathcal{I}_h[\phi](x) = \sum_{m \in \mathbb{Z}^d} \phi(x_m) \omega_m(x; h), \quad |\phi(x) - \mathcal{I}_h[\phi](x)| \leq Ch^2 |D^2 \phi|_0, \quad (4.7)$$

where $\{\omega_m(x; h)\}_m$ are some basis functions satisfying $0 \leq \omega_m(x; h) \leq 1$, $\sum_m \omega_m = 1$, $\omega_m(x; h) = \delta_{mi}$ and $\text{supp } \omega_m \subset B(x_m, 2h)$. In particular, one can choose \mathcal{I}_h to be the piecewise linear or multilinear interpolations on the spatial grid $\{x_i\}_{i \in \mathbb{Z}^d} = h\mathbb{Z}^d$, for which the desired properties have been established by equations (3.4) and (3.5) in [5]. Then we shall approximate the nonlocal terms (4.4) and (4.5) by

$$K_{r,h}^\alpha \phi(t_n, x_i) = \int_{|e| > r} \mathcal{I}_h[\phi(t_n, x_i + \cdot) - \phi(t_n, x_i)](\eta^\alpha(t_n, x_i, e)) \nu(de), \quad (4.8)$$

$$\begin{aligned} B_{r,h}^\alpha \phi(t_n, x_i) &= \int_{|e| > r} m(\mathcal{I}_h[\phi(t_n, x_i + \cdot) - \phi(t_n, x_i)](\eta^\alpha(t_n, x_i, e))) \gamma(t_n, x_i, e) \nu(de) \\ &= \int_{|e| > r} m\left(\sum_{j \in \mathbb{Z}^d} \omega_j(\eta^\alpha(t_n, x_i, e); h) [\phi(t_n, x_i + x_j) - \phi(t_n, x_i)]\right) \gamma(t_n, x_i, e) \nu(de), \end{aligned} \quad (4.9)$$

which in practice can be further evaluated by using consistent quadrature rules with positive weights, such as Gauss methods of appropriate order.

We remark that using (4.7), one can express (4.8) in the following monotone form:

$$K_{r,h}^\alpha \phi(t_n, x_i) = \sum_{j \in \mathbb{Z}^d} k_{r,h,j,i}^{\alpha,n} [\phi(t_n, x_i + x_j) - \phi(t_n, x_i)], \quad k_{r,h,j,i}^{\alpha,n} = \int_{|e| > r} \omega_j(\eta^\alpha(t_n, x_i, e); h) \nu(de), \quad (4.10)$$

while, as we will see in the subsequent analysis, the approximation (4.9) is closely related to the following coefficients

$$b_{r,h,j,i}^{\alpha,n} = \int_{|e| > r} \omega_j(\eta^\alpha(t_n, x_i, e); h) \gamma(t_n, x_i, e) \nu(de), \quad j \in \mathbb{Z}^d, \quad (4.11)$$

where we assume without loss of generality that $k_{r,h,0,i}^{\alpha,n} = b_{r,h,0,i}^{\alpha,n} = 0$ for all $i \in \mathbb{Z}^d$.

The boundedness of η (see Assumption 1 Item (1)) implies that the sums in (4.9) and (4.10) are finite. Moreover, one can deduce from (4.7) that for any fixed $r \in (0, 1)$, these approximations are consistent with the truncation error

$$\begin{aligned} |K_{r,h}^\alpha \phi(t_n, x_i) - K_r^\alpha \phi(t_n, x_i)| &\leq C |D^2 \phi|_0 h^2 \Gamma_1(r, \kappa), \\ |B_{r,h}^\alpha \phi(t_n, x_i) - B_r^\alpha \phi(t_n, x_i)| &\leq C |D^2 \phi|_0 h^2 \Gamma_2(r, \kappa), \end{aligned} \quad (4.12)$$

with some constant C independent of r and h , and

$$\Gamma_1(r, \kappa) = \int_{|e| > r} \nu(de) \leq \begin{cases} -\log r & \text{if } \kappa = 0, \\ r^{-\kappa} & \text{if } \kappa > 0, \end{cases} \quad (4.13)$$

$$\Gamma_2(r, \kappa) = \int_{|e| > r} (1 \wedge |e|) \nu(de) \leq \begin{cases} 1 & \text{if } \kappa \in [0, 1), \\ -\log r & \text{if } \kappa = 1, \\ r^{1-\kappa} & \text{if } \kappa \in (1, 2), \end{cases} \quad (4.14)$$

where we have used the density estimate (4.1). Since Godunov's Theorem in [21] asserts that one in general can expect a monotone scheme to be at most first order accurate, in the following, we shall choose $r = \max(h^{1/\kappa}, h)$ to ensure the truncation error (4.12) to be of the magnitude $O(h)$.

We now estimate the summations of coefficients $k_{r,h,j,i}^{\alpha,n}$ and $b_{r,h,j,i}^{\alpha,n}$, which will be essential for the stability of the scheme. The property $\sum_j \omega_j = 1$ leads immediately to the estimate

$$\sum_j k_{r,h,j,i}^{\alpha,n} \leq \Gamma_1(r, \kappa), \quad \sum_j b_{r,h,j,i}^{\alpha,n} \leq C\Gamma_2(r, \kappa), \quad \forall i \in \mathbb{Z}^d, \quad n = 0, \dots, N, \quad (4.15)$$

where Γ_1 and Γ_2 are defined as in (4.13) and (4.14), respectively. It is worth pointing out that other upper bounds of these summations can be derived using the approach in [5]. In fact, suppose the basis functions have the property that $|D\omega_j|_0 \leq C/h$, then we can deduce that

$$\begin{aligned} \sum_{j \neq 0} k_{r,h,j,i}^{\alpha,n} &= \sum_{j \neq 0} \int_{|e| > r} \omega_j(\eta^\alpha(t_n, x_i, e); h) - \omega_j(0; h) \nu(de) \\ &\leq \frac{C}{h} \int_{|e| > r} |\eta^\alpha(t_n, x_i, e)| \nu(de) \leq \frac{C}{h} \Gamma_2(r, \kappa). \end{aligned}$$

However, the relation $r \geq h$ clearly implies that (4.15) always gives a sharper upper bound.

We now proceed to consider the modified local operator A_r^α , which will be approximated by a consistent and monotone scheme $A_{r,h}^\alpha$, such that for any test function ϕ we have

$$|A_r^\alpha \phi - A_{r,h}^\alpha \phi| \leq C|D^2 \phi|_0 h \Gamma_2(r, \kappa), \quad (4.16)$$

$$A_{r,h}^\alpha \phi(t_n, x_i) = \sum_{j \in \mathbb{Z}^d} l_{r,h,j,i}^{\alpha,n} [\phi(t_n, x_j) - \phi(t_n, x_i)], \quad (4.17)$$

with some constant C independent of r and h , $\Gamma_2(r, \kappa)$ defined as in (4.14), and coefficients $l_{r,h,j,i}^{\alpha,n} \geq 0$ for all $i, j \in \mathbb{Z}^d$ and n . The construction of numerical approximations with the above properties has been discussed thoroughly in [6]. In particular, one can adopt the standard schemes of Kushner in [31] if the diffusion coefficient is diagonally dominant, and use the semi-Lagrangian scheme in [12] if the coefficient $\tilde{\sigma}^\alpha(\tilde{\sigma}^\alpha)^T$ is not diagonally dominant.

Finally, we construct numerical approximations for the Lipschitz nonlinearity of f^α on Du . For simplicity, we shall focus on the Lax-Friedrichs numerical flux, but it is straightforward to extend our schemes and analysis to other Lipschitz numerical fluxes, for instance the Godunov flux, which are monotone and consistent with f^α (see [11, 33]).

Let U_i^n be the discrete approximation of the solution to (3.1) at the node (t_n, x_i) , we denote by $\Delta_+^{(l)} U_i^n$ (resp. $\Delta_-^{(l)} U_i^n$) the one-step forward (resp. backward) difference of U along the l -th coordinate for each $l = 1, \dots, d$, and by $\Delta U_i^n = (\Delta_+^{(1)} U_i^n + \Delta_-^{(1)} U_i^n, \dots, \Delta_+^{(d)} U_i^n + \Delta_-^{(d)} U_i^n)^T$ the central difference of U at the node (t_n, x_i) . Then for any given $(y, k) \in \mathbb{R} \times \mathbb{R}$, the Lax-Friedrichs numerical flux is given by:

$$\bar{f}^\alpha(t_n, x_i, y, \Delta U_i^n, k) := f^\alpha\left(t_n, x_i, y, \sigma_r^\alpha(t_n, x_i)^T \frac{\Delta U_i^n}{2h}, k\right) + \sum_{l=1}^d \frac{\theta}{\lambda} \left(\frac{\Delta_+^{(l)} U_i^n - \Delta_-^{(l)} U_i^n}{h} \right), \quad (4.18)$$

where $\lambda = \Delta t/h$, d is the spatial dimension of (1.1), and $\theta > 0$ is a prescribed parameter for the Lax-Friedrichs numerical flux \bar{f} , which is essential for the numerical stability and convergence of the algorithm (see Section 4.2, in particular the CFL condition (4.22)).

With all these spatial approximations in hand, we are ready to write the fully-discrete scheme for (3.1). We shall adopt an implicit timestepping for the local term $A_{r,h}^\alpha$ and an explicit timestepping for the nonlocal term $K_{r,h}^\alpha$. This enables us to enjoy a less restrictive stability condition than that for fully explicit schemes and avoid solving the dense system resulting from the integral operator. For the nonlinear terms, we shall perform implicit timestepping for the u term and explicit timestepping for Du and $B^\alpha u$. As we will see later, by taking advantage of the monotonicity of the driver and the penalty term on u , our scheme can ensure stability with a less restrictive time stepsize, especially for a large penalty parameter ρ . Therefore, our semi-implicit scheme shall read as: $U_i^0 = g(x_i)$ for all $i \in \mathbb{Z}^d$ and for any given $n = 0, \dots, N-1$ and $i \in \mathbb{Z}^d$,

$$\begin{aligned} 0 &= G_h(t_{n+1}, x_i, U_i^{n+1}, \{U_a^{b+1}\}_{(a,b) \neq (i,n)}) \\ &= \inf_{\alpha \in \mathbf{A}} \left(\frac{U_i^{n+1} - U_i^n}{\Delta t} - A_{r,h}^\alpha U_i^{n+1} - K_{r,h}^\alpha U_i^n - \bar{f}^\alpha(t_n, x_i, U_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n) \right) \\ &\quad - \rho(\zeta(t_n, x_i) - U_i^{n+1})^+ \\ &= \inf_{\alpha \in \mathbf{A}} \left(\frac{U_i^{n+1} - U_i^n}{\Delta t} - A_{r,h}^\alpha U_i^{n+1} - K_{r,h}^\alpha U_i^n - \tilde{f}^\alpha(t_n, x_i, U_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n) \right), \end{aligned} \quad (4.19)$$

where for each $(t, x, y, z, k) \in \bar{\mathcal{Q}}_T \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$, $\bar{f}^\alpha(t, x, y, z, k)$ is the the Lax-Friedrichs numerical flux defined as in (4.18) and $\tilde{f}^\alpha(t, x, y, z, k) = \bar{f}^\alpha(t, x, y, z, k) + \rho(\zeta(t, x) - y)^+$.

4.2 Well-posedness and convergence analysis

In this section, we shall establish the well-posedness of the discrete equation (4.19) and perform its convergence analysis, which subsequently leads us to a constructive proof for the existence of bounded solutions to the penalized equation (3.1) and the HJBVI (1.1). We emphasize that the non-Lipschitz dependence of f^α on y requires novel analysis techniques for the well-posedness and stability of schemes, which are essentially different from the fixed-point arguments in most existing works (see e.g. [6, 12]). We remark that throughout this section we shall assume without loss of generality that f^α is strictly monotone in y with $\mu < 0$ (see Remark 1).

We start by recalling several important properties of the Lax-Friedrichs numerical flux for Lipschitz continuous Hamiltonian, which have been established in [11] and are essential for the subsequent analysis.

Lemma 4.1. *Let \bar{f} be defined as in (4.18) and $(t, x, u, k) \in \bar{\mathcal{Q}}_T \times \mathbb{R}^2$, and suppose Assumption 1 and the condition $\theta > C(\sup_{\alpha \in \mathbf{A}} |\sigma^\alpha|_0)\lambda$ hold, where C is the Lipschitz constant of the driver f^α in Assumption 1 Item (2c).*

(1) (Consistency.) *For any test function $\phi \in C^{1,2}(\bar{\mathcal{Q}}_T)$, we have*

$$|\bar{f}^\alpha(t, x, u, \Delta\phi_{j,i}^n, k) - \bar{f}^\alpha(t, x, u, D\phi(\mathbf{x}_{j,i}^n), k)| \leq O(h^2/\Delta t). \quad (4.20)$$

(2) (Monotonicity.) *If $U_i^n \geq V_i^n$ for all i, n , then we have*

$$\Delta t \bar{f}^\alpha(t, x, u, \Delta U_i^n, k) + 2d\theta U_i^n \geq \Delta t \bar{f}^\alpha(t, x, u, \Delta V_i^n, k) + 2d\theta V_i^n. \quad (4.21)$$

(3) (Stability.) *For any bounded functions U and V , we have*

$$|(\Delta t \bar{f}^\alpha(t, x, u, \Delta V_i^n, k) + 2d\theta V_i^n) - (\Delta t \bar{f}^\alpha(t, x, u, \Delta U_i^n, k) + 2d\theta U_i^n)| \leq 2d\theta |U - V|_0.$$

The next proposition presents the monotonicity of the scheme (4.19), which plays an important role in the stability and convergence analysis of the discrete equation. The proof is an extension of the standard case allowing for a potentially non-Lipschitz, monotone nonlinearity of the driver in u and nonlinearity in the jump term, and we include it in Appendix B for the convenience of the reader.

Proposition 4.2. *Under Assumptions 1 and 2, the discrete equation (4.19) is monotone, i.e., it holds for any functions U^{n+1} and $X_i^n \geq Y_i^n$, $\forall i, n$, that*

$$G_h(t_{n+1}, x_i, U_i^{n+1}, \{X_a^{b+1}\}_{(a,b) \neq (i,n)}) \leq G_h(t_{n+1}, x_i, U_i^{n+1}, \{Y_a^{b+1}\}_{(a,b) \neq (i,n)}),$$

provided that the following CFL conditions are satisfied:

$$1 - \Delta t \Gamma_1(r, \kappa) - 2d\theta \geq 0, \quad \theta > C \left(\sup_{\alpha \in \mathbf{A}} |\sigma^\alpha|_0 \right) \lambda, \quad (4.22)$$

where Γ_1 is defined in (4.13), and C is the Lipschitz constant of the driver f^α in Assumption 1 Item (2c).

The following proposition establishes a discrete comparison principle for (4.19), which subsequently implies the uniqueness of the solution to the discrete equation (4.19). We postpone the proof to Appendix B for the sake of readability.

Proposition 4.3. *Suppose Assumption 1 holds. Let $\{X_i^{n+1}\}_i$ and $\{Y_i^{n+1}\}_i$ be two bounded functions such that*

$$G_h(t_{n+1}, x_i, X_i^{n+1}, \{X_a^{n+1}, U^n\}_{a \neq i}) \leq G_h(t_{n+1}, x_i, Y_i^{n+1}, \{Y_a^{n+1}, U^n\}_{a \neq i}), \quad \forall i \in \mathbb{Z}^d,$$

then we have $X_i^{n+1} \leq Y_i^{n+1}$ for all $i \in \mathbb{Z}^d$. Consequently, the discrete equation (4.19) admits at most one bounded solution.

The next result provides an a priori estimate for the solution to (4.19), whose proof will be given in Appendix B.

Proposition 4.4. *Suppose Assumptions 1 and 2, and the CFL condition (4.22) hold. Let U^{n+1} be a bounded solution to (4.19), then we have the following a priori estimate:*

$$|U^{n+1}|_0 \leq \max\{|\zeta(t_n, \cdot)^+|_0, |U^n|_0 + \Delta t \sup_{\alpha \in \mathbf{A}} |f^\alpha(\cdot, \cdot, 0, 0, 0)|_0\}. \quad (4.23)$$

The next lemma shows that the discrete equation (4.19) admits a unique bounded solution provided that the driver f^α is Lipschitz in y , which has been established in [15] for $m(x) = x$ by reformulating the equation into a contraction mapping on the Banach space of bounded functions on $h\mathbb{Z}^d$ endowed with the sup-norm. The extension to general Lipschitz continuous m is straightforward and therefore the proof omitted.

Lemma 4.5. *Suppose Assumptions 1 and 2, and the CFL condition (4.22) hold. If we further assume f^α is globally Lipschitz continuous in y , i.e.,*

$$|f^\alpha(t, x, y, z, k) - f^\alpha(t, x, y', z, k)| \leq C|y - y'|, \quad \forall y, y' \in \mathbb{R}, (t, x, z, k) \in \bar{Q}_T \times \mathbb{R}^d \times \mathbb{R},$$

then the discrete equation (4.19) admits a unique bounded solution.

Now we are ready to demonstrate the existence of solutions to the discrete equation (4.19) with a general monotone driver. We shall adapt some arguments for monotone backward stochastic difference equations employed in [34], by approximating (4.19) with discrete equations with Lipschitz drivers, whose solutions subsequently enable us to construct the solution of (4.19).

Theorem 4.6. *Suppose Assumptions 1 and 2, and the CFL condition (4.22) hold, then the discrete equation (4.19) admits a unique bounded solution satisfying the a priori estimate (4.23).*

Proof. The uniqueness and the a priori bound have been established in Proposition 4.3 and 4.4, respectively. We now prove the existence of solution U^{n+1} to (4.19) with a given U^n in two steps.

Step 1: $f^\alpha(t, x, 0, z, k)$ is uniformly bounded for all $(\alpha, t, x, z, k) \in \mathbf{A} \times \bar{\mathcal{Q}}_T \times \mathbb{R}^d \times \mathbb{R}$.

For a family of mollifiers $\varrho_m : \mathbb{R} \rightarrow (0, \infty)$, which are smooth functions supported in $(-\frac{1}{m}, \frac{1}{m})$ with mass one, i.e. $\int_{\mathbb{R}} \varrho_m(s) ds = 1$ for all $m \in \mathbb{N}$, we define the regularized drivers:

$$f_m^\alpha(t, x, y, z, k) := (f^\alpha(t, x, \cdot, z, k) * \varrho_m)(y), \quad (4.24)$$

which are clearly monotone in y with $\mu \leq 0$ and globally Lipschitz continuous in x, z, k with the same Lipschitz constant as f^α . One can deduce from the uniform boundedness of $f^\alpha(t, x, 0, z, k)$ and the growth condition (2.2) that f_m^α is locally Lipschitz continuous in y , uniformly with respect to (α, t, x, z, k) . Therefore, for each $m, p \in \mathbb{N}$, by considering the truncated drivers $f_{m,p}^\alpha(t, x, y, z, k) := f_m^\alpha(t, x, \Pi_p(y), z, k)$ with

$$\Pi_p(s) = \frac{\inf(p, |s|)}{|s|} s, \quad s \in \mathbb{R},$$

we can construct a family of globally Lipschitz continuous and bounded drivers satisfying

$$y(f_{m,p}^\alpha(t, x, y, z, k) - f_m^\alpha(t, x, 0, z, k)) = \frac{|y|}{\inf(p, |y|)} \Pi_p(y) (f_{m,p}^\alpha(t, x, y, z, k) - f_m^\alpha(t, x, 0, z, k)) \leq 0.$$

Therefore, using Lemma 4.5 and following the proof of the a priori estimate (4.23), we know for each $m, p \in \mathbb{N}$, there exists a unique solution $U^{m,p}$ (where we omit the superscript $n+1$ for simplicity) solving (4.19) with the numerical flux associated to $f_{m,p}^\alpha(t, x, y, z, k)$ and satisfying the following estimate:

$$\begin{aligned} |U^{m,p}|_0 &\leq \max\{|\zeta(t_n, \cdot)^+|_0, |U^n|_0 + \Delta t \sup_{\alpha \in \mathbf{A}} |f_{m,p}^\alpha(\cdot, \cdot, 0, 0, 0)|_0\} \\ &= \max\{|\zeta(t_n, \cdot)^+|_0, |U^n|_0 + \Delta t \sup_{\alpha \in \mathbf{A}} |f_m^\alpha(\cdot, \cdot, 0, 0, 0)|_0\} \\ &= \max\{|\zeta(t_n, \cdot)^+|_0, |U^n|_0 + \Delta t \sup_{(\alpha, t, x) \in \mathbf{A} \times \bar{\mathcal{Q}}_T} \left| \int_{[-\frac{1}{m}, \frac{1}{m}]} f^\alpha(t, x, s, 0, 0) \varrho_m(-s) ds \right|\} \\ &\leq \max\{|\zeta(t_n, \cdot)^+|_0, |U^n|_0 + \Delta t \sup_{(\alpha, t, x, y) \in \mathbf{A} \times \bar{\mathcal{Q}}_T \times [-1, 1]} |f^\alpha(t, x, y, 0, 0)|\}, \end{aligned}$$

which is independent of p and m . In other words, for large enough p , the truncation of the driver f_m^α has no influence on the solution. Consequently, we can obtain a family of uniformly bounded functions $\{U_i^m\}_i$ such that for each $m \in \mathbb{N}$, $\{U_i^m\}_i$ solves the following regularized equation:

$$\inf_{\alpha \in \mathbf{A}} \left(\frac{U_i^m - U_i^n}{\Delta t} - A_{r,h}^\alpha U_i^m - K_{r,h}^\alpha U_i^n - \tilde{f}_m^\alpha(t_n, x_i, U_i^m, \Delta U_i^n, B_{r,h}^\alpha U_i^n) \right) = 0, \quad i \in \mathbb{Z}^d. \quad (4.25)$$

Now the uniform boundedness of the sequence of functions $\{U^m\}$ enables us to extract a subsequence, which by a slight abuse of notation is still denoted as $\{U^m\}$, such that for each

index i , U_i^m converges to some value U_i as m tends to ∞ . This defines a function $\{U_i\}_i$ in $\ell^\infty(\mathbb{Z}^d)$ satisfying the following estimate:

$$|U|_0 \leq \max\{|\zeta(t_n, \cdot)^+|_0, |U^n|_0 + \Delta t \sup_{(\alpha, t, x, y) \in \mathbf{A} \times \bar{\mathcal{Q}}_T \times [-1, 1]} |f^\alpha(t, x, y, 0, 0)|\}. \quad (4.26)$$

The properties of the mollifier ϱ_m and the continuity of f^α imply that $f_m^\alpha(t, x, y, z, k)$ converges to $f^\alpha(t, x, y, z, k)$ as m tends to infinity, uniformly on any compact subset of $\mathbf{A} \times \mathcal{Q}_T \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$. Also for each given $i \in \mathbb{Z}^d$, we know the number of terms summed in $A_{r,h}^\alpha U_i^m$ is finite uniformly in α . Therefore, we can conclude by using the inequality $|\inf_{\mathbf{A}} f - \inf_{\mathbf{A}} g| \leq \sup_{\mathbf{A}} |f - g|$ and passing $m \rightarrow \infty$ in (4.25) that $\{U_i\}$ satisfies the discrete equation (4.19).

Step 2: The general case.

We shall approximate the driver f^α by the following sequence (different from f_m^α earlier):

$$f_p^\alpha(t, x, y, z, k) = f^\alpha(t, x, y, z, k) - f^\alpha(t, x, 0, z, k) + \Pi_p(f^\alpha(t, x, 0, z, k)), \quad p \in \mathbb{N},$$

which clearly fulfils all the assumptions of Step 1, and converges locally uniformly to $f^\alpha(t, x, y, z, k)$. Thus for each $p \in \mathbb{N}$, there exists a function $\{U_i^p\}_i$ satisfying the following discrete equation:

$$\inf_{\alpha \in \mathbf{A}} \left(\frac{U_i^p - U_i^n}{\Delta t} - A_{r,h}^\alpha U_i^p - K_{r,h}^\alpha U_i^n - \tilde{f}_p^\alpha(t_n, x_i, U_i^p, \Delta U_i^n, B_{r,h}^\alpha U_i^n) \right) = 0, \quad i \in \mathbb{Z}^d. \quad (4.27)$$

Moreover, one can deduce from (4.26) and the uniform boundedness of $f^\alpha(t, x, 0, 0, 0)$ in Assumption 1 that for large enough p , the sequence of functions $\{U^p\}$ is uniformly bounded:

$$\begin{aligned} |U^p|_0 &\leq \max\{|\zeta(t_n, \cdot)^+|_0, |U^n|_0 + \Delta t \sup_{(\alpha, t, x, y) \in \mathbf{A} \times \bar{\mathcal{Q}}_T \times [-1, 1]} |f_p^\alpha(t, x, y, 0, 0)|\} \\ &= \max\{|\zeta(t_n, \cdot)^+|_0, |U^n|_0 + \Delta t \sup_{(\alpha, t, x, y) \in \mathbf{A} \times \bar{\mathcal{Q}}_T \times [-1, 1]} |f^\alpha(t, x, y, 0, 0)|\}. \end{aligned}$$

Then similar arguments as those in Step 1 enable us to extract a subsequence, which converges pointwise to a bounded function U . Then we can pass $p \rightarrow \infty$ in (4.27) and establish that U solves (4.19) for each $i \in \mathbb{Z}^d$, which consequently completes our proof for the existence of solutions. \square

The next result concludes the convergence of the discrete equation.

Theorem 4.7. *Suppose Assumptions 1 and 2 hold and the CFL condition (4.22) is satisfied, then for any fixed $\rho \geq 0$, the solution to the discrete equation (4.19) converges to the solution of (3.1) uniformly on compact sets as $h \rightarrow 0$.*

Proof. It is straightforward to derive from the one-step estimate (4.23) and Gronwall's lemma that the numerical solutions are bounded uniformly in r , Δt and h . Also it follows immediately from (4.6), (4.12), (4.16) and (4.20) that the scheme is consistent with (3.1) as $r, \Delta t, h \rightarrow 0$. Given the monotonicity of the scheme as ascertained by Proposition 4.2, we can conclude the convergence of the numerical solution by adapting the standard arguments in [4] to our current context. \square

Remark 2. *The convergence analysis in [4] does not assume the penalized equation (3.1) admits a solution. In fact, it considers the semicontinuous envelopes of the discrete solutions and demonstrates they are viscosity solutions to (3.1). Therefore, the convergence result in Theorem 4.7 and the a-priori estimate (4.23) subsequently provide us with a constructive proof for the existence of solutions of (3.1), which are bounded uniformly in ρ . Similar arguments can be carried out to demonstrate that the obstacle problem admits a bounded viscosity solution.*

5 Policy iteration for the discrete equation

In this section, we propose an efficient method for solving the discrete problem based on policy iteration. We shall further demonstrate local superlinear convergence by interpreting the scheme as a nonsmooth Newton method. Since in practice one usually truncates the discrete equation (4.19) by localizing it onto a chosen bounded computational domain, and specifying the behaviour of the solution outside the domain, we shall consider the following finite-dimensional problem: for any given $u^n \in \mathbb{R}^M$, we aim to find $u \in \mathbb{R}^M$ such that for each $i \in \mathcal{I} := \{1, \dots, M\}$,

$$0 = \mathcal{G}_h^{n+1}[u]_i = \inf_{\alpha \in \mathbf{A}} \left(u_i - u_i^n - \Delta t \left[(A_{r,h}^\alpha u)_i + (K_{r,h}^\alpha u^n)_i + \tilde{f}^\alpha(t_n, x_i, u_i, \Delta u_i^n, B_{r,h}^\alpha u_i^n) \right] \right) \quad (5.1)$$

where $\tilde{f}^\alpha(t, x, y, z, k) = \bar{f}^\alpha(t, x, y, z, k) + \rho(\zeta(t, x) - y)^+$ with the Lax-Friedrichs numerical flux \bar{f} . For simplicity, we shall denote (5.1) as $\mathcal{G}[u] = 0$ in the sequel.

We remark that the (finite-dimensional) discrete operators $A_{r,h}^\alpha$, $K_{r,h}^\alpha$ and $B_{r,h}^\alpha$ in (5.1) and the numerical flux \bar{f} are in general different from those in (3.1), where the first and last rows of these discrete operators will need to be modified to take the boundary conditions into consideration. However, without loss of generality, we can assume that

- (1) the operators $A_{r,h}^\alpha$, $K_{r,h}^\alpha$, $B_{r,h}^\alpha$ and \bar{f}^α are continuous in α , and
- (2) for each $\alpha \in \mathbf{A}$, the matrix $-A_{r,h}^\alpha$ admits positive diagonals, nonpositive off-diagonals and nonnegative row sums,

where the second property is motivated by truncating the monotone form (4.17) of the operator $A_{r,h}^\alpha$. Then the well-posedness of (5.1) follows from similar arguments as those in Section 4.2.

In order to design an efficient iterative scheme for solving (5.1), we need to impose the following regularity conditions on the nonlinear function f^α :

Assumption 3. *There exists a function $\partial_y^\alpha f : \mathbf{A} \times \bar{\mathcal{Q}}_T \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$, which is bounded above by some constant $\mu \in \mathbb{R}$, and satisfies the following properties:*

For any $(t, x, y) \in \bar{\mathcal{Q}}_T \times \mathbb{R}$ and compact subset \mathcal{K} of $\mathbf{A} \times \mathbb{R}^d \times \mathbb{R}$, there exists a neighbourhood \mathcal{U}_y of y and a constant $C_y > 0$, such that

- (1) *the family of functions $\{\partial_y^\alpha f(\cdot, t, x, u, \cdot, \cdot)\}_{u \in \mathcal{U}_y}$ is equicontinuous and uniformly bounded by the constant C_y with respect to $(\alpha, z, k) \in \mathcal{K}$;*
- (2) *the following identity holds uniformly with respect to $(\alpha, z, k) \in \mathcal{K}$:*

$$\lim_{h \rightarrow 0} \frac{f^\alpha(t, x, y + h, z, k) - f^\alpha(t, x, y, z, k) - \partial_y^\alpha f(\alpha, t, x, y + h, z, k)h}{h} = 0. \quad (5.2)$$

Remark 3. *An immediate consequence of Assumption 3 and the continuity of the operators on α is that for any given $u^n \in \mathbb{R}^M$, the nonlinear function $\mathcal{G}[u]$ defined in (5.1) is locally Lipschitz continuous on \mathbb{R}^M , which is crucial for our subsequent analysis.*

For notational simplicity, we shall denote by $\partial_y^\alpha f^\alpha$ the dependence of $\partial_y^\alpha f$ on the controls.

Assumption 3 is a generalized differentiability assumption of f^α with respect to the component y , which in general needs to be verified in a problem dependent way, as demonstrated in the following special cases. It is clear that Assumption 3 is satisfied if f^α is continuously differentiable and monotone in y . In this case, one can choose $\partial_y^\alpha f^\alpha$ in (5.2) to be the classical partial derivative of f^α with respect to the component y . However, unlike [41], we do not require $\partial_y^\alpha f^\alpha$ to be continuous

in y and f^α to be Fréchet-differentiable in order to enjoy the property (5.2). For example, let us assume for simplicity that f^α is of a separable form, i.e., $f^\alpha(t, x, y, z, k) = f_1^\alpha(t, x, z, k) + f_2(t, x, y)$ with functions $f_1^\alpha(t, x, z, k) : \bar{Q}_T \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ and $f_2(t, x, y) : \bar{Q}_T \times \mathbb{R} \rightarrow \mathbb{R}$. Then Assumption 3 Item (2) is satisfied if the map $\mathbb{R} \ni y \mapsto f_2(t, x, y) \in \mathbb{R}$ is slantly differentiable as defined in [9, 22], and $\partial_y^\circ f$ can be chosen as a slanting function of $\mathbb{R} \ni y \mapsto f_2(t, x, y) \in \mathbb{R}$. This allows us to consider f_2 with $\mathbb{R} \ni y \mapsto f_2(t, x, y) \in \mathbb{R}$ being a piecewise differentiable function, (possibly non-differentiable) convex/concave function or more generally semismooth function. We refer the reader to [9] for a detailed discussion on slantly differentiable functions and the associated slanting functions. Since most problems arising from finance are related to (5.1) with semismooth drivers (see e.g. [20, 18] and Section 6.1 for details), Assumption 3 applies to a wide range of optimal control and stopping problems that are of our interest.

Let u^n be a given solution at the previous discrete time point, $\alpha \in \mathbf{A}^M$ and $u \in \mathbb{R}^M$. We introduce the diagonal matrix $P^\alpha[u]$ with diagonal entries $\partial_y^\circ f^{\alpha_i}(t_n, x_i, u_i, \sigma_r^{\alpha_i}(t_n, x_i)^T \Delta u_i^n / 2h, B_{r,h}^{\alpha_i} u_i^n)$, $i \in \mathcal{I}$, which are bounded above by μ as stated in Assumption 3. Moreover, to handle the penalty term $\rho(\zeta - u)^+$, for any given $u = (u_1, \dots, u_M)^T \in \mathbb{R}^M$, we shall introduce the diagonal matrix $V^+[u] = \{v_{ij}[u]\}$ with $v_{ii}[u] = -\rho 1_{\{\zeta(t_n, x_i) - u_i > 0\}}$ for each $i \in \mathcal{I}$.

With these matrices in hand, we shall introduce the following mapping $\mathcal{L}^\alpha : \mathbb{R}^M \rightarrow \mathbb{R}^{M \times M}$ for any $\alpha \in \mathbf{A}^M$, which maps any given $u \in \mathbb{R}^M$ into a matrix $\mathcal{L}^\alpha[u]$, whose i -th row is defined as:

$$\mathcal{L}^\alpha[u]_i := (I - \Delta t A_{r,h}^{\alpha_i})_i - \Delta t (P^\alpha[u]_i + V^+[u]_i), \quad i = 1, \dots, M. \quad (5.3)$$

Now we are ready to present our policy iteration algorithm, which extends the classical Howard algorithm [7, 40] to the current nonlinear context.

Algorithm 1. Set $u^{(0)} = u^n$. Given $u^{(k)}$, $k \geq 0$, the next iterate $u^{(k+1)}$ is computed as follows:

Policy improvement step. Compute $\alpha^{(k+1)} = \{\alpha_i^{(k+1)}\}_{i=1}^M$ such that for each $i \in \mathcal{I}$,

$$\alpha_i^{(k+1)} \in \arg \min_{\alpha \in \mathbf{A}} \mathcal{G}[u^{(k)}]_i. \quad (5.4)$$

Policy evaluation step. Compute $u^{(k+1)} \in \mathbb{R}^M$ by solving

$$\mathcal{G}[u^{(k)}] + \mathcal{L}^{(k+1)}[u^{(k)}](u^{(k+1)} - u^{(k)}) = 0, \quad (5.5)$$

where $\mathcal{L}^{(k+1)}[u^{(k)}]$ is the matrix (5.3) evaluated at the control $\alpha^{(k+1)}$ and the iterate $u^{(k)}$.

We now proceed to investigate the convergence of Algorithm 1 by regarding it as a Newton's method to the nonlinear function \mathcal{G} , where $\mathcal{L}^{(k+1)}[u^{(k)}]$ plays the essential role of the derivative of \mathcal{G} at the point $u^{(k)}$. Since in general \mathcal{G} is not Fréchet-differentiable, we shall interpret the derivative in the sense of slant differentiability [9]. Recall that given two Banach spaces \mathcal{X} and \mathcal{Y} , a function $F : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be slantly differentiable in an open set $\mathcal{U} \subset \mathcal{X}$ if there exists a family of bounded linear operators $\{\mathcal{L}[u]\}_{u \in \mathcal{U}}$ from \mathcal{X} into \mathcal{Y} , called a slanting function for F in \mathcal{U} , such that for all $u \in \mathcal{U}$ we have

$$\lim_{|h|_{\mathcal{X}} \rightarrow 0} |h|_{\mathcal{X}}^{-1} |F(u+h) - F(u) - \mathcal{L}[u+h](h)|_{\mathcal{Y}} = 0.$$

We start with the convergence analysis of the optimal controls. For each vector $u \in \mathbb{R}^M$, we write $\mathbf{A}_{u,i} = \arg \min_{\alpha \in \mathbf{A}} \mathcal{G}[u]_i$, $i \in \mathcal{I}$, as the set of minimizers of $\mathcal{G}[u]_i$ at the index i . The following lemma shows the $\arg \min / \arg \max$ set-valued mapping is upper hemicontinuous.

Lemma 5.1 ([1], Theorem 17.31). *Let X, Y be topological spaces, $\Gamma \subset Y$ be a nonempty compact subset, and $g : X \times \Gamma \rightarrow \mathbb{R}$ be a continuous function. Define the value function $m : X \rightarrow \mathbb{R}$ by $m(x) = \max_{y \in \Gamma} g(x, y)$, and the set-valued map $\mu : X \rightrightarrows Y$ by $\mu(x) = \{y \in \Gamma \mid g(x, y) = m(x)\}$. Then μ has nonempty and compact values. Moreover, if Y is Hausdorff, then μ is upper hemicontinuous, i.e., for every $x \in X$ and every neighborhood U of $\mu(x)$, there is a neighborhood V of x such that $z \in V$ implies $\mu(z) \subset U$.*

With Lemma 5.1 in hand, one can demonstrate the convergence of sets of optimal controls.

Corollary 5.2. *Suppose Assumptions 1 and 3 hold. Then the set-valued mapping $u \in \mathbb{R}^M \rightrightarrows \prod_{i=1}^M \mathbf{A}_{u,i}$ is upper hemicontinuous.*

Proof. The result follows directly from the fact that \mathbf{A} is a compact subset of a metric space, and all the coefficients in (5.1) are continuous in α . \square

Remark 4. *A direct consequence of the above corollary is that for any index $i \in \mathcal{I}$ and sequences $(u^n)_{n \in \mathbb{N}} \in \mathbb{R}^M$ and $(\alpha^n)_{n \in \mathbb{N}} \in \mathbf{A}$ such that $u^n \rightarrow u \in \mathbb{R}^M$ as $n \rightarrow \infty$ and $\alpha^n \in \mathbf{A}_{u^n,i}$ for all n , we know all limit points of $(\alpha^n)_{n \in \mathbb{N}}$ belong to $\mathbf{A}_{u,i}$, the set of optimal controls at the index i . If we further assume $\mathbf{A}_{u,i}$ is a singleton, then we have $d_{\mathcal{H}}(\mathbf{A}_{u^n,i}, \mathbf{A}_{u,i}) \rightarrow 0$ as $n \rightarrow \infty$, where $d_{\mathcal{H}}$ denotes the Hausdorff distance.*

Lemma 5.3. *Let \mathcal{G} as in (5.1) and suppose Assumptions 1 and 3 hold. Then \mathcal{G} is slantly differentiable in \mathbb{R}^M with slanting function $\mathcal{L}^\alpha[u]$ defined in (5.3) and $\alpha \in \mathbf{A}_u$.*

Proof. Let u^n and u be fixed. Since $(\sigma_r^\alpha(t_n, x_i)^T \Delta u_i^n / 2h, B_{r,h}^\alpha u^n)_{\alpha \in \mathbf{A}}$ is contained in a compact subset of $\mathbb{R}^d \times \mathbb{R}$, there exists a neighbourhood \mathcal{U} of u such that (5.2) holds for this compact set. For notational brevity, we will denote $f^\alpha(\cdot, \cdot, \cdot, \Delta u_i^n, B_{r,h} u_i^n) = f^\alpha(\cdot, \cdot, \cdot, \sigma_r^\alpha(\cdot, \cdot)^T \Delta u_i^n / 2h, B_{r,h}^\alpha u_i^n)$. A similar notation applies to $\partial_y^\alpha f^\alpha(\cdot, \cdot, \cdot, \Delta u_i^n, B_{r,h} u_i^n)$.

Let $h \in \mathbb{R}^M$ with $u + h \in \mathcal{U}$. Then for any $\alpha^u \in \mathbf{A}_u$, $\alpha^{u+h} \in \mathbf{A}_{u+h}$ and $i \in \mathcal{I}$, we have

$$\begin{aligned} \mathcal{G}[u]_i &= (I - \Delta t A_{r,h}^{\alpha^u})(u)_i - \Delta t \tilde{f}^{\alpha^u}(t_n, x_i, u_i, \Delta u_i^n, B_{r,h}^{\alpha^u} u_i^n) - u_i^n - \Delta t K_{r,h}^{\alpha^u} u_i^n \\ &\geq \mathcal{G}[u + h]_i - \mathcal{L}^{\alpha^{u+h}}[u + h](h)_i + \Delta t (A_{r,h}^{\alpha^u} - A_{r,h}^{\alpha^{u+h}})(h)_i \\ &\quad + \Delta t [f^{\alpha^u}(t_n, x_i, u_i + h_i, \Delta u_i^n, B_{r,h} u_i^n) - f^{\alpha^u}(t_n, x_i, u_i, \Delta u_i^n, B_{r,h} u_i^n) \\ &\quad - \partial_y^\alpha f^{\alpha^u}(t_n, x_i, u_i + h_i, \Delta u_i^n, B_{r,h} u_i^n) h_i] \\ &\quad + \Delta t [\partial_y^\alpha f^{\alpha^u}(t_n, x_i, u_i + h_i, \Delta u_i^n, B_{r,h} u_i^n) - \partial_y^\alpha f^{\alpha^{u+h}}(t_n, x_i, u_i + h_i, \Delta u_i^n, B_{r,h} u_i^n)] h_i \\ &\quad + \Delta t \rho[(\zeta(t_n, x_i) - u_i - h_i)^+ - (\zeta(t_n, x_i) - u_i)^+ + 1_{\{\zeta(t_n, x_i) - u_i - h_i > 0\}} h_i]. \end{aligned} \quad (5.6)$$

Note that (5.6) vanishes for small enough h , so that we can conclude from the identity (5.2), the equicontinuity of $\{\partial_y^\alpha f\}_{u \in \mathcal{U}}$ in (α, z, k) , uniform continuity of $A_{r,h}^\alpha$ in α , and Corollary 5.2 that

$$\mathcal{G}[u + h]_i - \mathcal{G}[u]_i - \mathcal{L}^{\alpha^{u+h}}[u + h](h)_i \leq o(|h|_0).$$

On the other hand, we can start with $\mathcal{G}[u + h]_i$ and deduce the corresponding lower bound:

$$\mathcal{G}[u + h]_i - \mathcal{G}[u]_i - \mathcal{L}^{\alpha^{u+h}}[u + h](h)_i \geq o(|h|_0), \quad \text{as } |h|_0 \rightarrow 0, \quad (5.7)$$

which consequently leads to the desired slant differentiability of \mathcal{G} . \square

The following result concludes the local superlinear convergence of Algorithm 1.

Theorem 5.4. *Suppose Assumptions 1 and 3 hold. Then for all Δt with $1 - \mu\Delta t \geq c_0 > 0$, $\mathcal{L}^\alpha[u]$ is nonsingular and satisfies $|\mathcal{L}^\alpha[u]^{-1}|_0 \leq 1/c_0$. Consequently, the iterates $\{u^{(k)}\}_k$ generated by Algorithm 1 converge superlinearly to the solution u^* of (5.1) in a neighbourhood of u^* .*

Proof. We can deduce from Assumption 3 and the properties of the matrices $-A_{r,h}^\alpha$ and $-V^+[u]$ that for all $\alpha \in \mathbf{A}^M$, $u \in \mathbb{R}^M$ and Δt with $1 - \mu\Delta t > 0$, $\mathcal{L}^\alpha[u] = \{l_{ij}^{\alpha,u}\}$ defined in (5.3) is a strictly diagonally dominant matrix whose row sums satisfy:

$$\min_{1 \leq i \leq M} \left(l_{ii}^{\alpha,u} - \sum_{j=1, j \neq i}^M |l_{ij}^{\alpha,u}| \right) \geq 1 - \Delta t \mu \geq c_0 > 0,$$

from which, along with Theorem A in [39], we obtain the desired estimate for $|\mathcal{L}^\alpha[u]^{-1}|_0$. Then we can directly infer the local superlinear convergence of Algorithm 1 from [9, Theorem 3.4]. \square

6 Numerical experiments

In this section, we demonstrate the effectiveness of the schemes through numerical experiments. We present two examples, an optimal investment problem with model uncertainty, and a consumption-portfolio allocation problem with non-Lipschitz recursive utilities. Both examples are related to non-standard HJB equations, where the first example contains non-smooth convex/concave nonlinearities, while the second one involves monotone drivers of polynomial growth.

6.1 Optimal investment under ambiguity

We study first an optimal investment problem over a time interval $[0, T]$ in a financial market with a risk-free asset and a risky asset. For our numerical tests, we assume the interest rate is zero, and the price of the risky asset follows the jump-diffusion process:

$$dS_t = S_{t-} (bdt + \sigma dW_t + (1 \wedge |e|) \tilde{N}(dt, de)),$$

where W is a Brownian motion and $\tilde{N}(dt, de) = N(dt, de) - \nu(de)dt$ is an independent compensated Poisson process defined on a probability space $(\Omega, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$.

An investor with initial wealth $x > 0$ at time t can control their wealth process $X^{t,x,\alpha}$ through a selection of the portion α_t of wealth allocated in the risky asset, and also the duration of the investment via a stopping time $\tau \in [t, T]$, which leads to the following wealth process:

$$dX_s^{t,x,\alpha} = \alpha_s X_{s-}^{t,x,\alpha} (bds + \sigma dW_s + (1 \wedge |e|) \tilde{N}(ds, de)), \quad s \in [t, \tau]; \quad X_t^{t,x,\alpha} = x,$$

and the terminal payoff $\xi_\tau^{t,x,\alpha} = g(X_\tau^{t,x,\alpha})$.

The aim of the agent is to maximize the expected performance of the investment by taking ambiguity into account in the spirit of [38, 19]. More precisely, for given parameters $r, R, \kappa_1, \kappa_2 > 0$, we consider the following value function:

$$u_*(t, x) := \sup_{\tau \in \mathcal{T}_t} \sup_{\alpha \in \mathcal{A}_t} \mathcal{E}_{\tau,*}^t[\xi_\tau^{t,x,\alpha}] = \sup_{\tau \in \mathcal{T}_t} \sup_{\alpha \in \mathcal{A}_t} \inf_{\beta \in \mathcal{B}_t, \mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \left[\exp \left(- \int_t^\tau \beta_s ds \right) \xi_\tau^{t,x,\alpha} \right], \quad (6.1)$$

over all admissible choices of $(\alpha, \tau) \in \mathcal{A}_t \times \mathcal{T}_t$, where \mathcal{B}_t is a class of adapted processes $\beta = (\beta_s)_{s \in [t, T]}$ valued in $[r, R]$, which represent ambiguous discount rates, and \mathcal{M} is a family of absolutely continuous probability measures with respect to \mathbb{P} with density

$$dM_t^{\pi,\ell} = M_{t-}^{\pi,\ell} \left(\pi_t dW_t + \int_E \ell_t(e) \tilde{N}(de, dt) \right); \quad M_0^{\pi,\ell} = 1,$$

where (π, ℓ) are predictable processes satisfying $|\pi_t| \leq \kappa_1$ and $0 \leq \ell_t(e) \leq \kappa_2(1 \wedge |e|)$. In other words, the nonlinear expectation $\mathcal{E}_{\tau,*}^t[\cdot]$ represents the worst-case scenario in a market with uncertainty arising from the discount rate, the Brownian motion, and the random jump source (see [38, 19]). Similarly, we consider the value function associated to the best-case scenario:

$$u^*(t, x) := \sup_{\tau \in \mathcal{T}_t} \sup_{\alpha \in \mathcal{A}_t} \mathcal{E}_{\tau,*}^t[\xi_{\tau}^{t,x,\alpha}] = \sup_{\tau \in \mathcal{T}_t} \sup_{\alpha \in \mathcal{A}_t} \sup_{\beta \in \mathcal{B}_t, \mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}} \left[\exp \left(- \int_t^{\tau} \beta_s ds \right) \xi_{\tau}^{t,x,\alpha} \right]. \quad (6.2)$$

Using the dual representation of $\mathcal{E}_{\tau,*}^t[\cdot]$ (resp. $\mathcal{E}_{\tau,*}^t[\cdot]$), we can characterize the value function u^* (resp. u_*) as the viscosity solution to the following HJBVI (see [38, 19, 35, 13]): $u(0, x) = g(x)$ for $x \in \mathbb{R}$, and for $(t, x) \in (0, T] \times \mathbb{R}$,

$$\begin{aligned} \min \{ u(t, x) - g(x), \inf_{\alpha \in [0,1]} (u_t - L^{\alpha} u - Ru^{-} + ru^{+} - \alpha \kappa_1 \sigma |xu_x| - \kappa_2 B^{\alpha,*} u) \} &= 0, \\ (\text{resp. } \min \{ u(t, x) - g(x), \inf_{\alpha \in [0,1]} (u_t - L^{\alpha} u - ru^{-} + Ru^{+} + \alpha \kappa_1 \sigma |xu_x| + \kappa_2 B_*^{\alpha} u) \} &= 0,) \end{aligned} \quad (6.3)$$

where the nonlocal operators $L^{\alpha} = A^{\alpha} + K^{\alpha}$, $B^{\alpha,*}$ and B_*^{α} satisfy for $\phi \in C^{1,2}([0, T] \times \mathbb{R})$ that

$$\begin{aligned} A^{\alpha} \phi(t, x) &= \frac{1}{2} \alpha^2 \sigma^2 x^2 \phi_{xx}(t, x) + \alpha b x \phi_x(t, x), \\ K^{\alpha} \phi(t, x) &= \int_{\mathbb{R} \setminus \{0\}} (\phi(t, x + \alpha x \eta(e)) - \phi(t, x) - \alpha x \eta(e) \phi_x(t, x)) \nu(de), \\ B^{\alpha,*} \phi(t, x) &= \int_{\mathbb{R} \setminus \{0\}} (\phi(t, x + \alpha x \eta(e)) - \phi(t, x))^+ (1 \wedge |e|) \nu(de), \\ B_*^{\alpha} \phi(t, x) &= \int_{\mathbb{R} \setminus \{0\}} (\phi(t, x + \alpha x \eta(e)) - \phi(t, x))^- (1 \wedge |e|) \nu(de). \end{aligned} \quad (6.4)$$

We now specify the choice of parameters for our experiments. For the jump component, we shall consider a symmetric Variance Gamma model (see e.g. [32]) with a Lévy measure $\nu(de) = \exp(-\mu|e|)/|e|de$ on \mathbb{R} and intensity $\eta(e) = 1 \wedge |e|$, while for the initial condition and the obstacle of the HJBVI we use the exponential utility function $g(x) = 1 - 2e^{-2x}$, which implies the solution of (6.3) changes its sign on the domain and hence ensures both u^+ and u^- in (6.3) have effects on the solution. We consider the value functions at (T, x_0) with the model parameters in Table 1.

b	σ	μ	r	R	κ_1	κ_2	T	x_0
0.1	0.2	6	0.02	0.04	0.2	0.5	1	1

Table 1: Model parameters for the optimal investment problem under ambiguity.

Now we discuss the implementation details and discretization parameters. The HJBVIs (6.3) will be localized to the domain $(0, 2)$ with $u(\cdot, x) = g(x)$ for $x \in \mathbb{R} \setminus (0, 2)$. Since the singularity of the measure ν behaves like $\log(r)$, $r > 0$, around zero, we can deduce from the consistency and stability analysis in Section 4.2 along with the choice of parameters that choosing $r = h$, $\lambda = \Delta t/h = 1/5$, and $\theta = 1/5$ for the numerical flux will lead us to a consistent and stable scheme. To ensure the monotonicity of the scheme, we discretize the first-order and second-order derivative by the upwind scheme and the central-difference scheme, respectively, and evaluate the nonlocal operators by the mid-point quadrature formula. We further discretize the control set $\mathbf{A} = [0, 1]$ with a mesh $h_{\varepsilon} = \frac{1}{10}$. and stop the policy iteration (see Algorithm 1), at each timestep if the difference between two consecutive iterates is less than 10^{-10} . We remark that on the basis

of our experiments, this control discretization mesh seems to be sufficiently small, since further refinements lead to a relative difference less than 10^{-7} in the value functions, which is negligible compared to other discretization errors. The effect of the control discretization will be investigated more closely in the next example.

Note that we apply Algorithm 1 to solve penalized HJBVs (6.3) with different penalty parameters independently (without using a continuation procedure in the penalty parameter as suggested in [24, Section 5] for elliptic variational inequalities and in [37, Section 7] for elliptic HJB quasi-variational inequalities). Strictly speaking, Algorithm 1 is not guaranteed to be globally convergent due to the non-convex nonlinearity of (6.3). However, we have not observed such a divergence of the policy iteration in our experiments. This is mainly because for sufficiently small time step sizes, the numerical solution at the previous timestep produces a very good initial guess for the policy iteration, which subsequently enables us to solve the discrete penalized equations with a small number of iterations (see Table 2 for details).

Table 2 contains, for different mesh sizes and penalty parameters, the numerical solutions of the value function u_* at the point (T, x_0) and the maximal number of iterations among all time steps. The line (a) clearly indicates the efficiency of our policy iteration scheme, which solves the discrete equation (5.1) at the accuracy 10^{-10} with a small number of iterations. Moreover, we can infer from the line (b) that for a fixed penalty parameter ρ , the numerical solutions converge monotonically to the exact solution. The asymptotic magnitude of the approximation error can be deduced from line (d), which is of $O(h) + O(\Delta t)$, and seems to be independent of the size of the penalty parameter ρ . We remark that a similar first-order monotone convergence can be observed for u^* , for which a detailed discussion is omitted.

	h	1/40	1/80	1/160	1/320	1/640
$\rho = 10^3$	(a)	4	4	4	4	5
	(b)	0.7292780	0.7292918	0.7292987	0.7293021	0.7293038
	(c)		13.788	6.879	3.433	1.715
	(d)			2.004	2.004	2.002
$\rho = 16 \times 10^3$	(a)	4	4	4	5	4
	(b)	0.7293262	0.7293271	0.7293275	0.7293277	0.7293278
	(c)		0.8616	0.4300	0.2146	0.1068
	(d)			2.004	2.004	2.009

Table 2: Numerical solutions of the value function u_* for the optimal investment problem with different mesh sizes and penalty parameters. Shown are: (a) the maximal number of iterations among all time points; (b) the numerical solutions $U_{\rho,h}$ at (T, x_0) ; (c) the increments $U_{\rho,h} - U_{\rho,2h}$ (in 10^{-6}) ; (d) the rate of increments $(U_{\rho,2h} - U_{\rho,4h})/(U_{\rho,h} - U_{\rho,2h})$.

We proceed to analyze the impact of computational domains by performing computations on the domains $(0, 2)$ and $(0, 4)$ with $h = 1/640$ and $\rho = 64 \times 10^3$. It can be observed that this enlarged computational domain has a negligible effect on the numerical solution of value functions (a relative difference of $3.7 \cdot 10^{-7}$ for u^* and $2.5 \cdot 10^{-12}$ for u_*). Moreover, the maximal number of iterations remains to be 4 for both u^* and u_* , which seems to be independent of the size M of the discrete equation (5.1).

Finally we examine the convergence of value functions in terms of the penalty parameter ρ . Table 3 presents the numerical results obtained using the domain $(0, 2)$ with a fixed mesh size $h = 1/640$ and different penalty parameters. For both u^* and u_* , we can infer from lines (a) and (b) a monotone convergence of the numerical solutions, with an approximation error proportional to

ρ		10^3	4×10^3	16×10^3	64×10^3
u^*	(a)	0.75071151	0.75071215	0.75071231	0.75071235
	(b)		0.639	0.159	0.040
	(c)			3.9998	4.0006
u_*	(a)	0.72930381	0.72932303	0.72932783	0.72932903
	(b)		19.215	4.802	1.201
	(c)			4.0016	3.9976

Table 3: Numerical results of the value functions u^* and u_* for the optimal investment problem with different penalty parameters. Shown are: (a) the numerical solutions U_ρ at (T, x_0) ; (b) the increments $U_\rho - U_{\rho/4}$ (in 10^{-6}); (c) the rate of increments $(U_{\rho/4} - U_{\rho/16})/(U_\rho - U_{\rho/4})$.

the reciprocal of the penalty parameter, as asserted in Theorem 3.2 and 3.3. Then by performing linear regression of the values in line (a) against the reciprocal of penalty parameters, we can estimate the constant C_0 in (3.5) and construct a convergent approximation of the free boundary of (6.3) as suggested in (3.6). Figure 1 compares the feedback control strategies for u^* (i.e., the best-case scenario) and u_* (i.e., the worst-case scenario) with $\rho = 64 \times 10^3$, where the white region represents the sets in which the obstacle is active, and otherwise the colour indicates the value of the optimal control, as presented in the panel on the right. It clearly illustrates that the investor in general behaves more conservatively in the worst-case scenario.

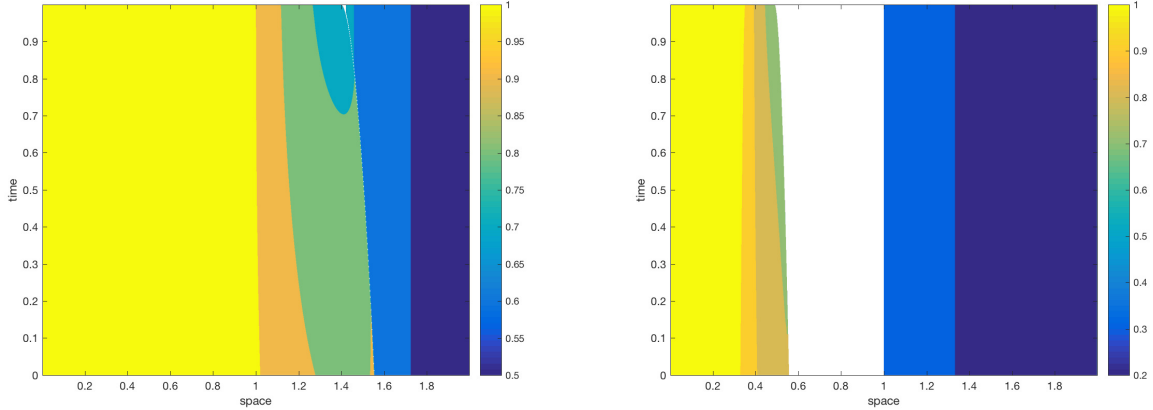


Figure 1: Feedback control strategies with $\rho = 16 \cdot 10^3$ for the best-case scenario (left) and the worst-case scenario (right), where the early stopping region is white.

6.2 Consumption-portfolio allocation with recursive utility

As a second example, we shall address a consumption-portfolio maximization problem in terms of recursive utilities, which extend the classical additive utilities by allowing one's current well-being to depend on the expected future utilities in a non-risk-neutral way, and play an important role in modern mathematical finance (see e.g. [30, 34] and references therein).

For our numerical examples, we shall consider an economy with a risk-free bond with constant interest rate $r > 0$, and a risky asset whose dynamics follows a stochastic volatility model:

$$\begin{aligned}
dS_t &= S_t[(r + \lambda v_t)dt + \sqrt{v_t}dW_t], \\
dv_t &= (\vartheta - \kappa v_t)dt + \beta\sqrt{v_t}(\rho dW_t + \sqrt{1 - \rho^2}d\hat{W}_t),
\end{aligned} \tag{6.5}$$

where $\vartheta, \kappa, \beta > 0$ are constants, and W, \hat{W} are two independent Brownian motions on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$.

An agent controls their wealth by deciding the portions invested in stocks and consumed, which implies the dynamics of the wealth follow the following equation:

$$dX_t^{x, \pi, c} = X_t^{x, \pi, c} [r + \pi_t \lambda v_t - (1 + r) c_t] dt + \sqrt{v_t} X_t^{x, \pi, c} \pi_t dW_t, \quad t \in [0, T]; \quad X_0^{x, \pi, c} = x_0,$$

where $\pi, c : \Omega \times [0, T] \rightarrow [0, 1]$ are the proportion for investment and consumption respectively, and x_0 is the initial wealth.

The preference of the agent between consumption and investment is described by the well-known (normalized) continuous-time Epstein-Zin utility suggested in [20]. More precisely, suppose the utility from the terminal wealth at the terminal time T is given by $g(X_T^{x, \pi, c})$, then the Epstein-Zin recursive utility is defined as

$$\mathcal{E}^c[g(X_T^{x, \pi, c})] = Y_t \quad \text{and} \quad Y_t = \mathbb{E} \left[\int_t^T f(c_s X_s^{x, \pi, c}, Y_s) ds + g(X_T^{x, \pi, c}) \mid \mathcal{F}_t \right], \quad t \in [0, T],$$

with the following driver:

$$f(c, y) := \frac{\delta}{1 - \frac{1}{\psi}} (1 - \gamma) y \left[\left(\frac{c}{((1 - \gamma)y)^{\frac{1}{1-\gamma}}} \right)^{1 - \frac{1}{\psi}} - 1 \right], \quad (6.6)$$

where we follow the standard parametrization by letting $\delta > 0$ be the rate of time preference and $0 < \psi \neq 1$ be the elasticity of intertemporal substitution. The objective of the agent is to maximize the recursive utility over all admissible choices of $\alpha = (\pi, c)$:

$$u(t, x) := \sup_{\alpha \in \mathcal{A}_t} \mathcal{E}^c[g(X_T^{x, \pi, c})]. \quad (6.7)$$

It has been demonstrated in [30, 34] that for certain empirically important parameters, for instance the coefficients in Table 4, which are taken from [30] and will be used for our numerical test, this driver (6.6) of the Epstein-Zin utility is non-Lipschitz but monotone and differentiable in the y -component (hence satisfies Assumptions 1 and 3). Moreover, one can identify the value function (6.7) (with a change of time variable) as the solution to the following HJB equation: $u(0, x, v) = g(x)$ for $(x, v) \in \mathbb{R}^2$, and for $(t, x, v) \in (0, T] \times \mathbb{R}^2$,

$$\begin{aligned} \inf_{(\pi, c) \in \mathbf{A}} \left(u_t - \frac{1}{2} \pi^2 x^2 v u_{xx} - \pi \beta \rho x v u_{xv} - \frac{1}{2} \beta^2 v u_{vv} - x[r + \pi \lambda v - (1 + r)c] u_x \right. \\ \left. - (\vartheta - \kappa v) u_v - f(cx, u) \right) = 0, \end{aligned} \quad (6.8)$$

with $\mathbf{A} = \{(\pi, c) \in [0, 1] \times [0, 1] \mid \pi + c \leq 1\}$.

For the purpose of numerical experiments, we shall take the negative exponential utility as the initial condition $g(x) = -e^{-x/2}$, and localize the equation on the domain $[0, 2] \times [0, 0.05]$. The following homogeneous Neumann boundary conditions will be imposed as suggested in [26]:

$$u_x(t, 2, v) = 0, \quad (t, v) \in [0, T] \times [0, 0.05]; \quad u_v(t, x, 0.05) = 0, \quad (t, x) \in [0, T] \times [0, 2],$$

while the equation (6.8) itself is set as the boundary condition at $x = 0$ and $v = 0$. We remark that based on our experiments with larger computational domains, the error of the value function caused by this domain truncation appears to be less than 10^{-7} .

γ	ψ	δ	r	ρ	λ	β	κ	ϑ	x_0	v_0	T
2	1.5	0.08	0.05	-0.5	0.5	0.25	5	0.1125	1	0.02	0.5

Table 4: Model parameters for the optimal consumption-portfolio allocation problem.

The localized HJB equation (6.8) is then discretized using the implicit linear interpolation Semi-Lagrangian scheme (Scheme 2 in [36]) with the mesh size $h_v = h_x = h$ and the time stepsize $\Delta t = 4h$, which is monotone and locally first-order accurate. We shall further discretize the control set \mathbf{A} with a mesh h_ε , and for each time step, terminate policy iteration once the sup-norm of two consecutive iterates is within the threshold 10^{-6} .

h	1/100	1/200	1/400	1/800	1/1600
(a)	3	4	4	3	3
(b)	-0.6604205	-0.6581355	-0.6580512	-0.6580101	-0.6579897
(c)		2.2851	0.0843	0.0411	0.0204
(d)			27.120	2.052	2.014

Table 5: Numerical solutions for the consumption-portfolio allocation problem with different mesh sizes. Shown are: (a) the maximal number of iterations among all time points; (b) the numerical solutions U_h at (T, x_0, v_0) ; (c) the increments $U_h - U_{2h}$ (in 10^{-3}); (d) the rate of increments $(U_{2h} - U_{4h})/(U_h - U_{2h})$.

Table 5 presents the numerical solutions of (6.8) at the grid point (T, x_0, v_0) with different spatial mesh size h and a fixed control discretization mesh $h_\varepsilon = 1/20$. We can observe from line (a) that our algorithm requires a small number of iterations to obtain an accurate solution. Moreover, lines (b) and (d) indicate the numerical solution converge monotonically with the convergence rate $O(h) + O(\Delta t)$, as the mesh size tends to zero.

We then investigate the effect of the control discretization by performing computations with a fixed mesh size $h = 1/800$ and different control meshes. Numerical results are given in Table 6, from which we can observe that the control discretization error decreases rapidly as the meshsize tends to zero, and the control mesh $h_\varepsilon = 1/20$ already leads to an accurate approximation with a negligible control discretization error. We further present the optimal investment and consumption allocation corresponding to (6.7) at $t = 0$ in Figure 2, where the colour indicates the value of the optimal feedback control, as shown in the panel on the right. It depicts that the optimal stock allocation in general decreases with respect to the initial wealth, but less sensitive than consumption as observed in [30]. Moreover, the consumption is insensitive to the volatility, while the investment allocation depends explicitly on the initial states of volatility and wealth.

h_ε	1/5	1/10	1/20	1/40
(a)	-0.668535135	-0.660102239	-0.658010097	-0.658005963
(b)		8.4329	2.0921	0.0041

Table 6: Numerical solutions for the consumption-portfolio allocation problem with different control refinements. Shown are: (a) the numerical solutions U_{h_ε} at (T, x_0, v_0) ; (b) the increments $U_{h_\varepsilon} - U_{2h_\varepsilon}$ (in 10^{-3}).

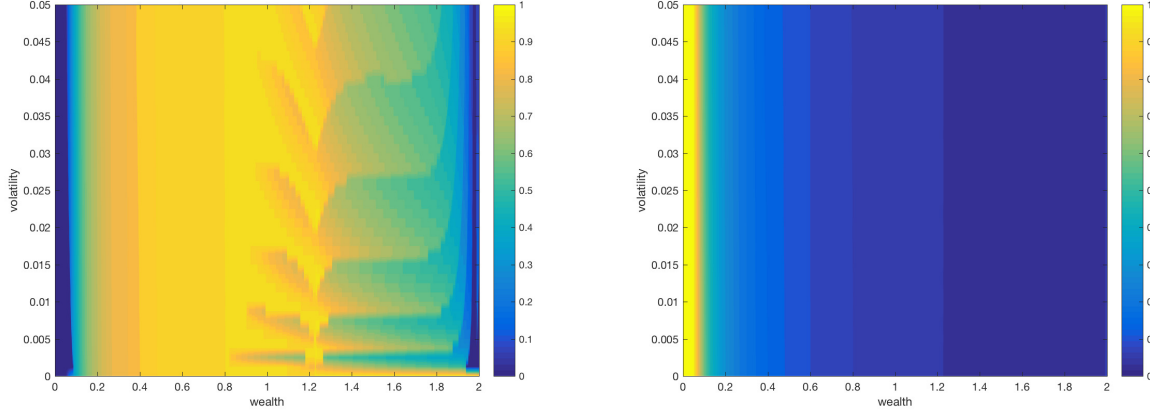


Figure 2: Optimal proportions of wealth for investment (left) and consumption (right).

7 Conclusions

This paper constructs numerical approximations to the solution and free boundary of HJB variational inequalities with monotone drivers, which arise from mixed stochastic control/optimal stopping problems with recursive preferences and infinite activity jumps. The schemes are based on the penalty method, monotone discretizations and policy iteration. We prove the convergence of the numerical scheme and illustrate the theoretical results with some numerical examples including an optimal investment under ambiguity problem and a recursive consumption-portfolio allocation problem.

To the best of our knowledge, this is the first paper which proposes numerical approximations for a HJBVI with a general monotone driver. Natural next steps would be to establish the theoretical convergence rate of the discretization schemes and to extend this approach to “double-obstacle” HJBVIs obtained in [14] and Hamilton-Jacobi-Bellman-Isaacs equations in [5].

A Proof of Proposition 3.1

Proof of Proposition 3.1. We shall focus on the the penalized problem (3.1), since the estimate for the obstacle problem (1.1) can be established by a similar argument. For notational simplicity, we shall omit the dependence of the solution u^ρ on the penalty parameter ρ . Moreover, we shall assume without loss of generality that $\mu \leq -1$ in Assumption 1, since otherwise one can consider the equations for $v_i^\rho = e^{-(1+\mu)^+ t} u_i^\rho$, $i = 1, 2$, and derive the desired estimates (see Remark 1).

For any given λ , θ and $\varepsilon > 0$, we define the following functions:

$$\phi(t, x, y) = \theta e^{\lambda t} |x - y|^2 + \varepsilon e^{\lambda t} (|x|^2 + |y|^2), \quad \psi(t, x, y) = u_1(t, x) - u_2(t, y) - \phi(t, x, y),$$

for all $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ and introduce the following quantities:

$$m_{\theta, \varepsilon}^0 = \sup_{\mathbb{R}^d \times \mathbb{R}^d} \psi(0, x, y)^+, \quad m_{\theta, \varepsilon} = \sup_{[0, T] \times \mathbb{R}^d \times \mathbb{R}^d} \psi(t, x, y) - m_{\theta, \varepsilon}^0.$$

The boundedness and semicontinuity of u_1 , u_2 , along with the penalization terms, imply that there exists $(t_0, x_0, y_0) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, depending on θ and ε , such that

$$\psi(t_0, x_0, y_0) = \sup_{[0, T] \times \mathbb{R}^d \times \mathbb{R}^d} \psi(t, x, y) = m_{\theta, \varepsilon} + m_{\theta, \varepsilon}^0.$$

We further introduce the several nonlinear operators, which are essential for the subsequent analysis. For any given $\alpha \in \mathbf{A}$, $\kappa \in (0, 1)$, $\phi \in C^{1,2}(\bar{\mathcal{Q}}_T)$ and bounded semicontinuous function u , we define for $\mathbf{x} = (t, x) \in \mathcal{Q}_T$ that

$$K_\kappa^\alpha[\phi](\mathbf{x}) = \int_{|e| \leq \kappa} (\phi(t, x + \eta^\alpha(\mathbf{x}, e)) - \phi(\mathbf{x}) - \eta^\alpha(\mathbf{x}, e) \cdot \nabla_x \phi(\mathbf{x})) \nu(de), \quad (\text{A.1})$$

$$\tilde{K}_\kappa^\alpha[u, p](\mathbf{x}) = \int_{|e| > \kappa} (u(t, x + \eta^\alpha(\mathbf{x}, e)) - u(\mathbf{x}) - \eta^\alpha(\mathbf{x}, e) \cdot p) \nu(de), \quad (\text{A.2})$$

$$B_\kappa^\alpha[\phi](\mathbf{x}) = \int_{|e| \leq \kappa} m(\phi(t, x + \eta^\alpha(\mathbf{x}, e)) - \phi(\mathbf{x})) \gamma(\mathbf{x}, e) \nu(de), \quad (\text{A.3})$$

$$\tilde{B}_\kappa^\alpha[u](\mathbf{x}) = \int_{|e| > \kappa} m(u(t, x + \eta^\alpha(\mathbf{x}, e)) - u(\mathbf{x})) \gamma(\mathbf{x}, e) \nu(de). \quad (\text{A.4})$$

We then focus on deriving an upper bound of $m_{\theta, \varepsilon}$ by first assuming $m_{\theta, \varepsilon} > 0$. This further implies that $t_0 > 0$ since otherwise we have $m_{\theta, \varepsilon} = \sup_{\mathbb{R}^d \times \mathbb{R}^d} \psi(0, x, y) - m_{\theta, \varepsilon}^0 \leq 0$.

Now applying the nonlocal version of Jensen Ishii's lemma [28, Theorem 2.2] and using the fact $\inf_\alpha(a) - \inf_\alpha(b) \geq \inf_\alpha(a - b)$, we obtain that for each $\kappa \in (0, 1)$ and $\theta' > \theta$, there exist two symmetric matrices $X, Y \in \mathbb{R}^{d \times d}$ satisfying

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 2\theta' e^{\lambda t_0} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\varepsilon e^{\lambda t_0} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad (\text{A.5})$$

such that the following inequality holds:

$$\begin{aligned} & \lambda \theta e^{\lambda t_0} |x_0 - y_0|^2 + \lambda \varepsilon e^{\lambda t_0} (|x_0|^2 + |y_0|^2) + \inf_{\alpha \in \mathbf{A}} [\text{tr}(-\sigma^\alpha(t_0, x_0)(\sigma^\alpha(t_0, x_0))^T X + \sigma^\alpha(t_0, y_0)(\sigma^\alpha(t_0, y_0))^T Y) \\ & - b^\alpha(t_0, x_0) \nabla_x \phi(t_0, x_0, y_0) - b^\alpha(t_0, y_0) \nabla_y \phi(t_0, x_0, y_0) - l_K^\alpha(t_0, x_0) + l_K^\alpha(t_0, y_0) \\ & - f^\alpha(t_0, x_0, u_1(t_0, x_0), \sigma^\alpha(t_0, x_0))^T \nabla_x \phi(t_0, x_0, y_0), l_B^\alpha(t_0, x_0)) \\ & + f^\alpha(t_0, y_0, u_2(t_0, x_0), \sigma^\alpha(t_0, y_0))^T (-\nabla_y \phi(t_0, x_0, y_0)), l_B^\alpha(t_0, y_0))] \\ & - \rho((\zeta_1(t_0, x_0) - \zeta_2(t_0, y_0)) - (u_1(t_0, x_0) - u_2(t_0, y_0)))^+ \leq 0, \end{aligned} \quad (\text{A.6})$$

where the nonlocal terms are defined as:

$$\begin{aligned} l_K^\alpha(t_0, x_0) &:= K_\kappa^\alpha[\phi(\cdot, \cdot, y_0)](t_0, x_0) + \tilde{K}_\kappa^\alpha[u_1, \nabla_x \phi(\cdot, \cdot, y_0)](t_0, x_0), \\ l_K^\alpha(t_0, y_0) &:= K_\kappa^\alpha[-\phi(\cdot, x_0, \cdot)](t_0, y_0) + \tilde{K}_\kappa^\alpha[u_2, -\nabla_y \phi(\cdot, x_0, \cdot)](t_0, y_0), \\ l_B^\alpha(t_0, x_0) &:= B_\kappa^\alpha[\phi(\cdot, \cdot, y_0)](t_0, x_0) + \tilde{B}_\kappa^\alpha[u_1](t_0, x_0), \\ l_B^\alpha(t_0, y_0) &:= B_\kappa^\alpha[-\phi(\cdot, x_0, \cdot)](t_0, y_0) + \tilde{B}_\kappa^\alpha[u_2](t_0, y_0). \end{aligned}$$

In the case that $(\zeta_1(t_0, x_0) - \zeta_2(t_0, y_0)) - (u_1(t_0, x_0) - u_2(t_0, y_0)) \geq 0$, we can deduce from Lipschitz continuity of ζ_1 and ζ_2 that

$$m_{\theta, \varepsilon} + m_{\theta, \varepsilon}^0 \leq u_1(t_0, x_0) - u_2(t_0, y_0) \leq |\zeta_1 - \zeta_2|_0 + C|x_0 - y_0|. \quad (\text{A.7})$$

Therefore, in the sequel, we shall discuss the case where the last term in (A.6) is equal to 0.

A straightforward computation gives us that

$$\nabla_x \phi(t_0, x_0, y_0) = 2\theta e^{\lambda t_0} (x_0 - y_0) + 2\varepsilon e^{\lambda t_0} x_0, \quad -\nabla_y \phi(t_0, x_0, y_0) = 2\theta e^{\lambda t_0} (x_0 - y_0) - 2\varepsilon e^{\lambda t_0} y_0,$$

from which, together with the Lipschitz continuity of the coefficients, we can deduce the following estimates for the local terms:

$$\begin{aligned}
& \text{tr}(\sigma^\alpha(t_0, x_0)(\sigma^\alpha(t_0, x_0))^T X - \sigma^\alpha(t_0, y_0)(\sigma^\alpha(t_0, y_0))^T Y) \\
& \leq \theta e^{\lambda t_0} C |x_0 - y_0|^2 + \varepsilon e^{\lambda t_0} C (1 + |x_0|^2 + |y_0|^2), \\
& |b^\alpha(t_0, x_0) \nabla_x \phi(t_0, x_0, y_0) + b^\alpha(t_0, y_0) \nabla_y \phi(t_0, x_0, y_0)| \\
& \quad + |\sigma^\alpha(t_0, x_0)^T \nabla_x \phi(t_0, x_0, y_0) - \sigma^\alpha(t_0, y_0)^T (-\nabla_y \phi(t_0, x_0, y_0))| \\
& \leq \theta e^{\lambda t_0} C |x_0 - y_0|^2 + \varepsilon e^{\lambda t_0} C (1 + |x_0|^2 + |y_0|^2).
\end{aligned} \tag{A.8}$$

Moreover, following the same arguments as those for Theorem 4.1 in [28], we derive that

$$l_K^\alpha(t_0, x_0) - l_K^\alpha(t_0, y_0) \leq \omega(\kappa) + \theta e^{\lambda t_0} C |x_0 - y_0|^2 + \varepsilon e^{\lambda t_0} C (1 + |x_0|^2 + |y_0|^2), \tag{A.9}$$

where $\omega(\kappa)$ denotes a quantity which possibly depends on θ, ε , and converges to zero as $\kappa \rightarrow 0$.

We then proceed to estimate the nonlinear term. For notational convenience, we denote $\sigma^\alpha(t_0, x_0)^T \nabla_x \phi(t_0, x_0, y_0)$ and $\sigma^\alpha(t_0, y_0)^T (-\nabla_y \phi(t_0, x_0, y_0))$ by p_1 and p_2 , respectively, then

$$\begin{aligned}
& f^\alpha(t_0, x_0, u_1(t_0, x_0), p_1, l_B^\alpha(t_0, x_0)) - f^\alpha(t_0, y_0, u_2(t_0, y_0), p_2, l_B^\alpha(t_0, y_0)) \\
& \leq f^\alpha(t_0, x_0, u_1(t_0, x_0), p_1, l_B^\alpha(t_0, x_0)) - f^\alpha(t_0, x_0, u_2(t_0, y_0), p_1, l_B^\alpha(t_0, x_0)) \\
& \quad + f^\alpha(t_0, x_0, u_2(t_0, y_0), p_1, l_B^\alpha(t_0, x_0)) - f^\alpha(t_0, y_0, u_2(t_0, y_0), p_2, l_B^\alpha(t_0, x_0)) \\
& \quad + f^\alpha(t_0, y_0, u_2(t_0, y_0), p_2, l_B^\alpha(t_0, x_0)) - f^\alpha(t_0, y_0, u_2(t_0, y_0), p_2, l_B^\alpha(t_0, y_0)) \\
& \leq \mu(m_{\theta, \varepsilon} + m_{\theta, \varepsilon}^0) + C[|x_0 - y_0| + \theta e^{\lambda t_0} |x_0 - y_0|^2 + \varepsilon e^{\lambda t_0} (1 + |x_0|^2 + |y_0|^2)] \\
& \quad + C(l_B^\alpha(t_0, x_0) - l_B^\alpha(t_0, y_0))^+,
\end{aligned}$$

where we have used the condition that $\mu \leq -1$, the fact that

$$u_1(t_0, x_0) - u_2(t_0, y_0) = \phi(t_0, x_0, y_0) + m_{\theta, \varepsilon} + m_{\theta, \varepsilon}^0 \geq m_{\theta, \varepsilon} + m_{\theta, \varepsilon}^0 \geq 0,$$

the estimate (A.8), and the monotonicity of f^α in u and k .

It now remains to bound $(l_B^\alpha(t_0, x_0) - l_B^\alpha(t_0, y_0))^+$. One can obtain from the integrability of the singular measures that

$$|B_\kappa^\alpha[\phi(\cdot, \cdot, y_0)](t_0, x_0)| + |B_\kappa^\alpha[-\phi(\cdot, x_0, \cdot)](t_0, y_0)| \leq \omega(\kappa).$$

Moreover, since ψ attains its maximum at (t_0, x_0, y_0) , we have for all $e \in E$ that

$$\psi(t_0, x_0, y_0) \geq \psi(t_0, x_0 + \eta^\alpha(t_0, x_0, e), y_0 + \eta^\alpha(t_0, y_0, e)).$$

which, together with the property $m(x) - m(y) \leq C(x - y)^+$ of the function m , gives that

$$\begin{aligned}
& \tilde{B}_\kappa^\alpha[u_1](t_0, x_0) - \tilde{B}_\kappa^\alpha[u_1](t_0, y_0) \\
& \leq \int_{|e| > \kappa} C[(u_1(t_0, x_0 + \eta^\alpha(t_0, x_0, e)) - u_1(t_0, x_0)) - (u_2(t_0, y_0 + \eta^\alpha(t_0, y_0, e)) - u_2(t_0, y_0))]^+ \gamma(t_0, x_0, e) \nu(de) \\
& \quad + \int_{|e| > \kappa} m(u_2(t_0, y_0 + \eta^\alpha(t_0, y_0, e)) - u_2(t_0, y_0)) (\gamma(t_0, x_0, e) - \gamma(t_0, y_0, e)) \nu(de).
\end{aligned}$$

Then, by using Assumption 1 and the boundedness of u_2 , we deduce that

$$\tilde{B}_\kappa^\alpha[u_1](t_0, x_0) - \tilde{B}_\kappa^\alpha[u_1](t_0, y_0) \leq C[\theta e^{\lambda t_0} |x_0 - y_0|^2 + \varepsilon e^{\lambda t_0} (1 + |x_0| + |y_0|) + |x_0 - y_0|]. \tag{A.10}$$

Now we are ready to derive the upper bound of $m_{\theta,\varepsilon}$. Substituting (A.7), (A.8), (A.9) and the above estimate of nonlinear terms into (A.6), we can obtain that

$$\begin{aligned} \lambda\theta e^{\lambda t_0}|x_0 - y_0|^2 + \lambda\varepsilon e^{\lambda t_0}(|x_0|^2 + |y_0|^2) &\leq \mu(m_{\theta,\varepsilon} + m_{\theta,\varepsilon}^0) + |\zeta_1 - \zeta_2|_0 + C|x_0 - y_0| \\ &\quad + C\theta e^{\lambda t_0}|x_0 - y_0|^2 + \varepsilon e^{\lambda t_0}C(1 + |x_0|^2 + |y_0|^2) + \omega(\kappa), \end{aligned}$$

from some constant C depends only on the coefficients (independent of θ, ε). Then, by letting $\kappa \rightarrow 0$, taking $\lambda = C + 1$ and maximizing over $|x_0 - y_0|$, we can obtain from $\mu \leq -1$ that

$$m_{\theta,\varepsilon} + m_{\theta,\varepsilon}^0 \leq -\mu(m_{\theta,\varepsilon} + m_{\theta,\varepsilon}^0) \leq C/\theta + \varepsilon C e^{(C+1)T} + |\zeta_1 - \zeta_2|_0.$$

We recall that the above estimate is based on the assumption that $m_{\theta,\varepsilon} > 0$. In the case where $m_{\theta,\varepsilon} \leq 0$, we obtain from the definition of $m_{\theta,\varepsilon}^0$ that

$$m_{\theta,\varepsilon} + m_{\theta,\varepsilon}^0 \leq \sup_{\mathbb{R}^d \times \mathbb{R}^d} [u_1(0, x) - u_2(0, y) - \phi(0, x, y)] = u_1(0, \tilde{x}_0) - u_2(0, \tilde{y}_0),$$

with some $(\tilde{x}_0, \tilde{y}_0) \in \mathbb{R}^d \times \mathbb{R}^d$, which depend on θ, ε and satisfy that $\lim_{\theta \rightarrow \infty} \theta|\tilde{x}_0 - \tilde{y}_0|^2 = 0$. Therefore for any given $(t, x) \in \bar{Q}_T$ and $\varepsilon, \theta > 0$, we can deduce from the definition of $m_{\theta,\varepsilon}$ that

$$\begin{aligned} u_1(t, x) - u_2(t, x) &\leq m_{\theta,\varepsilon} + m_{\theta,\varepsilon}^0 + 2\varepsilon e^{\lambda t}|x|^2 \\ &\leq |\zeta_1 - \zeta_2|_0 + O(\varepsilon) + C/\theta + u_1(0, \tilde{x}_0) - u_2(0, \tilde{y}_0), \end{aligned}$$

then by passing $\theta \rightarrow \infty$ and $\varepsilon \rightarrow 0$, and using the upper semicontinuity of $u_1 - u_2$, we can conclude the desired result. \square

B Proofs of Propositions 4.2, 4.3 and 4.4

Proof of Proposition 4.2. Let $X_i^n \geq Y_i^n$, $i \in \mathbb{Z}^d$, $n = 0, \dots, N-1$, we can deduce from the inequality $\inf_\alpha S - \inf_\alpha T \leq \sup_\alpha (S - T)$ and (4.17) that it suffices to establish for any given $\mathbf{x}_i^n = (t_n, x_i)$ that

$$\begin{aligned} X_i^n + \Delta t \sum_{j \neq i} l_{r,h,j,i}^{\alpha,n+1} X_j^{n+1} + \Delta t K_{r,h}^\alpha X_i^n + \Delta t \bar{f}^\alpha(\mathbf{x}_i^n, U_i^{n+1}, \Delta X_i^n, B_{r,h}^\alpha X_i^n) \\ - [Y_i^n + \Delta t \sum_{j \neq i} l_{r,h,j,i}^{\alpha,n+1} Y_j^{n+1} + \Delta t K_{r,h}^\alpha Y_i^n + \Delta t \bar{f}^\alpha(\mathbf{x}_i^n, U_i^{n+1}, \Delta Y_i^n, B_{r,h}^\alpha Y_i^n)] \geq 0, \end{aligned}$$

which by using (4.10) and the fact that $l_{r,h,j,i}^{\alpha,n+1}, k_{r,h,j,i}^{\alpha,n+1} \geq 0$ can be reduced to showing

$$\begin{aligned} (1 - \Delta t \sum_{j \neq 0} k_{r,h,j,i}^{\alpha,n+1} - 2d\theta)(X_i^n - Y_i^n) + \Delta t [\bar{f}^\alpha(\mathbf{x}_i^n, U_i^{n+1}, \Delta X_i^n, B_{r,h}^\alpha X_i^n) - \bar{f}^\alpha(\mathbf{x}_i^n, U_i^{n+1}, \Delta X_i^n, B_{r,h}^\alpha Y_i^n)] \\ + \Delta t [\bar{f}^\alpha(\mathbf{x}_i^n, U_i^{n+1}, \Delta X_i^n, B_{r,h}^\alpha Y_i^n) + 2d\theta X_i^n - \bar{f}^\alpha(\mathbf{x}_i^n, U_i^{n+1}, \Delta Y_i^n, B_{r,h}^\alpha Y_i^n) - 2d\theta Y_i^n] \geq 0. \quad (\text{B.1}) \end{aligned}$$

Suppose the condition $\theta > C(\sup_{\alpha \in \mathbf{A}} |\sigma^\alpha|_0)\lambda$ is satisfied, we can then deduce from the monotonicity (4.21) and the definition (4.18) of the numerical flux \bar{f} that it remains to obtain a lower bound of $f^\alpha(t, x, u, z, B_{r,h}^\alpha X_i^n) - f^\alpha(t, x, u, z, B_{r,h}^\alpha Y_i^n)$.

Since f^α is non-decreasing in k , we shall assume $B_h^\alpha X_i^n \leq B_h^\alpha Y_i^n$, otherwise the lower bound is 0. Using the Lipschitz continuity of f^α on k , we obtain for any $\mathbf{x}_i^n = (t_n, x_i)$ that

$$\begin{aligned}
& f^\alpha(t, x, u, z, B_{r,h}^\alpha Y_i^n) - f^\alpha(t, x, u, z, B_{r,h}^\alpha X_i^n) \leq C(B_{r,h}^\alpha Y_i^n - B_{r,h}^\alpha X_i^n) \\
& = C \int_{|e|>r} \left[m\left(\sum_{j \neq 0} \omega_j(\eta^\alpha(\mathbf{x}_i^n, e); h)[Y_{i+j}^n - Y_i^n]\right) - m\left(\sum_{j \neq 0} \omega_j(\eta^\alpha(\mathbf{x}_i^n, e); h)[X_{i+j}^n - X_i^n]\right) \right] \gamma(\mathbf{x}_i^n, e) \nu(de) \\
& \leq C \int_{|e|>r} \left[\sum_{j \neq 0} \omega_j(\eta^\alpha(\mathbf{x}_i^n, e); h)[(Y_{i+j}^n - X_{i+j}^n) - (Y_i^n - X_i^n)] \right]^+ \gamma(\mathbf{x}_i^n, e) \nu(de) \\
& \leq C \sum_{j \neq 0} \int_{|e|>r} \omega_j(\eta^\alpha(\mathbf{x}_i^n, e); h) \gamma(\mathbf{x}_i^n, e) \nu(de) \left[[(Y_{i+j}^n - X_{i+j}^n) - (Y_i^n - X_i^n)] \right]^+ \\
& = C \sum_{j \in \mathcal{A}_i} b_{r,h,j,i}^{\alpha,n} [(X_i^n - Y_i^n) - (X_{i+j}^n - Y_{i+j}^n)],
\end{aligned}$$

with $b_{r,h,j,i}^{\alpha,n}$ defined as in (4.11) and the index set $\mathcal{A}_i := \{j \in \mathbb{Z}^d \mid Y_{i+j}^n - X_{i+j}^n > Y_i^n - X_i^n\}$. Therefore, the above estimate together with (B.1) implies that the discrete equation (4.19) is monotone provided that

$$1 - \Delta t \left(\sum_{j \neq 0} k_{r,h,j,i}^{\alpha,n+1} + C \sum_{j \neq 0} b_{r,h,j,i}^{\alpha,n} \right) - 2d\theta \geq 0, \quad (\text{B.2})$$

which along with the estimate (4.15) and the fact $\Gamma_1(r, \kappa) > \Gamma_2(r, \kappa)$ for small enough r lead us to the desired CFL condition (4.22). \square

Proof of Proposition 4.3. We shall consider the quantity $m = \sup_i (X_i^{n+1} - Y_i^{n+1})$. Since the desired result holds if $m \leq 0$, we shall assume $m > 0$, which implies for each small enough ε , there exists an index $i \in \mathbb{Z}^d$ such that

$$0 < m - \varepsilon < X_i^{n+1} - Y_i^{n+1} \leq m,$$

from which we can deduce that $\{X_i^{n+1}\}_i$ and $\{Y_i^{n+1}\}_i$ satisfy the following inequality:

$$\begin{aligned}
& (X_i^{n+1} - Y_i^{n+1})[(X_i^{n+1} - U_i^n) + \Delta t \inf_{\alpha \in \mathbf{A}} (-A_{r,h}^\alpha X_i^{n+1} - K_{r,h}^\alpha U_i^n - \tilde{f}^\alpha(t_n, x_i, X_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n)) \\
& - (Y_i^{n+1} - U_i^n) - \Delta t \inf_{\alpha \in \mathbf{A}} (-A_{r,h}^\alpha Y_i^{n+1} - K_{r,h}^\alpha U_i^n - \tilde{f}^\alpha(t_n, x_i, Y_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n))] \leq 0,
\end{aligned}$$

which together with the inequality $-(\inf_\alpha S - \inf_\alpha T) \leq -\inf_\alpha (S - T)$ implies that

$$\begin{aligned}
|X_i^{n+1} - Y_i^{n+1}|^2 & \leq -\Delta t \inf_{\alpha \in \mathbf{A}} \{ (X_i^{n+1} - Y_i^{n+1}) [-(A_{r,h}^\alpha X_i^{n+1} - A_{r,h}^\alpha Y_i^{n+1}) \\
& - (\tilde{f}^\alpha(t_n, x_i, X_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n) - \tilde{f}^\alpha(t_n, x_i, Y_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n))] \} \\
& \leq \Delta t \sup_{\alpha \in \mathbf{A}} \left\{ -\sum_m l_{r,h,m,i}^{\alpha,n+1} |X_i^{n+1} - Y_i^{n+1}|^2 + \sum_m l_{r,h,m,i}^{\alpha,n+1} (X_m^{n+1} - Y_m^{n+1})(X_i^{n+1} - Y_i^{n+1}) \right. \\
& \quad \left. + (X_i^{n+1} - Y_i^{n+1})(\tilde{f}^\alpha(t_n, x_i, X_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n) - \tilde{f}^\alpha(t_n, x_i, Y_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n)) \right\} \\
& \leq \Delta t \sup_{\alpha \in \mathbf{A}} \left\{ -\sum_m l_{r,h,m,i}^{\alpha,n+1} (m - \varepsilon)^2 + \sum_m l_{r,h,m,i}^{\alpha,n+1} m^2 \right\} \leq O(\varepsilon),
\end{aligned}$$

and consequently we have $(m - \varepsilon)^2 \leq O(\varepsilon)$. Letting $\varepsilon \rightarrow 0$ yields $m = 0$, which leads to a contradiction. This completes the proof of the discrete comparison principle for (4.19), from which we can directly infer the uniqueness of bounded solutions to (4.19). \square

Proof of Proposition 4.4. Without loss of generality, we can assume $|U^{n+1}|_0 > |\zeta(t_n, \cdot)^+|_0$. Then for any small enough $\varepsilon > 0$, we can choose an index i such that $|U_i^{n+1}| \geq |U^{n+1}|_0 - \varepsilon > |\zeta(t_n, \cdot)^+|_0$.

Let us first assume $U_i^{n+1} > 0$, which implies $\rho(\zeta(t_n, x_i) - U_i^{n+1})^+ = 0$. We then deduce from (4.19) and $\mu \leq 0$ that

$$\begin{aligned}
|U_i^{n+1}|^2 &= U_i^{n+1} [U_i^n + \Delta t \sup_{\alpha \in \mathbf{A}} (A_{r,h}^\alpha U_i^{n+1} + K_{r,h}^\alpha U_i^n + \bar{f}^\alpha(t_n, x_i, U_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n))] \\
&\leq \Delta t \sup_{\alpha \in \mathbf{A}} [U_i^{n+1} \sum_m l_{r,h,m,i}^{\alpha,n+1} (U_m^{n+1} - U_i^{n+1}) + U_i^{n+1} (\bar{f}^\alpha(t_n, x_i, U_i^{n+1}, \Delta U_i^n, B_{r,h}^\alpha U_i^n) \\
&\quad - \bar{f}^\alpha(t_n, x_i, 0, \Delta U_i^n, B_{r,h}^\alpha U_i^n))] + U_i^{n+1} [U_i^n + \Delta t \sup_{\alpha \in \mathbf{A}} (K_{r,h}^\alpha U_i^n + \bar{f}^\alpha(t_n, x_i, 0, \Delta U_i^n, B_{r,h}^\alpha U_i^n))] \\
&\leq \Delta t \sup_{\alpha \in \mathbf{A}} [\sum_m l_{r,h,m,i}^{\alpha,n+1} |U^{n+1}|_0^2 - \sum_m l_{r,h,m,i}^{\alpha,n+1} (|U^{n+1}|_0 - \varepsilon)^2] + U_i^{n+1} \sup_{\alpha \in \mathbf{A}} \{ \Delta t \sum_{j \neq 0} k_{r,h,j,i}^{\alpha,n} [U_{i+j}^n - U_i^n] \\
&\quad + U_i^n + [\Delta t \bar{f}^\alpha(t_n, x_i, 0, \Delta U_i^n, B_h^\alpha U_i^n)] + 2d\theta U_i^n - \Delta t \bar{f}^\alpha(t_n, x_i, 0, 0, B_h^\alpha U_i^n) - 2d\theta U_i^n \\
&\quad + \Delta t \bar{f}^\alpha(t_n, x_i, 0, 0, B_h^\alpha U_i^n) - \Delta t \bar{f}^\alpha(t_n, x_i, 0, 0, 0) + \Delta t \bar{f}^\alpha(t_n, x_i, 0, 0, 0) \}. \tag{B.3}
\end{aligned}$$

Since the CFL condition (4.22) is satisfied, we know the numerical flux \bar{f} is stable in sup-norm, hence we can use the monotonicity of f^α on k and bound the above expression by

$$\begin{aligned}
O(\varepsilon) + U_i^{n+1} \sup_{\alpha \in \mathbf{A}} \{ (1 - 2d\theta - \Delta t \sum_{j \neq 0} k_{r,h,j,i}^{\alpha,n}) U_i^n + (\Delta t \sum_{j \neq 0} k_{r,h,j,i}^{\alpha,n} + 2d\theta) |U^n|_0 \\
+ \Delta t C(B_{r,h}^\alpha U_i^n)^+ + \Delta t f^\alpha(t_n, x_i, 0, 0, 0) \}. \tag{B.4}
\end{aligned}$$

Note by using the properties of m , we can obtain a positive upper bound for $B_h^\alpha U_i^n$:

$$B_h^\alpha U_i^n \leq C \int_{|e| > r} \left(\sum_{j \neq 0} \omega_j(\eta^\alpha(t_n, x_i, e); h) (U_{i+j}^n - U_i^n) \right)^+ \gamma(t_n, x_i, e) \nu(de) \leq C \sum_{j \neq 0} b_{r,h,j,i}^{\alpha,n} (U_{i+j}^n - U_i^n)^+,$$

from which, along with the index set $\mathcal{A}_i = \{j \in \mathbb{Z}^d \mid U_{i+j}^n > U_i^n\}$, we can further bound (B.4) by:

$$\begin{aligned}
U_i^{n+1} \sup_{\alpha \in \mathbf{A}} \{ [1 - 2d\theta - \Delta t (\sum_{j \neq 0} k_{r,h,j,i}^{\alpha,n} + C \sum_{j \in \mathcal{A}_i} b_{r,h,j,i}^{\alpha,n})] U_i^n + \\
[(\Delta t \sum_{j \neq 0} k_{r,h,j,i}^{\alpha,n} + C \sum_{j \in \mathcal{A}_i} b_{r,h,j,i}^{\alpha,n}) + 2d\theta] |U^n|_0 + \Delta t f^\alpha(t_n, x_i, 0, 0, 0) \} + O(\varepsilon),
\end{aligned}$$

then by using the CFL condition (4.22) and the estimate (B.3), we obtain for small enough r that

$$(|U^{n+1}|_0 - \varepsilon)^2 \leq |U_i^{n+1}|^2 \leq |U^{n+1}|_0 (|U^n|_0 + \Delta t \sup_{\alpha \in \mathbf{A}} |f^\alpha(\cdot, \cdot, 0, 0, 0)|_0) + O(\varepsilon). \tag{B.5}$$

For the case with $U_i^{n+1} < 0$, we have $\rho(\zeta(t_n, x_i) - U_i^{n+1})^+ U_i^{n+1} \leq 0$, and hence one can derive the same estimate (B.5) similarly by considering $-U^{n+1}$. Then letting $\varepsilon \rightarrow 0$ and dividing both sides by $|U^{n+1}|_0$ give us the desired result (4.23). \square

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