Analytic conformal bootstrap at large spin



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Abstract

The conformal bootstrap is a powerful method to study conformal field theories, relying only on the existence and associativity of the OPE algebra, and has been used to provide both numeric and analytic constraints on a wide variety of theories. This thesis reviews the ideas of the light-cone bootstrap and large spin perturbation theory, and applies them to constrain the spectrum and algebra of several weakly coupled conformal field theories.

The first part studies the Gross-Neveu and Gross-Neveu-Yukawa models, whose spectrum and algebra are analyzed perturbatively in $d = 2 + \epsilon$ and $d = 4 - \epsilon$ dimensions, respectively. This is achieved by combining the method of twist conformal blocks with constraints from crossing symmetry of correlators of composite operators. Known results for the anomalous dimensions of nearly conserved currents are reproduced, and new results are found for the OPE coefficients and anomalous dimensions of a larger class of operators.

The second part studies the critical O(N) model in two different perturbative regimes, the $d = 4 - \epsilon$ expansion and the large N expansion. In both cases, crossing symmetry of a correlator of fundamental fields is combined with the Lorentzian inversion formula to find anomalous dimensions and OPE coefficients for weakly broken currents to a higher order than previously computed in the literature. In particular these are used to predict central charges in the 3d Ising model, comparing favourably with numerical results and previous predictions.

Statement of Originality

This thesis contains results from a single-authored paper, and from work in collaboration with Luis Fernando Alday and Johan Henriksson, to which I contributed substantially.

Chapter 3 is based on my single-authored paper

[1] M. van Loon, *The Analytic Bootstrap in Fermionic CFTs*, *JHEP*, **01** 2018 104, [arXiv:1711.02099].

Chapter 4 is based on the jointly-authored papers

[2] L.F. Alday, J. Henriksson, and M. van Loon, *Taming the ϵ-expansion with large spin perturbation theory*, *JHEP* **07** 2018 131, [arXiv:1712.02314].

[3] J. Henriksson and M. van Loon, *Critical* O(N) *model to order ϵ^4 from analytic bootstrap*, *J. Phys* **A52** (2019) 025401, [arXiv:1801.03512].

Chapter 5 is based on the jointly-authored paper

[4] L.F. Alday, J. Henriksson, and M. van Loon, *An alternative to diagrams for the critical* O(N) *model: dimensions and structure constants to order $1/N^2$*, [arXiv:1907.02445].

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Chapter 1

Introduction

When describing physical systems by a quantum field theory, we use effective theories, which are a valid approximation at low enough energies but are expected to break down at sufficiently high energies. In fact, we can think of all quantum field theories as equipped with such a cut-off energy (or length) scale, with the precise theory used to describe a given physical system depending on the chosen scale. Renormalization, formulated most elegantly in the language of the renormalization group (RG) introduced by Wilson [5, 6], relates such equivalent theories with different energy scales by integrating out short-range (high energy) modes, thus inducing a flow from high energy theories to low energy theories.

Special positions are occupied by QFTs that are invariant under this flow, which describe phenomena that exhibit scale invariance. Examples of such phenomena are continuous phase transitions, in which the correlation length of the system, describing its characteristic length scale, becomes infinite. Under some mild additional assumptions, scale invariance of a QFT actually implies full conformal invariance of the theory [7].

One can then take a point of view in which every QFT is constructed by taking a CFT, perturbing it by some new operator, and following the RG flow. Such perturbations can be irrelevant, in which case the flow goes back to the original CFT; marginal, in which case the flow preserves conformal invariance of the theory, defining

new CFTs; or relevant, in which case the flow breaks conformal invariance and defines new non-conformal QFTs. It is therefore worthwhile to study the general properties of CFTs, as the signposts in the landscape of all QFTs.

Within this framework, many well-known QFTs, such as the standard model, lie in the subclass of those that can be obtained by perturbing the free theory through the addition of relevant operators in a Lagrangian. Perturbations of the free theory can, after RG flow, reach a new CFT, as for example happens at the Wilson-Fisher fixed point, where the relevant operator is φ^4. There are however many other CFTs and QFTs that cannot be reached by adding a small perturbation to the free theory Lagrangian. To see that this point of view is not merely academic, note that it contradicts ideas that were long considered standard lore, such as that there are no renormalizable interacting field theories in dimensions in $d > 4$ [8] (since one cannot write down a relevant term perturbing the Lagrangian of the free theory), or that there are no four-dimensional, inherently $\mathcal{N} = 3$ theories, which instead must always have enhanced $\mathcal{N} = 4$ supersymmetry [9]. Both of these arguments rely on a Lagrangian description of the theory, which need not exist; and indeed non-Lagrangian theories have been found. Perhaps the most famous of these is the maximally supersymmetric 6d $\mathcal{N} = (2, 0)$ theory [10,11]. It is closely related to string and M-theory, has no small couplings at all, and its compactifications have provided insights into many theories, for example giving a geometrical proof of the $\mathrm{SL}(2, \mathbb{Z})$ duality of $\mathcal{N} = 4$ super Yang-Mills theory [8, 12], as well as being the source of the AGT correspondence [13] and other 3d-3d and 2d-4d dualities.

Another motivation for the study of conformal field theories comes from condensed matter physics. Many physical systems can be described by lattice models, with a well-defined notion of rescaling and renormalization, and with a continuum limit described by a QFT. Often such systems exhibit scale invariance at a particular fine-tuning of the parameters of the model, so that the system can be described by a

CFT. Examples include phase transitions in ferromagnets, or water at the critical point, described by the 3d Ising CFT [14].

A further reason to study CFTs comes from the goal of uniting QFT with general relativity into a theory of quantum gravity. This unification ranks as one the most important outstanding problems in theoretical physics, but we do not understand how to directly construct general quantum gravity theories beyond the semiclassical description. However a way to access quantum gravity indirectly is through the holographic principle, the notion that there is a duality between theories of quantum gravity in some spacetime M, and QFTs living on the boundary ∂M. The AdS/CFT correspondence [15–17] is the particular case in which M is an asymptotically locally AdS space. In this case the relation between the gravity theory and dual field theory can be made more precise: for example type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is conjectured to be dual to $\mathcal{N} = 4$ SYM on the boundary of AdS_5. On the bulk side of the correspondence, the conformal group $\mathrm{SO}(d, 2)$ can be recognized as the isometry group of AdS_{d+1}, which is identified with the conformal invariance of the boundary theory. Therefore by studying well-defined CFTs we can learn more about the structure of quantum gravity theories.

Many techniques for analyzing perturbative CFTs have been developed over the years, such as those of the perturbative renormalization group [18,19], or the "old conformal bootstrap"[1], a method of organizing Feynman diagrams in the skeleton expansion to impose conformal symmetry [20]. However, how can we study CFTs that cannot be described as a perturbation of the free theory? Furthermore, given a particular Lagrangian, it is usually straightforward to do perturbation theory, but this may not be the most efficient method of computation, and it is very hard to make general

[1]Not to be confused with the modern conformal bootstrap, which imposes constraints from associativity of the operator product expansion, and is the subject of this thesis.

statements about CFTs, or particular subclasses of CFTs. This is where the (modern) conformal bootstrap, whose ideas date back to the 1970s [21–23], provides an answer: it is applicable to any CFT since it relies only on generic properties of CFTs, and demands that they satisfy these in a self-consistent way. Crucial in this idea is the existence in CFTs of a convergent and effectively computable operator product expansion (OPE), which allows one to replace a product of two suitably close operators by a sum of single operators. Using this, n-point functions can be reduced to $(n-1)$-point functions, so that the correlators of all local operators in the theory can be fixed in terms of the set of CFT data, consisting of the spectrum (the representation theoretic data, generally consisting of a set of scaling dimensions and spins), and the OPE coefficients, the structure constants in the OPE algebra. When reducing n-point to $(n-1)$-point functions, there are different choices of which pairs of operators to contract with the OPE, and the bootstrap exploits the fact that these different choices all produce the same value for the correlator. Non-trivial constraints are found for four-point correlators when demanding equality of the "s-channel" and "t-channel" OPEs:

$$\langle \overbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}\overbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)}\rangle = \langle \overbrace{\mathcal{O}_1(x_1)\overbrace{\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)}\mathcal{O}_4(x_4)}\rangle. \tag{1.0.1}$$

On either side of the equation, each operator appearing in both OPEs gives a contribution to the correlator. In fact the contributions of all operators in a particular conformal multiplet can be combined into a set of known functions, the conformal blocks, which can be computed using purely group theoretic methods. Therefore equation (1.0.1) demands that the expansion of the correlator in terms of two disparate sets of functions must agree.

For two-dimensional CFTs this idea already enjoyed great success in the 1980s, made possible because in two dimensions the conformal algebra enhances to an infinite dimensional one. The condition of conformality locally becomes equivalent to

the Cauchy-Riemann equations, so that after the identification $\mathbb{C} \cong \mathbb{R}^2$, all holomorphic functions generate conformal transformations.[2] Since the conformal algebra enhances, its (global) conformal multiplets recombine into Virasoro multiplets, making its spectrum and algebra much more constrained. The crossing equation also simplifies since conformal blocks combine together into Virasoro conformal blocks. A particular triumph was the full classification of the minimal models [25], whose spectrum is built from finitely many Virasoro multiplets. Conformal field theories with finitely many (global) conformal multiplets in fact do not exist [26, 27], significantly hindering developments in higher dimensions.

The application of the conformal bootstrap to dimension $d > 2$ had to wait until the 2008 seminal paper [28], which particularly relied on progress in the computation of conformal blocks [29, 30]. In its most direct form, crossing symmetry can be seen as a constraint on a suitable vector space spanned by conformal blocks $F_{\Delta,\ell}(z,\bar{z})$:[3]

$$\sum_{\mathcal{O}} a_{\mathcal{O}} F_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}(z,\bar{z}) = 0. \tag{1.0.2}$$

The key insight of [28] consisted of formulating this constraint in the dual space, i.e. constraining the theory by acting on equation (1.0.2) with a suitable set of linear functionals. Any sufficiently continuous functional that can be interchanged with the sum over operators, will give a constraint, or sum rule, on the spectrum of the CFT. This can be exploited by suggesting a putative spectrum, and attempting to show that this spectrum violates equation (1.0.2). For example, one can do this by finding a linear functional α such that $\alpha[F_{\Delta_{\mathcal{O}},\ell_{\mathcal{O}}}] > 0$ for all $(\Delta_{\mathcal{O}}, \ell_{\mathcal{O}})$ in the suggested spectrum.[4] The authors of [28] use a finite set of functionals that evaluate derivatives

[2] A subtlety is that in Euclidean signature the conformal group is not infinite dimensional (only its algebra is), while in Lorentzian signature the conformal group truly is infinite dimensional [24]. Either way we are interested in local transformations, and the algebra consists of two copies of the Witt algebra, of which the Virasoro algebra is the central extension.

[3] The two independent conformal cross-ratios (u, v) for a four-point correlator, and the related coordinates $(z, \bar{z})$, are defined as $u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$, $v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ with $x_{ij} = x_i - x_j$.

[4] Unitarity of the theory is crucial, since it implies that all $a_{\mathcal{O}} \geqslant 0$.

at the point $u = v = \frac{1}{4}$, whose action on the conformal blocks is evaluated numerically. By systematically varying the suggested spectrum and deciding whether it is allowed by their set of functionals, they numerically carve out a set of disallowed spectra. In particular, they find a numerical bound, valid in any unitary four-dimensional CFT, relating the scaling dimensions of the scalar operators of lowest and second-lowest dimension.

Since then the conformal bootstrap field has exploded, with many new results, both numerical and analytical, for a wide variety of theories in different dimensions. On the numerical side, the method outlined above has been improved, for example making it possible to search continuously over spectra [31] – even without having to discuss linear functionals [32] – or by analyzing multiple correlators simultaneously, which often makes constraints much more stringent [33]. Such numerical methods have been applied to many different theories [31, 33–42], a noteworthy result being the most precise determination of the critical exponents in the 3d Ising model [41].

A different numerical approach is the truncation method of [43], in which the OPE is truncated to include a finite number of primary operators, and a finite set of functionals, to create an overconstrained system. This system has solutions only if all of the minors of a suitable matrix vanish, which gives a set of equations that is solved numerically. Its main advantage is that it does not rely on unitarity of the theory, and has, for example, had some success in finding the CFT data of the non-unitary Yang-Lee edge model [43–45]. However, the method gives no control over error bounds.

Further numerical methods exist, such as those of [46–48], which analyze the crossing equations around points other than $u = v = \frac{1}{4}$. Another interesting method is that of [49–52], in which the functionals are contour integrals of the correlator against some suitable holomorphic functions, and which has produced both interesting numerical and analytical results for one-dimensional CFTs.

In a correlator of scalar operators all exchanged operators transform under spin-ℓ traceless symmetric representations of the Lorentz group; to deal with correlators of non-scalar operators, one needs to compute conformal blocks associated to more general Lorentz representations. Such conformal blocks are related via recursion relations to a set of "seed conformal blocks", thus allowing for them, or at least efficient numerical approximations, to be computed [53–60]. This has allowed for the analysis of correlators of spin-ℓ operators, such as conserved spin 1 currents [61] and stress tensors [42]. Fermions have also been analyzed in various dimensions, for example numerically in [62–64] and analytically in [65, 66].

A further development is the study of superconformal theories: in these theories conformal multiplets recombine into superconformal multiplets, and the associated conformal blocks recombine into superconformal blocks. Such blocks have been computed for various superconformal algebras, starting with [67–69], and have been used to constrain a wide range of theories. Examples include 4d $\mathcal{N} = 1$ theories [31, 34], 4d $\mathcal{N} = 2$ theories [70–73], 4d $\mathcal{N} = 3$ theories [74, 75], 4d $\mathcal{N} = 4$ theories [76–82], and 6d $\mathcal{N} = (2, 0)$ theories [83, 84].

In parallel with the above numeric bootstrap, analytic results for CFTs have been found from analyzing crossing symmetry. In the analytic bootstrap one usually takes a particular limit of the crossing equation and uses analytic properties of correlators and conformal blocks to constrain the CFT. Examples of interesting regimes are the u-channel Regge limit[5] $z, \bar{z} \to \mathrm{i}\infty$ with fixed $z/\bar{z}$, and the lightcone limit $z \ll 1 - \bar{z} \ll 1$, the latter getting its name from the fact that in this limit the pairs of operators $(\mathcal{O}_1, \mathcal{O}_2)$ and $(\mathcal{O}_2, \mathcal{O}_3)$ become nearly lightlike separated. A crucial observation from [87] is that at large spin, conformal gauge theories become essentially free,

[5]Note that the various Regge limits naively look the same as the corresponding OPE limits. However they differ because the OPE limits happen on the first sheet of the $(z, \bar{z})$ coordinates, while the Regge limits happen on the second sheet. For example the s-channel Regge limit differs from the s-channel OPE limit by first continuing the $\bar{z}$-coordinate around the branch point $\bar{z} = 1$ [85, 86].

in the sense that in the limit $\ell \to \infty$, the scaling dimensions and OPE coefficients of double trace operators approach those of the generalized free field theory.[6] The almost simultaneous papers [26, 27] used the lightcone bootstrap to understand how these corrections arise in more general CFTs as a consequence of crossing symmetry, and how they can be extracted from an expansion of the crossing equation in orders of z and $(1-\bar{z})$, to give the leading behaviour of CFT data in the limit $\ell \to \infty$. The method of large spin perturbation theory [88, 89] extended these ideas to allow for a systematic study of the crossing equation in powers of $1/\ell$.

Let us discuss the simple, but powerful, example of twist additivity:[7] if a CFT in dimension $d > 2$ contains two (scalar) primary operators of twists τ_1, τ_2, then it also contains a doubly infinite family of primary operators $[\mathcal{O}_1, \mathcal{O}_2]_{n,\ell}$, called the double field operators, whose twists satisfy $\tau_{n,\ell} = \tau_1 + \tau_2 + 2n + \gamma_{n,\ell}$, with $\lim_{\ell\to\infty} \gamma_{n,\ell} = 0$.[8] Note that the existence of this family of primary operators implies the earlier statement that there are no CFTs in dimension $d > 2$ with only a finite number of conformal primaries. The constraint $d > 2$ is necessary to ensure, via unitarity bounds, that there is a twist gap, i.e. that the twist of the lowest-twist operator not equal to the identity is strictly greater than 0, or more formally that $\inf\{\tau_{\mathcal{O}} : \mathcal{O} \neq \mathbb{1}\} > 0$.

To prove twist additivity, consider the correlators $\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\rangle$ and $\langle \mathcal{O}_2\mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\rangle$, which can be fully described by the functions $\mathcal{G}_{1122}(u,v)$ and $\mathcal{G}_{2112}(u,v)$ respectively. These obey a crossing relation

$$\mathcal{G}_{2112}(u,v) = \frac{u^{\frac{\tau_1+\tau_2}{2}}}{v^{\tau_1}}\mathcal{G}_{1122}(v,u). \tag{1.0.3}$$

Both sides of the equation can be expanded into a basis of conformal blocks, which give the contribution of each intermediate primary operator in the OPE. On the right-hand side, the identity operator appears in both the OPEs $\mathcal{O}_1 \times \mathcal{O}_1$ and $\mathcal{O}_2 \times \mathcal{O}_2$

[6] This is also referred to as the mean field theory; I will use both terms interchangeably.

[7] The twist τ of a primary operator of scaling dimension Δ and spin ℓ is defined as $\tau = \Delta - \ell$.

[8] This argument only applies to physical operators. For example in the context of gauge theories, this argument applies to double trace operators, but does not apply to operators that may naively look like double field operators built out of the fundamental field.

and gives a contribution of 1 to $\mathcal{G}_{1122}(u,v)$, which implies the existence of a term $u^{\frac{\tau_1+\tau_2}{2}}/v^{\tau_1}$ on the left-hand side. Since conformal blocks $G_{\tau,\ell}(u,v)$ behave like $u^{\tau/2}$ for small u, this implies that there must be primary operators of approximate twist $\tau_1+\tau_2$ in the OPE $\mathcal{O}_1\times\mathcal{O}_2$. Individual conformal blocks $G_{\tau,\ell}(u,v)$ have only a logarithmic divergence in v; therefore to produce the power-law divergence $u^{\frac{\tau_1+\tau_2}{2}}/v^{\tau_1}$ on the left-hand side of equation (1.0.3), there need to be an infinite number of operators whose twists accumulate at $\tau_1+\tau_2$.

In fact, since the analytic behaviour of conformal blocks is fully understood order by order in u, one can find the OPE coefficients and scaling dimensions necessary to reproduce the exact divergence $u^{\frac{\tau_1+\tau_2}{2}}/v^{\tau_1}$ that is implied by the presence of the identity operator. In [90, 91] this was used to find the full OPE coefficients in the generalized free scalar theory, where we can think of the double field operators as $[\varphi,\varphi]_{n,\ell}\sim\varphi\Box^n\partial_{\mu_1}\dots\partial_{\mu_\ell}\varphi$.[9]

More generally, assuming the existence of a twist gap, one can compute the contribution of a single operator of twist τ on the right-hand side of (1.0.3). In a large ℓ expansion, this contributes to the anomalous dimensions $\gamma_{n,\ell}$ of the double field operators as $\gamma(n,\ell)\sim\frac{\gamma_n}{\ell^\tau}+\dots$, where γ_n can be explicitly computed [26, 27, 87]. In particular, since in a local theory one always assumes the existence of a stress-tensor of protected twist $\tau=d-2$, this shows that anomalous dimensions in a general CFT decay like $1/\ell^{d-2}$ in a large ℓ series.[10]

This highlights the general strategy in the lightcone bootstrap: using conformal block expansions, find the precise divergences that appear on one side of the crossing equation, and demand that these are reproduced on the other side. Many different

[9] As a special case, in the free theory (with $\Delta_\varphi=\frac{d}{2}-1$), only the operators with $n=0$ appear with a non-zero OPE coefficient.

[10] In a perturbative setting this may not be the case if the stress-tensor does not couple to the operator at leading order. This happens for example in the large N approach to the critical O(N) model, discussed in chapter 5, in which leading twist operators in the traceless and antisymmetric O(N) representations have anomalous dimensions that decay like $1/\ell^2$ for every $2<d<4$ at leading order in $1/N$.

technical approaches have been used: for example [92–94] study the crossing equation in the limit $\ell \gg \tau \gg 1$ to find universal behaviour of scaling dimensions of double field operators; [95] uses the Casimir operator to find a recurrence relation for coefficients of the large ℓ expansion of CFT data; and [96] studies the 3d Ising model by using the fact that $\mathrm{SO}(d,2)$ conformal blocks decompose into $\mathrm{SL}(2,\mathbb{R})$ conformal blocks. The method of twist conformal blocks, developed in [89], forms the basis of the analysis in chapter 3. In theories with a degenerate twist spectrum, twist conformal blocks sum the contributions of intermediate operators with the same approximate twist. As sums of an infinite number of conformal blocks, they make the divergences in the crossing equation manifest, thus greatly aiding in the analysis of these divergences. A technical review of them is provided in section 2.3.1.

In all these computational schemes, the central questions in the lightcone bootstrap are:

1. What divergences of a four-point correlator are relevant for the computation of CFT data?

2. How does one compute these divergences?

3. What contribution to the CFT data does a given divergence produce?

A major advance was the Lorentzian inversion formula of [97], reviewed in section 2.3.2, which answers questions 1 and 3.[11] The formula says that the relevant information in the correlator is its double discontinuity, defined as the difference between the function and its two analytic continuations around the branch point $\bar{z} = 1$, and it provides an integral formula for extracting the CFT data from this double discontinuity. It is valid not just asymptotically in $1/\ell$, but even for finite spins $\ell > 1$, and proves, in an obvious manner, that the CFT data can be analytically continued in

[11] Computing the form of the divergences in any particular theory remains more of an art, in which a wide array of ideas and techniques is used, for example see the computations in chapters 4 and 5.

the spin. The inversion formula has since been proven using different methods in [86] and [98]. In particular, [86] explains it from the point of view of light-ray operators – non-local operators found from suitably integrating local operators along a null line – and it extends the method to give, for example, a new proof of the Averaged Null Energy Condition (ANEC). The inversion formula has also been applied to analyze a wide variety of CFTs [66, 82, 99–103], and is instrumental in recent work on crossing kernels and 6j symbols [66, 104–108]. Recently it has been extended to include situations like 1d CFTs [109], boundary CFTs [110–112], and defect CFTs [113, 114].

The Lorentzian inversion formula, in a particular limit, plays a fundamental role in chapters 4 and 5, where it is used to justify which operators contribute to, and how to invert, the double discontinuity of the correlator we study.

Another interesting method that provides analytic results both at finite and at large spin, is the Polyakov-Mellin bootstrap of [115–119]. In the approaches to the bootstrap described so far, the OPE expansion of the four-point function is manifest from an expansion in conformal blocks, while crossing symmetry is imposed as a constraint. The Polyakov-Mellin approach instead expands the four-point function in a basis of crossing-symmetric functions (often Witten exchange and contact diagrams in AdS_{d+1}), and imposes consistency with the OPE expansion by demanding that the resulting Mellin amplitude has poles only at the position of physical operators. This has been successfully used to find, for example, CFT data for leading twist operators in the critical $\mathrm{O}(N)$ model to order ϵ^3 in the $d = 4 - \epsilon$ expansion and order $1/N$ in the large N expansion [118], and provides an important consistency check for our results in chapters 4 and 5.

1.1 Outline of thesis

In this thesis we perturbatively analyze the spectrum of operators in several weakly coupled CFTs. First, chapter 2 presents a technical review of conformal field theory

and the conformal bootstrap, paying special attention to the method of twist conformal blocks and the Lorentzian inversion formula, both of which are used heavily throughout the rest of the thesis.

Chapter 3 analyzes, as a perturbation series in ϵ, the Gross-Neveu model in $d = 2+\epsilon$ dimensions, and the Gross-Neveu-Yukawa model in $d = 4-\epsilon$ dimensions. These are two theories with fundamental fermions as dynamical fields; to study them we consider correlators of composite bosonic operators. Since these have a much richer OPE than the fundamental fields, there are many intermediate operators contributing to the correlator, complicating the analysis. The crossing equation is analyzed using the method of twist conformal blocks, which makes the divergences of the correlator explicit; these divergences are then matched on both sides of the crossing equation. Since the analysis involves composite operators, it shows a similarity to that of 4d gauge theories [80], where the fundamental field is not part of the spectrum of physical operators. Results are presented to first order in both theories, matching known results. The non-integrality of the spacetime dimensions makes a higher order analysis based on twist conformal blocks prohibitive.

In chapters 4 and 5 we analyze the critical O(N) model, a theory with global O(N) symmetry living in $2 < d < 4$ dimensions, getting constraints from the four-point function $\langle\varphi_i\varphi_j\varphi_k\varphi_l\rangle$ of fundamental fields. This time the full force of the inversion formula is apparent, since the OPE of two fundamental fields is much simpler, so that contributions to the double discontinuity can be computed systematically and exactly order by order in the perturbative expansion parameter.

The theory is studied in two ways: firstly, in chapter 4 in a perturbation series in $d = 4 - \epsilon$. Because the first non-trivial corrections to the correlator are at order ϵ^2, the crossing equation can be straightforwardly studied to order ϵ^4. New results for the leading twist operators are found to this order, including a value for the correction to the central charge, and we comment on the relevance of these results to the 3d

Ising model. Interestingly, all free parameters in the theory are uniquely fixed if one assumes that the result from the Lorentzian inversion formula can be analytically continued to spin $\ell = 0$, beyond its usual range of validity. Furthermore, mirroring results in $\mathcal{N} = 4$ SYM, a transcendentality principle is found, i.e. the corrections to the correlator organize themselves in a basis of harmonic polylogarithms [120], and corrections to the CFT data in terms of a basis of harmonic sums [121].

An orthogonal approach to the theory exists, in which one introduces a Hubbard-Stratonovich auxiliary scalar field that allows one to define the theory around $N = \infty$ in a perturbation series in $1/N$. In chapter 5 we study the implications of crossing symmetry on the spectrum perturbatively to order $1/N^2$. Various known results are reproduced, and new results for OPE coefficients are presented. The results for the anomalous dimensions are particularly simple to obtain in our approach, whereas a Feynman diagram computation for the anomalous dimensions includes diagrams of loop order 4 [122]. Again there is an interesting analytic continuation of the large spin data down to spin $\ell = 0$, which completely fixes the parameters in the theory. The transcendentality principle is less easily defined in this large N limit, but manifests itself in the result that corrections organize themselves into hypergeometric functions whose complexity increases with the perturbation order.

Chapter 2

Review of the conformal bootstrap

In this chapter we review basic results about CFTs, the conformal bootstrap and large spin perturbation theory. Apart from a few minor computations, this is not original research but a summary of results in the literature.

The standard textbook on CFTs, particularly relevant in two dimensions, is [123]. Excellent recent reviews of higher-dimensional CFTs in the context of the conformal bootstrap include [124–126]. Central to this viewpoint of CFT is that one should think of it as defined by correlators of (local) operators, or equivalently that it is fully fixed by its operator product algebra.[1]

2.1 Conformal field theory

Conformal transformations $x \to x'(x)$ of spacetime are those that preserve angles. Locally they can be viewed as a combination of a rotation and scaling, and they are such that they leave the metric invariant up to a position-dependent scaling: $g'_{\mu\nu}(x') = \Omega(x)^2 g_{\mu\nu}(x)$. In general dimension d, the conformal group (in Euclidean signature) is isomorphic to the Lie group $\mathrm{SO}(d+1,1)$.[2] All conformal transformations

[1] By focusing only on local operators we may miss out on information of the theory contained in its non-local operators. Existence of an OPE for non-local operators is a highly non-trivial question: recent work [86, 127] has shown existence of an OPE for light-ray operators.

[2] In two dimensions, this symmetry is enhanced, as all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ locally define a conformal transformation. We shall not be concerned with the group of all such functions, but only with the "global conformal group", which is equivalent to the group of Möbius transformations.

can be obtained by composition of translations, Lorentz transformations (rotations), dilatations (scalings), and the inversion $x^\mu \mapsto x^\mu/\|x\|^2$. In particular, the composition of inversion, translations and inversion gives the set of special conformal transformations. The $\mathfrak{so}(d+1,1)$ Lie algebra is generated by the associated generators of these basic building blocks: P_μ for translations, $L_{\mu\nu}$ for Lorentz transformations, D for dilatations, and K_μ for special conformal transformations. Their commutation relations are as follows [125, 126]:

$$\begin{aligned}
[K_\mu, P_\nu] &= 2\delta_{\mu\nu} D - 2L_{\mu\nu}\,, & [L_{\mu\nu}, K_\rho] &= \delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu\,, \\
[L_{\mu\nu}, P_\rho] &= \delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu\,, & [D, P_\mu] &= P_\mu\,, \\
[L_{\mu\nu}, L_{\rho\sigma}] &= \delta_{\nu\rho} L_{\mu\sigma} - \delta_{\mu\rho} L_{\nu\sigma} + \delta_{\nu\sigma} L_{\rho\mu} - \delta_{\mu\sigma} L_{\rho\nu}\,, & [D, K_\mu] &= -K_\mu\,,
\end{aligned} \tag{2.1.1}$$

with all other commutators vanishing.

Operators in the theory are irreducible representations of the conformal group. Since the operator $\mathcal{O}(x)$ can be generated via an action of the generator P: $\mathcal{O}(x) = e^{-P\cdot x}\mathcal{O}(0)e^{P\cdot x}$, the representation theory of the conformal group is completely fixed by that of the stabilizer of the origin, which is generated by D, K_μ and $L_{\mu\nu}$. From the algebra it is clear that P and K act as raising/lowering operators with respect to the dilatation operator, whose spectrum is bounded below in unitary theories [128–131]. Therefore we may define primary operators to be those operators that are annihilated by K; all operators obtained from it by action of P are called descendant operators. Thus primary operators are fully classified according to the action of $D, L_{\mu\nu}$, which are given by the scaling dimension Δ (the eigenvalue under D) and the spin representation r. The operators studied in this thesis will be either Lorentz scalars or transform under a traceless symmetric representation, in which case the spin of the representation can simply be described by an integer $\ell = 0, 1, 2, \ldots$. It is convenient to describe the spinning operators using "index-free notation" [132–134], in which the Lorentz

indices are contracted with an auxiliary null polarization vector z:

$$\mathcal{O}_{\Delta,\ell}(x,z) = \mathcal{O}_{\Delta,\ell}^{\mu_1\dots\mu_\ell}(x) z_{\mu_1}\dots z_{\mu_\ell}. \tag{2.1.2}$$

The original indexed version $\mathcal{O}_{\Delta,\ell}^{\mu_1\dots\mu_\ell}$ can be recovered by action of the Thomas derivative $D_z^\mu = \left(\frac{d}{2} - 1 + z\cdot\partial_z\right)\partial_{z_\mu} - \frac{1}{2}z^\mu\partial_{z_\nu}^2$ [132, 133].

2.1.1 Conformal correlators

The transformation properties of operators under the action of the conformal group yield specific transformation rules for their correlators, severely restricting the form they can take. There are several frameworks to evaluate these forms for operators in generic representations, most notably the embedding formalism [133, 135, 136], in which one identifies spacetime with a section of a null cone in $\mathbb{R}^{d+1|1}$, where the conformal group $\mathrm{SO}(d+1,1)$ acts linearly, and the conformal frame formalism [137], which has been particularly useful in classifying possible tensor structures for correlators of tensor operators and fermionic operators [59].

Two-point functions of primary operators are completely fixed after normalizing operators appropriately[3]; specifically, the operators need to have equal scaling dimension and transform in conjugate Lorentz representations. Specializing to scalar and traceless symmetric operators, which will be the relevant cases for us, we have the following two-point functions:

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\rangle = \frac{1}{|x_{12}|^{2\Delta}}, \tag{2.1.3}$$

$$\langle \mathcal{O}_{\Delta,\ell}(x_1,z_1)\mathcal{O}_{\Delta,\ell}(x_2,z_2)\rangle = \frac{1}{|x_{12}|^{2\Delta}}\left(z_1\cdot z_2 - \frac{2}{x_{12}^2}(z_1\cdot x_{12})(z_2\cdot x_{12})\right)^\ell. \tag{2.1.4}$$

The dynamical information of the theory is contained in the three-point functions of its primary operators, in which the OPE coefficients C_{123} survive as free parameters.

[3] An exception are symmetry currents such as the stress-energy tensor. If the theory has an action, their normalization is fixed by their definition as the variation of the action; more generally they are normalized by demanding that their three-point function gives the right charge algebra. The two-point normalization of such a current is the associated central charge.

Because the thesis will focus on four-point functions of scalar operators, we only need the following three-point functions:[4]

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_{123}}|x_{23}|^{\Delta_{231}}|x_{13}|^{\Delta_{132}}}, \tag{2.1.5}$$

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3,\ell}(x_3,z)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_{123}+\ell}|x_{23}|^{\Delta_{231}+\ell}|x_{13}|^{\Delta_{132}+\ell}}\left(\frac{z\cdot x_{13}}{x_{13}^2}-\frac{z\cdot x_{23}}{x_{23}^2}\right)^{\ell}, \tag{2.1.6}$$

where $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$.

The four-point function of scalar primary operators is the first n-point function whose spacetime dependence is not completely fixed by symmetry:

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle = \mathbb{K}_4\ \mathcal{G}(u,v), \tag{2.1.7}$$

where the kinematical factor $\mathbb{K}_4$ has been split off, and where $\mathcal{G}(u,v)$ is a theory-dependent function of the conformally invariant cross-ratios (u,v):

$$\mathbb{K}_4 = \frac{1}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}}\frac{|x_{24}|^{\Delta_{12}}|x_{14}|^{\Delta_{34}}}{|x_{14}|^{\Delta_{12}}|x_{13}|^{\Delta_{34}}}, \qquad u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\ v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}, \tag{2.1.8}$$

with $\Delta_{ij} = \Delta_i - \Delta_j$.

To simplify computations, four-point functions are often written in terms of the coordinates $(z,\bar{z})$ defined by $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$. To see their physical interpretation, note that by a conformal transformation we can fix three operators along a line, so that the four operators lie in a plane. In fact we could fully gauge-fix the conformal symmetry to put them at the positions [126]

$$x_1 = (0,0,0,\dots),\quad x_2 = (\tfrac{z+\bar{z}}{2}, \tfrac{z-\bar{z}}{2\mathrm{i}}, 0,\dots),\quad x_3 = (1,0,0,\dots),\quad x_4 = (\infty,0,0,\dots), \tag{2.1.9}$$

giving the cross-ratios as above.

We will not discuss higher n-point functions, since the existence of the OPE, discussed below, implies that all such correlators are fixed by all lower-point correlators.

[4] In the case of more complicated external operators such as fermions, there can be several independent OPE coefficients corresponding to different tensor structures [59, 62, 63].

2.1.2 Operator product expansion

In any quantum field theory, one needs to discuss the quantization of the theory. The theory is usually quantized by specifying a foliation of spacetime by leaves, to which one associates isomorphic Hilbert spaces, and a unitary evolution operator U that interpolates between the Hilbert spaces [124–126]. For example, in Poincaré invariant theories, it is often convenient to quantize by equal-time slices $t = t_i$, with associated Hilbert spaces $\mathcal{H}_i$ between which one interpolates via the operator $U(t_i, t_j) = e^{\mathrm{i}P_0(t_j - t_i)}$, in which P_0 plays the role of Hamiltonian. Inserting operators in the past of a time-slice creates an "in-state" $|\Psi_{in}\rangle \in \mathcal{H}_i$, while inserting operators in the future creates an "out-state" in the dual space $\langle\Psi_{out}| \in \mathcal{H}_i^*$. For states living on different leaves, one interpolates between the two Hilbert spaces with the evolution operator U to compute the correlator $\langle\Psi_{out}|U|\Psi_{in}\rangle$.

In a conformal field theory it is most convenient to quantize via radial slices, in which the leaves are spheres S^{d-1} of fixed radius r, with the dilatation operator acting as the Hamiltonian. Via the path integral, one can associate in-states to insertions of operators on the interior of the sphere, and vice versa, with the vacuum state corresponding to the case of no insertions. One can scale the radius of the sphere to zero via a conformal transformation to see that the state $|\Psi_{in}\rangle$ can be expanded in a basis of operators inserted at the origin. This is the idea of the operator product expansion (OPE): two operators within a sphere can be viewed as equivalent to a particular sum of operators inserted at the origin of that sphere. Conformal invariance fixes the expansion parameters of descendant states in terms of those of primary states, so that one may write

$$\mathcal{O}_i(x_1) \times \mathcal{O}_j(x_2) = \sum_{\mathcal{O}_k \text{ primary}} C_{ijk}\, \hat{c}_{ijk}(x_1, x_2, y, \partial_y)\mathcal{O}_k(y), \tag{2.1.10}$$

with the function $\hat{c}$ completely fixed by conformal invariance. In practice, one can compute it by demanding consistency with the form (2.1.5), (2.1.6) of the three-point

function $\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k^\dagger \rangle$.

The OPE can be used to progressively turn n-point functions into $(n-1)$-point functions, fully determining all correlators of local operators, and thus it is the OPE which we shall view as the defining feature of a CFT.[5]. Constraining theories by demanding consistency of the OPE algebra is the main idea of the conformal bootstrap.

2.1.3 Unitarity bounds

Before we move on to discuss the conformal bootstrap, a few useful results arising from unitarity are in order. Unitarity of the theory imposes that the Hilbert spaces associated to any foliation of spacetime have a positive-definite norm. In particular: in radial quantization, to any in-state $|\Psi\rangle \in \mathcal{H}$ generated by an operator insertion $\mathcal{O}(x)$, one may associate a reflected out-state $\langle \Psi | \in \mathcal{H}^*$ generated by an operator insertion $\mathcal{O}^\dagger(y)$, where y has been obtained by inversion in the sphere defining $\mathcal{H}$. Analogously, in the equal-time plane quantization of Euclidean CFTs, given an in-state from an operator insertion $\mathcal{O}(x)$, one can define an out-state from an insertion $\mathcal{O}^\dagger(y)$ at the position reflected in the plane of quantization. Unitarity then demands that this correlator $\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \langle \Psi | \Psi \rangle$ is positive for all operator insertions. This property is known as reflection positivity and leads to some non-trivial implications.

Firstly, one can deduce that in a unitary theory all OPE coefficients are real, an especially important property for numerical bootstrap procedures (see also section 2.2.2). To show this, consider a six-point function $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_3^\dagger \mathcal{O}_2^\dagger \mathcal{O}_1^\dagger \rangle$ of three closely clustered operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$, and their mirror images under reflection in a plane. By reflection positivity this six-point function must be positive; also by the cluster decomposition theorem, it should factorize into two three-point functions as we send

[5] A mathematical formulation of 2d CFTs through vertex operator algebras is essentially an attempt to define the local behaviour of one of the chiral sectors by formalizing the OPE [138, 139] Additional work is required to combine two chiral CFTs into a full CFT [140, 141].

the clusters towards infinity, away from the plane of reflection [126]. These two three-point functions are clearly of the form $C_{123} \times$ (positive) and $C^*_{123} \times$ (positive), so that necessarily $|C_{123}|^2 \geqslant 0$.

Secondly, one can obtain bounds on scaling dimensions of operators in unitary theories [128–131]. Positivity of two-point functions of primary operators merely implies that the two-points functions have a positive normalization, but interesting bounds can be obtained by considering two-point functions of descendants. Since descendants are obtained from primary operators by the action of $P_\mu \sim \partial_\mu$, their two-point functions will be polynomials in the scaling dimension Δ of the primary operator. Considering all level 1 descendants, i.e. those obtained from a single action of P_μ, one finds the following constraint on the allowed scaling dimension of an operator in SO(d) representation R:

$$\Delta_R \geqslant \frac{1}{2}\left(c_2(R) + c_2(V) - c_2(R')\right), \tag{2.1.11}$$

where c_2 is the quadratic Casimir, V is the vector representation and R' the representation with smallest quadratic Casimir that appears in the Clebsch-Gordan decomposition of $R \otimes V$ [131][6]. This bound is saturated precisely when one of the descendant states becomes null.

More explicitly, if R is the SO(d) representation described by the state with highest weight $h = (h_1, \ldots, h_n)$, where $n = \lfloor d/2 \rfloor$, then the bound becomes

$$\Delta_R \geqslant |h_i| + d - i - 1, \tag{2.1.12}$$

where i is the smallest value such that $h_i > |h_{i+1}| + 1$, if such an i exists. If no such i exists, and all $h_i = 0$, the bound simply becomes $\Delta_R \geqslant 0$; while if all $h_i \geqslant 1$, equation (2.1.12) applies with $i = \lfloor d/2 \rfloor$.

[6]The paper [131] extends the 3d and 4d results of [128–130] to provide the equation (2.1.11), valid for general $d > 2$. In the physics community, it was realized only recently [55, 142] that the results for general dimensions had essentially already been computed in the mathematics literature in [143].

Scalar operators, with $h = (0, \ldots, 0)$, satisfy the bound $\Delta \geqslant 0$. This bound is saturated by the identity operator, the unique operator with $\Delta = 0$, all of whose descendants are null. For scalar operators other than the identity, a tighter bound can be derived by considering level 2 descendants, which gives the constraint $\Delta \geqslant \frac{d}{2} - 1$. If the state $|\mathcal{O}\rangle$ saturates the bound, it has a null descendant $P^2|\mathcal{O}\rangle = 0$, which implies, in operator language, that $\mathcal{O}(x)$ satisfies the Klein-Gordon equation [125]. Thus this bound is saturated only by the free boson.

Symmetric traceless operators of spin ℓ live in the representation with highest weight $h = (\ell, 0, \ldots, 0)$, so that the bound gives $\Delta_\ell \geqslant d - 2 + \ell$. The representation in $R \otimes V_\ell$ with minimal Casimir is the spin-$(\ell - 1)$ representation $V_{\ell-1}$: therefore any operator $\mathcal{O}^{\mu_1 \cdots \mu_\ell}$ that saturates this bound has a null descendant in the spin-$(\ell - 1)$ representation, implying that $\partial_{\mu_1} \mathcal{O}^{\mu_1 \mu_2 \cdots \mu_\ell} = 0$ [125]. Hence any operator saturating this bound is a conserved current, and conversely every conserved current saturates this bound. Cases of special interest are the stress-tensor, a conserved current of spin $\ell = 2$, and global symmetry currents of spin $\ell = 1$, both of which we encounter in our theories and whose conservation shall provide crucial constraints on the values of parameters in the theory. If conserved currents of spin $\ell \geqslant 3$ are present in the theory, then, modulo some mild assumptions, the theory must in fact be free [144].

2.2 The bootstrap problem

The OPE shows that the operators satisfy an algebra: given two suitably close (primary) operators[7], one can expand their "product" in a basis of other operators. This then allows one to compute an n-point correlator in terms of a set of $(n-1)$-point correlators. Given an n-point function, there are multiple choices of how to perform the OPE; demanding that these expansions produce the same n-point function imposes associativity of the operator product algebra. With all the two- and three-point

[7] Specifically, a sphere surrounding only these two operators needs to exist. For more details on OPE convergence, see [145].

functions fixed, all other higher order correlators are then fixed as well. Associativity of three-point functions is trivial and merely restates the symmetry properties of the OPE coefficients which are already evident from equation (2.1.6). However, demanding that the different ways of using the OPE to expand a four-point function yield equivalent results, also known as crossing symmetry, imposes highly non-trivial constraint equations on the possible values of the OPE coefficients and of the spectrum of operator dimensions and spins. The idea of the conformal bootstrap is to impose these equations systematically to constrain the CFT data, possibly even uniquely determining it.

Let us describe this in more detail. Consider a four-point function of (not necessarily identical) scalar operators. Using the OPE, we could contract the pairs of operators $(1,2)$ and $(3,4)$, called the direct or s-channel; or we could contract the operators $(2,3)$ and $(1,4)$, called the crossed or t-channel. Associativity of the OPE implies that both methods should give the same result for the four-point function:

$$\langle \overbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}\overbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)}\rangle = \langle \overbrace{\mathcal{O}_1(x_1)\overbrace{\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)}\mathcal{O}_4(x_4)}\rangle. \tag{2.2.1}$$

This leads to a non-trivial constraint on the function $\mathcal{G}(u,v)$ in equation (2.1.7):

$$v^{\frac{\Delta_2+\Delta_3}{2}}\mathcal{G}_{1234}(u,v) = u^{\frac{\Delta_1+\Delta_2}{2}}\mathcal{G}_{3214}(v,u). \tag{2.2.2}$$

The conformal bootstrap consists of the study of the consequences of this equation. The fact that a single equation can constrain CFTs so heavily is incredibly surprising!

From this OPE point of view, we can speak of the exchange of an operator in a particular channel if it appears in both of the OPEs in that channel.[8] By performing both OPEs explicitly, we can precisely compute the contribution of an exchanged operator in a particular channel. Systematically packaging the contributions to a

[8] Technically we need an operator in one OPE and its conjugate in the other, but we shall disregard this distinction and simply talk of an exchanged operator in a channel.

four-point function of a primary operator and all its descendants in a single function naturally leads to the idea of conformal blocks, reviewed below.

First a comment regarding global symmetries is in order, since in chapters 3, 4 and 5, we will encounter theories with global U(N) and O(N) symmetry respectively. In the presence of a global symmetry G, a correlator of n operators $\mathcal{O}_1, \ldots, \mathcal{O}_n$ transforming under G-representations $V_1, \ldots, V_n$ is non-vanishing if and only if there is a G-invariant tensor in the product representation, i.e. if and only if $(V_1 \otimes \cdots \otimes V_n)^G$ is non-empty. In particular, four point-functions are non-vanishing if and only if $(V_1 \otimes V_2 \otimes V_3 \otimes V_4)^G$ is non-empty.

A perhaps more useful perspective is that of exchanged operators in a channel: the only operators that can contribute in e.g. the s-channel are those that transform under a representation V such that $(V_1 \otimes V_2 \otimes V)^G$ and $(V_3 \otimes V_4 \otimes \bar{V})^G$ are non-empty, or equivalently such that $\bar{V}$ and V appear in the decomposition of $V_1 \otimes V_2$ and $V_3 \otimes V_4$ respectively. The crossing equation (2.2.2) then splits into a set of coupled equations, one for each exchanged representation. An example is the correlator of four fundamental O(N) fields in equation (4.2.1). The allowed exchanged operator representations are the singlet (S), traceless symmetric (T) and antisymmetric (A) rank-2 tensor representations, which leads in equation (4.2.3) to a set of three coupled crossing equations for the functions $\mathcal{G}_S(u,v)$, $\mathcal{G}_T(u,v)$, $\mathcal{G}_A(u,v)$, each containing the contribution of intermediate operators in a specific O(N) representation.

2.2.1 Conformal blocks

To extract useful information from the crossing symmetry equation (2.2.2), one needs to impose further constraints from conformality of the theory. As mentioned before, one can think of each side of the equation as being created by an exchange of operators in the OPEs of the fields in a certain channel. Each conformal multiplet, consisting of

a primary operator and its descendants, makes a specific contribution to the overall correlator: furthermore, this contribution can actually be computed explicitly from the OPE (2.1.10):

$$\mathbb{K}_4\ \mathcal{G}(u,v) = \sum_{\mathcal{O}_k} \hat{c}_{12k}(x_1,x_2,y,\partial_y)\hat{c}_{34k}(x_3,x_4,z,\partial_z)\langle \mathcal{O}_k(y)\mathcal{O}_k^\dagger(z)\rangle, \tag{2.2.3}$$

where the sum is over primary operators $\mathcal{O}_k$ exchanged in the s-channel. Therefore we can expand equation (2.2.3) as follows:

$$\mathcal{G}(u,v) = \sum_{\mathcal{O}} C_{12\mathcal{O}}C_{34\mathcal{O}}G^{\Delta_{12},\Delta_{34}}_{\tau_\mathcal{O},\ell_\mathcal{O}}(u,v), \tag{2.2.4}$$

where $G^{\Delta_{12},\Delta_{34}}_{\tau,\ell}(u,v)$ are the conformal blocks, counting precisely the contribution of a single conformal multiplet.

While the conformal blocks can in theory be computed directly from the OPE [29, 146], this is generally not an efficient method of computation. Since the form of the OPE is, up to the overall OPE coefficient, completely fixed by conformal symmetry and the representation of the primary operator, these contributions can be computed as group theoretic objects. Hence much of the structure of (2.2.3) is unnecessary for the computation of conformal blocks, and more efficient methods are available: one highly successful method uses the fact that the blocks satisfy Casimir differential equations [30, 147], and another uses the shadow formalism [22, 91, 135, 147], in which the blocks are expressed via an integral.

The main idea of the lightcone bootstrap and large spin perturbation theory, is to constrain the CFT by using the analytic properties of conformal blocks in a specific limit, combined with the analytical properties implied by the crossing equation (2.1.7). We list here some of the most relevant results that will be used time and again in the thesis.[9]

[9]Since all correlators in this thesis are of external scalars, we limit ourselves here to results for scalar conformal blocks.

Firstly, in 2d and 4d the conformal blocks are completely known, which are most easily expressed in terms of $(z, \bar{z})$. The 4d conformal blocks are then given by

$$G^{\Delta_{12},\Delta_{34}}_{\tau,\ell}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left(k^{\Delta_{12},\Delta_{34}}_{\tau+2\ell}(z)k^{\Delta_{12},\Delta_{34}}_{\tau-2}(\bar{z}) - k^{\Delta_{12},\Delta_{34}}_{\tau-2}(z)k^{\Delta_{12},\Delta_{34}}_{\tau+2\ell}(\bar{z})\right), \quad (2.2.5)$$

and the 2d conformal blocks by[10]

$$G^{\Delta_{12},\Delta_{34}}_{\tau,\ell}(z,\bar{z}) = \left(1-\frac{1}{2}\delta_{0,\ell}\right)\left(k^{\Delta_{12},\Delta_{34}}_{\tau+2\ell}(z)k^{\Delta_{12},\Delta_{34}}_{\tau}(\bar{z}) + k^{\Delta_{12},\Delta_{34}}_{\tau}(z)k^{\Delta_{12},\Delta_{34}}_{\tau+2\ell}(\bar{z})\right), \quad (2.2.6)$$

where

$$k^{\Delta_{12},\Delta_{34}}_{\beta}(x) = x^{\beta/2}\ {}_2F_1\left(\frac{\beta}{2}-\frac{\Delta_{12}}{2}, \frac{\beta}{2}+\frac{\Delta_{34}}{2}; \beta; x\right). \quad (2.2.7)$$

From the definition of the $(z, \bar{z})$ coordinates, it is clear that they provide a double covering of the (u, v) coordinates, evident in the symmetry $z \leftrightarrow \bar{z}$. Where appropriate, we make the choice of mapping the small u limit onto the small z limit, and the small v limit onto the small $(1-\bar{z})$ limit.

In the limit of small u, the conformal blocks are given by the collinear conformal blocks, associated to the 1-dimensional conformal group $\mathrm{SL}(2,\mathbb{R})$:

$$G^{\Delta_{12},\Delta_{34}}_{\tau,\ell}(u,v) = u^{\tau/2}(1-v)^{\ell}\ {}_2F_1\left(\tfrac{\tau}{2}+\ell-\tfrac{\Delta_{12}}{2}, \tfrac{\tau}{2}+\ell+\tfrac{\Delta_{34}}{2}; \tau+2\ell; 1-v\right) + O(u^{\tau/2+1}). \quad (2.2.8)$$

In the limit of small v, the conformal blocks have at most a logarithmic singularity:

$$G^{\Delta_{12},\Delta_{34}}_{\tau,\ell}(u,v) = u^{\tau/2}(f_1(u,v)\log v + f_0(u,v)), \quad (2.2.9)$$

with $f_1(u,v)$ and $f_0(u,v)$ finite in the limit $v \to 0$. Finally, the conformal blocks satisfy an eigenvalue equation under the quadratic Casimir $\mathcal{D}_2$ of the conformal group [30, 147]:

$$\mathcal{D}_2 G_{\tau,\ell}(u,v) = \left(\ell^2 + \ell(\tau-1) + \frac{1}{2}\tau(\tau-d)\right)G_{\tau,\ell}(u,v). \quad (2.2.10)$$

[10] These are the conformal blocks associated to the global conformal group in $2d$, i.e. they are not the Virasoro conformal blocks.

Introducing the shifted Casimir operator $\mathcal{C}_{\tau,d} = \mathcal{D}_2 + \frac{1}{4}\tau(2d - \tau - 2)$, this becomes $\mathcal{C}_{\tau,d}G_{\tau,\ell}(u,v) = J^2_{\tau,\ell}G_{\tau,\ell}(u,v)$, where $J^2_{\tau,\ell} = \left(\frac{\tau}{2} + \ell\right)\left(\frac{\tau}{2} + \ell - 1\right)$ is the conformal spin. This property is particularly important for the definition of twist conformal blocks in section 2.3.1.

2.2.2 The numeric bootstrap

While this thesis focuses completely on the analytic bootstrap, a review of the bootstrap would not be complete without a mention of the numeric bootstrap, which kicked off the modern bootstrap revival [28] and which has enjoyed great successes in recent years, such as the most precise determination of the 3d Ising model critical exponents [41]. We will explain the main ideas as they were used in [28].

In common with the analytic bootstrap, one takes the crossing equation (2.2.2) and expands both sides in conformal blocks as in equation (2.2.4). Focusing for simplicity on the case of identical external scalars φ, this leads to the equation

$$\sum_{\mathcal{O}_k} C^2_{\varphi\varphi\mathcal{O}_k} F^{\Delta_\varphi}_{\Delta_k,\ell_k}(u,v) = 0, \tag{2.2.11}$$

where we defined $F^{\Delta_\varphi}_{\Delta,\ell}(u,v) = v^{\Delta_\varphi}G^{0,0}_{\Delta,\ell}(u,v) - u^{\Delta_\varphi}G^{0,0}_{\Delta,\ell}(v,u)$. The s- and t-channel conformal blocks have different analytical properties, and as such, the functions $F^{\Delta_\varphi}_{\Delta,\ell}$ are defined only on the subspace $\mathcal{R} = \{(z,\bar{z}) \in \mathbb{C}^2 : \bar{z} \notin [1,+\infty), z \notin (-\infty,0]\}$ in which the s- and t-channels both converge [124]. One can then consider both sides of equation (2.2.11) as living in the vector space V consisting of all countable sums $\mathcal{K}(u,v) = \sum_k \lambda_k F^{\Delta_\varphi}_{\Delta_k,\ell_k}(u,v)$ which converge in all of $\mathcal{R}$, and satisfy appropriate physical constraints, such as constraints from unitarity.

Now posit a spectrum $\{\Delta_k, \ell_k\}$ and set of couplings $\{C_{\varphi\varphi\mathcal{O}_k}\}$, and pick a linear functional $\alpha \in V^*$. Defining $\alpha(\Delta,\ell) = \alpha\left(F^{\Delta_\varphi}_{\Delta,\ell}\right) \in \mathbb{R}$, and applying α to equation (2.2.11), we see that

$$\sum_{\mathcal{O}_k} C^2_{\varphi\varphi\mathcal{O}_k}\alpha(\Delta_k,\ell_k) = 0. \tag{2.2.12}$$

Of course we need to make sure there are no convergence issues when pulling α into the infinite sum, which may mean restricting the functionals to lie in some strict subspace of V^*.[11]

If, given a putative spectrum, it is possible to find a functional α with $\alpha(\Delta_k, \ell_k) > 0$ for all k, then this spectrum is not consistent with crossing symmetry of the $\langle\varphi\varphi\varphi\varphi\rangle$ correlator, and can be ruled out. Note that positivity of the squared OPE coefficients, a direct result of unitarity, is of crucial importance here, which is why the numeric bootstrap procedure has generally only been applied to unitary CFTs.[12]

By picking a suitable basis of functionals α, often taken to be derivatives $\alpha_{m,n} = \partial_z^m \partial_{\bar{z}}^n \big|_{z=\bar{z}=\frac{1}{2}} F(z, \bar{z})$, and truncating this infinite-dimensional space by imposing that $m, n \leqslant \Lambda < \infty$, one can systematically search for a functional α that is in conflict with equation (2.2.12) by recasting it as a linear programming problem, for which well-known and highly efficient numerical methods exist.

One then searches over possible spectra and couplings and rules out a large subset of these as inconsistent with crossing symmetry of the chosen correlator, leaving a space of "allowed CFTs" – or more correctly, a space of CFTs not disallowed by the specific set of functionals. These spaces often have interesting features, such as isolated "islands" as in the 3d Ising model [33], or "kinks" at points where interesting theories live [32, 37].

Of course one can consider many correlators simultaneously, and the main ideas stay the same. On a technical level, the problem is now recast as a semi-definite programming problem, for which again efficient numerical methods exist. Generally the numeric bootstrap approach gains tremendously in strength when applied to multiple correlators simultaneously. This gain in strength also happens in the analytic

[11] For an explicit example in the context of 1d CFTs, see e.g. [51], where this is a non-trivial constraint that makes contact with Regge physics.

[12] The truncation method of [43] gives numerical estimates (with no control over errors), but does not rely on unitarity.

bootstrap, as seen in section 3, where we consider multiple correlators in the Gross-Neveu and Gross-Neveu-Yukawa model.

2.3 Large spin perturbation theory

As outlined in the introduction, a study of the bootstrap equation in the limit $z \ll 1 - \bar{z} \ll 1$ can be used to find analytic results about the large spin behaviour of the spectrum, for example proving the result of twist additivity. The general philosophy is to show that the correlator has a particular divergence in one channel and to demand that this behaviour is reproduced by the conformal block expansion in the other channel. Therefore we should first ask precisely what divergences of the correlator contribute to the CFT data in a large spin expansion, and secondly what the precise contribution of such divergences to the CFT data is. Examples of further interesting questions would then be, e.g., the analytic properties of these contributions.

In this section we outline the two main techniques used in this thesis to answer these questions: twist conformal blocks [89] and the Lorentzian inversion formula [97].

2.3.1 Twist conformal blocks

In [88], it was shown that perturbatively, the CFT data can be expanded as a power series in the inverse of the conformal spin $J^2 = \left(\frac{\tau}{2} + \ell - 1\right)\left(\frac{\tau}{2} + \ell\right)$. Specifically, they showed that for a family of leading twist operators appearing in an OPE of two scalars, if one defines $\gamma(J)$ to be the anomalous dimensions and $\alpha(J)$ to be the multiplicative OPE corrections with respect to the free theory, then

$$\gamma(J) \qquad \text{and} \qquad \alpha(J)\left(1 - \frac{\sqrt{1+4J^2}}{4J}\gamma'(J)\right) \tag{2.3.1}$$

have an asymptotic expansion for large J that contains only even powers of $1/J$.[13]

[13] Note that this expansion may still contain factors of $\log J^2$.

Subsequently this result was used in [89] to introduce the concept of twist conformal blocks (TCBs), which was used to analyze the perturbative spectrum of CFTs with broken twist degeneracy in [89,148]. For a four-point correlator in a theory with a degenerate twist spectrum, i.e. where there are infinitely many operators of a given twist (such as the generalized free theory), the TCBs resum the contributions of all intermediate operators of a given twist:

$$\mathcal{G}^{(0)}(u,v) = \sum_{\tau} H^{(0)}_{\tau}(u,v), \tag{2.3.2}$$

where each twist conformal block $H^{(0)}_{\tau}(u,v)$ is realized as a sum over intermediate operators of twist τ

$$H^{(0)}_{\tau}(u,v) = \sum_{\ell} a^{(0)}_{\tau,\ell} G_{\tau,\ell}(u,v). \tag{2.3.3}$$

Here the $a^{(0)}_{\tau,\ell}$ are squared OPE coefficients of two external operators with the intermediate operators of twist τ and spin ℓ.

If one perturbs such a theory by a small parameter g, e.g. through coupling to a new operator, the correlator becomes a power series in g:

$$\mathcal{G}^{(0)}(u,v) \mapsto \mathcal{G}^{(0)}(u,v) + g\,\mathcal{G}^{(1)}(u,v) + g^2\,\mathcal{G}^{(2)}(u,v) + \dots. \tag{2.3.4}$$

To see how this happens, note that after the perturbation, the OPE coefficients and scaling dimensions become series in g:

$$\begin{aligned}
\mathcal{G}(u,v) &= \sum_{\tau,\ell} a_{\tau,\ell} G_{\tau,\ell}(u,v) \\
&= \sum_{\tau_0,\ell} \left(a^{(0)}_{\tau_0,\ell} + g a^{(1)}_{\tau_0,\ell} + \dots\right)\left(1 + g\gamma^{(1)}_{\tau_0,\ell}\partial_\tau + \dots\right) G_{\tau,\ell}(u,v)\big|_{\tau=\tau_0} \\
&= \mathcal{G}^{(0)}(u,v) + g \underbrace{\sum_{\tau_0,\ell} \left(a^{(1)}_{\tau_0,\ell} + a^{(0)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}\partial_\tau\right) G_{\tau,\ell}(u,v)\big|_{\tau=\tau_0}}_{=\,\mathcal{G}^{(1)}(u,v)} + O(g^2).
\end{aligned} \tag{2.3.5}$$

Since the CFT data is expandable in inverse powers of the conformal spin $J^2_{\tau,\ell}$ [88], the corrections to the correlator can be expressed in terms of twist conformal blocks

$H^{(\rho)}_{\tau}(u,v)$. For example: by expanding $a^{(1)}_{\tau_0,\ell} = a^{(0)}_{\tau_0,\ell}\sum_\rho A_{\tau_0,\rho}/J^{2\rho}_{\tau_0,\ell}$, we see in equation (2.3.5) that $\mathcal{G}^{(1)}(u,v)$ contains a correction of the form

$$\mathcal{G}^{(1)}(u,v) \supset \sum_{\tau_0,\ell} a^{(1)}_{\tau_0,\ell} G_{\tau_0,\ell}(u,v) = \sum_{\rho} A_{\tau_0,\rho} H^{(\rho)}_{\tau_0}(u,v), \tag{2.3.6}$$

where the twist conformal blocks $H^{(\rho)}_{\tau_0}(u,v)$ were defined as[14]

$$H^{(\rho)}_{\tau_0}(u,v) \equiv \sum_{\ell} \frac{1}{J^{2\rho}_{\tau_0,\ell}} a^{(0)}_{\tau_0,\ell} G_{\tau_0,\ell}(u,v). \tag{2.3.7}$$

The crucial observation is that these blocks inherit a Casimir differential equation from the conformal blocks:

$$\mathcal{C}_\tau G_{\tau,\ell}(u,v) = J^2_{\tau,\ell} G_{\tau,\ell}(u,v) \qquad \Rightarrow \qquad \mathcal{C}_\tau H^{(\rho+1)}_{\tau}(u,v) = H^{(\rho)}_{\tau}(u,v), \tag{2.3.8}$$

which makes the TCBs $H^{(\rho)}_{\tau}(u,v)$ effectively computable. Specifically, their analytic structure can be understood; matching appropriate singularities of such functions on either side of the crossing equation then allows one to constrain the spectrum.

This technology was first used in the papers [89, 148] to study generalized free theories, cubic models in $d = 6-\epsilon$ dimensions, the Wilson-Fisher and critical O(N) model in $d = 4-\epsilon$ dimensions to second order in ϵ, and the large N O(N) model to first order in $1/N$. It was also used to study $\mathcal{N} = 4$ SYM in a weak coupling expansion in [80]. The twist conformal blocks form the basis of the computations in my paper [1], on which chapter 3 is based.

Papers [2] and [3], which respectively analyze the Wilson-Fisher and critical O(N) model in $d = 4-\epsilon$ to order ϵ^4, and [4], which analyzes the critical O(N) model in large N to order $1/N^2$, form the basis of chapters 4 and 5. In these chapters, divergences of the crossing equation are analyzed using the Lorentzian inversion formula, explained in section 2.3.2 below. However, twist conformal blocks are still used to find some of the contributions that go into the inversion formula.

[14] By a small abuse of notation, we also use this to refer to twist conformal blocks with an insertion of the form $(\log^k J)/J^{2m}$, i.e. $H^{(m,\log^k J)}_{\tau_0}(u,v) \equiv \sum_\ell \log J_{\tau_0,\ell}\, J^{-2m}_{\tau_0,\ell} a^{(0)}_{\tau_0,\ell} G_{\tau_0,\ell}(u,v)$.

2.3.2 Lorentzian inversion formula

While twist conformal blocks have had great success in computing CFT data in specific theories, the analysis of their divergences in the crossing equation is generally done in an ad-hoc manner, matching particular terms on both sides of the equation. Furthermore, they give little understanding of the analytic properties of the CFT data at finite spin: a priori, the expansion in terms of the conformal spin $J^2_{\tau,\ell}$ is only an asymptotic one and it is not clear for what finite spin ℓ, if any, it should converge.

All this was solved by the breakthrough paper [97], which proved the Lorentzian inversion formula, in analogy with a Froissart-Gribov formula in S-matrix theory. The original paper uses a contour deformation argument; since then, the formula has been re-proven in [98] and [86] using other methods. These papers shed an interesting light on the physical origins of the formula: in particular, [86], explains the origin of the strange quantum numbers appearing in the conformal block that is integrated against in equation (2.3.30). We will first derive the Euclidean inversion formula, along the lines of [91], which also explains the conformal block expansion (2.2.4) from a group theoretic point of view. This Euclidean formula is then the starting point for the proof of the Lorentzian inversion formula in [98], which we summarize here.

The Euclidean inversion formula

The key is to study harmonic analysis on the conformal group $G = \mathrm{SO}(d+1,1)$, which can be viewed as a generalization of Fourier analysis to topological groups, with the regular Fourier transform corresponding to harmonic analysis on the additive group $\mathbb{R}^n$. Since G is locally compact, it has a Haar measure $\mathrm{d}g$. Given a representation π of G, and an infinitely differentiable function f with compact support on G, we can

define a trace class operator $\mathcal{T}_\pi$ and the character Θ_π of π by [149][15]

$$\mathcal{T}_\pi(f) = \int \mathrm{d}g\ f(g)\pi(g), \qquad \Theta_\pi(f) = \mathrm{Tr}\left[\mathcal{T}_\pi(f)\right]. \tag{2.3.9}$$

The space of unitary irreducible representations π on G is called the unitary dual $\widehat{G}$, which possesses a Plancherel measure $\mathrm{d}\pi$ satisfying

$$f(1) = \int_{\widehat{G}} \mathrm{d}\pi\ \Theta_\pi(f) \tag{2.3.10}$$

for C^∞ functions f with compact support on G. It then follows that [149]

$$f(g) = \int_{\widehat{G}} \mathrm{d}\pi\ \mathrm{Tr}\left[\pi(g)\widehat{f}(\pi)\right], \tag{2.3.11}$$

where $\widehat{f}(\pi) = \int_G \mathrm{d}g\ f(g)\pi(g)^{-1}$ is the analogue of the Fourier transform. To evaluate the trace in equation (2.3.11), one often inserts a complete basis of states. For example, if one has a discrete basis of states $\{|\psi\rangle\}$ for the representation π, it may be evaluated as

$$\mathrm{Tr}\left[\pi(g)\widehat{f}(\pi)\right] = \sum_{\psi,\psi'} \overbrace{\langle\psi'|\ \pi(g)\ |\psi\rangle}^{\text{Basis functions}} \underbrace{\langle\psi|\ \widehat{f}(\pi)\ |\psi'\rangle}_{\text{“Fourier coefficients”}}. \tag{2.3.12}$$

To see how this reduces to the regular Fourier transform when $G = \mathbb{R}$: note that unitary irreducible representations of $\mathbb{R}$ are labelled by a real number $k \in \mathbb{R}$, with $\pi_k(x) = e^{\mathrm{i}kx}$. Furthermore, the Haar measure on $\mathbb{R}$ is simply the Lebesgue measure, and the Plancherel measure is $\mathrm{d}\pi = \mathrm{d}k/(2\pi)$, together leading to the familiar Fourier transform.[16]

For the conformal group $\mathrm{SO}(d+1,1)$, the group $\widehat{G}$ consists of two families: the discrete series of representations, which do not contribute in the Plancherel formula to the correlator of scalar operators, and the principal series representations, which

[15] Generally, the character of a group element is defined as $\Theta_\pi(g) = \mathrm{Tr}\left[\pi(g)\right]$. We can heuristically think of this relation as $\Theta_\pi(f) = \int \mathrm{d}g\ \Theta_\pi(g)f(g)$ [149].

[16] The Haar measure on $\mathbb{R}$ is only defined up to a non-zero multiplicative constant, with the Plancherel measure compensating with exactly the inverse constant. This corresponds to the familiar normalization ambiguity in the definition of the Fourier transform: here we follow the standard convention in the physics literature.

are characterized by a scaling dimension $\Delta \in \frac{d}{2} + \mathrm{i}\mathbb{R}$, and a finite-dimensional representation ρ of $\mathrm{SO}(d)$. We will only need to focus on the principal series with ρ a spin J traceless symmetric representation, which we shall denote as $\mathcal{E}_{\Delta,J}$.

A physical correlator $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ is then regarded as the result of a group action on the first two operators, so that this correlator can be seen as a function on the group:

$$f(g) = \langle g\cdot(\phi(0)\phi(\infty))\ \phi(x_3)\phi(x_4)\rangle, \tag{2.3.13}$$

whence the Plancherel theorem may be applied. For the principal series representation $\mathcal{E}_{\Delta,J}$, a complete basis of states is given by operators of the form $|\mathcal{O}_{\mu_1,\ldots,\mu_J}(x)\rangle$, with a completeness relation of the form

$$\mathbb{1} = \int \mathrm{d}^d x |\mathcal{O}_{\mu_1,\ldots,\mu_J}(x)\rangle\langle\widetilde{\mathcal{O}}_{\mu_1,\ldots,\mu_J}(x)|, \tag{2.3.14}$$

where $\widetilde{\mathcal{O}}$ is the shadow operator to $\mathcal{O}$, with shadow scaling dimension $\widetilde{\Delta} = d - \Delta$.

After applying the Plancherel formula, one gets the expression

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = &\sum_{J=0}^{\infty}\int_{\frac{d}{2}}^{\frac{d}{2}+\mathrm{i}\infty}\frac{\mathrm{d}\Delta}{2\pi\mathrm{i}}\ \mu(\Delta,J)\times \\ &\int \mathrm{d}^d x\ \langle\phi(x_1)\phi(x_2)\mathcal{O}_{\mu_1,\ldots,\mu_J}(x)\rangle_S P^{\mu_1,\ldots,\mu_J}_{\Delta,J}(x,x_3,x_4),\end{aligned} \tag{2.3.15}$$

where by $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_S$ we do not mean a physical correlator, but the correct conformally invariant structure in a particular normalization.[17] Furthermore, $\mu(\Delta, J)$ is the Plancherel measure for the representation $\mathcal{E}_{\Delta,J}$ [91]:

$$\mu(\Delta,J) = \frac{\mathrm{Tr}_{\mathcal{E}_{\Delta,\rho}}[\mathbb{1}]\ \mathrm{vol}\,(\mathrm{SO}(1,1))}{\mathrm{vol}\,(\mathrm{SO}(d+1,1))}, \tag{2.3.16}$$

which should be considered as a regularized quotient. An explicit expression for $\mu(\Delta, J)$, omitted here because of its length, can be found in [91]. The $P^{\mu_1,\ldots,\mu_J}_{\Delta,J}(x, x_3, x_4)$

[17] For example for a three-point function of scalars we have that $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_S = 1/(|x_{12}|^{\Delta_{123}}|x_{23}|^{\Delta_{231}}|x_{13}|^{\Delta_{132}})$, without the OPE coefficient $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$.

can be computed from the formulae above. One can see however that it transforms as a conformally invariant 3-point function, so that it takes the form

$$P^{\mu_1,\ldots,\mu_J}_{\Delta,J}(x,x_3,x_4) = I(\Delta,J)\langle\phi(x_3)\phi(x_4)\mathcal{O}^{\mu_1,\ldots,\mu_J}(x)\rangle_S. \tag{2.3.17}$$

Putting this together, we find that

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{J=0}^{\infty}\int_{\frac{d}{2}}^{\frac{d}{2}+\mathrm{i}\infty}\frac{\mathrm{d}\Delta}{2\pi\mathrm{i}}\,\mu(\Delta,J)I(\Delta,J)\Psi^{\Delta_\phi}_{\Delta,J}(x_1,x_2,x_3,x_4), \tag{2.3.18}$$

where the Ψ are conformal partial waves satisfying

$$\begin{aligned}\Psi^{\Delta_\phi}_{\Delta,J}(x_1,x_2,x_3,x_4) &= \int \mathrm{d}^d x\ \langle\phi(x_1)\phi(x_2)\widetilde{\mathcal{O}}^{\mu_1,\ldots,\mu_J}(x)\rangle_S\langle\mathcal{O}_{\mu_1,\ldots,\mu_J}(x)\phi(x_3)\phi(x_4)\rangle_S \\ &= \frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\left(K^{\Delta_\phi}_{\widetilde{\Delta},J}G^{\Delta_\phi}_{\Delta,J}(u,v) + K^{\Delta_\phi}_{\Delta,J}G^{\Delta_\phi}_{\widetilde{\Delta},J}(u,v)\right),\end{aligned} \tag{2.3.19}$$

with $K^{\Delta_\phi}_{\Delta,J}$ a constant, and $G^{\Delta_\phi}_{\Delta,J}$ the regular conformal blocks for identical scalar operators of dimension Δ_ϕ.

Finally then we arrive at a formula expressing the correlator as an integral over conformal blocks with $\Delta \in \frac{d}{2}+\mathrm{i}\mathbb{R}$:

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}}\sum_{J=0}^{\infty}\int_{\frac{d}{2}+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\Delta}{2\pi\mathrm{i}}\,c(\Delta,J)G^{\Delta_\phi}_{\Delta,J}(u,v), \tag{2.3.20}$$

where $c(\Delta,J) = \mu(\Delta,J)I_{\Delta,J}K^{\Delta_\phi}_{\widetilde{\Delta},J}$, and where invariance of the integrand under $\Delta \to \widetilde{\Delta}$ was used to extend the integration region. Since the blocks $G^{\Delta_\phi}_{\Delta,J}(u,v)$ decay exponentially quickly in the right-half Δ-plane, we can close the contour on that side [91]. Assuming that the function $c(\Delta,J)$ is meromorphic in this region, we get a discrete sum of conformal blocks, with coefficients equal to residues at the poles. Assuming further that correlators can be decomposed uniquely into such sums of conformal blocks, we deduce that the residues of $c(\Delta,J)$ give the OPE coefficients of physical operators.[18] Specifically note that $c(\Delta,J)$ contains all the physical information of the intermediate operators: the spectrum through the position of its poles, and the OPE coefficients through its residues.

[18]It is possible that $c(\Delta,J)$ contains additional spurious poles whose contributions cancel. This happens for example even in a mean field theory correlator of scalars [91].

Equation (2.3.18) is particularly interesting because there exists an orthogonality relation for the conformal partial waves with $\Delta = \frac{d}{2} + ir$ in the principal series [98]:

$$\left(\Psi^{\Delta_\phi}_{\Delta,J}, \Psi^{\widetilde{\Delta}_\phi}_{\widetilde{\Delta}',J'}\right) = \int \frac{\mathrm{d}^d x_1 \dots \mathrm{d}^d x_4}{\mathrm{vol}(\mathrm{SO}(d+1,1))} \Psi^{\Delta_\phi}_{\Delta,J}(x_i) \Psi^{\widetilde{\Delta}_\phi}_{\widetilde{\Delta}',J'}(x_i) = 2\pi n(\Delta,J)\delta(r-r')\delta_{J,J'}, \tag{2.3.21}$$

with $n_{\Delta,J}$ a normalization constant. Therefore equation (2.3.18) can be easily inverted, giving the Euclidean inversion formula:

$$I_{\Delta,J} = \frac{n(\Delta,J)}{\mu(\Delta,J)} \int \frac{\mathrm{d}^d x_1 \dots \mathrm{d}^d x_4}{\mathrm{vol}(\mathrm{SO}(d+1,1))} \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \Psi^{\widetilde{\Delta}_\phi}_{\widetilde{\Delta},J}(x_i). \tag{2.3.22}$$

From Euclidean to Lorentzian inversion formula

To get the Lorentzian inversion formula from the Euclidean formula, one performs an appropriate contour deformation; we describe briefly here the proof in [98].

We first use the "shadow representation" (2.3.19) to write the partial wave as an integral over an additional point x_5, and explicitly find an expression for the integrand in terms of a Gegenbauer polynomial $\widehat{C}_J$. Furthermore we gauge fix three of the points to lie at $x_1 = (1,0,0,\dots), x_2 = 0, x_5 = \infty$, leaving an integral over only x_3, x_4. The Gegenbauer polynomial does not decay nicely everywhere, but this can be remedied using an integral representation over spatial directions. A rotational invariance of the integrand then allows us to fix a spatial direction, leading to the intermediate formula

$$I_{\Delta,J} = \widehat{C}_J(1) \int \frac{\mathrm{d}^d x_3 \mathrm{d}^d x_4}{\mathrm{vol}(\mathrm{SO}(d-1))} \frac{\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle}{|x_{34}|^{J+2d-2\Delta_\phi-\Delta}} (x^0_{34} + \mathrm{i}\, x^1_{34})^J. \tag{2.3.23}$$

Now we Wick rotate by performing a contour deformation. In the process, we encounter singularities when operators become null separated, and when crossing such a null line, the integrand picks up precisely the commutator between these operators. The contour has to be deformed into one of the half-planes depending on whether certain operators lie in the future or past lightcones of other operators, which leads

into a separation into two different integrals:

$$I_{\Delta,J} = -\frac{\widehat{C}_J(1)}{\text{vol}(S^{d-2})}\left[\int_{3>1,2>4}\frac{\mathrm{d}^d x_3 \mathrm{d}^d x_4}{\text{vol}(\text{SO}(d-2,1))}\frac{\langle[\phi_4,\phi_2][\phi_1,\phi_3]\rangle}{|x_{34}|^{2d-2\Delta_\phi-\Delta}}B_J(\eta)\right.$$
$$\left. + (-1)^J\int_{4>1,2>3}\frac{\mathrm{d}^d x_3 \mathrm{d}^d x_4}{\text{vol}(\text{SO}(d-2,1))}\frac{\langle[\phi_3,\phi_2][\phi_1,\phi_4]\rangle}{|x_{34}|^{2d-2\Delta_\phi-\Delta}}B_J(-\eta)\right], \quad (2.3.24)$$

where $\eta = (x_{12}\cdot x_{34})/(|x_{12}||x_{34}|)$ and where $B_J(\eta)$ is a hypergeometric function. This contour deformation is only valid when $J > 1$, so that the eventual Lorentzian inversion formula only holds for such spins.

We then unfix the gauge and perform the x_5 integral, which turns $B_J(\eta)$ into a conformal block $G^{\tilde{\Delta}_\phi}_{J+d-1,\Delta+d-1}(z,\bar{z})$. Noteworthy is that the spin J and dimension Δ have swapped positions, so that this block is naturally analytically continued in J. Finally we gauge fix the integral over (x_1,x_2,x_3,x_4) into an integral over the cross-ratios $(z,\bar{z})$, to find an integral of the form

$$I_{\Delta,J} = \alpha_{\Delta,J}\left[(-1)^J\int_0^1 \mathrm{d}z\int_0^1 \mathrm{d}\bar{z}\frac{|z-\bar{z}|^{d-2}}{(z\bar{z})^d}G^{\tilde{\Delta}_\phi}_{J+d-1,\Delta+d-1}(z,\bar{z})\frac{\langle[\phi_3,\phi_2][\phi_1,\phi_4]\rangle}{\mathbb{K}_4}\right.$$
$$\left. + \int_{-\infty}^0 \mathrm{d}z\int_{-\infty}^0 \mathrm{d}\bar{z}\frac{|z-\bar{z}|^{d-2}}{(z\bar{z})^d}G^{\tilde{\Delta}_\phi}_{J+d-1,\Delta+d-1}(z,\bar{z})\frac{\langle[\phi_4,\phi_2][\phi_1,\phi_3]\rangle}{\mathbb{K}_4}\right]. \quad (2.3.25)$$

Finally, to get the inversion formula of [97], one relates the double commutators to double discontinuities of the correlator $\langle\phi_1\phi_2\phi_3\phi_4\rangle = \mathcal{G}(z,\bar{z})\mathbb{K}_4$

$$\frac{\langle[\phi_3,\phi_2][\phi_1,\phi_4]\rangle}{\mathbb{K}_4} = -2\,\text{dDisc}\,\mathcal{G}(z,\bar{z}), \quad (2.3.26)$$
$$\frac{\langle[\phi_4,\phi_2][\phi_1,\phi_3]\rangle}{\mathbb{K}_4} = -2\,\text{dDisc}\,\mathcal{G}(z,\bar{z}), \quad (2.3.27)$$

where the double commutator is defined as

$$\text{dDisc}\,f(z,\bar{z}) = f(z,\bar{z}) - \frac{1}{2}f^{\circlearrowleft}(z,\bar{z}) - \frac{1}{2}f^{\circlearrowright}(z,\bar{z}), \quad (2.3.28)$$

where $f^{\circlearrowleft}(z,\bar{z})$ and $f^{\circlearrowright}(z,\bar{z})$ indicate that one should take $\bar{z}$ around its branch point ($\bar{z}=1$ in (2.3.26) and $\bar{z}=\infty$ in (2.3.27)), while leaving z fixed. After relating all the appropriate constants this reduces to the inversion formula of [97]:

$$c(\Delta,J) = c^t(\Delta,J) + (-1)^J c^u(\Delta,J) \quad (2.3.29)$$

with

$$c^t(\Delta,J)=\frac{1}{4}\kappa_{\Delta+J}\int_0^1 \mathrm{d}z\int_0^1 \mathrm{d}\bar{z}\,\underbrace{\frac{|z-\bar{z}|^{d-2}}{(z\bar{z})^d}}_{=:\mu(z,\bar{z})}G^{\Delta_i}_{J+d-1,\Delta+d-1}(z,\bar{z})\,\mathrm{dDisc}\left[\mathcal{G}(z,\bar{z})\right] \tag{2.3.30}$$

and with $c^u(\Delta,J)$ the same but with $z,\bar{z}\in(-\infty,0)$ and with the double discontinuity around $\bar{z}=\infty$, or equivalently, the same but with operators 1 and 2 swapped. For identical external operators $\kappa_\beta=\frac{\Gamma(\beta/2)^4}{2\pi^2\Gamma(\beta)\Gamma(\beta-1)}$.

Perturbative setting

In chapters 4 and 5 we shall be interested in the case where all external operators are identical, so that $c(\Delta,J)=2c^t(\Delta,J)$, and where the intermediate operators are double field operators $[\varphi,\varphi]_{n,\ell}$ of twist $\tau_{n,\ell}=2\Delta_\varphi+2n+\ell+\gamma_{n,\ell}$. This is a perturbative setting, and we can think of the $\gamma_{n,\ell}$ as the perturbative parameters.[19] To find the CFT data of the leading twist operators $[\varphi,\varphi]_{0,\ell}$, we rewrite the inversion formula as

$$c(\Delta,J)=\int_0^1\frac{\mathrm{d}z}{2z}z^{\frac{J-\Delta}{2}}\left(\int_0^1\mathrm{d}\bar{z}\,\kappa_{\Delta+J}\frac{\mu(z,\bar{z})G_{J+d-1,\Delta+d-1}(z,\bar{z})}{z^{\frac{J-\Delta}{2}-1}}\,\mathrm{dDisc}\left[\mathcal{G}(z,\bar{z})\right]\right). \tag{2.3.31}$$

Since the poles of $c(\Delta,J)$ encode the squared OPE coefficients $a_\mathcal{O}=-\operatorname*{Res}_{\Delta=\Delta_\mathcal{O}}c(\Delta,J_\mathcal{O})$ [97], this makes the twists of the intermediate operators clear. A term of the form $z^{\tau/2}$ in $\mathcal{G}(z,\bar{z})$ contributes to a pole of the form $\frac{1}{\tau+J-\Delta}$, corresponding to the exchange of an operator of twist τ, as expected.

In a perturbative setting the poles of the function $c(\Delta,J)$ can collide and develop poles of higher order, which encode powers of the anomalous dimensions [99]. To see this, consider an infinite family of operators $\mathcal{O}_J$ with twists that are the same at tree-level: $\tau_{\mathcal{O}_J}=2\Delta_\varphi+\gamma_J$. We expand $c(\Delta,J)$ around the corresponding poles to see higher order poles appearing:

$$c(\Delta,J)\sim\sum_{\mathcal{O}_J}\frac{-a_{\mathcal{O}_J}}{\Delta-J-2\Delta_\varphi-\gamma_J}+\cdots=\sum_{p=0}^{\infty}\sum_{\mathcal{O}_J}\frac{-a_{\mathcal{O}_J}\gamma_J^p}{(\Delta-J-2\Delta_\varphi)^{p+1}}+\ldots, \tag{2.3.32}$$

[19] To be more precise: there is a perturbative parameter g such that $\gamma_{n,\ell}=O(g)$.

where the right-hand side corresponds to a perturbative expansion.

In the inversion formula (2.3.31), such higher order poles develop when the correlator $\mathcal{G}(z,\bar{z})$ contains powers of $\log z$. This occurs naturally in a perturbative setting when anomalous dimensions are present, since conformal blocks $G_{\tau,\ell}(z,\bar{z})$ contain a factor $z^{\tau/2} = z^{\Delta_\varphi}\left(1+\frac{1}{2}\gamma\log z+\frac{1}{8}\gamma^2\log^2 z+\dots\right)$. From the inversion formula one can then find the perturbative corrections to the CFT data. First we expand the correlator in powers of z:

$$\begin{aligned}\mathcal{G}(z,\bar{z}) &= \sum_J a_\ell z^{\Delta_\varphi+\frac{1}{2}\gamma_J} k_{\frac{1}{2}(\Delta_J+J)}(\bar{z}) + O\left(z^{\Delta_\varphi+1}\right)\\ &= z^{\Delta_\varphi}\sum_p \frac{1}{2^p p!}\log^p z\, f_p(\bar{z}) + O\left(z^{\Delta_\varphi+1}\right),\end{aligned} \tag{2.3.33}$$

with $f_p(\bar{z}) := \sum_J a_J \gamma_J^p k_{\frac{1}{2}(\Delta_J+J)}(\bar{z})$. Defining

$$U_{\bar{h}}^{(p)} := \frac{\Gamma\left(\bar{h}\right)^2}{\pi^2\Gamma(2\bar{h}-1)}\int_0^1 \mathrm{d}\bar{z}\, k_{\bar{h}}(\bar{z})\,\mathrm{dDisc}\, f_p(\bar{z}), \tag{2.3.34}$$

we then find that the CFT data can be recovered as [99]

$$\hat{a}_{\mathcal{O}_J}\gamma^p_{\mathcal{O}_J} = U^{(p)}_{\bar{h}_b} + \frac{1}{2}\partial_{\bar{h}_b}U^{(p+1)}_{\bar{h}_b} + \frac{1}{8}\partial^2_{\bar{h}_b}U^{(p+2)}_{\bar{h}_b} + \dots, \tag{2.3.35}$$

where $\bar{h} = \bar{h}_f = \frac{1}{2}(\Delta_J+2J)$, $\bar{h}_b = \Delta_\varphi + J$ and $\hat{a}$ is the OPE coefficient in a different normalization: $\frac{\Gamma(\bar{h}_f)^2}{\Gamma(2\bar{h}_f)}\hat{a}_{\mathcal{O}_J} = a_{\mathcal{O}_J}$. Note the difference between the "full" $\bar{h}_f = \frac{1}{2}(\Delta_J+2J)$ and the "bare" $\bar{h}_b = \bar{h}_f - \frac{1}{2}\gamma_J$, a distinction that will be important in chapters 4 and 5. This formula is also derived in chapter 4 using a method based on twist conformal blocks.

Chapter 3

The critical Gross-Neveu and Gross-Neveu-Yukawa models

The conformal bootstrap, with its main idea of combining the constraints of crossing symmetry and the OPE algebra, has been successfully applied to find results about many CFTs, as reviewed in chapter 1. Since these computations rely on the specific form of conformal blocks, they mainly focus on theories of scalar operators, or sometimes on correlators in supersymmetric theories where the superconformal primary is a scalar, or on a subsector of scalar operators in a more complicated theory like $\mathcal{N} = 4$ SYM.[1]

In this chapter, we remedy this by studying theories which are perturbations of the theory of free Dirac fermions, namely the Gross-Neveu (GN) and Gross-Neveu-Yukawa (GNY) models. These two theories live in dimensions $2 < d < 4$, with the Gross-Neveu model perturbatively renormalizable in $d = 2 + \epsilon$ dimensions and the Gross-Neveu-Yukawa model perturbatively renormalizable in $d = 4 - \epsilon$ dimensions. They lie in the same universality class and are used to describe various condensed matter systems, such as phase transitions in graphene and 3d Weyl semimetals, as well as the random bond Ising model [150–152]. They also serve as a toy model for studying various effects that also occur, but are much harder to study, in QCD,

[1] Nevertheless, there has been increased interest in the study of correlators of non-scalar external operators since the publication of [1], on which this chapter is based – see e.g. [53–59, 62–66].

such as asymptotic freedom and dynamical mass generation through symmetry breaking [150, 153]. Furthermore, they are of interest because like the O(N) model in the next chapter, they are expected to have a gravity dual through the AdS/CFT correspondence [150, 154].

We study the GN model in $d = 2+\epsilon$ and the GNY model in $d = 4-\epsilon$ perturbatively in ϵ. The philosophy of our analysis is always to use twist conformal blocks to match divergent terms on each side of the crossing equation and hence constrain the CFT data. We are reasonably explicit in the computations, which have many subtleties due to the fact that the dimension of space is non-integral and that the dimensions of external operators are not equal; however it is useful to remember that these are just technical difficulties in the implementation of the general philosophy.

We find, as a large spin perturbation, leading order CFT data in the theories by analyzing the consequences of crossing symmetry of four-point functions of composite operators formed out of the fundamental fermions. To the orders in the ϵ-expansion to which we study these theories, the intermediate operators can be divided into bilinear operators (formed out of two fundamental fields) and quadrilinear operators (formed out of four fundamental fields). Known anomalous dimensions of bilinear currents are reproduced, and new results are found for OPE coefficient corrections of these bilinear currents, and for anomalous dimensions and OPE coefficient corrections of quadrilinear operators. Furthermore a non-trivial solution in the $d = 2 + \epsilon$ expansion is found for a fermionic theory in which anomalous dimensions scale logarithmically with the spin, which we conjecture to describe theories in which the fundamental field is not part of the spectrum, for example because it is not gauge-invariant. The results are perturbative in ϵ and valid to all orders in the spin, reproducing known results for operator dimensions and providing some new results for operator dimensions and OPE coefficients. They are summarized at the start of the relevant sections.

The structure of this chapter is as follows. Section 3.1 discusses the relevant back-

ground on crossing symmetry, twist conformal blocks and the free fermion theory. Section 3.2 introduces, in generality, the main method to study conformal field theories with weakly broken higher spin symmetry. These methods are then applied in sections 3.3 and 3.4 to study the fermionic theories in the $d = 2 + \epsilon$ and $d = 4 - \epsilon$ expansions, paying special attention to the Gross-Neveu and Gross-Neveu-Yukawa models.

3.1 Setup

Let us quickly recall the basics of crossing symmetry, as described in detail in chapter 2. Conformal invariance restricts the four-point function of four scalars φ_i to be of the following form [155]:

$$\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\varphi_4(x_4)\rangle = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} \frac{\mathcal{G}_{1234}(u,v)}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}}\,, \quad (3.1.1)$$

with crossing symmetry imposing that

$$v^{\frac{\Delta_2+\Delta_3}{2}}\mathcal{G}_{1234}(u,v) = u^{\frac{\Delta_1+\Delta_2}{2}}\mathcal{G}_{3214}(v,u)\,. \quad (3.1.2)$$

Furthermore, from the OPE we deduce that $\mathcal{G}(u,v)$ can be expanded in terms of a basis of conformal blocks:

$$\mathcal{G}_{1234}(u,v) = \sum_{\mathcal{O}_{\tau,\ell}} C_{12\mathcal{O}_{\tau,\ell}} C_{34\mathcal{O}_{\tau,\ell}} G_{\tau,\ell}^{\Delta_{12},\Delta_{34}}(u,v)\,. \quad (3.1.3)$$

Global $\mathrm{U}(N_f)$ symmetry

The operators in the theories in this chapter are representations of a global $\mathrm{U}(N_f)$ symmetry: as reviewed in chapter 2, in the presence of a global symmetry group G the contributions from intermediate primary operators will decompose according to their representations under G. In particular, four-point functions involving operators in the $\mathrm{U}(N_f)$ adjoint representation have multiple $\mathrm{U}(N_f)$ tensor structures, arising from the possible representations of the exchanged intermediate operators. If the operators

φ_i transform in representations V_i of G, then in the s-channel, defined by taking the OPEs $\varphi_1 \times \varphi_2$ and $\varphi_3 \times \varphi_4$, the representations V of intermediate operators are such that $\bar{V}$ and V appear in the decomposition of $V_1 \otimes V_2$ and $V_3 \otimes V_4$, respectively, with an analogous result for the t-channel.

Note that the intermediate operators on each side of the crossing equation may transform under different representations: an example from section 3.3.1.4 is that in which φ_1 and φ_2 are $\mathrm{U}(N_f)$ singlets, while φ_3 and φ_4 are $\mathrm{U}(N_f)$ adjoints, with singlet operators exchanged in one channel, and adjoint operators in the other.

The crossing equations decompose according to the exchanged representations: if we denote the representations in the s-channel and t-channel by $V_{s,i}$ and $V_{t,j}$, respectively, then

$$\mathcal{G}_{1234}(u,v) = \sum_i \mathcal{G}_{V_{s,i}}(u,v)\mathbf{T}_{s,i}\,, \qquad \mathcal{G}_{3214}(v,u) = \sum_j \mathcal{G}_{V_{t,j}}(v,u)\mathbf{T}_{t,j}\,, \tag{3.1.4}$$

where the $\mathbf{T}$ range over bases of tensor structures for the intermediate operators. The crossing equation relates these two:

$$v^{\frac{1}{2}(\Delta_2+\Delta_3)} \sum_i \mathcal{G}_{V_{s,i}}(u,v)\mathbf{T}_{s,i} = u^{\frac{1}{2}(\Delta_1+\Delta_2)} \sum_j \mathcal{G}_{V_{t,j}}(v,u)\mathbf{T}_{t,j}\,. \tag{3.1.5}$$

By projecting either side onto the other's basis of tensor structures, we find crossing relations of the form

$$v^{\frac{1}{2}(\Delta_2+\Delta_3)}\mathcal{G}_{V_{s,i}}(u,v) = u^{\frac{1}{2}(\Delta_1+\Delta_2)} \sum_j \beta_{ij}\mathcal{G}_{V_{t,j}}(v,u)\,. \tag{3.1.6}$$

An interesting case is the four-point function of four adjoints, whose correlator duly decomposes:[2]

$$\langle \mathcal{O}^{i_1}_{j_1}(x_1)\mathcal{O}^{i_2}_{j_2}(x_2)\mathcal{O}^{i_3}_{j_3}(x_3)\mathcal{O}^{i_4}_{j_4}(x_4)\rangle \propto \sum_{k=1}^{6} \mathcal{G}_k(u,v)\,(\mathbf{T}_k)^{i_1\ldots i_4}_{j_1\ldots j_4}\,. \tag{3.1.7}$$

[2]For $N_f \geqslant 4$ there are seven different irreducible representations in the tensor product of two adjoint representations: $(\mathbf{N_f^2-1})\otimes(\mathbf{N_f^2-1}) = \bigoplus_{i=1}^{7} V_i$. Two of these are conjugate representations and combine into one tensor structure.

Twist conformal blocks

Twist conformal blocks were reviewed in section 2.3.1: since corrections to CFT data can be expanded in the conformal spin [88], the TCBs allow us to effectively capture the divergences in the correlator generated perturbatively around a point of twist degeneracy. To reiterate the example in section 2.3.1: if after a perturbation by some small coupling g, the OPE coefficients in the theory get corrections from the twist-degenerate point: $a^{(0)}_{\tau_0,\ell} \mapsto a^{(0)}_{\tau_0,\ell} + g a^{(1)}_{\tau_0,\ell} + \ldots$, with $a^{(1)}_{\tau_0,\ell} = a^{(0)}_{\tau_0,\ell} \sum_\rho A_{\tau_0,\rho}/J^{2\rho}_{\tau_0,\ell}$, then the correlator at first order contains a correction

$$\mathcal{G}^{(1)}(u,v) \supset \sum_{\tau_0,\ell} a^{(1)}_{\tau_0,\ell} G_{\tau_0,\ell}(u,v) = \sum_\rho A_{\tau_0,\rho} H^{(\rho)}_{\tau_0}(u,v), \tag{3.1.8}$$

where the $H^{(\rho)}_{\tau_0}(u,v)$ are twist conformal blocks.

Our analysis is based on the study of enhanced divergences, i.e. those divergences generated by an infinite sum of conformal blocks, which the Lorentzian inversion formula (2.3.30) shows are those that have a non-zero double discontinuity. In particular, this includes the terms that [96] calls Casimir-singular, i.e. that can be made arbitrarily divergent in v by applying the Casimir many times. Enhanced divergences therefore include all terms of the form $u^\alpha v^\beta$ where $\beta > 0$ is not integer, and $u^\alpha \log^k v$ where $k \geqslant 2$ is integer.

In contrast with regular conformal blocks, twist conformal blocks generically all individually possess enhanced divergences, and do so in prescribed ways, thus making the singularities of the correlator explicit. Their behaviour in the limit $0 < u \ll v \ll 1$ is generally as follows:

$$H^{(m)}_\tau(u,v) \sim \frac{u^{\tau/2}}{v^{k_0}} v^m \left(h^{(m)}_0(u) + h^{(m)}_1(u) v + h^{(m)}_2(u) v^2 + \ldots \right), \tag{3.1.9}$$

for some $k_0 \geq 0$, and where $h^{(m)}_n(0) \neq 0$. This is consistent with the expectation that since $J^{-2}_{\tau,\ell} \sim \ell^{-2}$ for large ℓ, the twist conformal blocks should become less divergent

as one inserts more powers of $J_{\tau,\ell}^{-2}$. If the tree-level TCB $H_\tau^{(0)}(u,v)$ has such a power-law divergence of the form v^{-k_0}, then the higher-order TCBs can have two types of divergences:

- If $k_0 \notin \mathbb{N}$ is not an integer, then all $H_\tau^{(m)}(u,v)$ contain enhanced divergences from non-integer powers of v.
- If $k_0 \in \mathbb{N}_{>0}$ is a positive integer, then $H_\tau^{(0)}(u,v), \ldots, H_\tau^{(k_0-1)}(u,v)$ have a leading power-law divergence of the form $H_\tau^{(m)} \sim v^{m-k_0}$, while the twist conformal blocks $H_\tau^{(k_0)}(u,v), H_\tau^{(k_0+1)}(u,v), \ldots$ all develop enhanced divergences proportional $\log^2 v$. This shall prove to be one of our most powerful tools, since such $\log^2 v$ terms can often only be produced on one side of the crossing equation.

3.1.1 Free fermion CFT

The theory of N_f free massless Dirac fermions in d dimensions has the action

$$S = \int \mathrm{d}^d x \; \overline{\psi}_i \partial\!\!\!/ \psi^i \,. \tag{3.1.10}$$

From this it is clear that the fermions ψ have scaling dimension $\Delta_\psi = \frac{d-1}{2}$, and that there is a global $\mathrm{U}(N_f)$ symmetry, under which the fermions $\psi, \overline{\psi}$ transform in the fundamental, respectively anti-fundamental, representation.

The spectrum of bilinear primary operators in the theory consists of [154, 156]:

- A $\mathrm{U}(N_f)$ scalar operator $\mathcal{O} \equiv \overline{\psi}\psi = \overline{\psi}^i \psi_i$ of dimension $\Delta_\mathcal{O} = d-1$. Spinor indices have been traced over, so that $\mathcal{O}$ is a spacetime scalar.
- A $\mathrm{U}(N_f)$ adjoint operator $\mathcal{O}^i_j = \overline{\psi}^i \psi_j - \frac{1}{N_f}\delta^i_j \overline{\psi}\psi$ of dimension $\Delta_A = d-1$, again with spinor indices traced over. When suppressing $\mathrm{U}(N_f)$ indices, we shall also refer to this operator as $\mathcal{O}^A$.
- A tower of totally symmetric conserved tensors $J_{\mu_1 \ldots \mu_\ell} \sim \overline{\psi}\gamma_{\mu_\ell}\partial_{\mu_1}\cdots\partial_{\mu_{\ell-1}}\psi$ of dimension $\Delta_{J_\ell} = d-2+\ell$. These correspond to the traceless symmetric

representation of the d-dimensional Lorentz group, and the singlet or adjoint representation of the global $\mathrm{U}(N_f)$ symmetry. Suppressing spacetime indices, we shall refer to the $\mathrm{U}(N_f)$ singlet operators as $J_{S,\ell}$, and to the $\mathrm{U}(N_f)$ adjoint operators as $J_{A,\ell}$. The operator $J_{S,2}$ is the stress-energy tensor, while the operator $J_{A,1}$ is the global symmetry current.

- A tower of mixed symmetry conserved tensors $B_{\mu_1\cdots\mu_\ell\nu_1\ldots\nu_k}\sim\overline{\psi}\gamma_{\mu_1\nu_1\ldots\nu_k}\partial_{\mu_2}\cdots\partial_{\mu_\ell}\psi$ (with $1\leqslant k<d$), where $\gamma_{\mu_1\nu_1\ldots\nu_k}=\gamma_{[\mu_1}\gamma_{\nu_1}\ldots\gamma_{\nu_k]}$. They correspond to representations of $\mathrm{SO}(d)$ with highest weight $(\ell,1,\ldots,1,0,\ldots,0)$, i.e. to Young tableaux with ℓ boxes in the first row, and k rows in total, and can transform in either the singlet or adjoint representation of $\mathrm{U}(N_f)$. They also saturate the unitarity bound $\Delta_{B_\ell}=d-2+\ell$ [131].

The two-point function of the fermions is as follows:

$$\langle\overline{\psi}^i(x_1)\psi_j(x_2)\rangle=\delta^i_j\frac{C_{\psi\psi}\not{x}_{12}}{(x_{12}^2)^{\Delta_\psi+\frac{1}{2}}}\,,\qquad C_{\psi\psi}=\frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}}\,,\tag{3.1.11}$$

with the two-point functions $\langle\psi_i\psi_j\rangle$ and $\langle\overline{\psi}^i\overline{\psi}^j\rangle$ necessarily vanishing due to their spacetime representations. With this normalization, all other correlators can be calculated from judicious use of Wick's theorem.

Our primary point of study will be the four-point correlators formed out of $\mathcal{O}$ and $\mathcal{O}^A$. Using Wick contractions, and with the help of Mathematica and the package *Gamma* [157], we find the free theory result for $\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$:

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle=\ N^2C_{\psi\psi}^4\frac{1}{(x_{12}^2x_{34}^2)^{d-1}}\Big[1+\mathcal{G}_{bil}^{(0),(d)}(u,v)+\mathcal{G}_{quad}^{(0),(d)}(u,v)\Big],\tag{3.1.12}$$

with

$$\mathcal{G}_{bil}^{(0),(d)}(u,v)=\frac{1}{N}u^{\frac{d}{2}-1}\frac{(1-v)(1-v^{d/2})}{v^{d/2}}-\frac{1}{N}u^{\frac{d}{2}}\frac{1+v^{d/2}}{v^{d/2}}\,,\tag{3.1.13}$$

$$\mathcal{G}_{quad}^{(0),(d)}(u,v)=u^{d-1}\left(1+\frac{1}{v^{d-1}}-\frac{1}{N}\frac{1+v}{v^{d/2}}\right)+\frac{1}{N}u^d\frac{1}{v^{d/2}}\,,\tag{3.1.14}$$

where $N \equiv N_f \, \mathrm{Tr}(\mathbf{1})$, with the trace over spinor indices. Here we use the $^{(0)}$ to refer to the fact that this is the free theory, and $^{(d)}$ to indicate the dimension of spacetime. As a sanity check, we can look at the large N limit, in which the disconnected diagrams dominate, so that the four-point function should in the large N limit reduce to that of a free boson of dimension $\Delta_{\mathcal{O}} = d - 1$, as is indeed the case.

Using the leading u behaviour of the conformal blocks from equation (2.2.8), we see that the exchanged operators at least contain operators of twist $\tau = d - 2$, corresponding to the conserved currents, and operators of twist $\tau = 2d - 2$, corresponding to quadrilinear operators $\mathcal{O}_{quad} \sim \overline{\psi}\partial^{\ell_1}\psi\partial^{\ell_2}\overline{\psi}\partial^{\ell_3}\psi$ of spin $\ell = \ell_1 + \ell_2 + \ell_3$. The leading u behaviour can also be used to fix the squared OPE coefficients $a^{(0)}_{S,\tau,\ell}$ for these operators. For example, the squared OPE coefficients $a^{(0)}_{S,d-2,\ell}$ of the bilinear currents $J_{S,\ell}$ in the OPE $\mathcal{O} \times \mathcal{O}$, are given by

$$a^{(0)}_{S,d-2,\ell} = \left(1 + (-1)^{\ell}\right) \frac{\Gamma\left(\frac{d}{2} + \ell - 1\right)^2 \Gamma(d + \ell - 2)}{N\Gamma\left(\frac{d}{2}\right)^2 \Gamma(\ell)\Gamma\left(d - 3 + 2\ell\right)}, \qquad \ell \geqslant 2. \tag{3.1.15}$$

Only operators of even spins appear, as must be the case in the OPE of two identical operators.

The question remains whether the $u^{d/2}$ and u^d terms in equation (3.1.12) are the result of further operators appearing, or whether they arise from the subleading u contributions in the conformal blocks $G_{d-2,\ell}(u, v)$ and $G_{2d-2,\ell}(u, v)$. Checking in 2d and 4d, where we have closed form expressions for the conformal blocks, we find that the $u^{d/2}$ contribution is explained from the conformal blocks $G_{d-2,\ell}(u, v)$, while there needs to be an infinite tower of operators of twist $\tau = 2d - 2 + 2n$ for $n \in \mathbb{Z}_{\geqslant 0}$ to account for all the terms of the form u^{d-1+n}. That is, the parts of the correlator can be written in terms of twist conformal blocks as follows:

$$\mathcal{G}^{(0),(d)}_{bil}(u, v) = H^{(0),(d)}_{d-2}(u, v), \qquad \mathcal{G}^{(0),(d)}_{quad}(u, v) = \sum_{n=0}^{\infty} H^{(0),(d)}_{2d-2+2n}(u, v) \tag{3.1.16}$$

This is a general feature of all the $\mathcal{G}_k(u, v)$ encountered in this chapter.

3.2 Crossing analysis

In this section we perform a detailed analysis of the crossing equation in its general form. We study a perturbation of the free fermion theory in $d_0 > 2$, in which no additional operators enter at first loop order. Since the Gross-Neveu and Gross-Neveu-Yukawa theories we are interested in both violate one of these assumptions, this shall merely be a toy model to introduce the methods used in sections 3.3 and 3.4. We restrict ourselves to studying the correlator $\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ because its analysis already contains the main ideas used in the chapter.

The anomalous dimension of the external operator $\mathcal{O}$ is given by

$$\Delta_{\mathcal{O}} = d - 1 + \gamma_{\mathcal{O}} = d - 1 + \epsilon\gamma_{\mathcal{O}}^{(1)} + \epsilon^2\gamma_{\mathcal{O}}^{(2)} + \dots . \tag{3.2.1}$$

Furthermore, we assume that the correlator can be expanded in terms of ϵ:

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \epsilon\mathcal{G}^{(1)}(u,v) + \epsilon^2\mathcal{G}^{(2)}(u,v) + \dots . \tag{3.2.2}$$

From this the crossing equation $v^{\Delta_\mathcal{O}}\mathcal{G}(u,v) = u^{\Delta_\mathcal{O}}\mathcal{G}(v,u)$ can be expanded in powers of ϵ. Taking the results order by order in ϵ, the crossing relation decomposes into a set of equations

$$v^{d-1}\mathcal{G}^{(0)}(u,v) = u^{d-1}\mathcal{G}^{(0)}(v,u) \tag{3.2.3}$$

$$v^{d-1}\left(\gamma_{\mathcal{O}}^{(1)}\log v\,\mathcal{G}^{(0)}(u,v) + \mathcal{G}^{(1)}(u,v)\right) = u^{d-1}\left(\gamma_{\mathcal{O}}^{(1)}\log u\,\mathcal{G}^{(0)}(v,u) + \mathcal{G}^{(1)}(v,u)\right) \tag{3.2.4}$$

$$\vdots$$

Generally the crossing equations of lower order in ϵ can be used to simplify the crossing equations of higher orders. For example, substituting the order ϵ^0 equation (3.2.3) into the order ϵ^1 equation (3.2.4) simplifies the latter to

$$\gamma_{\mathcal{O}}^{(1)}\log v\,\mathcal{G}^{(0)}(u,v) + \mathcal{G}^{(1)}(u,v) = \gamma_{\mathcal{O}}^{(1)}\log u\,\mathcal{G}^{(0)}(u,v) + \frac{u^{d-1}}{v^{d-1}}\mathcal{G}^{(1)}(v,u)\,. \tag{3.2.5}$$

3.2.1 Dimension shift

In the theories in this chapter the perturbative parameter ϵ is related to the dimension in which the theories live: the Gross-Neveu model lives in $d = 2 + \epsilon$, while the Gross-Neveu-Yukawa model lives in $d = 4 - \epsilon$ [158]. In this case there will be corrections to the free theory correlators, OPE coefficients and scaling dimensions due to this dimensional shift. In an interacting theory living in e.g. $d = d_0 + \epsilon$, we want to define anomalous dimensions and OPE coefficient corrections with respect to the dimensions and OPE coefficients of the free theory in $d = d_0 + \epsilon$ dimensions – and not with respect to those of the free theory in d_0 dimensions. For computational purposes however, it is useful to calculate twist conformal blocks in d_0 dimensions, so that we need to carefully keep track of the changes to twist conformal blocks from this dimensional shift. This is tricky since the conformal blocks are not fully known in general dimensions: we will instead leverage the fact that the free field correlator is known exactly in every dimension.

For example, take the bilinear operators in the free fermion theory and consider a small change in dimension away from some fixed dimension: $d = d_0 \to d_0 + \epsilon$. This changes the OPE coefficients and conformal blocks, leading to a change in the TCB:

$$H^{(0),(d_0+\epsilon)}_{d_0-2+\epsilon}(u,v) = H^{(0),(d_0)}_{d_0-2}(u,v) + \epsilon\, \widetilde{H}^{(1),(d_0)}_{d_0-2}(u,v) + \mathcal{O}(\epsilon^2)\,. \tag{3.2.6}$$

The free theory OPE coefficients change because of their explicit dependence on the dimension d; this change can simply be found by substituting $d = d_0 + \epsilon$ and Taylor expanding. The changes to the conformal blocks are twofold: firstly, the blocks are simply different in different dimensions (even if the twist and spin are the same), and secondly, the free theory twist $\tau_0 = d - 2$ of the intermediate operators depends on the dimension.

Let us capture these changes as follows:

$$a^{(0)}_{d_0+\epsilon-2,\ell} = a^{(0)}_{d_0-2,\ell}\left(1+\epsilon\,\widetilde{\alpha}^{(0)}_{d_0-2,\ell}+\mathcal{O}(\epsilon^2)\right), \tag{3.2.7}$$

$$G^{(d_0+\epsilon)}_{d_0-2+\epsilon,\ell}(u,v) = G^{(d_0)}_{d_0-2,\ell}(u,v)+\epsilon\partial_d G^{(d)}_{d_0-2,\ell}(u,v)+\epsilon\partial_\tau G^{(d_0)}_{\tau,\ell}(u,v)+\mathcal{O}(\epsilon^2), \tag{3.2.8}$$

so that

$$\widetilde{H}^{(1),(d_0)}_{d_0-2}(u,v) = \sum_{\ell=0}^{\infty} a^{(0)}_{d_0-2,\ell}\left(\widetilde{\alpha}^{(0)}_{d_0-2,\ell}G^{(d_0)}_{d_0-2,\ell}(u,v)+\partial_\tau G^{(d_0)}_{\tau,\ell}(u,v)+\partial_d G^{(d_0)}_{d_0-2,\ell}(u,v)\right), \tag{3.2.9}$$

with all derivatives evaluated at $\tau = d_0-2, d = d_0$. A Taylor expansion of the squared OPE coefficients $a^{(0)}_{d-2,\ell}$ in equation (3.1.15) around $d = d_0$ gives the correction $\widetilde{\alpha}^{(0)}_{d_0-2,\ell}$. While we might also be able to compute $\partial_\tau G^{(d_0)}_{\tau,\ell}(u,v)\Big|_{\tau=d_0-2}$ in $d_0 = 2,4,6$, where the blocks are known in their full form, we are unable to compute $\partial_d G^{(d)}_{d_0-2,\ell}(u,v)\Big|_{d=d_0}$ since we do not know the full conformal blocks in arbitrary dimensions.

Instead we rely on an expansion of the free theory result in terms of the dimension to find the collective sum (3.2.9). Since the dependence of the correlator $\mathcal{G}^{(0),(d)}(u,v)$ on the dimension d is explicit, we can simply set $d = d_0+\epsilon$ and Taylor expand:

$$\mathcal{G}^{(0),(d_0+\epsilon)}(u,v) = \mathcal{G}^{(0),(d_0)}(u,v)+\epsilon\,\widetilde{\mathcal{G}}^{(1),(d_0)}(u,v)+\mathcal{O}(\epsilon^2), \tag{3.2.10}$$

where

$$\widetilde{\mathcal{G}}^{(1),(d_0)}(u,v) = \log u\left(\frac{1}{2}\mathcal{G}^{(0),(d_0)}_{bil}(u,v)+\mathcal{G}^{(0),(d_0)}_{\text{quad}}(u,v)\right)-\log v\,\widehat{\mathcal{G}}^{(d_0)}(u,v), \tag{3.2.11}$$

with $\widehat{\mathcal{G}}^{(d_0)}(u,v)$ satisfying the equation

$$\frac{u^{d_0-1}}{v^{d_0-1}}\widehat{\mathcal{G}}^{(d_0)}(v,u) = 1+\frac{1}{2}\mathcal{G}^{(0),(d_0)}_{bil}(u,v). \tag{3.2.12}$$

Let us now finally turn to crossing. The external operators $\mathcal{O}$ have anomalous dimensions $\gamma_{\mathcal{O}}$ defined with respect to the free theory in $d_0+\epsilon$ dimensions:

$$\Delta_{\mathcal{O}} = d-1+\gamma_{\mathcal{O}} = d_0-1+\epsilon+\gamma_{\mathcal{O}} = d_0-1+\epsilon\left(\gamma^{(1)}_{\mathcal{O}}+1\right)+\mathcal{O}(\epsilon^2). \tag{3.2.13}$$

The crossing equation takes the form

$$v^{d_0-1+\epsilon(\gamma^{(1)}_{\mathcal{O}}+1)+\ldots}\left(\mathcal{G}^{(0),(d_0)}(u,v)+\epsilon\,\widetilde{\mathcal{G}}^{(1),(d_0)}(u,v)+\epsilon\,\mathcal{G}^{(1)}(u,v)+\ldots\right)=\left(u\leftrightarrow v\right), \tag{3.2.14}$$

where $\mathcal{G}^{(1)}$ captures the corrections arising from the departure from the free theory. We now expand this to first order in ϵ, and simplify the result by using the free theory result $v^{d_0-1}\mathcal{G}^{(0),(d_0)}(u,v) = u^{d_0-1}\mathcal{G}^{(0),(d_0)}(v,u)$, together with equations (3.2.11) and (3.2.12), reducing the first order crossing equation to

$$\boxed{\gamma^{(1)}_{\mathcal{O}}\log v\,\mathcal{G}^{(0)}(u,v)+\mathcal{G}^{(1)}(u,v)=\gamma^{(1)}_{\mathcal{O}}\log u\,\mathcal{G}^{(0)}(u,v)+\frac{u^{d_0-1}}{v^{d_0-1}}\mathcal{G}^{(1)}(v,u)\,,} \tag{3.2.15}$$

where $\mathcal{G}^{(0)}(u,v)=\mathcal{G}^{(0),(d_0)}(u,v)$. Note that we performed an expansion $d\to d_0+\epsilon$; in the case of an expansion $d\to d_0-\epsilon$, some of the signs in the intermediate equations would change, but equation (3.2.15) would remain the same.

3.2.2 Analysis of divergences

To analyze the consequences of equation (3.2.15), consider the effect of some intermediate operators of twist τ_0 gaining an anomalous dimension $\tau_0\mapsto\tau_0+\epsilon\gamma^{(1)}_{\tau_0,\ell}+\ldots$ and an OPE coefficient correction $a^{(0)}_{\tau_0,\ell}\mapsto a^{(0)}_{\tau_0,\ell}\left(1+\epsilon\alpha^{(1)}_{\tau_0,\ell}+\ldots\right)$. This creates a correction to the first order correlator:

$$\mathcal{G}^{(1)}(u,v)=\sum_{\tau_0,\ell}a^{(0)}_{\tau_0,\ell}\left(\alpha^{(1)}_{\tau_0,\ell}+\gamma^{(1)}_{\tau_0,\ell}\partial_\tau\right)G_{\tau,\ell}(u,v)\,. \tag{3.2.16}$$

From the form of the conformal blocks it follows that this has a $\log u$ piece:

$$\sum_{\tau_0,\ell}a^{(0)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}\partial_\tau G_{\tau,\ell}(u,v)\Big|_{\log u}=\frac{1}{2}\sum_{\tau_0,\ell}a^{(0)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}G_{d_0-2,\ell}(u,v)=\frac{1}{2}\sum_{\tau_0,\rho}^{\infty}B^{(1)}_{\tau_0,\rho}H^{(\rho)}_{\tau_0}(u,v)\,, \tag{3.2.17}$$

where the $B^{(1)}_{\tau_0,\rho}$ are expansion coefficients of $\gamma^{(1)}_{\tau_0,\ell}$ in terms of the conformal spin: $\gamma^{(1)}_{\tau_0,\ell}=\sum_\rho B^{(1)}_{\tau_0,\rho}J^{-2\rho}_{\tau_0,\ell}$. The full $\log v$ part of equation (3.2.16) is hard to identify in general dimension d_0; therefore we consider dimensions $d_0=2,4$ and focus only on

enhanced divergent parts proportional to $\log v$, i.e. terms of the form $v^{-k}\log v$ and $\log^m v$ for $k \geqslant 1, m \geqslant 2$.

We follow the arguments in [80], recalling the special forms of the 2d and 4d conformal blocks in (2.2.5) and (2.2.6). Let us now define

$$\left(\widetilde{\tau}, \widetilde{\ell}\right) = \left(\tau, l + \frac{\tau}{2}\right), \qquad (\partial_\tau, \partial_l) = \left(\widetilde{\partial}_\tau + \frac{1}{2}\widetilde{\partial}_\ell, \widetilde{\partial}_\ell\right), \tag{3.2.18}$$

with $\widetilde{\partial}_\tau \equiv \partial_{\widetilde{\tau}}$, $\widetilde{\partial}_\ell \equiv \partial_{\widetilde{\ell}}$, so that we can rewrite the ∂_τ part of equation (3.2.16) as follows:

$$\begin{aligned}\sum_{\tau_0,\ell} a^{(0)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}\partial_\tau G_\tau(u,v) = \sum_{\tau_0,\ell} a^{(0)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}\widetilde{\partial}_\tau G_{\tau,\ell}(u,v) - \frac{1}{2}\sum_{\tau_0,\ell}\widetilde{\partial}_\ell\left(a^{(0)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}\right)G_{\tau,\ell}(u,v) \\ + \frac{1}{2}\sum_{\tau_0,\ell}\widetilde{\partial}_\ell\left(a^{(0)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}G_{\tau,\ell}(u,v)\right).\end{aligned} \tag{3.2.19}$$

The last term is a boundary term, so that it will not contain any enhanced divergences (see appendix A.1 for a more detailed discussion). By construction $\widetilde{\partial}_\tau k^{\Delta_{12},\Delta_{34}}_{\tau+2l}(z) = 0$ so that the first term contains no enhanced divergences of the form $\frac{\log v}{v^k}$ or $\log^2 v$.[3]

Plugging this back into equation (3.2.16), we find that, ignoring the $\log u \log v$ divergences:

$$\sum_{\tau_0,\ell} a^{(0)}_{\tau_0,\ell}\left(\alpha^{(1)}_{\tau_0,\ell} + \gamma^{(1)}_{\tau_0,\ell}\partial_\tau\right)G_{\tau,\ell}(u,v)\Big|_{\log v,\, enh.div.} = \sum_{\tau_0,\ell} a^{(0)}_{\tau_0,\ell}\widehat{\alpha}^{(1)}_{\tau_0,\ell}G_{\tau,\ell}(u,v)\Big|_{\log v,\, enh.div.}, \tag{3.2.20}$$

where $\widehat{\alpha}^{(1)}_{\tau_0,\ell}$ is defined by

$$\widehat{\alpha}^{(1)}_{\tau_0,\ell} = \alpha^{(1)}_{\tau_0,\ell} - \frac{1}{2a^{(0)}_{\tau_0,\ell}}\partial_\ell\left(a^{(0)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}\right) = \alpha^{(1)}_{\tau_0,\ell} - \frac{1}{2}\partial_\ell\gamma^{(1)}_{\tau_0,\ell} - \frac{1}{2}\gamma^{(1)}_{\tau_0,\ell}\partial_\ell\log a^{(0)}_{\tau_0,\ell}. \tag{3.2.21}$$

Let us now expand the anomalous dimensions and OPE coefficient corrections of the bilinears in terms of the conformal spin $J^2_{\tau,\ell}$, assuming there are no $\log J$ terms in the expansion of the anomalous dimensions:

$$\gamma^{(1)}_{\tau_0,\ell} = 2\sum_{m=0}^{\infty}\frac{B^{(1)}_{\tau_0,m}}{J^{2m}_{d-2,\ell}}, \qquad \widehat{\alpha}^{(1)}_{\tau_0,\ell} = 2\sum_{m=0}^{\infty}A^{(1)}_{\tau_0,m}\frac{\log J}{J^{2m}_{d-2,\ell}} + 2\sum_{m=0}^{\infty}\frac{\widetilde{A}^{(1)}_{\tau_0,m}}{J^{2m}_{d-2,\ell}}. \tag{3.2.22}$$

[3] The special form of the conformal blocks implies that such divergences would have to already be present in the tree-level twist conformal blocks, which is not the case.

These will contribute to $\log u$ terms in equation (3.2.16) as follows:

$$\mathcal{G}^{(1)}(u,v)\big|_{v^{-\bullet}\log u} \supseteq \sum_{m=0}^{\infty} B^{(1)}_{d_0-2,m} H^{(m)}_{d_0-2}(u,v)\,, \tag{3.2.23}$$

$$\mathcal{G}^{(1)}(v,u)\big|_{u^{-\bullet}\log u} \supseteq -\sum_{m=0}^{\infty} A^{(1)}_{d_0-2,m} H^{(m)}_{d_0-2}(v,u)\,. \tag{3.2.24}$$

Here we used the result $H_\tau^{(k,\log J)}(u,v)\Big|_{v^{-\bullet}\log v} = -\frac{1}{2}H_\tau^{(k)}(u,v)\big|_{v^{-\bullet}}$, which follows from analytically continuing in k, identifying

$$H_\tau^{(k_0,\log J)}(u,v) = -\frac{1}{2}\,\frac{\partial}{\partial k}\bigg|_{k=k_0} H_\tau^{(k)}(u,v)\,, \tag{3.2.25}$$

and the assumption that the k-dependence of $H_\tau^{(k)}(u,v)$ takes the form in equation (3.1.9).

To see how this can be used to constrain the anomalous dimensions and OPE coefficients, let us assume that the dimension satisfies $d_0 > 2$, so that there is a gap between $u^{d_0/2}$, the highest power of u in the bilinear TCB, and u^{d_0-1}, the lowest power of u in the quadrilinear TCBs. This gap allows for the bilinear enhanced divergences, i.e. those of the form[4]

$$\frac{u^{\frac{d_0}{2}-1}}{v^{\frac{d_0}{2}-m}}\,, \quad \frac{u^{\frac{d_0}{2}}}{v^{\frac{d_0}{2}-m}} \qquad \text{for } m \in \mathbb{Z}_{\geqslant 0}\,, \tag{3.2.26}$$

to be studied without reference to the quadrilinear operators, since crossing maps the set of these bilinear divergences onto itself. Define $h^{(m)}_{d_0-2}(u,v)$ by

$$H^{(m)}_{d_0-2}(u,v)\Big|_{enh.div.} = \frac{u^{\frac{d_0}{2}-1}}{v^{\frac{d_0}{2}}} h^{(m)}_{d_0-2}(u,v) \tag{3.2.27}$$

Focusing on these divergences in the $\log u$ part of equation (3.2.16) yields, after substitution of (3.2.23):

$$\sum_m B^{(1)}_{d_0-2,m} h^{(m)}_{d_0-2}(u,v) = \gamma^{(1)}_{\mathcal{O}} h^{(0)}_{d_0-2}(u,v) - \sum_m A^{(1)}_{d_0-2,m} h^{(m)}_{d_0-2}(v,u)\,. \tag{3.2.28}$$

[4]If d_0 is an even integer, then not all such terms are divergent, since $v^{-\frac{d_0}{2}+m}$ will become regular in v for large integer m; furthermore powers of u can recombine with the quadrilinear TCBs since the gap in powers is an integer. Special care needs to be taken in such dimensions.

From the tree-level result and our knowledge of the asymptotic behaviour of TCBs, we find that

$$\begin{cases} h^{(0)}_{d_0-2}(u,v) & = 1-u-v\,, \\ h^{(m)}_{d_0-2}(u,v) & \sim u^0 v^m \qquad \text{for small } u,v\,. \end{cases} \tag{3.2.29}$$

Equating terms of orders u^0v^0, $u^m v^0$ and $u^0 v^m$, we find that

$$\boxed{B^{(1)}_{d_0-2,0} + A^{(1)}_{d_0-2,0} = \gamma^{(1)}_{\mathcal{O}}\,, \qquad B^{(1)}_{d_0-2,m} = A^{(1)}_{d_0-2,m} = 0 \text{ for } m \geqslant 1\,.} \tag{3.2.30}$$

This result holds in perturbations of the free theory in which the intermediate operators do not change. As we shall see in the Gross-Neveu-Yukawa model, there can be non-zero $B^{(1)}_{d_0-2,m}$ if there is a coupling to new operators.

3.2.3 Subtleties: finite-support solutions and degeneracy

The study of enhanced divergences of twist conformal blocks as outlined above uses the assumption that the CFT data is analytic in the spin ℓ. However it is known that this analyticity can fail to hold for all spins: in this case we need to consider solutions with a finite support on the spin [148, 159].

Recently it has been shown that the OPE coefficients and anomalous dimensions, under some mild assumptions regarding Regge behaviour in the theory, are in fact analytic in the spin all the way down to spin $\ell = 2$ [97], thereby limiting the finite support solutions to $\ell = 0, 1$. In the theories in this chapter, operators of spin $\ell = 0$ often do not appear in the correlators we consider, leaving only spin $\ell = 1$ open to a finite support solution.

There is one caveat here: the argument in [97] shows that the CFT data for spin $\ell > 1$ is analytic in ℓ *non-perturbatively*, while our analysis is perturbative.[5] Perturbatively one expects, from the violation of Regge behaviour, that the CFT data will be analytic in the spin down to some minimal spin L proportional to the loop order.

[5] For a simple example of a statement that is true non-perturbatively but is not true perturbatively, consider the function e^{-gx} with $g > 0$. It is clearly bounded for all positive x, but each term in a perturbative expansion around $g = 0$ is unbounded.

We shall explicitly state in the rest of the chapter whenever we use this result.

Another subtlety arises from the possibility that there are multiple operators with the same tree-level twist and spin, so that they enter the crossing equation on the same footing. That is, if there are different operators $\mathcal{O}_i$ with the same spin ℓ, and with (possibly different) twists $\tau_i = \tau_0 + \gamma_{\tau_0,i}$ and OPE coefficients $a_{\tau,\ell,i}$, then they enter the crossing equation as

$$a_{\tau,\ell}G_{\tau,\ell}(u,v) \mapsto \sum_i a_{\tau,\ell,i}G_{\tau_i,\ell}(u,v)\,. \tag{3.2.31}$$

In such a case our analysis does not find the CFT data of the individual operators, but a weighted average. For example, the sum in equation (3.2.31) has an $\epsilon \log u$ part equal to:

$$\sum_i a_{\tau,\ell,i}G_{\tau_i,\ell}(u,v)\bigg|_{\epsilon\log u} = \left(\sum_i a^{(0)}_{\tau_0,\ell,i}\frac{\gamma^{(1)}_{\tau_0,\ell,i}}{2}\right)G_{\tau_0,\ell}(u,v)\,, \tag{3.2.32}$$

and as such, the crossing equation is only sensitive to the average[6]

$$\langle\gamma^{(1)}_{\tau_0,\ell}\rangle \equiv \frac{\sum_i a^{(0)}_{\tau_0,\ell,i}\gamma^{(1)}_{\tau_0,\ell,i}}{\sum_i a^{(0)}_{\tau_0,\ell,i}}\,. \tag{3.2.33}$$

We similarly define for any function the average $\langle f_{\tau_0,\ell}\rangle$ to be

$$\langle f_{\tau_0,\ell}\rangle \equiv \frac{\sum_i a^{(0)}_{\tau_0,\ell,i}f_{\tau_0,\ell,i}}{\sum_i a^{(0)}_{\tau_0,\ell,i}}\,. \tag{3.2.34}$$

For notational purposes, we shall also define the following sum over degenerate states:

$$\langle\!\langle f_{\tau_0,\ell}\rangle\!\rangle \equiv \sum_i f_{\tau_0,\ell,i}\,. \tag{3.2.35}$$

Specifically, note that knowledge of $\langle f_{\tau_0,\ell}\rangle$ does not determine $\langle f^2_{\tau_0,\ell}\rangle$, a problem that we will need to consider in section 3.3.2.[7]

[6] Theorems regarding analyticity or convergence of the OPE that rely on crossing symmetry, such as in [97], generally apply to these averages.

[7] For a different perspective: the $a^{(0)}_{\tau_0,\ell,i} \geqslant 0$ can be considered as a probability distribution on the different operators of fixed twist τ_0 and spin ℓ, with $f_{\tau_0,\ell}$ a random variable. Knowledge of the first moment $\mathbb{E}\left[f_{\tau_0,\ell}\right]$ does not fix the second moment $\mathbb{E}\left[f^2_{\tau_0,\ell}\right]$; to fully determine the values $f_{\tau_0,\ell,i}$, or equivalently all moments $\mathbb{E}\left[f^n_{\tau_0,\ell}\right]$, one needs access to at least as many moments as there are degenerate operators.

This type of degeneracy is present for the quadrilinear operators in both models. Furthermore, in the Gross-Neveu-Yukawa model there is a degeneracy in the bilinear currents, which is resolved in section 3.4.2 by considering multiple correlators simultaneously.

3.3 The $d = 2 + \epsilon$ expansion and the Gross-Neveu model

In this section we study fermionic CFTs that weakly break higher spin symmetry, in dimension $d = 2+\epsilon$, order by order in ϵ. We pay particular attention to the (critical) Gross-Neveu model, which can be described by the following action

$$S = \int \mathrm{d}^{2+\epsilon}x \left(\overline{\psi}^i \not{\partial} \psi_i + \frac{1}{2} g (\overline{\psi}^i \psi_i)^2 \right) , \tag{3.3.1}$$

where $g \sim \epsilon$, so that at $\epsilon = 0$ it reduces to the free fermion in 2 dimensions.

Section 3.3.1 finds first order corrections to the CFT data of both the bilinear and quadrilinear operators in the singlet representation, and of bilinears in the adjoint representation. Section 3.3.2 discusses the second order corrections to the anomalous dimensions of bilinear currents in the Gross-Neveu model.

3.3.1 First order

3.3.1.1 Results

To summarize the results in this section: through an analysis of the correlator $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle$, we find that a non-trivial solution to the first-order crossing equation exists, which reduces to the Gross-Neveu model upon demanding that the first order anomalous dimensions do not scale logarithmically with the spin. After demanding this, further results about the non-singlet operators are found through the study of the correlator $\langle \mathcal{O}^A \mathcal{O}^A \mathcal{O}^A \mathcal{O}^A \rangle$ and mixed correlators such as $\langle \mathcal{O}^A \mathcal{O}\mathcal{O}\mathcal{O}^A \rangle$.

The full solution for the singlets is of a similar form to that for 4d (bosonic) gauge theories studied in [80]. We find that the singlet bilinear operators of even spin $\ell \geqslant 2$

have the following anomalous dimensions and OPE coefficient corrections:[8]

$$\gamma^{(1)}_{S,0,\ell} = \beta\left(S_1(\ell-1)-1\right)\,, \tag{3.3.2}$$

$$\widehat{\alpha}^{(1)}_{S,0,\ell} = \left(2\gamma^{(1)}_{\mathcal{O}}+\beta\right)S_1(\ell-1)+\widehat{\xi}_{-1}\,, \tag{3.3.3}$$

where it is assumed that there is a unique spin 2 operator of twist $d-2$, corresponding to the stress-energy tensor, and where $\widehat{\xi}_{-1}$ is a constant related to the central charge. The full OPE coefficient correction $\alpha^{(1)}_{S,0,\ell}$ can then be found using the definition (3.2.21); we do not produce it here.

Almost all the $\mathrm{U}(N_f)$ singlet quadrilinear operators are degenerate, and we find the following results for their infinite support solution:

$$\langle\gamma^{(1),inf.}_{S,\tau_0,\ell}\rangle = \beta_{\tau_0}S_1\left(\ell+\frac{\tau_0}{2}-1\right)+\kappa_{\tau_0}, \tag{3.3.4}$$

$$\langle\widehat{\alpha}^{(1),inf.}_{S,\tau_0,\ell}\rangle = \widehat{\alpha}_{\tau_0}S_1\left(\ell+\frac{\tau_0}{2}-1\right)+\widehat{\xi}_{\tau_0}, \tag{3.3.5}$$

where

$$\beta_{\tau_0} = -\frac{\eta}{N+\eta}\beta\,, \tag{3.3.6}$$

$$\kappa_{\tau_0} = 2\gamma^{(1)}_{\mathcal{O}}\frac{N+2\eta}{N+\eta}+\frac{\eta\beta}{N+\eta}\left(2-S_1\left(\frac{\tau_0}{2}-1\right)+\frac{1}{2}\delta_{\tau_0,2}\right)\,, \tag{3.3.7}$$

$$\widehat{\alpha}_{\tau_0} = 2\gamma^{(1)}_{\mathcal{O}}+\frac{\eta\beta}{N+\eta}\left(1-2S_1\left(\frac{\tau_0}{2}-1\right)+S_1\left(\tau_0-2\right)+\frac{1}{4}\delta_{\tau_0,2}\right)\,, \tag{3.3.8}$$

$$\widehat{\xi}_{\tau_0} = \frac{\eta}{N+\eta}\left(\widehat{\xi}_{-1}+\beta\widehat{\xi}^{(\beta)}_{\tau_0}+\gamma^{(1)}_{\mathcal{O}}\widehat{\xi}^{(\gamma\mathcal{O})}_{\tau_0}\right)\,, \tag{3.3.9}$$

where we defined $\eta=(-1)^{\frac{\tau_0}{2}}$, and where, in equation (3.3.9):

$$\begin{aligned}\widehat{\xi}^{(\beta)}_{\tau_0} = {}& \zeta_2+3S_1\left(\frac{\tau_0}{2}-1\right)-S_1\left(\frac{\tau_0}{2}-1\right)^2-2S_1\left(\tau_0-2\right)\\ &+S_1\left(\frac{\tau_0}{2}-1\right)S_1\left(\tau_0-2\right)+\frac{1}{2}S_2\left(\frac{\tau_0}{2}-1\right)-\frac{5}{4}\delta_{\tau_0,2}\,,\end{aligned} \tag{3.3.10}$$

$$\widehat{\xi}^{(\gamma\mathcal{O})}_{\tau_0} = 6S_1\left(\frac{\tau_0}{2}-1\right)-4S_1\left(\tau_0-2\right)-\delta_{\tau_0,2}+N\eta\left(4S_1\left(\frac{\tau_0}{2}-1\right)-2S_1\left(\tau_0-2\right)\right)\,, \tag{3.3.11}$$

with $\zeta_2\equiv\zeta(2)=\pi^2/6$.

[8] $S_r(n)$ denotes the r-th order harmonic number: $S_r(n)=\sum_{k=1}^{n}\frac{1}{k^r}$.

We find that the above solution with $\beta \neq 0$ requires the existence of a solution with finite support on the spin, i.e. with $\gamma^{(1)}_\ell \neq 0$ only for $\ell = 0, \ldots, L$. As per the results of [97], we would expect $L = 1$ to first order in ϵ. Indeed such a solution exists, and for the $\mathrm{U}(N_f)$ quadrilinear singlets it takes the form

$$\langle \gamma^{(1),fin.}_{S,\tau_0,0} \rangle = \frac{N}{N+\eta} \frac{1}{\tau_0 - 1} \left(\gamma_{fin} + \frac{\beta}{4N} \left(1 - \delta_{\tau_0,2}\right) \right), \tag{3.3.12}$$

where γ_{fin} is a constant not fixed by our analysis. The part of the solution that is independent of the infinite support solution, which is found by setting $\beta = 0$, matches the form of the solutions found in [159].

Logarithmic scaling and results for Gross-Neveu

Logarithmic scaling of the anomalous dimensions with the spin is known to occur in CFTs, for example at order ϵ^3 in the Gross-Neveu model [156] or at order ϵ^2 in the critical nonlinear sigma model in $d = 2 + \epsilon$ [160]. This behaviour can be understood from the fact that the nearly conserved currents of twist $\tau = d - 2$ contribute a term of the form $1/\ell^{d-2}$ in the large spin expansion, which generates a logarithmic term in $d = 2 + \epsilon$. However, we are unaware of any known theories in which such behaviour already occurs at first order in ϵ, and we would expect this to correspond to a theory in which ψ is not part of the spectrum, for example because it is prohibited by gauge symmetry.

Assuming that the twist additivity property extends suitably to fermions, the twists of the bilinear currents would have to accumulate at $2\tau_\psi$, implying that logarithmic scaling of the anomalous dimensions with the spin is prohibited. This sets $\beta = 0$ and reduces our results to those in the Gross-Neveu model in $d = 2 + \epsilon$.[9] The analysis shows parallels to the study of gauge theories in 4d such as in [80], which studies the implications of crossing symmetry of the correlator $\langle \varphi^2 \varphi^2 \varphi^2 \varphi^2 \rangle$ for the

[9] An interesting point to note is that in this case $\tau_{J_{S,\ell}} = d - 2 + O(\epsilon^2)$. Demanding that these accumulate at $2\tau_\psi$ then implies that $\gamma_\psi = O(\epsilon^2)$, as is indeed the case in the Gross-Neveu model [150].

CFT data. Their results include the possibility of $\log J$ terms in the anomalous dimensions, and in fact such terms appear in $\mathcal{N}=4$ SYM. However in theories in which the fundamental field φ appears, such as the Wilson-Fisher model, an analysis like that of [148] shows that no such terms may appear; the reason that the corrections in $\mathcal{N}=4$ SYM could have logarithmic behaviour is because gauge symmetry prevented the field ϕ from appearing in the spectrum. A similar thing is likely happening here, where a full analysis of the correlator $\langle\overline{\psi}\psi\overline{\psi}\psi\rangle$ of fundamental fields would be able to conclude that $\beta=0$ if ψ is part of the spectrum.

When $\beta=0$ our results reduce to

$$\gamma^{(1)}_{S,0,\ell}=0\,, \tag{3.3.13}$$

$$\alpha^{(1)}_{S,0,\ell}=2\gamma^{(1)}_{\mathcal{O}}S_1(\ell-1)+\widehat{\xi}_{-1}\,. \tag{3.3.14}$$

Furthermore the quadrilinear operators have corrections of the form

$$\langle\gamma^{(1)}_{S,\tau_0,\ell}\rangle=\kappa_{\tau_0}\,, \tag{3.3.15}$$

$$\langle\widehat{\alpha}^{(1)}_{S,\tau_0,\ell}\rangle=2\gamma^{(1)}_{\mathcal{O}}S_1\left(\ell+\frac{\tau_0}{2}-1\right)+\widehat{\xi}_{\tau_0}\,, \tag{3.3.16}$$

where

$$\kappa_{\tau_0}=2\gamma^{(1)}_{\mathcal{O}}\frac{N+2\eta}{N+\eta}\,, \tag{3.3.17}$$

$$\widehat{\xi}_{\tau_0}=\frac{\eta}{N+\eta}\left(\widehat{\xi}_{-1}+\gamma^{(1)}_{\mathcal{O}}\widehat{\xi}^{(\gamma\mathcal{O})}_{\tau_0}\right)\,, \tag{3.3.18}$$

with $\widehat{\xi}^{(\gamma\mathcal{O})}_{\tau_0}$ as before. To fix $\widehat{\xi}_{-1}$, we use the relation for the OPE coefficient of an operator $\mathcal{O}$ with the stress-energy tensor T [27]:

$$a_{\mathcal{O}\mathcal{O}T}=\frac{1}{c_T}\frac{d^2}{(d-1)^2}\Delta^2_{\mathcal{O}}\,, \tag{3.3.19}$$

with c_T the central charge of the theory. Evaluating at $d=2+\epsilon$, with $c^{(0)}_T=N$, and subtracting the free theory correction to the OPE coefficient, one deduces that

$$\alpha^{(0)}_{S,0,2}=2\gamma^{(1)}_{\mathcal{O}}-\frac{1}{N}c^{(1)}_T=2\gamma^{(1)}_{\mathcal{O}}\,, \tag{3.3.20}$$

where the full central charge of the theory is $c_T = c_T^{(0)} + \epsilon\, c_T^{(1)} + \ldots$, and where we used the result that the central charge corrections in the Gross-Neveu model only start at order ϵ^3 [161].

From these results for the singlet operators in the Gross-Neveu model, we deduce results for the non-singlet operators. Specifically, we find for the bilinear adjoint operators of even spin $\ell \geqslant 2$:

$$\gamma^{(1)}_{A,0,\ell} = 0\,, \tag{3.3.21}$$

$$\lambda^{(1)}_{AA\ell} = \gamma^{(1)}_{\mathcal{O}^A} S_1(\ell - 1) + k_A\,, \tag{3.3.22}$$

$$\lambda^{(1)}_{SA\ell} = \frac{1}{2}\left(\gamma^{(1)}_{\mathcal{O}} + \gamma^{(1)}_{\mathcal{O}^A}\right) S_1(\ell - 1) + k_{SA}\,. \tag{3.3.23}$$

Here the $\lambda^{(1)}_{\bullet\bullet\bullet}$ are the (multiplicative) corrections to the (non-squared) OPE coefficients $c_{\bullet\bullet\bullet}$, i.e. they satisfy

$$c_{\bullet\bullet\bullet} = c^{(0)}_{\bullet\bullet\bullet} + \epsilon c^{(1)}_{\bullet\bullet\bullet} + \cdots = c^{(0)}_{\bullet\bullet\bullet}\left(1 + \epsilon\lambda^{(1)}_{\bullet\bullet\bullet} + \ldots\right)\,, \tag{3.3.24}$$

with $c_{AA\ell} = c_{\mathcal{O}^A\mathcal{O}^A J_{S,\ell}}$ and $c_{SAl} = c_{\mathcal{O}\mathcal{O}^A J_{A,\ell}}$.

Furthermore, we have found that for bilinear currents of odd spin ℓ:

$$\gamma^{(1)}_{A,0,\ell} = 0\,. \tag{3.3.25}$$

The odd spin singlets cannot be directly accessed by our method since they do not appear as intermediate operators in the four-point correlators we consider.

The bilinear anomalous dimensions $\gamma^{(1)}_{S,0,\ell}$ and $\gamma^{(1)}_{A,0,\ell}$ match known results for the Gross-Neveu model in $2+\epsilon$ dimensions, found for example in [156].

3.3.1.2 Singlets - $\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$

We first analyze the bilinear currents on their own and later add the quadrilinear operators to the analysis.

Bilinear currents

From the general tree-level result, equation (3.1.15), the tree-level squared OPE coefficients of the $\mathrm{U}(N_f)$ singlet currents can be found in $d_0 = 2$:

$$a^{(0)}_{S,0,\ell} = \frac{\left(1+(-1)^\ell\right)}{N} \frac{\Gamma(\ell)^2}{\Gamma\left(2\ell-1\right)} \,. \tag{3.3.26}$$

We repeat here the first-order crossing equation (3.2.15), and evaluate it at $d_0 = 2$:

$$\gamma^{(1)}_{\mathcal{O}} \log v \, \mathcal{G}^{(0)}(u,v) + \mathcal{G}^{(1)}(u,v) = \gamma^{(1)}_{\mathcal{O}} \log u \, \mathcal{G}^{(0)}(u,v) + \frac{u}{v} \mathcal{G}^{(1)}(v,u) \,. \tag{3.3.27}$$

As in section 3.2.2, corrections to the OPE coefficients and anomalous dimensions create a correction to the correlator:

$$\mathcal{G}^{(1)}(u,v) = \sum_{\tau_0,\ell} a^{(0)}_{\tau_0,\ell} \left(\alpha^{(1)}_{\tau_0,\ell} + \gamma^{(1)}_{\tau_0,\ell} \partial_\tau\right) G_{\tau,\ell}(u,v) \,. \tag{3.3.28}$$

When we expand the anomalous dimensions and OPE coefficient corrections in terms of the conformal spin $J^2_{0,\ell} = \ell(\ell-1)$, these corrections will organize themselves in terms of the twist conformal blocks $H^{(\rho)}_0(u,v)$; let us therefore investigate their analytical properties. The free theory result is

$$H^{(0)}_0(u,v) = \frac{1}{N}\left(\frac{(1-v)^2}{v} - u\frac{(1+v)}{v}\right) . \tag{3.3.29}$$

This has enhanced divergences of the form $\frac{1}{v}$ and $\frac{u}{v}$, so that we expect $\log^2 v$ divergences in $H^{(1)}_0(u,v), H^{(2)}_0(u,v), \ldots$. Indeed this is what happens: in terms of the $z, \bar{z}$ coordinates, for $m \geqslant 1$, there is a term of the form

$$H^{(m)}_0(z,\bar{z}) \supseteq h^{(m)}(\bar{z}) \log(1-\bar{z})^2 \,, \qquad h^{(m)}(\bar{z}) \sim (1-\bar{z})^{m-1} \qquad \text{for } (1-\bar{z}) \ll 1 \,. \tag{3.3.30}$$

To see the consequences of this consider terms in the crossing equation (3.3.27) of the form $\log^2 v$. These cannot be produced on the right-hand side, and hence must also be absent on the left-hand side. Consider now an expansion of $\widehat{\alpha}_{0,\ell}$, which we

recall encodes all the enhanced divergences proportional to $\log v$ in equation (3.3.28) (excluding $\log u \log v$ divergences), in terms of the conformal spin $J^2 = J_{0,\ell}^2$

$$\widehat{\alpha}_{0,\ell}^{(1)} = \sum_{m=0}^{\infty} \left(A_{0,m}^{(1)} \frac{\log J}{J^{2m}} + \widetilde{A}_{0,m}^{(1)} \frac{1}{J^{2m}} \right) . \tag{3.3.31}$$

This contributes a term

$$\mathcal{G}^{(1)}(u,v) \supseteq \sum_{l} a_{0,\ell}^{(0)} \widehat{\alpha}_{0,\ell}^{(1)} G_{0,\ell}(u,v) = \sum_{m=0}^{\infty} \left(A_{0,m}^{(1)} H_0^{(m,\log J)}(u,v) + \widetilde{A}_{0,m}^{(1)} H_0^{(m)}(u,v) \right) . \tag{3.3.32}$$

We immediately conclude that $A_{0,m}^{(1)} = 0$ for $m \geqslant 1$, since these would produce terms of the form $\log^3 v$. Furthermore, note that a non-zero $A_{0,0}^{(1)}$ produces $\log^2 v$ terms that can be cancelled by the $\widetilde{A}_{0,m}^{(1)}$ with $m \geqslant 1$. In fact, we find that this imposes that they are of the following form

$$\frac{\widetilde{A}_{0,1}^{(1)}}{A_{0,0}^{(1)}} = \frac{1}{6} , \quad \frac{\widetilde{A}_{0,2}^{(1)}}{A_{0,0}^{(1)}} = -\frac{1}{30} , \quad \frac{\widetilde{A}_{0,3}^{(1)}}{A_{0,0}^{(1)}} = \frac{4}{315} , \quad \frac{\widetilde{A}_{0,4}^{(1)}}{A_{0,0}^{(1)}} = -\frac{1}{105} , \quad \dots \tag{3.3.33}$$

which the astute reader may recognize as the coefficients of the expansion of the harmonic number $S_1(\ell-1)$ in terms of the conformal spin $J^2 = \ell(\ell-1)$:

$$S_1(\ell-1) = \gamma_E + \log J + \frac{1}{6J^2} - \frac{1}{30J^4} + \frac{4}{315J^6} - \frac{1}{105J^8} + \dots , \tag{3.3.34}$$

with γ_E the Euler-Mascheroni constant. Hence we conclude that the general form of the OPE correction is

$$\widehat{\alpha}_{S,0,\ell}^{(1)} = \widehat{\alpha}_{-1} S_1(\ell-1) + \widehat{\xi}_{-1} , \tag{3.3.35}$$

where $\widehat{\alpha}_{-1}$ and $\widehat{\xi}_{-1}$ are constants to be fixed. Similarly we demand absence of terms of the form $u^0 \log u \log^2 v$, which arise as follows:

$$\mathcal{G}^{(1)}(u,v)\Big|_{\log u} \supseteq \sum_{\ell} a_{0,\ell}^{(0)} \gamma_{0,\ell}^{(1)} \partial_\tau \big|_{\tau=0} G_{\tau,\ell}(u,v)\Big|_{\log u} = \sum_{\ell} a_{0,\ell}^{(0)} \frac{\gamma_{0,\ell}^{(1)}}{2} G_{0,\ell}(u,v) . \tag{3.3.36}$$

In the last line we used the fact that conformal blocks are of the form $G_{\tau,\ell}(u,v) = u^{\tau/2} g_{\tau,\ell}(u,v)$, with $g_{\tau,\ell}(u,v)$ analytic in u. In the crossing equation, we then come to

the same conclusion as for the $\widehat{\alpha}^{(1)}_{0,\ell}$, i.e. that the anomalous dimensions are of the form:

$$\gamma^{(1)}_{S,0,\ell} = \beta\left(S_1(\ell-1)-1\right)\,, \tag{3.3.37}$$

where exact conservation of the stress-energy tensor was used to fix $\gamma^{(1)}_{S,0,2}=0$.

Let us now fix the constant $\widehat{\alpha}_{-1}$. We look at the $u^0 v^{-1}\log v$ part of the crossing equation:

$$\gamma^{(1)}_{\mathcal{O}}\mathcal{G}^{(0)}(u,v)\Big|_{\frac{u^0\log v}{v}} + \mathcal{G}^{(1)}(u,v)\Big|_{\frac{u^0\log v}{v}} = \frac{u}{v}\left(\mathcal{G}^{(1)}(v,u)\Big|_{\frac{v^0\log v}{u}}\right)\,, \tag{3.3.38}$$

and use the known expansions of $\widehat{\alpha}^{(1)}_{S,0,\ell}$ and $\gamma^{(1)}_{S,0,\ell}$ to find that

$$\mathcal{G}^{(1)}(u,v)\Big|_{\frac{u^0\log v}{v}} = -\frac{1}{2}\widehat{\alpha}_{-1}H^{(0)}_0(u,v)\Big|_{\frac{u^0}{v}} = -\frac{\widehat{\alpha}_{-1}}{2N}\,, \tag{3.3.39}$$

$$\mathcal{G}^{(1)}(v,u)\Big|_{\frac{v^0\log v}{u}} = -\frac{1}{2}\beta H^{(0)}_0(v,u)\Big|_{\frac{v^0}{u}} = -\frac{\beta}{2N}\,. \tag{3.3.40}$$

Combining this with the tree-level result, we find that equation (3.3.38) reduces to $\gamma^{(1)}_{\mathcal{O}} - \frac{1}{2}\widehat{\alpha}_{-1} = -\frac{1}{2}\beta$, so that

$$\widehat{\alpha}_{-1} = 2\gamma^{(1)}_{\mathcal{O}} + \beta\,. \tag{3.3.41}$$

At this point we can fix no further constants using our analysis.

Summation

Given the form of the anomalous dimensions and OPE coefficient corrections, it is natural to consider the full bilinear part of the correction (3.3.28) to $\mathcal{G}^{(1)}(u,v)$. While the full correction can in fact be found, for our analysis we shall only need its leading u behaviour, because on the left-hand side of the crossing equation, the infinite support CFT data of the quadrilinear operators is fully determined by the part that has an enhanced divergence of the form $\frac{1}{v}$. Since the crossing equation is of the form $\mathcal{G}^{(1)}(u,v) = \frac{u}{v}\mathcal{G}^{(1)}(v,u) + \ldots$, on the right-hand side these terms are fully determined by the leading behaviour of corrections from the bilinear currents.

Following the notation in [80], let us write the bilinear CFT data as

$$\gamma^{(1)}_{S,0,\ell} = \alpha_{1,1} S_1(\ell-1) + \alpha_{1,0}\,, \tag{3.3.42}$$

$$\widehat{\alpha}^{(1)}_{S,0,\ell} = \alpha_{0,1} S_1(\ell-1) + \alpha_{0,0}\,. \tag{3.3.43}$$

The leading u behaviour of the correction to the four-point correlator will then be a function $f(u,v)$ that can be expanded as

$$f(u,v) = \alpha_{1,1} f_{1,1}(u,v) + \alpha_{1,0} f_{1,0}(u,v) + \alpha_{0,1} f_{0,1}(u,v) + \alpha_{0,0} f_{0,0}(u,v)\,. \tag{3.3.44}$$

Explicitly summing, the functions are as follows:

$$f_{1,1}(u,v) = \left(\frac{-1+v^2}{4Nv}\right)\log u \log v + \left(\frac{-1+v^2}{2Nv}\right)\mathrm{Li}_2(1-v) + \zeta_2 \frac{(1-v)^2}{2Nv}\,, \tag{3.3.45}$$

$$f_{1,0}(u,v) = \frac{(1-v)^2}{2Nv}\log u + \left(\frac{1-v}{2N}\right)\log v\,, \tag{3.3.46}$$

$$f_{0,1}(u,v) = \frac{(1-v)^2}{2Nv}\log v\,, \tag{3.3.47}$$

$$f_{0,0}(u,v) = \frac{(1-v)^2}{Nv}\,, \tag{3.3.48}$$

where $\frac{(1-v)^2}{Nv}$ is the leading u part of the tree-level $H_0^{(0)}(u,v)$, and where $\frac{-1+v^2}{2Nv}$ is the leading u part of $H_0^{(0),harm}(u,v) \equiv \sum_\ell a^{(0)}_{S,0,\ell} S_1(\ell-1) G_{0,\ell}(u,v)$.

Quadrilinears

Recall that generally the quadrilinear operators are degenerate: there are multiple operators with the same twist and spin. Let us define $\tau_0(n) = 2+2n$ for $n \geqslant 0$, and denote by $\langle \gamma^{(1)}_{n,\ell} \rangle$ the (average) first-order anomalous dimension of the quadrilinear operators of twist $\tau_0(n)$ and spin ℓ.[10] Then the free theory twist conformal blocks $H^{(0)}_{\tau_0(n)}(u,v)$ can be found from the free theory correlator, equation (3.1.14). Decomposing these, the following formula can be found for the OPE coefficients:

$$\langle\!\langle a^{(0)}_{n,\ell} \rangle\!\rangle = \left(1 + \frac{(-1)^{n+1}}{N}\right) \frac{2\Gamma(n+1)^2 \Gamma(\ell+n+1)^2}{\Gamma(2n+1)\,\Gamma(2\ell+2n+1)}\,, \tag{3.3.49}$$

[10] Since we only discuss quadrilinears that are $U(N_f)$ singlets, we shall often omit the subscript S, and for further ease of notation we write terms like $\gamma^{(1)}_{n,\ell}$ to denote $\gamma^{(1)}_{S,\tau_0(n),\ell}$.

where $\langle\!\langle a^{(0)}_{n,\ell} \rangle\!\rangle$ is defined as in equation (3.2.35). Like their bilinear counterpart, the quadrilinear TCBs have the property that for $m \geqslant 1$:

$$H^{(m)}_{\tau_0(n)}(z,\bar{z}) \supseteq z^{\tau_0(n)/2} h^{(m)}(\bar{z}) \log^2(1-\bar{z})\,, \qquad h^{(m)}(\bar{z}) \sim (1-\bar{z})^{m-1} \quad \text{for } (1-\bar{z}) \ll 1\,. \tag{3.3.50}$$

so that no linear combination of them is free of $\log^2 v$ divergences. Another property they share with $H^{(0)}_0(u,v)$ is that if one demands that the sum

$$H^{(0,\log J)}_{\tau_0(n)}(u,v) + \sum_{m=0}^{\infty} B^{(m)}_{\tau_0(n)} H^{(m)}_{\tau_0(n)}(u,v) \tag{3.3.51}$$

is free of $\log^2 v$ divergences, then the $B^{(m)}_{\tau_0(n)}$ must be coefficients in the expansion of the harmonic number $S_1(\ell+n)$ in terms of $J^{-2}_{\tau_0(n),\ell}$, i.e. $\sum_\ell a^{(0)}_{n,\ell} S_1(\ell+n) G_{\tau_0(n),\ell}(u,v)$ is free of $\log^2 v$ divergences. From this we deduce that the CFT data of the quadrilinear operators must have the following form

$$\langle \gamma^{(1)}_{n,\ell} \rangle = \beta_n S_1(\ell+n) + \kappa_n\,, \tag{3.3.52}$$

$$\langle \widehat{\alpha}^{(1)}_{n,\ell} \rangle = \widehat{\alpha}_n S_1(\ell+n) + \widehat{\xi}_n\,. \tag{3.3.53}$$

On the other hand, the quadrilinear TCBs have an interesting property not shared by their bilinear counterpart: namely, while their sum $\mathcal{G}^{(0)}_{quad}(u,v) = \sum_{n=0}^{\infty} H^{(0)}_{\tau_0(n)}(u,v)$ is free of $\log v$ terms, each $H^{(0)}_{\tau_0(n)}(u,v)$ contains $\log v$ terms, in such a way that only the following combination is free of them:

$$\alpha_0 \sum_{n\in 2\mathbb{N}} H^{(0)}_{\tau_0(n)}(u,v) + \alpha_1 \sum_{n\in 2\mathbb{N}+1} H^{(0)}_{\tau_0(n)}(u,v)\,, \tag{3.3.54}$$

restricting the form that anomalous dimensions and OPE coefficients may take. For example, since there can be no terms of the form $\log u \log^2 v$ in the first-order crossing equation, the anomalous dimensions must be of the following form:

$$\langle \gamma^{(1)}_{n,\ell} \rangle = \begin{cases} \beta_0 S_1(\ell+n) + \kappa_n\,, & \text{if } n \text{ even}, \\ \beta_1 S_1(\ell+n) + \kappa_n\,, & \text{if } n \text{ odd}. \end{cases} \tag{3.3.55}$$

Consider the part of the crossing equation that has a power-law divergence in v:

$$\gamma_{\mathcal{O}}^{(1)}(\log v - \log u)\mathcal{G}^{(0)}(u,v)\Big|_{\frac{1}{v}} + \mathcal{G}^{(1)}(u,v)\Big|_{\frac{1}{v}} = \frac{u}{v}f(v,u)\,. \tag{3.3.56}$$

By looking at specific terms in this equation order by order in u, we are able to fix the CFT data of the quadrilinear operators.[11] Specifically, we fix:

- β_n by looking at the $\log u \log v$ part of equation (3.3.56).
- κ_n by looking at the $\log u$ part of equation (3.3.56). Note we do not need to consider the $\log u \log v$ part, which has already been fixed by the β_n.
- $\widehat{\alpha}_n$ by looking at the $\log v$ part of equation (3.3.56). Again the $\log u \log v$ part has already been fixed by the β_n.
- $\widehat{\xi}_n$ by looking at the remaining part of equation (3.3.56), i.e. the terms without logarithms.

Doing this full computation we find the results in equations (3.3.4)-(3.3.11), which we do not reproduce here due to their length. This completely fixes the infinite support solution in terms of three constants: $(\beta, \gamma_{\mathcal{O}}^{(1)}, c_T^{(1)})$, after using the relation (3.3.19) to exchange dependence on $\widehat{\xi}_{-1}$ into dependence on the central charge correction $c_T^{(1)}$.

Note that finite support solutions may exist. Specifically, assuming the argument from section 3.2.3, one would expect to find a finite support solution for operators of spin $\ell = 0$. We analyze this possibility by looking at the full $\log u \log v$ part of the crossing equation, and truncating to a finite order in u, v. We find that such a finite support solution must exist and that it takes the form

$$\langle \gamma_{S,\tau_0,0}^{(1),fin.} \rangle = \frac{N}{N+\eta}\frac{1}{\tau_0 - 1}\left(\gamma_{fin} + \frac{\beta}{4N}\left(1 - \delta_{\tau_0,2}\right)\right), \tag{3.3.57}$$

[11] For technical reasons we used the coordinates $(z, \bar{z})$ in our computation, and looked at power-law divergent terms in $(1-\bar{z})$, order by order in z. This only affects some technical details of the computation: the idea is the same if one replaces $\log u \mapsto \log z$ and $\log v \mapsto \log(1-\bar{z})$ in the discussion below.

for some undetermined constant γ_{fin}.

The part of this finite support solution that is independent of the infinite support solution, i.e. that is not proportional to β, takes the same form as equation (4.13) in [159] after fixing the dimension of the external operator to $\Delta = 1$ and writing $\tau_0 = \tau_0(k) = 2 + 2k$. Similarly to the results in [159] our analysis shows that there are further finite support solutions in which the spin cutoff L satisfies $L \geqslant 2$, which we would not expect to see given the analyticity results of [97].

3.3.1.3 Adjoints: $\langle \mathcal{O}^{i_1}_{j_1} \mathcal{O}^{i_2}_{j_2} \mathcal{O}^{i_3}_{j_3} \mathcal{O}^{i_4}_{j_4} \rangle$

Having determined the anomalous dimensions and OPE coefficients of the singlets, and having found the Gross-Neveu model as the theory in which $\beta = 0$, let us focus on the anomalous dimensions of the adjoint currents by considering the four-point function of adjoint operators in the Gross-Neveu model.

Recall that the four-point function of adjoints decomposes according to the $\mathrm{U}(N_f)$ representations of the intermediate operators. There are three representations containing bilinear operators: the singlet representation of even spin operators, and two adjoint representations, one containing even spin operators and the other containing odd spin operators. Furthermore there are four representation containing quadrilinear operators. They consist of tensors with four indices and can be classified according to the symmetry properties of their associated tensors: $\mathbf{T}^{(i_1 i_2)}_{(j_1 j_2)}$, $\mathbf{T}^{(i_1 i_2)}_{[j_1 j_2]}$, $\mathbf{T}^{[i_1 i_2]}_{(j_1 j_2)}$, $\mathbf{T}^{[i_1 i_2]}_{[j_1 j_2]}$. Note that the second and third representations in this list are conjugate representations.

Let us denote the function corresponding to the symmetric adjoint representation by $\mathcal{G}_{A,S}(u,v)$, and that of the antisymmetric adjoint by $\mathcal{G}_{A,A}(u,v)$. Let us also denote the function corresponding to the singlet representation by $\mathcal{G}_S(u,v)$.

Even spin

The bilinear part of $\mathcal{G}^{(0)}_{A,S}(u,v)$, i.e. the conformal block $H^{(0)}_{d-2,A,S}(u,v)$, is the same

as the conformal block $H^{(0)}_{d-2}(u,v)$ in the correlator $\langle\mathcal{OOOO}\rangle$ up to an overall normalization. The crossing relation is of the form

$$v^{\Delta_A}\mathcal{G}_{A,S}(u,v) = u^{\Delta_A}\sum_{j=1}^{6}\beta_{2j}\mathcal{G}_j(v,u)\,, \tag{3.3.58}$$

where the β_{2j} are constants fixed by the representation theory of $\mathrm{U}(N_f)$.

Analyzing crossing to first order in ϵ yields

$$\gamma^{(1)}_A \log v\, \mathcal{G}^{(0)}_{A,S}(u,v) + \mathcal{G}^{(1)}_{A,S}(u,v) = \gamma^{(1)}_A \log u\, \mathcal{G}^{(0)}_{A,S}(u,v) + \frac{u}{v}\sum_{j=1}^{6}\beta_{2j}\mathcal{G}^{(1)}_j(v,u)\,. \tag{3.3.59}$$

Since the tree-level result is the same as for the bilinear twist conformal block in section 3.3.1.2, we conclude that the expansion of $\gamma^{(1)}_{A,\ell}$ must be constant to ensure that there is no $\log u \log^2 v$ term in the crossing equation:

$$\boxed{\gamma^{(1)}_{A,0,\ell} = \gamma^{(1)}_{A,0,2}, \qquad \ell \geqslant 2 \quad \text{even}\,.} \tag{3.3.60}$$

Odd spin

The bilinear part of $\mathcal{G}^{(0)}_{A,A}(u,v)$ is different from that in the correlator $\langle\mathcal{OOOO}\rangle$: up to an overall normalization, we find it to be

$$H^{(0)}_{d-2,AA}(u,v) = u^{\frac{d}{2}-1}\frac{(1-v)\left(1+v^{d/2}\right)}{v^{d/2}} + u^{\frac{d}{2}}\,\frac{1-v^{d/2}}{v^{d/2}}\,, \tag{3.3.61}$$

so that in 2 dimensions

$$H^{(0)}_{0,AA}(u,v) = \frac{1-v^2}{v} + u\frac{1-v}{v}\,. \tag{3.3.62}$$

The crossing relation at first order in ϵ gives a very similar result to that for $\mathcal{G}_{A,S}$:

$$\gamma^{(1)}_A \log v\, \mathcal{G}^{(0)}_{A,A}(u,v) + \mathcal{G}^{(1)}_{A,A}(u,v) = \gamma_A \log u\, \mathcal{G}^{(0)}_{A,A} + \frac{u}{v}\sum_{j=1}^{6}\beta_{3j}\mathcal{G}^{(1)}_j(v,u)\,. \tag{3.3.63}$$

From the explicit form of the twist conformal block $H^{(0)}_{0,AA}(u,v)$ in equation (3.3.62), all higher TCBs $H^{(m)}_{0,AA}(u,v)$ can be calculated. We find that for $m \geqslant 1$, they have $\log^2(1-\bar{z})$ contributions that are precisely the same as for the even spin case; the

standard argument of demanding no $\log u \log^2 v$ divergences then implies that the $\gamma^{(1)}_{A,0,\ell}$ are constant for odd ℓ.

Whether this constant can be non-zero depends on the existence of solutions with finite support on the spin.[12] Since the solution in the quadrilinears can be unbounded (in both spin and twist), we cannot rule out this possibility easily, and we shall revisit this issue in section 3.3.1.5. For now, we can only make the following ansatz:

$$\gamma^{(1)}_{A,0,\ell} = \omega_\infty + \sum_{m=1}^{L} \omega_m \delta_{m,\ell}\,, \qquad \ell \geqslant 1 \text{ odd}. \tag{3.3.64}$$

Imposing exact conservation of the global symmetry current, which is a spin 1 current in the $\mathrm{U}(N_f)$ adjoint representation, then shows that $\omega_\infty + \omega_1 = 0$. We shall show in section 3.3.1.5 that there is no such finite support solution, so that in fact

$$\boxed{\gamma^{(1)}_{A,0,\ell} = 0\,, \qquad \ell \geqslant 1 \text{ odd}.} \tag{3.3.65}$$

3.3.1.4 Mixed correlators

There is a plethora of mixed four-point correlators to consider:

$$\langle \mathcal{O}^A \mathcal{O}\mathcal{O}\mathcal{O}^A \rangle \quad \to \quad \mathcal{G}_A(u,v) \tag{3.3.66}$$

$$\langle \mathcal{O}\mathcal{O}^A \mathcal{O}^A \mathcal{O} \rangle \quad \to \quad \mathcal{G}'_A(u,v) \tag{3.3.67}$$

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}^A \mathcal{O}^A \rangle \quad \to \quad \mathcal{G}_S(u,v) \tag{3.3.68}$$

$$\langle \mathcal{O}\mathcal{O}^A \mathcal{O}\mathcal{O}^A \rangle \quad \to \quad \widetilde{\mathcal{G}}_A(u,v) \tag{3.3.69}$$

In the free theory all these $\mathcal{G}$ have the same bilinear contribution. Using the property of conformal blocks [147]:

$$G^{\Delta_{ij},\Delta_{kl}}_{\tau,s}(u,v) = v^{\frac{\Delta_{ij}-\Delta_{kl}}{2}} G^{-\Delta_{ij},-\Delta_{kl}}_{\tau,s}(u,v)\,, \tag{3.3.70}$$

and the fact that the same even spin bilinear operators appear in the relevant OPEs, it follows that to all orders in ϵ:

$$v^{\Delta_\mathcal{O}}\, \mathcal{G}_A(u,v)\big|_{bil.} = v^{\Delta_A} \mathcal{G}'_A(u,v)\big|_{bil.}\,. \tag{3.3.71}$$

[12] Note that an expansion in inverse powers of the conformal spin $J^2_{0,\ell} = \ell(\ell-1)$ makes no sense at $\ell = 1$.

Since the external dimensions are not identical, there are extra contributions to the conformal blocks. In $\mathcal{G}_A(u,v)$ this affects the blocks as:

$$G_{\tau,\ell}^{\Delta_{AS},\Delta_{SA}}(u,v) = G_{\tau_0,\ell}^{0,0}(u,v) + \epsilon\,\widetilde{G}_{\tau_0,\ell}(u,v) + \epsilon\,G'_{\tau_0,\ell}(u,v) + \dots\,, \tag{3.3.72}$$

where $\widetilde{G}_{\tau_0,\ell}$ is the change due to the fact that the possible corrections to the dimensions $\Delta_{AS} \equiv \Delta_A - \Delta_{\mathcal{O}}$ and $\Delta_{SA} = -\Delta_{AS}$, and where $G'_{\tau_0,\ell}(u,v)$ captures the changes due to a correction to the spacetime dimension, as well as the changes due to possibly non-zero anomalous dimensions of the intermediate operators. Similarly there are changes to the blocks in $\mathcal{G}'_A(u,v)$:

$$G_{\tau,\ell}^{\Delta_{SA},\Delta_{AS}}(u,v) = G_{\tau_0,\ell}^{0,0}(u,v) - \epsilon\,\widetilde{G}_{\tau_0,\ell}(u,v) + \epsilon\,G'_{\tau_0,\ell}(u,v) + \dots\,. \tag{3.3.73}$$

Expanding equation (3.3.71) in ϵ and matching the order ϵ part, we find that

$$\sum_\ell a_{SA,\ell}^{(0)}\widetilde{G}_{0,\ell}(u,v) = \frac{\gamma_A^{(1)} - \gamma_{\mathcal{O}}^{(1)}}{2}\log v\, H_{0,A}^{(0)}(u,v)\,, \tag{3.3.74}$$

where $H_{0,A}^{(0)}(u,v) = \sum_\ell a_{SA,\ell}^{(0)} G_{0,\ell}^{0,0}(u,v)$ is the tree-level part of both $\mathcal{G}_A(u,v)\big|_{bil.}$ and $\mathcal{G}'_A(u,v)\big|_{bil.}$. This identity shall prove useful later.

Crossing for $\langle\mathcal{O}\mathcal{O}^A\mathcal{O}\mathcal{O}^A\rangle$

The crossing relation for $\widetilde{\mathcal{G}}_A(u,v)$ maps it onto itself:

$$v^{\frac{\Delta_{\mathcal{O}}+\Delta_A}{2}}\widetilde{\mathcal{G}}_A(u,v) = u^{\frac{\Delta_{\mathcal{O}}+\Delta_A}{2}}\widetilde{\mathcal{G}}_A(v,u) \tag{3.3.75}$$

This makes the analysis slightly simpler; however the conformal blocks have unequal external dimensions. The relevant conformal blocks are

$$G_{\tau,\ell}^{\Delta_{SA},\Delta_{SA}}(u,v) = G_{\tau,\ell}^{0,0}(u,v) + \epsilon\,\widetilde{F}_{0,\ell}(u,v) + \dots\,, \tag{3.3.76}$$

where $\widetilde{F}$ is the correction due to the fact that Δ_{SA} may acquire an anomalous dimension at order ϵ. However the identity (3.3.70) implies that $G_{\tau,\ell}^{\Delta_{SA},\Delta_{SA}}(u,v) = G_{\tau,\ell}^{-\Delta_{SA},-\Delta_{SA}}(u,v)$, so that the order ϵ part of the correction has to vanish, i.e. $\widetilde{F} = 0$.

We may therefore ignore the added subtleties of different external dimensions, and simply get the first order crossing equation

$$\frac{\gamma_{\mathcal{O}}^{(1)}+\gamma_A^{(1)}}{2}\log v\,\widetilde{\mathcal{G}}_A^{(0)}(u,v)+\widetilde{\mathcal{G}}_A^{(1)}(u,v)=\frac{\gamma_{\mathcal{O}}^{(1)}+\gamma_A^{(1)}}{2}\log u\,\widetilde{\mathcal{G}}_A^{(0)}(u,v)+\frac{u}{v}\,\widetilde{\mathcal{G}}_A^{(1)}(v,u)\,. \tag{3.3.77}$$

From the standard argument of being free of $\log^2 v$, we find that the bilinear anomalous dimensions are all constant.[13] Projecting onto the $u^0v^{-1}\log v$ part of the equation to isolate the bilinear operators, we find that

$$\begin{aligned}\frac{\gamma_A^{(1)}+\gamma_{\mathcal{O}}^{(1)}}{2}\;\widetilde{\mathcal{G}}_A^{(0)}(u,v)\Big|_{u^0v^{-1}}+\sum_\ell a_{SA\ell}^{(0)}\widehat{\alpha}_{SA\ell}^{(1)}G_{0,\ell}^{0,0}(u,v)\Big|_{u^0v^{-1}\log v}&\\ =\frac{u}{v}\frac{\gamma_{A,0,2}^{(1)}}{2}\;\widetilde{\mathcal{G}}_A^{(0)}(v,u)\Big|_{v^0u^{-1}}\,.&\end{aligned} \tag{3.3.78}$$

Crossing for $\langle\mathcal{O}^A\mathcal{O}\mathcal{O}\mathcal{O}^A\rangle$

The order ϵ part of the crossing equation relating $\mathcal{G}_A$ and $\mathcal{G}_S$ reads:

$$v\Big(\gamma_{\mathcal{O}}^{(1)}\log v\,\mathcal{G}_A^{(0)}(u,v)+\mathcal{G}_A^{(1)}(u,v)\Big)=u\Big(\tfrac{1}{2}\big(\gamma_{\mathcal{O}}^{(1)}+\gamma_A^{(1)}\big)\log u\,\mathcal{G}_S^{(0)}(v,u)+\mathcal{G}_S^{(1)}(v,u)\Big). \tag{3.3.79}$$

Let us analyze it by taking different parts of this equation.

$\log(v)$ term We take the $\log v$ term in equation (3.3.79) and use the fact that $\gamma_{S,0,\ell}^{(1)}=0$ to conclude that $\mathcal{G}_S^{(1)}(v,u)\Big|_{v^0\log v}=0$. Projecting out the bilinear part of the crossing equation again, we see that

$$\gamma_{\mathcal{O}}^{(1)}\mathcal{G}_A^{(0)}(u,v)\Big|_{u^0v^{-1}}+\mathcal{G}_A^{(1)}(u,v)\Big|_{u^0v^{-1}\log v}=0\,. \tag{3.3.80}$$

[13] Harmonic number terms can be excluded by the crossing equation for $\langle\mathcal{O}^A\mathcal{O}\mathcal{O}\mathcal{O}^A\rangle$, since they would imply the presence of harmonic number terms in the singlet operators, which is not the case in the Gross-Neveu model.

We find this term using equation (3.3.74):

$$\begin{aligned}\mathcal{G}_A^{(1)}(u,v)\Big|_{u^0v^{-1}\log v} &= \sum_\ell a_{SA\ell}^{(0)}\widetilde{G}_{0,\ell}(u,v)\Big|_{u^0v^{-1}\log v} + \sum_\ell a_{SA\ell}^{(0)}\widehat{\alpha}_{SA\ell}^{(1)}G_{0,\ell}^{0,0}(u,v)\Big|_{u^0v^{-1}\log v} \\ &= \frac{\gamma_A^{(1)}-\gamma_{\mathcal{O}}^{(1)}}{2}\,\mathcal{G}_A^{(0)}(u,v)\Big|_{u^0v^{-1}} + \sum_\ell a_{SA\ell}^{(0)}\widehat{\alpha}_{SA\ell}^{(1)}G_{0,\ell}^{0,0}(u,v)\Big|_{u^0v^{-1}\log v}\,.\end{aligned} \tag{3.3.81}$$

Putting this in the crossing equation (3.3.80), we find that

$$\frac{\gamma_A^{(1)}+\gamma_{\mathcal{O}}^{(1)}}{2}\,\mathcal{G}_A^{(0)}(u,v)\Big|_{u^0v^{-1}} + \sum_\ell a_{SA\ell}^{(0)}\widehat{\alpha}_{SA\ell}^{(1)}G_{0,\ell}^{0,0}(u,v)\Big|_{u^0v^{-1}\log v} = 0\,. \tag{3.3.82}$$

Comparing equations (3.3.78) and (3.3.82), and using the fact that $\mathcal{G}_A^{(0)}(u,v)$ and $\widetilde{\mathcal{G}}_A^{(0)}(u,v)$ have the same bilinear part, we conclude that

$$\boxed{\gamma_{A,0,\ell}^{(1)} = 0\,, \quad \ell \geqslant 2 \text{ even.}} \tag{3.3.83}$$

Specifically note that then $\alpha_{SA\ell}^{(1)} = \widehat{\alpha}_{SA\ell}^{(1)}$. Furthermore, matching terms of the form $\frac{u^0}{v}$ in equation (3.3.82), we find that

$$\boxed{\alpha_{SA\ell}^{(1)} = 2\lambda_{SA\ell}^{(1)} = \left(\gamma_{\mathcal{O}}^{(1)}+\gamma_A^{(1)}\right)S_1(\ell-1)+k_{SA}\,,} \tag{3.3.84}$$

for some constant k_{SA}.

$\log(u)$ term Let us take the $\log u$ term in equation (3.3.79). We use the fact that $\gamma_{A,0,\ell}^{(1)} = 0$ to conclude that $\mathcal{G}_A^{(1)}(u,v)\Big|_{u^0\log u} = 0$, so that

$$\frac{\gamma_{\mathcal{O}}^{(1)}+\gamma_A^{(1)}}{2}\mathcal{G}_S^{(0)}(v,u)\Big|_{u^{-1}} + \mathcal{G}_S^{(1)}(v,u)\Big|_{u^{-1}\log u} = 0\,. \tag{3.3.85}$$

Recall that the exchanged currents in $\mathcal{G}_S$ are singlets, which have trivial first-order anomalous dimensions, and which produce conformal blocks with equal external dimensions, so that

$$\mathcal{G}_S^{(1)}(v,u)\Big|_{v^0u^{-1}\log u} = \sum_\ell c_{SS\ell}^{(0)}c_{AA\ell}^{(0)}\left(\lambda_{SS\ell}^{(1)}+\lambda_{AA\ell}^{(1)}\right)G_{0,\ell}^{0,0}(v,u)\Big|_{v^0u^{-1}\log u}\,. \tag{3.3.86}$$

The standard analysis then implies that

$$\lambda^{(1)}_{SS\ell} + \lambda^{(1)}_{AA\ell} = \left(\gamma^{(1)}_{\mathcal{O}} + \gamma^{(1)}_{A}\right) S_1(\ell - 1) + k_S + k_A \,, \tag{3.3.87}$$

for some constant k_A. From this we deduce that

$$\boxed{\lambda^{(1)}_{AA\ell} = \gamma^{(1)}_{A} S_1(\ell - 1) + k_A \,.} \tag{3.3.88}$$

3.3.1.5 Bilinear finite support solutions

We would like to discount the possibility of finite support solutions for the bilinear $\gamma^{(1)}_{S,0,\ell}$. From the analysis in section 3.3.1.2, it is clear that in the singlet correlator $\mathcal{G}^{(1)}(u,v)$, there are no terms of the form $\frac{\log u \log v}{v^k}$ with $k \geq 0$, since there are no $\log J$ terms in the expansion of any anomalous dimensions.

By considering crossing for the mixed correlator $\langle \mathcal{O}\mathcal{O}\mathcal{O}^A\mathcal{O}^A \rangle$, which relates CFT data for even spin singlet and adjoint operators, the same conclusions hold for the even spin adjoint operators: there are no $\log J$ terms in the expansions of bilinear anomalous dimensions.

For the odd spin adjoint operators, we need to consider the crossing equation for the correlator $\langle \mathcal{O}^A\mathcal{O}^A\mathcal{O}^A\mathcal{O}^A \rangle$, which is

$$\mathcal{G}^{(1)}_i(u,v)\Big|_{\log u \log v} = \frac{u}{v} \sum_{j=1}^{6} \beta_{ij} \mathcal{G}^{(1)}_j(v,u)\Big|_{\log u \log v} \qquad i = 1, \ldots, 6 \,. \tag{3.3.89}$$

From the above: $\mathcal{G}^{(1)}_1(u,v)\Big|_{u^0 \log u \log v} = \mathcal{G}^{(1)}_2(u,v)\Big|_{u^0 \log u \log v} = 0$. Therefore we find that

$$\mathcal{G}^{(1)}_3(u,v)\Big|_{u^0 \log u \log v} = \frac{u}{v} \sum_{j=3}^{6} \beta_{3j}\, \mathcal{G}^{(1)}_j(v,u)\Big|_{u^{-1} \log u \log v} \tag{3.3.90}$$

$$\mathcal{G}^{(1)}_i(u,v)\Big|_{u^0 \log u \log v} = \frac{u}{v} \sum_{j=3}^{6} \beta_{ij}\, \mathcal{G}^{(1)}_j(v,u)\Big|_{u^{-1} \log u \log v} = 0 \qquad \text{for } i \neq 3 \,. \tag{3.3.91}$$

We may view the last equation as a set of five linear constraints on four functions $\mathcal{G}^{(1)}_j(u,v)\Big|_{v^{-1} \log u \log v}$. Generally we expect this to have no non-trivial solutions, and

indeed we find that the β_{ij} are such that all $\mathcal{G}_j^{(1)}(u,v)\Big|_{v^{-1}\log u\log v}$ vanish. Hence we conclude that

$$\mathcal{G}_3^{(1)}(u,v)\Big|_{u^0\log u\log v}=0\,, \tag{3.3.92}$$

so that there are no finite support solutions, nor any $\log J$ terms in the expansion of the anomalous dimensions of any of the bilinear currents.

3.3.2 Second order

Once again we need to take into account the dimensional shift due to the theory living in $d=2+\epsilon$. Let us consider a fixed TCB of twist τ_0 and expand the various contributions to it. For ease of notation, we shall omit the fact that all functions and derivatives are to be evaluated at $d=d_0=2$ and $\tau=\tau_0$. Consider the free theory TCB in $d=d_0=2$:

$$H_{\tau_0}^{(0)}(u,v)=\sum_l a_{\tau_0,\ell}^{(0)}G_{\tau_0,\ell}^{(d_0)}(u,v)\,. \tag{3.3.93}$$

In the Gross-Neveu model in $d=2+\epsilon$ there are corrections:

$$a_{\tau_0,\ell}^{(0)}\to a_{\tau_0,\ell}^{(0)}\left(1+\epsilon\,\widetilde{\alpha}_{\tau_0,\ell}^{(1)}+\epsilon\,\alpha_{\tau_0,\ell}^{(1)}+\epsilon^2\widetilde{\alpha}_{\tau_0,\ell}^{(2)}+\epsilon^2\alpha_{\tau_0,\ell}^{(2)}+\dots\right)\,, \tag{3.3.94}$$

$$\begin{aligned}G_{\tau_0,\ell}^{(d_0)}(u,v)\to\Bigg[&1+\epsilon\partial_d+\epsilon\left(\gamma_{\tau_0,\ell}^{(1)}+\zeta\right)\partial_\tau+\epsilon^2\gamma_{\tau_0,\ell}^{(2)}\partial_\tau+\\&+\frac{1}{2}\epsilon^2\left(\left(\gamma_{\tau_0,\ell}^{(1)}+\zeta\right)^2\partial_\tau^2+\left(\gamma_{\tau_0,\ell}^{(1)}+\zeta\right)\partial_\tau\partial_d+\partial_d^2\right)+\dots\Bigg]G_{\tau,\ell}^{(d)}(u,v)\,,\end{aligned} \tag{3.3.95}$$

where $\widetilde{\alpha}_{\tau_0,\ell}^{(1)},\widetilde{\alpha}_{\tau_0,\ell}^{(2)}$ are corrections to the free theory due to the dimensional shift, and $\alpha_{\tau_0,\ell}^{(1)},\alpha_{\tau_0,\ell}^{(2)}$ are corrections due to the departure from the free theory in $d=2+\epsilon$. Furthermore $\zeta=\frac{\partial\tau_0}{\partial d}$ is the explicit dimensional dependence of the twists in the free theory, i.e. $\zeta=1$ for the bilinear currents and $\zeta=2$ for the quadrilinear operators. Gathering the terms in equation (3.3.93) we find to order ϵ the combination with which we are familiar:

$$\underbrace{\sum_\ell a_{\tau_0,\ell}^{(0)}\left(\widetilde{\alpha}_{\tau_0,\ell}^{(1)}+\zeta\partial_\tau+\partial_d\right)G_{\tau,\ell}^{(d)}(u,v)}_{\text{Free theory correction }\widetilde{\mathcal{G}}^{(1)}(u,v)}+\underbrace{\sum_\ell a_{\tau_0,\ell}^{(0)}\left(\alpha_{\tau_0,\ell}^{(1)}+\gamma_{\tau_0,\ell}^{(1)}\partial_\tau\right)G_{\tau,\ell}^{(d)}(u,v)}_{\text{Gross-Neveu data }\mathcal{G}^{(1)}(u,v)}\,. \tag{3.3.96}$$

To order ϵ^2 we find the following correction:

$$\underbrace{\sum_\ell a^{(0)}_{\tau_0,\ell}\left[\widetilde{\alpha}^{(2)}_{\tau_0,\ell}+\widetilde{\alpha}^{(1)}_{\tau_0,\ell}(\zeta\partial_\tau+\partial_d)+\frac{1}{2}\left(\zeta^2\partial_\tau^2+2\zeta\partial_\tau\partial_d+\partial_d^2\right)\right]G^{(d)}_{\tau,\ell}(u,v)}_{\text{Free theory correction }\widetilde{\mathcal{G}}^{(2)}(u,v)}$$
$$+\underbrace{\sum_\ell a^{(0)}_{\tau_0,\ell}\left[\alpha^{(2)}_{\tau_0,\ell}+\gamma^{(2)}_{\tau_0,\ell}\partial_\tau+\alpha^{(1)}_{\tau_0,\ell}\gamma^{(1)}_{\tau_0,\ell}\partial_\tau+\frac{1}{2}\left(\gamma^{(1)}_{\tau_0,\ell}\right)^2\partial_\tau^2\right]G^{(d)}_{\tau,\ell}(u,v)}_{\text{Gross-Neveu data }\mathcal{G}^{(2)}(u,v)}$$
$$+\underbrace{\sum_\ell a^{(0)}_{\tau_0,\ell}\left[\alpha^{(1)}_{\tau_0,\ell}(\zeta\partial_\tau+\partial_d)+\gamma^{(1)}_{\tau_0,\ell}\left(\widetilde{\alpha}^{(1)}_{\tau_0,\ell}+\partial_d+\zeta\partial_\tau\right)\partial_\tau\right]G^{(d)}_{\tau,\ell}(u,v)}_{\text{Cross-term }C^{(2)}(u,v)}\,. \tag{3.3.97}$$

The free theory correction $\widetilde{\mathcal{G}}^{(2)}(u,v)$ can again be calculated by expanding the free theory correlator in $2+\epsilon$ dimensions. The novelty at this order is the appearance of a cross term $C^{(2)}(u,v)$ that combines first-order corrections to the free theory, and first-order departures from the free theory.

Expanding the full crossing relation for the singlets to second order, we find after much simplification the relation

$$\gamma^{(2)}_{\mathcal{O}}\mathcal{G}^{(0)}(u,v)\log v+\left(\gamma^{(1)}_{\mathcal{O}}+1\right)\mathcal{G}^{(1)}(u,v)\log v+\gamma^{(1)}_{\mathcal{O}}\widetilde{\mathcal{G}}^{(1)}(u,v)\log v+$$
$$+\gamma^{(1)}_{\mathcal{O}}\left(\frac{1}{2}\gamma^{(1)}_{\mathcal{O}}+1\right)\mathcal{G}^{(0)}(u,v)\log^2 v+\mathcal{G}^{(2)}(u,v)+C^{(2)}(u,v)=\frac{u}{v}(u\leftrightarrow v)\,. \tag{3.3.98}$$

Let us consider the $u^0\log u\log^2 v$ terms on both sides. Firstly, we shall rule out any such terms appearing in the cross-term. On the left-hand side of equation (3.3.98) the relevant term would be

$$C^{(2)}(u,v)\big|_{\log u}=\sum_\ell a^{(0)}_{\tau_0,\ell}\alpha^{(1)}_{\tau_0,\ell}\partial_\tau G^{(d_0)}_{\tau,\ell}(u,v)\Big|_{\log u}=\frac{1}{2}\sum_\ell a^{(0)}_{\tau_0,\ell}\alpha^{(1)}_{\tau_0,\ell}G^{(d_0)}_{\tau,\ell}(u,v)\,, \tag{3.3.99}$$

where we used the fact that the bilinear currents satisfy $\gamma^{(1)}_{S,0,\ell}=0$. Recall that $\alpha^{(1)}_{\tau_0,\ell}$ has been specifically constructed so that this sum is free of $\log^2 v$ divergences, so that $C^{(2)}(u,v)$ does not contain a $\log u\log^2 v$ term. On the right-hand side, the relevant

term is due to the quadrilinear operators:

$$C^{(2)}(v,u)\big|_{\log u\log^2 v} = \sum_{n,\ell}\langle\!\langle a^{(0)}_{n,\ell}\rangle\!\rangle\langle\gamma^{(1)}_{n,\ell}\rangle\partial_\tau^2 G^{(d_0)}_{\tau,\ell}(v,u)\Big|_{\log u\log^2 v}$$
$$= \sum_{n,\ell}\frac{1}{4}\langle\!\langle a^{(0)}_{n,\ell}\rangle\!\rangle\langle\gamma^{(1)}_{n,\ell}\rangle G^{(d_0)}_{\tau,\ell}(v,u)\Big|_{\log u}\,. \tag{3.3.100}$$

However recall that also the $\langle\gamma^{(1)}_{n,\ell}\rangle$ are precisely of the form guaranteeing that this sum is free of $\log u$ terms, so that we can ignore the cross-term in this analysis.

Taking the $u^0\log u\log^2 v$ term in equation (3.3.98), we then find the constraint

$$\mathcal{G}^{(2)}(u,v)\big|_{u^0\log u\log^2 v} = \frac{u}{v}\,\mathcal{G}^{(2)}(v,u)\big|_{\frac{\log u\log^2 v}{u}} \tag{3.3.101}$$

On the left-hand side, $\mathcal{G}^{(2)}(u,v)$ may contain a $u^0\log u\log^2 v$ term, generated by $\gamma^{(2)}_{S,0,\ell}$, if its expansion in terms of J^{-2} is not constant:

$$\mathcal{G}^{(2)}(u,v)\big|_{\log u\log^2 v} = \sum_\rho B^{(2)}_{0,\rho}H^{(\rho)}_0(u,v)\,, \qquad \gamma^{(2)}_{S,0,\ell} = 2\sum_\rho B^{(2)}_{0,\rho}J^{-2\rho}_{0,\ell}\,. \tag{3.3.102}$$

On the right-hand side the contribution must be from the quadrilinear operators

$$\frac{u}{v}\,\mathcal{G}^{(2)}(v,u)\big|_{\frac{\log u\log^2 v}{u}} = \frac{u}{v}\sum_{n,\ell}\frac{1}{8}\langle\!\langle a^{(0)}_{n,\ell}\rangle\!\rangle\langle(\gamma^{(1)}_{n,\ell})^2\rangle G_{\tau_0(n),\ell}(v,u)\Big|_{\frac{\log u\log^2 v}{u}}\,. \tag{3.3.103}$$

Note that we had previously found $\langle\gamma^{(1)}_{n,\ell}\rangle$; however this does not determine $\langle(\gamma^{(1)}_{n,\ell})^2\rangle$ due to the degeneracy. Nevertheless, we can expand the CFT data in terms of the conformal spin

$$\frac{1}{8}\sum_i\langle\!\langle a^{(0)}_{n,\ell}\rangle\!\rangle\langle(\gamma^{(1)}_{n,\ell})^2\rangle = \sum_\rho C^{(\rho)}_{2n+2}J^{-2\rho}_{2n+2,\ell}, \tag{3.3.104}$$

to get in equation (3.3.103) a contribution

$$\frac{u}{v}\,\mathcal{G}^{(2)}(v,u)\big|_{\frac{\log u\log^2 v}{u}} = \frac{u}{v}\sum_{n,\rho}C^{(\rho)}_{2n+2}H^{(\rho)}_{2n+2}(v,u)\Big|_{\frac{\log u}{u}}\,. \tag{3.3.105}$$

This term in fact vanishes, which follows from the fact that none of the $H^{(\rho)}_{2n+2}(v,u)$ have $\frac{\log u}{u}$ divergences. Hence it follows that there is no such term and $\gamma^{(2)}_{S,0,\ell}$ is constant. Imposing stress-energy tensor conservation then implies that

$$\boxed{\gamma^{(2)}_{S,0,\ell} = 0\,.} \tag{3.3.106}$$

3.3.2.1 Adjoints

The above analysis applies in the same way to the four-point function of adjoints, and hence we find that the adjoint anomalous dimensions are constant. Assuming analyticity down to spin $\ell = 2$, there may be a finite support solution for $\gamma^{(1)}_{A,0,1}$, so that our ansatz for the adjoint anomalous dimensions is:

$$\begin{aligned} \gamma^{(2)}_{A,0,\ell} &= \gamma^{(2)}_{A,0,2} && \ell \geqslant 2 \text{ even}, \\ \gamma^{(2)}_{A,0,\ell} &= \gamma^{(2)}_{A,0,3}\,(1-\delta_{\ell,1}) && \ell \geqslant 1 \text{ odd}\,. \end{aligned} \tag{3.3.107}$$

These results match the known results in [156].

3.4 The Gross-Neveu-Yukawa model in $d = 4 - \epsilon$

The Gross-Neveu-Yukawa model is a CFT in $d = 4 - \epsilon$ dimensions providing a perturbation of the free fermion theory. It has the following action

$$S = \int \mathrm{d}^{4-\epsilon}x \left(\overline{\psi}^i \not{\partial} \psi_i + \frac{1}{2}\left(\partial\sigma\right)^2 + g_1 \overline{\psi}^i \psi_i \sigma + g_2 \sigma^4 \right), \tag{3.4.1}$$

where $\psi, \overline{\psi}$ are conjugate Dirac fermions and σ is a scalar field. The theory is conformal for a specific value of the pair (g_1, g_2), satisfying $g_1 \sim \sqrt{\epsilon}$ and $g_2 \sim \epsilon$, so that at $\epsilon = 0$ it reduces to a 4-dimensional theory of a decoupled free boson and N_f free fermions.

Our results are for the first order anomalous dimensions of the bilinear currents. For the adjoint bilinear currents, these are:

$$\gamma^{(1)}_{A,2,\ell} = 2\gamma^{(1)}_{\psi}\left(1 - \frac{2}{\ell(\ell+1)}\right), \tag{3.4.2}$$

for both odd and even spin ℓ.

The singlet bilinear currents $J_{\psi,\ell} \sim \overline{\psi}\gamma\partial^{\ell-1}\psi$ mix with the currents $J_{\sigma,\ell} \sim \sigma\partial^{\ell}\sigma$. The anomalous dimensions of the resulting primary operators are, for even spin ℓ,

$$\gamma^{(1)}_{\pm,\ell} = 2\gamma^{(1)}_{\psi}\left(\frac{N+1}{2} - \frac{1}{\ell(\ell+1)} \pm \frac{\sqrt{4 + (-4+20N)\ell(\ell+1) + (N-1)^2\ell^2(\ell+1)^2}}{2\ell(\ell+1)}\right), \tag{3.4.3}$$

which were found as the eigenvalues of the following matrix

$$H = 2\gamma_\psi^{(1)} \begin{pmatrix} 1 - \frac{2}{\ell(\ell+1)} & \pm\frac{2\sqrt{N}}{\sqrt{\ell(\ell+1)}} \\ \pm\frac{2\sqrt{N}}{\sqrt{\ell(\ell+1)}} & N \end{pmatrix} . \tag{3.4.4}$$

This reproduces the known literature results [156].

3.4.1 Crossing analysis

Let us first look at the four-point correlator of singlets. As per the discussion in section 3.2, the order ϵ crossing equation reads:

$$\gamma_{\mathcal{O}}^{(1)} \log v \, \mathcal{G}^{(0)}(u,v) + \mathcal{G}^{(1)}(u,v) = \gamma_{\mathcal{O}}^{(1)} \log u \, \mathcal{G}^{(0)}(u,v) + \frac{u^3}{v^3}\mathcal{G}^{(1)}(v,u) \, . \tag{3.4.5}$$

A naive analysis of the power-law divergences in v in equation (3.4.5), as per section 3.2.2, would conclude that

$$\gamma_{\tau_0,\ell}^{(1)} = 2\beta \, , \qquad \widehat{\alpha}_{\tau_0,\ell}^{(1)} = 2\left(\gamma_{\mathcal{O}}^{(1)} - \beta\right) \log J + 2\sum_{m=0}^{\infty} \frac{\widetilde{A}_{\tau_0,m}^{(1)}}{J_{d-2,\ell}^{2m}} \, . \tag{3.4.6}$$

There are now some additional subtleties because the dimension $d_0 = 4$ is even, so that the relevant twist conformal blocks can develop $\log^2 v$ divergences; we find that, up to the addition of a constant, this forces the $\widetilde{A}_{\tau_0,m}^{(1)}$ to be expansion coefficients of the harmonic number $S_1(\ell-1)$ in inverse powers of $J^2 = \ell(\ell-1)$. Stress-energy conservation imposes that $\gamma_{\tau_0,\ell=2}^{(1)} = 0$, so that

$$\gamma_{S,2,\ell}^{(1)} = 0 \, , \qquad \widehat{\alpha}_{S,2,\ell} = 2\gamma_{\mathcal{O}}^{(1)} S_1(\ell) + K \, . \tag{3.4.7}$$

Furthermore the argument in section 3.3.1.4 establishing that the anomalous dimensions of the even singlet and adjoint currents are the same, in facts holds true in any dimension, so that the solution would have the same form as in the Gross-Neveu model.

Coupling

The results above would hold in a CFT which is perturbation of the free fermion theory in which no additional operators appear in the OPE. We are however not aware of any such CFT but will instead be interested in the Gross-Neveu-Yukawa model. The Yukawa interaction in the action (3.4.1) shows that at order ϵ one should expect additional operators to appear in the OPE of $\mathcal{O}\times\mathcal{O}$. This leads to two effects: firstly, the anomalous dimensions and OPE coefficients in the previous section acquire corrections. Secondly, there is a another set of bilinear currents $J_{\sigma,\ell}\sim\sigma\partial^\ell\sigma$ of twist 2 which mix with the bilinear currents $J_{\psi,\ell}$.

Let us therefore do a more conservative analysis and focus on ensuring that the crossing relation contains no terms of the form $\log u\log^m v$ for $m\geqslant 2$. From the free theory result for the twist conformal block $H_2^{(0)}(u,v)$ of the bilinears, we can calculate all $H_2^{(m)}(u,v)$. We find that $H_2^{(m)}(u,v)$ contains a term of the form $h^{(m)}(v)\log^2 v$ for $m\geqslant 2$, with $h^{(m)}(v)\sim v^{m-2}$ for small v and therefore we exclude any terms of the form J^{-2m} with $m\geqslant 2$ in the expansion of $\gamma^{(1)}_{S,2,\ell}$. In fact we claim that the expansion can also not contain any terms of the form $\frac{\log^k J}{J^{2m}}$, fixing it to

$$\gamma^{(1)}_{S,2,\ell}=A\left(1-\frac{B}{\ell(\ell+1)}\right)\,. \tag{3.4.8}$$

To see why this claim holds, consider the case $k=1$. From the above, we see that any terms $\frac{\log J}{J^{2m}}$ with $m\geqslant 2$ will contain divergences of the form $\log^3 v$, and must therefore be discarded, leaving only the possibility of $\log J$ and $\frac{\log J}{J^2}$. To exclude the latter we focus on the $\log u\log v$ part of the crossing equation:

$$\mathcal{G}^{(1)}(u,v)\big|_{\log u\log v}=\frac{u^3}{v^3}\,\mathcal{G}^{(1)}(v,u)\big|_{\log u\log v}\,, \tag{3.4.9}$$

and specifically terms of the form $u^1v^{-2},u^1v^{-1},u^2v^{-2},u^2v^{-1}$. These terms arise only from bilinears with infinite support on the spin, and are taken onto themselves under crossing symmetry. Specifically additional operators appearing in the OPE cannot

give a $\log u \log v$ term, so we may ignore them. An analysis of the precise divergences then shows that no term of the form $\frac{\log J}{J^2}$ can appear.

To show there is no term of the form $\gamma^{(1)}_{S,2,\ell} = 2\beta S_1(\ell) + \dots$, we analyze all terms in the crossing equation of the form $\frac{u^n}{v^m} \log u \log v$, where $m > 0$. These terms can only arise from harmonic number behaviour of anomalous dimensions. Note that the quadrilinear operators of $\tau_0(n) = 6 + 2n$ cannot have a $\log J$ behaviour in their anomalous dimensions, since under crossing this would map to a correction to the identity operator. We therefore make the ansatz

$$\gamma^{(1)}_{S,2,\ell} = 2\beta S_1(\ell) + \dots , \qquad \gamma^{(1)}_{S,6+2n,\ell} = 2\beta_n \frac{S_1(\ell+n+2)}{J^2_{6+2n,\ell}} + \dots . \tag{3.4.10}$$

Matching the $\log u \log v$ part of the crossing equation then yields:

$$\beta H_2^{(0),harm}(u,v)\Big|_{\log v} + \sum_{n=0}^{\infty} \beta_n H^{(1),harm}_{6+2n}(u,v)\Big|_{\log v} \doteq \beta \frac{u^3}{v^3} H_2^{(0),harm}(v,u)\Big|_{\log u} , \tag{3.4.11}$$

where by $\doteq$ we mean that only power-divergent terms in v are matched, and where $H^{(m),harm}_{\tau_0}(u,v)$ is the m-th twist conformal block with a harmonic number insertion:

$$H^{(m),harm}_{\tau_0}(u,v) \equiv \sum_{\ell} a^{(0)}_{\tau_0,\ell} J^{-2m}_{\tau_0,\ell} S_1\left(\ell + \frac{\tau_0}{2} - 1\right) G_{\tau_0,\ell}(u,v) . \tag{3.4.12}$$

We find that this equation has only the trivial solution $\beta = \beta_n = 0$.

We therefore conclude that no $\log J$ terms appear in the expansions of the anomalous dimensions of any of the intermediate operators. Terms with higher powers of $\log J$ are excluded by a similar argument.

Note that the argument relies on the assumption that the CFT data is analytic in the spin down to spin $\ell = 2$, so that finite support solutions for the bilinear operators could be excluded.

3.4.1.1 Finite support solutions

Like for the 2d Gross-Neveu model, we would like to discount the possibility of finite support solutions for the bilinear $\gamma^{(1)}_{A,2,\ell}$. The discussion is isomorphic to that in

section 3.3.1.5, the only difference being that some of the powers of u and v change. Specifically, consideration of terms of the form $u^k \log u \log v$ for $k = 0, 1, 2$ shows that there are no terms of the form $v^{-k} \log u \log v$ for $k = 1, 2, 3$, i.e. there are no enhanced divergences proportional to $\log u \log v$. We therefore conclude that there are no finite support solutions for the anomalous dimensions of the bilinear currents.

3.4.2 Resolving the mixing of singlet currents

In the Gross-Neveu-Yukawa model, there are two sets of singlet operators of twist $\tau = d - 2$: the currents $J_{\psi,\ell}$, and the currents $J_{\sigma,\ell} \sim \sigma\partial^\ell\sigma$. These currents mix due to a coupling between ψ and σ, so that they are no longer eigenstates of the Hamiltonian/dilatation operator. Instead, the eigenstates of the Hamiltonian will be

$$\Sigma_{-,\ell} = A_- J_{\psi,\ell} + B_- J_{\sigma,\ell}\,, \tag{3.4.13}$$

$$\Sigma_{+,\ell} = A_+ J_{\psi,\ell} + B_+ J_{\sigma,\ell}\,, \tag{3.4.14}$$

with the corresponding eigenvalues $\Delta_{\pm,\ell} = d - 2 + \gamma_{\pm,\ell}$ as their scaling dimensions.

Phrased differently: in the basis $\{J_{\psi,\ell}, J_{\sigma,\ell}\}$ the Hamiltonian does not act diagonally. Let us represent it as a matrix in this basis:

$$H = \begin{pmatrix} \langle J_{\psi,\ell}|\hat{H}|J_{\psi,\ell}\rangle & \langle J_{\sigma,\ell}|\hat{H}|J_{\psi,\ell}\rangle \\ \langle J_{\psi,\ell}|\hat{H}|J_{\sigma,\ell}\rangle & \langle J_{\sigma,\ell}|\hat{H}|J_{\sigma,\ell}\rangle \end{pmatrix} = H_0 + \epsilon H_\epsilon + \cdots = H_0 + \epsilon \begin{pmatrix} a_\ell & c_\ell \\ c_\ell & d_\ell \end{pmatrix} + \dots\,, \tag{3.4.15}$$

which is symmetric since the Hamiltonian is self-adjoint. Our goal is to find a_ℓ, c_ℓ, d_ℓ, from which we can deduce the anomalous dimensions $\gamma^{(1)}_{\pm,\ell}$.

Since $\mathcal{O}$ only couples to σ at order ϵ we find that

$$\langle \mathcal{O}\mathcal{O}\Sigma_\pm\rangle = A_\pm \langle \mathcal{O}\mathcal{O}J_\psi\rangle + O(\epsilon)\,, \tag{3.4.16}$$

so that to zeroth order in ϵ, $A_\pm = \langle \mathcal{O}\mathcal{O}\Sigma_\pm\rangle / \langle \mathcal{O}\mathcal{O}J_\psi\rangle$. The $\gamma^{(1)}_{S,2,\ell}$ calculated in section 3.4.1 is actually the average over both eigenstates:

$$\gamma^{(1)}_{S,2,\ell} \langle \mathcal{O}\mathcal{O}J_\psi\rangle^2 = \sum_\pm \langle \mathcal{O}\mathcal{O}\Sigma_\pm\rangle^2 \gamma^{(1)}_{\pm,\ell}\,. \tag{3.4.17}$$

Since the $\Sigma_\pm$ are precisely the vectors that diagonalize the symmetric matrix H_ϵ, the above is simply an entry of H_ϵ

$$\gamma^{(1)}_{S,2,\ell} = \sum_{\pm} A^2_\pm \gamma^{(1)}_{\pm,\ell} = a_\ell \,. \tag{3.4.18}$$

Furthermore, recall that at $\epsilon = 0$, σ is a free boson, so that the analysis from [148] applies. It shows that the first order correction d_ℓ to $\langle J_{\sigma,\ell}|\hat{H}|J_{\sigma,\ell}\rangle$ satisfies $d_\ell = 2\gamma^{(1)}_\sigma$. Then H_ϵ is of the form

$$H_\epsilon = \begin{pmatrix} A\left(1 - \frac{B}{\ell(\ell+1)}\right) & C_\ell \\ C_\ell & 2\gamma^{(1)}_\sigma \end{pmatrix}. \tag{3.4.19}$$

Assuming as in section 3.3 that the twist additivity property of [26, 27] extends to fermions, it follows that

$$\lim_{\ell\to\infty} \{\gamma_{-,\ell}, \gamma_{+,\ell}\} = \left\{2\gamma^{(1)}_\psi, 2\gamma^{(1)}_\sigma\right\} . \tag{3.4.20}$$

Assuming that $\gamma^{(1)}_\psi \neq \gamma^{(1)}_\sigma$, this imposes that $A = 2\gamma^{(1)}_\psi$, and that $\lim_{\ell\to\infty} C_\ell = 0$, from which it follows that

$$H_\epsilon = \begin{pmatrix} 2\gamma^{(1)}_\psi\left(1 - \frac{B}{\ell(\ell+1)}\right) & C_\ell \\ C_\ell & 2\gamma^{(1)}_\sigma \end{pmatrix} \tag{3.4.21}$$

with $\lim_{\ell\to\infty} C_\ell = 0$.

In order to proceed we need to find the off-diagonal terms $C_\ell = \langle J_{\psi,\ell}|H|J_{\sigma,\ell}\rangle$. This naturally leads us to consider the mixed correlator $\langle\mathcal{O}\mathcal{O}\sigma\sigma\rangle$, which maps to $\langle\sigma\mathcal{O}\mathcal{O}\sigma\rangle$ under crossing:

$$\mathcal{G}_{\sigma\mathcal{O}\mathcal{O}\sigma}(u,v) = \frac{u^2}{v^3}\mathcal{G}_{\mathcal{O}\mathcal{O}\sigma\sigma}(u,v) \,. \tag{3.4.22}$$

Using Wick contractions, we can calculate the free theory results:

$$\mathcal{G}^{(0)}_{\mathcal{O}\mathcal{O}\sigma\sigma}(u,v) = 1 \,, \qquad \mathcal{G}^{(0)}_{\sigma\mathcal{O}\mathcal{O}\sigma}(u,v) = \frac{u^2}{v^3} \,. \tag{3.4.23}$$

We will also need the free theory result for the related correlator $\langle\mathcal{O}\sigma\mathcal{O}\sigma\rangle$, which satisfies

$$\mathcal{G}^{(0)}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v) = u^2 \,, \qquad \mathcal{G}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v) = \frac{u^2}{v^2}\mathcal{G}_{\mathcal{O}\sigma\mathcal{O}\sigma}(v,u) \,. \tag{3.4.24}$$

In the s-channel expansion of $\langle \mathcal{O}\mathcal{O}\sigma\sigma\rangle$, we simply see the contribution of the identity. In particular, we see no contribution from the bilinear currents $J_{\psi,\ell} \sim \overline{\psi}\gamma\partial^{\ell-1}\psi$ and $J_{\sigma,\ell} \sim \sigma\partial^{\ell}\sigma$. This is expected, since at tree-level these currents only couple to one of $\mathcal{O}$ and σ, and thus cannot function as intermediate states. At tree-level the operators $J_{\psi,\ell}$ and $J_{\sigma,\ell}$ have the same scaling dimension, so the Hamiltonian simply acts as a multiple of the identity in the space spanned by them. Therefore, with the benefit of foresight, let us define the rotated states $\Sigma_{\pm,\ell}$ by

$$\Sigma_{-,\ell} = \cos\theta_\ell \, J_{\psi,\ell} - \sin\theta_\ell \, J_{\sigma,\ell} \,, \tag{3.4.25}$$

$$\Sigma_{+,\ell} = \sin\theta_\ell \, J_{\psi,\ell} + \cos\theta_\ell \, J_{\sigma,\ell} \,, \tag{3.4.26}$$

and note that the Hamiltonian still acts diagonally on these states. As ϵ turns on we choose the θ_ℓ so that these are the eigenstates of $\hat{H}$. Note the (potential) ℓ-dependence of the angle θ_ℓ.[14]

We may view these as two different intermediate states propagating in the direct channel, which give cancelling contributions of

$$\pm \, \tilde{H}_2^{(0)}(u,v) = \pm \sum_{\ell\in 2\mathbb{N}} \sin\theta_\ell \cos\theta_\ell c^{(0)}_{\mathcal{O}\mathcal{O}J_{\psi,\ell}} c^{(0)}_{\sigma\sigma J_{\sigma,\ell}} G_{2,\ell}(u,v) \tag{3.4.27}$$

to the tree-level result.

In the interacting theory there will be some θ_ℓ for which $\Sigma_{\pm,\ell}$ are the eigenstates of the Hamiltonian. These states may acquire different anomalous dimensions, so that the contribution above will generally change to include an order ϵ term

$$\sum_{\ell\in 2\mathbb{N}} \frac{1}{2} \sin\theta_\ell \cos\theta_\ell \left(\gamma^{(1)}_{+,\ell} - \gamma^{(1)}_{-,\ell}\right) c^{(0)}_{\mathcal{O}\mathcal{O}J_{\psi,\ell}} c^{(0)}_{\sigma\sigma J_{\sigma,\ell}} \partial_\tau\big|_{\tau=2} G_{\tau,\ell}(u,v) \,. \tag{3.4.28}$$

We would like to consider this sum in the language of twist conformal blocks; hence it would be useful to calculate

$$H_2^{(0)}(u,v) \equiv \sum_{\ell\in 2\mathbb{N}} c^{(0)}_{\mathcal{O}\mathcal{O}J_{\psi,\ell}} c^{(0)}_{\sigma\sigma J_{\sigma,\ell}} G_{2,\ell}(u,v) \,. \tag{3.4.29}$$

[14] We assume here that these bases of states can be related by a rotation. There is also the possibility of the transformation being a reflection; however this will not affect the discussion in this section.

An interesting point to note here is that the OPE coefficients $c^{(0)}_{\mathcal{O}\mathcal{O}J_{\psi,\ell}}$ and $c^{(0)}_{\sigma\sigma J_{\sigma,\ell}}$ in the free fermion and boson theories are related:

$$c^{(0)}_{\mathcal{O}\mathcal{O}J_{\psi,\ell}} = \sqrt{\ell(\ell+1)}c^{(0)}_{\sigma\sigma J_{\sigma,\ell}} \tag{3.4.30}$$

which is the reason why in the free fermion theory correlator $\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$, the twist conformal block $H^{(0)}_{2,F}$ is equal to the $H^{(-1)}_{2,B}$ of the free boson correlator $\langle\sigma\sigma\sigma\sigma\rangle$. Phrased differently, the free fermion TCB is simply the result of applying the Casimir operator $\mathcal{C}$ to the free boson TCB. Therefore equation (3.4.30) implies that

$$H^{(0)}_2(u,v) = H^{\left(-\frac{1}{2}\right)}_{2,B}(u,v) = H^{\left(\frac{1}{2}\right)}_{2,F}(u,v)\,, \tag{3.4.31}$$

where $H^{(m)}_{2,B}$ and $H^{(m)}_{2,F}$ are the free boson, respectively free fermion, twist conformal blocks of twist 2.

Using the method in [88], we have been able to evaluate the small u limit:[15]

$$H^{(0)}_2(u,v) = \frac{\pi}{4}\frac{u}{v^{3/2}}\left(1 - v - \frac{v^2}{2} - \frac{7v^3}{18} - \frac{203v^4}{600} + \dots\right) + \mathcal{O}(u^2)\,. \tag{3.4.32}$$

Let us investigate the implications of this. The twist conformal blocks have the following behaviour:

$$H^{(\rho)}_2(u,v) \equiv \sum_{\ell\in 2\mathbb{N}} J^{-2\rho}_{2,\ell} c^{(0)}_{\mathcal{O}\mathcal{O}J_{\psi,\ell}} c^{(0)}_{\sigma\sigma J_{\sigma,\ell}} G_{2,\ell}(u,v) \sim \frac{u}{v^{\frac{3}{2}-\rho}}(1+O(v)) + O(u^2)\,. \tag{3.4.33}$$

Expanding, as usual:

$$\frac{1}{2}\sin\theta_\ell\cos\theta_\ell\left(\gamma^{(1)}_{+,\ell} - \gamma^{(1)}_{-,\ell}\right) = \sum_\rho B_\rho J^{-2\rho}_{2,\ell}\,, \tag{3.4.34}$$

we are interested in the sum

$$\sum_\rho B_\rho H^{(\rho)}_2(u,v) \subset \mathcal{G}^{(1)}_{\mathcal{O}\mathcal{O}\sigma\sigma}(u,v)\,, \tag{3.4.35}$$

[15] We have been able to compute it to a finite order in v. In principle we can calculate to arbitrarily high order in v, but computation time becomes an issue. Regardless, the discussion that follows only relies on the structure of this function, not its precise coefficients.

where the sum over ρ now includes half-integers to allow for the possibility of odd powers of $J_{2,\ell}^{-1}$ in the expansion of $\sin\theta_\ell \cos\theta_\ell \left(\gamma^{(1)}_{+,\ell} - \gamma^{(1)}_{-,\ell}\right)$.

From equation (3.4.31) and the earlier discussion in section 3.4.1, one deduces that $H_2^{(\rho)}(u,v)$ contains $\log^2 v$ terms for $\rho = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$, in such a way that no linear combination of them is free of $\log^2 v$ terms. Thus

$$B_\rho = 0\,, \qquad \text{for } \rho = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots\,. \tag{3.4.36}$$

We shall now show that $B_\rho = 0$ for all $\rho = 0, 1, 2, 3\ldots$ by chasing divergences through many different mixed correlators. We start with the crossing equation

$$\mathcal{G}_{\sigma\mathcal{O}\mathcal{O}\sigma}(u,v) = \frac{u^2}{v^3}\mathcal{G}_{\mathcal{O}\mathcal{O}\sigma\sigma}(v,u) \supset \epsilon \frac{u^2}{v^3} \sum_{\rho=0,\frac{1}{2},1,2,3,\ldots} B_\rho H_2^{(\rho)}(v,u)\log v \tag{3.4.37}$$

and note that

$$\frac{u^2}{v^3} H_2^{(\rho)}(v,u) \sim \frac{u^{\rho+\frac{1}{2}}}{v^2}\,. \tag{3.4.38}$$

Therefore $\mathcal{G}_{\sigma\mathcal{O}\mathcal{O}\sigma}(u,v)$ contains terms of the form $u^{\rho+\frac{1}{2}}v^{-2}\log v$. Then $\mathcal{G}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v)$ will necessarily contain a term of the form $u^{\rho+\frac{1}{2}}v^{-2+k}\log v$ for some integer k.[16] The crossing equation for $\mathcal{G}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v)$ implies that then also

$$\mathcal{G}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v) \supset \frac{u^k}{v^{\frac{3}{2}-\rho}}\log u\,. \tag{3.4.39}$$

This term must arise from a non-zero anomalous dimension for any operator that is already present in the free theory.[17] It is easy to see that in the free theory $\langle \mathcal{O}\sigma\Sigma_{\pm,\ell}\rangle = 0$, so that $k \geqslant 2$. The tree-level result $\mathcal{G}^{(0)}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v) = u^2$ implies that there are operators of twists $4, 6, \ldots$. Decomposing the tree-level result, we see that

[16] This follows from the fact that both correlators have the same intermediate operators with the OPE coefficients differing only by a power $(-1)^\ell$. Splitting the correlator into an even spin and odd spin contribution, as in equation (3.4.40) for the tree-level correlator, then shows that such terms will appear.

[17] In the definition of present we also allow mixed operators such as $\Sigma_{\pm,\ell}$ that give cancelling contributions to the tree-level result, since they can yield terms of the form $\epsilon\log u$ in $\mathcal{G}^{(1)}(u,v)$.

the intermediate operators of odd and even spin give similar, almost cancelling contributions that have a power-law divergence of the form v^{-2}:

$$\begin{aligned}\mathcal{G}^{(0)}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v) &= \mathcal{G}^{(0),even}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v) + \mathcal{G}^{(0),odd}_{\mathcal{O}\sigma\mathcal{O}\sigma}(u,v) \\ &= u^2\left(1+\left(\frac{1}{4v^2}+\frac{1}{4v}-\frac{1}{2}\right)\right) - u^2\left(\frac{1}{4v^2}+\frac{1}{4v}-\frac{1}{2}\right) + \mathcal{O}(u^3)\,. \quad (3.4.40)\end{aligned}$$

We have shown this property here for the twist 4 operators, but due to the special factorization of the 4d conformal blocks, it in fact holds for all higher twist operators as well. If the even and odd spin intermediate operators acquire different anomalous dimensions, then there can be $\log u$ corrections to the correlator of the form $u^n v^m$ for integer $n \geqslant 2, m \geqslant -2$. Specifically note that integrality of the powers of v forces, by virtue of equation (3.4.39), that $\frac{3}{2} - \rho \in \mathbb{Z}$, so that all integer ρ are excluded. Therefore $\rho = \frac{1}{2}$ is the only remaining possibility, from which it follows that

$$\boxed{\frac{1}{2}\sin\theta_\ell \cos\theta_\ell \left(\gamma^{(1)}_{+,\ell} - \gamma^{(1)}_{-,\ell}\right) = \frac{B_{1/2}}{J_{2,\ell}}\,.} \quad (3.4.41)$$

Looking at the diagonalization of the Hamiltonian H, this is precisely the off-diagonal entry! That is, we have found that in the basis $\{J_{\psi,\ell}, J_{\sigma,\ell}\}$, the order ϵ correction to the Hamiltonian takes the form

$$H_\epsilon = 2\gamma^{(1)}_\psi \begin{pmatrix} 1-\frac{B}{J^2} & \frac{C}{J} \\ \frac{C}{J} & \omega \end{pmatrix}\,, \quad (3.4.42)$$

where $\omega = \gamma^{(1)}_\sigma / \gamma^{(1)}_\psi$.

Fixing constants from the central charge

We now use a central charge argument to fix B and C in terms of ω and N. Recall that a 4d free theory of N_B free scalar fields and N_f free Dirac fermions has a central charge

$$C_T = c_B + c_F = \frac{4}{3}N_B + 8N_f = \frac{4}{3}N_B + 2N\,, \quad (3.4.43)$$

where $N = N_f \mathrm{tr}\mathbb{1} = 4N_f$. Furthermore, recall the relation between the central charge c_T and OPE coefficients with the stress-energy tensor: for any operator $\mathcal{O}$ in a

d-dimensional theory with a (unique) stress-tensor T, the following relation holds [27]:

$$a_{\mathcal{O}\mathcal{O}T} \equiv \frac{C^2_{\mathcal{O}\mathcal{O}T}}{C_{TT}C^2_{\mathcal{O}\mathcal{O}}} = \frac{1}{c_T}\frac{d^2}{(d-1)^2}\Delta^2_{\mathcal{O}}\,, \tag{3.4.44}$$

where the $C_{\bullet\bullet}$ are two-point normalizations. Thus, in the theory of free fermions and free bosons:

$$\frac{1}{c_F}\frac{d^2}{(d-1)^2}\Delta^2_{\psi^2} = \frac{C^2_{\psi^2\psi^2 T_F}}{C_{T_FT_F}C^2_{\psi^2\psi^2}}\,, \qquad \frac{1}{c_B}\frac{d^2}{(d-1)^2}\Delta^2_{\sigma} = \frac{C^2_{\sigma\sigma T_B}}{C_{T_BT_B}C^2_{\sigma\sigma}}\,. \tag{3.4.45}$$

In the GNY model at $\epsilon = 0$ there are two decoupled free theories with central charges $c_B = \frac{4}{3}$, $c_F = 2N$, and with two separate stress-energy tensors $T_B = J_{\sigma,2}$, $T_F = J_{\psi,2}$ satisfying equation (3.4.45). As we turn on ϵ, there will be a unique stress tensor $T = \Sigma_{-,2}$, and a single central charge c_T satisfying

$$\lim_{\epsilon\to 0} c_T = c_B + c_F = \frac{4}{3} + 2N, \tag{3.4.46}$$

and

$$\frac{1}{c_T}\frac{d^2}{(d-1)^2}\Delta^2_{\psi^2} = a^2_{\psi^2\psi^2 T} = \frac{C^2_{\psi^2\psi^2 T}}{C_{TT}C^2_{\psi^2\psi^2}}\,, \tag{3.4.47}$$

$$\frac{1}{c_T}\frac{d^2}{(d-1)^2}\Delta^2_{\sigma} = a^2_{\sigma\sigma T} = \frac{C^2_{\sigma\sigma T}}{C_{TT}C^2_{\sigma\sigma}}\,. \tag{3.4.48}$$

We find that the last two equations give the same constraint, so we shall only use equation (3.4.47). Furthermore, we shall only be interested in the free theory limit $\epsilon \to 0$.

Assuming that $J_{\psi,\ell}$ and $J_{\sigma,\ell}$ are normalized such that $C_{J_{\bullet,\ell}J_{\bullet,\ell}} = 1$, we can write

$$\begin{pmatrix}\Sigma_{-,\ell}\\ \Sigma_{+,\ell}\end{pmatrix} = \begin{pmatrix}\alpha_\ell & \beta_\ell\\ \gamma_\ell & \delta_\ell\end{pmatrix}\begin{pmatrix}J_{\psi,\ell}\\ J_{\sigma,\ell}\end{pmatrix}, \tag{3.4.49}$$

where both the columns and rows of the matrix form orthonormal vectors. Note that $\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell$ are functions of B, C, ω and ℓ. Considering only ϵ^0 terms, we find that

$$\langle\psi^2\psi^2\Sigma_{-,2}\rangle^2 = \alpha_2^2\langle\psi^2\psi^2 J_{\psi,2}\rangle^2 \tag{3.4.50}$$

and that

$$\langle \Sigma_{-,2}\Sigma_{-,2}\rangle = \alpha_2^2\langle J_{\psi,2}J_{\psi,2}\rangle^2 + \beta_2^2\langle J_{\sigma,2}J_{\sigma,2}\rangle^2 = \alpha_2^2+\beta_2^2. \tag{3.4.51}$$

Then equation (3.4.47) yields

$$\frac{1}{c_F+c_B}\frac{d^2}{(d-1)^2}\Delta_{\psi^2}^2 = \frac{\alpha_2^2 C^2_{\psi^2\psi^2 T_F}}{C^2_{\psi^2\psi^2}(\alpha_2^2+\beta_2^2)} = \frac{1}{c_F}\frac{d^2}{(d-1)^2}\frac{1}{1+x^2}\Delta_{\psi^2}^2\,, \tag{3.4.52}$$

where $x(B,C,\omega)\equiv\frac{\beta_2}{\alpha_2}$. Thus we get a constraint on B,C: $x(B,C,\omega)^2=\frac{c_B}{c_F}$.

Additionally conservation of the stress-tensor $\Sigma_{-,2}$ imposes that $\gamma_{-,2}(B,C,\omega)=0$. These two constraints are independent and yield the following solutions for B,C:

$$B = 6-4\frac{\omega}{N}, \qquad\qquad C = \pm 2\frac{\omega}{\sqrt{N}}\,. \tag{3.4.53}$$

Unfortunately we are not able to fix the sign on C; however note that the anomalous dimensions are not sensitive to the sign of C.

Finally, we are unable to fix ω from a bootstrap argument and instead get the value $\omega=N$ from the literature, so that

$$\langle J_{\psi,\ell}|\hat{H}|J_{\psi,\ell}\rangle = 2\gamma_\psi^{(1)}\left(1-\frac{2}{\ell(\ell+1)}\right)\,. \tag{3.4.54}$$

The argument from section 3.3.1.4 that the even spin singlet and even spin adjoint currents have the same anomalous dimensions,[18] is in fact independent of the dimension of the space, so that the adjoint currents of even spin have an anomalous dimension as in equation (3.4.54). Furthermore, the conservation of the global symmetry current implies that the anomalous dimensions of the odd spin adjoint currents take the same form, so that for both odd and even ℓ:

$$\boxed{\gamma^{(1)}_{A,2,\ell} = 2\gamma_\psi^{(1)}\left(1-\frac{2}{\ell(\ell+1)}\right)} \qquad \ell\geqslant 1\,. \tag{3.4.55}$$

Using the minus sign for C in equation (3.4.53), we reproduce the known result for the singlet currents [156]:

$$\boxed{H = 2\gamma_\psi^{(1)}\begin{pmatrix} 1-\frac{2}{\ell(\ell+1)} & -\frac{2\sqrt{N}}{\sqrt{\ell(\ell+1)}} \\ -\frac{2\sqrt{N}}{\sqrt{\ell(\ell+1)}} & N\end{pmatrix},} \tag{3.4.56}$$

[18] To be more precise: the argument that $\langle J_{\psi,\ell}|\hat{H}|J_{\psi,\ell}\rangle = \langle J^A_{\psi,\ell}|\hat{H}|J^A_{\psi,\ell}\rangle$ for even spin $\ell\geqslant 2$.

which gives the anomalous dimensions as

$$\gamma^{(1)}_{\pm,\ell} = 2\gamma^{(1)}_{\psi}\left(\frac{N+1}{2} - \frac{1}{\ell(\ell+1)} \pm \frac{\sqrt{4+(-4+20N)\ell(\ell+1)+(N-1)^2\ell^2(\ell+1)^2}}{2\ell(\ell+1)}\right). \tag{3.4.57}$$

3.5 Discussion

In this chapter we have used crossing symmetry to constrain fermionic CFTs that weakly break higher spin symmetry by studying the analytic properties of the twist conformal blocks occurring in the four-point correlators of composite operators. Novel to the use of composite operators is that, in contrast to the studies of correlators of fundamental scalar fields, quadrilinear operators appear as intermediate states already in the free theory correlator. Their CFT data mixes under crossing with that of the bilinear operators, making it harder to isolate the CFT data of the bilinear operators. As has been found in previous work (see e.g. [33]), the bootstrap gains tremendously in power when several different correlators are studied simultaneously, demonstrated in this chapter by the study of mixed correlators such as $\langle \mathcal{O}\mathcal{O}^A\mathcal{O}^A\mathcal{O}\rangle$ in both models, and the study of $\langle \mathcal{O}\mathcal{O}\sigma\sigma\rangle$ in the Gross-Neveu-Yukawa model. Our method reproduces known results for the anomalous dimensions of bilinear currents [156], produces some new results for bilinear OPE coefficients, and finds CFT data of the quadrilinear operators. Furthermore it finds a solution in 2 dimensions for a potential fermionic CFT in which the fundamental field ψ is not in the spectrum. Some extensions of our work are clear.

Compared with the analysis of correlators of fundamental scalar fields [148], our method suffers from the obvious drawback that there are a larger number of intermediate operators, making it harder to isolate contributions of any particular intermediate operator to the four-point correlator. It would be interesting to extend the method of the large spin bootstrap to include correlators of non-scalar operators to facilitate the study of the correlator of four fundamental fermion fields. For example, in [59]

a formalism to study such correlators in four dimensions is established; it would be interesting to study the Gross-Neveu-Yukawa model using these methods.

The Gross-Neveu model can be studied in any dimension $2 < d < 4$ through the large N expansion [162], and results for CFT data of leading twist operators are known perturbatively in $1/N$ [156]. They have the interesting property that, at least to leading order, they are essentially identical to corrections in the bosonic critical large N model and it would be interesting to apply the method of the large spin bootstrap to try to understand this. In the bootstrap approach, large N computations tend to be more difficult because the spacetime dimension is properly non-integral as opposed to perturbatively non-integral. For example, the analysis of the ϵ-expansion of the critical O(N) model in chapter 4 is significantly simpler than that in 5, partly because the dimension $d = 4 - \epsilon$ allows one to use 4d conformal blocks in some computations, and partly because in the large N approach, the coupling to the Hubbard-Stratonovich field σ introduces many new operators into the OPE. Furthermore, this computation would likely be simplified by considering the correlator of fundamental fermions, but since such an approach has to analytically continue in the dimension it is complicated by having to deal with fermionic representations.

Finally we have dealt with the dimensional shift arising from the non-integer dimension of spacetime in a rather ad-hoc manner, such as by exploiting the free theory result and by deriving some identities for sums of conformal blocks with unequal external dimensions. At higher orders in ϵ these problems get more severe and we are unable to apply the same ideas. The development of a systematic method to deal with these issues should simplify calculations and hopefully allow for an analysis of higher-order corrections.

Chapter 4

The critical O(N) model in $d = 4 - \epsilon$

4.1 Introduction

The strongly interacting critical O(N) models in $2 < d < 4$ form a set of theories that are well-known in applications in condensed matter theory, the most famous case being of course the 3d Ising model, describing such diverse systems as ferromagnets, boiling water and self-avoiding random walks. Other examples of applications of the O(N) models in three dimensions are the XY universality class ($N = 2$) for the superfluid transition of ^{4}He, the Heisenberg class ($N = 3$) for the critical behaviour of isotropic magnets, and the O(4) class that is expected to describe a phase transition in QCD [163].[1]

The theory in three dimensions has no weak coupling regime, though there are various perturbative regimes one can consider, such as $d = 2 + \epsilon$, $d = 4 - \epsilon$ and large N expansions. Furthermore it is a relatively simple theory with a global symmetry, its field content consisting only of bosonic fields, and it has been an excellent testing ground for a variety of methods to study QFT.

In this chapter we study the O(N) model perturbatively in the $d = 4-\epsilon$ expansion. In this description, its field content is built up from fundamental fields φ^i transforming

[1] For an extensive review of the physical applications, see [14].

in the fundamental representation of a global O(N) symmetry. From a Lagrangian point of view the theory can be considered by perturbing N free fields with a quartic interaction $\lambda(\varphi^i\varphi^i)^2$. In $d = 4 - \epsilon$ dimensions this theory has an IR fixed-point, the Wilson-Fisher fixed point, with $\lambda \sim \epsilon$, at which the theory becomes conformal and is referred to as the critical O(N) model [164]. In the literature, the scaling dimensions for the scalars φ^i, φ_S^2 and φ_S^4 have been computed to order ϵ^6 using diagrammatic methods [165], and similarly for singlet double field operators $\varphi_{S,\ell}^2$ to order ϵ^4 [122, 166]. Our study of the theory, however, will employ the large spin bootstrap idea and make no reference at all to the Lagrangian description or Feynman diagrams. We will assume conformal and global O(N) symmetry, and make some very general assumptions about the spectrum of operators – essentially that the spectrum is a perturbation from generalized free fields. Using crossing symmetry and general properties of CFTs, we will then be able to show that the results that follow from these assumptions agree with only the critical O(N) model.

To cubic order the relevant divergences of the correlator arise, via crossing symmetry, from just two operators in the crossed channel: the identity operator and the bilinear scalar operator φ^2. This makes our derivation very simple: in the present framework it essentially involves a first order computation. At fourth order the divergences in the correlator are more involved and arise from infinite towers of operators with arbitrarily large spins, the computation being further complicated by the appearance of new operators in the OPE at quadratic order. We find that these operators give a contribution to the correlator that can be expressed purely in terms of harmonic polylogarithms, which enables us to make an appropriate ansatz for this contribution, and hence to determine the CFT data of weakly broken currents to fourth order.

A remarkable feature of our computation is that the convergence properties of the inversion integral allow for the results to be extrapolated down to spin zero. Conservation of the stress tensor and global symmetry current, together with matching

conditions for spin zero, lead to six non-trivial constraints that allow us to fix all free parameters in the theory, including the dimensions of the external operator φ and the scalar operator φ^2. We have therefore shown that (at least to this order in perturbation theory) the critical O(N) model is the only CFT, other than the free theory, to be consistent with our assumptions on the spectrum.

The quartic order results that we derive are new, except the anomalous dimensions in the O(N) singlet, where we reproduce the recent diagrammatic calculation [122, 166]. Furthermore all our results agree with known large N results previously computed in the literature [167–169]. Finally we combine our results for the O(N) model in $d = 4 - \epsilon$ dimensions with the known results for the non-linear sigma model in $d = 2 + \epsilon$ dimensions [161] to provide Padé approximations to the central charge in the strongly interacting 3d models. These provide an improvement on previous predictions and show excellent agreement with numerical results.

Section 4.2 gives a brief overview of the general method of large spin perturbation theory in the presence of a global O(N) symmetry. Section 4.3 applies this method to find the CFT data in the O(N) model, focusing on the contribution from quadrilinear operators of approximate twist 4. In section 4.4 we constrain the various parameters in the theory using matching conditions for leading twist operators of low spins, and we pay particular attention to the subtle analytic continuation of the CFT data to spin $\ell = 0$. In section 4.5 we derive new predictions for the central charges C_T and C_J associated to the stress-energy tensor and global symmetry current respectively, and we use Padé approximants to provide predictions for these charges in 3d.

Appendix B.2 provides some useful background on the small z behaviour of conformal blocks that is used in our computation. Appendices B.3 and B.4 present the full results of our analysis. These results can also be found in the Mathematica file included with [3], one of the two papers on which this chapter is based.

4.2 Large spin perturbation theory with global symmetry

We study the correlator of four fundamental fields in general dimension d in the presence of a global $\mathrm{O}(N)$ symmetry

$$\langle \varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\varphi_l(x_4)\rangle = \frac{\mathcal{G}_{ijkl}(z,\bar{z})}{|x_{12}|^{2\Delta_\varphi}|x_{34}|^{2\Delta_\varphi}}. \tag{4.2.1}$$

Crossing symmetry implies that $\mathcal{G}_{ijkl}(z,\bar{z})$ satisfies

$$\left(\frac{1-z}{z}\right)^{\Delta_\varphi}\mathcal{G}_{ijkl}(z,\bar{z}) = \left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\varphi}\mathcal{G}_{kjil}(1-\bar{z},1-z). \tag{4.2.2}$$

The intermediate operators transform in the singlet (S), traceless symmetric (T) and antisymmetric (A) rank two tensor representations of the global $\mathrm{O}(N)$ symmetry, and as such the correlator decomposes

$$\mathcal{G}_{ijkl}(z,\bar{z}) = \mathcal{G}_S(z,\bar{z})\mathbf{T}^S_{ijkl} + \mathcal{G}_T(z,\bar{z})\mathbf{T}^T_{ijkl} + \mathcal{G}_A(z,\bar{z})\mathbf{T}^A_{ijkl}\,, \tag{4.2.3}$$

where we defined the following basis of tensor structures

$$\mathbf{T}^S_{ijkl} = \delta_{ij}\delta_{kl}, \qquad \mathbf{T}^T_{ijkl} = \frac{\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}}{2} - \frac{1}{N}\delta_{ij}\delta_{kl}, \qquad \mathbf{T}^A_{ijkl} = \frac{\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}}{2}. \tag{4.2.4}$$

The four-point functions can be decomposed into conformal blocks as follows

$$\mathcal{G}_R(z,\bar{z}) = \sum_{\mathcal{O}_{R,\Delta,\ell}} a_{R,\Delta,\ell}(z\bar{z})^{\tau/2}G^{(d)}_{\Delta,l}(z,\bar{z}), \tag{4.2.5}$$

where the sum is over all intermediate operators $\mathcal{O}_{R,\Delta,\ell}$ in the respective $\mathrm{O}(N)$ representation, parametrised by scaling dimension Δ and spin ℓ. In this equation, $a_{R,\Delta,\ell}$ is the relevant (squared) OPE coefficient, $\tau = \Delta - \ell$ is the twist, and $G^{(d)}_{\Delta,\ell}(z,\bar{z})$ is the d-dimensional conformal block.

The crossing relation (4.2.2) decomposes as

$$\begin{aligned}
\mathcal{G}_S(u,v) &= \left(\frac{u}{v}\right)^{\Delta_\varphi}\left(\frac{1}{N}\mathcal{G}_S(v,u)+\frac{(N+2)(N-1)}{2N^2}\mathcal{G}_T(v,u)+\frac{1-N}{2N}\mathcal{G}_A(v,u)\right),\\
\mathcal{G}_T(u,v) &= \left(\frac{u}{v}\right)^{\Delta_\varphi}\left(\mathcal{G}_S(v,u)+\frac{N-2}{2N}\mathcal{G}_T(v,u)+\frac{1}{2}\mathcal{G}_A(v,u)\right),\\
\mathcal{G}_A(u,v) &= \left(\frac{u}{v}\right)^{\Delta_\varphi}\left(-\mathcal{G}_S(v,u)+\frac{N+2}{2N}\mathcal{G}_T(v,u)+\frac{1}{2}\mathcal{G}_A(v,u)\right).
\end{aligned} \tag{4.2.6}$$

We shall establish an ϵ-expansion around the generalized free theory in $d=4-\epsilon$ dimensions. The correlator in the generalized free theory is simply

$$\mathcal{G}^{(0)}_{ijkl}(z,\bar{z}) = \delta_{ij}\delta_{kl}+u^{\Delta_\varphi}\delta_{ik}\delta_{jl}+\left(\frac{u}{v}\right)^{\Delta_\varphi}\delta_{il}\delta_{jk}, \tag{4.2.7}$$

which implies that

$$\begin{aligned}
\mathcal{G}^{(0)}_S(z,\bar{z}) &= 1+\frac{1}{N}u^{\Delta_\varphi}\left(1+\frac{1}{v^{\Delta_\varphi}}\right),\\
\mathcal{G}^{(0)}_T(z,\bar{z}) &= u^{\Delta_\varphi}\left(1+\frac{1}{v^{\Delta_\varphi}}\right),\\
\mathcal{G}^{(0)}_A(z,\bar{z}) &= u^{\Delta_\varphi}\left(1-\frac{1}{v^{\Delta_\varphi}}\right).
\end{aligned} \tag{4.2.8}$$

The intermediate operators have dimensions $\Delta_{n,\ell}=2\Delta_\varphi+2n+\ell$ for $n=0,1,\ldots$. The OPE coefficients $a_{T,\Delta,\ell}$ of operators in the traceless symmetric representation match the OPE coefficients in the $\langle\varphi\varphi\varphi\varphi\rangle$ correlator in the theory of a single generalized free field; for this reason we shall refer to them as the generalized free field OPE coefficients $a^{GFF}_{\Delta,\ell}$. They are (see e.g. [90, 170][2])

$$a^{GFF}_{\Delta_{n,\ell},\ell} = \frac{2(\Delta_\varphi-\frac{d}{2}+1)^2_n(\Delta_\varphi)^2_{\ell+n}}{\ell!n!(\frac{d}{2}+\ell)_n(2\Delta_\varphi+n-d+1)_n(2\Delta_\varphi+2n+\ell-1)_\ell(2\Delta_\varphi+n+\ell-\frac{d}{2})_n}, \tag{4.2.9}$$

for even ℓ, with the odd spin OPE coefficients vanishing. Here $(a)_b\equiv\Gamma(a+b)/\Gamma(a)$ is the Pochhammer symbol. The OPE coefficients of operators in the antisymmetric

[2]Different normalizations of conformal blocks, depending on whether one includes factors of the form $2^{-\ell}$ and $\frac{(d/2-1)_\ell}{(d-2)_\ell}$, are used in [170] and [90], both of which differ from the normalization of the blocks in this note.

representation are of the exact same form, except that the formula is valid only for odd ℓ and that the coefficients vanish for even ℓ.

Perturbing to the interacting O(N) model breaks higher spin symmetry, and the intermediate operators acquire corrections. We define anomalous dimensions and OPE coefficient corrections with respect to their generalized free field values: that is, the anomalous dimensions of the intermediate operators are defined by[3]

$$\Delta_{R,n,\ell} = 2\Delta_\varphi + 2n + \ell + \gamma_{R,n,\ell}, \tag{4.2.10}$$

with R denoting the representation of the operator under the global O(N) symmetry. Furthermore it was shown in [148] that at first order in ϵ, crossing symmetry admits a non-trivial solution only in $d = 4$, which consists of a finite support solution at spin $\ell = 0$. Therefore amongst the intermediate operators of leading twist, only the scalar operators acquire first order anomalous dimensions. Such operators exist in the O(N) singlet and traceless symmetric representations and we shall denote the corrections to their scaling dimensions by g_S and g_T respectively. Thus we define[4]

$$\Delta_{\varphi_S^2} = 2\Delta_\varphi + g_S, \qquad \Delta_{\varphi_T^2} = 2\Delta_\varphi + g_T, \tag{4.2.11}$$

and use g_S and g_T as perturbative expansion parameters, which is possible since $g_S, g_T \sim \epsilon$.

To find the CFT data of the leading twist operators, we take the conformal block expansion of the correlators $\mathcal{G}_R(z,\bar{z})$, expand as series in ϵ up to order ϵ^4, and take the limit of small z, to reduce the crossing equations (4.2.2) to

$$\sum_\ell z^{\Delta_\varphi}\left(U^{(0)}_{R,\bar{h}} + \frac{1}{2}\log z\, U^{(1)}_{R,\bar{h}} + \frac{1}{8}\log^2 z\, U^{(2)}_{R,\bar{h}}\right) f_{\Delta,\ell}(\bar{z}) = Q_R(z,\bar{z}), \tag{4.2.12}$$

[3]Note that this convention, which is the most natural in the context of crossing symmetry, differs from most of the literature where the anomalous dimension is defined with respect to the free theory value $2\Delta^{(0)}_\varphi = d - 2$.

[4]In the notation of (4.2.10), we have defined $g_S = \gamma_{S,0,\ell}$ and $g_T = \gamma_{T,0,\ell}$.

where each $Q_R(z,\bar{z})$ is the small z limit of the right-hand side of the crossing equation (4.2.6) for representation R, up to order ϵ^4. We also defined $\bar{h} \equiv \ell + \Delta_\varphi$, and defined the functions $f_{\Delta,\ell}(\bar{z})$ to be the $\mathrm{SL}(2,\mathbb{R})$ conformal blocks, normalized as follows:

$$f_{\Delta,\ell}(\bar{z}) = r_{\frac{\Delta+\ell}{2}} k_{\frac{\Delta+\ell}{2}}(\bar{z}), \qquad r_h = \frac{\Gamma(h)^2}{\Gamma(2h)}, \tag{4.2.13}$$

with $k_\beta(\bar{z}) = \bar{z}^\beta \, {}_2F_1(\beta,\beta;2\beta;\bar{z})$ as before. Furthermore the $U^{(m)}_{R,\bar{h}}$ are related to the CFT data as follows [2, 99]:

$$\hat{a}_R(\bar{h})\,(\gamma_{R,\ell})^p = U^{(p)}_{R,\bar{h}} + \frac{1}{2}\partial_{\bar{h}} U^{(p+1)}_{R,\bar{h}} + \frac{1}{8}\partial^2_{\bar{h}} U^{(p+2)}_{R,\bar{h}} + \ldots, \tag{4.2.14}$$

where the $\hat{a}_R(\bar{h})$ correspond to the OPE coefficients as $a_{R,\Delta,\ell} = r_{\frac{\Delta+\ell}{2}} \hat{a}_R(\bar{h})$.

The problem is then to find the $U^{(p)}_{R,\bar{h}}$ from (4.2.12), given a known $Q_R(z,\bar{z})$. This is where we use the Lorentzian inversion formula, which we revisit in the section below.

4.2.1 From large spin perturbation theory to a collinear inversion formula

To find the CFT data of the leading twist operators we use an appropriate limit of the Lorentzian inversion formula, which we derive here. While this discussion is not a rigorous derivation of the inversion formula, it explains its relation to large spin perturbation theory.

Consider a basis of $SL(2,\mathbb{R})$ conformal blocks $f_{\Delta,\ell}(\bar{z})$, normalized as in (4.2.13). We are then interested in solving the following inversion problem: given $G(\bar{z})$ containing a double discontinuity around $\bar{z}=1$, find $\hat{a}_\ell$ such that

$$\sum_{\substack{\Delta = 2\Delta_\varphi + \ell, \\ \ell = 0,2,\cdots}} \hat{a}_\ell f_{\Delta,\ell}(\bar{z}) = G(\bar{z}), \tag{4.2.15}$$

with the double discontinuity as defined in 2.3.28. Now we use the ideas of [89] as follows. First recall that the $SL(2,\mathbb{R})$ conformal blocks are eigenfunctions of a

quadratic Casimir operator

$$\overline{D} f_{\Delta,\ell}(\bar{z}) = J^2 f_{\Delta,\ell}(\bar{z}), \qquad \overline{D} = (1-\bar{z})\bar{z}^2\partial_{\bar{z}} - \bar{z}^2\partial_{\bar{z}}^2, \tag{4.2.16}$$

where J^2 is the conformal spin. We assume that $\hat{a}_\ell \equiv \hat{a}(J)$ admits an expansion in inverse powers of the conformal spin

$$\hat{a}(J) = \sum_m \frac{a_m}{J^{2m}} \tag{4.2.17}$$

and define the following family of functions

$$h^{(m)}(\bar{z}) = \sum_{\substack{\Delta=2\Delta_\varphi+\ell,\\ \ell=0,2,\cdots}} \frac{f_{\Delta,\ell}(\bar{z})}{J^{2m}}. \tag{4.2.18}$$

From the explicit form of the blocks we can compute

$$h^{(0)}(\bar{z}) = \sum_{\substack{\Delta=2\Delta_\varphi+\ell,\\ \ell=0,2,\cdots}} f_{\Delta,\ell}(\bar{z}) = \frac{\pi}{4}\left(\frac{\bar{z}}{1-\bar{z}}\right)^{1/2} + \text{regular}, \tag{4.2.19}$$

where the regular terms do depend on Δ_φ but are not important for us. The sequence of functions $h^{(m)}(\bar{z})$ can then be generated by the inverse action of the Casimir

$$\overline{D} h^{(m+1)}(\bar{z}) = h^{(m)}(\bar{z}). \tag{4.2.20}$$

The inversion problem (4.2.15) then amounts to decomposing $G(\bar{z})$ in the basis of functions $h^{(m)}(\bar{z})$, with the precise range of m depending on the specific form of $G(\bar{z})$. The recursion (4.2.20) can be used to systematically construct the functions $h^{(m)}(\bar{z})$ and hence find the coefficients a_m: specifically, one matches the double discontinuity on both sides of (4.2.15). To make contact with the inversion formula of section 2.3.2, assume that there exists a family of projectors $K^{(m)}(\bar{z})$ such that

$$\int_0^1 \mathrm{d}\bar{z}\; K^{(m)}(\bar{z})\, \mathrm{dDisc}\left[h^{(n)}(\bar{z})\right] = \delta^{mn}. \tag{4.2.21}$$

Using the projectors $K^{(m)}(\bar{z})$ we can write

$$\hat{a}(J) = \int_0^1 \mathrm{d}\bar{z}\; K(\bar{z}, J)\, \mathrm{dDisc}\left[G(\bar{z})\right], \qquad K(\bar{z}, J) = \sum_m \frac{K^{(m)}(\bar{z})}{J^{2m}}. \tag{4.2.22}$$

As will be clear momentarily, the precise form of these projectors will not be necessary. Acting on both sides of (4.2.15) with the Casimir operator $\bar{D}$ and integrating by parts we obtain

$$\left(\overline{D}^{\dagger} - J^2\right) K(\bar{z}, J) = 0 \tag{4.2.23}$$

where we have assumed the absence of boundary terms and $\overline{D}^{\dagger} = \bar{\partial}(1-\bar{z})\bar{\partial}\bar{z}^2$. Introducing the notation $J^2 = \bar{h}(\bar{h}-1)$, we find two independent solutions related by $\bar{h} \leftrightarrow 1-\bar{h}$; imposing regularity for positive $\bar{h}$ singles out one of them. Finally, requiring that the inversion formula give $\hat{a}(J) = 1$ for $G(\bar{z}) = h^{(0)}(\bar{z})$ fixes the overall normalization, leading to the following result

$$\hat{a}(\bar{h}) = \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)} \int_0^1 \mathrm{d}\bar{z}\; z^{\bar{h}-2}\, {}_2F_1(\bar{h}, \bar{h}; 2\bar{h}; \bar{z})\, \mathrm{dDisc}\left[G(\bar{z})\right]. \tag{4.2.24}$$

In appendix B.1 we give several results relevant for our computations, i.e. for a set of $G(\bar{z})$, we give the result of the inversion integral (4.2.24). In all cases the integral is convergent in the region $\bar{h} > 1$, which for our applications means that the integral converges, and is expected to give the right answer, for $\ell > 0$. We discuss the case $\ell = 0$ in more detail in section 4.4 .

4.3 Inversion of the $\mathrm{O}(N)$ model

Let us now discuss the precise inversion problem we solve. Take for instance the first crossing relation in (4.2.6),

$$\mathcal{G}_S(u,v) = \left(\frac{u}{v}\right)^{\Delta_\varphi} \left(\frac{1}{N}\mathcal{G}_S(v,u) + \frac{(N+2)(N-1)}{2N^2}\mathcal{G}_T(v,u) + \frac{1-N}{2N}\mathcal{G}_A(v,u)\right), \tag{4.3.1}$$

and consider the terms on the right-hand-side that create a double discontinuity. Taking the small z limit, we arrive at the following inversion problem for singlet operators of leading twist:

$$\sum_{\Delta_{S,\ell}=2\Delta_\varphi+\ell} \hat{a}_\ell z^{\tau_{S,\ell}/2} f_{\Delta_{S,\ell},\ell}(\bar{z}) = z^{\Delta_\varphi}\left(\frac{1}{N}D_S + \frac{(N+2)(N-1)}{2N^2}D_T + \frac{1-N}{2N}D_A\right)\Big|_{\text{small } z}, \tag{4.3.2}$$

where we have defined

$$D_R = \text{dDisc}\left[\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\varphi} \mathcal{G}_R(1-\bar{z}, 1-z)\right]. \tag{4.3.3}$$

To order $O(\epsilon^4)$, the double discontinuities take the form

$$\begin{aligned} D_S &= \text{dDisc}\Big[I_{\mathbb{1}} + I_{\varphi_S^2} + I_2^S + I_4^S\Big], \\ D_T &= \text{dDisc}\Big[\quad I_{\varphi_T^2} + I_2^T + I_4^T\Big], \\ D_A &= \text{dDisc}\Big[\qquad\quad I_2^A + I_4^A\Big]. \end{aligned} \tag{4.3.4}$$

Here $I_{\mathbb{1}}$ is of order $O(\epsilon^0)$ and corresponds to the exchange of the identity operator in the crossed channel, $I_{\varphi_S^2}$ and $I_{\varphi_T^2}$ are of order $O(\epsilon^2)$ and correspond to the exchange of bilinear scalars, whereas I_2^R and I_4^R are of order $O(\epsilon^4)$ and come from infinite towers of operators of approximate twists 2 and 4 respectively. At this order in ϵ, there are no contributions from operators of higher twist. Let us discuss in more detail the precise forms of the discontinuities in (4.3.4).

4.3.1 Contributions to order ϵ^3

At zeroth order in ϵ there is only the double discontinuity $I_{\mathbb{1}}$ from the identity operator in the cross-channel, which gives the generalized free field theory OPE coefficients:

$$I_{\mathbb{1}} = \left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\varphi}. \tag{4.3.5}$$

At second order in ϵ the only new contributions to the double discontinuity come from scalar operators in the singlet and traceless symmetric representations, whose anomalous dimensions g_S and g_T (4.2.11) are of order ϵ. For example,

$$\begin{aligned} I_{\varphi_S^2} &= \left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\varphi} a_{S,0}\,(1-\bar{z})^{\frac{1}{2}\Delta_{S,0}} G^{(d)}_{\Delta_{S,0},0}(1-\bar{z}, 1-z) + O(\epsilon^4) \\ &= \bar{z}^{\Delta_\varphi} a_{S,0}\,(1-\bar{z})^{g_S/2} G^{(d)}_{\Delta_{S,0},0}(1-\bar{z}, 1-z) + O(\epsilon^4), \end{aligned} \tag{4.3.6}$$

where $\Delta_{S,0} = \Delta_{\varphi_S^2}$ The terms with a non-zero double discontinuity arise in two ways: firstly from expanding the term $(1-\bar{z})^{g_S/2}$ in powers of g_S, and secondly from Taylor

expanding the conformal block

$$G^{(d)}_{\Delta_{S,0},0}(1-\bar{z},1-z) = G^{(4-\epsilon)}_{2\Delta_\varphi+g_{S,0},0}(1-\bar{z},1-z). \tag{4.3.7}$$

Notice that the conformal block $G^{(d)}_{\Delta,0}(1-\bar{z},1-z)$ itself is regular as $\bar{z}\to 1$, and therefore the double-discontinuity of $I_{\varphi_S^2}$ will start at order g_S^2. This is a general feature that explains the orders at which the double discontinuities (4.3.4) appear: the contributions in the crossed channel from operators with dimensions $2\Delta_\varphi + n + \gamma$ for integer n have a double discontinuity that starts at order γ^2. To evaluate the expansion (4.3.7) of the conformal blocks, we use the precise form of the d-dimensional conformal block for the scalar exchange between two identical scalar external operators [29]:

$$G^{(d)}_{\Delta,0}(1-\bar{z},1-z) = \sum_{m,n=0}^{\infty} \frac{(\Delta/2)_m^2\,(\Delta/2)_{m+n}^2}{m!n!\,(\Delta+1-d/2)_m\,(\Delta)_{2m+n}}(1-z)^m(1-\bar{z})^m(1-z\bar{z})^n, \tag{4.3.8}$$

By first performing the sum over n and then expansing for small z, the contributions $I_{\varphi_S^2}$ and $I_{\varphi_T^2}$ can be readily computed. Notice that in the expression (4.3.6), and in the analogous expression for $I_{\varphi_T^2}$, we need to allow for the OPE coefficients to acquire corrections. Therefore we write

$$a_{S,0} = \frac{2}{N}\left(1+\alpha_S\right), \qquad a_{T,0} = 2\left(1+\alpha_T\right). \tag{4.3.9}$$

The corrections α_S and α_T start at order ϵ. In section 4.4 we will be able to compute them to order ϵ^2, which is the maximal order with which they appear in our ϵ^4 results.

4.3.2 Contributions to order ϵ^4

At fourth order in ϵ we get two additional contributions: I_2^R and I_4^R, coming from bilinear and quadrilinear operators, respectively.

Leading twist (bilinear) operators

Since the anomalous dimensions of the leading twist operators start at order ϵ^2, the sums of their squares give contributions I_2^R to the double discontinuity at order ϵ^4. Therefore at this order we may evaluate the conformal blocks in four dimensions. This leads to, for example, the singlet contribution

$$I_2^S = \frac{1}{8}\log^2(1-\bar{z}) \sum_{\ell=0,2,\ldots} a_{S,\ell}\,(\gamma_{S,\ell})^2\, G^{(4d)}_{2+\ell,\ell}(1-\bar{z},1-z)\Big|_{\text{small } z} + O(\epsilon^5). \tag{4.3.10}$$

This sum is most easily found using the language of twist conformal blocks of section 2.3.1. The individual 4d conformal blocks have the particular form:

$$G^{(4d)}_{\tau+\ell,\ell}(1-\bar{z},1-z) = \frac{(1-z)(1-\bar{z})}{z-\bar{z}}\Big(k_{\frac{\tau}{2}+\ell}(1-\bar{z})k_{\frac{\tau}{2}-1}(1-z) - (z\leftrightarrow\bar{z})\Big), \tag{4.3.11}$$

whence the small z part of the sum factorizes. By noticing that $\gamma_{R,\ell} \sim 1/J_b^2$ to leading order, where $J_b^2 = \ell(\ell+1)$, one can subsequently relate the sums I_2^R to sums of the form

$$h^{(m)}_{R,2}(\bar{z}) \equiv \sum_\ell \frac{a^{(0)}_{R,\ell}}{J_b^{2m}} k_{\ell+1}(1-\bar{z}), \tag{4.3.12}$$

with $m=2$, where $\ell = 0,2,4,\ldots$ for the singlet and traceless symmetric representations, and $\ell = 1,3,5,\ldots$ for the antisymmetric representation. The $a^{(0)}_{R,\ell}$ are the results for generalized free fields for $\Delta_\varphi = 1$, $d=4$ in the appropriate representation:

$$a^{(0)}_{S,\ell} = \frac{2}{N}\frac{\Gamma(\ell+1)^2}{\Gamma(2\ell+1)}, \quad a^{(0)}_{T,\ell} = \frac{2\,\Gamma(\ell+1)^2}{\Gamma(2\ell+1)}, \quad a^{(0)}_{A,\ell} = -\frac{2\Gamma(\ell+1)^2}{\Gamma(2\ell+1)}. \tag{4.3.13}$$

Note that the $h^{(0)}_{R,2}(\bar{z})$ are precisely the sums that appears in the GFF result and are therefore already known. The precise relation between the $h^{(m)}$ is through a Casimir differential equation:

$$\overline{D}\, h^{(m+1)}_{R,2}(1-\bar{z}) = h^{(m)}_{R,2}(1-\bar{z}), \tag{4.3.14}$$

where $\overline{D} = (1-\bar{z})\bar{z}^2\bar{\partial}^2 - \bar{z}^2\bar{\partial}$, which has eigenvalue J_b^2 on $k_{\ell+1}(\bar{z})$. Solving these equations with the correct boundary conditions, we find the results for I_2^R listed in B.3.

Quadrilinear operators

The discontinuities I_4^R arise because the Lagrangian contains a quartic coupling, so that new quadrilinear operators of approximate twist four appear. These are formed out of four fundamental fields and derivatives and appear with (squared) OPE coefficients at order ϵ^2. At this order, no operators of approximate twists $\tau = 6, 8, \ldots$ appear. The twist four operators acquire anomalous dimensions at first order in ϵ, so that they contribute to the double discontinuity at order ϵ^4. For example, the singlet contribution takes the form

$$I_4^S = \frac{1}{8}\log^2(1-\bar{z})\sum_\ell \langle\!\langle a_{S,4,\ell}(\gamma_{S,4,\ell})^2 \rangle\!\rangle G_{4,\ell}^{(4d)}(1-\bar{z}, 1-z) + O(\epsilon^5). \tag{4.3.15}$$

These operators are degenerate for spins $\ell \geqslant 2$. As in chapter 3, we therefore use the notation $\langle\!\langle \cdot \rangle\!\rangle$ for a sum over degenerate states:

$$\langle\!\langle a_{S,4,\ell}(\gamma_{S,4,\ell})^2 \rangle\!\rangle \equiv \sum_i a_{S,4,\ell,i}(\gamma_{S,4,\ell,i})^2. \tag{4.3.16}$$

If there were no degeneracy we could have computed first the OPE coefficients and anomalous dimensions using an order ϵ^3 inversion, and from this compute the sum (4.3.15). We will employ this strategy later for the non-degenerate operators of spin 0 and 1. To compute the full I_4^R, however, we will instead need to employ another strategy. We use the special form of the 4d conformal blocks, equation (4.3.11), and take the small z limit to deduce that it must be of the form

$$I_4^R = (h_{R,4}(\bar{z})\log z - h_{R,4}(z)\log\bar{z})\log^2(1-\bar{z}), \tag{4.3.17}$$

with

$$h_{R,4}(\bar{z}) = \sum_\ell \frac{1}{8}\langle\!\langle a_{R,4,\ell}(\gamma_{R,4,\ell})^2 \rangle\!\rangle k_{2+\ell}(\bar{z}). \tag{4.3.18}$$

Guided by the structure of perturbation theory, and the results for double discontinuities to lower order, we make an ansatz for each $h_{R,4}(\bar{z})$ in terms of functions of

pure transcendentality:

$$\left\{1,\, \log\bar{z},\, \log^2\bar{z},\, \mathrm{Li}_2(1-\bar{z}),\, \log^3\bar{z},\, \log\bar{z}\,\mathrm{Li}_2(1-\bar{z}),\, \mathrm{Li}_3(1-\bar{z}),\, \mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right)\right\}. \tag{4.3.19}$$

These form a basis of harmonic polylogarithms $H(\boldsymbol{m}_w;\bar{z})$ that are regular at $\bar{z}\to 1$, whose weight w is at most 3, and whose weight vectors $\boldsymbol{m}_w$ have entries in $\{0,1\}$ [120]. We now fix the 24 constants in the ansatz. Demanding, order by order in $(1-\bar{z})$, that $h_{R,4}(\bar{z})$ arises as the sum in equation (4.3.18) leaves only nine undetermined constants in the total ansatz. Next note that the I_4^R are the only terms on the right-hand side of the crossing equations that give terms of the form $\log^2 z$ and $\log^3 z$. Since there are no $\log^3 z$ terms on the left-hand side, this gives us three constraints on the constants in the ansatz. The $\log^2 z$ terms on the left-hand side are generated by sums of $\hat{a}_{R,\ell}\big(\gamma^{(1)}_{R,\ell}\big)^2$, which are known from our lower order computations, yielding three further constraints. To see in more detail how this works, consider for example the crossing equation for the singlet representation, equation (4.3.2). The order ϵ^2 result for $\gamma_{S,\ell}$ immediately gives

$$U^{(2)}_{S,\bar{h}} = \frac{\left(2g_S^2+g_T^2(N-1)(N+2)\right)^2\left(2\bar{h}-1\right)}{2N^3(\bar{h}-1)^2\bar{h}^2}+O(\epsilon^5). \tag{4.3.20}$$

The $\log^2 z$ piece of this equation takes the form

$$\sum_\ell \frac{1}{8}\log^2 z\,\hat{a}_{S,\ell}\left(\gamma_{S,\ell}\right)^2 z^{\Delta_\varphi} f_{\Delta,\ell}(\bar{z}) = z^{\Delta_\varphi}\left(\frac{1}{N}I_4^S+\frac{N^2+N-2}{2N^2}I_4^T+\frac{1-N}{2N}I_4^A\right)\Big|^{\text{small } z}_{\log^2 z} \tag{4.3.21}$$

On the right-hand side, each I_4^R has a $\log^2 z$ piece that can be readily computed in terms of the constants in the ansatz. We then invert these terms and match to the known CFT data $\frac{1}{8}\hat{a}_{S,\ell}\big(\gamma_{S,\ell}\big)^2$ on the left-hand side to gain an equation for the constants. After also matching this for the T and A representations we are left with three undetermined constants.

Fixing the constants using low spin quadrilinear operators

The last three constants in the ansatz will be fixed by using the fact that we have three non-degenerate quadrilinear operators, namely the singlet and traceless symmetric spin 0 operators, and the antisymmetric spin 1 operator. As briefly mentioned above, we will first compute the OPE coefficients and anomalous dimensions of these operators by performing a projection of the order ϵ^3 inversion onto twist four operators. With $a_{R,4,\ell}(\gamma_{R,4,\ell})^2$ at hand for the non-degenerate operators the last three constants in I_4^R are fixed by the decomposition (4.3.18).

Let us explain in more detail how to use the inversion formula to find the OPE coefficients and anomalous dimensions we need. Recall that the conformal blocks in the small z limit behave as $z^{\tau/2}$. In the usual inversion problem, for leading twist operators, we can therefore focus on just the leading z dependence. When considering the inversion problem for operators with higher twists we will need to study also the subleading z dependence. We will therefore make use of a projection, defined below, and of expressions for the subleading z behaviour of the conformal blocks, as derived in appendix B.2. For simplicity we will focus on just the singlet representation, since the method for the other representations is completely analogous.

On the right-hand side of the crossing equation, to cubic order, there are only the contributions from the identity operator $\mathbb{1}$ and the operators φ_S^2 and φ_T^2. On the left-hand side, there is a 'spillover' from the leading twist operators. To compute this, we use the special form (4.3.11) of the 4d blocks. Since a double discontinuity can only arise from an infinite sum over the spin, we need only consider the piece

$$G_{\tau,\ell}^{(4d),\text{enh.}}(z,\bar{z}) = -\frac{z\bar{z}}{z-\bar{z}} k_{\frac{\tau}{2}-1}(z) k_{\frac{\tau}{2}+\ell}(\bar{z}). \tag{4.3.22}$$

With this in mind, to find the CFT data of the leading twist operators we could have projected the crossing equation onto $-\frac{z\bar{z}}{z-\bar{z}} k_{\Delta_\varphi - 1}(z)$; however this is simply equivalent to considering the small z limit. For the CFT data of the quadrilinear operators, we

do need to actually perform this projection onto $-\frac{z\bar{z}}{z-\bar{z}}k_{\Delta_\varphi}(z)$, which can be achieved by using the orthogonality relation

$$\frac{1}{2\pi \mathrm{i}}\oint \frac{\mathrm{d}z}{z^2}k_\alpha(z)k_{1-\beta}(z)=\delta_{\alpha\beta}. \tag{4.3.23}$$

To order ϵ^2, the double discontinuity I_{φ^2} has a vanishing projection, while $I_{\mathbb{1}}$ simply gives the GFF contribution to the OPE coefficients. However, on the left-hand side of the crossing equation there is a non-zero contribution from the anomalous dimensions of the leading twist operators. In particular, the following sum has a non-zero projection onto $-\frac{z\bar{z}}{z-\bar{z}}k_{\Delta_\varphi}(z)$:

$$\sum_\ell a^{(0)}_{S,\ell}\gamma_{S,\ell}\partial_\tau G^{(4d),\text{enh.}}_{\tau,\ell}(z,\bar{z})\Big|_{\tau=2}\xrightarrow{\text{projects}}-\frac{1}{4}\frac{z\bar{z}}{z-\bar{z}}k_1(z)\sum_\ell a^{(0)}_{S,\ell}\gamma_{S,\ell}k_{1+\ell}(\bar{z}), \tag{4.3.24}$$

where we used the 4d value $\Delta_\varphi=1$ since the expression is already of order ϵ^2, and where the factor of $\frac{1}{4}$ arises from the projection of

$$\partial_\tau k_{\frac{\tau}{2}-1}(z)\big|_{\tau=2}=\frac{1}{4}k_1(z)+\frac{1}{2}k_0(z)\log z\,. \tag{4.3.25}$$

Implementing this we find that the correction to the OPE coefficients $\langle\!\langle a_{S,4,\ell}\rangle\!\rangle = a^{GFF}_{S,4,\ell}+a^{(2)}_{S,4,\ell}+\mathcal{O}(\epsilon^3)$ satisfies

$$-\frac{z\bar{z}}{z-\bar{z}}k_1(z)\sum_{\ell=0,2,\ldots}\left(a^{(2)}_{S,4,\ell}k_{2+\ell}(\bar{z})+\frac{1}{4}a^{(0)}_{S,\ell}\gamma_{S,\ell}k_{1+\ell}(z)\right)\dot{=}\,0\,, \tag{4.3.26}$$

where by $\dot{=}$ we mean an equality of double discontinuities. Note that both $a^{GFF}_{S,4,\ell}$ and $a^{(2)}_{S,4,\ell}$ are of order ϵ^2, and we have ignored higher order terms. From the kernel method [95], it follows that the double discontinuity of this sum can be computed by replacing the sum with an integral over $(0,\infty)$. Furthermore making a change $\ell\to\ell+1$ in one of the terms gives the condition

$$\int_0^\infty \mathrm{d}\ell\left(a^{(2)}_{S,4,\ell}+\frac{1}{4}a^{(0)}_{S,\ell+1}\gamma_{S,\ell+1}\right)k_{2+\ell}(\bar{z})\,\dot{=}\,0, \tag{4.3.27}$$

which yields the result

$$a^{(2)}_{S,4,\ell}=-\frac{1}{4}a^{(0)}_{S,\ell+1}\gamma_{S,\ell+1}\,. \tag{4.3.28}$$

By adding the GFF contribution, we then find the following results:

$$\begin{aligned}
\langle\!\langle a_{S,4,\ell} \rangle\!\rangle &= \frac{\Gamma(\ell+2)^2}{\Gamma(2\ell+3)} \left(\frac{1}{N}\gamma_\varphi + \frac{2g_S^2 + (N+2)(N-1)g_T^2}{4N^2(\ell+1)(\ell+2)} \right) + O(\epsilon^3), \\
\langle\!\langle a_{T,4,\ell} \rangle\!\rangle &= \frac{\Gamma(\ell+2)^2}{\Gamma(2\ell+3)} \left(\gamma_\varphi + \frac{2g_S^2 + (N-2)g_T^2}{4N(\ell+1)(\ell+2)} \right) + O(\epsilon^3), \\
\langle\!\langle a_{A,4,\ell} \rangle\!\rangle &= \frac{\Gamma(\ell+2)^2}{\Gamma(2\ell+3)} \left(-\gamma_\varphi + \frac{-2g_S^2 + (N+2)g_T^2}{4N(\ell+1)(\ell+2)} \right) + O(\epsilon^3),
\end{aligned} \tag{4.3.29}$$

where, as for the leading twist operators, the OPE coefficients of singlet and traceless operators vanish for odd spin, while those of the antisymmetric operators vanish for even spin. In these expressions $\gamma_\varphi = \Delta_\varphi - \frac{d-2}{2}$ is the anomalous dimension of φ, which starts at order ϵ^2.

To find the sums $\langle\!\langle a_{R,4,\ell}\gamma_{R,4,\ell} \rangle\!\rangle$, we perform the same procedure at cubic order. In this case it is convenient to specialize to terms in the crossing equation that contain a factor of $\log z$. On the right-hand side, the double discontinuity I_{φ^2} contains such terms, while on the left-hand side there is a 'spillover' contribution from the leading twist operators that arises as a dimensional correction to the conformal blocks:

$$\log z \sum_\ell a^{(0)}_{R,\ell} \frac{1}{2} \gamma_{R,\ell} \partial_d G^{(d)}_{2\Delta_\varphi,\ell}(z,\bar{z})\Big|_{d=4}. \tag{4.3.30}$$

To find this contribution, we make use of the known form of the leading z and subleading z behaviour of the conformal blocks[5], which allows us to find the projections onto $-\frac{z\bar{z}}{z-\bar{z}} k_{\Delta_\varphi}(z)$. Inverting the right-hand side with the inversion formula, and making the appropriate shifts in the summation over ℓ on the left-hand side, we then find the results (valid at order ϵ^3)

$$\begin{aligned}
\langle\!\langle a_{S,4,\ell}\gamma_{S,4,\ell} \rangle\!\rangle &= \frac{\Gamma(\ell+2)^2}{(\ell+1)(\ell+2)\Gamma(2\ell+3)N} \left(\frac{g_S^2\,(\epsilon - g_S)}{N} + \frac{(N-1)(N+2)}{2N} g_T^2\,(\epsilon - g_T) \right), \\
\langle\!\langle a_{T,4,\ell}\gamma_{T,4,\ell} \rangle\!\rangle &= \frac{\Gamma(\ell+2)^2}{(\ell+1)(\ell+2)\Gamma(2\ell+3)N} \left(g_S^2\,(\epsilon - g_S) + \frac{N-2}{2} g_T^2\,(\epsilon - g_T) \right), \\
\langle\!\langle a_{A,4,\ell}\gamma_{A,4,\ell} \rangle\!\rangle &= \frac{\Gamma(\ell+2)^2}{(\ell+1)(\ell+2)\Gamma(2\ell+3)N} \left(-g_S^2\,(\epsilon - g_S) + \frac{N+2}{2} g_T^2\,(\epsilon - g_T) \right).
\end{aligned} \tag{4.3.31}$$

[5]We give some details of this computation in appendix B.2.

Combining equations (4.3.29) with (4.3.31), the anomalous dimensions of the non-degenerate operators can be found, which then allows us to fix the final constants in the I_4^R, given in appendix B.3.

As a check of this computation one can substitute the leading order expansions of $(g_S, g_T, \gamma_\varphi)$, which can be computed from the double discontinuities in section 4.3.1, to find that[6]

$$\gamma_{S,4,0} = \epsilon + \mathcal{O}(\epsilon^2), \qquad \gamma_{T,4,0} = \frac{8}{N+8}\epsilon + \mathcal{O}(\epsilon^2), \qquad \gamma_{A,4,1} = \frac{2}{N+8}\epsilon + \mathcal{O}(\epsilon^2), \tag{4.3.32}$$

in perfect agreement with the literature [170–172][7].

4.4 Constraints from low spins

As explained in the previous section, we have computed all double-discontinuities needed to find the CFT data of leading twist operators to order ϵ^4. Applying the inversion procedure thus gives expressions for $U^{(p)}_{R,\bar{h}}$, and hence all CFT data to order ϵ^4 in terms of ϵ and the following parameters:

$$g_S, \quad g_T, \quad \gamma_\varphi, \quad \alpha_S, \quad \alpha_T. \tag{4.4.1}$$

Notice that all these parameters depend perturbatively on ϵ. In this section we will see how all these constants can be determined in terms of ϵ, to leading and subleading order. This will be achieved by writing down consistency conditions for operators with low spin.

Firstly, there are two symmetry currents amongst the leading twist operators, namely the stress-energy tensor (S, $\ell = 2$) and the global symmetry current (A, $\ell = 1$). Their scaling dimensions are protected, and therefore we get

$$\Delta_{S,2} = d, \qquad \Delta_{A,1} = d - 1, \tag{4.4.2}$$

[6] Recall that in our conventions, these anomalous dimensions are defined by $\Delta_{R,4,\ell} = 2\Delta_\varphi + 2 + \ell + \gamma_{R,4,\ell}$. Other conventions exist in the literature.

[7] The result for the antisymmetric spin $\ell = 1$ operator was computed using the method described in [172]. We thank P. Liendo for making us aware of this method, which can be used to find the anomalous dimension of any operator to first order in the ϵ-expansion.

valid to all orders in perturbation theory. For example, the first of these equations gives at leading order $\gamma_\varphi = (2g_S^2 + (N+2)(N-1)g_T^2)/(24N) + \mathcal{O}(\epsilon^3)$. Using this in the second equation then gives the leading order relation $g_T = \pm 2g_S/(N+2) + \mathcal{O}(\epsilon^2)$.

Secondly, the assumption that the results for the singlet and traceless symmetric representations can be analytically continued all the way down to spin zero gives the following self-consistency conditions

$$\Delta_{S,0} = 2\Delta_\varphi + g_S, \quad a_{S,0} = \frac{2}{N}(1+\alpha_S), \quad \Delta_{T,0} = 2\Delta_\varphi + g_T, \quad a_{T,0} = 2(1+\alpha_T). \tag{4.4.3}$$

Here the left-hand sides of all these equations are computed from the $U^{(p)}_{R,\bar{h}}$, and therefore depend on ϵ and the constants in (4.4.1).

The extrapolation to spin zero is subtle and works in the following way. Let us recall that the CFT data was computed in the variable $\bar{h} = \Delta_\varphi + \ell$, which is the most convenient variable for perturbative computations. However, the CFT data is most naturally expressed in terms of the (full) conformal spin $J_{\mathrm{f}}^2 = \bar{h}_{\mathrm{f}}(\bar{h}_{\mathrm{f}} - 1)$ [173], conveniently written in terms of $\bar{h}_{\mathrm{f}} = \bar{h} + \gamma_\ell/2 = (\Delta+\ell)/2$. Similar to the computation in [2], the expressions for the CFT data are analytically continued past the pole $\bar{h}_{\mathrm{f}} = 1$, and evaluated for spin zero. For example, the equation for the leading correction to the singlet scalar operators becomes

$$\Delta_{S,0} = 2\Delta_\varphi - \left.\frac{2g_S^2 + (N+2)(N-1)g_T^2}{2N\bar{h}_{\mathrm{f}}(\bar{h}_{\mathrm{f}}-1)}\right|_{\bar{h}_{\mathrm{f}} = \Delta_\varphi + \frac{g_S}{2}} = 2\Delta_\varphi + g_S, \tag{4.4.4}$$

which is solved at order ϵ.[8] Together with the completely analogous equation for g_T we get a coupled system of equations, which has two solutions:

$$(g_S, g_T) = (0,0) \qquad \text{and} \qquad (g_S, g_T) = \left(\frac{N+2}{N+8}, \frac{2}{N+8}\right)\epsilon + \mathcal{O}(\epsilon^2). \tag{4.4.5}$$

This result shows that, in the $4-\epsilon$ expansion, any CFT with global O(N) symmetry agrees to leading order with either the free theory or the critical O(N) model! We

[8] Notice that in this equation $\bar{h}_{\mathrm{f}} = 1 + \mathcal{O}(\epsilon)$, so that the denominator in the anomalous dimension is of order ϵ. γ_φ decouples from the equation at this order.

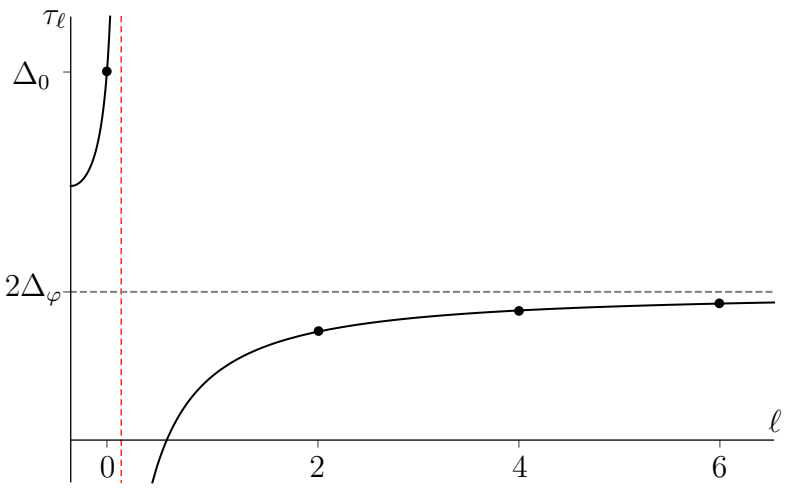


Figure 4.1: Schematic graph of τ_ℓ. As we move from spin two to spin zero we move to the left of the pole at $\bar{h}_f = 1$, denoted by a red line. Note the change of sign in the correction. Assuming the standard continuation in (4.4.4), we reproduce the correct dimension on both sides.

emphasize that this result has followed purely from conformal and global symmetry with no input from Feynman diagrams. A similar equation has been derived in the large N expansion of O(N) symmetric CFTs in $2 < d < 4$ [148].

By using the full fourth order results for both anomalous dimensions and OPE coefficients, we can solve all four equations (4.4.3) to subleading order. The results are

$$g_S = \frac{N+2}{N+8}\epsilon + \frac{6(N+2)(N+3)}{(N+8)^3}\epsilon^2 + O(\epsilon^3), \tag{4.4.6}$$

$$g_T = \frac{2}{N+8}\epsilon + \frac{36+4N-N^2}{(N+8)^3}\epsilon^2 + O(\epsilon^3), \tag{4.4.7}$$

$$\gamma_\varphi = \frac{N+2}{4(N+8)^2}\epsilon^2 + \frac{(N+2)(272+56N-N^2)}{16(N+8)^4}\epsilon^3 + O(\epsilon^4), \tag{4.4.8}$$

$$\alpha_S = -g_S - \frac{3}{N+2}g_S^2 + O(\epsilon^3), \tag{4.4.9}$$

$$\alpha_T = -g_T - \frac{N+6}{4}g_T^2 + O(\epsilon^3), \tag{4.4.10}$$

all in agreement with known results for the critical O(N) model. From this we write down the complete results for the CFT data, given in appendix B.4. We have also checked that the results agree with known results in the large N expansion [167, 169].

4.5 Central charges and Padé approximants

From the results in the previous section we can extract the values to order ϵ^4 of the central charges C_T and C_J. They are defined by the intrinsic normalization of the stress-energy tensor (S, $\ell = 2$) and the global symmetry current (A, $\ell = 1$) respectively and can be computed from their respective OPE coefficients using conformal Ward identities. Compared to the free theory in $d = 4 - \epsilon$ dimensions they take the values[9]

$$\begin{aligned}\frac{C_T}{C_{T,\text{free}}} = 1 &- \frac{5(N+2)}{12(N+8)^2}\epsilon^2 - \frac{(N+2)\left(7N^2+382N+1708\right)}{36(N+8)^4}\epsilon^3 \\ &- \frac{(N+2)\left(65N^4+5998N^3+309036N^2+2396800N+5440832\right)}{1728(N+8)^6}\epsilon^4 \\ &+ \frac{(N+2)\left(2N^3+43N^2+922N+3488\right)\zeta_3}{12(N+8)^5}\epsilon^4 + O(\epsilon^5),\end{aligned} \tag{4.5.1}$$

and

$$\begin{aligned}\frac{C_J}{C_{J,\text{free}}} = 1 &- \frac{3(N+2)}{4(N+8)^2}\epsilon^2 - \frac{(N+2)\left(N^2+132N+632\right)}{8(N+8)^4}\epsilon^3 \\ &+ \frac{(N+2)\left(11N^4+246N^3-13124N^2-126976N-310976\right)}{64(N+8)^6}\epsilon^4 \\ &+ \frac{(N+2)\left(7N^2+442N+1792\right)\zeta_3}{4(N+8)^5}\epsilon^4 + O(\epsilon^5).\end{aligned} \tag{4.5.2}$$

The expressions up to order ϵ^3 are in perfect agreement with the ones computed in [117], and the ϵ^4 values are new results. For $N = 1$ there is no global symmetry current, and the central charge C_T result reduces to the one in [2]. Expanding the results (4.5.1) and (4.5.2) for large N we also get agreement with the $1/N$ term in the O(N) model given in [168].

With the ϵ^4 results at hand we can compute some approximations for the value of the central charges in the strongly coupled theories in 3d. A direct approach is to consider the expressions (4.5.1) and (4.5.2) evaluated at $\epsilon = 1$. Including the

[9] To facilitate a comparison with results in the literature, we have substituted the results of the previous section, as well as the expressions (B.4.3)–(B.4.4) for g_S and g_T at order ϵ^3.

Table 4.1: Values for $C_T/C_{T,\text{free}}$ in three dimensions using different methods of computation. The numeric values are taken from [174].

N	ϵ^3 trunc.	ϵ^4 trunc.	Padé$_{[4,2]}$	Padé$_{[4,3]}$	Padé$_{[5,2]}$	Num. bootstrap
1	0.957933	0.953972	0.948725*	0.951076*	0.951751*	0.946600(22)
2	0.955556	0.951008	0.946610*	0.948404*	0.948791*	0.94365(13)
3	0.955111	0.950485	0.933861	0.939515	0.946438	0.94418(43)
4	0.955729	0.951342	0.938871	0.943453	0.948923	0.94581(71)
5	0.956919	0.952943	0.944714	0.948313	0.952052	0.9520(40)
6	0.958397	0.954911	0.949952	0.952772	0.955282	0.9547(41)
10	0.964792	0.963212	0.964425	0.965578	0.966169	
20	0.976230	0.977070	0.979761	0.980002	0.980052	

*The $N = 1, 2$ values denote the Padé$_{[3,1]}$, Padé$_{[3,2]}$ and Padé$_{[4,1]}$ respectively.

fourth order improves the ϵ^3-truncation results in [117], as can be seen in the first two columns of tables 4.1 and 4.2.

A more refined method to give approximate results in three dimensions is to use Padé approximants. To construct a Padé approximant for the central charge in the interval $2 < d < 4$, we define

$$\text{Padé}_{[m,n]}(d) = \frac{P_0 + P_1 d + P_2 d^2 + \ldots + P_m d^m}{1 + Q_1 d + Q_2 d^2 + \ldots + Q_n d^n} \tag{4.5.3}$$

and subsequently fix the $m + n + 1$ constants such that Padé$_{[m,n]}(d)$ smoothly interpolates between the known expansions at $d = 4 - \epsilon$ and at $d = 2 + \epsilon$. Our results for the central charges allow for the construction of Padé approximants of one order higher than previously available in the literature.

Let us start by discussing the central charge C_T associated to the normalization of the stress-energy tensor. The expression (4.5.1) in $d = 4 - \epsilon$ is valid for all N. For $N > 2$, the critical O(N) model is perturbatively described by the UV fixed-point of the O(N) symmetric non-linear sigma model of [175, 176][10]. This is perturbatively renormalizable in $d = 2 + \epsilon$, and the central charge has the expansion [161]

$$\left.\frac{C_T}{C_{T,\text{free}}}\right|_{d=2+\epsilon} = 1 - \frac{1}{N} + \frac{3(N-1)}{4N(N-2)}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{4.5.4}$$

[10]See [162] for an excellent review. In particular, it gives a description of this model in terms of the Lagrangian $\mathcal{L} = (\partial_\mu \phi^i)^2 + \lambda(\phi^i \phi_i - N)$, with $i = 1, \ldots, N$, and where λ is an auxiliary field.

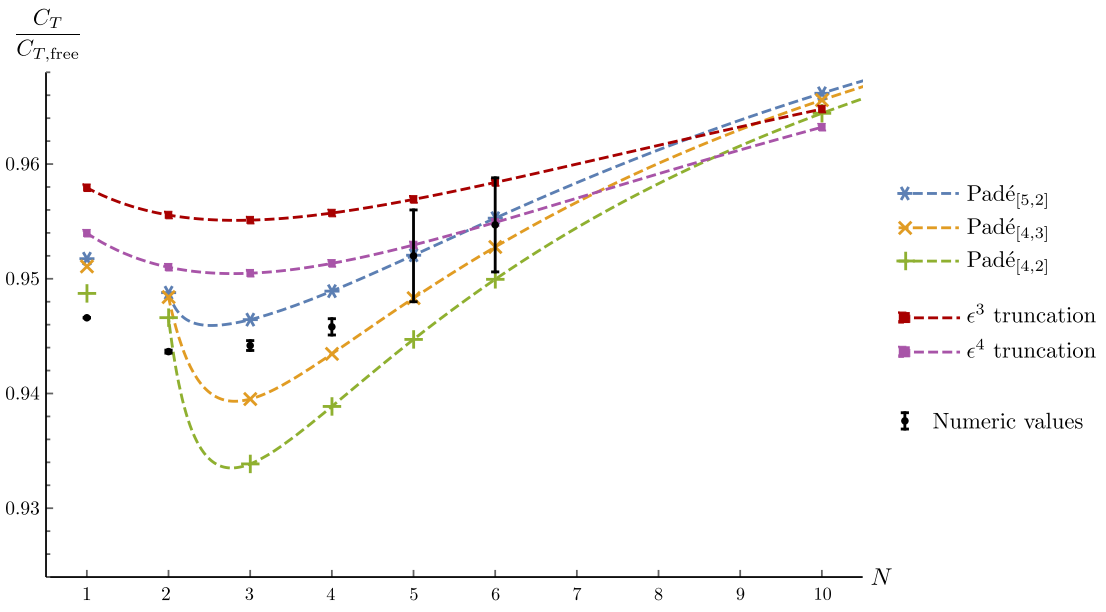


Figure 4.2: Padé approximants for $C_T/C_{T,\text{free}}$ in three dimensions. For $N = 1, 2$, the crosses show the values from the corresponding Padé approximants of lower orders, as given in table 4.1.

For $N = 1, 2$, we use the values for the two-dimensional Ising model and the XY model respectively. In both these cases we have $C_T/C_{T,\text{free}} = 1/2$ [177]. We are not aware of any perturbative expansions in $2 + \epsilon$ dimensions for these values of N,[11] and therefore the corresponding Padé approximants will be of two orders lower. In table 4.1 we present the values at $d = 3$ for different Padé approximants. One can see that for $3 \leqslant N \leqslant 6$ both new Padé approximants give values of the central charge closer to the values from numeric bootstrap than Padé$_{[4,2]}$, which is computed using only the ϵ^3 result in $d = 4 - \epsilon$.

In figure 4.2 we show the results in three dimensions plotted against N. It is interesting to note that $\text{Padé}_{[m,n]}(d)\big|_{N\to 2} = \text{Padé}_{[m-1,n-1]}(d)\big|_{N=2}$, namely that the Padé approximant constructed for $N > 2$ using (4.5.3) reduces in the limit $N \to 2$ to the Padé approximant for $N = 2$ of two orders lower. Furthermore, note that the Padé$_{[5,2]}$ approximation provides only a small improvement over the naive truncation

[11] Some numeric estimates are done in [178] for the $N = 1$ case in non-integer $2 < d < 4$, but there is no known analytic expression.

Table 4.2: Values for $C_J/C_{J,\text{free}}$ in three dimensions using different methods of computation. Numeric values are only available in the literature for $N = 2, 3, 20$ [180].

N	ϵ^3 trunc.	ϵ^4 trunc.	Padé$_{[3,2]}$	Padé$_{[3,3]}$	Padé$_{[4,2]}$	Num. bootstrap
2	0.925	0.919049				0.9050(16)
3	0.924740	0.919029	0.879552	0.896922	0.918319	0.9065(27)
4	0.926215	0.921103	0.893426	0.906422	0.920658	
5	0.928587	0.924237	0.905072	0.915022	0.924905	
6	0.931383	0.927839	0.914751	0.922546	0.929686	
10	0.942901	0.942106	0.940299	0.943830	0.946478	
20	0.962525	0.964566	0.966540	0.967555	0.968256	0.9674(8)

at order ϵ^4. We would expect however that as one increases the order, the Padé approximants would continue to improve in accuracy, while a naive truncation would not.[12]

Finally we also construct Padé approximants for $C_J/C_{J,\text{free}}$, whose values in three dimensions are shown in table 4.2 together with the truncated $4-\epsilon$ expansions. To construct these, we have used the expansion [161]

$$\left.\frac{C_J}{C_{J,\text{free}}}\right|_{d=2+\epsilon} = \frac{N-2}{N} + \frac{1}{N}\epsilon + \mathcal{O}(\epsilon^2) \tag{4.5.5}$$

from the non-linear sigma model for $N \geqslant 3$.

4.6 Discussion

In this chapter we used large spin perturbation theory to find the full CFT data of the leading twist operators in the O(N) model, to fourth order in the ϵ-expansion. An advantage of this approach is that the OPE coefficients, and the associated central charge corrections, are as easy to compute as anomalous dimensions, while they are generally significantly harder to compute using other methods.

Our derivation relies crucially on the inversion integrals from [97], which are guaranteed to converge for spin $\ell > 1$. However, by assuming that the expression for the

[12] An example is the anomalous dimension γ_φ, which is known to order ϵ^5 [179]. For small N, the size of the coefficients in this series actually has a minimum at order ϵ^4, suggesting that a simple truncation method may have an optimal value at or near the fourth order approximation.

CFT data can be continued to spin zero by re-expanding in the full conformal spin, we found that all CFT data agrees with that of either the free O(N) theory or the critical O(N) model. It would be nice to gain a better understanding of precisely when such a continuation is possible.

We also gave a new result for the central charge corrections. Using Padé approximants, we provided predictions for these corrections in the strongly coupled 3d models discussed in the introduction, and our predictions compare favourably with previous predictions and with the known numerical results.

It would of course be interesting to extend our method to compute CFT data at order ϵ^5. At this order, we do not expect that new operators contribute to the double discontinuity: the generalized free field theory OPE coefficients of the intermediate operators of approximate twists $\tau = 6, 8, 10, \ldots$ are all of order ϵ^4, so that they are expected to contribute to the double discontinuity first at order ϵ^6. We expect this pattern to persist in the interacting theory, though we have not ruled out the possibility that there are corrections to the OPE coefficients at order ϵ^3: a full analysis of the crossing equation to order ϵ^3 would be necessary to see if these coefficients indeed vanish to this order. If so, the main challenge of extending our results to ϵ^5 would be the technical problem of computing and inverting the double discontinuities of the existing operators. In particular, the contributions I_2 and I_4 can no longer be written as sums of purely four-dimensional conformal blocks, so one would need to take into account dimensional corrections to these blocks.

Finally, all our results are found to be compatible with known $1/N$ expansions, and it would be interesting to see if the expansions in terms of ϵ and $1/N$ also commute at higher orders than currently available. This problem is tackled partly in the next chapter, where we perform a bootstrap computation for CFT data of leading twist operators in the large N expansion.

Chapter 5

The critical O(N) model at large N to order $1/N^2$

5.1 Introduction

The purpose of this chapter is to study the critical O(N) model in the large N expansion, using large spin perturbation theory and the Lorentzian inversion formula. The critical O(N) model can be understood in the following way, as summarized in figure 5.1. In each space-time dimension $2 < d < 4$ and at each $N = 1, 2, \ldots$ there is an interacting conformal field theory with global O(N) symmetry and a field φ^i transforming in the fundamental (**N**) representation. These theories are continuously connected by varying the parameters d and N.[1] The case $N = 1$ corresponds to the Ising model, where we can continue down to $d = 2$ and arrive at the 2d Ising model. The upper limit $d = 4$ is the critical dimension for ϕ^4 theory, whereas in the lower limit $d = 2$ the twist gap between the stress tensor and the identity operator vanishes. Notice that at any d there is also the theory of N free scalars, and that below the multicritical dimensions $d = 3, \frac{8}{3}, \frac{5}{2}, \ldots$ multicritical O(N) symmetric CFTs also appear.[2]

[1] For generic, non-integer values of d and N, the theories are not expected to be unitary [144,181]. This manifests itself by the appearance of evanescent operators that may acquire complex dimensions and OPE coefficients. These operators do not appear at the order to which we are working in the $1/N$ expansion.

[2] It is believed that these theories cannot be reached from the large N expansion [182].

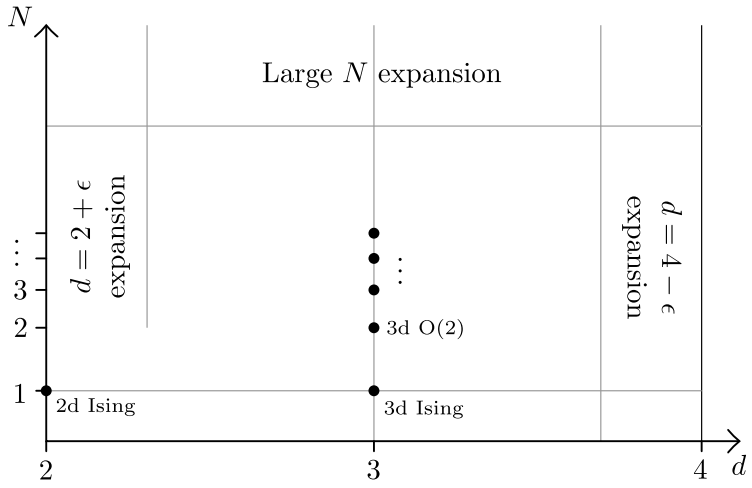


Figure 5.1: The O(N) model for various d and N, with the different perturbative regimes highlighted.

The O(N) model in general dimension can be described by perturbing the free theory Lagrangian by the operator φ^4 and following the RG flow to the IR. Performing this procedure in $d = 3$ and at fixed N, the model becomes strongly coupled in the IR [183], so that λ is not a useful parameter to study the model perturbatively. However, the theory can be studied perturbatively at the edges of figure 5.1, i.e. in $d = 2 + \epsilon$ or $d = 4 - \epsilon$ dimensions, or perturbatively around large N via the Hubbard-Stratonovich method described below.

As in the previous chapter, we will study the CFT data of the primary operators appearing in the OPE $\varphi^i \times \varphi^j$ of two fundamental fields. They transform in irreducible representations in the product

$$\mathbf{N} \otimes \mathbf{N} = S \oplus T \oplus A, \tag{5.1.1}$$

where $S = \mathbf{1}$ is the singlet representation, $T = \mathbf{N(N+1)/2 - 1}$ is the rank two symmetric traceless representation, and $A = \mathbf{N(N-1)/2}$ is the rank two antisymmetric representation of O(N). Since we consider the OPE of scalars, all relevant operators transform in the traceless symmetric Lorentz representation for integer spin ℓ, and we can therefore label the conformal primary operator content by (R, Δ, ℓ), where R

is S, T or A, and Δ is the scaling dimension.

We will primarily study the operators of leading twist, which include an infinite family of double field operators in each $\mathrm{O}(N)$ representation, parametrized by spin ℓ. The CFT data of these operators is described by functions analytic in spin, and the purpose of this paper is to compute these functions in the large N expansion.

In the $d = 4 - \epsilon$ expansion, the critical $\mathrm{O}(N)$ model can be described by a Lagrangian with coupling $\lambda(\varphi^i\varphi^i)^2$, where λ takes the value such that the β-function vanishes. This value is called the Wilson–Fisher fixed-point and is given perturbatively in ϵ. Primary operators can be constructed from the field φ^i and this process is subject to multiplet recombination [171].

The large N expansion, for any $2 < d < 4$, can be described in a Lagrangian picture using the Hubbard–Stratonovich method. To the action of N free fields φ^i, one adds the interaction terms

$$S_{\mathrm{I}} = \int \mathrm{d}^d x \left(\frac{1}{2\sqrt{N}} \sigma \varphi^i \varphi^i - \frac{1}{4\lambda N} \sigma^2 \right), \tag{5.1.2}$$

where σ is an auxiliary field. Integrating out σ gives an interaction $\lambda(\varphi^i\varphi^i)^2$. However, as described in detail in [184], for large N, σ gets promoted to a dynamical field and one can derive its effective action. Flowing to the IR, the second term in (5.1.2) becomes irrelevant, and the first term can be used to develop a perturbation theory with the coupling constant proportional to $N^{-1/2}$. The operator $\varphi^i\varphi^i$ is no longer part of the spectrum and has been replaced by σ.

Let us stress that the operator content is the same where the expansions overlap. For instance, $\Delta_\sigma = 2 + \gamma_\sigma^{(1)}/N + \ldots$, is known to order N^{-2} [185],[3] and when expanded in $d = 4 - \epsilon$, it agrees with the expansion of

$$\Delta_{\varphi_S^2} = d - 2 + \frac{N+2}{N+8}\epsilon + \ldots = 2 - 6\frac{\epsilon}{N} + \ldots \tag{5.1.3}$$

[3]Notice that there is a typo in the definition of the parameter α in [185]; as pointed out in [184] it should properly read $\alpha = \mu - 2 + \eta/2$.

to order ϵ^6/N^2, i.e. to the full overlap of known orders.

It has also been suggested that, at least for large N, one can extend the O(N) model to dimensions $4 \leqslant d < 6$. In particular the O(N) model can be matched to the $6-\epsilon$ expansion of a theory with interactions $\varphi^i\varphi^i\sigma$ and σ^3 [184].

For completeness, let us also mention that for $N \geqslant 3$ one can develop a perturbative expansion around $d = 2+\epsilon$, starting from the O(N) symmetric non-linear sigma model [162,175,176] of section 4.5, where some scaling dimensions are known to order ϵ^2 [160]. In three dimensions at finite N, the numeric conformal bootstrap provides results for some CFT data to very high precision [41], as well as numeric estimates for a larger set of operators [96]. Furthermore, in [186] the operator spectrum was numerically traced along the $N = 1$ line from $d = 3$ towards $d = 2$, where it was shown how the spectrum reorganizes into appropriate Virasoro primaries as one approaches the 2d Ising model.

The structure of this chapter is as follows. In what is left of this introduction we spell out exactly the assumptions about the O(N) model that we use to derive our results. In section 5.2 we briefly review the large spin perturbation theory framework in the context of the O(N) model, and derive results for all CFT data of leading twist operators at order N^{-1}. Already at this level, the results are non-trivial, and can be checked against known results computed by other methods. In particular, the N^{-1} correction to the OPE coefficients in the S representation is a new result. In section 5.3 we extend our results to order N^{-2} in the T and A representations. We find full results for the anomalous dimensions, while the OPE coefficients at this order are computed up to an undetermined function for which we give numeric estimates. As a special case, we derive a prediction in three dimensions for the N^{-2} correction to the charge C_J, associated to the global symmetry current:

$$\left.\frac{C_J}{C_{J,\text{free}}}\right|_{d=3} = 1 - \frac{64}{9\pi^2}\frac{1}{N} + \frac{1.14230(2)}{N^2} + O(N^{-3}). \tag{5.1.4}$$

5.1.1 Assumptions on the theory

In this chapter we use a minimal set of assumptions about the theory, from which all our results will follow. These assumptions mainly relate to the orders in N^{-1} at which operators appear in the OPE and acquire anomalous dimensions, as well as assuming the existence of a conserved stress tensor and global current. They are as follows:

- The fundamental field φ^i transforms in the fundamental representation of $\mathrm{O}(N)$ and has scaling dimension $\Delta_\varphi = \mu - 1 + \frac{1}{N}\gamma_\varphi^{(1)} + \frac{1}{N^2}\gamma_\varphi^{(2)} + O(N^{-3})$.

- The OPE of two fundamental fields contains the following operators:[4]

$$\begin{aligned}\varphi^i \times \varphi^j \sim \delta^{ij} &\left[\mathbb{1} + \frac{1}{N}\varphi^2_{S,\ell} + \frac{1}{N^2}[\varphi^2_S]_{n,\ell} + \frac{1}{N}\sigma + \frac{1}{N^2}[\sigma,\sigma]_{n,\ell} + O\left(\frac{1}{N^3}\right)\right] \\ &+ \left[(\varphi^2)^{\{ij\}}_{T,\ell} + \frac{1}{N}[\varphi^2_T]^{\{ij\}}_{n,\ell} + O\left(\frac{1}{N^2}\right)\right] \\ &+ \left[(\varphi^2)^{[ij]}_{A,\ell} + \frac{1}{N}[\varphi^2_A]^{[ij]}_{n,\ell} + O\left(\frac{1}{N^2}\right)\right]\end{aligned} \tag{5.1.5}$$

  Here the operators $[\varphi^2_R]_{n,\ell}$ are the tower of generalized free field operators with scaling dimensions $\Delta_{R,n,\ell} = 2\Delta_\varphi + 2n + \ell + \gamma_{R,n,\ell}$. Since we are primarily interested in the leading twist operators, we have separated the $n = 0$ case, so that $n = 1, 2, \ldots$. Furthermore ℓ is even for S and T, and odd for A. The fact that the operators with $n \geqslant 1$ appear at one order higher in N^{-1} follows from the explicit form of the result for the GFF OPE coefficients [90], since they are proportional to $\Delta_\varphi - (\mu - 1) = \gamma_\varphi \sim N^{-1}$. Alternatively, we can argue that they appear with vanishing OPE coefficients in the free theory and therefore must appear at order N^{-1} or higher.

- We will assume that the leading twist operators are non-degenerate and of the following form

[4] In this equation we have written out the operators with the appropriate squared OPE coefficients. That is, for instance $c^2_{\varphi,\varphi,\varphi^2_{S,\ell}} \sim N^{-1}$.

- Singlet (S): $\varphi^2_{S,\ell}$ for $\ell = 2, 4, 6, \ldots$. The scaling dimensions are $\Delta_{S,\ell} = 2\Delta_\varphi + \ell + \frac{1}{N}\gamma^{(1)}_{S,\ell} + O(N^{-2})$, and the OPE coefficients are $a_{S,\ell} := c^2_{\varphi\varphi\varphi^2_{S,\ell}} = \frac{1}{N}a^{(0)}_{S,\ell} + \frac{1}{N^2}a^{(1)}_{S,\ell} + O(N^{-3})$. Furthermore the $\ell = 2$ operator is the stress tensor $T_{\mu\nu}$ with $\Delta_{S,2} = d$ and OPE coefficient related to the central charge C_T through the conformal Ward identity $a_{S,2} = \frac{\mu^2\Delta^2_\varphi}{(2\mu-1)^2 C_T}$ [161].
- Traceless symmetric (T): $\varphi^2_{T,\ell}$ for $\ell = 0, 2, 4, \ldots$ with scaling dimensions $\Delta_{T,\ell} = 2\Delta_\varphi + \ell + \frac{1}{N}\gamma^{(1)}_{T,\ell} + \frac{1}{N^2}\gamma^{(2)}_{T,\ell} + O(N^{-3})$ and the OPE coefficients $a_{T,\ell} := a^{(0)}_{T,\ell} + \frac{1}{N}a^{(1)}_{T,\ell} + \frac{1}{N^2}a^{(2)}_{T,\ell} + O(N^{-3})$.
- Antisymmetric (A): $\varphi^2_{A,\ell}$ for $\ell = 1, 3, 5, \ldots$. The perturbative structure is the same as for T. The $\ell = 1$ operator is the global symmetry current $J^{[ij]}_\mu$, whose dimension is protected: $\Delta_{A,1} = d - 1$, and whose OPE coefficient is related to the charge C_J through the conformal Ward identity $a_{A,1} = -\frac{1}{2C_J}$ [161].

- We will assume that the scalar singlet operator σ appears in the OPE with $a_\sigma = \frac{1}{N}a^{(0)}_\sigma + \frac{1}{N^2}a^{(0)}_\sigma + O(N^{-3})$, and that it has dimension $\Delta_\sigma = 2 + \frac{1}{N}\gamma^{(1)}_\sigma + O(N^{-2})$. In the O($N$) model, both of these properties follow from the process of coupling to the Hubbard-Stratonovich field σ, but we take them as assumptions.

Furthermore, the existence of σ guarantees the existence of double field operators $[\sigma, \sigma]_{n,\ell}$ for $n = 0, 1, \ldots$ and $\ell = 0, 2, \ldots$. These are non-degenerate and their squared OPE coefficients $c^2_{\varphi\varphi[\sigma,\sigma]_{n,\ell}}$ start at order N^{-2}. In section 5.3.3 we need the explicit form of their OPE coefficients, and we extract them from the $1/N$ contribution to the $\langle\varphi\varphi\sigma\sigma\rangle$ correlator, which was first computed in [187].[5]

[5]This computation was done via Feynman diagrams, but could in principle also be done using purely bootstrap methods.

5.2 Large spin perturbation theory for the O(N) model

As in chapter 4, we aim to study the implications of crossing symmetry of the $\langle\varphi_i\varphi_j\varphi_k\varphi_l\rangle$ correlator on the CFT data of leading twist operators in the singlet, traceless symmetric and antisymmetric representations. The setup for the study of the crossing equation is similar to before: here we repeat some of the important parts.

We study the correlator of four fundamental fields, whose definition (4.2.1) we repeat here:

$$\langle\varphi_i(x_1)\varphi_j(x_2)\varphi_k(x_3)\varphi_l(x_4)\rangle = \frac{\mathcal{G}_{ijkl}(z,\bar{z})}{|x_{12}|^{2\Delta_\varphi}|x_{34}|^{2\Delta_\varphi}}, \tag{5.2.1}$$

with $\mathcal{G}_{ijkl}(z,\bar{z})$ satisfying the crossing symmetry equation (4.2.2). As before, the crossing equation decomposes into a set of coupled equations according to the O(N) representations of the intermediate operators, leading to equation (4.2.3), repeated here for convenience:

$$\begin{aligned}
\mathcal{G}_S(u,v) &= \left(\frac{u}{v}\right)^{\Delta_\varphi}\left(\frac{1}{N}\mathcal{G}_S(v,u) + \frac{(N+2)(N-1)}{2N^2}\mathcal{G}_T(v,u) + \frac{1-N}{2N}\mathcal{G}_A(v,u)\right),\\
\mathcal{G}_T(u,v) &= \left(\frac{u}{v}\right)^{\Delta_\varphi}\left(\mathcal{G}_S(v,u) + \frac{N-2}{2N}\mathcal{G}_T(v,u) + \frac{1}{2}\mathcal{G}_A(v,u)\right),\\
\mathcal{G}_A(u,v) &= \left(\frac{u}{v}\right)^{\Delta_\varphi}\left(-\mathcal{G}_S(v,u) + \frac{N+2}{2N}\mathcal{G}_T(v,u) + \frac{1}{2}\mathcal{G}_A(v,u)\right).
\end{aligned} \tag{5.2.2}$$

In each representation $R = S, T, A$, the operators of leading twist (excluding the identity) are the double field operators $\varphi^2_{R,\ell}$ with dimension $\Delta_{R,\ell} = 2\Delta_\varphi + \gamma_{R,\ell}$, where $\Delta_\varphi = \mu - 1 + \ell + \gamma_\varphi$ with $d = 2\mu$. In the small z limit, their contributions dominate the conformal block expansion

$$\mathcal{G}_R(z,\bar{z}) = z^{\Delta_\varphi}\sum_\ell\left(U^{(0)}_{R,\bar{h}} + \frac{1}{2}U^{(1)}_{R,\bar{h}}\log z + \frac{1}{8}U^{(2)}_{R,\bar{h}}\log^2 z + \dots\right) r_{\bar{h}}k_{\bar{h}}(\bar{z}) + \dots. \tag{5.2.3}$$

From these functions, the OPE coefficients and anomalous dimensions can be extracted using

$$\hat{a}_R(\bar{h})\,(\gamma_{R,\ell})^p = U^{(p)}_{R,\bar{h}} + \frac{1}{2}\partial_{\bar{h}}U^{(p+1)}_{R,\bar{h}} + \frac{1}{8}\partial^2_{\bar{h}}U^{(p+2)}_{R,\bar{h}} + \dots, \tag{5.2.4}$$

where $\hat{a}_R(\bar{h})$ is related to the squared OPE coefficient a_ℓ as

$$a_R(\ell) = c^2_{\varphi\varphi\varphi^2_{R,\ell}} = r_{\frac{\Delta_\ell+\ell}{2}} \hat{a}_R(\bar{h}). \tag{5.2.5}$$

Finally, we use the collinear inversion formula derived in chapter 4: if

$$G(\bar{z}) = \sum_\ell U_{\bar{h}}\, r_{\bar{h}}\, k_{\bar{h}}(\bar{z}), \qquad \bar{h} = \Delta_\varphi + \ell, \tag{5.2.6}$$

then the function $U_{\bar{h}}$ can be computed using

$$U_{\bar{h}} = \mathrm{INV}\left[G(\bar{z})\right] := \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)} \int_0^1 \frac{\mathrm{d}\bar{z}}{\bar{z}^2}\, k_{\bar{h}}(\bar{z})\, \mathrm{dDisc}\left[G(\bar{z})\right], \tag{5.2.7}$$

which we refer to as the inversion of $G(\bar{z})$.

To find the CFT data of the double field operators, we expand each of the equations (5.2.2) in the direct channel (left-hand side) as in (5.2.3). We then find the relevant double discontinuities by analyzing the crossed channel (right-hand side) of the equations in an expansion around infinite N. At each order in N^{-1}, only a few (families of) operators appear with a non-zero double discontinuity, which makes it possible to find the double discontinuities exactly order by order in N^{-1}.

5.2.1 Contributions from the identity operator

Let us start by considering the large N expansion of the crossing equation for the traceless symmetric (T) representation:

$$\mathcal{G}_T(u,v) = \left(\frac{u}{v}\right)^{\Delta_\varphi} \left(\mathcal{G}_S(v,u) + \frac{\mathcal{G}_T(v,u) + \mathcal{G}_A(v,u)}{2} - \frac{1}{N}\mathcal{G}_T(v,u)\right). \tag{5.2.8}$$

The only contributions up to order N^{-1} come from the identity operator and from σ, both appearing in the S representation. The contribution from these operators takes the form

$$\mathcal{G}_S(v,u) = 1 + \frac{a^{(0)}_\sigma}{N} v^{\Delta_\sigma/2} g^{(d)}_{\Delta_\sigma,0}(v,u) + O(N^{-2}), \tag{5.2.9}$$

where $G^{(d)}_{\tau,\ell}(u,v) = u^{\tau/2} g^{(d)}_{\tau,\ell}(u,v)$ is the conformal block in d dimensions.

The inversion of the identity contribution is straightforward, and involves computing the integral

$$A[p](\bar{h}) := \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)}\int_0^1 \frac{d\bar{z}}{\bar{z}^2}k_{\bar{h}}(\bar{z})\,\mathrm{dDisc}\left[\left(\frac{\bar{z}}{1-\bar{z}}\right)^p\right] = \frac{2(2\bar{h}-1)\Gamma(\bar{h}+p-1)}{\Gamma(p)^2\Gamma(\bar{h}-p+1)} \tag{5.2.10}$$

for $p=\Delta_\varphi$. The result $A[p](\bar{h})$ will be heavily used in the following and can easily be found by noting that $\mathrm{dDisc}[(1-\bar{z})^{-p}] = 2\sin^2(\pi p)(1-\bar{z})^{-p}$ and by using an integral representation for the hypergeometric function in $k_{\bar{h}}(\bar{z})$.

Using the different crossing prefactors for the S, T and A representations, we readily get[6]

$$NU^{(0)}_{S,\bar{h}} = U^{(0)}_{T,\bar{h}} = -U^{(0)}_{A,\bar{h}} = A[\Delta_\varphi](\bar{h}) + O(N^{-1}). \tag{5.2.11}$$

Extracting the OPE coefficients from this using (5.2.5), we find that to this order they agree with the result from a conformal block decomposition of the generalized free theory correlator

$$\mathcal{G}_S(u,v) = 1 + \frac{1}{N}u^{\Delta_\varphi} + \frac{1}{N}\left(\frac{u}{v}\right)^{\Delta_\varphi}, \qquad G_{T/A}(u,v) = u^{\Delta_\varphi} \pm \left(\frac{u}{v}\right)^{\Delta_\varphi}. \tag{5.2.12}$$

5.2.2 Contribution from the scalar σ

Let us now turn to the second term in (5.2.9). Although at this order in N we have $\Delta_\sigma = 2$, it is worth considering the general problem of inverting a scalar operator. To be precise, we would like to compute the contributions to $U^{(0)}_{\bar{h}}$ and $U^{(1)}_{\bar{h}}$ due to a scalar operator with dimension Δ appearing with OPE coefficient $a_\Delta = c^2_{\varphi\varphi\mathcal{O}_\Delta}$ in the crossed channel of the φ four-point function, i.e.

$$U^{(0)}_{\bar{h}}|_\Delta + \frac{1}{2}U^{(1)}_{\bar{h}}|_\Delta \log z = \frac{a_\Delta\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)}\int_0^1 \frac{d\bar{z}}{\bar{z}^2}k_{\bar{h}}(\bar{z})\,\mathrm{dDisc}\left[F_\Delta(z,\bar{z})\right]. \tag{5.2.13}$$

where

$$F_\Delta(z,\bar{z}) = \bar{z}^{\Delta_\varphi}(1-\bar{z})^{\frac{\Delta}{2}-\Delta_\varphi}\, g^{(d)}_{\Delta,0}(v,u)\Big|_{\text{small } z}. \tag{5.2.14}$$

[6] We have chosen a normalization for the conformal blocks that differs by a factor $(-2)^\ell$ from e.g. [29]. In our convention, this means that the squared OPE coefficients of the intermediate operators in the antisymmetric representation are negative, since they have odd spin.

The scalar conformal block for identical external scalars in $d = 2\mu$ dimensions was given in [29]:

$$g^{(d)}_{\Delta,0}(v,u) = \sum_{m,n=0}^{\infty} \frac{(\Delta/2)_m^2 \, (\Delta/2)_{m+n}^2}{m!n! \, (\Delta+1-\mu)_m \, (\Delta)_{2m+n}} v^m (1-u)^n, \tag{5.2.15}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol. Performing the sum over n and expanding for small z gives

$$g^{(d)}_{\Delta,0}(v,u)\Big|_{\text{small } z} = \sum_{m=0}^{\infty} -\frac{\Gamma(2m+\Delta)\left(\frac{\Delta}{2}\right)_m^2 \left(2S_1(m+\frac{\Delta}{2}-1)+\log(z\bar{z})\right)}{\Gamma\left(\frac{\Delta}{2}\right)^2 (\Delta)_{2m} (\Delta-\mu+1)_m m!} (1-\bar{z})^m, \tag{5.2.16}$$

where $S_1(n)$ denotes the analytic continuation of the harmonic numbers. If we perform the sum over m and use some identities for hypergeometric functions, we arrive at the closed-form expression

$$\begin{aligned} g^{(d)}_{\Delta,0}(v,u)\Big|_{\text{small } z} = -\frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^2} \Big[&\left(2S_1(\tfrac{\Delta}{2}-1)+\log(z\bar{z})\right) {}_2F_1\left(\tfrac{\Delta}{2},\tfrac{\Delta}{2};\Delta+1-\mu;1-\bar{z}\right) \\ &+ D_a \, {}_2F_1\left(\tfrac{\Delta}{2}+a,\tfrac{\Delta}{2};\Delta+1-\mu;1-\bar{z}\right)\Big], \end{aligned} \tag{5.2.17}$$

where $D_a f = \frac{\partial f}{\partial a}\big|_{a=0}$. We defer the solution of the inversion problem for general Δ until section 5.3.2, and focus now on the contribution from the operator σ. At leading order in N^{-1} we can take $a_\Delta = c^2_{\varphi\varphi\sigma} = \frac{a^{(0)}_\sigma}{N}$ and $\Delta = \Delta^{(0)}_\sigma = 2$. Furthermore, we may put $\Delta_\varphi = \mu - 1$. The expression inside the double discontinuity in (5.2.13) becomes

$$-\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1} (1-\bar{z}) \left[\log(z\bar{z}) \, {}_2F_1\left(1,1;3-\mu;1-\bar{z}\right) + 2D_a \, {}_2F_1\left(1+a,1;3-\mu;1-\bar{z}\right)\right]. \tag{5.2.18}$$

The integral in (5.2.13) now involves products of hypergeometric functions. In the remaining part of this paper we will encounter a variety of similar, and even more complicated, functions that enter the inversion integral. We develop different methods for computing these inversion integrals. In the body of the paper we will describe briefly these methods as we use them, but we refer further details to the comprehensive appendix C.1.

One method of finding the inversion of (5.2.18), which will also be useful in the inversion integrals that appear in the subsequent sections, is to make use of the collinear Casimir $\overline{D} = (1-\bar{z})\bar{z}^2\partial_{\bar{z}} - \bar{z}^2\partial_{\bar{z}}^2$, defined in equation (4.2.16). In the inversion integral (5.2.7), $\overline{D}$ is self-adjoint, so that we may act with it on $k_{\bar{h}}(\bar{z})$ instead, to produce $\overline{D}k_{\bar{h}}(\bar{z}) = \bar{h}(\bar{h}-1)k_{\bar{h}}(\bar{z})$ [188]. Therefore, if we know the inversion of $\overline{D}g(\bar{z})$, the inversion of $g(\bar{z})$ follows from simply dividing by $J^2 = \bar{h}(\bar{h}-1)$.[7] For instance, the $\log z$ piece of the above inversion satisfies

$$\overline{D}\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1}(\bar{z}-1)_2F_1\left(1,1;3-\mu;1-\bar{z}\right) = -(\mu-2)^2\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1}, \tag{5.2.19}$$

from which we immediately conclude that

$$U^{(1)}_{T/A,\bar{h}} = \mp 2(\mu-2)^2\frac{a^{(0)}_\sigma}{N}\frac{A[\mu-1](\bar{h})}{J^2}. \tag{5.2.20}$$

Here, as in (5.2.12), the upper sign refers to the T representation, and we used that the exact form of the crossing equation guarantees that the results in the A representation are equal up to a sign to this order in N^{-1}. Similarly for the non-$\log z$ piece we get

$$\overline{D}\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1}(\bar{z}-1)\left[\log\bar{z}\,{}_2F_1\left(1,1;3-\mu;1-\bar{z}\right) + 2D_a\,{}_2F_1\left(1+a,1;3-\mu;1-\bar{z}\right)\right] \tag{5.2.21}$$

$$= -\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1}\left((\mu-2)^2\log\bar{z} + (1-\bar{z})_2F_1\left(1,1;3-\mu;1-\bar{z}\right)\right). \tag{5.2.22}$$

The first term can now be integrated directly, for instance by replacing the $k_{\bar{h}}(\bar{z})$ by its defining infinite sum and integrating term by term. This gives $A[\mu-1](\bar{h})\mathbf{S_1}[\mu-1](\bar{h})$, where

$$\mathbf{S_1}[\alpha](\bar{h}) = 2S_1(\bar{h}-1) - S_1(\bar{h}+\alpha-2) - S_1(\bar{h}-\alpha) \tag{5.2.23}$$

is a combination of harmonic numbers with a large J expansion that contains only even powers of J and is free from $\log J$ terms. The second term is again of the form

[7] This holds up to the addition of functions in the kernel of $\overline{D}$. For a detailed discussion see appendix C.1.1.

(5.2.19) and will thus contribute with a term proportional to $A[\mu-1](\bar{h})/J^4$. In summary, this gives

$$U^{(0)}_{T/A,\bar{h}} = \pm A[\Delta_\varphi](\bar{h}) \pm (\mu-2)^2 \frac{a^{(0)}_\sigma}{N} \frac{A[\mu-1](\bar{h})}{J^2} \left(\mathbf{S_1}[\mu-1](\bar{h}) - \frac{1}{J^2} \right). \tag{5.2.24}$$

5.2.3 CFT data for T and A at N^{-1}

As discussed above, at order N^{-1}, the only operators that contribute to the CFT data in the T and A representations are the identity $\mathbb{1}$ and the auxiliary field σ. The CFT data at this order can therefore be computed from the functions $U^{(p)}_{T/A}$ as given in (5.2.20), (5.2.24) using the general relation (5.2.4).

For the anomalous dimensions we only need the N^0 term in (5.2.24), which can be evaluated at $\Delta_\varphi = \mu - 1$. This directly gives

$$\gamma_{T,\ell} = \gamma_{A,\ell} = -\frac{2(\mu-2)^2}{J^2} \frac{a^{(0)}_\sigma}{N} + O(N^{-2}), \tag{5.2.25}$$

where $J^2 = (\ell+\mu-1)(\ell+\mu-2)+O(N^{-1})$. The results for T (A) take even (odd) values for ℓ. Specifically, the $\ell = 1$ operator is the conserved global symmetry current $J^{[ij]}_\mu$, which has $\Delta_{A,1} = d-1$. Since $\Delta_{T/A,\ell} = 2\Delta_\varphi+\ell+\gamma_{T/A,\ell} = d-2+\ell+\gamma^{(1)}_\varphi/N+\gamma^{(1)}_{T/A,\ell}/N$, this imposes the relation

$$a^{(0)}_\sigma = \frac{\mu(\mu-1)}{(\mu-2)^2} \gamma^{(1)}_\varphi, \tag{5.2.26}$$

which we will use in the following.

The OPE coefficients up to order $1/N$ are extracted using (5.2.4) and (5.2.5),

$$\begin{aligned} a_{T/A,\ell} = \pm \frac{2\Gamma(\ell+\mu-1)^2\Gamma(\ell+2\mu-3)}{\Gamma(\mu-1)^2\Gamma(\ell+1)\Gamma(2\ell+2\mu-3)} \Big[1 + \frac{2\gamma^{(1)}_\varphi}{N} \Big(S_1(\ell+\mu-2) - S_1(\mu-2) \\ + \frac{\mu(\mu-1)}{J^2(\ell+\mu-1)} + \frac{(\ell-1)(\ell+2\mu-2)}{J^2} \left(S_1(\ell+2\mu-4) - S_1(2\ell+2\mu-4) \right) \Big) \Big] + \ldots, \end{aligned} \tag{5.2.27}$$

where, as always, the upper sign refers to T and the lower sign to A. The OPE coefficient for the global symmetry current is related to the charge C_J through the conformal Ward identity $a_{A,1} = -\frac{1}{2C_J}$ [161], from which we get

$$\frac{C_J}{C_{J,\text{free}}} = 1 - \frac{2(2\mu-1)}{\mu(\mu-1)} \frac{\gamma^{(1)}_\varphi}{N} + O(N^{-2}). \tag{5.2.28}$$

5.2.4 CFT data for S at N^{-1}

Let us consider the crossing equation (5.2.2) for the singlet representation in large N:

$$\mathcal{G}_S(u,v) = \left(\frac{u}{v}\right)^{\Delta_\varphi}\left(\frac{1}{N}\mathcal{G}_S(v,u) + \frac{1}{2}\left(\mathcal{G}_T(v,u) - \mathcal{G}_A(v,u)\right) + \frac{1}{2N}\left(\mathcal{G}_T(v,u) + \mathcal{G}_A(v,u)\right) + O(N^{-2})\right). \tag{5.2.29}$$

Now recall that the singlet tree-level OPE coefficients are of order N^{-1}, so that we study corrections to their CFT data by analyzing equation (5.2.29) at order N^{-2}.

On the RHS of this equation, the contributions to the double discontinuity of $\mathcal{G}_S(v,u)$ are from the identity and from σ, yielding the same contributions as in the previous section. The new term is part of the combination $\frac{1}{2}\left(\mathcal{G}_T - \mathcal{G}_A\right)$: let us all call this contribution I_{T-A}. It arises from the fact that the leading twist operators in the T and A have order N^{-1} corrections to their scaling dimensions, as found in section 5.2.3. These then generate a contribution with double discontinuity at order N^{-2} from expanding $(1-\bar{z})^{\tau_\ell/2} = (1-\bar{z})^{\tau_\ell^{(0)}}\left(1 + \frac{1}{2}\gamma_\ell \log(1-\bar{z}) + \frac{1}{8}\gamma_\ell^2 \log^2(1-\bar{z}) + \dots\right)$ and noting that $\log^2(1-\bar{z})$ has a non-trivial double discontinuity. Explicitly, the relevant contribution is

$$I_{T-A} = \left(\frac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\varphi}\frac{1}{2}\left(\sum_{\ell=0,2,\dots} a_{T,\ell}\gamma_{T,\ell}^2 G^{(d)}_{2\mu-2+\ell,\ell}(1-\bar{z},1-z) - \sum_{\ell=1,3,\dots} a_{A,\ell}\gamma_{A,\ell}^2 G^{(d)}_{2\mu-2+\ell,\ell}(1-\bar{z},1-z)\right)\frac{\log^2(1-\bar{z})}{8} + O(N^{-3}). \tag{5.2.30}$$

Now we make the following observation [148]: the anomalous dimensions in the representations T and A take the same functional form:

$$\gamma^{(1)}_{T,\ell} = \gamma^{(1)}_{A,\ell} = -\frac{2(\mu-1)\mu}{J^2}\gamma^{(1)}_\varphi. \tag{5.2.31}$$

Furthermore, since $\left(\gamma_{T/A,\ell}\right)^2 \sim \frac{1}{N^2}$, all other parts can be evaluated at tree level. Specifically we can use the tree level values of $a_{T,\ell}$ and $a_{A,\ell}$, which also have the same

functional form $a_{T,\ell} = -a_{A,\ell}$, so that

$$I_{T-A} = \frac{1}{N^2}\frac{\log^2(1-\bar{z})}{8}\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1}\sum_{\ell=0,1,2,\ldots}\frac{a^{(0)}_{T,\ell}}{2}\frac{4(\mu-1)^2\mu^2}{J^4}\left(\gamma^{(1)}_{\varphi}\right)^2 G^{(d)}_{2\mu-2+\ell,\ell}(1-\bar{z},1-z)$$
$$= \frac{1}{N^2}\frac{\log^2(1-\bar{z})}{8}\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1}\kappa\, H^{(2)}_{T-A}(1-\bar{z},1-z), \tag{5.2.32}$$

where $\kappa = 4(\mu-1)^2\mu^2\left(\gamma^{(1)}_{\varphi}\right)^2$, and where

$$H^{(m)}_{T-A}(z,\bar{z}) := \sum_{\ell=0,1,2,\ldots}\frac{a^{(0)}_{T,\ell}}{2J^{2m}}G^{(d)}_{2\mu-2+\ell,\ell}(z,\bar{z}) = \frac{1}{2}\left(H^{(m)}_{T}(z,\bar{z}) - H^{(m)}_{A}(z,\bar{z})\right) \tag{5.2.33}$$

are the twist conformal blocks defined in [89, 148]. In particular they satisfy the Casimir relation

$$\mathcal{C}^m H^{(m)}_{T-A}(z,\bar{z}) = H^{(0)}_{T-A}(z,\bar{z}) = \left(\frac{u}{v}\right)^{\mu-1}. \tag{5.2.34}$$

where $\mathcal{C}$ is the shifted conformal Casimir with eigenvalue is $J^2 = (\ell+\frac{\tau}{2})(\ell+\frac{\tau}{2}-1)$, given by [148]

$$\mathcal{C} = D_z + D_{\bar{z}} + (2\mu-2)\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right) - \frac{\tau(\tau+2-4\mu)}{4} \tag{5.2.35}$$

where $D_x = x^2(1-x)\partial_x^2 - x^2\partial_x$. In (5.2.34) we have $\tau = 2\mu-2$. Furthermore, since the intermediate operators are weakly broken conserved currents, i.e. their dimension in the free theory saturates the unitarity bound $\Delta_\ell = d-2+\ell$, this contribution can be expanded in terms of unitarity bound blocks, defined and computed in appendix C.3. Note that since we are only computing the CFT data of the leading twist operators, we only need to know the small z limit of $H^{(2)}_{T-A}(1-\bar{z},1-z)$. These facts can be combined to constrain $H^{(2)}_{T-A}(1-\bar{z},1-z)$ such that its small z limit can be found fully. For the full details of this computation we refer the reader to appendix C.4. The result is that

$$H^{(2)}_{T-A}(1-\bar{z},1-z)\Big|_{z^0} = \xi^{\mu-1}\left(g_0^{(-)}(\xi)\log z + f_0^{(-)}(\xi)\right), \tag{5.2.36}$$

where $\xi \equiv (1-\bar{z})/\bar{z}$, where

$$g_0^{(-)}(\xi) = -\frac{\pi\csc(\pi\mu)}{\mu-2}, \tag{5.2.37}$$

and where

$$f_0^{(-)}(\xi) = \pi \csc(\pi\mu)\left[\frac{\pi\cot(\pi\mu)}{\mu-2} - \frac{1}{(\mu-2)^2} + (\mu-1)\xi\,{}_4F_3\left(\begin{matrix}1,1,3-\mu,\mu\\2,2,2\end{matrix}\,\middle|\,-\xi\right)\right.$$
$$\left. - \frac{{}_2F_1(2-\mu,\mu-1;1;-\xi)-1}{(\mu-2)^2}\right]. \tag{5.2.38}$$

It is now trivial to solve for the contribution to $U_S^{(1)}$:

$$\begin{aligned} U_{S,\bar{h}}^{(1)} &= \frac{1}{N}U_{T,\bar{h}}^{(1)} + \frac{2}{N^2}\,\mathrm{INV}\left[\frac{\kappa}{8}\log^2(1-\bar{z})g_0^{(-)}(\xi)\right] \\ &= \frac{1}{N}U_{T,\bar{h}}^{(1)} - (2\bar{h}-1)\frac{4\pi\csc(\pi\mu)(\mu-1)^2\mu^2}{(\mu-2)J^2N^2}\left(\gamma_\varphi^{(1)}\right)^2. \end{aligned} \tag{5.2.39}$$

From this we can extract the first order correction to the anomalous dimensions

$$\gamma_{S,\ell}^{(1)} = -\frac{2\gamma_\varphi^{(1)}}{J^2}\left((\mu-1)\mu + \gamma_\varphi^{(1)}\frac{\pi\csc(\pi\mu)\Gamma(\mu+1)^2\Gamma(\ell+1)}{(\mu-2)\Gamma(\ell+2\mu-3)}\right), \tag{5.2.40}$$

where $J^2 = (\ell+\mu-1)(\ell+\mu-2)+O\left(N^{-1}\right)$. Conservation of the stress tensor implies that $\Delta_{S,2} = d$, i.e. $2\gamma_\varphi^{(1)} + \gamma_{S,2}^{(1)} = 0$. This leads to a quadratic equation for $\gamma_\varphi^{(1)}$, whose solutions are

$$\gamma_\varphi^{(1)} = 0, \quad \gamma_\varphi^{(1)} = \frac{(\mu-2)\Gamma(2\mu-1)}{\Gamma(\mu+1)\Gamma(\mu)^2\Gamma(1-\mu)}. \tag{5.2.41}$$

We recognize this as the known results for the free theory and the interacting critical O(N) theory. Hence we have shown that, among interacting perturbative large N theories with O(N) symmetry in dimensions $2 < d < 4$, the leading twist spectrum is uniquely determined to be that of the critical O(N) model. Using the non-trivial solution for $\gamma_\varphi^{(1)}$ from (5.2.41), we can check that all results computed above agree with their literature values. In particular the singlet anomalous dimensions (5.2.40) agree with the value given in [122] and the non-singlet OPE coefficients (5.2.27) agree with the results of [117].[8]

It is interesting to note that if we assume that the the result for $\Delta_{S,\ell}$ can be extrapolated down to spin 0, and that the shadow relation $\Delta_\sigma = d - \Delta_{S,0}$ holds also

[8]Notice that there is a typo in equation (4.21) of [117]. We thank the authors for sharing with us a file with the correct result.

at order N^{-1}, then we can compute

$$\gamma_\sigma^{(1)} = \frac{4(\mu-1)(2\mu-1)}{\mu-2}\gamma_\varphi^{(1)} \tag{5.2.42}$$

in agreement with the literature value [122, 189].

For the OPE coefficients, we invert the expression (5.2.38). Using the inversion methods from appendix C.1, we find $U_S^{(0)}$:

$$\begin{aligned} U_{S,\bar{h}}^{(0)} &= \frac{1}{N}U_{T,\bar{h}}^{(0)} + \frac{1}{N^2}\text{INV}\left[\frac{\kappa}{8}\log^2(1-\bar{z})f_0^{(-)}(\xi)\right] \\ &= \frac{1}{N}U_{T,\bar{h}}^{(0)} + \frac{2(2\bar{h}-1)\mu^2(\mu-1)^2\pi\csc(\pi\mu)}{(\mu-2)^2N^2}\left(\gamma_\varphi^{(1)}\right)^2\left[\frac{(\mu-2)\pi\cot(\pi\mu)}{J^2}\right. \\ &\qquad \left. -\frac{1}{J^2} + (\mu-2)\frac{\mathbf{S_1}[\mu-1](\bar{h})-1/J^2}{J^2}\right]. \end{aligned} \tag{5.2.43}$$

Hence we find the full OPE coefficients:

$$\begin{aligned} a_{S,\ell} = \frac{1}{N}a_{\mu,\ell}^{\text{free}} + \frac{1}{N^2}\Bigg[& 2\,a_{\mu,\ell}^{\text{free}}\gamma_\varphi^{(1)}\Bigg(S_1(\ell+\mu-2) + \frac{\ell^2+2\mu^2+2(\mu-1)\ell-3\mu+1}{(\ell+\mu-1)^2(2\ell+2\mu-3)} \\ & - S_1(\mu-2) + \frac{(\ell-1)(2\mu+\ell-2)\left(S_1(\ell+2\mu-4)-S_1(2\ell+2\mu-3)\right)}{(\ell+\mu-1)(\ell+\mu-2)}\Bigg) \\ & + \frac{2\pi(\mu-1)^2\mu^2\csc(\pi\mu)\Gamma(\ell+\mu-1)^2}{(\mu-2)(\ell+\mu-2)(\ell+\mu-1)\Gamma(2\ell+2\mu-3)}\left(\gamma_\varphi^{(1)}\right)^2\Bigg(\frac{2(\ell+\mu-2)}{(\ell+\mu-1)(2\ell+2\mu-3)} \\ & + \frac{1}{2-\mu} + \pi\cot(\pi\mu) + 2S_1(2\ell+2\mu-3) - S_1(\ell+2\mu-4) - S_1(\ell)\Bigg)\Bigg] + O(N^{-3}), \end{aligned} \tag{5.2.44}$$

where $a_{\mu,\ell}^{\text{free}} = \frac{2\Gamma(\ell+\mu-1)^2\Gamma(\ell+2\mu-3)}{\Gamma(\ell+1)\Gamma(\mu-1)^2\Gamma(2\ell+2\mu-3)}$ are the OPE coefficients in the theory of a single free scalar field.

From this result we can also compute the order $1/N$ correction to the central charge:

$$\frac{C_T}{C_{T,\text{free}}} = 1 + \frac{\gamma_\varphi^{(1)}}{\mu(\mu+1)}\left(\frac{4}{\mu-2} - \mu - 2\pi\mu\cot(\pi\mu) - 2\mu\,S_1(2\mu-2)\right)\frac{1}{N} + O(N^{-2}), \tag{5.2.45}$$

matching the known result [168].

All the results for the CFT data at order N^{-1} can be checked to agree perfectly with the known fourth-order results in $d = 4-\epsilon$ [3].

5.3 Non-singlet CFT data at order N^{-2}

In this section we move on to computing the CFT data at order N^{-2}. We will consider only the T and A representations, whose crossing equations read

$$\mathcal{G}_{T/A}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\varphi}\left(\pm\mathcal{G}_S(v,u) + \frac{\mathcal{G}_T(v,u)+\mathcal{G}_A(v,u)}{2} \mp \frac{1}{N}\mathcal{G}_T(v,u)\right) \tag{5.3.1}$$

On the right-hand side, the singlet (S) representation now contains three different contributions. Firstly the identity operator gives $A[\Delta_\varphi](\bar{h})$ in $U^{(0)}_{\bar{h}}$ to any order. Secondly, there are subleading corrections to the contribution from σ: we denote this contribution I_σ and compute it in section 5.3.2. Thirdly the operators $[\sigma,\sigma]_{n,\ell}$ will contribute at this order. To find their contribution $I_{[\sigma,\sigma]_{n,\ell}}$ we need the compute a double sum over $n = 0, 1, 2, \ldots$ and $\ell = 0, 2, 4, \ldots$. As discussed in section 5.3.3, this sum can be given in a closed form as an integral in Mellin space, but the inversion is highly non-trivial.

The operators contributing in the T and A representations on the right-hand side of (5.3.1) are the leading twist operators themselves, whose contribution is suppressed by $\gamma^2_{T/A} \sim N^{-2}$. We can therefore ignore the last term in (5.3.1), and note that the contribution from the T and A representation takes a special form I_{T+A}, similar to what we saw in section 5.2.4. This means that we will be able to determine this contribution in a similar way to I_{T-A}, which we do first.

5.3.1 Inverting $H^{(2)}_{T+A}$

Consider the contributions from the T and A representations in (5.3.1), which at order N^{-2} will arise from the squares of anomalous dimensions, exactly as for I_{T-A} in section 5.2.4. The computation of the double discontinuity is almost identical to section 5.2.4, with a contribution

$$I_{T+A} = \frac{1}{N^2}\frac{\log^2(1-\bar{z})}{8}\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1}\kappa\, H^{(2)}_{T+A}(1-\bar{z},1-z), \tag{5.3.2}$$

where the twist conformal blocks $H^{(m)}_{T+A}$ are defined exactly analogously to the $H^{(m)}_{T-A}$:

$$H^{(m)}_{T+A}(z,\bar{z}) := \sum_{\ell=0,1,2,\dots} \frac{1}{2}\left(a^{(0)}_{T,\ell}\delta_{\ell,\text{even}} + a^{(0)}_{A,\ell}\delta_{\ell,\text{odd}}\right)\frac{1}{J^{2m}}G^{(d)}_{2\mu-2+\ell,\ell}(z,\bar{z}), \tag{5.3.3}$$

which satisfy the equations

$$\mathcal{C}^m H^{(m)}_{T+A}(z,\bar{z}) = H^{(0)}_{T+A}(z,\bar{z}) = u^{\mu-1}. \tag{5.3.4}$$

for $\mathcal{C}$ defined in (5.2.35).

The twist conformal block $H^{(2)}_{T+A}(1-\bar{z},1-z)$ is found in a very similar way to the TCB $H^{(2)}_{T-A}(1-\bar{z},1-z)$, and details of the computation, which relies crucially on the fact that the intermediate operators saturate the unitarity bound, can be found in appendix C.4. The result is that

$$H^{(2)}_{T+A}(1-\bar{z},1-z)\Big|_{z^0} = \xi^{\mu-1}\left(g^{(+)}_0(\xi)\log z + f^{(+)}_0(\xi)\right), \tag{5.3.5}$$

where

$$g^{(+)}_0(\xi) = -\frac{1}{(\mu-2)^2}, \tag{5.3.6}$$

and[9]

$$\begin{aligned} f^{(+)}_0(\xi) = \frac{1}{(\mu-2)^2}\Bigg[&\frac{(\mu-2)^2(S_2(\mu-2)-\zeta_2)+\mu-3}{\mu-2}\,{}_2F_1(2-\mu,\mu-1;1;-\xi)\\ &+\frac{1}{2}(\mu-2)(\mu-1)\mu\left(\overline{D}-K^2\right)^{-1}\left[\xi\,{}_3F_2\left(\begin{matrix}1,1,\mu+1\\2,3\end{matrix}\,\middle|\,-\xi\right)\right]\\ &-\frac{\mu-3}{\mu-2}-S_1(\mu-2)\Bigg], \end{aligned} \tag{5.3.7}$$

where $K^2=(\mu-1)(\mu-2)$, and where $\overline{D}$ was defined in (4.2.16). The contribution to $U^{(1)}_{T/A,\bar{h}}$ is straightforwardly found as the inversion of $g^{(+)}_0$:

$$\begin{aligned} U^{(1),(T+A)}_{T/A,\bar{h}} &= 2\,\text{INV}\left[\frac{\kappa}{8}\log^2(1-\bar{z})g^{(+)}_0(\xi)\right]\\ &= -(2\bar{h}-1)\frac{4(\mu-1)^2\mu^2}{J^2(\mu-2)^2}\left(\gamma^{(1)}_\varphi\right)^2. \end{aligned} \tag{5.3.8}$$

[9]Here $\left(\overline{D}-K^2\right)^{-1}[h(\xi)]$ is defined to be the solution q to the linear ODE $\overline{D}\,q(\xi)-K^2q(\xi)=h(\xi)$ with a boundary condition that is described in appendix C.4.2.

The contribution to $U^{(0)}_{T/A,\bar{h}}$ is more complicated, since it involves the inversion of a complicated hypergeometric ${}_3F_2$. The result is

$$\begin{aligned} U^{(0),(T+A)}_{T/A,\bar{h}} &= \frac{1}{N^2}\text{INV}\left[\frac{\kappa}{8}\log^2(1-\bar{z})f_0^{(+)}(\xi)\right] \\ &= \frac{2\bar{h}-1}{N^2}\frac{\mu^2(\mu-1)^2}{(\mu-2)^3}\left(\gamma_\varphi^{(1)}\right)^2\left(\frac{2\left(3-\mu-(\mu-2)S_1(\mu-2)\right)}{J^2}\right. \\ &\left.+\frac{2\left(\mu-3+(\mu-2)^2\left(S_2(\mu-2)-\zeta_2\right)\right)}{J^2-K^2}+\frac{\mu(\mu-1)(\mu-2)^2B(\bar{h},\mu)}{J^2-K^2}\right), \end{aligned} \tag{5.3.9}$$

where we defined

$$(2\bar{h}-1)B(\bar{h},\mu)=\text{INV}\left[\frac{1}{4}\log^2(1-\bar{z})\,\xi\,{}_3F_2\left(\begin{matrix}1,1,\mu+1\\2,3\end{matrix}\,\middle|\,-\xi\right)\right]. \tag{5.3.10}$$

The inversion of this can be found via various methods, e.g. via the Mellin space method of appendix C.1.4, or by hitting it once with the Casimir $\overline{D}$, collecting powers of $\bar{z}$ and $(1-\bar{z})$, and inverting term-by-term as in appendix C.1.3.

5.3.2 Contribution from scalar block revisited

The contribution from σ at order N^{-2} will contain subleading corrections for both Δ_φ and Δ_σ, compared with what we saw at tree-level in section 5.2.2.

To compute the CFT data at order N^{-2} for the T and A representations, we need to evaluate the $U^{(p)}_{\bar{h}}|_\Delta$ for $\Delta_\varphi=\mu-1+\gamma^{(1)}_\varphi/N$ and $\Delta=2+\gamma^{(1)}_\sigma/N$. We can truncate these expansions to this order since a_σ is already of order N^{-1}. Inserting these expansions in (5.2.17) gives a well-defined inversion problem at $O(N^{-2})$.

The $\log z$ computation can be done explicitly, for example by identifying the asymptotic J^2 series. The result is

$$\begin{aligned} U^{(1)}_{\bar{h}}|_{\Delta_\sigma} = 2(\mu-2)^2&\left(\frac{a^{(0)}_\sigma}{N}+\frac{a^{(1)}_\sigma}{N^2}\right)\frac{A[\Delta_\varphi](\bar{h})}{J^2}\left[-1+\frac{\gamma^{(1)}_\varphi}{N}\left(\frac{1}{J^2}-\frac{2}{\mu-2}\right)\right. \\ &\left.+\frac{\gamma^{(1)}_\sigma}{N}\left(\frac{3-\mu}{\mu-2}+S_1(\bar{h}-1)-S_1(\mu-2)\right)\right]+O(N^{-3}) \end{aligned} \tag{5.3.11}$$

where $a_\sigma^{(0)}$ is given by (5.2.26) and $\gamma_\sigma^{(1)}$ and $a_\sigma^{(1)}$ are yet to be determined. Here we have multiplied out the factor $A[\Delta_\varphi](\bar{h})$ rather than its tree-level counterpart $A[\mu-1](\bar{h})$, which simplifies the term proportional to $\gamma_\varphi^{(1)}$.

For $U_{\bar{h}}^{(0)}|_{\Delta_\sigma}$, the Mellin space inversion method fails due to the appearance of the $\log \bar{z}$ term. By expanding in ξ we can generate a J^2 series for this, which we again recognize:

$$\begin{aligned} U_{\bar{h}}^{(0)}|_{\Delta_\sigma} = (\mu-2)^2 \left(\frac{a_\sigma^{(0)}}{N} + \frac{a_\sigma^{(1)}}{N^2}\right) \frac{A[\Delta_\varphi](\bar{h})}{J^2} \Bigg[& \mathbf{S_1}[\Delta_\varphi](\bar{h}) - \frac{1}{J^2} \\ & - \frac{\gamma_\sigma^{(1)}}{N} \left(\zeta_2 + \left(\mathbf{S_1}[\Delta_\varphi](\bar{h}) - \frac{1}{J^2}\right)\left(\frac{3-\mu}{\mu-2} + S_1(\bar{h}-1) - S_1(\mu-2)\right)\right) \\ & + \frac{\gamma_\varphi^{(1)}}{N} \left(\frac{2\mathbf{S_1}[\Delta_\varphi](\bar{h})}{\mu-2} - \frac{\mathbf{S_1}[\Delta_\varphi](\bar{h})}{J^2} + \frac{2}{J^4} + \frac{2(\mu-3)}{(\mu-2)J^2}\right) \Bigg] + O(N^{-3}) \end{aligned} \tag{5.3.12}$$

where $\mathbf{S_1}$ was defined in (5.2.23).

5.3.2.1 Relation to crossing kernels

Let us be a bit more general and consider the problem of inverting the contribution from a single scalar operator with generic dimension Δ appearing in the crossed channel of the correlator of identical scalars with dimension Δ_φ.

> *What contribution to the CFT data of leading twist operators follows from the appearance of a single scalar operator in the crossed channel?*

Recall that this problem amounts to computing (5.2.13), where the scalar block in the small z limit is given by (5.2.17). In the case of $U_{\bar{h}}^{(1)}|_\Delta$ one can compute the integral in terms of two ${}_4F_3$ functions using the Mellin space methods in appendix C.1.4. Here we will proceed with an alternative approach that also generalizes to $U_{\bar{h}}^{(0)}|_\Delta$. Replacing the hypergeometric functions by their defining infinite sums in both the inversion kernel $\bar{z}^{-2}k_{\bar{h}}(\bar{z})$ and the one in the $\log z$ piece of (5.2.16), we get the following

expression for $U^{(1)}_{\bar{h}}|_\Delta$ after exchanging the order of summation and integration

$$U^{(1)}_{\bar{h}}|_\Delta = -4a_\Delta \sin^2\left(\pi(\Delta_\varphi - \tfrac{\Delta}{2})\right) \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)} \frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^2} \times \sum_{m,n=0}^{\infty} \frac{(\bar{h})_n^2}{(2\bar{h})_n (1)_n} \frac{(\frac{\Delta}{2})_m^2}{(1+\Delta-\mu)_m (1)_m} \int_0^1 \mathrm{d}\bar{z}\ \bar{z}^{\Delta_\varphi+\bar{h}-2+n}(1-\bar{z})^{\frac{\Delta}{2}-\Delta_\varphi+m}. \tag{5.3.13}$$

The integral mixes the m and n dependence, which means that the result is a double infinite sum that defines a *Kampé de Fériet function*:

$$U^{(1)}_{\bar{h}}|_\Delta = -4a_\Delta \sin^2\left(\pi(\Delta_\varphi - \tfrac{\Delta}{2})\right) \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)} \frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^2} \frac{\Gamma(1+\frac{\Delta}{2}-\Delta_\varphi)\Gamma(\Delta_\varphi+\bar{h}-1)}{\Gamma(\frac{\Delta}{2}+\bar{h})} \times F^{03}_{11}\left(\frac{\frac{\Delta}{2}+\bar{h}}{} \middle| \begin{matrix} \{\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{\Delta}{2}-\Delta_\varphi+1\}, \{\bar{h}, \bar{h}, \bar{h}+\Delta_\varphi-1\} \\ \{\Delta-\mu+1\}, \{2\bar{h}\} \end{matrix} \middle| 1, 1 \right). \tag{5.3.14}$$

The Kampé de Fériet function is a hypergeometric function of two variables defined by [190]

$$F^{pq}_{rs}\left(\begin{matrix} a_1, \ldots, a_p \\ c_1, \ldots c_r \end{matrix} \middle| \begin{matrix} \{b_1, \ldots, b_q\}, \{b'_1, \ldots, b'_q\} \\ \{d_1, \ldots, d_s\}, \{d'_1, \ldots, d'_s\} \end{matrix} \middle| x, y \right) = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n} \cdots (a_p)_{m+n}}{(c_1)_{m+n} \cdots (c_r)_{m+n}} \frac{(b_1)_m (b'_1)_n \cdots (b_q)_m (b'_q)_n}{(d_1)_m (d'_1)_n \cdots (d_s)_m (d'_s)_n} \frac{x^m y^n}{m! n!}. \tag{5.3.15}$$

The problem for $U^{(0)}_{\bar{h}}|_\Delta$ is analogous. The terms $S_1(m+\frac{\Delta}{2}-1)$ and $\log \bar{z}$ can be rewritten in terms of derivatives of a Gamma function and $\bar{z}$ power respectively, and after some simplifications we get

$$U^{(0)}{}_{\bar{h}}|_\Delta = -2a_\Delta \sin^2\left(\pi(\Delta_\varphi - \tfrac{\Delta}{2})\right) \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)} \frac{\Gamma(\Delta)}{\Gamma(\frac{\Delta}{2})^2} \frac{\Gamma(1+\frac{\Delta}{2}-\Delta_\varphi)\Gamma(\Delta_\varphi+\bar{h}-1)}{\Gamma(\frac{\Delta}{2}+\bar{h})} \times \Bigg\{ D_a F^{03}_{11}\left(\frac{\frac{\Delta}{2}+\bar{h}+a}{} \middle| \begin{matrix} \{\frac{\Delta}{2}, \frac{\Delta}{2}+2a, \frac{\Delta}{2}-\Delta_\varphi+1\}, \{\bar{h}, \bar{h}, \bar{h}+\Delta_\varphi-1+a\} \\ \{\Delta-\mu+1\}, \{2\bar{h}\} \end{matrix} \middle| 1, 1 \right) + \left[2S_1(\tfrac{\Delta}{2}-1) - S_1(\tfrac{\Delta}{2}+\bar{h}-1) + S_1(\bar{h}+\Delta_\varphi-2)\right] \times F^{03}_{11}\left(\frac{\frac{\Delta}{2}+\bar{h}}{} \middle| \begin{matrix} \{\frac{\Delta}{2}, \frac{\Delta}{2}, \frac{\Delta}{2}-\Delta_\varphi+1\}, \{\bar{h}, \bar{h}, \bar{h}+\Delta_\varphi-1\} \\ \{\Delta-\mu+1\}, \{2\bar{h}\} \end{matrix} \middle| 1, 1 \right) \Bigg\}. \tag{5.3.16}$$

Let us emphasize that the expressions (5.3.14) and (5.3.16) for $U^{(0)}_{\bar{h}}|_\Delta$ are completely general, valid within any four-point function of identical scalar operators, and without reference to any perturbative parameters.

The problem above is closely related to crossing kernels and $6j$ symbols [66, 104–108]. In [106] expressions were given for the leading order anomalous dimension of a double field operator from a generic operator in the crossed channel. In order to compare with these results we have to make some further assumptions.

If we assume that the $\varphi\times\varphi$ OPE contains the identity and the scalar of dimensions Δ and OPE coefficient a_Δ, as well as the GFF operators $[\varphi,\varphi]_{n,\ell}$, but no further operators until order a_Δ^2, then at order a_Δ the anomalous dimensions are given by

$$\gamma_\ell|_\Delta = \frac{U_{\bar{h}}^{(1)}|_\Delta}{A[\Delta_\varphi](\bar{h})} + O\big(a_\Delta^2\big). \tag{5.3.17}$$

Furthermore, up to this order, the contributions from several scalar operators are additive provided their OPE coefficients are of the same order in expansion. One can check that the expression (5.3.17) agrees with the result for leading order anomalous dimensions of the leading twist operators [106].

Notice, however, that the result (5.3.17) is only valid for the anomalous dimensions to leading orders in a_Δ. If, for instance, $\Delta = \Delta^{(0)} + \Delta^{(1)}a_\Delta$, the anomalous dimensions at order a_Δ^2 are not given by just expanding the parameter Δ in (5.3.17), since even with the most minimal assumptions, further contributions need to be considered. Firstly, the denominator would have to include the terms $U_{\bar{h}}^{(0)}|_\Delta + \frac{1}{2}\partial_{\bar{h}}U_{\bar{h}}^{(1)}|_\Delta$. Secondly, more operators will now necessarily contribute to the numerator. These are the double field operators themselves which contribute with a non-zero double discontinuity proportional to $\gamma_\ell \sim a_\Delta^2$ in $U_{\bar{h}}^{(1)}$, and operators $[\mathcal{O},\mathcal{O}]_{n,\ell}$ which contribute to both $U_{\bar{h}}^{(1)}$ and $U_{\bar{h}}^{(2)}$ [10]. Hence the computation of anomalous dimensions at leading order in a_Δ in the crossing kernel literature is equivalent to our computation of $A[\Delta_\varphi](\bar{h})U_{\bar{h}}^{(1)}|_\Delta$, and indeed one can check that this result agrees with equation (1.41) of [106].

[10] It is clear from the left-hand side that we must have a non-zero $U_{\bar{h}}^{(2)}$ at order a_Δ^2. Our minimal assumptions guarantee the existence of $[\mathcal{O},\mathcal{O}]_{n,\ell}$ [26, 27], and furthermore their inversion gives the correct large J expansion $U_{\bar{h}}^{(2)} \sim a_\Delta^2/J^{2\Delta}$. In a more generic theory the origin of $U_{\bar{h}}^{(2)}$ could be more complicated.

5.3.3 Contribution from $[\sigma,\sigma]_{n,\ell}$

The final contribution to the CFT data at this order is the contribution $I_{[\sigma,\sigma]_{n,\ell}}$ from the double field operators $[\sigma,\sigma]_{n,\ell}$, which we can heuristically think of as the operators $\sigma\Box^n\partial^{\mu_1}\dots\partial^{\mu_\ell}\sigma$. This contribution takes the form

$$I_{[\sigma,\sigma]_{n,\ell}} = \xi^{1-\mu}\, S(v,u)|_{z^0}\,, \qquad S(v,u) = \sum_{n,\ell} c^2_{\varphi^i\varphi^i[\sigma,\sigma]_{n,\ell}} G^{(d)}_{4+2n,\ell}(1-\bar{z},1-z). \quad (5.3.18)$$

The fact that the OPE coefficient $c_{\varphi\varphi\sigma}\sim N^{-1/2}$, suggests that the double field operators couple to φ with OPE coefficients $c_{\varphi\varphi[\sigma,\sigma]_{n,\ell}}\sim N^{-1}$, which implies they contribute to our correlator $\langle\varphi\varphi\varphi\varphi\rangle$ at order N^{-2}. To find these OPE coefficients, we study the mixed correlator $\langle\varphi\varphi\sigma\sigma\rangle$, in which these operators appear with OPE coefficients $c_{\varphi\varphi[\sigma,\sigma]_{n,\ell}}c_{\sigma\sigma[\sigma,\sigma]_{n,\ell}}\sim N^{-1}$. This correlator was computed to first order in [187]. Using $\overline{D}$ functions [29], this result reads, in our conventions[11]

$$\begin{aligned}\langle\varphi^i\varphi^j\sigma\sigma\rangle\big|_{\frac{1}{N}} = \frac{\delta^{ij}}{x_{12}^{2\Delta_\varphi}x_{34}^{2\Delta_\sigma}}\gamma^{(1)}_\varphi\Bigg[&\frac{2\mu(\mu-1)(2\mu-3)}{(\mu-2)^2\Gamma(\mu-2)}u^{\mu-1}\overline{D}_{\mu-1,\mu-1,1,1}(u,v)\\ &+\frac{\Gamma(\mu+1)}{\Gamma(\mu-1)^2}\Big(u^{\mu-1}\overline{D}_{\mu-2,\mu-1,1,2}(u,v)+u^2\overline{D}_{1,2,\mu-1,\mu-2}(u,v)\Big)\Bigg]. \quad (5.3.19)\end{aligned}$$

A conformal block decomposition of this can be performed using the subleading corrections to the conformal blocks $G_{\tau,\ell}(z,\bar{z})$. The functional form of these block corrections is known [96], and by imposing the Casimir equation, they can be computed systematically order by order in z; for a short summary, see appendix B.2. We have computed the OPE coefficients order by order, and found that they satisfy

$$c^2_{\varphi^i\varphi^i[\sigma,\sigma]_{n,\ell}} = \frac{1}{N^2}K(\mu)^2\frac{(\mu-1-n)_n^2}{(\mu-3-n)_n^2}\frac{1}{J^4_{2n+4,\ell}}\,a^{GFF}_{n,\ell}\big|_{\Delta=2} + O\left(N^{-3}\right), \quad (5.3.20)$$

where the $a^{GFF}_{n,\ell}$ are the generalized free field OPE coefficients of a free scalar of dimension 2, where $J^2_{2n+4,\ell} = (n+\ell+2)(n+\ell+1)$ is the conformal spin of the operator $[\sigma,\sigma]_{n,\ell}$, and where $K(\mu) = \frac{\mu(\mu-1)}{(\mu-2)(\mu-3)}\gamma^{(1)}_\varphi$.

[11] Notice that this correlator was translated into $\overline{D}$ functions already in [95]. The (u,v) there are defined by other conventions, namely $u_{\text{there}} = \frac{x_{13}^2x_{24}^2}{x_{12}^2x_{34}^2} = \frac{1}{u}$, $v_{\text{there}} = \frac{x_{14}^2x_{23}^2}{x_{12}^2x_{34}^2} = \frac{v}{u}$. This can be transformed into the form (5.3.19) by using identities for $\overline{D}$ [29].

Mellin amplitude

From (5.3.20), and the known expressions for the subleading corrections to the conformal blocks, we can find the function $S(v,u)$ order by order in v. Unfortunately we have not been able to find a closed form for it in position space, instead finding a closed form in Mellin space. We follow the conventions in [191], borrowing the language to describe holographic correlators. For a correlator of identical external operators of dimension Δ_φ, we define the Mellin amplitude $\mathcal{M}(s,t)$ by

$$\mathcal{G}(u,v)=\frac{1}{4}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{dsdt}{(2\pi\mathrm{i})^2}\Gamma\left(\Delta_\varphi-\tfrac{s}{2}\right)^2\Gamma\left(\Delta_\varphi-\tfrac{t}{2}\right)^2\Gamma\left(\Delta_\varphi-\tfrac{\tilde{u}}{2}\right)^2 u^{\frac{s}{2}}v^{\frac{t}{2}-\Delta_\varphi}\,\mathcal{M}(s,t), \tag{5.3.21}$$

where $s+t+\tilde{u}=4\Delta_\varphi$ are the "Mandelstam variables". Notice that the integrand has double poles at the dimensions $s,t=2\Delta_\varphi+2n+\ell$. To shift the poles to the dimensions of the double field operators $[\sigma,\sigma]_{n,\ell}$, we therefore define

$$\mathcal{M}_{[\sigma,\sigma]_{n,\ell}}(s,t)=\frac{\Gamma\left(\Delta_\sigma-\frac{s}{2}\right)}{\Gamma\left(\Delta_\varphi-\frac{s}{2}\right)^2}\mathsf{M}_{[\sigma,\sigma]_{n,\ell}}(s,t). \tag{5.3.22}$$

By demanding that we reproduce the series expansion of $S(v,u)$ computed previously, we can fix the poles of $\mathsf{M}_{[\sigma,\sigma]_{n,\ell}}(s,t)$ completely. In fact, we find that fixing the $\log^2 u$ part of $S(v,u)$, together with demanding that the expression is invariant under the swap $t\leftrightarrow\tilde{u}$, completely fixes $\mathsf{M}_{[\sigma,\sigma]_{n,\ell}}$ to

$$\mathsf{M}_{[\sigma,\sigma]_{n,\ell}}(s,t)=K(\mu)^2\sum_{k=2}^{\infty}\left(\frac{1}{t+4-2k-2\Delta_\varphi}+\frac{1}{\tilde{u}+4-2k-2\Delta_\varphi}\right)\tilde{R}(k,s), \tag{5.3.23}$$

where $\tilde{u}=4\Delta_\varphi-s-t$, and where

$$\tilde{R}(k,s)=\frac{2\pi\csc(\pi\mu)\Gamma(\mu-2)}{\Gamma(\mu-3)^2}\frac{(-1)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}-1\right)}{\Gamma\left(\frac{s}{2}+2-\mu\right)^2}\sum_{i=0}^{\frac{s}{2}-2}\frac{\Gamma(i+3-\mu)}{\Gamma(i+1)(3+i-k-\frac{s}{2})}. \tag{5.3.24}$$

Note that the integrand has only single poles in s, but triple poles in t. This implies that $S(v,u)$ has no terms proportional to $\log v$, but has terms of the form $\log^2 u,\log^1 u,\log^0 u$, contributing to the CFT data $U^{(2)}$, $U^{(1)}$ and $U^{(0)}$ respectively.

By carefully computing the residues, an infinite sum expression for $S(v,u)$ can be found – for details see appendix C.5. In particular, we are able to compute the $u^0 \log^2 u$ piece explicitly. After changing to the coordinates $(z,\bar{z})$, we find that

$$\begin{aligned} S(v,u)|_{z^0\log^2 z} &= K(\mu)^2 w^2 \left[{}_2F_1\left(\begin{matrix}1,2\\4-\mu\end{matrix}\middle|1-\bar{z}\right) + (\mu-3)\, D_a\, {}_2F_1\left(\begin{matrix}1,2\\4-\mu+a\end{matrix}\middle|1-\bar{z}\right)\right] \\ &= \frac{1}{8} 4K(\mu)^2 \xi^2\, {}_3F_2\left(\begin{matrix}2,3-\mu,3-\mu\\4-\mu,4-\mu\end{matrix}\middle|-\xi\right), \end{aligned} \tag{5.3.25}$$

where, as before, $\xi = (1-\bar{z})/\bar{z}$. This has a particularly simple inversion, and gives

$$U^{(2)}_{T/A,\bar{h}} = \pm \frac{4\mu^2(\mu-1)^2}{N^2} \left(\gamma^{(1)}_\varphi\right)^2 \frac{A[\mu-1](\bar{h})}{J^4} + O\left(N^{-3}\right), \tag{5.3.26}$$

in perfect agreement with the result (5.2.25) for the squared anomalous dimension $\gamma^2_{T/A,\ell}$.

We have also found the $z^0 \log z$ piece, which contributes to $U^{(1)}_{T/A,\bar{h}}$, by expanding it to very high order in ξ and recognizing the function as

$$\begin{aligned} S(v,u)|_{z^0\log z} = \frac{1}{2} 2K(\mu)^2 \xi^2 D_a \Bigg[&(\mu-2)\, {}_3F_2\left(\begin{matrix}1+a,3-\mu-a,3-\mu\\4-\mu,4-\mu\end{matrix}\middle|-\xi\right) \\ &+ (\mu-3)\, {}_2F_1\left(\begin{matrix}1-a,3-\mu+2a\\4-\mu\end{matrix}\middle|-\xi\right)\Bigg]. \end{aligned} \tag{5.3.27}$$

This gives the following contribution to $U^{(1)}_{T/A,\bar{h}}$:

$$\begin{aligned} U^{(1),[\sigma,\sigma]_{n,\ell}}_{T/A,\bar{h}} &= 2\,\mathrm{INV}\left[\xi^{-\Delta_\varphi} S(v,u)|_{z^0\log z}\right] \\ &= \frac{1}{N^2} 2\mu^2(\mu-1)^2 \left(\gamma^{(1)}_\varphi\right)^2 A[\Delta_\varphi](\bar{h}) \left(\frac{1}{J^4} - \frac{R(\bar{h},\mu)}{J^2} - \frac{\mathbf{S_1}[\Delta_\varphi](\bar{h}) - 1/J^2}{J^4}\right), \end{aligned} \tag{5.3.28}$$

where $\mathbf{S_1}$ was defined in (5.2.23) and where we defined the function[12]

$$R(\bar{h},\mu) = \frac{1}{\bar{h}(\bar{h}+2-\mu)}\, {}_3F_2\left(\begin{matrix}1,\bar{h},3-\mu\\\bar{h}+1,\bar{h}+3-\mu\end{matrix}\middle|1\right). \tag{5.3.29}$$

[12]This function, up to an additional term, appeared first in [169] as part of a double integral. In [95] the explicit form used here was given.

Finally, we have not been able to find the full $z^0 \log^0 z$ piece of $S(v,u)$ in a closed form; however in appendix C.5 we find a closed form expression for the piece proportional to ζ_2 in a power series in z and $(1-\bar{z})$. It takes the form

$$S(v,u)|_{z^0\zeta_2} = K(\mu)^2(1-\bar{z})^2\Bigg[\frac{{}_2F_1\left(\begin{smallmatrix}1,2\\4-\mu\end{smallmatrix}\middle|\,1-\bar{z}\right)}{\mu-2} - \frac{{}_3F_2\left(\begin{smallmatrix}1,2,2\\4-\mu,4-\mu\end{smallmatrix}\middle|\,1-\bar{z}\right)}{\mu-2} + (\mu-3)D_a\,{}_2F_1\left(\begin{matrix}1,2\\4-\mu+a\end{matrix}\middle|\,1-\bar{z}\right)\Bigg], \tag{5.3.30}$$

which can be straightforwardly inverted. The other piece of $S(v,u)$, not proportional to ζ_2, can be evaluated to arbitrarily high order in ξ, which yields a large J expansion to arbitrarily high order.

The contribution to $U^{(0)}_{T/A}$ then takes the form

$$U^{(0),[\sigma,\sigma]_{n,\ell}}_{T/A,\bar{h}} = K(\mu)^2\frac{A[\Delta_\varphi]}{N^2}\left(\rho_1(\bar{h},\mu)+\zeta_2\rho_2(\bar{h},\mu)\right), \tag{5.3.31}$$

where

$$\rho_2(\bar{h},\mu) = (\mu-2)^3(\mu-3)^2\left(\frac{R(\bar{h},\mu)}{J^2-K^2} - \frac{1}{J^2(J^2-K^2)} + \frac{1-\mu}{J^4(J^2-K^2)}\right), \tag{5.3.32}$$

and where $\rho_1(\bar{h},\mu)$ can be computed to arbitrarily high order in $1/J^2$. For example, the first few terms read

$$\rho_1(\bar{h},\mu) = (\mu-1)(\mu-2)^3(\mu-3)^2\Bigg(\frac{\mu^2-5\mu+12}{2J^8} + \frac{\mu^4-10\mu^3+39\mu^2-34\mu-48}{2J^{10}} + \frac{11\mu^6-159\mu^5+943\mu^4-2817\mu^3+6686\mu^2-16648\mu+20448}{24J^{12}} + O\big(J^{-14}\big)\Bigg). \tag{5.3.33}$$

In section 5.3.4.1 we use this series to provide numeric estimates for OPE coefficients.

5.3.4 CFT data at order N^{-2}

By combining the contributions I_σ, I_{T+A} and $I_{[\sigma,\sigma]_{n,\ell}}$, we can now give the full CFT data $U^{(p)}_{T/A,\bar{h}}$ up to order N^{-2}.

Firstly, $U^{(2)}_{T/A,\bar{h}}$ was already given in (5.3.26), since it receives a contribution only from $[\sigma,\sigma]_{n,\ell}$. To this order $U^{(2)}_{T/A,\bar{h}} = \hat{a}_\ell \gamma_\ell^2$ gives the squares of anomalous dimensions, so that it is fully fixed by $U^{(1)}_{T/A,\bar{h}}$ and $U^{(0)}_{T/A,\bar{h}}$ at previous orders, a non-trivial consistency condition that it satisfies.

For $U^{(1)}_{T/A,\bar{h}}$ we have the following closed form expressions:

$$U^{(1)}_{T,\bar{h}} = U^{(1),\sigma}_{T/A,\bar{h}} + U^{(1),[\sigma,\sigma]_{n,\ell}}_{T/A,\bar{h}} + U^{(1),(T+A)}_{T/A,\bar{h}}, \tag{5.3.34}$$

$$U^{(1)}_{A,\bar{h}} = -U^{(1),\sigma}_{T/A,\bar{h}} - U^{(1),[\sigma,\sigma]_{n,\ell}}_{T/A,\bar{h}} + U^{(1),(T+A)}_{T/A,\bar{h}}, \tag{5.3.35}$$

where the various contributions were given in equations (5.3.8), (5.3.11) and (5.3.28).[13]

Using this, the anomalous dimensions $\gamma_{T/A,\ell}$ can be computed to order N^{-2}. They are most easily expressed in terms of $\bar{h}$:[14]

$$\begin{aligned}\gamma_{T/A}(\bar{h}) = -2\gamma^{(1)}_\varphi \frac{\mu(\mu-1)}{J^2}\frac{1}{N} + 2\left(\gamma^{(1)}_\varphi\right)^2 \frac{1}{N^2}\Bigg[\mu^2(\mu-1)^2\Bigg(-\frac{\left(2\bar{h}-1\right)}{J^6} + \frac{(\mu^2-\mu+1)}{J^4(\mu-1)\mu} \\ - \frac{2\left(4\mu^3-18\mu^2+21\mu-8\right)}{J^2(\mu-2)^2(\mu-1)\mu} - \frac{R\left(\bar{h},\mu\right)}{J^2} + \frac{4(2\mu-1)\left(S_1\left(\bar{h}-1\right)-S_1(\mu-2)\right)}{J^2(\mu-2)\mu}\Bigg) \\ - \frac{(\mu-2)^2 a^{(1)}_\sigma \Big/ \left(\gamma^{(1)}_\varphi\right)^2}{J^2} \mp \frac{\Gamma(\mu+1)^2\Gamma\left(\bar{h}+2-\mu\right)}{J^2(\mu-2)^2\Gamma\left(\bar{h}-2+\mu\right)}\Bigg],\end{aligned} \tag{5.3.36}$$

with the upper sign for T and the lower sign for A. The conservation of the global symmetry current implies that $2\Delta_\varphi + \gamma_{A,\ell=1} \stackrel{!}{=} d-1$, which fixes the OPE coefficient $a^{(1)}_\sigma$ in terms of $\gamma^{(2)}_\varphi$

$$a^{(1)}_\sigma = \frac{(\mu-1)\mu}{(\mu-2)^2}\left(\gamma^{(2)}_\varphi + \left(\gamma^{(1)}_\varphi\right)^2\left(-8\mu + \frac{17}{\mu-2} + \frac{14}{(\mu-2)^2} + \frac{1}{\mu-1} - \frac{1}{\mu} + 12\right)\right). \tag{5.3.37}$$

We expect that $\gamma^{(2)}_\varphi$ can be fixed by demanding conservation of the stress energy tensor, similar to how $\gamma^{(1)}_\varphi$ was fixed by this conservation in section 5.2.4. Since this is a computation at next order, we will instead use the literature value of $\gamma^{(2)}_\varphi$ [122,189]

[13] Note that the order $1/N$ result (5.2.20) has been absorbed into $U^{(1),\sigma}_{T/A,\bar{h}}$.

[14] To extract the anomalous dimensions as functions of ℓ one has to substitute $\bar{h} = \mu - 1 + \ell + \gamma_\varphi$.

to find that

$$a_\sigma^{(1)} = 2\left(\gamma_\varphi^{(1)}\right)^2 \frac{\mu}{(\mu-2)^4}\Big(-4\mu^4+23\mu^3-43\mu^2+31\mu-6 \\ -(\mu-2)(\mu-1)\left(2\mu^2-3\mu+2\right)\left(\pi\cot(\pi\mu)+S_1(2\mu-3)\right)\Big). \quad (5.3.38)$$

With this value, we find that our result (5.3.36) for $\gamma_{T,A}(\bar{h})$ matches that found in [122, 166].[15]

5.3.4.1 OPE coefficients and C_J correction

The final piece of CFT data that we can compute is the OPE coefficients in the T and A representations to order N^{-2}. They are computed from the $U^{(p)}_{T/A,\bar{h}}$ using (5.2.4) and (5.2.5). The resulting expressions are lengthy and we give the explicit form in appendix C.6. Notice that they depend on the unknown function $\rho_1(\bar{h},\mu)$ defined in (5.3.31). The result has been checked against the ϵ expansion [3] and gives complete agreement to order ϵ^4/N^2.[16]

If one is interested in a particular OPE coefficient in a given dimension, one can use the series expansion (5.3.33) for $\rho_1(\bar{h},\mu)$ to generate numeric estimates. In particular, we are interested in computing the corrections to the central charge C_J, which follows from the OPE coefficient in the antisymmetric representation, with $\ell=1$ and $\mu=3/2$. Thus we need a numeric estimate for $\rho_1(\mu,\mu)$. Let us define

$$s_\mu(\bar{z}) := S(v,u)|_{z^0\log^0 z} = \sum_{n=3}^{\infty} c_n(\mu) w^n, \qquad w = 1-\bar{z}. \quad (5.3.39)$$

We are able to compute the first 140 coefficients $c_n(\mu)$. In any specific dimension we can quickly compute more: in three dimensions we have computed the first 4300

[15] Note that one of the terms in the expression for the $1/N^2$ correction to the anomalous dimensions differs between these two papers by a factor of (-1). We find agreement with the results of [166].

[16] Notice that the contribution from $\rho_1(\bar{h},\mu)$ does not appear until order ϵ^5/N^2. This is due to the explicit prefactor $(\gamma_\varphi^{(1)})^2/(\mu-2)^2 \sim \epsilon^2$ in (C.6.1) and the additional factor $(\mu-2)^3 \sim \epsilon^3$ in the series (5.3.33).

coefficients. For reference we give the first orders here:

$$s_{\mu=3/2}(\bar{z}) = K\left(\tfrac{3}{2}\right)^2 \left(\frac{4\zeta_2}{25} w^3 + \left(\frac{48\zeta_2}{175} - \frac{27}{2450}\right) w^4 + \left(\frac{3904\zeta_2}{11025} - \frac{79}{3150}\right) w^5 + \dots\right). \tag{5.3.40}$$

However the convergence is very slow. In order to get better control of the tail $n \to \infty$, we study the large n behaviour of the coefficients $c_n(\mu)$ and find that it takes the following form:

$$\begin{aligned} c_n(\mu) = \frac{\Gamma(4-\mu)(3-\mu)\Gamma(n)}{\Gamma(n+2-\mu)} \left[a_2(\mu) S_1(n)^2 + a_1(\mu) S_1(n) + a_0(\mu) + O\left(\frac{1}{n}\right)\right] \\ + \frac{1}{n}\left[\tilde{a}_{0,1}(\mu) S_1(n) + \tilde{a}_{0,0}(\mu) + \frac{\tilde{a}_{1,1}(\mu) S_1(n)}{n} + \frac{\tilde{a}_{1,0}(\mu)}{n} + O\left(\frac{1}{n^2}\right)\right]. \end{aligned} \tag{5.3.41}$$

This form was derived by finding recursion relations for the $c_n(\mu)$ in various integer μ, and studying the solutions around large n. Notice that the first term scales as $n^{\mu-2}\log^2 n$ and thus dominates at large positive μ, whereas the second term scales as $n^{-1}\log n$ and dominates at negative μ. At negative integer μ the terms $\tilde{a}_{k,1}(\mu)\log n + \tilde{a}_{k,2}(\mu)$ develop poles that exactly cancel with the corresponding poles induced by the $a_2(\mu)S_1(n)^2 + a_1(\mu)S_1(n) + a_0(\mu)$ order by order. This leads to a determination of $a_2(\mu)$, $a_1(\mu)$ and $a_0(\mu)$.[17]

The analytic continuation of the coefficients $a_p(\mu)$ away from integer μ now means that we can derive the leading large n dependence in any dimension. This will map to the small $\bar{z}$ expansion

$$\begin{aligned} s_\mu(\bar{z}) \sim \frac{\pi(\mu-2)(\mu-3)^2 \bar{z}^{1-\mu}}{\sin(\pi\mu)} \Big[\left((S_1(\mu-2) - \log\bar{z})^2 - S_2(\mu-2) + \zeta_2\right) a_2(\mu) \\ + \left(S_1(\mu-2) - \log\bar{z}\right) a_1(\mu) + a_0(\mu)\Big] + \dots \end{aligned} \tag{5.3.42}$$

Under the inversion integral (5.2.7) this expansion elucidates the structure of the poles in $\bar{h}$: a term of the form $\bar{z}^\alpha \log^\beta \bar{z}$ maps to a pole as follows:

$$\text{INV}\left[\left(\tfrac{\bar{z}}{1-\bar{z}}\right)^{\Delta_\varphi} \bar{z}^\alpha \log^\beta \bar{z}\right] = 2\sin^2(\pi\Delta_\varphi) \frac{(-1)^\beta \beta!}{(\bar{h} - (1 - \Delta_\varphi - \alpha))^{\beta+1}} + \dots \tag{5.3.43}$$

[17] We get $a_2(\mu) = \frac{1}{2}\left(S_1(2-\mu) + S_1(\mu-2)\right)$ and some lengthy expressions for $a_1(\mu)$ and $a_0(\mu)$, which are given in appendix C.5.1. We have determined many of the $\tilde{a}_{k,p}$, the first ones are $\tilde{a}_{0,1} = \frac{(\mu-3)^2}{(\mu-1)^2}$ and $\tilde{a}_{0,0} = \frac{(\mu-3)^2(5\mu-8)}{(\mu-2)(\mu-1)^3}$.

In the case at hand, with $\Delta_\varphi = \mu - 1$, we get a triple pole at $\bar{h} = 1$, or equivalently at $J^2 = 0$. From the expressions (5.3.42) we can thus work out the expansion near $J^2 = 0$ of the inversion of $s_\mu(\bar{z})$. In a similar manner, the second line of (5.3.41) will give poles as $J^2 = K^2$. Thus we have full control of the pole structure of this inversion:

$$\rho_1(\bar{h},\mu) + \zeta_2\rho_2(\bar{h},\mu) \sim \frac{r_3(\mu)}{J^6} + \frac{r_2(\mu)}{J^4} + \frac{r_1(\mu)}{J^2} \sim \frac{\tilde{r}_2(\mu)}{(J^2-K^2)^2} + \frac{\tilde{r}_1(\mu)}{J^2-K^2} \tag{5.3.44}$$

where

$$r_3(\mu) = 2(\mu-2)^2(\mu-3)^2 a_2(\mu), \tag{5.3.45}$$

$$r_2(\mu) = (\mu-2)^2(\mu-3)^2\big(2(S_2(2-\mu)+1)a_2(\mu) + a_1(\mu)\big), \tag{5.3.46}$$

$$r_1(\mu) = (\mu-2)^2(\mu-3)^2\big((S_1(2-\mu)^2 - S_2(2-\mu) - \zeta_2)a_2(\mu) + S_1(2-\mu)a_1(\mu) + a_0(\mu)\big) \tag{5.3.47}$$

and

$$\tilde{r}_2(\mu) = (2\mu-3)^2\tilde{a}_{0,1}(\mu) \tag{5.3.48}$$

$$\tilde{r}_1(\mu) = \big(1 + (2\mu-3)S_1(2-2\mu) - 2(2\mu-3)S_1(1-\mu)\big)\tilde{a}_{0,1}(\mu) - (2\mu-3)\tilde{a}_{0,0}(\mu) \tag{5.3.49}$$

In order to interpolate between these poles at finite J^2 and the asymptotic expansion around infinite J^2, we construct Padé approximants of the form

$$\text{Padé}_n[\rho_1 + \zeta_2\rho_2](J^2) = \frac{\sum_{k=0}^{n}\alpha_k J^{2k}}{J^6(J^2-K^2)^2\left(1+\sum_{k=1}^{n-2}\beta_k J^{2k}\right)}. \tag{5.3.50}$$

After determining the constants α_k, β_k we get estimates for $\rho(\mu,\mu)$. These can be converted to estimates for the C_J corrections:

$$\frac{C_J}{C_{J,\text{free}}} = 1 + \frac{c_J^{(1)}}{N} + \frac{c_J^{(2)}}{N^2} + O(N^{-3}) \tag{5.3.51}$$

where $c_J^{(1)}$ matches (5.2.28) computed before, and $c_J^{(2)}$ is extracted from (C.6.1). In figure 5.2 we present graphically the numeric results for the second order correction $c_J^{(2)}$ in different dimensions based on Padé approximants of order $n = 30$. In particular,

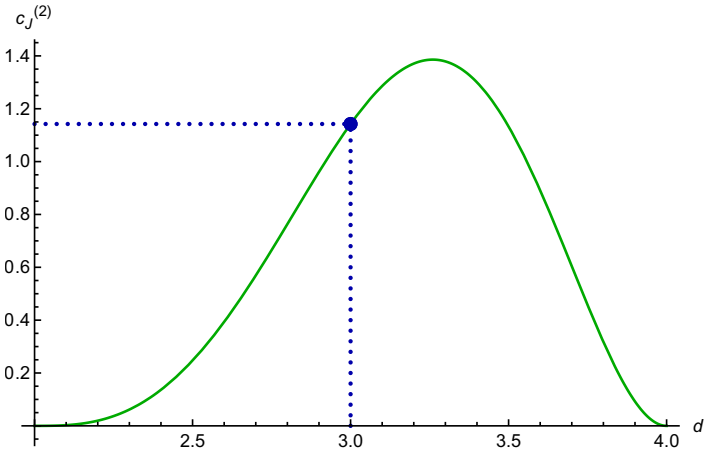


Figure 5.2: The numeric estimate for the $1/N^2$ correction $c_J^{(2)}$ in (5.3.51) to the ratio of the global current charge $C_J/C_{J,\text{free}}$ in the interacting and in the free theory.

note that the behaviour of the graph is consistent with the known expansions in $d = 4 - \epsilon$ and $d = 2 + \epsilon$, where both the value and the slope of $c_J^{(2)}$ vanish. In $d = 3$ we can compute the large J series to order 800, to provide the more precise estimate:

$$\rho_1\left(\tfrac{3}{2}, \tfrac{3}{2}\right) = -0.3422(1) \tag{5.3.52}$$

This was derived by computing Padé approximants (5.3.50) in three dimensions for n of increasing order, and matching the asymptote. We then compute the correction to the central charge C_J in three dimensions:

$$\begin{aligned}\left.\frac{C_J}{C_{J,\text{free}}}\right|_{\mu=3/2} &= 1 - \frac{1}{N}\frac{64}{9\pi^2} + \frac{1}{N^2}\frac{16}{9\pi^4}\Big(- \rho_1\left(\tfrac{3}{2}, \tfrac{3}{2}\right) + 8\log(2)^2 + \tfrac{100}{3}\log 2 + 6\pi\log 2 \\ &\qquad - 7\pi + \tfrac{11}{2}\pi^2 + \tfrac{3}{16}\pi^3 - 24\beta(2) + \tfrac{55}{9}\Big) + O(N^{-3}) \\ &= 1 - \frac{0.720506}{N} + \frac{1.14230(2)}{N^2} + O(N^{-3}),\end{aligned} \tag{5.3.53}$$

where $\beta(2) = 0.91597\ldots$ is Catalan's constant. In this expression we used the fact that $\partial_{\bar{h}} R\left(\frac{3}{2}, \frac{3}{2}\right) = 16 + 4\pi - 32\beta(2)$. Notice that the only unknown quantity here is $\rho_1\left(\frac{3}{2}, \frac{3}{2}\right)$, for which we used the estimate in (5.3.52). In principle, one can write down a closed form expression for $\rho(\bar{h}, \mu)$ in terms of the inversion integral of a nested sum, or as the inversion integral of a limit of an inverse Mellin transform, and therefore compute $\rho_1\left(\frac{3}{2}, \frac{3}{2}\right)$ to arbitrary precision.

Results in three dimensions for C_J are only given for $N = 2, 3, 20$ in the literature [180], computed using numeric conformal bootstrap. When we compare their results to our result (5.3.53), we find that for all these three values, adding the N^{-2} correction gives an estimate closer to these numeric bootstrap results than previous estimates using only N^{-1}. For $N = 20$ our value lies within the error margin $C_J/C_{J,\text{free}} = 0.9674(8)$.

5.4 Discussion

In this chapter we analyzed the critical O(N) model perturbatively in large N to order $1/N^2$, confirming known results for anomalous dimensions and finding new results for OPE coefficients of singlet operators, for the OPE coefficient $a_{\varphi\varphi\sigma}$, and for the OPE coefficients in the traceless symmetric and antisymmetric representations, including in particular a numerical prediction for the correction to the global symmetry charge C_J. Several questions arise from this work.

For general d and N not necessarily integer, the theory is non-unitary [144, 181], and new evanescent operators may appear with complex dimensions and OPE coefficients. These operators do not appear in our analysis in the $1/N$ expansion, but their existence may be relevant in attempts to extrapolate the perturbative large N data to finite integer N. It would be interesting to extend the $d = 4 - \epsilon$ work of [181] to understand the exact way in which such operators appear and how they affect CFT data of the physical operators.

As mentioned in chapter 1, the inversion formula gave a precise way to extract CFT data from correlator divergences; however as evident from the computations in this chapter, calculating these divergences is less systematic, relying on a variety of techniques such as solving Casimir differential equations and using Mellin amplitudes. In the non-perturbative crossing kernel approach of [66, 104–108], double discontinuities can be found systematically since the double discontinuity from each

conformal block can be found individually; however it is generally harder to combine their inversions into the full CFT data. It would be good to try to find a systematic approach to the computation of correlator contributions arising from twist conformal blocks, for example by exploiting a transcendentality principle or by doing large spin perturbation theory in Mellin space.

In planar $\mathcal{N} = 4$ SYM, the reciprocity and transcendentality principle have been used to great success to find CFT data to very high order, see for example [192] for a recent 7-loop result. This idea of transcendentality also provided a very fruitful ansatz in chapter 4, where it allowed us to find the double discontinuity from operators of approximate twist 4, and hence the order ϵ^4 corrections to the CFT data of leading twist operators. In this chapter we also see that the CFT data and double discontinuities arrange themselves in particular (hypergeometric) functions. It would be useful, both from a conceptual and a computational point of view, to formulate a transcendentality principle in general dimension and to understand its underlying principles.

An interesting observation in section 5.3.3 is that we were unable to find the relevant contribution to the double discontinuity in position space, but were able to do this in Mellin space. Furthermore this Mellin space expression was fixed completely by demanding the correct $\log^2 u$ term, and symmetry under interchange of the Mellin space variables $(t, \tilde{u})$. It would be nice to formulate large spin perturbation theory in a way that makes direct contact with Mellin amplitudes while avoiding the need to discuss position space, for example by providing a "Mellin space inversion formula" analogous to the formula from [97]. Some interesting progress towards this goal can be found in [100, 105, 193].

Chapter 6

Conclusions

In many ways conformal field theories are at the heart of modern mathematical and theoretical physics, with intimate connections to condensed matter theory, high-energy physics, quantum gravity and mathematics. Since its rebirth roughly a decade ago, the conformal bootstrap has proven to be a powerful new tool with which to analyze conformal field theories. The bootstrap also provides a perspective on CFTs in which operators and their correlators, as opposed to fields and path integrals, are put centre stage. It has at this point far surpassed the "proof-of-concept" stage and has been used to derive many new results, ranging from general statements about CFTs, such as twist additivity [26, 27] and spin analyticity [97]; to precise numerical predictions in known theories, such as the critical exponents of the 3d Ising model [41]; to the study of largely unknown theories such as 4d $\mathcal{N} = 3$ and 6d $\mathcal{N} = (2, 0)$ theories [74, 83]; and to higher order perturbative computations in known theories such as $\mathcal{N} = 4$ SYM [76–82]. This thesis fits in the latter camp, using the large spin perturbation approach to the conformal bootstrap to analyze the Gross-Neveu and critical O(N) models, for which we find new CFT data in different perturbative regimes. Such results are interesting not only because of the direct relation of these theories to various physical systems such as boiling water at the critical point and self-avoiding random walks [14], or models of graphene and semimetals [150–152], but also as a playground to study phenomena in other theories, such as dynamical mass generation

in QCD, or the study of type IIB string theory from $\mathcal{N} = 4$ SYM and AdS/CFT.

Nevertheless the conformal bootstrap is still young and many open questions remain. Here we will discuss some questions that arise from the work in this thesis.

Our analyses use large spin perturbation theory, combining the powerful Lorentzian inversion formula with the twist conformal block approach. Of course other approaches to the conformal bootstrap exist: even within the context of large spin, the Polyakov-Mellin bootstrap approach [115–119] and the crossing kernel/6j symbol approach [66, 104–108] have proven to be effective methods to study CFTs. It would be of great interest to understand the precise relations between these different approaches. For example, in our approach we are generally in a perturbative setting, where the operators naturally organize into families, and we first find the double discontinuity due to such families of operators, before inverting them. In contrast, the inherently non-perturbative crossing kernel approach finds CFT data by computing the contributions of individual intermediate operators to the double discontinuity, inverting these one by one, and summing up the inversions. It would be interesting to understand when these operations commute and when both approaches are valid. In particular, we would like to understand how in a perturbative setting the crossing kernels and 6j symbols combine into the full CFT data, which in our examples organizes itself into transcendental functions such as hypergeometric functions and harmonic sums.

The general approach in the numeric bootstrap, relying on the application of linear functionals to the crossing equation, is very different from our approach in the analytic lightcone bootstrap. Recent work in 1d [49–52], utilizing functionals that are contour integrals in cross-ratio space, has been able to make the connections between these approaches more precise, and it would be interesting to extend these ideas to CFTs in higher dimensions.

Most work in the bootstrap so far, including this work, has focused on correlators

of local operators in CFTs in flat space, missing out on the rich structure due to for example defects, boundaries and Wilson lines. Furthermore, the work in chapter 3 highlights the potential benefits from studying correlators involving non-scalar operators. There has recently been a flurry of activity, with results such as conformal blocks for non-scalar operators [53–59], and in boundary and defect CFTs [111,112,114,194,195]. A promising development in this direction has been that of [113, 114, 196–199], in which harmonic analysis on the conformal group is used to relate conformal blocks to integrable Calogero-Sutherland models, in turn relating them to the mathematical theory of Heckman-Opdam hypergeometric functions. These results are then used to constrain the spectrum of CFTs with such additional structure.

The results in this thesis also raise interesting questions. An example is the transcendentality principle, i.e. the result that the correlator corrections and CFT data organize themselves into particular families of functions. This is most strikingly apparent in the $d = 4-\epsilon$ expansion of the critical O(N) model, where the correlator and CFT data corrections are found in terms of harmonic polylogarithms and harmonic sums, respectively. This phenomenon has been found in many CFTs, with examples ranging from Witten diagrams in AdS_2 [51] to $\mathcal{N} = 4$ SYM [80,82,99,200,201]. There exists a Feynman diagrammatic approach to understanding the appearance of polylogarithms and their generalizations in certain correlators, see e.g. [202–205]. Since the harmonic polylogarithms are a subset of all hypergeometric functions, and conformal blocks in four dimensions are simply products of hypergeometric functions, it is tempting to suggest that this may be understood from a bootstrap perspective, or from some other purely CFT approach.

Another interesting problem is found in our analysis of the critical O(N) model in $d = 4 - \epsilon$: at least to order ϵ^2, the critical O(N) model is singled out as the unique non-free theory after extending the large spin result down to spin $\ell = 0$, beyond the usual range of validity of the inversion formula. This limit was subtle

and involved an analytic continuation through a pole, which was moved away from $\ell = 0$ by expressing the anomalous dimension in terms of the full conformal spin $J_f^2 = \bar{h}_f(\bar{h}_f - 1)$.[1] Similarly we found constraints in the large N approach to the O(N) model by continuing the anomalous dimensions of the singlet operators down to spin $\ell = 0$ and demanding that this yields the dimension $d - \Delta_\sigma$ of the shadow operator of the scalar σ. It would be useful to understand why these a priori unjustified procedures give correct results for the critical O(N) model, and in what other cases they may be justified.

A further idea is to try to use the conformal bootstrap to learn about non-conformal field theories, for example those that can be described as a perturbation of a CFT, or that can otherwise be related to some CFT. One way this could be done is by relating flat space physics to physics in AdS spaces in the limit of infinite curvature radius [206–208]. For example [206, 207] considers non-conformal theories on an AdS space, which has a dual boundary conformal theory. They study correlation functions in this boundary theory using conformal bootstrap methods, relate this to physics of the QFT in AdS, before finally recovering results for a massive QFT in flat space by sending the curvature radius of AdS to infinity.

Finally, there have been many other approaches to the analytic study of conformal field theories in recent years, and a lot is likely to be gained by understanding the intersection of these methods with the conformal bootstrap. A particularly interesting recent idea is that of [209]: in ideas reminiscent of the amplituhedron, it finds that in 1d CFTs, unitarity constrains CFT data to lie in a cyclic polytope, and that crossing symmetry imposes that it lies on a certain hyperplane. It then characterizes the intersection of these using combinatorial methods, finding new results for the behaviour of 1d CFTs. It would be interesting to explore these ideas further, for example by extending the results to higher dimensions.

[1] Since the conformal spin itself depends on the anomalous dimension, this essentially sets up an equation for $\gamma_{\bar{h}_f}$ that can be solved order by order in ϵ.

Appendix A

Appendices to chapter 3

A.1 Boundary term

In this appendix we try to make precise the statement from section 3.2.2 that

$$\sum_{\tau_0,\ell} \partial_\ell \left(a^{(0)}_{\tau_0,\ell} \gamma^{(1)}_{\tau_0,\ell} G_{\tau,\ell}(u,v) \right) \tag{A.1.1}$$

is a boundary term that does not contain any enhanced divergences in v.

To this end, consider a function $f : \mathbb{R} \to \mathbb{R}$, arising from some function $\tilde{f} : \mathbb{N} \to \mathbb{R}$ that has been suitably analytically continued to have some desirable behaviour at infinity, and decays suitably quickly at infinity. We show that under some reasonable assumptions, $\sum_\ell f'(\ell)$ is a boundary term.

Fix an $N \in \mathbb{N}$ and consider $f(N) - f(0) = \sum_{l=0}^{N-1} [f(\ell+1) - f(\ell)]$. By the intermediate value theorem, for each ℓ there exists a $\xi_\ell \in (\ell, \ell+1)$ such that $f'(\xi_\ell) = f(\ell+1) - f(\ell)$. Therefore

$$\lim_{N\to\infty} f(N) - f(0) = \sum_{\ell=0}^{\infty} f'(\xi_\ell)\,. \tag{A.1.2}$$

The left-hand side is clearly a boundary term, so that we are done if we can relate the right-hand side to $\sum_\ell f'(\ell)$. This can be done for example if $f'(\ell)$ is monotonic; in fact, since we are interested in enhanced divergences, we do not care about finite sums and may in fact only demand that $f'(\ell)$ is monotonic for some $\ell > L$. From numerical explorations we indeed find that this holds for the sums $\sum_\ell \partial_\ell \left(a^{(0)}_{\tau_0,\ell} \gamma^{(1)}_{\tau_0,\ell} G_{\tau,\ell}(u,v) \right)$ we encounter.

Appendix B

Appendices to chapter 4

B.1 Some inversion integrals

In the body of chapter 4 we arrived at the following inversion integral

$$\hat{a}(\bar{h}) = \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)} \int_0^1 \mathrm{d}\bar{z}\; z^{\bar{h}-2}\, {}_2F_1(\bar{h},\bar{h};2\bar{h};\bar{z})\, \mathrm{dDisc}\left[G(\bar{z})\right]. \tag{B.1.1}$$

where $J^2 = \bar{h}(\bar{h}-1)$. As an example we considered

$$G(\bar{z}) = \left(\frac{\bar{z}}{1-\bar{z}}\right)^p \quad \to \quad \mathrm{dDisc}\, G(\bar{z}) = 2\sin^2(\pi p)\left(\frac{\bar{z}}{1-\bar{z}}\right)^p, \tag{B.1.2}$$

in which case

$$\hat{a}_p(J) = \frac{2(2\bar{h}-1)\Gamma(\bar{h}+p-1)}{\Gamma(p)^2\Gamma(\bar{h}-p+1)}. \tag{B.1.3}$$

This precise inversion problem was also considered in [96], with exactly the same result. Other inversions used in this paper are presented in tables B.1–B.2. In these tables we use the nested harmonic sums $S_{\mathbf{a}} = S_{\mathbf{a}}(\bar{h}-1)$ which for integer arguments take the values

$$S_{a_1,a_2,\ldots}(n) = \sum_{b_1=1}^{n} \frac{(\mathrm{sgn}\, a_1)^{b_1}}{b_1^{|a_1|}} \sum_{b_2=1}^{b_1} \frac{(\mathrm{sgn}\, a_2)^{b_2}}{b_2^{|a_2|}} \sum_{b_3=1}^{b_2} \frac{(\mathrm{sgn}\, a_3)^{b_3}}{b_3^{|a_3|}} \cdots . \tag{B.1.4}$$

For non-integer values of $\bar{h}$ we make the standard analytic continuation from even arguments n, see *e.g.* [121], so that for instance

$$S_{-2}(x) = \frac{1}{4}\left(\psi^{(1)}\left(\tfrac{x+1}{2}\right) - \psi^{(1)}\left(\tfrac{x+2}{2}\right)\right) - \frac{\zeta_2}{2}, \tag{B.1.5}$$

where $\psi^{(1)}(x)$ is the trigamma function.

Evaluating these inversion integrals is non-trivial, but one can proceed as follows. Expanding the function to invert in powers of $\frac{1-\bar{z}}{\bar{z}}$ we are led to the integral entering in (B.1.2). We then obtain an expression for the inverted function, as a power expansion for large $\bar{h}$, which can be resummed in simple cases, or more generally one can fit an ansatz to this expansion.[1] The final result can then be checked numerically, for finite values of $\bar{h}$, to very high precision.

Note that we use the notation $J^2 = \bar{h}(\bar{h}-1)$ in the tables.

$G(\bar{z})$	$\hat{a}(J)/(2\bar{h}-1)$
$\log^2(1-\bar{z})$	$4/J^2$
$\log^3(1-\bar{z})$	$-24\,S_1/J^2$
$\log^4(1-\bar{z})$	$96\big(S_1^2-\zeta_2-S_{-2}\big)/J^2$
$\log^2(1-\bar{z})\mathrm{Li}_2(1-\bar{z})$	$4\big(\zeta_2+2S_{-2}\big)/J^2$
$\log^3(1-\bar{z})\mathrm{Li}_2(1-\bar{z})$	$24\big((S_{-3}-2S_{-2,1})-3(S_{-3}-2S_{1,-2})+3\zeta_2S_1-2S_3\big)/J^2$
$\log^2(1-\bar{z})\mathrm{Li}_3(1-\bar{z})$	$4\big(-2\,(S_{-3}-2S_{1,-2})+\zeta_3+2\zeta_2S_1-2S_3\big)/J^2$
$\log^2(1-\bar{z})\mathrm{Li}_3\left((\bar{z}-1)/\bar{z}\right)$	$4\big(2S_3-2\zeta_3-J^{-6}-2J^{-4}\big)/J^2$

Table B.1: Inversions for $G(\bar{z})$ not containing explicit powers of $\log\bar{z}$.

[1] For an in-depth discussion, see appendix C.1 of chapter 5.

$G(\bar{z})$	$\hat{a}(J)/(2\bar{h}-1)$
$\log^2(1-\bar{z})\log\bar{z}$	$-4/J^4$
$\log^2(1-\bar{z})\log^2\bar{z}$	$8\left(-\zeta_2+J^{-4}+J^{-2}-2S_{-2}\right)J^2$
$\log^2(1-\bar{z})\log^3\bar{z}$	$24\big(2(S_{-3}-2S_{1,-2})+2S_{-2}\,J^{-2}-2\zeta_2 S_1$ $+\zeta_3+\zeta_2 J^{-2}-J^{-6}-2J^{-4}\big)/J^2$
$\log^3(1-\bar{z})\log\bar{z}$	$24\big(-2S_{-2}+S_1\,J^{-2}-\zeta_2\big)/J^2$
$\log^4(1-\bar{z})\log\bar{z}$	$48\big(2S_3-4(S_{-3}-2S_{-2,1})+6(S_{-3}-2S_{1,-2})$ $+2(S_{-2}-S_1^2+\zeta_2)J^{-2}-6\zeta_2 S_1+3\zeta_3\big)/J^2$
$\log^2(1-\bar{z})\mathrm{Li}_2(1-\bar{z})\log\bar{z}$	$4\big(6\zeta_2 S_1-6(S_{-3}-2S_{1,-2})-(2S_{-2}+\zeta_2)J^{-2}-3\zeta_3\big)/J^2$

Table B.2: Inversions for $G(\bar{z})$ containing explicit powers of $\log\bar{z}$.

B.2 Conformal blocks in the small z limit

The conformal blocks $G^{(d)}_{\tau,\ell}(z,\bar{z})$ are eigenfunctions of the shifted quadratic Casimir operator $\mathcal{C}_{\tau,d}$ with eigenvalues equal to the conformal spin $J^2_{\tau,\ell}=\left(\frac{\tau}{2}+\ell-1\right)\left(\frac{\tau}{2}+\ell\right)$ [30]. Since the Casimir $\mathcal{C}_{\tau,d}$ respects orders in z, i.e.

$$\mathcal{C}_{\tau,d}\left(z^\alpha f_0(\bar{z})+\mathcal{O}(z^{\alpha+1})\right)=z^\alpha\overline{\mathcal{C}}_{\tau,d,\alpha}\left(f_0(\bar{z})\right)+\mathcal{O}(z^{\alpha+1}) \tag{B.2.1}$$

for some reduced differential operator $\overline{\mathcal{C}}_{\tau,d,\alpha}$, the Casimir equation can be solved order by order in z.

Expanding the conformal block as

$$G^{(d)}_{\tau,\ell}(z,\bar{z})=z^{\frac{\tau}{2}}\sum_{k=0}^{\infty}z^k f^{(d)}_{k,\tau,\ell}(\bar{z}), \tag{B.2.2}$$

we find that the leading-z behaviour, given by $f_{0,\tau,d,\ell}(\bar{z})$, solves a hypergeometric equation, with solution

$$f_{0,\tau,d,\ell}(\bar{z})=k_{\frac{\tau}{2}+\ell}(\bar{z})\equiv\bar{z}^{\frac{\tau}{2}+\ell}\,{}_2F_1\left(\frac{\tau}{2}+\ell,\frac{\tau}{2}+\ell;\tau+2\ell;\bar{z}\right), \tag{B.2.3}$$

reproducing the well-known $\mathrm{SL}(2,\mathbb{R})$ conformal blocks.[2]

Inspired by [95], we make the following ansatz for the functions $f^{(d)}_{n,\tau,\ell}$:

$$f^{(d)}_{n,\tau,\ell}(\bar{z}) = \sum_{m=-n}^{n} c_{n,m}(d,\tau,\ell) k_{\frac{\tau}{2}+\ell+m}(\bar{z}). \tag{B.2.4}$$

Considering simultaneously the leading and subleading terms in the Casimir equation, we solve for the $n=1$ coefficients

$$c_{1,-1} = \frac{(d-2)\ell}{d-4+2\ell}, \quad c_{1,0} = \frac{\tau}{4}, \quad c_{1,1} = \frac{(d-2)(\ell+\tau-1)(2\ell+\tau)^2}{16(2\ell+\tau-1)(2\ell+\tau+1)(2(\ell+\tau+1)-d)}. \tag{B.2.5}$$

This allows us to find in section 4.3.2 the dimensional corrections $\partial_d G^{(d)}_{\tau,\ell}(z,\bar{z})$ to subleading order in z. Solving the Casimir equation for higher values of n we find increasingly complicated functions $c_{n,m}(\tau,d,\ell)$, which will not be needed in this paper.

B.3 Results for I_2^R and I_4^R

The results for the I_2^R are as follows:

$$I_2^S\Big|_{\text{small } z} = \frac{\log^2(1-\bar{z})\left((N+2)(N-1)g_T^2+2g_S^2\right)^2}{192N^3}\Bigg[6\mathrm{Li}_3(1-\bar{z}) - 6\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + \log^3(\bar{z}) + 12\log(\bar{z}) - 6\zeta_3 + 6\zeta_2\log(z) - 12\log(z)\Bigg] + \mathcal{O}(\epsilon^5), \tag{B.3.1}$$

$$I_2^T\Big|_{\text{small } z} = \frac{\log^2(1-\bar{z})\left((N-2)g_T^2+2g_S^2\right)^2}{192N^2}\Bigg[6\mathrm{Li}_3(1-\bar{z}) - 6\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + \log^3(\bar{z}) + 12\log(\bar{z}) - 6\zeta_3 + 6\zeta_2\log(z) - 12\log(z)\Bigg] + \mathcal{O}(\epsilon^5), \tag{B.3.2}$$

$$I_2^A\Big|_{\text{small } z} = \frac{\log^2(1-\bar{z})\left((N+2)g_T^2-2g_S^2\right)^2}{64N^2}\Bigg[-6\mathrm{Li}_3(1-\bar{z}) - 6\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + 4\mathrm{Li}_2(1-\bar{z})\log(\bar{z}) + \log^3(\bar{z}) + 6\zeta_3 + 2\zeta_2\log(z)\Bigg] + \mathcal{O}(\epsilon^5). \tag{B.3.3}$$

[2] The hypergeometric equation has two solutions, but we discard the unphysical solution that blows up as $\bar{z}\to 0$.

We do not give the results for the I_4^R directly, but give the functions $h_{R,4}(\bar{z})$ defined in (4.3.17):

$$h_{S,4}(\bar{z}) = \frac{1}{4}\Gamma_{S,0}\Big[-6\mathrm{Li}_3(1-\bar{z}) - 6\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + 4\mathrm{Li}_2(1-\bar{z})\log(\bar{z}) + \log^3(\bar{z})\Big] - \frac{(N+2)(N-1)g_T^4 + 2g_S^4}{32N^2}\Big[-12\mathrm{Li}_3(1-\bar{z}) - 12\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + 8\mathrm{Li}_2(1-\bar{z})\log(\bar{z}) + 2\log^3(\bar{z}) - \log^2(\bar{z})\Big] + \mathcal{O}(\epsilon^5), \quad \text{(B.3.4)}$$

$$h_{T,4}(\bar{z}) = \frac{1}{4}\Gamma_{T,0}\Big[-6\mathrm{Li}_3(1-\bar{z}) - 6\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + 4\mathrm{Li}_2(1-\bar{z})\log(\bar{z}) + \log^3(\bar{z})\Big] - \frac{\big((N-2)(N+4)g_T^2 + 8g_S^2\big)g_T^2}{64N}\Big[-12\mathrm{Li}_3(1-\bar{z}) - 12\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + 8\mathrm{Li}_2(1-\bar{z})\log(\bar{z}) + 2\log^3(\bar{z}) - \log^2(\bar{z})\Big] + \mathcal{O}(\epsilon^5), \quad \text{(B.3.5)}$$

$$h_{A,4}(\bar{z}) = -\frac{9}{4}\Gamma_{A,1}\Big[6\mathrm{Li}_3(1-\bar{z}) - 6\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + \log^3(\bar{z}) + 12\log(\bar{z})\Big] - \frac{1}{64}(N+2)g_T^4\Big[4\mathrm{Li}_2(1-\bar{z}) + 12\mathrm{Li}_3(1-\bar{z}) - 12\mathrm{Li}_3\left(\frac{\bar{z}-1}{\bar{z}}\right) + 2\log^3(\bar{z}) + \log^2(\bar{z}) + 28\log(\bar{z})\Big] + \mathcal{O}(\epsilon^5), \quad \text{(B.3.6)}$$

where we defined $\Gamma_{R,\ell} = a_{R,4,\ell}\gamma_{R,4,\ell}^2$, which were computed in section 4.3.2 with the following result:

$$\Gamma_{S,0} = \frac{\big((N-1)(N+2)g_T^2(\epsilon - g_T) + 2\epsilon g_S^2 - 2g_S^3\big)^2}{4N^2\big((N-1)(N+2)g_T^2 + 2g_S^2 + 8N\gamma_\varphi\big)} + \mathcal{O}(\epsilon^5), \quad \text{(B.3.7)}$$

$$\Gamma_{T,0} = \frac{\big((N-2)g_T^2(\epsilon - g_T) + 2\epsilon g_S^2 - 2g_S^3\big)^2}{4N\big((N-2)g_T^2 + 2g_S^2 + 8N\gamma_\varphi\big)} + \mathcal{O}(\epsilon^5), \quad \text{(B.3.8)}$$

$$\Gamma_{A,1} = \frac{\big((N+2)g_T^2(\epsilon - g_T) - 2\epsilon g_S^2 + 2g_S^3\big)^2}{36N\big((N+2)g_T^2 - 2g_S^2 - 24N\gamma_\varphi\big)} + \mathcal{O}(\epsilon^5). \quad \text{(B.3.9)}$$

B.4 Results for CFT data

In this appendix we present the results for the functions $U_{R,\bar{h}}^{(p)}$, which are related to the anomalous dimensions and OPE coefficients through equation (4.2.14):

$$\hat{a}_{R,\Delta}(\bar{h})\,(\gamma_{R,\ell})^p = U_{R,\bar{h}}^{(p)} + \frac{1}{2}\partial_{\bar{h}}U_{R,\bar{h}}^{(p+1)} + \frac{1}{8}\partial_{\bar{h}}^2 U_{R,\bar{h}}^{(p+2)} + \dots, \quad \text{(B.4.1)}$$

where the $\hat{a}(\bar{h})$ are in turn defined by[3]

$$a_{R,\Delta,\ell} = r_{\frac{\Delta+\ell}{2}} \hat{a}_R(\bar{h}), \qquad \bar{h} = \Delta_\varphi + \ell, \qquad r_h = \frac{\Gamma(h)^2}{\Gamma(2h)}. \tag{B.4.2}$$

The results for the $U^{(p)}_{R,\bar{h}}$, as well as explicit results for the scaling dimensions and OPE coefficients, can be found in the Mathematica file included with [3]. In the functions $U^{(p)}_{R,\bar{h}}$ below we have substituted the second order solutions for the parameters of the theory according to equations (4.4.6)–(4.4.10). In order to compare with the literature, one needs to substitute also the ϵ^3 terms of g_S and g_T, given by [210,211][4]

$$g_S^{(3)} = -\frac{(N+2)\left(N^3-202N^2-976N-1568\right)}{4(N+8)^5} - \frac{12(N+2)(5N+22)}{(N+8)^4}\zeta_3, \tag{B.4.3}$$

$$g_T^{(3)} = -\frac{N^4+45N^3+190N^2-144N-1568}{2(N+8)^5} - \frac{24(5N+22)}{(N+8)^4}\zeta_3, \tag{B.4.4}$$

where $\zeta_k \equiv \zeta(k)$ denote values of the Riemann zeta function.

In the equations below, the harmonic sums have argument $S_a = S_a(\bar{h}-1)$. For S_{-2} we always use the analytic continuation from even argument, $S_{-2}(x) = \frac{1}{4}\left(\psi^{(1)}\left(\frac{x+1}{2}\right) - \psi^{(1)}\left(\frac{x+2}{2}\right)\right) - \frac{1}{2}\zeta_2$. For ease of notation, we also write $J^2 = \bar{h}(\bar{h}-1)$.

Results for $U^{(2)}_{R,\bar{h}}$:

$$U^{(2)}_{S,\bar{h}} = \frac{18(N+2)^2(2\bar{h}-1)}{N(N+8)^4 J^4}\epsilon^4, \tag{B.4.5}$$

$$U^{(2)}_{T,\bar{h}} = \frac{2(N+6)^2(2\bar{h}-1)}{(N+8)^4 J^4}\epsilon^4, \tag{B.4.6}$$

$$U^{(2)}_{A,\bar{h}} = -\frac{2(N+2)^2(2\bar{h}-1)}{(N+8)^4 J^4}\epsilon^4. \tag{B.4.7}$$

[3]Note that since our definition (4.3.11) of the conformal blocks differ from [29] by a factor $(-2)^{-\ell}$, the OPE coefficients for the antisymmetric representation are negative.

[4]The fourth order correction to Δ_φ also enters the results, but is fixed in terms of g_S and g_T using stress-energy tensor conservation at order ϵ^4. In the explicit expressions for the CFT-data in the Mathematica file, we have done this substitution.

Results for $U^{(1)}_{R,\bar{h}}$:

$$U^{(1)}_{S,\bar{h}} = \frac{(2\bar{h}-1)(N+2)}{N(N+8)^2J^2}\Bigg[-6\epsilon^2+\epsilon^3\frac{2\left(5N^2+26N+68+(N+8)^2S_1\right)}{(N+8)^2}$$
$$+\epsilon^4\Bigg(\frac{9(N+2)}{J^4(N+8)^2}+\frac{16(5N+22)S_{-2}+21(N+2)}{2J^2(N+8)^2}-\frac{4(N+8)(g_S^{(3)}+(N-1)g_T^{(3)})}{N}$$
$$-\frac{\zeta_2}{2}-\frac{\left(N^2+6N+20\right)S_1^2+2(N+2)^2S_1+2(5N+22)S_{-2}}{(N+8)^2}$$
$$+\frac{-6N^4-51N^3+132N^2+2856N+7464}{(N+8)^4}\Bigg)\Bigg], \tag{B.4.8}$$

$$U^{(1)}_{T,\bar{h}} = -\frac{2\bar{h}-1}{2J^2(N+8)^2}\Bigg[4(N+6)\epsilon^2-\epsilon^3\Bigg(\frac{4(N(N+6)+16)S_1}{N+8}$$
$$+\frac{4(N(N(N+12)+64)+136)}{(N+8)^2}\Bigg)+\epsilon^4\Bigg(-\frac{(N+6)(3N+14)+32(5N+22)S_{-2}}{J^2(N+8)^2}$$
$$-\frac{2(N+6)^2}{J^4(N+8)^2}+\frac{4(N(N+16)+44)S_{-2}-2(N^3+8N^2+24N+40)S_1^2}{(N+8)^2}$$
$$-\frac{22N^4+492N^3+4512N^2+18704N+29856}{(N+8)^4}+\frac{\zeta_2(16-(N-2)N)}{N+8}$$
$$+\frac{4(N+2)^2\left(N^2+16\right)S_1}{(N+8)^3}+\frac{8(N+8)((N+2)g_S^{(3)}+(N-2)g_T^{(3)})}{N}\Bigg)\Bigg], \tag{B.4.9}$$

$$U^{(1)}_{A,\bar{h}} = \frac{(2\bar{h}-1)(N+2)}{2J^2(N+8)^2}\Bigg[4\epsilon^2-4\epsilon^3\frac{(N+4)(N+8)S_1+N(N+2)+12}{(N+8)^2}$$
$$+\epsilon^4\Bigg(-\frac{2(N+2)}{J^4(N+8)^2}-\frac{3(N+2)}{J^2(N+8)^2}-\frac{46N^3+696N^2+3984N+7280}{(N+8)^4}$$
$$+\frac{4(N-2)(N+2)^2S_1}{(N+8)^3}+\frac{2(N(N+6)+12)S_1^2}{(N+8)^2}+\frac{4(N-2)S_{-2}}{(N+8)^2}-\frac{\zeta_2(N(N+4)+8)}{(N+8)^2}$$
$$+\frac{8(N+8)(g_S^{(3)}-g_T^{(3)})}{N}\Bigg)\Bigg]. \tag{B.4.10}$$

Results for $U^{(0)}_{R,\bar{h}}$:

$$U^{(0)}_{S,\bar{h}} = \frac{1}{N}\hat{a}^{GFF}(\bar{h})-\frac{(2\bar{h}-1)(N+2)}{4N(N+8)^2J^2}\Bigg[\frac{12}{J^2}\epsilon^2-4\epsilon^3\left(2\zeta_2+\frac{N(5N+26)+68}{J^2(N+8)^2}+\frac{S_1}{J^2}\right)$$
$$+\epsilon^4\Bigg(-\frac{6(N+2)}{J^4(N+8)^2}+\frac{12(N(N(N(N+8)-31)-524)-1308)}{J^2(N+8)^4}+6S_3$$
$$+\frac{\zeta_2((N-2)N+28)+4(N+2)^2S_1+2(N(N+6)+20)S_1^2+4(5N+22)(6\zeta_3-S_{-2})}{J^2(N+8)^2}$$

$$+\frac{2\zeta_2\left(4(5N+22)S_1+N(8N+47)+134\right)}{(N+8)^2}-\frac{6\zeta_3(N(N+21)+86)}{(N+8)^2}$$
$$+\frac{8(N+8)(g_S^{(3)}+(N-1)g_T^{(3)})}{J^2N}\Bigg)\Bigg], \tag{B.4.11}$$

$$U^{(0)}_{T,\bar{h}}=\hat{a}^{GFF}(\bar{h})+\frac{2\bar{h}-1}{4J^2(N+8)^2}\Bigg[-\frac{4(N+6)}{J^2}\epsilon^2+\epsilon^3\Bigg(\frac{4(N(N(N+12)+64)+136)}{J^2(N+8)^2}$$
$$+\frac{4(N(N+6)+16)S_1}{J^2(N+8)}+\frac{32\zeta_2(N+4)}{N+8}\Bigg)+\epsilon^4\Bigg(\frac{2(N+2)(N+6)}{J^4(N+8)^2}+\frac{N^2\left(\zeta_2-2S_1^2\right)}{J^2(N+8)}$$
$$+\frac{12(N(N(N(2N+45)+410)+1676)+2616)}{J^2(N+8)^4}-\frac{8\zeta_2(N(3N+20)+44)S_1}{(N+8)^2}$$
$$-\frac{4(N+2)^2\left(N^2+16\right)S_1}{J^2(N+8)^3}+\frac{2\zeta_3(N(N(N+25)+208)+516)}{(N+8)^2}-2(N+6)S_3$$
$$+\frac{4\left((N(N+16)+44)S_{-2}-2\zeta_2(N+7)-4(3N+5)S_1^2\right)}{J^2(N+8)^2}-\frac{48\zeta_3(5N+22)}{J^2(N+8)^2}$$
$$-\frac{2\zeta_2(N(N(33N+296)+1244)+2144)}{(N+8)^3}$$
$$-\frac{8(N+8)((N+2)g_S^{(3)}+(N-2)g_T^{(3)})}{J^2N}\Bigg)\Bigg], \tag{B.4.12}$$

$$U^{(0)}_{A,\bar{h}}=-\hat{a}^{GFF}(\bar{h})+\frac{(2\bar{h}-1)(N+2)}{4J^2(N+8)^2}\Bigg[\frac{4}{J^2}\epsilon^2-\epsilon^3\Bigg(\frac{4(N(N+2)+12)}{J^2(N+8)^2}+\frac{4(N+4)S_1}{J^2(N+8)}$$
$$+\frac{16\zeta_2}{N+8}\Bigg)+\epsilon^4\Bigg(\frac{-2(N+2)}{J^4(N+8)^2}-\frac{12\left(4N^3+61N^2+348N+628\right)}{J^2(N+8)^4}+\frac{8\zeta_2(3N+10)S_1}{(N+8)^2}$$
$$+\frac{4(N-2)(N+2)^2S_1}{J^2(N+8)^3}+\frac{(N(N+6)+12)\left(2S_1^2-\zeta_2\right)+4(N-2)S_{-2}}{J^2(N+8)^2}+2S_3$$
$$+\frac{2\zeta_2(N(17N+74)+224)}{(N+8)^3}-\frac{2\zeta_3(N(N+13)+70)}{(N+8)^2}$$
$$+\frac{8(N+8)(g_S^{(3)}-g_T^{(3)})}{J^2N}\Bigg)\Bigg]. \tag{B.4.13}$$

In these expressions

$$\hat{a}^{GFF}(\bar{h})=\frac{2(2\bar{h}-1)\Gamma(\bar{h}+\Delta_\varphi-1)}{\Gamma(\Delta_\varphi)^2\Gamma(\bar{h}-\Delta_\varphi+1)}. \tag{B.4.14}$$

Appendix C

Appendices to chapter 5

C.1 Inversion methods

In this appendix we elaborate on some of the methods used to evaluate the inversion integral

$$\widehat{\text{INV}}\left[g(\bar{z})\right] := \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)}\int_0^1 \frac{d\bar{z}}{\bar{z}^2}k_{\bar{h}}(\bar{z})g(\bar{z}), \tag{C.1.1}$$

where the $\widehat{\text{INV}}$ indicates that we are not taking the double discontinuity, i.e. we have the defining relation:

$$\text{INV}\left[G\right] = \widehat{\text{INV}}\left[\text{dDisc}(G)\right]. \tag{C.1.2}$$

C.1.1 Action of the collinear Casimir

The inversion integral above solves the following problem:

$$\sum_\ell a_\ell f_{2\Delta_\varphi+\ell,\ell}(\bar{z}) = G(\bar{z}) \quad \Rightarrow \quad a_\ell = \text{INV}\left[G(\bar{z})\right], \tag{C.1.3}$$

where the $f_{\Delta,\ell}(\bar{z}) = r\left(\frac{\Delta+\ell}{2}\right)k_{\frac{\Delta+\ell}{2}}(\bar{z})$ are collinear conformal blocks in a convenient normalization. We now apply $\overline{D}$ to both sides of equation (C.1.3), assume it is appropriate to interchange $\overline{D}$ and the infinite sum over spins, and use the fact that the f are eigenfunctions under the collinear Casimir $\overline{D}$ with eigenvalue J^2, to see that we should expect

$$a_\ell = \text{INV}\left[G(\bar{z})\right] \qquad \Rightarrow \qquad (J^2+\alpha)a_\ell = \text{INV}\left[(\overline{D}+\alpha)G(\bar{z})\right], \quad \alpha\in\mathbb{R}. \tag{C.1.4}$$

This can be very useful since $(\overline{D}+\alpha)G(\bar{z})$ is often easier to invert that $G(\bar{z})$ itself.

At the level of the double discontinuities this can be made more precise since the inversion integral (C.1.1) can be seen as an inner product on an appropriate function space V [188]:

$$\widehat{\mathrm{INV}}\left[g(\bar{z})\right] = \langle k_{\bar{h}}(\bar{z}), g(\bar{z})\rangle, \qquad \langle f(\bar{z}), h(\bar{z})\rangle := \int_0^1 \frac{\mathrm{d}\bar{z}}{\bar{z}^2} f(\bar{z})h(\bar{z}). \tag{C.1.5}$$

Furthermore, $\overline{D}$ is a self-adjoint operator in this space [188], so that it easily follows that

$$\widehat{\mathrm{INV}}\left[\overline{D}g(\bar{z})\right] = \langle k_{\bar{h}}(\bar{z}), \overline{D}g(\bar{z})\rangle = \langle \overline{D}k_{\bar{h}}(\bar{z}), g(\bar{z})\rangle = J^2\widehat{\mathrm{INV}}\left[g(\bar{z})\right], \tag{C.1.6}$$

assuming that both $g(\bar{z})$ and $\overline{D}g(\bar{z})$ lie within V. In practice this means that this works up to a contribution $\frac{2\bar{h}-1}{J^2}$, which is the inversion of $\log^2(1-\bar{z})$. To see this explicitly:

$$\mathrm{INV}\left[\log^2(1-\bar{z})\right] = \widehat{\mathrm{INV}}\left[4\pi^2\right] = \frac{4(2\bar{h}-1)}{J^2}, \tag{C.1.7}$$

so that

$$\mathrm{INV}\left[\frac{2\bar{z}^2}{1-\bar{z}}\right] = \mathrm{INV}\left[\overline{D}\log^2(1-\bar{z})\right] = 4(2\bar{h}-1) \neq 0 = \widehat{\mathrm{INV}}\left[\overline{D}(4\pi^2)\right]. \tag{C.1.8}$$

This does not pose any real problem, since for any inversion we can always check whether a term of the form $\frac{2\bar{h}-1}{J^2}$ needs to be added by comparing to the asymptotic large J expansion from appendix C.1.2, or to a numeric evaluation of the integral.

C.1.2 Asymptotic expansions for large J

The basic idea behind this method is as follows: in this paper the functions $g(\bar{z})$ in equation (C.1.1) can always be expanded, up to an overall power, as a power series in $(1-\bar{z})$, or equivalently a power series in $\xi = (1-\bar{z})/\bar{z}$, i.e. $g(\bar{z}) = \xi^\alpha \sum_n g_n \xi^n$. Let us make a change of variables in the integral to ξ, using the identity

$${}_2F_1\left(a, c-b; c; \frac{\bar{z}}{\bar{z}-1}\right) = (1-\bar{z})^a\, {}_2F_1(a, b; c; \bar{z}), \tag{C.1.9}$$

to rewrite it as

$$\widehat{\mathrm{INV}}\left[g(\bar{z})\right] = -\frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)}\int_0^\infty \mathrm{d}\xi\ \xi^{-\bar{h}}\,{}_2F_1(\bar{h},\bar{h};2\bar{h};-\xi^{-1})g(\xi), \tag{C.1.10}$$

where by a small abuse of notation, we have written $g(\xi)$ to mean $g(\bar{z}(\xi))$. For large $\bar{h}$, this integral is dominated by the contribution from the region $\xi \ll 1$. Therefore, while the power series and integral may not generally be interchanged outside the radius of convergence, a term-by-term integration of the power series in ξ will give an expansion of the inversion that is asymptotic in large $\bar{h}$, or equivalently in large $J^2 = \bar{h}(\bar{h}-1)$:

$$\widehat{\mathrm{INV}}\left[g(\bar{z})\right] \sim \sum_{n=0}^{\infty} g_n\ \widehat{\mathrm{INV}}\left[\xi^{\alpha+n}\right]. \tag{C.1.11}$$

The inversion of a single power $\widehat{\mathrm{INV}}\left[\xi^p\right]$ occurs when discussing the generalized free theory: by definition $A[-p](\bar{h}) := \mathrm{INV}\left[\xi^p\right] = 2\sin^2(\pi p)\ \widehat{\mathrm{INV}}\left[\xi^p\right]$. This inversion can be found in many various ways, e.g. via an integral representation for the hypergeometric function, or using either of the methods in sections C.1.3 and C.1.4, and gives

$$\frac{\widehat{\mathrm{INV}}\left[\xi^p\right]}{(2\bar{h}-1)} = \frac{\Gamma(1+p)^2\Gamma(\bar{h}-p-1)}{\Gamma(\bar{h}+p+1)} \quad \Rightarrow \quad \frac{\mathrm{INV}\left[\xi^{\alpha+n}\right]}{\mathrm{INV}\left[\xi^{\alpha}\right]} = \frac{(-1)^n(\alpha+1)_n^2}{\left(\alpha-\bar{h}+2\right)_n\left(\alpha+\bar{h}+1\right)_n} \tag{C.1.12}$$

so that

$$\mathrm{INV}\left[g(\bar{z})\right] \sim A[-\alpha](\bar{h})\sum_{n=0}^{\infty} g_n\,\frac{(-1)^n(\alpha+1)_n^2}{\left(\alpha-\bar{h}+2\right)_n\left(\alpha+\bar{h}+1\right)_n} \tag{C.1.13}$$

provides an asymptotic expansion in $1/\bar{h}$ for $\mathrm{INV}\left[g(\bar{z})\right]$.

In simple cases this asymptotic expansion may be recognized as that of a known function, such as rational functions of J^2 and simple combinations of harmonic numbers $S_1(\bar{h}-1)$. More generally an ansatz of appropriate functions can be fitted to a sufficiently high order asymptotic expansion, and the final result can then be checked for finite $\bar{h}$ through a numeric evaluation of the integral.

C.1.3 Power series in $\bar{z}$ and $1-\bar{z}$

The approach in section C.1.2 is always able to provide an asymptotic expansion around $J=\infty$, but unfortunately misses out on one of the main advantages of the inversion formula: the fact that it converges for finite spins down to $\ell=2$. However, a simpler approach would be to not make the change of variables to ξ but expand both the integration kernel and the double discontinuity $g(\bar{z})$ as power series in $\bar{z}$ and $1-\bar{z}$ respectively. The kernel has radius of convergence 1, as do many of the double discontinuities encountered in this paper. Since it is fine to integrate a power series term-by-term inside its radius of convergence, we expect that the resulting infinite sum gives the full non-asymptotic answer for the inversion integral.[1]

To be more explicit, we expand the kernel and double discontinuity as power series:

$$\bar{z}^{\bar{h}-2}\,{}_2F_1(\bar{h},\bar{h};2\bar{h};\bar{z})=\sum_{n=0}^{\infty}\frac{(\bar{h})_n^2}{n!(2\bar{h})_n}\bar{z}^{\bar{h}-2+n},\qquad g(\bar{z})=\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\alpha}\sum_{k=0}^{\infty}g_k(1-\bar{z})^k. \tag{C.1.14}$$

Then we interchange the orders of summation and integration, and perform the simple beta-integral to find

$$\frac{\text{INV}\left[g(\bar{z})\right]}{\text{INV}\left[\xi^{-\alpha}\right]}=\frac{\Gamma(\bar{h})\Gamma(\bar{h}-\alpha+1)}{\Gamma(1-\alpha)\Gamma(2\bar{h})}\sum_{n,k=0}^{\infty}\frac{(\bar{h})_n^2(\bar{h}+\alpha-1)_n(1-\alpha)_k\,g_k}{n!\,(2\bar{h})_n(\bar{h})_{k+n}}. \tag{C.1.15}$$

Depending on the form of the g_k, this sum may be used to give a nice expression for the inversion. For example, if $g(\bar{z})$ is a hypergeometric function of $(1-\bar{z})$, equation C.1.15 becomes a Kampé de Feriet function, which we use to derive the expressions for the crossing kernels in section 5.3.2.1.

In some special cases, the sum may even be evaluated to give an elementary function. An example is the case $P_0(\bar{z})=\left(\frac{\bar{z}}{1-\bar{z}}\right)^{\mu-1}(1-\bar{z})$, i.e. $\alpha=\mu-1$ and

[1] There may be an issue involving convergence at the radius of convergence, which is not guaranteed. If the final result is convergent though, one expects that this interchange is valid by Abel's theorem.

$g_k = \delta_{k,1}$. The sum over k is trivial and the sum over n yields a ${}_3F_2$ hypergeometric function at 1, to which we apply an identity [212] to find

$$\text{INV}\left[P_0(\bar{z})\right] = (\mu-2)^2 \text{INV}\left[\xi^{1-\mu}(1-\bar{z})\right] = \frac{A[\mu-1](\bar{h})}{\bar{h}(\bar{h}+2-\mu)}\, {}_3F_2\left(\begin{matrix}1, \bar{h}, 3-\mu \\ \bar{h}+1, \bar{h}+3-\mu\end{matrix}\middle| 1\right). \tag{C.1.16}$$

We recognize this as a multiple of the function $R(\bar{h}, \mu)$, defined in equation (5.3.29).

Similarly one can incorporate logarithms like $\log(\bar{z})$ in the power series of $g(\bar{z})$. A function of particular interest to us is the function $Q_\alpha(\bar{z}) = -\left(\frac{\bar{z}}{1-\bar{z}}\right)^\alpha \log(\bar{z})$, which again trivializes the sum over k. The sum over n evaluates to derivatives of hypergeometric ${}_2F_1$ functions at 1. Since these hypergeometric functions are simply a ratio of gamma functions [212], we can explicitly evaluate the result to find that

$$\text{INV}\left[Q_\alpha(\bar{z})\right] = A[\alpha](\bar{h})\ \mathbf{S_1}[\Delta](\bar{h}), \tag{C.1.17}$$

where $\mathbf{S_1}[\alpha](\bar{h}) = 2S_1(\bar{h}-1) - S_1(\bar{h}+\alpha-2) - S_1(\bar{h}-\alpha)$, as defined in (5.2.23).

C.1.4 Mellin space methods

The idea of this method is to replace both the kernel $\bar{z}^{\bar{h}}\, {}_2F_1(\bar{h}, \bar{h}; 2\bar{h}; \bar{z})$ and the double discontinuity $g(\bar{z})$ by their Mellin representation [107, 188]. In many cases, such as when g is a hypergeometric function, this makes the $\bar{z}$ integral into a simple Beta-function integral, which can be performed with some suitable regularization. What remains are two inverse Mellin transforms: for a suitable class of functions, discussed below, one of these can be performed using Barnes' second lemma [188]. For simple enough functions, the final inverse Mellin transform may then be performed again using known lemmas, or a sum over poles may be performed to give an infinite sum representation of the inversion.

We will actually deviate from the method above slightly in a way that avoids both the need for regularization and for using Barnes' second lemma, instead directly

giving the inversion integral as a single inverse Mellin transform. First we define the Mellin transform of a function $f(x)$ as

$$F(s) = \mathcal{M}[f](s) = \int_0^\infty \mathrm{d}x\ x^{s-1} f(x). \tag{C.1.18}$$

Defining $F(s) = \int_0^\infty \mathrm{d}x\ x^{s-1} f(x)$ and $G(s) = \int_0^\infty \mathrm{d}x\ x^{s-1} g(x)$ to be the Mellin transforms of f and g respectively, we get that

$$\int_0^\infty \mathrm{d}x\ x^{s-1} f(x) g(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}t\ F(t) G(s-t). \tag{C.1.19}$$

To see the relevance to the inversion problem, we use the substitution $\bar{z} \to \chi = \xi^{-1} = \frac{\bar{z}}{1-\bar{z}}$ and some identities of hypergeometric functions to recast the inversion integral as

$$\widehat{\mathrm{INV}}\,[g] = \frac{\Gamma(\bar{h})^2}{\pi^2 \Gamma(2\bar{h}-1)} \int_0^\infty \frac{\mathrm{d}\chi}{\chi^2}\, \chi^{\bar{h}}\, {}_2F_1(\bar{h}, \bar{h}; 2\bar{h}; -\chi) g(\chi), \tag{C.1.20}$$

which is now of the form (C.1.19).

There are two free parameters in this method: we can shift around the powers of χ between f and g, and we can evaluate the Mellin transform at different s. The latter does not affect the integrand, but the former does: however the poles and residues do not change, so that the inversion is insensitive to this.[2] For most inversions, it has been useful to pick $f(\chi) = \frac{\Gamma(\bar{h})^2}{\pi^2\Gamma(2\bar{h}-1)}\, {}_2F_1(\bar{h}, \bar{h}; 2\bar{h}; -\chi)$, and absorb the rest into $g(\chi)$. For example, to invert $\xi^p = \chi^{-p}$, we use $g(\chi) = \chi^{-p+\bar{h}-1}$, so that

$$F(s) = \frac{2\bar{h}-1}{\pi^2} \frac{\Gamma(s)\Gamma(\bar{h}-s)^2}{\Gamma(2h-s)}, \qquad G(s) = 2\pi\delta(\mathrm{i}(-p+\bar{h}-1+s)) \tag{C.1.21}$$

Therefore we find that

$$\begin{aligned} \widehat{\mathrm{INV}}[\xi^p] &= \frac{1}{\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}t\ \delta(\mathrm{i}(-p+\bar{h}-1+t)) \frac{\Gamma(-t)\Gamma(\bar{h}+t)^2}{\Gamma(2\bar{h}+t)} \\ &= \frac{2\bar{h}-1}{\pi^2} \frac{\Gamma(1+p)^2 \Gamma(\bar{h}-p-1)}{\Gamma(\bar{h}+p+1)}. \end{aligned} \tag{C.1.22}$$

[2]The contour depends slightly on this choice: it needs to be chosen such that certain series of poles lie to its right, and others to its left.

A general class of functions that can be inverted in this way, are the hypergeometric functions of $-\xi$, i.e. functions of the form

$$g(\xi) = \xi^{-k}\, {}_{p+1}F_p\left(\begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix} \middle| -\xi\right). \tag{C.1.23}$$

We find that

$$\frac{\pi^2}{2\bar{h}-1}\widehat{\mathrm{INV}}[g(\xi)] = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}t\ \frac{\Gamma(-t)\Gamma(1-k+t)^2\Gamma(\bar{h}+k-t-1)}{\Gamma(\bar{h}-k+t+1)}\frac{\prod_{j=1}^{p+1}\Gamma(a_j+t)}{\prod_{j=1}^{p}\Gamma(b_j+t)}. \tag{C.1.24}$$

Closing the contour to the right, we find poles at $t \in \mathbb{N}$ and $t \in \bar{h}+k-1+\mathbb{N}$. Computing the residues carefully, we find two sums corresponding to hypergeometric functions, leading to the final result

$$\begin{aligned}\frac{\pi^2}{2\bar{h}-1}\widehat{\mathrm{INV}}[g(\xi)] = {}& \Gamma(1-\bar{h}-k)\frac{\prod_{j=1}^{p}\Gamma(b_j)/\Gamma(\tilde{b}_j)}{\prod_{j=1}^{p+1}\Gamma(a_j)/\Gamma(\tilde{a}_j)}\, {}_{p+3}F_{p+2}\left(\begin{matrix} \bar{h}, \bar{h}, \tilde{a}_1\ \dots,\ \tilde{a}_{p+1} \\ 2\bar{h}, \bar{h}+k, \tilde{b}_1, \dots, \tilde{b}_p \end{matrix} \middle| 1\right) \\ & + \frac{\Gamma(2\bar{h})\Gamma(1-k)^2\Gamma(\bar{h}+k-1)}{\Gamma(\bar{h})^2\Gamma(1+\bar{h}-k)}\, {}_{p+3}F_{p+2}\left(\begin{matrix} 1-k, 1-k, a_1\ \dots,\ a_{p+1} \\ 2-\bar{h}-k, 1+\bar{h}-k, b_1, \dots, b_p \end{matrix} \middle| 1\right),\end{aligned} \tag{C.1.25}$$

where we defined $\tilde{a}_j = a_j + \bar{h} + k - 1$, and similarly $\tilde{b}_j = b_j + \bar{h} + k - 1$.

This is a non-asymptotic result that is valid for finite $\bar{h}$. The analytic structure of this result is quite interesting. The first hypergeometric function in equation (C.1.25) can be expanded in $1/\bar{h}$ from the definition of the hypergeometric function as an infinite sum; in fact, it is precisely the hypergeometric function that one would naively obtain if one expanded $g(\xi)$ as a power series in ξ and integrated term-by-term, as in appendix C.1.2. This single hypergeometric function indeed gives the correct asymptotic expansion in large $\bar{h}$, but is incorrect for finite $\bar{h}$: it even has an infinite series of poles in $\bar{h}$, which are precisely cancelled by the other hypergeometric function.

An example of such a function is the special function $B(\bar{h}, \mu)$ in equation (5.3.10).

Using the inversion method above, we see that it satisfies[3]

$$B(\bar{h},\mu) = \frac{1}{J^2\left(J^2-2\right)}\,{}_4F_3\left(\begin{matrix}1,1,2,\mu+1\\3,3-\bar{h},\bar{h}+2\end{matrix}\middle|\,1\right) - \frac{2\pi\Gamma\left(\bar{h}\right)\Gamma\left(\mu+\bar{h}-1\right)}{J^2\Gamma(\mu+1)\sin\left(\pi\bar{h}\right)\Gamma\left(2\bar{h}\right)}\,{}_3F_2\left(\begin{matrix}\bar{h}-1,\bar{h},\bar{h}+\mu-1\\2\bar{h},\bar{h}+1\end{matrix}\middle|\,1\right). \tag{C.1.26}$$

It also has a large J expansion of the form:

$$B(\bar{h},\mu) \sim \frac{1}{J^4} - \frac{2(\mu-2)}{3J^6} + \frac{\mu^2-\frac{7\mu}{3}+\frac{2}{3}}{J^8} + \dots. \tag{C.1.27}$$

C.2 Inversion table

We present here, in tables C.1 and C.2, a collection of inversion results, i.e. for a collection of different $G(\bar{z})$, we give $\text{INV}\left[G(\bar{z})\right]$. For the reader's convenience, we recall the definitions of frequently recurring functions:

$$A[p](\bar{h}) = \frac{2(2\bar{h}-1)\Gamma(\bar{h}+p-1)}{\Gamma(p)^2\Gamma(\bar{h}-p+1)} \tag{C.2.1}$$

$$\mathbf{S_1}[\alpha](\bar{h}) = 2S_1(\bar{h}-1) - S_1(\bar{h}+\alpha-2) - S_1(\bar{h}-\alpha) \tag{C.2.2}$$

$$R(\bar{h},\mu) = \frac{1}{\bar{h}(\bar{h}+2-\mu)}\,{}_3F_2\left(\begin{matrix}1,\bar{h},3-\mu\\\bar{h}+1,\bar{h}+3-\mu\end{matrix}\middle|\,1\right). \tag{C.2.3}$$

$$B(\bar{h},\mu) = \frac{1}{J^2\left(J^2-2\right)}\,{}_4F_3\left(\begin{matrix}1,1,2,\mu+1\\3,3-\bar{h},\bar{h}+2\end{matrix}\middle|\,1\right) - \frac{2\pi\Gamma(\bar{h})\Gamma(\mu+\bar{h}-1)}{J^2\Gamma(\mu+1)\sin(\pi\bar{h})\Gamma(2\bar{h})}\,{}_3F_2\left(\begin{matrix}\bar{h}-1,\bar{h},\bar{h}+\mu-1\\2\bar{h},\bar{h}+1\end{matrix}\middle|\,1\right). \tag{C.2.4}$$

From the inversions in these tables we can deduce many other inversions through the use of the collinear Casimir $\overline{D}$. An example that is relevant to the first inversion in this paper, giving the first-order the anomalous dimensions $\gamma^{(1)}_{T/A,\ell}$ is the following: table C.2 says that $\text{INV}\left[\xi^{1-\mu}\right] = A[\mu-1](\bar{h})$. Since

$$\overline{D}\left[\frac{1}{(\mu-2)^2}\xi^{1-\mu}\,(1-\bar{z})\,{}_2F_1\left(\begin{matrix}1,1\\3-\mu\end{matrix}\middle|\,1-\bar{z}\right)\right] = \xi^{1-\mu}, \tag{C.2.5}$$

[3] Another expression for B, valid for finite $\bar{h}$, can be found by acting once with the Casimir and inverting as a power series in $(1-\bar{z})$, as in appendix C.1.3. The resulting expression is manifestly expandable in large $\bar{h}$, but involves derivatives of hypergeometric functions and due to its length we do not present it here.

it follows that

$$\text{INV}\left[\frac{1}{(\mu-2)^2}\xi^{1-\mu}\,(1-\bar{z})\,{}_2F_1\left(\begin{matrix}1,1\\3-\mu\end{matrix}\middle|\,1-\bar{z}\right)\right]=\frac{A[\mu-1](\bar{h})}{J^2}. \tag{C.2.6}$$

$G(\bar{z})$	$\text{INV}\,[G(\bar{z})]$
$\log^2(1-\bar{z})$	$4\left(2\bar{h}-1\right)\frac{1}{J^2}$
$\log^2(1-\bar{z})\,\bar{z}^{3-\mu}$	$2\pi^2\left(2\bar{h}-1\right)R(\bar{h},\mu)$
$\frac{1}{4}\log^2(1-\bar{z})\,\xi\,{}_3F_2\left(\begin{matrix}1,1,\mu+1\\2,3\end{matrix}\middle|\,-\xi\right)$	$(2\bar{h}-1)B(\bar{h},\mu)$

Table C.1: Inversions for $G(\bar{z})$ not containing non-integer powers of $(1-\bar{z})$

$G(\bar{z})$	$\text{INV}\,[G(\bar{z})]$
$\left(\frac{\bar{z}}{1-\bar{z}}\right)^p$	$A[p](\bar{h})$
$\frac{1}{(\mu-2)^2}\xi^{1-\mu}\,(1-\bar{z})$	$A[\mu-1](\bar{h})R(\bar{h},\mu)$
$\xi^{1-\mu}\log\bar{z}$	$A[\mu-1](\bar{h})\mathbf{S_1}[\mu-1](\bar{h})$

Table C.2: Inversions for $G(\bar{z})$ containing non-integer powers of $(1-\bar{z})$

For completeness, we also include an explicit form for $R(\bar{h},\mu)$ and $B(\bar{h},\mu)$ in integer dimensions $2,3,4$:

$$R(\bar{h},1)=1-J^2\left(2S_{-2}(\bar{h}-1)+\zeta_2\right) \tag{C.2.7}$$

$$R(\bar{h},\tfrac{3}{2})=4+2(2\bar{h}-1)\left(S_1(\bar{h}/2-1)-S_1(\bar{h}/2-1/2)\right) \tag{C.2.8}$$

$$R(\bar{h},2)=2S_{-2}(\bar{h}-1)+\zeta_2, \tag{C.2.9}$$

where $S_{-2}(n) = \frac{1}{4}\left(S_2(n/2) - S_2(n/2-1/2)\right) - \zeta_2/2$. Furthermore

$$B(\bar{h},1) = \frac{2}{J^4} - \frac{2}{J^2} - 4S_3(\bar{h}-1) + 4\zeta_3 \tag{C.2.10}$$

$$B(\bar{h},\tfrac{3}{2}) = \frac{1}{J^4} + \frac{\left(1-2\bar{h}+2J^2S_1(\bar{h}/2-1) - 2J^2S_1(\bar{h}/2-1/2)\right)^2}{J^4} \tag{C.2.11}$$

$$B(\bar{h},2) = \frac{1}{J^4}. \tag{C.2.12}$$

To compare with the ϵ-expansion, one also needs

$$\partial_\mu B(\bar{h},2) = \frac{2S_{-2}(\bar{h}-1)+\zeta_2}{J^2} - 2\left(S_3(\bar{h}-1) - \zeta_3\right) - \frac{1}{J^2} - \frac{1}{2J^4}. \tag{C.2.13}$$

C.3 Sums of unitarity bound conformal blocks

The contributions $H^{(2)}_{T\pm A}$ to the double discontinuity in sections 5.2.4 and 5.3.1 are sums of conformal blocks of twist $\tau = d-2$, corresponding to operators that saturate the unitarity bound. To find these contributions, we are interested in computing sums of unitarity bound blocks, $G^{(d)}_{\tau=d-2,\ell}(z,\bar{z})$. In this appendix, we will find an explicit expression for these blocks and sums thereof.

C.3.1 A solution basis

Any unitarity bound conformal block is annihilated by the operator [148]

$$\mathcal{D}_{\text{sat}} = 2(\mu-1)\left(z^2\frac{\partial}{\partial z} - \bar{z}^2\frac{\partial}{\partial \bar{z}}\right) + 2z\bar{z}(\bar{z}-z)\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}. \tag{C.3.1}$$

The twist conformal blocks $H^{(m)}_{T\pm A}(z,\bar{z})$, as sums of such blocks, are also annihilated by $\mathcal{D}_{\text{sat}}$:

$$\mathcal{D}_{\text{sat}}\, H^{(m)}_{T\pm A}(z,\bar{z}) = 0. \tag{C.3.2}$$

Recall that they also solve a recursion relation using the Casimir in (5.2.35):

$$\mathcal{C}^m H^{(m)}_{T\pm A}(z,\bar{z}) = H^{(0)}_{T\pm A}(z,\bar{z}). \tag{C.3.3}$$

We shall find the twist conformal blocks by solving the $\mathcal{D}_{\text{sat}}$ equation, and then imposing the Casimir equation on the solution. To do this, we make the following expansion:

$$f(z,\bar{z}) = (z\bar{z})^{\mu-1}\sum_k z^k g_k(y), \qquad y = \frac{z}{\bar{z}}. \tag{C.3.4}$$

Acting with $\mathcal{D}_{\text{sat}}$ on $f(z,\bar{z})$, the $\mathcal{D}_{\text{sat}}$ action respects the powers in z and reduces to a linear ordinary differential equation for each $g_k(y)$:

$$\begin{aligned} 0 &\overset{!}{=} \mathcal{D}_{\text{sat}}\left((z\bar{z})^{\mu-1}z^k g_k(y)\right) \\ &= 2(z\bar{z})^{\mu-1}z^{k+1}\left(\frac{k(\mu-1)g_k(y)}{y} + (\mu-2-k+(\mu+k)y)g_k'(y) + y(y-1)g_k''(y)\right). \end{aligned} \tag{C.3.5}$$

The general solution to this linear ODE is

$$g_k(y) = c_k\, y^{-k}\,{}_2F_1\left(\mu-1,-k;2-\mu-k;y\right) + \tilde{c}_k\, y^{\mu-1}\,{}_2F_1\left(\mu-1,2\mu+k-2;\mu+k;y\right). \tag{C.3.6}$$

In general, solutions to $\mathcal{D}_{\text{sat}} f = 0$ will have k ranging over $\mathbb{Z}$. However, sums of unitarity bound blocks will have only positive $k = 0,1,2,\ldots$, since each conformal block behaves like $G^{(d)}_{\tau=d-2,\ell}(z,\bar{z}) \sim z^{\mu-1}$ in the limit of small z. The small z limit also shows that $\tilde{c}_k = 0$ for all k, and therefore sums of unitarity bound blocks can be written as a sum

$$\sum_{k=0}^{\infty} c_k (z\bar{z})^{\mu-1} T_k(z,\bar{z}), \qquad T_k(z,\bar{z}) = \bar{z}^k\,{}_2F_1\left(\mu-1,-k;2-\mu-k;\frac{z}{\bar{z}}\right). \tag{C.3.7}$$

Notice that each T_k is a symmetric polynomial in $z,\bar{z}$ of homogeneous degree k.

C.3.2 Examples

Scalar block at $\Delta = d-2$

The scalar conformal blocks $g^{(d)}_{\Delta,\ell}(z,\bar{z})$ are given in (5.2.15). In the particular case where $\Delta = d-2$, this is

$$G^{(d)}_{d-2,0} = (z\bar{z})^{\mu-1} g^{(d)}_{d-2,0}(z,\bar{z}) = (z\bar{z})^{\mu-1}\sum_{k=0}^{\infty} c_k T_k(z,\bar{z}), \tag{C.3.8}$$

where

$$c_k = \frac{(\mu-1)_k^2}{k!\,(2\mu-2)_k}. \tag{C.3.9}$$

This is readily found by performing the sum over n in (5.2.15) and then expanding both sides for small z.

Tree-level correlator $H^{(0)}_{T\pm A}(z,\bar{z})$

Also the tree-level twist conformal block $H^{(0)}_{T-A}(z,\bar{z})$ is easily decomposed into the basis of functions $T_k(z,\bar{z})$

$$H^{(0)}_{T-A}(z,\bar{z}) = \left(\frac{z\bar{z}}{(1-z)(1-\bar{z})}\right)^{\mu-1} = (z\bar{z})^{\mu-1}\sum_{k=0}^{\infty}\frac{(\mu-1)_k}{k!}T_k(z,\bar{z}). \tag{C.3.10}$$

It is clear that $H^{(0)}_{T+A}(z,\bar{z})$ is simply equal to a single such function:

$$H^{(0)}_{T+A}(z,\bar{z}) = (z\bar{z})^{\mu-1}\sum_{k=0}^{\infty}\delta_{k,0}T_k(z,\bar{z}) = (z\bar{z})^{\mu-1}, \tag{C.3.11}$$

Spinning unitarity bound conformal block

We provide here the explicit results for the unitarity bound blocks for spinning operators, as an infinite sum. As far as we are aware, this is a new result.

$$G^{(d)}_{\tau=d-2,\ell}(z,\bar{z}) = (z\bar{z})^{\mu-1}\sum_{k=0}^{\infty}c_{k,\ell}T_k(z,\bar{z}) \tag{C.3.12}$$

where

$$c_{k,\ell} = \frac{\Gamma\left(\mu-1+k\right)^2\Gamma(2\mu+2\ell-2)}{\Gamma\left(\mu-1+\ell\right)^2\Gamma(k-\ell+1)\Gamma(2\mu+k+\ell-2)}. \tag{C.3.13}$$

To find this, we look for simultaneous solutions of

$$\mathcal{D}_{\text{sat}}\,G^{(d)}_{\tau=d-2,\ell}(z,\bar{z}) = 0, \qquad \mathcal{C}\,G^{(d)}_{\tau=d-2,\ell}(z,\bar{z}) = J^2_{d-2,\ell}\,G^{(d)}_{\tau=d-2,\ell}(z,\bar{z}), \tag{C.3.14}$$

where $J^2_{\tau,\ell} = (\ell+\tau/2)\,(\ell+\tau/2-1)$.

For consistency we have checked that the solutions (C.3.12) reduce to the well-known spinning conformal blocks in four dimensions.

C.4 Computing $H^{(2)}_{T\pm A}$

In this appendix we provide some more details on the computation of the contributions $H^{(2)}_{T\pm A}$ used in sections 5.2.4 and 5.3.1. As described in detail in these sections, they are special cases of the following twist conformal blocks at the unitarity bound:

$$H^{(m)}_{T\pm A}(z,\bar{z}) = \sum_{\ell=0,1,2,\dots} \frac{a^{\text{free}}_{\mu,\ell}}{J^{2m}_{d-2,\ell}} (\mp 1)^\ell G^{(d)}_{\tau=d-2,\ell}(z,\bar{z}), \tag{C.4.1}$$

where $a^{\text{free}}_{\mu,\ell}$ is OPE coefficients for a free scalar field in $d = 2\mu$ dimensions.

In section C.4.1 we will use the expression (C.3.7) for sums of spinning unitarity bound blocks to write down $H^{(2)}_{T\pm A}$ in terms of the so-called Kampé de Feriet functions [190]. These functions give an expression for the full $H^{(2)}_{T\pm A}(z,\bar{z})$ as an infinite sum in powers of $z,\bar{z}$; however we are interested in the limit $w = 1-\bar{z} \ll 1$, which is difficult to extract from this expression. In section C.4.2 we describe a method using coupled differential equations that allows us to find the small w limit explicitly, with some integration constants fixed by the Kampé de Feriet expansion.

C.4.1 As Kampé de Feriet functions

Notice that the conformal blocks satisfy the following relation

$$G^{(d)}_{\tau,\ell}(z,\bar{z}) = (-1)^\ell G^{(d)}_{\tau,\ell}\left(\tfrac{z}{z-1}, \tfrac{\bar{z}}{\bar{z}-1}\right), \tag{C.4.2}$$

from which it follows that

$$H^{(m)}_{T-A}(z,\bar{z}) = H^{(m)}_{T+A}(z,\bar{z})\Big|_{z,\bar{z}\to \frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}}. \tag{C.4.3}$$

This can clearly be seen for $m = 0$ in equations (C.3.10) and (C.3.11). This means that we can focus our attention on computing for instance $H^{(2)}_{T+A}(z,\bar{z})$, from which $H^{(2)}_{T-A}(z,\bar{z})$ will follow immediately.

$H^{(m)}_{T+A}(z,\bar{z})$ is a sum of unitarity bound blocks, and we therefore look for an expression of the form (C.3.7)

$$H^{(m)}_{T+A}(z,\bar{z}) = (z\bar{z})^{\mu-1} \sum_{k=0}^{\infty} c^{(m)}_k T_k(z,\bar{z}), \qquad c^{(0)}_k = \delta_{k,0}. \tag{C.4.4}$$

The action of the Casimir (5.2.35) gives a recursion relation amongst the basis functions:

$$\mathcal{C}\left((z\bar{z})^{\mu-1}T_k(z,\bar{z})\right)=(z\bar{z})^{\mu-1}(\mu-1+k)(\mu-2+k)T_k(z,\bar{z})+(z\bar{z})^{\mu-1}(\mu-1+k)^2T_{k+1}(z,\bar{z}). \tag{C.4.5}$$

With the ansatz and initial condition (C.4.4), the recursion relation leads to a difference equation that can be solved case by case in m. The first two solutions are

$$c_k^{(1)}=\frac{(-1)^k}{(\mu-2)(\mu-1+k)},\qquad c_k^{(2)}=\frac{(-1)^k(k+1)}{(\mu-2)^2(\mu-1+k)^2},\qquad k=0,1,2,\ldots \tag{C.4.6}$$

Using the finite sum representation for $T_k(z,\bar{z})$, we can express the $H^{(m)}_{T+A}(z,\bar{z})$ as double sums over $z,\bar{z}$. After some manipulations, the explicit results are

$$\begin{aligned}H^{(1)}_{T+A}(z,\bar{z})&=\frac{(z\bar{z})^{\mu-1}}{(\mu-1)(\mu-2)}F^{11}_{10}\left(\begin{matrix}1\\ \mu\end{matrix}\middle|\{\mu-1\},\{\mu-1\}\middle|z,\bar{z}\right)\\&=\frac{(z\bar{z})^{\mu-1}}{(\mu-1)(\mu-2)}\mathrm{F}_1(1;\mu-1,\mu-1;\mu;z,\bar{z}),\end{aligned} \tag{C.4.7}$$

$$H^{(2)}_{T+A}(z,\bar{z})=\frac{(z\bar{z})^{\mu-1}}{(\mu-1)^2(\mu-2)^2}F^{21}_{20}\left(\begin{matrix}2, & \mu-1\\ \mu, & \mu\end{matrix}\middle|\{\mu-1\},\{\mu-1\}\middle|z,\bar{z}\right), \tag{C.4.8}$$

where F^{pq}_{rs} is the Kampé de Fériet function defined in (5.3.15), and F_1 is Appell's hypergeometric function [213, 214]. $H^{(2)}_{T-A}(z,\bar{z})$ follows from (C.4.8) upon replacing $z,\bar{z}\to\frac{z}{z-1},\frac{\bar{z}}{\bar{z}-1}$.

C.4.2 Differential equation approach

Here we present a different approach, in which the small z limit of $H^{(2)}_{T\pm A}(1-\bar{z},1-z)$ is evident from the start, and which is based on the observation that both the Casimir $\mathcal{C}$ and $\mathcal{D}_{\text{sat}}$ preserve integrality and non-integrality of powers of $z,(1-\bar{z})$. We can therefore expand $H^{(2)}_{T\pm A}(z,1-w)$ in powers of $w=1-\bar{z}$ and treat the differential equations for integer-power part completely separately. That is, we write:

$$H^{(2)}_{T\pm A,\text{int}}(1-\bar{z},1-z)=\xi^{\mu-1}\sum_{k=0}^{\infty}z^k\left(f_k^{(\pm)}(\xi)+g_k^{(\pm)}(\xi)\log z\right), \tag{C.4.9}$$

where $\xi = \frac{1-\bar{z}}{\bar{z}}$, and we impose the $\mathcal{D}_{\text{sat}}$ and Casimir equations order by order in z.

By isolating the $\log z$ piece of these equations, we can find all the $g_k^{(\pm)}$ order by order:

$$g_0^{(-)}(\xi) = -\frac{\pi \csc(\pi\mu)}{\mu-2}, \qquad g_1^{(-)}(\xi) = -\frac{(\mu-1)^2\pi \csc(\pi\mu)}{\mu-2}\xi, \qquad \ldots, \tag{C.4.10}$$

$$g_0^{(+)}(\xi) = -\frac{1}{(\mu-2)^2}, \qquad g_1^{(+)}(\xi) = -\frac{1}{\mu-2} - \frac{(\mu-1)^2}{(\mu-2)^2}\xi, \qquad \ldots, \tag{C.4.11}$$

where the overall constant was fixed from the Kampé de Feriet expansions (C.4.8).

Plugging this back into the differential equations, we get a set of two coupled differential equations for $f_0(\xi)$ and $f_1(\xi)$. We can eliminate $f_1(\xi)$ from the Casimir equation and put this in the $\mathcal{D}_{\text{sat}}$ equation to get a single, third order differential equation for $f_0(\xi)$. This is an inhomogeneous differential equation, with inhomogeneous part determined by $\xi^{1-\mu} H^{(1)}_{T\pm A}(1-\bar{z}, 1-z)\big|_{z^0}$, which can be extracted from (C.4.7)[4]. Specifically, we find the equations:

$$\xi(\xi+1)\partial_\xi^3 f_0^{(-)}(\xi) + 2(2\xi+1)\partial_\xi^2 f_0^{(-)}(\xi) - \mu(\mu-3)\partial_\xi f_0^{(-)}(\xi) = \frac{1 - {}_2F_1\left(3-\mu, \mu; 2; -\xi\right)}{4\xi}, \tag{C.4.12}$$

$$\xi(\xi+1)\partial_\xi^3 f_0^{(+)}(\xi) + 2(2\xi+1)\partial_\xi^2 f_0^{(+)}(\xi) - \mu(\mu-3)\partial_\xi f_0^{(+)}(\xi) = \frac{(1+\xi)^{1-\mu} + (\mu-1)\xi - 1}{(\mu-2)\xi^2}. \tag{C.4.13}$$

The equation (C.4.12) is easily solved exactly. Demanding that the solution has no logarithmic singularity at $\xi = 0$, and that it is compatible with the Kampé de Feriet expansions, fixes the integration constants to give

$$f_0^{(-)}(\xi) = \pi \csc(\pi\mu)\left[\frac{\pi\cot(\pi\mu)}{\mu-2} - \frac{1}{(\mu-2)^2} + (\mu-1)\xi\, {}_4F_3\left(\begin{matrix}1,1,3-\mu,\mu\\2,2,2\end{matrix}\,\middle|\, -\xi\right) - \frac{{}_2F_1(2-\mu,\mu-1;1;-\xi)-1}{(\mu-2)^2}\right]. \tag{C.4.14}$$

[4] One has

$$H^{(1)}_{T-A}(1-\bar{z},1-z)\Big|_{z^0} = -\frac{\pi\csc(\pi\mu)}{\mu-2}\xi^{\mu-1}\,{}_2F_1\left(2-\mu,\mu-1;1;-\xi\right),$$

$$H^{(1)}_{T+A}(1-\bar{z},1-z)\Big|_{z^0} = -\frac{1}{\mu-2}\xi^{\mu-1}\log z - \frac{S_1(\mu-2)}{\mu-2}\xi^{\mu-1} - \frac{\mu-1}{\mu-2}\xi^{\mu}\,{}_3F_2\left(\begin{matrix}1,1,\mu\\2,2\end{matrix}\,\middle|\,-\xi\right).$$

As for $f_0^{(+)}(\xi)$, we are unable to solve the differential equation directly, but we can find it as a series in ξ and match this up to known functions. This way we find that

$$f_0^{(+)}(\xi) = \frac{1}{(\mu-2)^2}\Bigg[\frac{(\mu-2)^2(S_2(\mu-2)-\zeta_2)+\mu-3}{\mu-2}\,{}_2F_1(2-\mu,\mu-1;1;-\xi) + \frac{1}{2}(\mu-2)(\mu-1)\mu\left(\overline{D}-K^2\right)^{-1}\left[\xi\,{}_3F_2\left(\begin{matrix}1,1,\mu+1\\2,3\end{matrix}\,\middle|\,-\xi\right)\right] - \frac{\mu-3}{\mu-2} - S_1(\mu-2)\Bigg]. \tag{C.4.15}$$

Here $\overline{D}$ is the collinear Casimir operator defined in (4.2.16), and $K^2=(\mu-1)(\mu-2)$. Note that $\left(\overline{D}-K^2\right)^{-1}[g(\xi)]$ is the solution to an ODE with kernel equal to ${}_2F_1(2-\mu,\mu-1;1;-\xi)$. In the expression above, the coefficient of the kernel element is chosen such that in a small ξ expansion $\left(\overline{D}-K^2\right)^{-1}[g(\xi)]$ starts at one order higher than $g(\xi)$, i.e.

$$\left(\overline{D}-K^2\right)^{-1}\left[\xi\,{}_3F_2\left(\begin{matrix}1,1,\mu+1\\2,3\end{matrix}\,\middle|\,-\xi\right)\right] = c_2\xi^2+c_3\xi^3+c_4\xi^4+\dots. \tag{C.4.16}$$

This choice of boundary condition guarantees that the inversion integrals of these functions are related by

$$\widehat{\text{INV}}\left[\left(\overline{D}-K^2\right)^{-1}[g(\xi)]\right] = \frac{1}{J^2-K^2}\widehat{\text{INV}}\,[g(\xi)]\,. \tag{C.4.17}$$

C.5 Infinite sum for $S(v,u)$ from Mellin amplitude

From the Mellin representation of this contribution, we can deduce a closed form expression:

$$S(v,u) = K(\mu)^2\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\frac{2\Gamma(j+k-2)^2}{\Gamma(k-1)^2\Gamma(j-1)^2}c_{j,k}v^j u^{k-2}, \tag{C.5.1}$$

where

$$\begin{aligned} c_{j,k} = {}& \frac{1}{8}\log^2 v\; R_{k,j} \\ & + \frac{1}{2}\log v\left[\left(S_1(k+j-3)-S_1(k-2)\right)R_{k,j}+\Sigma_1^{(0)}(k,j)+\Sigma_2^{(0)}(k,j)\right] \\ & + \left(S_1(k+j-3)-S_1(k-2)\right)\left(\Sigma_1^{(0)}(k,j)+\Sigma_2^{(0)}(k,j)\right)+\Sigma_1^{(1)}(k,j)+\Sigma_2^{(1)}(k,j) \\ & + \frac{1}{4}\left[2\zeta_2+2\left(S_1(j+k-3)-S_1(k-2)\right)^2-\left(S_2(j+k-3)-S_2(k-2)\right)\right]R_{k,j}, \end{aligned} \tag{C.5.2}$$

where

$$R_{k,j} = 2\Gamma(j-1)^2 \operatorname*{Res}_{s=2j}\left[\frac{1}{4}\Gamma\left(2-\frac{s}{2}\right)\tilde{R}(k,s)\right] \tag{C.5.3}$$
$$= \frac{2(\mu-3)\Gamma(j-1)^2\Gamma(\mu-j-1)^2}{\Gamma(\mu-3)}\sum_{i=0}^{j-2}\frac{(-1)^i}{(-i+j+k-3)\Gamma(i+1)\Gamma(\mu-i-2)},$$

$$\Sigma_1^{(0)}(k,j) = \frac{1}{2}\sum_{\substack{\tilde{k}\geqslant 2\\ \tilde{k}\neq k}}\frac{R_{\tilde{k},j}}{k-\tilde{k}}, \qquad \Sigma_1^{(1)}(k,j) = -\frac{1}{4}\sum_{\substack{\tilde{k}\geqslant 2\\ \tilde{k}\neq k}}\frac{R_{\tilde{k},j}}{(k-\tilde{k})^2}, \tag{C.5.4}$$

$$\Sigma_2^{(0)}(k,j) = \frac{1}{2}\sum_{\tilde{k}\geqslant 2}\frac{R_{\tilde{k},j}}{4-(j+\tilde{k}+k)}, \qquad \Sigma_2^{(1)}(k,j) = \frac{1}{4}\sum_{\tilde{k}\geqslant 2}\frac{R_{\tilde{k},j}}{(4-(j+\tilde{k}+k))^2}. \tag{C.5.5}$$

We have been able to evaluate some, but unfortunately not all, of these sums by suitably swapping orders of summation. Recall that in the end, we only need the small u limit of $S(v,u)$, so that we only need to know the sum $\sum_{j\geqslant 2}\frac{2\Gamma(j)^2}{\Gamma(j-1)^2}c_{j,2}v^j$.

The simplest one is simply $R_{2,j}$, which equals

$$R_{2,j} = 2(\mu-3)\frac{\Gamma(j-1)}{(j-1)(4-\mu)_{j-2}}\left(S_1(2-\mu)-S_1(j-\mu+1)\right). \tag{C.5.6}$$

This implies a contribution to $S(v,u)|_{u^0}$ of the form

$$\sum_{j\geqslant 2}\frac{2\Gamma(j)^2}{\Gamma(j-1)^2}R_{2,j}v^j = 4v^2\left[{}_2F_1\left(\begin{matrix}1,2\\4-\mu\end{matrix}\middle|\, v\right)+(\mu-3)\,\frac{\partial}{\partial a}\,{}_2F_1\left(\begin{matrix}1,2\\4-\mu+a\end{matrix}\middle|\, v\right)\right]_{a=0}. \tag{C.5.7}$$

Up to an overall factor, this is essentially the $\log^2 v$ piece, and thus completely determines $U_{\bar{h}}^{(2)}$.

The next sum that we have been able to find, is $\Sigma_2^{(0)}$:

$$\Sigma_2^{(0)}(2,j) = -\frac{(\mu-3)}{(\mu-2)}\frac{\Gamma(j-1)}{(j-1)(4-\mu)_{j-2}}\left(S_1(2-\mu)-S_1(j-\mu+1)+S_1(j-1)\right). \tag{C.5.8}$$

Therefore this contributes with a sum

$$\sum_{j\geqslant 2}\frac{2\Gamma(j)^2}{\Gamma(j-1)^2}\Sigma_2^{(0)}(2,j)v^j = -2v^2\left[{}_2F_1\left(\begin{matrix}1,2\\4-\mu\end{matrix}\middle|\, v\right)+\frac{\mu-3}{\mu-2}D_a\,{}_2F_1\left(\begin{matrix}1,2+a\\4-\mu+a\end{matrix}\middle|\, v\right)\right]. \tag{C.5.9}$$

We have partial results for the other sums: specifically, we have been able to sum the ζ_2 pieces of the sums $\Sigma_2^{(1)}(1,j), \Sigma_2^{(1)}(2,j)$. These are, respectively:

$$\sum_{j\geqslant 2} \frac{2\Gamma(j)^2}{\Gamma(j-1)^2}\Sigma_1^{(0)}(2,j)v^j\Big|_{\zeta_2} = -v^2\left[{}_2F_1\left(\begin{matrix}1,2\\4-\mu\end{matrix}\middle| v\right) + (\mu-3)D_a\,{}_2F_1\left(\begin{matrix}1,2\\4-\mu+a\end{matrix}\middle| v\right)\right], \tag{C.5.10}$$

$$\sum_{j\geqslant 2} \frac{2\Gamma(j)^2}{\Gamma(j-1)^2}\Sigma_2^{(0)}(2,j)v^j\Big|_{\zeta_2} = -\frac{v^2}{\mu-2}\left[{}_3F_2\left(\begin{matrix}1,2,2\\4-\mu,4-\mu\end{matrix}\middle| v\right) + (\mu-3)\,{}_2F_1\left(\begin{matrix}1,2\\4-\mu\end{matrix}\middle| v\right)\right]. \tag{C.5.11}$$

Together with the explicit ζ_2 term in (C.5.2), this gives the full ζ_2 piece of the small u limit of $S(v,u)$ as

$$\begin{aligned} S(v,u)|_{u^0\zeta_2} = K(\mu)^2 v^2 \Bigg[&\frac{1}{\mu-2}\left({}_2F_1\left(\begin{matrix}1,2\\4-\mu\end{matrix}\middle| v\right) - {}_3F_2\left(\begin{matrix}1,2,2\\4-\mu,4-\mu\end{matrix}\middle| v\right)\right) \\ &+ (\mu-3)D_a\,{}_2F_1\left(\begin{matrix}1,2\\4-\mu+a\end{matrix}\middle| v\right)\Bigg]. \end{aligned} \tag{C.5.12}$$

C.5.1 Series expansion for $s_\mu(\bar{z})$

The Taylor coefficients for the function $s_\mu(1-w) := S(V,U)|_{z^0,\log^0 z}$ around small $w = 1-\bar{z}$ have the large n dependence given by (5.3.41), where

$$a_2(\mu) = \frac{1}{2}\left(S_1(2-\mu) + S_1(\mu-2)\right), \tag{C.5.13}$$

$$a_1(\mu) = -S_1(1-\mu)^2 + S_1(\mu-2)S_1(1-\mu) + \frac{2S_1(2-\mu)}{\mu-2} - S_2(1-\mu) - 2S_2(\mu-2) - \zeta_2, \tag{C.5.14}$$

$$\begin{aligned} a_0(\mu) = &\frac{31}{2}\zeta_2 S_1(1-\mu) - \frac{11\zeta_2}{2(\mu-2)} + \frac{3}{2}\zeta_2 S_1(\mu-2) + \frac{25}{6}S_1(1-\mu)^3 \\ &- \frac{3S_1(1-\mu)^2}{2(\mu-2)} + \frac{7}{2}S_1(\mu-2)^2 S_1(1-\mu) + \frac{5S_1(\mu-2)S_1(1-\mu)}{\mu-2} \\ &+ 5S_2(\mu-2)S_1(1-\mu) + \frac{1}{12}S_1(\mu-2)^3 - \frac{S_1(\mu-2)^2}{\mu-2} - \frac{2S_1(\mu-2)}{(\mu-2)^2} \\ &- \frac{3}{4}S_1(\mu-2)S_2(1-\mu) - \frac{3S_2(1-\mu)}{2(\mu-2)} - \frac{5}{4}S_1(\mu-2)S_2(\mu-2) - \frac{2S_2(\mu-2)}{\mu-2} \\ &+ \frac{1}{3}S_3(1-\mu) + \frac{5}{3}S_3(\mu-2) - \frac{35}{4}S_1(\mu-2)S_1(1-\mu)^2 + \frac{9}{2}S_2(1-\mu)S_1(1-\mu). \end{aligned} \tag{C.5.15}$$

C.6 Results for T/A OPE coefficients

Below are the results to order $1/N^2$ for the OPE coefficients in the traceless symmetric and antisymmetric representations. The functions $R(\bar{h},\mu)$ and $B(\bar{h},\mu)$ are given in appendix C.2. The result is as follows:

$$
\begin{aligned}
a_{T/A,\ell} &= \pm a^{\text{free}}_{\mu,\ell} \pm 2a^{\text{free}}_{\mu,\ell}\left(\frac{\gamma_\varphi^{(1)}}{N}+\frac{\gamma_\varphi^{(2)}}{N^2}\right)\left[S_1(\mu+\ell-2)+\frac{2\mu^2+\mu(2\ell-3)+(\ell-1)^2}{(\mu+\ell-1)^2(2\mu+2\ell-3)}\right.\\
&\quad\left.-S_1(\mu-2)+\frac{(\ell-1)(2\mu+\ell-2)\left(S_1(2\mu+\ell-4)-S_1(2\mu+2\ell-3)\right)}{J_0^2}\right]\\
&\quad+\frac{\left(\gamma_\varphi^{(1)}\right)^2}{N^2}\frac{(\mu-1)^2\mu^2\Gamma(\ell+\mu-1)^2}{(\mu-2)^3\Gamma(2\ell+2\mu-2)}\left[\frac{2(\mu-2)\left(2(\mu+\ell-1)^2-2(\mu+\ell-1)+1\right)}{J_0^4}\right.\\
&\quad+\frac{4(\mu-2)(2\mu+2\ell-3)\left(S_1(2\mu+2\ell-3)-S_1(\mu+\ell-2)\right)}{J_0^2}\\
&\quad+(2\mu+2\ell-3)\left(\frac{(\mu-2)^2(\mu-1)\mu B(\mu+\ell-1,\mu)}{\ell(2\mu+\ell-3)}-\frac{2\left(\mu+(\mu-2)S_1(\mu-2)-3\right)}{J_0^2}\right.\\
&\quad\left.\left.+\frac{2\left(\mu+(\mu-2)^2S_2(\mu-2)-(\mu-2)^2\zeta_2-3\right)}{\ell(2\mu+\ell-3)}\right)\right]\\
&\pm a^{\text{free}}_{\mu,\ell}\frac{\left(\gamma_\varphi^{(1)}\right)^2}{N^2}\left(A_3+A_2+A_1+A_0\right)+O(N^{-3}),
\end{aligned}
\tag{C.6.1}
$$

where the upper sign corresponds to T and the lower to A. Here

$$
a^{\text{free}}_{\mu,\ell}=\frac{2\Gamma(\ell+\mu-1)^2\Gamma(\ell+2\mu-3)}{\Gamma(\ell+1)\Gamma(\mu-1)^2\Gamma(2\ell+2\mu-3)},\quad J_0^2=(\ell+\mu-1)(\ell+\mu-2),
\tag{C.6.2}
$$

and the expressions A_i are given by

$$
\begin{aligned}
A_3 &= -\frac{(\mu-1)^2\mu^2\left(\mu^2-3\mu+\ell^2+(2\mu-3)\ell+3\right)}{J_0^6}\left(S_1(\ell)+S_1(2\mu+2\ell-3)\right)\\
&\quad+\frac{S_1(2\mu+2\ell-3)-S_1(2\mu+\ell-4)}{J_0^6(\mu-2)^2(2\mu+2\ell-3)}\Big[4\left(8\mu^4-35\mu^3+67\mu^2-76\mu+40\right)\ell^5\\
&\quad-4(\mu-2)^2\ell^6+2\left(80\mu^5-444\mu^4+1005\mu^3-1248\mu^2+920\mu-324\right)\ell^4\\
&\quad+2\left(159\mu^6-1110\mu^5+3203\mu^4-5016\mu^3+4690\mu^2-2600\mu+680\right)\ell^3\\
&\quad+\left(314\mu^7-2651\mu^6+9472\mu^5-18763\mu^4+22642\mu^3-17040\mu^2+7584\mu-1560\right)\ell^2\\
&\quad+(\mu-1)^2\left(154\mu^6-1210\mu^5+3897\mu^4-6680\mu^3+6596\mu^2-3680\mu+928\right)\ell
\end{aligned}
$$

$$
\begin{aligned}
&+ (\mu-2)^2(\mu-1)^2\left(30\mu^5-155\mu^4+327\mu^3-371\mu^2+232\mu-56\right)\Big]\\
&-\frac{2S_1(\mu+\ell-2)}{J_0^6(\mu-2)(2\mu+2\ell-3)}\Big[-2(\mu-2)\ell^6+2\left(4\mu^4-26\mu^3+86\mu^2-127\mu+66\right)\ell^4\\
&-6\left(2\mu^2-7\mu+6\right)\ell^5+2\left(15\mu^5-88\mu^4+246\mu^3-403\mu^2+355\mu-126\right)\ell^3\\
&+3(\mu-1)^2\left(14\mu^4-71\mu^3+148\mu^2-168\mu+88\right)\ell^2\\
&+\left(26\mu^7-226\mu^6+835\mu^5-1728\mu^4+2195\mu^3-1718\mu^2+760\mu-144\right)\ell\\
&+(\mu-1)^2\left(6\mu^6-51\mu^5+173\mu^4-305\mu^3+302\mu^2-160\mu+32\right)\Big],
\end{aligned}
\tag{C.6.3}
$$

$$
\begin{aligned}
A_2=&\frac{(\mu-1)^2\mu^2\left(2\mu^2-6\mu+2\ell^2+4\mu\ell-6\ell+5\right)}{J_0^4(2\mu+2\ell-3)}R(\mu+\ell-1,\mu)\\
&-\frac{(\mu-1)^2\mu^2}{2J_0^4}\left(2J_0^2\partial_{\bar{\hbar}}R(\mu+\ell-1,\mu)+S_2(\ell)\right)-\left(S_2(2\mu+\ell-4)-S_1(2\mu+\ell-4)^2\right)\times\\
&\times\frac{\left(-\mu^2-3\mu+4\mu\ell+2(\ell-3)\ell+4\right)\left(\mu^2-5\mu+4\mu\ell+2(\ell-3)\ell+4\right)}{2J_0^4}\\
&+\frac{2(\ell-1)^2(2\mu+\ell-2)^2\left(S_1(2\mu+2\ell-3)^2+S_2(2\mu+2\ell-3)\right)}{J_0^4}\\
&-\frac{(\mu-1)^2\mu^2\left(S_1(\ell)-2J_0^2R(\mu+\ell-1,\mu)-4S_1(\mu+\ell-2)+2S_1(2\mu+\ell-4)\right)S_1(\ell)}{2J_0^4}\\
&+\frac{2(\ell-1)(2\mu+\ell-2)\left(2(\mu-1)^2+\ell^2+(2\mu-3)\ell\right)}{J_0^4}S_1(\mu+\ell-2)^2\\
&-\frac{4(\ell-1)^2(2\mu+\ell-2)^2}{J_0^4}S_1(2\mu+\ell-4)S_1(2\mu+2\ell-3)\\
&+S_1(\mu+\ell-2)S_1(2\mu+\ell-4)\Big[\frac{2(\mu-1)^2\left(8\mu^3-11\mu^2+8\right)+4\ell^4+8(2\mu-3)\ell^3}{J_0^4}\\
&+\frac{4\left(4\mu^4-5\mu^3-19\mu^2+45\mu-26\right)\ell^2+4(\mu-1)(2\mu-3)\left(4\mu^3-5\mu^2-4\mu+8\right)\ell}{J_0^4(\mu-2)}\Big],
\end{aligned}
\tag{C.6.4}
$$

$$
\begin{aligned}
A_1=&-\frac{(\mu-1)^2\mu^2}{J_0^2}R(\mu+\ell-1,\mu)\left(2S_1(\mu+\ell-2)+S_1(2\mu+\ell-4)-2S_1(2\mu+2\ell-3)\right)\\
&-\frac{(\mu-1)^3\mu^2\zeta_2}{J_0^2\ell(2\mu+\ell-3)}+\frac{8(\mu-1)\left((2\mu-3)\mu^2+2\right)+4(\mu-2)\ell^2+4(\mu-2)(2\mu-3)\ell}{J_0^2(\mu-2)}\times\\
&\times\Big[S_1(\mu-2)\left(S_1(2\mu+2\ell-3)-S_1(2\mu+\ell-4)\right)-S_1(\mu+\ell-2)S_1(2\mu+2\ell-3)\Big]
\end{aligned}
\tag{C.6.5}
$$

and

$$A_0 = \frac{(\mu-2)(\mu-1)^2\mu^2\zeta_2}{\ell(2\mu+\ell-3)}R(\mu+\ell-1,\mu) + \frac{(\mu-1)^2\mu^2\rho_1(\mu+\ell-1,\mu)}{(\mu-3)^2(\mu-2)^2}$$
$$- \frac{4\left((\mu-2)\ell^2+2\left(\mu^2-3\mu+2\right)\ell-4\mu^4+12\mu^3-15\mu^2+9\mu-2\right)}{(\mu-2)(\mu+\ell-1)^2(2\mu+2\ell-3)}S_1(\mu-2)$$
$$+ \frac{2(\mu-1)^2((\mu-2)\mu(\mu(5\mu-7)+8)+4)+2\left((\mu-2)\mu\left(8\mu^2-13\mu+8\right)+4\right)\ell^2}{(\mu-2)^2(\mu+\ell-1)^4(2\mu+2\ell-3)}$$
$$+ \frac{4(\mu-1)\left(8\mu^4-29\mu^3+34\mu^2-16\mu+4\right)\ell}{(\mu-2)^2(\mu+\ell-1)^4(2\mu+2\ell-3)} - 4S_1(\mu-2)S_1(\mu+\ell-2) + 2S_1(\mu-2)^2$$
$$+ \left(\frac{\mu^2(\mu-1)^2}{J_0^4} - \frac{4\mu(2\mu-1)(\mu-1)^2}{J_0^2(\mu-2)} - 1\right)S_2(\mu+\ell-2) + S_2(\mu-2). \qquad \text{(C.6.6)}$$

Bibliography

[1] M. van Loon, *The Analytic Bootstrap in Fermionic CFTs*, *JHEP* **01** (2018) 104, [arXiv:1711.02099].

[2] L. F. Alday, J. Henriksson, and M. van Loon, *Taming the ϵ-expansion with large spin perturbation theory*, *JHEP* **07** (2018) 131, [arXiv:1712.02314].

[3] J. Henriksson and M. van Loon, *Critical O(N) model to order ϵ^4 from analytic bootstrap*, *J. Phys.* **A52** (2019), no. 2 025401, [arXiv:1801.03512].

[4] L. F. Alday, J. Henriksson, and M. van Loon, *An alternative to diagrams for the critical O(N) model: dimensions and structure constants to order $1/N^2$*, arXiv:1907.02445.

[5] K. G. Wilson and J. B. Kogut, *The Renormalization group and the epsilon expansion*, *Phys. Rept.* **12** (1974) 75–199.

[6] K. G. Wilson, *The renormalization group and critical phenomena*, *Rev. Mod. Phys.* **55** (1983) 583–600.

[7] A. Dymarsky, Z. Komargodski, A. Schwimmer, and S. Theisen, *On Scale and Conformal Invariance in Four Dimensions*, *JHEP* **10** (2015) 171, [arXiv:1309.2921].

[8] E. Witten, *Conformal Field Theory In Four And Six Dimensions*, in *Topology, geometry and quantum field theory. Proceedings, Symposium in the honour of the 60th birthday of Graeme Segal, Oxford, UK, June 24-29, 2002*, pp. 405–419, 2007. arXiv:0712.0157.

[9] S. Weinberg, *The quantum theory of fields. Vol. 3: Supersymmetry*. Cambridge University Press, 2013.

[10] E. Witten, *Some comments on string dynamics*, in *Future perspectives in string theory. Proceedings, Conference, Strings'95, Los Angeles, USA, March 13-18, 1995*, pp. 501–523, 1995. hep-th/9507121.

[11] P. Claus, R. Kallosh, and A. Van Proeyen, *M five-brane and superconformal (0,2) tensor multiplet in six-dimensions*, *Nucl. Phys.* **B518** (1998) 117–150, [hep-th/9711161].

[12] E. P. Verlinde, *Global aspects of electric - magnetic duality*, *Nucl. Phys.* **B455** (1995) 211–228, [hep-th/9506011].

[13] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett. Math. Phys.* **91** (2010) 167–197, [arXiv:0906.3219].

[14] A. Pelissetto and E. Vicari, *Critical phenomena and renormalization group theory*, *Phys. Rept.* **368** (2002) 549–727, [cond-mat/0012164].

[15] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133, [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)].

[16] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109].

[17] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150].

[18] J. L. Cardy, *Scaling and Renormalization in Statistical Physics*. Cambridge Lecture Notes in Physics. Cambridge University Press, 1996.

[19] Z. Komargodski and D. Simmons-Duffin, *The Random-Bond Ising Model in 2.01 and 3 Dimensions*, *J. Phys.* **A50** (2017), no. 15 154001, [arXiv:1603.04444].

[20] A. Petkou, *Conserved currents, consistency relations and operator product expansions in the conformally invariant O(N) vector model*, *Annals Phys.* **249** (1996) 180–221, [hep-th/9410093].

[21] S. Ferrara, A. F. Grillo, and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, *Annals Phys.* **76** (1973) 161–188.

[22] S. Ferrara, A. F. Grillo, G. Parisi, and R. Gatto, *Covariant expansion of the conformal four-point function*, *Nucl. Phys.* **B49** (1972) 77–98. [Erratum: Nucl. Phys.B53,643(1973)].

[23] A. M. Polyakov, *Nonhamiltonian approach to conformal quantum field theory*, *Zh. Eksp. Teor. Fiz.* **66** (1974) 23–42. [Sov. Phys. JETP39,9(1974)].

[24] M. Schottenloher, *A Mathematical Introduction to Conformal Field Theory*. Lecture Notes in Physics. Springer Berlin Heidelberg, 2009.

[25] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, *Nucl. Phys.* **B241** (1984) 333–380.

[26] A. L. Fitzpatrick, J. Kaplan, D. Poland, and D. Simmons-Duffin, *The Analytic Bootstrap and AdS Superhorizon Locality*, *JHEP* **12** (2013) 004, [arXiv:1212.3616].

[27] Z. Komargodski and A. Zhiboedov, *Convexity and Liberation at Large Spin*, *JHEP* **11** (2013) 140, [arXiv:1212.4103].

[28] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031, [arXiv:0807.0004].

[29] F. A. Dolan and H. Osborn, *Conformal four point functions and the operator product expansion*, *Nucl. Phys.* **B599** (2001) 459–496, [hep-th/0011040].

[30] F. A. Dolan and H. Osborn, *Conformal partial waves and the operator product expansion*, *Nucl. Phys.* **B678** (2004) 491–507, [hep-th/0309180].

[31] D. Poland, D. Simmons-Duffin, and A. Vichi, *Carving Out the Space of 4D CFTs*, *JHEP* **05** (2012) 110, [arXiv:1109.5176].

[32] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents*, *J. Stat. Phys.* **157** (2014) 869, [arXiv:1403.4545].

[33] F. Kos, D. Poland, and D. Simmons-Duffin, *Bootstrapping Mixed Correlators in the 3D Ising Model*, *JHEP* **11** (2014) 109, [arXiv:1406.4858].

[34] D. Poland and D. Simmons-Duffin, *Bounds on 4D Conformal and Superconformal Field Theories*, *JHEP* **05** (2011) 017, [arXiv:1009.2087].

[35] R. Rattazzi, S. Rychkov, and A. Vichi, *Bounds in 4D Conformal Field Theories with Global Symmetry*, *J. Phys.* **A44** (2011) 035402, [arXiv:1009.5985].

[36] A. Vichi, *Improved bounds for CFT's with global symmetries*, *JHEP* **01** (2012) 162, [arXiv:1106.4037].

[37] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Solving the 3D Ising Model with the Conformal Bootstrap*, *Phys. Rev.* **D86** (2012) 025022, [arXiv:1203.6064].

[38] D. Gaiotto, D. Mazac, and M. F. Paulos, *Bootstrapping the 3d Ising twist defect*, *JHEP* **03** (2014) 100, [arXiv:1310.5078].

[39] Y. Nakayama and T. Ohtsuki, *Five dimensional $O(N)$-symmetric CFTs from conformal bootstrap*, *Phys. Lett.* **B734** (2014) 193–197, [arXiv:1404.5201].

[40] Y. Nakayama and T. Ohtsuki, *Conformal Bootstrap Dashing Hopes of Emergent Symmetry*, *Phys. Rev. Lett.* **117** (2016), no. 13 131601, [arXiv:1602.07295].

[41] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, *Precision Islands in the Ising and $O(N)$ Models*, *JHEP* **08** (2016) 036, [arXiv:1603.04436].

[42] A. Dymarsky, F. Kos, P. Kravchuk, D. Poland, and D. Simmons-Duffin, *The 3d Stress-Tensor Bootstrap*, *JHEP* **02** (2018) 164, [arXiv:1708.05718].

[43] F. Gliozzi, *More constraining conformal bootstrap*, *Phys. Rev. Lett.* **111** (2013) 161602, [arXiv:1307.3111].

[44] F. Gliozzi and A. Rago, *Critical exponents of the 3d Ising and related models from Conformal Bootstrap*, *JHEP* **10** (2014) 042, [arXiv:1403.6003].

[45] S. Hikami, *Conformal bootstrap analysis for the Yang–Lee edge singularity*, *PTEP* **2018** (2018), no. 5 053I01, [arXiv:1707.04813].

[46] M. Hogervorst and S. Rychkov, *Radial Coordinates for Conformal Blocks*, *Phys. Rev.* **D87** (2013) 106004, [arXiv:1303.1111].

[47] A. Castedo Echeverri, B. von Harling, and M. Serone, *The Effective Bootstrap*, *JHEP* **09** (2016) 097, [arXiv:1606.02771].

[48] W. Li, *New method for the conformal bootstrap with OPE truncations*, arXiv:1711.09075.

[49] D. Mazac, *Analytic bounds and emergence of AdS_2 physics from the conformal bootstrap*, *JHEP* **04** (2017) 146, [arXiv:1611.10060].

[50] D. Mazac and M. F. Paulos, *The analytic functional bootstrap. Part I: 1D CFTs and 2D S-matrices*, *JHEP* **02** (2019) 162, [arXiv:1803.10233].

[51] D. Mazac and M. F. Paulos, *The analytic functional bootstrap. Part II. Natural bases for the crossing equation*, *JHEP* **02** (2019) 163, [arXiv:1811.10646].

[52] M. F. Paulos and B. Zan, *A functional approach to the numerical conformal bootstrap*, arXiv:1904.03193.

[53] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, *Spinning Conformal Blocks*, *JHEP* **11** (2011) 154, [arXiv:1109.6321].

[54] A. Castedo Echeverri, E. Elkhidir, D. Karateev, and M. Serone, *Deconstructing Conformal Blocks in 4D CFT*, *JHEP* **08** (2015) 101, [arXiv:1505.03750].

[55] J. Penedones, E. Trevisani, and M. Yamazaki, *Recursion Relations for Conformal Blocks*, *JHEP* **09** (2016) 070, [arXiv:1509.00428].

[56] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin, and R. Yacoby, *Fermion-Scalar Conformal Blocks*, *JHEP* **04** (2016) 074, [arXiv:1511.01497].

[57] A. Castedo Echeverri, E. Elkhidir, D. Karateev, and M. Serone, *Seed Conformal Blocks in 4D CFT*, *JHEP* **02** (2016) 183, [arXiv:1601.05325].

[58] D. Karateev, P. Kravchuk, and D. Simmons-Duffin, *Weight Shifting Operators and Conformal Blocks*, *JHEP* **02** (2018) 081, [arXiv:1706.07813]. [,91(2017)].

[59] G. F. Cuomo, D. Karateev, and P. Kravchuk, *General Bootstrap Equations in 4D CFTs*, *JHEP* **01** (2018) 130, [arXiv:1705.05401]. [,57(2017)].

[60] J.-F. Fortin and W. Skiba, *A recipe for conformal blocks*, arXiv:1905.00036.

[61] A. Dymarsky, J. Penedones, E. Trevisani, and A. Vichi, *Charting the space of 3D CFTs with a continuous global symmetry*, arXiv:1705.04278.

[62] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, D. Simmons-Duffin, and R. Yacoby, *Bootstrapping 3D Fermions*, *JHEP* **03** (2016) 120, [arXiv:1508.00012].

[63] L. Iliesiu, F. Kos, D. Poland, S. S. Pufu, and D. Simmons-Duffin, *Bootstrapping 3D Fermions with Global Symmetries*, *JHEP* **01** (2018) 036, [arXiv:1705.03484].

[64] D. Karateev, P. Kravchuk, M. Serone, and A. Vichi, *Fermion Conformal Bootstrap in 4d*, arXiv:1902.05969.

[65] E. Elkhidir and D. Karateev, *Scalar-Fermion Analytic Bootstrap in 4D*, arXiv:1712.01554.

[66] S. Albayrak, D. Meltzer, and D. Poland, *More Analytic Bootstrap: Nonperturbative Effects and Fermions*, arXiv:1904.00032.

[67] F. A. Dolan and H. Osborn, *Superconformal symmetry, correlation functions and the operator product expansion*, *Nucl. Phys.* **B629** (2002) 3–73, [hep-th/0112251].

[68] F. A. Dolan, L. Gallot, and E. Sokatchev, *On four-point functions of 1/2-BPS operators in general dimensions*, *JHEP* **09** (2004) 056, [hep-th/0405180].

[69] F. A. Dolan and H. Osborn, *Conformal partial wave expansions for N=4 chiral four point functions*, *Annals Phys.* **321** (2006) 581–626, [hep-th/0412335].

[70] C. Beem, M. Lemos, P. Liendo, L. Rastelli, and B. C. van Rees, *The $\mathcal{N} = 2$ superconformal bootstrap*, *JHEP* **03** (2016) 183, [arXiv:1412.7541].

[71] P. Liendo, I. Ramirez, and J. Seo, *Stress-tensor OPE in $\mathcal{N} = 2$ superconformal theories*, *JHEP* **02** (2016) 019, [arXiv:1509.00033].

[72] M. Lemos and P. Liendo, *Bootstrapping $\mathcal{N} = 2$ chiral correlators*, *JHEP* **01** (2016) 025, [arXiv:1510.03866].

[73] M. Lemos and P. Liendo, *$\mathcal{N} = 2$ central charge bounds from 2d chiral algebras*, *JHEP* **04** (2016) 004, [arXiv:1511.07449].

[74] M. Lemos, P. Liendo, C. Meneghelli, and V. Mitev, *Bootstrapping $\mathcal{N} = 3$ superconformal theories*, *JHEP* **04** (2017) 032, [arXiv:1612.01536].

[75] M. Cornagliotto, M. Lemos, and V. Schomerus, *Long Multiplet Bootstrap*, *JHEP* **10** (2017) 119, [arXiv:1702.05101].

[76] C. Beem, L. Rastelli, and B. C. van Rees, *The $\mathcal{N} = 4$ Superconformal Bootstrap*, *Phys. Rev. Lett.* **111** (2013) 071601, [arXiv:1304.1803].

[77] L. F. Alday and A. Bissi, *The superconformal bootstrap for structure constants*, *JHEP* **09** (2014) 144, [arXiv:1310.3757].

[78] L. F. Alday and A. Bissi, *Generalized bootstrap equations for $\mathcal{N} = 4$ SCFT*, *JHEP* **02** (2015) 101, [arXiv:1404.5864].

[79] C. Beem, L. Rastelli, and B. C. van Rees, *More $\mathcal{N} = 4$ superconformal bootstrap*, *Phys. Rev.* **D96** (2017), no. 4 046014, [arXiv:1612.02363].

[80] J. Henriksson and T. Łukowski, *Perturbative Four-Point Functions from the Analytic Conformal Bootstrap*, *JHEP* **02** (2018) 123, [arXiv:1710.06242].

[81] L. F. Alday and A. Bissi, *Loop Corrections to Supergravity on $AdS_5 \times S^5$*, *Phys. Rev. Lett.* **119** (2017), no. 17 171601, [arXiv:1706.02388].

[82] L. F. Alday, A. Bissi, and E. Perlmutter, *Genus-One String Amplitudes from Conformal Field Theory*, arXiv:1809.10670.

[83] C. Beem, M. Lemos, L. Rastelli, and B. C. van Rees, *The (2, 0) superconformal bootstrap*, *Phys. Rev.* **D93** (2016), no. 2 025016, [arXiv:1507.05637].

[84] T. Abl, P. Heslop, and A. E. Lipstein, *Recursion relations for anomalous dimensions in the 6d $(2,0)$ theory*, *JHEP* **04** (2019) 038, [arXiv:1902.00463].

[85] M. S. Costa, V. Goncalves, and J. Penedones, *Conformal Regge theory*, *JHEP* **12** (2012) 091, [arXiv:1209.4355].

[86] P. Kravchuk and D. Simmons-Duffin, *Light-ray operators in conformal field theory*, *JHEP* **11** (2018) 102, [arXiv:1805.00098]. [,236(2018)].

[87] L. F. Alday and J. M. Maldacena, *Comments on operators with large spin*, *JHEP* **11** (2007) 019, [arXiv:0708.0672].

[88] L. F. Alday, A. Bissi, and T. Łukowski, *Large spin systematics in CFT*, *JHEP* **11** (2015) 101, [arXiv:1502.07707].

[89] L. F. Alday, *Large Spin Perturbation Theory for Conformal Field Theories*, *Phys. Rev. Lett.* **119** (2017), no. 11 111601, [arXiv:1611.01500].

[90] A. L. Fitzpatrick and J. Kaplan, *Unitarity and the Holographic S-Matrix*, *JHEP* **10** (2012) 032, [arXiv:1112.4845].

[91] D. Karateev, P. Kravchuk, and D. Simmons-Duffin, *Harmonic Analysis and Mean Field Theory*, arXiv:1809.05111.

[92] A. Kaviraj, K. Sen, and A. Sinha, *Analytic bootstrap at large spin*, *JHEP* **11** (2015) 083, [arXiv:1502.01437].

[93] A. Kaviraj, K. Sen, and A. Sinha, *Universal anomalous dimensions at large spin and large twist*, *JHEP* **07** (2015) 026, [arXiv:1504.00772].

[94] P. Dey, A. Kaviraj, and K. Sen, *More on analytic bootstrap for O(N) models*, *JHEP* **06** (2016) 136, [arXiv:1602.04928].

[95] L. F. Alday and A. Zhiboedov, *An Algebraic Approach to the Analytic Bootstrap*, *JHEP* **04** (2017) 157, [arXiv:1510.08091].

[96] D. Simmons-Duffin, *The Lightcone Bootstrap and the Spectrum of the 3d Ising CFT*, *JHEP* **03** (2017) 086, [arXiv:1612.08471].

[97] S. Caron-Huot, *Analyticity in Spin in Conformal Theories*, *JHEP* **09** (2017) 078, [arXiv:1703.00278].

[98] D. Simmons-Duffin, D. Stanford, and E. Witten, *A spacetime derivation of the Lorentzian OPE inversion formula*, *JHEP* **07** (2018) 085, [arXiv:1711.03816].

[99] L. F. Alday and S. Caron-Huot, *Gravitational S-matrix from CFT dispersion relations*, *JHEP* **12** (2018) 017, [arXiv:1711.02031].

[100] L. F. Alday, *On Genus-one String Amplitudes on $AdS_5 \times S^5$*, arXiv:1812.11783.

[101] S. Caron-Huot and A.-K. Trinh, *All tree-level correlators in $AdS_5 \times S_5$ supergravity: hidden ten-dimensional conformal symmetry*, *JHEP* **01** (2019) 196, [arXiv:1809.09173].

[102] P. Kraus and A. Sivaramakrishnan, *Light-state Dominance from the Conformal Bootstrap*, arXiv:1812.02226.

[103] L. Iliesiu, M. Koloğlu, R. Mahajan, E. Perlmutter, and D. Simmons-Duffin, *The Conformal Bootstrap at Finite Temperature*, *JHEP* **10** (2018) 070, [arXiv:1802.10266].

[104] C. Sleight and M. Taronna, *Spinning Mellin Bootstrap: Conformal Partial Waves, Crossing Kernels and Applications*, *Fortsch. Phys.* **66** (2018), no. 8-9 1800038, [arXiv:1804.09334].

[105] C. Cardona and K. Sen, *Anomalous dimensions at finite conformal spin from OPE inversion*, *JHEP* **11** (2018) 052, [arXiv:1806.10919].

[106] C. Sleight and M. Taronna, *Anomalous Dimensions from Crossing Kernels*, *JHEP* **11** (2018) 089, [arXiv:1807.05941].

[107] J. Liu, E. Perlmutter, V. Rosenhaus, and D. Simmons-Duffin, *d-dimensional SYK, AdS Loops, and* $6j$ *Symbols*, *JHEP* **03** (2019) 052, [arXiv:1808.00612].

[108] C. Cardona, S. Guha, S. K. Kanumilli, and K. Sen, *Resummation at finite conformal spin*, *JHEP* **01** (2019) 077, [arXiv:1811.00213]. [JHEP19,077(2020)].

[109] D. Mazáč, *A Crossing-Symmetric OPE Inversion Formula*, arXiv:1812.02254.

[110] P. Liendo, L. Rastelli, and B. C. van Rees, *The Bootstrap Program for Boundary* CFT_d, *JHEP* **07** (2013) 113, [arXiv:1210.4258].

[111] A. Kaviraj and M. F. Paulos, *The Functional Bootstrap for Boundary CFT*, arXiv:1812.04034.

[112] D. Mazáč, L. Rastelli, and X. Zhou, *An Analytic Approach to* $BCFT_d$, arXiv:1812.09314.

[113] M. Isachenkov, P. Liendo, Y. Linke, and V. Schomerus, *Calogero-Sutherland Approach to Defect Blocks*, *JHEP* **10** (2018) 204, [arXiv:1806.09703].

[114] P. Liendo, Y. Linke, and V. Schomerus, *A Lorentzian inversion formula for defect CFT*, arXiv:1903.05222.

[115] R. Gopakumar, A. Kaviraj, K. Sen, and A. Sinha, *Conformal Bootstrap in Mellin Space*, *Phys. Rev. Lett.* **118** (2017), no. 8 081601, [arXiv:1609.00572].

[116] R. Gopakumar, A. Kaviraj, K. Sen, and A. Sinha, *A Mellin space approach to the conformal bootstrap*, *JHEP* **05** (2017) 027, [arXiv:1611.08407].

[117] P. Dey, A. Kaviraj, and A. Sinha, *Mellin space bootstrap for global symmetry*, *JHEP* **07** (2017) 019, [arXiv:1612.05032].

[118] P. Dey, K. Ghosh, and A. Sinha, *Simplifying large spin bootstrap in Mellin space*, *JHEP* **01** (2018) 152, [arXiv:1709.06110].

[119] R. Gopakumar and A. Sinha, *On the Polyakov-Mellin bootstrap*, *JHEP* **12** (2018) 040, [arXiv:1809.10975].

[120] E. Remiddi and J. A. M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys.* **A15** (2000) 725–754, [hep-ph/9905237].

[121] S. Albino, *Analytic Continuation of Harmonic Sums*, *Phys. Lett.* **B674** (2009) 41–48, [arXiv:0902.2148].

[122] A. N. Manashov, E. D. Skvortsov, and M. Strohmaier, *Higher spin currents in the critical $O(N)$ vector model at $1/N^2$*, *JHEP* **08** (2017) 106, [arXiv:1706.09256].

[123] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[124] S. Rychkov, *EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions*. SpringerBriefs in Physics. Springer International Publishing, 2016.

[125] D. Simmons-Duffin, *The Conformal Bootstrap*, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015*, pp. 1–74, 2017. arXiv:1602.07982.

[126] D. Poland, S. Rychkov, and A. Vichi, *The Conformal Bootstrap: Theory, Numerical Techniques, and Applications*, *Rev. Mod. Phys.* **91** (2019), no. 1 15002, [arXiv:1805.04405]. [Rev. Mod. Phys.91,015002(2019)].

[127] M. Kologlu, P. Kravchuk, D. Simmons-Duffin, and A. Zhiboedov, *The light-ray OPE and conformal colliders*, arXiv:1905.01311.

[128] N. T. Evans, *Discrete series for the universal covering group of the 3 + 2 de sitter group*, *Journal of Mathematical Physics* **8** (1967), no. 2 170–184, [https://doi.org/10.1063/1.1705183].

[129] S. Ferrara, R. Gatto, and A. F. Grillo, *Positivity Restrictions on Anomalous Dimensions*, *Phys. Rev.* **D9** (1974) 3564.

[130] G. Mack, *All unitary ray representations of the conformal group SU(2,2) with positive energy*, *Commun. Math. Phys.* **55** (1977) 1.

[131] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, *Adv. Theor. Math. Phys.* **2** (1998) 783–851, [hep-th/9712074].

[132] V. K. Dobrev, V. B. Petkova, S. G. Petrova, and I. T. Todorov, *Dynamical Derivation of Vacuum Operator Product Expansion in Euclidean Conformal Quantum Field Theory*, *Phys. Rev.* **D13** (1976) 887.

[133] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, *Spinning Conformal Correlators*, *JHEP* **11** (2011) 071, [arXiv:1107.3554].

[134] S. Giombi, S. Prakash, and X. Yin, *A Note on CFT Correlators in Three Dimensions*, *JHEP* **07** (2013) 105, [arXiv:1104.4317].

[135] D. Simmons-Duffin, *Projectors, Shadows, and Conformal Blocks*, *JHEP* **04** (2014) 146, [arXiv:1204.3894].

[136] M. S. Costa and T. Hansen, *Conformal correlators of mixed-symmetry tensors*, *JHEP* **02** (2015) 151, [arXiv:1411.7351].

[137] P. Kravchuk and D. Simmons-Duffin, *Counting Conformal Correlators*, *JHEP* **02** (2018) 096, [arXiv:1612.08987].

[138] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster*. Pure and Applied Mathematics. Elsevier Science, 1989.

[139] Y.-Z. Huang, *Vertex operator algebras, the Verlinde conjecture and modular tensor categories*, *Proc. Nat. Acad. Sci.* **102** (2005) 5352–5356, [math/0412261].

[140] I. Runkel, J. Fjelstad, J. Fuchs, and C. Schweigert, *Topological and conformal field theory as Frobenius algebras*, *Contemp. Math.* **431** (2007) 225–248, [math/0512076].

[141] L. Kong, *Conformal field theory and a new geometry*, arXiv:1107.3649.

[142] M. Yamazaki, *Comments on Determinant Formulas for General CFTs*, *JHEP* **10** (2016) 035, [arXiv:1601.04072].

[143] J. C. Jantzen, *Kontravariante formen auf induzierten darstellungen halbeinfacher lie-algebren*, *Mathematische Annalen* **226** (Feb, 1977) 53–65.

[144] J. Maldacena and A. Zhiboedov, *Constraining Conformal Field Theories with A Higher Spin Symmetry*, *J. Phys.* **A46** (2013) 214011, [arXiv:1112.1016].

[145] D. Pappadopulo, S. Rychkov, J. Espin, and R. Rattazzi, *OPE Convergence in Conformal Field Theory*, *Phys. Rev.* **D86** (2012) 105043, [arXiv:1208.6449].

[146] J.-F. Fortin and W. Skiba, *Conformal Bootstrap in Embedding Space*, *Phys. Rev.* **D93** (2016), no. 10 105047, [arXiv:1602.05794].

[147] F. A. Dolan and H. Osborn, *Conformal Partial Waves: Further Mathematical Results*, arXiv:1108.6194.

[148] L. F. Alday, *Solving CFTs with Weakly Broken Higher Spin Symmetry*, *JHEP* **10** (2017) 161, [arXiv:1612.00696].

[149] V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov, *Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory*, *Lect. Notes Phys.* **63** (1977) 1–280.

[150] J. A. Gracey, T. Luthe, and Y. Schroder, *Four loop renormalization of the Gross-Neveu model*, *Phys. Rev.* **D94** (2016), no. 12 125028, [arXiv:1609.05071].

[151] B. Ihrig, L. N. Mihaila, and M. M. Scherer, *Critical behavior of Dirac fermions from perturbative renormalization*, *Phys. Rev.* **B98** (2018), no. 12 125109, [arXiv:1806.04977].

[152] N. Zerf, L. N. Mihaila, P. Marquard, I. F. Herbut, and M. M. Scherer, *Four-loop critical exponents for the Gross-Neveu-Yukawa models*, *Phys. Rev.* **D96** (2017), no. 9 096010, [arXiv:1709.05057].

[153] J. A. Gracey, *Large N_f quantum field theory*, *Int. J. Mod. Phys.* **A33** (2019), no. 35 1830032, [arXiv:1812.05368].

[154] S. Giombi, *Higher Spin — CFT Duality*, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015*, pp. 137–214, 2017. arXiv:1607.02967.

[155] L. F. Alday and A. Bissi, *Crossing symmetry and Higher spin towers*, *JHEP* **12** (2017) 118, [arXiv:1603.05150].

[156] S. Giombi, V. Kirilin, and E. Skvortsov, *Notes on Spinning Operators in Fermionic CFT*, *JHEP* **05** (2017) 041, [arXiv:1701.06997].

[157] U. Gran, *GAMMA: A Mathematica package for performing gamma matrix algebra and Fierz transformations in arbitrary dimensions*, hep-th/0105086.

[158] J. Zinn-Justin, *Four fermion interaction near four-dimensions*, *Nucl. Phys.* **B367** (1991) 105–122.

[159] I. Heemskerk, J. Penedones, J. Polchinski, and J. Sully, *Holography from Conformal Field Theory*, *JHEP* **10** (2009) 079, [arXiv:0907.0151].

[160] S. Giombi and V. Kirilin, *Anomalous dimensions in CFT with weakly broken higher spin symmetry*, *JHEP* **11** (2016) 068, [arXiv:1601.01310].

[161] K. Diab, L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, *On C_J and C_T in the Gross–Neveu and O(N) models*, *J. Phys.* **A49** (2016), no. 40 405402, [arXiv:1601.07198].

[162] M. Moshe and J. Zinn-Justin, *Quantum field theory in the large N limit: A Review*, *Phys. Rept.* **385** (2003) 69–228, [hep-th/0306133].

[163] T. Bhattacharya *et al.*, *QCD Phase Transition with Chiral Quarks and Physical Quark Masses*, *Phys. Rev. Lett.* **113** (2014), no. 8 082001, [arXiv:1402.5175].

[164] K. G. Wilson and M. E. Fisher, *Critical exponents in 3.99 dimensions*, *Phys. Rev. Lett.* **28** (1972) 240–243.

[165] M. V. Kompaniets and E. Panzer, *Minimally subtracted six loop renormalization of $O(n)$-symmetric ϕ^4 theory and critical exponents*, *Phys. Rev.* **D96** (2017), no. 3 036016, [arXiv:1705.06483].

[166] S. E. Derkachov, J. A. Gracey, and A. N. Manashov, *Four loop anomalous dimensions of gradient operators in ϕ^4 theory*, *Eur. Phys. J.* **C2** (1998) 569–579, [hep-ph/9705268].

[167] K. Lang and W. Rühl, *Critical O(N) vector nonlinear sigma models: A Resume of their field structure*, *Adv. Appl. Clifford Algebras* **4** (1994), no. S1 203–216, [hep-th/9311046].

[168] A. C. Petkou, *C(T) and C(J) up to next-to-leading order in $1/N$ in the conformally invariant $O(N)$ vector model for $2 < d < 4$*, *Phys. Lett.* **B359** (1995) 101–107, [hep-th/9506116].

[169] S. E. Derkachov and A. N. Manashov, *The Simple scheme for the calculation of the anomalous dimensions of composite operators in the 1/N expansion*, *Nucl. Phys.* **B522** (1998) 301–320, [hep-th/9710015].

[170] F. Gliozzi, A. L. Guerrieri, A. C. Petkou, and C. Wen, *The analytic structure of conformal blocks and the generalized Wilson-Fisher fixed points*, *JHEP* **04** (2017) 056, [arXiv:1702.03938].

[171] S. Rychkov and Z. M. Tan, *The ϵ-expansion from conformal field theory*, *J. Phys.* **A48** (2015), no. 29 29FT01, [arXiv:1505.00963].

[172] P. Liendo, *Revisiting the dilatation operator of the Wilson–Fisher fixed point*, *Nucl. Phys.* **B920** (2017) 368–384, [arXiv:1701.04830].

[173] B. Basso and G. P. Korchemsky, *Anomalous dimensions of high-spin operators beyond the leading order*, *Nucl. Phys.* **B775** (2007) 1–30, [hep-th/0612247].

[174] F. Kos, D. Poland, and D. Simmons-Duffin, *Bootstrapping the $O(N)$ vector models*, *JHEP* **06** (2014) 091, [arXiv:1307.6856].

[175] E. Brezin and J. Zinn-Justin, *Renormalization of the nonlinear sigma model in 2 + epsilon dimensions. Application to the Heisenberg ferromagnets*, *Phys. Rev. Lett.* **36** (1976) 691–694.

[176] W. A. Bardeen, B. W. Lee, and R. E. Shrock, *Phase Transition in the Nonlinear σ Model in 2 + ϵ Dimensional Continuum*, *Phys. Rev.* **D14** (1976) 985.

[177] P. di Francesco, H. Saleur, and J. B. Zuber, *Relations between the Coulomb Gas Picture and Conformal Invariance of Two-Dimensional Critical Models*, *J. Stat. Phys.* **49** (1987) 57.

[178] S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin, and A. Vichi, *Conformal Field Theories in Fractional Dimensions*, *Phys. Rev. Lett.* **112** (2014) 141601, [arXiv:1309.5089].

[179] H. Kleinert and V. Schulte-Frohlinde, *Critical properties of phi**4-theories*. 2001.

[180] F. Kos, D. Poland, D. Simmons-Duffin, and A. Vichi, *Bootstrapping the O(N) Archipelago*, *JHEP* **11** (2015) 106, [arXiv:1504.07997].

[181] M. Hogervorst, S. Rychkov, and B. C. van Rees, *Unitarity violation at the Wilson-Fisher fixed point in $4-\epsilon$ dimensions*, *Phys. Rev.* **D93** (2016), no. 12 125025, [arXiv:1512.00013].

[182] S. Yabunaka and B. Delamotte, *Surprises in $O(N)$ Models: Nonperturbative Fixed Points, Large N Limits, and Multicriticality*, *Phys. Rev. Lett.* **119** (2017), no. 19 191602, [arXiv:1707.04383].

[183] L. Fei, *Conformal Field Theories in the Epsilon and 1/N Expansions*. PhD thesis, Princeton U., 2017.

[184] L. Fei, S. Giombi, and I. R. Klebanov, *Critical $O(N)$ models in $6-\epsilon$ dimensions*, *Phys. Rev.* **D90** (2014), no. 2 025018, [arXiv:1404.1094].

[185] A. N. Vasiliev, Yu. M. Pismak, and Yu. R. Khonkonen, *1/n Expansion: Calculation of the Exponents η and ν in the Order $1/n^2$ for Arbitrary Number of Dimensions*, *Theor. Math. Phys.* **47** (1981) 465–475. [Teor. Mat. Fiz.47,291(1981)].

[186] A. Cappelli, L. Maffi, and S. Okuda, *Critical Ising Model in Varying Dimension by Conformal Bootstrap*, *JHEP* **01** (2019) 161, [arXiv:1811.07751].

[187] K. Lang and W. Rühl, *The Critical O(N) sigma model at dimensions $2 < d < 4$: A List of quasi-primary fields*, *Nucl. Phys.* **B402** (1993) 573–603.

[188] M. Hogervorst and B. C. van Rees, *Crossing symmetry in alpha space*, *JHEP* **11** (2017) 193, [arXiv:1702.08471].

[189] A. N. Vasiliev, Yu. M. Pismak, and Yu. R. Khonkonen, *Simple Method of Calculating the Critical Indices in the 1/N Expansion*, *Theor. Math. Phys.* **46** (1981) 104–113. [Teor. Mat. Fiz.46,157(1981)].

[190] H. Exton, *Handbook of hypergeometric integrals: theory, applications, tables, computer programs*. Ellis Horwood series in mathematics and its applications. Ellis Horwood, 1978.

[191] L. Rastelli and X. Zhou, *How to Succeed at Holographic Correlators Without Really Trying*, *JHEP* **04** (2018) 014, [arXiv:1710.05923].

[192] C. Marboe and V. Velizhanin, *Twist-2 at seven loops in planar $\mathcal{N} = 4$ SYM theory: full result and analytic properties*, *JHEP* **11** (2016) 013, [arXiv:1607.06047].

[193] C. Cardona, *OPE inversion in Mellin space*, arXiv:1803.05086.

[194] P. Liendo and C. Meneghelli, *Bootstrap equations for $\mathcal{N} = 4$ SYM with defects*, *JHEP* **01** (2017) 122, [arXiv:1608.05126].

[195] E. Lauria, M. Meineri, and E. Trevisani, *Spinning operators and defects in conformal field theory*, arXiv:1807.02522.

[196] V. Schomerus, E. Sobko, and M. Isachenkov, *Harmony of Spinning Conformal Blocks*, *JHEP* **03** (2017) 085, [arXiv:1612.02479].

[197] V. Schomerus and E. Sobko, *From Spinning Conformal Blocks to Matrix Calogero-Sutherland Models*, *JHEP* **04** (2018) 052, [arXiv:1711.02022].

[198] M. Isachenkov and V. Schomerus, *Integrability of conformal blocks. Part I. Calogero-Sutherland scattering theory*, *JHEP* **07** (2018) 180, [arXiv:1711.06609].

[199] I. Buric, V. Schomerus, and E. Sobko, *Superconformal Blocks: General Theory*, arXiv:1904.04852.

[200] P. Liendo, C. Meneghelli, and V. Mitev, *Bootstrapping the half-BPS line defect*, *JHEP* **10** (2018) 077, [arXiv:1806.01862].

[201] L. J. Dixon, C. Duhr, and J. Pennington, *Single-valued harmonic polylogarithms and the multi-Regge limit*, *JHEP* **10** (2012) 074, [arXiv:1207.0186].

[202] J. M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601, [arXiv:1304.1806].

[203] S. Bloch, M. Kerr, and P. Vanhove, *A Feynman integral via higher normal functions*, *Compos. Math.* **151** (2015), no. 12 2329–2375, [arXiv:1406.2664].

[204] E. Panzer, *Feynman integrals and hyperlogarithms.* PhD thesis, Humboldt U., Berlin, Inst. Math., 2015. arXiv:1506.07243.

[205] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, *Elliptic polylogarithms and Feynman parameter integrals*, arXiv:1902.09971.

[206] M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees, and P. Vieira, *The S-matrix bootstrap. Part I: QFT in AdS*, *JHEP* **11** (2017) 133, [arXiv:1607.06109].

[207] A. Homrich, J. Penedones, J. Toledo, B. C. van Rees, and P. Vieira, *The S-matrix Bootstrap IV: Multiple Amplitudes*, arXiv:1905.06905.

[208] E. Hijano, *Flat space physics from AdS/CFT*, arXiv:1905.02729.

[209] N. Arkani-Hamed, Y.-T. Huang, and S.-H. Shao, *On the Positive Geometry of Conformal Field Theory*, arXiv:1812.07739.

[210] A. Brezin, J. C. Le Guillou, J. Zinn-Justin, and B. J. Nickel, *Higher order contributions to critical exponents*, *J. Phys.* **A44** (1973), no. 3 227.

[211] V. M. Braun and A. N. Manashov, *Evolution equations beyond one loop from conformal symmetry*, *Eur. Phys. J.* **C73** (2013) 2544, [arXiv:1306.5644].

[212] W. Bühring, *Generalized hypergeometric functions at unit argument*, *Proceedings of the American Mathematical Society* **114** (1992), no. 1 145–153.

[213] P. Appell, *Sur les séries hypergéométriques de deux variables et sur dés équations différentielles linéaires aux dérivés partielles.*, *C. R. Acad. Sci., Paris* **90** (1880) 296–299, 731–735.

[214] P. Appell, *Sur les fonctions hypergométriques de plusieurs variables, les polynômes d'hermite et autres fonctions sphériques dans l'hyperespace*, *Mémorial des Sciences Mathématiques* **3** (1925) 82.