Topological phases of matter, symmetries, and K-theory

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Abstract

This thesis contains a study of topological phases of matter, with a strong emphasis on symmetry as a unifying theme. We take the point of view that the "topology" in many examples of what is loosely termed "topological matter", has its origin in the symmetry data of the system in question. From the fundamental work of Wigner, we know that topology resides not only in the group of symmetries, but also in the cohomological data of projective unitary-antiunitary representations. Furthermore, recent ideas from condensed matter physics highlight the fundamental role of charge-conjugation symmetry. With these as physical motivation, we propose to study the topological features of gapped phases of free fermions through a $\mathbb{Z}_2$-graded C^*-algebra encoding the symmetry data of their dynamics.

In particular, each combination of time reversal and charge conjugation symmetries can be associated with a Clifford algebra. K-theory is intimately related to topology, representation theory, Clifford algebras, and $\mathbb{Z}_2$-gradings, so it presents itself as a powerful tool for studying gapped topological phases. Our basic strategy is to use various K-theoretic invariants of the symmetry algebra to classify symmetry-compatible gapped phases. The *super-representation group* of the algebra classifies such gapped phases, while its K-theoretic *difference-group* classifies the obstructions in passing between two such phases. Our approach is a noncommutative version of the twisted K-theory approach of Freed–Moore, and generalises the K-theoretic classification first suggested by Kitaev. It has the advantage of conceptual simplicity in its uniform treatment of all symmetries. Physically, it encompasses phenomena which require noncommutative algebras in their description; mathematically, it clarifies and provides rigour to the meaning of "homotopic phases", and easily explains the salient features of Kitaev's Periodic Table.

To my family
for your unwavering support,
and Sylvia
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Contents

Chapter 1

Introduction

Wigner's theorem says that the symmetry group G of a quantum mechanical system is represented projectively on a complex Hilbert space[1] $\mathscr{H}$ as unitary or antiunitary operators. This means that the topological group G comes with a continuous homomorphism $\phi : G \to \{\pm 1\}$, with the kernel G^u represented unitarily and $G^a = G - G^u$ represented antiunitarily. Projectivity of the unitary-antiunitary representation is encoded in a generalised 2-cocycle $\sigma : G \times G \to \mathbb{T} \cong \mathrm{U}(1)$ satisfying

$$\sigma(x,y)\sigma(xy,z) = \sigma(y,z)^x \sigma(x,yz), \tag{1.1}$$

where for $\lambda \in \mathbb{T}$, $\lambda^x := \lambda$ if $\phi(x) = +1$ and $\lambda^x := \overline{\lambda}$ if $\phi(x) = -1$. A projective unitary-antiunitary representation [70, 81] (PUA-rep) θ of (G, ϕ, σ) is a map $x \mapsto \theta_x$ from G to the unitary-antiunitary group of $\mathscr{H}$, such that $\theta_x \theta_y = \sigma(x,y)\theta_{xy}$, and θ_x is unitary (resp. antiunitary) if $\phi(x) = +1$ (resp. $\phi(x) = -1$).

As we will see, a PUA-rep of (G, ϕ, σ) is a special case of a twisted covariant representation of a twisted C^*-dynamical system $(G, \mathcal{A}, \alpha, \sigma)$. Here, $\mathcal{A}$ is a real or complex C^*-algebra, and α is some action of G on $\mathcal{A}$, twisted by a cocycle σ. A PUA-rep is simply the case where $\mathcal{A} = \mathbb{C}$ as a real C^*-algebra, and G acts on $\mathcal{A}$ trivially or by complex conjugation. There are physical reasons for considering such generalisations of PUA-reps. For instance, $\mathcal{A}$ could be the algebra of the canonical commutation or anticommutation relations, or it might be used as a model for disorder. Mathematically, twisted covariant representations arise even when we are only interested in PUA-reps, for instance, when we consider the relationship between the PUA-reps of (G, ϕ, σ) and those of some subgroup of G. Motivated by constructions in the quantization of charged fields, as well as charge-conjugation and particle-hole symmetries in condensed matter systems, we will study, more generally, $\mathbb{Z}_2$-graded[2] twisted covariant representations of $\mathbb{Z}_2$-graded twisted C^*-dynamical systems. This reflects one of

[1] All Hilbert spaces are assumed to be separable.

[2] Unless otherwise stated, the grading group will always be taken to be $\mathbb{Z}_2$.

the central themes of this thesis, namely, the treatment of charge-conjugation symmetry as a symmetry with the same conceptual status as the other charge-preserving symmetries.

The first general problem at hand is to classify the irreducible twisted covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$. In physical applications, G often has a normal subgroup N, whose representations are better understood, or demanded on physical grounds. We wish to build the twisted covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$ from those of N and their associated "little groups", using a generalisation of Mackey's theory of induced unitary representations to the setting of graded twisted covariant representations.

The second problem, occupying the main body of this thesis as well as [86], is to clarify the link between the symmetry classes of free fermions as discussed in [2, 37, 77, 79], and the K-theoretic classification of gapped free-fermion phases and topological insulators first suggested in [50], and expanded upon using twisted K-theory in [30]. In both situations, a tenfold classification was shown to arise. The former classified single-electron Hamiltonians with given time reversal or charge-conjugation/particle-hole symmetries according to the compact symmetric spaces. The large-N versions of the symmetric spaces are related to the loop spaces for the stable unitary and orthogonal groups, and hence to Bott periodicity and the topological K-theory classification of the latter.

Clifford algebras play an important role in Bott periodicity in K-theory, as illustrated in the algebraic treatment of K-theory in [39, 40, 43, 42], as well as the Atiyah–Bott–Shapiro [4] construction of the higher K-theory groups of a point. On the physics side, the compatibility of a gapped Hamiltonian with the symmetries of time reversal and charge conjugation can be rephrased in terms of Clifford algebras — an observation which was made for certain examples in [50], and expanded upon in several other papers [1, 30, 84] but in inequivalent ways. The precise connection depends crucially on the definitions of time/charge reversal, as well as the role of the one-particle Hamiltonian in the passage to second quantization. We find it most convenient to distinguish between the neutral and charged cases, and our definitions of time/charge reversal are closest to those of [30].

The overarching idea in all of the proposed classification schemes, is symmetry and its projective unitary-antiunitary realization, in the spirit of Wigner's theorem. As pointed out in [30], the tenfold way is not exhaustive, but assumes a certain splitting between the distinguished time/charge reversal symmetries and the rest of the symmetry group. To avoid such assumptions, we set up our formalism in a fairly general manner, and restrict to special cases to recover the tenfold way, as well as Kitaev's Periodic Table [50]. Our work can be viewed as an "algebraic" version, or noncommutative generalisation, of the treatment in [30], dealing mostly with twisted group C^*-algebras, and their representations, decompositions and K-theory. Besides recovering the main results in these papers, the algebraic language also offers numerous advantages, and clarifies some issues that are obscured by taking a topological (commutative) point of view. In particular, the periodicities and dimension shifts in Kitaev's Periodic Table are directly explained using various isomorphisms in the K-theory of C^*-algebras. Our formalism is also useful for studying systems such integer quantum Hall systems [10], where a topological picture may be unnatural or unavailable.

Outline

We begin with a treatment of free-fermion dynamics and its symmetries and quantization, in Chapter 2, distinguishing carefully between the charged and neutral cases. This motivates the definitions of graded covariant representations, graded twisted C^*-dynamical systems, and graded twisted crossed products in Chapter 3, where the connection to the study of gapped free-fermion Hamiltonians is also explained. We make a detour in Chapter 4 to sketch how Mackey's theory of induced representations can be generalised to twisted covariant representations. This is applied to find graded PUA-reps of the full Poincaré group with charge-conjugation symmetry in Section 5.2, following closely the approach of [82]. The PUA-reps of some physically relevant groups are studied in Chapter 5, where as a side result, a canonical decomposition of central cocycles for a semidirect product $N \rtimes H$ with well-behaved N is given, generalising the calculations of Mackey in [58]. More relevant to the rest of the thesis is the second half of Chapter 5, where the crucial link between Clifford algebras, graded PUA-reps of parity groups, and the ubiquitous tenfold way is explained.

The transition into a detailed and rigorous K-theoretic treatment of symmetry-compatible gapped Hamiltonians is made in Chapter 6. Drawing inspiration from various pictures of K-theory, the super-representation group and difference-group are introduced as invariants of symmetry classes of gapped Hamiltonians. The computation of these groups, using decomposition results for twisted crossed products, is illustrated with physically motivated examples in Chapter 7. The equivalence with a picture in terms of vector bundles, Clifford modules, and topological equivariant K-theory, where available, is also explained. Chapter 8 is devoted to physical applications, where we explain how our formalism encompasses and unifies many approaches to topological band insulators in a consistent way. In particular, all the features of the Periodic Table of Kitaev are deduced from very general results in K-theory.

Chapter 2

Positive energy quantization of linear dynamics with symmetries

Following the general arguments of [30, 60], elements of the symmetry (topological) group G for the unitary dynamics of a quantum mechanical system are presumed to be endowed with Hamiltonian and/or time preserving/reversing properties. These properties are encoded by a pair of continuous homomorphisms $c, \tau : G \to \{\pm 1\}$. An element $g \in G$ preserves (resp. reverses) the arrow of time if $\tau(g) = +1$ (resp. $\tau(g) = -1$); similarly, it commutes (resp. anticommutes) with the Hamiltonian H if $c(g) = +1$ (resp. $c(g) = -1$). A third homomorphism $\phi : G \to \{\pm 1\}$ specifies whether g is implemented unitarily ($\phi(g) = +1$) or antiunitarily ($\phi(g) = -1$), as in Wigner's Theorem. Writing u_t for the unitary dynamics generated by the Hamiltonian H, and $\mathtt{g}$ for the unitary/antiunitary representative of g, the time-reversal equation $\mathtt{g} u_t \mathtt{g}^{-1} = u_{\tau(g)t}$ leads to $\phi \cdot \tau \cdot c \equiv 1$, so any two of ϕ, τ, c specifies the third. Often, $c \equiv 1$ is assumed, i.e., all symmetries commute with the Hamiltonian. This is often a reasonable assumption, for example, when the Hamiltonian is bounded below but unbounded above. Since $\mathtt{g} H \mathtt{g}^{-1} = c(g) H$, a unitary/antiunitary g with $c(g) = -1$ needs to both preserve and flip the specturm of H, which is not possible. Then $\phi = \tau$ and antiunitarity becomes synonymous with time-reversal. However, we want to consider Hamiltonians which are compatible with symmetries that may effect charge-conjugation, so we allow for $c \not\equiv 1$. Then ϕ and τ may be independently specified. We also allow the symmetries to be projectively realised in Hilbert space, i.e., there may be a non-trivial cocycle σ.

The possibility of charge-reversing symmetries (present or otherwise) for free-fermion dynamics requires, logically, a notion of *charged* dynamics and *charged* representations of the canonical anticommutation relations (CARs), as opposed to *neutral* dynamics and *neutral* CAR representations. Here, we are borrowing the terminology of Dereziński and Gérard [19, 20], as well as the algebraic formalism of charged field quantization as can be found in [31]. Neutral orthogonal dynamics more correctly describes neutral (Majorana) fermions. Similar remarks can be made for free bosons and the canonical commutation relations (CCRs), but we will not deal with bosons in this thesis. In both the charged and neutral cases, non-degeneracy of the dynamics allows us to distinguish between particle and antiparticle sectors. We would also like both species to have positive energy in (second) quantization. For instance, in condensed matter applications, the Fermi level (which may be set to 0) of a band insulator lies in a gap of the Hamiltonian, and provides the particle-hole distinction. Both particles and holes/antiparticles have positive energy in the second quantized picture. With these considerations in mind, we outline the formalism of positive energy quantization, following [19, 20]. We also establish our conventions for describing symmetries of the dynamics, including those of time and charge reversal, which differ slightly from the conventions in [19, 20].

2.1 Neutral and charged CAR representations

Let $\mathcal{Y}$ be a real Hilbert space with inner product b. A *neutral CAR representation* over $(\mathcal{Y}, b)$ in a complex Hilbert space $\mathscr{H}$ is a linear map φ from $\mathcal{Y}$ to the bounded self-adjoint operators on $\mathscr{H}$, such that

$$\{\varphi(y_1), \varphi(y_2)\} = 2b(y_1, y_2)1_{\mathscr{H}}, \qquad y_1, y_2 \in \mathcal{Y}.$$

If $\mathcal{Y}$ is instead a complex Hilbert space with inner product h, a *charged CAR representation* over $(\mathcal{Y}, h)$ in a complex Hilbert space $\mathscr{H}$ is a linear map a^* from $\mathcal{Y}$ to the bounded operators on $\mathscr{H}$, such that

$$\begin{aligned} \{\mathrm{a}(y_1), \mathrm{a}(y_2)\} &= 0 = \{\mathrm{a}^*(y_1), \mathrm{a}^*(y_2)\}, \\ \{\mathrm{a}(y_1), \mathrm{a}^*(y_2)\} &= h(y_1, y_2)1_{\mathscr{H}}, \qquad\qquad y_1, y_2 \in \mathcal{Y}. \end{aligned}$$

Here, $\mathrm{a}(y)$ is the adjoint of $\mathrm{a}^*(y)$, and thus a is an antilinear map from $\mathcal{Y}$ to the bounded operators on $\mathscr{H}$.

A complex Hilbert space $(\mathcal{Y}, h)$ is equivalently a real Hilbert space $(\mathcal{Y}, b)$ with $b = \mathrm{Re}(h)$, along with an orthogonal complex structure J (i.e. $J^2 = -1$) playing the role of multiplication by i. The complex inner product h may be recovered from b and J by setting $h(u, v) = b(u, v) + \mathrm{i}b(Ju, v)$. There is a similar equivalence of viewpoints at the level of CAR representations. A charged CAR representation over $(\mathcal{Y}, h)$ gives rise to a neutral CAR representation over $(\mathcal{Y}, b = \mathrm{Re}\, h)$ via

$$\varphi(y) = \mathrm{a}(y) + \mathrm{a}^*(y),$$

along with a U(1) symmetry $\theta \mapsto e^{\mathrm{i}\theta}$ of charge 1 on $(\mathcal{Y}, b)$ as orthogonal operators. Conversely, given a neutral CAR representation φ over $(\mathcal{Y}, b)$, and an orthogonal complex structure J on $\mathcal{Y}$ (so that $\theta \mapsto e^{J\theta}$ is a U(1) symmetry of charge 1), we can define a charged CAR representation over $(\mathcal{Y}, h)$, where h is the complex inner product determined by J. We first identify $(\mathcal{Y}, h)$ with the J-holomorphic subspace $\mathcal{Z} = \{(1 - iJ)y : y \in \mathcal{Y}\} \subset \mathcal{Y} \otimes_{\mathbb{R}} \mathbb{C}$, via $y \leftrightarrow \frac{1}{\sqrt{2}}(1 - iJ)y =: z$. Note that $\mathcal{Z}$ is the i-eigenspace of the complex-linear extension of J to $\mathcal{Y} \otimes_{\mathbb{R}} \mathbb{C}$. The *charged fermionic fields* are then

$$\mathrm{a}^*(y) = \frac{\varphi(z)}{\sqrt{2}}, \quad \mathrm{a}(y) = \frac{\varphi(\overline{z})}{\sqrt{2}},$$

where φ has been extended complex-linearly to $\mathcal{Y} \otimes_{\mathbb{R}} \mathbb{C}$.

2.2 Quantization of linear dynamics and Fock space representations

For a complex Hilbert space $\mathcal{Z}$, the symbols $a^*(z)$ and $a(z)$, with $z \in \mathcal{Z}$, will be used to refer to the usual creation and annihilation operators on the fermionic Fock space $\bigwedge^* \mathcal{Z}$. They provide an example of a charged CAR representation over $\mathcal{Z}$. We shall be considering linear dynamics u_t which are either orthogonal on a real Hilbert space (neutral), or unitary on a complex Hilbert space (charged). There is also a notion of algebraic quantization at the CAR-algebra level, including the definitions of time and/or charge reversal therein, but we will not consider this. Instead, we work in the concrete setting of Fock quantization and Fock space representations. The idea is that we are seeking a CAR representation on a complex Hilbert space $\mathscr{H}$, for which there exists a *positive* self-adjoint operator $\mathcal{H}$ such that $e^{it\mathcal{H}}$ implements u_t. We have left out many technical details, especially in the infinite-dimensional case, for which we refer to [19, 20]. A more detailed discussion of CAR-algebras and Fock space representations can be found in many references, such as [19, 20, 72, 31].

2.2.1 Quantization of non-degenerate orthogonal dynamics

We first consider the neutral case, in which we have a strongly continuous 1-parameter group $u_t \subset \mathrm{O}(\mathcal{Y}, b)$ generated by an antisymmetric A, i.e., $b(y_1, Ay_2) = -b(Ay_1, y_2)$, $y_1, y_2 \in \mathcal{Y}$. We assume non-degeneracy[1], i.e., $\ker(A) = \{0\}$. The operator $A = -A^*$ has a polar decomposition (see Chapter 2.4 of [20])

$$A = HJ = JH, \qquad \text{where } H = |A| = \sqrt{AA^*} = \sqrt{A^*A} = \sqrt{-A^2} > 0.$$

[1] In practice, we often impose a stronger *gapped condition*, namely, that 0 is in the resolvent set of A, so A has a (bounded) inverse.

In this decomposition, J is an orthogonal complex structure, and $u_t = e^{tJH}$ commutes with J. Extend H and J complex linearly to $\mathcal{Y}\otimes\mathbb{C}$, and let $\mathcal{Z}$ be its J-holomorphic subspace (on which J acts by multiplication by i). Since H commutes with J, the restriction $H_{\mathcal{Z}} \coloneqq H|_{\mathcal{Z}}$ makes sense, and is even positive, since H itself is. Associated with this choice of J is the CAR representation over $(\mathcal{Y}, b)$ on the Fock space $\bigwedge^* \mathcal{Z}$, defined as

$$\varphi(y) = a^*(z) + a(z),$$

where $z \coloneqq \frac{1}{\sqrt{2}}(1 - \mathrm{i}J)y$. Defining[2] $\mathcal{H} = \mathrm{d}\Lambda(H_{\mathcal{Z}})$, we find that

$$e^{\mathrm{i}t\mathcal{H}}\varphi(y)e^{-\mathrm{i}t\mathcal{H}} = \varphi(u_t y),$$

so u_t is implemented unitarily on Fock space, with positive generator $\mathcal{H}$. We call φ the *positive energy Fock quantization* for the non-degenerate orthogonal dynamics u_t.

Time reversal in non-degenerate orthogonal dynamics

Symmetries of the dynamics u_t are required to preserve the CARs, so G is assumed to have a (possibly projective) realization as orthogonal operators on $(\mathcal{Y}, b)$. In addition, we require $g \in G$ to be time-preserving or reversing according to the value of $\tau(g)$. There is no notion of charge-reversal under consideration, so $c \equiv 1$, which means that $\phi \equiv \tau$. Writing g for the representative of g on $\mathcal{Y}$, we need

$$\mathsf{g}u_t = u_{\tau(g)t}\mathsf{g},$$

or equivalently, $\mathsf{g}A = \tau(g)A\mathsf{g}$. A short computation then leads to

$$\mathsf{g}H = H\mathsf{g}, \quad \mathsf{g}J = \tau(g)J\mathsf{g}. \tag{2.1}$$

We extend g linearly (resp. antilinearly) to $\mathcal{Y}\otimes\mathbb{C}$ if $\tau(g) = +1$ (resp. $\tau(g) = -1$), and check that g preserves $\mathcal{Z}$ (as well as $\overline{\mathcal{Z}}$):

$$\mathsf{g}\left(\frac{(1-\mathrm{i}J)y}{2}\right) = \frac{\mathsf{g}y}{2} - \frac{\mathrm{i}}{2}\tau(g)\mathsf{g}Jy = \frac{\mathsf{g}y}{2} - \frac{\mathrm{i}}{2}J\tau(g)^2\mathsf{g}y = \left(\frac{1-\mathrm{i}J}{2}\right)(\mathsf{g}y),$$

so we may define $\mathsf{g}_{\mathcal{Z}} \coloneqq \mathsf{g}|_{\mathcal{Z}}$. Furthermore, $\mathsf{g}_{\mathcal{Z}}$ is (anti)unitary and commutes with $H_{\mathcal{Z}}$. Passing to Fock space, g is amplified to the (anti)unitary $\hat{\mathsf{g}} = \Lambda(\mathsf{g}_{\mathcal{Z}})$, and satisfies

$$\begin{aligned}
\hat{\mathsf{g}}\mathcal{H}\hat{\mathsf{g}}^{-1} &= \mathcal{H}, \\
\hat{\mathsf{g}}e^{\mathrm{i}t\mathcal{H}}\hat{\mathsf{g}}^{-1} &= e^{\tau(g)\mathrm{i}t\mathcal{H}}, \\
\hat{\mathsf{g}}\varphi(y)\hat{\mathsf{g}}^{-1} &= \varphi(\mathsf{g}y).
\end{aligned}$$

[2] $\mathrm{d}\Lambda(\cdot)$ refers to the operation of "second quantization" of a self-adjoint operator into one acting on Fock space.

Remark 2.2.1. If we assume that A is invertible, then so is $|A|$. In this case, J is determined by the dynamics u_t only through the "flattened" generator $\tilde{A} = A/|A|$. This ties in nicely with the notions of homotopy equivalence and spectrally-flattened Hamiltonians suggested by Kitaev [50] as well as [30]. In the general case, we can understand J by complexifying and looking at the polar decomposition for the self-adjoint operator $\mathrm{i}A = \mathrm{sgn}(\mathrm{i}A)|\mathrm{i}A|$. Then $-\mathrm{i}\,\mathrm{sgn}(\mathrm{i}A)$ is the complexification of J, and the former can be restricted back to the real subspace $\mathcal{Y}$ (see Section 2.2.2 for the meaning of $\mathrm{sgn}(\mathrm{i}A)$).

2.2.2 Quantization of non-degenerate unitary dynamics

Suppose the orthogonal dynamics u_t on $(\mathcal{Y}, b)$ is compatible with a symmetry of charge 1. This means that $\mathrm{U}(1) \ni \theta \mapsto s_\theta := e^{J\theta} \in \mathrm{O}(\mathcal{Y}, b)$ for some orthogonal J with $J^2 = -1$, and J commutes with u_t. If we regard $\mathcal{Y}$ as a complex Hilbert space with the inner product h induced via J, the dynamics u_t becomes a 1-parameter group of unitaries. In this case, we say that the dynamics is *charged.*

So let $(\mathcal{Y}, h)$ be a complex Hilbert space and u_t be a strongly continuous 1-parameter unitary group, with self-adjoint generator H. We assume that u_t is non-degenerate, meaning that $\ker(H) = \{0\}$. In the polar decomposition of the injective self-adjoint operator H (see Example 7.1 in [78]), there is a canonical decomposition $H = H^+ \oplus H^-$ into the direct sum of a positive operator and a negative operator, acting on a corresponding orthogonal direct sum $\mathcal{Y} = \mathcal{Y}^+ \oplus \mathcal{Y}^-$. We define $\mathrm{sgn}(H)$ to be the operator which is $+1$ on $\mathcal{Y}^+$ and -1 on $\mathcal{Y}^-$, as well as the operators

$$Q = \mathrm{sgn}(H), \quad J = \mathrm{i}\,\mathrm{sgn}(H) = \mathrm{i}Q, \quad |H| = \sqrt{H^2} > 0.$$

Then $H = \mathrm{sgn}(H)|H|$, and we can rewrite u_t as $e^{tJ|H|}$. Also, J is unitary, skew-adjoint and commutes with H, Q and $|H|$. Furthermore, Q is the grading operator for $\mathcal{Y} = \mathcal{Y}^+ \oplus \mathcal{Y}^-$, and J acts as multiplication by $\pm\mathrm{i}$ on $\mathcal{Y}^\pm$. If we now write $\mathcal{Z}$ for the space $\mathcal{Y}$ equipped with the complex structure J instead of i, then we have $\mathcal{Z} = \mathcal{Y}^+ \oplus \overline{\mathcal{Y}^-}$, where $\overline{\mathcal{Y}^-}$ is given the inner product dual to $h|_{\mathcal{Y}^-}$. The subspaces $\mathcal{Y}^\pm$ are invariant for Q, H and $|H|$, so we may regard the latter as self-adjoint operators on $\mathcal{Z}$, in which case a subscript is appended, e.g., $H_\mathcal{Z}$.

On the Fock space $\bigwedge^* \mathcal{Z}$, we can define, for each $y = (y^+, y^-) \in \mathcal{Y}$, the *charged fields*

$$\begin{aligned} \mathrm{a}^*(y) &= a^*(y^+) + a(\overline{y^-}), \\ \mathrm{a}(y) &= a(y^+) + a^*(\overline{y^-}), \end{aligned}$$

which furnish a (charged) CAR representation over $(\mathcal{Y}, h)$, called the *positive energy Fock quantization* for the non-degenerate unitary dynamics u_t. As before, we can define the second quantized versions of $|H|_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$ to be

$$\mathcal{H} = \mathrm{d}\Lambda(|H|_{\mathcal{Z}}), \qquad \mathcal{Q} = \mathrm{d}\Lambda(Q_{\mathcal{Z}}).$$

These implement the dynamics and charge symmetry on Fock space,

$$e^{\mathrm{i}t\mathcal{H}}\mathrm{a}(y)e^{-\mathrm{i}t\mathcal{H}} = \mathrm{a}(u_t y), \quad e^{\mathrm{i}\theta\mathcal{Q}}\mathrm{a}(y)e^{-\mathrm{i}\theta\mathcal{Q}} = \mathrm{a}(e^{\mathrm{i}\theta}y).$$

Charge and/or time reversal in non-degenerate unitary dynamics

Since there is now a notion of charge and its reversal, we allow for $c \not\equiv 1$, or equivalently, $\phi \not\equiv \tau$. We look for representations of G as unitary/antiunitary operators on $(\mathcal{Y}, h)$, which are required to be time preserving/reversing according to the value of $\tau(g)$ (see Section 3.2.2 for a precise definition), i.e.,

$$\mathsf{g}u_t = u_{\tau(g)t}\mathsf{g}.$$

The homomorphism ϕ determines whether g is implemented as a unitary or antiunitary operator on $\mathcal{Y}$. A short computation leads to the following commutation relations

$$\begin{aligned}
\mathsf{g}H\mathsf{g}^{-1} &= \phi(g)\tau(g)H \equiv c(g)H, \\
\mathsf{g}|H|\mathsf{g}^{-1} &= |H|, \\
\mathsf{g}Q\mathsf{g}^{-1} &= \phi(g)\tau(g)Q \equiv c(g)Q, && (2.2\text{a}) \\
\mathsf{g}J\mathsf{g}^{-1} &= \tau(g)J. && (2.2\text{b})
\end{aligned}$$

From (2.2b), we find that $\mathsf{g}_{\mathcal{Z}}$ (i.e. the map g considered as an operator on $\mathcal{Z}$) is unitary or antiunitary according to $\tau(g)$. As before, we may amplify $\mathsf{g}_{\mathcal{Z}}$ to the (anti)unitary $\hat{\mathsf{g}} = \Lambda(\mathsf{g}_{\mathcal{Z}})$, which satisfies

$$\begin{aligned}
\hat{\mathsf{g}}\mathcal{H}\hat{\mathsf{g}}^{-1} &= \mathcal{H}, \\
\hat{\mathsf{g}}e^{\mathrm{i}t\mathcal{H}}\hat{\mathsf{g}}^{-1} &= e^{\tau(g)\mathrm{i}t\mathcal{H}}, \\
\hat{\mathsf{g}}\mathcal{Q}\hat{\mathsf{g}}^{-1} &= c(g)\mathcal{Q}, \\
\hat{\mathsf{g}}e^{\mathrm{i}\theta\mathcal{Q}}\hat{\mathsf{g}}^{-1} &= e^{\phi(g)\mathrm{i}\theta\mathcal{Q}}, \\
\hat{\mathsf{g}}\mathrm{a}^{\pm}(y)\hat{\mathsf{g}}^{-1} &= \mathrm{a}^{\pm c(g)}(\mathsf{g}y), \qquad \text{where } \mathrm{a}^{+} := \mathrm{a}^{*}, \mathrm{a}^{-} := \mathrm{a}.
\end{aligned}$$

Remark 2.2.2. As in the neutral case, the modified imaginary unit J depends on the dynamics u_t only through the spectrally-flattened generator $Q = \mathrm{sgn}(H)$. Physically, the modification $\mathrm{i} \mapsto J$ corresponds to the reinterpretation of negative-energy "particles" as postive-energy "antiparticles". The standard charge reversal symmetry C in quantum field theory preserves the arrow of time, but reverses H and is antiunitary in first quantization, i.e. $(\phi, \tau, c)(C) = (-1, +1, -1)$. It is, however, unitary in second quantization (see Section 29 in [64]).

Remark 2.2.3. Equations (2.2a)–(2.2b), along with the possibility that G is realised projectively, say that we have a graded projective unitary-antiunitary representation (graded PUA-rep) of (G, c, ϕ, σ) as defined in Section 3.2.2.

2.3 Special cases — CT-subgroups and a tenfold way

As in much of the literature, we could *assume* that there is a distinguished charge *reversal* symmetry $C \in G$, with $(\phi, \tau, c)(C) = (-1, +1, -1)$, and antiunitary representative C on $\mathcal{Y}$ satisfying $\mathsf{C}^2 = \pm 1$. Similarly, we could assume that there is a distinguished time reversal symmetry $T \in G$ with $(\phi, \tau, c)(T) = (-1, -1, +1)$, and antiunitary representative $\mathsf{T}^2 = \pm 1$. Dereziński argues that C and T, realized on the observable algebra (gauge invariant part of the CAR algebra) as $\hat{\mathsf{C}}$ and $\hat{\mathsf{T}}$, should obey $(\hat{\mathsf{C}}\hat{\mathsf{T}})^2 = +1$. This suggests that $(\mathsf{CT})^2 = \pm 1$, from which it follows that C and T either commute or anticommute. The freedom to multiply C or T by i means that we can assume that they commute. There are then four possibilities corresponding to the independent choices $\mathsf{C}^2 = \pm 1$ and $\mathsf{T}^2 = \pm 1$. Similarly, two possibilities arise in each of the two cases where only one of C and T is present[3] in G. Thus, there are eight "real" cases in which some antiunitary symmetries are present.

When all symmetries are unitary, so $(\phi, \tau, c)(g) \neq (-1, +1, -1)$ or $(-1, -1, +1)$ for any $g \in G$, there is still the possibility of a distinguished unitary symmetry $S \in G$ with $(\phi, \tau, c)(S) = (1, -1, -1)$. S is sometimes called a sublattice symmetry. Its representative S is only determined up to a phase factor, and we may take $\mathsf{S}^2 = +1$, so only one distinct case arises. One more case arises when $\phi = \tau = c = 1$ for all $g \in G$, so there are two "complex" cases in which no antiunitary symmetries are present.

What we have described above is an instance of the so-called ten-fold way $(8 + 2)$ [23, 2, 37, 77, 60]. A more detailed discussion can be found in Section 5.4.2. Similarly, there is a three-fold way [23] in the neutral setting, given by the two choices $\mathsf{T}^2 = \pm 1$ ("real" and "quaternionic"), along with the case where T is not present in G ("complex"). In either case, *it is important to keep in mind that there are a number of different classification questions to which the answer is a ten-fold or three-fold way.*

We stress that in our formalism, the presence of a time/charge revers*ing* symmetry does not imply the presence of a distinguished charge/time revers*al* operator in the above sense. For example, if the homomorphism τ is surjective, but the corresponding exact sequence $1 \to \ker(\tau) \to G \xrightarrow{\tau} \{\pm 1\} \to 1$ is not split, there may not be a distinguished group element T of order 2 in G, whose representative T is a projective involution. In other words, there may be no time-revers*al* operators T such that $\mathsf{T}^2 = \pm 1$, despite time-revers*ing* symmetries

[3] Nevertheless, we are considering charged fermionic fields, even if there are no charge-reversing symmetries.

being present in G. Indeed, Freed–Moore [30, 60] have pointed out that there are physically relevant examples that do not fit into the tenfold or threefold way. We prefer to work more generally, and think of time/charge reversal as properties of a symmetry $g \in G$; under certain splitting assumptions on G, we can recover the usual distinguished T and/or C symmetries as elements of G.

2.4 Remarks on conventions for free-fermion dynamics

The definition of symmetries of a Hamiltonian, including those of charge-conjugation and time-reversal, differ between authors. Consequently, the physical meaning of a particular "tenfold" way classification [1, 2, 30, 37, 50, 77, 79] depends on the conventions being adopted.

One of the approaches is to consider the single-particle "Hamiltonian" as an operator on a *Nambu space* $W = V \oplus \overline{V}$, rather than a bare Hilbert space V. Here, V is some complex Hilbert space, and $\overline{V}$ is its dual. A Nambu space has a canonical real structure $\Sigma : (v_1, \overline{v_2}) \mapsto (v_2, \overline{v_1})$, which is sometimes taken as a charge-conjugation operator. The fixed points of Σ form the real mode space $\mathcal{M}$ of Majorana operators, and $\mathcal{M}$ inherits a real inner product b from W by restriction of h. The real symmetric form b can then be extended complex bilinearly to a symmetric form $b^{\mathbb{C}}$ on W. The above description of Nambu space is essentially that of Heinzner–Huckleberry–Zirnbauer [37], who then consider the space of "good" Hamiltonians to be $\mathfrak{so}(W, b^{\mathbb{C}}) \cap \mathrm{i}\mathfrak{u}(W, h)$. Note that the authors make no explicit mention of a gapped condition for such Hamiltonians. We also mention some other papers with closely related approaches: Abramovici–Kalugin [1], and Zirnbauer [99].

The operator $J = \mathrm{i} \oplus -\mathrm{i}$ on a Nambu space $W = V \oplus \overline{V} = \mathcal{M} \otimes \mathbb{C}$ restricts to an orthogonal complex structure on $\mathcal{M}$, and $(\mathcal{M}, J|_{\mathcal{M}}) \cong V$. One begins with second-quantized dynamics on Fock space $\bigwedge^* V$, generated by a Hamiltonian H_{F} which is required to be quadratic in the creation and annihilation operators $a^*(v), a(v), v \in V$. Such dynamics can be reformulated on Nambu space $V \oplus \overline{V}$, with generating "Hamiltonian" H_{N} subject to certain symmetry constraints. Alternatively, the dynamics can be specified by a skew-symmetric operator A on $\mathcal{M}$, whose complexification is $\mathrm{i}H_{\mathrm{N}}$. An example is the Bogoliubov–de Gennes (BdG) Hamiltonian for the quasi-particle dynamics of a superconducting system. It is important to note that the polarization J, and thus the Fock space in second quantization, is already implicit in the Nambu space formulation, whereas they are determined by H (or A) in positive energy Fock quantization. Also, particle number is not necessarily conserved (because $a(v)a(v')$ and $a^*(v)a^*(v')$ terms are allowed in the second-quantized Hamiltonian H_{F}), so the BdG Hamiltonian may not have a U(1)-symmetry (i.e. it may not commute with J).

Freed–Moore conventions, and symmetries in Nambu space

Freed–Moore begin with the mode space $(\mathcal{M}, b)$, assumed to be an even-dimensional real vector space $\mathcal{M} = \mathbb{R}^{2N}$ with a positive definite bilinear form b. The operator algebra $\mathscr{O}$ is the complexified Clifford algebra $Cl(\mathcal{M}, b) \otimes \mathbb{C}$ with $Cl(\mathcal{M}, b)$ generated by orthonormal basis vectors $\{c_1, \ldots, c_{2N}\}$ satisfying the canonical anticommutation relations (CARs) $\{c_i, c_j\} = 2\delta_{ij}$. The antilinear involution $*$ is such that the c_i are self-adjoint. A *choice* of orthogonal complex structure I on $\mathcal{M}$ turns it into a unitary space $V = \mathbb{C}^N$ with hermitian inner product $\langle u \,|v\rangle_V := b(u, v) + \mathrm{i}b(Iu, v)$. The exterior algebra $\bigwedge^* V$ is a fermionic Fock space, and plays the role of an $\mathscr{O}$-module. In more detail, $\bigwedge^* V$ is the orthogonal direct sum $\bigoplus_{n=0}^{N} \bigwedge^n V$, where the inner product on $\bigwedge^n V$ is determined by the formula $\langle u_1 \wedge \ldots \wedge u_n \,|v_1 \wedge \ldots \wedge v_n\rangle = \mathrm{Det}\,[\langle u_i \,|v_j\rangle]$. Further details on Fock representations can be found in [72] and Chapter 5.3 of [31].

A free-fermion Hamiltonian H is a self-adjoint degree-two element of $\mathscr{O}$, so it may be written as $-\mathrm{i}A^{ij}c_ic_j$ for some real antisymmetric matrix A^{ij}. Freed–Moore define Nambu space to be $\mathcal{M} \otimes \mathbb{C}$, with b extended sesquilinearly to a complex inner product h, and A extended complex-linearly. If H is gapped, A is invertible and we may flatten it into an orthogonal complex structure J on $\mathcal{M}$; H is similarly flattened into the self-adjoint grading operator $\Gamma = -\mathrm{i}J$ on $\mathcal{M}\otimes\mathbb{C}$. We can use J to turn $\mathcal{M}$ into a complex Hilbert space V in the usual way. The complex-linear extension of J (denoted with the same symbol) is unitary and skew-adjoint, and splits $\mathcal{M} \otimes \mathbb{C}$ into the orthogonal direct sum of the $\pm\mathrm{i}$ eigenspaces $W^{\pm}$ of J. Then, the summands $W^{\pm}$ are conjugate, with W^+ isomorphic to V as a complex Hilbert space via $v \mapsto \frac{1}{\sqrt{2}}(v - \mathrm{i}Jv)$. Thus, $\mathcal{M}\otimes\mathbb{C} = W^+ \oplus W^- \cong V \oplus \overline{V}$, making contact with the earlier definition of Nambu space in this section. Complex conjugation, which is the real structure on Nambu space fixing $\mathcal{M}$, exchanges W^+ and W^- (it anticommutes with Γ), and commutes with J. A symmetry element $x \in G$ is required to preserve the CARs, so there should be a homomorphism $\beta : G \to O(\mathcal{M}, b)$. We extend β_x to a unitary or antiunitary operator on the Nambu space $\mathcal{M} \otimes \mathbb{C}$, according to the sign of $\phi(x)$. As before, the second homomorphism c encodes whether β_x commutes or anticommutes with Γ. Equivalently, the homomorphism $\tau = c \cdot \phi$ determines whether β_x commutes or anticommutes with J, according to whether $\tau(x) = +1$ or $\tau(x) = -1$.

Freed–Moore regard the Nambu space $\mathcal{M}\otimes\mathbb{C}$ as an analogue of a single-particle quantum mechanical Hilbert space, with symmetries realised in Nambu space as described above. In Fock space, there is another possibility — the orthogonal symmetries β_x can be extended to a (inner) Bogoliubov automorphism $\hat{\beta}_x$ of $Cl(\mathcal{M}, b)$, via $\hat{\beta}_x(u_1 \ldots u_n) = (\beta_x u_1) \ldots (\beta_x u_n)$. These are (since we are in the finite-dimensional situation) implementable by unitary operators on Fock space, which may be allowed to combine projectively at this level. Indeed, the entire (special) orthogonal group (S)O$(\mathcal{M}, b)$ acts projectively as the (s)pin representation on Fock space, where there is even room for a representation of the (s)pin^c groups. However, it is not clear whether more general projective representations are allowed at the Fock space level — see, for instance, [1]. In any case, we will restrict ourselves to projective symmetries on Nambu spaces or single-particle Hilbert spaces.

2.4.1 Charged versus neutral dynamics

We do not work with the Nambu space formalism. Instead, our conventions based on graded PUA-reps (see Remark 2.2.3) are closest to those in the later parts of [30], and that in [77]. We find them to be well-motivated by the Shale–Stinespring formalism for second quantization of (charged) fermionic fields (see Chapter 6 of [31], especially the concluding remarks), and the closely related formalism of positive energy quantization of linear fields. An important feature of our formalism is the distinction between the charged and neutral cases.

Neutral case

The role of the generator A of non-degenerate *neutral* free-fermion (orthogonal) dynamics is to provide an orthogonal complex structure J on the real Hilbert space $\mathcal{Y}$, through the decomposition $A = HJ$. With the help of the polarisation J, we obtain the complex one-particle Hilbert space $\mathcal{Z}$ as a subspace of $\mathcal{Y} \otimes \mathbb{C}$. On $\mathcal{Z}$, the dynamics is unitarily generated by a positive Hamiltonian $H_{\mathcal{Z}}$. We regard dynamics given by A and $A/|A|$ (assuming A is invertible) to be equivalent in a "homotopy" sense. Note that H gets flattened to the identity operator when $A \mapsto A/|A|$, so J suffices to specify the homotopy class[4] of orthogonal dynamics. The data of $(\mathcal{Y}, b)$ and J is equivalent to that of a *complex* Hilbert space $(\mathcal{Y}, h)$. On $\mathcal{Y}$, the symmetries in G are required to obey equation (2.1), which is precisely the statement that $g \in G$ is realised as unitary or antiunitary operators on $(\mathcal{Y}, h)$ according to $\phi(g) = \tau(g) = \pm 1$. Therefore, the specification of (a homotopy class of) non-degenerate neutral free-fermion dynamics respecting the symmetry data (G, ϕ, σ) is achieved by the provision of a projective unitary-antiunitary representation for (G, ϕ, σ) (see Section 3.2.1). Note that despite the fact that we are dealing with non-degenerate (or even "gapped") dynamics, the Hamiltonian $H_{\mathcal{Z}}$ is not gapped, and is even positive. The "gap" is manifested in the self-adjoint operator $\mathrm{i}A = \mathrm{sgn}(\mathrm{i}A)|A|$ on the complexification $\mathcal{Y} \otimes \mathbb{C}$. The operator $\mathrm{i}A$ is sometimes referred to as the "Hamiltonian" as well.

[4]Here, and in the charged case, the generator A or H of the dynamics may be unbounded, and care must be taken in order to interpret spectral-flattening as a homotopy in a precise sense, see Appendix D of [30].

Suppose there is a distinguished time-reversal symmetry $T \in G$ of order 2, with $\phi(T) = -1$ and antiunitary representative T on $(\mathcal{Y}, h)$. By the general arguments of Section 5.4.2, we may assume that $\mathsf{T}^2 = \pm 1$. Then there is a $Cl_{1,1} = M_2(\mathbb{R})$ or $Cl_{2,0} = \mathbb{H}$-action on the underlying real Hilbert space $(\mathcal{Y}, b)$, generated by $\{J, \mathsf{T}\}$. If T is absent, there is simply a $Cl_{1,0} = \mathbb{C}$-action due to J. This is the well-known threefold Wigner–Dyson classification [23] of projective unitary-antiunitary representations into Wigner types I, II and III, based on $\mathbb{R}, \mathbb{H}$ and $\mathbb{C}$. However, note that Wigner's threefold classification may also be applied to projective unitary-antiunitary representations on complex Hilbert spaces understood as ordinary quantum mechanical Hilbert spaces [95].

Charged case

In the charged case, the Hamiltonian H generating the non-degenerate unitary dynamics on the complex Hilbert space $(\mathcal{Y}, h)$, determines the particle-antiparticle distinction in second quantization. In practice, we impose a stronger *gapped* condition on H; namely, we require 0 to lie in a gap of the spectrum of H. In this case, we call H a *gapped* Hamiltonian. In addition to allowing antiunitary symmetries, there is the possibility of charge-reversing symmetries which reverse $Q = \mathrm{sgn}(H)$, as indicated by $c(g) = -1$. We wish to identify two symmetry-compatible gapped Hamiltonians if they have the same spectral flattening, i.e., if they result in the same grading operator Q on $\mathcal{Y}$. Then, the specification of (a homotopy class of) charged free-fermion dynamics respecting the symmetry data (G, c, ϕ, σ), is precisely the specification of a *graded* projective unitary-antiunitary representation for (G, c, ϕ, σ) (see Section 3.2.2). This provides the physical motivation behind the constructions in Chapter 3.

As in the neutral case, a graded PUA-rep for (G, c, ϕ, σ) also has another interpretation as an ordinary quantum mechanical system, with flattened gapped Hamiltonian compatible with the symmetry data. We usually combine such quantum mechanical systems using the tensor product. On the other hand, at the level of one-particle Hilbert spaces, we combine free-fermion systems using the direct sum operation, which gets translated into the tensor product at the Fock space level. We are only interested in describing free-fermion dynamics and its symmetries at the one-particle level, so the direct sum applies. This allows us to construct commutative monoids of free-fermion systems, paving the way for the use of K-theoretic methods in their classification.

Chapter 3

Twisted crossed products, covariant representations, and symmetry classes of gapped Hamiltonians

We give an outline of the basic definitions and constructions of twisted covariant representations of twisted C^*-dynamical systems [13, 55, 66]. We make a simple generalisation to $\mathbb{Z}_2$-graded twisted covariant representations, and show that they arise naturally in the context of gapped quantum systems with time/charge-reversing symmetries. All gradings in this section are $\mathbb{Z}_2$-gradings.

3.1 Ungraded covariant representations

Let $\mathcal{A}$ be a separable, possibly non-unital, real or complex C^*-algebra[1]. We denote its multiplier algebra by $\mathcal{MA}$, and its group of unitary elements $\{u \in \mathcal{MA} : u^*u = uu^* = 1_{\mathcal{MA}}\}$ by $\mathcal{UMA}$. If $\mathbb{F}$ is the ground field of $\mathcal{A}$, we write $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ for the group of $\mathbb{F}$-linear $*$-automorphisms of $\mathcal{A}$. Let G be a locally compact, second countable group, with left Haar measure μ and identity element e. As in Section 2 of [66], we give $\mathcal{UMA}$ the strict topology, which coincides with the more familiar strong operator topology when $\mathcal{UMA}$ is the group of unitary operators on a separable Hilbert space. The topology on $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ is taken to be the point-norm topology, i.e., $\alpha_{(n)} \to \alpha$ iff $\alpha_{(n)}(a) \to \alpha(a)\ \forall a \in \mathcal{A}$.

Definition 3.1.1 (Twisted C^*-dynamical system [13, 55])**.** A pair (α, σ) of Borel maps

[1] Basic facts about real C^*-algebras can be found in Chapter 1 of [80] and Chapter 5 of [56]. For our purposes, we need the fact that the (Banach space) complexification of a real C^*-algebra is a complex C^*-algebra.

$\alpha : G \to \mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ and $\sigma : G \times G \to \mathcal{UMA}$ satisfying

$$\alpha(x)\alpha(y) = \mathrm{Ad}(\sigma(x, y)) \circ \alpha(xy), \tag{3.1a}$$

$$\sigma(x, y)\sigma(xy, z) = \alpha(x)(\sigma(y, z))\sigma(x, yz), \tag{3.1b}$$

$$\sigma(x, e) = 1 = \sigma(e, x), \tag{3.1c}$$

$$\alpha(e) = \mathrm{id}_{\mathcal{A}}, \qquad x, y, z \in G, \tag{3.1d}$$

is called a *twisting pair* for $(G, \mathcal{A})$. The map σ is called a *2-cocycle* with values in $\mathcal{UMA}$, or simply a *cocycle*, and the quadruple $(G, \mathcal{A}, \alpha, \sigma)$ is called a *twisted C^*-dynamical system*, or simply a *twisted dynamical system*.

For notational ease, we will often write $\alpha_x := \alpha(x)$ and $a^x := \alpha_x(a) \equiv \alpha(x)(a)$.

Definition 3.1.2 (Twisted covariant representation)**.** A *twisted covariant representation* of a twisted C^*-dynamical system $(G, \mathcal{A}, \alpha, \sigma)$ is a non-degenerate $*$-representation of $\mathcal{A}$ as bounded operators on a separable Hilbert space $\mathscr{H}$ over $\mathbb{F}$, along with a compatible Borel map $\theta : x \mapsto \theta_x$ from G to the unitary[2] operators on $\mathscr{H}$, in the sense that

$$\theta_x\theta_y = \sigma(x, y)\theta_{xy}, \tag{3.2a}$$

$$\theta_x(am) = a^x(\theta_x m), \qquad x, y, \in G, a \in \mathcal{A}, m \in \mathscr{H}. \tag{3.2b}$$

Note that (3.2b) can be restated as $a^x = \mathrm{Ad}(\theta_x)(a)$, and then we see that (3.2a) is consistent with (3.1a).

In the untwisted case, i.e. $\sigma(x, y) = 1$ for all $x, y \in G$, the map α is a Borel homomorphism $G \to \mathrm{Aut}_{\mathbb{F}}\mathcal{A}$, which is automatically continuous since G and $\mathrm{Aut}_{\mathbb{F}}\mathcal{A}$ are Polish groups (see Theorem D.11 of [97]). Then $(G, \mathcal{A}, \alpha, 1)$ is a C^*-dynamical system $(G, \mathcal{A}, \alpha)$ in the usual sense (e.g. 7.4.1 of [71], 2.1 of [97], or 10.1 of [11]). Also, a twisted covariant representation θ of $(G, \mathcal{A}, \alpha, 1)$ on $\mathscr{H}$ is, in particular, a Borel homomorphism from G to the unitary operators $\mathrm{U}(\mathscr{H})$ on $\mathscr{H}$, which is also automatically continuous if $\mathrm{U}(\mathscr{H})$ is given the strong operator topology. Thus, θ is a covariant representation of $(G, \mathcal{A}, \alpha)$ in the usual sense (e.g. 7.4.8 of [71] or 10.1 of [11]). No harm is done by dropping the adjective “twisted” and using it only for emphasis. We say that two twisted covariant representations $(\theta, \mathscr{H}), (\theta', \mathscr{H}')$ of $(G, \mathcal{A}, \alpha, \sigma)$ are *equivalent* if there is a unitary $\mathcal{A}$-linear intertwiner $U : \mathscr{H} \to \mathscr{H}'$ such that $U\theta_x U^{-1} = \theta'_x$ for all $x \in G$.

Definition 3.1.3 (Exterior equivalence of twisting pairs (Section 3 of [66]))**.** The action of a Borel function $\lambda : G \to \mathcal{UMA}$ on a twisting pair, $\lambda \cdot (\alpha, \sigma) \equiv (\lambda \cdot \alpha, \lambda \cdot \sigma) \equiv (\alpha', \sigma')$, is

[2] When $\mathbb{F} = \mathbb{R}$, we also use “orthogonal” for emphasis.

defined by

$$\alpha'(x) = \mathrm{Ad}(\lambda(x)) \circ \alpha(x), \tag{3.3a}$$

$$\sigma'(x,y) = \lambda(x)\alpha_x(\lambda(y))\sigma(x,y)\lambda(xy)^{-1}. \tag{3.3b}$$

Two twisting pairs (α, σ) and (α', σ') are *exterior equivalent* if they are related by such a transformation, and there is a 1–1 correspondence between the covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$ and those of $(G, \mathcal{A}, \alpha', \sigma')$, via the adjustments $\theta_x \mapsto \lambda(x)\theta_x$.

One can verify that (3.3a)–(3.3b) defines an action of the group of Borel functions $\lambda : G \to \mathcal{UMA}$. Definition 3.1.3 generalises the familiar notion of projective equivalence of ordinary projective unitary group representations (i.e. $\mathcal{A} = \mathbb{C}$). However, it is important to consider α and σ *concurrently* when considering (exterior) equivalence under an adjustment $\theta_x \mapsto \lambda(x)\theta_x$.

Central cocycles and cocyle classes

If the cocycle σ is assumed to be central (e.g. for ordinary projective unitary representations, or when $\mathcal{A}$ is commutative), there is no effect of λ on α in (3.3a). The conjugation in (3.1a) and the condition (3.1d) are then redundant, and we also have $\alpha(x^{-1}) = \alpha(x)^{-1}$. The cocycles also gain some additional structure, for example, they form an abelian group[3] $\mathscr{M}(G)$ under pointwise operations on $G \times G$. A central cocycle is said to be *trivial* if there is a Borel function $\lambda : G \to \mathcal{Z}(\mathcal{UMA})$ such that $\sigma(x, y) = \lambda(x)\lambda(y)^x\lambda(xy)^{-1}$. Here, the notation $\mathcal{Z}(\cdot)$ indicates restriction to the central elements of $\mathcal{MA}$. A central cocycle σ is a 2-cocycle in the sense of cohomology; if we define the coboundary maps $\mathrm{d}^1, \mathrm{d}^2$ by

$$\begin{aligned} \mathrm{d}^1\lambda(x,y) &:= \lambda(x)\lambda(y)^x\lambda(xy)^{-1} \\ \mathrm{d}^2\sigma(x,y,z) &:= \sigma(xy,z)^{-1}\sigma(x,y)^{-1}\sigma(y,z)^x\sigma(x,yz), \end{aligned}$$

then $\mathrm{d}^2\sigma = 1$ as required, while a trivial cocycle is precisely a coboundary $\sigma = \mathrm{d}^1\lambda$. The set of trivial cocycles form a subgroup $\mathscr{M}_0(G)$ of $\mathscr{M}(G)$, and the quotient group $\mathscr{M}(G)/\mathscr{M}_0(G)$ is the set of cocycle classes for G. We denote $\mathscr{M}(G)/\mathscr{M}_0(G)$ by $H^2_\alpha(G, \mathcal{Z}(\mathcal{UMA}))$ in analogy to the usual group cohomology $H^2(G, \mathbb{T})$. Two central cocycles σ_1, σ_2 are *equivalent*, or in the same *cocycle class*, if $\sigma_1\sigma_2^{-1}$ is a trivial cocycle.

Let σ, σ' be two central cocycles from the same cocycle class $[\sigma]$. Two covariant representations $(\theta, \mathscr{H}), (\theta', \mathscr{H}')$, of $(G, \mathcal{A}, \alpha, \sigma)$ and $(G, \mathcal{A}, \alpha, \sigma')$ respectively, are said to be *projectively equivalent* if there exists a unitary $\mathcal{A}$-linear map $U : \mathscr{H} \to \mathscr{H}'$ such that $U\theta_xU^{-1} = \lambda(x)\theta'_x$ for some Borel function $\lambda : G \to \mathcal{Z}(\mathcal{UMA})$. If we fix a representative

[3]We suppress the dependence on the data $\mathcal{A}$ and α in $\mathscr{M}(G)$.

cocycle σ once and for all, projective equivalence requires λ to be a generalised character, $\lambda(x)\lambda(y)^x = \lambda(xy)$. Note that two projectively equivalent covariant representations (with the same representative σ) may not be equivalent in the stronger sense that $U\theta_x U^{-1} = \theta'_x$. The latter notion of non-projective equivalence, which we simply call *equivalence*, will be needed for generalised inducing later on (also see Section 1.3 of [81]), and is important in itself. A natural problem is to identify the classes of (central) cocycles for a given $(G, \mathcal{A}, \alpha)$, and then to classify the equivalence classes of covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$ for a σ in a cocycle class. More generally, we can look for exterior equivalence classes of twisting pairs, and classify the covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$, up to equivalence, for a (α, σ) in each class of twisting pairs.

3.1.1 Graded covariant representations

The definitions in Section 3.1 can be generalised slightly to accommodate a notion of graded covariant representations. Let $\mathcal{A}$ be a *graded* real or complex C^*-algebra (see Chapter 14 of [11]). This means that $\mathcal{A}$ has a direct sum decomposition into two self-adjoint closed subspaces $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, satisfying $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j\,(\mathrm{mod}\,2)}$. Let $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ now denote its group of *even* $\mathbb{F}$-linear $*$-automorphisms, i.e., $*$-automorphisms that preserve the decomposition $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$. We assume that the cocycles σ take values in the *even* elements $\mathcal{UMA}_0$ of $\mathcal{UMA}$. These restrictions are consistent with equations (3.1a)–(3.1b) for a twisting pair (α, σ), and in particular, the notion of exterior equivalence still makes sense.

Suppose that the group G is also equipped with a continuous homomorphism $c : G \to \{\pm 1\}$. The quintuple $(G, c, \mathcal{A}, \alpha, \sigma)$ is called a *graded twisted C^*-dynamical system.*

Definition 3.1.4 (Graded twisted covariant representation)**.** A *graded* covariant representation of a graded twisted C^*-dynamical system $(G, c, \mathcal{A}, \alpha, \sigma)$ is a non-degenerate graded $*$-representation of $\mathcal{A}$ on a graded Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ over $\mathbb{F}$ (i.e. $\mathcal{A}_i\mathscr{H}_j \subset \mathscr{H}_{i+j\,(\mathrm{mod}\,2)}$), along with a compatible Borel map $\theta : G \to \mathrm{U}(\mathscr{H})$, in the sense of (3.2a)–(3.2b), with the additional condition that θ_x is an even (resp. odd) operator if $c(x) = +1$ (resp. $c(x) = -1$).

Two graded covariant representations $(\theta, \mathscr{H}), (\theta', \mathscr{H}')$ for $(G, c, \mathcal{A}, \alpha, \sigma)$ are *graded equivalent*, or simply *equivalent*, if there is an *even* unitary $\mathcal{A}$-linear map $U : \mathscr{H} \to \mathscr{H}'$ intertwining θ with θ'. Note that graded equivalence is distinct from ordinary ungraded equivalence. For instance, a graded Hilbert space over $\mathbb{F}$ and its parity-reverse are equivalent as ungraded representations of $\mathbb{F}$, but are inequivalent in the graded sense unless the two homogeneous subspaces have the same dimension. When the cocycles σ are restricted to be central, we again have the weaker notion of projective graded equivalence, which only requires that

$U\theta_x U^{-1} = \lambda(x)\theta'_x$ for all $x \in G$, for some $\lambda : G \to \mathcal{Z}(\mathcal{UMA}_0)$. There is also a distinct notion of *super-equivalence* of graded covariant representations, which we will consider in Section 6.2.

In many of our applications, $\mathcal{A}$ is trivially graded, i.e. purely even, and the only complication comes from requiring G to be realised as even or odd operators according to c. In fact, when considering symmetry data (G, c, ϕ, σ), we will often take $\mathcal{A}$ to be the purely even algebra $\mathbb{C}$, regarded as a real algebra if ϕ is surjective and a complex algebra otherwise. Our definitions of graded covariant representations generalise natural ones that arise from physical considerations, as the following examples illustrate.

3.2 Examples

3.2.1 Projective unitary-antiunitary representations

Let $\phi : G \to \{\pm 1\}$ be a continuous homomorphism, and σ be a $\mathbb{T}$-valued 2-cocycle as in (1.1). A projective unitary-antiunitary representation[4] (PUA-rep) θ of (G, ϕ, σ) on a complex Hilbert space $(\mathscr{H}, h)$ is a map $x \mapsto \theta_x$ such that θ_x is a unitary (resp. antiunitary) operator on $(\mathscr{H}, h)$ if $\phi(x) = +1$ (resp. $\phi(x) = -1$), and $\theta_x\theta_y = \sigma(x, y)\theta_{xy}$. By regarding $\mathscr{H}$ as a real Hilbert space with real inner product $b = \mathrm{Re}(h)$, and i as an orthogonal complex structure J on $(\mathscr{H}, b)$, we can equivalently define a PUA-rep of (G, ϕ, σ) as a map θ from G to the orthogonal operators on $(\mathscr{H}, b)$, subject to

$$\begin{aligned} \theta_x\theta_y &= \sigma(x, y)\theta_{xy}, \\ \theta_x J &= \phi(x) J\theta_x \qquad x, y \in G. \end{aligned}$$

The complex Hilbert space $(\mathscr{H}, h)$ is recovered from $(\mathscr{H}, b)$ by taking J as the imaginary unit, and setting $h(u, v) = b(u, v) + \mathrm{i}b(Ju, v)$. Note that h and b induce the same norm on $\mathscr{H}$, and that a b-orthogonal operator is h-(anti)-unitary iff it (anti)-commutes with J.

Suppose ϕ is surjective, and let $\mathcal{A} = \mathbb{C}$ as a *real* C^*-algebra. Thus $\mathcal{A} = \mathbb{R} \oplus \mathrm{i}\mathbb{R}$ as a real vector space, with basis $\{1, \mathrm{i}\}$, $\mathrm{i}^2 = -1$, and $\mathrm{i}^* = -\mathrm{i}$. There are two elements of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$, namely complex conjugation K and the identity $\mathrm{id}_{\mathbb{C}}$. A $*$-representation of $\mathcal{A} = \mathbb{C}$ is a real Hilbert space $(\mathscr{H}, b)$ along with a real-linear operator J representing i, such that $J^2 = -1$ and $J^* = -J$, i.e. J is an orthogonal complex structure. Define the map $\alpha : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ by

$$\alpha_x \equiv \alpha(x) \coloneqq \begin{cases} \mathrm{id}_{\mathbb{C}} & \text{if } \phi(x) = +1, \\ K & \text{if } \phi(x) = -1. \end{cases} \tag{3.4}$$

[4] See [70, 97], as well as Appendix B.2, for some technicalities.

Equations (3.2a)–(3.2b) say that a covariant representation θ of $(G, \mathbb{C}, \alpha, \sigma)$ on a real Hilbert space $(\mathscr{H}, b)$, is precisely a PUA-rep of (G, ϕ, σ) on the complex Hilbert space $(\mathscr{H}, h)$. Two PUA-reps $\theta^{(i)}$ for $(G, \phi, \sigma^{(i)})$, are considered to be projectively equivalent if the representatives $\theta_x^{(i)}$ are related, up to a unitary, by multiplication by a phase factor $\lambda(x) \in \mathbb{T}$. Note that $\sigma^{(1)}$ and $\sigma^{(2)}$ are necessarily from the same cocycle class. Thus, our concern is with PUA-reps for cocycle classes $[\sigma] \in H^2_\alpha(G, \mathbb{T})$, and we may fix a convenient choice σ in such a class.

3.2.2 Gapped Hamiltonians and graded projective unitary-antiunitary representations

Following Section 2.4, as well as [30], we assume that a PUA-rep for (G, ϕ, σ) has, additionally, a gapped self-adjoint Hamiltonian H, and that G has a second continuous homomorphism $c : G \to \{\pm 1\}$ such that

$$\theta_x H = c(x) H \theta_x, \qquad \forall x \in G. \tag{3.5}$$

We say that such a gapped Hamiltonian H is compatible with the symmetries specified by the data (G, c, ϕ, σ). By functional calculus, we can continuously deform H to $\Gamma = \mathrm{sgn}(H)$, which has spectrum in $\{\pm 1\}$, while preserving the commutation/anticommutation relations with G as expressed in (3.5). Note that H is gapped, so that $\mathrm{sgn}(\cdot)$ is a continuous function on the spectrum of H homotopic to the identity function. We will not distinguish a (G, c, ϕ, σ)-compatible Hamiltonian with its spectrally-flattened version. As discussed in Section 2.4.1, this is one aspect of the notion of "homotopy" equivalence of symmetry-compatible Hamiltionians, first suggested by Kitaev [50].

We are thus led to study *graded* PUA-reps θ of (G, c, ϕ, σ) on a graded complex Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$, with self-adjoint grading operator Γ satisfying $\Gamma^2 = 1_{\mathscr{H}}$. The operators θ_x in a graded PUA-rep are required to be even or odd according to whether $c(x) = +1$ or -1. The grading operator Γ is interpreted as a compatible gapped Hamiltonian, or more correctly, a representative of a class of (G, c, ϕ, σ)-compatible Hamiltonians on $\mathscr{H}$, whose spectral flattenings are Γ. The notion of equivalence of graded PUA-reps is that of *graded* unitary equivalence, i.e., the intertwining unitary map U must be even.

Suppose ϕ is surjective. Let $\mathcal{A}$ be the purely even real $*$-algebra $\mathbb{C}$, and define the automorphisms α_x as in (3.4). Then each α_x is an even automorphism, the cocycle σ is even, and a graded covariant representation of $(G, c, \mathbb{C}, \alpha, \sigma)$ is precisely a graded PUA-rep of (G, c, ϕ, σ).

3.2.3 Symmetries in Nambu space

Recall the Nambu space description of the symmetries of free-fermion dynamics in Section 2.4. A Nambu space realisation of the symmetry data (G, c, ϕ, σ) has an equivalent description as a covariant representation. Let $\mathcal{A}$ be the real C^*-algebra $Cl_{1,1} \otimes Cl_{1,0} \cong Cl_{2,1} \cong M_2(\mathbb{C})$, generated by i, e and f, with $\mathrm{i}^2 = -1 = f^2$, $e^2 = 1, \mathrm{i}^* = -\mathrm{i}, f^* = -f, e^* = e$, and the commutation relations $\mathrm{i}e = -e\mathrm{i}, \mathrm{i}f = f\mathrm{i}, ef = fe$. In a covariant representation for $(G, \mathcal{A}, \alpha, \sigma)$, the algebra element i plays the role of the imaginary unit, e is represented by complex conjugation, and f is the operator J corresponding to a flattened Hamiltonian. The automorphisms α_x of $\mathcal{A}$ take i to $\pm\mathrm{i}$ and f to $\pm f$ according to whether $\phi(x) = \pm 1$ and $\tau(x) \equiv c(x)\phi(x) = \pm 1$ respectively. Equations (3.2a)–(3.2b) then say that a covariant representation θ of $(G, c, M_2(\mathbb{C}), \alpha, \sigma)$ is a representation of (G, c, ϕ, σ) on a Nambu space, as symmetries of a gapped free-fermion Hamiltonian. Note that a finite-dimensional Nambu space is necessarily even-dimensional.

3.3 Twisted crossed products and covariant representations

The symmetry data (G, ϕ, σ), or more generally, a twisted C^*-dynamical system $(G, \mathcal{A}, \alpha, \sigma)$, is associated with a twisted crossed product C^*-algebra $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$. This is a C^*-algebra constructed in such way that its ordinary representation theory coincides with the covariant representation theory of $(G, \mathcal{A}, \alpha, \sigma)$. Let $L^1(G, \mathcal{A}, \alpha, \sigma)$ be the Banach $*$-algebra of integrable functions[5] $F : G \to \mathcal{A}$ with the L^1-norm $\|F\|_1 = \int_G \|F(x)\| \,\mathrm{d}x$, which is equipped with a (α, σ)-twisted convolution product $\star$ and involution $*$,

$$(F_1 \star F_2)(y) := \int_G F_1(x) \left(F_2(x^{-1}y)\right)^x \sigma(x, x^{-1}y) \,\mathrm{d}x, \tag{3.6a}$$

$$F^*(x) := \sigma(x, x^{-1})^* \left(F(x^{-1})^*\right)^x \Delta(x^{-1}), \tag{3.6b}$$

where Δ is the modular function on G. The rather messy verification that $F^{**} = F$ and $(F_1 \star F_2)^* = F_2^* \star F_1^*$ can be found in Section 6 of [13]. There is a 1–1 correspondence between covariant representations θ of $(G, \mathcal{A}, \alpha, \sigma)$, and non-degenerate $*$-representations of $L^1(G, \mathcal{A}, \alpha, \sigma)$, given by taking the "integrated form" $\tilde{\theta}$ of θ,

$$\tilde{\theta}(F) := \int_G F(x)\theta_x \,\mathrm{d}x, \qquad F \in L^1(G, \mathcal{A}, \alpha, \sigma),$$

see Theorem 3.3 of [13] and Remark 2.6 of [66]. A pre-C^*-norm is defined on $L^1(G, \mathcal{A}, \alpha, \sigma)$ by

$$\begin{aligned} \|F\|_{\max} &= \sup\{\|\pi(F)\| \,:\, \pi \text{ is a non-degenerate } *\text{-representation of } L^1(G, \mathcal{A}, \alpha, \sigma)\} \\ &= \sup\{\|\tilde{\theta}(F)\| \,:\, \theta \text{ is a covariant representation of } (G, \mathcal{A}, \alpha, \sigma)\}. \end{aligned}$$

[5] The integral of a $\mathcal{A}$-valued function on G is a Bochner integral with respect to μ, for which a detailed discussion can be found in Appendix B of [97], and a shorter version in the preliminary section of [66].

Definition 3.3.1 (Twisted crossed product C^*-algebra [13]). Let $(G, \mathcal{A}, \alpha, \sigma)$ be a twisted dynamical system. The *twisted crossed product C^*-algebra* associated with $(G, \mathcal{A}, \alpha, \sigma)$, denoted by $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$, is defined to be the completion of $L^1(G, \mathcal{A}, \alpha, \sigma)$ in the norm $\|\cdot\|_{\max}$.

When $\sigma \equiv 1$, we recover the untwisted crossed product C^*-algebra associated with the untwisted C^*-dynamical system $(G, \mathcal{A}, \alpha)$. If $\alpha \equiv 1$, we call $\mathbb{R} \rtimes_{(1,\sigma)} G$ (resp. $\mathbb{C} \rtimes_{(1,\sigma)} G$) the real (resp. complex) *twisted group C^*-algebra* of (G, σ). If $\sigma \equiv 1$ as well, we use a shortened notation[6] $\mathbb{R} \rtimes G$ (resp. $\mathbb{C} \rtimes G$) for the real (resp. complex) *group C^*-algebra* of G. More generally, when $\alpha \equiv 1$ and $\sigma \equiv 1$, we will write $\mathcal{A} \rtimes G := \mathcal{A} \rtimes_{(1,1)} G$ to ease notation.

The group G is embedded, not necessarily homomorphically, in the multiplier algebra of the twisted crossed product via a Borel map $j_G : G \to \mathcal{UM}(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$; the multipliers $j_G(g)$ are defined on functions $F \in L^1(G, \mathcal{A}, \alpha, \sigma)$ by

$$(j_G(g)(F))(x) := (F(g^{-1}x))^g \sigma(g, g^{-1}x), \qquad x \in G.$$

Likewise, the algebra $\mathcal{A}$ is embedded in $\mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$ through the $*$-homomorphism $j_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$, defined by

$$(j_{\mathcal{A}}(a)(F))(x) := a(F(x)), \qquad F \in L^1(G, \mathcal{A}, \alpha, \sigma), x \in G.$$

In Definition 2.4 of [66], the authors defined a twisted crossed product for a twisted dynamical system $(G, \mathcal{A}, \alpha, \sigma)$ abstractly, as a C^*-algebra $\mathcal{B}$, along with a non-degenerate $*$-homomorphism[7] $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{MB}$ and a Borel map $i_G : G \to \mathcal{UMB}$, such that

1. $(i_{\mathcal{A}}, i_G)$ is covariant, i.e.,

$$\begin{aligned} i_{\mathcal{A}}(a^x) &= i_G(x) i_{\mathcal{A}}(a) i_G(x)^{-1}, && (3.7a) \\ i_G(x) i_G(y) &= i_{\mathcal{A}}(\sigma(x,y)) i_G(xy), && (3.7b) \end{aligned}$$

2. for every covariant representation θ of $(G, \mathcal{A}, \alpha, \sigma)$ on a Hilbert space $\mathscr{H}$, there is a (non-degenerate) $*$-representation $\tilde{\theta}$ of $\mathcal{B}$ on $\mathscr{H}$ such that $a(v) = (\tilde{\theta} \circ i_{\mathcal{A}}(a))(v)$ and $\theta_x(v) = (\tilde{\theta} \circ i_G(x))(v)$ for all $a \in \mathcal{A}$, $x \in G$, $v \in \mathscr{H}$; the representation $\tilde{\theta}$ is called the *integrated* form of θ,

3. for every function $F \in L^1(G, \mathcal{A}, \alpha, \sigma)$, the integral $(i_{\mathcal{A}} \times i_G)(F) := \int_G i_{\mathcal{A}}(F(x)) i_G(x)\, \mathrm{d}x$ is an element of $\mathcal{B}$, and the image of $(i_{\mathcal{A}} \times i_G)$ is a dense subspace of $\mathcal{B}$.

[6] Unfortunately, it is also standard to write $\mathbb{R} \rtimes G$ for the semidirect product of the *group* $\mathbb{R}$ with G. Nevertheless, the correct meaning should be clear from the context.

[7] A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{MB}$ is *non-degenerate* if $\pi(\mathcal{A})\mathcal{B}$ is dense in $\mathcal{B}$. This is equivalent to the statement that $\pi(e_\lambda) \to 1$ strictly for some approximate unit (e_λ) for $\mathcal{A}$. In this case, there is a unique extension of π to a $*$-homomorphism from $\mathcal{MA}$ to $\mathcal{MB}$. See, e.g., II.7.3 of [12] and Section 1 of [66].

The covariant representation theory of $(G, \mathcal{A}, \alpha, \sigma)$ is thus the same as the representation theory of a twisted crossed product $\mathcal{B}$ for $(G, \mathcal{A}, \alpha, \sigma)$. As pointed out in Proposition 2.7 of [66], a twisted crossed product $(\mathcal{B}, i_{\mathcal{A}}, i_G)$ is unique up to isomorphisms which respect the embedding maps. In more detail, if $(\mathcal{C}, i'_{\mathcal{A}}, i'_G)$ is another twisted crossed product for $(G, \mathcal{A}, \alpha, \sigma)$, then there is an isomorphism $\xi : \mathcal{B} \to \mathcal{C}$ such that $\xi \circ i_{\mathcal{A}} = i'_{\mathcal{A}}$ and $\xi \circ i_G = i'_G$.

In fact, there is a stronger universal property [67] which characterises the twisted crossed product $(\mathcal{B}, i_{\mathcal{A}}, i_G)$ for $(G, \mathcal{A}, \alpha, \sigma)$. The authors define a *covariant homomorphism* of $(G, \mathcal{A}, \alpha, \sigma)$ into the multiplier algebra $\mathcal{MC}$ of a C^*-algebra $\mathcal{C}$ to be a pair (η, ν), where $\eta : \mathcal{A} \to \mathcal{MC}$ is a non-degenerate $*$-homomorphism, and $\nu : G \to \mathcal{UMC}$ is a Borel map, satisfying

1. $\nu_x \nu_y = \eta(\sigma(x, y))\nu_{xy}, \;\; x, y \in G,$

2. $\eta(a^x) = \nu_x \eta(a) \nu_x^{-1}, \;\; x \in G, a \in \mathcal{A}.$

For example, if $(\mathcal{B}, i_{\mathcal{A}}, i_G)$ is a twisted crossed product for $(G, \mathcal{A}, \alpha, \sigma)$, then the pair $(i_{\mathcal{A}}, i_G)$ is a covariant homomorphism into $\mathcal{M}(\mathcal{B})$. Also, a covariant representation θ of $(G, \mathcal{A}, \alpha, \sigma)$ on a Hilbert space $\mathscr{H}$ defines a covariant homomorphism (η, θ) into $\mathcal{L}(\mathscr{H}) = \mathcal{M}(\mathcal{K}(\mathscr{H}))$, where η is the $\mathcal{A}$-action on $\mathscr{H}$. Theorem 1.2 of [67] says that a twisted crossed product $(\mathcal{B}, i_{\mathcal{A}}, i_G)$ has the following property (call it Property A): every covariant homomorphism (ν, η) into $\mathcal{MC}$ gives rise to a unique non-degenerate homomorphism $(\eta \times \nu)$ of the twisted crossed product $\mathcal{B}$ into $\mathcal{MC}$, such that $\eta = (\eta \times \nu) \circ i_{\mathcal{A}}$ and $\nu = (\eta \times \nu) \circ i_G$. Furthermore, if $\mathcal{B}'$ is another C^*-algebra, $(i'_{\mathcal{A}}, i'_G)$ is a covariant homomorphism of $(G, \mathcal{A}, \alpha, \sigma)$ into $\mathcal{MB}'$, and $(\mathcal{B}', i'_{\mathcal{A}}, i'_G)$ has Property A, then there is an isomorphism $\xi : \mathcal{B}' \to \mathcal{B}$ such that $\xi \circ i'_{\mathcal{A}} = i_{\mathcal{A}}$ and $\xi \circ i'_G = i_G$. Therefore, a twisted crossed product for $(G, \mathcal{A}, \alpha, \sigma)$ can alternatively be defined as a triple $(\mathcal{B}, i_{\mathcal{A}}, i_G)$ comprising a C^*-algebra $\mathcal{B}$ and a covariant homomorphism $(i_{\mathcal{A}}, i_G)$ into $\mathcal{MB}$, which is an initial object for such triples.

For our purposes, we do not need the details of these universal characterisations of the twisted crossed product, but only the following important property of $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$:

Proposition 3.3.2 ([13, 66, 67]). *The algebra $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$, along with the maps $(j_{\mathcal{A}}, j_G)$, is a twisted crossed product for $(G, \mathcal{A}, \alpha, \sigma)$ in the sense of [66]. There is a one-to-one correspondence between the covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$ and the non-degenerate $*$-representations of $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$.*

This link between $*$-representations of $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ and covariant representations θ of $(G, \mathcal{A}, \alpha, \sigma)$ is analogous to the link between representations for the group algebra of a (finite) group and representations of the group. We carry out a short calculation to illustrate the similarity. We assume, for now, that G is discrete, and use the symbol δ_g to denote the element in $\mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$ whose value is 1 on g and 0 otherwise[8]. Taking $\mathrm{d}x$ to be the counting measure, the computation

$$\begin{aligned}(\delta_g \star \delta_h)(y) &= \int_G \delta_g(x) \left(\delta_h(x^{-1}y)\right)^x \sigma(x, x^{-1}y)\, \mathrm{d}x \\ &= \delta_h(g^{-1}y)\sigma(g, g^{-1}y) \\ &= \left(\delta_{gh}\sigma(g,h)\right)(y), \end{aligned} \tag{3.8}$$

says that the representatives of δ_g in a $*$-representation of $\mathcal{A}\rtimes_{(\alpha,\sigma)}G$ verifies (3.2a). Similarly,

$$\begin{aligned}(\delta_g \star a\delta_e)(y) &= \int_G \delta_g(x) \left(a\delta_e(x^{-1}y)\right)^x \sigma(x, x^{-1}y)\, \mathrm{d}x \\ &= \delta_g(y)a^y\sigma(y,e) \\ &= a^g\delta_g(y)\sigma(e,y) \\ &= \int_G a^g\delta_e(x) \left(\delta_g(x^{-1}y)\right)^x \sigma(x, x^{-1}y)\, \mathrm{d}x \\ &= \left(a^g\delta_e \star \delta_g\right)(y)\end{aligned}$$

verifies (3.2b). We also find that the algebra $\mathcal{A}$ is embedded in $\mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$ via $a \mapsto a\delta_e$, or more correctly, through the homomorphism $a \mapsto j_{\mathcal{A}}(a)$.

Finally, let us recall the action (3.3a)–(3.3b) of a Borel map $\lambda : G \to \mathcal{UMA}$ implementing exterior equivalence between two twisting pairs (α,σ) and (α',σ'). Then Lemma 3.3 of [66] says that there is a "projective isomorphism" between the twisted crossed products $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ and $\mathcal{A} \rtimes_{(\alpha',\sigma')} G$, i.e., there is an isomorphism $\xi : \mathcal{A} \rtimes_{(\alpha,\sigma)} G \to \mathcal{A} \rtimes_{(\alpha',\sigma')} G$, such that $\xi \circ j_{\mathcal{A}} = j_{\mathcal{A}}$ and $\xi \circ j_G(x) = j_{\mathcal{A}}(\lambda(x)^{-1}) j_G(s)$. Here, the maps $j_{\mathcal{A}}, j_G$ refer to the standard embeddings of $\mathcal{A}$ and G inside the crossed products. Writing $j'_G(s) = j_{\mathcal{A}}(\lambda(x)^{-1}) j_G(x)$, this says that $(\mathcal{A} \rtimes_{(\alpha',\sigma')} G, j_{\mathcal{A}}, j'_G)$ is another twisted crossed product for $(G, \mathcal{A}, \alpha, \sigma)$. It follows that if θ is a covariant representation of $(G, \mathcal{A}, \alpha, \sigma)$, then $\theta' : x \mapsto \lambda(x)\theta_x$ is a covariant representation for $(G, \mathcal{A}, \alpha', \sigma')$. Thus the covariant representations of two twisted dynamical systems with exterior equivalent twisting pairs are "projectively" related.

3.3.1 Graded twisted crossed products

One can also define the *graded* twisted crossed product C^*-algebra associated with a graded twisted C^*-dynamical system $(G, c, \mathcal{A}, \alpha, \sigma)$, by assigning an appropriate grading to the ungraded twisted crossed product associated with $(G, \mathcal{A}, \alpha, \sigma)$. Let $G_0 := \ker(c)$ and $G_1 := G - G_0$. We define the even subalgebra of $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ to be the completion in $\|\cdot\|_{\max}$ of $L^1(G_0, \mathcal{A}_0, \alpha, \sigma) \oplus L^1(G_1, \mathcal{A}_1, \alpha, \sigma)$, and the odd subspace to be the completion of $L^1(G_0, \mathcal{A}_1, \alpha, \sigma) \oplus L^1(G_1, \mathcal{A}_0, \alpha, \sigma)$. Note that (3.6a) and (3.6b) respect this grading due to

[8] For non-discrete G, the function δ_g is not actually an element of $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$, and we should use $j_G(g)$ instead of δ_g.

the restriction to even automorphisms α_x and even cocycles σ. A graded $*$-representation of the graded twisted crossed product $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ then corresponds to a graded covariant representation of $(G, c, \mathcal{A}, \alpha, \sigma)$.

Often, $\mathcal{A}$ is a purely even algebra, in which case we can identify the even subalgebra of $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ with $\mathcal{A} \rtimes_{(\alpha,\sigma)} G_0$. If $c \equiv 1$ as well, then the graded twisted crossed product is purely even. In this case, an ungraded $*$-representation for $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ on $\mathscr{H}$ can be promoted to a graded $*$-representation, by giving $\mathscr{H}$ either the even or odd grading, and we *do* distinguish between these two options as *graded* $*$-representations of $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$.

Chapter 4

Induced twisted covariant representations

The fundamental work of Mackey showed how the unitary representations of a large class of groups G can be obtained by first choosing an appropriate normal subgroup N whose representation theory is known, and then applying a procedure which is now called the "Mackey machine"; see the original works of Mackey [57, 58, 59], as well as the expository articles [76, 91] for details and applications to quantum physics. A central construction is that of an *induced representation*, first described for unitary representations of locally compact groups in [57], then generalised to projective unitary representations in [58]. If the (generalised) dual of N and its canonical G-action are nice enough, then all irreducible projective unitary representations of G may be obtained through Mackey's theory. Mackey's ideas proved to be very general, and lent themselves to significant further generalisation. For instance, the inducing process for projective unitary-antiunitary representations was made in [70, 81]. A different generalisation was made for covariant representations of a dynamical system $(G, \mathcal{A}, \alpha)$, with $\mathcal{A}$ a complex C^*-algebra, by Takesaki [85], and extended by Green [32]. However, note that the twisted crossed products, or *twisted covariance algebras*, of Green are not the same as the Leptin–Busby–Smith twisted crossed products which we had introduced in Chapter 3. The latter version of twisted crossed products is formally more general, although the analogue of the Mackey machine is better developed for Green's version. The relationship between the two notions of twisted crossed products is explained in [66, 74].

Using the constructions in the above references as a guide, we generalise the inducing procedure to twisted covariant representations of a twisted dynamical system $(G, \mathcal{A}, \alpha, \sigma)$, where $\mathcal{A}$ may be a real C^*-algebra. Graded twisted covariant representations can be similarly induced. In order for the full Mackey machinery to work, various measure-theoretic conditions need to be met, and we have not worked out what these are in the general case.

Therefore, we focus on the algebraic aspects of a generalised Mackey machine for twisted dynamical systems; *no rigorous theorems are stated or proven in this regard.* Nevertheless, the specific twisted dynamical systems $(G, \mathcal{A}, \alpha, \sigma)$ which we are interested in, typically have $\ker(\alpha)$ being a subgroup of G with finite index, as well as $\mathbb{T}$-valued cocycles. Taking $N = \ker(\alpha)$ and assuming that the covariant representations of $(N, \mathcal{A}, 1, \sigma)$ are already known (e.g. using one of the available generalisations of Mackey's theory), we do not imagine that the complications arising from the finite G/N and the twisting pair (α, σ) will violate these conditions in a serious way.

4.1 Generalised inducing — the ungraded case

Let G be a locally compact, second countable group, which we assume to be unimodular for simplicity of notation, and let $(G, \mathcal{A}, \alpha, \sigma)$ be a twisted dynamical system as in Section 3.1. Let H be a closed subgroup of G, and θ be a covariant representation for $(H, \mathcal{A}, \alpha, \sigma)$ on a Hilbert space M over $\mathbb{F}$, where (α, σ) are understood to be restricted to H. The induced covariant representation of $(G, \mathcal{A}, \alpha, \sigma)$, which we denote by $\Theta \equiv \theta \uparrow G$, is constructed on a H-equivariant subspace of Borel functions $\psi : G \to M$, where equivariance means

$$\psi(g) = \sigma(h, h^{-1}g)^{-1}\theta_h\psi(h^{-1}g), \quad \forall g \in G, h \in H. \tag{4.1}$$

We check that the cocycles work out consistently when we apply the equivariance condition repeatedly,

$$\begin{aligned}
\psi(g) &= \sigma(h_1, h_1^{-1}g)^{-1}\theta_{h_1}\psi(h_1^{-1}g) \\
&= \sigma(h_1, h_1^{-1}g)^{-1}(\sigma(h_2, h_2^{-1}h_1^{-1}g)^{-1})^{h_1}\theta_{h_1}\theta_{h_2}\psi(h_2^{-1}h_1^{-1}g) \\
&= \big(\sigma(h_1, h_2)\sigma(h_1h_2, (h_1h_2)^{-1}g)\big)^{-1}\sigma(h_1, h_2)\theta_{h_1h_2}\psi((h_1h_2)^{-1}g) \\
&= \sigma(h_1h_2, (h_1h_2)^{-1}g)^{-1}\theta_{h_1h_2}\psi((h_1h_2)^{-1}g),
\end{aligned}$$

where we have used the cocycle condition (3.1b) in the third equality. Note that $g \mapsto \|\psi(g)\|_M^2$ is constant on the right cosets p in $H\backslash G$, and may be regarded as a function on the cosets. Following Mackey [57, 58], a quasi-invariant measure μ can be defined on $H\backslash G$. Identifying functions which are equal almost everywhere, the set of ψ satisfying (4.1), as well as

$$\|\psi\|^2 := \int_{H\backslash G} \|\psi(g)\|_M^2 \,\mathrm{d}\mu < \infty,$$

forms a Hilbert space M_H^G.

Definition 4.1.1 (Induced covariant representation)**.** Let $(G, \mathcal{A}, \alpha, \sigma)$ be a twisted dynamical system, H be a closed subgroup of G, and θ be a covariant representation for $(H, \mathcal{A}, \alpha, \sigma)$

on a Hilbert space M. The *induced covariant representation* $\theta \uparrow G \equiv \Theta$ for $(G, \mathcal{A}, \alpha, \sigma)$ is defined on the Hilbert space M_H^G constructed above, and the actions of $a \in \mathcal{A}$ and $x \in G$ on M_H^G are defined to be

$$\begin{aligned} (a\psi)(g) &\coloneqq a^g \psi(g), && (4.2a) \\ (\Theta_x \psi)(g) &\coloneqq \sigma(g,x)\psi(gx). && (4.2b) \end{aligned}$$

We check that $a\psi$ and $\Theta_x\psi$ remain H-equivariant:

$$\begin{aligned} (a\psi)(g) = a^g\psi(g) &= a^g \sigma(h, h^{-1}g)^{-1}\theta_h \psi(h^{-1}g) \\ &= \sigma(h, h^{-1}g)^{-1}((a^g)^{h^{-1}})^h \theta_h \psi(h^{-1}g) \\ &= \sigma(h, h^{-1}g)^{-1}\theta_h a^{h^{-1}g}\psi(h^{-1}g) \\ &= \sigma(h, h^{-1}g)^{-1}\theta_h (a\psi)(h^{-1}g), \end{aligned}$$

and

$$\begin{aligned} (\Theta_x\psi)(g) = \sigma(g,x)\psi(gx) &= \sigma(g,x)\sigma(h, h^{-1}gx)^{-1}\theta_h\psi(h^{-1}gx) \\ &= \sigma(h, h^{-1}g)^{-1}\sigma(h^{-1}g, x)^h \theta_h \psi(h^{-1}gx) \\ &= \sigma(h, h^{-1}g)^{-1}\theta_h \sigma(h^{-1}g, x)\psi(h^{-1}gx) \\ &= \sigma(h, h^{-1}g)^{-1}\theta_h (\Theta_x\psi)(h^{-1}g). \end{aligned}$$

In order for Θ_x to be unitary, a (real) factor $\sqrt{\varrho_x(g)}$ should be included, where $g \mapsto \varrho_x(g)$ is the Radon–Nikodym derivative of the x translate of μ with respect to μ [57, 58], but we omit these factors to reduce clutter. It is easy to see that (4.2a) is $*$-preserving, and that $(a_1a_2)\psi = a_1(a_2\psi)$. The computation

$$\begin{aligned} (\Theta_x\Theta_y\psi)(g) &= \sigma(g,x)(\Theta_y\psi)(gx) \\ &= \sigma(g,x)\sigma(gx,y)\psi(gxy) \\ &= \sigma(x,y)^g\sigma(g,xy)\psi(gxy) \\ &= \sigma(x,y)^g(\Theta_{xy}\psi)(g) \\ &= (\sigma(x,y)\Theta_{xy}\psi)(g) \quad \text{see (4.2a)}, \end{aligned}$$

verifies that $\Theta_x\Theta_y = \sigma(x,y)\Theta_{xy}$ as is required for a covariant representation with cocycle σ. Finally, Θ_x has the correct commutation relation with $a \in \mathcal{A}$, since

$$\begin{aligned} (\Theta_x a\psi)(g) &= \sigma(g,x)(a\psi)(gx) \\ &= \sigma(g,x)a^{gx}\psi(gx) \\ &= (a^x)^g\,\sigma(g,x)\psi(gx) \\ &= (a^x)^g\,(\Theta_x\psi)(g) \\ &= (a^x\Theta_x\psi)(g), \end{aligned}$$

which says that $\Theta_x a = a^x \Theta_x$, in accordance with (3.2b).

4.1.1 Induced covariant representation space as space of functions on cosets

We will sometimes call a $*$-representation space for $\mathcal{A}$ an $\mathcal{A}$-module for brevity. Given any $*$-automorphism $\beta \in \mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$, we may regard any $*$-representation space M for $\mathcal{A}$ as a $*$-representation space for the algebra $\beta\mathcal{A} \coloneqq \beta(\mathcal{A})$, by replacing multiplication by a with multiplication by $\beta(a)$. We denote the identity map on M, regarded as a map from an $\mathcal{A}$-module to a $(\beta\mathcal{A})$-module, by $\mathcal{K}_\beta$. Thus, $\mathcal{K}_\beta(am) \equiv \beta(a)(\mathcal{K}_\beta(m))$ for all $a \in \mathcal{A}, m \in M$. For automorphisms $\beta = \alpha_x$, $x \in G$, we replace the β subscript with x, so for example, $\mathcal{K}_x(am) \equiv a^x(\mathcal{K}_x(m))$. *We will now assume that the cocycles σ are central.* Then α is a homomorphism, so that $\alpha_{x^{-1}} = \alpha_x^{-1} \Rightarrow (a^x)^{x^{-1}} = a \;\; \forall a \in \mathcal{A}$, and $\mathcal{K}_{x^{-1}} = \mathcal{K}_x^{-1}$.

Example 4.1.2. Consider the setting of PUA-reps in Section 3.2.1, with $\mathcal{A} = \mathbb{C}$. Let $x \in G$ be a group element such that $\phi(x) = -1$. Then $\mathcal{K}_x : m \mapsto \overline{m}$ turns a $\mathbb{C}$-module M into its conjugate $\alpha_x(\mathbb{C}) = \overline{\mathbb{C}}$-module $\overline{M}$. Multiplication by $\lambda \in \mathbb{C}$ on m turns into multiplication by $\overline{\lambda}$ on $\overline{m}$, i.e. $\overline{\lambda m} = \overline{\lambda}\overline{m}$.

We can view the set of H-equivariant functions $\psi : G \to M$ as functions $\varphi : H\backslash G \to M$, as follows. Choose a Borel section[1] $s : H\backslash G \ni p \mapsto s_p \in G$ for the quotient map $q : G \to H\backslash G$, with $s(He) = e$. Then the correspondence is

$$\varphi(p) \;=\; \psi(s_p), \tag{4.3a}$$
$$\psi(x) = \psi(hs_p) \;=\; \sigma(h, s_p)^{-1}\theta_h\varphi(p). \tag{4.3b}$$

Note that every $x \in G$ can be written uniquely as hs_p, with $p = q(x)$ and $h = xs_p^{-1} \in H$, so (4.3b) specifies ψ from φ completely. There is a right action of $x \in G$ on the cosets, given by $p \cdot x \coloneqq q(s_p x)$. The translation of the induced action of $x \in G$ on ψ into an action on the coset functions φ, which we also denote by Θ_x, is

$$(\Theta_x\varphi)(p) \;\coloneqq\; [\sigma(s_p, x)/\sigma(h, s_{p\cdot x})]\theta_{h_x}\varphi(p \cdot x), \tag{4.4}$$

where $h_x \coloneqq s_p x s_{p\cdot x}^{-1}$. The transformation law for φ is somewhat more complicated than that for ψ. The $\mathcal{A}$-action is similarly translated,

$$(a \cdot \varphi)(p) \coloneqq a^{s_p}\left(\varphi(p)\right).$$

Heuristically, we can imagine the induced representation space as a direct integral of Hilbert spaces over the coset space $H\backslash G$, each isomorphic (as a Hilbert space, but not

[1] Such sections exist by Lemma 1.1 of [57].

necessarily as a $\mathcal{A}$-module) to M,

$$\mathscr{H} = \int_{p\in H\backslash G}^{\oplus} \mathscr{H}_p, \qquad \mathscr{H}_p = \mathcal{K}_{s_p} M. \tag{4.5}$$

The operator Θ_x can then be thought of as a direct integral of operators labelled by the cosets p,

$$\Theta_x = \int_{p\in H\backslash G}^{\oplus} \Theta_x^{(p)}, \qquad [\sigma(s_p, x)/\sigma(h, s_{p\cdot x})]\theta_{h_x} =: \Theta_x^{(p)} : \mathscr{H}_{p\cdot x} \to \mathscr{H}_p. \tag{4.6}$$

A different choice of section $s'_p = h_p s_p$, $h_p \in H$ might change the $\mathcal{A}$-module structure on each $\mathscr{H}_p$, as well as $\Theta_x^{(p)}$, but always in such a way as to preserve the relation $\Theta_x^{(p)} a = a^x \Theta_x^{(p)}$. Furthermore, the induced representation Θ' determined by s' is equivalent to the induced representation Θ determined by s, the intertwiner being $\int_p^{\oplus} \mathcal{K}_{h_p}^{-1} \theta_{h_p}$.

4.2 A sketch of a generalised Mackey machine

According to Mackey, the irreducible unitary representations of a large class of groups G can all be constructed by first fixing a suitable normal subgroup N, and then inducing up from irreducible representations of the "little groups" corresponding to the irreducible representations of N. We would like to generalise this procedure to twisted covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$. The generalisation of the usual induced representation construction has already been covered in Section 4.1. We proceed to generalise the notion of "little groups".

The action of a group on the generalised dual of a normal subgroup

The *generalised dual* $\hat{G}^\sigma$ is defined to be the set of equivalence classes of irreducible covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$. Similarly, for a closed normal subgroup $N \subset G$, $\hat{N}^\sigma$ is the generalised dual of $(N, \mathcal{A}, \alpha, \sigma)$ where (α, σ) are understood to be restricted to N. Note that we have left the dependence on $\mathcal{A}$ and α implicit in the notation $\hat{G}^\sigma$.

In order for the algebra to work out properly, *we will assume that σ is central, and that $\sigma(x, y)^n = \sigma(x, y)$ for all $n \in N, x, y \in G$.* As a preliminary calculation, we state the following identity,

$$\begin{aligned} \sigma(x,n)^{-1}\Theta_x\Theta_n &= \Theta_{xn} = \sigma(xnx^{-1},x)^{-1}\Theta_{xnx^{-1}}\Theta_x \\ \iff \quad \Theta_x\Theta_n\Theta_x^{-1} &= [\sigma(x,n)/\sigma(xnx^{-1},x)]\Theta_{xnx^{-1}}, \end{aligned} \tag{4.7}$$

which is required to hold for any $\Theta \in \hat{G}^\sigma$. When $\sigma = 1$ and $\alpha = 1$, the right action of $x \in G$ on $\vartheta \in \hat{N}$ is usually defined by conjugation as $(\vartheta \cdot x)_n = \vartheta_x \vartheta_n \vartheta_x^{-1} = \vartheta_{xnx^{-1}}$. Allowing for non-trivial (α,σ), and using (4.7) as a guide, we define for $\vartheta \in \hat{N}^\sigma$ the *conjugate covariant representation* $\vartheta \cdot x$ on the $(\alpha_x^{-1}\mathcal{A})$-module $\mathcal{K}_x^{-1}(M)$ by

$$\begin{aligned} (\vartheta \cdot x)_n &:= \mathcal{K}_x^{-1}[\sigma(x,n)/\sigma(xnx^{-1},x)]\vartheta_{xnx^{-1}}\mathcal{K}_x && \text{(4.8a)} \\ &= \mathcal{K}_x^{-1}\vartheta_x\vartheta_n\vartheta_x^{-1}\mathcal{K}_x. && \text{(4.8b)} \end{aligned}$$

It is clear that $(\vartheta \cdot x)_n(am) = a^n(\vartheta \cdot x)_n(m)$. Also, this definition is well-defined on the equivalence classes in $\hat{N}^\sigma$, since if $\vartheta' \simeq \vartheta$ via U, i.e., $\vartheta'_n = U\vartheta_n U^{-1}$, then $\vartheta' \cdot x \simeq \vartheta \cdot x$ via $\mathcal{K}_x^{-1}U\mathcal{K}_x$.

One can perform a direct, albeit slightly tedious computation to check that the cocycles work out properly, i.e. $(\vartheta \cdot x)_{n_1}(\vartheta \cdot x)_{n_2} = \sigma(n_1,n_2)(\vartheta \cdot x)_{n_1 n_2}$, so that $x \cdot \vartheta$ is indeed a covariant representation of $(N,\mathcal{A},\alpha,\sigma)$, and that $(\vartheta \cdot x) \cdot y = \vartheta \cdot (xy)$. In this manner, $\hat{N}^\sigma$ becomes a G-space[2]. Alternatively, using (4.8b), it follows quickly that

$$\begin{aligned} (\vartheta \cdot x)_{n_1}(\vartheta \cdot x)_{n_2} &= \mathcal{K}_x^{-1}\vartheta_x\vartheta_{n_1}\vartheta_x^{-1}\mathcal{K}_x^{-1}\mathcal{K}_x\vartheta_x\vartheta_{n_2}\vartheta_x^{-1}\mathcal{K}_x \\ &= \mathcal{K}_x^{-1}\vartheta_x\sigma(n_1,n_2)\vartheta_{n_1n_2}\vartheta_x^{-1}\mathcal{K}_x \\ &= \sigma(n_1,n_2)^{x^{-1}x}\mathcal{K}_x^{-1}\vartheta_x\vartheta_{n_1n_2}\vartheta_x^{-1}\mathcal{K}_x && \text{(4.9)} \\ &= \sigma(n_1,n_2)(\vartheta \cdot x)_{n_1n_2}, \end{aligned}$$

and that

$$\begin{aligned} ((\vartheta \cdot x)\cdot y)_n &= \mathcal{K}_y^{-1}(\vartheta \cdot x)_y(\vartheta \cdot x)_n(\vartheta \cdot x)_y^{-1}\mathcal{K}_y \\ &= \mathcal{K}_y^{-1}\mathcal{K}_x^{-1}\vartheta_x\vartheta_y\vartheta_n\vartheta_y^{-1}\vartheta_x^{-1}\mathcal{K}_x\mathcal{K}_y \\ &= \mathcal{K}_{xy}^{-1}\sigma(x,y)\vartheta_{xy}\vartheta_n\vartheta_{xy}^{-1}\sigma(x,y)^{-1}\mathcal{K}_{xy} \\ &= \mathcal{K}_{xy}^{-1}\left[\frac{\sigma(x,y)}{\sigma(x,y)^n}\right]\vartheta_{xy}\vartheta_n\vartheta_{xy}^{-1}\mathcal{K}_{xy} && \text{(4.10)} \\ &= (\vartheta \cdot (xy))_n. \end{aligned}$$

Equations (4.9) and (4.10) make it clear why we have included the conjugation by $\mathcal{K}_x^{-1}$ in (4.8a), and why we have assumed that $\sigma(x,y)^n = \sigma(x,y)$.

Building a Mackey machine for twisted covariant representations

Suppose we are given an irreducible covariant representation Θ of $(G,\mathcal{A},\alpha,\sigma)$ on the Hilbert space $\mathscr{H}$, and a closed normal subgroup N of G. The restriction of Θ to $(N,\mathcal{A},\alpha,\sigma)$, denoted by $\Theta|_N$, is in general reducible. Let $\vartheta \in \hat{N}^\sigma$ be an irreducible subrepresentation

[2]The action of G on $\hat{N}^\sigma$ needs to sufficiently "regular" to ensure the good orbit structure needed for the Mackey machine. This needs the specification of a suitable topology on $\hat{N}^\sigma$, which we do not attempt to describe.

of $\Theta|_N$ on a closed Hilbert subspace $V \subset \mathscr{H}$, and let G_ϑ be its stabilizer group[3] under the action (4.8a). Using (4.8b), it is easy to see that $\vartheta \simeq \vartheta \cdot n$ for all $n \in N$, via the unitary $\mathcal{A}$-module map $\mathcal{K}_n^{-1}\vartheta_n$. Thus N is a normal subgroup of G_ϑ. By considering the restriction to the subspace $\Theta_x^{-1}V$ of (4.7), we find that the decomposition of $\Theta|_N$ into irreducibles in $\hat{N}^\sigma$ comprises all the conjugates $\vartheta \cdot x, x \in G$. The irreducible Θ is thus associated with a single G-orbit $\mathcal{O}$ in $\hat{N}^\sigma$, where $\mathcal{O}$ can be identified as the coset space $G_\vartheta \backslash G$. Roughly speaking, we have the decomposition

$$\Theta|_N = \int_{p\in\mathcal{O}}^{\oplus} m_\Theta(\vartheta \cdot s_p),$$

where s_p are coset representatives of p, and m_Θ is a multiplicity function. Let $\mathscr{H}'$ be the sum of the subspaces $\Theta_h^{-1}V$, $h \in G_\vartheta$, which furnishes an irreducible covariant representation ρ of $(G_\vartheta, \mathcal{A}, \alpha, \sigma)$, with ρ being the restriction of $\Theta|_{G_\vartheta}$ to the subspace $\mathscr{H}'$. The restriction of ρ to $(N, \mathcal{A}, \alpha, \sigma)$ is a multiple of the irreducible ϑ. This analysis says that Θ is a system of imprimitivity based on the transitive G-space $\mathcal{O} \simeq G_\vartheta \backslash G$. By an appropriate generalisation of Mackey's Imprimitivity Theorem[4], Θ is obtained by generalised inducing from ρ. Conversely, if ρ is an irreducible covariant representation of $(G_\vartheta, \mathcal{A}, \alpha, \sigma)$ whose restriction $\rho|_N$ to $(N, \mathcal{A}, \alpha, \sigma)$ is a multiple of ϑ, then the induced covariant representation $\Theta = \rho \uparrow G$ is an irreducible covariant representation of G.

Scalar cocycles and the Mackey obstruction

A generalised Mackey machine should include a mechanism for determining the irreducible covariant representations of $(G_\vartheta, \mathcal{A}, \alpha, \sigma)$ whose restriction to $(N, \mathcal{A}, \alpha, \sigma)$ is a multiple of ϑ. Such representations can be sought in an ad-hoc manner, but in some cases, a systematic procedure is available. In view of the need to use Schur's lemma[5] in what follows, *we will now restrict attention to* $\mathcal{A}$ *which are also complex* C^**-algebras, assume that cocycles are scalar* (i.e. $\mathbb{T}$-valued), *that* α *restricts to the scalars* (i.e. $\alpha_x(\mathbb{C}) = \mathbb{C} \subset \mathcal{MA}$ for all $x \in G$), *and that* $N \subset \ker(\alpha)$. In particular, this means that the representatives of $n \in N$ in a covariant representation of $(N, \mathcal{A}, \alpha, \sigma)$ must be complex-linear. An irreducible covariant representation of $(N, \mathcal{A}, \alpha, \sigma)$ is equivalently an irreducible representation of the *complex* twisted crossed product C^*-algebra $\mathcal{A} \rtimes_{(\alpha,\sigma)} N$ (see Section 3.3). Intertwiners between two equivalent irreducible covariant representations of $(N, \mathcal{A}, \alpha, \sigma)$ are thus unitary scalar multiples of each other.

[3] Appropriate assumptions on $\hat{N}$ and the G-action ensure that G_ϑ is closed, see [58, 85].

[4] We have not made this generalisation rigorously, but it is known in the case of PUA-reps [81, 70].

[5] There are some situations in which Schur's lemma need not be invoked, and the subsequent assumptions may be relaxed.

The first step is to extend ϑ to a covariant representation θ of $(G_\vartheta, \mathcal{A}, \alpha, \overline{\omega'}\sigma)$, where $\overline{\omega'}\sigma$ is the product cocycle of σ with the dual/inverse cocycle to ω', and ω' is canonically determined by an application of Schur's lemma as follows. By the definition of G_ϑ, there exists, for each $h \in G_\vartheta$, some unitary $\mathcal{A}$-module map U_h intertwining $\vartheta \cdot h$ with ϑ. This means that

$$(\vartheta \cdot h)_n = \mathcal{K}_h^{-1}[\sigma(h,n)/\sigma(hnh^{-1},h)]\vartheta_{hnh^{-1}}\mathcal{K}_h = U_h\vartheta_n U_h^{-1}, \tag{4.11}$$

with U_h determined up to a phase factor. Defining $\theta_h \coloneqq \mathcal{K}_h U_h$, we obtain the identity

$$\vartheta_{hnh^{-1}} = [\sigma(hnh^{-1},h)/\sigma(h,n)]\theta_h\vartheta_n\theta_h^{-1}, \qquad n \in N, h \in G_\vartheta. \tag{4.12}$$

For $h \in N$, we can choose the intertwiners $U_h = \mathcal{K}_h^{-1}\vartheta_h$ so that $\theta_h = \vartheta_h$. Also, $\theta_h a = a^h\theta_h$ for all $h \in G_\vartheta$, so θ is indeed an extension of the covariant representation ϑ. Using (4.12) repeatedly, we obtain

$$\theta_{hh'}\vartheta_n\theta_{hh'}^{-1} \approx \vartheta_{hh'n(hh')^{-1}} \approx \theta_h\vartheta_{h'nh'^{-1}}\theta_h^{-1} \approx \theta_h\theta_{h'}\vartheta_n\theta_{h'}^{-1}\theta_h^{-1},$$

where $\approx$ means equality up to some products of $\sigma(\cdot,\cdot)$. A straightforward computation shows that the $\sigma(\cdot,\cdot)$ factors cancel, i.e. $\theta_{hh'}^{-1}\theta_h\theta_{h'}$ commutes with each ϑ_n. By Schur's lemma, $\theta_h\theta_{h'}$ is some unitary scalar multiple of $\theta_{hh'}$, which we call $\overline{\omega'}\sigma(h,h')$. In other words, θ furnishes a covariant representation of $(G_\vartheta, \mathcal{A}, \alpha, \overline{\omega'}\sigma)$, where $(\overline{\omega'}\sigma)$ is a $\mathbb{T}$-valued cocycle for $(G_\vartheta, \mathcal{A}, \alpha)$. Multiplying by $\bar{\sigma}$ and taking the dual (inverse) cocycle yields a cocycle ω' for $(G_\vartheta, \mathcal{A}, \alpha)$. Actually, α_h, $h \in G_\vartheta$ restricts to the scalars $\mathbb{C}$, so we can view (α, ω') as a twisting pair for $(G_\vartheta, \mathbb{C})$. Furthermore, since α_h is either the identity or complex conjugation on $\mathbb{C}$, covariant representations of $(G_\vartheta, \mathbb{C}, \alpha, \omega')$ are really PUA-reps for $(G_\vartheta, \phi, \omega')$, where $\phi(h)$ equals -1 if α_h complex-conjugates the scalars and equals $+1$ otherwise.

There is some flexibility in the choice of ω', which can be traced to the freedom in the choice of U_h (or θ_h) in (4.11). We can actually choose θ_h such that ω' turns out to be a lift of some cocycle ω on G_ϑ/N, by mimicking the argument on pp. 273-274 of [81]. Note that α, and thus ϕ, descends to G_ϑ/N due to our assumption on N, and so the ω which we are about to construct is a generalised cocycle with respect to $\alpha|_{G_\vartheta/N}$. Recall that ϑ and θ coincide on N, so the cocycles σ and $\overline{\omega'}\sigma$ agree on $N \times N$, being scalar-valued there[6]. Thus, $\omega'(n,n') = 1$ for all $n, n' \in N$. Next, (4.12) continues to hold with ϑ replaced by θ and σ replaced by $\overline{\omega'}\sigma$, since it then reduces to the general identity (4.7) for covariant representations. This leads to

$$\omega'(hnh^{-1},h) = \omega'(h,n), \qquad h \in G_\vartheta, n \in N. \tag{4.13}$$

[6]If σ is not restricted to be scalar, we only know that the (necessarily scalar) representatives of σ and $\overline{\omega'}\sigma$ in the covariant representation coincide on $N \times N$.

Let $h_p \in G_\theta$ be coset representatives for each $p \in G_\vartheta/N$. Then by multiplying each θ_{nh_p} by $\omega'(n, h_p)$, we can set $\omega'(n, h_p) = 1$ for all $n \in N, p \in G_\vartheta/N$. Using the cocycle identity, we obtain

$$\omega'(n, n'h_p) = \omega'(n', h_p)\omega'(n, n'h_p) = \omega'(nn', h_p)\omega'(n, n') = 1.$$

Since every element $h \in G_\vartheta$ can be written as $n'h_p$ for some n' and h_p, we have shown that $\omega'(n, h) = 1$ for all $n \in N, h \in G_\vartheta$, and by (4.13), we have $\omega'(h, n) = 1$ as well. We apply these two results in the following instances of the cocycle identity,

$$\omega'(h, h')\omega'(hh', n) = \omega'(h, h'n)\omega'(h', n)^h \quad \Rightarrow \quad \omega'(h, h') = \omega'(h, h'n) \tag{4.14a}$$

$$\begin{aligned} \omega'(h', n)\omega'(h'n, h) = \omega'(h', nh)\omega'(n, h)^h \quad \Rightarrow \quad \omega'(h'n, h) &= \omega'(h', nh) = \omega'(h', hh^{-1}nh) \\ &= \omega'(h', h), \end{aligned} \tag{4.14b}$$

where the last equality follows from another application of the cocycle identity along with $\omega'(\,\cdot\,, n) = 1$. The two relations (4.14a) and (4.14b) mean that the cocycle ω' depends only on the cosets G_ϑ/N. Therefore, it descends to a cocycle ω on G_ϑ/N. The cocycle class $[\omega] \in H^2_\alpha(G_\vartheta/N, \mathbb{T})$ is the so-called *Mackey obstruction* to choosing θ_h such that $\theta_h\theta_{h'} = \theta_{hh'}$.

Each PUA-rep W of $(G_\vartheta/N, \phi, \omega)$ lifts to a PUA-rep W' of $(G_\vartheta, \phi, \omega')$ on the same space, via $W' = W \circ q$, where q is the quotient map onto the cosets. The tensor product $\theta \otimes W'$ (over $\mathbb{C}$) is then a covariant representation of $(G_\vartheta, \mathcal{A}, \alpha, \sigma)$; $a \in \mathcal{A}$ acts as $a \otimes \mathrm{id}$ and its cocycle is the product $(\overline{\omega'}\sigma)\omega' = \sigma$. Note that the scalars $\lambda \in \mathbb{C}$ can be considered to act as $\lambda \otimes \mathrm{id}$ or $\mathrm{id} \otimes \lambda$, with both multiplications consistent with

$$(\theta_h \otimes W'_h)\lambda(v \otimes w) = \lambda^h(\theta_h \otimes W'_h)(v \otimes w), \qquad h \in G_\vartheta.$$

Furthermore, the restriction of $\theta \otimes W'$ to $(N, \mathcal{A}, \alpha, \sigma)$ is $\theta \otimes \mathrm{id}$, which is a multiple of ϑ. The commutant of $\theta \otimes W'$ consists of those operators of the form $\mathrm{id} \otimes X$, where X is in the commutant of W'. It follows that $\theta \otimes W'$ is irreducible iff W' is irreducible. Generalised inducing yields a covariant representation $\Theta \coloneqq (\theta \otimes W') \uparrow G$ of $(G, \mathcal{A}, \alpha, \sigma)$, "lying over" the orbit $\mathcal{O} \simeq G_\vartheta \backslash G$ of G in $\hat{N}^\sigma$, which is irreducible if W' is irreducible. In other words, having fixed a scalar cocycle class $[\sigma]$ for $(G, \mathcal{A}, \alpha)$, a representative cocycle σ, and a normal subgroup $N \subset G$, under appropriate assumptions on $\hat{N}^\sigma$ and the action of G on $\hat{N}^\sigma$, the irreducible covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$ having orbit $\mathcal{O} \ni \vartheta$ correspond 1–1 to the irreducible PUA-reps W of $(G_\vartheta/N, \phi, \omega)$, via

$$W \leftrightarrow (\theta \otimes W') \uparrow G.$$

4.3 Generalised inducing — the graded case

Let $(G, c, \mathcal{A}, \alpha, \sigma)$ be a graded twisted C^*-dynamical system in the sense of Section 3.1.1. We can also define a notion of induced graded covariant representations for $(G, c, \mathcal{A}, \alpha, \sigma)$. Let H be a closed subgroup of G, and write $G_0 = \ker(c)$, $G_1 = G - G_0$. We induce a graded covariant representation for $(G, c, \mathcal{A}, \alpha, \sigma)$ from a graded covariant representation θ for $(H, c, \mathcal{A}, \alpha, \sigma)$ on M as follows. Take the Hilbert space space M_H^G as in the ungraded case, and give it a $\mathbb{Z}_2$-grading by setting the homogeneous subspaces $(M_H^G)_i$, $i = 0, 1$, to be

$$(M_H^G)_i = \bigoplus_{j=0,1} \{H\text{-equivariant}\, \psi : G \to M \,|\, \mathrm{supp}(\psi) \subset G_j, \mathrm{Im}(\psi) \subset M_{i+j\,(\mathrm{mod}\,2)}\}.$$

The action of G on (M_H^G) is defined by (4.2b) as before, and one verifies that the various gradings work out correctly for this to be a graded action.

There is an involutary operation of (right) parity reversal $\pi_R : (\Theta, \mathscr{H}) \to (\Theta^\Pi, \mathscr{H}^\Pi)$ on a graded covariant representation Θ of $(G, c, \mathcal{A}, \alpha, \sigma)$ on $\mathscr{H}$ (see Section 6.2 for more details) The vectors in $\mathscr{H}^\Pi$ are the same as those in $\mathscr{H}$ and have the same covariant action of $\mathcal{A}$ and G, but their gradings are reversed. Thus $(\Theta, \mathscr{H})$ and $(\Theta^\Pi, \mathscr{H}^\Pi)$ are equivalent in the ungraded sense, but may be inequivalent in the graded sense.

If the even cocycle σ is central, we can understand the induced representation space as functions on cosets $H\backslash G$ as in Section 4.1.1. Upon choosing a section $s : H\backslash G \to G$, the cosets split into an "even" subset $(H\backslash G)_0 \coloneqq \{p \in H\backslash G : c(s_p) = +1\}$ and an "odd" subset $(H\backslash G)_1 \coloneqq \{p \in H\backslash G : c(s_p) = -1\}$. We may regard M_H^G as a space $\mathscr{H}$ of functions $\varphi : H\backslash G \to M$, as in (4.3b)–(4.3a), which is graded in the natural way, the homogeneous subspaces being

$$\mathscr{H}_i = \bigoplus_{j=0,1} \{\varphi : \mathrm{supp}(\varphi) \subset (H\backslash G)_j, \mathrm{Im}(\varphi) \subset M_{j+i\,(\mathrm{mod}\,2)}\}.$$

Writing $\mathscr{H}_p = \mathcal{K}_{s_p} M$ if p is even and $\mathscr{H}_p = (\mathcal{K}_{s_p} M)^\Pi$ if p is odd, we can regard $\mathscr{H}$ as $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 = \int_p^\oplus \mathscr{H}_p$ as in (4.5). The action of $x \in G$ defined by (4.4), or (4.6), is correctly graded. This can be seen by comparing the parity of $h_x = s_p x s_{p\cdot x}^{-1}$, which determines the parity of the operators $\Theta_x^{(p)} : \mathscr{H}_{p\cdot x} \to \mathscr{H}_p$, with those of $p\cdot x$ and p. A different section s' with $s'_p = h_p s_p$ leads to an equivalent induced representation in the graded sense, with even intertwiner $\int_p^\oplus (\pi_R)^{h_p} \mathcal{K}_{h_p}^{-1} \theta_{h_p}$, where $(\pi_R)^{h_p}$ is the right parity reversal operation if $c(h_p) = -1$ and the identity map otherwise.

Some modifications need to be made for the graded version of the Mackey machine as sketched in Section 4.2. The generalised dual $\hat{G}^\sigma$ should be understood in the graded sense, i.e. equivalence requires an *even* intertwiner. When defining the action of $x \in G$ on $\vartheta \in \hat{N}^\sigma$, the formula (4.8a) should be accompanied by conjugation with the parity reversal operation π_R whenever $x \in G_1$. This ensures that $\vartheta \cdot n \simeq \vartheta$ in the graded sense for all $n \in N$. A graded version of Schur's lemma for graded covariant representations on graded Hilbert spaces is also required. In the finite-dimensional case, this says the following: let $\mathcal{A}$ be a collection of even/odd $\mathbb{F}$-linear operators acting irreducibly on a graded vector space $V = V_0 \oplus V_1$ over an algebraically closed field $\mathbb{F}$ (e.g. V is an irreducible graded projective unitary representation of $(G, c, \phi = 1, \sigma)$). Then the only *even* $\mathcal{A}$-equivariant endomorphisms of V are scalar multiples of the identity (e.g. see [53] and references therein).

4.4 Induced twisted covariant representations and twisted crossed products

At the beginning of this chapter, we constructed M_H^G, the $\mathcal{A}$-module of H-equivariant functions $\psi : G \to M$, which hosts the covariant representation of $(G, \mathcal{A}, \alpha, \sigma)$ induced from a given covariant representation of $(H, \mathcal{A}, \alpha, \sigma)$. We show that this is equivalently the construction of a $*$-representation of $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ on the same space, via

$$(F\psi)(g) := \int_G (F(x))^g \, \sigma(g, x)\psi(gx) \, \mathrm{d}x, \quad F \in \mathcal{A} \rtimes_{(\alpha,\sigma)} G.$$

We check that H-equivariance is preserved by the action of F,

$$\begin{aligned}
(F\psi)(g) &= \int_G (F(x))^g \, \sigma(g, x)\psi(gx) \, \mathrm{d}x \\
&= \int_G (F(x))^{h(h^{-1}g)} \, \sigma(h, h^{-1}g)^{-1}\sigma(h^{-1}g, x)^h \sigma(h, h^{-1}gx) \\
&\qquad\qquad\qquad\qquad \times \, \sigma(h, h^{-1}gx)^{-1}\theta_h\psi(h^{-1}gx) \, \mathrm{d}x \\
&= \int_G \sigma(h, h^{-1}g)^{-1} \left((F(x))^{h^{-1}g} \right)^h \sigma(h^{-1}g, x)^h \theta_h \psi(h^{-1}gx) \, \mathrm{d}x \\
&= \sigma(h, h^{-1}g)^{-1}\theta_h \int_G (F(x))^{h^{-1}g} \, \sigma(h^{-1}g, x)\psi(h^{-1}gx) \, \mathrm{d}x \\
&= \sigma(h, h^{-1}g)^{-1}\theta_h (F\psi)(h^{-1}g),
\end{aligned}$$

and verify the $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$-module property by computing

$$\begin{aligned}
(F_1(F_2(\psi))\,(g) &= \int_G (F_1(x))^g\,\sigma(g,x)(F_2\psi)(gx))\,\mathrm{d}x \\
&= \int_G\int_G (F_1(x))^g\,\sigma(g,x)\,(F_2(y))^{gx}\,\sigma(gx,y)\psi(gxy)\,\mathrm{d}x\mathrm{d}y \\
&= \int_G\int_G (F_1(x))^g\,\sigma(g,x)\,\big(F_2(x^{-1}y)\big)^{gx}\,\sigma(gx,x^{-1}y)\psi(gy)\,\mathrm{d}x\mathrm{d}y \\
&= \int_G\int_G \big(F_1(x)(F_2(x^{-1}y))^x\big)^g\,\sigma(g,x)\sigma(gx,x^{-1}y)\psi(gy)\,\mathrm{d}x\mathrm{d}y \\
&= \int_G\int_G \big(F_1(x)(F_2(x^{-1}y))^x\sigma(x,x^{-1}y)\big)^g\,\sigma(g,y)\psi(gy)\,\mathrm{d}x\mathrm{d}y \\
&= \int_G\left(\int_G \big(F_1(x)(F_2(x^{-1}y))^x\sigma(x,x^{-1}y)\big)\,\mathrm{d}x\right)^g\sigma(g,y)\psi(gy)\,\mathrm{d}y \\
&= \int_G ((F_1\star F_2)(y))^g\,\sigma(g,y)\psi(gy)\,\mathrm{d}y \\
&= ((F_1\star F_2)\psi)\,(g),
\end{aligned}$$

where we have used the translational invariance of the measure on G, as well as (3.1a) and (3.1b). Finally, the verification that $\langle\zeta\,|F^*\phi\rangle = \langle F\zeta\,|\phi\rangle$, i.e. the operators F and F^* acting on M_H^G are Hilbert space adjoints, requires untangling the definition of the quasi-invariant measure μ used in the definition of $\langle\cdot\,|\cdot\rangle$ in M_H^G, as well as the translational invariance of the Haar measure on G.

Chapter 5

Projective unitary-antiunitary representations of spacetime symmetry groups and parity groups

The projective unitary-antiunitary representation theory of the Poincaré group plays a fundamental role in the classification of elementary particles. The original treatment of Wigner [94] predates the more general Mackey theory, and treated the projective unitary representations of the identity component of the Poincaré group. It was found that the discrete symmetries of space inversion P and time reversal T can be quite tricky to handle [96, 70, 82]. From the point of view of this thesis, we would like to treat charge-conjugation symmetry C on the same level as P and T, so we will investigate the Poincaré group supplemented with a C symmetry. The P, C, T symmetries generate parity groups $\{\pm 1\}^n$ whose graded PUA-representations can be systematically studied using Clifford algebras. Since this relationship between graded PUA-reps of $\{\pm 1\}^n$ and Clifford algebras is crucial for later Chapters, especially in their role in K-theory, we explain the connection in some detail in Section 5.4–5.5.

5.1 Cocycles for semidirect products

In this section, we investigate the algebraic structure of the $\mathcal{UMA}$-valued cocycles for groups G formed from the *topological semidirect product*[1] $N \rtimes H$ of two locally compact and second countable groups N and H. Let $h \in H$ act on $n \in N$ continuously via automorphisms, which we denote by $h : n \mapsto h(n) =: n^h$. The underlying set of the semidirect product is $N \times H$, and we write an element of this set as nh, *in that order*. The semidirect group product is

[1] A detailed treatment of topological groups, semidirect products, and exact sequences of topological groups in the context of crossed products can be found in [97].

$n_1h_1 \cdot n_2h_2 := n_1n_2^{h_1}h_1h_2$, and the inverse is $(nh)^{-1} = (n^{-1})^{h^{-1}}h^{-1}$. These operations turn $N \rtimes H$ into a locally compact group. If there is no ambiguity, we will sometimes abuse notation and use n and h to denote, respectively, the elements ne and eh in the semidirect product. In particular, $n \cdot h \equiv ne \cdot eh = nh$ so the notation nh is consistent with the group multiplication law; *however*, $h \cdot n = n^h h$. Note that N and H are closed subgroups of $N \rtimes H$, with N normal, and $N \cap H = \{e\}$. Also, $h \cdot n \cdot h^{-1} = n^h e \equiv n^h$. Conversely, if we begin with a locally compact second countable group G with closed subgroups N and H, such that N is normal, $N \cap H = \{e\}$ and $N \times H = G$, then we can define an action of H on N by conjugation, $h(n) = hnh^{-1}$; the resulting semidirect product $N \rtimes H$ is isomorphic to G.

The subsequent calculations are generalised versions of those carried out for projective unitary representations in [58, 69]. Let σ be a cocycle for $N \rtimes H$, which we may assume to satisfy $\sigma(e, nh) = 1 = \sigma(nh, e)$. We would like to factorize σ into a product involving a cocycle for N, a cocycle for H, and function on $N \times H$. The first step is to compute, using the cocycle identity,

$$\begin{aligned}
\sigma(n_1h_1, n_2h_2) &= \sigma(n_1, h_1)^{-1}\sigma(h_1, n_2h_2)^{n_1}\sigma(n_1, n_2^{h_1}h_1h_2) \\
&= \sigma(n_1, h_1)^{-1}\sigma(h_1, n_2h_2)^{n_1}\left(\sigma(n_2^{h_1}, h_1h_2)^{n_1}\right)^{-1}\sigma(n_1, n_2^{h_1})\sigma(n_1n_2^{h_1}, h_1h_2) \\
&= \sigma(n_1, h_1)^{-1}\left(\sigma(n_2, h_2)^{n_1h_1}\right)^{-1}\sigma(h_1, n_2)^{n_1}\cancel{\sigma(n_2^{h_1}h_1, h_2)^{n_1}} \\
&\quad \times \cancel{\left(\sigma(n_2^{h_1}h_1, h_2)^{n_1}\right)^{-1}}\left(\sigma(n_2^{h_1}, h_1)^{n_1}\right)^{-1}\sigma(h_1, h_2)^{(n_1n_2^{h_1})} \\
&\quad \times \sigma(n_1, n_2^{h_1})\sigma(n_1n_2^{h_1}, h_1h_2).
\end{aligned}$$

Defining the function $\lambda : N \rtimes H \to \mathcal{UMA}$ by $\lambda(nh) := \sigma(n, h)^{-1}$, we can rewrite the above identity as

$$\begin{aligned}
\sigma(n_1h_1, n_2h_2) &= \lambda(n_1h_1)\lambda(n_2h_2)^{n_1h_1}\sigma(h_1, n_2)^{n_1}\left(\sigma(n_2^{h_1}, h_1)^{n_1}\right)^{-1} \\
&\quad \times \sigma(h_1, h_2)^{(n_1n_2^{h_1})}\sigma(n_1, n_2^{h_1})\left(\lambda(n_1h_1 \cdot n_2h_2)\right)^{-1},
\end{aligned}$$

which recognise as the action of λ on σ defined in (3.3b). Therefore, we may pass to the equivalent[2] cocycle σ' defined by

$$\begin{aligned}
\sigma'(n_1h_1, n_2h_2) &:= \left[\sigma(h_1, n_2)\sigma(n_2^{h_1}, h_1)^{-1}\right]^{n_1}\left[\sigma(h_1, h_2)^{(n_1n_2^{h_1})}\right]\left[\sigma(n_1, n_2^{h_1})\right] \\
&=: \varsigma(n_2, h_1)^{n_1} \cdot \sigma|_H(h_1, h_2)^{(n_1n_2^{h_1})} \cdot \sigma|_N(n_1, n_2^{h_1}) \qquad (5.1)
\end{aligned}$$

where $\varsigma : N \times H \to \mathcal{UMA}$ is defined by the first square bracket, while $\sigma|_N$ and $\sigma|_H$ are the restrictions of the original cocycle σ (and also the equivalent cocycle σ') to $N \times N$ and $H \times H$ respectively. Thus $\sigma|_N$ and $\sigma|_H$ are, respectively, cocycles for N and H.

[2]When the cocycles are not central, equivalence really means *exterior equivalence*, under which the automorphisms α_x become $\alpha'_x = \mathrm{Ad}(\lambda(x)) \circ \alpha_x$ according to (3.3a). Since $\lambda(n) = 1$, we can take $\sigma(\cdot,\cdot)^n$ to mean either $\alpha_n(\sigma(\cdot,\cdot))$ or $\alpha'_n(\sigma(\cdot,\cdot))$ and no ambiguity arises from our abbreviated notation.

The function ς must satisfy some restrictive properties due to the cocycle condition on σ'. First, note that $\varsigma(n,e) = 1 = \varsigma(e,h)$ for all $n \in N, h \in H$, and that we can write $\varsigma(n,h) = \sigma'(h,n)$. Then, the cocycle identity for σ' gives, for all $n, n_1, n_2 \in N, h, h_1, h_2 \in H$,

$$\begin{aligned}\varsigma(n_1n_2,h) = \sigma'(h,n_1n_2) &= \left(\sigma'(n_1,n_2)^h\right)^{-1}\sigma'(h,n_1)\sigma'(n_1^h h, n_2) \\ &= \left(\sigma|_N(n_1,n_2)^h\right)^{-1}\varsigma(n_1,h)\varsigma(n_2,h)^{n_1^h}\sigma|_N(n_1^h,n_2^h), \end{aligned} \tag{5.2}$$

as well as

$$\begin{aligned}\varsigma(n,h_1h_2) = \sigma'(h_1h_2,n) &= \sigma'(h_1,h_2)^{-1}\sigma'(h_2,n)^{h_1}\sigma'(h_1,n^{h_2}h_2) \\ &= \sigma|_H(h_1,h_2)^{-1}\varsigma(n,h_2)^{h_1}\varsigma(n^{h_2},h_1)\sigma|_H(h_1,h_2) \end{aligned} \tag{5.3}$$

This does not look very enlightening, but a converse statement to (5.1) can be made if we assume central cocycles, as well as $\alpha(n) = \mathrm{id}_{\mathcal{A}}$ for all $n \in N$, c.f. Theorem 9.4 of [58].

Proposition 5.1.1 (Central cocycles for semidirect products)**.** *Let $(N \rtimes H, \mathcal{A}, \alpha, \sigma)$ be a twisted dynamical system with σ central and $\alpha(n) = \mathrm{id}_{\mathcal{A}}$ for all $n \in N$. Then σ is equivalent to a central cocycle σ' of the form*

$$\sigma'(n_1h_1, n_2h_2) = \varsigma(n_2,h_1)\cdot\eta(h_1,h_2)\cdot\nu(n_1,n_2^{h_1}), \tag{5.4}$$

where ν is a cocycle for N, η is a cocycle for H, and ς is a Borel function $N\times H \to \mathcal{Z}(\mathcal{UMA})$ satisfying the simplified versions of (5.2)–(5.3)*,*

$$\varsigma(n_1n_2,h) = \nu(n_1^h,n_2^h)\left(\nu(n_1,n_2)^h\right)^{-1}\varsigma(n_1,h)\varsigma(n_2,h), \tag{5.5a}$$

$$\varsigma(n,h_1h_2) = \varsigma(n^{h_2},h_1)\varsigma(n,h_2)^{h_1}, \tag{5.5b}$$

for all $n, n_1, n_2 \in N, h, h_1, h_2 \in H$. Conversely, if ν and η are central cocycles for N and H respectively, and ς satisfies the above conditions, then the function σ' defined by (5.4) *is a central cocycle for $N \rtimes H$.*

Proof. By the preceding discussion, we only need to prove the converse. Define σ' by the right-hand-side of (5.4). Since η is assumed to be a cocycle, $n_1h_1 \cdot n_2h_2 = n_1n_2^{h_1}h_1h_2$, and η only appears in the form $\eta(h_1,h_2)$, we only need to keep track of the ς and ν terms when checking the cocycle identity for σ'. Using (5.5b), we compute, modulo η terms,

$$\begin{aligned}&\sigma'(n_1h_1,n_2h_2)\sigma'(n_1n_2^{h_1}h_1h_2,n_3h_3) \\ &\quad= \varsigma(n_2,h_1)\nu(n_1,n_2^{h_1})\varsigma(n_3,h_1h_2)\nu(n_1n_2^{h_1},n_3^{h_1h_2}) \\ &\quad= \varsigma(n_2,h_1)\nu(n_1,n_2^{h_1})\varsigma(n_3^{h_2},h_1)\varsigma(n_3,h_2)^{h_1}\nu(n_1n_2^{h_1},n_3^{h_1h_2}). \end{aligned} \tag{5.6}$$

Using (5.5a), we have, modulo η terms,

$$\begin{aligned}
&\sigma'(n_2h_2, n_3h_3)^{h_1}\sigma'(n_1h_1, n_2n_3^{h_2}h_2h_3) \\
&\qquad = \varsigma(n_3, h_2)^{h_1}\nu(n_2, n_3^{h_2})^{h_1}\varsigma(n_2n_3^{h_2}, h_1)\nu(n_1, n_2^{h_1}n_3^{h_1h_2}) \\
&\qquad = \varsigma(n_3, h_2)^{h_1}\varsigma(n_2, h_1)\varsigma(n_3^{h_2}, h_1)\nu(n_2^{h_1}, n_3^{h_1h_2})\nu(n_1, n_2^{h_1}n_3^{h_1h_2}). \qquad (5.7)
\end{aligned}$$

Since ν is assumed to be a cocycle for N, the right-hand-sides of (5.6) and (5.7) are equal. The equality of their left-hand-sides is precisely the desired cocycle condition for σ',

$$\sigma'(n_1h_1, n_2h_2)\sigma'(n_1h_1 \cdot n_2h_2, n_3h_3) = \sigma'(n_2h_2, n_3h_3)^{n_1h_1}\sigma'(n_1h_1, n_2h_2 \cdot n_3h_3).$$

□

Remark 5.1.2. Let $r : N \to \mathcal{Z}(\mathcal{UMA})$ and $s : H \to \mathcal{Z}(\mathcal{UMA})$ be Borel functions implementing the equivalences $\nu \sim \tilde{\nu} = r \cdot \nu$ and $\eta \sim \tilde{\eta} = s \cdot \eta$. When ν, η are replaced by equivalent cocycles $\tilde{\nu}, \tilde{\eta}$, we can still preserve the factorisation in (5.4). For notational ease, we omit the primes on σ. Define the function $t(n, h) = r(n)s(h)$, and pass to the equivalent cocycle $\tilde{\sigma} = t \cdot \sigma$, then

$$\begin{aligned}
\frac{\tilde{\sigma}(n_1h_1, n_2h_2)}{\sigma(n_1h_1, n_2h_2)} &= \frac{t(n_1, h_1)t(n_2, h_2)}{t(n_1n_2^{h_1}, h_1h_2)} = \frac{r(n_1)s(h_1)r(n_2)s(h_2)}{r(n_1n_2^{h_1})s(h_1h_2)} \\
&= \left[\frac{r(n_2)}{r(n_2^{h_1})}\right]\left[\frac{s(h_1)s(h_2)}{s(h_1h_2)}\right]\left[\frac{r(n_1)r(n_2^{h_1})}{r(n_1n_2^{h_1})}\right].
\end{aligned}$$

By defining $\tilde{\varsigma}(n, h) = \frac{r(n)}{r(n^h)}\varsigma(n, h)$, we achieve the factorisation

$$\tilde{\sigma}(n_1h_1, n_2h_2) = \tilde{\varsigma}(n_2, h_1) \cdot \tilde{\eta}(h_1, h_2) \cdot \tilde{\nu}(n_1, n_2^{h_1}).$$

For the rest of this section, we assume that cocycles are central and $\alpha(n) = \mathrm{id}_{\mathcal{A}}$ for all $n \in N$. It is sometimes known that all cocycles on N are equivalent to the trivial cocycle, In such situations, (5.2) reduces to

$$\varsigma(n_1n_2, h) = \varsigma(n_1, h)\varsigma(n_2, h),$$

which means that $\varsigma(\cdot, h)$ is a ($\mathcal{Z}(\mathcal{UMA})$-valued) character on N, for every $h \in H$. If we also know that N has no non-trivial characters, for instance, when N is a connected non-compact simple Lie group and cocycles are complex-valued, then $\varsigma \equiv 1$, so a cocycle σ for $N \rtimes H$ may be assumed to be trivial between elements of H and N.

For direct products $N \times H$, there is a further simplification.

Corollary 5.1.3 (Central cocycles for direct products). *If $N \rtimes H$ is replaced by a direct product $N \times H$ in Proposition 5.1.1, then σ is equivalent to the product*

$$\sigma(n_1h_1, n_2h_2) = \varsigma(n_2, h_1) \cdot \eta(h_1, h_2) \cdot \nu(n_1, n_2) \tag{5.8}$$

of a cocycle ν for N, a cocycle η for H, and a generalised bicharacter ς, i.e., a Borel function $N \times H \to \mathcal{Z}(\mathcal{UMA})$ such that

$$\begin{aligned} \varsigma(n_1n_2, h) &= \varsigma(n_1, h)\varsigma(n_2, h), & \text{(5.9a)} \\ \varsigma(n, h_1h_2) &= \varsigma(n, h_1)\varsigma(n, h_2)^{h_1}, \qquad n, n_1, n_2 \in N, h, h_1, h_2 \in H. & \text{(5.9b)} \end{aligned}$$

Conversely, if ν, η, ς are as described above, the function σ defined by (5.8) *is a cocycle for $N \times H$.*

5.2 Projective unitary-antiunitary representations of the Poincaré group

The PUA-reps of the full Poincaré group $\mathscr{P}$ were studied by Parthasarathy [70], and then by Shaw–Lever [82]. We provide an overview of the approach in [82], which is based on their theory of induced representations as generalised to PUA-reps. This should be compared with Wigner's original approach [94] of first finding the irreducible projective unitary representations of the restricted Poincaré group $\mathscr{P}_+^\uparrow$, and then adjoining the space inversion and time reversal symmetries. The outline in this section contains only minor modifications in notation and conventions to the Shaw–Lever work. They are made in order to facilitate our inclusion of charge-conjugation symmetry in Section 5.2.1.

The identity component $\mathscr{P}_+^\uparrow$ of the Poincaré group is the semidirect product of the spacetime translations $\mathscr{T} \cong \mathbb{R}^4$ with the identity component of the homogeneous Lorentz group $\mathscr{L}_+^\uparrow$; similarly for the full Poincaré group $\mathscr{P}$ and the full Lorentz group $\mathscr{L} \equiv \mathrm{O}(1,3)$,

$$\mathscr{P}_+^\uparrow = \mathscr{T} \rtimes \mathscr{L}_+^\uparrow, \qquad \mathscr{P} = \mathscr{T} \rtimes \mathscr{L}.$$

The action of the Lorentz group as automorphisms of $\mathscr{T}$ is the obvious defining one. The universal cover $\tilde{\mathscr{L}}_+^\uparrow$ of $\mathscr{L}_+^\uparrow$ is $\mathrm{SL}(2,\mathbb{C})$ with the usual 2–1 covering homomorphism $\Lambda : \mathrm{SL}(2,\mathbb{C}) \to \mathscr{L}_+^\uparrow$. Concretely, we can make the vector space identification between $\mathscr{T} \cong \mathbb{R}^4$ and complex self-adjoint 2×2 matrices via

$$x \equiv (x_0, \mathbf{x}) \equiv (x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 - \mathrm{i}x_2 \\ x_1 + \mathrm{i}x_2 & x_0 - x_3 \end{pmatrix} =: X.$$

For each $A \in \mathrm{SL}(2,\mathbb{C})$, the map $X \mapsto AXA^\dagger$ preserves $\det(X) = x_0^2 - x_1^2 - x_2^2 - x_3^2$, so it defines a corresponding element $\Lambda(A) \in \mathscr{L}_+^\uparrow$. One verifies that $A \mapsto \Lambda(A)$ is a 2–1 homomorphism onto $\mathscr{L}_+^\uparrow$. The action of $A \in \mathrm{SL}(2,\mathbb{C})$ on $x \in \mathscr{T}$ is taken to be that of $\Lambda(A)$. Then, the simply-connected cover of $\mathscr{P}_+^\uparrow$ is the semidirect product

$$\tilde{\mathscr{P}}_+^\uparrow = \mathscr{T} \rtimes \mathrm{SL}(2,\mathbb{C}).$$

The full Lorentz group is itself a semidirect product with the Klein 4-group,

$$\mathscr{L} = \mathscr{L}_+^\uparrow \rtimes F_4, \qquad F_4 = \{1, T, P, PT\} \cong \{\pm 1\} \times \{\pm 1\},$$

where P and T are respectively the space inversion and time reversal elements. The discrete subgroup elements $F \in F_4$ act on $\mathscr{L}_+^\uparrow$ by conjugation, $\Lambda \mapsto F\Lambda F^{-1}$, and this lifts to an automorphic action $A \mapsto A^F$ of F_4 on $\mathrm{SL}(2,\mathbb{C})$ which satisfies $\Lambda(A^F) = F\Lambda(A)F^{-1}$. More explicitly, the action of $F \in F_4$ on $\mathrm{SL}(2,\mathbb{C})$ is [82]

$$A^P = A^T = (A^\dagger)^{-1}, \qquad A^{TP} = A.$$

This allows us to define a double cover $\tilde{\mathscr{L}}$ of the full Lorentz group $\mathscr{L}$,

$$\tilde{\mathscr{L}} = \tilde{\mathscr{L}}_+^\uparrow \rtimes F_4 = \mathrm{SL}(2,\mathbb{C}) \rtimes F_4.$$

Similarly, the full Poincare group $\mathscr{P}$ also has a double cover $\tilde{\mathscr{P}}$, defined by

$$\tilde{\mathscr{P}} = \mathscr{T} \rtimes \tilde{\mathscr{L}} = \mathscr{T} \rtimes (\mathrm{SL}(2,\mathbb{C}) \rtimes F_4) = (\mathscr{T} \rtimes \mathrm{SL}(2,\mathbb{C})) \rtimes F_4 = \tilde{\mathscr{P}}_+^\uparrow \rtimes F_4,$$

where the action of $(A, F) \in \tilde{\mathscr{L}}$ on $\mathscr{T}$ takes $x \mapsto A(F(x))$. The covering homomorphism $\Lambda : \tilde{\mathscr{P}} \to \mathscr{P}$ is $(x, A, F) \mapsto (x, \Lambda(A), F)$.

The homomorphism $\phi = \tau : \mathscr{P} \to \{\pm 1\}$, which keeps track of unitarity/antiunitarity, is first specified for F_4, with $\phi(T) = -1$, $\phi(P) = +1$. Then, it is lifted to $\mathscr{P}$ (and denoted by the same symbol), so that $\ker(\phi) = \mathscr{P}^\uparrow \equiv \mathscr{P}_+^\uparrow \cup \mathscr{P}_-^\uparrow$ acts unitarily and $\mathscr{P}^\downarrow \equiv \mathscr{P}_+^\downarrow \cup \mathscr{P}_-^\downarrow$ acts antiunitarily. We are looking for complex Hilbert spaces on which the unitary/antiunitary representatives of elements in $\mathscr{P}$ act projectively in a manner specified by some generalised cocycle σ. In other words, we are in the situation of Section 3.2.1. Note that ϕ may be further lifted to a homomorphism from the double cover $\tilde{\mathscr{P}}$ to $\{\pm 1\}$.

There are only four equivalence classes of generalised cocycles [70] for (F_4, ϕ), labelled by $\alpha, \beta \in \{\pm 1\}$, as in Table 5.1. We will describe one method of establishing these classes in Section 5.4.1. It is known that $\tilde{\mathscr{P}}_+^\uparrow$ has no non-trivial characters (e.g. 8.1B Theorem 2 of [9]) or complex cocycles (pp. 51 Corollary 1 of [69], or [94]), so Proposition 5.1.1 tells us that the generalised cocycles of $\tilde{\mathscr{P}} = \tilde{\mathscr{P}}_+^\uparrow \rtimes F_4$ are, up to equivalence,

$$\sigma^{\alpha\beta}((x, A, F), (x', A', F')) = \sigma^{\alpha\beta}(F, F'),$$

$\sigma^{\alpha\beta}$	1	P	T	PT
1	1	1	1	1
P	1	1	1	1
T	1	$\alpha\beta$	α	β
PT	1	$\alpha\beta$	α	β

Table 5.1: The four cocycle classes for (F_4, ϕ).

α	β	$U^{(\pm)}_{\alpha\beta}(P)$	$U^{(\pm)}_{\alpha\beta}(T)$	$U^{(\pm)}_{\alpha\beta}(PT)$	Wigner type
$+1$	$+1$	± 1	$\pm\kappa$	κ	I
-1	-1	$\pm 1_2$	$\mp \mathrm{i} Y\kappa$	$-\mathrm{i} Y\kappa$	II
$+1$	-1	Z	$-X\kappa$	$-\mathrm{i} Y\kappa$	III
-1	$+1$	Z	$\mathrm{i} Y\kappa$	$X\kappa$	III

Table 5.2: Irreducible projective unitary-antiunitary representations of $(F_4, \phi, \sigma^{\alpha\beta})$.

where we have abused notation slightly. By the arguments in Section 2.1 of [82], the irreducible PUA-reps V of $\mathscr{P}$ are in 1–1 correspondence with the irreducible PUA-reps U of $\tilde{\mathscr{P}}$, via $U = V \circ \Lambda$, so we shall concern ourselves with the latter.

As an intermediate step, we will require, for each of the four $\sigma^{\alpha\beta}$, the irreducible PUA-reps for $(F_4, \phi, \sigma^{\alpha\beta})$. When $\alpha = \beta$, there are two unitarily inequivalent irreducible PUA-reps (each of these are of course projectively equivalent), while there is only one unitary equivalence class of irreducible PUA-reps when $\alpha \neq \beta$. This is summarised in Table 5.2, which is a reproduction of Table 2 of [82]. There, X, Y, Z refer to the Pauli matrices in their standard form with respect to some basis, and κ is complex conjugation in that basis.

The irreducible PUA-reps of $(\tilde{\mathscr{P}}, \phi, \sigma^{\alpha\beta})$ can be obtained via generalised inducing with respect to the abelian normal subgroup $\mathscr{T}$. The irreducible PUA-reps[3] of $\mathscr{T}$ are the one-dimensional representations $\chi_p(x) = e^{\mathrm{i}p\cdot x} = e^{\mathrm{i}(x_0 p_0 - \mathbf{x}\cdot\mathbf{p})}$, with the characters χ_p labelled by $p \equiv (p_0, \mathbf{p}) \in \hat{\mathscr{T}} \cong \hat{\mathbb{R}}^4$. The hat on $\hat{R}^4$ serves to distinguish the dual group of $\mathbb{R}^4$ from $\mathbb{R}^4$ itself. For the action of $\tilde{\mathscr{P}}$ on $\hat{\mathscr{T}}$, it is clear that $\mathscr{T}$ stabilises each χ_p, while the action of $g \in \tilde{\mathscr{L}}$ is

$$(\chi_p \cdot g)(x) = (\chi_p(\Lambda(g)x))^g \,,$$

where $(\,\cdot\,)^g$ indicates complex conjugation when $g \in \tilde{\mathscr{L}}^{\downarrow}$ and the identity otherwise, and we have written $\Lambda(g) \coloneqq (\Lambda(A), F)$, $g = (A, F) \in \tilde{\mathscr{L}}$. In particular,

$$\chi_{(p_0,\mathbf{p})} \cdot T = \overline{\chi}_{(-p_0,\mathbf{p})} = \chi_{(p_0,-\mathbf{p})} = \chi_{(p_0,\mathbf{p})} \cdot P.$$

[3]Ordinary irreducible representations, since $\sigma^{\alpha\beta}$ is trivial on $\mathscr{T}$.

For χ_p with $p^2 = p_0^2 - \mathbf{p}\cdot\mathbf{p} > 0, p_0 > 0$, the orbit of the $\tilde{\mathscr{P}}$-action is a positive energy hyperbola. Its generalised isotropy group is $\mathscr{T} \rtimes L_p$, where L_p is the generalised little group for χ_p, and is (conjugate to) $\mathrm{SU}(2) \times F_4$.

We proceed as in Section 4.2. Since $\mathscr{T}$ is abelian, $\chi_p(hxh^{-1}) = \chi_p(x)$ for all $h \in \mathscr{T} \rtimes L_p$, and we can take the extension of χ_p to $\mathscr{T} \rtimes L_p$ to be $\chi_p((x,h)) = \chi_p(x)$ in (4.11). The Mackey obstruction vanishes (we can take $\omega \equiv 1$), so we need to find the irreducible PUA-reps of $(\mathscr{T} \rtimes L_p, \phi, \sigma^{\alpha\beta})$. These can be obtained from the table of irreducible PUA-reps for $(F_4, \phi, \sigma^{\alpha\beta})$, using the procedure outlined in Section 3.2 of [82]. The idea is to use the inducing construction with $\mathrm{SU}(2)$ as the normal subgroup of $L_p = \mathrm{SU}(2) \times F_4$, on which ϕ and $\sigma^{\alpha\beta}$ are trivial. The irreducible representations of $\mathrm{SU}(2)$ are the usual spin-j representations D^j of complex dimension $2j+1$. From (4.8a), we see that the stabiliser of D^j is the entire group $\mathrm{SU}(2) \times F_4$. The D^j are known to be unitarily equivalent to their complex conjugates. Furthermore, the half-integer representations are quaternionic while the integer representations are real. This means that there is an antiunitary operator K_j, for each j, such that

$$\begin{aligned} K_j D^j(A) K_j^{-1} &= D^j(A), \quad A \in \mathrm{SU}(2), \\ K_j^2 &= (-1)^{2j}. \end{aligned}$$

Using this fact in the construction outlined in Section 4.2, the extension $\mathbf{D}^j$ of D^j to the stabiliser group $\mathrm{SU}(2) \times F_4$ is

$$\mathbf{D}^j(A,F) = \begin{cases} D^j(A), & \text{if } F \in \{1, P\}, \\ D^j(A)K_j, & \text{if } F \in \{T, PT\}, \end{cases}$$

which has the cocycle

$$\overline{\omega}'_j \sigma^{\alpha\beta}((A,F),(A',F')) = \sigma^{(-1)^{2j}(-1)^{2j}}(F,F').$$

Accordingly, we need to find the irreducible PUA-reps for (F_4, ϕ, ω_j), where $\omega_j = \sigma^{\alpha_j\beta_j}$ and

$$\alpha_j \coloneqq (-1)^{2j}\alpha, \quad \beta_j \coloneqq (-1)^{2j}\beta,$$

then tensor with $\mathbf{D}^j$ to arrive at the desired irreducible PUA-reps $\mathbf{D}^{j(\pm)}_{\alpha\beta} = \mathbf{D}^j \otimes U^{(\pm)}_{\alpha_j\beta_j}$ of $(\mathrm{SU}(2) \times F_4, \phi, \sigma^{\alpha\beta})$. The results are summarised in Table 5.3, which corresponds to Table 3 of [81].

Tensoring again with χ_p (or simply multiplying by χ_p, since χ_p is one-dimensional) yields the irreducible PUA-reps of $(\mathscr{T} \rtimes L_p, \phi, \sigma^{\alpha\beta})$. The final step is to induce up to an irreducible PUA-rep $(\chi_p \mathbf{D}^{j(\pm)}_{\alpha\beta}) \uparrow \tilde{\mathscr{P}}$ of $(\tilde{\mathscr{P}}, \phi, \sigma^{\alpha\beta})$. Most physical examples seem to assume $\alpha = \beta = 1$, but it may be interesting to see what the other cases correspond to.

α_j	β_j	$U^{(\pm)}_{\alpha\beta}(P)$	$U^{(\pm)}_{\alpha\beta}(T)$	$U^{(\pm)}_{\alpha\beta}(PT)$
$+1$	$+1$	$\pm 1_{2j+1}$	$\pm K_j$	K_j
-1	-1	$\pm 1_{2(2j+1)}$	$\pm\begin{pmatrix} 0 & -K_j \\ K_j & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -K_j \\ K_j & 0 \end{pmatrix}$
$+1$	-1	$Z \otimes 1_{2j+1}$	$\begin{pmatrix} 0 & -K_j \\ -K_j & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -K_j \\ K_j & 0 \end{pmatrix}$
-1	$+1$	$Z \otimes 1_{2j+1}$	$\begin{pmatrix} 0 & K_j \\ -K_j & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & K_j \\ K_j & 0 \end{pmatrix}$

Table 5.3: Irreducible PUA-reps for (F_4, ϕ, ω_j). Note that the first two columns denote the values of α_j, β_j, rather than α, β as in Table 5.2.

The original and more commonly used approach uses the 1–1 correspondence between projective unitary representations of $\mathscr{P}_+^\uparrow$ and the ordinary[4] unitary representations of $\tilde{\mathscr{P}}_+^\uparrow$. The latter can be found using Mackey's theory for unitary representations, while the subsequent inclusion of P and T is treated in [96].

5.2.1 Graded projective unitary-antiunitary representations of the Poincaré group with C symmetry

We may also wish to consider the enlarged symmetry group ${}^C\mathscr{P} := \mathscr{P} \times \{1, C\}$ including charge conjugation, with ϕ extended such that $\phi(C) = -1$, along with an additional homomorphism c with $c(\mathscr{P}) = \{+1\}$ and $c(C) = -1$. As we discuss later, this extension becomes relevant when passing to quantum field theory. We look for *graded* PUA-reps of $({}^C\mathscr{P}, c, \phi, \sigma)$, with σ a generalised 2-cocycle. Let $F_8 := F_4 \times \{1, C\}$, so that ${}^C\mathscr{P} = \mathscr{P}_+^\uparrow \rtimes F_8$, and similarly for the double cover ${}^C\tilde{\mathscr{P}} = \tilde{\mathscr{P}} \rtimes F_8$. A similar argument to that in the previous section says that we only have to specify the values of σ between the elements of F_8. A systematic treatment of a more general version this problem is undertaken in Section 5.4. For now, we consider only the case where σ is simply $\sigma^{\alpha\beta}$ of the previous section extended trivially to F_8. Thus, C is required to act as an odd antiunitary involution which commutes with all the operators representing $\mathscr{P}$ (or $\tilde{\mathscr{P}}$).

We start the induction process with respect to $N = \mathscr{T}$, whose irreducible *graded* unitary representations are the one-dimensional *graded* characters $\chi_p^\pm$, which are the usual ungraded characters χ_p with an additional $\pm$ label specifying whether the representation space has grading $+1$ or -1. Note that equivalence now requires an *even* unitary intertwiner, so χ_p^+ and χ_p^- are not equivalent. Recall that the grading operator is meant to correspond to the flattened Hamiltonian $\mathrm{sgn}(H)$, and that the Hamiltonian is the self-adjoint generator of the

[4]That a projective unitary representation of G has a lift to a unitary representation of a universal covering group $\tilde{G}$, is not generally true. For a connected Lie group, a central extension of the universal cover is sometimes needed, see [6, 91].

one-parameter group of unitaries corresponding to time translation symmetries. That is, $\Gamma = \mathrm{sgn}(H) \leftrightarrow \mathrm{sgn}(p_0)$, so for consistency, for timelike p, we only admit χ_p^+ when $p_0 > 0$ and χ_p^- when $p_0 < 0$. Consider χ_p^+ with $p^2 = m^2 > 0, p_0 > 0$. The action of C is computed to be

$$\begin{aligned} \chi^+_{(p_0,\mathbf{p})} \cdot C &= \chi^-_{(-p_0,-\mathbf{p})} = \overline{\chi}^-_{(p_0,\mathbf{p})}, \\ \chi^-_{(-p_0,\mathbf{p})} \cdot C &= \chi^+_{(p_0,-\mathbf{p})} \quad = \overline{\chi}^+_{(-p_0,\mathbf{p})}. \end{aligned} \tag{5.10}$$

The generalised little group of χ_p^+ is, as before, $L_p = \mathrm{SU}(2) \times F_4$, but due to the additional action of C, its orbit under the ${}^C\tilde{\mathscr{P}}$-action now consists of *both* branches of the hyperbola $p^2 = m^2$, with the grading operator taking the values ± 1 on the $p_0 \gtrless 0$ branches. The orbit can be identified with a coset space as in Section 4.1.1.

The (massive) irreducible graded PUA-reps of $({}^C\tilde{\mathscr{P}}, c, \phi, \sigma^{\alpha\beta})$ are induced from those of $(\mathscr{T} \rtimes L_p, c = 1, \phi, \sigma^{\alpha\beta})$, in a manner analogous to that of the previous section[5]. The total representation space is a direct integral

$$\mathscr{H} = \int_{\oplus_{p:p^2=m^2}} \mathscr{H}_p,$$

where each $\mathscr{H}_p$ is isomorphic to the representation space $\mathscr{H}^{\mathbf{D}}$ of the inducing PUA-rep $\mathbf{D}$ of[6] $(L_p, 1, \phi, \sigma^{\alpha,\beta})$ when $p_0 > 0$, while $\mathscr{H}_p$ is grading-reversed and anti-isomorphic to $\mathscr{H}^{\mathbf{D}}$ when $p_0 < 0$. Roughly speaking, elements (A, F) of L_p act on $\mathscr{H}_p$ according to $\mathbf{D}$ or its conjugate representation, depending on whether $p_0 > 0$ or $p_0 < 0$. Other elements $g \in {}^C\tilde{\mathscr{P}}$ act as operators from $\mathscr{H}_{g^{-1}p}$ to $\mathscr{H}_p$, where $g^{-1}p$ is the translation of the action of g on the characters $\chi_p^\pm \in \hat{N}$ into an action on $p \in \hat{\mathbb{R}}^4$ labelling the orbit. For g in the (covering group of the) Lorentz group, $g^{-1}p$ is just the point in the orbit obtained from applying the Lorentz transformation $\Lambda(g)^{-1}$ to p. Similarly, C acts as an odd antiunitary taking $\mathscr{H}_{(p_0,\mathbf{p})}$ to $\mathscr{H}_{(-p_0,-\mathbf{p})}$. Thus, the picture of $\mathscr{H}$ is a bundle of Hilbert spaces over the base space $\{p \in \hat{\mathscr{T}} = \hat{\mathbb{R}}^4 : p^2 = m^2\}$. The ${}^C\tilde{\mathscr{P}}$-action on the base space includes a discrete $\mathbb{Z}_2$-action by C, turning the bundle (when regarded as an ungraded complex vector bundle) into a Real $\tilde{\mathscr{P}}$-equivariant bundle in the sense of Section 7.3.3.

An alternative way to "fibre" $\mathscr{H}$ is to form the direct integral of *graded* Hilbert spaces over the positive energy hyperbola, labelled by $\mathbf{p} \in \hat{\mathbb{R}}^3$. Writing $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, this is the decomposition

$$\mathscr{H} = \int_{\mathbf{p}}^{\oplus} \mathscr{H}^+_{\mathbf{p}} \oplus \overline{\mathscr{H}}^-_{\mathbf{p}} := \int_{\mathbf{p}}^{\oplus} \mathscr{H}^+_{(\omega_{\mathbf{p}},\mathbf{p})} \oplus \mathscr{H}^-_{(-\omega_{\mathbf{p}},-\mathbf{p})}, \tag{5.11}$$

[5]The irreducible graded PUA-reps of $(L_p, c = 1, \phi, \sigma^{\alpha\beta})$ are the irreducible PUA-reps of $(L_p, \phi, \sigma^{\alpha\beta})$ from the previous section, with an additional ± 1 label. We choose the positive grading to retain consistency with $\Gamma = \mathrm{sgn}(H)$

[6]We have used $\mathbf{D}$ to denote $\mathbf{D}^{j(\pm)}_{\alpha\beta}$ to reduce congestion in the notation.

where we have written $\mathscr{H}^+_{\mathbf{p}} \equiv \mathscr{H}^+_{(\omega_{\mathbf{p}},\mathbf{p})} \cong \mathscr{H}^{\mathbf{D}}$ and so $\overline{\mathscr{H}^-_{\mathbf{p}}} \equiv \mathscr{H}^-_{(-\omega_{\mathbf{p}},-\mathbf{p})} \cong \overline{\mathscr{H}^{\mathbf{D}}}$; thus each graded fibre is the direct sum of a copy of $\mathscr{H}^{\mathbf{D}}$ with its parity-reversed conjugate $\overline{\mathscr{H}^{\mathbf{D}}}$. In this decomposition, C acts fibrewise as an antiunitary operator exchanging the two summands.

Indeed, this is the picture that we get if we had chosen to induce with respect to the abelian subgroup $N = \mathscr{T} \times \{1, C\}$ instead. Then, with $\sigma|_{N\times N}$ trivial, the graded generalised dual $\hat{N}^\sigma$ consists of direct sums $\chi^+_{(p_0,\mathbf{p})} \oplus \chi^-_{(-p_0,-\mathbf{p})} = \chi^+_{(p_0,\mathbf{p})} \oplus \overline{\chi}^-_{(p_0,\mathbf{p})}$ of graded characters of $\mathscr{T}$, with C represented as the odd antiunitary $\begin{pmatrix} 0 & \kappa \\ \kappa & 0 \end{pmatrix}$. From (5.10), we see that the action of C on $\hat{N}^\sigma$ is trivial, as it should be. The actions of P and T on $\hat{N}^\sigma$ are

$$(\chi^+_{(p_0,\mathbf{p})} \oplus \overline{\chi}^-_{(p_0,\mathbf{p})}) \cdot P = (\chi^+_{(p_0,-\mathbf{p})} \oplus \overline{\chi}^-_{(p_0,\mathbf{p})}) \cdot T = \chi^+_{(p_0,-\mathbf{p})} \oplus \overline{\chi}^-_{(p_0,-\mathbf{p})}.$$

Let us concentrate on the massive orbits ($p^2 = m^2 > 0, p_0 > 0$) of the ${}^C\tilde{\mathscr{P}}$-action. We write $\chi^\pm_{\mathbf{p}} \coloneqq \chi^\pm_{(\omega_{\mathbf{p}},\mathbf{p})}$ for $\mathbf{p} \in \hat{\mathbb{R}}^3$. The orbit of $\chi^+_{\mathbf{0}} \oplus \overline{\chi}^-_{\mathbf{0}}$ is $\{\chi^+_{\mathbf{p}} \oplus \overline{\chi}^-_{\mathbf{p}} : \mathbf{p} \in \mathbb{R}^3\}$ and its little group is again $L_p = \mathrm{SU}(2) \times F_4$. We can extend the irreducible PUA-reps $\mathbf{D}$ of $(L_p, 1, \phi, \sigma^{\alpha\beta})$ to an ungraded PUA-rep of $((\mathscr{T} \rtimes L_p) \times \{1, C\}, \phi, \sigma^{\alpha\beta})$ with C acting as complex conjugation. Then tensoring $\chi^+_{\mathbf{0}} \oplus \overline{\chi}^-_{\mathbf{0}}$ with $\mathbf{D}$ (in the graded sense) gives an irreducible graded PUA-rep of $((\mathscr{T} \rtimes L_p) \times \{1, C\}, c, \phi, \sigma^{\alpha\beta})$, on a representation space isomorphic to the graded fibre $\mathscr{H}^+_{\mathbf{0}} \oplus \overline{\mathscr{H}^-_{\mathbf{0}}}$ in (5.11). This can then be induced up to a graded PUA-rep of $({}^C\tilde{\mathscr{P}}, c, \phi, \sigma^{\alpha\beta})$.

5.3 Relation to relativistic wave equations

The relativistic wave equations, such as the Klein–Gordon and Dirac equations, are closely connected to the irreducible projective representations of $\mathscr{P}^\uparrow_+$, as explained in detail in [63, 64]. For example, the positive energy solutions to the Klein–Gordon equation form an irreducible representation of $\mathscr{P}^\uparrow_+$. This is clear in momentum space, where the solutions are supported on the positive energy hyperbola of $p^2 = m^2$. They also form an irreducible UA-rep of $\mathscr{P}$, according to the first line of Table 5.2 (alongside the spin-0 representation of SU(2)). Similar considerations apply to spin-$\frac{1}{2}$ fields, with the positive energy Dirac spinors furnishing an irreducible PUA-rep of $\mathscr{P}$. The positive energy condition $p_0 > 0$, is non-local (in configuration space), and necessitates a doubling of the orbit to include the $p_0 < 0$ hyperbola as well [64]. The representation space is then reducible for $\mathscr{P}$, graded by $\mathrm{sgn}(p_0)$, with the two $\mathscr{P}$-irreducible components intertwined by the antiunitary operation of charge-conjugation C. In this way, the full symmetry group is augmented to ${}^C\mathscr{P}$. What we have done above is to assume this larger symmetry group from the outset, instead of discovering the C symmetry afterwards; then we proceeded to look for its irreducible *graded* PUA-reps.

5.4 Graded PUA-representations of parity groups $\{\pm 1\}^n$

In [8, 7], a procedure for finding the irreducible PUA-reps of a direct product $\{\pm 1\}^n$, where m of the $\{\pm 1\}$ generators U_i are represented unitarily ($\phi(U_i) = +1$) and the other $n - m$ generators A_j are represented antiunitarily ($\phi(A_j) = -1$), was described. The basic strategy was to rephrase the problem into one of finding the irreducible representations of an associated algebra generated by the representatives of the group generators subject to certain commutation/anticommutation relations. The authors used the complex Clifford algebras $\mathbb{C}l_n$ and their representation theory to simplify some special cases. We describe how to generalise this to graded PUA-reps, using the representation theory of real Clifford algebras $Cl_{r,s}$. We had already faced a particular instance of this problem in the earlier sections, when considering the PUA-reps of F_4. In general, we may be interested in graded PUA-reps of F_8, or some other group involving other types of parities, like G-parity or R-parity, or perhaps other $\{\pm 1\}$ subgroups appearing in some point group of a crystal.

Let $G = \{\pm 1\}^n, n \geq 0$, $\{\pm 1\}^0 \equiv \{1\}$, where $0 \leq m \leq n$ of the $\{\pm 1\}$ generators $U_i, 1 \leq i \leq m$, are to be represented unitarily ($\phi(U_i) = +1$), while the other $n - m$ generators $A_k, 1 \leq k \leq n - m$, are to be represented antiunitarily ($\phi(A_k) = -1$). Let $c : \{\pm 1\}^n \to \{\pm 1\}$ be the homomorphism which indicates whether θ_g, $g \in \{\pm 1\}^n$ is even or odd, and θ be a graded PUA-rep for (G, c, ϕ, σ). For brevity, we will also write $\mathsf{U}_i := \theta_{U_i}$ and $\mathsf{A}_k := \theta_{A_k}$ for the unitary/antiunitary representatives of U_i and A_k respectively.

Lemma 5.4.1. *We may assume, without loss of generality, that there are at most two antiunitaries* $\mathsf{A}_1, \mathsf{A}_2$.

Proof. Let A be the image of the homomorphism $(\phi, c) : \{\pm 1\}^n \to \{\pm 1\}\times\{\pm 1\}$, and B be its kernel. Every non-identity element of A, B and $\{\pm 1\}^n$ has order 2. Regarding the groups as finite-dimensional vector spaces over the two-element field $\mathbb{F}_2$, we have $\{\pm 1\}^n \cong B \times A$. Any $\mathbb{F}_2$-basis for B provides a set of even unitarily implemented generators U_i. Since $\dim(A) \leq 2$, there are at most two $A_k \in A$ with $\phi(A_k) = -1$ providing antiunitary operators A_k. $\square$

A basis for A can be chosen to be one of the following: (i) empty, (ii) $\{$odd $U_n\}$, (iii) $\{$even $A_1\}$, (iv) $\{$odd $A_1\}$, or (v) $\{$even A_1, odd $A_2\}$. We proceed to study the possible cocycles σ for $G = \{\pm 1\}^n$, and to fix representatives for their cocycle classes.

Since $\mathsf{U}_i^2 = \sigma(U_i, U_i)\theta_e = \sigma(U_i, U_i)$, we can make the modification $\mathsf{U}_i \mapsto \pm\sigma(U_i, U_i)^{-1/2}\mathsf{U}_i$ to fix $\mathsf{U}_i^2 = +1$. This does not work for A_k, since $(\lambda\mathsf{A}_k)(\lambda\mathsf{A}_k) = \lambda\overline{\lambda}\mathsf{A}_k^2 = \mathsf{A}_k^2$ for any $\lambda \in \mathbb{T}$. Setting $x = y = z = A_k$ in (1.1) leads to $\sigma(A_k, A_k) = \overline{\sigma(A_k, A_k)}$, so $\mathsf{A}_k^2 = \sigma(A_k, A_k) = \pm 1$ are invariants of the cocycle class of σ. Next, we look at the commutation relations amongst

U_i and A_k. For two unitaries $\mathsf{U}_i, \mathsf{U}_j$, $i < j$, we write $\lambda_{ij} \coloneqq \sigma(U_i, U_j)/\sigma(U_j, U_i)$ so that $\mathsf{U}_i\mathsf{U}_j = \lambda_{ij}\mathsf{U}_j\mathsf{U}_i$. Then

$$\mathsf{U}_j = \mathsf{U}_i^2\mathsf{U}_j = \lambda_{ij}^2\mathsf{U}_j\mathsf{U}_i^2 = \lambda_{ij}^2\mathsf{U}_j,$$

so fixing $\mathsf{U}_i^2 = +1$ leads us to $\lambda_{ij} = \pm 1$. For any two complex scalars a_1, a_2, we have $(a_1\mathsf{A}_1)(a_2\mathsf{A}_2) = a_1\bar{a}_2\sigma(A_1, A_2)\theta_{A_1A_2}$, as well as $(a_2\mathsf{A}_2)(a_1\mathsf{A}_1) = a_2\bar{a}_1\sigma(A_2, A_1)\theta_{A_1A_2}$. By choosing $a_1 = \pm\sigma(A_2, A_1)^{1/2}, a_2 = \pm\sigma(A_1, A_2)^{1/2}$, and making the adjustments $\mathsf{A}_1' = a_1\mathsf{A}_1, \mathsf{A}_2' = a_2\mathsf{A}_2$ and $\theta'_{A_1A_2} = a_1a_2\theta_{A_1A_2}$, we obtain $\mathsf{A}_1'\mathsf{A}_2' = \theta'_{A_1A_2} = \mathsf{A}_2'\mathsf{A}_1'$. Finally, we look at $\mathsf{U}_i\mathsf{A}_k = \nu_{ik}\mathsf{A}_k\mathsf{U}_i$, where $\nu_{ik} \coloneqq \sigma(U_i, A_k)/\sigma(A_k, U_i)$. The equation

$$\mathsf{A}_k = \mathsf{U}_i^2\mathsf{A}_k = \nu_{ik}^2\mathsf{A}_k\mathsf{U}_i^2 = \nu_{ik}^2\mathsf{A}_k$$

means that $\nu_{ik} = \pm 1$, which cannot be fixed by utilising the residual ± 1 phase freedom in $\mathsf{U}_i, \mathsf{A}_k$. We summarise this discussion in the following Proposition:

Proposition 5.4.2. *Let θ be a graded PUA-rep of $(\{\pm 1\}^n, c, \phi, \sigma)$, with $0 \leq m \leq n$ unitarily implemented group generators U_i and $0 \leq n - m \leq 2$ antiunitarily implemented group generators A_k, as in Lemma 5.4.1. We can adjust U_i and A_k, while staying in the same cocycle class, so that $\mathsf{U}_i^2 = +1$, $\mathsf{A}_k^2 = \pm 1$, $\mathsf{A}_k\mathsf{A}_l = \mathsf{A}_l\mathsf{A}_k$, $\mathsf{U}_i\mathsf{U}_j = \pm\mathsf{U}_j\mathsf{U}_i$, and $\mathsf{U}_i\mathsf{A}_k = \pm\mathsf{A}_k\mathsf{U}_i$, for all $1 \leq i, j \leq m$ and $1 \leq k, l \leq n - m$.*

Let $\mathbf{i} = \{i_1, \ldots, i_p\}$ and $\mathbf{k} = \{k_1, \ldots, k_q\}$ be (possibly empty) ordered subsets of $\{1, \ldots, n\}$ and $\{1, \ldots, n - m\}$ respectively, with the elements listed in increasing order. Let $U_{\mathbf{i}}$ and $A_{\mathbf{k}}$ denote the group elements $U_{i_1} \ldots U_{i_p}$ and $A_{k_1} \ldots A_{k_q}$ respectively, with $U_\emptyset = 1 = A_\emptyset$. Every element of $\{\pm 1\}^n$ can clearly be written uniquely as $U_{\mathbf{i}}A_{\mathbf{k}}$ for some $\mathbf{i}, \mathbf{k}$. We have already used the phase freedom in $\mathsf{U}_i, \mathsf{A}_k$ and $\theta_{A_1A_2}$ to obtain the standardisations in Proposition 5.4.2. We can use the phase freedom for the remaining group elements to fix the representatives $\theta_{U_{\mathbf{i}}A_{\mathbf{k}}} = \mathsf{U}_{i_1} \ldots \mathsf{U}_{i_p}\mathsf{A}_{k_1} \ldots \mathsf{A}_{k_q}$ for all $\mathbf{i}, \mathbf{k}$, without changing the cocycle class of σ. The resulting representative cocycle for this standardised θ is then completely determined by the set of ± 1 in Proposition 5.4.2.

Note that there is a residual freedom to modify $\mathsf{U}_i \mapsto \pm\mathsf{U}_i$, $\mathsf{A}_k \mapsto \pm\mathsf{A}_k$ (and thereby all the $\theta_{U_{\mathbf{i}}A_{\mathbf{k}}}$). This is the freedom to multiply θ by a generalised character[7], which does not change the cocycle function $\sigma : \{\pm 1\}^n \times \{\pm 1\}^n \to \mathbb{T}$. These discrete modifications may, however, change the (non-projective) unitary equivalence class of the (graded) PUA-representation of $(\{\pm 1\}^n, c, \phi, \sigma)$.

Remark 5.4.3. The above arguments can be specialised to handle ungraded PUA-reps of $(\{\pm 1\}^n, \phi, \sigma)$. In Lemma 5.4.1, there is zero or one antiunitary A depending on the image of ϕ, and Proposition 5.4.2 holds with $n - m \leq 1$.

[7]A generalised character on (G, ϕ) is a phase function $\chi : G \to \mathbb{T}$ which satisfies $\chi(x)\chi(y)^x\chi(xy)^{-1} = 1$ for all $x, y \in G$.

Algebras associated to cocycle classes

Suppose $\phi \not\equiv 1$. Each set of ± 1 in Proposition 5.4.2 specifies a standardised cocycle σ in some cocycle class, and is associated to the unital real $*$-algebra generated by $\{\mathsf{U}_i, \mathsf{A}_k, \mathrm{i}\}$, subject to relations specified by their squares $(= \pm 1)$, $\mathsf{U}_i^*\mathsf{U}_i = 1 = \mathsf{A}_k^*\mathsf{A}_k$, $\mathrm{i}^* = -\mathrm{i}$, $\mathrm{i}^2 = -1$, as well as the commutation relations between them. In the graded case, we introduce a grading on the algebra by taking i to be even, and taking the generators $\mathsf{U}_i, \mathsf{A}_k$ to be even/odd according to the signs of $c(U_i), c(A_k)$. Since the algebra generators square to some real scalar, this algebra grading is well-defined. The *graded* $*$-representations of this *graded* $*$-algebra then correspond to the graded PUA-reps for $(\{\pm 1\}^n, c, \phi, \sigma)$. Note that i is orthogonal in such representations, so real $*$-representations can be transcribed back into complex $*$-representations with U_i unitary and A_k antiunitary. If $\phi \equiv 1$, we do not include i as a real algebra generator, but look instead at the unital complex $*$-algebra generated by unitary elements U_i, subject to $\mathsf{U}_i^2 = \pm 1$ and $\mathsf{U}_i\mathsf{U}_j = \pm \mathsf{U}_j\mathsf{U}_i$. For a given (standardised) cocycle, there may be more than one equivalence class of $*$-representations for the associated algebra — this corresponds to the possibility that utilizing the ± 1 residual phase freedom within a PUA-rep can change its unitary equivalence class.

An alternative approach in the graded case is to supplement the set of ungraded algebra generators $\{\mathsf{U}_i, \mathsf{A}_k, \mathrm{i}\}$ with an additional algebra generator Γ, which satisfies $\Gamma^*\Gamma = 1 = \Gamma^2$, commutes with i, and commutes or anticommutes with $\mathsf{U}_i, \mathsf{A}_k$ according to the values of $c(U_i), c(A_k)$. A $*$-representation for this larger ungraded algebra is, in particular, a PUA-rep for $(\{\pm 1\}^n, \phi, \sigma)$, on which Γ is represented as a self-adjoint grading operator furnishing a graded PUA-rep for $(\{\pm 1\}^n, c, \phi, \sigma)$.

One motivation for passing to the associated (graded) real/complex algebra, is that the algebras turn out to resemble the real/complex Clifford algebras $Cl_{r,s}$ and $\mathbb{C}l_n$, whose (graded) representation theory is well known [54, 31]. Here, $Cl_{r,s}$ is the real Clifford algebra with $r+s$ mutually anticommuting generators, where r generators square to -1 and s generators square to $+1$. We will sometimes write $Cl_n \coloneqq Cl_{n,0}$. The Clifford algebras have standard involutions and gradings when regarded as graded $*$-algebras; namely, the Clifford generators are odd, and positive (resp. negative) Clifford generators are self-adjoint (resp. skew-adjoint). Equivalently, they are odd and orthogonal/unitary and square to ± 1. Let us work out how the Clifford algebras arise in specific examples.

5.4.1 Clifford algebras associated to PUA-reps of the PT-group F_4

Let $F_4 = \{1, P\} \times \{1, T\}$, $\phi(P) = +1, \phi(T) = -1$, which is the discrete subgroup of space and time reflections in the full Poincaré group $\mathscr{P}$. We use P and T to denote, respectively, the representatives of P and T in a PUA-rep θ of (F_4, ϕ, σ). In the canonical form of

α	β	T^2	$(\mathsf{iT})^2$	P^2	$(\mathsf{iP})^2$	Clifford algebra	Morita class
$+1$	$+1$	$+1$	$+1$		-1	$Cl_{1,2} = M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$	Cl_7
-1	-1	-1	-1		-1	$Cl_{3,0} = \mathbb{H} \oplus \mathbb{H}$	Cl_3
$+1$	-1	$+1$	$+1$	$+1$		$Cl_{0,3} = M_2(\mathbb{C})$	Cl_5
-1	$+1$	-1	-1	$+1$		$Cl_{2,1} = M_2(\mathbb{C})$	Cl_1

Table 5.4: Cocycle classes for $F_4 = \{1, P\} \times \{1, T\}$, their associated Clifford algebras, and their generators.

Proposition 5.4.2, $\mathsf{P}^2 = +1$, $\mathsf{T}^2 = \alpha, \mathsf{PT} = \alpha\beta\mathsf{TP}$, where $\alpha, \alpha\beta = \pm 1$, so there are four cocycle classes labelled by $\alpha, \beta = \pm 1$. Furthermore, we can choose $\theta_{PT} = \mathsf{PT}$. With these choices, the values of $\sigma(\cdot, \cdot)$ for each of the four cocycle classes can be calculated, and are summarised in Table 5.1.

The real unital $*$-algebra associated to each choice of α, β is generated by orthogonal elements P, T, i which square to ± 1. For $\alpha = \beta$, we may use the alternative anticommuting generating set $\{\mathsf{T}, \mathsf{iT}, \mathsf{iP}\}$; for $\alpha \neq \beta$, we use $\{\mathsf{T}, \mathsf{iT}, \mathsf{P}\}$. Then it is clear that a Clifford algebra is generated in each case, as summarised in Table 5.4.

We make a few observations from this presentation. The Clifford algebras are either real simple algebras, or a direct sum of two real simple algebras. Thus, there are two inequivalent irreducible representations when $\alpha = \beta$, but only one irreducible representation when $\alpha \neq \beta$ — this accounts for the extra $\pm$ label in Table 5.2. In the two $\alpha = \beta$ cases, the central element formed from the product of the three Clifford generators is (a multiple of) P. It follows that the two inequivalent representations in the $\alpha = \beta$ cases are distinguished by whether $\mathsf{P} = \pm 1$. The (real) dimension of the irreducible representation(s) in all cases can be read off directly from the forms of the Clifford algebras. The matrices in Table 5.2 can be obtained by rewriting the real representations as complex PUA-reps. Finally, their commutants, which are real division algebras, can be read off from the type of matrix algebra in the corresponding Clifford algebra. This immediately yields the respective Wigner corepresentaion types in Table 5.2.

5.4.2 Clifford algebras associated to graded PUA-reps of CT-subgroups — the tenfold way

We now consider subgroups A of the CT-group $\{\pm 1\}^2 = \{1, T, C, S\}$, which has $(\phi, c)(T) = (-1, +1), (\phi, c)(C) = (-1, -1), (\phi, c)(S) = (+1, -1)$. The elements T, C and $S = CT = TC$ refer to time-reversal, charge-conjugation, and sublattice symmetries respectively. We wish to study the graded PUA-reps of (A, σ), where the homomorphisms ϕ and c are implicit. We will identify A with its image under (ϕ, c), and denote the representatives of T, C and S (where present) by T, C and S respectively. First, we consider the full CT-group

Generators of A	C^2	T^2	Associated algebra	Ungraded Clifford algebra	Graded Morita class
T		$+1$	$M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$	$Cl_{1,2}$	$Cl_{0,0}$
C, T	-1	$+1$	$M_4(\mathbb{R})$	$Cl_{2,2}$	$Cl_{1,0}$
C	-1		$M_2(\mathbb{C})$	$Cl_{2,1}$	$Cl_{2,0}$
C, T	-1	-1	$M_2(\mathbb{H})$	$Cl_{3,1}$	$Cl_{3,0}$
T		-1	$\mathbb{H} \oplus \mathbb{H}$	$Cl_{3,0}$	$Cl_{4,0}$
C, T	$+1$	-1	$M_2(\mathbb{H})$	$Cl_{0,4}$	$Cl_{5,0}$
C	$+1$		$M_2(\mathbb{C})$	$Cl_{0,3}$	$Cl_{6,0}$
C, T	$+1$	$+1$	$M_4(\mathbb{R})$	$Cl_{1,3}$	$Cl_{7,0}$
N/A	N/A		$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}l_1$	$\mathbb{C}l_0$
S	$\mathsf{S}^2 = +1$		$M_2(\mathbb{C})$	$\mathbb{C}l_2$	$\mathbb{C}l_1$

Table 5.5: The ten classes of graded PUA-reps of (A, σ), where A is a subgroup of the CT-group, along with their corresponding Clifford algebras.

$A = \{1, T, C, S\}$. In the standard form of Proposition 5.4.2, there are four choices $\mathsf{T}^2 = \pm 1, \mathsf{C}^2 = \pm 1$, and we may assume that $\mathsf{TC} = \mathsf{S} = \mathsf{CT}$. In each of these four cases, there is an associated real algebra generated by $\{\mathrm{i}, \mathsf{T}, \mathsf{C}, \Gamma\}$, with commutation relations determined by ϕ and c, as well as $\mathrm{i}\Gamma = \Gamma\mathrm{i}$. Equivalently, we can choose the generating set $\{\Gamma, \mathsf{C}, \mathrm{i}\mathsf{C}, \mathrm{i}\mathsf{CT}\}$, whose elements are mutually anticommuting, and are self-adjoint (resp. skew-adjoint) if they square to $+1$ (resp. -1). The latter choice shows explicitly that a graded PUA-rep for the full CT-group is precisely an ungraded $*$-representation of a certain real Clifford algebra $Cl_{r,s}$, where r and s are determined by the squares of the algebra generators in $\{\Gamma, \mathsf{C}, \mathrm{i}\mathsf{C}, \mathrm{i}\mathsf{CT}\}$.

For $A = \{1, T\}$, there are two choices for the cocycle class of σ according to $\mathsf{T}^2 = \pm 1$. The anticommuting set $\{\mathrm{i}, \mathsf{T}, \mathrm{i}\mathsf{T}\Gamma\}$ generates the associated real algebra $Cl_{1,2}$ or $Cl_{3,0}$. For $A = \{1, C\}$, there are again two choices $\mathsf{C}^2 = \pm 1$. The associated real algebra is generated by the anticommuting set $\{\Gamma, \mathsf{C}, \mathrm{i}\mathsf{C}\}$, leading to $Cl_{0,3}$ or $Cl_{2,1}$. For the subgroup $A = \{1, S\}$, there is only one standard choice $\mathsf{S}^2 = +1$. The associated *complex* algebra is generated by $\{\Gamma, \mathsf{S}\}$, with $\Gamma\mathsf{S} = -\mathsf{S}\Gamma$ — this is the complex Clifford algebra $\mathbb{C}l_2$. Finally, the trivial subgroup $A = \{1\}$ leads easily to the complex Clifford algebra $\mathbb{C}l_1$, generated by Γ. We summarise this discussion in Table 5.5.

The Morita equivalence classes are obtained by the isomorphisms

$$Cl_{r,s} \otimes Cl_{1,1} \cong Cl_{r+1,s+1} \tag{5.12a}$$

$$Cl_{n,0} \otimes Cl_{0,2} \cong Cl_{0,n+2} \tag{5.12b}$$

$$Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{n+2,0}, \tag{5.12c}$$

and the periodicities

$$\begin{aligned}\mathbb{C}l_n \otimes M_2(\mathbb{C}) &\cong \mathbb{C}l_{n+2} &\quad (5.13a)\\ Cl_{n,0} \otimes M_{16}(\mathbb{R}) &\cong Cl_{n+8,0} &\quad (5.13b)\\ Cl_{0,n} \otimes M_{16}(\mathbb{R}) &\cong Cl_{0,n+8}. &\quad (5.13c)\end{aligned}$$

The graded projective unitary-antiunitary representation theory of a CT-subgroup is therefore identical to the representation theory of the appropriate (ungraded) Clifford algebra $\mathbb{C}l_n$ or $Cl_{r,s}$.

There is an equivalence between ungraded $Cl_{r,s+1}$-modules and graded $Cl_{r,s}$-modules, where in the latter, the convention is that the degree 1 (mod 2) elements of $Cl_{r,s}$ are represented as odd operators. Similarly, there is an equivalence between the categories of ungraded $\mathbb{C}l_n$-modules and graded $\mathbb{C}l_{n+1}$-modules (and also $\mathbb{C}l_{n-1}$-modules, by the periodicity (5.13a)). Thus, an alternative approach excludes Γ as a Clifford generator, and regards the algebra generated by the relevant subset of the operators $\{\mathrm{i}, \mathsf{T}, \mathsf{C}, \mathsf{S}\}$ as a *graded* algebra. Except for the case $A = \{1, T\}$, the generating set of *odd*, mutually anticommuting operators may be taken from $\{\mathsf{C}, \mathrm{i}\mathsf{C}, \mathrm{i}\mathsf{CT}\}$ or $\{\mathsf{S}\}$, which generates a *graded* Clifford algebra with one fewer positive Clifford generator compared to the first approach. Then, a graded PUA-rep for (A, σ) is precisely a *graded* $*$-representation of the corresponding *graded* Clifford algebra, with grading operator Γ. However, the subgroup $A = \{1, T\}$ cannot be handled directly in this way, see Remark 5.4.4.

Remark 5.4.4. In the two cases for $A = \{1, T\}$, the grading operator is not explicitly given as one of the Clifford generators in the associated ungraded Clifford algebra. In fact, $\{\mathrm{i}, \mathsf{T}\}$ generates the purely even algebra $M_2(\mathbb{R})$ or $\mathbb{H}$, which *commutes* with Γ in a graded PUA-rep of (A, σ). We can fix this by recalling the Morita equivalence between $Cl_{r,s}$ and $Cl_{r,s} \otimes Cl_{1,1} \cong Cl_{r+1,s+1}$. We introduce two extra Clifford generators e, f with $\mathsf{e}^2 = -1, \mathsf{f}^2 = +1$, so that $\{\mathrm{i}, \mathsf{T}, \mathrm{i}\mathsf{T}\Gamma, \mathsf{e}, \mathsf{f}\}$ forms a mutually anticommuting set of Clifford generators for either $Cl_{1,2} \otimes Cl_{1,1} = Cl_{2,3}$ (when $\mathsf{T}^2 = +1$) or $Cl_{3,0} \otimes Cl_{1,1} = Cl_{4,1}$ (when $\mathsf{T}^2 = -1$). An alternative generating set of mutually anticommuting operators is $\{\Gamma, \mathsf{e}, \mathrm{i}\mathsf{e}, \mathsf{eT}, \mathrm{i}\mathsf{fT}\}$. We can now exclude Γ and let $\{\mathsf{e}, \mathrm{i}\mathsf{e}, \mathsf{eT}, \mathrm{i}\mathsf{fT}\}$ generate the graded Clifford algebra $Cl_{2,2}$ or $Cl_{4,0}$. We can now look for graded $*$-representations of $Cl_{2,2}$ of $Cl_{4,0}$ as a graded Clifford algebra. In such a representation, the self-adjoint involutory operator (ef) commutes with each of $\Gamma, \mathrm{i}, \mathsf{T}$, so the restriction of the latter to the $(+1)$-eigenspace of (ef) gives a graded PUA-rep of $(\{1, T\}, \sigma)$.

The above is an important observation: when we deal with K-theory later on, we will be interested in the various ways in which a fixed ungraded action of a graded Clifford algebra can be supplemented with a compatible grading operator Γ.

$\mathsf{PC} = \pm\mathsf{CP}$	$\mathsf{PT} = \pm\mathsf{TP}$	Ξ
$+1$	$+1$	$\Gamma\mathsf{PCT}$
$+1$	-1	$\mathrm{i}\mathsf{PCT}$
-1	$+1$	PCT
-1	-1	$\mathrm{i}\Gamma\mathsf{PCT}$

Table 5.6: Classes of graded PUA-reps of the full PCT-group. The extra Clifford generator Ξ is given in the last column.

5.4.3 Graded PUA-reps of the PCT-subgroups

A similar analysis can be carried out for the full PCT-group $\{1, P\} \times \{1, C\} \times \{1, T\}$. Here, the space inversion element P is taken to act as an even unitary operator, i.e., $(\phi, c)(P) = (+1, +1)$. In standard form, the cocycle classes are specified by the squares $\mathsf{C}^2 = \pm 1, \mathsf{T}^2 = \pm 1, \mathsf{P}^2 = +1$, along with the commutation relations $\mathsf{PC} = \pm\mathsf{CP}, \mathsf{PT} = \pm\mathsf{TP}$. We can get the associated algebras from those constructed in the case of the CT-group, by incorporating an additional generator P with the appropriate commutation relations with the set $\{\Gamma, \mathsf{C}, \mathrm{i}\mathsf{C}, \mathrm{i}\mathsf{CT}\}$. It turns out that for each cocycle class, we can construct, in place of P, an alternative generator Ξ which anticommutes with $\{\Gamma, \mathsf{C}, \mathrm{i}\mathsf{C}, \mathrm{i}\mathsf{CT}\}$, and squares to ± 1. This extra Clifford generator is listed in Table 5.6. The associated algebra is then the graded tensor product of $Cl_{0,1}$ (if $\Xi^2 = +1$) or $Cl_{1,0}$ (if $\Xi^2 = -1$), with the original algebra $Cl_{r,s}$ associated to the CT-group. This new algebra is another Clifford algebra, according to the identities

$$Cl_{r,s} \hat{\otimes} Cl_{0,1} \cong Cl_{r,s+1}$$
$$Cl_{r,s} \hat{\otimes} Cl_{1,0} \cong Cl_{r+1,s}.$$

A similar analysis can be carried out for proper subgroups of the PCT group. We have already handled subgroups of the CT-group in Section 5.4.2. The remaining order-2 subgroups are easy to handle. The order-4 subgroups $\{1, PC\} \times \{1, T\}$ and $\{1, PT\} \times \{1, C\}$ are identical to the case of the CT or PT-group, with PC and PT taking on the roles of C and T. The remaining order-4 subgroups, $\{1, P\} \times \{1, C\}$ and $\{1, P\} \times \{1, CT\} \equiv \{1, P\} \times \{1, S\}$, are a little more tricky. To illustrate, let us work out the $\{1, P\} \times \{1, C\}$ case. Here, the associated algebras can be identified with complex Clifford algebras, provided we take the imaginary unit to be $\mathrm{i}\Gamma$. The complex-linear (with respect to $\mathrm{i}\Gamma$) *positive* generators of the Clifford algebra, $\mathbb{C}l_3$ are then given as in Table 5.7.

$\mathsf{PC} = \pm\mathsf{CP}$	C^2	Clifford algebra generators
$+1$	$+1$	$\mathsf{C}, \mathrm{i}\mathsf{C}, \Gamma\mathsf{P}$
$+1$	-1	$\Gamma\mathsf{C}, \mathrm{i}\Gamma\mathsf{C}, \Gamma\mathsf{P}$
-1	$+1$	$\mathsf{C}, \mathrm{i}\mathsf{C}, \mathsf{P}$
-1	-1	$\Gamma\mathsf{C}, \mathrm{i}\Gamma\mathsf{C}, \mathsf{P}$

Table 5.7: Graded PUA-reps of the PC-group $\{1, P\} \times \{1, C\}$. The associated Clifford algebra $Cl_{r,s}$ is determined by the squares of the anticommuting generators in the last column.

5.5 Clifford algebras as (graded) twisted group algebras

While we have constructed explicit identifications between the (graded) PUA-reps of the (subgroups of) CT and PCT groups, and (graded) Clifford algebra representations, it is instructive to recall how the Clifford algebras are themselves twisted group algebras over $\mathbb{R}$ of some parity group $\mathbb{Z}_2^n$, twisted by some cocycle σ. For the moment, we use additive notation for the elements of $\mathbb{Z}_2^n = \{0,1\}^n$. Then the real Clifford algebras $Cl_{r,s}$ are isomorphic to the twisted group algebras $\mathbb{R} \rtimes_{(1,\sigma_{r,s})} \mathbb{Z}_2^{r+s}$, where the cocycles $\sigma_{r,s}$ are given by the formula

$$\sigma_{r,s}(\mathbf{x}, \mathbf{y}) = \big(-1\big)^{\sum\limits_{j<i} x_i y_j + \sum\limits_{i\leq r} x_i y_i}, \tag{5.14}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_2^{r+s}$ are $(r+s)$-component binary vectors.

Similarly, the complex Clifford algebras have a description as twisted group algebras $\mathbb{C}l_n \cong \mathbb{C} \rtimes_{(1,\sigma_n)} \mathbb{Z}_2^n$ over the complex field, where

$$\sigma_n(\mathbf{x}, \mathbf{y}) = \big(-1\big)^{\sum\limits_{j<i} x_i y_j}.$$

We can push this description of the Clifford algebras further by identifying them with graded twisted group algebras of $\mathbb{Z}_2^n$. This is done by defining the homomorphism c on $\mathbb{Z}_2^n$ to be -1 on each generator of a $\mathbb{Z}_2$ factor. The twisted group algebras $\mathbb{R} \rtimes_{(1,\sigma_{r,s})} \mathbb{Z}_2^{r+s}$ and $\mathbb{C} \rtimes_{(1,\sigma_n)} \mathbb{Z}_2^n$ then acquire a grading according to c, and they can be identified as the graded Clifford algebras $Cl_{r,s}$ and $\mathbb{C}l_n$ with their standard gradings.

In the examples of Section 5.4.2, we start off with a symmetry group A, assumed to be some subgroup of the CT-group $\mathbb{Z}_2^2$. The constructions showing the equivalence of graded PUA-reps of (A, c, ϕ, σ) with graded Clifford modules, is precisely the standardisation of the cocycle σ into a form which explicitly demonstrates the graded twisted group algebra $\mathbb{C} \rtimes_{(\alpha,\sigma)} A$ as a real or complex Clifford algebra. One of the crucial steps in this identification is the inclusion of i as a real Clifford algebra generator, or as an extra parity group generator, whenever $\phi \not\equiv 1$. Similarly, Γ is included as an extra parity group or Clifford algebra generator where it is convenient to do so. To summarise: *we may interpret a (graded) Clifford algebra as the (graded) twisted group algebra of some subgroup of the CT-group.*

As an aside, we note that the Clifford algebras are also related to linear representations of the Clifford groups. The *Clifford group* $CL_{r,s}$ can be presented as the finite group with generators $\{(-1), e_1, \ldots, e_{r+s}\}$, subject to the relation that (-1) is central[8], as well as the relations

$$\begin{aligned} e_i^2 &= (-1), && 1 \leq i \leq r, \\ e_j^2 &= 1, && r+1 \leq j \leq r+s, \\ e_i e_j &= (-1) e_j e_i, && i \neq j. \end{aligned}$$

A real $Cl_{r,s}$ representation gives rise to a real linear representation of $CL_{r,s}$ for which the group element (-1) acts as -1, the converse being true as well. Thus, $Cl_{r,s}$ is almost the (ordinary untwisted) group algebra of $CL_{r,s}$, the correct relationship being

$$Cl_{r,s} = \mathbb{R}[CL_{r,s}]/\mathbb{R} \cdot \{(-1) + 1\},$$

where $\mathbb{R} \cdot \{(-1) + 1\}$ is the ideal generated by the relation $(-1) + 1 = 0$.

[8] (-1) denotes a group generator, which is not (yet) the additive inverse in $\mathbb{R}$.

Chapter 6

Symmetry-compatible Hamiltonians, topological triviality, and K-theory

The twisted group algebra $\mathbb{C} \rtimes_{(\alpha,\sigma)} G$ has a K-theory which is closely linked to the PUA-reps of (G, ϕ, σ). In Section 6.1, we recall this connection and motivate the study of the K-theory of twisted crossed products in general. To handle charge-conjugating symmetries, we need to work with graded crossed products and their graded modules. We stress that *there is no single "K-theory classification" of topological phases; rather, different types of K-theory objects are needed depending on the precise classification problem at hand.*

In Section 6.2, we define the notion of a *super-representation group* for graded algebras, which is motivated by and formalises the notion of "topological triviality" [50, 30], as explained in Section 6.2.3. In the case of the graded Clifford algebras, which may we interpret as twisted group algebras of CT-subgroups (Section 5.5), the super-representation groups are related to the K-theory groups of a point through the Atiyah–Bott–Shapiro isomorphisms [4]. A detour is made in Section 6.3 to investigate multiplicative structures on super-representation groups, where super-division algebras play a prominent role; the latter are viewed as a unifying idea underlying the tenfold way in [30, 60, 61].

Section 6.4 contains our most important definitions, including that of the *difference-group* of symmetry-compatible gapped Hamiltonians. These definitions are crucial for the application of various K-theoretic tools in Chapter 8. They provide a precise and natural connection between Karoubi's picture of K-theory [42], and the notion of "homotopic phases" which often appears in the literature.

6.1 Prelude: Ungraded representation rings and group algebras

Let G be a finite group. Its representation theory can be studied in various ways, but we choose to look at group algebras and their modules, in order to make the passage to more general groups and K-theory more natural. By the theorems of Maschke and Artin–Wedderburn, the real and complex group algebras $\mathbb{F}[G] := \mathbb{F} \rtimes G,\ \mathbb{F} = \mathbb{R}, \mathbb{C}$ are semisimple, and admit a decomposition into a direct sum of simple algebras

$$\mathbb{F}[G] \cong \bigoplus_{i=1}^{r} M_{n_i}(D_i), \tag{6.1}$$

where the D_i are division algebras over $\mathbb{F}$. This decomposition arises from considering $\mathbb{F}[G]$ as a left module over itself (or alternatively, the left regular representation of G), and its isotypical decomposition $\mathbb{F}[G] \cong \oplus_i n_i V_i$ into simple modules (or irreducible G-representations), with $V_i \ncong V_j$ as $\mathbb{F}[G]$-modules if $i \neq j$. Then $(\mathbb{F}[G])^{\mathrm{op}} \cong \mathrm{End}_{\mathbb{F}[G]}\mathbb{F}[G]$ and $D_i \cong D_i^{\mathrm{op}}$ for the real or complex division algebras leads to (6.1). The simple modules for $\mathbb{F}[G]$ are then the simple modules for each direct sum factor, namely, $V_i = D_i^{n_i}$. Furthermore, every simple $\mathbb{F}[G]$-module V_i is cyclic and appears as a direct summand of the module $\mathbb{F}[G]$. Any finitely-generated (automatically projective) $\mathbb{F}[G]$-module V is semisimple (or completely reducible), and isomorphic to a direct sum $V \cong \oplus_{i=1}^{s} m_i V_i$, where m_i are the multiplicities of the simple submodules V_i.

The (real or complex) *representation group*[1] $\mathcal{R}_{\mathbb{F}}(G)$ of G is defined to be the Grothendieck group of the monoid of isomorphism classes of finite-dimensional representations of G over $\mathbb{F}$ under the direct sum. In this definition, we can replace finite-dimensional representations of G with finitely-generated (projective) $\mathbb{F}[G]$-modules. By complete reducibility, $\mathcal{R}_{\mathbb{F}}(G)$ is freely-generated by the simple modules V_i. Note that finite-dimensional representations of a finite group are unitarisable, so we could have used G-invariant inner products on the representation spaces to construct invariant complements, and then deduce complete reducibility. We can thus restrict to unitary representations in the definition of $\mathcal{R}_{\mathbb{F}}(G)$. To emphasise that we are interested in unitary representations and to generalise to infinite compact groups, we recall that $\mathbb{F}[G]$ can be given the structure of a C^*-algebra, and (6.1) gives a decomposition into a direct sum of C^*-algebras. The K-theoretic formulation of the (unitary) representation group is then the statement that $\mathcal{R}_{\mathbb{F}}(G) \cong K_0(\mathbb{F}[G])$, with the K-theory group of the C^*-algebra $\mathbb{F}[G]$ on the right-hand-side defined in the usual way, e.g.,

[1]This is sometimes called the representation ring, but we will not be investigating possible product structures.

through finitely-generated projective (f.g.p.) $\mathbb{F}[G]$-modules, or through projections in the stabilised algebra $M_\infty(\mathbb{F}[G])$.

If G is compact, its (complex) unitary representation theory is described by the Peter–Weyl theorem. In C^*-algebraic language, this says that the group C^*-algebra $\mathbb{C} \rtimes G$ decomposes as a (possibly countably infinite) direct sum of matrix algebras indexed by the unitary dual $\hat{G}$ (equivalence classes of irreducible unitary representations of G),

$$\mathbb{C} \rtimes G \cong \bigoplus_{[V]\in\hat{G}} M_{\dim(V)}(\mathbb{C}). \tag{6.2}$$

For a proof, see Proposition 3.4 of [97]. The complex representation ring $\mathcal{R}_\mathbb{C}(G)$ of G is, as before, the Grothendieck group of the monoid of isomorphism classes of finite-dimensional unitary representations of G under the direct sum. By complete reducibility, $\mathcal{R}_\mathbb{C}(G)$ is freely-generated by the elements of the unitary dual $\hat{G}$. In terms of K-theory, there is an isomorphism $\mathcal{R}_\mathbb{C}(G) \cong K_0(\mathbb{C} \rtimes G) \cong K_0^G(\mathbb{C})$ (see Section 11.1 of [11]), where K_0^G denotes the equivariant K-theory group. If $\mathbb{C} \rtimes G$ is unital, its K_0-group has a direct description as the Grothendieck group of the monoid of f.g.p. $\mathbb{C} \rtimes G$-modules. Although $\mathbb{C} \rtimes G$ is non-unital when G is not discrete, it is *stably unital*, and we can still use the same picture for its K_0-group (see Propositions 5.5.5 and 11.1.1 of [11]). The real representation ring $\mathcal{R}_\mathbb{R}(G)$ requires a modification to the simple matrix algebras appearing in (6.2). For projective unitary representations of (G, σ), there is a twisted Peter–Weyl theorem,

$$\mathbb{C} \rtimes_{(1,\sigma)} G \cong \bigoplus_{[V]\in\hat{G}^\sigma} M_{\dim(V)}(\mathbb{C}),$$

where $\hat{G}^\sigma$ is the generalised dual: the set of equivalence classes of irreducible projective unitary representations of (G, σ). In view of this, we can define the twisted representation group $\mathcal{R}_\mathbb{C}(G, \sigma) := K_0(\mathbb{C} \rtimes_{(1,\sigma)} G)$. Note that for compact G, the monoid of f.g.p. $(\mathbb{C} \rtimes_{(1,\sigma)} G)$-modules has cancellation, so it embeds naturally into its Grothendieck completion. For general algebras, such as $\mathcal{A} = C(S^2)$, cancellation does not hold for the monoid of f.g.p. $\mathcal{A}$-modules, so non-isomorphic modules can become isomorphic in $K_0(\mathcal{A})$. In general, the images of two f.g.p. $\mathcal{A}$-modules in $K_0(\mathcal{A})$ are equal iff they are isomorphic upon the addition of some free module $\mathcal{A}^n$.

If G is locally compact but not compact, its (twisted) representation group is difficult to define. To avoid technical distinctions between full and reduced crossed products, we will assume that G is *amenable*[2]. We choose the K-theoretic option: the *complex twisted representation group*, or simply *representation group* $\mathcal{R}_\mathbb{C}(G, \sigma)$, is *defined* to be the K_0-group of the twisted group C^*-algebra $\mathbb{C} \rtimes_{(1,\sigma)} G$. With this definition, an element of

[2]This assumption holds in all the physical examples that we consider.

the representation group need not correspond to a (virtual) finite-dimensional projective unitary representation of (G, σ). Furthermore, not every irreducible projective unitary representation of (G, σ) appears in the representation group. For instance, consider the representation group of $G = \mathbb{Z}$. The complex group C^*-algebra is $C(\mathbb{T})$, and its K_0-group comprises f.g.p. $C(\mathbb{T})$-modules — these are not finite-dimensional representations for $\mathbb{Z}$. Also, the irreducible unitary representations of $\mathbb{Z}$ are one-dimensional and are labelled by a phase $e^{\mathrm{i}\theta} \in \mathbb{T}$, but these are not f.g.p. $C(\mathbb{T})$-modules. Nevertheless, there is good physical motivation for studying f.g.p. modules. The Serre–Swan theorem identifies a f.g.p. $C(\mathbb{T})$-module with the sections of some finite-rank vector bundle over $\mathbb{T}$, which is reminiscent of electronic band theory in condensed matter physics. The representation group of $\mathbb{Z}$ thus has a physical interpretation as a means of classifying band insulators. Indeed, this is a motivation for choosing the K-theoretic definition of representation groups in the first place. Further motivation and information about the K-theory of twisted group algebras can be found in [24].

Finally, for a general twisted dynamical system $(G, \mathcal{A}, \alpha, \sigma)$ (Section 3.1), and a C^*-algebra $\mathcal{B}$, we make the following definitions:

Definition 6.1.1 (General notion of a representation group)**.** The *representation group* of a twisted dynamical system $(G, \mathcal{A}, \alpha, \sigma)$ is defined to be $K_0(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$. For a general C^*-algebra $\mathcal{B}$, not necessarily arising from a twisted crossed product, we will also call $K_0(\mathcal{B})$ the (ungraded) *representation group of* $\mathcal{B}$, which we denote by $\mathcal{R}(\mathcal{B})$.

Definition 6.1.1 subsumes all the earlier definitions given in this section. The introduction of $\mathcal{R}(\mathcal{B})$ is to emphasise the module-theory aspect of $K_0(\mathcal{B})$, and to distinguish $K_0(\mathcal{B})$ from the difference-group $\mathbf{K}_0(\mathcal{B})$ (Definition 6.4.4) when $\mathcal{B}$ has the added structure of a graded algebra.

6.2 Super-representation groups of graded algebras

We recall a few basic definitions for graded algebras and their modules, which can be found in [11, 90, 53, 18, 60, 36, 65]. In particular, the notions of left and right parity reversals are emphasised, leading to our definition of a super-representation group for a graded C^*-algebra. It is essentially the same as the super-representation ring introduced in [52], with slight modifications and generalised to C^*-algebras.

A graded module for a graded algebra $\mathcal{A}$ is an (ungraded) $\mathcal{A}$-module W which admits a direct sum decomposition into $W = W_0 \oplus W_1$, such that $A_i W_j \subset W_{i+j\,(\mathrm{mod}\,2)}$. Thus, even elements of $\mathcal{A}$ act as even (parity-preserving) maps on W, while odd elements of $\mathcal{A}$ act as odd (parity-reversing) maps on W.

Definition 6.2.1 (Left and right parity reversals)**.** Let W be a graded module for a graded algebra $\mathcal{A}$. The *right* parity-reversal operation $\pi_R : W \to W^\Pi$ takes W to the *right* parity-reversed module W^Π, which has the same underlying vectors as W fixed under π_R, and the same action of $\mathcal{A}$, but the grading on vectors is reversed: $W_0^\Pi = W_1, W_1^\Pi = W_0$. The *left* parity reversal operation $\pi_L : W \to {}^\Pi W$ and *left* parity reversed module ${}^\Pi W$ are defined similarly, with ${}^\Pi W_0 = W_1, {}^\Pi W_1 = W_0$, but there is a modified $\mathcal{A}$-action on ${}^\Pi W$, defined for homogeneous elements to be

$$a \cdot (\pi_L(w)) = \pi_L((-1)^{|a|} a \cdot w), \qquad a \in \mathcal{A}_0 \cup \mathcal{A}_1, w \in W,$$

where $|a| \in \mathbb{Z}_2$ denotes the parity of a.

It is easy to see that both π_R and π_L are odd involutory operations on graded $\mathcal{A}$-modules, $\pi_R^2 = \mathrm{id} = \pi_L^2$. We will denote $\pi_R(w)$ by w^π and $\pi_L(w)$ by ${}^\pi w$, which distinguishes them from $w \in W$.

Proposition 6.2.2 (Graded equivalence of left and right parity reversals)**.** *The left and right parity reversals of W are equivalent as graded $\mathcal{A}$-modules.*

Proof. Define the even map $\varphi : W^\Pi \to {}^\Pi W$ by

$$\varphi : W^\Pi \ni w^\pi \equiv w_0^\pi + w_1^\pi \mapsto {}^\pi w_0 - {}^\pi w_1 \in {}^\Pi W.$$

We check that φ is indeed an $\mathcal{A}$-module map,

$$\begin{aligned} \varphi(a(w_0^\pi + w_1^\pi)) &= (-1)^{|aw_0|}({}^\pi(aw_0)) + (-1)^{|aw_1|}({}^\pi(aw_1)) \\ &= (-1)^{|w_0|} a \cdot {}^\pi w_0 + (-1)^{|w_1|} a \cdot {}^\pi w_1 \\ &= a(\varphi(w_0^\pi + w_1^\pi)), \qquad\qquad w_0^\pi + w_1^\pi \in w^\pi. \end{aligned}$$

The inverse $\mathcal{A}$-module map $\varphi^{-1} : {}^\Pi W \to W^\Pi$ has a similar formula ${}^\pi w_0 + {}^\pi w_1 \mapsto w_0^\pi - w_1^\pi$. □

Despite the equivalence $W^\Pi \cong {}^\Pi W$, there is an important difference in that π_R *commutes* with $\mathcal{A}$, whereas π_L *graded commutes* with $\mathcal{A}$.

Definition 6.2.3. Let $\mathcal{A}$ be a graded unital algebra. A *graded finitely-generated free* $\mathcal{A}$-module is one of the form $\mathcal{A}^m \oplus (\mathcal{A}^\Pi)^n =: \mathcal{A}^{m|n}$, where $\mathcal{A}$ is regarded as a graded $\mathcal{A}$-module by left multiplication on itself, and $\mathcal{A}^\Pi$ is its right parity reverse. A graded finitely-generated projective (f.g.p.) $\mathcal{A}$-module is defined to be a graded $\mathcal{A}$-module which is a direct summand of $\mathcal{A}^{m|n}$ for some pair of natural numbers (m, n).

Note that the left parity reverse ${}^{\Pi}\mathcal{A}$ could have been used in the definition of a free module. It can be shown [36] that a graded f.g.p. $\mathcal{A}$-module is the same thing as a graded $\mathcal{A}$-module which is a f.g.p. $\mathcal{A}$-module in the ungraded sense (i.e. a direct summand of an ungraded free module $\mathcal{A}^n$). As mentioned in the previous section, f.g.p. $\mathcal{A}$-modules are, in general, distinct from finite-dimensional $*$-representations of a C^*-algebra on a Hilbert space, although they turn out to coincide for the twisted group C^*-algebras of a compact group. For brevity, *all modules in this and subsequent chapters are assumed to be f.g.p. unless otherwise stated.*

Let $\mathcal{V}(\mathcal{A})$ (resp. $\mathcal{GV}(\mathcal{A})$) be the commutative monoid of equivalence classes of ungraded (resp. graded) $\mathcal{A}$-modules, under the direct sum operation.

Definition 6.2.4 (Trivial graded module)**.** A graded module for a unital graded C^*-algebra $\mathcal{A}$ is *trivial* if it admits an odd involution $\mathcal{I}$, with $\mathcal{I}^2 = +1$, which graded commutes with the action of $\mathcal{A}$,

$$a \cdot \mathcal{I}w = (-1)^{|a|}\mathcal{I}a \cdot w, \qquad a \in \mathcal{A}_0 \cup \mathcal{A}_1, w \in W.$$

We define $\mathcal{T}(\mathcal{A})$ to be the set of graded equivalence classes of trivial graded $\mathcal{A}$-modules.

The set $\mathcal{T}(\mathcal{A})$ is closed under direct sum and so forms a submonoid of $\mathcal{GV}(\mathcal{A})$. It generates an equivalence relation on $\mathcal{GV}(\mathcal{A})$ as follows: $W \sim W'$ iff there exists $T, T' \in \mathcal{T}(\mathcal{A})$ such that $W \oplus T \cong W' \oplus T'$. Note that this relation is a *congruence* on $\mathcal{GV}(\mathcal{A})$, i.e., $W \sim W'$ and $Y \sim Y'$ implies that $W \oplus Y \sim W' \oplus Y'$.

Definition 6.2.5 (Super-representation group)**.** The *super-representation* group $\mathcal{SR}(\mathcal{A})$ of a graded unital C^*-algebra $\mathcal{A}$ is the quotient set $\mathcal{GV}(\mathcal{A})/\sim$.

Proposition 6.2.6. *The super-representation group $\mathcal{SR}(\mathcal{A})$ is an abelian group.*

Proof. It is easy to see that ${}^{\Pi}W$ is the inverse of W in the quotient $\mathcal{SR}(\mathcal{A})$, since $W \oplus {}^{\Pi}W$ admits the odd trivialising involution $\mathcal{I} = \pi_L \oplus \pi_L$. It follows that $[{}^{\Pi}W] = -[W]$ in $\mathcal{SR}(\mathcal{A})$. □

Definition 6.2.5 captures the notion that a graded $\mathcal{A}$-module is "cancelled" by its left parity reverse. Furthermore, $\mathcal{T}(\mathcal{A})$ is closed under π_L: suppose $\mathcal{I}$ is an odd trivialising involution for T, then the operator $\tilde{\mathcal{I}} := \pi_L \circ \mathcal{I} \circ \pi_L$ is an odd trivialising involution for ${}^{\Pi}T$. Note, however, that a trivial T need not take the obvious trivial form $T = W \oplus {}^{\Pi}W$.

Remark 6.2.7. Finitely-generated projective modules do not have a Hilbert space structure in general, but they can be endowed with the structure of a Hilbert $\mathcal{A}$-module, see Theorem 15.4.2 in [93]. A *Hilbert $\mathcal{A}$-module*, or a *Hilbert C^*-module over $\mathcal{A}$*, is an $\mathcal{A}$-module with an $\mathcal{A}$-valued "inner product" $\langle \cdot, \cdot \rangle$, whose associated norm $|||x||| = \|\langle x, x\rangle\|^{1/2}$ is complete.

A f.g.p. $\mathcal{A}$-module can be endowed with the structure of a Hilbert $\mathcal{A}$-module, see Theorem 15.4.2 in [93]. Other references for Hilbert C^*-modules include [11, 51]. One is typically interested in the class of *adjointable* module homomorphisms, which need not exhaust all the bounded module homomorphisms. A finitely-generated $\mathcal{A}$-module with its Hilbert $\mathcal{A}$-module structure is *self-dual*, with the consequence that bounded module homomorphisms between Hilbert $\mathcal{A}$-modules are adjointable and complementable. The adjoint operation on the set of adjointable module homomorphisms behaves much like the Hilbert space adjoint. In terms of Hilbert C^*-modules, we may choose the trivialising involution in a trivial $\mathcal{A}$-module to be self-adjoint.

When $\mathcal{A}$ is ungraded, it has its usual representation group $\mathcal{R}(\mathcal{A})$ and K-theory group $K_0(\mathcal{A})$, which are identical by definition. We can also regard $\mathcal{A}$ as a purely even graded algebra. Then, a graded $\mathcal{A}$-module is trivial iff it is the direct sum of two isomorphic ungraded $\mathcal{A}$-modules. By definition, an element in $\mathcal{SR}(\mathcal{A})$ can be represented by a graded $\mathcal{A}$-module $W = W_0 \oplus W_1$, where the summands W_0 and W_1 are ungraded $\mathcal{A}$-modules. On the other hand, an element of $\mathcal{R}(\mathcal{A}) \equiv K_0(\mathcal{A})$ can be represented by a virtual module $W_0 \ominus W_1$.

Proposition 6.2.8 (Super-representation groups of purely-even unital algebras)**.** *The super-representation group $\mathcal{SR}(\mathcal{A})$ of an ungraded unital C^*-algebra $\mathcal{A}$, when regarded as a purely-even algebra, is isomorphic to $K_0(\mathcal{A})$, under the correspondence $[W_0 \oplus W_1] \leftrightarrow [W_0 \ominus W_1]$.*

Proof. We need to check that the correspondence is well-defined on classes. Suppose $[W_0 \ominus W_1] = [W_0' \ominus W_1']$ in $\mathcal{R}(\mathcal{A}) \equiv K_0(\mathcal{A})$. Then there exists $X \in \mathcal{V}(\mathcal{A})$ such that $W_0 \oplus W_1' \oplus X \cong W_0' \oplus W_1 \oplus X =: T$. Let $W = W_0 \oplus W_1$, $W' = W_0' \oplus W_1' \in \mathcal{GV}(\mathcal{A})$, and let $S = W_0 \oplus W_1 \oplus X$. Define the trivial graded $\mathcal{A}$-modules $\hat{S}, \hat{T}$, which have homogeneous subspaces $\hat{S}_0 = \hat{S}_1 = S$ and $\hat{T}_0 = \hat{T}_1 = T$. It is straightforward to see that $W \oplus \hat{T} \cong W' \oplus \hat{S}$ in $\mathcal{GV}(\mathcal{A})$, which implies that $[W] = [W']$ in $\mathcal{SR}(\mathcal{A})$. Conversely, if $[W] \equiv [W_0 \oplus W_1] = [W'] \equiv [W_0' \oplus W_1']$ in $\mathcal{SR}(\mathcal{A})$, there are trivial graded modules $\hat{S} = \hat{S}_0 \oplus \hat{S}_1 = S \oplus S$, $\hat{T} = \hat{T}_0 \oplus \hat{T}_1 = T \oplus T \in \mathcal{T}(\mathcal{A})$, such that $W \oplus \hat{T} \cong W' \oplus \hat{S}$ as graded $\mathcal{A}$-modules. This is equivalent to the two isomorphisms of ungraded $\mathcal{A}$-modules, $W_0 \oplus T \cong W_0' \oplus S$ and $W_1 \oplus T \cong W_1' \oplus S$, which implies that $W_0 \oplus W_1' \oplus (S \oplus T) \cong W_0' \oplus W_1 \oplus (S \oplus T)$. Therefore, $[W_0 \ominus W_1] = [W_0' \ominus W_1']$ in $K_0(\mathcal{A})$. ☐

It is interesting to see what the zero elements of $\mathcal{R}(\mathcal{A})$ and $\mathcal{SR}(\mathcal{A})$ are when $\mathcal{A}$ is purely even. $[W] \equiv [W_0 \oplus W_1] = 0$ in $\mathcal{SR}(\mathcal{A})$ iff $W_0 \oplus V$ is isomorphic to $W_1 \oplus V$ as ungraded $\mathcal{A}$-modules for some V, iff $[W_0 \ominus W_1] = 0$ in $\mathcal{R}(\mathcal{A})$. When cancellation holds in the monoid $\mathcal{V}(\mathcal{A})$, the zero elements of $\mathcal{R}(\mathcal{A})$ and $\mathcal{SR}(\mathcal{A})$ are, respectively, exactly of the form $[U \ominus V]$ and $[U \oplus V]$ for isomorphic ungraded $\mathcal{A}$-modules $U \cong V$.

Although we get nothing new when considering $\mathcal{SR}(\mathcal{A})$ for a purely even $\mathcal{A}$, it becomes distinct from the ungraded representation group $\mathcal{R}(\mathcal{A}) \equiv K_0(\mathcal{A})$, when $\mathcal{A}$ has a non-trivial grading and is regarded as an ungraded algebra. In fact, there are many distinct "representation groups" that can be defined for a graded algebra $\mathcal{A}$; for a sample of possible definitions of such groups in the case of finite-dimensional associative graded algebras and Lie superalgebras, see [52, 53, 36]. Definition 6.2.5 is in fact based closely on the definitions given in these papers. Besides $\mathcal{SR}(\mathcal{A})$ and $\mathcal{R}(\mathcal{A})$, there is a third possibility:

Definition 6.2.9. Let $\mathcal{A}$ be a graded unital C^*-algebra. Its *graded representation group* $\mathcal{GR}(\mathcal{A})$ is defined to be the Grothendieck completion of the monoid $\mathcal{GV}(\mathcal{A})$.

If $\mathcal{A}$ is purely even, it is easy to see that $\mathcal{GR}(\mathcal{A})$ is simply two copies of $\mathcal{R}(\mathcal{A})$; for a general graded $\mathcal{A}$ this is no longer the case.

6.2.1 Super-representation groups of non-unital graded algebras

So far, we have defined three representation groups for graded unital algebras $\mathcal{A}$. In the non-unital case, we will define the representation groups through a procedure similar to the definition of K_0 for non-unital algebras. For this, we need the unitisation $\mathcal{A}$ of a graded non-unital C^*-algebra. This is the usual C^*-algebra unitisation $\mathcal{A}^+ = \{(a, \lambda) : a \in \mathcal{A}, \lambda \in \mathbb{F}\}$ with its unique C^*-norm, component-wise sum and adjoint, and multiplication given by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. We assign the grading

$$\begin{aligned} \mathcal{A}_0^+ &= \{(a, \lambda) : a \in \mathcal{A}_0, \lambda \in \mathbb{F}\}, \\ \mathcal{A}_1^+ &= \{(a, 0) : a \in \mathcal{A}_1\}. \end{aligned}$$

Then $\mathcal{A}$ is a graded two-sided ideal in $\mathcal{A}^+$, i.e., $\mathcal{A}$ is a two-sided ideal in the ungraded sense, and $\mathcal{A}_i = \mathcal{A} \cap \mathcal{A}_i^+$. The quotient algebra $\mathcal{A}^+/\mathcal{A} = \mathbb{F}$ is purely even. Next, we need to define the map on graded modules induced by an even homomorphism $\mathcal{A} \xrightarrow{f} \mathcal{B}$ between unital graded algebras. The graded algebra $\mathcal{B}$ becomes a graded $\mathcal{A}$-bimodule via

$$\begin{aligned} a \cdot b &:= f(a)b, \\ b \cdot a &:= bf(a), \qquad a \in \mathcal{A},\, b \in \mathcal{B}. \end{aligned}$$

For a graded $\mathcal{A}$-module W, the graded $\mathcal{B}$-module induced by the homomorphism f is defined to be[3] $f_*(W) := \mathcal{B} \hat{\otimes}_{\mathcal{A}} W$. Its homogeneous subspaces $(f_*(W))_i$ are spanned by

$$\{b \hat{\otimes} w : b \in \mathcal{B},\, w \in W,\, |b| + |w| = i \,(\mathrm{mod}\, 2)\}, \quad i = 0, 1.$$

[3]To see that $f_*(W)$ is still a finitely-generated projective module, consider a (split) exact sequence $0 \to W^\perp \to W \oplus W^\perp = \mathcal{A}^{m|n} \to W \to 0$. Then apply the additive functor $\mathcal{B} \hat{\otimes}_{\mathcal{A}} (\cdot)$ and note that $\mathcal{B} \hat{\otimes}_{\mathcal{A}} \mathcal{A}^{m|n} \cong \mathcal{B}^{m|n}$.

We then verify that $\mathcal{V}(\cdot)$ and $\mathcal{GV}(\cdot)$, along with $f \mapsto f_*$, are covariant functors from (graded) unital C^*-algebras to abelian monoids. The universal property of the Grothendieck completion then yields induced homomorphisms $\mathcal{R}(\mathcal{A}) \xrightarrow{f_*} \mathcal{R}(\mathcal{B})$ and $\mathcal{GR}(\mathcal{A}) \xrightarrow{f_*} \mathcal{GR}(\mathcal{B})$.

Definition 6.2.10 (Representation groups for non-unital graded algebras)**.** The representation group and graded representation group of a non-unital graded C^*-algebra $\mathcal{A}$ are defined as the kernels

$$\begin{aligned}\mathcal{R}(\mathcal{A}) &:= \ker(\mathcal{R}(\mathcal{A}^+) \xrightarrow{p_*} \mathcal{R}(\mathbb{F})),\\ \mathcal{GR}(\mathcal{A}) &:= \ker(\mathcal{GR}(\mathcal{A}^+) \xrightarrow{p_*} \mathcal{GR}(\mathbb{F}))\end{aligned}$$

induced by the projection $\mathcal{A}^+ \xrightarrow{p} \mathbb{F}$.

As for the super-representation group, we need the fact that the induced monoid homomorphism $\mathcal{GV}(\mathcal{A}) \xrightarrow{f_*} \mathcal{GV}(\mathcal{B})$ descends to the respective quotients by the trivial submonoids. So suppose that $T \in \mathcal{T}(\mathcal{A})$ is a trivial graded $\mathcal{A}$-module, with the odd trivialising involutary operator $\mathcal{I}$ which graded commutes with $\mathcal{A}$. We claim that the operator $\mathcal{I}'$ defined on $f_*(T) = \mathcal{B}\,\hat{\otimes}_{\mathcal{A}} T$ by

$$\mathcal{I}'(b \,\hat{\otimes}\, t) := (-1)^{|b|} b \,\hat{\otimes}\, \mathcal{I}t, \qquad b\,\hat{\otimes} t \in \mathcal{B}\,\hat{\otimes}_{\mathcal{A}} T$$

makes $f_*(T)$ trivial. $\mathcal{I}'$ is certainly a $\mathbb{F}$-linear operator, and squares to the identity. It is odd, because

$$|\mathcal{I}'(b \,\hat{\otimes}\, t)| = |b \,\hat{\otimes}\, \mathcal{I}t| = |b| + |\mathcal{I}t| = |b| + |t| + 1 = |b \,\hat{\otimes}\, t| + 1 \pmod 2.$$

Also, $\mathcal{I}'$ graded commutes with the $\mathcal{B}$-action, since

$$\mathcal{I}'(b(b' \,\hat{\otimes}\, t)) = (-1)^{|bb'|} bb' \,\hat{\otimes} \mathcal{I}t = (-1)^{|b|+|b'|} b (-1)^{|b'|} (\mathcal{I}'(b' \,\hat{\otimes}\, t)) = (-1)^{|b|} b (\mathcal{I}'(b' \,\hat{\otimes}\, t)),$$

so $f_*(T)$ is a trivial graded $\mathcal{B}$-module. If $[W] = [W']$ in $\mathcal{SR}(\mathcal{A})$, i.e., $W \oplus T \cong W' \oplus T'$ for some $T, T' \in \mathcal{T}(\mathcal{A})$, then

$$\begin{aligned}[f_*(W')] &= [f_*(W') \oplus f_*(T')] = [f_*(W' \oplus T')]\\ &= [f_*(W \oplus T)] = [f_*(W) \oplus f_*(T)]\\ &= [f_*(W)]\end{aligned}$$

in $\mathcal{SR}(\mathcal{B})$, so $f_* : [W] \mapsto [f_*(W)]$ is a well-defined homomorphism between the super-representation groups $\mathcal{SR}(\cdot)$.

Definition 6.2.11. The super-representation group $\mathcal{SR}(\mathcal{A})$ of a non-unital graded C^*-algebra $\mathcal{A}$ is defined to be the kernel

$$\mathcal{SR}(\mathcal{A}) := \ker(\mathcal{SR}(\mathcal{A}^+) \xrightarrow{p_*} \mathcal{SR}(\mathbb{F})).$$

Remark 6.2.12. The distinction between the left and right parity reversals can be understood as a special case of the more general notion of the graded tensor product of a graded $\mathcal{A}$-module V with a graded $\mathcal{B}$-module W, where $\mathcal{A}$ and $\mathcal{B}$ are graded algebras over the same field[4]. The graded tensor product $V\hat{\otimes}W$ becomes a graded module for the graded tensor product algebra $\mathcal{A}\hat{\otimes}_{\mathbb{F}}\mathcal{B}$ as follows:

$$(a\hat{\otimes}b)\cdot(v\hat{\otimes}w) := (-1)^{|b||v|}(a\cdot v)\hat{\otimes}(b\cdot w), \qquad a\in\mathcal{A}, b\in\mathcal{B}, v\in V, w\in W. \tag{6.3}$$

The left parity reversal of a graded $\mathcal{A}$-module W can be viewed as the graded $\mathbb{F}_+\hat{\otimes}_{\mathbb{F}}\mathcal{A}\cong\mathcal{A}$-module $\mathbb{F}_-\hat{\otimes}W$, where $\mathbb{F}_+$ is $\mathbb{F}$ regarded as a purely even algebra, and $\mathbb{F}_-$ is $\mathbb{F}$ regarded as a purely odd $\mathbb{F}_+$-module. The action of $a\in\mathcal{A}$ on $v\hat{\otimes}w\in\mathbb{F}_-\hat{\otimes}W$ acquires an extra factor of $(-1)^{|a||v|}=(-1)^{|a|}$ according to (6.3). On the other hand, the right parity reversal of a graded $\mathcal{A}$-module W can be viewed as the graded $\mathcal{A}\hat{\otimes}_{\mathbb{F}}\mathbb{F}_+\cong\mathcal{A}$-module $W\hat{\otimes}\mathbb{F}_-$, with the action of $a\in\mathcal{A}$ on $w\hat{\otimes}v\in W\hat{\otimes}\mathbb{F}_-$ picking up a trivial sign factor $(-1)^{|1_{\mathbb{F}}||w|}=1$.

In [52], the author used the right parity reversal in his definition of $\mathcal{SR}(\cdot)$, in order to make contact with Theorem 6.2.14 and $KO^{-n}(\star), K^{-n}(\star)$ (the real and complex K-theory of a point) more directly; see Remark 4.1 in his paper. An important difference between his conventions and ours, is that his odd trivialising involution is required to commute, rather than graded commute, with the $\mathcal{A}$-action. This has the effect of identifying $\mathcal{SR}(Cl_{n,0})$ with $K^{-n}(\star)$, whereas our definition leads to $\mathcal{SR}(Cl_{0,n})\cong K^{-n}(\star)$, see Remark 6.2.17.

We have chosen our convention for two reasons. First, when $\mathcal{A}$ is the Clifford algebra associated with the symmetry data (A,σ) of a CT-subgroup, we recover the periodic table of Kitaev in the correct order. Second, the trivialising operator $\mathcal{I}$ has an interpretation in terms of the K-theory of $\mathcal{A}$, which we explain in Remark 6.4.8. We also note that the authors in [30] defined their "twisted K-theory groups" and "reduced topological phases" via a related notion of trivial representations, which only requires the existence of an odd operator $\mathcal{I}$ that graded commutes with the projective unitary-antiunitary group action. They do not impose an involution condition on $\mathcal{I}$, so their convention differs from ours.

6.2.2 Examples of (super)-representation groups

The ungraded algebra $\mathbb{R}$

Consider the trivial ungraded algebra $\mathcal{A}=\mathbb{R}$, whose modules are just (finite-dimensional) real vector spaces. Its ungraded representation group is $\mathcal{R}(\mathbb{R})=K_0(\mathbb{R})=\mathbb{Z}$, generated by the one-dimensional vector space $\mathbb{R}$. If we consider $\mathbb{R}$ as a purely even algebra, its graded

[4]For general C^*-algebras the tensor product needs to be specified more carefully, but we are only dealing with the case in which one of the two algebras is the trivial algebra $\mathbb{F}$. In this case, there is no ambiguity in taking tensor products.

modules are graded real vector spaces $\mathbb{R}^{m|n}$, whose (graded) isomorphism classes are given by $(m,n) \in \mathbb{N} \oplus \mathbb{N}$. Thus, $\mathcal{GR}(\mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$. A graded $\mathbb{R}$-module is trivial iff $m = n$. After quotienting by triviality, the equivalence class of $\mathbb{R}^{m|n}$ can be represented by the purely even vector space $\mathbb{R}^{m-n|0}$ or the purely odd vector space $\mathbb{R}^{0|n-m}$, depending on whether $m - n \geq 0$ or $n - m > 0$. Thus the super-representation group $\mathcal{SR}(\mathbb{R})$ is $\mathbb{Z}$, as expected.

The Clifford algebra $Cl_{0,1}$

Next, consider the graded algebra $\mathcal{A} = Cl_{0,1}$ which is $\mathbb{R} \oplus \mathbb{R}$ as an ungraded algebra. The even subalgebra is $\mathcal{A}_0 = \{(x,x) : x \in \mathbb{R}\}$, while the odd subspace is $\mathcal{A}_1 = \{(x,-x) : x \in \mathbb{R}\}$. Its ungraded representation group is $\mathcal{R}(Cl_{0,1}) = K_0(\mathbb{R} \oplus \mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z}$. Graded $Cl_{0,1}$-modules correspond to ungraded $Cl_{0,2} \cong M_2(\mathbb{R})$-modules, which are direct sums of the simple $M_2(\mathbb{R})$-module $\mathbb{R}^2$. Passing to the Grothendieck completion yields $\mathcal{GR}(Cl_{0,1}) = \mathbb{Z}$, generated by the graded vector space $\mathbb{R}^{1|1}$. An odd trivialising involution for a graded $Cl_{0,1}$-module is nothing other than the action of an extra odd *positive* Clifford algebra generator. Thus, a trivial graded $Cl_{0,1}$-module (equiv. ungraded $Cl_{0,2}$-module) is one which is also a graded $Cl_{0,2}$-module (equiv. ungraded $Cl_{0,3}$-module). The ungraded $Cl_{0,2}$-modules (i.e. even-dimensional vector spaces) which admit an ungraded $Cl_{0,3} \cong M_2(\mathbb{C})$-module structure, are those with real dimension a multiple of 4. It follows that $\mathcal{SR}(Cl_{0,1}) = \mathbb{Z}_2$.

The Clifford algebras $Cl_{r,s}$

More generally, the Clifford algebras $Cl_{r,s}$ and $\mathbb{C}l_n$ are naturally graded C^*-algebras, with odd (resp. even) products of Clifford generators being odd (resp. even), and positive (resp. negative) generators being self-adjoint (resp. skew-adjoint). The Clifford algebras $Cl_{r,s}$ are semisimple, so finite-dimensional Clifford modules are finitely-generated projective Clifford modules. The same remarks hold for graded Clifford modules due to the equivalence between graded $Cl_{r,s}$-modules and ungraded $Cl_{r,s+1}$-modules, and similarly for the complex Clifford algebras. The ungraded representation group of $Cl_{r,s}$ (resp. $\mathbb{C}l_n$) is freely generated by its equivalence classes of irreducible real (resp. complex) representations, and we have (see Chapter I.5 of [54])

$$\begin{aligned}
\mathcal{R}(Cl_{r,s}) &= \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & r - s = 3, 7 \,(\mathrm{mod}\, 2), \\ \mathbb{Z} & \text{otherwise}; \end{cases} \\
\mathcal{R}(\mathbb{C}l_n) &= \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \text{ odd}, \\ \mathbb{Z} & n \text{ even}. \end{cases}
\end{aligned}$$

The graded representation groups are easily found using the isomorphisms (see, for instance, Proposition 5.20 of [54])

$$\mathcal{GR}(Cl_{r,s}) \cong \mathcal{R}(Cl_{r-1,s}) \cong \mathcal{R}(Cl_{r,s+1}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & r-s = 0, 4 \pmod 2, \\ \mathbb{Z} & \text{otherwise;} \end{cases}$$
$$\mathcal{GR}(\mathbb{C}l_n) \cong \mathcal{R}(\mathbb{C}l_{n-1}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n \text{ even}, \\ \mathbb{Z} & n \text{ odd}. \end{cases}$$

where the indices are taken modulo 8 in the real case and modulo 2 in the complex case.

More interesting are their super-representation groups. A trivial graded $Cl_{r,s}$-module is a graded $Cl_{r,s+1}$-module with the action of the $(s+1)^{\text{th}}$ positive Clifford generator forgotten. Recall that $\mathcal{GR}(Cl_{r,s})$ and $\mathcal{GR}(Cl_{r,s+1})$ are free abelian, generated by the classes of irreducible graded $Cl_{r,s}$-modules and $Cl_{r,s+1}$-modules respectively. The inclusion $i : Cl_{r,s} \hookrightarrow Cl_{r,s+1}$ induces a "restriction-of-scalars" homomorphism $i^* : \mathcal{GR}(Cl_{r,s+1}) \to \mathcal{GR}(Cl_{r,s})$. Similar remarks hold for the complex Clifford algebras.

Lemma 6.2.13. *There are group isomorphisms*

$$\begin{aligned} \mathcal{SR}(Cl_{r,s}) &\cong \mathcal{GR}(Cl_{r,s})/i^*\mathcal{GR}(Cl_{r,s+1}), \\ \mathcal{SR}(\mathbb{C}l_n) &\cong \mathcal{GR}(\mathbb{C}l_n)/i^*\mathcal{GR}(\mathbb{C}l_{n+1}). \end{aligned}$$

Proof. Let $[W]$ denote an element of $\mathcal{SR}(Cl_{r,s})$ represented by a graded $Cl_{r,s}$-module W. For a class $[W \ominus W']$ in $\mathcal{GR}(Cl_{r,s})$ represented by a virtual graded $Cl_{r,s}$-module $W \ominus W'$, we denote its image in the quotient $\mathcal{GR}(Cl_{r,s})/i^*\mathcal{GR}(Cl_{r,s+1})$ by $[[W \ominus W']]$. The required isomorphism is $p : [W] \mapsto [[W \ominus 0]]$, with the inverse map $q : [[W \ominus W']] \mapsto [W] - [W'] = [W] + [{}^{\Pi}W']$. We check that p is well-defined. So suppose $W \oplus T \cong W' \oplus T'$ for some $T, T' \in \mathcal{T}(Cl_{r,s})$. Then

$$p([W']) = [[W' \ominus 0]] = [[(W \oplus T) \ominus T']] = [[W \ominus 0] + [T \ominus T']] = [[W \ominus 0]] = p([W]),$$

where we have used the fact that $[T \ominus T'] \in i^*\mathcal{GR}(Cl_{r,s+1})$ and is thus zero in the quotient $\mathcal{GR}(Cl_{r,s})/i^*\mathcal{GR}(Cl_{r,s+1})$. Similarly, q is well-defined since an element of $i^*\mathcal{GR}(Cl_{r,s+1})$ has the form $[T \ominus T']$ for some trivial graded $Cl_{r,s}$-modules T, T', so that $q([[T \ominus T']]) = [T] - [T'] = 0$ in $\mathcal{SR}(Cl_{r,s})$. It is easy to see that p and q are homomorphisms and that $q \circ p = \mathrm{id}$. Since $W \oplus {}^{\Pi}W$ is a graded $Cl_{r,s+1}$-module for any graded $Cl_{r,s}$-module W, it follows that

$$\begin{aligned} q \circ p([[W \ominus W']]) &= q([W] + [{}^{\Pi}W']) = [[W \ominus 0]] + [[{}^{\Pi}W]] + [[W' \ominus W']] \\ &= [[W \ominus W']] + [[(W' \oplus {}^{\Pi}W) \ominus 0]] = [[W \ominus W']], \end{aligned}$$

so p and q are inverses of each other. The complex case is similarly proved. $\square$

Lemma 6.2.13 is reminiscent of the Atiyah–Bott–Shapiro isomorphisms, which say that

Theorem 6.2.14 (Atiyah–Bott–Shapiro isomorphisms [4])**.**

$$\begin{aligned} K_n(\mathbb{R}) \cong & KO^{-n}(\star) \cong \mathcal{GR}(Cl_{r,s})/i^*\mathcal{GR}(Cl_{r+1,s}), \quad n = r - s \,(\mathrm{mod}\, 8) \\ K_n(\mathbb{C}) \cong & \;\; K^{-n}(\star) \;\cong \mathcal{GR}(\mathbb{C}l_n)/i^*\mathcal{GR}(\mathbb{C}l_{n+1}). \end{aligned}$$

A graded $Cl_{r,s}$-module W can be turned into a graded $Cl_{s,r}$-module, by a simple adjustment of the actions of the Clifford generators. If Γ is the grading operator on W, and e_i and f_j are the mutually anticommuting odd operators representing the r negative and s positive $Cl_{r,s}$ generators, we define the operators $e'_j := \Gamma f_j$, $f'_i := \Gamma e_i$. They have commutation relations and squares given by $\Gamma e'_j = -e'_j\Gamma$, $\Gamma f'_i = -f'_i\Gamma$, $e'^2_j = -\Gamma^2 f_j^2 = -1$, $f'^2_i = -\Gamma^2 e_i^2 = +1$, and $e'_j f'_i = \Gamma f_j \Gamma e_i = -\Gamma e_i \Gamma f_j = -f'_i e'_j$. In other words, there is an odd action of s negative Clifford generators e'_j and r positive Clifford generators f'_i on W. Therefore, there is a 1–1 correspondence between graded $Cl_{r,s}$-modules and graded $Cl_{s,r}$-modules. Furthermore, if W is a graded $Cl_{r,s}$-module which admits the action of an extra negative Clifford generator e_{r+1}, the operator $f'_{r+1} := \Gamma e_{r+1}$ acts as an extra *positive* Clifford generator if W is regarded as a graded $Cl_{s,r}$-module.

Proposition 6.2.15. *There is an isomorphism $\mathcal{GR}(Cl_{r,s}) \cong \mathcal{GR}(Cl_{s,r})$ which respects the homomorphisms i^*, i.e.,*

$$\mathcal{GR}(Cl_{r,s})/i^*\mathcal{GR}(Cl_{r,s+1}) \cong \mathcal{GR}(Cl_{s,r})/i^*\mathcal{GR}(Cl_{s+1,r}). \tag{6.5}$$

Corollary 6.2.16.

$$\mathcal{SR}(Cl_{r,s}) \cong K_{s-r}(\mathbb{R}) \cong KO^{r-s}(\star), \tag{6.6a}$$

$$\mathcal{SR}(\mathbb{C}l_n) \cong \; K_n(\mathbb{C}) \; \cong K^{-n}(\star) \tag{6.6b}$$

Remark 6.2.17. For a complex graded algebra $\mathcal{A}$, it does not matter whether the odd trivialising operator $\mathcal{I}$ is defined to be an involution $\mathcal{I}^2 = 1$, or an anti-involution $\mathcal{I}^2 = -1$; multiplication by i turns one into the other. However, this does not work in the case of real graded algebras because the imaginary unit i is unavailable. Indeed, if we had asked for an extra action of a negative Clifford generator in the definition of a trivial graded $Cl_{r,s}$-module, we would have arrived at $\mathcal{SR}(Cl_{r,s}) \cong K_{r-s}(\mathbb{R})$ instead. Similarly, if we had required the trivialising involution $\mathcal{I}$ to commute, rather than anticommute, with $Cl_{r,s}$, then $\mathcal{I}\Gamma$ becomes an anti-involution anticommuting with $Cl_{r,s}$, so we will get the same reversal in the K-theory degree.

Algebras of the form $\mathcal{B}\hat{\otimes}Cl_{r,s}$

We can generalise the arguments leading to Corollary 6.2.16 for algebras of the form $\mathcal{A} = \mathcal{B}\hat{\otimes}Cl_{r,s}$. A trivial graded $\mathcal{A}$-module T is a graded $\mathcal{A}$-module with the action of an extra positive Clifford generator which graded commutes with the $\mathcal{A}$ action, so it is also a graded $\mathcal{A}\hat{\otimes}Cl_{0,1} = \mathcal{B}\hat{\otimes}Cl_{r,s+1}$-module. The inclusion $i : \mathcal{A} \to \mathcal{A}\hat{\otimes}Cl_{0,1}$ induces a homomorphism $i^* : \mathcal{GR}(\mathcal{B}\hat{\otimes}Cl_{r,s+1}) \to \mathcal{GR}(\mathcal{B}\hat{\otimes}Cl_{r,s})$ of Grothendieck groups,

$$i^*([W \ominus W']) \coloneqq [W \ominus W'], \qquad W,\, W' \in \mathcal{GV}(\mathcal{B}\hat{\otimes}Cl_{r,s+1}),$$

where W, W' are regarded as graded $\mathcal{B}\hat{\otimes}Cl_{r,s}$-modules on the right-hand-side. We can check that the homomorphism is well-defined on classes. In particular, for $T,\, T' \in \mathcal{T}(\mathcal{A}) = \mathcal{T}(\mathcal{B}\hat{\otimes}Cl_{r,s})$, the element $[T \ominus T']$ is in the image of i^*, so it is zero in $\mathcal{GR}(\mathcal{B}\hat{\otimes}Cl_{r,s})/i^*\mathcal{GR}(\mathcal{B}\hat{\otimes}Cl_{r,s+1})$. Then the arguments leading to (6.2.13) continue to hold upon replacing $Cl_{r,s}$ with $\mathcal{B}\hat{\otimes}Cl_{r,s}$, giving

$$\begin{aligned} \mathcal{SR}(\mathcal{B}\hat{\otimes}Cl_{r,s}) &\cong \mathcal{GR}(\mathcal{B}\hat{\otimes}Cl_{r,s})/i^*\mathcal{GR}(\mathcal{B}\hat{\otimes}Cl_{r,s+1}) \\ &\cong \mathcal{GR}(\mathcal{B}\hat{\otimes}Cl_{s,r})/i^*\mathcal{GR}(\mathcal{B}\hat{\otimes}Cl_{s+1,r}), \end{aligned}$$

where we have identified graded $B\hat{\otimes}Cl_{r,s}$-modules with graded $B\hat{\otimes}Cl_{s,r}$-modules as before.

If $\mathcal{B} = \mathcal{A}^{\mathrm{ev}}$ is purely even, there is a further simplification. A graded $\mathcal{A} = \mathcal{A}^{\mathrm{ev}}\hat{\otimes}Cl_{r,s}$-module has commuting graded actions of $Cl_{r,s}$ and $\mathcal{A}^{\mathrm{ev}}$, and decomposes into a direct sum of irreducibles of the form $V_i\hat{\otimes}W_j$, where V_i and W_j are irreducible for $\mathcal{A}^{\mathrm{ev}}$ and $Cl_{r,s}$ respectively. The super-representation group for $\mathcal{A}$ then decomposes as

$$\begin{aligned} \mathcal{SR}(\mathcal{A}) \equiv \mathcal{SR}(\mathcal{A}^{\mathrm{ev}}\hat{\otimes}Cl_{r,s}) &\cong \mathcal{SR}(\mathcal{A}^{\mathrm{ev}}) \otimes_{\mathbb{Z}} \mathcal{SR}(Cl_{r,s}) \\ &= K_0(\mathcal{A}^{\mathrm{ev}}) \otimes_{\mathbb{Z}} K_{s-r}(\mathbb{R}). \end{aligned} \tag{6.7}$$

6.2.3 Quotienting symmetry-compatible gapped Hamiltonians by topological triviality

Our interest in studying gapped Hamiltonians compatible with a given set of symmetry data (G, c, ϕ, σ) has led us to construct the graded twisted group C^*-algebra $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$. Then a graded f.g.p. module for $\mathcal{A}$ is interpreted as a gapped phase compatible with the symmetries. Here, we have identified gapped phases up to a notion of homotopy, namely, the spectral flattening of H into Γ. It was suggested by Kitaev [50], in the context of band insulators, that we should consider two gapped phases as being "equivalent", if they become equivalent (in the ordinary sense) upon adding some "trivial" bands. He argued that one can generally augment a given system by "... a set of local, disjoint modes, like inner atomic shells. This corresponds to adding an extra flat band on an insulator." An admissible system

(X) is supposed to be "...effectively cancelled by its particle-hole conjugate $(-X)$, resulting in a trivial system". He also suggested looking at *differences* between systems rather than systems themselves, motivating his K-theoretic classification of band insulators. We have taken these suggestions a little further, with some differences in interpretation. In Definition 6.2.4, we introduced a notion of trivial graded modules for $\mathcal{A}$, which may be added to a given graded $\mathcal{A}$-module without changing its equivalence class in $\mathcal{SR}(\mathcal{A})$. In particular, the sum of a graded module with its particle-hole conjugate (left parity reversal) becomes trivial in $\mathcal{SR}(\mathcal{A})$. Thus, Definition 6.2.5 formalises algebraically the notion of "quotienting by topological triviality" introduced by Kitaev, as well as Freed–Moore [30]. In some special cases, the super-representation group reduces to an ordinary K-theory group. There is another physical interpretation of the K-theoretic invariants of $\mathcal{A}$, hinted at in [50], in terms of *differences* of symmetry-compatible phases. This is discussed in detail in Section 6.4.

6.3 Higher super-representation groups, super-Brauer groups, and super-division algebras

For this section, all algebras are assumed to be *finite-dimensional*[5] unless otherwise stated. As is conventional in the literature, we will use the prefix "super-" interchangeably with "$\mathbb{Z}_2$-graded", except when referring to the distinct notions of the graded representation group and the super-representation group of a super-algebra $\mathcal{A}$. Many of the ideas used in the representation theory of semisimple ungraded algebras are applicable to the graded representation theory for semisimple super-algebras, see [90] for a comprehensive treatment, and [21] for connections to twisted K-theory. The main definitions and theorems can be found in these references, which we list here in a concise way for reference. They are needed for our definitions of higher super-representation groups in Section 6.3.1.

Ungraded simple algebras

In the ungraded case, the Artin–Wedderburn theorem shows that division algebras (or more generally, rings) are fundamental building blocks for semisimple algebras (or rings). Since there is only one division algebra over $\mathbb{C}$, namely, $\mathbb{C}$ itself, the simple algebras over $\mathbb{C}$ are matrix algebras over $\mathbb{C}$. These are also central, i.e., the centre is the ground field $\mathbb{C}$. Over $\mathbb{R}$, there are three division algebras $\mathbb{R}, \mathbb{H}, \mathbb{C}$, with the first two being central. The simple real algebras are matrix algebras over $\mathbb{R}, \mathbb{H}$ or $\mathbb{C}$, with the central ones being matrix algebras over $\mathbb{R}$ or $\mathbb{H}$. Central simple algebras are closed under taking tensor products and opposites. If we

[5]The generalisation to general $\mathbb{Z}_2$-graded C^*-algebras requires a much more careful treatment of Morita equivalence in that setting.

assign each central simple algebra $\mathcal{A}$ of the form $M_n(\mathcal{D})$ to the class denoted by $[\mathcal{D}]$, where $\mathcal{D}$ is some central division algebra over $\mathbb{F}$, we can define *Brauer multiplication* on the classes $[\mathcal{A}]$ via $[\mathcal{A}_1] \cdot [\mathcal{A}_2] \coloneqq [\mathcal{A}_1 \otimes \mathcal{A}_2]$. This multiplication is well-defined, commutative, and one even has $[\mathcal{D}] \cdot [\mathcal{D}^{\mathrm{op}}] = [\mathbb{F}]$, showing that $\mathrm{Br}(\mathbb{F}) \coloneqq \{[\mathcal{D}] : \mathcal{D} \text{ is a central division algebra over } \mathbb{F}\}$ is a group, called the *Brauer group* of $\mathbb{F}$. The identity element is $[\mathbb{F}]$, and $[\mathcal{D}]^{-1} = [\mathcal{D}^{\mathrm{op}}]$. For instance, $\mathrm{Br}(\mathbb{C})$ is the trivial group, while $\mathrm{Br}(\mathbb{R})$ is $\mathbb{Z}_2$, generated by $[\mathbb{H}]$.

Simple super-algebras

A simple super-algebra is a super-algebra which has no non-zero proper graded ideals. The *super-centre* $\mathrm{sctr}(\mathcal{A})$ of a super-algebra $\mathcal{A}$ is the sub-super-algebra of $\mathcal{A}$ whose homogeneous elements x are defined by

$$ax = (-1)^{|a||x|} xa, \qquad a \in \mathcal{A}_0 \cup \mathcal{A}_1.$$

This is to be contrasted with the ordinary centre $\mathrm{ctr}(\mathcal{A})$ of $\mathcal{A}$, which is also a sub-super-algebra of $\mathcal{A}$. Nevertheless, the even subalgebras of the centre and super-centre of $\mathcal{A}$ coincide. As an example, $\mathrm{sctr}(\mathbb{C}l_n) = \mathbb{C}$, whereas $\mathrm{ctr}(\mathbb{C}l_n)$ is $\mathbb{C}$ if n is even and is isomorphic to $\mathbb{C}l_1$ if n is odd.

A *central super-algebra* over $\mathbb{F}$ is a super-algebra with super-centre $\mathbb{F}$. A *super-division algebra* $\mathcal{D}$ is a super-algebra in which every non-zero homogeneous element has an inverse. Note that the super-centre of a super-division algebra $\mathcal{D}$ must be a field containing $\mathbb{F}$. This is because a non-zero odd super-central element of $\mathcal{D}$ is invertible and squares to 0, so it must itself be 0 — a contradiction. Thus, $\mathrm{sctr}(\mathcal{D})$ is purely-even. For non-zero $x \in \mathrm{sctr}(\mathcal{D})$, its inverse is easily seen to be in $\mathrm{sctr}(\mathcal{D})$ as well. Therefore, $\mathrm{sctr}(\mathcal{D})$ is a commutative division algebra, i.e., a field.

The central simple super-algebras over $\mathbb{F}$ can be characterised as the super-algebras of the form $M_{r|s}(\mathbb{F}) \hat{\otimes} \mathcal{D}$, where $M_{r|s}(\mathbb{F})$ is the super-algebra of linear endomorphisms of the super-vector space $\mathbb{F}^{r|s}$, and $\mathcal{D}$ is a central super-division algebra over $\mathbb{F}$ (see, for instance, Theorem 6.2.5 of [90]). In more detail, the super-Artin–Wedderburn theorem says that a simple (not necessarily central) super-algebra has the form $M_n(\mathcal{D})$ as an ungraded algebra, where $\mathcal{D}$ is a super-division algebra over $\mathbb{R}$. If $\mathcal{D}$ is not purely even, the even (resp. odd) elements of $\mathcal{A} = M_n(\mathcal{D})$ can be assumed to be the $n \times n$-matrices with entries in the even (resp. odd) elements of $\mathcal{D}$. If $\mathcal{D}$ is purely even, $\mathcal{A} = M_n(\mathcal{D})$ can be assumed to be $M_{r|n-r}(\mathcal{D})$, $0 \le r \le \lceil \frac{n}{2} \rceil$ as a super-algebra, with even elements being block-diagonal (the diagonal blocks are $r \times r$ and $(n-r) \times (n-r)$), and odd elements being off-diagonal. Conversely, a super-algebra $M_n(\mathcal{D})$ graded in one of the above ways is simple[6]. The building

[6] This is actually a special case of a more general graded Artin–Wedderburn structure theorem, which holds for algebras graded by more general groups, see Remark 1.4.8 of [36] and [65].

blocks of semisimple super-algebras over $\mathbb{F}$ are thus the super-division algebras over $\mathbb{F}$. Note that amongst the simple super-algebras over $\mathbb{F}$, the central ones are those with $\mathcal{D}$ being a central super-division algebra over $\mathbb{F}$; there may be others which are central over a field extension of $\mathbb{F}$.

As in the ungraded case, we assign the central simple super-algebras $\mathcal{A} = M_{r|s}(\mathbb{F})\hat{\otimes}\mathcal{D}$ to the class $[\mathcal{D}]$, and define a *super-Brauer multiplication* $[\mathcal{A}_1] \cdot [\mathcal{A}_2] := [\mathcal{A}_1\hat{\otimes}\mathcal{A}_2]$ on classes. This turns $\mathrm{sBr}(\mathbb{F}) := \{[\mathcal{D}] \,:\, \mathcal{D}$ is a central super-division algebra over $\mathbb{F}\}$ into an Abelian group, called the *super Brauer group* of $\mathbb{F}$. It is known that $\mathrm{sBr}(\mathbb{C}) = \mathbb{Z}_2$ and $\mathrm{sBr}(\mathbb{R}) = \mathbb{Z}_8$ [92].

For $\mathbb{F} = \mathbb{C}$, there are two super-division algebras which are both central, and they represent the two classes in $\mathrm{sBr}(\mathbb{C})$. We write them as $\mathcal{D}_0^{\mathbb{C}} = \mathbb{C} = \mathbb{C}l_0$ and $\mathcal{D}_1^{\mathbb{C}} = \mathbb{C}l_1$, with the subscript indicating the corresponding element in $\mathrm{sBr}(\mathbb{C}) = \mathbb{Z}_2$. The two super-division algebras over $\mathbb{C}$ can also be regarded as the *non-central* super-division algebras over $\mathbb{R}$. There are eight remaining (central) super-division algebras $\mathcal{D}_i^{\mathbb{R}}$ over $\mathbb{R}$, each representing an element of $\mathrm{sBr}(\mathbb{R}) = \mathbb{Z}_8$. Except for $\mathcal{D}_4^{\mathbb{R}} = \mathbb{H}$, they are the real Clifford algebras $\mathcal{D}_i^{\mathbb{R}} = Cl_{0,i}$, $i = 0, 1, 2, 3$, $\mathcal{D}_i^{\mathbb{R}} = Cl_{8-i,0}$, $i = 5, 6, 7$. Since $Cl_{0,4} \cong M_{1|1}(\mathbb{H})$, the super-Brauer class $[\mathcal{D}_4^{\mathbb{R}}]$ can be represented by the Clifford algebra $Cl_{0,4}$. In summary, there are $2+8 = 10$ super-division algebras over $\mathbb{R}$, and they represent the elements of the super-Brauer groups $\mathrm{sBr}(\mathbb{C})$ and $\mathrm{sBr}(\mathbb{R})$. Each element of the super-Brauer groups can be represented by a Clifford algebra, and apart from $[\mathcal{D}_4^{\mathbb{R}}]$, we can choose the representative Clifford algebra to be a super-division algebra over $\mathbb{R}$.

Remark 6.3.1. A connection between the ten super-division algebras over $\mathbb{R}$, and the tenfold way was made in [30], and explained in more detail in a set of lecture notes by Gregory Moore [60]. A connection between Wall's classification of central simple super-algebras, the Altland–Zirnbauer tenfold classification of free-fermions, and the classification of topological phases of time-reversal (TR) invariant Majorana fermions in one dimension was made in [28].

6.3.1 Higher super-representation groups and rings

Definition 6.3.2. For a graded C^*-algebra (not necessarily finite-dimensional) over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we define the higher super-representation groups $\mathcal{SR}^{-n}(\mathcal{A})$ to be

$$\mathcal{SR}^{-n}(\mathcal{A}) := \mathcal{SR}(\mathcal{A}\hat{\otimes}\mathcal{D}_n^{\mathbb{F}}).$$

Note that we can replace $\mathcal{D}_n^{\mathbb{F}}$ in this definition by a Morita-equivalent Clifford algebra. For complex super-algebras, there are two higher super-representation groups, whereas for real super-algebras, there are eight higher super-representation groups $\mathcal{SR}^{-n}(\mathcal{A})$, $n = 0, \ldots, 7$.

In certain cases, there are an additional structures on the direct sum of the higher super-representation groups $\mathcal{SR}_{\text{total}}(\mathcal{A}) := \oplus_n \mathcal{SR}^{-n}(\mathcal{A})$.

Proposition 6.3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be graded C^*-algebras over the same field, at least one of which is nuclear. On pairs of classes $([V],[W])$ in $\mathcal{SR}^{-i}(\mathcal{A}) \times \mathcal{SR}^{-j}(\mathcal{B})$, there is a well-defined product*

$$\begin{aligned} \otimes : \mathcal{SR}^{-i}(\mathcal{A}) \times \mathcal{SR}^{-j}(\mathcal{B}) &\longrightarrow \mathcal{SR}^{-(i+j)}(\mathcal{A}\hat{\otimes}\mathcal{B}) \\ ([V],[W]) &\mapsto [V]\otimes[W] := [V\hat{\otimes}W], \end{aligned} \tag{6.8}$$

where V and W are, respectively, any representative graded modules in the classes $[V]$ and $[W]$.

Proof. Suppose $\mathcal{I}_{\mathcal{A}}$ is an odd involutory operator verifying the triviality of $V \in \mathcal{GV}(\mathcal{A})$, so $\mathcal{I}_{\mathcal{A}}$ graded commutes with the action of $\mathcal{A}$ on V. Let $W \in \mathcal{GV}(\mathcal{B})$. Then $\mathcal{I}_{\mathcal{A}}\hat{\otimes}1_W$ is an odd involutory operator on $V\hat{\otimes}W \in \mathcal{GV}(\mathcal{A}\hat{\otimes}\mathcal{B})$ which graded commutes with the action of $\mathcal{A}\hat{\otimes}\mathcal{B}$, so $V\hat{\otimes}W \in \mathcal{T}(\mathcal{A}\hat{\otimes}\mathcal{B})$. This conclusion also holds when $\mathcal{A},\mathcal{B}$ are replaced by $\mathcal{A}\hat{\otimes}D_i^{\mathbb{F}}, \mathcal{B}\hat{\otimes}D_j^{\mathbb{F}}$, and when the roles of V and W are reversed, so the product respects triviality. □

For example, consider the simplest case where $\mathcal{A} = \mathbb{R}$. Then

$$\mathcal{SR}^{-n}(\mathbb{R}) = \mathcal{SR}(\mathcal{D}_n^{\mathbb{R}}) \cong \mathcal{SR}(Cl_{0,n}) \cong K_n(\mathbb{R}) \cong KO^{-n}(\star), \tag{6.9}$$

where we have used the Morita equivalence of $\mathcal{D}_n^{\mathbb{R}}$ with $Cl_{0,n}$, and Corollary 6.2.16. By taking $\mathcal{A} = \mathcal{D}_i^{\mathbb{R}}$ and $\mathcal{B} = \mathcal{D}_j^{\mathbb{R}}$, and recalling the Morita equivalences between $Cl_{0,n}$ and $Cl_{0,n+8}$, we find that the direct sum

$$\mathcal{SR}_{\text{total}}(\mathbb{R}) := \bigoplus_n \mathcal{SR}^{-n}(\mathbb{R}) = \bigoplus_n \mathcal{SR}(Cl_{0,n})$$

becomes a graded ring with multiplication $\otimes$, and grading group $\text{sBr}(\mathbb{R}) = \mathbb{Z}_8$.

Proposition 6.3.4. *$\mathcal{SR}_{\text{total}}(\mathbb{R})$ is isomorphic as a $\mathbb{Z}_8$-graded ring to the KO-theory ring of a point.*

Proof. It is clear from Lemma 6.2.13 and Proposition 6.2.15 that $\oplus_n \mathcal{GR}(Cl_{n,0})/i^*\mathcal{GR}(Cl_{n+1,0})$, with graded tensor product as the multiplication, is isomorphic to $\oplus_n \mathcal{SR}^{-n}(\mathbb{R})$ as a graded ring. The second isomorphism in

$$\mathcal{SR}_{\text{total}}(\mathbb{R}) \cong \bigoplus_n \mathcal{GR}(Cl_{n,0})/i^*\mathcal{GR}(Cl_{n+1,0}) \cong \bigoplus_n KO^{-n}(\star), \tag{6.10}$$

is the graded ring isomorphism of Atiyah–Bott–Shapiro in Theorem 11.5 of [4], where the product on the right-hand-side is the tensor product of bundles (c.f. Chapter I.9 of [54]). Bott periodicity and the Morita equivalence of the Clifford algebras allow us to interpret (6.10) as an isomorphism of $\mathbb{Z}_8$-graded rings. □

Remark 6.3.5. More generally, taking $\mathcal{A} = \mathbb{R}$ and $\mathcal{B}$ a super-algebra over $\mathbb{R}$ in Proposition 6.3.3, the direct sum

$$\mathcal{SR}_{\text{total}}(\mathcal{B}) := \bigoplus_n \mathcal{SR}^{-n}(\mathcal{B})$$

becomes a sBr($\mathbb{R}$)-graded module over $\mathcal{SR}_{\text{total}}(\mathbb{R})$.

In the complex case, we can also define $\mathcal{SR}_{\text{total}}(\mathcal{A})$ for a super-algebra $\mathcal{A}$ over $\mathbb{C}$, and obtain a sBr($\mathbb{C}$) $= \mathbb{Z}_2$-graded ring structure on $\mathcal{SR}_{\text{total}}(\mathbb{C})$, as well as a $\mathbb{Z}_2$-graded module structure on $\mathcal{SR}_{\text{total}}(\mathcal{A})$ over the graded ring $\mathcal{SR}_{\text{total}}(\mathbb{C})$. These are the (super)-algebraic analogues of the well-known graded module structures on $\oplus_n KO^{-n}(X)$ and $\oplus_n K^{-n}(X)$, over the respective graded rings $\oplus_n KO^{-n}(\star)$ and $\oplus_n K^{-n}(\star)$.

We may also regard $\mathcal{SR}_{\text{total}}(\mathbb{R})$ as a $\mathbb{Z}_2$-graded ring, with even subring $\oplus_{n\,\text{even}}\mathcal{SR}^{-n}(\mathbb{R})$.

Proposition 6.3.6. *$\mathcal{SR}_{\text{total}}(\mathbb{R})$ and $\mathcal{SR}_{\text{total}}(\mathbb{C})$ are super-commutative rings.*

Proof. Consider two graded modules S, T for $Cl_{0,n}$, $n \geq 2$, which are isomorphic as graded vector spaces under the invertible linear map $\iota : S \to T$, but have the actions of two Clifford generators f_i, f_j, $i \neq j$ swapped and identical actions of the remaining $n-2$ Clifford generators. Thus, for all $s \in S$, we have

$$\iota(f_i s) = f_j(\iota(s)), \qquad \iota(f_j s) = f_i(\iota(s)), \qquad \iota(f_k s) = f_k(\iota(s)),\ k \neq i, j.$$

We claim that $S \cong {}^{\Pi}T$ as graded $Cl_{0,n}$-modules. Define the even linear map $\alpha := \pi_L \circ \frac{1}{\sqrt{2}}(f_i - f_j) \circ \iota : S \to {}^{\Pi}T$. Then for $s \in S$,

$$\begin{aligned} f_i(\alpha s) &= \frac{1}{\sqrt{2}} f_i{}^{\pi}((f_i - f_j)\iota s) \\ &= -\frac{1}{\sqrt{2}}{}^{\pi}(f_i(f_i - f_j)\iota s) \\ &= -\frac{1}{\sqrt{2}}{}^{\pi}((f_j^2 - f_i f_j)\iota s) \\ &= \frac{1}{\sqrt{2}}{}^{\pi}((f_i - f_j)(\iota(f_i(s))) \\ &= \alpha(f_i s), \end{aligned}$$

and similarly, $f_j(\alpha s) = \alpha(f_j s)$. It is also clear that α intertwines the actions of the remaining Clifford generators, and that $\alpha^{-1} = \iota^{-1} \circ \frac{1}{\sqrt{2}}(f_i - f_j) \circ \pi_L$. Therefore, swapping the actions of any two Clifford generators takes $T \mapsto {}^{\Pi}T$; in terms of $\mathcal{SR}(Cl_{0,n}) \cong \mathcal{SR}^{-n}(\mathbb{R})$, it takes $[T] \mapsto -[T]$. Now consider graded modules V and W for $Cl_{0,i}$ and $Cl_{0,j}$, respectively. Both $V \hat{\otimes} W$ and $W \hat{\otimes} V$ are graded $Cl_{0,i+j}$-modules, and we get from one to the other through an interchange of ij pairs of Clifford generators. It follows that

$$[V] \otimes [W] = (-1)^{ij}[W] \otimes [V], \qquad [V] \in \mathcal{SR}^i(\mathbb{R}),\ [W] \in \mathcal{SR}^j(\mathbb{R}). \tag{6.11}$$

Since the sign factor $(-1)^{ij}$ only depends on the ($\mathbb{Z}_2$) parity of i and j, (6.11) says that the multiplication $\otimes$ on $\mathcal{SR}_{\text{total}}(\mathbb{R})$, regarded as a $\mathbb{Z}_2$-graded ring, is super-commutative. A similar argument shows that $\mathcal{SR}_{\text{total}}(\mathbb{C})$ is also a super-commutative ring. $\square$

Ring structure on $\mathcal{SR}_{\text{total}}(\mathcal{A})$

We remark that $\mathcal{SR}_{\text{total}}(\mathcal{A})$ can be given a sBr($\mathbb{F}$)-graded ring structure if there is a comultiplication $\Delta : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$ which is a morphism of super-algebras. The comultiplication is needed to pull-back a $(\mathcal{A}\hat{\otimes}\mathcal{D}_i^{\mathbb{F}}\hat{\otimes}\mathcal{A}\hat{\otimes}\mathcal{D}_j^{\mathbb{F}})$-module to a $(\mathcal{A}\hat{\otimes}\mathcal{D}_{i+j}^{\mathbb{F}})$-module. Such a comultiplication exists, for instance, if $\mathcal{A}$ is the universal enveloping super-algebra of a Lie super-algebra, see [52] for a related construction and discussion.

Another example, more relevant for our purposes, is when $\mathcal{A} = \mathbb{F}[G]$ is the purely even group algebra of a finite group G, with the comultiplication Δ defined on the group-like basis $\Delta(g) = g \otimes g$ and extended $\mathbb{F}$-linearly. If V, W are graded modules for $\mathbb{R}[G]\hat{\otimes}Cl_{0,i}$ and $\mathbb{R}[G]\hat{\otimes}Cl_{0,j}$ respectively, then $V\hat{\otimes}W$ is a graded module for $\mathbb{R}[G]\hat{\otimes}Cl_{0,i}\hat{\otimes}\mathbb{R}[G]\hat{\otimes}Cl_{0,j} \cong (\mathbb{R}[G] \otimes \mathbb{R}[G])\hat{\otimes}Cl_{0,i+j}$. Pulling this back via Δ turns $V\hat{\otimes}W$ into a graded $\mathbb{R}[G]\hat{\otimes}Cl_{0,i+j}$-module. In other words, we can define a multiplication $\otimes$ on $\mathcal{SR}_{\text{total}}(\mathbb{R}[G])$, by $[V] \otimes [W] = [\Delta^*(V\hat{\otimes}W)]$, making it a sBr($\mathbb{R}$)-graded ring. The arguments leading to (6.11) continue to hold when $\mathbb{R}$ is replaced by $\mathbb{R}[G]$, since there is simply an extra action of $\mathbb{R}[G]$ which is even and commutes with the Clifford actions. Thus, we may also regard $\mathcal{SR}_{\text{total}}(\mathbb{R}[G])$, and indeed $\mathcal{SR}_{\text{total}}(\mathbb{C}[G])$, as $\mathbb{Z}_2$-graded rings which are super-commutative.

A final example is when $\mathcal{A}$ is purely even, complex, and commutative (but not necessarily finite-dimensional), so $\mathcal{A} \cong C_0(\hat{A})$ where $\hat{A}$ is the Gelfand spectrum of $\mathcal{A}$. Then the tensor product of two $\mathcal{A}$-modules is another $\mathcal{A}$-module, by taking the (internal) tensor product of the vector bundles that they correspond to (and taking unitisations/compactifications where required). We then obtain a sBr($\mathbb{C}$)-graded ring structure on $\mathcal{SR}_{\text{total}}(C_0(\hat{A}))$. This should be compared to the much richer graded ring structure on (topological) "K-theory with local coefficients" as defined in [21], which has a larger grading group containing sBr($\mathbb{C}$) as a direct product factor. However, for real commutative C^*-algebras $\mathcal{A}$, the vector bundle construction requires Real bundles over the Real spectrum, so the sBr($\mathbb{R}$)-graded ring structure on $\mathcal{SR}_{\text{total}}(\mathcal{A})$ is not really directly related to the KO-theory with local coefficients in [21].

6.4 K-theory and the difference-group of symmetry-compatible Hamiltonians

While the super-representation groups of $\mathcal{A}$ have the two or eight-fold periodicities inherited from the Clifford algebras used in their definition, it is not clear that they coincide with the usual (higher) K-theory groups of $\mathcal{A}$ when $\mathcal{A}$ is not the trivial algebra $\mathbb{F}$. The K-theory of (ungraded) C^*-algebras has been well-studied, with many important results for crossed product algebras in particular. To motivate the use of this rich theory, we give the K-theory groups some physical interpretation relevant to the classification of symmetry-compatible gapped Hamiltonians. To achieve this, we utilise a version of K-theory introduced by Karoubi [42], which is defined for graded C^*-algebras, and is equivalent to the more standard Grothendieck group-type definitions, e.g., in [11, 93], for ungraded C^*-algebras. These K-theory groups may be regarded as special cases of Kasparov's bivariant KK-theory groups. We describe the constructions for unital algebras $\mathcal{A}$; the definitions for non-unital algebras follow the usual procedure of unitisation followed by taking the kernel of the map in K-theory induced by the projection $\mathcal{A}^+ \to \mathbb{F}$.

6.4.1 Symmetry-compatible gapped Hamiltonians

Recall that the grading operator Γ of a graded f.g.p. module W for the graded algebra $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$ can be interpreted as a spectrally-flattened gapped Hamiltonian compatible with the symmetry data $(G, c, \mathbb{C}, \phi, \sigma)$. Strictly speaking, this interpretation requires W to be a graded Hilbert space with a graded $*$-representation of $\mathcal{A}$. Although the K-theory of $\mathcal{A}$ will be defined in terms of f.g.p. graded modules W for $\mathcal{A}$ (or its unitisation), we may regard the latter as graded Hilbert $\mathcal{A}$-modules (see Remark 6.2.7), on which there is a notion of self-adjointness for the adjointable (and bounded) operators $\mathscr{B}(W)$. In fact, $\mathscr{B}(W)$, along with the grading operator Γ, form a (evenly) graded C^*-algebra under the operator norm. The self-adjoint unitary grading operator for $\mathscr{B}(W)$ is Γ, and it makes sense to talk about continuous functions of Γ and their homotopies. As an example, when $\mathcal{A} = \mathbb{C} \rtimes \mathbb{Z}^d \cong C(\mathbb{T}^d)$, a graded f.g.p. $C(\mathbb{T}^d)$-module is, as a Hilbert $C(\mathbb{T}^d)$-module, the set of continuous sections of some graded Hermitian vector bundle over $\mathbb{T}^d$, on which there is a continuous (i.e., $C(\mathbb{T}^d)$-valued) fibre-wise inner product. The restriction of the grading operator to a fibre can be viewed as the flattened version of a gapped Bloch Hamiltonian on that fibre. The positively-graded sub-bundle is the conduction band, while the negatively-graded sub-bundle is the valence band. The usual Bloch–Floquet picture of a direct integral decomposition of $L^2(\mathbb{R}^d)$ over the character space $\mathbb{T}^d$, can be recovered by passing to the GNS representation induced by a faithful trace on $C(\mathbb{T}^d)$ (e.g., integration over the Haar measure on $\mathbb{T}^d$). For a general, possibly noncommutative, graded algebra $\mathcal{A}$, we interpret the grading operator Γ on a f.g.p. graded $\mathcal{A}$-module W as a flattened gapped Hamiltonian on the noncommutative graded "vector bundle" corresponding to the graded $\mathcal{A}$-module W. For more details regarding the use of Hilbert C^*-modules to formalise a noncommutative version of Bloch theory, see [33].

Given an ungraded $\mathcal{A}$-module W, we can consider the set $\mathrm{Grad}_{\mathcal{A}}(W)$ of possible grading operators[7] on W turning it into a graded $\mathcal{A}$-module. There is a standard Banach space structure on W (either from its Hilbert $\mathcal{A}$-module structure or induced from the free module $\mathcal{A}^n$ which it is a direct summand of), which determines a norm topology on the bounded linear maps $W \to W$. Thus, there is an induced topology on $\mathrm{Grad}_{\mathcal{A}}(W) \subset \mathrm{End}_{\mathcal{A}}(W) \subset \mathrm{End}(W)$ (e.g. see I.6.22 of [42], 11.2 of [11], or Chapter 15 of [93]). We can then talk about the homotopy classes of symmetry-compatible flattened Hamiltonians on W:

Definition 6.4.1 (Symmetry compatible gapped Hamiltonians)**.** Let (G, c, ϕ, σ) be symmetry data as described in Section 3.2.2, and let $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$ be its associated graded twisted crossed product. Suppose $\mathcal{A}$ is unital, and let W be an ungraded f.g.p. $\mathcal{A}$-module. We call $\mathrm{Grad}_{\mathcal{A}}(W)$ the set of (G, c, ϕ, σ)*-compatible*, or $\mathcal{A}$*-compatible*, or simply *symmetry-compatible (flattened) gapped Hamiltonians on* W. Two grading operators $\Gamma_1, \Gamma_2 \in \mathrm{Grad}_{\mathcal{A}}(W)$ are said to be *homotopic* if there is a norm-continuous path between Γ_1 and Γ_2 within $\mathrm{Grad}_{\mathcal{A}}(W)$; in this case, we write $\Gamma_1 \sim_h \Gamma_2$.

Remark 6.4.2. More generally, we can also consider $\mathrm{Grad}_{\mathcal{A}}(W)$ for the twisted crossed product $\mathcal{A}$ associated to a twisted C^*-dynamical system $(G, c, \mathcal{B}, \alpha, \sigma)$, and apply Definition 6.4.1.

Recall that $\mathrm{End}_{\mathcal{A}}(W)$ can be given the structure of a C^*-algebra. We may then assume that the Hamiltonians Γ_i are self-adjoint and unitary, and that a homotopy between Γ_1 and Γ_2 takes place within such self-adjoint grading operators (see 4.6 of [11]). Intuitively, $\Gamma_1 \sim_h \Gamma_2$ means that the two Hamiltonians can be continuously deformed into one another, while respecting the symmetries encoded by the algebra $\mathcal{A}$, and maintaining the gapped condition. The set of homotopy classes of symmetry-compatible gapped Hamiltonians on W has been of interest in the literature [50, 84], and can be identified with the set of path-components $\pi_0(\mathrm{Grad}_{\mathcal{A}}(W))$. However, these sets do not have a group structure yet, nor are they easily computable in general. The idea of Kitaev [50] and Stone et al [84] is to identify, at least for some special $\mathcal{A}$ and families of modules $W^{(N)}$, the sets $\mathrm{Grad}_{\mathcal{A}}(W^{(N)})$ with a family of symmetric spaces, take a "large-N" limit, and then identify the resulting space with a loop space of the stable orthogonal or unitary group. The latter loop spaces are classifying spaces C_n and R_n for (topological) K-theory and KO-theory respectively, in the sense that $K^{-n}(X) = [X, C_n]$ and $KO^{-n}(X) = [X, R_n]$ for a compact Hausdorff space X (Theorem 4.29 of [42]). For example, $K^{-n}(\star)$ and $KO^{-n}(\star)$ can be identified with the path-components of C_n and R_n. One might be tempted to identify the K-theory group $K^{-n}(\star) = \pi_0(C_n)$ as the set of "stable" homotopy classes of symmetry-compatible Hamiltonians, but

[7] Karoubi calls these *gradations*.

the precise connection between K-theory and homotopy classes of symmetry-compatible Hamiltonians is somewhat more subtle (see Section 8.3.2 for some further comments on this matter). Furthermore, the correct theory to use in the presence of antiunitary symmetries is Atiyah's KR-theory [3], rather than the ordinary KO-theory.

6.4.2 Differences between symmetry-compatible gapped Hamiltonians

Using a slightly modified version of Karoubi's definitions for K-theory, the elements of the K-theory of $\mathcal{A}$ can be represented by *differences* between $\mathcal{A}$-compatible Hamiltonians, in a sense which we will now describe.

Definition 6.4.3 (Trivial differences between Hamiltonians)**.** Let W be a graded f.g.p. module for a graded unital C^*-algebra $\mathcal{A}$, and let $\Gamma_1, \Gamma_2 \in \mathrm{Grad}_{\mathcal{A}}(W)$ be a pair of compatible gapped Hamiltonians on W. We call (W, Γ_1, Γ_2) a *trivial triple* if $\Gamma_1 \sim_h \Gamma_2$ in $\mathrm{Grad}_{\mathcal{A}}(W)$.

A triple (W, Γ_1, Γ_2) is meant to represent the (ordered) difference between two $\mathcal{A}$-compatible gapped Hamiltonians on W, and Definition 6.4.3 says that we do not wish to distinguish between two Hamiltonians which can be continuously deformed into one another. We want to be able to consider all graded modules concurrently, and to combine two or more systems together. The direct sum operation gives a natural commutative monoid structure to the collection $\mathrm{Grad}_{\mathcal{A}}$ of all triples, where some obvious identifications have been made to ensure commutativity and associativity. The collection of trivial triples forms a submonoid $\mathrm{Grad}^t_{\mathcal{A}}$.

Definition 6.4.4 (Difference-group of compatible Hamiltonians)**.** Let $\mathbf{K}_0(\mathcal{A})$ be the quotient monoid of $\mathrm{Grad}_{\mathcal{A}}$ by the congruence generated by $\mathrm{Grad}^t_{\mathcal{A}}$, i.e., $[W, \Gamma_1, \Gamma_2] = [W', \Gamma'_1, \Gamma'_2]$ in $\mathbf{K}_0(\mathcal{A})$ if and only if there are trivial triples (F, ζ_1, ζ_2) and (F', ζ'_1, ζ'_2) in $\mathrm{Grad}^t_{\mathcal{A}}$, such that $(W \oplus F, \Gamma_1 \oplus \zeta_1, \Gamma_2 \oplus \zeta_2) = (W' \oplus F', \Gamma'_1 \oplus \zeta'_1, \Gamma'_2 \oplus \zeta'_2)$ in $\mathrm{Grad}_{\mathcal{A}}$. We call $\mathbf{K}_0(\mathcal{A})$ the *difference-group of $\mathcal{A}$-compatible gapped Hamiltonians.*

Note the use of boldface notation for the difference group $\mathbf{K}_0(\mathcal{A})$. The non-boldface $K_0(\mathcal{A})$ will always refer to the ordinary K-theory group of an *ungraded* $\mathcal{A}$ (or a graded $\mathcal{A}$ regarded as an ungraded algebra). We now consider some of the structural properties of the $\mathbf{K}_0(\mathcal{A})$. The proofs are similar to those of a related construction in Chapter III of [42].

Proposition 6.4.5. *$\mathbf{K}_0(\mathcal{A})$ is an abelian group, with $[W, \Gamma_1, \Gamma_2] = -[W, \Gamma_2, \Gamma_1]$. Furthermore, two isomorphic triples (in the natural sense) define the same class in $\mathbf{K}_0(\mathcal{A})$.*

Proof. $\Gamma_1 \oplus \Gamma_2 \sim_h \Gamma_2 \oplus \Gamma_1$ in $\mathrm{Grad}_{\mathcal{A}}(W \oplus W)$ via the homotopy

$$\Gamma(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, \frac{\pi}{2}], \tag{6.12}$$

so $(W\oplus W,\Gamma_1\oplus\Gamma_2,\Gamma_2\oplus\Gamma_1)$ is trivial, and we can write $[W,\Gamma_2,\Gamma_1]=-[W,\Gamma_1,\Gamma_2]$ in $\mathbf{K}_0(\mathcal{A})$. For isomorphic triples (W,Γ_1,Γ_2) and (W',Γ_1',Γ_2'), let $\alpha: W\to W'$ be the isomorphism of ungraded $\mathcal{A}$-modules, such that $\Gamma_i'=\alpha\Gamma_i\alpha^{-1}$, $i=1,2$. Then $\Gamma_2\oplus\Gamma_1'\sim_h\Gamma_1\oplus\Gamma_2'$ in $\mathrm{Grad}_{\mathcal{A}}(W\oplus W')$ via the homotopy

$$\Gamma(\theta)=\begin{pmatrix}\cos\theta & -\alpha^{-1}\sin\theta\\ \alpha\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\Gamma_1 & 0\\ 0 & \Gamma_2'\end{pmatrix}\begin{pmatrix}\cos\theta & \alpha^{-1}\sin\theta\\ -\alpha\sin\theta & \cos\theta\end{pmatrix},\qquad \theta\in[0,\frac{\pi}{2}].$$

Therefore, $0=[W\oplus W',\Gamma_1\oplus\Gamma_2',\Gamma_2\oplus\Gamma_1']=[W,\Gamma_1,\Gamma_2]-[W',\Gamma_1',\Gamma_2']$. Note that $\mathcal{A}$ acts diagonally, so it is easy to see that it graded commutes with $\Gamma(\theta)$ in both cases. □

Physically, this is the statement that going from one Hamiltonian Γ_1 to another Hamiltonian Γ_2 is "opposite" to going from Γ_2 to Γ_1.

Proposition 6.4.6 (Path-independence and homotopy-independence of differences)**.** *The equation* $[W,\Gamma_1,\Gamma_2]+[W,\Gamma_2,\Gamma_3]=[W,\Gamma_1,\Gamma_3]$ *holds in* $\mathbf{K}_0(\mathcal{A})$*. Furthermore,* $[W,\Gamma_1,\Gamma_2]$ *depends only on the homotopy class of* Γ_i *in* $\mathrm{Grad}_{\mathcal{A}}(W)$*.*

Proof. We need to show that $[W\oplus W\oplus W,\Gamma_1\oplus\Gamma_2\oplus\Gamma_3,\Gamma_2\oplus\Gamma_3\oplus\Gamma_1]$ is trivial. Since $\Gamma_2\oplus\Gamma_3\oplus\Gamma_1$ can be obtained from $\Gamma_1\oplus\Gamma_2\oplus\Gamma_3$ by conjugation with the permutation matrix

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\in\mathrm{SO}(3),$$

and SO(3) is path-connected, it follows that $\Gamma_2\oplus\Gamma_3\oplus\Gamma_1\sim_h\Gamma_1\oplus\Gamma_2\oplus\Gamma_3$ in $\mathrm{Grad}_{\mathcal{A}}(W\oplus W\oplus W)$. If $\Gamma_i'\sim_h\Gamma_i$, $i=1,2$, then (W,Γ_1',Γ_1) and (W,Γ_2,Γ_2') are trivial, so $[W,\Gamma_1',\Gamma_2']=[W,\Gamma_1',\Gamma_1]+[W,\Gamma_1,\Gamma_2]+[W,\Gamma_2,\Gamma_2']=[W,\Gamma_1,\Gamma_2]$. □

The pleasant "path-independence" result says that it does not matter whether we go from Γ_1 to Γ_3 directly, or via an intermediate Γ_2. The homotopy-independence property says that there is no difference between two compatible gapped Hamiltonians which can be continuously deformed into one another.

Remark 6.4.7 (Difference-group of non-unital algebras)*.* For non-unital graded C^*-algebras $\mathcal{A}$, we define $\mathbf{K}_0(\mathcal{A})$ to be the kernel of the homomorphism $\mathbf{K}_0(\mathcal{A}^+)\to\mathbf{K}_0(\mathbb{F})\cong\mathbb{Z}$ induced by the projection $p:\mathcal{A}\to\mathbb{F}$. Note that a triple (W,Γ_1,Γ_2) gets mapped to a triple $(p_*(W),p_*(\Gamma_1),p_*(\Gamma_2))$, where $p_*(W)=\mathbb{F}\hat{\otimes}_{\mathcal{A}^+}W$, and $p_*(\Gamma_i)=1\hat{\otimes}\Gamma_i$; then $\Gamma_1\sim_h\Gamma_2$ implies $p_*(\Gamma_1)\sim_h p_*(\Gamma_2)$ as is required for consistency.

Remark 6.4.8*.* The classes in the difference-group which can be written as $[W,\Gamma,-\Gamma]$ form a subgroup of $\mathbf{K}_0(\mathcal{A})$. Such classes can be "represented" by a *single* graded $\mathcal{A}$-module (W,Γ). The right parity reversal $W^{\Pi}=(W,-\Gamma)$ "represents" the class $[W,-\Gamma,\Gamma]$, which "cancels"

(W,Γ) in the sense that $[W,\Gamma,-\Gamma]=-[W,-\Gamma,\Gamma]$. The left parity reversal $({}^{\Pi}W,-\Gamma)$ "represents" $[{}^{\Pi}W,-\Gamma,\Gamma]$, which also "cancels" (W,Γ) using, again, the homotopy (6.12). Of course, if W is homotopic to W^{Π}, i.e., $\Gamma\sim_h-\Gamma$, then $[W,\Gamma,-\Gamma]=0$ automatically. This occurs, for example, if (W,Γ) admits a trivialising operator $\mathcal{I}$ in the sense of Definition 6.2.4. Then $\Gamma(\theta)=(\cos\theta)\Gamma+(\sin\theta)\mathcal{I}$, $\theta\in[0,\pi]$ provides a homotopy between Γ and $-\Gamma$, showing that $[W,\Gamma,-\Gamma]=0$. Thus, we can define a homomorphism $\mathcal{SR}(\mathcal{A})\to\mathbf{K}_0(\mathcal{A})$ which takes a class $[W]$, represented by a graded $\mathcal{A}$-module W with grading operator Γ, to the class of the triple $[W,\Gamma,-\Gamma]$.

6.4.3 Relation between difference-groups and ordinary K-theory groups

For purely-even $\mathcal{A}^{\rm ev}$, our definition of $\mathbf{K}_0(\mathcal{A}^{\rm ev})$ is equivalent to either of Karoubi's two definitions of $K'^{0,0}(\mathcal{A}^{\rm ev})$ in III.4.15 and III.4.19 of [42], and his $K'^{0,0}(\mathcal{A}^{\rm ev})$ is itself isomorphic to his $K^{0,0}(\mathcal{A}^{\rm ev})$ as defined in III.4.11 and II.2.13 of the same reference. As pointed out in [44], Karoubi's definitions also work for *graded* algebras, and as we explain below, they continue to coincide with our $\mathbf{K}_0(\mathcal{A})$. We provide them here for convenience.

Karoubi's $K^{0,0}(\mathcal{A})$ is generated by (isomorphism classes of) triples (W,W',α), where W,W' are graded $\mathcal{A}$-modules and α is an isomorphism of W and W' as ungraded $\mathcal{A}$-modules. A triple (W,W',α) is called *elementary* if α is homotopic to the identity map within the automorphisms of W, and $K^{0,0}(\mathcal{A})$ is the group that results from quotienting out the elementary triples. Karoubi's $K'^{0,0}(\mathcal{A})$ is, like our $\mathbf{K}_0(\mathcal{A})$, generated by the set of triples (W,Γ_1,Γ_2), but is quotiented by the relation $(W,\Gamma_1,\Gamma_2)\sim(W',\Gamma_1',\Gamma_2')$ iff there is a triple of the form (F,ζ,ζ), such that $\Gamma_1\oplus\Gamma_2'\oplus\zeta\sim_h\Gamma_2\oplus\Gamma_1'\oplus\zeta$.

Proposition 6.4.9. *Karoubi's $K'^{0,0}(\mathcal{A})$ is isomorphic to our difference-group $\mathbf{K}_0(\mathcal{A})$.*

Proof. Suppose $[W,\Gamma_1,\Gamma_2]=[W,\Gamma_1',\Gamma_2']$ in $\mathbf{K}_0(\mathcal{A})$, so there exist (F,ζ_1,ζ_2) and (F',ζ_1',ζ_2') with $\zeta_1\sim_h\zeta_2$ and $\zeta_1'\sim_h\zeta_2'$, such that $W\oplus F=W'\oplus F'$, $\Gamma_1\oplus\zeta_1=\Gamma_1'\oplus\zeta_1'$, $\Gamma_2\oplus\zeta_2=\Gamma_2'\oplus\zeta_2'$. Then $\Gamma_1\oplus\Gamma_2'\oplus\zeta_1\oplus\zeta_2'$ and $\Gamma_2\oplus\Gamma_1'\oplus\zeta_1\oplus\zeta_2'$ are homotopic grading operators on $W\oplus W'\oplus(F\oplus F')$, so $(W,\Gamma_1,\Gamma_2)\sim(W',\Gamma_1',\Gamma_2')$ in Karoubi's sense. Conversely, let $(W,\Gamma_1,\Gamma_2)\sim(W',\Gamma_1',\Gamma_2')$ so that $\Gamma_1\oplus\Gamma_2'\oplus\zeta\sim_h\Gamma_2\oplus\Gamma_1'\oplus\zeta$ for some elementary (F,ζ,ζ). There are two decompositions

$$\begin{aligned}
&(W\oplus W'\oplus W\oplus W'\oplus F,\Gamma_1\oplus\Gamma_2'\oplus\Gamma_2\oplus\Gamma_1'\oplus\zeta,\Gamma_2\oplus\Gamma_2'\oplus\Gamma_1\oplus\Gamma_2'\oplus\zeta)\\
&\quad=(W,\Gamma_1,\Gamma_2)+(W',\Gamma_2',\Gamma_2')+(W\oplus W'\oplus F,\Gamma_2\oplus\Gamma_1'\oplus\zeta,\Gamma_1\oplus\Gamma_2'\oplus\zeta)\\
&\quad=(W',\Gamma_1',\Gamma_2')+(W,\Gamma_2,\Gamma_2)+(W\oplus W'\oplus F,\Gamma_1\oplus\Gamma_2'\oplus\zeta,\Gamma_1\oplus\Gamma_2'\oplus\zeta),
\end{aligned}$$

where both of the last two triples in each of the last two lines are trivial. It follows that $[W,\Gamma_1,\Gamma_2]=[W',\Gamma_1',\Gamma_2']$ in $\mathbf{K}_0(\mathcal{A})$. $\square$

Note that we could have used a weaker equivalence in our definition of $\mathbf{K}_0(\mathcal{A})$, namely, declare that $(W,\Gamma_1,\Gamma_2) \sim' (W,\Gamma_1',\Gamma_2')$ iff there exist trivial (F,ζ_1,ζ_2) and (F',ζ_1',ζ_2') such that the triple $(W\oplus F,\Gamma_1\oplus\zeta_1,\Gamma_2\oplus\zeta_2)$ is *isomorphic*[8] to the triple $(W'\oplus F',\Gamma_1'\oplus\zeta_1',\Gamma_2'\oplus\zeta_2')$. Assuming this isomorphism, we can write

$$\begin{aligned}
&(W\oplus F\oplus W'\oplus F'\oplus W'\oplus F', \Gamma_1\oplus\zeta_1\oplus\Gamma_2'\oplus\zeta_2'\oplus\Gamma_1'\oplus\zeta_1', \Gamma_2\oplus\zeta_2'\oplus\Gamma_1'\oplus\zeta_1\oplus\Gamma_2'\oplus\zeta_2)\\
=\;& (W,\Gamma_1,\Gamma_2)+(F,\zeta_1,\zeta_2)\\
&\quad +(W'\oplus F'\oplus W'\oplus F', \Gamma_2'\oplus\zeta_2'\oplus\Gamma_1'\oplus\zeta_1', \Gamma_1'\oplus\zeta_1'\oplus\Gamma_2'\oplus\zeta_2')
\end{aligned}$$

where on the right-hand-side, the second term is a trivial triple, while the third term is the sum of $(W'\oplus F',\Gamma_2'\oplus\zeta_2',\Gamma_1'\oplus\zeta_1')$ with its inverse $(W'\oplus F',\Gamma_1'\oplus\zeta_1',\Gamma_2'\oplus\zeta_2')$, which is also trivial by Proposition 6.4.5. We can present the same triple as a sum in a second way,

$$\begin{aligned}
&(W\oplus F\oplus W'\oplus F'\oplus W'\oplus F', \Gamma_1\oplus\zeta_1\oplus\Gamma_2'\oplus\zeta_2'\oplus\Gamma_1'\oplus\zeta_1', \Gamma_2\oplus\zeta_2'\oplus\Gamma_1'\oplus\zeta_1\oplus\Gamma_2'\oplus\zeta_2)\\
&\quad = (W\oplus F\oplus W'\oplus F', \Gamma_1\oplus\zeta_1\oplus\Gamma_2'\oplus\zeta_2', \Gamma_2\oplus\zeta_2\oplus\Gamma_1'\oplus\zeta_1')\\
&\qquad + (W',\Gamma_1',\Gamma_2')+(F',\zeta_1',\zeta_2').
\end{aligned}$$

On the right-hand-side, the first term is the sum of the triple $(W\oplus F,\Gamma_1\oplus\zeta_1,\Gamma_2\oplus\zeta_2)$ with the inverse of its isomorphic triple $(W'\oplus F',\Gamma_1'\oplus\zeta_1',\Gamma_2'\oplus\zeta_2')$, which is trivial by Proposition 6.4.5. The third term is also trivial. Therefore, $[W,\Gamma_1,\Gamma_2]=[W',\Gamma_1',\Gamma_2']$ in the original sense of Definition 6.4.4.

In the purely-even case, both $K'^{0,0}(\mathcal{A}^{\mathrm{ev}})$ and the equivalent $K^{0,0}(\mathcal{A}^{\mathrm{ev}})$ were shown by Karoubi to be isomorphic to the ordinary K-theory group $K_0(\mathcal{A}^{\mathrm{ev}})$ (Theorem III.4.12 of [42]). In particular, a virtual module $[W_0\ominus W_1]\in\mathcal{R}(\mathcal{A}^{\mathrm{ev}})\cong K_0(\mathcal{A}^{\mathrm{ev}})$ corresponds to the element $[W_0\oplus W_1, 1\oplus -1, -1\oplus 1]$ in $\mathbf{K}_0(\mathcal{A}^{\mathrm{ev}})$. Since both Karoubi's $K'^{0,0}(\cdot)$ and our $\mathbf{K}_0(\cdot)$ continue to make sense for graded algebras $\mathcal{A}$, we shall take $\mathbf{K}_0(\mathcal{A})$ to be a *definition* of the K_0-group of a graded algebra $\mathcal{A}$ [44], generalising the ordinary $K_0(\cdot)$.

Higher difference-groups

Definition 6.4.10 (Higher difference-groups)**.** The higher difference-groups of a graded real C^*-algebra $\mathcal{A}$ are defined, for $r,s,n\geq 0$, to be

$$\begin{aligned}
\mathbf{K}_{s,r}(\mathcal{A}) &:= \mathbf{K}_0(\mathcal{A}\hat{\otimes}Cl_{r,s}),\\
\mathbf{K}_n(\mathcal{A}) &:= \mathbf{K}_0(\mathcal{A}\hat{\otimes}Cl_{0,n}).
\end{aligned}$$

Similarly, for a complex C^*-algebra $\mathcal{A}$, we define

$$\mathbf{K}_n(\mathcal{A}) := \mathbf{K}_0(\mathcal{A}\hat{\otimes}\mathbb{C}l_n).$$

[8] This definition is closer to yet another variant of Karoubi's K-theory groups, defined on pp. 71 of [39].

We remark that for purely-even $\mathcal{A}^{\mathrm{ev}}$, the above are alternative definitions for Karoubi's $K^{r,s}(\mathcal{A}^{\mathrm{ev}})$ (or $K'^{r,s}(\mathcal{A}^{\mathrm{ev}})$) as defined in III.4.11 of [42]. Due to the periodicity properties of the Clifford algebras, the $\mathbf{K}_{s,r}(\mathcal{A})$ (and $K'^{r,s}(\mathcal{A})$) groups depend only on $(r-s) \pmod 8$, while $\mathbf{K}_n(\mathcal{A})$ depends only on $n \pmod 8$ in the real case and $n \pmod 2$ in the complex case. Karoubi went on to prove the difficult result that the "Clifford suspension" operation $\mathcal{A} \mapsto \mathcal{A}\hat{\otimes}Cl_{0,1}$ (or $\mathcal{A} \mapsto \mathcal{A}\hat{\otimes}\mathbb{C}l_1$ in the complex case) is K-theoretically compatible with the usual notion of suspension, in the sense that $\mathbf{K}_1(\mathcal{A}) \cong K'^{0,0}(\mathcal{A}\hat{\otimes}Cl_{0,1}) \cong K'^{0,0}(C_0(\mathbb{R},\mathcal{A})) \cong \mathbf{K}_0(C_0(\mathbb{R},\mathcal{A}))$ [39, 44], and similarly in the complex case. Iterating this result, we have $\mathbf{K}_n(\mathcal{A}) \coloneqq \mathbf{K}_0(\mathcal{A}\hat{\otimes}Cl_{0,n}) \cong \mathbf{K}_0(C_0(\mathbb{R}^n,\mathcal{A}))$.

Recall that for purely-even algebras $\mathcal{A}^{\mathrm{ev}}$, the ordinary suspension operation raises the ordinary K-theory degree (or defines the higher $K_n(\cdot)$); i.e., $K_0(C_0(\mathbb{R}^n,\mathcal{A}^{\mathrm{ev}})) \equiv K_n(\mathcal{A}^{\mathrm{ev}})$. Thus, $\mathbf{K}_n(\mathcal{A}^{\mathrm{ev}}) \cong \mathbf{K}_0(C_0(\mathbb{R}^n,\mathcal{A}^{\mathrm{ev}})) \equiv K_0(C_0(\mathbb{R}^n,\mathcal{A}^{\mathrm{ev}})) \equiv K_n(\mathcal{A}^{\mathrm{ev}})$, so the higher difference groups $\mathbf{K}_n(\cdot)$ generalise the ordinary higher K-theory groups $K_n(\cdot)$ to the graded setting.

Chapter 7

Decomposition of twisted crossed products

The $\mathbf{K}_0(\cdot)$ group for a graded twisted crossed product $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ can be difficult to compute or understand directly. One possible strategy is to first decompose the crossed product with respect to some normal subgroup of G. Such decompositions are quite well-known for special cases (e.g. for untwisted group algebras of semidirect products), but it is useful to understand the general case. Decomposition results can be combined with various general results about the K-theory of crossed products, so as to relate the K-theory of a complicated crossed product with that of a simpler one appearing in the decomposition. Furthermore, one can often use the decomposition results to isolate the commutative part of a crossed product, which arises from an abelian subgroup N of G. This special case allows us to make contact with the *topological* K-theory picture used in the physics literature on topological phases. It also illustrates the advantages of working in the noncommutative setting, as we will see in several examples.

7.1 The Packer–Raeburn decomposition theorem

The following is a straightforward generalisation of a fundamental result of [66], proved there for ungraded complex twisted crossed products.

Theorem 7.1.1 (Packer–Raeburn decomposition theorem [66])**.** *Let $(G, c, \mathcal{A}, \alpha, \sigma)$ be a graded twisted C^*-dynamical system, and let N be a closed normal subgroup of G in the kernel of c. There is an isomorphism of graded C^*-algebras*

$$\mathcal{A} \rtimes_{(\alpha,\sigma)} G \cong (\mathcal{A} \rtimes_{(\alpha,\sigma)} N) \rtimes_{(\beta,\nu)} G/N, \tag{7.1}$$

where the twisting pair (β, ν) is determined by a choice of Borel section $s : G/N \ni p \mapsto s_p \in$

G such that $s_{eN} = 1$. For each $x \in G$, there is a $\gamma_x \in \mathrm{Aut}_{\mathbb{F}}(\mathcal{A} \rtimes_{(\alpha,\sigma)} N)$ such that

$$\gamma_x(a) = \alpha_x(a) \equiv a^x, \qquad a \in \mathcal{A}, \tag{7.2a}$$

$$\gamma_x(n) = \sigma(x,n)\sigma(xnx^{-1},x)^{-1}xnx^{-1}, \qquad n \in N, \tag{7.2b}$$

where the canonical embeddings $j_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} N)$ and $j_N : N \to \mathcal{UM}(\mathcal{A} \rtimes_{(\alpha,\sigma)} N)$ are implied. The formulae for (β, ν) are, for $p, p_1, p_2 \in G/N$,

$$\beta_p \coloneqq \beta(p) = \gamma_{s_p}, \tag{7.3a}$$

$$\nu(p_1,p_2) = \sigma(s_{p_1}, s_{p_2})\sigma(s_{p_1}s_{p_2}s_{p_1p_2}^{-1}, s_{p_1p_2})^{-1}s_{p_1}s_{p_2}s_{p_1p_2}^{-1}, \tag{7.3b}$$

where the canonical embeddings of $\sigma(\cdot,\cdot)$ and $s_{p_1}s_{p_2}s_{p_1p_2}^{-1} \in N$ in $\mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} N)$ are again implied.

To see that the γ_x in (7.2a)–(7.2b) exist, we define $j_{\mathcal{A}}^{(x)} : \mathcal{A} \to \mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} N)$ and $j_N^{(x)} : N \to \mathcal{UM}(\mathcal{A} \rtimes_{(\alpha,\sigma)} N)$ by the right-hand sides of (7.2a)–(7.2b), and compute that [66] the embeddings $j_{\mathcal{A}}^{(x)}, j_N^{(x)}$ are covariant for $(\mathcal{A}, N, \alpha, \sigma)$ in the sense of (3.7a) and (3.7b). This means that for each $x \in G$, the algebra $\mathcal{A} \rtimes_{(\alpha,\sigma)} N$, along with $j_{\mathcal{A}}^{(x)}, j_N^{(x)}$, is another twisted crossed product for the twisted dynamical system $(N, \mathcal{A}, \alpha, \sigma)$ in the sense of [66], and is thereby isomorphic to $\mathcal{A} \rtimes_{(\alpha,\sigma)} N$ with its original embeddings $j_{\mathcal{A}}, j_N$. The automorphism of $\mathcal{A} \rtimes_{(\alpha,\sigma)} N$ corresponding to this isomorphism of twisted crossed products is the desired γ_x. Furthermore, a very lengthy calculation [66] verifies that (β, ν) satisfies (3.1a) and (3.1b), as a twisting pair should, and also verifies the (ungraded) isomorphism (7.1).

Part of the theorem says that up to isomorphism, the iterated crossed product does not depend on the choice of section s. Note that we have required $N \subset \ker(c)$ to ensure that c descends to the quotient group G/N. Then, the automorphisms γ_x and β_p, as well as the cocycle $\nu(\cdot,\cdot)$ are even, independently of s. For the graded isomorphism in Equation (7.1), one only needs to check that the standard grading on either side of (7.1) agrees with the other, and that a different choice of s does not change this grading.

7.2 Finitely-generated projective modules in equivariant K-theory

We make a short digression to define the notion of a finitely-generated projective $(G, \mathcal{A}, \alpha)$-module W, where $(G, \mathcal{A}, \alpha)$ is a dynamical system, with $\mathcal{A}$ unital and G compact. Recall that α is actually a continuous homomorphism from the compact group G to $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$. Such modules are needed to define the G-equivariant K-theory of $\mathcal{A}$, and when $\mathcal{A}$ is commutative, they provide the link to the corresponding topological equivariant K-theory. We first recall some facts about finitely-generated projective $\mathcal{A}$-modules, following[1] Chapter 11.2 of [11]. If W is a f.g.p. $\mathcal{A}$-module, it is equivalently a direct summand of the free module $\mathcal{A}^n$ for some natural number n. Explicitly, there is a projection p in $M_n(\mathcal{A})$ such that $W \cong \mathcal{A}^n p$. There are many equivalent Banach norms on W, but they determine a unique topology on W, independently of its realisation as a direct summand. We write $\mathscr{L}(W)$ for the set of bounded linear operators on W, which has a strong operator topology alongside its norm topology. We also write $\mathscr{GL}(W)$ for the subgroup of invertible operators, and $\mathscr{B}(W)$ for the subalgebra of module maps. $\mathscr{B}(W)$ is a C^*-algebra isomorphic to $pM_n(\mathcal{A})p$.

Definition 7.2.1. Let $(G, \mathcal{A}, \alpha)$ be a dynamical system with G compact and $\mathcal{A}$ unital. A finitely-generated projective $(G, \mathcal{A}, \alpha)$-module is a finitely-generated projective $\mathcal{A}$-module W, along with a strongly continuous homomorphism $\theta : G \to \mathscr{GL}(W)$, such that

$$\theta_x(aw) = a^x(\theta_x w), \qquad x \in G,\ a \in \mathcal{A},\ w \in W. \tag{7.4}$$

Definition 7.2.1 resembles our definition of covariant representations, a difference being that W is not necessarily a Hilbert space (although it can be made into a Hilbert $\mathcal{A}$-module). This definition of a f.g.p. $(G, \mathcal{A}, \alpha)$-module is in fact the usual one, i.e., a direct summand of a "free module": every f.g.p. $(G, \mathcal{A}, \alpha)$-module is equivariantly isomorphic to a direct summand of $V \otimes \mathcal{A}$ for some finite-dimensional representation space V for G (see Proposition 11.2.3 of [11]). We recall the definition of equivariant K-theory groups, c.f. Chapter 11.5 of [11]:

Definition 7.2.2. Let $(G, \mathcal{A}, \alpha)$ be a dynamical system with G compact. For unital $\mathcal{A}$, the equivariant K-theory group $K_0^G(\mathcal{A})$ is defined to be the Grothendieck group of the monoid (under the direct sum) of the equivalence classes of finitely-generated projective $(G, \mathcal{A}, \alpha)$-modules. If $\mathcal{A}$ is non-unital, $K_0^G(\mathcal{A})$ is defined to be the kernel of the the induced map $p_* : K_0^G(\mathcal{A}^+) \to K_0^G(\mathbb{F})$, where $\mathcal{A}^+$ has the induced action from α, $\mathbb{F}$ has the trivial action, and p is the equivariant projection $\mathcal{A}^+ \to \mathbb{F}$.

The Green–Julg theorem relates equivariant K-theory groups to the ordinary K-theory groups of a crossed product, see Theorem 11.7.1 of [11] and original references therein.

Theorem 7.2.3 (Green–Julg theorem)**.** *Let $(G, \mathcal{A}, \alpha)$ be a dynamical system with G compact. The equivariant $K_0^G(\mathcal{A})$-group is isomorphic to the ordinary K_0-group of the crossed product,*

$$K_0^G(\mathcal{A}) \cong K_0(\mathcal{A} \rtimes_{(\alpha,1)} G). \tag{7.5}$$

[1]The author worked with right modules, but we prefer to use left modules.

If G is finite and $\mathcal{A}$ is unital, the crossed product $\mathcal{A} \rtimes_{(\alpha,1)} G$ is unital and its f.g.p. modules can be translated into f.g.p. $(G, \mathcal{A}, \alpha)$-modules directly. For more general compact groups G and possibly non-unital $\mathcal{A}$, the isomorphism (7.5) is demonstrated in a less direct manner (e.g. see the proof of Theorem 11.7.1 of [11]).

In the graded case, we define graded f.g.p. $(G, c, \mathcal{A}, \alpha)$-modules W in a similar way, noting that the linear operators on W acquire a natural grading. The operators θ_x are required to be odd or even according to $c(x)$, and W must be a graded module for $\mathcal{A}$.

7.3 Special cases of crossed product decompositions

7.3.1 I: Decomposing ordinary group algebras over an abelian normal subgroup

Consider the case where $\alpha \equiv 1, \sigma \equiv 1$, N is a discrete abelian group, G/N is compact, and $G = N \times G/N$. Then $\mathbb{C} \rtimes N$ is just the complex group C^*-algebra, which is isomorphic to $C(\hat{N})$ via the Fourier transform, where $\hat{N}$ is the Pontryagin dual of N appropriately topologised. Choosing the standard section $s_p = (e, p)$, the twisting pair (β, ν) is trivial, and $\mathbb{C} \rtimes G \cong C(\hat{N}) \rtimes G/N$. We are interested in the K-theory of $C(\hat{N}) \rtimes G/N$, which by the Green–Julg theorem, is the G/N-equivariant K-theory of $C(\hat{N})$. A f.g.p. $(G/N, C(\hat{N}), 1)$-module is, first of all, a f.g.p. $C(\hat{N})$-module, which by the Serre–Swan theorem, corresponds to the sections of some finite-rank complex vector bundle over $\hat{N}$. The G/N-action on the sections commute with the $C(\hat{N})$-action, so each fibre is a finite-dimensional representation space for G/N.

Generalising very slightly, we can consider (topological) semidirect products $G = N \rtimes G/N$. We choose the standard section $s : p \mapsto (e, p) \in N \rtimes G/N$, which is a homomorphism. Note that $(e, p)(n, e)(e, p^{-1}) = (p \cdot n, e)$, where $n \mapsto p \cdot n$ is the defining automorphic action of $p \in G/N$ on N. Using (7.2b) and (7.3a), we find that the automorphisms β_p act on the canonical generators $\delta_n \in \mathbb{C} \rtimes N$ by $\beta_p(\delta_n) = \delta_{p \cdot n}$; in terms of functions $f : N \to \mathbb{C}$, this is $\beta_p(f)(n) = f(p^{-1} \cdot n)$. Under the Fourier transform $f \mapsto \hat{f} \in C(\hat{N})$, the automorphism β_p becomes $\hat{\beta}_p$, defined by $\hat{\beta}_p(\hat{f})(\chi) := \hat{f}(p^{-1} \cdot \chi)$, where $(p \cdot \chi)(n) := \chi(p^{-1} \cdot n)$ is the dual G/N-action on $\hat{N}$. Also, (7.3b) gives $\nu \equiv 1$ for our canonical choice of section, so we may rewrite $\mathbb{C} \rtimes (N \rtimes G/N)$ as $C(\hat{N}) \rtimes_{(\hat{\beta},1)} G/N$. The equivariant Serre–Swan theorem gives a 1–1 correspondence between isomorphism classes of f.g.p. $(G/N, C(\hat{N}), \hat{\beta})$-modules, and isomorphism classes of G/N-equivariant vector bundles over $\hat{N}$.

Non-semidirect product case

Things complicate significantly when G is not a semidirect product of N and G/N. According to (7.3b), in the decomposition $\mathbb{C} \rtimes G \cong C(\hat{N}) \rtimes_{(\hat{\beta},\nu)} G/N$, there is a non-trivial cocycle $\nu(p_1, p_2) = s_{p_1} s_{p_2} s_{p_1 p_2}^{-1}$, regarded as a unitary element of $C(\hat{N})$. Explicitly, $\nu(p_1, p_2)$ is the function in $C(\hat{N})$ whose value at a character $\chi \in \hat{N}$ is $\chi(s_{p_1} s_{p_2} s_{p_1 p_2}^{-1})$. Abusing notation, we write $\nu(p_1, p_2)(\chi) = \chi(\nu(p_1, p_2))$. Next, the automorphism $\hat{\beta}_p$ is still the dual of conjugation by s_p, i.e., $\hat{\beta}_p(\hat{f})(\chi) = \hat{f}(p^{-1} \cdot \chi)$, where $(p \cdot \chi)(n) := \chi(s_p^{-1} n s_p)$. A different section with $s'_p = n_p s_p, n_p \in N$ leads again to $(p \cdot \chi)(n) = \chi(n_p^{-1} s_p^{-1} n s_p n_p) = \chi(s_p^{-1} n s_p)$, so $\hat{\beta}_p$ is well-defined. Next, we check that $\hat{N}$ remains a G/N-space,

$$\begin{aligned}
(p_1 \cdot (p_2 \cdot \chi))(n) &= (p_2 \cdot \chi)(s_{p_1}^{-1} n s_{p_1}) \\
&= \chi(s_{p_2}^{-1} s_{p_1}^{-1} n s_{p_1} s_{p_2}) \\
&= \chi(s_{p_1 p_2}^{-1} \nu(p_1, p_2)^{-1} n \nu(p_1, p_2) s_{p_1 p_2}) \\
&= (p_1 p_2 \cdot \chi)(n) \\
&= ((p_1 p_2) \cdot \chi))(n),
\end{aligned}$$

This suggests that we can try to build a G/N-equivariant vector bundle picture similar to that in the semidirect product case. Ignoring possible continuity issues, we can interpret, loosely speaking, a "f.g.p. $(G/N, C(\hat{N}), \hat{\beta}, \nu)$-module" as the sections of a vector bundle E over $\hat{N}$, equipped with a "ν-twisted equivariant G/N-action" on the fibres. The bundle sections v have an action of $C(\hat{N})$ by pointwise multiplication as usual. The action of p on v is required to be covariant in the sense that $p \cdot \hat{f} \cdot v = \hat{\beta}_p(\hat{f}) \cdot p \cdot v$ for all $\hat{f} \in C(\hat{N})$. If we define the fibrewise maps $U_p^\chi : E_{p^{-1} \cdot \chi} \to E_\chi$ by $(p \cdot v)(\chi) =: U_p^\chi v(p^{-1} \cdot \chi)$, then

$$\begin{aligned}
U_p^\chi (\hat{f} \cdot v)(p^{-1} \cdot \chi) &= p \cdot (\hat{f} \cdot v)(\chi) \\
&= \left(\hat{\beta}_p(\hat{f}) \cdot p \cdot v\right)(\chi) \\
&= \hat{\beta}_p(\hat{f})(\chi)(p \cdot v)(\chi) \\
&= \hat{f}(p^{-1} \cdot \chi)(p \cdot v)(\chi) \\
&= \hat{f}(p^{-1} \cdot \chi) U_p^\chi v(p^{-1} \cdot \chi),
\end{aligned}$$

so the U_p^χ are, in particular, linear maps. Alternatively, a set of linear fibrewise maps $U_p^\chi : E_{p^{-1} \cdot \chi} \to E_\chi$ defines an action of p on sections, $(p \cdot v)(\chi) := U_p^\chi v(p^{-1} \cdot \chi)$, which is covariant.

Next, we investigate how the maps U_p^χ compose. To ease notation, we shall omit the superscript labelling the codomain of U_p^χ when it is understood. In the semidirect product case, composable linear fibrewise maps $U_{p_1}^\chi U_{p_2}^{p_1^{-1} \cdot \chi}$ satisfy $U_{p_1} U_{p_2} = U_{p_1 p_2}$, and E becomes a true G/N-equivariant bundle. In the general case, the maps U_p^χ furnish a "projective representation" of G/N, as follows. In the twisted crossed product $C(\hat{N}) \rtimes_{(\hat{\beta}, \nu)} G/N$, the elements of G/N (more precisely, their images under $j_{G/N}$) satisfy $p_1 \star p_2 = \nu(p_1, p_2) \cdot p_1 p_2$ (see (3.8)). On the module of bundle sections, we should have

$$\begin{aligned}
U_{p_1} U_{p_2} v((p_2^{-1} \cdot p_1^{-1} \cdot \chi)) &= (p_1 \cdot (p_2 \cdot v))(\chi) \\
&= ((p_1 \star p_2) \cdot v)(\chi) \\
&= (\nu(p_1, p_2) \cdot p_1 p_2 \cdot v))(\chi) \\
&= \nu(p_1, p_2)(\chi) U_{p_1 p_2} v(p_2^{-1} \cdot p_1^{-1} \cdot \chi) \\
&= \chi(\nu(p_1, p_2)) U_{p_1 p_2} v(p_2^{-1} \cdot p_1^{-1} \cdot \chi).
\end{aligned}$$

Thus, for each pair of maps between basepoints $p_2^{-1} \cdot p_1^{-1} \cdot \chi \xrightarrow{p_2} p_1^{-1} \cdot \chi \xrightarrow{p_1} \chi$ in $\hat{N}$, the corresponding set of composable fibre maps U_{p_1}, U_{p_2} satisfies

$$U_{p_1} U_{p_2} = \chi(\nu(p_1, p_2)) U_{p_1 p_2}. \tag{7.6}$$

Note that the extra phase factor depends on final basepoint χ. If there is a third map $p_3^{-1} \cdot p_2^{-1} \cdot p_1^{-1} \cdot \chi \xrightarrow{p_3} p_2^{-1} \cdot p_1^{-1} \cdot \chi$ with the associated fibre map U_{p_3}, then there are two phase factors arising from $(U_{p_1} U_{p_2}) U_{p_3}$, and two (a priori different) phase factors from $U_{p_1}(U_{p_2} U_{p_3})$. We show that the product of the two phase factors is the same in either case, so that 3-composable fibre maps do so in an associative manner.

Composing in the first way gives

$$\begin{aligned}
(U_{p_1} U_{p_2}) U_{p_3} &= \chi(\nu(p_1, p_2)) U_{p_1 p_2} U_{p_3} \\
&= \chi(\nu(p_1, p_2)) \chi(\nu(p_1 p_2, p_3)) U_{p_1 p_2 p_3},
\end{aligned} \tag{7.7}$$

with an overall phase factor of

$$\chi(s_{p_1} s_{p_2} s_{p_1 p_2}^{-1}) \chi(s_{p_1 p_2} s_{p_3} s_{p_1 p_2 p_3}^{-1}) = \chi(s_{p_1} s_{p_2} s_{p_3} s_{p_1 p_2 p_3}^{-1}). \tag{7.8}$$

Composing in the second way gives

$$\begin{aligned}
U_{p_1}(U_{p_2} U_{p_3}) &= U_{p_1}(p_1^{-1} \cdot \chi)(\nu(p_2, p_3)) U_{p_2 p_3} \\
&= \chi(\nu(p_1, p_2 p_3))(p_1^{-1} \cdot \chi)(\nu(p_2, p_3)) U_{p_1 p_2 p_3} && (7.9\text{a}) \\
&= \chi(s_{p_1^{-1}}^{-1} s_{p_2} s_{p_3} s_{p_2 p_3}^{-1} s_{p_1^{-1}}) \chi(\nu(p_1, p_2 p_3)) U_{p_1 p_2 p_3}. && (7.9\text{b})
\end{aligned}$$

Since $s_{p^{-1}} s_p \in N$, we can write $s_{p^{-1}} = s_p^{-1} n_p$ for some $n_p \in N$. Then the overall phase factor in (7.9b) is

$$\begin{aligned}
\chi(n_{p_1}^{-1} s_{p_1} s_{p_2} s_{p_3} s_{p_2 p_3}^{-1} s_{p_1}^{-1} n_{p_1}) \chi(s_{p_1} s_{p_2 p_3} s_{p_1 p_2 p_3}^{-1}) &= \chi(s_{p_1} s_{p_2} s_{p_3} s_{p_2 p_3}^{-1} s_{p_1}^{-1}) \chi(s_{p_1} s_{p_2 p_3} s_{p_1 p_2 p_3}^{-1}) \\
&= \chi(s_{p_1} s_{p_2} s_{p_3} s_{p_1 p_2 p_3}^{-1}),
\end{aligned} \tag{7.10}$$

which is indeed equal to (7.8). Thus, the vector bundle picture of a f.g.p. module for $\mathbb{C} \rtimes G$ is that of a "ν-twisted G/N-equivariant bundle over $\hat{N}$".

Groupoid approach of Freed–Moore

In Section 9 of [30], Freed–Moore study group extensions $1 \to N \to G \xrightarrow{q} G/N \to 1$, where N is compact or a discrete lattice. They define a *right* action of G on $\hat{N}$ by $(\chi \cdot g)(n) = \chi(gng^{-1})$, which descends to a well-defined G/N-action on equivalence classes in $\hat{N}$, denoted by $[\chi] \cdot q(g)$. This action gives rise to a quotient groupoid[2] $\hat{N}//(G/N)$. They then construct a "central extension" ν for this groupoid, which is supposed to give a correspondence betwen representations of G and ν-twisted G/N-equivariant *sheaves* over $\hat{N}$ — this is Theorem 9.8 of [30]. Their construction is as follows. A representative χ is chosen for each class $[\chi] \in \hat{N}$. For each pair $(g, [\chi])$, they define, using Schur's lemma, the line of intertwiners $\tilde{L}_{g,[\chi]} := \mathrm{Hom}_N(g \cdot \chi, \chi')$, where χ' is the pre-chosen representative of the class $[\chi'] = [g \cdot \chi] = [q(g) \cdot \chi] =: q(g) \cdot [\chi]$, and $G \xrightarrow{q} G/N$ is the quotient map. These lines have an associative composition law

$$\begin{aligned} \tilde{L}_{g_1,q(g_2)\cdot[\chi]} \otimes \tilde{L}_{g_2,[\chi]} &\to \tilde{L}_{g_1g_2,[\chi]}, \\ l_1 \otimes l_2 &\mapsto l_1 \circ l_2. \end{aligned} \tag{7.11}$$

The map $l_1 \circ l_2$ should really be replaced by $l_1 \circ g_1 \circ l_2 \circ g_1^{-1}$ for this composition to make sense. It is subsequently shown that lines $L_{p,[\chi]}$ can be defined for each pair $(p, [\chi]) \in G/N \times \hat{N}$ (i.e. for each arrow $[\chi] \xrightarrow{p} p \cdot [\chi]$), and that the composition in (7.11) descends to an associative composition $L_{p_1,[\chi]} \otimes L_{p_1,p_2\cdot[\chi]} \to L_{p_1p_2,[\chi]}$ for these lines — this is their central extension ν of the groupoid $\hat{N}//(G/N)$. There is also notion of a ν-twisted vector bundle E over the groupoid $\hat{N}//(G/N)$ (Definition 7.23 of [30]), which is an ordinary vector bundle $E \to \hat{N}$, along with a linear map $S_{p.[\chi]}$ for each arrow $\chi \xrightarrow{p} \chi'$,

$$S_{p,[\chi]} : (L_{p,[\chi]} \otimes E_{[\chi]}) \longrightarrow E_{[\chi']}.$$

See also, examples 1.10-1.13 of [29], and Chapter 4.4–4.5 of [61].

In the case where N is discrete and abelian, so $\hat{N}$ is the compact dual group of characters of N, it is possible to define a groupoid version of a 2-cocycle condition on the arrows $(p, [\chi])$. Let $\rho\left((p_1, p_2 \cdot [\chi]), (p_2, [\chi])\right) := p_1p_2 \cdot [\chi](\nu(p_1, p_2)) \in \mathrm{U}(1)$, where the right-hand side means the evaluation of (a pre-chosen representative of) $p_1p_2 \cdot [\chi] = [p_1p_2 \cdot \chi]$ on the element $\nu(p_1, p_2)$. The equality between (7.7) and (7.9a) may be expressed as

$$\begin{aligned} &\rho\left((p_1, p_2p_3 \cdot [\chi]), (p_2, p_3 \cdot [\chi])\right) \rho\left((p_1p_2, p_3 \cdot [\chi]), (p_3, [\chi])\right) \\ = \; &\rho\left((p_2, p_3 \cdot [\chi]), (p_3, [\chi])\right)^{p_1} \rho\left((p_1, p_2p_3 \cdot [\chi]), (p_2p_3, [\chi])\right), \end{aligned}$$

where

$$\rho\left((p_2, p_3 \cdot [\chi]), (p_3, [\chi])\right)^{p_1} := p_1p_2p_3 \cdot [\chi](\nu(p_2, p_3)).$$

[2] The authors used *right* actions for their quotient groupoid, but we use shall use left actions throughout.

Link to topological equivariant K-theory

According to our definitions in Section 6.1, the complex representation group $\mathcal{R}_{\mathbb{C}}(G)$ of G is the K-theory group $K_0(\mathbb{C} \rtimes_{(1,1)} G) = K_0(C(\hat{N}) \rtimes_{(\hat{\beta},\nu)} G/N)$. Under the assumption that G is a semidirect product $G = N \rtimes G/N$, the representation group becomes $\mathcal{R}_{\mathbb{C}}(G) = K_0^{G/N}(C(\hat{N})) \cong K^0_{G/N}(\hat{N})$. The expression on the right-hand-side is the topological G/N-equivariant K-theory of the compact space $\hat{N}$. A typical example arising in solid-state physics is the symmetry group G of a crystal. This may include some compact internal symmetries such as spin SU(2). There is a discrete subgroup of translations $N = \mathbb{Z}^d$, corresponding, for example, to the atomic positions in a crystal lattice. The dual $\hat{N} = \mathbb{T}^d$ is the Brillouin torus, over which bands in electronic band (Bloch) theory reside. In general, $G \to G/N$ need not be split. If it is split (corresponding to the symmorphic crystallographic groups), we can identify $\mathcal{R}_{\mathbb{C}}(G)$ as the topological G/N-equivariant K-theory group of the Brillouin torus.

Remark 7.3.1 (Non-discrete abelian N)*.* If we drop the assumption of discreteness on the abelian subgroup N, its dual $\hat{N}$ is locally compact but not compact, and $\mathbb{C} \rtimes N$ is the non-unital algebra $C_0(\hat{N})$. The Packer–Raeburn decomposition continues to hold, and we have $\mathbb{C} \rtimes G \cong C_0(\hat{N}) \rtimes_{(\hat{\beta},\nu)} G/N$, where $\hat{\beta}$ and ν are computed in the same way as before. If $\nu \equiv 1$ (e.g. if G is a semidirect product of N with G/N), we obtain $\mathcal{R}_{\mathbb{C}}(G) = K_0^{G/N}(C_0(\hat{N})) \cong K^0_{G/N}(\hat{N})$. Since $\hat{N}$ is not compact, the topological equivariant K-theory group must be interpreted using vector bundles trivialised outside a compact subspace of $\hat{N}$, i.e., K-theory with "compact supports". An example of N is the group $\mathbb{R}^n$ (e.g. of translational symmetries), which has a very different topological nature to $\mathbb{Z}^n$.

Remark 7.3.2*.* Since $\mathcal{A} = \mathbb{C} \rtimes G$ is ungraded (or purely even), its super-representation group $\mathcal{SR}(\mathcal{A})$, which classifies gapped phases compatible with G, is identical to its ungraded representation group $\mathcal{R}_{\mathbb{C}}(G) = K_0(\mathcal{A})$. As in Section 6.4, the K-theory group $K_0(\mathcal{A}) \cong \mathbf{K}_0(\mathcal{A})$ can also be interpreted in terms of differences of gapped phases compatible with G.

7.3.2 II: Decomposing twisted group algebras over an abelian normal subgroup

We move on to twisted group algebras. Thus, we allow $\sigma \not\equiv 1$, but assume that $\sigma(n,\cdot) = 1 = \sigma(\cdot,n)$ for all n in an abelian normal subgroup N. In other words, N is to be represented non-projectively, although the full group G is represented projectively. We have the decomposition $\mathbb{C} \rtimes_{(1,\sigma)} G \cong C(\hat{N}) \rtimes_{(\hat{\beta},\nu)} G/N$, where $(\hat{\beta},\nu)$ are determined by (7.2b), (7.3a) and (7.3b), along with the Fourier transformation of β to $\hat{\beta}$ acting on $C(\hat{N})$. We obtain

$\beta_p(\delta_n) = \sigma(s_p, n)\sigma(s_p n s_p^{-1}, s_p)^{-1}\delta_{s_p n s_p^{-1}}$, which translates to the dual action of $p \in G/N$ on the characters χ of N given by

$$\begin{aligned} (p \cdot \chi)(n) &= \sigma(s_p^{-1}, n)\sigma(s_p^{-1} n s_p, s_p^{-1})^{-1}\chi(s_p^{-1} n s_p) \\ &= \chi(s_p^{-1} n s_p), \end{aligned} \tag{7.12}$$

turning $\hat{N}$ into a G/N-space. All the results in the previous section follow, except that $\nu(p_1, p_2)$ now acquires an additional phase factor according to the formula (7.3b), so that we have to modify (7.6) to take this into account.

If we lift the condition that $\sigma(n, \cdot\,) = 1 = \sigma(\,\cdot\,, n)$, we no longer have $\mathbb{C} \rtimes_{(1,\sigma)} N \cong C_0(\hat{N})$. Instead, we need the generalised dual $\hat{N}^\sigma$ as in Section 4.2, i.e., the space of equivalence classes of projective representations χ of N satisfying $\chi(n_1)\chi(n_2) = \sigma(n_1, n_2)\chi(n_1 n_2)$. The formula (7.12) for $p \cdot \chi$ is the same as the right action of s_p^{-1} on χ as defined in (4.8a). Thus, $(p \cdot \chi)$ is another element of $\hat{N}^\sigma$. However, $\hat{\beta}$ is not necessarily a genuine action of G/N on $\mathbb{C} \rtimes_{(1,\sigma)} N$. Setting aside the topological issues, there are two situations in which we can recover an associative G/N action on the algebra $\mathbb{C} \rtimes_{(1,\sigma)} N$: (1) G is a semidirect product and s can be chosen to be a homomorphism, or (2) $\sigma(n, g) = 1 = \sigma(g, n)$ whenever g is not in N. In the latter situation, $n \in N$ has non-trivial cocycle only with other elements of N.

Since $\mathbb{C} \rtimes_{(1,\sigma)} N$ is a *noncommutative* algebra, the equivariant vector bundle picture developed in Section 7.3.1 is not available. This situation occurs in the Integer Quantum Hall Effect [10], so it is important to have techniques that can handle such situations. We have developed everything in this section from a C^*-algebra (i.e. noncommutative topology) point of view, and it still makes sense to study the K-theory of the twisted group algebra. If we wish to, we can loosely interpret a f.g.p. $\mathbb{C} \rtimes_{(1,\sigma)} G$-module as the "space of sections" of some ν-twisted G/N-equivariant "bundle" over the noncommutative space corresponding to $\mathbb{C} \rtimes_{(1,\sigma)} N$.

7.3.3 III: $N \lhd G$ acts unitarily but G may act antiunitarily

Let us see what happens if we fix $\sigma \equiv 1$ but now allow $\alpha_g = K$ (complex conjugation) for some $g \in G$. Thus, we are interested in the *real* C^*-algebra $\mathbb{C} \rtimes_{(\alpha,1)} G$. A $\mathbb{C} \rtimes_{(\alpha,1)} G$-module is a linear-antilinear representation of G with respect to the decomposition $G^u = \ker(\alpha)$, $G^a = G - G^u$. By intersecting with G^u, we may assume that $N \subset \ker(\alpha)$. We retain the assumption that G/N is compact. The map α is a homomorphism and descends to the quotient G/N, which we denote again by α. We will think of α as being determined by the homomorphism $\phi : G \to \{\pm 1\}$ in the symmetry data (G, ϕ) for UA-reps, then ϕ descends to G/N as well. We also assume the splitting $G = N \rtimes G/N$, then $\mathbb{C} \rtimes_{(\alpha,1)} G \cong C_0(\hat{N}) \rtimes_{(\hat{\beta},1)} G/N$, where the *real-linear* automorphisms $\hat{\beta}_p$ are determined as follows. From

(7.3a), we obtain $\beta_p(\lambda) = \alpha_p(\lambda) \equiv \lambda^p$ for $\lambda \in \mathbb{C}$. Thus p complex-conjugates the complex numbers when $\phi(p) = -1$. Also, $\beta_p(n) = pnp^{-1} = p \cdot n$. These say that p acts on $\hat{f} \in C_0(\hat{N})$ by

$$\hat{\beta}_p(\hat{f})(\chi) = \begin{cases} \hat{f}(p^{-1} \cdot \chi), & \text{if } \phi(p) = +1, \\ \overline{\hat{f}(p^{-1} \cdot \chi)}, & \text{if } \phi(p) = -1, \end{cases}$$

where $(p \cdot \chi)(n) = \chi(p^{-1} \cdot n)$ if $\phi(p) = +1$ and $(p \cdot \chi)(n) = \overline{\chi(p^{-1} \cdot n)}$ if $\phi(p) = -1$.

For discrete N, we can understand a f.g.p. module for $\mathbb{C} \rtimes_{(\alpha,1)} G$ as the sections of some G/N-equivariant complex vector bundle E over the compact space $\hat{N}$, equipped with some additional structure arising from the non-trivial α, or equivalently, ϕ. The quotient group G/N no longer acts complex-linearly on the bundle sections. Instead, the action of $p \in G/N$, and hence the fibre maps U_p^χ, are *antilinear* when $\phi(p) = -1$. As a complex bundle, E is determined by the underlying $C(\hat{N})$-module structure of a f.g.p. $(G/N, C(\hat{N}), \hat{\beta})$-module. In addition, E is a "Real" $(G/N, \phi)$-bundle over the G/N-space $\hat{N}$, in the sense of [41]; this is defined to be a complex vector bundle $E \xrightarrow{\varpi} \hat{N}$, on which the total space has a G/N action such that:

1. The G/N-action commutes with the bundle projection ϖ, so G/N acts on the base space $\hat{N}$, and gives rise to fibre maps $V_{p^{-1}\cdot\chi} \xrightarrow{U_p^\chi} V_\chi$.
2. For every $\chi \in \hat{N}$, the fibre maps are linear/antilinear according to whether $\phi(p) = +1$ or -1.

Bundle maps between "Real" bundles are required to be complex bundle maps that commute with the G/N-structure.

Splitting assumption and time reversal element

The following splitting assumption is often made on G/N. Suppose that there is a continuous $\{\pm 1\}$-action $\epsilon : \{\pm 1\} \to \mathrm{Aut}(Q := \ker(\phi))$, so we can write G/N itself as a semidirect product $G/N = Q \rtimes \{\pm 1\}$[3]. Then ϕ is simply the projection onto the $\{\pm 1\}$ factor. Note that $\epsilon(-1)$ is an involutary automorphism of Q, and we denote this involution by $q \mapsto \bar{q} := \epsilon(-1)(q)$. Let T (suggestive of a time-reversal element) denote both (-1) and its lift to an element of order 2 in G/N. Then T has an involutary action (via $\hat{\beta}$) on $\hat{N}$, written as $T \cdot \chi =: \bar{\chi}$. It further lifts to an order 2 element in G (also denoted by T). The group Q is also embedded in $G \cong N \rtimes (Q \rtimes \{1, T\})$, and we also use the same symbol to denote both $q \in Q$ and its lift to G. Note that $Tq = \bar{q}T$ in the semidirect product $G/N = Q \rtimes \{1, T\}$,

[3]Some authors call (Q, ϵ) a Real group.

which of course persists when T and q are regarded as elements of G. We then have the relation

$$\overline{q \cdot \chi} \equiv T \cdot (q \cdot \chi) = \bar{q} \cdot (T \cdot \chi) \equiv \bar{q} \cdot \bar{\chi}, \qquad q \in Q, \chi \in \hat{N}.$$

Such a space $\hat{N}$ is sometimes called a Real Q-space for the Real group (Q, ϵ). Recall that the total space E has a $G/N = Q \rtimes \{1, T\}$-action commuting with the bundle projection ϖ. In particular, the element T has an involutary action on E, which is antilinear between fibres. Such an E is sometimes called a Real[4] Q-equivariant bundle over the Real Q-space $\hat{N}$. The group $Q \subset G/N$ also acts on $C(\hat{N})$ by complex-linear automorphisms according to $\hat{\beta}_q$. Since $Tq = \bar{q}T$, this action of Q on $C(\hat{N})$ is Real in the sense that $\overline{\hat{\beta}_q(\hat{f})} = \hat{\beta}_{\bar{q}}(\bar{\hat{f}})$, where $\bar{\hat{f}} := \hat{\beta}_T(\hat{f})$. The sections of E form a f.g.p. module for $C(\hat{N})$, which is Real in the sense that there is an antilinear involution on sections $v \mapsto \bar{v}$, defined by $\bar{v}(\bar{\chi}) := \overline{v(\chi)}$, which satisfies $\overline{\hat{f} \cdot v} = \bar{\hat{f}} \cdot \bar{v}$, $\hat{f} \in C(\hat{N})$. The Real $C(\hat{N})$-module action is complemented by a Real Q-action $q \cdot v(\chi) := v(q^{-1} \cdot \chi)$ which satisfies $\overline{q \cdot v} = \bar{q} \cdot \bar{v}$.

Remark 7.3.3. We can also view $C(\hat{N}) \rtimes_{(\hat{\beta},1)} Q$ itself as a Real C^*-algebra, with antilinear involution on the functions $\xi : Q \to C(\hat{N})$ given by $\bar{\xi}(q) := \overline{\xi(\bar{q})}$. The sections of E then furnish a f.g.p. Real module for $C(\hat{N}) \rtimes_{(\hat{\beta},1)} Q$, under the Real actions of $C(\hat{N})$ and Q defined above.

Remark 7.3.4. Freed–Moore arrived at Real Q-vector bundles in the sense of Atiyah [3], by assuming that G/N is a direct product of Q with $\{\pm 1\}$. We are using a more general notion of Real Q-equivariant bundles, where Q need not commute with the bundle involution. More generally, there may not even be a distinguished involution if G/N is not a semidirect product of Q with $\{\pm 1\}$; in that case we have to deal with "Real" bundles.

Remark 7.3.5. As before, without the assumption that $G = N \rtimes G/N$, the twisting by ν causes the fibrewise G/N-action to be "projective". The G/N-action on the sections is "Real" in the sense that $p \in G/N$ acts complex linearly (resp. antilinearly) when $\phi(p) = +1$ (resp. $\phi(p) = -1$). We may understand such a "Real" bundle as a "ν-twisted $(G/N, \phi)$-equivariant Real bundle over $\hat{N}$".

Remark 7.3.6. If we allow $\sigma \not\equiv 1$, but only for $\sigma(T, T) = -1$, we can use "Quaternionic" bundles [16] for a vector bundle picture. Unfortunately, the geometrical picture in terms of "Quaternionic", "Real" or Real bundles can become rather unwieldy without splitting assumptions on G, and it seems better to work at the algebraic level.

To summarise: given the symmetry data (G, ϕ), the compatible gapped Hamiltonians or their difference classes are classified by $K_0(\mathcal{A}) \cong \mathcal{SR}(\mathcal{A}) \cong \mathbf{K}_0(\mathcal{A})$, where $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,1)} G$.

[4] Note the differences between "Real", Real, and real. The last two are fairly standard in the literature.

With the assumptions that $\phi(N) = +1$, G/N is compact, and $G = N \rtimes (Q \times \{1, T\})$, we can identify $K_0(\mathcal{A})$ with a topological Q-equivariant KR-group, $KR_Q^{0,0}(\hat{N})$, in the sense of Atiyah [3]. We may drop the assumption that $G/N = Q \times \{1, T\}$, or even $G/N = Q \rtimes \{1, T\}$, but more general versions of Real (or "Real") equivariant K-theory [27, 41] are required in order to handle the lack of a distinguished antilinear involution.

7.3.4 IV: A Clifford algebra factorises: Higher K-theory groups

We now consider the symmetry data given by (G, c, ϕ, σ), and study its associated twisted crossed product. We assume that G is a semidirect product $G = N \rtimes G/N$, where $N \subset \ker(\phi, c)$ is abelian, and G/N is compact. The homomorphism (ϕ, c) descends to G/N, and we denote its image by $A \subset \{\pm 1\}^2$. We can think of A as one of the subgroups of the CT-group, by identifying $\{\pm 1\}^2$ with $\{1, T, C, S\}$ as in Section 5.4.2. We assume that (ϕ, c), as a homomorphism on G/N, is split, so G/N is a semidirect product $G/N = Q \rtimes A$ with $Q := \ker(\phi, c)$. Thus $G \cong N \rtimes (Q \rtimes A) \cong (N \rtimes Q) \rtimes A$. Where present, we denote the lifts of C, T, S to G/N and G by the same symbols. We also assume that $\sigma(nq, \cdot) = 1 = \sigma(\cdot, nq)$ for all nq in $N \rtimes Q$, then σ is simply specified by its restriction to A.

Let $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$ as before, and denote the images, under $j_{\mathcal{A}}$, of C, T, S in $\mathcal{MA}$ by $\mathsf{C}, \mathsf{T}, \mathsf{S}$. Note that the commutation relations amongst $\mathsf{C}, \mathsf{T}, \mathsf{S}$ are the same as those amongst the representatives of C, T, S in a graded PUA-rep of (G, c, ϕ, σ), which is why we have used the same symbols for both. As in Section 5.4.2, we can choose σ such that $\mathsf{CT} = \mathsf{TC} = \mathsf{S}$, then σ is simply specified by $\mathsf{T}^2 = \pm 1, \mathsf{C}^2 = \pm 1$, while $\mathsf{S}^2 = +1$ can be assumed. There are ten possibilities for (A, σ), each with a corresponding Clifford algebra, as listed in Table 5.5. Note that if A is the trivial group, we are back in the situation of Section 7.3.1. We have also handled the case $A = \{1, T\}$, $\mathsf{T}^2 = +1$ in Section 7.3.3.

We are interested in graded f.g.p. modules for the graded twisted crossed product $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$, with the (even) automorphisms α_g on $\mathbb{C}$ determined by $\phi(g)$, and the grading on $\mathcal{A}$ determined by c. We first decompose $\mathcal{A}$ with respect to the subgroup $N \rtimes Q$, noting that ν reduces to σ in (7.3b). We obtain

$$\mathcal{A} \cong (\mathbb{C} \rtimes (N \rtimes Q)) \rtimes_{(\gamma,\sigma)} A \;\cong\; (\mathbb{C} \rtimes (N \rtimes Q)) \rtimes_{(\gamma,\sigma)} A \tag{7.13a}$$

$$\cong\; \left(C_0(\hat{N}) \rtimes_{(\hat{\beta},1)} Q\right) \rtimes_{(\hat{\gamma},\sigma)} A \tag{7.13b}$$

for some automorphisms $\gamma_r, \hat{\gamma}_r$, $r \in A$. Here, the automorphisms γ_r, $r \in A$ are computed as in (7.3a), and $\hat{\gamma}_r$ are their Fourier transformed versions. Also, the cocycle $\nu(r_1, r_2)$, as determined through (7.3b), is simply the complex scalar $\sigma(r_1, r_2)$, so we continue to use the symbol σ for it.

The explicit expression for $\hat{\gamma}$ is complicated by the fact that A acts on Q in the semidirect product $N \rtimes Q$. Nevertheless, its action on (the copy of) $C_0(\hat{N})$ is manageable:

$$\begin{aligned}
\hat{\gamma}_T(\hat{f})(\chi) = &\overline{\hat{f}(T^{-1}\cdot\chi)} = \overline{\hat{f}(T\cdot\chi)},\\
\hat{\gamma}_C(\hat{f})(\chi) = &\overline{\hat{f}(C^{-1}\cdot\chi)} = \overline{\hat{f}(C\cdot\chi)},\\
\hat{\gamma}_S(\hat{f})(\chi) = &\hat{f}(S^{-1}\cdot\chi) = \hat{f}(S\cdot\chi), \qquad\qquad \hat{f}\in C_0(\hat{N}).
\end{aligned} \tag{7.14}$$

Note that each of the dual actions of T, C, S on $\hat{N}$ is involutary, since they act as involutions on N, and that $\hat{\gamma}_T$ and $\hat{\gamma}_C$ are antilinear, whereas $\hat{\gamma}_S$ is complex-linear. However, T and C need not define the same involution on $\hat{N}$, although it is true that $C\cdot(T\cdot\chi) = S\cdot\chi$, so S acts trivially on $\hat{N}$ and $C_0(\hat{N})$ if the actions of T and C do coincide.

Suppose $A = \{1, T, C, S\} = \{1, T\}\times\{1, S\}$, and we move $\{1, S\}$ into the "unitary" group $\tilde{Q} \coloneqq Q \rtimes \{1, S\}$. Then can write

$$\mathcal{A} \equiv \mathbb{C} \rtimes_{(\alpha,\sigma)} \Big((N \rtimes \tilde{Q}) \rtimes \{1, T\}\Big) \cong \Big(C_0(\hat{N}) \rtimes_{(\hat{\beta},1)} \tilde{Q}\Big) \rtimes_{(\hat{\gamma},\sigma)} \{1, T\},$$

where $C_0(\hat{N}) \rtimes_{(\hat{\beta},1)} \tilde{Q}$ is now a *graded complex* algebra, T anticommutes with i in $\mathcal{MA}$, and $\mathsf{T}^2 = \pm 1$. Therefore, in any graded $\mathcal{A}$-module, we can choose (the representative of) T to be a distinguished even antilinear involution or anti-involution, i.e., a real or quaternionic structure. It is possible to use graded Real or "Quaternionic" bundles to understand graded f.g.p. $\mathcal{A}$-modules [17, 16], and we have already done this for $\mathsf{T}^2 = +1$ in Section 7.3.3. However, such bundles need to accommodate grading-reversing operations due to the presence of S, and can be quite cumbersome to describe. Furthermore, it is desirable to understand all the ten cases for (A, σ) in a unified manner, and to relate them to Clifford algebras a la Section 5.4.2.

To make progress in this direction, *we shall assume that* $G = (N \rtimes Q) \times A$. The decompositions (7.13a)–(7.13b) simplify, because γ_T, γ_C do not act on N or Q but simply effects complex conjugation on $\mathbb{C}$, and γ_S is just the identity. In the complex case, i.e. $A \subset \{1, S\}$, we have

$$\begin{aligned}
\mathcal{A} \cong (\mathbb{C} \rtimes (N \rtimes Q)) \rtimes_{(\gamma,\sigma)} A &= (\mathbb{C} \rtimes (N \rtimes Q)) \otimes_{\mathbb{C}} \big(\mathbb{C} \rtimes_{(1,\sigma)} A\big),\\
&= (\mathbb{C} \rtimes (N \rtimes Q)) \,\hat{\otimes}\, \mathbb{C}l_n,\\
&= \Big(C_0(\hat{N}) \rtimes_{(\hat{\beta},1)} Q\Big) \,\hat{\otimes}\, \mathbb{C}l_n,
\end{aligned} \tag{7.15}$$

where the complex Clifford algebra is $\mathbb{C}l_1$ if $A = \{1, S\}$ and $\mathbb{C}l_0$ if $A = \{1\}$. For discrete N, we can understand a graded f.g.p. $(Q, C(\hat{N}), \hat{\beta})$-module as in Section 7.3.1. It is the set of continuous sections of some graded Q-equivariant complex vector bundle over $\hat{N}$, and the latter is just the direct sum of two ungraded Q-equivariant bundles over $\hat{N}$. When $n = 1$, there is an additional graded action of $\mathbb{C}l_1$ on the fibres which commutes with the Q-action.

In the real case, i.e., either of C and T is present, we first write $\mathbb{C} \rtimes (N \rtimes Q) = (\mathbb{R} \rtimes (N \rtimes Q)) \otimes_{\mathbb{R}} \mathbb{C}$. Then we obtain

$$\begin{aligned}\mathcal{A} \cong ((\mathbb{R} \rtimes (N \rtimes Q)) \otimes_{\mathbb{R}} \mathbb{C}) \rtimes_{(\gamma,\sigma)} A &= (\mathbb{R} \rtimes (N \rtimes Q)) \otimes_{\mathbb{R}} \left(\mathbb{C} \rtimes_{(\alpha,\sigma)} A\right), \\ &= (\mathbb{R} \rtimes (N \rtimes Q)) \hat{\otimes} Cl_{r,s}, \end{aligned} \tag{7.16}$$

where the Clifford algebra $Cl_{r,s}$ is determined by (A, σ), according to Section 5.4.2. Actually, when $A = \{1, T\}$, the factor $\mathbb{C} \rtimes_{(\alpha,\sigma)} A$ is purely even, so we have actually made a modification (see Remark 5.4.4), replacing it by a graded Clifford algebra $Cl_{r,s}$. This detail is actually quite important, see Section 8.1. Unfortunately, the Clifford-module bundle picture is much less clear in the real case. This is because the real C^*-algebra $\mathbb{R} \rtimes N$ does not simply translate into $C_0(\hat{N}, \mathbb{R})$ under the Fourier transform. Instead, we have to consider $\hat{N}$ as a Real space with involution given by the map taking χ to the complex conjugate character $\bar{\chi}$. The algebra $\mathbb{R} \rtimes N$ is the real subalgebra of $\mathbb{C} \rtimes N$ which is fixed under complex conjugation. Upon taking the Fourier transform $\mathbb{C} \rtimes N \cong C_0(\hat{N})$, complex conjugation turns into the antilinear involution $\bar{\hat{f}}(\chi) := \overline{\hat{f}(\bar{\chi})}$, and the correct isomorphism is

$$\mathbb{R} \rtimes N \cong C_0(\mathrm{i}\hat{N}) := \left\{\hat{f} \in C_0(\hat{N}) \,:\, \overline{\hat{f}(\chi)} = \hat{f}(\bar{\chi})\right\}. \tag{7.17}$$

With (7.15) and (7.16), we have an expression for the graded twisted group algebra $\mathcal{A}$ as a tensor product of a purely even algebra with a graded Clifford algebra. According to Section 6.4, tensoring by a Clifford algebra simply produces a degree shift in K-theory, so the difference-group $\mathbf{K}_0(\mathcal{A})$ is easy to compute. In the complex case, we have

$$\begin{aligned}\mathbf{K}_0(\mathcal{A}) &= \mathbf{K}_0((C_0(\hat{N}) \rtimes_{(\hat{\beta},1)} Q)\hat{\otimes}\mathbb{C}l_n) \\ &= K_n(C_0(\hat{N}) \rtimes_{(\hat{\beta},1)} Q) \\ &\cong K_Q^{-n}(\hat{N}), \end{aligned} \tag{7.18}$$

where the last expression is an equivariant topological complex K-theory group. In the real case, we have

$$\begin{aligned}\mathbf{K}_0(\mathcal{A}) &= K_{s-r}(\mathbb{R} \rtimes (N \rtimes Q)) \\ &\cong K_{s-r}(C_0(\mathrm{i}\hat{N}) \rtimes_{(\hat{\beta},1)} Q). \end{aligned}$$

To obtain a topological K-group in the real case, we need to move to the Real category, suited for spaces with involution. Indeed, spaces with involution are naturally interpreted as the spectra of real commutative C^*-algebras. The topological KR-theory approach was used in [30], and is explained in more detail in Section 7.3.5. The final KR-theory expression for the difference-group is

$$\mathbf{K}_0(\mathcal{A}) \cong KR_Q^{r-s\,(\mathrm{mod}\,8)}(\hat{N}), \tag{7.19}$$

We summarise our computations in the following proposition:

Proposition 7.3.7. *In the graded twisted dynamical system $(G, c, \mathbb{C}, \alpha, \sigma)$, suppose that $G = (N \rtimes Q) \times A$, where N is a normal abelian subgroup, Q is compact, $N \rtimes Q \in \ker(\phi, c)$, A is a CT-subgroup, and (ϕ, c) descends to the standard homomorphisms on A. We assume that σ satisfies $\sigma(nq, \cdot) = 1 = \sigma(\cdot, nq)$ for all $nq \in N \rtimes Q$, so that it is completely specified by a cocycle for A. Let $\mathcal{A}$ be the graded twisted group algebra $\mathbb{C} \rtimes_{(\alpha,\sigma)} G$. Then (A, σ) determines a Clifford algebra $Cl_{r,s}$ or $\mathbb{C}l_n$ according to Table 5.5, and the difference group $\mathbf{K}_0(\mathcal{A})$ can be identified with an equivariant topological K-theory group,*

$$\mathbf{K}_0(\mathcal{A}) \cong KR_Q^{r-s\,(\mathrm{mod}\,8)}(\hat{N}), \qquad \text{real case,}$$
$$\mathbf{K}_0(\mathcal{A}) \cong K_Q^{-n}(\hat{N}), \qquad \text{complex case.}$$

7.3.5 V: Real Clifford algebras and Real vector bundles

In this section, we attempt to recover a Real vector bundle interpretation for the modules for the graded real twisted group algebras $\mathcal{A}$ considered at the end of Section 7.3.4. It is not crucial for the rest of the thesis, but leads us to two important observations: (1) the topological vector bundle picture can be quite complicated even when it is available; (2) KR-theory is *not* the most natural setting to study topological phases in.

Our assumptions on (G, τ, c, σ) are the same as those in Proposition 7.3.7. Since we are interested in the real case, we further assume that ϕ is surjective, or equivalently, that τ and/or c is surjective. We first recall some facts about Real Clifford algebras, which can be found in Chapter I.10 of [54]

Real Clifford algebras

The *Real Clifford algebra* $Cl(\mathbb{R}^{r,s})$ is defined to be the Clifford algebra $Cl_{r+s,0}$, with canonical negative generators $e_1, \ldots, e_{r+s}$, equipped with an algebra involution $\overline{(\cdot)}$ such that $\bar{e}_i = e_i$, $1 \le i \le r$ and $\bar{e}_j = -e_j$, $r+1 \le j \le r+s$. A *Real module* over the algebra $Cl(\mathbb{R}^{r,s})$ is a *complex* $Cl_{r,s}$-module W with a $\mathbb{C}$-antilinear involution such that

$$\overline{z \cdot w} = \bar{z} \cdot \bar{w}, \qquad z \in Cl(\mathbb{R}^{r,s}),\ w \in W. \tag{7.21}$$

Note that there are two *distinct* multiplications in W — multiplication by $z \in Cl(\mathbb{R}^{r,s})$ is indicated by $z \cdot w$, whereas multiplication by $z \in Cl_{r,s}$ is denoted by zw. The complexification $\mathbb{C}l(\mathbb{R}^{r,s}) := Cl(\mathbb{R}^{r,s}) \otimes_{\mathbb{R}} \mathbb{C}$ is a Real C^*-algebra with the natural Real involution $\overline{(\cdot)}$ defined by extending the involution on $Cl(\mathbb{R}^{r,s})$ antilinearly. A Real $Cl(\mathbb{R}^{r,s})$-module W is also a Real $\mathbb{C}l(\mathbb{R}^{r,s})$-module, by extending module multiplication to $\lambda z \equiv z \otimes \lambda \in \mathbb{C}l(\mathbb{R}^{r,s})$, i.e.,

$$(\lambda z) \cdot w := \lambda(z \cdot w), \qquad \lambda \in \mathbb{C},\, z \in Cl(\mathbb{R}^{r,s}),\, w \in W.$$

With this understanding, equation (7.21) continues to hold for $z \in \mathbb{C}l(\mathbb{R}^{r,s})$. Conversely, a Real $\mathbb{C}l(\mathbb{R}^{r,s})$-module is clearly a Real $Cl(\mathbb{R}^{r,s})$-module by restricting module multiplication to $z \in Cl(\mathbb{R}^{r,s})$. Thus it makes little difference whether we use Real $\mathbb{C}l(\mathbb{R}^{r,s})$-modules or Real $Cl(\mathbb{R}^{r,s})$-modules. Now, all $\mathbb{C}l(\mathbb{R}^{r,s})$ with $r+s=n$ are isomorphic as complex algebras to $\mathbb{C}l_n$. The real subalgebra of $\mathbb{C}l(\mathbb{R}^{r,s})$ fixed under ordinary complex conjugation is of course $Cl(\mathbb{R}^{r,s})$, which is $Cl_{n,0}$ as a real algebra. However, if we use the Real involution $\overline{(\cdot)}$ on $\mathbb{C}l(\mathbb{R}^{r,s})$, which are distinct for each choice of (r,s) such that $r+s=n$, then the fixed points form the real subalgebra $Cl_{r,s}$, which are *not* isomorphic to each other. This is the significant property of the involutions $\overline{(\cdot)}$ on the Real Clifford algebras.

According to Proposition I.10.9 of [54], there is an equivalence between the categories of real $Cl_{r,s}$-modules and Real $Cl(\mathbb{R}^{r,s})$-modules, or equivalently, Real $\mathbb{C}l(\mathbb{R}^{r,s})$-modules. In one direction, a real $Cl_{r,s}$-module W is extended to the complexification $W \otimes_{\mathbb{R}} \mathbb{C}$ with $Cl_{r,s}$-multiplication extended complex-linearly, and antilinear involution given by complex conjugation. Note that the involution *commutes* with $Cl_{r,s}$-multiplication. On $W \otimes_{\mathbb{R}} \mathbb{C}$, the $Cl(\mathbb{R}^{r,s})$ or $\mathbb{C}l(\mathbb{R}^{r,s})$-multiplication is defined by

$$\begin{aligned} e_i \cdot w =& e_i w, & 1 \le i \le r, \\ e_j \cdot w =& \mathrm{i}(e_j w), & r+1 \le j \le r+s, \end{aligned} \tag{7.22}$$

and extended $\mathbb{R}$-linearly to $z \in Cl(\mathbb{R}^{r,s})$ or $\mathbb{C}$-linearly to $z \in \mathbb{C}l(\mathbb{R}^{r,s})$. It is easy to verify that the $Cl(\mathbb{R}^{r,s})$ or $\mathbb{C}l(\mathbb{R}^{r,s})$-multiplication is Real, i.e., (7.21) holds. The reverse construction takes a Real $Cl(\mathbb{R}^{r,s})$ or $\mathbb{C}l(\mathbb{R}^{r,s})$-module W to the real subspace fixed by the conjugation in W, with $Cl_{r,s}$ multiplication defined by

$$\begin{aligned} e_i w =& e_i \cdot w, & 1 \le i \le r, \\ e_j w =& -\mathrm{i} e_j \cdot w, & r+1 \le j \le r+s. \end{aligned} \tag{7.23}$$

The equivalence between real $Cl_{r,s}$-modules, Real $Cl(\mathbb{R}^{r,s})$-modules and Real $\mathbb{C}l(\mathbb{R}^{r,s})$-modules means that the Morita classes of $Cl(\mathbb{R}^{r,s})$ and $\mathbb{C}l(\mathbb{R}^{r,s})$ inherit the $(1,1)$-periodicity of $Cl_{r,s}$. Thus, the Morita classes of $Cl(\mathbb{R}^{r,s})$ and $\mathbb{C}l(\mathbb{R}^{r,s})$ depend only on $r-s \,(\mathrm{mod}\, 8)$. In contrast, if we forget the involution, $\mathbb{C}l(\mathbb{R}^{r,s}) \cong \mathbb{C}l_{r+s}$ for any $r+s=n$, so that the Morita class of $\mathbb{C}l_{r+s}$ depends on $r+s \,(\mathrm{mod}\, 2)$. The Real Clifford algebras can be given their natural gradings, namely, e_i, e_j are taken to be odd, and this choice respects the involution. Then there is a natural notion of Real graded modules for the Real Clifford algebras — W is required to be a graded complex vector space, with an even involution, and the Real Clifford algebra action is required to be graded. There is again a 1–1 correspondence between real graded $Cl_{r,s}$-modules, Real graded $Cl(\mathbb{R}^{r,s})$-modules and Real graded $\mathbb{C}l(\mathbb{R}^{r,s})$-modules, given by the same formulae (7.22) and (7.23). Finally, there is also a 1–1 correspondence between Real graded $\mathbb{C}l(\mathbb{R}^{r,s})$-modules and Real $\mathbb{C}l(\mathbb{R}^{r,s+1})$-modules.

Real bundles of Clifford modules

We return to the problem described at the beginning of this Section. The kernel of (ϕ, c), as a homomorphism on $G/N = Q \times A$, is Q, and we want all its elements to be represented as even and complex-linear operators on the fibres of some graded complex vector bundle over $\hat{N}$. With the aim of obtaining the groups $KR_Q^{\star,\star}(\hat{N})$ of Real KR-theory [3] in mind, we define $\mathcal{V}_Q^{r,s}(\hat{N})$ to be the set of (equivalence classes of) Q-equivariant Real vector bundles $\tilde{E} \to \hat{N}$ over a compact Real Q-space $\hat{N}$, whose fibres are Real $\mathbb{C}l(\mathbb{R}^{r,s})$-modules. Such a Real bundle comes with a lift of the involution $\overline{(\cdot)}$ on $\hat{N}$ to $\tilde{E}$ which is antilinear between fibres, and commutes[5] with Q. The Real action of $\mathbb{C}l(\mathbb{R}^{r,s})$ is required to commute with Q, and satisfy $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ for all $w \in \tilde{E}$. There is a corresponding Real $\mathbb{C}l(\mathbb{R}^{r,s})$-action on the sections (equipped with the usual involution $\bar{v}(\bar{\chi}) = \overline{v(\chi)}$), denoted by $v \mapsto (z \cdot v)$, which is defined by $(z \cdot v)(\chi) := z \cdot (v(\chi))$. The computation

$$\overline{(z \cdot v)}(\chi) = \overline{(z \cdot v)(\bar{\chi})} = \overline{z \cdot (v(\bar{\chi}))} = \bar{z} \cdot \overline{v(\bar{\chi})} = \bar{z} \cdot (\bar{v}(\chi)) = (\bar{z} \cdot \bar{v})(\chi)$$

then says that $\overline{(z \cdot v)} = (\bar{z} \cdot \bar{v})$, as is required for a Real action. The direct (Whitney) sum gives $\mathcal{V}_Q^{r,s}(\hat{N})$ the structure of a commutative monoid. Similarly, we define $\mathcal{GV}_Q^{r,s}(\hat{N})$ to be the monoid of the (even equivalence classes of) graded Q-equivariant Real vector bundles $\tilde{E} \to \hat{N}$ over a compact Real Q-space $\hat{N}$, with a compatible Real graded action of $\mathbb{C}l(\mathbb{R}^{r,s})$ on the fibres. Note that both Q and the involution $\overline{(\cdot)}$ are required to be even on the fibres. Using the equivalence of Real graded $\mathbb{C}l(\mathbb{R}^{r,s})$-modules with ungraded Real $\mathbb{C}l(\mathbb{R}^{r,s+1})$-modules, we see that there is an isomorphism $\mathcal{GV}_Q^{r,s}(\hat{N}) \cong \mathcal{V}_Q^{r,s+1}(\hat{N})$. Furthermore, from the $(1,1)$-periodicity of the Real Clifford algebras, $\mathcal{GV}_Q^{r,s}(\hat{N})$ only depends on $(r-s) \pmod 8$. Also, we will use Real graded $Cl(\mathbb{R}^{r,s})$-modules in place of Real graded $\mathbb{C}l(\mathbb{R}^{r,s})$-modules in the definition of $\mathcal{GV}_Q^{r,s}(\hat{N})$, when it is more convenient to so do.

According to (7.13b), for discrete N, we have

$$\mathcal{A} \cong \left(C(\hat{N}) \rtimes_{(\hat{\beta},1)} Q\right) \rtimes_{(\hat{\gamma},\sigma)} A$$

Graded f.g.p. $\mathcal{A}$-modules are, in particular, graded f.g.p. $C(\hat{N}) \rtimes_{(\hat{\beta},1)} Q$-modules. So, guided by the Green–Julg theorem, we look for graded f.g.p. $(Q, C(\hat{N}), \hat{\beta})$-modules, which are the

[5]More generally, we can let Q be a Real group, and require $\overline{q \cdot \chi} = \bar{q} \cdot \bar{\chi}$ for all $q \in Q$, but the $\overline{(\cdot)}$ that we will use comes from C or T, so it commutes with Q due to our assumption of a direct product $G/N = Q \times A$.

sections of some graded Q-equivariant complex vector bundle E over $\hat{N}$. In addition, such modules of sections must be compatible with A according to the formula in (7.4), so E must be endowed with some extra structure. Recall (7.14) and (7.17), which say that $\hat{\gamma}_T$ and $\hat{\gamma}_C$ act as the same antilinear involution on $C(\hat{N})$, taking $\hat{f} \mapsto \bar{\hat{f}}$, where $\bar{\hat{f}}(\chi) = \overline{\hat{f}(\bar{\chi})}$. The base space $\hat{N}$ is a Real space under the map $\chi \mapsto \bar{\chi}$, and Q respects this involution. The elements $\mathsf{T}, \mathsf{C} \in \mathcal{MA}$ are represented as antilinear maps on the module of sections of E, which square to ± 1 and commute with the Q-action on the sections. Their corresponding bundle maps, denoted by $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}$, square to ± 1, and map the fibre E_χ antilinearly to $E_{\bar{\chi}}$. They also commute with the Q-action on the bundle. Note that the squares and commutation relations amongst T, C, and amongst $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}$, are the same — they are all determined in the same way by σ. Also, on a graded $\mathcal{A}$-module of bundle sections, T is even while C is odd, so I_T and I_C are respectively even and odd on the fibres of E. We define S and I_S similarly, and note that $\mathsf{TC} = \mathsf{S} = \mathsf{CT}$ implies $\mathsf{I}_\mathsf{T}\mathsf{I}_\mathsf{C} = \mathsf{I}_\mathsf{S} = \mathsf{I}_\mathsf{C}\mathsf{I}_\mathsf{T}$. The bundle map I_S is odd, complex-linear, and squares to $+1$.

Our aim now is to construct, for each of the eight choices for (A, σ), a 1–1 correspondence between (isomorphism classes of) Q-equivariant complex vector bundles $E \to \hat{N}$ with a compatible action of $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}, \mathsf{I}_\mathsf{S}$ as described in the previous paragraph, and elements $\bar{E}$ of and appropriate $\mathcal{GV}_Q^{r,s}(\hat{N})$. In this correspondence, (r, s) is determined by the Clifford algebra $Cl_{r,s}$ associated with (A, σ). As a complex vector bundle, $\tilde{E}$ will be either E itself, or $E \oplus E$. Starting from such a bundle E, we need to construct $\tilde{E}$, as well as an even involution on $\tilde{E}$ which lifts that on $\hat{N}$ and commutes with Q. Then we need to describe the $\mathbb{C}l(\mathbb{R}^{r,s})$-action on the fibres of $\tilde{E}$. Finally, we also need to describe the reverse construction.

Besides making the correspondence $E \leftrightarrow \tilde{E}$, we should also check that the notion of a trivial $\mathcal{A}$-module (sections of E) corresponds to a trivial Real graded bundle. For the latter, we define triviality to mean the existence of an odd complex-linear bundle automorphism which squares to 1, commutes with the involution and Q-action, and graded commutes with the Real graded Clifford action. In other words, we want the existence of an extra compatible positive Real Clifford generator. So suppose the sections of E are trivial as a $\mathcal{A}$-module. Then there is a complex-linear bundle map $\mathcal{I}$ which squares to 1, commutes with Q and I_T, and anticommutes with I_C and Γ. We need to construct a corresponding trivialising operator (extra positive Clifford generator) $\tilde{\mathcal{I}}$ on the Real bundle $\tilde{E}$. Conversely, given a trivial Real bundle $\tilde{E}$, we need to show that the corresponding E is also trivial, by constructing a suitable operator $\mathcal{I}$.

We shall proceed on a case-by-case basis. It will be more convenient to regard the Real multiplication by $z \in Cl(\mathbb{R}^{r,s})$, which satisfies (7.21), as a (complex-linear) multiplication by $z \in Cl_{r,s}$ which *commutes* with $\overline{(\cdot)}$. We will sometimes use ungraded bundles, and utilise the isomorphisms $\mathcal{V}_Q^{r,s+1}(\hat{N}) \cong \mathcal{GV}_Q^{r,s}(\hat{N}) \cong \mathcal{GV}_Q^{r-s\,(\mathrm{mod}\,8),0}(\hat{N})$.

The simplest case is when $A = \{1, T\}$ and $\mathsf{T}^2 = +1$. Then I_T squares to 1, and thus provides an even antilinear involution E. So take $\tilde{E} = E$ with the involution $\overline{(\cdot)} = \mathsf{I}_\mathsf{T}$, to obtain an element of $\mathcal{GV}_Q^{0,0}(\hat{N})$. If C is also in A, with $\mathsf{C}^2 = \pm 1$, we can additionally take $e_1 = \mathsf{I}_\mathsf{S}$, which is odd, $\mathbb{C}$-linear, and commutes with $\overline{(\cdot)}$. Accordingly, we obtain an element of $\mathcal{GV}_{Q_0}^{0,1}(\hat{N}) \cong \mathcal{GV}_Q^{7,0}(\hat{N})$ or $\mathcal{GV}_Q^{1,0}(\hat{N})$, according to whether $\mathsf{C}^2 = +1$ or $\mathsf{C}^2 = -1$ respectively. In these three cases, we can use the same operator $\mathcal{I}$ which trivialises a trivial E, as a trivialising operator $\tilde{\mathcal{I}}$ on $\tilde{E}$.

In the remaining five cases, we treat E as an ungraded bundle, and take $\tilde{E} = E \oplus E$ as an ungraded bundle. We denote the original grading operator on E by Γ, which anticommutes with I_C and commutes with $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{S}$. If E is trivial, we need a trivialising operator $\tilde{\mathcal{I}}$ for $\tilde{E}$. In the case $A = \{1, C\}$ and $\mathsf{C}^2 = +1$, we can take

$$\overline{(\cdot)} = \begin{pmatrix} 0 & \mathsf{I}_\mathsf{C} \\ \mathsf{I}_\mathsf{C} & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \tilde{\mathcal{I}} = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix},$$

which gives an element of $\mathcal{V}_Q^{0,3}(\hat{N}) \cong \mathcal{GV}_Q^{6,0}(\hat{N})$. When $A = \{1, C\}$ and $\mathsf{C}^2 = -1$, we take

$$\overline{(\cdot)} = \begin{pmatrix} 0 & \mathsf{I}_\mathsf{C} \\ -\mathsf{I}_\mathsf{C} & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\mathcal{I}} = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix},$$

which gives an element of $\mathcal{V}_Q^{2,1}(\hat{N}) \cong \mathcal{GV}_Q^{2,0}(\hat{N})$.

If $A = \{1, T, C, CT\}$, with $\mathsf{C}^2 = \pm 1$ and $\mathsf{T}^2 = -1$, we take $\overline{(\cdot)}, e_1, e_2, e_3$ to be as above, and we take the additional Clifford generator and trivialising operator on $\tilde{E}$ to be

$$e_4 = \begin{pmatrix} \Gamma \mathsf{I}_\mathsf{S} & 0 \\ 0 & -\Gamma \mathsf{I}_\mathsf{S} \end{pmatrix}, \quad \tilde{\mathcal{I}} = \begin{pmatrix} \mathrm{i}\mathcal{I}\Gamma & 0 \\ 0 & -\mathrm{i}\mathcal{I}\Gamma \end{pmatrix}.$$

This leads to an element of $\mathcal{V}_Q^{0,4}(\hat{N}) \cong \mathcal{GV}_Q^{5,0}(\hat{N})$ in the $\mathsf{C}^2 = +1$ case, and $\mathcal{V}_Q^{3,1}(\hat{N}) \cong \mathcal{GV}_Q^{3,0}(\hat{N})$ in the $\mathsf{C}^2 = -1$ case.

Finally, there is the $A = \{1, T\}, \mathsf{T}^2 = -1$ case, where we can take

$$\overline{(\cdot)} = \begin{pmatrix} 0 & \mathsf{I}_\mathsf{T} \\ -\mathsf{I}_\mathsf{T} & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & \Gamma \\ -\Gamma & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad \tilde{\mathcal{I}} = \begin{pmatrix} 0 & \Gamma\mathcal{I} \\ -\Gamma\mathcal{I} & 0 \end{pmatrix},$$

which gives an element of $\mathcal{V}_Q^{3,0}(\hat{N}) \cong \mathcal{GV}_Q^{4,0}(\hat{N})$. Note that all eight values of $r - s\,(\mathrm{mod}\,8)$ occur. Note that this is the only case where there is no explicit choice of compatible grading operator $\tilde{\Gamma}$for $\tilde{E}$, either by using $\tilde{\Gamma} = \Gamma$ (when $\tilde{E} = E$) or $\tilde{\Gamma} = e_1$ (when $\tilde{E} = E \oplus E$). This can be fixed as in Remark 5.4.4.

The reverse construction proceeds as follows. For each $r - s\,(\mathrm{mod}\,8)$, we may assume that we have an element of $\mathcal{V}_Q^{r,s+1}(\hat{N})$ or $\mathcal{GV}_Q^{r,s}(\hat{N})$ with (r, s) being one of the eight pairs found above, modulo 8. Thus, we have a (graded) Real bundle $\tilde{E}$ with a compatible Real (graded) action of some $\mathbb{C}l(\mathbb{R}^{r,s+1})$ ($\mathbb{C}l(\mathbb{R}^{r,s})$). As mentioned earlier, we look at the complex-linear $Cl_{r,s}$-action instead — this commutes with the involution $\overline{(\cdot)}$ on $\tilde{E}$. For $(r,s) = (0,0),(0,1),(1,0)$, we take $E = \tilde{E}$ as a complex bundle. In the $(r,s) = (0,0)$ case, we take $\mathsf{I_T} = \overline{(\cdot)}$. In the $(r,s) = (0,1)$ and $(r,s) = (1,0)$ cases, regard the graded Real bundle as a graded complex bundle and take $\mathsf{I_T} = \overline{(\cdot)}$, $\mathsf{I_C} = \overline{(\cdot)}e_1$. If $\tilde{\mathcal{I}}$ trivialises $\tilde{E}$, we use the same operator $\mathcal{I} = \tilde{\mathcal{I}}$ to make E trivial.

In the remaining five cases, we take the (initally ungraded) complex bundle E to be the i-eigenspace of e_2e_3, where e_2, e_3 are positive Clifford generators in the $(r, s+1) = (0,3)$ or $(0,4)$ cases, and negative generators in the $(r,s+1) = (3,0),(2,1),(3,1)$ cases. For the $(r,s+1) = (3,0)$ case, there is a leftover negative Clifford generator e_1, and we take $\Gamma = \mathrm{i}e_1$, $\mathsf{I_T} = \overline{(\cdot)}e_2$ and $\mathcal{I} = \tilde{\mathcal{I}}e_1$, restricted to E. For the $(r,s+1) = (0,3)$ and $(2,1)$ cases, there is a leftover positive Clifford generator e_1, and we choose $\Gamma = e_1$, $\mathsf{I_C} = \overline{(\cdot)}e_2$ and $\mathcal{I} = \tilde{\mathcal{I}}$, restricted to E. When $(r,s+1) = (0,4)$, there is a leftover positive Clifford generator e_1, and we take $\Gamma = e_1$, $\mathsf{I_C} = \overline{(\cdot)}e_2$, $\mathsf{I_T} = \overline{(\cdot)}e_2e_1e_4 = \mathsf{I_C}e_1e_4$ and $\mathcal{I} = \mathrm{i}e_1\tilde{\mathcal{I}}$. Finally, when $(r,s+1) = (3,1)$, there is a leftover negative Clifford generator e_1, and we take $\Gamma = \mathrm{i}e_1$, $\mathsf{I_T} = \overline{(\cdot)}e_2$, $\mathsf{I_C} = \overline{(\cdot)}e_2e_1e_4 = \mathsf{I_T}e_1e_4$ and $\mathcal{I} = e_1\tilde{\mathcal{I}}$. A fairly straightforward computation shows that E is a graded complex Q-equivariant bundle with the correct actions of $\mathsf{I_T}$ and $\mathsf{I_C}$, the correct commutation relations, as well as the correct values for their squares. Furthermore, E is trivial if $\tilde{E}$ is.

Thus, we have shown that graded $\mathcal{A}$-modules are, for each of the eight choices for (A,σ), in 1–1 correspondence with elements of $\mathcal{GV}_Q^{r,s}(\hat{N})$, and that this correspondence respects the notion of triviality on each side. Furthermore, a grading operator Γ for the ungraded $\mathcal{A}$-module underlying a graded $\mathcal{A}$-module W becomes a grading operator $\tilde{\Gamma}$ for the underlying ungraded Real bundle of the corresponding bundle $\tilde{E}$, and vice-versa. Here, a grading operator for $\tilde{E}$ is defined in the natural way as an involutive Q-equivariant Real bundle automorphism (i.e. complex bundle automorphism which commutes with the involution and Q-action) which graded commutes with the fibrewise $Cl_{r,s}$-action (equivalently $Cl(\mathbb{R}^{r,s})$)-action. This means that a triple (W,Γ_1,Γ_2) entering the definition of $\mathbf{K}_0(\mathcal{A})$ has a 1–1 translation to a triple $(\tilde{E},\tilde{\Gamma}_1,\tilde{\Gamma}_2)$. Using a Karoubi-type definition (in terms of grading operators) for the Q-equivariant $KR^{r,s}$-groups of $\hat{N}$, as in III.7.14 of [42], we find that

$$\mathbf{K}_0(\mathcal{A}) \cong KR'^{r,s}_Q(\hat{N}) \cong KR_Q^{r-s\,(\mathrm{mod}\,8)}(\hat{N}), \tag{7.24}$$

giving a topological interpretation[6] to the group $\mathbf{K}_0(\mathcal{A})$.

[6]In the usual definitions of the doubly-indexed topological $KR^{s,r}(\cdot)$-groups, as in Appendix A.3 or [54], the positions of the two indices are *opposite* to those of Kaboubi's $KR'^{r,s}(\cdot)$. To emphasise the distinction,

If $\mathcal{B}$ is a Real graded C^*-algebra (Appendix A.3.1), we write $\mathcal{GV}(\mathcal{B})$ for the monoid of equivalence classes of f.g.p. Real graded $\mathcal{B}$-modules, $\mathcal{T}(\mathcal{B})$ for its submonoid of trivial modules, and $\mathcal{GR}(\mathcal{B})$ for the Grothendieck completion of $\mathcal{GV}(\mathcal{B})$. Then the KR-theory version of the Atiyah–Bott–Shapiro isomorphisms [54] is

$$\mathcal{GV}(\mathbb{C}l(\mathbb{R}^{r,s}))/\sim_{\mathcal{T}(\mathbb{C}l(\mathbb{R}^{r,s}))}\cong \mathcal{GR}(\mathbb{C}l(\mathbb{R}^{s,r}))/i^*\mathcal{GR}(\mathbb{C}l(\mathbb{R}^{s+1,r}))\cong KR^{s,r}(\star).$$

Note that $KR^{s,r}(\star) = KR'^{r,s}(\star) \cong KO^{r-s\,(\mathrm{mod}\,8)}(\star)$. Using these isomorphisms, we arrive at an alternative expression for the super-representation groups of $\mathcal{A}$ in terms of topological KR-groups,

$$\begin{aligned} \mathcal{SR}(\mathcal{A}) &\cong \mathcal{SR}\left(\left(C(\hat{N})\rtimes_{(\hat{\beta},1)}Q\right)\rtimes_{(\hat{\gamma},\sigma)}A\right)\\ &\cong \mathcal{GV}_Q^{r,s}(\hat{N})/i^*\mathcal{GV}_Q^{r,s+1}(\hat{N})\\ &\cong KR^0_Q(\hat{N})\otimes_{\mathbb{Z}} KR^{s,r}(\star)\\ &\cong KR^0_Q(\hat{N})\otimes_{\mathbb{Z}} KO^{r-s\,(\mathrm{mod}\,8)}(\hat{N}), \end{aligned} \tag{7.25}$$

where the indices (r,s) are determined by (A,σ), as presented in Table 5.5.

In practice, the real algebras $\mathcal{A}$ given by the expression (7.4), and their associated complex bundles E are more natural objects, but we have made the detour to the Real bundles $\tilde{E}$ in order to exhibit (7.24) and (7.25).

Remark 7.3.8. There is a similar construction in the proof of Corollary 10.25 of [30]. We also point out a recent work [17] on the study "Real" Bloch bundles with applications to topological insulators of type AI. Note that the "Real" bundles of the authors are Real bundles in our terminology.

Remark 7.3.9. If N is a subgroup of translation symmetries and spatial inversion symmetry P is also present as a direct product factor of Q, it is tempting to incorporate it into A to form a subgroup of the PCT-group, and use the analysis of Section 5.4.3 to find an associated Clifford algebra. There is a genuine non-trivial action of P on N, its complex group algebra $C_0(\hat{N})$, as well as its real group algebra $C_0(\mathrm{i}\hat{N})$. In contrast, C and T do not act on N or $C_0(\mathrm{i}\hat{N})$, and only acts on the $\mathbb{C}$ factor in $C_0(\hat{N})\cong C_0(\mathrm{i}\hat{N})\otimes_{\mathbb{R}}\mathbb{C}$. The apparent action of C or T on $\hat{N}$ is really an artefact of the Fourier transformation of the complex-conjugation operation, and thinking of $C_0(\hat{N})$ as $C_0(\hat{N},\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$ instead of $C_0(\mathrm{i}\hat{N})\otimes_{\mathbb{R}}\mathbb{C}$. Nevertheless, the fact that P, C, and T all effect $\chi\mapsto\overline{\chi}$ in $\hat{N}$ is sometimes useful in computations, see Section 11 of [30]. Also, see Appendix A of [17] for an example of the different roles of T and P in defining extra structure on complex vector bundles over $\hat{N}$.

we have added a prime to the latter. Both are isomorphic (by respective definitions) to the singly-indexed $KR'^{r-s}(\cdot)$.

Chapter 8

Applications

We have been interpreting the grading operator on a graded $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$-module as a spectrally-flattened gapped Hamiltonian compatible with the symmetry data (G, c, ϕ, σ). The classification of such gapped Hamiltonians, up to the topological triviality introduced in Section 6.2.3, is achieved by taking the super-representation group $\mathcal{SR}(\mathcal{A})$. Differences of symmetry-compatible gapped Hamiltonians, up to a homotopy equivalence and stabilisation introduced in Section 6.4, are classified by the $\mathbf{K}_0$-group of $\mathcal{A}$. At this stage, our classification may look quite abstract, and it is not clear what the elements of the groups $\mathcal{SR}(\mathcal{A})$ and $\mathbf{K}_0(\mathcal{A})$ look like. Nevertheless, we will see that it generalises the notion of symmetry classes of topological band insulators, as commonly discussed in the literature.

8.1 Band insulators and topological K-theory

Consider the symmetry data (G, c, ϕ, σ) as usual, and let $\mathbb{Z}^d = N \triangleleft G$ be a lattice of spatial translations. As in Sections 7.3.1–7.3.4, we assume that $\mathbb{Z}^d \subset \ker(\phi, c)$ and that $\sigma(\mathbf{x}, \cdot)$, $\sigma(\cdot, \mathbf{x})$ are trivial for all $\mathbf{x} \in \mathbb{Z}^d$. Thus, the translations are realised non-projectively, complex-linearly, and do not have any charge-reversing or time-reversing properties. The characters of $\mathbb{Z}^d$ are labelled by d phases, i.e., the quasi-momenta in the Brillouin torus $\hat{N} = \mathbb{T}^d$ familiar from solid-state physics. To obtain a vector bundle picture for the graded f.g.p. modules for $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$, we make a splitting assumption $G = \mathbb{Z}^d \rtimes G/\mathbb{Z}^d$ with $G/\mathbb{Z}^d$ compact, so $\mathcal{A} \cong C(\mathbb{T}^d) \rtimes_{(\hat{\beta},\nu)} G/\mathbb{Z}^d$. The underlying graded $C(\mathbb{T}^d)$-module structure in a graded f.g.p. $\mathcal{A}$-module gives a graded complex vector bundle $E \to \mathbb{T}^d$, which we call a graded *Bloch bundle*. There is a (possibly ν-twisted) equivariant *real*-linear action of $G/\mathbb{Z}^d$ between fibres, which is complex-linear if $\phi \equiv 1$. The positively-graded subbundle can be interpreted as the conduction band lying above the Fermi level $\mathcal{E}_F$ (which is taken to be 0), while the negatively-graded subbundle is the valence band, lying below $\mathcal{E}_F$. The symmetries in $G/\mathbb{Z}^d$ respect this particle-hole distinction if $c(p) = +1$, and reverses it if

$c(p) = -1$. They are also complex-linear/antilinear according to $\phi(p) = \pm 1$, and if $\phi \not\equiv 1$, we can make E a "Real" graded bundle in the sense of Section 7.3.3. As always, the time preserving/reversing nature of $p \in G/\mathbb{Z}^d$ is determined by $\tau \equiv \phi \cdot c$.

We call a ("Real") graded Bloch bundle with the above $G/\mathbb{Z}^d$ action a (G, c, ϕ, σ)-*compatible band insulator*. In the Bloch bundle picture, a trivial graded $\mathbb{C} \rtimes_{(\alpha,\sigma)} G$-module is a ("Real") graded $G/\mathbb{Z}^d$-equivariant Bloch bundle $E \to \mathbb{T}^d$, such that there exists an odd complex bundle automorphism $\mathcal{I}$ satisfying $\mathcal{I}^2 = +1$, and $\mathcal{I}p = c(p)p\mathcal{I}$ for all $p \in G/\mathbb{Z}^d$. Such a Bloch bundle is called a *topologically trivial* (G, c, ϕ, σ)-*compatible band insulator*, and is the commutative version of our general formulation of topological triviality in Definition 6.2.4. The super-representation group of $\mathbb{C} \rtimes_{(\alpha,\sigma)} G$ classifies the (G, c, ϕ, σ)-compatible band insulators, modulo the topologically trivial ones.

In Section 7.3.4, we found that a significant simplification occurs when G has the form $G = (\mathbb{Z}^d \rtimes Q) \times A$ with $A \cong (\phi, c)(G) \subset \{1, T, C, S\}$, and σ is only non-trivial between elements of A. The data of (A, σ) is associated with one of ten possible Clifford algebras, which factorises in $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$. In the two complex cases (i.e. $A \subset \{1, S\}$), the graded Bloch bundle picture of a graded f.g.p. $\mathcal{A}$-module is a graded Q-equivariant complex bundle over $\mathbb{T}^d$, corresponding to the factor $C(\mathbb{T}^d) \rtimes_{(\hat{\beta},1)} Q$ in (7.13b). In the $A = \{1, S\}$ case, there is an additional *commuting* graded $\mathbb{C}l_1$-action on each fibre, arising from the complex-linear, odd, and involutary map I_S representing $\mathsf{S} \in \mathcal{MA}$. A topologically trivial (G, c, ϕ, σ)-compatible band insulator (in a complex class) is one which admits an odd Q-equivariant complex bundle automorphism $\mathcal{I}$ satisfying $\mathcal{I}^2 = +1$, and also $\mathcal{I}\mathsf{I}_\mathsf{S} = -\mathsf{I}_\mathsf{S}\mathcal{I}$ if S is present.

As for the remaining eight cases corresponding to the real Clifford algebras, the graded Bloch bundle is again a complex graded Q-equivariant bundle E over $\mathbb{T}^d$. The elements T, C of A, where present, act through the antilinear bundle maps $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}$ representing $\mathsf{T}, \mathsf{C} \in \mathcal{MA}$, which are even and odd respectively. The squares of I_T and I_C are ± 1 according to $\sigma(T, T)$ and $\sigma(C, C)$. Unlike I_S, the bundle maps $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}$ do not commute with the bundle projection, but they take the fibre over $\chi \in \mathbb{T}^d$ to the fibre over $\bar{\chi}$. The map I_T is the standard time-reversal operation on the Bloch bundle, and both cases $\mathsf{I}_\mathsf{T}^2 = \pm 1$ are considered in the literature on "time-reversal invariant topological insulators". Similarly, I_C is the standard particle-hole conjugation operation on the graded Bloch bundle. A topologically-trivial (G, c, ϕ, σ)-compatible band insulator is one which admits an odd Q-equivariant complex bundle automorphism, satisfying $\mathcal{I}^2 = +1$, $\mathcal{I}\mathsf{I}_\mathsf{T} = \mathsf{I}_\mathsf{T}\mathcal{I}$, and $\mathcal{I}\mathsf{I}_\mathsf{C} = -\mathsf{I}_\mathsf{C}\mathcal{I}$. Note that the Clifford algebra action generated by the appropriate subset of $\{\mathrm{i}, \mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}\}$ does *not* commute with that of $C(\mathbb{T}^d)$, so the graded Bloch bundle is *not* a bundle of graded Clifford modules. In Section 7.3.5, we used the extra structure on the Bloch bundles to construct a corresponding

Real graded Q-equivariant bundle $\tilde{E}$ over $\mathbb{T}^d$, which has a Real graded $\mathbb{C}l(\mathbb{R}^{r,s})$-action on the fibres which commutes with Q. This construction allows us to use KR-theory to classify the Real bundles $\tilde{E}$ up to triviality, and hence, the Bloch bundles E up to triviality.

Under the assumptions of Section 7.3.4, the classification of gapped Hamiltonians compatible with (G, c, ϕ, σ) is greatly simplified. In particular, the difference-groups of (G, c, ϕ, σ)-compatible band insulators are topological K-theory groups, which can be read off from Proposition 7.3.7:

$$\mathbf{K}_0(\mathcal{A}) = \begin{cases} K_Q^{-n}(\mathbb{T}^d) & \text{complex case,} \\ KR_Q^{r-s\,(\mathrm{mod}\,8)}(\mathbb{T}^d) & \text{real case,} \end{cases}$$

with n, r, s determined by (A, σ) and Table 5.5. Their super-representation groups can be read off from (6.7):

$$\mathcal{SR}(\mathcal{A}) = \begin{cases} K_Q^0(\mathbb{T}^d) \otimes_{\mathbb{Z}} K^{-n}(\star) & \text{complex case,} \\ KR_Q^0(\mathbb{T}^d) \otimes_{\mathbb{Z}} KO^{r-s\,(\mathrm{mod}\,8)}(\star) & \text{real case.} \end{cases}$$

The three purely-even cases

Although $\mathcal{SR}(\mathcal{A})$ and $\mathbf{K}_0(\mathcal{A})$ do not generally coincide for graded $\mathcal{A}$, they *do* when $\mathcal{A}$ is purely even, since both are then isomorphic to the ordinary K_0-group of $\mathcal{A}$ (Proposition 6.2.8). Important examples are when G is of the form $(\mathbb{Z}^d \rtimes Q) \times A$, where $A = \{1\}$ or $A = \{1, T\}$, and only $\sigma(T, T)$ may be non-trivial. In each of these cases, $\mathbb{C} \rtimes_{(\alpha,\sigma)} A$ is purely even; it is either the complex algebra $\mathbb{C}$, the real algebra $M_2(\mathbb{R})$, or the real algebra $\mathbb{H}$. The latter two real algebras are generated by i and T. This means that (7.15) and (7.16) should give

$$\mathbb{C} \rtimes_{(\alpha,\sigma)} G = \begin{cases} \mathbb{C} \rtimes (N \rtimes Q) & A = \{1\}, \\ \mathbb{R} \rtimes (N \rtimes Q) \otimes_{\mathbb{R}} M_2(\mathbb{R}) & A = \{1, T\}, \mathsf{T}^2 = +1, \\ (\mathbb{R} \rtimes (N \rtimes Q)) \otimes_{\mathbb{R}} \mathbb{H} & A = \{1, T\}, \mathsf{T}^2 = -1. \end{cases}$$

Recall that in the third case, we had replaced $\mathbb{H}$ by the graded Clifford algebra $Cl_{4,0} \cong Cl_{0,4}$ in (7.16), which we used to arrive at (7.19). We could also use Theorem III.4.12 of [42], which says that $K'^{0,0}(\mathcal{B} \otimes_{\mathbb{R}} \mathbb{H}) \cong K'^{0,4}(\mathcal{B})$ for a purely even $\mathcal{B}$. Taking $\mathcal{B} = \mathbb{R} \rtimes (N \rtimes Q)$, we obtain $\mathbf{K}_0(\mathcal{B} \otimes_{\mathbb{R}} \mathbb{H}) \equiv K'^{0,0}(\mathcal{B} \otimes_{\mathbb{R}} \mathbb{H}) \cong K'^{0,4}(\mathcal{B}) \equiv \mathbf{K}_0(\mathcal{B} \hat{\otimes} Cl_{0,4})$, which justifies our replacement. The (super)-representation groups for the three cases in which $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$ is purely even, are

$$\mathcal{R}(\mathcal{A}) \cong \mathcal{SR}(\mathcal{A}) \cong \mathbf{K}_0(\mathcal{A}) = \begin{cases} K_Q^0(\hat{N}) & A = \{1\}, \\ KR_Q^0(\hat{N}) & A = \{1, T\}, \mathsf{T}^2 = +1, \\ KR_Q^{-4}(\hat{N}) & A = \{1, T\}, \mathsf{T}^2 = -1, \end{cases}$$

which should be compared with Corollary 10.28 of [30]. Specialising to $N = \mathbb{Z}^d$ yields the (super)-representation groups of (G, ϕ, σ)-compatible band insulators.

Finally, we remark that the somewhat awkward treatment of the third case ($\mathsf{T}^2 = -1$), which resulted in a KR^{-4} group, can be modified to resemble the first two cases more closely. In the third case, we can identify a f.g.p. $\mathcal{A}$-module with the sections of a Q-equivariant "Quaternionic" vector bundle over $\hat{N}$. Like Real vector bundles, "Quaternionic" vector bundles are complex vector bundles over a Real space (X, ς), but with a lift of ς to an antilinear *anti-involution* Θ such that $\Theta^2 = -1$. Such bundles were considered in a classification scheme for Type AII topological insulators in [16], in which detailed definitions and references for "Quaternionic" bundles can be found. The construction of the "Quaternionic" vector bundle corresponding to a $\mathcal{A}$-module is quite straightforward. Recall that $\mathcal{A} \cong \left(C(\hat{N}) \rtimes_{(\hat{\beta},1)} Q\right) \rtimes_{(\hat{\gamma},\sigma)} \{1, T\}$ by (7.13b), with $N = \mathbb{Z}^d$ for band insulators. Then, we simply form the Q-equivariant complex vector bundle corresponding to the underlying $\left(C(\hat{N}) \rtimes_{(\hat{\beta},1)} Q\right)$-module structure of a f.g.p. $\mathcal{A}$-module, and define $\Theta = \mathsf{I}_{\mathsf{T}}$. The topological K-theory for "Quaternionic" bundles is called KQ-theory, and there is a useful isomorphism $KR^{-4}(X, \varsigma) \cong KQ^0(X, \varsigma)$ (see [22] and the Appendix of [16]). This establishes a KR^{-4} group as a Grothendieck group KQ^0 of "Quaternionic" bundles; the latter is somewhat easier to picture, and allows us to describe $\mathbf{K}_0(\mathcal{A})$ directly as a Grothendieck group of vector bundles in all three cases where $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$ is purely even, *but not the other seven.*

8.2 Finite versus infinite rank bundles

In many realistic Hamiltonians compatible with a subgroup $N = \mathbb{Z}^d$ or $N = \mathbb{R}^d$ of translational symmetries of a lattice, the Bloch Hamiltonians are unbounded operators on a bundle of infinite-dimensional Hilbert spaces over the Brillouin torus $\hat{N} = \mathbb{T}^d$. In our application of K-theory, we have been confined to the finite-rank situation. We can motivate this constraint physically by imposing an energy cut-off, or by assuming that there is a finite-rank conduction band which we are interested in studying. As such, Γ refers not to the flattened version of the full Hamiltonian, but to the flattened version of its restriction to the relevant low energy modes. Likewise, the symmetry group G acts only on this restricted subspace. In certain discretised models, the Bloch Hamiltonians do have finite-rank, so the K-theory approach addresses the full symmetry-compatible Hamiltonians.

With appropriate modifications, the infinite-rank situation can still be handled, and it is, in any case, the correct setting for some physical systems studied in the literature. In the Hilbert C^*-module description of Bloch theory [33], the Bloch Hamiltonians act on a continuous field of infinite-dimensional separable Hilbert spaces over $\mathbb{T}^d$, whose sections form a countably-generated Hilbert $C(\mathbb{T}^d)$-module. In [30], the authors considered graded bundles $E^+ \oplus E^-$ over $\mathbb{T}^d$, such that E^+ is infinite-dimensional, while E^- is a finite-rank bundle. They called such bundles "Type I" insulators, while graded bundles with both E^+ and E^- having finite rank were called "Type F" insulators. Let us formalise their idea in noncommutative language.

Definition 8.2.1 (Type I and Type F insulators)**.** Let $\mathcal{A}$ be a graded unital C^*-algebra. A *Type I insulator* is a graded $\mathcal{A}$-module $E = E_0 \oplus E_1 =: E^+ \oplus E^-$ such that E^- is a f.g.p. $\mathcal{A}$-module and E^+ is a countably-generated (but not finitely-generated) Hilbert $\mathcal{A}$-module. A *Type F insulator* is a graded f.g.p. $\mathcal{A}$-module.

Suppose $\mathcal{A}$ is the graded twisted crossed product C^*-algebra arising from the symmetry data (G, c, ϕ, σ). For Type I insulators, there is no possibility of an invertible odd operator taking E^+ to the f.g.p. submodule E^-, so $c \equiv 1$ and $\mathcal{A}$ is purely even. We define $\mathcal{GV}^I(\mathcal{A})$ to be the commutative monoid, under the direct sum, of equivalence classes of Type I insulators. Note that we have to formally introduce the zero module as a zero element, since it is not actually in $\mathcal{GV}^I(\mathcal{A})$. This monoid is really the direct sum of the monoid $\mathcal{V}^+(\mathcal{A})$ of ungraded countably-generated Hilbert $\mathcal{A}$-modules E^+, and the monoid $\mathcal{V}^-(\mathcal{A}) := \mathcal{V}(\mathcal{A})$ of ungraded f.g.p. $\mathcal{A}$-modules.

For Type I insulators, the notion of triviality in Definition 6.2.4 is vacuous, and we cannot construct a "super-representation group" using Definition 6.2.5. Instead, the passage to an abelian group is via the Grothendieck completion $\mathcal{GR}^I(\mathcal{A})$ of $\mathcal{GV}^I(\mathcal{A}) = \mathcal{V}^+(\mathcal{A}) \oplus \mathcal{V}^-(\mathcal{A})$. We can perform an "Eilenberg swindle" on the $\mathcal{V}^+$ part of $\mathcal{GV}^I(\mathcal{A})$ as follows.

Proposition 8.2.2. *The Grothendieck completion of $\mathcal{V}^+(\mathcal{A})$ is the trivial group.*

Proof. There is a "standard" (countably-generated) Hilbert $\mathcal{A}$-module $\mathscr{H}_\mathcal{A}$ (see 15.1.7 of [93]), defined by

$$\mathscr{H}_\mathcal{A} := \left\{ (a_k) \in \prod_{k=1}^{\infty} A \,:\, \sum_{k=1}^{\infty} a_k^* a_k \text{ converges in norm in } \mathcal{A} \right\},$$

with component-wise operations and $\mathcal{A}$-valued inner product $\langle (a_k), (b_k) \rangle := \sum a_k^* b_k$. The Kasparov stabilisation theorem (see 15.4.6 of [93] for a proof and original references), says that $\mathscr{H}_\mathcal{A}$ is absorbing in the sense that every countably-generated Hilbert $\mathcal{A}$-module E^+ satisfies $E^+ \oplus \mathscr{H}_\mathcal{A} \cong \mathscr{H}_\mathcal{A}$. Then any virtual module $[E^+ \ominus E'^+]$ in the Grothendieck completion of $\mathcal{V}^+(\mathcal{A})$ satisfies $[E^+ \ominus E'^+] = [(E^+ \oplus \mathscr{H}_\mathcal{A}) \ominus (E'^+ \oplus \mathscr{H}_\mathcal{A})] = [\mathscr{H}_\mathcal{A} \ominus \mathscr{H}_\mathcal{A}] = 0.$ □

Corollary 8.2.3. *$\mathcal{GR}^I(\mathcal{A})$ is isomorphic to the Grothendieck completion of $\mathcal{V}^-(\mathcal{A}) = \mathcal{V}(\mathcal{A})$, i.e., $\mathcal{GR}^I(\mathcal{A}) \cong \mathcal{R}(\mathcal{A}) \cong K_0(\mathcal{A})$.*

Recall that we also have $K_0(\mathcal{A}) \cong \mathcal{SR}(\mathcal{A}) \cong \mathbf{K}_0(\mathcal{A})$ when $\mathcal{A}$ is purely even. Despite this, an element of $K_0(\mathcal{A})$, for a purely even $\mathcal{A}$, has a very different interpretation depending on whether one is in the Type I or Type F situation. For Type F insulators, we can think of the (class of the) virtual module $[E \ominus E'] \in K_0(\mathcal{A})$ as the (class of the) graded $\mathcal{A}$-module $[E \oplus E']$, with the grading operator interpreted as a flattened gapped Hamiltonian $\mathrm{sgn}\,(H)$. Alternatively, a triple $[E, \Gamma_1, \Gamma_2] \in \mathbf{K}_0(\mathcal{A}) \cong K_0(\mathcal{A})$ is a difference-class between two Type F insulators (E, Γ_1) and (E, Γ_2). However, for Type I insulators, a $K_0(\mathcal{A}) \cong \mathcal{R}(\mathcal{A})$ element $[E^- \ominus E'^-]$ should *not* be thought of as an $\mathcal{A}$-module graded by $\mathrm{sgn}\,(H)$. Rather, a gapped Type I Hamiltonian determines a "bundle of negative eigenstates[1]" E^-, which represents the virtual module $[E^- \ominus 0]$ in $\mathcal{GV}^I(\mathcal{A}) \cong K_0(\mathcal{A})$. This is actually more familiar than it looks. For instance, the Hall conductivity in the Integer Quantum Hall Effect is related to the K-theory element $[E^- \ominus 0]$ arising from a Type I Landau Hamiltonian [10]. Here, it is important to realise that there is no commutative Brillouin zone, so our definition of a "noncommutative" Type I insulator has genuine physical relevance. Similarly, the Kane–Mele $\mathbb{Z}_2$ invariant for T-invariant insulators is determined by the negative energy bundle of a Type I insulator, and was studied in [30]. It is important to note that representing $K_0(\mathcal{A})$ by Type I insulators only makes sense for three of the ten possible CT-symmetries.

8.3 Integer Quantum Hall Effect

The abbreviated treatment of the Integer quantum Hall effect (IQHE) in this section is adapted from [10, 98, 35], where the relation to non-commutative geometry and K-theory are emphasised. Some of the ideas in these papers have been generalised to the other complex class of topological insulators in [73], while the appropriate generalisation to the real classes remains an interesting open question.

In a simplified model of the IQHE in two dimensions, the motion of a single free electron with mass m and charge e in the x_1-x_2 plane, and constant magnetic field $\boldsymbol{B} = B\boldsymbol{k}$ in the x_3-direction, can be modelled by the Landau Hamiltonian on $L^2(\mathbb{R}^2, \mathbb{C})$,

$$H_L = |\boldsymbol{P} - e\boldsymbol{A}|^2/2m.$$

Here, we have chosen the vector potential $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\wedge\boldsymbol{X}$, and the operators $\boldsymbol{P}$ and $\boldsymbol{X}$ are the usual quantum mechanical position and momentum operators in two-dimensional Euclidean space. Introducing the two "quasi-momentum" operators $\boldsymbol{K}_\pm \equiv (K^1_\pm, K^2_\pm) := \boldsymbol{P} \mp e\boldsymbol{A}$, we find that the components of $\boldsymbol{K}_+$ commute with those of $\boldsymbol{K}_-$, whereas $[K^1_\pm, K^2_\pm] = \pm i\hbar eB$.

[1] The authors of [84] study the "bundle of negative eigenstates" of gapped insulators, but their $8+2$-fold way has a different interpretation from ours, especially in the $c \not\equiv 1$ cases.

Since $H_L = \frac{1}{2m}[(K_+^1)^2 + (K_+^2)^2]$, its spectrum is that of the corresponding harmonic oscillator, namely, $\{(n+\frac{1}{2})\hbar eB/m \,:\, n \in \mathbb{N}\}$. However, H_L commutes with the components of $\boldsymbol{K}_-$, so each energy level in the spectrum is infinitely degenerate. These are the *Landau levels*, which can be imagined as a sequence of equally-spaced "bands", but not in the sense of Bloch bundle bands — there is no Brillouin torus for instance. Although the vector potential breaks translational symmetry, the Landau Hamiltonian is invariant under *magnetic* translations defined as follows. The vector potential $\boldsymbol{A}$ gives rise to the connection $\nabla_j = \partial_j + \mathrm{i}eA_j/\hbar$, which we exponentiate to the magnetic translations

$$\mathsf{U}(\boldsymbol{a}) = e^{-\mathrm{i}\boldsymbol{a}\cdot\nabla} = e^{-\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{K}_-/\hbar}, \qquad \boldsymbol{a} \in \mathbb{R}^2.$$

The Landau Hamiltonian commutes with these $\mathsf{U}(\boldsymbol{a})$, so we can regard $\mathbb{R}^2$ as a subgroup of symmetries. The representation $\boldsymbol{a} \mapsto \mathsf{U}(\boldsymbol{a})$ is, however, a *projective* one. This arises from the non-zero curvature of the connection, $[\nabla_1, \nabla_2] = B$. More explicitly, defining $\varphi \equiv \varphi(\boldsymbol{a}, \boldsymbol{b}) := e\boldsymbol{B}\cdot(\boldsymbol{a}\wedge\boldsymbol{b})/\hbar$, we have

$$\mathsf{U}(\boldsymbol{a})\mathsf{U}(\boldsymbol{b}) = e^{\frac{1}{2}\mathrm{i}\varphi}\mathsf{U}(\boldsymbol{a}+\boldsymbol{b}) = e^{\mathrm{i}\varphi}\mathsf{U}(\boldsymbol{b})\mathsf{U}(\boldsymbol{a}), \tag{8.1}$$

which shows that $\boldsymbol{a} \mapsto \mathsf{U}(\boldsymbol{a})$ represents $\mathbb{R}^2$ projectively, with cocycle $\sigma(\boldsymbol{a}, \boldsymbol{b}) = e^{\frac{1}{2}\mathrm{i}\varphi}$.

Assuming that the Fermi energy $\mathcal{E}_F$ lies in a gap between the Landau levels (i.e. an integer filling factor), we can understand $\mathrm{sgn}(H_L - \mathcal{E}_F)$ as defining the grading operator in a graded projective unitary representation of $\mathbb{R}^2$, or equivalently, a graded unitary representation of the twisted group algebra $\mathscr{A} := \mathbb{C} \rtimes_{(1,\sigma)} \mathbb{R}^2$, on the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^2, \mathbb{C})$. The projection P_F onto the states with energy less than E_F appears in the Kubo–Greenwood formula for the conductivity matrix, which is

$$M_{kj} = \mathrm{i}(e^2/\hbar)\mathcal{T}(P_F[\partial_j P_F, \partial_k P_F]),$$

where $\mathcal{T}(\cdot)$ is a trace-per-unit-volume. A related formula by Thouless–Kohmoto–Nightingale–Den Nijs [87] holds in a discretised model based on a square lattice and rational flux per unit cell, with the Hall conductivity related to the Chern character of the vector bundle associated with P_F (integrated over a primitive cell of the Brillouin zone $\mathbb{T}^2$). The algebra $\mathscr{A}$ in their model is $C(\mathbb{T}^2)$, which requires the assumption of a rational flux per unit cell in the lattice. To avoid making this unphysical assumption, we need to use a non-commutative Chern character. In any case, the projection P_F defines an element of the K_0-theory group[2] of $\mathscr{A}$, and it is the latter which determines the Hall conductivity. In fact, the projection

[2]For instance, $\mathscr{A} = \mathbb{C} \rtimes_{(1,\sigma)} \mathbb{R}^2$ can be realised explicitly as an algebra of functions on $\mathbb{R}^2$, acting on $L^2(\mathbb{R}^2, \mathbb{C})$ as integral operators. The projection operator onto the lowest Landau level has integral kernel as in Eq. 50 of [10] with $\Omega = \{\star\}$, and has the correct properties to be an element of $\mathscr{A}$.

onto each Landau level defines a projection in $\mathscr{A}$, so the projection onto the first n Landau levels defines a projection in $M_n(\mathscr{A})$. Since there are infinitely many Landau levels, the projection onto the unfilled Landau bands defines an infinite-type projection, and we are in the Type I situation. The formula for the Hall conductivity can thus be understood as a map from the *virtual class* in K-theory defined by the filled "bands" of the Landau Hamiltonian.

A more realistic description of the IQHE should include a model for the effects of impurities and localisation, which are in fact crucial for explaining finer details of the phenomenon. One way to do this is to include in the Hamiltonian a random bounded potential $V_\omega \equiv V_\omega(\boldsymbol{X})$, with ω an element of a compact probability space (Ω, μ) [98]. Elements of $C(\Omega)$ can be viewed as random variables, with the expectation functional $\mathrm{tr}(\cdot)$ providing a trace on $C(\Omega)$. There is then a family of Hamiltonians $H_\omega = H_L + V_\omega$. The magnetic translations do not commute with V_ω in general, but we assume that they give rise to a homeomorphic action on Ω which keeps μ invariant[3], and whose flow is assumed to have certain nice properties (e.g. ergodic and minimal). This can be considered as an action α by automorphisms of the C^*-algebra $C(\Omega)$ which keeps $\mathrm{tr}(\cdot)$ invariant. Instead of the twisted group algebra, which is just the case where $\Omega = \{\star\}$, we form the twisted crossed product $\mathscr{A} = C(\Omega) \rtimes_{(\alpha,\sigma)} \mathbb{R}^2$. In its explicit realisation as functions $f : \mathbb{R}^2 \to C(\Omega)$, there is an induced trace[4] $\mathrm{Tr}(f) := \mathrm{tr}(f(\mathbf{0}))$, as well as the derivations[5] $(\partial_k f)(\boldsymbol{x}) := -\mathrm{i}x_k f(\boldsymbol{x})$, but we can think of these abstractly as well. In the abstract algebraic setting, the Hall conductivity can be defined as

$$M_{21} = \mathrm{i}(e^2/\hbar)\mathrm{Tr}(p[\partial_2 p, \partial_1 p]),$$

where p is a projection in $\mathscr{A}$ determined by the spectral projection P_F onto states with energy (with respect to H_ω) below the Fermi level[6]. This formula only makes mathematical sense for a restricted class of projections. In [10], a Sobolev-type condition is imposed on p, and attention is restricted to a dense subalgebra $\mathscr{A}_0$ of $\mathscr{A}$ which has the same K_0-group. In [98], one assumes that the Fermi level is in a gap of the spectrum of H_ω, and shows that $p \in \mathscr{A}_0$. The formula for the Hall conductivity can be written as a pairing between a cyclic 2-cocycle c_{Tr} on $\mathscr{A}_0$, and the $K_0(\mathscr{A})$ element $[p]$. Namely, c_{Tr} is the trilinear form

$$c_{\mathrm{Tr}}(a_0, a_1, a_2) = \mathrm{Tr}(a_0(\partial_1 a_1 \partial_2 a_2 - \partial_2 a_1 \partial_1 a_2)),$$

[3]Bellissard [10] constructs Ω and its $\mathbb{R}^2$-action as the "hull" of a Schrödinger operator $H_0 = H_L + V_0$, which is "homogeneous" with respect to the projective representation (8.1) of $\mathbb{R}^2$.

[4]This is a trace-per-unit-area, see Proposition 1 of [10].

[5]These are momentum space derivatives, as can be seen after performing a Fourier transform. As position-space operators on $L^2(\mathbb{R}^2, \mathbb{C})$, they are the commutators $-\mathrm{i}[X_k, \cdot]$.

[6]The choice of ω turns out to be inconsequential. Each $\omega \in \Omega$ determines a faithful representation π_ω of $\mathscr{A}$ on $L^2(\mathbb{R}^2, \mathbb{C})$, and they are all equivalent and related by a covariance relation, see [98, 10] for details. The Hamiltonian $H_0 = H + V_0$ is affiliated to $\mathscr{A}$ in the sense that $f(H_0) \in \mathscr{A}$ for any $f \in C_0(\mathbb{R})$.

and the Hall conductivity can be expressed as $M_{21} = \mathrm{i}(e^2/\hbar)c_{\mathrm{Tr}}([p],[p],[p])$. The latter can be understood as a noncommutative Chern character. As with the TKNN invariant defined by the ordinary (commutative) Chern character, it is the K-theory element associated with P_F which enters the formula for the Hall conductivity.

At this stage, the image of c_{Tr} is not yet known, but it can be shown to be integral by relating it to some analytic [10] or topological [98] index. In the latter paper, a different gauge was used in the Landau Hamiltonian, which facilitated writing $\mathscr{A}$ explicitly as a repeated untwisted crossed product of $C(\Omega)$ by $\mathbb{R}$. Expressed in this manner, the Connes–Thom isomorphism (see [15] and Theorem 8.5.1) can be used to identify $K_0(\mathscr{A})$ with $K_0(C(\Omega)) \cong K^0(\Omega)$. Alternatively, we can utilise the stabilisation trick of Packer and Raeburn [66] (Theorem 8.5.2), which says that there is an isomorphism

$$(\mathcal{A} \rtimes_{(\alpha,\sigma)} G) \otimes \mathcal{K} \cong (\mathcal{A} \otimes \mathcal{K}) \rtimes_{(\alpha',1)} G,$$

where $\mathcal{K}$ denotes the compact operators on the Hilbert space $L^2(G)$, and α' is some untwisted action of G on $\mathcal{A} \otimes \mathcal{K}$ determined by (α,σ). When $G = \mathbb{R}^2$ and σ is any cocycle for G, we have the isomorphisms

$$\begin{aligned}
K_0(\mathscr{A}) \cong K_0(\mathscr{A} \otimes \mathcal{K}) &= K_0\left((\mathcal{A} \rtimes_{(\alpha,\sigma)} \mathbb{R}^2) \otimes \mathcal{K}\right) \\
&= K_0\left((\mathcal{A} \otimes \mathcal{K}) \rtimes_{(\alpha',1)} \mathbb{R}^2\right) \\
&\cong K_0\left(\left((\mathcal{A} \otimes \mathcal{K}) \rtimes_{(\alpha_1',1)} \mathbb{R}\right) \rtimes_{(\alpha_2',1)} \mathbb{R}\right) \\
&\cong K_0(\mathcal{A} \otimes \mathcal{K}) \\
&\cong K_0(\mathcal{A}).
\end{aligned}$$

Since $\mathcal{A} = C(\Omega)$ in our case, we have $K_0(\mathscr{A}) \cong K^0(\Omega)$. When there is no disorder, i.e. $\mathcal{A} = C(\{\star\}) \cong \mathbb{C}$, then $[p] \in K_0(\mathscr{A}) = \mathbb{Z}$. In general, the topology of the disorder space complicates matters. For instance, a reasonable choice of Ω is $\mathbb{T}^2$, which already leads to $K_0(\mathscr{A}) = \mathbb{Z} \oplus \mathbb{Z}$. To show that the image of $K_0(\mathscr{A})$ under the pairing with c_{Tr} is an integer, Xia uses the Thom isomophisms in cyclic cohomology [26] to identify this pairing with a pairing involving $K_0(C(\Omega))$ and a 0-cocycle given by integration over (Ω,μ). The latter is then easily seen to be integer-valued.

It is also possible to directly rewrite $\mathscr{A} = \mathcal{A} \rtimes_{(\alpha,\sigma)} \mathbb{R}^2$ as an iterated untwisted crossed product by $\mathbb{R}$, following the example of [98] with some modifications. Recall the standard construction of a twisted crossed product as functions $f : \mathbb{R}^2 \to \mathcal{A}$, equipped with a twisted convolution product and involution, as in equations (3.6a) and (3.6b). Explicity, the product is

$$(f \star g)(\boldsymbol{y}) := \int_{\mathbb{R}^2} f(\boldsymbol{x})\,(g(\boldsymbol{y}-\boldsymbol{x}))^{\boldsymbol{x}}\, e^{\frac{\mathrm{i}}{2}\zeta \boldsymbol{x}\wedge\boldsymbol{y}}\,\mathrm{d}\boldsymbol{x},$$

where $\zeta := eB/\hbar$, while the involution is

$$f^*(\boldsymbol{x}) = (f(-\boldsymbol{x}))^{\boldsymbol{x}}.$$

The map $f \mapsto f'$, where

$$f'(\boldsymbol{x}) \equiv f'(x_1, x_2) := f(\boldsymbol{x}) e^{-\frac{\mathrm{i}}{2}\zeta x_1 x_2}, \tag{8.2}$$

provides an isomorphism from $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$ onto the algebra comprising the same functions $f' : \mathbb{R}^2 \to \mathcal{A}$, but equipped with the modified product and involution (denoted by the same symbols $\star$ and *)

$$\begin{aligned} (f' \star g')(\boldsymbol{y}) &= \int_{\mathbb{R}^2} f'(\boldsymbol{x}) \left(g'(\boldsymbol{y} - \boldsymbol{x})\right)^{\boldsymbol{x}} e^{-\mathrm{i}\zeta x_1 (y_2 - x_2)} \, \mathrm{d}\boldsymbol{x}, & \text{(8.3a)} \\ f'^*(\boldsymbol{x}) &= e^{-\mathrm{i}\zeta x_1 x_2} \left(f'(-\boldsymbol{x})\right)^{\boldsymbol{x}}. & \text{(8.3b)} \end{aligned}$$

The goal is to write this algebra as a crossed product of $\mathcal{A}$ by $\mathbb{R}_2$, followed by a crossed product by $\mathbb{R}_1$, where $\mathbb{R}^2 = \mathbb{R}_1 \times \mathbb{R}_2$. For notational ease, we will drop the primes on f', and write $\mathcal{A} \rtimes_\alpha G \equiv \mathcal{A} \rtimes_{(\alpha,\sigma)} G$ for untwisted crossed products. The first crossed product is simply that due to the restriction $\alpha^{(2)} \equiv \alpha|_{\mathbb{R}_2}$, which is untwisted, since $\sigma|_{\mathbb{R}_2 \times \mathbb{R}_2} = 1$. For the second crossed product, the action $\alpha^{(1)}$ of $\mathbb{R}_1$ on $\mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2$ is defined by

$$(\alpha^{(1)}_{x_1} \tilde{f})(x_2) = e^{-\mathrm{i}\zeta x_1 x_2} (\tilde{f}(x_2))^{(x_1, 0)}, \qquad \tilde{f} \in \mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2, x_1 \in \mathbb{R}_1.$$

This is an untwisted action as well. If $f, g \in (\mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2) \rtimes_{\alpha^{(1)}} \mathbb{R}_1$, we can consider f, g as $\mathcal{A}$-valued functions of $\mathbb{R}_1 \times \mathbb{R}_2$, by writing $f(\boldsymbol{x}) = f(x_1, x_2) = (f(x_1))(x_2)$. Writing $\star_1$ and $\star_2$ for the products in $(\mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2) \rtimes_{\alpha^{(1)}} \mathbb{R}_1$ and $\mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2$ respectively, the product of f and g is

$$\begin{aligned} (f \star_1 g)(\boldsymbol{y}) &= \int_{\mathbb{R}_1} \Big(f(x_1) \star_2 \left(\alpha^{(1)}_{x_1}(g(y_1 - x_1))\right) \Big)(y_2) \, \mathrm{d}x_1 \\ &= \int_{\mathbb{R}_1} \int_{\mathbb{R}_2} f(x_1, x_2) \left(\alpha^{(1)}_{x_1}(g(y_1 - x_1))(y_2 - x_2)\right)^{(0, x_2)} \, \mathrm{d}x_2 \mathrm{d}x_1 \\ &= \int_{\mathbb{R}_1} \int_{\mathbb{R}_2} f(x_1, x_2) \left(g(y_1 - x_1, y_2 - x_2)\right)^{(x_1, x_2)} e^{-\mathrm{i}\zeta x_1 (y_2 - x_2)} \, \mathrm{d}x_2 \mathrm{d}x_1 \\ &= \int_{\mathbb{R}_1 \times \mathbb{R}_2} f(\boldsymbol{x}) \left(g(\boldsymbol{y} - \boldsymbol{x})\right)^{\boldsymbol{x}} e^{-\mathrm{i}\zeta x_1 (y_2 - x_2)} \, \mathrm{d}\boldsymbol{x}, \end{aligned}$$

which is the same as the formula for $f \star g$ in Eq. (8.3a). We have thus exhibited the isomorphism

$$\mathscr{A} \equiv \mathcal{A} \rtimes_{(\alpha,\sigma)} \mathbb{R}^2 \cong (\mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2) \rtimes_{\alpha^{(1)}} \mathbb{R}_1.$$

Under this identification, the trace Tr retains the same expression $\mathrm{Tr}(f) = \mathrm{tr}(f(\mathbf{0}))$, and similarly for the conductivity cocycle.

When $\mathcal{A} = C(\Omega)$ and tr is the 0-cocyle $[\mu]$ of integration over the ergodic measure μ (i.e. the expectation functional described at the beginning of this section), Xia shows that the Thom isomorphisms in cyclic cohomology lead to (Lemma 4.2 of [98])

$$\#_{\alpha^{(1)}}\#_{\alpha^{(2)}}\mu(f_0, f_1, f_2) = -c_{\mathrm{Tr}}(f_0, f_1, f_2), \qquad f_i \in (\mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2) \rtimes_{\alpha^{(1)}} \mathbb{R}_1.$$

With a more careful treatment involving smooth crossed products, and utilising the results of [26], Xia proved his Theorem 3.3, namely,

$$\langle [c_{\mathrm{Tr}}], [p] \rangle = -\left\langle [\mu], \xi^{-1}[p] \right\rangle. \tag{8.4}$$

Here, ξ is the composition of the two Connes–Thom isomorphisms associated to $\alpha^{(1)}, \alpha^{(2)}$,

$$K_0(C(\Omega)) \longrightarrow K_1(C(\Omega) \rtimes_{\alpha^{(2)}} \mathbb{R}_2) \longrightarrow K_0((C(\Omega) \rtimes_{\alpha^{(2)}} \mathbb{R}_2) \rtimes_{\alpha^{(1)}} \mathbb{R}_1).$$

For connected Ω, the pairing on the right-hand-side of (8.4) is certainly integer-valued.

8.3.1 No quantum Hall effect with time reversal symmetry

A necessary condition for a non-trivial quantised Hall conductivity is $K_0(\mathscr{A}) \neq \{0\}$. As found above, this is the case when a uniform magnetic field perpendicular to a 2-dimensional sample is present. Equivalently, the relevant symmetry group is $\mathbb{R}^2$ with non-trivial cocycle φ. In particular, the Landau Hamiltonian does not admit a time-reversal symmetry, which would otherwise be present if the magnetic field is zero. There are various arguments saying that time-reversal symmetry must be "broken" in order to exhibit the quantum Hall effect. We provide an alternative argument, based on K-theoretic considerations, which is applicable very generally.

Suppose the symmetry group is $G = \mathbb{R}^2 \times \{\pm 1\}$, corresponding to translations in two-dimensional space and time-reversal. The cocycle σ is taken to be trivial between elements of $\mathbb{R}^2$, reflecting the absence of a magnetic field. We will also take $\sigma(T, T) = \pm 1$, which reflects the property $\mathsf{T}^2 = +1$ for spin-0 particles and $\mathsf{T}^2 = -1$ for spin-$\frac{1}{2}$ particles. For simplicity, we ignore disorder, and look for Type I insulators compatible with these symmetries. According to the discussion in Section 8.1, and taking $N = \mathbb{R}^2$, we find that the virtual symmetry classes are given by

$$[p] \in \begin{cases} KR^0(\hat{\mathbb{R}}^{0,2}), & \mathsf{T}^2 = +1, \\ KR^{-4}(\hat{\mathbb{R}}^{0,2}), & \mathsf{T}^2 = -1. \end{cases}$$

Here, we are using KR-theory with compact supports, since $\mathbb{C} \rtimes \mathbb{R}^2 \cong C_0(\hat{\mathbb{R}}^2)$. Also, we are treating the Pontryagin dual $\hat{\mathbb{R}}^2$ as a Real space $\hat{\mathbb{R}}^{0,2}$ with involution $\boldsymbol{p} \mapsto -\boldsymbol{p}$. The action of I_T takes a fibre above $\boldsymbol{p}$ antilinearly to the fibre above $-\boldsymbol{p}$, and squares to either

+1 or −1. The construction of the higher KR-theory groups via suspensions by $\hat{\mathbb{R}}^{r,s}$ (see e.g., [3] or Chapter I.10 of [54]) relates these groups to the $KR^{-n}(\star)$ groups, and hence to the $KO^{-n}(\star)$ groups of a point:

$$[p] \in \begin{cases} KO^{-6}(\star) = \{0\}, & \mathsf{T}^2 = +1, \\ KO^{-2}(\star) = \mathbb{Z}_2, & \mathsf{T}^2 = -1. \end{cases}$$

When modelling electrons as spin-0 particles, as was done in the quantum Hall effect, we see that the restoration of time-reversal symmetry by turning off the magnetic field leads to trivial K-theory invariants. On the other hand, by considering (more correctly) electrons as spin-$\frac{1}{2}$ particles, a non-trivial $\mathbb{Z}_2$-invariant arises. This invariant arises, for example, in the quantum *spin* Hall effect (QSHE) [38]. A K-theoretic treatment of a lattice model for the QSHE is given in Section 11 of [30]. There is a similar treatment in [50], although the author's "Q-symmetry" has the meaning of "absence of C-symmetry" under our conventions.

Similarly, we can imagine time-reversal symmetry being broken, but not necessarily by a magnetic field. In other words, $G = \mathbb{R}^2$ with trivial cocycle. The virtual symmetry classes come from $K^0(\hat{\mathbb{R}}^2) \cong K^0(\star) = \mathbb{Z}$, where we have used Bott periodicity, so non-trivial K-theory invariants are possible. A related example is Haldane's model [34], which exhibits a quantised Hall conductivity in the absence of Landau levels[7]. His model is a discretised model, with a non-uniform magnetic field to break time-reversal symmetry, but zero net magnetic flux through each unit cell. There is then a $\mathbb{Z}^2$ translational symmetry, with possibly an additional space inversion symmetry. Since $K^0(\mathbb{C} \rtimes \mathbb{Z}^2) = K_0(C(\mathbb{T}^2)) = \mathbb{Z} \oplus \mathbb{Z}$, non-trivial K-theory invariants can potentially be found. The $\mathbb{Z}$-subgroup generated by the trivial line bundle over $\mathbb{T}^2$ is uninteresting, and, indeed, the first Chern class only recognises the other $\mathbb{Z}$ factor[8]. Upon integrating over the fundamental class, this gives rise to an integer-valued invariant (first Chern number), which is identified as the Hall conductivity. Haldane's model had two levels for each point in the Brillouin torus, so the first Chern class is that of the line bundle of filled states, which for his model, has first Chern number $-1, 0$ or $+1$.

In general, the first Chern number is *additive*, and can be applied to filled bands of any finite rank. For instance, in the TKNN model for the quantum Hall effect with rational flux per unit cell, there is the same (non-projective) $\mathbb{Z}^2$ symmetry. The Brillouin torus makes sense in this case, and the Chern–Kubo formula identifies the Hall conductivity with a suitable multiple of the sum of the Chern numbers of the filled Landau bands, each of

[7] This, and related models, are also called "Chern insulators" in the condensed-matter literature.

[8] The generator of this other subgroup can be identified with $[H] - 1$ under the isomorphism $\tilde{K}^0(\mathbb{T}^2) \cong \tilde{K}^0(S^2)$, where $[H]$ is the Bott element.

which is an integer. Line bundles over the 2-torus can also be highly twisted, by taking tensor products for example, and noting that the Chern character is also multiplicative, and even a *ring* homomorphism $K^0(X) \xrightarrow{\text{ch}} H^{\text{even}}(X)$ for compact X. In any case, only the *virtual class* of the vector bundle is relevant, and the distinction between, e.g., $[L]$ and $[L \oplus 1_n] - [1_n]$ is not too important. To paraphrase Kitaev [50], adding "trivial bands" should not change the topological invariants that we are concerned with.

Remark 8.3.1. Haldane argues that charge-conjugation symmetry needs to be broken as well [34]. This is automatic when we are in the Type I situation, e.g. in the standard setup of the IQHE. In a Type F situation, the absence of charge-conjugation or sublattice symmetry is an *additional* requirement, and one needs to keep track of both the bundle E^+ of unfilled states, and the bundle E^- of filled states. The graded bundle $E^+ \oplus E^-$ is an actual (flattened) symmetry-compatible gapped Hamiltonian, and it determines a class in the relevant super-representation group or difference-group. Suppose a sublattice symmetry is present, so there is a complex-linear isomorphism between E^+ and E^-. Then the virtual bundle $E^+ \ominus E^-$ necessarily has vanishing Chern class.

Remark 8.3.2. Our arguments are model-independent, referring only to the presence or absence of T in the symmetry group G, as well as the (possibly projective) translational symmetries. Similar arguments can be made in a different number of spatial dimensions d, using the abstract machinery of K-theory, see Section 8.5.

8.3.2 Additivity of topological invariants and K-theory

The additivity of the Chern number is actually a very important point, because it tells us that the natural monoidal or group structure underlying a "topological invariant" should be that of the direct sum. A popular, if somewhat mathematically misguided, point of view, is to regard $\tilde{K}^{-n}(X)$ as the "homotopy group" of (based) maps $[X, C_n]$, where C_n is a classifying space for complex K-theory; similarly for R_n in the real case. Then if X is the m-sphere, $[S^m, C_n]$ is identified with $\pi_m(C_n)$. However, it is only possible to form a group from the *set* of homotopy classes of (based) maps $[X, Y]$, when Y is a H-group (homotopy-associative) or when X is a H-cogroup. The two possible ways of defining a composition on $[X, Y]$ from the H-group or H-cogroup structures are a priori different. For example, when X is an m-sphere with $m \geq 1$, which is a H-cogroup (as are all suspensions), the group operation on $[X, Y] = [S^m, Y]$ is the one which appears in the usual definition of $\pi_m(Y)$. On the other hand, when Y is C_n or R_n, which is a loop space and therefore a H-space, there is another possible composition on $[X, Y] = [X, C_n]$. Nevertheless, if both choices are available, the resulting groups are isomorphic (pp. 44 of [83]). Not all X are H-cogroups, but all the C_n are H-groups, so $\tilde{K}^{-n}(X) = [X, C_n]$ is well-defined as a group for all X, with respect to the H-group structure on C_n. In particular, *the group structure on* $K^0(X)$ *is the direct sum of virtual bundles over* X*, whereas the* $\pi_m(C_0) = [X, C_0]$ *interpretation of* $\tilde{K}^0(S^m)$ *only makes sense for* $X = S^m$; yet the case of $X = \mathbb{T}^d$ is most certainly of interest in the study to topological insulators.

Chern-type invariants are really invariants of K-theory, and are additive with respect to the group structure of K-theory. Contrary to the impression that one gets from the literature on topological insulators, K-theory is not (apart from the K^0 case) simply a way to classify vector bundles up to stable equivalence, but is intimately linked to homotopy, *although not necessarily in the sense of homotopy groups* $\pi_m(\cdot)$. While it is true that the *isomorphism* classes of rank-k vector bundles over S^m are classified by the homotopy classes of maps from S^m to a classifying space $BU(k)$, the natural homotopy group multiplication of two maps in $\pi_m(BU(k))$ does not translate straightforwardly to the direct sum of the vector bundles associated with those maps. Indeed, the latter does not make sense if the rank of the vector bundles are fixed. Furthermore, $\pi_0(\cdot)$ does not have a canonical group structure, essentially because S^0 (unlike $S^{m\geq 1}$) is not a suspension, so it is strange to talk about "zeroth homotopy groups". For instance, $\pi_0(C_0)$, is really $[S^0, C_0] = [S^0, \mathrm{Gr}_{\mathbb{C}}(\infty) \times \mathbb{Z}] = \tilde{K}^0(S^0) = K^0(\star) = \mathbb{Z}$, which is non-trivial because of the extra $\mathbb{Z}$ factor in the classifying space. This factor is related to Bott periodicity, and certainly does not arise directly from the connected Grassmannian. In general, one needs to be careful when defining topological phases in zero dimensions, where the "extraordinary" cohomological aspect of K-theory manifests itself.

8.3.3 Euclidean invariance in the quantum Hall effect

The classical Hall effect has Euclidean invariance, and we would like to see whether this persists in the quantum Hall effect. Already, the translations are represented projectively, so we look for a projective representation of $\mathrm{SE}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$. For convenience, we choose units such that $\hbar = e = B = 1$, so $K_-^1 = -\mathrm{i}\partial_1 - X_2/2$ and $K_-^2 = -\mathrm{i}\partial_2 + X_1/2$. A short calculation shows that

$$-\frac{1}{2}\left((K_+^1)^2 + (K_+^2)^2 - (K_-^1)^2 - (K_-^2)^2\right) = \boldsymbol{X} \wedge \boldsymbol{P} \cdot \boldsymbol{k} = -\mathrm{i}(X_1\partial_2 - X_2\partial_1) = L_3 \tag{8.5}$$

is the angular momentum operator about the x_3-axis. The left-hand-side of (8.5) commutes with H due to the commutation relations amongst the $K_\pm^i$. There is a non-projective representation of $\theta \in \mathrm{SO}(2)$ on $L^2(\mathbb{R}^2)$, via $\mathsf{U}(\theta) = e^{\mathrm{i}\theta L_3}$. To check that this is consistent with a projective representation of SE(2), it is easiest to work with the Lie algebras. The commutators between the operators $\mathrm{i}K_-^1, \mathrm{i}K_-^2, \mathrm{i}L_3$ are

$$\begin{aligned}[\mathrm{i}K^1_-, \mathrm{i}K^2_-] &= \mathrm{i}\\ [\mathrm{i}L_3, \mathrm{i}K^1_-] &= -\mathrm{i}K^2_-,\\ [\mathrm{i}L_3, \mathrm{i}K^2_-] &= \mathrm{i}K^1_-.\end{aligned}$$

Therefore, $\mathrm{i}K^1_-, \mathrm{i}K^2_-, \mathrm{i}L_3, \mathrm{i}$ generate a four-dimensional Lie algebra $\mathfrak{c}$, with centre $\mathrm{i}\mathbb{R}$. In comparison, the Euclidean Lie algebra $\mathfrak{e}(2)$ has the Lie brackets

$$\begin{aligned}[t_1, t_2] &= 0\\ [l, t_1] &= -t_2\\ [l, t_2] &= t_1.\end{aligned}$$

The linear map $\mathrm{i}L_3 \mapsto l, \mathrm{i}K^1_- \mapsto t_1, \mathrm{i}K^2_- \mapsto t_2, \mathrm{i} \mapsto 0$ is a surjective Lie algebra homomorphism $\mathfrak{c} \to \mathfrak{e}(2)$, whose kernel is $\mathrm{i}\mathbb{R} \cong \mathfrak{u}(1)$. In other words, there is a central extension of $\mathfrak{e}(2)$ by $\mathfrak{u}(1)$,

$$0 \to \mathfrak{u}(1) \to \mathfrak{c} \to \mathfrak{e}(2) \to 0,$$

which does not split.

To see explicitly that L_3, K^1_-, K^2_- exponentiate to give a projective representation of SE(2), we define the map $\tilde{\mathsf{U}}(\boldsymbol{a}, \theta) := \mathsf{U}(\boldsymbol{a})\mathsf{R}(\theta)$, where $\mathsf{R}(\theta) = e^{\mathrm{i}\theta L_3}$. Note that $\tilde{\mathsf{U}}$ restricts to the magnetic translations for $\mathbb{R}^2$, and the ordinary planar rotations for SO(2). According to Proposition 5.1.1, we should look for a map $\varsigma : \mathbb{R}^2 \times \mathrm{SO}(2) \to \mathrm{U}(1)$, which would appear in the factorisation of a cocycle $\tilde{\sigma}$ for the semidirect product SE(2). Chasing through our definitions,

$$\begin{aligned}\varsigma(\boldsymbol{a}, \theta) &= \tilde{\sigma}((0, \theta), (\boldsymbol{a}, 0))\\ &= \tilde{\mathsf{U}}(\boldsymbol{a}^\theta, \theta)\mathsf{R}^{-1}(\theta)\mathsf{U}^{-1}(\boldsymbol{a})\\ &= \mathsf{U}(\boldsymbol{a}^\theta)\mathsf{U}^{-1}(\boldsymbol{a})\\ &= e^{-\mathrm{i}\boldsymbol{A}\cdot(\boldsymbol{a}^\theta - \boldsymbol{a})}.\end{aligned}$$

A short computation verifies that ς is a bicharacter in the sense of (5.9a)–(5.9b),

$$\begin{aligned}\varsigma(\boldsymbol{a}, \theta_1 + \theta_2) &= \varsigma(\boldsymbol{a}^{\theta_2}, \theta_1)\varsigma(\boldsymbol{a}, \theta_2),\\ \varsigma(\boldsymbol{a}_1 + \boldsymbol{a_2}, \theta) &= \varsigma(\boldsymbol{a}_1, \theta)\varsigma(\boldsymbol{a}_2, \theta).\end{aligned}$$

The U(1)-valued function $\tilde{\sigma}$ defined by

$$\tilde{\sigma}((\boldsymbol{a}_1, \theta_1), (\boldsymbol{a}_2, \theta_2)) := \varsigma(\boldsymbol{a}_2, \theta_1)\sigma(\boldsymbol{a}_1, \boldsymbol{a}_2^{\theta_1}),$$

is thus a cocycle for $\mathrm{SE}(2) = \mathbb{R}^2 \rtimes \mathrm{SO}(2)$, and $\tilde{\mathsf{U}}$ is a projective unitary representation of $(\mathrm{SE}(2), \tilde{\sigma})$.

Remark 8.3.3. The rotational symmetry of the Landau Hamiltonian makes it clear that the choice of orthogonal axes $\mathbb{R}_1$ and $\mathbb{R}_2$ in the isomorphism (8.2) is irrelevant. This is a manifestation of the gauge invariance in the choice of vector potential for the magnetic field. A distinguished choice of $\mathbb{R}_2$ might be singled out, for instance, by an edge of a macroscopic Hall sample. Then, the perpendicular $\mathbb{R}_1$-direction (assuming there is a fixed orientation on $\mathbb{R}^2$), is also determined.

8.4 Relating bulk and edge conductivities

An interesting way to relate bunk and edge conductivities, and to demonstrate the quantisation of the latter from the quantisation of the former, is described in the works of Kellendonk et al [47, 49, 48]. One expects that the algebra of observables in the bulk, $\mathscr{A} = (\mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2) \rtimes_{\alpha^{(1)}} \mathbb{R}_1$, differs from the algebra $\mathscr{J}$ associated to an edge. To relate $\mathscr{A}$ and $\mathscr{J}$ in a "continuous" manner, we follow the approach of [47, 48].

Suppose that there is an (infinite) edge along the $\mathbb{R}_2$-direction, which may be anywhere in the $\mathbb{R}_1$-direction. The edge algebra $\mathscr{J}$ may be expected to be related to $\mathcal{B} \coloneqq \mathcal{A} \rtimes_{\alpha^{(2)}} \mathbb{R}_2$, at least stably. We make an allowance for "infinitesimal" operators — compact operators in non-commutative parlance — in the perpendicular $\mathbb{R}_1$-direction, which describe the x_1-coordinate of the edge as well as its translational covariance. The isomorphism (e.g. using Takai duality, although this is not necessary)

$$\mathcal{K}(L^2(\mathbb{R}_1)) \cong (\mathbb{C} \rtimes_{\hat{\varrho}} \hat{\mathbb{R}}_1) \rtimes_{\varrho} \mathbb{R}_1 \cong C_0(\mathbb{R}_1) \rtimes_{\varrho} \mathbb{R}_1 = S\mathbb{C} \rtimes_{\varrho} \mathbb{R}_1,$$

where ϱ is the translational action of $\mathbb{R}_1$ and $S(\cdot) = C_0(\mathbb{R}_1, (\cdot)) \cong C_0(\mathbb{R}_1) \otimes (\cdot)$ is the usual suspension operation, gives us what we need. In other words, we take

$$\begin{aligned} \mathscr{J} &= \mathcal{K}(L^2(\mathbb{R}_1)) \otimes \mathcal{B} \\ &\cong (C_0(\mathbb{R}_1) \rtimes_{\varrho} \mathbb{R}_1) \otimes \mathcal{B} \\ &\cong S\mathcal{B} \rtimes_{\varrho \otimes 1} \mathbb{R}_1 \\ &\cong S\mathcal{B} \rtimes_{\varrho \otimes \alpha^{(1)}} \mathbb{R}_1. \end{aligned}$$

The last isomorphism takes $f \in C_c(\mathbb{R}_1, S\mathcal{B})$ to the function $\tilde{f}$, defined by

$$\tilde{f}(x)(s) = \alpha_x^{(1)} \left(f(x)(s) \right),$$

and is needed in order to exhibit the total algebra $\mathscr{E}$ of the system-with-boundary as a Weiner–Hopf extension of $\mathscr{A}$ by $\mathscr{J}$. Let us write $\tilde{\mathbb{R}}_1 = \mathbb{R}_1 \cup \{+\infty\}$, and extend the translation action ϱ to $\tilde{\mathbb{R}}_1$ with $\{+\infty\}$ as a fixed point. The cone of $\mathcal{B}$ can then be defined as $C\mathcal{B} \coloneqq C_0(\tilde{\mathbb{R}}_1, \mathcal{B}) \cong C_0(\tilde{\mathbb{R}}_1) \otimes \mathcal{B}$, then the suspension $S\mathcal{B}$ is the kernel of the evaluation map $\mathrm{ev}_\infty : C\mathcal{B} \twoheadrightarrow \mathcal{B}$ at $+\infty$. Defining $\mathscr{E} := C\mathcal{B} \rtimes_{\varrho\otimes\alpha^{(1)}} \mathbb{R}_1$ and recalling that $\mathscr{A} = \mathcal{B} \rtimes_{\alpha^{(1)}} \mathbb{R}_1$, we see that $\mathscr{J}$ is the kernel of the evaluation (of the cone/suspension variable) map at $+\infty$. The exact sequence of algebras

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{E} \overset{\mathrm{ev}_\infty}{\twoheadrightarrow} \mathscr{A} \longrightarrow 0 \tag{8.6}$$

is therefore exhibited as a Weiner–Hopf extension [75]

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{B} \longrightarrow C\mathcal{B} \rtimes_{\varrho\otimes\alpha^{(1)}} \mathbb{R}_1 \overset{\mathrm{ev}_\infty}{\twoheadrightarrow} \mathcal{B} \rtimes_{\alpha^{(1)}} \mathbb{R}_1 \longrightarrow 0. \tag{8.7}$$

According to Kellendonk et al., the total algebra $\mathscr{E}$ as defined above is the algebra of observables for the continuous family, over $s \in \tilde{\mathbb{R}}_1$, of (disordered) quantum Hall systems confined to the half-spaces[9] $(-\infty, s]\times\mathbb{R}_2$. Each $s \in \tilde{\mathbb{R}}_1$ determines, canonically, a non-faithful representation π_s of $\mathscr{E}$ on $L^2(\mathbb{R}^2)$. Their direct sum is, however, faithful. The restriction of the bulk Hamiltonian H_0 to the half-space $(-\infty, s]\times\mathbb{R}_2$ with Dirichlet boundary conditions is denoted by $H^{(s)}$, with $H^{(\infty)} \equiv H_0$. Then the family of Hamiltonians $\{H^{(s)}\}_{s\in\tilde{\mathbb{R}}_1}$ is affiliated to $\mathscr{E}$ in the sense that there exists, for each $f \in C_0(\mathbb{R})$, an element $F \in \mathscr{E}$ such that $f(H^{(s)}) = \pi_s(F)$. The family of restricted Hamiltonians $(H^{(s)} : s \in \mathbb{R}_1)$ is thereby related to the bulk Hamiltonian in a continuous manner.

From a mathematical viewpoint, exhibiting $\mathscr{E}$ as an extension of the bulk algebra by the edge algebra allows one to relate the K-groups of $\mathscr{A}$ with those of $\mathscr{J}$, via the six-term exact sequence,

$$\begin{array}{ccccc} K_0(\mathscr{J}) & \longrightarrow & K_0(\mathscr{E}) & \longrightarrow & K_0(\mathscr{A}) \\ \uparrow \mathrm{ind} & & & & \downarrow \exp \\ K_1(\mathscr{A}) & \longleftarrow & K_1(\mathscr{E}) & \longleftarrow & K_1(\mathscr{J}) \end{array}.$$

Here, the horizontal maps are the induced maps in K-theory corresponding to the homomorphisms in (8.6), while the vertical maps are the boundary maps in K-theory. Furthermore, $\mathscr{E}$ is the crossed product of the contractible algebra $C\mathcal{B}$ by $\mathbb{R}$, so after using the Connes–Thom isomorphism (Theorem 8.5.1), we obtain $K_0(\mathscr{E}) = 0 = K_1(\mathscr{E})$. It follows that the boundary maps are actually *isomorphisms* in this case. Recall that the formula for the bulk Hall conductivity involves a pairing between the 2-cocycle c_{Tr} and the $K_0(\mathscr{A})$ element corresponding to the Fermi projection $[p]$, provided the Fermi level $\mathcal{E}_F$ lies in a gap of the bulk Hamiltonian H_0. The strategy of [47] is to identify the edge conductivity with the pairing between the "dual" cocycle on $\mathscr{J}$, and the image $[u]$ of $[p]$ in $K_1(\mathscr{J})$ under the boundary isomorphism exp.

An explicit formula for $[u] = \exp([p])$ can be given as follows. Let $\Delta = [a, b]$ be an interval containing $\mathcal{E}_F$, lying in a gap of the spectrum of H_0. Thus $\chi_\Delta(H_0) = 0$, while for

[9]For $s = +\infty$, the boundary is at infinity, so we have the algebra of a quantum Hall system *without* boundary.

$s \in \mathbb{R}_1$, the operator $\chi_\Delta(H^{(s)})$ is the projection onto the edge states with energies in the interval $[a, b]$. Define the bounded continuous function $v : \mathbb{R} \to \mathbb{C}$ by

$$v(t) - 1 = \chi_\Delta(t) \left(\exp\left(-\frac{2\pi \mathrm{i}(t-a)}{b-a} \right) - 1 \right).$$

The function $v - 1$ vanishes at infinity, so there exists an element $u - 1 \in \mathscr{E}$ such that $\pi_s(u) = v(H^{(s)})$. Kellendonk shows that u belongs to (the unitisation of) $\mathscr{J}$, and so defines an element $[u]$ in $K_1(\mathscr{J})$ which is precisely $\exp([p])$.

To define the cocycle "dual" to c_{Tr}, Kellendonk considers "higher n-traces", which are constructed out of an ordinary trace $\mathscr{T}$ on an algebra $\mathcal{A}$, together with n commuting derivations $\{\partial_k\}_{k=1}^n$ which are each annihilated by the trace. Together, they define, via Connes' pairing, two additive functionals. For even n and $[p] \in K_0(\mathcal{A})$, the pairing is, up to some constants,

$$\langle (\mathscr{T}; \partial_1, \ldots, \partial_n), [p] \rangle \coloneqq \sum_{\sigma \in S_n} \mathscr{T}\left(p \partial_{\sigma(1)}(p) \ldots \partial_{\sigma(n)}(p) \right),$$

while for odd n and $[u] \in K_1(\mathcal{A})$, the pairing is

$$\langle (\mathscr{T}; \partial_1, \ldots, \partial_n), [u] \rangle \coloneqq \sum_{\sigma \in S_n} \mathscr{T}\left((u^* - 1) \partial_{\sigma(1)}(u) \partial_{\sigma(2)}(u^*) \ldots \partial_{\sigma(n)}(u) \right).$$

The trace and derivations are extended to matrix algebras over $\mathcal{A}$ in order for these formulae to make sense. As an example, the bulk Hall conductivity is a 2-trace on $\mathscr{A}$ defined by $(\mathrm{Tr}; \partial_1, \partial_2)$, where the derivations[10] are $\partial_k = -\mathrm{i}[X_k, \cdot]$. For the edge algebra $\mathscr{J}$, we define a trace $\mathscr{T}$ on tensors $f_\perp \otimes f_\parallel \in \mathcal{K} \otimes \mathcal{B} = \mathscr{J}$ by the product

$$\mathscr{T}(f_\perp \otimes f_\parallel) \coloneqq tr(f_\perp) \mathrm{tr}(f_\parallel(0)).$$

Here, tr is the standard (possibly infinite) trace on the compact operators. The first factor can be interpreted as an average (per-unit-length) over the position s of the edge[11], while the second factor is a trace-per-unit-length. The 1-cocycle dual to $(\mathrm{Tr}; \partial_1, \partial_2)$ under the boundary map exp (in the sense of preserving the Connes' pairing) turns out to be $-(\mathscr{T}; \partial_2)$ [48, 47]. In other words, there is an equality

$$\langle (\mathrm{Tr}; \partial_1, \partial_2), [p] \rangle = -\frac{1}{2\pi} \langle (\mathscr{T}; \partial_2), [u] \rangle, \tag{8.8}$$

where p corresponds to the Fermi projection and $[u] = \exp([p])$.

[10] $\mathscr{A}$ is represented on $L^2(\mathbb{R}^2)$ here.

[11] The isomorphism $\mathcal{K}(L^2(\mathbb{R}_1)) \cong C_0(\mathbb{R}_1) \rtimes_\varrho \mathbb{R}_1$ can be described explicitly by taking an element $f \in C_c(\mathbb{R}_1, C_0(\mathbb{R}_1))$ to the kernel $\tilde{f}(x, s) = f(s - x)(s)$ of an integral operator on $L^2(\mathbb{R}_1)$. Chasing through the isomorphisms, one finds that the trace of such an integral operator in $\mathcal{K}$ corresponds to $\int \tilde{f}(s, s)\, \mathrm{d}s = \int f(0)(s)\, \mathrm{d}s$.

It remains to give a formula for the edge "conductivity[12]" in terms of $[u], \mathscr{T}, \partial_2$. This is defined to be the pairing $-\frac{1}{2\pi}\frac{\mathrm{i}e^2}{\hbar}\langle(\mathscr{T};\partial_2),[u]\rangle$, which can be understood as the average of the operator (see, for instance, Theorem 3 in [49]) $\frac{1}{|\Delta|}\frac{\mathrm{i}e}{\hbar}\chi_\Delta(H^{(s)})[H^{(s)},Q_2]\chi_\Delta(H^{(s)})$. This operator is $\frac{1}{|\Delta|}$ times of the current operator in the x_2-direction, restricted to the edge states with energies lying in Δ. The duality expressed in (8.8) is then the statement that the bulk and edge conductivities coincide. The chiral nature of the edge current is also apparent from the freedom in choosing the x_1, x_2-axes. If we had considered an edge on the left instead, the edge current would run in the opposite direction.

8.4.1 Boundary maps, Connes–Thom isomorphisms, and suspensions

In the discussion of the Weiner–Hopf extension, the Connes–Thom isomorphism was used, along with Takai duality[13] and Bott periodicity, to establish that the two boundary maps are isomorphisms. On the other hand, the isomorphism property of the index map for the Weiner–Hopf extension (along with Takai duality) is precisely the Connes–Thom isomorphism for $\star = 1$. Indeed, the Connes–Thom isomorphism can be proven by demonstrating that ind is an isomorphism, and for this, the surjectivity of ind actually suffices, see Chapter 10.9 of [11]. Suspension, which commutes with crossed products, shows that all the boundary maps in the long exact sequence are also isomorphisms. The map exp is actually a boundary map followed by the Bott periodicity isomorphism, so it is an isomorphism as well. Altogether, these ideas give a link between $\mathbb{R}$-covariance in an additional spatial dimention, and a *decrease* by one degree in K-theory. The definition of higher K-groups via suspension, namely, $K_{\star+1}(\mathcal{A}) = K_\star(\mathcal{S}\mathcal{A})$, shows that suspension *increases* the K-theory degree by one. Complex K-theory is 2-periodic, so taking the suspension and taking the crossed product with $\mathbb{R}$ produce the same effect on the K-theory groups. In particular, if the $\mathbb{R}$-action is trivial, there is the isomorphism

$$\mathcal{A}\rtimes_{\mathrm{id}}\mathbb{R}\cong C_0(\hat{\mathbb{R}},\mathcal{A}) \coloneqq \bar{S}\mathcal{A}\cong C_0(\mathbb{R},\mathcal{A})\equiv S\mathcal{A}, \tag{8.9}$$

where the middle isomorphism involves choosing an isomorphism $\hat{\mathbb{R}}\cong\mathbb{R}$. It is useful to distinguish a suspension variable in $\mathbb{R}$ from one in $\hat{\mathbb{R}}$, which is why we have defined $\bar{S}\mathcal{A}\coloneqq C_0(\hat{\mathbb{R}},\mathcal{A})\cong\mathcal{A}\rtimes_{\mathrm{id}}\mathbb{R}$ in (8.9).

For complex algebras $\mathcal{A}$, the two suspensions $S\mathcal{A}$ and $\bar{S}\mathcal{A}$ are isomorphic. However, an important difference arises in the real and Real cases. In general, if $\mathcal{A}$ is a real/complex C^*-algebra, we write $C_0(\mathbb{R},\mathcal{A})$ for the real/complex C^*-algebra of continuous functions

[12]Conductivity may not be the most appropriate term, since no electric field or perturbation theory is actually used to compute the edge currents.

[13]Actually, we only used the result $\mathcal{K}(L^2(\mathbb{R}))\cong C_0(\mathbb{R})\rtimes_\varrho\mathbb{R}$, which continues to hold for real suspensions, i.e., $\mathcal{K}(L^2(\mathbb{R},\mathbb{R}))\cong C_0(\mathbb{R},\mathbb{R})\rtimes_\varrho\mathbb{R}$.

$f : \mathbb{R} \to \mathcal{A}$ vanishing at infinity, with the supremum norm. For instance, $C_0(\mathbb{R}, \mathbb{C})$ (resp. $C_0(\mathbb{R}, \mathbb{R})$) is the complex (resp. real) C^*-algebra of continuous complex-valued (resp. real-valued) functions on $\mathbb{R}$ vanishing at infinity. $C_0(\mathbb{R}, \mathbb{C})$ is obviously the complexification of $C_0(\mathbb{R}, \mathbb{R})$, so it is naturally a Real C^*-algebra with Real structure given by point-wise complex conjugation. It has a second natural Real structure, given by $\bar{f}(x) = \overline{f(-x)}$. So $C_0(\mathbb{R}, \mathbb{C}) = C_0(\mathrm{i}\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ where

$$C_0(\mathrm{i}\mathbb{R}) \coloneqq \{f \in C_0(\mathbb{R}, \mathbb{C}) \,:\, f(x) = \overline{f(-x)}\}. \tag{8.10}$$

More generally, if $\mathcal{A}$ is a real C^*-algebra, its complexification $\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$ has a natural involution $\overline{(\cdot)}$ which fixes $\mathcal{A}$, and we define

$$C_0(\mathrm{i}\mathbb{R}, \mathcal{A}) \coloneqq \{f \in C_0(\mathbb{R}, \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}) \,:\, f(x) = \overline{f(-x)}\}.$$

For a real C^*-algebra, the two suspensions $S\mathcal{A}$ and $\bar{S}\mathcal{A}$ are defined to be

$$\begin{aligned} S\mathcal{A} &\coloneqq C_0(\mathbb{R}, \mathbb{R}) \otimes \mathcal{A} \cong C_0(\mathbb{R}, \mathcal{A}), \\ \bar{S}\mathcal{A} &\coloneqq \quad C_0(\mathrm{i}\mathbb{R}) \otimes \mathcal{A} \cong C_0(\mathrm{i}\mathbb{R}, \mathcal{A}). \end{aligned}$$

The complexifications of $S\mathcal{A}$ and $\bar{S}\mathcal{A}$ are both isomorphic as complex C^*-algebras to $S(\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}) = C_0(\mathbb{R}, \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C})$, but not as Real C^*-algebras. Also, while a crossed product of a complex C^*-algebra by $\mathbb{R}$ with the trivial action is isomorphic to its suspension operation, $\mathcal{A} \rtimes_{\mathrm{id}} \mathbb{R} \cong S\mathcal{A}$, we need to use the modified suspension $\bar{S}$ in the real case. To see this, recall that $\mathbb{R} \rtimes_{\mathrm{id}} \mathbb{R}$ is generated by real-valued functions $\mathbb{R} \to \mathbb{R}$ with the convolution product. Fourier transformation turns such functions into complex-valued functions on $\mathbb{R}$ satisfying $f(x) = \overline{f(-x)}$, with the pointwise product, i.e. $\mathbb{R} \to \mathbb{R} \cong C_0(\mathrm{i}\mathbb{R}) \cong \bar{S}\mathbb{R}$. Replacing the algebra $\mathbb{R}$ by a general real algebra $\mathcal{A}$ gives $\mathcal{A} \rtimes_{\mathrm{id}} \mathbb{R} \cong C_0(\mathrm{i}\mathbb{R}, \mathcal{A}) = \bar{S}\mathcal{A}$.

The higher K-groups for a real C^*-algebra may also be defined through suspensions,

$$K_n(\mathcal{A}) \coloneqq K_0(C_0(\mathbb{R}^n, \mathbb{R}) \otimes \mathcal{A}) = K_0(S^n \mathcal{A}).$$

We can actually define the doubly-indexed K-theory groups,

$$K_{r,s}(\mathcal{A}) \coloneqq K_0(S^r \bar{S}^s \mathcal{A}),$$

which have a $(1,1)$-periodicity (see (A.6))

$$K_{r,s}(\mathcal{A}) \cong K_{r-s,0}(\mathcal{A}) =: K_{r-s}(\mathcal{A}), \tag{8.11}$$

where $r - s$ is taken modulo 8. These double-indexed groups are really special cases of KKO-theory groups, see [80]. The analogy with the suspension definition of the higher topological KR-groups should be noted: $KR^{r,s}(X) := KR^0(X \times \mathbb{R}^{r,s})$, where the Real space $\mathbb{R}^{r,s}$ is the space $\mathbb{R}^{r+s}$ with involution $\overline{(\mathbf{x},\mathbf{y})} = (\mathbf{x}, -\mathbf{y})$.

Now, consider a non-trivial $\mathbb{R}$-action on a real algebra $\mathcal{A}$. We obtain a Weiner–Hopf extension for real algebras in a similar fashion to the derivation of (8.7) for complex algebras. The $\mathbb{R}$-action is extended, as before, to an action $\varrho \otimes \alpha$ on $S\mathcal{A}$ and $C\mathcal{A}$, where ϱ is the translation of the suspension/cone variable with fixed point at $\{+\infty\}$. This extension gives rise to the boundary maps of the corresponding long exact sequence,

$$K_{r+1,s}(\mathcal{A} \rtimes_\alpha \mathbb{R}) \xrightarrow{\partial_r} K_{r,s}(S\mathcal{A} \rtimes_{\varrho\otimes\mathrm{id}} \mathbb{R}) \cong K_{r,s}(\mathcal{A} \otimes \mathcal{K}(L^2(\mathbb{R}))) \cong K_{r,s}(\mathcal{A}). \tag{8.12}$$

There is a real version of the Connes–Thom isomorphisms[14],

$$K_{n+1}(\mathcal{A} \rtimes_\alpha \mathbb{R}) \cong K_n(\mathcal{A}). \tag{8.13}$$

Using (8.13) and (8.11), we see that the boundary maps ∂_r are isomorphisms. In particular, if the $\mathbb{R}$-action is trivial, the left-hand side of (8.12) is just $K_{q+1}(\bar{S}\mathcal{A}) \cong K_q(S\bar{S}\mathcal{A}) \cong K_q(\mathcal{A})$, which is immediate from the definition of the K-groups and their $(1,1)$-periodicity, and does not require the Connes–Thom isomorphism.

Heuristically, the bulk algebra $\mathscr{A}$ has an extra $\mathbb{R}$ symmetry compared to the algebra $\mathcal{A}$ of a cross-section/boundary at a particular point in the perpendicular direction $\mathbb{R}_\perp$. Therefore, we expect that $\mathscr{A} = \mathcal{A} \rtimes_\alpha \mathbb{R}_\perp$. The precise choice of cross-section/boundary should not matter, up to isomorphism, and is somewhat arbitrary in the bulk. On the other hand, if a cross-section is actually a boundary, its spatial position is a relevant classical observable, as are all (reasonable) functions of it. The suspension $S\mathcal{A} = C_0(\mathbb{R}_\perp, \mathcal{A})$ encodes this information, and the extension of α to $S\mathcal{A} \rtimes_{\varrho\otimes\alpha} \mathbb{R}_\perp$ to include translation in the suspension variable encodes the required covariance. The algebra $S\mathcal{A} \rtimes_{\varrho\otimes\alpha} \mathbb{R}_\perp$ is actually isomorphic to $\mathcal{A} \otimes \mathcal{K}(L^2(\mathbb{R}_\perp))$, and consists of observables "localised" on a cross-section/boundary.

8.5 A unified explanation for the periodicities in the classification of gapped topological phases

In Section 5.4.2, we showed that the graded PUA-reps of each of the ten choices of (A, σ) are in 1–1 correspondence with the graded representations of a real or complex Clifford algebra according to Table 5.5. The super-representation group of $\mathbb{C} \rtimes_{(\alpha,\sigma)} A$ and that of its corresponding Clifford algebra coincide, and is precisely a K-theory group of a point, according to equations (6.6a)–(6.6b). Therefore, we say that the classes of 0-dimensional gapped topological phases compatible with the symmetries specified by (A, σ), modulo

[14] For a proof, see Corollary 2.5.3 of [80]. Kasparov has even more general KK-theoretic results in [45, 46].

n	Cartan label	Generators of A	C^2	T^2	$Cl_{0,n}$ or $\mathbb{C}l_n$	$K_n(\mathbb{R})$ or $K_n(\mathbb{C})$
0	AI	T		$+1$	$Cl_{0,0}$	$\mathbb{Z}$
1	BDI	C, T	$+1$	$+1$	$Cl_{0,1}$	$\mathbb{Z}_2$
2	D	C	$+1$		$Cl_{0,2}$	$\mathbb{Z}_2$
3	$DIII$	C, T	$+1$	-1	$Cl_{0,3}$	0
4	AII	T		-1	$Cl_{0,4}$	$\mathbb{Z}$
5	CII	C, T	-1	-1	$Cl_{0,5}$	0
6	C	C	-1		$Cl_{0,6}$	0
7	CI	C, T	-1	$+1$	$Cl_{0,7}$	0
0	A	N/A	N/A		$\mathbb{C}l_0$	$\mathbb{Z}$
1	$AIII$	S	$\mathsf{S}^2 = +1$		$\mathbb{C}l_1$	0

Table 8.1: Classification of 0-dimensional gapped topological phases and their difference-classes. The next-to-last column lists the Clifford algebra $Cl_{0,n}$ or $\mathbb{C}l_n$ in the graded Morita class of the algebra $\mathcal{A}$ associated with (A, σ). The K-theory group in the last column is isomorphic to the super-representation group $\mathcal{SR}(\mathcal{A})$, as well as the difference-group $\mathbf{K}_0(\mathcal{A})$. The second column lists the Cartan label of the symmetric space of quantum mechanical time evolutions with corresponding C, T symmetries [77, 37].

topologically trivial phases, is given by $K^{-n}(\star) \cong K_n(\mathbb{C})$ or $KO^{-n}(\star) \cong K_n(\mathbb{R})$, according to Table 8.1. The same K-theory groups also classify the differences of (A, σ)-compatible gapped Hamiltonians, due to the isomorphisms $\mathbf{K}_0(Cl_{r,s}) \cong K_{s-r}(\mathbb{R})$ and $\mathbf{K}_0(\mathbb{C}l_n) \cong K_n(\mathbb{C})$.

In the literature, it is often claimed [50, 84, 77] that the K-theoretic classification of gapped topological phases in d spatial dimensions is the same as that in 0 dimensions, except for a shift in the K-theory index by d. This is typically shown in a heuristic manner, using specific models with Hamiltonians parameterised by $\mathbb{R}^d$. We show that the dimension shifts can be explained in a robust and rigorous manner using several powerful results in the K-theory of crossed product C^*-algebras.

Theorem 8.5.1 (Connes–Thom isomorphism [15, 46, 80])**.** *Let $(\mathbb{R}, \mathcal{A}, \alpha)$ be a C^*-dynamical system, with $\mathcal{A}$ a real or complex (ungraded) C^*-algebra. Then $K_n(\mathcal{A} \rtimes_{(\alpha,1)} \mathbb{R}) \cong K_{n-1}(\mathcal{A})$.*

Remarkably, Theorem 8.5.1 holds for any continuous $\alpha : \mathbb{R} \to \mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$.

Theorem 8.5.2 (Packer–Raeburn stabilisation trick [66])**.** *Let $(G, \mathcal{A}, \alpha, \sigma)$ be a twisted C^*-dynamical system, and let $\mathcal{K}$ denote the compact operators on the Hilbert space $L^2(G)$. There is an isomorphism*

$$(\mathcal{A} \rtimes_{(\alpha,\sigma)} G) \otimes \mathcal{K} \cong (\mathcal{A} \otimes \mathcal{K}) \rtimes_{(\alpha',1)} G,$$

for some untwisted action α' of G on $\mathcal{A} \otimes \mathcal{K}$.

Corollary 8.5.3 (Dimension shifts)**.** *Let* $(\mathbb{R}^d, \mathcal{A}, \alpha, \sigma)$ *be a twisted* C^**-dynamical system. Then* $K_n(\mathcal{A} \rtimes_{(\alpha,\sigma)} \mathbb{R}^d) \cong K_{n-d}(\mathcal{A})$.

Proof. Iterating Theorem 7.1.1 yields

$$\begin{aligned}
\mathcal{A} \rtimes_{(\alpha,\sigma)} \mathbb{R}^d &\cong (\mathcal{A} \rtimes_{(\alpha_1,\sigma_1)} \mathbb{R}) \rtimes_{(\alpha_2,\sigma_2)} \mathbb{R}^{d-1} \\
&\cong \big((\mathcal{A} \rtimes_{(\alpha_1,\sigma_1)} \mathbb{R}) \rtimes_{(\alpha_2,\sigma_2)} \mathbb{R}\big) \rtimes_{(\alpha_3,\sigma_3)} \mathbb{R}^{d-2} \\
&\vdots \\
&\cong \big(\ldots \big((\mathcal{A} \rtimes_{(\alpha_1,\sigma_1)} \mathbb{R}) \rtimes_{(\alpha_2,\sigma_2)} \mathbb{R}\big) \rtimes_{(\alpha_3,\sigma_3)} \ldots\big) \rtimes_{(\alpha_d,\sigma_d)} \mathbb{R},
\end{aligned}$$

for some sequence of twisting pairs (α_i, σ_i), $i = 1, \ldots, d$. Then, Theorem 8.5.2 says that we can untwist each of the iterated crossed products, provided we stabilise the algebras, i.e., tensor by $\mathcal{K}$. Since K-theory is invariant under stabilisation, we have

$$\begin{aligned}
K_n(\mathcal{A} \rtimes_{(\alpha,\sigma)} \mathbb{R}^n) &\cong K_n\Big(\Big(\ldots\Big((\mathcal{A} \rtimes_{(\alpha'_1,1)} \mathbb{R}) \rtimes_{(\alpha'_2,1)} \mathbb{R}\Big) \rtimes_{(\alpha'_3,1)} \ldots\Big) \rtimes_{(\alpha'_d,1)} \mathbb{R}\Big) \\
&\cong K_{n-d}(\mathcal{A}),
\end{aligned}$$

so the effect on K-theory is to lower the index by d. □

Suppose (G, c, ϕ, σ) describes the symmetries of a physical system, $G = \ker(\phi, c) \times A \equiv G_0 \times A$, and $\sigma(g_0, y) = 1 = \sigma(y, g_0)$ for all $g_0 \in G_0$, $y \in A$. In the real case, the graded twisted group C^*-algebra takes the form $\mathcal{A} = \mathcal{A}^{\mathrm{ev}} \hat{\otimes} Cl_{r,s}$ where $\mathcal{A}^{\mathrm{ev}} = \mathbb{R} \rtimes_{(1,\sigma)} G_0$; in the complex case, it is $\mathcal{A} = \mathcal{A}^{\mathrm{ev}} \hat{\otimes} \mathbb{C}l_n$, where $\mathcal{A}^{\mathrm{ev}} = \mathbb{C} \rtimes_{(1,\sigma)} G_0$. In Section 6.4, we found the difference-group of $\mathcal{A}$ to be an ordinary K-theory group, $\mathbf{K}_0(\mathcal{A}) \cong K_{s-r}(\mathcal{A}^{\mathrm{ev}})$ (real case) or $\mathbf{K}_0(\mathcal{A}) \cong K_n(\mathcal{A}^{\mathrm{ev}})$ (complex case). Suppose there are now d additional translational symmetries, $\tilde{G} = G \times \mathbb{R}^d$, so that $\tilde{G} = (G_0 \times \mathbb{R}^d) \times A =: \tilde{G}_0 \times A$. We may assume that the homomorphisms c and ϕ are extended trivially to $\mathbb{R}^d$ by continuity. We also assume that the extension $\tilde{\sigma}$ of σ remains trivial between $\tilde{G}_0$ and A, although we allow $\tilde{\sigma}(\mathbf{x}, g_0)$ and $\tilde{\sigma}(g_0, \mathbf{x})$ to be non-trivial for $\mathbf{x} \in \mathbb{R}^d, g_0 \in \tilde{G}_0$. Thus the extra $\mathbb{R}^d$ symmetries may be projectively realised. Then the graded twisted group C^*-algebra $\tilde{\mathcal{A}}$ of $(\tilde{G}, c, \phi, \tilde{\sigma})$ is $\tilde{\mathcal{A}} = (\mathcal{A}^{\mathrm{ev}} \rtimes_{(\beta,\nu)} \mathbb{R}^d) \hat{\otimes} Cl_{r,s}$, for some twisting pair (β, ν). Note that $\mathcal{A}^{\mathrm{ev}} \rtimes_{(\beta,\nu)} \mathbb{R}^d$ is purely even as well. It follows from Corollary 8.5.3 that

$$\mathbf{K}_0(\tilde{\mathcal{A}}) \cong K_{s-r}(\mathcal{A}^{\mathrm{ev}} \rtimes_{(\beta,\nu)} \mathbb{R}^d) \cong K_{s-r-d}(\mathcal{A}^{\mathrm{ev}}) \qquad \text{real case} \tag{8.14a}$$

$$\mathbf{K}_0(\tilde{\mathcal{A}}) \cong K_n(\mathcal{A}^{\mathrm{ev}} \rtimes_{(\beta,\nu)} \mathbb{R}^d) \cong K_{n-d}(\mathcal{A}^{\mathrm{ev}}) \qquad \text{complex case} \tag{8.14b}$$

In fact, to arrive at (8.14a)–(8.14b), we only need the factor $\tilde{G}_0$ to be an extension (of topological groups) of G_0 by $\mathbb{R}^d$,

$$1 \longrightarrow G_0 \longrightarrow \tilde{G}_0 \longrightarrow \mathbb{R}^d \longrightarrow 1. \tag{8.15}$$

Furthermore, the argument still works for general graded twisted C^*-dynamical systems, i.e., with $\mathbb{C}$ replaced by a general purely even algebra $\mathcal{B}^{\text{ev}}$. For example, $\mathcal{B}^{\text{ev}}$ could be used as a model for disorder [10] . Therefore, under fairly general assumptions about additional $\mathbb{R}^d$ symmetries, we find that the difference-group $\mathbf{K}_0(\cdot)$ becomes that for $d = 0$, provided we *lower* the K-theory index by d. It is also important to note that the suspension operation *raises* the degree in K-theory. This does not matter in the complex case because of the 2-periodic Bott periodicity, but it matters in the real case, where there are in fact two distinct notions of suspension, see Appendix A. The degree shifts can also be accounted for if we use the correct notion of suspension, and is a special case of our approach.

There is a similar discretised version of the dimension shift phenomenon. To understand this, we first consider the simplest example of the graded twisted group C^*-algebra of $G = \mathbb{Z}^d \times A$, where $\mathbb{Z}^d$ acts trivially on $\mathbb{C}$, and σ is only non-trivial between elements of A. Then

$$\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} (\mathbb{Z}^d \times A) \cong \begin{cases} (\mathbb{R} \rtimes \mathbb{Z}^d)\hat{\otimes} Cl_{r,s} \cong C(\mathbb{T}^d, \varsigma)\hat{\otimes} Cl_{r,s} & \text{real case,} \\ (\mathbb{C} \rtimes \mathbb{Z}^d)\hat{\otimes} \mathbb{C}l_n \cong C(\mathbb{T}^d, \mathbb{C})\hat{\otimes} \mathbb{C}l_n & \text{complex case,} \end{cases}$$

where $C(\mathbb{T}^d, \varsigma)$ is defined in Appendix A.1.1. The $\mathbf{K}_0(\mathcal{A})$ groups reduce to ordinary K-theory groups, and when $d = 1$, we can use (A.8a)–(A.8b) to obtain

$$\mathbf{K}_0(\mathbb{C} \rtimes_{(\alpha,\sigma)} (\mathbb{Z} \times A)) \cong \begin{cases} K_{s-r}(\mathbb{R}) \oplus K_{s-r-1}(\mathbb{R}) & \text{real case,} \\ K_n(\mathbb{C}) \oplus K_{n-1}(\mathbb{C}) & \text{complex case.} \end{cases} \tag{8.16}$$

Thus, a trivial crossed product with $\mathbb{Z}$ results in $\mathbf{K}_0(\cdot)$ acquiring an extra K-theory group with index *lowered* by 1. In the complex case, equation (8.16) continues to hold with $\mathbb{C}$ replaced by some other purely even complex algebra $\mathcal{B}^{\text{ev}}$ which is trivially acted upon by $\mathbb{Z}$. Similarly, in the real case, we may replace $\mathbb{C} = \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ by some purely even complexfied algebra $\mathcal{B}^{\text{ev}} = \mathcal{B}^{\text{ev}}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Using (A.8a)–(A.8b) repeatedly, we obtain, assuming $\alpha|_{\mathbb{Z}^d} = 1$,

$$\mathbf{K}_0\left(\mathcal{B}^{\text{ev}} \rtimes_{(\alpha,\sigma)} (\mathbb{Z}^d \times A)\right) \cong \begin{cases} \bigoplus_{k=0}^{d} \binom{d}{k} K_{s-r-k}(\mathcal{B}^{\text{ev}}_{\mathbb{R}}) & \text{real case,} \\ \bigoplus_{k=0}^{d} \binom{d}{k} K_{n-k}(\mathcal{B}^{\text{ev}}) & \text{complex case.} \end{cases} \tag{8.17}$$

Note that there is always a single extra $K_{s-r-d}(\mathcal{B}^{\text{ev}}_{\mathbb{R}})$ or $K_{n-d}(\mathcal{B}^{\text{ev}})$ factor, just as in the case of a crossed product with $\mathbb{R}^d$. It is with respect to this factor, that the "discretised" dimension shift phenomenon occurs. Equation (8.17) generalises the topological KR-theory calculations of Kitaev in equations 25–27 of [50]. In particular, if we take $A = \{1, T\}$, $\mathsf{T}^2 = -1$, $\mathcal{B}^{\text{ev}} = \mathbb{C}$ and $d = 3$, we obtain the difference-group for 3D T-invariant insulators,

$$\begin{aligned}\mathbf{K}_0(\mathbb{C} \rtimes_{(\alpha,\sigma)} (\mathbb{Z}^d \times \{1,T\})) &\cong \bigoplus_{k=0}^{3} \binom{3}{k} K_{4-k}(\mathbb{R}) \\ &= K_4(\mathbb{R}) \oplus 3K_3(\mathbb{R}) \oplus 3K_2(\mathbb{R}) \oplus K_1(\mathbb{R}) \\ &= \mathbb{Z} \oplus 4\mathbb{Z}_2,\end{aligned}$$

a result obtained by KR-theory methods in Theorem 11.14 of [30]. According to [50], the Kane–Mele invariant detects the $\mathbb{Z}_2$ factor coming from $K_1(\mathbb{R})$, the remaining three $\mathbb{Z}_2$ factors coming from $K_2(\mathbb{R})$ refer to "weak" topological insulators, and the $\mathbb{Z}$ factor due to $K_4(\mathbb{R})$ counts the number of Kramers degenerate valence bands. We stress that the expression (8.17) assumes some specific structure of the symmetry data (G, c, ϕ, σ), in particular, the way in which $\mathbb{Z}^d$ sits inside G. These assumptions do *not* hold, for instance, when there is spatial inversion symmetry, which acts on $\mathbb{Z}^d$ non-trivially.

Nevertheless, in the complex case (no antiunitaries), there is a "robustness" in the discretised version of the dimension shift phenomenon, in the sense that we may twist the group algebra $\mathbb{C} \rtimes \mathbb{Z}^d \cong C(\mathbb{T}^d)$ by any cocycle σ, without changing its K-theory groups. This is the well-known result for the K-theory groups of "non-commutative tori" [25],

$$K_\star(\mathbb{C} \rtimes_{(1,\sigma)} \mathbb{Z}^d) \cong K_*(C(\mathbb{T}^d)) \cong K^*(\mathbb{T}^d). \tag{8.18}$$

Thus, for $\mathcal{B}^{\mathrm{ev}} = \mathbb{C}$ in (8.17), we need not assume that $\sigma|_{\mathbb{Z}^d \times \mathbb{Z}^d} \equiv 1$. This allows discretised models of the quantum hall effect (with any flux, not necessarily rational) to be handled in our framework. Equation (8.18) is a special case of the results in [68] for discrete subgroups of solvable Lie groups, and the latter was used to study the quantum hall effect on the hyperbolic plane [14].

One conceptual advantage of our approach is that all symmetries are treated on an equal footing. These include time/charge preserving/reversing symmetries, projectively-realised symmetries, $\mathbb{Z}^d$-symmetries underlying band insulators, and translations $\mathbb{R}^d$ in extra spatial dimensions. Furthermore, the arguments leading to Equation (8.14a)–(8.14b) suggest that the phenomenon of "dimension shift" in the classification schemes for topological insulators observed in [50, 79, 77] is very robust, and does not depend on the details of how the extra dimensions come into play. Table 8.5 summarises the periodicities in the difference-groups, in the special case where $G = A$. It is similar to the results in [77, 79], and to Kitaev's table [50], after a suitable re-interpretation. Note, however, that the groups appearing in the table need to be modified if $G_0 \neq 1$ in $G = G_0 \times A$. The $d = 0$ column should be replaced by the G_0-equivariant K-theory groups of a point. Then, assuming the extra $\mathbb{R}^d$ symmetries enter the picture as in (8.15), these groups will fill up the $d > 0$ columns with

n	Cartan label	C^2	T^2	$\mathbf{K}_0(\mathcal{A}) \cong K_{n-d}(\mathbb{R})$ or $K_{n-d}(\mathbb{C})$			
				$d=0$	$d=1$	$d=2$	$d=3$
0	AI		$+1$	$\mathbb{Z}$	0	0	0
1	BDI	$+1$	$+1$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
2	D	$+1$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
3	$DIII$	$+1$	-1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
4	AII		-1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
5	CII	-1	-1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
6	C	-1		0	0	$\mathbb{Z}$	0
7	CI	-1	$+1$	0	0	0	$\mathbb{Z}$
0	A	N/A		$\mathbb{Z}$	0	$\mathbb{Z}$	0
1	$AIII$	$\mathsf{S}^2 = +1$		0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 8.2: Periodic Table of difference-groups for gapped topological phases in d dimensions. The $n = 0$ and $n = 4$ rows may also be interpreted as super-representation groups, or as K-theory groups classifying Type-I insulators. The K-theory degree shifts in both the vertical and horizontal directions are manifestations of isomorphisms in K-theory. In the vertical direction, it is due to the effect on $\mathbf{K}_0(\mathcal{A})$ of tensoring with a Clifford algebra; in the horizontal direction, it is due to the Connes–Thom isomorphism. The twofold and eightfold periodicities are due to Bott periodicity in the Clifford algebras and K-theory.

the appropriate dimension shifts. If we only have extra discrete $\mathbb{Z}^d$ symmetries, the shifts require additional assumptions, as well as the understanding that only the $k = d$ terms in (8.17) are considered.

It is crucial that we do not use the assumption that the extra $\mathbb{R}$ symmetry acts trivially on $\mathbb{C}$ (or more generally, on the algebra $\mathcal{A}$). As we have seen with the physically important example of the IQHE, there is generally an algebra *covariance* under the additional $\mathbb{R}$ symmetry. Furthermore, despite the projectively-realised $\mathbb{R}^2$ symmetry in the IQHE, it fits nicely into our version of the periodic table. We also see why time-reversal symmetry needs to be broken, by a magnetic field or otherwise, in order to exhibit the Integer Quantum Hall effect. For Type I insulators in two dimensions with $\mathsf{T}^2 = +1$ (assuming spin-0), the relevant K-theory group is trivial; however, a $\mathbb{Z}_2$-invariant is possible if the spin-$\frac{1}{2}$ nature of electrons are taken into account, so that $\mathsf{T}^2 = -1$. Finally, by identifying the K-theoretic origins of these features, we also make contact with the boundary isomorphisms relating the bulk phenomena with edge phenomena, as discussed in Section 8.4.1.

Appendix A

Some K-theory definitions and results

A.1 K-theory definitions

We begin with a definition of K-theory groups which combines, in some sense, the topological K-groups for spaces, and the K-groups for C^*-algebras. It follows closely the treatment in [80]. Let $\mathcal{A}$ be a real or complex unital C^*-algebra, and X be a compact Hausdorff space. We can consider the category of $\mathcal{A}$-bundles over X, which are locally trivial Banach bundles over X such that the fibres are finitely-generated projective (left[1]) $\mathcal{A}$-modules. The morphisms are Banach bundle morphisms that are morphisms of $\mathcal{A}$-modules between fibres. Direct sums of isomorphism classes of $\mathcal{A}$-bundles over X makes sense, and we define $K(X;\mathcal{A}) \equiv K^0(X;\mathcal{A}) \equiv K_0(X;\mathcal{A})$ to be the Grothendieck group of the monoid of $\mathcal{A}$-bundles over X with the direct sum operation. If $\mathcal{A}$ is a real C^*-algebra, we sometimes write KO in place of K for emphasis, although the nature of $\mathcal{A}$ should make it clear whether we are dealing with real or complex K-theory. When $\mathcal{A}$ is just the ground field, assumed to be $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $K(X;\mathbb{C})$ and $KO(X;\mathbb{R})$ reduce to the usual topological K-groups $K^0(X) \equiv K^0_{\rm top}(X)$ and $KO^0(X) \equiv KO^0_{\rm top}(X)$. When X is a point $\star$, we recover the operator algebra K-groups, i.e., $K(\star;\mathcal{A}) = K_0(\mathcal{A})$ and $KO(\star;\mathcal{A}) = KO_0(\mathcal{A})$. Alternatively, we can move X to the algebra side and define $K(X;\mathcal{A})$ to be the operator algebra K-group $K_0(C(X,\mathcal{A}))$.

To extend these definitions to locally compact X, we need to first define relative K-groups. For compact pairs $Y \subset X$, the relative K-group $K(X,Y;\mathcal{A})$ is defined as in Definition 1.4.3 of [80]. Namely, we consider triples (E_i, F_i, α_i), $i = 1, 2$, where E_i, F_i, are $\mathcal{A}$-bundles over X, and $\alpha_i : E_i|_Y \to F_i|_Y$ are $\mathcal{A}$-bundle isomorphisms on the restrictions to

[1]It seems to be more usual to use right modules in operator algebra K-theory. The choice is irrelevant for commutative algebras, and is largely unimportant even for noncommutative algebras due to the restriction to projective modules, see 1.7.2 of [11]. However, if one uses a KK-theory definition for K-theory groups, then *graded* modules are indispensible, and it does matter, for example, whether we want a left or right graded $Cl_{r,s}$-module in the definitions.

Y. We say that (E_1, F_1, α_1) is equivalent to (E_2, F_2, α_2) if there are $\mathcal{A}$-bundle isomorphisms $f : E_1 \to E_2$ and $g : F_1 \to F_2$, such that $\alpha_2 \circ f|_Y = g|_Y \circ \alpha_1$. An elementary triple is a triple (E, E, α) such that α homotopic to id_E in the set of $\mathcal{A}$-bundle isomorphisms. We can form the direct sum of triples in the natural way, and we say that two triples are *stably isomorphic* if they are isomorphic upon adding elementary triples. $K(X, Y; \mathcal{A})$ is then defined to be the group of stable isomorphism classes of triples $[E, F, \alpha]_{\mathrm{stab}}$.

Then, for a locally compact space X, we define $K(X; \mathcal{A}) := K(X^+, \{\infty\}; \mathcal{A})$. This can also be understood as K-theory with "compact supports", see e.g., Chapter I.9 of [54]. Alternatively, taking the one-point compactification X^+ as a compact basepointed space, $K(X; \mathcal{A})$ can be defined as the reduced K-group $\tilde{K}(X^+; \mathcal{A})$, which is the kernel of the natural projection $K(X^+; \mathcal{A}) \to K(\star; \mathcal{A}) \equiv K(\mathcal{A})$, induced by the inclusion of the basepoint $\star$ into X^+. The higher relative K-groups are defined to be $K^{-n}(X, Y; \mathcal{A}) := K((X - Y) \times \mathbb{R}^n; \mathcal{A})$. When $Y = \phi$, we get the higher K-groups for a locally compact space $K^{-n}(X; \mathcal{A}) = K(X \times \mathbb{R}^n; \mathcal{A})$. For a closed subset Y of a locally compact space X, the higher relative K-groups are defined to be $K^{-n}(X, Y; \mathcal{A}) \cong K((X - Y) \times \mathbb{R}^n; \mathcal{A})$, generalising the expression for the compact case.

On the operator algebra side, for unital $\mathcal{A}$, we obtain the higher operator K-groups $K_n(\mathcal{A})$ by taking $X = \star$:

$$K_n(\mathcal{A}) := K^{-n}(\star; \mathcal{A}) = K(\mathbb{R}^n; \mathcal{A}) = K(B^n, S^{n-1}; \mathcal{A}).$$

More generally, for unital C^*-algebras of the form $C_0(X, \mathcal{A})$, we can define $K_n(C_0(X, \mathcal{A})) = K^{-n}(X; \mathcal{A})$. Taking $X = \mathbb{R}^n$ then gives $K_n(\mathcal{A}) = K(\mathbb{R}^n; \mathcal{A}) = K(C_0(\mathbb{R}^n, \mathcal{A}))$, the last expression being the usual definition[2] of $K_n(\mathcal{A})$ via the n-th suspension of $\mathcal{A}$. In particular, for the commutative algebras of the form $C_0(X, \mathbb{F})$, we have

$$K_n(C_0(X, \mathbb{F})) = K^{-n}(X; \mathbb{F}) = K(X \times \mathbb{R}^n; \mathbb{F}) = K_{\mathrm{top}}(X \times \mathbb{R}^n) = K^{-n}(X).$$

Finally, for non-unital algebras $\mathcal{A}$, we define their K-groups as the kernel

$$K_n(\mathcal{A}) := \ker(K_n(\mathcal{A}^+) \to K_n(\mathbb{F}))$$

induced by the projection $\mathcal{A}^+ \to \mathbb{F}$.

There are a number of less well-known definitions for graded C^*-algebras $\mathcal{A}$, which reduce to ordinary K theory when $\mathcal{A}$ is purely even. For instance, there is the K-theory of graded Banach algebras in [88, 89], which we do not use. Instead, we have used Karoubi's definitions

[2]For complex C^*-algebras, $K_1(\mathcal{A})$ is sometimes defined as $GL_\infty(\mathcal{A})/GL_\infty(\mathcal{A})_0$, where $GL_n(\mathcal{A}) = \{x \in GL_n(\mathcal{A}^+) : x \equiv 1_n \bmod M_n(\mathcal{A})\}$. $GL_\infty(\mathcal{A})$ is the direct limit of the natural embedding $x \mapsto \mathrm{diag}(x, 1)$ of $GL_n(\mathcal{A})$ into $GL_{n+1}(\mathcal{A})$, and $GL_\infty(\mathcal{A})_0$ is the identity component of $GL_\infty(\mathcal{A})_0$. $K_1(\mathcal{A})$, defined in this way, is then shown to be equal to $K_0(C_0(\mathbb{R}, \mathcal{A})) = K_0(S\mathcal{A})$.

of K-theory in terms of gradations. We also remark that in Kasparov's KK-theory [45, 46], groups $KK(\mathcal{B}, \mathcal{A})$ are defined for graded C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, and reduce to the ordinary K-groups for $\mathcal{B} = \mathbb{F}$ and purely even $\mathcal{A}$. The full KK-theory is extremely general, encompassing the complex, real, Real, graded, and equivariant cases, but is technically difficult. Instead, in Chapter 6, we prefer to define the super-representation group $\mathcal{SR}(\mathcal{A})$ and a Karoubi-type $\mathbf{K}_0(\mathcal{A})$ for graded C^*-algebras $\mathcal{A}$. Both $\mathcal{SR}(\mathcal{A})$ and $\mathbf{K}_0(\mathcal{A})$ reduce to the ordinary K-theory group $K_0(\mathcal{A})$ when $\mathcal{A}$ is purely even.

There is also a homotopy-theoretic version of the K-theory groups of a C^*-algebra $\mathcal{A}$. We merely outline the definitions, and refer to Chapter III.1.6 of [42] for the complex case, and Theorem 1.4.6 of [80] for the real case. For a unital C^*-algebra, we write $GL_k(\mathcal{A})$ for the group of invertible elements in the matrix algebra $M_n(\mathcal{A})$ over $\mathcal{A}$. For each $l > k$, there is a natural embedding of $GL_k(\mathcal{A})$ into $GL_l(\mathcal{A})$ by $u \mapsto \mathrm{diag}(u, 1_{l-k})$. We write $GL_\infty(\mathcal{A})$ for the direct limit $\varinjlim GL_l(\mathcal{A})$ with the inductive limit topology. Then there is an isomorphism

$$K_n(\mathcal{A}) \cong \varinjlim \pi_{n-1}(GL_k(\mathcal{A})) = \pi_{n-1}(GL_\infty(\mathcal{A})), \quad n > 0.$$

The isomorphism takes a homotopy class $[\alpha] \in \pi_{n-1}(GL_k(\mathcal{A}))$ with representative

$$\alpha \in C(S^{n-1}, GL_k(\mathcal{A})),\, \alpha(1) = 1_k,$$

to the equivalence class of triples

$$[\alpha] \mapsto [(C(B^n, \mathcal{A})^k, C(B^n, \mathcal{A})^k, \alpha)] \in K(B^n, S^{n-1}; \mathcal{A}) = K_n(\mathcal{A}),$$

where $C(B^n, \mathcal{A})^k$ denotes the trivial $\mathcal{A}$-bundle over B^n with fibres $\mathcal{A}^k$.

A.1.1 A collection of results in real and complex K-theory

The main important results in complex K-theory can be found in [11, 93], while their generalisations to real K-theory can be found in Chapter 1.4 of [80]. In both complex and real K-theory, there is a long exact sequence of K-groups associated to a short exact sequence of algebras $0 \to \mathcal{J} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{A}/\mathcal{J} \to 0$. There are connecting maps $\partial_n : K_n(\mathcal{A}/\mathcal{J}) \to K_{n-1}(\mathcal{J})$, making the following long exact sequence exact,

$$\ldots \to K_n(\mathcal{J}) \xrightarrow{i_*} K_n(\mathcal{A}) \xrightarrow{p_*} K_n(\mathcal{A}/\mathcal{J}) \xrightarrow{\partial_n} K_{n-1}(\mathcal{J}) \xrightarrow{i_*} K_{n-1}(\mathcal{A}) \xrightarrow{p_*} K_{n-1}(\mathcal{A}/\mathcal{J}) \to \ldots \tag{A.1}$$

Because of Bott periodicity in K-theory,

$$\begin{aligned} K_n(\mathcal{A}) \cong& K_{n+8}(\mathcal{A}) \qquad \text{real case,} \\ K_n(\mathcal{A}) \cong& K_{n+2}(\mathcal{A}) \qquad \text{complex case,} \end{aligned}$$

the long exact sequence in (A.1) becomes cyclic, with six terms in the complex case and 24 terms in the real case.

Let $(\mathbb{T}^d, \varsigma)$ be the Real space $\mathbb{T}^d$ with involution $\mathbf{z} \mapsto \varsigma(\mathbf{z}) := \bar{\mathbf{z}}$. Let us write $C(\mathbb{T}^d, \varsigma)$ for the real C^*-algebra of continuous functions $f : \mathbb{T}^d \to \mathbb{C}$ such that $f(\varsigma(\mathbf{z})) = \overline{f(\mathbf{z})}$. The real and complex Toeplitz algebras $\mathcal{T}^{\mathbb{R}}$, $\mathcal{T}$ fit into non-split exact sequences

$$0 \to \mathcal{K}_{\mathbb{R}} \to \mathcal{T}^{\mathbb{R}} \to C(\mathbb{T}, \varsigma) \to 0 \tag{A.2}$$

$$0 \to \mathcal{K}_{\mathbb{C}} \to \mathcal{T} \to C(\mathbb{T}, \mathbb{C}) \to 0. \tag{A.3}$$

Let q denote the composition of the quotient map $\mathcal{T} \to C(\mathbb{T}, \mathbb{C})$ with evaluation at 1, and similarly in the real case. The kernels $\mathcal{T}_0$ and $\mathcal{T}_0^{\mathbb{R}}$ of q are K-contractible in the sense that for all n,(Corollary 1.5.3 of [80])

$$K_n(\mathcal{A} \otimes \mathcal{T}_0^{\mathbb{R}}) = 0 \qquad \text{real case,} \tag{A.4a}$$

$$K_n(\mathcal{A} \otimes \mathcal{T}_0) = 0 \qquad \text{complex case.} \tag{A.4b}$$

Recall the definition of $C_0(\mathrm{i}\mathbb{R})$ in (8.10). Applying (A.4a)–(A.4b) to the long exact sequence for

$$\begin{aligned} &0 \to \mathcal{A} \otimes \mathcal{K}_{\mathbb{R}} \to \mathcal{A} \otimes \mathcal{T}_0^{\mathbb{R}} \to \mathcal{A} \otimes C_0(\mathrm{i}\mathbb{R}) \to 0 \qquad \text{real case,} \\ &0 \to \mathcal{A} \otimes \mathcal{K}_{\mathbb{C}} \to \mathcal{A} \otimes \mathcal{T}_0 \to \mathcal{A} \otimes C_0(\mathbb{R}, \mathbb{C}) \to 0 \qquad \text{complex case,} \end{aligned}$$

we arrive at the isomorphisms

$$\begin{aligned} K_n(\bar{S}\mathcal{A}) \equiv K_n(\mathcal{A} \otimes C_0(\mathrm{i}\mathbb{R})) \cong K_{n-1}(\mathcal{A} \otimes \mathcal{K}_{\mathbb{R}}) \cong K_{n-1}(\mathcal{A}), \qquad \text{real case,} \\ K_n(S\mathcal{A}) \equiv K_n(\mathcal{A} \otimes C_0(\mathbb{R}, \mathbb{C})) \cong K_{n-1}(\mathcal{A} \otimes \mathcal{K}_{\mathbb{C}}) \cong K_{n-1}(\mathcal{A}), \qquad \text{complex case.} \end{aligned} \tag{A.5}$$

Thus, we see that in the real case, $\bar{S}$ is the K-theoretic "inverse" of S,

$$K_n(S\bar{S}\mathcal{A}) \cong K_n(\bar{S}S\mathcal{A}) = K_n(\mathcal{A} \otimes C_0(\mathrm{i}\hat{\mathbb{R}}) \otimes C_0(\mathbb{R}, \mathbb{R})) \cong K_n(\mathcal{A}). \tag{A.6}$$

The complex group algebra of $\mathbb{Z}^d$ is, upon taking the Fourier transform, isomorphic to the complex C^*-algebra $C(\mathbb{T}^d, \mathbb{C})$, which has a real subalgebra $C(\mathbb{T}^d, \varsigma)$ isomorphic to the *real* group C^*-algebra of $\mathbb{Z}$. In other words,

$$\begin{aligned} \mathbb{R} \rtimes_{(1,1)} \mathbb{Z}^d &\cong C(\mathbb{T}^d, \varsigma), \\ \mathbb{C} \rtimes_{(1,1)} \mathbb{Z}^d &\cong C(\mathbb{T}^d, \mathbb{C}). \end{aligned}$$

If we apply (A.5) to the *split* short exact sequences

$$\begin{aligned} &0 \to \bar{S}\mathcal{A} \to \mathcal{A} \otimes C(\mathbb{T}^1, \varsigma) \xrightarrow{\mathrm{ev}_1} \mathcal{A} \to 0 \qquad \text{real case,} \\ &0 \to S\mathcal{A} \to \mathcal{A} \otimes C(\mathbb{T}^1, \mathbb{C}) \xrightarrow{\mathrm{ev}_1} \mathcal{A} \to 0 \qquad \text{complex case,} \end{aligned}$$

we find that

$$K_n(\mathcal{A} \otimes C(\mathbb{T}^1, \varsigma)) \cong K_n(\mathcal{A}) \oplus K_{n-1}(\mathcal{A}), \qquad \text{real case,} \tag{A.8a}$$

$$K_n(\mathcal{A} \otimes C(\mathbb{T}^1, \mathbb{C})) \cong K_n(\mathcal{A}) \oplus K_{n-1}(\mathcal{A}), \qquad \text{complex case.} \tag{A.8b}$$

Equations (A.8a)–(A.8b) are really special cases of the Pimsner–Voiculescu (PV) exact sequence (Theorem 10.2.1 of [11] and Theorem 1.5.5 of [80]) for the K-groups of crossed products of $\mathcal{A}$ by $\mathbb{Z}$. Let $\alpha \in \mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ generate an action of $\mathbb{Z}$ on $\mathcal{A}$. The PV-sequence is the exact sequence

$$\ldots \longrightarrow K_n(\mathcal{A}) \xrightarrow{\mathrm{id}-\alpha_*} K_n(\mathcal{A}) \xrightarrow{i_{\mathcal{A}*}} K_n(\mathcal{A} \rtimes_{(\alpha,1)} \mathbb{Z}) \longrightarrow K_{n-1}(\mathcal{A}) \xrightarrow{\mathrm{id}-\alpha_*} \ldots. \tag{A.9}$$

By Bott periodicity, the PV-sequence reduces to a cyclic exact sequence, with six terms in the complex case and 24 terms in the real case. By setting $\alpha = \mathrm{id}$ in (A.9), we recover the result (A.8a)–(A.8b).

A.2 The Atiyah–Bott–Shapiro construction

The Atiyah–Bott–Shapiro isomorphisms [4] highlight the fundamental role of the Clifford algebras in K-theory. The Bott periodicity of K-theory is given an algebraic description in terms of Clifford modules, which we find extremely useful in our work. We outline the Atiyah–Bott–Shapiro construction in real K-theory. Let $W = W_0 \oplus W_1$ be a graded Cl_n-module, and let E_i be the trivial real vector bundles $B^n \times W_i$ for $i = 1, 2$. We can associate to W the relative K-group element

$$\varphi_n(W) = [E_0, E_1, \alpha] \in K(B^n, S^{n-1}; \mathbb{R}),$$

where α is the isomorphism over S^{n-1},

$$\alpha(x, w) = (x, x \cdot w), \quad x \in S^{n-1},\ w \in W_0.$$

One checks that φ_n is well-defined on isomorphism classes and is additive, so that there is an induced homomorphism $\varphi_n : \mathcal{GR}(Cl_n) \to K(B^n, S^{n-1}; \mathbb{R})$. Now suppose that W extends to a graded (left) Cl_{n+1}-module. Let e_{n+1} denote the extra Clifford generator, also regarded as an element of $\mathbb{R}^{n+1}$ orthogonal to $\mathbb{R}^n$. Then we can extend the isomorphism α to the whole of B^n by setting

$$\alpha(x, w) = \Big(x, (x + \sqrt{1 - \|x\|^2}\, e_{n+1}) \cdot w\Big), \quad x \in B^n,\ w \in W_0.$$

In this case, E_0 and E_1 are isomorphic, so $\varphi_n(W)$ is equivalent to the zero element in $K(B^n, S^{n-1}; \mathbb{R}) = K^{-n}(\star; \mathbb{R})$. The homomorphism φ_n thus descends to a homomorphism

$$\varphi_n : \mathcal{GR}(Cl_n)/i^*\mathcal{GR}(Cl_{n+1}) \longrightarrow K^{-n}(\star; \mathbb{R}).$$

That φ_n is actually an isomorphism is one of the main results in [4]. Similarly, in the complex case, we have the ABS isomorphisms

$$\mathcal{GR}(\mathbb{C}l_n)/i^*\mathcal{GR}(\mathbb{C}l_{n+1}) \longrightarrow K^{-n}(\star),$$

A second related way in which Clifford algebras enter the description of higher K-groups is through the approach of Karoubi, and can be found in Chapter III of [42]. We have followed his definitions closely in defining $\mathbf{K}_n(\cdot)$ in Section 6.4. Next, there is the KK-theory description (Definition 2.4.11 of [80]); in the real case, this is

$$K_n(\mathcal{A}) \cong KKO_n(\mathbb{R},\mathcal{A}) \equiv KKO_{n,0}(\mathbb{R},\mathcal{A}) \coloneqq KKO(\mathbb{R}, Cl_{n,0}\hat{\otimes}\mathcal{A}) \cong KKO(Cl_{0,n},\mathcal{A}),$$

while in the complex case, it is

$$K_n(\mathcal{A}) \cong KK(\mathbb{C},\mathcal{A}) \coloneqq KK_n(\mathbb{C},\mathbb{C}l_n\hat{\otimes}\mathcal{A}) \cong KK_n(\mathbb{C}l_n,\mathcal{A}).$$

Note that the standard definitions of $KKO(\mathcal{B},\mathcal{A})$ and $KKO(\mathcal{B},\mathcal{A})$ involve *right* Hilbert $\mathcal{A}$-modules.

For commutative $\mathcal{A} \cong C(X)$, Karoubi also developed [40] his $\bar{K}^{r,s}(X)$ groups in terms of homotopy classes of graded bundles of $Cl_{r,s}$-antilinear (anticommutes with the Clifford generators) odd self-adjoint Fredholm operators over X, and showed that his $\bar{K}^{r,s}(X)$ are isomorphic to his $K'^{r,s}(C(X))$ and hence our $\mathbf{K}_0(C(X))$. This should be compared to the use of $Cl_{n-1,0}$-antilinear skew-adjoint Fredholm operators (or odd $Cl_{n,0}$-linear self-adjoint Fredholm operators) as classifying spaces for $K^{-n}(X)$ in [5]. Some of the connections between the ABS constructions in [3], Karoubi's K-theory, as well as the twisted K-theory of [21], are decribed in [43].

A.3 KR-theory

Atiyah's KR-theory [3] is defined for Real spaces (X,ς), with ς being a homeomorphism of X such that $\varsigma^2 = 1$. We write $\bar{x} \coloneqq \varsigma(x)$ for brevity. Complex vector bundles over X are replaced by Real bundles over X, which are complex vector bundles equipped with their own involutions $\tilde{\varsigma}$, which respect the projection onto X (so $\pi \circ \tilde{\varsigma} = \varsigma \circ \pi$), and are antilinear on fibres. The higher KR-theory groups are doubly-indexed, and defined through suspensions, see (A.10). There is a $(1,1)$-periodicity in these groups, and one also writes $KR^{s-r}(X,\varsigma) \equiv KR^{r-s,0}(X,\varsigma) \cong KR^{r,s}(X,\sigma)$, with indices taken modulo 8. KR-theory is also defined for Real C^*-algebras $(\mathcal{A},\varsigma)$, where $(\mathcal{A},\varsigma)$-modules M must now have the extra structure of a Real module, namely, an antilinear involution such that $\overline{a\cdot m} = \bar{a}\cdot\bar{m}$ for all $a \in \mathcal{A}$ and $m \in M$. In the commutative case, there is an isomorphism $KR^0(X,\varsigma) \cong KR_0(C_0(X,\varsigma))$, where $C_0(X,\varsigma)$ is the Real C^*-algebra of continuous complex-valued functions on X vanishing at infinity, with the antilinear involution $\bar{f}(x) = \overline{f(\bar{x})}$. Evidently, $C_0(X,\varsigma)$ is the complexification of the real subalgebra $C_0^{\mathbb{R}}(X,\varsigma) \coloneqq \{f \in C_0(X,\varsigma) : f(x) = \overline{f(\bar{x})}\}$, so one even has $KR^0(X,\varsigma) \cong KO_0(C_0^{\mathbb{R}}(X,\varsigma))$. In this way, topological KR-theory can be dealt with using real operator K-theory. In general, replacing "complex" with "Real", along with modifications respecting the new structures, gives the definitions of $KR(X;\mathcal{A})$, where we have dropped the explicit reference to the involutions on X and $\mathcal{A}$. They are understood to be present whenever when the notation "KR" is used.

Gelfand–Naimark duality in the real case states that a real C^*-algebra is isometrically *-isomorphic to the real C^*-algebra of functions $C_0(X,\varsigma)$ for some Real space (X,ς). Recall that the real Clifford algebras $Cl_{r,s}$ can be abstractly characterised as $Cl(V,q)$, where V is a $r+s$ dimensional real vector space and q is a non-degenerate quadratic form with signature $(+,\ldots,+,-,\ldots,-)$, the plus sign appearing r times and the minus sign appearing s times. Here, the unital algebra $Cl(V,q)$ is generated by $V \subset Cl(V,q)$, subject to $v^2 = -q(v)1$, and there is an isomorphism $Cl(V,q) \cong Cl_{r,s}$. Also recall that $Cl_{r,s}$ is the real fixed-point subalgebra of the Real algebra $Cl(\mathbb{R}^{r,s}) \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathbb{C}l(\mathbb{R}^{r,s})$ with the appropriate antilinear involution, namely, complex conjugation along with the involution of $Cl(\mathbb{R}^{r,s})$ (see Section 7.3.5). Now consider $\mathbb{R}^{r,s} \equiv \mathbb{R}^r \oplus \mathrm{i}\mathbb{R}^s$, and make it a Real space by defining the involution $(\mathbf{x},\mathbf{y}) \mapsto (\mathbf{x},-\mathbf{y})$. We can understand $\mathbb{C}l(\mathbb{R}^{r,s})$ as the Real algebra associated to the Real space $\mathbb{R}^{r,s}$. The Real subspaces $B^{r,s}$ and $S^{r,s}$ are defined to be the unit ball and unit sphere in $\mathbb{R}^{r,s}$ (equipped with the standard Euclidean norm in $\mathbb{R}^{r+s}$). Note that $S^{r,s}$ has dimension $r+s-1$; for example, $S^{2,0} = S^1$ is a circle. For a compact Real pair $Y \subset X$, the doubly-indexed relative KR-groups are defined as

$$KR^{r,s}(X,Y) \coloneqq KR((X-Y)\times\mathbb{R}^{r,s}) = KR(X\times B^{r,s}, X\times S^{r,s}\cup Y\times B^{r,s}),$$

and are extended to the locally compact case in the same manner as the relative K^{-n} groups. When $s=0$, we also write $KR^{-r} \coloneqq KR^{r,0}$. For a general locally compact Real space, we thus have

$$KR^{r,s}(X) = KR(X\times\mathbb{R}^{r,s}) = KR(X\times B^{r,s}, X\times S^{r,s}) \tag{A.10}$$

The ABS construction also works in KR-theory (see Theorem I.10.11 in [54], and [4, 3]). Given a Real graded module $W = W_0 \oplus W_1$ for $Cl(\mathbb{R}^{r,s})$ (equivalently, for $\mathbb{C}l(\mathbb{R}^{r,s})$), we associate the element $\varphi_{r,s}(W) = [E_0, E_1, \alpha]$, where $E_i = B^{r,s}\times W_i$ are trivial Real bundles, and α is the isomorphism over $S^{r,s}$ given by

$$\alpha(z,w) = (z, z\cdot w), \quad z=(x,y)\in\mathbb{R}^{r,s},\ w\in W_0.$$

This yields, as before, the homomorphisms $\varphi_{r,s} : \mathcal{GR}(\mathbb{C}l(\mathbb{R}^{r,s})) \to KR(B^{r,s}, S^{r,s}) = KR^{r,s}(\star)$. If W extends to a $\mathbb{C}l(\mathbb{R}^{r+1,s})$-module, the same argument shows that $\varphi_{r,s}(W)$ represents the zero element in $KR^{r,s}(\star)$, and one even gets isomorphisms

$$\varphi_{r,s} : \mathcal{GR}(\mathbb{C}l(\mathbb{R}^{r,s}))/i^*\mathcal{GR}(\mathbb{C}l(\mathbb{R}^{r+1,s})) \longrightarrow KR^{r,s}(\star).$$

A.3.1 Super-representation groups and KR-theory of Real graded algebras

In the algebraic setting, there is a KR-theory for Real graded C^*-algebras [62] (which is subsumed by Kasparov's KKR-theory). Recall that a Real graded C^*-algebra $\mathcal{A}$ is a complex graded C^*-algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ with an even antilinear involution $\overline{(\cdot)}$ preserving this decomposition. Examples are the complexifications of the Real Clifford algebras $\mathbb{C}l(\mathbb{R}^{r,s}) \equiv Cl(\mathbb{R}^{r,s}) \otimes_{\mathbb{R}} \mathbb{C}$, with the involution on $Cl(\mathbb{R}^{r,s})$ extended antilinearly. For instance, $\mathbb{C} \oplus \mathbb{C}$ can be viewed as the complexification of $Cl(\mathbb{R}^{0,1}) = \mathbb{R} \oplus \mathbb{R}$, the latter being graded by the decomposition $\mathcal{A}_0 = \{(\lambda, \lambda) : \lambda \in \mathbb{C}\}$ and $\mathcal{A}_1 = \{(\lambda, -\lambda) : \lambda \in \mathbb{C}\}$, and the involution being defined by $(\lambda, \lambda) \mapsto (\bar{\lambda}, \bar{\lambda})$ and $(\lambda, -\lambda) \mapsto (-\bar{\lambda}, \bar{\lambda})$. Note that the Real graded $Cl(\mathbb{R}^{r,s})$-modules defined in the previous subsection (with the obvious extensions to the graded case) are precisely the Real graded modules for the Real graded algebras $Cl(\mathbb{R}^{r,s})$ as defined here. Another example is the purely even Real C^*-algebra $C_0(X)$ of continuous functions on a Real space (X, ς). A Real graded module over $\mathcal{A}$ is a graded complex $\mathcal{A}$-module $W = W_0 \oplus W_1$, along with an antilinear involution $\overline{(\cdot)}$ such that

$$\begin{aligned} \overline{W_i} &= W_i, \\ A_i \cdot W_j &\subset W_{i+j (\mathrm{mod}\, 2)}, \\ \overline{a \cdot w} &= \bar{a} \cdot \bar{w}, \qquad\qquad a \in \mathcal{A},\, w \in W. \end{aligned}$$

The algebra $\mathrm{End}_{\mathcal{A}}(W)$ is itself a Real graded C^*-algebra, where an element is of degree 0 if it preserves the degree of $w \in W$, and is of degree 1 if it changes the degree of w by 1 (mod 2). The involution on $\mathrm{End}_{\mathcal{A}}(W)$, and more generally, linear maps $W \to W$, is defined by $\bar{L}(w) = \overline{L(\bar{w})}$. A *degenerate* Real graded $\mathcal{A}$-module is one for which there exists an odd (i.e. degree 1) $L \in \mathrm{End}_{\mathcal{A}}(W)$, such that $L^2 = 1$ and $L = \bar{L}$. We define the group $\bar{\mathcal{S}}\mathcal{R}(\mathcal{A})$ as the quotient of the monoid of equivalence classes of Real graded finitely-generated projective $\mathcal{A}$-modules, by the congruence generated by the degenerate ones. This is only slightly different from what we called the super-representation group of real or complex graded algebras in Definition 6.2.5. Instead of requiring $L \in \mathrm{End}_{\mathcal{A}}(W)$, we want a trivialising operator $\mathcal{I}$ which is a super-endomorphism, i.e., it *graded* commutes with $\mathcal{A}$. Then a Real graded $\mathcal{A}$-module is *trivial* if there is an odd super-endomorphism $\mathcal{I}$ such that $\mathcal{I}^2 = 1$ and $\mathcal{I} = \bar{\mathcal{I}}$. Then, the super-representation group of $\mathcal{A}$ is defined to be the quotient of the monoid of equivalence classes of Real graded finitely-generated projective $\mathcal{A}$-modules, by the congruence generated by the trivial ones. Note that when $\mathcal{A}$ is purely even, the degenerate and trivial modules coincide, since graded commutation with $\mathcal{A}$ is the same thing as ordinary commutation with $\mathcal{A}$. In this case, $\bar{\mathcal{SR}}(\mathcal{A})$ and $\mathcal{SR}(\mathcal{A})$ are isomorphic.

An operator L making W degenerate can be turned into the action of an extra *negative* Clifford algebra generator. If we write $L = \begin{pmatrix} 0 & \mathrm{L} \\ \mathrm{L}^{-1} & 0 \end{pmatrix}$ and L commutes with $\mathcal{A}$, then $e = \begin{pmatrix} 0 & \mathrm{L} \\ -\mathrm{L}^{-1} & 0 \end{pmatrix}$ and e graded commutes with $\mathcal{A}$, and $e^2 = -1$. In contrast, the operator $\mathcal{I}$ defining a trivial W acts like a *positive* Clifford algebra generator. For a purely even Real C^*-algebra, an alternative definition of $KR_0(\mathcal{A})$ is $\mathcal{SR}(\mathcal{A})$ or $\bar{\mathcal{SR}}(\mathcal{A})$. More generally, we find that $\bar{\mathcal{SR}}(\mathbb{C}l(\mathbb{R}^{r,s})) \cong \mathcal{SR}(\mathbb{C}l(\mathbb{R}^{s,r}))$. This follows from the fact that a $\mathbb{C}l(\mathbb{R}^{r,s})$-module is a $\mathbb{C}l(\mathbb{R}^{s,r})$-module by redefining the Clifford generators $e_i \mapsto \Gamma e_i =: e_i'$, and that L anticommutes with e_i' iff L commutes with e_i. We can also define a Karoubi-type group $\mathbf{KR}_0(\mathcal{A})$ for Real graded C^*-algebras using grading operators on Real $\mathcal{A}$-modules, which must additionally respect the Real structure. The higher super-representation groups and $\mathbf{KR}_{s,r}(\mathcal{A})$ groups can similarly be defined using $\mathbb{C}l(\mathbb{R}^{r,s})$, but we do not use them in our work.

Appendix B

Some technical remarks on projective unitary-antiunitary representations

B.1 Projective unitary representations

This section is based on Appendix D.3 of [97], where detailed proofs can be found. Recall that a (strongly-continuous) *unitary representation* of a topological group G on a separable complex Hilbert space $\mathscr{H}$ is a continuous homomorphism $\mathsf{U} : G \to \mathcal{U}(\mathscr{H})$, where $U(\mathscr{H})$ is the unitary group of $\mathscr{H}$ equipped with the strong operator topology. The unitary scalars $\mathbb{T}$ form the centre of $\mathcal{U}(\mathscr{H})$, and the quotient $\mathcal{P}(\mathscr{H}) := \mathcal{U}(\mathscr{H})/\mathbb{T}$, with the quotient topology, is the *projective unitary group* of $\mathscr{H}$. By a *projective unitary representation* of G, on $\mathscr{H}$, we mean a continuous homomorphism $G \to \mathcal{P}(\mathscr{H})$. Now, each unitary $U \in \mathcal{U}(\mathscr{H})$ determines a $*$-automorphism of the C^*-algebra of compact operators $\mathcal{K} \equiv \mathcal{K}(\mathscr{H})$ via the adjoint action Ad_U. The group $\mathrm{Aut}(\mathcal{K})$, like the automorphism group $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ of any C^*-algebra $\mathcal{A}$, can be given the point-norm topology. With our separability assumption on $\mathcal{A}$, it turns out that $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ with this topology is a Polish group. Furthermore, $\mathrm{Ad} : \mathcal{U}(\mathscr{H}) \to \mathrm{Aut}(\mathcal{K})$ factors through $\mathcal{P}(\mathscr{H})$, and induces a *homeomorphism* between $\mathcal{P}(\mathscr{H})$ and $\mathrm{Aut}(\mathcal{K})$. Thus, a projective unitary representation of G on $\mathscr{H}$ can equivalently be defined as a continuous homomorphism from G to $\mathrm{Aut}(\mathcal{K})$.

In Section 3.1, we defined a projective unitary representation θ in a different manner, as a special case of a (twisted) covariant representation θ of $(G, \mathbb{C}, 1, \sigma)$. Here, $\mathbb{C}$ is regarded as a complex C^*-algebra, and θ is a *Borel* map from G to $\mathcal{U}(\mathscr{H})$, satisfying (3.2a)–(3.2b). If we replace $\theta : x \mapsto \theta_x$ by $\mathrm{Ad}_\theta : x \mapsto \mathrm{Ad}_{\theta_x}$, we obtain a Borel homomorphism $\mathrm{Ad}_\theta : G \to \mathrm{Aut}(\mathcal{K})$. Since we always assume that G is locally compact and second countable, it is also Polish, and by Theorem D.11 of [97], Ad_θ is continuous. Therefore, the covariant representation θ gives rise to a projective unitary representation of G in the sense of the previous paragraph.

Suppose θ and θ' are, respectively, covariant representations of $(G, \mathbb{C}, 1, \sigma)$ and $(G, \mathbb{C}, 1, \sigma')$ on $\mathscr{H}$, and that they define the same projective unitary representations $\mathrm{Ad}_\theta = \mathrm{Ad}_{\theta'}$. Then for each $x \in G$, the operators θ_x and θ'_x differ by some unitary scalar, i.e. there is a Borel map $\lambda : G \to \mathbb{T}$ such that $\theta_x = \lambda_x \theta'_x$. A short computation verifies that $\sigma = \mathrm{d}^1\lambda \cdot \sigma'$, so that σ and σ' are really from the same cocycle class. It is also clear that the choice of σ from a cocycle class $[\sigma]$ does not affect the map Ad_θ. Thus, there is a well-defined map $[\theta] \mapsto \mathrm{Ad}_\theta$ taking a projective equivalence class $[\theta]$ of covariant representations of $(G, \mathbb{C}, 1, \sigma)$, to a projective unitary representation $\pi = \mathrm{Ad}_\theta$ of G. Conversely, given a projective unitary representation $\pi : G \to \mathrm{Aut}(\mathcal{K})$, the proof of Lemma D.24 of [97] provides the construction of a Borel map $\theta : G \to \mathcal{U}(\mathscr{H})$ such that $\mathrm{Ad}_\theta = \pi$. The map θ satisfies $\theta_x\theta_y = \sigma(x, y)\theta_{xy}$, $\forall x, y \in G$ for some Borel cocycle σ, so θ is a covariant representation of $(G, \mathbb{C}, 1, \sigma)$. A different choice θ' satisfying $\mathrm{Ad}_{\theta'} = \pi$ leads to a cocycle σ' differing from σ by a coboundary, so θ and θ' are projectively equivalent covariant representations of G.

A unitary representation U of G on $\mathscr{H}$ is of course a projective unitary representation π which can be realised as Ad_θ for some Borel θ with trivial cocycle σ — just take $\theta = \mathsf{U}$. Conversely, suppose a projective unitary representation π of G can be realised as Ad_θ, with the cocycle for θ in the trivial class, i.e., $\theta = \mathrm{d}^1\lambda$ for some Borel map $\lambda : G \to \mathbb{T}$. Then $\theta'_x := \lambda_x^{-1}\theta_x$ also satisfies $\mathrm{Ad}_{\theta'} = \pi$, and is furthermore a Borel (and thus continuous) *homomorphism* $G \to \mathcal{U}(\mathscr{H})$. Therefore, our general notion of covariant representations reduces to unitary representations in the special case of the dynamical system $(G, \mathbb{C}, 1, 1)$.

Everything here applies to real unitary (orthogonal) representations of G on a real Hilbert space, with $\mathbb{T}$ replaced by $\{\pm 1\}$, and $\mathbb{C}$ replaced by the real C^*-algebra $\mathbb{R}$.

B.2 Projective unitary-antiunitary representations

Following [70, 69], we generalise the notion of projective unitary representations on a complex Hilbert space $\mathscr{H}$ to include antiunitary operators. This involves taking $\mathcal{U}(\mathscr{H})$ to mean the group of all unitary and antiunitary operators on $\mathscr{H}$, with the strong (or weak) operator topology. The subgroup $\mathbb{T}$ of unitary scalars is not central, but is still a compact normal subgroup of $\mathcal{U}(\mathscr{H})$, and we can form the quotient group $\mathcal{P}(\mathscr{H}) = \mathcal{U}(\mathscr{H})/\mathbb{T}$ as before, with the quotient topology. The subgroup $\mathcal{U}^u(\mathscr{H})$ of unitary operators and the subset $\mathcal{U}^a(\mathscr{H})$ of antiunitary operators are clopen, as are their respective images $\mathcal{P}^u(\mathscr{H})$ and $\mathcal{P}^a(\mathscr{H})$ in $\mathcal{P}(\mathscr{H})$ under the quotient map q.

A projective unitary-antiunitary representation of G on a $\mathscr{H}$ is a continuous homomorphism[1] $\pi : G \to \mathcal{P}(\mathscr{H})$. The subset $G^u := \{x \in G : \pi_x \in \mathcal{P}^u(\mathscr{H})\}$ of unitarily implemented

[1] We can also define it as a continuous homomorphism $\pi : G \to \mathrm{Aut}_{\mathbb{R}}(\mathcal{K})$, in analogy to the definition of a projective unitary representation.

elements forms a clopen normal subgroup of G, and its coset space G/G^u has one or two elements depending on whether any element of G is represented antiunitarily. In the latter case, the identity coset can be identified with G^u, and the other coset with the antiunitarily implemented elements $G^a := \{x \in G : \pi_x \in \mathcal{P}^a(\mathscr{H})\}$. These cosets express G as a disjoint union of closed subsets $G = G^u \cup G^a$, which we call the UA-decomposition of G associated with π.

In the language of covariant representations, we take $\mathcal{A} = \mathbb{C}$ as a real C^*-algebra as in Section 3.2.1, and define $\phi : G \to \{\pm 1\}$ to be the surjective continuous homomorphism with kernel G^u, which in turn defines α_x as either complex conjugation or the identity automorphism on $\mathbb{C}$. According to Theorem 3.1 of [70], one can construct a Borel map $\theta : G \to \mathcal{U}(\mathscr{H})$, such that $q(\theta_x) = \pi_x$ for all $x \in G$. The map θ satisfies $\theta_x \theta_y = \sigma(x, y)\theta_{xy}$, $\forall x, y \in G$, for some generalised cocycle σ (equation (1.1)). Thus, π gives rise to a covariant representation θ of $(G, \mathbb{C}, \alpha, \sigma)$. Furthermore, any two θ with $q(\theta_x) = \pi_x$ have cocycles in the same cocycle class, so the covariant representations that they determine are projectively equivalent. Conversely, suppose θ is a covariant representation of $(G, \mathbb{C}, \alpha, \sigma)$, where α_x is either complex conjugation or the identity on $\mathbb{C}$ as determined by a surjective continuous homomorphism $\phi : G \to \{\pm 1\}$, and let $G^u = \ker(\alpha) = \ker(\phi)$ and $G^a = G - G^u$. Then $\pi_x := q(\theta_x)$ defines a Borel homomorphism $\pi : G \to \mathcal{P}(\mathscr{H})$, which is automatically continuous, and therefore a projective unitary-antiunitary representation of G. Furthermore, the UA-decomposition of G with respect to π is precisely $G = G^u \cup G^a$.

Remark B.2.1. In the main body of the thesis, we use the term "projective unitary-antiunitary representation of (G, ϕ, σ)" with a specified homomorphism ϕ and cocycle σ, in the sense of covariant representations. This is slightly different from a "projective unitary-antiunitary" representation of G as defined above.

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