

Square-Full Numbers in Short Intervals

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A positive integer n is called square-full if $p^2|n$ for every prime factor p of n . Let $Q(x)$ denote the number of square-full integers up to x . It was shown by Bateman and Grosswald [1] that

$$Q(x) = \frac{\zeta(3/2)}{\zeta(3)}x^{\frac{1}{2}} + \frac{\zeta(2/3)}{\zeta(2)}x^{\frac{1}{3}} + o(x^{\frac{1}{6}}). \quad (1)$$

Bateman and Grosswald also remarked that any improvement in the exponent $\frac{1}{6}$ would imply a “quasi Riemann Hypothesis” of the type $\zeta(s) \neq 0$ for $\Re(s) \geq 1 - \delta$. Thus (1) is essentially as sharp as one can hope for at present. From (1) it follows that, for the number of square-full integers in a short interval, we have

$$Q(x + x^{\frac{1}{2}}y) - Q(x) \sim \frac{\zeta(3/2)}{2\zeta(3)}y \quad (2)$$

when $y \geq x^{\frac{1}{6}}$ and $y = o(x^{\frac{1}{2}})$. (It seems more suggestive to write the interval as $(x, x + x^{\frac{1}{2}}y]$ than $(x, x + y]$, since only intervals of length $x^{\frac{1}{2}}$ or more can be of relevance here.) It was shown by Shiu [5] that (2) in fact holds on the longer range $y \geq x^{0.1526}$ (and $y = o(x^{\frac{1}{2}})$). The exponent of x was later lowered to 0.1490342 by Jia Chao-hua [2], and to 0.14254 by Liu Hongquan [3]. These last two authors used more delicate exponential sum estimates than were employed by Shiu.

The purpose of the present note is to indicate how a simple observation during the preliminary part of the argument leads immediately to a further improvement in these results. Before stating our theorem it is convenient to introduce the notations

$$y' = (x + x^{\frac{1}{2}}y)^{\frac{1}{2}} - x^{\frac{1}{2}}$$

and

$$S(x) = \sum_{a^2b^3 \leq x} 1.$$

It follows of course that $y' \sim y/2$ for $y = o(x^{\frac{1}{2}})$. As we shall see later we have

$$S(x) \sim \zeta\left(\frac{3}{2}\right)x^{\frac{1}{2}}.$$

We can now state our results.

Theorem Let θ_0 be a positive constant with the property that for any $\theta > \theta_0$ there exists a $\delta = \delta(\theta) > 0$ such that

$$S(x + x^{\frac{1}{2}}y) - S(x) = \zeta(3/2)y'(1 + O(x^{-\delta})) \quad (3)$$

uniformly for $x^\theta \leq y \leq x^{\frac{1}{2}}$. Then for any $\theta > \theta_0$ there exists $\eta = \eta(\theta) > 0$ such that

$$Q(x + x^{\frac{1}{2}}y) - Q(x) = \frac{\zeta(3/2)}{\zeta(3)}y'(1 + O(x^{-\eta})) \quad (4)$$

uniformly for $x^\theta \leq y \leq x^{\frac{1}{2}}$.

Corollary The asymptotic formula (2) holds for $y \geq x^{0.1318162}$ if $y = o(x^{\frac{1}{2}})$.

For the proof of the theorem we begin by giving ourselves a $\theta > \theta_0$. As in Shiu's work we start with the fact that

$$Q(x + x^{\frac{1}{2}}y) - Q(x) = \sum_{x < a^2 b^3 m^6 \leq x + x^{\frac{1}{2}}y} \mu(m). \quad (5)$$

We first consider the terms $m \leq M$, where

$$M = x^{(\theta - \theta_0)/3}.$$

This choice ensures that

$$\left(\frac{x}{m^6}\right)^{\theta'} \leq \frac{y}{m^3} \leq \left(\frac{x}{m^6}\right)^{\frac{1}{2}}$$

for $m \leq M$, with

$$\theta' = \frac{\theta_0}{1 - 2(\theta - \theta_0)} > \theta_0.$$

According to our hypothesis (3), applied with x replaced by x/m^6 and y by y/m^3 , the values $m \leq M$ contribute to (5) a total

$$\begin{aligned} \sum_{m \leq M} \mu(m) \left\{ S\left(\frac{x + x^{\frac{1}{2}}y}{m^6}\right) - S\left(\frac{x}{m^6}\right) \right\} &= y' \zeta\left(\frac{3}{2}\right) \sum_{m \leq M} \frac{\mu(m)}{m^3} \\ &\quad + O(y' x^{-\delta} \sum_{m \leq M} m^{-3+6\delta}) \\ &= y' \left(\frac{\zeta(\frac{3}{2})}{\zeta(3)} + O(M^{-2}) \right) + O(y' x^{-\delta}) \end{aligned}$$

for a suitably small constant $\delta > 0$. This gives us the main term of (4), together with acceptable error terms.

On the other hand, the contribution to (5) arising from numbers $m \geq M$ is at most

$$\begin{aligned}
\left| \sum_{\substack{x < a^2 b^3 m^6 \leq x + x^{\frac{1}{2}} y \\ m \geq M}} \mu(m) \right| &\leq \sum_{\substack{x < a^2 b^3 m^6 \leq x + x^{\frac{1}{2}} y \\ m \geq M}} 1 \\
&\leq \sum_{\substack{x < a^2 c^3 \leq x + x^{\frac{1}{2}} y \\ c \geq M^2}} d(c), \quad (6)
\end{aligned}$$

where $d(c)$ is the usual divisor function. We define

$$D = \max_{c \leq x} d(c).$$

Then (6) is

$$\begin{aligned}
&\leq D \sum_{\substack{x < a^2 c^3 \leq x + x^{\frac{1}{2}} y \\ c \geq M^2}} 1 \\
&= D \{S(x + x^{\frac{1}{2}} y) - S(x) - \sum_{\substack{x < a^2 c^3 \leq x + x^{\frac{1}{2}} y \\ c \leq M^2}} 1\}. \quad (7)
\end{aligned}$$

Our hypothesis (3) yields

$$S(x + x^{\frac{1}{2}} y) - S(x) = \zeta(3/2) y' (1 + O(x^{-\delta})),$$

while

$$\begin{aligned}
\sum_{\substack{x < a^2 c^3 \leq x + x^{\frac{1}{2}} y \\ c \leq M^2}} 1 &= \sum_{c \leq M^2} \{[(\frac{x + x^{\frac{1}{2}} y}{c^3})^{\frac{1}{2}}] - [(\frac{x}{c^3})^{\frac{1}{2}}]\} \\
&= \sum_{c \leq M^2} \{c^{-3/2} y' + O(1)\} \\
&= y' \{\zeta(\frac{3}{2}) + O(M^{-1/2})\} + O(M^2).
\end{aligned}$$

If θ has been chosen sufficiently close to θ_0 we can deduce that (7) is

$$\ll D y' x^{-\delta},$$

with a new value of δ . Since $D \ll x^\varepsilon$ for any $\varepsilon > 0$, the theorem follows.

Despite the “short interval” form of our hypothesis (3), there seems to be no advantage over a direct estimation of the error term $\Delta(x)$ in the asymptotic formula

$$S(x) = \zeta\left(\frac{3}{2}\right)x^{\frac{1}{2}} + \zeta\left(\frac{2}{3}\right)x^{\frac{1}{3}} + \Delta(x).$$

It was shown by Richert [4] that

$$\Delta(x) \ll x^{2/15},$$

by a simple exponential sum method, and Shiu [5] improved this slightly to

$$\Delta(x) \ll x^{0.1318161\dots},$$

by using two dimensional sums. This estimate provides our corollary. However it is apparent that further small reductions in the exponent are possible by more complicated exponential sum techniques.

References

- [1] P.T. Bateman and E. Grosswald, On a theorem of Erdős and Szekeres, *Illinois J. Math.*, 2 (1958), 88-98.
- [2] C.-H. Jia, The square-full integers in the short interval, *Acta Math. Sinica*, 5 (1987), 614-621.
- [3] H. Liu, On square-full numbers in short intervals, *Acta Math. Sinica*, 6 (1990), 148-164.
- [4] H.-E. Richert, Über die Anzahl Abelscher Gruppen gegebener Ordnung. I, *Math. Zeit.*, 56 (1952), 21-32.
- [5] P. Shiu, On square-full integers in a short interval, *Glasgow Math. J.*, 25 (1984), 127-134.