Pointwise polarization tensor bounds, and applications to voltage perturbations caused by thin inhomogeneities.

Yves Capdeboscq and Michael S. Vogelius *

April 20, 2006

Abstract

The main result of this paper establishes optimal pointwise bounds for the polarization tensors that appear in representation formulae for the voltage perturbations caused by low volume fraction inhomogeneities. Furthermore it is demonstrated how these pointwise bounds may be used to derive a particularly simple version of the representation formulae in the case of thin inhomogeneities.

1. Introduction and statement of main result
2. Some preliminary results
3. Proof of Theorem 3
4. Applications of the pointwise trace bounds

1 Introduction and statement of main result

In earlier work [8] we have derived a very general representation formula for the voltage perturbation caused by volumetrically small inhomogeneities in an otherwise known conductor. The ingredients of this formula are (1) a limiting probability measure, (2) a "background" fundamental solution, and (3) an "effective" polarization tensor. In [9] we proved optimal average bounds on the very essential polarization tensor, and we used these bounds to obtain sharp estimates for the volume of the inhomogeneities in terms of a few boundary voltage measurements. The principal goal of this paper is to prove that these optimal average bounds actually hold in a pointwise sense. A second goal is to show how the optimal pointwise bounds may be used to establish a particular, very specific form of the representation formula for thin inhomogeneities. This "thin inhomogeneity

*Laboratoire de Mathématiques de Versailles, Université de Versailles Saint-Quentin-en-Yvelines, 45, avenue des États Unis, 78035 Versailles cedex, France, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

representation formula" generalizes the one derived under much more restrictive smoothness assumptions in [5]; the proof given here also significantly simplifies the proof given in [4]. We now briefly recall the situation and the main result from [8].

We consider a conducting object that occupies a bounded, smooth domain $\Omega \subset I\!R^m$. For simplicity we take $\partial\Omega$ to be C^∞, but this assumption could be considerably weakened. $\gamma_0(\cdot)$ denotes the smooth background conductivity, that is, the conductivity in the absence of any inhomogeneities. We suppose that

$$0 < c_0 \leq \gamma_0(x) \leq C_0 < \infty, \quad x \in \Omega$$

for some fixed constants c_0 and C_0. For simplicity, we assume that γ_0 is $C^\infty(\bar{\Omega})$, but this latter assumption could also be considerably weakened. The function ψ denotes the imposed boundary current. It suffices that $\psi \in H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega} \psi \; d\sigma \; = 0$. The background voltage potential, U, is the solution to the boundary value problem

$$\begin{aligned} \nabla \cdot (\gamma_0(x) \nabla U) &= 0 \quad \text{in } \Omega \; , \\ \gamma_0(x) \frac{\partial U}{\partial n} &= \psi \quad \text{on } \partial\Omega \quad . \end{aligned} \tag{1}$$

Here n denotes the unit outward normal to the domain Ω.

Let ω_ϵ denote a set of inhomogeneities inside Ω. The geometric assumptions about the set of inhomogeneities are very simple: we suppose the set ω_ϵ is measurable, and separated away from the boundary, (*i.e.*, $\text{dist}(\omega_\epsilon, \partial\Omega) > d_0 > 0$). Most importantly, we suppose that $0 < |\omega_\epsilon|$ gets arbitrarily small, where $|\omega_\epsilon|$ denotes the Lebesgue measure of ω_ϵ. Let $\hat{\gamma}_\epsilon$ denote the conductivity profile in the presence of the inhomogeneities. The function $\hat{\gamma}_\epsilon$ is equal to γ_0, except on the set of inhomogeneities; on the set of inhomogeneities we suppose that $\hat{\gamma}_\epsilon$ equals the restriction of some other smooth function, $\gamma_1 \in C^\infty(\overline{\Omega})$, with

$$0 < c_1 \leq \gamma_1(x) \leq C_1 < \infty, \quad x \in \Omega \quad .$$

In other words

$$\hat{\gamma}_\epsilon(x) = \begin{cases} \gamma_0(x), & x \in \Omega \setminus \omega_\epsilon \\ \gamma_1(x), & x \in \omega_\epsilon \end{cases}$$

The voltage potential in the presence of the inhomogeneities is denoted $u_\epsilon(x)$. It is the solution to

$$\begin{aligned} \nabla \cdot (\hat{\gamma}_\epsilon(x) \nabla u_\epsilon) &= 0 \quad \text{in } \Omega \; , \\ \hat{\gamma}_\epsilon(x) \frac{\partial u_\epsilon}{\partial n} &= \psi \quad \text{on } \partial\Omega \quad . \end{aligned} \tag{2}$$

We normalize both U and u_ϵ by requiring that

$$\int_{\partial\Omega} U d\sigma = 0 \; , \quad \text{and} \quad \int_{\partial\Omega} u_\epsilon d\sigma = 0 \quad .$$

We note that the individual voltages U and u_ϵ need not be smooth (or even continuous) on $\partial\Omega$, however, the difference $u_\epsilon - U$ is smooth in a neighborhood of $\partial\Omega$, due to the regularity of γ_0, and the fact that ω_ϵ is strictly interior.

Before stating the main theorem from [8] we shall make some preliminary observations. Let 1_{ω_ϵ} denote the characteristic function corresponding to the set ω_ϵ, *i.e.,* the function which takes the value 1 on the set and the value 0 outside. Since the family of functions $|\omega_\epsilon|^{-1}1_{\omega_\epsilon}$ is bounded in $L^1(\Omega)$, it follows from a combination of the Banach-Alaoglu Theorem and the Riesz Representation Theorem that we may find a regular, positive Borel measure μ, and a subsequence ω_{ϵ_n}, with $|\omega_{\epsilon_n}| \to 0$, such that

$$|\omega_{\epsilon_n}|^{-1}1_{\omega_{\epsilon_n}}\, dx \to d\mu \quad . \tag{3}$$

The convergence refers to the weak* topology of the dual of $C^0(\overline{\Omega})$. More precisely, for any $\phi \in C^0(\overline{\Omega})$

$$|\omega_{\epsilon_n}|^{-1} \int_{\omega_{\epsilon_n}} \phi \, dx \to \int_\Omega \phi \, d\mu \quad .$$

The measure μ satisfies $\int_\Omega d\mu = 1$, so it is indeed a probability measure. Due to the fact that the sets ω_ϵ stay uniformly bounded away from the boundary, there exists a fixed compact set $K_0 \subset \Omega$ which strictly contains ω_ϵ, in the sense that

$$\omega_\epsilon \subset K_0 \subset \Omega \quad , \quad \text{and} \quad \operatorname{dist}(\omega_\epsilon, \Omega \setminus K_0) > \delta_0 > 0 \quad . \tag{4}$$

The support of μ lies inside the same compact set K_0. We shall need the so called Neumann function $N(x,y)$ for the operator $\nabla \cdot (\gamma_0 \nabla \quad)$. For $y \in \Omega$, $N(\cdot, y)$ is the solution to the boundary value problem

$$\begin{aligned} &\nabla_x \cdot (\gamma_0(x) \nabla_x N(x,y)) = \delta_y \quad \text{in } \Omega \ , \\ &\gamma_0(x) \frac{\partial N}{\partial n_x} = \frac{1}{|\partial\Omega|} \quad \text{on } \partial\Omega \quad , \end{aligned}$$

normalized by $\int_{\partial\Omega} N(x,y)\, d\sigma_x = 0$. The function $N(x,y)$ may be extended by continuity to $y \in \overline{\Omega}$. For $y \in \partial\Omega$ the function $N(\cdot, y)$ may also be interpreted as the solution to the boundary value problem

$$\begin{aligned} &\nabla_x \cdot (\gamma_0(x) \nabla_x N(x,y)) = 0 \quad \text{in } \Omega \ , \\ &\gamma_0(x) \frac{\partial N}{\partial n_x} = -\delta_y + \frac{1}{|\partial\Omega|} \quad \text{on } \partial\Omega \quad . \end{aligned}$$

The main result from [8] may now be formulated as follows.

Theorem 1. *Let ω_{ϵ_n} be a sequence of measurable subsets, with $|\omega_{\epsilon_n}| \to 0$, for which (3) and (4) hold. Given any $\psi \in H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega} \psi \, d\sigma = 0$, let U and*

u_{ϵ_n} denote the solutions to (1) and (2), respectively. There exists a subsequence, also denoted ω_{ϵ_n}, and a matrix valued function $M \in L^2(\Omega, d\mu)$ such that

$$\begin{aligned}(u_{\epsilon_n} - U)(y) &= |\omega_{\epsilon_n}| \int_\Omega (\gamma_1 - \gamma_0)(x) M_{ij}(x) \frac{\partial U}{\partial x_i} \frac{\partial N}{\partial x_j}(x, y)\ d\mu(x) \\ &\quad + o(|\omega_{\epsilon_n}|) \qquad y \in \partial\Omega \quad . \end{aligned} \tag{5}$$

The values of the function $M(\cdot)$ are symmetric, positive definite matrices in the sense that

$$\begin{aligned} M_{ij}(x) &= M_{ji}(x) \quad , \qquad \textit{and} \\ \min\{1, \frac{\gamma_0(x)}{\gamma_1(x)}\}\ |\xi|^2 \le M_{ij}(x)\xi_i\xi_j &\le \max\{1, \frac{\gamma_0(x)}{\gamma_1(x)}\}\ |\xi|^2 \quad , \\ \xi \in \mathbf{R}^m \quad , &\quad \mu \textit{ almost everywhere in the set } \{\ x\ :\ \gamma_0(x) \neq \gamma_1(x)\ \} \quad . \end{aligned} \tag{6}$$

The subsequence ω_{ϵ_n} and the matrix valued function $M \in L^2(\Omega, d\mu)$ are independent of the boundary flux ψ. The term $o(|\omega_{\epsilon_n}|)$ is such that $\|o(|\omega_{\epsilon_n}|)\|_{L^\infty(\partial\Omega)}/|\omega_{\epsilon_n}|$ converges to 0 for any fixed $\psi \in H^{-1/2}$, and uniformly on $\{\psi\ :\ \int_{\partial\Omega} \psi\ d\sigma = 0\ ,\quad \|\psi\|_{L^2(\partial\Omega)} \le 1\}$.

Given any $\xi \in I\!R^m$ let V and v_ϵ denote the solutions to

$$\nabla \cdot (\gamma_0 \nabla V) = \nabla \cdot (\gamma_0 \xi) \text{ in } \Omega, \quad \gamma_0 \frac{\partial V}{\partial n} = \gamma_0 \xi \cdot n \text{ on } \partial\Omega\ , \tag{7}$$

and

$$\nabla \cdot (\hat{\gamma}_\epsilon \nabla v_\epsilon) = \nabla \cdot (\gamma_0 \xi) \text{ in } \Omega, \quad \hat{\gamma}_\epsilon \frac{\partial v_\epsilon}{\partial n} = \gamma_0 \xi \cdot n \text{ on } \partial\Omega\ , \tag{8}$$

respectively. We note that $V(x)$ has a very simple formula: $V(x) = \xi \cdot x + \text{cst}$ – for convenience we take the constant to be zero. In [8] and [9] we proved the following result, which provides a fairly explicit characterization of the polarization tensor M, whose existence is asserted by Theorem 1.

Proposition 1. *Let ω_{ϵ_n} be a sequence of measurable subsets, with $|\omega_{\epsilon_n}| \to 0$, for which (3) and (4) hold. Let V and v_ϵ denote solutions to (7) and (8), respectively. There exists a subsequence, also denoted ω_{ϵ_n}, and a matrix valued function $M \in L^2(\Omega, d\mu)$, characterized by*

$$\int_\Omega M_{ij}\xi_i\xi_j\phi d\mu = \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} \nabla v_{\epsilon_n} \cdot \nabla V \phi dx \ +\ o(1)\ , \tag{9}$$

for any $\phi \in C^0(\overline{\Omega})$ and any $\xi \in I\!R^m$. The matrix valued function $M(\cdot)$ and the subsequence ω_{ϵ_n} have the exact properties stated in Theorem 1, in particular the representation formula (5) holds for any boundary flux $\psi \in H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega} \psi\ d\sigma = 0$.

We refer the reader to [8] for remarks concerning variations and immediate generalizations of the representation formula (5) for u_{ϵ_n}. Explicit representation formulas of this type are of significant interest from an imaging point of view. For instance: if one has very detailed knowledge of the "voltage boundary signatures" of internal inhomogeneities, then it becomes possible to design very effective numerical methods to identify the volume of the inhomogeneities and their location. We refer the reader to [1], [2], [7], [9], and [10] for examples of numerical methods based on such formulas. The optimal estimates for the volume of ω_ϵ given in [9] and [10] rely on the following "average" characterization of the polarization tensor M.

Theorem 2. *Suppose the conductivities γ_0 and γ_1 are constants, with $\gamma_0 \neq \gamma_1$. Let μ and $M_{ij}(\cdot) \in L^2(\Omega, d\mu)$ be the limiting probability measure and the polarization tensor from Proposition 1 (and Theorem 1), and let $a(s)$ denote the average of $m-1$ copies of 1 and one copy of s, i.e.,*

$$a(s) = \frac{1}{m}((m-1)+s) \quad .$$

Then

$$Trace\left(\int_\Omega M \; d\mu\right) \leq m \; a\left(\frac{\gamma_0}{\gamma_1}\right) \quad ,$$

and

$$Trace\left(\int_\Omega M \; d\mu\right)^{-1} \leq m \; a\left(\frac{\gamma_1}{\gamma_0}\right) \quad .$$

As stated earlier one goal of this paper is to prove that these bounds actually hold in a pointwise sense. We formulate this as the following theorem.

Theorem 3. *Let γ_0 and γ_1 be C^∞ conductivities. Let μ and $M_{ij}(\cdot) \in L^2(\Omega, d\mu)$ be the limiting probability measure and the polarization tensor from Proposition 1 (and Theorem 1), and let $a(s)$ denote the average of $m-1$ copies of 1 and one copy of s, i.e.,*

$$a(s) = \frac{1}{m}((m-1)+s) \quad .$$

Then

$$Trace\; M(x) \leq m \; a\left(\frac{\gamma_0(x)}{\gamma_1(x)}\right) \quad ,$$

and

$$Trace\; M^{-1}(x) \leq m \; a\left(\frac{\gamma_1(x)}{\gamma_0(x)}\right) \quad ,$$

μ almost everywhere in the set $\{\; x \;:\; \gamma_0(x) \neq \gamma_0(x) \;\}$.

2 Some preliminary results

The purpose of this section is to derive alternate characterizations of the polarization tensor M, alternate, that is, to the one already provided by Proposition 1. The principal results in that direction are the variational characterizations given in Lemma 3 and Lemma 4. We first note that we could equally well have defined M, through the solution of auxiliary Dirichlet problems, and we also note that the resulting M is independent of the bounded domain Ω (as long as $\omega_{\epsilon_n} \subset K_0 \subset \Omega$). These statements are an immediate consequence of the last identities in following lemma.

Lemma 1. *Suppose $\omega_\epsilon \subset K_0 \subset \Omega \subset\subset \Omega_1$. Suppose the bounded domains Ω and Ω_1 are either smooth (C^2) or Lipschitz and convex, and suppose γ_0 and γ_1 have been extended to positive C^∞ functions on all of $\overline{\Omega}_1$. For fixed $\xi \in I\!R^m$ let $V = \xi \cdot x$, and let v_ϵ be a solution to (8). Let $v_\epsilon^{(1)}$, $\tilde{v}_\epsilon^{(1)}$ be solutions to*

$$\nabla \cdot \left(\hat{\gamma}_\epsilon \nabla v_\epsilon^{(1)}\right) = \nabla \cdot (\gamma_0 \xi) \ \text{ in } \Omega_1, \quad \hat{\gamma}_\epsilon \frac{\partial v_\epsilon^{(1)}}{\partial n} = \gamma_0 \xi \cdot n \ \text{ on } \partial\Omega_1 \ . \tag{10}$$

and

$$\nabla \cdot \left(\hat{\gamma}_\epsilon \nabla \tilde{v}_\epsilon^{(1)}\right) = \nabla \cdot (\gamma_0 \xi) \ \text{ in } \Omega_1, \quad \tilde{v}_\epsilon^{(1)} = V \ (= \xi \cdot x) \ \text{ on } \partial\Omega_1 \ . \tag{11}$$

Suppose v_ϵ and $v_\epsilon^{(1)}$, are "normalized" by

$$\int_\Omega v_\epsilon \, dx = \int_\Omega V \, dx \quad \text{and} \quad \int_{\Omega_1} v_\epsilon^{(1)} \, dx = \int_{\Omega_1} V \, dx \ .$$

Then

$$\|\tilde{v}_\epsilon^{(1)} - V\|_{H^1(\Omega_1)} + \|v_\epsilon^{(1)} - V\|_{H^1(\Omega_1)} + \|v_\epsilon - V\|_{H^1(\Omega)} \leq C|\omega_\epsilon|^{\frac{1}{2}} \ ,$$

and, for any $\eta > 0$, there exists a constant C_η such that

$$\|\tilde{v}_\epsilon^{(1)} - V\|_{L^2(\Omega_1)} + \|v_\epsilon^{(1)} - V\|_{L^2(\Omega_1)} + \|v_\epsilon - V\|_{L^2(\Omega)} \leq C_\eta |\omega_\epsilon|^{\frac{1}{2} + \frac{1}{m^*} - \eta} \ ,$$

with $m^* = \max\{m, 2\}$ (m being the dimension of the ambient space). Furthermore

$$\int_{\omega_\epsilon} \nabla \tilde{v}_\epsilon^{(1)} \cdot \nabla V \ \phi \, dx = \int_{\omega_\epsilon} \nabla v_\epsilon^{(1)} \cdot \nabla V \ \phi \, dx + o(|\omega_\epsilon|) = \int_{\omega_\epsilon} \nabla v_\epsilon \cdot \nabla V \ \phi \, dx + o(|\omega_\epsilon|) \ ,$$

for any $\phi \in C^0(\overline{\Omega})$.

Proof. A proof of the inequalities

$$\|v_\epsilon^{(1)} - V\|_{H^1(\Omega_1)} + \|v_\epsilon - V\|_{H^1(\Omega)} \leq C|\omega_\epsilon|^{\frac{1}{2}}$$

and

$$\|v_\epsilon^{(1)} - V\|_{L^2(\Omega_1)} + \|v_\epsilon - V\|_{L^2(\Omega)} \leq C_\eta |\omega_\epsilon|^{\frac{1}{2} + \frac{1}{m^*} - \eta}$$

is found in [8] (*cf.* Lemma 1 of [8]). Similarly a proof of the identity

$$\int_{\omega_\epsilon} \nabla v_\epsilon^{(1)} \cdot \nabla V \ \phi \ dx = \int_{\omega_\epsilon} \nabla v_\epsilon \cdot \nabla V \ \phi \ dx + o(|\omega_\epsilon|)$$

is found in [9] (*cf.* Proposition 1 of [9]). The remaining inequalities

$$\|\tilde{v}_\epsilon^{(1)} - V\|_{H^1(\Omega_1)} \leq C|\omega_\epsilon|^{\frac{1}{2}} \quad , \qquad \|\tilde{v}_\epsilon^{(1)} - V\|_{L^2(\Omega_1)} \leq C_\eta |\omega_\epsilon|^{\frac{1}{2}+\frac{1}{m^*}-\eta} \quad , \tag{12}$$

and the remaining identity

$$\int_{\omega_\epsilon} \nabla \tilde{v}_\epsilon^{(1)} \cdot \nabla V \ \phi \ dx = \int_{\omega_\epsilon} \nabla v_\epsilon \cdot \nabla V \ \phi \ dx + o(|\omega_\epsilon|) \tag{13}$$

is proved in ways that are very similar to those presented in [8] and [9], but for the sake of completeness we provide an outline of the arguments. From the variational formulations of (7) and (11) it follows directly that

$$\begin{aligned} \left| \int_{\Omega_1} \hat{\gamma}_\epsilon \nabla(\tilde{v}_\epsilon^{(1)} - V) \cdot \nabla w \ dx \right| &= \left| \int_{\Omega_1} (\gamma_0 - \hat{\gamma}_\epsilon) \nabla V \cdot \nabla w \ dx \right| \\ &\leq C|\omega_\epsilon|^{1/2} \|\nabla V\|_{L^\infty(\omega_\epsilon)} \|\nabla w\|_{L^2(\Omega_1)} \\ &\leq C|\xi||\omega_\epsilon|^{1/2} \|\nabla w\|_{L^2(\Omega_1)} \quad , \end{aligned}$$

for any $w \in H_0^1(\Omega_1)$. This verifies the first inequality of (12). We also have

$$\int_{\Omega_1} \gamma_0 \nabla(\tilde{v}_\epsilon^{(1)} - V) \cdot \nabla w \ dx = \int_{\Omega_1} (\gamma_0 - \hat{\gamma}_\epsilon) \nabla \tilde{v}_\epsilon^{(1)} \cdot \nabla w \ dx \quad , \quad \forall w \in H_0^1(\Omega_1) \quad . \tag{14}$$

Select w_0 as the solution to

$$\nabla \cdot (\gamma_0 \nabla w_0) = V - \tilde{v}_\epsilon^{(1)} \quad \text{in } \Omega_1 \quad , \qquad w_0 = 0 \quad \text{on } \partial\Omega_1 \quad .$$

Elliptic estimates show that $\|w_0\|_{H^2(\Omega_1)} \leq C\|V - \tilde{v}_\epsilon^{(1)}\|_{L^2(\Omega_1)}$. This is true if Ω_1 is C^2, and it is also true if Ω_1 is only Lipschitz, but convex (cf. [13] Theorem 3.2.1.2). A similar estimate holds for the corresponding Neumann problem for both C^2 and convex domains (cf. [13] Theorem 3.2.1.3). After insertion of w_0 into (14) we obtain

$$\begin{aligned} \int_{\Omega_1} (\tilde{v}_\epsilon^{(1)} - V)^2 \ dx &= \int_{\Omega_1} \gamma_0 \nabla(\tilde{v}_\epsilon^{(1)} - V) \cdot \nabla w_0 \ dx \\ &= \left| \int_{\Omega_1} (\gamma_0 - \hat{\gamma}_\epsilon) \nabla \tilde{v}_\epsilon^{(1)} \cdot \nabla w_0 \ dx \right| \\ &\leq C \left(\int_{\omega_\epsilon} |\nabla \tilde{v}_\epsilon^{(1)}|^q \ dx \right)^{1/q} \left(\int_{\Omega_1} |\nabla w_0|^p \ dx \right)^{1/p} \\ &\leq C_q \left(\int_{\omega_\epsilon} |\nabla \tilde{v}_\epsilon^{(1)}|^q \ dx \right)^{1/q} \|w_0\|_{H^2(\Omega_1)} \\ &\leq C_q \left(\int_{\omega_\epsilon} |\nabla \tilde{v}_\epsilon^{(1)}|^q \ dx \right)^{1/q} \|V - \tilde{v}_\epsilon^{(1)}\|_{L^2(\Omega_1)} \quad , \end{aligned} \tag{15}$$

provided p and q are related by $\frac{1}{q}+\frac{1}{p}=1$, and provided we require that $q>\frac{2m^*}{m^*+2}$ (so that $1<p<\frac{2m^*}{m^*-2}$, and therefore, by Sobolev's Imbedding Theorem $(\int_{\Omega_1}|\nabla w_0|^p\ dx)^{1/p}\leq C_p\|w_0\|_{H^2(\Omega_1)}$, *cf.* [12], p. 155). For any $1<q<2$

$$\begin{aligned}\|\nabla\tilde{v}_\epsilon^{(1)}\|_{L^q(\omega_\epsilon)} &\leq \|\nabla(\tilde{v}_\epsilon^{(1)}-V)\|_{L^q(\omega_\epsilon)}+\|\nabla V\|_{L^q(\omega_\epsilon)}\\ &\leq \left(\int_{\omega_\epsilon}1\ dx\right)^s\|\nabla(\tilde{v}_\epsilon^{(1)}-V)\|_{L^2(\omega_\epsilon)}+|\xi||\omega_\epsilon|^{1/q}\\ &\leq C(|\omega_\epsilon|^{s+1/2}+|\omega_\epsilon|^{1/q})\ ,\end{aligned}\tag{16}$$

with $s=\frac{1}{q}-\frac{1}{2}$. A combination of (15) and (16) yields

$$\|\tilde{v}_\epsilon^{(1)}-V\|_{L^2(\Omega_1)}\leq C_q\left(\int_{\omega_\epsilon}|\nabla(\tilde{v}_\epsilon^{(1)}|^q\ dx\right)^{1/q}\leq C_q|\omega_\epsilon|^{1/q}\ ,$$

for any $\frac{2m^*}{m^*+2}<q<2$. We note that $\frac{1}{q}$ approaches $\frac{m^*+2}{2m^*}=\frac{1}{2}+\frac{1}{m^*}$ from below as q approaches $\frac{2m^*}{m^*+2}$ from above. The previous estimate now immediately implies that, given any $\eta>0$, there exists a constant C_η such that

$$\|\tilde{v}_\epsilon^{(1)}-V\|_{L^2(\Omega_1)}\leq C_\eta|\omega_\epsilon|^{\frac{1}{2}+\frac{1}{m*}-\eta}\ ,$$

exactly the second inequality in (12). It only remains to prove the identity (13). To that end we introduce $D=\{x\in\Omega_1\ :\ \mathrm{dist}(x,\partial\Omega)<\delta\}$, a sufficiently small neighborhood of $\partial\Omega$. Since $\tilde{v}_\epsilon^{(1)}-V$ is γ_0-harmonic outside K_0, elliptic estimates, in combination with the second inequality of (12), imply that

$$\|\tilde{v}_\epsilon^{(1)}-V\|_{C^1(D)}\leq C\|\tilde{v}_\epsilon^{(1)}-V\|_{L^2(\Omega_1)}\leq C_\eta|\omega_\epsilon|^{\frac{1}{2}+\frac{1}{m*}-\eta}\ .$$

Hence

$$\|\hat{\gamma}_\epsilon\frac{\partial\tilde{v}_\epsilon^{(1)}}{\partial n}-\hat{\gamma}_\epsilon\frac{\partial v_\epsilon}{\partial n}\|_{C^0(\partial\Omega)}=\|\hat{\gamma}_\epsilon\frac{\partial\tilde{v}_\epsilon^{(1)}}{\partial n}-\hat{\gamma}_\epsilon\frac{\partial V}{\partial n}\|_{C^0(\partial\Omega)}\leq C_\eta|\omega_\epsilon|^{\frac{1}{2}+\frac{1}{m*}-\eta}\ ,$$

and therefore, by energy estimates

$$\|\nabla\tilde{v}_\epsilon^{(1)}-\nabla v_\epsilon\|_{L^2(\Omega)}\leq\|\hat{\gamma}_\epsilon\frac{\partial\tilde{v}_\epsilon^{(1)}}{\partial n}-\hat{\gamma}_\epsilon\frac{\partial v_\epsilon}{\partial n}\|_{H^{-1/2}(\partial\Omega)}\leq C_\eta|\omega_\epsilon|^{\frac{1}{2}+\frac{1}{m*}-\eta}\ .$$

The last inequality immediately implies that

$$\left|\int_{\omega_\epsilon}(\nabla\tilde{v}_\epsilon^{(1)}-\nabla v_\epsilon)\cdot\nabla V\ \phi\ dx\right|\leq C_{\eta,\phi}|\omega_\epsilon|^{1/2}\|\nabla\tilde{v}_\epsilon^{(1)}-\nabla v_\epsilon\|_{L^2(\Omega)}\leq C_{\eta,\phi}|\omega_\epsilon|^{1+\frac{1}{m^*}-\eta}\ ,$$

for any $\phi\in C^0(\overline{\Omega})$, exactly as desired in (13). □

We now introduce $w_\epsilon = v_\epsilon - V$, $w_\epsilon^{(1)} = v_\epsilon^{(1)} - V$, and $\tilde{w}_\epsilon^{(1)} = \tilde{v}_\epsilon^{(1)} - V$. With ϕ replaced by $\phi(\gamma_1 - \gamma_0)$ (9) takes the form

$$\begin{aligned}\int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j\phi d\mu &= \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0)\,|\xi|^2\,\phi\; dx \\ &\quad + \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0)\,\nabla w_{\epsilon_n}\cdot\xi\phi\; dx \;+\; o(1)\;. \end{aligned} \tag{17}$$

The functions $w_\epsilon^{(1)}$ and $\tilde{w}_\epsilon^{(1)}$ are solutions to

$$\nabla\cdot\left(\hat{\gamma}_\epsilon\nabla w_\epsilon^{(1)}\right) = \nabla\cdot\left((\gamma_0 - \gamma_1)\,1_{\omega_\epsilon}\xi\right) \text{ in } \Omega_1, \quad \hat{\gamma}_\epsilon\frac{\partial w_\epsilon^{(1)}}{\partial n} = 0 \text{ on } \partial\Omega_1 \;\;, \tag{18}$$

and

$$\nabla\cdot\left(\hat{\gamma}_\epsilon\nabla \tilde{w}_\epsilon^{(1)}\right) = \nabla\cdot\left((\gamma_0 - \gamma_1)\,1_{\omega_\epsilon}\xi\right) \text{ in } \Omega_1, \quad \tilde{w}_\epsilon^{(1)} = 0 \text{ on } \partial\Omega_1 \;\;, \tag{19}$$

respectively.

From now on, unless otherwise stated, we only consider smooth functions $\phi \in C^\infty(\overline{\Omega})$ that are uniformly positive, *i.e.*, satisfy $\min_{x\in\overline{\Omega_1}}\phi(x) > 0$. We introduce the functions $w_{\epsilon,\phi}^{(1)}$ and $\tilde{w}_{\epsilon,\phi}^{(1)}$ as solutions to

$$\nabla\cdot\left(\hat{\gamma}_\epsilon\phi\nabla w_{\epsilon,\phi}^{(1)}\right) = \nabla\cdot\left((\gamma_0 - \gamma_1)\,1_{\omega_\epsilon}\xi\phi\right) \text{ in } \Omega_1, \quad \hat{\gamma}_\epsilon\phi\frac{\partial w_{\epsilon,\phi}^{(1)}}{\partial n} = 0 \text{ on } \partial\Omega_1 \;\;, \tag{20}$$

normalized by $\int_{\Omega_1} w_{\epsilon,\phi}^{(1)}\; dx = 0$, and

$$\nabla\cdot\left(\hat{\gamma}_\epsilon\phi\nabla \tilde{w}_{\epsilon,\phi}^{(1)}\right) = \nabla\cdot\left((\gamma_0 - \gamma_1)\,1_{\omega_\epsilon}\xi\phi\right) \text{ in } \Omega_1, \quad \tilde{w}_{\epsilon,\phi}^{(1)} = 0 \text{ on } \partial\Omega_1 \;\;. \tag{21}$$

Lemma 1 asserts that

$$\|\nabla\tilde{w}_\epsilon^{(1)}\|_{L^2(\Omega_1)} + \|\nabla w_\epsilon^{(1)}\|_{L^2(\Omega_1)} = O(|\omega_\epsilon|^{1/2}) \;\;, \tag{22}$$

$$\|\tilde{w}_\epsilon^{(1)}\|_{L^2(\Omega_1)} + \|w_\epsilon^{(1)}\|_{L^2(\Omega_1)} = o(|\omega_\epsilon|^{1/2}) \;\;,$$

and that

$$\begin{aligned}\int_{\omega_\epsilon}\nabla\tilde{w}_\epsilon^{(1)}\cdot\xi(\gamma_1 - \gamma_0)\;\phi\; dx &= \int_{\omega_\epsilon}\nabla w_\epsilon^{(1)}\cdot\xi(\gamma_1 - \gamma_0)\;\phi\; dx + o(|\omega_\epsilon|) \\ &= \int_{\omega_\epsilon}\nabla w_\epsilon\cdot\xi(\gamma_1 - \gamma_0)\;\phi\; dx + o(|\omega_\epsilon|) \;\;. \end{aligned} \tag{23}$$

Since $w_{\epsilon,\phi}^{(1)}$ and $\tilde{w}_{\epsilon,\phi}^{(1)}$ are the same as $w_\epsilon^{(1)}$ and $\tilde{w}_\epsilon^{(1)}$ in the case when γ_0 and γ_1 are replaced by $\phi\gamma_0$ and $\phi\gamma_1$, it follows immediately from Lemma 1 that

$$\|\nabla\tilde{w}_{\epsilon,\phi}^{(1)}\|_{L^2(\Omega_1)} + \|\nabla w_{\epsilon,\phi}^{(1)}\|_{L^2(\Omega_1)} = O(|\omega_\epsilon|^{1/2}) \;\;,$$

$$\|\tilde{w}^{(1)}_{\epsilon,\phi}\|_{L^2(\Omega_1)} + \|w^{(1)}_{\epsilon,\phi}\|_{L^2(\Omega_1)} = o(|\omega_\epsilon|^{1/2}) \quad , \tag{24}$$

and that

$$\int_{\omega_\epsilon} \nabla\tilde{w}^{(1)}_{\epsilon,\phi} \cdot \xi(\gamma_1 - \gamma_0)\ \phi\ dx = \int_{\omega_\epsilon} \nabla w^{(1)}_{\epsilon,\phi} \cdot \xi(\gamma_1 - \gamma_0)\ \phi\ dx + o(|\omega_\epsilon|) \quad . \tag{25}$$

Using the estimate (24), the weak formulation of (18), the estimate (22), and the weak formulation of (20), in that order, we calculate

$$\begin{aligned}
\int_{\omega_\epsilon} \nabla w^{(1)}_{\epsilon,\phi} \cdot \xi(\gamma_1 - \gamma_0)\ \phi\ dx &= \int_{\omega_\epsilon} \nabla\left(\phi w^{(1)}_{\epsilon,\phi}\right) \cdot \xi(\gamma_1 - \gamma_0)\ dx + o(|\omega_\epsilon|) \\
&= -\int_{\Omega_1} \hat{\gamma}_\epsilon \nabla\left(\phi w^{(1)}_{\epsilon,\phi}\right) \cdot \nabla w^{(1)}_{\epsilon}\ dx + o(|\omega_\epsilon|) \\
&= -\int_{\Omega_1} \hat{\gamma}_\epsilon \phi \nabla w^{(1)}_{\epsilon,\phi} \cdot \nabla w^{(1)}_{\epsilon}\ dx + o(|\omega_\epsilon|) \\
&= \int_{\omega_\epsilon} \nabla w^{(1)}_{\epsilon} \cdot \xi(\gamma_1 - \gamma_0)\ \phi\ dx + o(|\omega_\epsilon|) \\
&= \int_{\omega_\epsilon} \nabla w_\epsilon \cdot \xi(\gamma_1 - \gamma_0)\ \phi\ dx + o(|\omega_\epsilon|) \quad .
\end{aligned} \tag{26}$$

For the last identity we used (23). Combining the weak formulations of the equations for $w^{(1)}_{\epsilon,\phi}$ and $\tilde{w}^{(1)}_{\epsilon,\phi}$ with (25) and (26) we now obtain the following lemma.

Lemma 2. *Let Ω_1, and $w_\epsilon = v_\epsilon - V$ be as in Lemma 1. Let $w^{(1)}_{\epsilon,\phi}$, and $\tilde{w}^{(1)}_{\epsilon,\phi}$ be as above, for some smooth, uniformly positive ϕ. Then*

$$\begin{aligned}
\int_{\Omega_1} \hat{\gamma}_\epsilon \phi \nabla w^{(1)}_{\epsilon,\phi} \cdot \nabla w^{(1)}_{\epsilon,\phi}\ dx &= -\int_{\Omega_1} \nabla w^{(1)}_{\epsilon,\phi} \cdot (\gamma_1 - \gamma_0) 1_{\omega_\epsilon} \xi\, \phi\ dx \\
&= -\int_{\Omega} \nabla w_\epsilon \cdot (\gamma_1 - \gamma_0) 1_{\omega_\epsilon} \xi\, \phi\ dx + \ o(|\omega_\epsilon|)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega_1} \hat{\gamma}_\epsilon \phi \nabla \tilde{w}^{(1)}_{\epsilon,\phi} \cdot \nabla \tilde{w}^{(1)}_{\epsilon,\phi}\ dx &= -\int_{\Omega_1} \nabla \tilde{w}^{(1)}_{\epsilon,\phi} \cdot (\gamma_1 - \gamma_0) 1_{\omega_\epsilon} \xi\, \phi\ dx \\
&= -\int_{\Omega} \nabla w_\epsilon \cdot (\gamma_1 - \gamma_0) 1_{\omega_\epsilon} \xi\, \phi\ dx + \ o(|\omega_\epsilon|) \quad .
\end{aligned}$$

A simple computation, based on Lemma 2, gives

$$\begin{aligned}
&\int_{\Omega_1} \hat{\gamma}_\epsilon \left| \nabla w^{(1)}_{\epsilon,\phi} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi\ dx \\
&\quad = \int_{\omega_\epsilon} \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1} |\xi|^2\, \phi\ dx + \int_{\Omega_1} \hat{\gamma}_\epsilon \nabla w^{(1)}_{\epsilon,\phi} \cdot \nabla w^{(1)}_{\epsilon,\phi}\, \phi\ dx \\
&\qquad +2 \int_{\Omega_1} \nabla w^{(1)}_{\epsilon,\phi} \cdot (\gamma_1 - \gamma_0) 1_{\omega_\epsilon} \xi \phi\ dx \\
&\quad = \int_{\omega_\epsilon} \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1} |\xi|^2\, \phi\ dx + \int_{\Omega} \nabla w_\epsilon \cdot (\gamma_1 - \gamma_0) 1_{\omega_\epsilon} \xi \phi\ dx \ + \ o(|\omega_\epsilon|) \quad ,
\end{aligned}$$

or, after rearrangement,

$$\int_\Omega \nabla w_\epsilon \cdot (\gamma_1 - \gamma_0) 1_{\omega_\epsilon} \xi \phi \, dx \tag{27}$$
$$= -\int_{\omega_\epsilon} \frac{(\gamma_1 - \gamma_0)^2}{\gamma_1} |\xi|^2 \, \phi \, dx + \int_{\Omega_1} \hat{\gamma}_\epsilon \left| \nabla w^{(1)}_{\epsilon,\phi} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi \, dx \; + \; o(|\omega_\epsilon|) \quad .$$

Insertion of (27) into the expression (17) now yields

$$\int_\Omega (\gamma_1 - \gamma_0) \, M_{ij} \xi_i \xi_j \phi d\mu$$
$$= \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \, \frac{\gamma_0}{\gamma_1} \, |\xi|^2 \, \phi \, dx$$
$$+ \frac{1}{|\omega_{\epsilon_n}|} \int_{\Omega_1} \hat{\gamma}_{\epsilon_n} \left| \nabla w^{(1)}_{\epsilon_n,\phi} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_n}} \xi \right|^2 \phi \, dx \; + \; o(1) \quad .$$

A completely similar calculation gives

$$\int_\Omega (\gamma_1 - \gamma_0) \, M_{ij} \xi_i \xi_j \phi d\mu$$
$$= \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \, \frac{\gamma_0}{\gamma_1} \, |\xi|^2 \, \phi \, dx$$
$$+ \frac{1}{|\omega_{\epsilon_n}|} \int_{\Omega_1} \hat{\gamma}_{\epsilon_n} \left| \nabla \tilde{w}^{(1)}_{\epsilon_n,\phi} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_n}} \xi \right|^2 \phi \, dx \; + \; o(1) \quad .$$

The equations (20) and (21) for $w^{(1)}_{\epsilon,\phi}$ and $\tilde{w}^{(1)}_{\epsilon,\phi}$ guarantee that these are the minimizers for the expression

$$\int_{\Omega_1} \hat{\gamma}_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi \, dx$$

in $H^1(\Omega_1)$ and $H^1_0(\Omega_1)$, respectively, and so we obtain the following lemma.

Lemma 3. *Let ω_{ϵ_n} and M be a subsequence and a polarization tensor as introduced by Proposition 1 (and Theorem 1). Let Ω_1 be as in Lemma 1, and suppose ϕ is a uniformly positive, smooth function. Then*

$$\int_\Omega (\gamma_1 - \gamma_0) \, M_{ij} \xi_i \xi_j \phi d\mu$$
$$= \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0) \, \frac{\gamma_0}{\gamma_1} \, |\xi|^2 \, \phi \, dx \tag{28}$$
$$+ \frac{1}{|\omega_{\epsilon_n}|} \min_{w \in H^1(\Omega_1)} \int_{\Omega_1} \hat{\gamma}_{\epsilon_n} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_n}} \xi \right|^2 \phi \, dx \; + \; o(1) \quad ,$$

and

$$\int_\Omega (\gamma_1-\gamma_0)\, M_{ij}\xi_i\xi_j\phi d\mu$$
$$= \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\,\frac{\gamma_0}{\gamma_1}\,|\xi|^2\,\phi\;dx \tag{29}$$
$$+\frac{1}{|\omega_{\epsilon_n}|}\min_{w\in H^1_0(\Omega_1)}\int_{\Omega_1}\hat{\gamma}_{\epsilon_n}\left|\nabla w+\frac{\gamma_1-\gamma_0}{\gamma_1}1_{\omega_{\epsilon_n}}\xi\right|^2\phi\;dx\;+\;o(1)\quad.$$

Remark 1. *The identities (28) and (29) hold for any (C^2 or convex) Ω_1 with $\omega_{\epsilon_n}\subset K_0\subset\Omega_1$, and not just any Ω_1 with $\Omega\subset\subset\Omega_1$. To see this let Ω_1 be any, C^2 or convex, bounded domain with $\omega_{\epsilon_n}\subset K_0\subset\Omega_1$, and select $\tilde{\Omega}$ (smooth) such that $\omega_{\epsilon_n}\subset K_0\subset\tilde{\Omega}\subset\subset\Omega_1$ and let $\tilde{M}$ denote the polarization tensor associated to $\tilde{\Omega}$ (and some subsequence of the sequence $\{\epsilon_n\}$). We already know that $\tilde{M}d\mu = Md\mu$, for instance because, due to Lemma 3,*

$$\int_\Omega (\gamma_1-\gamma_0)\, M_{ij}\xi_i\xi_j\phi d\mu$$
$$= \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\,\frac{\gamma_0}{\gamma_1}\,|\xi|^2\,\phi\;dx$$
$$+\frac{1}{|\omega_{\epsilon_n}|}\min_{w\in H^1(\Omega_2)}\int_{\Omega_2}\hat{\gamma}_{\epsilon_n}\left|\nabla w+\frac{\gamma_1-\gamma_0}{\gamma_1}1_{\omega_{\epsilon_n}}\xi\right|^2\phi\;dx\;+\;o(1)$$
$$=\int_{\tilde{\Omega}}(\gamma_1-\gamma_0)\,\tilde{M}_{ij}\xi_i\xi_j\phi d\mu\;+\;o(1)\quad,$$

for any uniformly positive, smooth ϕ, and any domain Ω_2, which compactly contains Ω and $\tilde{\Omega}$. Note that this identity also implies that $\tilde{M}$ ($=M$, μ a.e.) is the "effective" polarization tensor associated to the entire sequence $\{\epsilon_n\}$ and the domain $\tilde{\Omega}$. Again, due to Lemma 3,

$$\int_{\tilde{\Omega}} (\gamma_1-\gamma_0)\, \tilde{M}_{ij}\xi_i\xi_j\phi d\mu$$
$$= \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\,\frac{\gamma_0}{\gamma_1}\,|\xi|^2\,\phi\;dx$$
$$+\frac{1}{|\omega_{\epsilon_n}|}\min_{w\in H^1(\Omega_1)}\int_{\Omega_1}\hat{\gamma}_{\epsilon_n}\left|\nabla w+\frac{\gamma_1-\gamma_0}{\gamma_1}1_{\omega_{\epsilon_n}}\xi\right|^2\phi\;dx\;+\;o(1)\quad,$$

since Ω_1 compactly contains $\tilde{\Omega}$. From the previous identity it now follows that

$$\int_\Omega (\gamma_1-\gamma_0)\, M_{ij}\xi_i\xi_j\phi d\mu$$
$$= \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\,\frac{\gamma_0}{\gamma_1}\,|\xi|^2\,\phi\;dx$$
$$+\frac{1}{|\omega_{\epsilon_n}|}\min_{w\in H^1(\Omega_1)}\int_{\Omega_1}\hat{\gamma}_{\epsilon_n}\left|\nabla w+\frac{\gamma_1-\gamma_0}{\gamma_1}1_{\omega_{\epsilon_n}}\xi\right|^2\phi\;dx\;+\;o(1)\quad,$$

as desired.

From the first of the formulae in this lemma we may immediately obtain the pointwise bounds of Theorem 1, as follows. Firstly, the right-hand side of the identity (28) is the sum of two non-negative terms, and thus

$$\int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j \ \phi \ d\mu \geq \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0)\, \frac{\gamma_0}{\gamma_1}\, |\xi|^2 \ \phi \ dx \ + \ o(1) \ . \quad (30)$$

Secondly, by taking $\nabla w = 0$ in (28), we obtain

$$\int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j \ \phi \ d\mu \leq \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0)\, |\xi|^2 \ \phi \ dx \ + \ o(1) \ . \quad (31)$$

In limit as $|\omega_{\epsilon_n}| \to 0$, (30)-(31) give

$$\int_\Omega (\gamma_1 - \gamma_0)\, \frac{\gamma_0}{\gamma_1}\, |\xi|^2 \ \phi \ d\mu \leq \int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j \ \phi \ d\mu \leq \int_\Omega (\gamma_1 - \gamma_0)\, |\xi|^2 \ \phi \ d\mu \ ,$$

for any smooth, uniformly positive ϕ. A simple limiting argument insures that these inequalities hold for all non-negative, continuous functions ϕ, and this in turn insures that

$$\min\left(1, \frac{\gamma_0(x)}{\gamma_1(x)}\right) |\xi|^2 \leq M(x)\xi \cdot \xi \leq \max\left(1, \frac{\gamma_0(x)}{\gamma_1(x)}\right) |\xi|^2$$

for all $\xi \in I\!R^m$, μ almost everywhere in the set $\{x \in \Omega \ : \ \gamma_0(x) \neq \gamma_1(x)\}$. In order to verify the stronger, optimal bounds of Theorem 3, it will be convenient to have the following characterization of M.

Lemma 4. *Let ω_{ϵ_n} and M be a subsequence and a polarization tensor as introduced by Proposition 1 (and Theorem 1). Suppose $Q = (-L, L)^m$ is a m-dimensional cube, with $\omega_{\epsilon_n} \subset K_0 \subset Q$, and suppose ϕ is a uniformly positive, smooth function. Then*

$$\begin{aligned}
&\int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j\phi d\mu \\
&\qquad = \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0)\, \frac{\gamma_0}{\gamma_1}\, |\xi|^2 \phi \ dx \\
&\qquad\quad + \frac{1}{|\omega_{\epsilon_n}|} \min_{w \in H^1_{per}(Q)} \int_Q \hat{\gamma}_{\epsilon_n} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_n}} \xi \right|^2 \phi \ dx \ + \ o(1) \ ,
\end{aligned}$$

where $H^1_{per}(Q)$ denotes the set of Q-periodic elements of $H^1(Q)$.

Proof. Since $H^1_0(Q) \subset H^1_{per}(Q) \subset H^1(Q)$

$$\begin{aligned}
&\min_{w \in H^1(Q)} \int_Q \hat{\gamma}_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi \ dx \\
&\qquad \leq \min_{w \in H^1_{per}(Q)} \int_Q \hat{\gamma}_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi \ dx \\
&\qquad \leq \min_{w \in H^1_0(Q)} \int_Q \hat{\gamma}_\epsilon \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi \right|^2 \phi \ dx \ .
\end{aligned}$$

Since Q is convex with $\omega_{\epsilon_n} \subset K_0 \subset Q$ a combination of these inequalities with Lemma 3 (and Remark 1) now gives

$$\begin{aligned}
&\int_\Omega (\gamma_1 - \gamma_0)\, M_{ij}\xi_i\xi_j \phi d\mu \ + o\ (1) \\
&\qquad \leq \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\,\frac{\gamma_0}{\gamma_1}\,|\xi|^2\,\phi\ dx \\
&\qquad\quad + \frac{1}{|\omega_{\epsilon_n}|}\min_{w\in H^1_{per}(Q)}\int_Q \hat{\gamma}_{\epsilon_n}\left|\nabla w + \frac{\gamma_1-\gamma_0}{\gamma_1}1_{\omega_{\epsilon_n}}\xi\right|^2\phi\ dx \\
&\qquad \leq \int_\Omega (\gamma_1-\gamma_0)\, M_{ij}\xi_i\xi_j\phi d\mu \ + \ o(1) \quad ,
\end{aligned}$$

as desired. □

3 Proof of Theorem 3

In the course of proving this theorem the following lemma will be very useful.

Lemma 5. *Let $Q = (-L, L)^m$ be an m-dimensional cube with $\omega_\epsilon \subset K_0 \subset Q$. Let a be a uniformly positive, smooth function on $\overline{Q}$, and let f be a continuous function on $\overline{Q}$. For any $\xi \in I\!R^m$, let u^ξ be the Q-periodic solution to*

$$-\nabla\cdot(a\nabla u^\xi) = \nabla\cdot(f(x)1_{\omega_\epsilon}(x)\xi) \ \text{in } I\!R^m \quad , \tag{32}$$

with $\int_Q u^\xi\ dx = 0$. Here we have also used a, f, and 1_{ω_ϵ} to denote the Q-periodic extensions of these same functions. The symmetric matrix S_ϵ, defined by

$$\begin{aligned}
S_\epsilon\xi\cdot\xi &= -\int_Q a\nabla u^\xi\cdot\nabla u^\xi\ dx \\
&= \int_Q f(x)1_{\omega_\epsilon}(x)\xi\cdot\nabla u^\xi\ dx \quad \text{for } \ \xi\in I\!R^m \quad ,
\end{aligned}$$

is negative, and satisfies

$$\mathit{Trace}(S_\epsilon) = -\int_{\omega_\epsilon}\frac{f^2(x)}{a(x)}\ dx \ + \ o(|\omega_\epsilon|) \quad .$$

Proof. Since a is positive it follows that $S_\epsilon \leq 0$. Multiplication of (32) by u_ξ, and integration by parts immediately yield

$$\|\nabla u^\xi\|_{L^2(Q)} \leq C|\xi|\,|\omega_\epsilon|^{\frac{1}{2}} \quad . \tag{33}$$

Let v^ξ denote the Q-periodic solution to the following auxiliary problem

$$-\Delta v^\xi = \nabla\cdot(f(x)1_{\omega_\epsilon}(x)\xi) \ \text{ in } I\!R^m \quad , \tag{34}$$

with $\int_Q v^\xi dx = 0$. A variant of the duality argument from the proof of Lemma 1 will give that

$$\|v^\xi\|_{L^2(Q)} \leq C_\eta |\xi|\, |\omega_\epsilon|^{\frac{1}{2}+\frac{1}{m*}-\eta} \quad , \tag{35}$$

for any $\eta > 0$. The argument goes as follows. Let w^ξ denote the Q-periodic solution to

$$\Delta w^\xi = v^\xi \quad \text{in} \quad I\!R^m \quad ,$$

"normalized" by $\int_Q w^\xi \, dx = 0$. We have that

$$\|w^\xi\|_{H^2(Q)} \leq C \|v^\xi\|_{L^2(Q)} \quad ,$$

and so

$$\begin{aligned} \int_Q (v^\xi)^2 \, dx &= -\int_Q \nabla w^\xi \cdot \nabla v^\xi \, dx \\ &= \int_Q f(x) \; 1_{\omega_\epsilon}(x) \xi \cdot \nabla w^\xi \, dx \\ &\leq C|\xi| |\omega_\epsilon|^{1/q} \left(\int_Q |\nabla w^\xi|^p \, dx \right)^{1/p} \quad , \end{aligned} \tag{36}$$

provided $1 < q < \infty$, and $1/p + 1/q = 1$. If we select $q > \frac{2m*}{m*+2}$, $m* = \max\{2, m\}$ (so that $1 < p < \frac{2m*}{m*-2}$, and therefore, by Sobolev's Imbedding Theorem ($\int_Q |\nabla w^\xi|^p \, dx)^{1/p} \leq C_p \|w^\xi\|_{H^2(Q)}$) it immediately follows that

$$\|v^\xi\|^2_{L^2(Q)} \leq C_q |\xi| \, |\omega_\epsilon|^{\frac{1}{q}} \, \|v^\xi\|_{L^2(Q)} \quad .$$

By taking q sufficiently close to $\frac{2m*}{m*+2}$ we obtain (35). We now compute

$$\begin{aligned} S_\epsilon \xi \cdot \xi &= \int_Q f(x) 1_{\omega_\epsilon}(x) \xi \cdot \nabla u^\xi \, dx \\ &= -\int_Q \nabla v^\xi \cdot \nabla u^\xi \, dx \\ &= -\int_Q a \nabla \left(v^\xi \frac{1}{a} \right) \cdot \nabla u^\xi \, dx + \int_Q a v^\xi \nabla \left(\frac{1}{a} \right) \cdot \nabla u^\xi \, dx \\ &= -\int_Q a \nabla \left(v^\xi \frac{1}{a} \right) \cdot \nabla u^\xi dx \; + \; o(|\omega_\epsilon|) \quad , \end{aligned}$$

thanks to (33) and (35). From the weak form of (32) we deduce that

$$\begin{aligned} S_\epsilon \xi \cdot \xi &= \int_Q f(x) 1_{\omega_\epsilon}(x) \xi \cdot \nabla \left(v^\xi \frac{1}{a} \right) dx \; + \; o(|\omega_\epsilon|) \\ &= \int_Q \frac{f(x)}{a(x)} 1_{\omega_\epsilon}(x) \xi \cdot \nabla v^\xi dx \; + \; o(|\omega_\epsilon|) \quad , \end{aligned}$$

where, for the last identity, we again used (35). As a consequence,

$$\mathrm{Trace}(S_\epsilon) = \int_Q \frac{f(x)}{a(x)} 1_{\omega_\epsilon}(x) \sum_{i=1}^m \frac{\partial v^i}{\partial x_i}\ dx \ + \ o(|\omega_\epsilon|) \quad ,$$

v^i being the Q-periodic solution to

$$-\Delta v^i = \frac{\partial}{\partial x_i}(f(x)1_{\omega_\epsilon}(x)) \quad \text{in } I\!R^m \quad ,$$

with $\int_Q v^i\ dx = 0$. Note that, because of the periodic boundary conditions,

$$\frac{\partial}{\partial x_j} v^i = -\frac{\partial^2}{\partial x_j \partial x_i} \Delta^{-1} \left(f(x) 1_{\omega_\epsilon}(x) - \frac{1}{|Q|} \int_{\omega_\epsilon} f(x) dx \right) \quad ,$$

for all $1 \leq i, j \leq m$, and therefore

$$\sum_{i=1}^m \frac{\partial v^i}{\partial x_i} = -f(x) 1_{\omega_\epsilon}(x) + O(|\omega_\epsilon|) \quad .$$

It follows immediately that

$$\mathrm{Trace}(S_\epsilon) = -\int_Q \frac{f^2(x)}{a(x)} 1_{\omega_\epsilon}(x)\ dx \ + \ o(|\omega_\epsilon|) \quad .$$

□

We are now ready for the

Proof of Theorem 3. We use the Hashin-Shtrikman variational technique as described in detail by Kohn and Milton (cf. [14]). For related polarization tensor bounds see also [3] and [15]. To simplify the notations we introduce $p_\epsilon = \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon} \xi$. According to Lemma 4 we have

$$\begin{aligned} \int_\Omega (\gamma_1 - \gamma_0)\, M_{ij} \xi_i \xi_j \phi\ d\mu \quad &= \quad \frac{1}{|\omega_{\epsilon_n}|} \int_{\omega_{\epsilon_n}} (\gamma_1 - \gamma_0)\, \frac{\gamma_0}{\gamma_1}\, |\xi|^2\, \phi\ dx \\ &\quad + \frac{1}{|\omega_{\epsilon_n}|} \min_{w \in H^1_{per}(Q)} \int_Q \hat{\gamma}_{\epsilon_n}\, |\nabla w + p_{\epsilon_n}|^2\, \phi\ dx \ + \ o(1) \quad . \end{aligned} \tag{37}$$

After introduction of a smooth reference medium, c, with $0 < c < \min\{\gamma_0, \gamma_1\}$, we may write

$$\begin{aligned} \int_Q \hat{\gamma}_\epsilon\, |\nabla w + p_\epsilon|^2\, \phi\ dx \quad &= \quad \sup_{\eta \in L^2(Q)} -\int_\Omega \frac{1}{(\hat{\gamma}_\epsilon - c)} |\eta|^2 \phi dx + 2 \int_Q (\nabla w + p_\epsilon) \cdot \eta \phi\ dx \\ &\qquad + \int_Q c\, |\nabla w + p_\epsilon|^2\, \phi\ dx \\ &= \quad \sup_{\eta \in L^2(Q)} -\int_\Omega \left(\frac{1}{(\hat{\gamma}_\epsilon - c)} + \frac{1}{c} \right) |\eta|^2 \phi\ dx \\ &\qquad + \int_Q c \left| \nabla w + p_\epsilon + \frac{1}{c} \eta \right|^2 \phi\ dx \quad . \end{aligned} \tag{38}$$

Consequently, by using the inequality "min sup $\geq$ sup min", we obtain

$$\begin{aligned}
&\min_{w\in H^1_{per}(Q)} \int_Q \hat{\gamma}_\epsilon \left|\nabla w + p_\epsilon\right|^2 \phi \; dx \\
&\qquad \geq \sup_{\eta\in L^2(Q)} \left(-\int_Q \left(\frac{1}{(\hat{\gamma}_\epsilon - c)} + \frac{1}{c}\right) |\eta|^2 \phi \; dx \right. \qquad (39) \\
&\qquad\qquad \left. + \min_{w\in H^1_{per}(Q)} \int_Q c \left|\nabla w + p_\epsilon + \frac{1}{c}\eta\right|^2 \phi \; dx \right) \quad .
\end{aligned}$$

The minimum of the last right hand side is attained at the Q-periodic solution of

$$-\nabla\cdot(c\phi\nabla w) = \nabla\cdot(\phi\,(cp_\epsilon+\eta)) \quad \text{in } I\!R^m \quad ,$$

i.e., at

$$w_0 = (-\Delta_{c\phi})^{-1}\nabla\cdot(\phi\,(cp_\epsilon+\eta)) \quad ,$$

where $\Delta_{c\phi}$ denotes the operator $\nabla\cdot(c\phi\nabla\ \ )$ operating on Q-periodic functions (with average zero). For that w_0,

$$\begin{aligned}
&-\int_Q \left(\frac{1}{(\hat{\gamma}_\epsilon - c)} + \frac{1}{c}\right) |\eta|^2\phi \; dx + \int_Q c\left|\nabla w_0 + p_\epsilon + \frac{1}{c}\eta\right|^2 \phi \; dx \\
&\qquad = -\int_Q \left(\frac{1}{(\hat{\gamma}_\epsilon - c)} + \frac{1}{c}\right) |\eta|^2\phi \; dx + \int_Q c\left|p_\epsilon + \frac{1}{c}\eta\right|^2 \phi \; dx \\
&\qquad\qquad - \int_Q c\left|\nabla w_0\right|^2 \phi \; dx \\
&\qquad = \int_Q c\left|p_\epsilon\right|^2 \phi \; dx - \int_Q \frac{1}{(\hat{\gamma}_\epsilon - c)} |\eta|^2\phi \; dx + 2\int_Q \eta\cdot p_\epsilon\phi \; dx \\
&\qquad\qquad + \int_Q (cp_\epsilon+\eta)\,\phi\cdot L_{c\phi}\left((cp_\epsilon+\eta)\,\phi\right) \; dx \quad , \qquad (40)
\end{aligned}$$

where $L_{c\phi}$ is the linear operator on $L^2_{per}(Q)^m$ (L^2 vector fields on Q that have been periodically extended to all of $I\!R^m$) defined by

$$L_{c\phi}\eta = \nabla\left(-\Delta_{c\phi}\right)^{-1}\nabla\cdot\eta \quad .$$

If we introduce

$$G_{c,\epsilon}(\eta) = -\int_Q \frac{1}{\hat{\gamma}_\epsilon - c}\; |\eta|^2\phi \; dx + 2\int_Q \frac{\hat{\gamma}_\epsilon}{\hat{\gamma}_\epsilon - c}\; p_\epsilon\cdot\eta\phi \; dx + \int_Q \eta\phi\cdot L_{c\phi}(\eta\phi) \; dx \quad ,$$

then a simple calculation yields

$$\begin{aligned}
G_{c,\epsilon}(cp_\epsilon+\eta) \;\; = \;\; &\int_Q \left(\frac{c\hat{\gamma}_\epsilon}{\hat{\gamma}_\epsilon - c} + c\right) |p_\epsilon|^2\phi \; dx - \int_Q \frac{1}{\hat{\gamma}_\epsilon - c} |\eta|^2\phi \; dx + 2\int_Q \eta\cdot p_\epsilon\phi \; dx \\
&+ \int_Q (cp_\epsilon+\eta)\,\phi\cdot L_{c\phi}\left((cp_\epsilon+\eta)\,\phi\right) \; dx \quad . \qquad (41)
\end{aligned}$$

Therefore, by a combination of (37), (39), (40), and (41), and use of the formula $p_\epsilon = \frac{\gamma_1-\gamma_0}{\gamma_1} 1_{\omega_\epsilon}\xi$, we obtain

$$\begin{aligned}
&\int_\Omega (\gamma_1-\gamma_0)\, M_{ij}\xi_i\xi_j\phi \; d\mu \\
&\quad \geq \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\,\frac{\gamma_0}{\gamma_1}\,|\xi|^2\,\phi\; dx - \frac{1}{|\omega_{\epsilon_n}|}\int_Q \frac{c\hat{\gamma}_\epsilon}{(\hat{\gamma}_\epsilon - c)}|p_\epsilon|^2\phi\; dx \\
&\qquad + \frac{1}{|\omega_{\epsilon_n}|}\sup_{\eta\in L^2(Q)} G_{c,\epsilon_n}(cp_{\epsilon_n}+\eta) \;+\; o(1) \quad , \qquad (42)\\
&\quad = \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\,\frac{\gamma_0-c}{\gamma_1-c}\,|\xi|^2\,\phi\; dx + \frac{1}{|\omega_{\epsilon_n}|}\sup_{\sigma\in L^2(Q)} G_{c,\epsilon_n}(\sigma) \;+\; o(1) \quad ,
\end{aligned}$$

for $0 < c < \min(\gamma_0,\gamma_1)$. The inequality in (42) may actually be shown to be an equality (*i.e.*, in deriving (39) we have that "min sup = sup min") but for our purpose (to prove bounds) we do not need this fact. In the case of a smooth reference medium $c > \max(\gamma_0,\gamma_1)$ we may proceed identically, except that the "sup" (in (38) and forward) gets replaced by an "inf". In that case the interchange of the order of the "inf" and the "min" is automatically associated with an equality, and so we end up with

$$\begin{aligned}
&\int_\Omega (\gamma_1-\gamma_0)\, M_{ij}\xi_i\xi_j\phi \; d\mu \qquad (43)\\
&\quad = \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\,\frac{\gamma_0-c}{\gamma_1-c}\,|\xi|^2\,\phi\; dx + \frac{1}{|\omega_{\epsilon_n}|}\inf_{\sigma\in L^2(Q)} G_{c,\epsilon_n}(\sigma) \;+\; o(1) \quad ,
\end{aligned}$$

for $c > \max(\gamma_0,\gamma_1)$. A simple calculation gives,

$$\begin{aligned}
&G_{c,\epsilon}(\lambda\gamma_1 p_\epsilon) + \int_{\omega_\epsilon} (\gamma_1-\gamma_0)\,\frac{\gamma_0-c}{\gamma_1-c}\;\phi\; dx\; |\xi|^2 \\
&= G_{c,\epsilon}(\lambda(\gamma_1-\gamma_0)1_{\omega_\epsilon}\xi) + \int_{\omega_\epsilon}(\gamma_1-\gamma_0)\,\phi\; dx\; |\xi|^2 - \int_{\omega_\epsilon}\frac{(\gamma_1-\gamma_0)^2}{\gamma_1-c}\phi\; dx\; |\xi|^2 \\
&= \int_{\omega_\epsilon}(\gamma_1-\gamma_0)\,\phi\; dx\; |\xi|^2 - \int_{\omega_\epsilon}(\lambda-1)^2\frac{(\gamma_1-\gamma_0)^2}{\gamma_1-c}\phi\; dx\; |\xi|^2 \\
&\qquad\qquad + \int_Q (\gamma_1-\gamma_0)\lambda\phi 1_{\omega_\epsilon}\xi\cdot L_{c\phi}\left((\gamma_1-\gamma_0)\lambda\phi 1_{\omega_\epsilon}\xi\right)\; dx,
\end{aligned}$$

for any continuous function λ. We specifically choose $\lambda_c = 1 + |c-\gamma_1|$, and note that whenever c tends to γ_1, λ_c tends to 1. With this choice of λ

$$\int_{\omega_\epsilon}\frac{(\gamma_1-\gamma_0)^2}{\gamma_1-c}(\lambda_c-1)^2\phi\; dx = \int_{\omega_\epsilon}(\gamma_1-\gamma_0)^2\,(\gamma_1-c)\phi\; dx \quad . \qquad (44)$$

If we only use test functions of the form $\sigma = \lambda_c\gamma_1 p_\epsilon$, then (42) gives

$$\begin{aligned}
\int_\Omega(\gamma_1-\gamma_0)M\xi\cdot\xi\;\phi d\mu \;&\geq\; \left(\int_\Omega(\gamma_1-\gamma_0)\,\phi\; d\mu - \int_\Omega(\gamma_1-\gamma_0)^2\,(\gamma_1-c)\phi\; d\mu\right)\; |\xi|^2 \\
&+\; \frac{1}{|\omega_{\epsilon_n}|}A_{\epsilon_n}\xi\cdot\xi + o(1) \quad , \qquad (45)
\end{aligned}$$

for $0 < c < \min(\gamma_1, \gamma_0)$. Here the negative definite, symmetric matrix A_ϵ is given by

$$A_\epsilon \xi \cdot \xi = \int_Q (\gamma_1 - \gamma_0)\lambda_c \phi 1_{\omega_\epsilon} \xi \cdot L_{c\phi}\left((\gamma_1 - \gamma_0)\lambda_c \phi 1_{\omega_\epsilon} \xi\right)\, dx$$

According to Lemma 5 (with $a = c\phi$ and $f = (\gamma_1 - \gamma_0)\lambda_c \phi$) we have that

$$\text{Trace}(A_\epsilon) = -\int_\Omega (\gamma_1 - \gamma_0)^2 \frac{\lambda_c^2}{c} \phi 1_{\omega_\epsilon}\, dx \;+\; o(|\omega_\epsilon|) \quad . \tag{46}$$

A combination of (45) and (46), and passage to the limit $|\omega_{\epsilon_n}| \to 0$ gives

$$\begin{aligned} \int_\Omega (\gamma_1 - \gamma_0)\text{Trace}(M)\, \phi\, d\mu \;\; \geq \;\; & m \int_\Omega (\gamma_1 - \gamma_0)\, \phi\, d\mu \\ & - \int_\Omega (\gamma_1 - \gamma_0)^2 \left(m(\gamma_1 - c) + \frac{\lambda_c^2}{c} \right) \phi\, d\mu \quad , \end{aligned}$$

for $0 < c < \min(\gamma_0, \gamma_1)$. Letting c tend to $\min(\gamma_0, \gamma_1)$ we now obtain

$$\begin{aligned} & \int_\Omega (\gamma_1 - \gamma_0)\text{Trace}(M)\, \phi\, d\mu \\ & \qquad \geq \int_\Omega (\gamma_1 - \gamma_0) \left(m - 1 + \frac{\gamma_0}{\gamma_1} \right) \chi_0(x) \phi\, d\mu \\ & \qquad\quad + \int_\Omega (\gamma_1 - \gamma_0) \left(m - m(\gamma_1 - \gamma_0)^2 - \frac{\gamma_1 - \gamma_0}{\gamma_0} \lambda_{\gamma_0}^2 \right) (1 - \chi_0(x)) \phi\, d\mu \quad , \end{aligned} \tag{47}$$

where χ_0 is the indicator function of the set $\{x : \gamma_1(x) < \gamma_0(x)\}$. Similarly if, for $c > \max(\gamma_0, \gamma_1)$, we only use test functions of the form $\sigma = \lambda_c \gamma_1 p_\epsilon$ in (43) then a passage to the limit $|\omega_{\epsilon_n}| \to 0$, followed by a passage to the limit $c \to \max(\gamma_0, \gamma_1)$ gives

$$\begin{aligned} & \int_\Omega (\gamma_1 - \gamma_0)\text{Trace}(M)\, \phi\, d\mu \\ & \qquad \leq \int_\Omega (\gamma_1 - \gamma_0) \left(m - 1 + \frac{\gamma_0}{\gamma_1} \right) \chi_1(x) \phi\, d\mu \\ & \qquad\quad + \int_\Omega (\gamma_1 - \gamma_0) \left(m - m(\gamma_1 - \gamma_0)^2 - \frac{\gamma_1 - \gamma_0}{\gamma_0} \lambda_{\gamma_0}^2 \right) (1 - \chi_1(x)) \phi\, d\mu \quad , \end{aligned} \tag{48}$$

where χ_1 is the indicator function of the set $\{x : \gamma_0(x) < \gamma_1(x)\}$. We recall that ϕ is an arbitrary, smooth, uniformly positive function. By a limiting argument the inequalities (47)-(48 are seen to hold for an arbitrary, continuous $\phi \geq 0$. Using (47) for ϕ whose support lies inside the support of χ_0 (*i.e.*, inside $\{x : \gamma_1(x) < \gamma_0(x)\}$) and (48) for ϕ whose support lies inside the support of χ_1 (*i.e.*, inside $\{x : \gamma_0(x) < \gamma_1(x)\}$) we arrive at the first of the bounds of Theorem 3.

Let η be a constant vector. By a simple calculation

$$G_{c,\epsilon}(1_{\omega_\epsilon}\eta) = -\int_{\omega_\epsilon} \frac{1}{\gamma_1 - c}\phi \; dx \; |\eta|^2 + 2\int_{\omega_\epsilon} \frac{\gamma_1 - \gamma_0}{\gamma_1 - c}\phi \; dx \; \xi\cdot\eta + B_\epsilon\eta\cdot\eta \quad ,$$

where B_ϵ is the negative definite matrix given by

$$B_\epsilon\eta\cdot\eta = \int_Q \phi 1_{\omega_\epsilon}\eta \cdot L_{c\phi}\left(\phi 1_{\omega_\epsilon}\eta\right) \; dx \quad .$$

According to Lemma 5,

$$\text{Trace}(B_\epsilon) = -\int_{\omega_\epsilon} \frac{1}{c}\phi \; dx \; + \; o(|\omega_\epsilon|) \quad . \tag{49}$$

If we only use test functions of the form $\sigma = 1_{\omega_\epsilon}\eta$, then (42) yields

$$\begin{aligned}\int_\Omega (\gamma_1 - \gamma_0) M\xi\cdot\xi\phi d\mu - 2\frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} \frac{\gamma_1-\gamma_0}{\gamma_1 - c}\phi \; dx \; \xi\cdot\eta \\ \geq \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} (\gamma_1-\gamma_0)\frac{\gamma_0 - c}{\gamma_1 - c}|\xi|^2\,\phi \; dx - \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}} \frac{1}{\gamma_1 - c}\phi \; dx \; |\eta|^2 \\ + \frac{1}{|\omega_{\epsilon_n}|}B_\epsilon\eta\cdot\eta \; + \; o(1) \quad ,\end{aligned}$$

for any $\xi, \eta \in I\!R^m$, provided $0 < c < \min(\gamma_0, \gamma_1)$. By taking $\eta = e_k$ (the $k'th$ standard basis vector) and $\xi = \xi_k$, for $k = 1, \; \ldots \; , \; m$, then summing these m inequalities, and passing to the limit as $|\omega_{\epsilon_n}| \to 0$, we arrive at

$$\begin{aligned}\int_\Omega (\gamma_1 - \gamma_0)\sum_{k=1}^m M\xi_k\cdot\xi_k\phi d\mu - 2\int_\Omega \frac{\gamma_1-\gamma_0}{\gamma_1-c}\phi \; d\mu \; \sum_{k=1}^m \xi_k\cdot e_k \\ \geq \int_\Omega (\gamma_1-\gamma_0)\frac{\gamma_0-c}{\gamma_1-c}\phi \; d\mu \sum_{k=1}^m |\xi_k|^2 - m\int_\Omega \frac{1}{\gamma_1-c}\phi \; d\mu \\ - \int_\Omega \frac{1}{c}\phi \; d\mu \quad ,\end{aligned} \tag{50}$$

provided $0 < c < \min(\gamma_0, \gamma_1)$. Similarly, starting from (43) we obtain

$$\begin{aligned}\int_\Omega (\gamma_1 - \gamma_0)\sum_{k=1}^m M\xi_k\cdot\xi_k\phi d\mu - 2\int_\Omega \frac{\gamma_1-\gamma_0}{\gamma_1-c}\phi \; d\mu \; \sum_{k=1}^m \xi_k\cdot e_k \\ \leq \int_\Omega (\gamma_1-\gamma_0)\frac{\gamma_0-c}{\gamma_1-c}\phi \; d\mu \sum_{k=1}^m |\xi_k|^2 - m\int_\Omega \frac{1}{\gamma_1-c}\phi \; d\mu \\ - \int_\Omega \frac{1}{c}\phi \; d\mu \quad ,\end{aligned} \tag{51}$$

provided $c > \max(\gamma_0, \gamma_1)$. By a limiting argument the inequalities (50)- (51) hold for all continuous $\phi \geq 0$.

Using (50) for ϕ whose support lies inside $\{x : \gamma_0(x) < \gamma_1(x)\}$ we see that for any smooth function $c < \min(\gamma_0, \gamma_1)$ there exists a set $E_c \subset \{x : \gamma_0(x) < \gamma_1(x)\}$, with $\mu(E_c) = 0$, such that

$$\begin{aligned}\sum_{k=1}^{m} M(x)\xi_k \cdot \xi_k - \frac{2}{\gamma_1(x) - c(x)} \sum_{k=1}^{m} \xi_k \cdot e_k \\ \geq \frac{\gamma_0(x) - c(x)}{\gamma_1(x) - c(x)} \sum_{k=1}^{m} |\xi_k|^2 \\ - \frac{1}{\gamma_1(x) - \gamma_0(x)} \left(\frac{m}{\gamma_1(x) - c(x)} + \frac{1}{c(x)} \right) \quad ,\end{aligned}$$

for all $x \in \{x : \gamma_0(x) < \gamma_1(x)\} \setminus E_c$ and all $\xi_k \in I\!R^m$, $1 \leq k \leq m$. By picking a sequence of smooth functions, c, that tend pointwise to $\min(\gamma_0, \gamma_1)$ we conclude that

$$\begin{aligned}\sum_{k=1}^{m} M(x)\xi_k \cdot \xi_k - \frac{2}{\gamma_1(x) - \gamma_0(x)} \sum_{k=1}^{m} \xi_k \cdot e_k \\ \geq \frac{-1}{\gamma_1(x) - \gamma_0(x)} \left(\frac{m}{\gamma_1(x) - \gamma_0(x)} + \frac{1}{\gamma_0(x)} \right) \quad , \qquad (52)\end{aligned}$$

μ almost everywhere in the set $\{x : \gamma_0(x) < \gamma_1(x)\}$.

Using (51) for ϕ whose support lies inside $\{x : \gamma_1(x) < \gamma_0(x)\}$ we see that for any smooth function $c > \max(\gamma_0, \gamma_1)$ there exists a set $\tilde{E}_c \subset \{x : \gamma_1(x) < \gamma_0(x)\}$, with $\mu(\tilde{E}_c) = 0$, such that

$$\begin{aligned}\sum_{k=1}^{m} M(x)\xi_k \cdot \xi_k - \frac{2}{\gamma_1(x) - c(x)} \sum_{k=1}^{m} \xi_k \cdot e_k \\ \geq \frac{\gamma_0(x) - c(x)}{\gamma_1(x) - c(x)} \sum_{k=1}^{m} |\xi_k|^2 \\ - \frac{1}{\gamma_1(x) - \gamma_0(x)} \left(\frac{m}{\gamma_1(x) - c(x)} + \frac{1}{c(x)} \right) \quad ,\end{aligned}$$

for all $x \in \{x : \gamma_0(x) < \gamma_1(x)\} \setminus \tilde{E}_c$ and all $\xi_k \in I\!R^m$, $1 \leq k \leq m$. By picking a sequence of smooth functions, c, that tend pointwise to $\max(\gamma_0, \gamma_1)$ we conclude that

$$\begin{aligned}\sum_{k=1}^{m} M(x)\xi_k \cdot \xi_k - \frac{2}{\gamma_1(x) - \gamma_0(x)} \sum_{k=1}^{m} \xi_k \cdot e_k \\ \geq \frac{-1}{\gamma_1(x) - \gamma_0(x)} \left(\frac{m}{\gamma_1(x) - \gamma_0(x)} + \frac{1}{\gamma_0(x)} \right) \quad , \qquad (53)\end{aligned}$$

μ almost everywhere in the set $\{x : \gamma_1(x) < \gamma_0(x)\}$. A combination of (52) and

(53) shows that there exists a set E, with $\mu(E) = 0$, such that

$$\sum_{k=1}^{m} M(x)\xi_k \cdot \xi_k - \frac{2}{\gamma_1(x) - \gamma_0(x)} \sum_{k=1}^{m} \xi_k \cdot e_k \\ \geq \frac{-1}{\gamma_1(x) - \gamma_0(x)} \left(\frac{m}{\gamma_1(x) - \gamma_0(x)} + \frac{1}{\gamma_0(x)} \right) \quad , \qquad (54)$$

for all x in $\{x : \gamma_0(x) \neq \gamma_0(x)\} \setminus E$, and all $\xi_k \in I\!R^m$, $1 \leq k \leq m$. For a fixed $x \in \{x : \gamma_0(x) \neq \gamma_0(x)\} \setminus E$ we now take

$$\xi_k = \frac{1}{\gamma_1(x) - \gamma_0(x)} M(x)^{-1} e_k \quad ,$$

and insert it into (54), to obtain

$$\text{Trace}\left(M(x)^{-1}\right) \leq m + \frac{\gamma_1(x) - \gamma_0(x)}{\gamma_0(x)} = m - 1 + \frac{\gamma_1(x)}{\gamma_0(x)} \quad ,$$

for all x in $\{x : \gamma_0(x) \neq \gamma_0(x)\} \setminus E$, which is exactly the second inequality of Theorem 3. □

4 Applications of the pointwise trace bounds

In this section, we will show that the pointwise trace bounds of Theorem 3 allow us to give an explicit formula for the polarization tensor corresponding to asymptotically thin domains. After combination with Theorem 1 we thus quite simply obtain an explicit representation formula for the voltage perturbation $u_\epsilon - U$. This representation formula was previously studied by Beretta et. al. [4, 5]. The analysis in those papers rely on fine regularity estimates for the gradient of solutions of linear elliptic problems. This analysis in fact provides precise information on the asymptotic rate of the error of the representation formula. Here we simply verify its asymptotic correctness, however, due to the fact that we do not rely on the aforementioned elliptic estimates, we are able to deal with "thin" sets ω_ϵ with very "rapidly oscillating" boundaries. The key observation is the following.

Lemma 6. *Let ω_{ϵ_n}, M and μ be as in Proposition 1 (and Theorem 1), and suppose γ_0 and γ_1 are smooth conductivities. Let Ω' be a smooth domain with $\omega_{\epsilon_n} \subset K_0 \subset \Omega' \subset \Omega$. Suppose there exists a sequence of functions $w_{\epsilon_n} \in H^1(\Omega')$ such that*

$$\lim_{|\omega_{\epsilon_n}| \to 0} \frac{1}{|\omega_{\epsilon_n}|} \left\| \nabla w_{\epsilon_n} - 1_{\omega_{\epsilon_n}} \nabla x_m \right\|^2_{L^2(\Omega')} = 0,$$

where x_m is the last coordinate of x in the orthonormal base $e_1, \ldots, e_m$. Then

$$M(x) = \sum_{j=1}^{m-1} e_j \otimes e_j + \frac{\gamma_0(x)}{\gamma_1(x)} e_m \otimes e_m \quad ,$$

μ almost everywhere in $\Omega' \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \}$ (and $\Omega \cap \{ x : \gamma_0(x) \neq \gamma_1(x) \}$).

Proof. We already know from Lemma 3 (and Remark 1) that the polarization tensor M satisfies

$$\begin{aligned}\int_\Omega (\gamma_1-\gamma_0)\, M_{ij}\xi_i\xi_j\phi d\mu &= \int_\Omega (\gamma_1-\gamma_0)\,\frac{\gamma_0}{\gamma_1}\,|\xi|^2\,\phi d\mu \\ &\quad + \frac{1}{|\omega_{\epsilon_n}|}\min_{w\in H^1(\Omega')}\int_{\Omega'}\hat{\gamma}_{\epsilon_n}\left|\nabla w + \frac{\gamma_1-\gamma_0}{\gamma_1}1_{\omega_{\epsilon_n}}\xi\right|^2\phi\ dx \\ &\quad +\ o(1)\quad ,\end{aligned}$$

for any smooth, uniformly positive function ϕ. Here $o(1)$ tends to zero as $|\omega_{\epsilon_n}|\to 0$. From our hypothesis we have that

$$\begin{aligned}&\frac{1}{|\omega_{\epsilon_n}|}\min_{w\in H^1(\Omega')}\int_{\Omega'}\hat{\gamma}_{\epsilon_n}\left|\nabla w + \frac{\gamma_1-\gamma_0}{\gamma_1}1_{\omega_\epsilon}\nabla x_m\right|^2\phi\ dx \\ &\qquad\le \frac{C}{|\omega_{\epsilon_n}|}\left\|\nabla w_{\epsilon_n} - 1_{\omega_{\epsilon_n}}\nabla x_m\right\|^2_{L^2(\Omega')} =\ o(1)\quad .\end{aligned}$$

Therefore, in the limit as $|\omega_{\epsilon_n}|\to 0$,

$$\int_\Omega (\gamma_1-\gamma_0)\, M_{mm}\ \phi\ d\mu = \int_\Omega (\gamma_1-\gamma_0)\,\frac{\gamma_0}{\gamma_1}\phi\ d\mu\quad ,$$

for any smooth, uniformly positive function ϕ. By a limiting argument we get that the same identity holds for any continuous $\phi\ge 0$. It follows immediately that $M(x)e_m\cdot e_m = M_{mm}(x) = \gamma_0(x)/\gamma_1(x)$, μ almost everywhere in $\Omega\cap\{\ x\ :\ \gamma_0(x)\neq\gamma_1(x)\ \}$. We already know that $\min\{1,\gamma_0(x)/\gamma_1(x)\}\le M(x)\le\max\{1,\gamma_0(x)/\gamma_1(x)\}$, μ almost everywhere in $\Omega\cap\{\ x\ :\ \gamma_0(x)\neq\gamma_1(x)\ \}$, *i.e.*,

$$\min\{1,\gamma_0(x)/\gamma_1(x)\}\le\min_{|\xi|=1}M(x)\xi\cdot\xi\le\max_{|\xi|=1}M(x)\xi\cdot\xi\le\max\{1,\gamma_0(x)/\gamma_1(x)\}\quad ,$$

and we may therefore conclude that, μ almost everywhere in $\Omega\cap\{\ x\ :\ \gamma_0(x)\neq\gamma_1(x)\ \}$, either

$$M(x)e_m\cdot e_m=\min_{|\xi|=1}M(x)\xi\cdot\xi=\gamma_0(x)/\gamma_1(x)\quad(\text{and }\gamma_0(x)<\gamma_1(x))\quad ,$$

or

$$M(x)e_m\cdot e_m=\max_{|\xi|=1}M(x)\xi\cdot\xi=\gamma_0(x)/\gamma_1(x)\quad(\text{and }\gamma_0(x)>\gamma_1(x))\quad .$$

In any event we have that e_m is an eigenvector of $M(x)$ corresponding to the eigenvalue $\lambda_m=\gamma_0(x)/\gamma_1(x)$, μ almost everywhere in $\Omega\cap\{\ x\ :\ \gamma_0(x)\neq\gamma_1(x)\ \}$. The trace bounds from Theorem 3 now assert that the other $m-1$ eigenvalues $\lambda_j(x)$, $1\le j\le m-1$, of $M(x)$ satisfy

$$\sum_{j=1}^{m-1}\lambda_j(x)\le m-1\quad ,\qquad\text{and}\qquad\sum_{j=1}^{m-1}\lambda_j^{-1}(x)\le m-1\quad ,$$

μ almost everywhere in $\Omega \cap \{\ x\ :\ \gamma_0(x) \neq \gamma_1(x)\ \}$. Since we also have that

$$\min\{1, \gamma_0(x)/\gamma_1(x)\} \leq \lambda_j(x) \leq \max\{1, \gamma_0(x)/\gamma_1(x)\}\ \ ,$$

(due to the fact that $\min\{1, \gamma_0(x)/\gamma_1(x)\} \leq M(x) \leq \max\{1, \gamma_0(x)/\gamma_1(x)\}$) we conclude that the $m-1$ eigenvalues $\lambda_j(x)$, $1 \leq j \leq m-1$, all are equal to one, μ almost everywhere in $\Omega \cap \{\ x\ :\ \gamma_0(x) \neq \gamma_1(x)\ \}$. On the hyperplane orthogonal to e_m, the polarization tensor is thus, μ almost everywhere in $\Omega \cap \{\ x\ :\ \gamma_0(x) \neq \gamma_1(x)\ \}$, the identity matrix, and so any orthogonal family in that plane is a base of eigenvectors, corresponding to the eigenvalue 1. □

The analysis presented in this section would apply to more general "thin" domains, but for simplicity from now on we only consider symmetric ones with a flat mid-surface. We select a coordinate system $x = (x', x_m)$ so that the mid-surface S_0 lies in the hyperplane $x_m = 0$. In other words, we consider ω_ϵ of the form

$$\omega_\epsilon = \{(x', x_m)\ :\ x' \in S_0\ \ ,\ \ |x_m| < \epsilon h_\epsilon(x')\} \subset\subset \Omega\ \ ,\ \ \ 0 < \epsilon << 1\ \ , \tag{55}$$

where S_0 is a bounded smooth subdomain of $I\!R^{m-1}$, and $h_\epsilon \in C^1(\overline{S}_0)$ satisfies

$$0 < c_0 \leq \min_{x' \in \overline{S}_0} h_\epsilon(x') \leq \max_{x' \in \overline{S}_0} h_\epsilon(x') < C_0\ \ \ , \qquad \|h_\epsilon\|_{C^1(\overline{S}_0)} \leq C\epsilon^{-a}\ \ , \tag{56}$$

for some positive constants c_0, C_0, C, and $a < 1$, all independent of ϵ. From the uniform bound on h_ϵ it follows immediately that we may find a subsequence that converges weak* in L^∞ to some limit $h_0 \in L^\infty(S_0)$. We shall without loss of generality therefore assume that

$$\int_{S_0} h_\epsilon(x') g(x')\ dx' \to \int_{S_0} h_0(x') g(x')\ dx' \quad \forall g \in L^1(S_0)\ \ , \tag{57}$$

as $\epsilon \to 0$.

Proposition 2. *Let ω_ϵ be as in (55) with h_ϵ satisfying (56) and (57). Suppose γ_0 and γ_1 are smooth conductivities. Given any $\psi \in H^{-1/2}(\partial\Omega)$, with $\int_{\partial\Omega} \psi\ d\sigma\ = 0$, let U and u_ϵ denote the solutions to (1) and (2), respectively. Then*

$$(u_\epsilon - U)(y) = |\omega_\epsilon| \int_\Omega (\gamma_1 - \gamma_0) M \nabla U \cdot \nabla_x N(x, y)\ d\mu_x\ +\ o(|\omega_\epsilon|)\ \ , \tag{58}$$

for $y \in \partial\Omega$, with

$$d\mu = \frac{1}{\int_{S_0} h_0(x') dx'} h_0(x') dx'_{S_0} \qquad \textit{and} \quad M(x) = \sum_{j=1}^{m-1} e_j \otimes e_j + \frac{\gamma_0(x)}{\gamma_1(x)} e_m \otimes e_m\ \ .$$

Proof. Due to the assumptions (56) and (57) about h_ϵ it is straightforward to show that for any $\phi \in C^0(\overline{\Omega})$

$$\frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon} \phi \; dx \to \frac{1}{\int_{S_0} h_0(x') \; dx'} \int_{S_0} \phi(x',0) h_0(x') \; dx' \quad ,$$

as $|\omega_\epsilon| \to 0$, in other words

$$\frac{1}{|\omega_\epsilon|} 1_{\omega_\epsilon} \to \frac{1}{\int_{S_0} h_0(x') dx'} h_0(x') dx'_{S_0} \tag{59}$$

in the sense of measures, as $|\omega_\epsilon| \to 0$. Let $\phi_\epsilon(x')$ be a cutoff function on $I\!R^{m-1}$, with

$$\phi_\epsilon \equiv 1 \quad \text{in } S_0 \, , \quad \text{supp}(\phi_\epsilon) \subset\subset S_0^\epsilon = \{x' \, : \, \text{dist}(x', S_0) < \epsilon\} \, , \quad \text{and } \|\phi_\epsilon\|_{C^1} \leq C\epsilon^{-1} \quad .$$

Similarly let $\psi_\epsilon(\cdot)$ be a cutoff function on $I\!R$, with

$$\psi_\epsilon \equiv 1 \quad \text{in } (-\epsilon C_0, \epsilon C_0) \, , \quad \text{supp}(\psi_\epsilon) \subset\subset (-\epsilon^a C_0, \epsilon^a C_0) \, , \quad \text{and } \|\psi_\epsilon\|_{C^1} \leq C\epsilon^{-a} \quad ,$$

C_0 and $a < 1$ being the constants from (56). For ϵ sufficiently small, we extend h_ϵ to S_0^ϵ, in such a way that

$$0 < \frac{1}{2} c_0 \leq \min_{x' \in \overline{S_0^\epsilon}} h_\epsilon(x') \leq \max_{x' \in \overline{S_0^\epsilon}} h_\epsilon(x') < 2C_0 \quad , \qquad \|h_\epsilon\|_{C^1(\overline{S_0^\epsilon})} \leq C\epsilon^{-a} \quad .$$

We now define

$$v_{m,\epsilon}(x) = \begin{cases} \epsilon h_\epsilon(x') & \text{for} \quad \epsilon h_\epsilon(x') < x_m \\ x_m & \text{for} \quad -\epsilon h_\epsilon(x') < x_m < \epsilon h_\epsilon(x') \\ -\epsilon h_\epsilon(x') & \text{for} \quad x_m < -\epsilon h_\epsilon(x') \end{cases} \quad ,$$

and

$$w_\epsilon(x) = \phi_\epsilon(x') \psi_\epsilon(x_m) v_{m,\epsilon}(x) \quad ,$$

for all $x \in S_0^\epsilon \times I\!R$. We set $w_\epsilon(x) = 0$ for all $x \in \Omega$, for which x' lies outside S_0^ϵ. A simple calculation gives

$$\begin{aligned} \int_\Omega |\nabla w_\epsilon - 1_{\omega_\epsilon} \nabla x_m|^2 \; dx &= \int_{\Omega \setminus \omega_\epsilon} |\nabla w_\epsilon|^2 \; dx \\ &\leq \int_{(S_0^\epsilon \setminus S_0) \times (-\epsilon^a C_0, \epsilon^a C_0)} |\nabla \left(\phi_\epsilon(x') \psi_\epsilon(x_m) \right) v_{m,\epsilon}|^2 \; dx \\ &\quad + \int_{(S_0^\epsilon \setminus S_0) \times (-\epsilon^a C_0, \epsilon^a C_0)} |\phi_\epsilon(x') \psi_\epsilon(x_m) \nabla v_{m,\epsilon}|^2 \; dx \\ &\quad + \int_{(S_0 \times (-\epsilon^a C_0, \epsilon^a C_0)) \setminus \omega_\epsilon} |\nabla (\phi_\epsilon(x') \psi_\epsilon(x_m)) v_{m,\epsilon}|^2 \; dx \\ &\quad + \int_{(S_0 \times (-\epsilon^a C_0, \epsilon^a C_0)) \setminus \omega_\epsilon} |\phi_\epsilon(x') \psi_\epsilon(x_m) \nabla v_{m,\epsilon}|^2 \; dx \\ &\leq C_1 \epsilon^{1+a} + C_2 \epsilon^{1+a} + C_3 \epsilon^{2-a} + C_4 \epsilon^{2-a} \quad . \end{aligned}$$

As a consequence

$$\frac{1}{|\omega_\epsilon|}\,\|\nabla w_\epsilon - 1_{\omega_\epsilon}\nabla x_m\|^2_{L^2(\Omega)} = O(|\omega_\epsilon|^a \;+\; |\omega_\epsilon|^{1-a}) \to 0 \quad , \tag{60}$$

as $|\omega_\epsilon| \to 0$. From Theorem 1 we know that, after extraction of a subsequence, the representation formula (58) holds with μ being the limit (in the sense of measures) of $\frac{1}{|\omega_\epsilon|}1_{\omega_\epsilon}$, *i.e.*, according to (59)

$$d\mu = \frac{1}{\int_{S_0} h_0(x')dx'}h_0(x')dx'_{S_0} \quad .$$

Due to the estimate (60) we may apply Lemma 6 to conclude that for this particular subsequence

$$M(x) = \sum_{j=1}^{m-1} e_j \otimes e_j + \frac{\gamma_0(x)}{\gamma_1(x)}e_m \otimes e_m \quad ,$$

μ almost everywhere in $\{\; x \;:\; \gamma_0(x) \neq \gamma_1(x) \;\}$. Conditioned on the extraction of a subsequence we have now established the representation formula (58) , but since all elements of the formula are explicit, and independent of the particular subsequence it follows that the formula holds, provided only $|\omega_\epsilon| \to 0$, as stated in this proposition. □

In contrast to [4, 5], we have allowed the surfaces of the thin domains to exhibit oscillations of amplitude ϵ, on a scale of order ϵ^a, $a < 1$. Some regularity of h_ϵ is essential for the result of Proposition 2 to hold; assuming just that h_ϵ is measurable and bounded will not suffice. We are also convinced that the condition $a < 1$ is essential – if we permit $a \geq 1$ we expect the emergence of other polarization tensors M than that in Proposition 2.

We complete this section by providing a simple, two-dimensional example of a family of multiply connected domains, the limiting measure, $\lim \frac{1}{|\omega_\epsilon|}1_{\omega_\epsilon}$, of which is the surface measure on a flat hyper-surface, but for which the "effective" polarization tensor is in general different from that of Proposition 2. By appropriately connecting the individual components of these domains with very thin "sheets", and regularizing the surfaces, our construction would give rise to a family of the type $a = 1$ (referring to the notation of Proposition 2).

Proposition 3. *Assume that the conductivities γ_0 and γ_1 are constants, with $\gamma_0 \neq \gamma_1$, and assume that the domain Ω contains the square $[-1,1] \times [-1,1]$. Let δ be a fixed parameter, $0 < \delta < \infty$, and define $\delta^* = \delta + \sqrt{\delta}$. For any $0 < \epsilon < \min\{\delta, 1/\delta^*\}$ let $\omega_\epsilon^\delta \subset [-1,1] \times [-1,1]$ denote the open set*

$$\omega_\epsilon^\delta = \cup_{k=0}^{N_{\epsilon,\delta}} \left(Q_\epsilon^\delta + \begin{pmatrix} k\epsilon\delta^* \\ 0 \end{pmatrix} \right) \quad ,$$

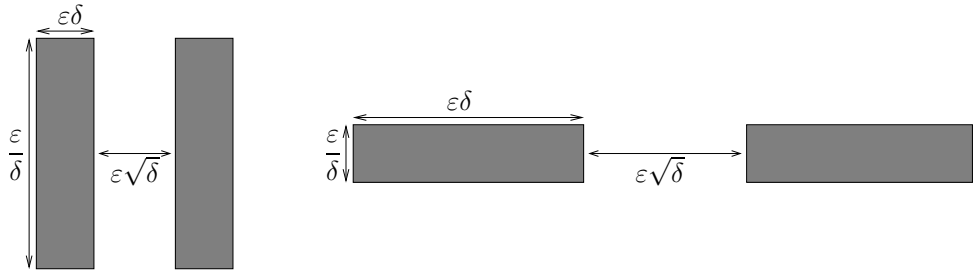


Figure 1: Two adjacent "cells" of ω_ϵ^δ, for $\delta = \dfrac{1}{2}$ on the left, and $\delta = 2$ on the right.

with

$$N_{\epsilon,\delta} = [\frac{1}{\epsilon\delta^*}] - 1 \quad , \qquad \textit{and} \qquad Q_\epsilon^\delta = \left(-\frac{\epsilon\delta}{2}, \frac{\epsilon\delta}{2}\right) \times \left(-\frac{\epsilon}{2\delta}, \frac{\epsilon}{2\delta}\right) \quad .$$

Then

$$\frac{\epsilon}{\delta^*} - \epsilon^2 \leq |\omega_\epsilon^\delta| \leq \frac{\epsilon}{\delta^*} \quad ,$$

and

$$\lim_{\epsilon\to 0} \int_\Omega \frac{1}{|\omega_\epsilon^\delta|} 1_{\omega_\epsilon^\delta}(x)\ \phi(x)\ dx = \lim_{\epsilon\to 0} \int_\Omega \frac{\delta^*}{\epsilon} 1_{\omega_\epsilon^\delta}(x)\ \phi(x)\ dx = \int_0^1 \phi(x_1, 0)\ dx_1 \quad ,$$

for all $\phi \in C^0(\overline{\Omega})$. Let $\omega_{\epsilon_n}^\delta$, $M^\delta(\cdot) \in L^2((0,1), dx)$ be a subsequence and a polarization tensor, the existence of which are guaranteed by Proposition 1 (and Theorem 1). The polarization tensor necessarily has the form

$$M^\delta = \begin{bmatrix} m_1^\delta & 0 \\ 0 & m_2^\delta \end{bmatrix} \quad ,$$

with

$$\lim_{\delta\to 0} m_1^\delta = \frac{\gamma_0}{\gamma_1} \quad , \qquad \lim_{\delta\to 0} m_2^\delta = 1 \quad ,$$

and

$$\lim_{\delta\to\infty} m_1^\delta = 1 \quad , \qquad \lim_{\delta\to\infty} m_2^\delta = \frac{\gamma_0}{\gamma_1} \quad ,$$

the limits being, e.g., in $L^2((0,1), dx)$.

Proof. The verification of the bounds on the Lebesgue measure of ω_ϵ^δ, and the proof of the limiting identity

$$\lim_{\epsilon\to 0} \frac{1}{|\omega_\epsilon^\delta|} 1_{\omega_\epsilon^\delta}\ dx = \lim_{\epsilon\to 0} \frac{\delta^*}{\epsilon} 1_{\omega_\epsilon^\delta}\ dx = dx_1|_{(0,1)}$$

are quite straightforward, and are left to the reader. The fact that M^δ has eigendirections that are parallel (and orthogonal) to the x_1 axis, follows immediately from (9)-(8) and natural solution symmetries, if we enlarge the domain Ω to

be symmetric in the x_1 axis (and note that the polarization tensor is independent of Ω). It thus only remains to verify the limiting identities concerning m_1^δ and m_2^δ. We note that it suffices to verify that

$$\lim_{\delta\to 0} m_1^\delta = \frac{\gamma_0}{\gamma_1} \quad , \qquad \text{and } \lim_{\delta\to\infty} m_2^\delta = \frac{\gamma_0}{\gamma_1} \qquad \text{in } L^2((0,1),dx) \quad . \tag{61}$$

If for instance we have verified the first limiting statement in (61), then it follows immediately (since $\min\{1,\frac{\gamma_0}{\gamma_1}\} \le m_1^\delta \le \max\{1,\frac{\gamma_0}{\gamma_1}\}$) that

$$\lim_{\delta\to 0} \frac{1}{m_1^\delta} = \frac{\gamma_1}{\gamma_0} \qquad \text{in } L^2((0,1),dx) \quad .$$

From integration of the inequalities of Theorem 3 we now get

$$\limsup_{\delta\to 0} \int_0^1 m_2^\delta(x)\ dx \le 1 \quad , \qquad \text{and} \quad \limsup_{\delta\to 0} \int_0^1 \frac{1}{m_2^\delta}(x)\ dx \le 1 \quad .$$

Use of these inequalities and Cauchy-Schwarz's inequality yields

$$1 = (\int_0^1 1\ dx)^2 \le \liminf_{\delta\to 0} \int_0^1 m_2^\delta(x)\ dx \cdot \limsup_{\delta\to 0} \int_0^1 \frac{1}{m_2^\delta}(x)\ dx \le 1 \quad ,$$

based on which we immediately conclude that

$$\lim_{\delta\to 0} \int_0^1 m_2^\delta(x)\ dx = \liminf_{\delta\to 0} \int_0^1 m_2^\delta(x)\ dx = \limsup_{\delta\to 0} \int_0^1 m_2^\delta(x)\ dx = 1 \quad .$$

Since $m_2^\delta - 1$ is either a.e. positive, or a.e. negative, it follows that

$$\lim_{\delta\to 0} \int_0^1 |m_2^\delta(x) - 1|\ dx = 0 \quad ,$$

and so

$$\int_0^1 |m_2^\delta(x) - 1|^2\ dx \le \left(\max(1,\frac{\gamma_0}{\gamma_1}) + 1\right) \int_0^1 |m_2^\delta(x) - 1|\ dx \to 0 \qquad \text{as } \delta \to 0 \quad ,$$

i.e., $\lim_{\delta\to 0} m_2^\delta = 1$ in $L^2((0,1),dx)$. Similarly, the second limiting statement in (61) implies that $\lim_{\delta\to\infty} m_1^\delta = 1$ in $L^2((0,1),dx)$. We now proceed to verify the two limiting statements of (61). In that regard it suffices to verify that

$$\lim_{\delta\to 0} \int_0^1 m_1^\delta\ dx = \frac{\gamma_0}{\gamma_1} \quad , \qquad \text{and } \lim_{\delta\to\infty} \int_0^1 m_2^\delta\ dx = \frac{\gamma_0}{\gamma_1} \quad . \tag{62}$$

The expressions $m_1^\delta - \frac{\gamma_0}{\gamma_1}$ and $m_2^\delta - \frac{\gamma_0}{\gamma_1}$ are namely either a.e. positive, or a.e. negative, and (62) thus implies

$$\lim_{\delta\to 0} \int_0^1 |m_1^\delta(x) - \frac{\gamma_0}{\gamma_1}|\ dx = 0 \quad , \qquad \text{and } \lim_{\delta\to\infty} \int_0^1 |m_2^\delta(x) - \frac{\gamma_0}{\gamma_1}|\ dx = 0 \quad .$$

Due to the a.e. boundedness of the expressions $m_1^\delta - \frac{\gamma_0}{\gamma_1}$ and $m_2^\delta - \frac{\gamma_0}{\gamma_1}$ the L^2 convergence, as stated in (61), follows. From Lemma 3 (and Remark 1) we know that

$$\int_0^1 m_i^\delta(x)\, dx$$
$$= \frac{\gamma_0}{\gamma_1} + \frac{1}{\gamma_1 - \gamma_0} \lim_{\epsilon_n \to 0} \min_{w \in H^1(\Omega)} \frac{\delta^*}{\epsilon_n} \int_\Omega \hat{\gamma}_{\epsilon_n} \left| \nabla w + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_{\epsilon_n}^\delta} \nabla x_i \right|^2 dx \quad .$$

In order to arrive at (62) it is thus sufficient to find two families of functions $w_{i,\epsilon}^\delta \in H^1(\Omega)$, $0 < \epsilon < \min\{\delta, 1/\delta^*\}$, $i = 1, 2$, so that

$$\frac{\delta^*}{\epsilon} \int_\Omega \hat{\gamma}_\epsilon \left| \nabla w_{i,\epsilon}^\delta + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega_\epsilon^\delta} \nabla x_i \right|^2 dx \le g_i(\delta) \quad , \tag{63}$$

for δ sufficiently close to zero (for $i = 1$), and δ sufficiently large (for $i = 2$), where g_1 is such that $\lim_{\delta \to 0} g_1(\delta) = 0$, and g_2 is such that $\lim_{\delta \to \infty} g_2(\delta) = 0$.

We first consider $\delta < 1$. We define a function f_ϵ with compact support in $[-(\delta/2 + d)\epsilon, (\delta/2 + d)\epsilon] \times [-(1/(2\delta) + d)\epsilon, (1/(2\delta) + d)\epsilon]$, d being a positive constant, a precise value of which will be chosen later. The function f_ϵ is constructed in the first quadrant as follows.

For $0 \le x_2 \le \epsilon/(2\delta)$, we choose

$$\begin{cases} f_\epsilon(x) = x_1 & \text{for } x_1 \in \left[0, \delta\frac{\epsilon}{2}\right], \\ f_\epsilon(x) = \delta\frac{\epsilon}{2}\left(1 + \frac{\delta\epsilon - 2x_1}{2d\epsilon}\right) & \text{for } x_1 \in \left[\delta\frac{\epsilon}{2}, \delta\frac{\epsilon}{2} + d\epsilon\right]. \end{cases}$$

For $\epsilon/(2\delta) \le x_2 \le \epsilon/(2\delta) + d\epsilon$, and $0 \le x_1 \le \delta\frac{\epsilon}{2}$, we choose

$$f_\epsilon(x) = x_1 \left(1 + \frac{\epsilon/\delta - 2x_2}{2d\epsilon}\right).$$

Finally, for $x_2 \ge \epsilon/(2\delta)$, $x_1 \ge \delta\epsilon/2$ and $\left(\frac{\delta\epsilon - 2x_1}{2d\epsilon}\right)^2 + \left(\frac{\epsilon/\delta - 2x_2}{2d\epsilon}\right)^2 \le 1$, we choose

$$f_\epsilon(x) = \delta\frac{\epsilon}{2}\left(1 - \sqrt{\left(\frac{\delta\epsilon - 2x_1}{2d\epsilon}\right)^2 + \left(\frac{\epsilon/\delta - 2x_2}{2d\epsilon}\right)^2}\right).$$

The function f_ϵ is then extended by zero elsewhere and to the other quadrants by $f_\epsilon(-x_1, x_2) = -f_\epsilon(x_1, x_2)$, $f_\epsilon(x_1, -x_2) = f_\epsilon(x_1, x_2)$. A straightforward computation shows that

$$\int_\Omega \left| \nabla f_\epsilon - 1_{Q_\epsilon^\delta} \nabla x_1 \right|^2 dx = \epsilon^2 \left(\frac{\delta}{2d} + \frac{\delta^3}{6d} + \frac{2d\delta}{3} + \frac{\pi}{4}\delta^2\right) \quad .$$

Now select $d = \sqrt{\delta}/2$. Then f_ϵ is compactly supported in $[-\frac{\epsilon\delta^*}{2}, \frac{\epsilon\delta^*}{2}] \times \left[-\left(\frac{1}{2\delta} + \frac{\sqrt{\delta}}{2}\right)\epsilon, \left(\frac{1}{2\delta} + \frac{\sqrt{\delta}}{2}\right)\epsilon\right] \subset [-1, 1,] \times [-1, 1]$. Therefore, introducing

$$w^\delta_{1,\epsilon}(x) = -\sum_{k=0}^{N_{\epsilon,\delta}} \frac{\gamma_1 - \gamma_0}{\gamma_1} f_\epsilon(x_1 + k\epsilon\delta^*, x_2)$$

we have

$$\begin{aligned}\frac{\delta^*}{\epsilon}\int_\Omega \hat{\gamma}_\epsilon \left|\nabla w^\delta_{1,\epsilon} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega^\delta_\epsilon} \nabla x_1\right|^2 dx \\ \leq \gamma_0 \left(\frac{\gamma_1 - \gamma_0}{\gamma_1}\right)^2 \left(\sqrt{\delta} + \frac{\delta^{5/2}}{3} + \frac{\delta^{3/2}}{3} + \frac{\pi}{4}\delta^2\right) \quad .\end{aligned}$$

Since the right-hand side tends to zero with δ, the proof of (63) is complete in the case $\delta \to 0$.

We now turn the attention to $\delta > 1$. By exchanging the roles of x_1 and x_2, and replacing δ by $1/\delta$ in the previous construction of f_ϵ, we obtain a function $\tilde{f}_\epsilon(x)$ with compact support in $[-\left(\frac{\delta}{2} + d\right)\epsilon, \left(\frac{\delta}{2} + d\right)\epsilon] \times [-\left(\frac{1}{2\delta} + d\right)\epsilon, \left(\frac{1}{2\delta} + d\right)\epsilon]$, such that

$$\int_\Omega \left|\nabla \tilde{f}_\epsilon - 1_{Q^\delta_\epsilon} \nabla x_2\right|^2 dx = \epsilon^2 \left(\frac{1}{2d\delta} + \frac{1}{6d\delta^3} + \frac{2d}{3\delta} + \frac{\pi}{4\delta^2}\right) \quad .$$

Now select $d = 1/2\sqrt{\delta}$. Then the function $\tilde{f}_\epsilon$ is compactly supported in $[-\frac{\epsilon\delta^*}{2}, \frac{\epsilon\delta^*}{2}] \times \left[-\left(\frac{1}{2\delta} + \frac{1}{2\sqrt{\delta}}\right)\epsilon, \left(\frac{1}{2\delta} + \frac{1}{2\sqrt{\delta}}\right)\epsilon\right] \subset [-1, 1,] \times [-1, 1]$. Therefore, introducing

$$w^\delta_{2,\epsilon}(x) = -\sum_{k=0}^{N_{\epsilon,\delta}} \frac{\gamma_1 - \gamma_0}{\gamma_1} \tilde{f}_\epsilon(x_1 + k\epsilon\delta^*, x_2)$$

we have

$$\begin{aligned}\frac{\delta^*}{\epsilon}\int_\Omega \hat{\gamma}_\epsilon \left|\nabla w^\delta_{2,\epsilon} + \frac{\gamma_1 - \gamma_0}{\gamma_1} 1_{\omega^\delta_\epsilon} \nabla x_2\right|^2 dx \\ \leq \gamma_0 \left(\frac{\gamma_1 - \gamma_0}{\gamma_1}\right)^2 \left(\frac{1}{\sqrt{\delta}} + \frac{1}{3\delta^{5/2}} + \frac{1}{3\delta^{3/2}} + \frac{\pi}{4\delta^2}\right) \quad .\end{aligned}$$

Since the right-hand side tends to zero as $\delta \to \infty$, the proof of (63) is complete in the case $\delta \to \infty$. This completes the proof of Proposition 3 □

Remark 2. *Due to the construction of ω^δ_ϵ we expect the matrices M^δ to be constant along the interval $(0, 1)$. In that case the L^2 convergence in Proposition 3 is really the convergence of constants. Proposition 3 asserts that the (set of all possible) polarization tensors M^δ converge to*

$$\begin{pmatrix} \gamma_0/\gamma_1 & 0 \\ 0 & 1 \end{pmatrix} \tag{64}$$

as $\delta \to 0$, and converge to

$$\begin{pmatrix} 1 & 0 \\ 0 & \gamma_0/\gamma_1 \end{pmatrix} \tag{65}$$

as $\delta \to \infty$. It is thus clear that for δ sufficiently close to zero M^δ will be different from the polarization tensor of Proposition 2. Let us also note that we do expect the matrix M^δ to be independent of the particular subsequence ϵ_n, i.e., we do expect a unique polarization tensor M^δ (as $\epsilon \to 0$ for fixed δ). Since we also expect the matrices M^δ to vary continuously with δ, there should exist a nonempty open interval of $\delta's$ for which we have "effective" polarization tensors M^δ that are different from both the matrix (64) and the matrix (65).

References

[1] H. Ammari, E. Beretta, and E. Francini, Reconstruction of thin conductivity imperfections. *Appl. Anal.* **83** (2004) pp. 63–76.

[2] H. Ammari, E. Beretta, and E. Francini, Reconstruction of thin conductivity imperfections. II. The case of multiple segments. *Appl. Anal.* **85** (2006) pp. 87–105.

[3] A.Yu. Belyaev, and S.M. Kozlov, Hierarchical structures and estimates for homogenized coefficients. *Russian J. Math. Phys.* **1** (1992) pp. 3–15.

[4] E. Beretta, E. Francini, and M.S. Vogelius, Asymptotic formulas for steady state voltage potentials in the presence of thin inhomogeneities. A rigorous error analysis. *Journal Mathematiques Pures et Appliquees* **82** (2003) pp. 1277–1301.

[5] E. Beretta, and A. Mukherjee, and M.S. Vogelius, Asymptotic formulas for steady state voltage potentials in the presence of conductivity imperfections of small area. *Z. Angew. Math. Phys.* **52** (2001) pp. 543–572.

[6] M.F. Ben Hassen, E. Bonnetier, Asymptotic formulas for the voltage potential in a composite medium containing close or touching disks of small diameter. *Multiscale Model. Simul.* **4** (2005) pp. 250–277.

[7] M. Brühl, M. Hanke, and M.S. Vogelius, A direct impedance tomography algorithm for locating small inhomogeneities. *Numer. Math.* **93** (2003) pp. 635–654.

[8] Y. Capdeboscq, and M.S. Vogelius, A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction. *M2AN* **37** (2003) pp. 159–173.

[9] Y. Capdeboscq, and M.S. Vogelius, Optimal asymptotic estimates for the volume of internal inhomogeneities in terms of multiple boundary measurements. *M2AN* **37** (2003) pp. 227–240.

[10] Y. Capdeboscq, and M.S. Vogelius, A review of some recent work on impedance imaging for inhomogeneities of low volume fraction. Contemporary Mathematics **362** (Eds. C. Conca, R. Manasevich, G. Uhlmann, and M.S. Vogelius). AMS (2004) pp. 69–88.

[11] D.J. Cedio-Fengya, S. Moskow, and M.S. Vogelius, Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction. *Inverse Problems* **14** (1998) pp. 553–595.

[12] D. Gilbarg, N.S Trudinger, *Elliptic Partial Differential Equations of Second Order*. Grundleheren der mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin, Heidelberg, New York (1983).

[13] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*. Monographs and Studies in Mathematics, Vol. 24. Pitman Publishing, Boston, London, Melbourne (1985).

[14] R.V. Kohn, and G.W. Milton, On bounding the effective conductivity of anisotropic composites. *Homogenization and Effective Moduli of Materials and Media, Eds. J.L. Ericksen, D. Kinderlehrer, R. Kohn, and J.-L. Lions*, IMA Volumes in Mathematics and its Applications, **1** pp. 97–125. Springer Verlag, Berlin, Heidelberg, New York (1986).

[15] R. Lipton, Inequalities for electric and elastic polarization tensors with applications to random composites. *J. Mech. Phys. Solids* **41** (1993) pp. 809-833.

Acknowledgments

The research of Y. Capdeboscq was partially supported by "Action Concertée Incitative" 991ZVA24. The research of M.S. Vogelius was partially supported by NSF grants DMS–0307119 and INT–0003788.