

Edge-dominating cycles, k -walks and Hamilton prisms in $2K_2$ -free graphs

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To the memory of Sergei Duzhin.

Abstract. We show that an edge-dominating cycle in a $2K_2$ -free graph can be found in polynomial time; this implies that every $\frac{1}{k-1}$ -tough $2K_2$ -free graph admits a k -walk, and it can be found in polynomial time. For this class of graphs, this proves a long-standing conjecture due to Jackson and Wormald (1990). Furthermore, we prove that for any $\epsilon > 0$ every $(1 + \epsilon)$ -tough $2K_2$ -free graph is prism-Hamiltonian and give an effective construction of a Hamiltonian cycle in the corresponding prism, along with few other similar results.

1 Introduction

A graph G is called β -*tough*, for a real $\beta > 0$, if for any $p \geq 2$ it cannot be split into p components by removing less than $p\beta$ vertices. This concept, a measure of graph connectivity and “resilience” under vertex subsets removal, was introduced in 1973 by Chvátal [7], while studying Hamiltonicity of graphs. For a survey of results on graph toughness till 2006 see [3].

In general, toughness of a graph is NP-hard to compute [2]. Considerable work went into investigating this computational problem for various classes of graphs. In particular, recently, Broersma, Patel and Pyatkin proved [6] that toughness of a $2K_2$ -free graph, i.e. a graph that does not contain an induced copy of the disjoint union of two edges, can be found in polynomial time.

Note that $2K_2$ -free graphs are an interesting class from algorithmic complexity point of view; most classical algorithmic problems for them are hard, with a notable exception of the maximum weighted independent set problem [1], [13, Graphclass: $2K_2$ -free]. In particular Hamiltonian cycle problem is NP-complete already for a subclass of $2K_2$ -free graphs, the *split* graphs—graphs for which the set of vertices can be partitioned into a clique and an independent set [9, Exercise 6.2]. Due to the latter, for $2K_2$ -free graphs it makes sense to study computational complexity of concepts which are generalisations of the Hamiltonian cycle problem, such as k -walk.

Let $p \times G$ denote the multigraph obtained from G by taking each edge p times. A k -walk is a spanning subgraph W of $2k \times G$ such that each vertex of W has even degree at most $2k$. In particular a graph has a 1-walk if and only if it is K_2 (i.e. one edge) or Hamiltonian. For a survey of results on walks in graphs till 2005 see [12]. In 1990 Jackson and Wormald conjectured [10] that for any integer $k \geq 2$ a $\frac{1}{k-1}$ -tough graph G admits a k -walk.

In this paper, we prove that Jackson and Wormald's Conjecture is true under the assumption that G is $2K_2$ -free.

Theorem 1. *For any integer $k \geq 2$, every $\frac{1}{k-1}$ -tough $2K_2$ -free graph G admits a k -walk. Moreover, the latter can be found in time polynomial in $|V(G)|$.*

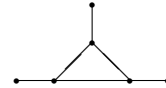
If for $k \geq 2$ we let the toughness value $\frac{1}{k-1}$ increase to $\frac{1}{k-2}$ then one does not need $2K_2$ -freeness. Indeed, it is shown in [10] that every $\frac{1}{k-2}$ -tough graph has a k -walk.

Clearly, if G is Hamiltonian, then G is 1-tough. More generally, if G has a k -walk, then G is $\frac{1}{k}$ -tough [10]. However, the converse is not true already for $k = 1$ (there even exist 2-tough graphs which are not Hamiltonian, cf. [4]).

This more or less summarises the situation with t -tough graphs, $t \leq 1$. On the $t > 1$ side a famous conjecture of Chvátal [7] claims that there exists a constant β such that every β -tough graph is Hamiltonian. Towards this, Ellingham and Zha [8] proved that every 4-tough graph has a 2-walk (cf. Theorem 2 below).

It was recently shown [6] that every 25-tough $2K_2$ -free graph on at least three vertices is Hamiltonian. Our Theorem 1 was inspired by this result. However, our approach is technically quite different.

Our next result concerns a structure that is half-way between 1- and 2-walks. The *prism* over a graph G is the Cartesian product $G \square K_2$ of G with the complete graph K_2 . G is called *prism-Hamiltonian* if $G \square K_2$ is Hamiltonian. If G is Hamiltonian, then $G \square K_2$ is also Hamiltonian, but the converse does not hold in general, cf. [11]. As well, this property is stronger than having a 2-walk: cf. figure on the right, where we have a $2K_2$ -free graph with a 2-walk, but without Hamiltonian prism.



Theorem 2. *Every $(1 + \epsilon)$ -tough $2K_2$ -free graph G is prism-Hamiltonian, for any $\epsilon > 0$. Moreover, a Hamiltonian cycle in the prism over G can be found in time polynomial in $|V(G)|$.*

It is worth mentioning that the toughness constant in Theorem 2 is much better than $\frac{3}{2}$, the lower bound on toughness of a $2K_2$ -free graph needed for its Hamiltonicity, see [6, Sect. 4].

To prove Theorem 1 and Theorem 2, we first prove a result on edge-dominating subgraphs (a subgraph S of G is called *edge-dominating* if each edge of G contains at least one vertex from $V(S)$).

Theorem 3. *Let G be a $2K_2$ -free graph. Then*

1. G admits an edge-dominating cycle (or an edge, or a vertex) C ;
2. if G contains a triangle, then G admits an edge-dominating cycle C , with three successive vertices on C forming a triangle in G .

Moreover, C can be found in time polynomial in $|V(G)|$.

In fact, in 1983 Veldman [14] has proved the existence of edge-dominating cycles for $2K_2$ -free graphs. However, his proof is based on contraposition, so it neither tells how to find C in (1), nor allows to restrict C as in (2).

In the remainder of the paper we provide the proofs, and then discuss related open questions.

2 Proof of Theorem 3

2.1 The proof of the first part of Theorem 3

If G is a tree, then, as it is $2K_2$ -free, it must either have an edge-dominating vertex, or an edge-dominating edge. Indeed, in this case, if any two edges intersect, then, as G has no cycles, they all must intersect in a vertex, which will be dominating. Now, assume that there are two non-intersecting edges, say xy and uv . As they cannot form a $2K_2$, they are on a 3-path, $xyuv$, without loss of generality. Now we claim that yu is a dominating edge. Indeed, suppose to the contrary that an edge ab does not intersect yu . Then, as they cannot form $2K_2$, $abyu$ is a 3-path, without loss of generality. As G is a tree, by is the only edge connecting vertices of ab and yu . Thus either ab and uv form a $2K_2$, or b lies in a cycle; in both cases this is a contradiction, proving that yu is a dominating edge.

Otherwise, G has a cycle, say $C = x_1x_2 \cdots x_kx_1$, where $k \geq 3$. If C is edge-dominating, then we are done. Otherwise there must be an edge v_1v_2 (assume there are $t > 0$ such edges), with neither v_1 nor v_2 on C . Since G is $2K_2$ -free, v_1 and v_2 have at least two distinct neighbours on C . Let $x_1v_1 \in E(G)$ without loss of generality;

1. if $x_2v_1 \in E(G)$, then $C' = x_1v_1x_2x_3 \cdots x_kx_1$ is a longer cycle;
2. if $x_2v_2 \in E(G)$, then $x_1v_1v_2x_2x_3 \cdots x_kx_1$ is a longer cycle;
3. if $x_2v_1, x_2v_2 \notin E(G)$, then applying $2K_2$ -freeness to v_1v_2 and x_2x_3 , we get either $x_3v_1 \in E(G)$ or $x_3v_2 \in E(G)$.
 - (a) If $x_3v_2 \in E(G)$, then $C' = x_1v_1v_2x_3 \cdots x_kx_1$ is a longer cycle;
 - (b) if $x_3v_2 \notin E(G)$, then $x_3v_1 \in E(G)$.
 - i. if x_2 is adjacent to no vertex outside C , then use $C' = x_1v_1x_3 \cdots x_kx_1$ instead of C . We know that C and C' have the same length, but C' dominates all the edges that are dominated by C , and C' also dominates v_1v_2 , which is not dominated by C . So replacing C by C' decreases t .
 - ii. Otherwise x_2 is adjacent to a vertex outside C , say z . As x_2 is not adjacent to v_1 or v_2 , we have z adjacent to either v_1 or v_2 . If $zv_1 \in E(G)$, then $C' = x_1v_1zx_2x_3 \cdots x_kx_1$ is a longer cycle. Otherwise $C' = x_1v_1v_2zx_2x_3 \cdots x_kx_1$ is a longer cycle.

Repeat the process above. At each iteration either $|V(C)|$ increases, or t decreases. Thus the process will stop, with $t = 0$, in at most $|E(G)|^2$ steps. \square

2.2 The proof of the second part of Theorem 3

The algorithmic procedure for the second part is almost the same, requiring only a minor modification described below.

Let G contain a triangle $w_1w_2w_3$. If $w_1w_2w_3$ is edge-dominating, then there is nothing to prove. Otherwise, there is $u_1u_2 \in E(G)$, with neither u_1 nor u_2 on $w_1w_2w_3$. Then, by the $2K_2$ -freeness, we either can connect $w_1w_2w_3$ and u_1u_2 together, to get a 5-cycle C , with $w_{\pi(1)}$, $w_{\pi(2)}$ and $w_{\pi(3)}$ successive on C for some permutation π of $\{1, 2, 3\}$, or else (without loss in generality) u_1 is adjacent to w_1 and w_2 . In the latter case set $C = u_1w_1w_3w_2$.

If C is edge-dominating, then we are done. Otherwise, we proceed by induction on $k := |V(C)|$. Suppose $k \geq 4$, and there are three successive vertices on C , namely X' , X and X'' forming a triangle in G . Let $v_1v_2 \in E(G)$ such that neither v_1 nor v_2 is on C . We claim that then we can find a cycle C' such that C' dominates more edges than C (perhaps all), and X' , X and X'' are also successive on C' .

Now we dispense with the case $k = 4$. By construction, there is an edge u_1u_2 with u_2 not in $C = u_1w_1w_3w_2$. If u_2 is joined to a vertex $w \neq u_1$ on C , then we use this extra edge to turn C into a 5-cycle through u_1u_2 and w_j ($j = 1, 2, 3$). Indeed, as w_j ($j = 1, 2, 3$) form a triangle, they will give, in some order, three successive vertices $X'XX''$ forming a triangle, as required. Namely, if $w = w_1$ or $w = w_2$, we will have $X'XX'' = w_1w_3w_2$, whereas for $w = w_3$ we will have $X'XX'' = w_3w_2w_1$. Thus, we are left with the case where the only edge joining u_2 and C is u_1u_2 .

First, consider the case $v_2 = u_2$. Then v_1 must be adjacent to the edge w_1w_2 , otherwise v_1u_2 and w_1w_2 form a $2K_2$. Thus we obtain a 5-cycle $\Omega(u_2)$ through u_1u_2 , v_1 , w_1 , and w_2 , with $\{X, X', X''\} = \{u_1, w_1, w_2\}$.

It remains to consider the case of $v_i \neq u_2$ ($i = 1, 2$). As u_1u_2 and v_1v_2 cannot form a $2K_2$, and as we already considered the case of u_2 adjacent to a vertex not on C , we may assume that v_2 is adjacent to u_1 . This gives us the already considered configuration, with u_2 replaced by v_2 , from which we obtain a 5-cycle $\Omega(v_2)$.

From now on we can assume $k \geq 5$. By $2K_2$ -freeness, v_1 and v_2 are adjacent to at least two of $\{X, X', X''\}$, and thus to at least one of $\{X', X''\}$. Suppose, without loss of generality, that $v_1X' \in E(G)$; label the vertices in C in the following way: X' is labeled by x_1 ; the neighbour of x_1 on C distinct from X is labeled by x_2 ; the other vertices on C are labeled successively, see Figure 1.

Note that for $k \geq 5$ the operation used in the proof of the first part of Theorem 3 of replacing C by C' (enlarging $|V(C)|$ or reducing t) does not touch the edges $x_{k-1}x_k$ and x_kx_1 . Thus the triangle-forming vertices X'' , X and X' are always successive on C in our process. Then they are on the edge-dominating cycle we obtain there. \square

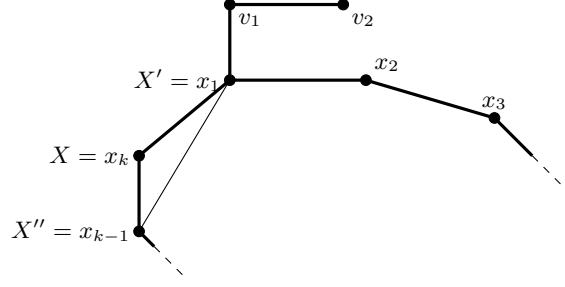


Fig. 1. The cycle C of length k

3 Proof of Theorem 1

Combining Theorem 3 and the following Lemma 1, we obtain Theorem 1.

Lemma 1. *Let $k \geq 2$. If G has an edge-dominating cycle C (or an edge, or a vertex) and if G is $\frac{1}{k-1}$ -tough then G admits a k -walk. Moreover, the latter can be found in time polynomial in $|V(G)|$.*

Proof. The induced subgraph $D = G - C$ is an independent set. Define an auxiliary bipartite graph Γ , with one part $V(D)$ and the other consisting of $k-1$ copies of $V(C)$, so that each edge $dc \in V(D) \times V(C)$ corresponds to $k-1$ edges dc_1, \dots, dc_{k-1} of Γ . Let Φ denote the $(k-1)$ -to-1 map $\Phi : E(\Gamma) \rightarrow E(G)$ sending each dc_j , for $1 \leq j \leq k-1$, to $dc = \Phi(dc_j)$.

For any $D' \subset D$, by $\frac{1}{k-1}$ -toughness, D' has at least $\lceil \frac{|D'|}{k-1} \rceil$ neighbours in C . Thus D' has at least $|D'|$ neighbours in Γ . By Hall's Theorem [5, Theorem 16.4] applied to Γ , it has a matching M saturating D ; i.e. each $v \in V(D)$ is incident to an $e \in M$, and each v in the other part of Γ is incident to at most one $e \in M$.

Hence for each $e \in \Phi(M) \subset E(G)$ we have $e \in D \times C$. Moreover, each $v \in V(D)$ is incident to exactly one $e \in \Phi(M)$, while each $v \in V(C)$ is incident to at most $k-1$ edges in $\Phi(M)$. Then the (doubled) edges in $\Phi(M)$ and the edges in the edge-dominating cycle (respectively, the doubled edge in the case of existence of a dominating edge) C form a k -walk in G .

To show the last claim, it suffices to notice that M (a maximum matching in a bipartite graph) can be found in time polynomial in $|V(G)|$. \square

4 Proof of Theorem 2

The following lemma is the key technique in the proof of Theorem 2.

Lemma 2. *Let G be $(1 + \epsilon)$ -tough, for some $\epsilon > 0$.*

1. *If G contains an edge-dominating cycle C with even number of vertices, then the prism over G is Hamiltonian.*

2. If G contains an edge-dominating cycle $C = v_1 v_2 \cdots v_{2p+1} v_1$ of odd length, and there are three vertices v_1 , v_{2q} and v_{2q+1} , for some $1 \leq q \leq p$, inducing a triangle in G , then the prism over G is Hamiltonian.

Proof. For the first part (see Figure 2), denote $C = v_1 v_2 \cdots v_{2p} v_1$. The set $D = V(G) - V(C)$ of vertices outside C is an independent set. By Hall's Theorem and 1-toughness, there is a matching M between D and C . That means that for any vertex u_j in D , there is a vertex v_{i_j} on C adjacent to u_j in M .

Obviously, we have a Hamiltonian cycle in \bar{C} , the prism over C , namely

$$v_1 v'_1 v'_2 v_2 \cdots v_{2p-1} v'_{2p-1} v'_{2p} v_{2p} v_1.$$

Now, we change every $v_{i_j} v'_{i_j}$ (or $v'_{i_j} v_{i_j}$) into $v_{i_j} u_j u'_j v'_{i_j}$ (or $v'_{i_j} u'_j u_j v_{i_j}$) to get a Hamilton cycle in \bar{G} .

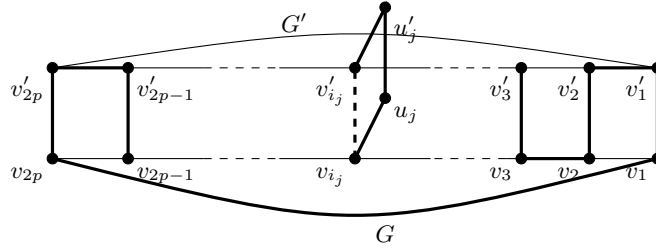


Fig. 2. G has an edge-dominating cycle of even length

For the second part, denote $C = v_1 v_2 \cdots v_{2p+1} v_1$ (see Figure 3). The set $D = V(G) - V(C)$ of vertices outside C is an independent set. By Hall's Theorem, and $(1 + \epsilon)$ -toughness, there is a matching M between D and $C - \{v_1\}$. This means that for any vertex u_j in D , there is a vertex v_{i_j} in $C - \{v_1\}$ adjacent to u_j in M .

Clearly, we have a Hamiltonian cycle in \bar{C} , namely

$$v_1 v_2 v'_2 v'_3 v_3 \cdots v_{2q-1} v_{2q} v'_{2q} v'_{2q+1} v_{2q+1} \cdots v_{2p+1} v_1.$$

Now, we change every $v_{i_j} v'_{i_j}$ (or $v'_{i_j} v_{i_j}$) into $v_{i_j} u_j u'_j v'_{i_j}$ (or $v'_{i_j} u'_j u_j v_{i_j}$) to get a Hamilton cycle in \bar{G} . \square

Now we complete the proof of Theorem 2. Suppose G is a triangle-free $2K_2$ -free graph. By [6, Theorem 4], if $|V(G)| \geq 3$ then G is Hamiltonian,³ and so

³ We only need to rely on [6, Theorem 4] for the case of G not having a dominating cycle C of even length, for otherwise we have case 1 of Lemma 2 at our disposal; $|C|$ can be odd only in the case of G not bipartite; such a G is, by [6, Lemma 2], of C_5^* -type, i.e. it has a rather special structure.

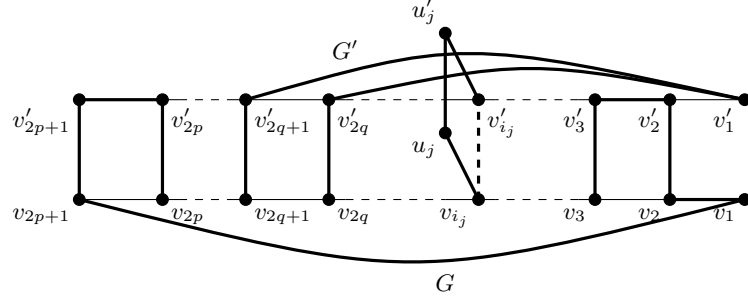


Fig. 3. G has an edge-dominating cycle of odd length

prism-Hamiltonian. A polynomial-time algorithm to construct a Hamiltonian cycle in G is implicit in the proof of [6, Theorem 4], and building up a Hamiltonian cycle in the prism over G from a Hamiltonian cycle $C = v_1 v_2 \dots v_m v_1$ in G is trivially (and in time polynomial in $|V(G)|$) done by concatenating the path $v_1 \dots v_{m-1} v_m$, i.e. C without the last edge, with the path $v'_m v'_{m-1} \dots v'_1$.

If $|V(G)| = 2$, i.e. G is a single edge, and obviously prism-Hamiltonian. Finally, if G is not triangle-free then we are done by Theorem 3 (2) and Lemma 2, and noting that the corresponding construction can be done in time polynomial in $|V(G)|$. \square

5 Concluding remarks

Lemma 1 can be used to prove existence of 2-walks in classes of graphs wider than $2K_2$ -free. For instance, it is immediate from [14, Corollary 3.2] that each 2-connected $3K_2$ -free graph admits an edge-dominating cycle. From the latter and Lemma 1, it is easy to obtain the following.

Theorem 4. *Let G be a 1-tough $3K_2$ -free graph. Then G admits a 2-walk.* \square

It would be interesting to find out whether Theorem 4 and similar results of this type can be made effective. Towards this end, we would like to propose the following

Conjecture 1. Let $\ell \geq 2$ be a fixed constant. Then for the $\ell - 1$ -connected ℓK_2 -free graphs there is a polynomial time algorithm finding an edge-dominating cycle.

Of independent interest would be finding out whether more general results from [14], in particular Theorem 5, can be made algorithmic.

Theorem 5. [14, Theorem 3]. *Let G be an $\ell - 1$ -connected graph such that for every induced ℓK_2 -subgraph H of G one has the sum of degrees of vertices in H at least $\frac{(\ell-1)(|V(G)|-\ell+1)}{2}$. Then G has an edge-dominating cycle.* \square

Acknowledgements.

The authors thank Nick Gravin, Edith Elkind, and the anonymous referee for helpful comments on versions of this text. Research supported by Singapore MOE Tier 2 Grant MOE2011-T2-1-090 (ARC 19/11) and by the EU Horizon 2020 research and innovation programme, grant agreement OpenDreamKit No 676541.

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