Hypertoric manifolds and hyperKähler moment maps

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To Simon Salamon on the occasion of his 60th birthday

Abstract We discuss various aspects of moment map geometry in symplectic and hyperKähler geometry. In particular, we classify complete hyperKähler manifolds of dimension $4n$ with a tri-Hamiltonian action of a torus of dimension n, without any assumption on the finiteness of the Betti numbers. As a result we find that the hyperKähler moment in these cases has connected fibres, a property that is true for symplectic moment maps, and is surjective. New examples of hypertoric manifolds of infinite topological type are produced. We provide examples of non-Abelian tri-Hamiltonian group actions of connected groups on complete hyperKähler manifolds such that the hyperKähler moment map is not surjective and has some fibres that are not connected. We also discuss relationships to symplectic cuts, hyperKähler modifications and implosion constructions.

1 Introduction

A symplectic structure on a (necessarily even-dimensional) manifold is a closed non-degenerate two-form. Several Riemannian and pseudo-Riemannian geometries have been developed over the years which give rise to a symplectic structure as part of their data. The most famous example is that of a *hyperKähler structure*, where we have a Riemannian metric g and complex structures I, J, K obeying the quaternionic multiplication relations, and such that g is Kähler with respect to I, J, K. We therefore obtain a triple $(\omega_I, \omega_J, \omega_K)$ of symplectic forms.

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One of the foundational results of symplectic geometry is the Darboux Theorem, which says that locally a symplectic structure can be put into a standard form $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. Many of the interesting questions in symplectic geometry are therefore global in nature, giving the subject a more topological flavour.

Geometries involving a metric of course do not have a Darboux-type theorem, because the metric contains local information through its curvature tensor. However, there is one area of symplectic geometry, that concerning *moment maps* where a rich theory has been developed for other geometries by analogy with the symplectic situation. In this paper we shall discuss some aspects of this, especially related to hypertoric manifolds, cutting and implosion.

2 Hypertoric manifolds

Let M be a hyperKähler manifold M of dimension $4n$. We say that an action of a group G on M is *tri-symplectic* if it preserves each of the symplectic forms ω_I, ω_J and ω_K. This is equivalent to G preserving both the metric g and each of the associated complex structures I, J and K; so the action is isometric and *tri-holomorphic*. We will usually assume that G is connected and that the action is effective.

Because hyperKähler metrics are Ricci-flat, we have that if M is compact, then any Killing field X is parallel and so G is Abelian. As the complex structures are also parallel the distribution $\mathbb{H}X = \mathrm{Span}_{\mathbb{R}}\{X, IX, JX, KX\}$ is integrable and flat. Up to finite covers, an M is a product $T^{4m} \times M_0$, with G acting trivially on M_0.

Thus the interesting cases are when M is non-compact. From the Riemannian perspective is now natural to consider complete metrics. Note that by Alekseevskiĭ & Kimel'fel'd [1], any homogeneous hyperKähler manifold is flat; such a manifold is necessarily complete, so its universal cover is $\mathbb{R}^{4n}$ with the flat metric. Thus one should consider actions on M with orbits of dimension strictly less than $4n$.

One says that a tri-holomorphic action of G on M is *tri-Hamiltonian* if it is Hamiltonian for each symplectic structure, meaning that there are equivariant moment maps

$$\mu_I, \mu_J, \mu_K \colon M \to \mathfrak{g}^*, \tag{1}$$

$$d\mu_A^X = X \lrcorner \omega_A, \tag{2}$$

where $\mu_A^X = \langle \mu_A, X \rangle$. Here we write X both for the element of $\mathfrak{g}$ and the corresponding vector field $x \mapsto X_x$ on M. Also $\langle \alpha, X \rangle = \alpha(X)$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

As each $X \in \mathfrak{g}$ preserves ω_A, we have $0 = L_X\omega_A = X \lrcorner d\omega_A + d(X \lrcorner \omega_A) = d(X \lrcorner \omega_A)$, so $X \lrcorner \omega_A$ is exact. Thus if M is simply-connected then, equation (2) has a solution $\mu_A^X \in C^\infty(M)$ that is unique up to an additive constant.

2.1 Abelian actions

For an Abelian group G, equivariance of μ_A is the just the condition $L_X\mu_A^Y = 0$ for each $X, Y \in \mathfrak{g}$. But $L_X\mu_A^Y = X \lrcorner d\mu_A^Y = -\omega_A(X, Y) = -g(AX, Y)$ and $d(\omega_A(X, Y)) = L_Y(X \lrcorner \omega_A) = 0$. So $L_X\mu_A^Y$ is constant and the action is tri-Hamiltonian only if for each A we have $\mathcal{G} \perp A\mathcal{G}$, where $\mathcal{G}_x = \{X_x \mid X \in \mathfrak{g}\} \subset T_xM$. This last condition is equivalent to $\dim \mathbb{H}\mathcal{G}_x = 4 \dim \mathcal{G}_x$ for each $x \in M$.

Proposition 1. *Suppose G is a connected Abelian group that has an effective tri-Hamiltonian action on a connected hyperKähler manifold M of dimension $4n$. Then the dimension of G is at most n.*

Proof. For each $x \in M$, the discussion above shows that the tri-Hamiltonian condition gives $\dim \mathcal{G}_x \leqslant n$. We thus need to show that there is some $x \in M$ such that the map $\mathfrak{g} \to \mathcal{G}_x$, $X \mapsto X_x$, is injective.

Fix a point $x \in M$ such that $\dim \operatorname{stab}_G(x)$ is the least possible. Note that $H = \operatorname{stab}_G(x)$ is a compact subgroup of $\mathrm{Sp}(n) \leqslant \mathrm{SO}(4n)$. We may therefore H-invariantly write $T_xM = T_x(G \cdot x) \oplus W$ as an orthogonal direct sum of the tangent space to the orbit through x and its orthogonal complement W. Now consider the map $F \colon G \times W \to M$ given by

$$F(g, w) = g \cdot (\exp_x w) = \exp_{gx} g_* w.$$

At $(e, 0) \in G \times W$ this has differential $(F_*)_{(e,0)}(X, w) = X_x + w$ and $F(gh, (h_*)^{-1}w) = F(g, w)$ for each $h \in H$. Thus F descends to a diffeomorphism from a neighbourhood of $(e, 0) \in G \times_H W$ to a neighbourhood U of $x \in M$ which is equivariant for the action of $\mathfrak{g}$. In particular $\operatorname{stab}_G F(e, w) \subset H$ when $F(e, w) \in U$.

As G acts effectively, we have for each $X \in \mathfrak{g} \setminus \{0\}$ there is some point y with $X_y \neq 0$. But M is Ricci-flat, so the Killing vector field X is analytic, thus the set $\{y \in M \mid X_y \neq 0\}$ is open and dense.

If $\dim H = \dim \operatorname{stab}_G(x)$ is non-zero, then there is a non-zero element $X \in \mathfrak{h}$. Now X is non-zero at some point $y = F(g, w)$ of U, and $z = g^{-1}y = F(e, w)$ has $X_z = (g_*)^{-1}X_y \neq 0$ too, since G is Abelian. So $\operatorname{Lie} \operatorname{stab}_G(z)$ is a subspace of $\mathfrak{h}$ not containing X. It follows that $\dim \operatorname{stab}_G(z) < \dim \operatorname{stab}_G(x)$, contradicting our choice of x.

We conclude that $\dim \mathfrak{h} = 0$, so $\operatorname{stab}_G(x)$ is finite and $\mathfrak{g} \mapsto \mathcal{G}_x$ is a bijection. Thus $\dim \mathfrak{g} \leqslant n$. □

Any connected Abelian group of finite dimension is of the form $G = \mathbb{R}^m \times T^k$ for some $m, k \geqslant 0$. If M is simply-connected then the tri-symplectic T^k-action is necessarily tri-Hamiltonian: each μ_A^Y obtains its maximum on each T^k-orbit, and so $L_X\mu_A^Y = X \lrcorner d\mu_A^Y$ is zero at these points, and hence on all of M. If $\dim G = n$ and the G-action is tri-Hamiltonian, Bielawski [5] proves that the $\mathbb{R}^m$ factor acts freely and any discrete subgroup of $\mathbb{R}^m$ acts properly discontinuously, so a discrete quotient of M has a tri-Hamiltonian T^n-action. In general, a hyperKähler manifold of dimension $4n$ with a tri-Hamiltonian T^n action is called *hypertoric*.

Bielawski [5] classified the hypertoric manifolds in any dimension under the assumption that M has finite topological type, meaning that the Betti numbers of M are finite. For $\dim M = 4$, this classification is extended to general hypertoric M in [24]. Here we wish to provide the full classification of hypertoric manifolds in arbitrary dimension, without any restriction on the topology. First let us recall some of the four-dimensional story.

2.2 Dimension four

Let M be a four-dimensional hyperKähler manifold with an effective tri-Hamiltonian S^1-action of period 2π. Let X be the corresponding vector field on M. Note that the only special orbits for the action are fixed points: if $g \in S^1$ stabilises the point x and $X_x \neq 0$, then g fixes $T_x M = \operatorname{Span}\{X_x, IX_x, JX_x, KX_x\}$ and hence a neighbourhood of x, so by analyticity $g = e$.

The hyperKähler moment map

$$\mu = (\mu_I, \mu_J, \mu_K)\colon M \to \mathbb{R}^3$$

is a local diffeomorphism away from the fixed point set M^X. Locally on $M' = M \setminus M^X$, the hyperKähler metric may be written as

$$g = \frac{1}{V}\beta_0^2 + V(\alpha_I^2 + \alpha_J^2 + \alpha_K^2),$$

where $\alpha_A = X \lrcorner \omega_A = d\mu_A$, $V = 1/g(X, X)$ and $\beta_0 = \alpha_0/\|X\| = g(X, \cdot)V^{1/2}$. The hyperKähler condition is now equivalent to the monopole equation $d\beta_0 = -{*_3}dV$, which implies that locally V is a harmonic function on $\mathbb{R}^3$.

Theorem 2 ([24]). *Let M be a complete connected hyperKähler manifold of dimension 4 with a tri-Hamiltonian circle action of period 2π. Then the hyperKähler moment map $\mu\colon M \to \mathbb{R}^3$ is surjective with connected fibres and induces a homeomorphism $\overline{\mu}\colon M/S^1 \to \mathbb{R}^3$. The metric on M is specified by any harmonic function $V\colon \mathbb{R}^3 \setminus Z \to (0, \infty)$ of the form*

$$V(p) = c + \frac{1}{2}\sum_{q \in Z} \frac{1}{\|p - q\|} \tag{3}$$

where $c \geqslant 0$ is constant and $Z \subset \mathbb{R}^3$ is finite or countably infinite. □

The set $Z = \mu(M^X)$ is the image of the fixed-point set M^X. The metrics with $c = 0$ and Z finite are the Gibbons-Hawking metrics [11]; $c > 0$ and Z finite gives the older multi-Taub-NUT metrics [16]. For $Z \neq \varnothing$, the hyperKähler manifold M is simply-connected, with $b_2(M) = |Z| - 1$; for $c > 0$, $Z = \varnothing$, we have $M = S^1 \times \mathbb{R}^3$ with μ projection to the second factor. For Z infinite and $c = 0$, the hyperKähler metrics are of type A_∞ as constructed by Anderson, Kronheimer and LeBrun [3] and

Goto [13], concentrating on the case $Z = \{(n^2, 0, 0) \mid n \in \mathbb{N}_{>0}\}$, and written down for general Z in Hattori [15]. These are the examples of infinite topological type.

To understand and extend Hattori's general formulation of these structures, we need to study when (3) gives a finite sum on an open subset of $\mathbb{R}^3$. By Harnack's Principle (see e.g. [4]) if (3) is finite at one point of $\mathbb{R}^3$, then it is finite on all of $\mathbb{R}^3 \setminus Z$. This may be seen in a elementary way via the following result.

Lemma 3. *Suppose $(q_n)_{n\in\mathbb{N}}$ is a sequence of points in $\mathbb{R}^3$ is given. Then the series $S_1 = \sum_{n\in\mathbb{N}} \|p - q_n\|^{-1}$ converges at some $p \in \mathbb{R}^3 \setminus \{q_n \mid n \in \mathbb{N}\}$ if and only if the series $S_2 = \sum_{n\in\mathbb{N}} (1 + \|q_n\|)^{-1}$ converges.*

Proof. First note that if there is a compact subset C of $\mathbb{R}^3$ containing infinitely many points of the sequence (q_n), then neither sum converges: there is some subsequence $(q_i)_{i\in I}$ that converges to a $q \in \mathbb{R}^3$, and so infinitely many terms are greater than some strictly positive lower bound.

Now putting $c = 1 + \|p\|$, we have $\|p - q\| \leqslant \|p\| + \|q\| \leqslant c(1 + \|q\|)$. It follows that convergence of S_1 implies converges of S_2.

For the converse, we consider $q \in \mathbb{R}^3 \setminus \overline{B}(0; R)$ for $R = 1 + 2\|p\|$ and have

$$\begin{aligned} \|p - q\| &\geqslant \|q\| - \|p\| = \tfrac{1}{2}\|q\| + (\tfrac{1}{2}\|q\| - \|p\|) \\ &> \tfrac{1}{2}(\|q\| + 1). \end{aligned} \tag{4}$$

If S_2 converges, then $\{n \in \mathbb{N} \mid q_n \in \overline{B}(0; R)\}$ is finite, so the inequality (4) implies convergence of S_1. □

Finally let us remark that scaling the hyperKähler metric g by a constant C scales V as a function on M by C^{-1}. However the hyperKähler moment map μ also scales by C, so the induced function $V(p) = V(\mu(x))$ on $\mathbb{R}^3$ has the same form, with a new constant term c/C and the points q replaced by q/C. On the other hand scaling the vector field X by a constant, so that the action it generates is no longer of period 2π, scales V on M and μ by different weights. In particular, such a change alters the factors $1/2$ in (3).

2.3 Construction of hypertoric manifolds

Bielawski and Dancer [6] provided a general construction of hypertoric manifolds in all dimensions with finite topological type. Goto [13] gave a particular construction of examples of in arbitrary dimension of infinite topological type. Let us now build on Hattori's four-dimensional description [15], to combine these two constructions.

Let $\mathbb{L}$ be a finite or countably infinite set. Choose $\Lambda = (\Lambda_k)_{k\in\mathbb{L}} \in \mathbb{H}^{\mathbb{L}}$ and define $\lambda = (\lambda_k)_{k\in\mathbb{L}}$ by $\lambda_k = -\frac{1}{2}\overline{\Lambda}_k i \Lambda_k \in \operatorname{Im}\mathbb{H}$. For each $k \in \mathbb{L}$, let $u_k \in \mathbb{R}^n$ be a non-zero vector and put $\hat{\lambda}_k = \lambda_k / \|u_k\|$. Suppose

$$\sum_{k\in\mathbb{L}} (1 + |\hat{\lambda}_k|)^{-1} < \infty. \tag{5}$$

Consider the Hilbert manifold $M_\Lambda = \Lambda + \mathbb{L}^2(\mathbb{H})$, where

$$\mathbb{L}^2(\mathbb{H}) = \Big\{ v \in \mathbb{H}^{\mathbb{L}} \;\Big|\; \sum_{k\in\mathbb{L}} |v_k|^2 < \infty \Big\}.$$

Let T_λ be the Hilbert group

$$T_\lambda = \Big\{ g \in T^{\mathbb{L}} = (S^1)^{\mathbb{L}} \;\Big|\; \sum_{k\in\mathbb{L}} (1+|\lambda_k|)\,|1-g_k|^2 < \infty \Big\}.$$

If $\|u_k\|$ is bounded away from 0, then $g \in T_\lambda$ implies g_k is arbitrarily close to 1 except for a finite number of $k \in \mathbb{L}$. As $|1-\exp(it)|^2 = 2 - 2\cos(t) \leqslant t^2$ for all $t \in \mathbb{R}$ and $|1-\exp(it)|^2 \geqslant 2t^2/\pi^2$ on $(-\pi/2, \pi/2)$, we see that the Lie algebra of T_λ is

$$\mathfrak{t}_\lambda = \Big\{ t \in \mathbb{R}^{\mathbb{L}} \;\Big|\; \|t\|^2_{\lambda,\mathfrak{t}} = \sum_{k\in\mathbb{L}} (1+|\lambda_k|)\,|t_k|^2 < \infty \Big\}.$$

Now consider the linear map $\beta\colon \mathfrak{t}_\lambda \to \mathbb{R}^n$ given by $\beta(e_k) = u_k$, where $e_i = (\delta_i^k)_{k\in\mathbb{L}}$, where $\delta_i^k \in \{0,1\}$ is Kronecker's delta. Supposing β is continuous then we define $\mathfrak{n}_\beta = \ker\beta \subset \mathfrak{t}_\lambda$. If $u_k \in \mathbb{Z}^n \subset \mathbb{R}^n$ for each $k \in \mathbb{L}$, we may define a Hilbert subgroup N_β of T_λ by

$$N_\beta = \ker(\exp\circ\beta\circ\exp^{-1}\colon T_\lambda \to T^n).$$

This gives exact sequences

$$0 \longrightarrow \mathfrak{n}_\beta \xrightarrow{\ \iota\ } \mathfrak{t}_\lambda \xrightarrow{\ \beta\ } \mathbb{R}^n \longrightarrow 0, \tag{6}$$

$$0 \longrightarrow N_\beta \longrightarrow T_\lambda \longrightarrow T^n \longrightarrow 0. \tag{7}$$

Our aim now is to construct hypertoric manifolds of dimension $4n$ as hyperKähler quotients of M_Λ by N_β.

Remark 4. The construction of Hattori [15] corresponds to $n = 1$ and $u_k = 1 \in \mathbb{R}$ for each k. For general dimension $4n$, Goto's construction [13] corresponds to $\mathbb{L} = (\mathbb{Z}\setminus\{0\}) \amalg \{1,\dots,n\}$,

$$\Lambda_k = \begin{cases} k\mathbf{i}, & \text{for } k \in \mathbb{Z}_{>0},\\ k\mathbf{k}, & \text{for } k \in \mathbb{Z}_{<0},\\ 0, & \text{for } k \in \{1,\dots,n\}, \end{cases} \qquad \text{with} \qquad u_k = \begin{cases} \mathbf{e}_1, & \text{for } k \in \mathbb{Z}\setminus\{0\},\\ \sum_{i=1}^n \mathbf{e}_i, & \text{for } k = 1 \in \{1,\dots,n\},\\ -\mathbf{e}_r, & \text{for } k = r \in \{2,\dots,n\}. \end{cases}$$

Thus Goto's construction is for one concrete choice of $(\lambda_k)_{k\in\mathbb{L}}$ and only one of the u_k's is repeatedly infinitely many times.

Returning to the general situation, note that the integrality of u_k implies $\|u_k\| \geqslant 1$, so the convergence condition (5) implies

$$\sum_{k\in\mathbb{L}}(1+|\lambda_k|)^{-1} < \infty. \tag{8}$$

The group T_λ acts on M_Λ via $gx = (g_k x_k)_{k\in\mathbb{L}}$: indeed for $g \in T_\lambda$ and $x = \Lambda + v \in M_\Lambda$, we have $gx = g\Lambda + gv = \Lambda - (1-g)\Lambda + gv$, but $gv \in \mathbb{L}^2(\mathbb{H})$ and $\|(1-g)\Lambda\|^2 = \sum_{k\in\mathbb{L}} \frac{1}{2}|\lambda_k||1-g_k|^2 \leqslant \frac{1}{2}\sum_{k\in\mathbb{L}}(1+|\lambda_k|)|1-g_k|^2$, which is finite by the definition of T_λ, so $(1-g)\Lambda \in \mathbb{L}^2(\mathbb{H})$ too. The action preserves the flat hyperKähler structure with $\mathbb{L}^2$-metric and complex structures obtained by regarding $\mathbb{L}^2(\mathbb{H})$ as a right $\mathbb{H}$-module. Identifying $\mathbb{R}^3$ with $\operatorname{Im}\mathbb{H} = \mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$, a corresponding hyperKähler moment map is given by $\langle \mu_\Lambda(x), t\rangle = \frac{1}{2}\sum_{k\in\mathbb{L}}(\overline{x}_k\mathbf{i}t_k x_k - \overline{\Lambda}_k\mathbf{i}t_k\Lambda_k)$. The terms $-\frac{1}{2}\overline{\Lambda}_k\mathbf{i}t_k\Lambda_k$ ensure that the sum in μ_Λ converges, but otherwise are arbitrary linear terms in t_k with values in $\operatorname{Im}\mathbb{H}$. With our definition of λ_k, we have

$$\mu_\Lambda(x) = \sum_{k\in\mathbb{L}}\bigl(\lambda_k + \tfrac{1}{2}\overline{x}_k\mathbf{i}x_k\bigr)e_k^*.$$

The hyperKähler moment map for the subgroup N_β is then $\mu_\beta = \iota^*\mu_\Lambda\colon M_\Lambda \to \operatorname{Im}\mathbb{H}\otimes\mathfrak{n}_\beta^*$. We define

$$M = M(\beta,\lambda) = M_\Lambda /\!/\!/ N_\beta = \mu_\beta^{-1}(0)/N_\beta.$$

Since (6) is exact, we have dually $\ker\iota^* = \operatorname{im}\beta^*$ and hence the following characterisation of $\mu_\beta^{-1}(0)$.

Lemma 5. *A point $x \in M_\Lambda$ lies in the zero set of the hyperKähler moment map μ_β for N_β if and only if there is an $a \in \operatorname{Im}\mathbb{H}\otimes(\mathbb{R}^n)^*$ with*

$$a(u_k) = \lambda_k + \tfrac{1}{2}\overline{x}_k\mathbf{i}x_k \tag{9}$$

for each $k \in \mathbb{L}$, where $u_k = \beta(e_k)$. □

In equation (9), note that $x_k = 0$ if and only if $a(u_k) = \lambda_k$. Indeed, the map $\mathbb{H} \to \operatorname{Im}\mathbb{H}$, $v \mapsto \overline{v}\mathbf{i}v$, is surjective with $|\overline{v}\mathbf{i}v| = |v|^2$. As in [6], we define affine subspaces $H_k \subset \operatorname{Im}\mathbb{H}\otimes(\mathbb{R}^n)^*$ of real codimension 3 by

$$H_k = H(u_k,\lambda_k) = \{a \in \operatorname{Im}\mathbb{H}\otimes(\mathbb{R}^n)^* \mid a(u_k) = \lambda_k\} \tag{10}$$

which we call *flats*. Note that $T^n = T_\lambda/N_\beta$ acts on M and that if M is smooth this action preserves induced the hyperKähler structure and has moment map $\phi\colon M \to \operatorname{Im}\mathbb{H}\otimes(\mathbb{R}^n)^*$ induced by μ_Λ: indeed $\operatorname{Lie}T^n = \mathfrak{t}_\lambda/\mathfrak{n}_\beta$ implies $(\operatorname{Lie}T^n)^* = (\mathfrak{n}_\beta)^0$, the annihilator of n_β in $\mathfrak{t}_\lambda^*$, so on $\mu_\beta^{-1}(0)$ the map μ_Λ takes values in $(\mathfrak{n}_\beta)^0 = (\operatorname{Lie}T^n)^*$ and descends to M as ϕ. It follows, as in [6], that ϕ induces a homeomorphism $M/T^n \to \operatorname{Im}\mathbb{H}\otimes(\mathbb{R}^n)^*$ and that for $p \in M$, the stabiliser $\operatorname{stab}_{T^n}(p)$ is the subtorus with Lie algebra spanned by the u_k such that $\phi(p) \in H_k$.

Theorem 6. *Suppose $u_k = \beta(e_k) \in \mathbb{Z}^n$, $k \in \mathbb{L}$, are primitive and span $\mathbb{R}^n$. Let $\lambda_k \in \operatorname{Im}\mathbb{H}$, $k \in \mathbb{L}$, be given such that the convergence condition* (5) *holds and the flats*

$H_k = H(u_k, \lambda_k)$, $k \in \mathbb{L}$, are distinct. Then the hyperKähler quotient $M = M(\beta, \lambda)$ is smooth if

(a) *any set of $n+1$ flats H_k has empty intersection, and*
(b) *whenever n distinct flats $H_{k(1)}, \dots, H_{k(n)}$ have non-empty intersection the corresponding vectors $u_{k(1)}, \dots, u_{k(n)}$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.*

We break the proof in to several steps.

Proposition 7. *Suppose $u_k \in \mathbb{Z}^n$, $k \in \mathbb{L}$, are primitive, span $\mathbb{R}^n$ and satisfy condition* (b) *of Theorem 6. Then $\mathcal{U} = \{u_k \mid k \in \mathbb{L}\}$ is finite.*

Proof. Note that if $u_{k(1)}, \dots, u_{k(n)}$ are linearly independent then $\bigcap_{j=1}^n H_{k(j)}$ is a single point. Thus (b) implies that $\mathcal{U}$ contains a $\mathbb{Z}$-basis $v_1, \dots, v_n$ for $\mathbb{Z}^n$. Then matrix A with columns $v_1, \dots, v_n$ is invertible with inverse in $M_n(\mathbb{Z})$, so A lies in $\mathrm{GL}(n, \mathbb{Z}) = \{B \in M_n(\mathbb{Z}) \mid \det B = \pm 1\}$. Multiplying with A^{-1} we may thus assume for the purpose of this proof that $\mathcal{U}$ contains the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n$.

Suppose $u = (u_1, \dots, u_n) \in \mathcal{U}$ is different from $\mathbf{e}_i$, for all $i = 1, \dots, n$. For $j \in \{1, \dots, n\}$, consider the matrix A_j with columns $u, \mathbf{e}_1, \dots, \widehat{\mathbf{e}_j}, \dots, \mathbf{e}_n$, so $\mathbf{e}_j$ is omitted. We have $\det A_j = \pm u_j$. If $\det A_j$ is non-zero, then its columns are linearly independent and the discussion above gives $\det A_j = \pm 1$. It follows that $u_j \in \{-1, 0, 1\}$ for each $j \in \{1, \dots, n\}$. In particular, there are only finitely many such u's. □

It follows that under condition (b), the set $\{\|u_k\| \mid k \in \mathbb{L}\}$ is bounded, so (8) and (5) are equivalent. Now the map $\beta\colon \mathfrak{t}_\lambda \to \mathbb{R}^n$ has Riesz representation $\beta(t) = \langle t, \gamma \rangle_{\lambda,\mathfrak{t}} = \sum_{k \in \mathbb{L}} (1 + |\lambda_k|) t_k \gamma_k$ given by $\gamma_k = u_k / (1 + |\lambda_k|)$. As $\|\gamma\|_{\lambda,\mathfrak{t}}^2 = \sum_{k \in \mathbb{L}} (1 + |\lambda_k|)^{-1} \|u_k\|^2$, boundedness of $\|u_k\|$ and (8) show that γ lies in $\mathfrak{t}_\lambda$. Thus β is continuous and it follows that N_β is a Hilbert subgroup of T_λ of codimension n.

Lemma 8. *If conditions* (a) *and* (b) *hold then the group N_β acts freely on $\mu_\beta^{-1}(0)$.*

Proof. Given a subset $\mathbb{K} \subset \mathbb{L}$, we define the subgroup $T_\mathbb{K}$ subgroup of T_λ by

$$T_\mathbb{K} = \{(g_k)_{k \in \mathbb{L}} \in T_\lambda \mid g_\ell = 1\ \forall k \notin \mathbb{K}\},$$

so that the Lie algebra of $T_\mathbb{K}$ is spanned by $\{e_k \mid k \in \mathbb{K}\}$. For $x \in M_\Lambda$, the stabiliser $\mathrm{stab}_{T_\lambda}(x)$ is $T_\mathbb{K}$ where $\mathbb{K} = \{k \mid x_k = 0\}$.

Now consider $x \in \mu_\beta^{-1}(0)$. Equation (9) implies that $x_k = 0$ if and only if $\phi(\pi(x)) \in H_k$, where $\pi\colon \mu_\beta^{-1}(0) \to M$ is the quotient map. Thus $\mathrm{stab}_{T_\lambda}(x) = T_{\mathbb{K}(x)}$, where $\mathbb{K}(x) = \{k \mid \phi(\pi(x)) \in H_k\}$. Condition (a) implies that $\mathbb{K}(x)$ contains at most n elements. The stabiliser of x under N_β consists of those elements in $\mathrm{stab}_{T_\lambda}(x)$ that lie in the kernel of the map $T_\lambda \to T^n$ induced by β. But $\beta(e_k) = u_k$ and implies that $(g_k)_{k \in \mathbb{L}} \in T_{\mathbb{K}(x)}$ maps to $h = \prod_{k \in \mathbb{K}(x)} g_k \exp(u_k) = \prod_{k \in \mathbb{K}(x)} \exp(i\theta_k u_k) \in T^n$, where $g_k = e^{i\theta_k}$.

Condition (b) implies that the u_k, $k \in \mathbb{K}(x)$, are part of a $\mathbb{Z}$-basis for $\mathbb{Z}^n$, so we may change basis via an element of $\mathrm{GL}(n, \mathbb{Z})$ so that the u_k become the first r

basis elements. Then h becomes $\mathrm{diag}(e^{i\theta(1)}, \dots, e^{i\theta(r)}, 1, \dots, 1)$, where $\theta(j)$ is a corresponding relabelling of the θ_k's. It follows that $\theta_k \in 2\pi\mathbb{Z}$ and so $g_k = 1$ for each $k \in \mathbb{K}(x)$. Thus $\mathrm{stab}_{N_\beta}(x)$ is trivial, as claimed. □

As in [13], for $x \in M_\Lambda$, let $X_x : \mathfrak{t}_\lambda \to T_x M_\Lambda$ be the map sending an element to the corresponding tangent vector at x generated by the action:

$$X_x(t) = \frac{d}{ds}(\exp(st)x)\bigg|_{s=0} = (\mathbf{i}t_k x_k)_{k\in\mathbb{L}}.$$

Lemma 9. *For $x \in \mu_\beta^{-1}(0)$, the map X_x induces a linear homeomorphism from $\mathfrak{n}_\beta$ to the tangent space $T_x(N_\beta \cdot x)$ of the N_β-orbit through x.*

Proof. For general $x \in M_\Lambda$, we have $\|X_x(t)\|^2 = \|(t_k x_k)_{k\in\mathbb{L}}\|^2 = \sum_{k\in\mathbb{L}} |t_k|^2 |\Lambda_k + v_k|^2$. Except for finitely many $k \in \mathbb{L}$, we have $|\Lambda_k| \geqslant 2|v_k|$, so for these k, we have $|\Lambda_k + v_k|^2 \leqslant |3\Lambda_k/2|^2 = 9|\lambda_k|/8$. It follows that there is a constant C_x, independent of t, such that $\|X_x(t)\| \leqslant C_x \|t\|_{\lambda,\mathfrak{t}}$. Thus $X_x : \mathfrak{t}_\lambda \to T_x M_\Lambda$ is continuous.

Now let $\mathbb{K}_1 = \mathbb{L} \setminus \{k \in \mathbb{L} \mid |\Lambda_k| \geqslant 1 \geqslant 2|v_k|\}$, which is a finite set by (8) and the condition that $v \in \mathbb{L}^2(\mathbb{H})$. For $k \notin \mathbb{K}_1$, we have $|\lambda_k| \geqslant 1/2$, so $|x_k|^2 = |\Lambda_k + v_k|^2 \geqslant |\Lambda_k/2|^2 = |\lambda_k|/8 \geqslant (1 + |\lambda_k|)/32$. It follows that for $k \notin \mathbb{K}(x) = \{k \mid x_k = 0\}$, there is a constant $c_x > 0$ such that $|t_k x_k|^2 \geqslant c_x(1 + |\lambda_k|)|t_k|^2$.

For $x \in \mu_\beta^{-1}(0)$, the set $\mathbb{K}(x)$ coincides with the previous definition $\mathbb{K}(x) = \{k \mid \phi(\pi(x)) \in H_k\}$ and so contains at most n elements. Let $V_x = \mathrm{Span}\{e_k \mid k \in \mathbb{K}(x)\} \leqslant \mathfrak{t}_\lambda$ and write $\mathrm{pr}^\perp : \mathfrak{t}_\lambda \to V_x^\perp$ for the orthogonal projection away from V_x. Then β is injective on V_x, so $\mathrm{pr}^\perp$ is a continuous linear bijection $\mathrm{pr}_\beta : \mathfrak{n}_\beta \to \mathrm{pr}^\perp(\mathfrak{n}_\beta)$. The image is the orthogonal complement to $V_x \oplus \beta^\dagger(\beta(V_x)^\perp)$, where $\beta(V_x)^\perp$ is the orthogonal complement in $\mathbb{R}^n$. As β is surjective, its adjoint $\beta^\dagger$ is injective, so $\mathrm{pr}^\perp(\mathfrak{n}_\beta)$ is of finite codimension and thus a Hilbert subspace of $\mathfrak{t}_\lambda$. By the Open Mapping Theorem, we conclude that pr_β^{-1} is continuous, and we note that its norm is non-zero.

Now for $x \in \mu_\beta^{-1}(0)$ and $t \in \mathfrak{n}_\beta$, we have

$$\begin{aligned}\|X_x(t)\|^2 = \|(t_k x_k)_{k\in\mathbb{L}}\|^2 &\geqslant c_x \sum_{k\notin\mathbb{K}(x)} (1 + |\lambda_k|)|t_k|^2 = c_x \|\mathrm{pr}_\beta(t)\|^2_{\lambda,\mathfrak{t}} \\ &\geqslant \frac{c_x}{\|\mathrm{pr}_\beta^{-1}\|^2} \|t\|^2_{\lambda,\mathfrak{t}},\end{aligned}$$

showing that X_x has continuous inverse on $T_x(N_\beta \cdot x)$. □

It now follows, as in [13], that for $x \in \mu_\beta^{-1}(0)$, the differential $d\mu_\beta : T_x M_\Lambda \to \mathrm{Im}\,\mathbb{H} \otimes \mathfrak{n}_\beta^*$ is split, with right inverse the $\mathbb{R}$-linear map given by $\mathbf{a} \otimes \delta = AX_x(t)$, where $\delta = \langle t, \cdot \rangle_{\lambda,\mathfrak{t}}$ and $A = a_1 I + a_2 J + a_3 K$ for $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. This implies that $\mu_\beta^{-1}(0)$ is a smooth Hilbert submanifold of M_Λ. On $\mu_\beta^{-1}(0)$, Goto's construction [13] of slices S_x goes through unchanged: one considers the map $F_x : \mu_\beta^{-1}(0) \to \mathfrak{n}_\beta^*$ given by $F_x(x + w)(t) = \langle w, X_x(t)\rangle$ and puts $S_x = F_x^{-1}(0) \cap U$, for a sufficiently small neighbourhood U of x. Thus $M(\beta, \lambda) = \mu_\beta^{-1}(0)/N_\beta$ is a smooth manifold.

Fix a point $q \in \operatorname{Im}\mathbb{H} \otimes (\mathbb{R}^n)^* \setminus \bigcup_{k\in\mathbb{L}} H_k$. For $x \in \mu_\beta^{-1}(0)$ with $\phi\pi(x) = q$, we have that $x_k \neq 0$ for all $k \in \mathbb{L}$. Thus T_λ acts freely on x_k. As $e_k \in \mathfrak{t}_\lambda$ for each $k \in \mathbb{L}$, it follows that $e_k \in F = T_x(T_\lambda \cdot x)$, and that $T_X M_\Lambda = F \oplus IF \oplus JF \oplus KF$. As β is surjective, we conclude that $F/T_x(N_\beta \cdot x)$ is of dimension n and that $M(\beta, \lambda) = \mu_\beta^{-1}(0)/N_\beta$ is of dimension $4n$.

We observe that the quotient is Hausdorff as follows. Suppose $g^{(i)}x \in \mu_\beta^{-1}(0)$, $g^{(i)} \in N_\beta$, is a sequence of points converging to $y \in \mu_\beta^{-1}(0)$. Lemma 9 gives a $c_x > 0$ such that $|x_k|^2 \geqslant c_x(1 + |\lambda_k|)$ for each $x_k \neq \mathbb{K}(x) = \{k \in \mathbb{L} \mid x_k = 0\}$. Thus $\sum_{k\in\mathbb{L}}(1+|\lambda_k|)|g_k^{(i)} - g_k^{(j)}|^2 \leqslant c_x^{-1}\|g^{(i)}x - g^{(j)}x\|^2 + \sum_{k\in\mathbb{K}(x)}(1+|\lambda_k|)|g_k^{(i)} - g_k^{(j)}|^2$, for all i, j. Taking a subsequence of the $g^{(i)}$ so that g_k^i converges in S^1 for all $k \in \mathbb{K}(x)$, it follows that this subsequence is Cauchy in N_β and has a limit $g \in N_\beta$ with $gx = y$, as required.

The standard considerations of the hyperKähler quotient construction show that $M(\beta, \lambda)$ inherits a smooth hyperKähler structure, completing the proof of Theorem 6.

Just as in Hattori [15], one may use the T_λ action to show that different choices of $(\Lambda_k)_{k\in\mathbb{L}}$ yielding the same $(\lambda_k)_{k\in\mathbb{L}}$ result in hyperKähler structures that are isometric via a tri-holomorphic map.

2.4 *Classification of complete hypertoric manifolds*

Now suppose that M is an arbitrary complete connected hypertoric manifold of dimension $4n$ and write $G = T^n$. Bielawski [5, §4] shows that locally M has much of the structure of the hypertoric manifolds constructed above.

Indeed for each $p \in M$, we may find a G-invariant neighbourhood of the form $U = G \times_H W$, where $H = \operatorname{stab}_G(p)$ and $W = T_p(G\cdot p)^\perp \subset T_x M$. Now H acts trivially on $W_1 = (\operatorname{Im}\mathbb{H})T_p(G \cdot p)$, and effectively as an Abelian subgroup of $\operatorname{Sp}(r)$ on the orthogonal complement $W_2 = W \cap W_1^\perp \cong \mathbb{H}^r$. Counting dimensions, it follows that H acts as T^r on W_2, and hence H is connected. The image of the singular orbits in U is a union of distinct flats $H_k = H(u_k, \lambda_k)$ as in (10), where u_k may be chosen to lie in $\mathbb{Z}^n \subset \mathbb{R}^n = \mathfrak{g}$ and be primitive vectors. The collection $\{u_k \mid \mu(p) \in H_k\}$ is then part of a $\mathbb{Z}$-basis for $\mathbb{R}^n$ spanning the Lie algebra of H. Furthermore, examining the structure of μ on such a neighbourhood U, Bielawski shows that μ induces a local homeomorphism $M/G \to \operatorname{Im}\mathbb{H} \otimes \mathfrak{g}^* \cong \mathbb{R}^{3n}$. In particular, the hyperKähler moment map $\mu\colon M \to \operatorname{Im}\mathbb{H} \otimes \mathfrak{g}^*$ is an open map.

Let $\{H_k \mid k \in \mathbb{L}\}$ be the collection of all flats that arise in this way. The index set $\mathbb{L}$ is finite or countably infinite, since M is second countable.

Lemma 10. *Suppose $\alpha \in (\mathbb{Z}^n)^* \subset (\mathbb{R}^n)^* = \mathfrak{g}^*$ is non-zero. Let T_α be the subtorus of $G = T^n$ whose Lie algebra is spanned by* $\ker\alpha = \{u \in \mathfrak{g} \mid \alpha(u) = 0\}$. *For $a \in \operatorname{Im}\mathbb{H} \otimes \mathfrak{g}^*$, write $[a]_\alpha = a + \operatorname{Im}\mathbb{H} \otimes \mathbb{R}\alpha$ for the equivalence class of a in* $\operatorname{Im}\mathbb{H} \otimes (\mathfrak{g}^*/\mathbb{R}\alpha)$. *Then except for countably many choices of $[a]_\alpha$, the group T_α acts freely on $\mu^{-1}([a]_\alpha)$.*

Note that $(\ker\alpha)^* = \mathfrak{g}^*/\mathbb{R}\alpha$.

Proof. Consider the intersection $[a]_\alpha \cap H_k$. A general point of $[a]_\alpha$ is $a + q \otimes \alpha$, $q \in \operatorname{Im}\mathbb{H}$, which lies in $H_k = H(u_k, \lambda_k)$ only if $a(u_k) + q\alpha(u_k) = \lambda_k$. If $\alpha(u_k) \neq 0$ this equation has a unique solution for q; if $\alpha(u_k) = 0$ then there is a solution only if $a(u_k) = \lambda_k$ and then $[a]_\alpha \subset H_k$. Thus choosing $a(u_k) \neq \lambda_k$ for each $k \in \mathbb{L}$, ensures that $[a]_\alpha \cap H_k$ is empty for every k for which $u_k \in \ker\alpha$. It follows that T_α acts almost freely on $\mu^{-1}([a]_\alpha)$, but as each stabiliser of the T^n-action is connected, we find that T_α acts freely. □

Corollary 11. *For α, T_α and a as in Lemma 10, the hyperKähler quotient $M(a,\alpha) = \mu^{-1}([a]_\alpha)/T_\alpha$ is a complete hyperKähler manifold of dimension four. Furthermore, $M(a,\alpha)$ carries an effective tri-Hamiltonian circle action.*

Proof. As T_α is compact and acts freely on $\mu^{-1}([a]_\alpha)$, it follows that $M(a,\alpha)$ is hyperKähler [17]. Completeness of M implies completeness of the level set $\mu^{-1}([a]_\alpha)$ and hence of the hyperKähler quotient.

As μ is T^n-invariant, the level set $\mu^{-1}([a]_\alpha)$ is preserved by T^n and we get an action of the circle T^n/T_α on the quotient. Identifying $[a]_\alpha$ with $\operatorname{Im}\mathbb{H}$ via $a + q \otimes \alpha \mapsto q$, the restriction of μ to $\mu^{-1}([a]_\alpha)$ descends to a hyperKähler moment map for this action. □

From the four-dimensional classification Theorem 2, we find have that the moment map of $M(a,\alpha)$ surjects on to $\operatorname{Im}\mathbb{H}$. Interpreting this in terms of the moment map μ of M, we have that $[a]_\alpha$ lies in the image of μ. But on M the moment map μ is an open map. And, as $\bigcup\{[a]_\alpha \mid a(u_k) \neq \lambda_k \forall k : u_k \in \ker\alpha\}$ is dense in $\operatorname{Im}\mathbb{H} \otimes \mathbb{R}^n$, we conclude that μ is a surjection.

Furthermore, the metric on $M(a,\alpha)$ is given by a potential of the form (3) up to an overall positive scale. The elements of Z are just the intersection points of $[a]_\alpha \cap H_k$, for $u_k \notin \ker\alpha$. These are the points $a_k = a + q_k \otimes \alpha$ with $q_k = (\lambda_k - a(u_k))/\alpha(u_k)$. Now for $p \in \mu^{-1}([a]_\alpha)$, writing $\mu(p) = a + q \otimes \alpha$, we have

$$\begin{aligned}\mu(p) - a_k = (q - q_k) \otimes \alpha &= \frac{q\alpha(u_k) - (\lambda_k - a(u_k))}{\alpha(u_k)} \otimes \alpha \\ &= (\langle \mu(p), u_k \rangle - \lambda_k) \otimes \frac{\alpha}{\alpha(u_k)}.\end{aligned}$$

Thus the potential for $M(a,\alpha)$ is proportional to

$$V_\alpha(p) = c + \frac{1}{2} \sum_{k \in \mathbb{L}} \frac{1}{\|\langle \mu(p), u_k \rangle - \lambda_k\|} \frac{|\alpha(u_k)|}{\|\alpha\|},$$

where we may include the terms with $u_k \in \ker\alpha$, since they contribute zero, and $\|\alpha\|$ is the norm of α with respect to the standard inner product from the identification $\mathfrak{g} = \mathbb{R}^n$.

Using this inner product we may identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Then the function

$$r_k(b) = \|b(u_k) - \lambda_k\| / \|u_k\| = \|b(\hat{u}_k) - \hat{\lambda}_k\|,$$

where $\hat{u}_k = u_k/\|u_k\|$ and $\hat{\lambda}_k = \lambda_k/\|u_k\|$, corresponds to the distance of b from the flat H_k. We may thus write

$$V_\alpha(p) = c + \frac{1}{2}\sum_{k\in\mathbb{L}} \frac{|\hat{\alpha}(\hat{u}_k)|}{r_k(\mu(p))}, \tag{11}$$

for $\hat{\alpha} = \alpha/\|\alpha\|$.

Now choose a so that $a(u_k) \neq \lambda_k$ for all $k \in \mathbb{L}$. Then for $p \in \mu^{-1}(a)$ we have that V_α in (11) is finite for each non-zero integral $\alpha \in \mathfrak{g}^*$. As each unit vector $\hat{u} \in \mathfrak{g} \cong \mathbb{R}^n$ has $\langle \hat{u}, \mathbf{e}_i\rangle \geqslant 1/\sqrt{n}$ for some $i \in \{1,\dots,n\}$, we conclude that $\sum_{k\in\mathbb{L}} 1/r_k(\mu(p))$ converges. In particular the distance of $\mu(p)$ to H_k is bounded below by a uniform constant. It follows that there is an open neighbourhood U of $\mu(p)$ in $\mu(M) \subset \operatorname{Im}\mathbb{H}\otimes\mathfrak{g}^*$ for which $a(u_k) \neq \lambda_k$ for all $a \in U$.

Let M_U be a connected component of $\mu^{-1}(U)$. Then T^n acts freely on M_U and the hyperKähler structure on M_U is uniquely determined via a polyharmonic function F on $U \subset \operatorname{Im}\mathbb{H}\otimes\mathfrak{g}^*$ as follows. The hyperKähler metric is of the form

$$g = \sum_{i,j=1}^{n} (V^{-1})_{ij}\beta_0^i\beta_0^j + V_{ij}(\alpha_I^i\alpha_I^j + \alpha_J^i\alpha_J^k + \alpha_K^i\alpha_K^j),$$

where for $X_1,\dots,X_n$ is a basis for $\mathbb{R}^n = \mathfrak{g}$, we have $\alpha_A^i = X_i \lrcorner \omega_A$, $(V^{-1})_{ij} = (g(X_i,X_j))$ and $\beta_0^1,\dots,\beta_0^n$ are the $C^\infty(M_U)$-linear combinations of $g(X_1,\cdot),\dots, g(X_n,\cdot)$ such that $\beta_0^i(X_j) = \delta_j^i$. As a result of Pedersen and Poon [22] and the Legendre transform of Lindström and Roček [20, 17] the functions V_{ij} on $U \subset \operatorname{Im}\mathbb{H}\otimes\mathfrak{g}^*$ are *polyharmonic* meaning that they are harmonic on each affine subspace $a + \operatorname{Im}\mathbb{H}\otimes\mathbb{R}\alpha$, $\alpha \in \mathfrak{g}^*\setminus\{0\}$. Furthermore this matrix of functions is given by a single polyharmonic function $F\colon U \to \mathbb{R}$ via $V_{ij} = F_{x_ix_j}$, where we choose a unit vector $\mathbf{e} \in \operatorname{Im}\mathbb{H}$ and $(x_1,\dots,x_n)$ are standard coordinates on $\mathbb{R}^n = \mathbb{R}\mathbf{e}\otimes\mathfrak{g}^* \subset \operatorname{Im}\mathbb{H}\otimes\mathfrak{g}^*$. We write

$$s_k(b) = \langle \mathbf{e}, b(\hat{u}_k) - \hat{\lambda}_k\rangle.$$

As $\mathbf{e}$ acts on $\mathbf{e}^\perp \subset \operatorname{Im}\mathbb{H}$ as a complex structure, we may choose corresponding standard complex coordinates $(z_1,\dots,z_n)$ on $\mathbf{e}^\perp\otimes\mathfrak{g}^*$. A potential V of the form (3) is then $V = F_{xx}$ with

$$F(x,z) = \frac{1}{4}c(2x^2 - |z|^2) + \frac{1}{2}\sum_{k\in\mathbb{L}}(s_k\log(s_k + r_k) - r_k).$$

As in Bielawski [5], we now deduce that for M_U the function F has the form

$$F = \sum_{k\in\mathbb{L}} a_k(s_k\log(s_k + r_k) - r_k) + \sum_{i,j=1}^{n} c_{ij}(4x_ix_j - z_i\bar{z}_j - z_j\bar{z}_i)$$

for some real constants a_k, c_{ij}. As Bielawski explains the c_{ij} terms are from a *Taub-NUT deformation* of a metric determined by the first sum, that is there is a hypertoric manifold M_2 and $M_U = M_2 \times (S^1 \times \mathbb{R}^3)^m /\!/\!/ T^m$ with T^m acting effectively on the product of S^1-factors, trivially on the $\mathbb{R}^3$-factors and as a subgroup of T^n on M_2. By analyticity, the hyperKähler metric on M_U determines the hyperKähler metric on M.

Using Bielawski's techniques and the computations of [6], one may now conclude that M_2 comes from the construction of the previous section. In particular, we note that the a_k's are bounded and convergence of $V_\alpha(p)$ in (11) for each non-zero integral α corresponds to the condition (5). To see this first we remark that convergence of the $V_\alpha(p)$'s corresponds to convergence of $R(b) = \sum_{k\in\mathbb{L}} 1/r_k(b)$ for $b = \mu(p)$: this follows from $|\hat{\alpha}(\hat{u}_k)| \leqslant 1$ and $\langle \hat{u}_k, \mathbf{e}_i \rangle \geqslant 1/\sqrt{n}$ for some i. Now $r_k(b) \leqslant |b(u_k)| + |\hat{\lambda}_k| \leqslant (1+\|b\|)(1+|\hat{\lambda}_k|)$ gives that convergence of $R(b)$ implies (5). Conversely, note that (5) implies that $|\hat{\lambda}_k| < 1 + 2\|b\|$ for only finitely many $k \in \mathbb{L}$. But for $|\hat{\lambda}_k| \geqslant 1+2\|b\|$, we have $r_k(b) \geqslant (|\hat{\lambda}_k|+1)/2$, as in (4), so we get convergence of $R(b)$.

We have thus proved the following result.

Theorem 12. *Let M be a connected hypertoric manifold of dimension $4n$. Then M is a product $M = M_2 \times (S^1 \times \mathbb{R}^3)^m$ with M_2 a hypertoric manifold of the type constructed in §2.3, i.e., the hyperKähler quotient of flat Hilbert hyperKähler manifold by an Abelian Hilbert Lie group. The hyperKähler metric in M is either the product hyperKähler metric or a Taub-NUT deformation of this metric.* □

From the proof and the construction of the previous section, we have the following properties of the hyperKähler moment map of in this situation.

Corollary 13. *If M is a connected complete hyperKähler manifold of dimension $4n$ with an effective tri-Hamiltonian action of T^n, then the hyperKähler moment map $\mu\colon M \to \operatorname{Im}\mathbb{H} \otimes (\mathbb{R}^n)^* \cong \mathbb{R}^{3n}$ is surjective with connected fibres.* □

3 Cuts and modifications

Symplectic cutting was introduced by Lerman in 1995 [19]. The construction starts with a symplectic manifold with Hamiltonian circle action, and produces a new manifold of the same type, but with different topology. Explicitly, given M with circle action and associated moment map μ, we form the symplectic quotient at level ϵ,

$$M^\epsilon_{\text{cut}} = (M \times \mathbb{C}) /\!/_\epsilon S^1$$

where the S^1 is the antidiagonal of the product action obtained from the given action on M and the standard rotation on $\mathbb{C}$. Note that the moment map for the action on $\mathbb{C}$ is $\phi\colon z \mapsto |z|^2$.

The new space M_{cut} is of the same dimension as M and inherits a circle action from the diagonal action on $M \times \mathbb{C}$. Moreover, as

$$M_{\text{cut}} = \{(m, z) \mid \mu(m) - |z|^2 = \epsilon\}/S^1$$

we see that the points $\{m \mid \mu(m) < \epsilon\}$ are removed. Moreover, because $\phi\colon z \mapsto |z|^2$ is a trivial circle fibration over $(0, \infty)$ with the circle fibre collapsing to a point at the origin, we see that the region $\{m \mid \mu(m) > \epsilon\}$ remains unchanged, while the hypersurface $\mu^{-1}(\epsilon)$ is collapsed by a circle action.

We can generalise this to the case of torus actions, by replacing $\mathbb{C}$ by a toric variety associated to a polytope Δ. The region $\mu^{-1}(\mathbb{R}^n \setminus (\Delta + \epsilon))$ will be removed, the preimage of the ϵ-translate of the interior of Δ is unchanged, while collapsing by tori takes place on the preimage of lower-dimensional faces of the translated polytope.

For general geometries, we want to mimic this construction by looking at the appropriate quotient of $M \times N$ by an Abelian group G, where N is a space whose reduction by G is a point. The topological change in M will be controlled by the geometry of the moment map for the G action on N.

The simplest example is that of a hyperKähler manifold with circle action. We now explain the hyperKähler analogue of a cut in this situation, which we call a *modification* [10]. The natural choice of N is now the quaternions $\mathbb{H}$. Several new features now emerge, because the hyperKähler moment map $\mu\colon \mathbb{H} \to \mathbb{R}^3$ has very different properties from that for $\mathbb{C}$. In particular, it is *surjective*, and is a *non-trivial* circle fibration over $\mathbb{R}^3 \setminus \{0\}$; in fact ϕ is given by the Hopf map on each sphere. This means that in forming the hyperKähler quotient $M_{\mathrm{mod}} = (M \times \mathbb{H})/\!/\!/S^1$ we do *not* discard any points in M (hence the use of the terminology modification rather than cut!). We still collapse the locus $\mu^{-1}(\epsilon)$ by a circle action, because ϕ is injective over the origin. The complements $M \setminus \mu^{-1}(\epsilon)$ and $M_{\mathrm{mod}} \setminus (\mu^{-1}(\epsilon)/S^1)$ can no longer be identified, because of the non-triviality of the Hopf fibration. Instead, the topology has been given a 'twist'. An example of this is if we start with $M = \mathbb{H}$. Now iterating the above construction generates the Gibbons-Hawking A_k multi-instanton spaces, where the spheres at large distance are replaced by lens spaces $S^3/\mathbb{Z}_{k+1}$.

We can make this more precise by observing that the space

$$M_1 = \{(m, q) \in M \times \mathbb{H} \mid \mu(m) - \phi(q) = \epsilon\}$$

projects onto both M and M_{mod}. The first map is just projection onto the first factor (onto as ϕ is surjective), while the second is just the quotient map $M_1 \to M_{\mathrm{mod}}$. Note that the first map is not quite a fibration: it has S^1 fibres generically but over $\mu^{-1}(\epsilon)$ the fibres collapse to a point.

On the open sets where both maps are fibrations, note that $M_1 \to M_{\mathrm{mod}}$ is a Riemannian submersion, but that the projection to first factor is not. As shown in [24], the metric $\tilde{g}$ induced on M by M_1 has the form

$$\tilde{g} = g + V(\mu) g_{\mathbb{H}}, \tag{12}$$

where $g_{\mathbb{H}} = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2$, with $\alpha_0 = g(X, \cdot\,)$, etc., and $V(\mu) = 1/2\|\mu - \epsilon\|$ is the potential of the flat hyperKähler metric on $\mathbb{H}$.

One may now generalise the hyperKähler modification, by replacing $\mathbb{H}$ by any hypertoric manifold N of dimension 4. The modification changes the metric as in (12) with $V(\mu)$ now the potential function of N, so one of the functions (3). In [24] it is proved that metric changes of the form (12) with V now an arbitrary smooth invariant function on M, so called ‘elementary deformations’, only lead to new hyperKähler metrics when V is $\pm V(\mu)$ for some hypertoric N^4. The case of negative V corresponds precisely to inverting a modification via a positive V.

For general torus actions one can take N to be a hypertoric manifold and we get a similar picture to that above.

4 Non-Abelian moment maps

One can also consider cutting constructions for non-Abelian group actions. Now, because the diagonal and anti-diagonal actions no longer commute, one considers the product of a K-manifold M with a space N with $K \times K$ action and then reduces by the antidiagonal action formed from the action on M and (say) the left action on N.

In the symplectic case, following Weitsman [25], with $K = \mathrm{U}(n)$ one can take $N = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ with $K \times K$ action $A \mapsto UAV^{-1}$. The moment map for the right action is $\mu\colon A \mapsto iA^*A$, with image Δ the set of non-negative Hermitian matrices. We have a picture that is quite reminiscent of the Abelian case, essentially because the fibres of ϕ are orbits of the *left* action. The right moment map ϕ gives a trivial fibration with fibres $\mathrm{U}(n)$ over the interior of the image, while over the lower-dimensional faces of Δ (corresponding to non-negative Hermitian matrices that are not strictly positive), the fibres are $\mathrm{U}(n)/\mathrm{U}(n-k)$ where k is the number of positive eigenvalues. This gives a nice non-Abelian generalisation of the toric cuts described above. To form the cut space we remove the complement of $\mu^{-1}(\Delta + \epsilon)$ and perform collapsing by the appropriate unitary groups on the preimages of the lower-dimensional faces of $\Delta + \epsilon$.

In the hyperKähler case life becomes more complicated. An obvious choice of N is $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)^*$, with action

$$(U, V)\colon (A, B) \mapsto (UAV^{-1}, VBU^{-1}).$$

The hyperKähler moment map for the right $\mathrm{U}(n)$-action is now

$$\mu\colon (A, B) \mapsto (\frac{\mathbf{i}}{2}(A^*A - BB^*), BA) \in \operatorname{Im}\mathbb{H} \otimes \mathfrak{u}(n)^* \cong \mathfrak{u}(n) \oplus \mathfrak{gl}(n, \mathbb{C}).$$

In contrast to the symplectic case, or the Abelian hyperKähler case, the fibres of μ are no longer group orbits in general; in particular the left $U(n)_L$ action need not be transitive and indeed the quotient of a fibre by this action may have positive dimension. The result is that when we perform the non-Abelian hyperKähler modification, blowing up of certain loci occurs.

This is closely related to the phenomenon that the fibres of hyperKähler moment maps over *non-central* elements may have larger than expected dimension, even on the locus where the group action is free. This is because the kernel of the differential of a moment map μ is the orthogonal of $I\mathcal{G} + J\mathcal{G} + K\mathcal{G}$, where $\mathcal{G}$, as in §2.1, denotes the tangent space to the orbits of the group action. For the fibre over a central element this sum is direct (because $\mathcal{G}$ is tangent to the fibre so is orthogonal to the sum, and hence the three summands are mutually orthogonal), but over non-central elements this is no longer necessarily true. The dimension of the fibre is now no longer determined by the dimension of $\mathcal{G}$, and hence not determined by the dimension of the stabiliser. Note that in the Kähler situation the kernel of $d\mu$ is just the orthogonal of $I\mathcal{G}$, so here the dimension of the fibre is completely controlled by the dimension of the stabiliser, even in the non-Abelian case.

Another example of the unexpected behaviour of hyperKähler moment maps over non-central elements is the phenomenon of disconnected fibres. This is in contrast to symplectic moment maps, whose level sets enjoy many connectivity properties, see [23] and the references therein. As a simple hyperKähler example we may consider the $SU(2)$ action on $\mathbb{H}^2 = \mathbb{C}^2 \times (\mathbb{C}^2)^* = T^*\mathbb{C}^2$

$$A\colon (z, w) \mapsto (Az, wA^{-1}) \tag{13}$$

with hyperKähler moment map

$$\mu = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}})\colon (z, w) \mapsto \Big(\frac{\mathbf{i}}{2}(zz^\dagger - w^\dagger w)_0, (zw)_0\Big), \tag{14}$$

where $\cdot_0$ denotes trace-free part, and $\cdot^\dagger$ the conjugate transpose. (This calculation arose in discussions with S. Tolman).

We are interested in finding the fibre of μ over $(\alpha, \beta) \in \mathfrak{su}(2) \oplus \mathfrak{sl}(2, \mathbb{C}) \cong \operatorname{Im}\mathbb{H} \otimes \mathfrak{su}(2)^*$. Using the $SU(2)$-equivariance, we may take

$$\beta = \begin{pmatrix} \lambda & \mu \\ 0 & -\lambda \end{pmatrix}.$$

We find that the fibre is:

(1) empty or a disjoint pair of circles if λ, μ are both non-zero;
(2) empty, a circle or a disjoint pair of circles if λ is zero and μ non-zero;
(3) empty or a disjoint pair of circles if μ is zero and λ non-zero;
(4) a disjoint pair of circles or a point if $\lambda = \mu = 0$ (the point fibre occurs exactly over the origin $\alpha = \beta = 0$.

Now let us turn to the the case of SU(3). This acts on $T^*\mathbb{C}^3 = \mathbb{C}^3 \oplus (\mathbb{C}^3)^*$ via (13) and has the same formula (14) for the hyperKähler moment map $\mu\colon T^*\mathbb{C}^3 \to \operatorname{Im}\mathbb{H} \otimes \mathfrak{su}(3)^* \cong \mathfrak{su}(3) \oplus \mathfrak{sl}(3, \mathbb{C})$. In particular $\mu_{\mathbb{C}}(z, w)$ has off-diagonal (i, j)-entries $z_i w_j$ whilst the diagonal entries are of the form $2z_i w_i - \sum_{k\neq i} z_k w_k$. Similarly, for $j > i$, $(\mu_{\mathbb{R}}(z, w))_{ij} = \frac{\mathbf{i}}{2}(z_i \overline{z_j} - \overline{w_i} w_j)$ and

$$(\mu_{\mathbb{R}}(z, w))_{ii} = \frac{\mathbf{i}}{6}\Big\{2(|z_i|^2 - |w_i|^2) - \sum_{k\neq i}(|z_k|^2 - |w_k|^2)\Big\}.$$

We consider the fibre $\mu^{-1}(\alpha,\beta)$, for $\alpha \in \mathfrak{su}(3)$, $\beta \in \mathfrak{sl}(3,\mathbb{C})$. Note that $\dim_{\mathbb{C}} \mathfrak{sl}(3,\mathbb{C}) = 8$ is strictly greater than $\dim_{\mathbb{C}} T^*\mathbb{C}^3 = 6$ so there are now restrictions on β to lie in the image of $\mu_{\mathbb{C}}$. In particular, we see that μ is not surjective.

For more detail, note that the map μ is SU(3)-equivariant and we may use the action to put β in to the canonical upper triangular form

$$\beta = \begin{pmatrix} \lambda_1 & \xi_1 & \zeta \\ 0 & \lambda_2 & \xi_2 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 0$. This gives the three constraints

$$z_2 w_1 = 0, \quad z_3 w_1 = 0, \quad z_3 w_2 = 0. \tag{15}$$

Case 1

Both $\xi_i \neq 0$: we have $z_1 w_2 \neq 0 \neq z_2 w_3$ implying that z_1, z_2, w_2, w_3 are non-zero and so $\zeta \neq 0$. Equation (15) gives $w_1 = 0 = z_3$. This implies

$$\lambda_1 = \lambda_3 = -\tfrac{1}{2}\lambda_2 = -\tfrac{1}{3} z_2 w_2.$$

Now z_1 determines the remaining variables via

$$w_2 = \frac{\xi_1}{z_1}, \quad w_3 = \frac{\zeta}{z_1}, \quad z_2 = \frac{\xi_2}{w_3} = \frac{\xi_2}{\zeta} z_1$$

giving the relation

$$3\lambda_2 \xi_2 = 2\xi_1 \zeta,$$

so $\lambda_2 \neq 0$. Constraints on z_1 come from $\mu_{\mathbb{R}}$ and using the above relations they are seen to only involve linear combinations of $x = |z_1|^2$ and $1/x$. Thus there are at most 2 values for $|z_1|$. However, closer inspection reveals that the entries above the diagonal in $\mu_{\mathbb{R}}$ are

$$\begin{pmatrix} * & p|z_1|^2 & 0 \\ * & * & q/|z_1|^2 \\ * & * & * \end{pmatrix},$$

with $p = \overline{\xi_2/\zeta}$, $q = \overline{\zeta}\xi_1$. Thus $\alpha_{13} = 0$, $4\alpha_{12}\alpha_{23} = -pq = -\overline{\xi_2}\xi_1$ and there is at most one solution for $|z_1|$, which together with β specifies the diagonal entries of α. Thus the fibre is either a circle or empty.

Case 2

$\xi_1 \neq 0, \xi_2 = 0$: we have $z_1 \neq 0 \neq w_2$ and thus $z_3 = 0$. Now $z_2 w_1 = 0 = z_2 w_3$ divides into cases.

(a) $z_2 \neq 0, w_1 = 0 = w_3$: gives $\lambda_2 = \frac{2}{3} z_2 w_2 = -2\lambda_1 = -2\lambda_3 \neq 0$. $z_2 = 3\lambda_2/2w_2$, $z_1 = \xi_1/w_2$. In α, the $(1,2)$-entry determines $z_1\overline{z}_2 \neq 0$ which specifies $|w_2|$ uniquely. So the fibre is either a circle or empty.

(b) $w_1 \neq 0, z_2 = 0$: gives $\lambda_1 = \frac{2}{3} z_1 w_1 = -2\lambda_2 = -2\lambda_3 \neq 0$, $\zeta = z_1 w_3$. So $w_1 = 3\lambda/2z_1$, $w_2 = \xi_1/z_1$, $w_3 = \zeta/z_1$. The $(1,2)$ entry of α then specifies $|z_1|$ uniquely and the fibre is a single circle.

(c) $w_1 = 0 = z_2$: gives $\lambda_i = 0$ for all i and $\zeta = z_1 w_3$, so $w_2 = \xi_1/z_1$, $w_3 = \zeta/z_1$. For $\zeta \neq 0$, $|z_1|$ is determined by α_{23} and the fibre is a circle or empty. For $\zeta = 0$, the only non-zero entries are the diagonal ones, with a the first entry -2 times the other two which are equal. These give a single quadratic equation for $|z_1|$, so the fibre is either 2 circles, 1 circle or empty. For example, the first diagonal entry in $\mu_{\mathbb{R}}$ is proportional to $2|z_1|^2 + |\xi_1|^2/|z_1|^2$, which attains any sufficiently large positive value at two different values of $|z_1|$. In particular, the fibre of μ can be disconnected.

Case 3

$\xi_1 = 0 = \xi_2 = \zeta$: there are three types of case:

(a) two z's non-zero: $z_1 \neq 0 \neq z_2$ implies $w \equiv 0$ and $\beta = 0$. The off-diagonal entries of α determine z_2 and z_3 in terms of z_1. If $\alpha_{23} \neq 0$, then this determines $|z_1|$ and the fibre is empty or a circle. Otherwise the diagonal entries lead to a quadratic constraint in $|z_1|$. The fibres are thus 2 circles, one circle or empty.

(b) one z and one w non-zero: then these must have the same index, say $z_1 \neq 0 \neq w_1$. So $w_2 = 0 = w_3 = z_2 = z_3$. β is diagonal with two repeated eigenvalues, $w_1 = -3\lambda_1/2z_1$. α is necessarily diagonal, the diagonal entries give a quadratic constraint on $|z_1|$. The fibres are 2 circles, one circle or empty.

(c) one z or w non-zero: gives $\beta = 0$, α diagonal with the sign of the entries in $i\alpha$ determined by whether it is z or w that is non-zero. The fibre is either a circle, a point or empty.

5 Implosion

Implosion arose as an abelianisation construction in symplectic geometry [14]. Given a Hamiltonian K-manifold M, one forms a new Hamiltonian space M_{impl} with an action of the maximal torus T of K, such that the symplectic reductions agree

$$M /\!/_{\lambda} K = M_{\mathrm{impl}} /\!/_{\lambda} T$$

for λ in the closed positive Weyl chamber. In most cases the implosion is a singular stratified space, even in M is smooth.

The key example is the implosion $(T^*K)_{\mathrm{impl}}$ of T^*K by (say) the right K action. Because T^*K has a $K \times K$ action, the implosion has a $K \times T$ action, and in fact we may implode a general Hamiltonian K-manifold by forming the reduction of

$M \times (T^*K)_{\mathrm{impl}}$ by the diagonal K action. In this sense $(T^*K)_{\mathrm{impl}}$ is a universal example for imploding K-manifolds. Concretely, $(T^*K)_{\mathrm{impl}}$ is obtained from the product of K with the closed positive Weyl chamber $\bar{\mathfrak{t}}^*$ by stratifying by the face of the Weyl chamber (i.e. by the centraliser C of points in $\bar{\mathfrak{t}}^*$) and then collapsing by the commutator of C.

There is also a more algebraic description of the universal implosion $(T^*K)_{\mathrm{impl}}$, as the non-reductive Geometric Invariant Theory (GIT) quotient $K_{\mathbb{C}}/\!/N$, where N is the maximal unipotent subgroup. Note that in general non-reductive quotients need not exist as varieties, due to the possible failure of finite generation for the ring of N-invariants. In the above case (and in the hyperKähler and holomorphic symplectic situation described below) it is a non-trivial result that we do have finite generation, so the quotient does exist as an affine variety.

In a series of papers [8, 9, 7] the authors and Kirwan described an analogue of implosion in hyperKähler geometry. There is a hyperKähler metric (due to Kronheimer [18]) with $K \times K$ action on the cotangent bundle $T^*K_{\mathbb{C}}$, and the idea is that the analogue of the universal symplectic implosion should be the complex-symplectic quotient (in the GIT sense) of $T^*K_{\mathbb{C}}$ by N. Explicitly, this quotient is $(K_{\mathbb{C}} \times \mathfrak{n}^{\circ})/\!/N$, which it is often convenient to identify with $(K_{\mathbb{C}} \times \mathfrak{b})/\!/N$, where $\mathfrak{b}$ is the Borel subalgebra.

In the case of $K = SU(n)$ it was shown in [8] that $(K_{\mathbb{C}} \times \mathfrak{n}^{\circ})/\!/N$ arises, via a quiver construction, as a hyperKähler quotient with residual $K \times T$ action. Moreover the hyperKähler quotients by T may be identified with the Kostant varieties, that is the subvarieties of $\mathfrak{k}_{\mathbb{C}}$ obtained by fixing the values of a generating set of invariant polynomials. For example, reducing at zero gives the nilpotent variety. For general semi-simple K, results of Ginzburg and Riche [12] give the existence of the complex-symplectic quotient $(K_{\mathbb{C}} \times \mathfrak{n}^{\circ})/\!/N$ as an affine variety, and the complex-symplectic quotients by the $T_{\mathbb{C}}$ action again give the Kostant varieties.

There is a link here with some intriguing work by Moore and Tachikawa [21]. They propose a category HS whose objects are complex semi-simple groups, and where elements of $\mathrm{Mor}(G_1, G_2)$ are complex-symplectic manifolds with $G_1 \times G_2$ action (together with a commuting circle action that acts on the complex-symplectic form with weight -2). We compose morphisms $X \in \mathrm{Mor}(G_1, G_2)$ and $Y \in \mathrm{Mor}(G_2, G_3)$ by taking the complex-symplectic reduction of $X \times Y$ by the diagonal G_2 action. The Kronheimer space T^*G, where $G = K_{\mathbb{C}}$, gives a canonical element of $\mathrm{Mor}(G, G)$—in fact this functions as the identity in $\mathrm{Mor}(G, G)$. The implosion now may be viewed as giving an element of $\mathrm{Mor}(G, T_{\mathbb{C}})$, where $G = K_{\mathbb{C}}$ and $T_{\mathbb{C}}$ is the complex maximal torus in G.

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