2-tactics in the Choquet game, and the Filter Dichotomy



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Dedicated to my family, wishing them all the best in a year already full of so many changes.

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Statement of Originality

All of the material that follows, unless otherwise indicated, is my own work. All of the material, unless otherwise indicated, and as far as I am aware, is original.

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Introduction

This thesis is comprised of two parts. The first part (Chapter 1) concerns a problem born from descriptive set theory, and is motivated by the desire to understand a particular topological game, in the presence of examples which exhibit interesting (arguably counterintuitive) behaviour. The work develops an understanding of the limits of this behaviour, and indicates where one might look for interesting examples. While set-theoretic techniques may appear in the analysis here, the focus is mostly topological.

The second part (Chapter 2) is much more set-theoretic. The work there developed from an interest in characterising closed subsets of $\beta\omega$, and focuses on developing tools which are promising for generalising the current relative consistency proofs of a duality principle for closed subsets of $\beta\omega \setminus \omega$. Knowledge of forcing comparable to that contained in Kunen [18] and Jech [16] is assumed.

The appendices contain results which are worthy of inclusion, and help provide additional perspective on the rest of the text. The results there are probably all already known, if not all recorded in the literature.

Chapter 1

The Choquet Game: 2-tactics

1.1 Introduction

1.1.1 Notation

Most of the notation used is standard, but we clarify some here. We will often make use of finite length sequences; $(a_0, \ldots, a_n)$ is shorthand for the function $(i \mapsto a_i)$ on domain $n+1$. $\emptyset$ denotes the empty sequence (the empty function), which we might otherwise write $(\,)$. $^\frown$ is the sequence concatenation operator; that is

$$(a_0, \ldots, a_n)^\frown (b_0, \ldots, b_m) = (a_0, \ldots, a_n, b_0, \ldots, b_m)\,.$$

$^\frown$ is clearly associative so parentheses are omitted.

Unless otherwise made clear, (finite or transfinite) sequences will be said to be descending if and only if they are descending with respect to $\subseteq$. That is, a sequence s is descending if and only if for all $\alpha, \beta \in \mathrm{dom}\,(s)$,

$$(\alpha < \beta \to s(\beta) \subseteq s(\alpha))\,.$$

The cardinal $2^{\aleph_0}$ is also written $\mathfrak{c}$. The successor cardinal to a cardinal κ is written κ^+. The successor ordinal to an ordinal α is written $\alpha + 1$. The cofinality of an ordinal α is written $\mathrm{cf}\,(\alpha)$.

ZFC is used as the base theory throughout, and additional axioms are indicated next to theorems that use them.

1.1.2 The Choquet game

The Choquet game is a useful tool in descriptive set theory for describing a particular subclass of the Baire spaces. This subclass is defined by the existence of a winning strategy for a particular player in a two player game defined in terms of the topology on a space. The game is of infinite length, and classically both players have access to all of the previous moves of the game in order to determine their next move. It is however not understood precisely which of the previous moves are crucial for the existence of winning strategies for our players. This forms the subject of the following.

We begin by introducing the standard terminology for the subject. Nothing in this section is new (except possibly Example 1.1.8 which is my own, but examples with the same properties are known). Section 1.2 begins the new material, and introduces a new abstraction, that of a transfer map, which has proved powerful in many of the proofs that follow. Section 1.2.2 consists of more known material; the results as far as I know are mostly folklore but the proofs my own — some of the proofs may be new, in particular the proof that Player II has a winning 1-tactic in the Choquet game on any pseudocompact space. The subsection on Choquet initial topologies is also an exception, the material there is entirely new. Everything in section 3 and onwards is entirely my own work unless otherwise noted. There we also relate Theorem 1.3.2 to an unpublished result of Galvin, see Theorem 1.4.13, Theorem 1.4.3 and [1] Theorem 39. Section 5 is a reconstruction of known results given descriptions of the spaces in question. Section 6 provides new proofs for the counterexample space in [10]. These new proofs are more modular and intuitive, and make clearer the development from the simpler spaces described in the previous section. A more detailed overview of the material in this chapter is provided at the end of this section.

We begin with the obvious, and define precisely what we mean by the Choquet game, and in turn Choquet spaces. The concepts here are intuitive, but the descriptions laborious.

Definition 1.1.1. *Let (X, τ) be a topological space. The Choquet game on (X, τ), $G_{(X,\tau)}$, is the ω-length game where two players, Player I and II, take turns to play non-empty open sets $\langle U_i : i \in \omega \rangle$ and $\langle V_i : i \in \omega \rangle$ respectively, with the properties*

$$V_i \subseteq U_i$$

$$U_{i+1} \subseteq V_i$$

for each $i \in \omega$. Player I begins by playing U_0. Player II is declared to win if and only if

$$\bigcap_{i \in \omega} V_i \neq \emptyset.$$

We often write G_X for the Choquet game on X when the topology is clear. This game is often referred to as the Banach-Mazur game. Notice, in the notation above

$$\bigcap_{i \in \omega} V_i = \bigcap_{i \in \omega} U_i.$$

In the fashion of [17], we can think of states of play of G_X as being elements of a tree, and runs of the game being infinite branches in this tree.

Definition 1.1.2. *For a set A we write $A^{<\omega}$ for the set of finite length sequences in A, that is $A^{<\omega}$ is the set of functions $s \colon n \to A$ where $n \in \omega$. A tree on a set A is a subset T of $A^{<\omega}$ such that if $s \in T$ and $t \in A^{<\omega}$ with $t \subseteq s$, then $t \in T$. So a tree is a subset of $A^{<\omega}$ closed under initial sequences. If S, T are trees on A, and $S \subseteq T$ then we say S is a subtree of T.*

If T is a tree on a set A then the body of T, written $[T]$ is the set of ω-length sequences in A with every finite initial segment in T, i.e.

$$[T] = \{x \in A^{\omega} \colon \forall n \in \omega \, (x \restriction n \in T)\}$$

A tree T is said to be pruned if and only if for any $s \in T$ there is a $t \in T$ with $s \subset t$ but $s \neq t$. That is, every element of T has a proper extension in T.

Remark 1.1.3. *For the purposes of this chapter, this definition of tree suffices. In the second chapter, we resume use of the more general set-theoretic definition of a partial order with the property that predecessors are well-ordered by the ordering.*

We can then think of G_X as being the tree

$$\left\{s \in (\tau \setminus \{\emptyset\})^{<\omega} : s \text{ is decreasing}\right\}$$

and runs of the game as $[G_X]$. We will often write elements of $[G_X]$ as

$$\langle (U_i, V_i) \ : \ i \in \omega \rangle ,$$

where this is intended as shorthand for the sequence $(U_0, V_0, U_1, V_1, \ldots)$. The U_i are then Player I's plays, and the V_i are Player II's plays. Our interest in this game centres around when either of the players has a winning strategy.

Definition 1.1.4. *A strategy for Player II in G_X is a pruned subtree σ of G_X with the following properties*

1. $\emptyset \in \sigma$
2. *If $(U_0, V_0, \ldots, V_n) \in \sigma$ then for any non-empty open $U_{n+1} \subseteq V_n$ we have*

$$(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma.$$

   Moreover, for any non-empty open set U_0, $(U_0) \in \sigma$.

3. *If $(U_0, V_0, \ldots, U_n) \in \sigma$ then there is a unique non-empty open $V_n \subseteq U_n$ with $(U_0, V_0, \ldots, U_n, V_n) \in \sigma$. We write $V_n = \sigma(U_0, V_0, \ldots, U_n)$.*

A play $\langle (U_i, V_i) \ : \ i \in \omega \rangle$ of G_X is said to be compatible with a strategy σ, or σ-compatible, if and only if $\langle (U_i, V_i) \ : \ i \in \omega \rangle \in [\sigma]$. σ is said to be a winning strategy for Player II if and only if every σ-compatible play of G_X is winning, i.e.

$$\forall \langle (U_i, V_i) \ : \ i \in \omega \rangle \left(\langle (U_i, V_i) \ : \ i \in \omega \rangle \in [\sigma] \to \bigcap_{n \in \omega} V_n \neq \emptyset \right)$$

Strategies and winning strategies for Player I are defined similarly, by making the appropriate changes.

We will be interested in special kinds of strategy, those which require only limited memory on the part of one of the players. These will be known as tactics.

Definition 1.1.5. *A strategy σ for Player II is called a k-tactic ($k > 0$) if and only if σ depends upon only the last k moves of Player I. Formally, σ is a k-tactic if and only if when $n_1, n_2 \geq k$,*

$$s_1 = \left(U'_0, V'_0, \ldots, V'_{n_1-k}, U'_{n_1-k+1}, V'_{n_1-k+1}, \ldots, V'_{n_1-1}, U'_{n_1}\right) \in \sigma$$

and

$$s_2 = \left(U''_0, V''_0, \ldots V''_{n_2-k}, U''_{n_2-k+1}, V''_{n_2-k+1}, \ldots, V''_{n_2-1}, U''_{n_2}\right) \in \sigma$$

with $U'_{n_1-k+i} = U''_{n_2-k+i}$ for $i \in \{1, \ldots, k\}$, then

$$\sigma\left(s_1\right) = \sigma\left(s_2\right).$$

A k-tactic σ is said to be winning for Player II if and only if σ is a winning strategy for Player II. The analogous concepts for Player I are defined in the obvious way.

Although trees provide a tidy way of describing and building strategies and tactics, it is often equally convenient to think of them as functions, taking some portion of the history of play so far, and providing the appropriate response for the player. From now on, whichever way is most convenient will be used in the arguments at hand.

The following is well known.

Theorem 1.1.6 (Oxtoby, see Kechris [17], Chapter I, Theorem 8.11)**.** *X is a Baire space if and only if Player I does not have a winning strategy in G_X.*

It is well known, using the Axiom of Choice, that it is possible to construct spaces X where neither player has a winning strategy in G_X. This gives rise to a special strict subclass of the Baire spaces.

Definition 1.1.7. *A topological space (X, τ) is called a Choquet space if and only if Player II has a winning strategy in G_X.*

Choquet spaces exhibit better behaviour than the more general Baire spaces — for example, a product of spaces is Choquet if and only if each of the factors is Choquet, while a product of Baire spaces need not be Baire. The property of being a

Baire space encodes a certain amount of completeness; complete metric spaces and locally compact Hausdorff spaces are the prototypical Baire spaces. Work has been conducted seeking completeness properties which have desirable closure properties, like productivity, including complete metric and locally compact Hausdorff spaces as examples. The property of being a Choquet space is one such property.

However, nonempty G_δ subspaces of Choquet spaces need not be Choquet, indeed they need not even be Baire! Non-empty open subspaces of Choquet spaces are Choquet: if U is a non-empty open subset of X, then given a winning strategy σ for Player II in G_X then σ is also a winning strategy for Player II in G_U. There is a class of Baire spaces closed under nonempty G_δ subspaces, the class of strong Choquet spaces. A strong Choquet space is a space where Player II has a winning strategy in the strong Choquet game, a modification of the Choquet game where for each move Player I may in addition specify a point which must be an element of Player II's following move. The winning condition is the same. Complete metric spaces and locally compact Hausdorff spaces are strong Choquet. The following example illustrates some of these points.

Example 1.1.8. *Let X be the disjoint union of ω and $\mathbb{Q}$. Equip X with the topology generated by the base*

$$\{\{n\} : n \in \omega\} \cup \{(a, b) \cup C \colon a, b \in \mathbb{Q},\ C \text{ a cofinite subset of } \omega\}.$$

ω is then a dense set of isolated points in X, and hence it is easily seen that X is Choquet (indeed Player II trivially has a winning 1-tactic in G_X). However, X is T_1, so $\mathbb{Q}$ (with the usual metric topology) is a nonempty G_δ subspace of X which is not Baire.

As an aside, we note that X cannot be strong Choquet. This is because strong Choquet spaces are Choquet, and the property of being strong Choquet is preserved under nonempty G_δ subspaces. Since X has a nonempty G_δ subspace which is not Choquet, it cannot therefore be strong Choquet. So X is also an example of a Choquet space which is not strong Choquet.

Let us demonstrate the converse direction of Theorem 1.1.6. More precisely, suppose (X, τ) is not a Baire space, then we show Player I has a winning strategy in G_X. Since (X, τ) is not Baire, there is a non-empty open set U and dense open sets W_n for $n \in \omega$ with

$$U \cap \bigcap_{n \in \omega} W_n = \emptyset.$$

Let Player I's first move be U. Suppose on some subsequent turn Player II has previously played the non-empty open set V. Find the least n such that $V \not\subseteq W_n$ and observe that $V \cap W_n \neq \emptyset$ as W_n is dense and $V \neq \emptyset$. Let Player I respond with $V \cap W_n$. This describes a strategy σ for Player I depending on only the previous play of Player II. That is, σ is a 1-tactic. We claim the strategy σ is winning. If $\langle (U_i, V_i) : i \in \omega \rangle$ is a σ-compatible sequence of plays, then by induction $U_{n+1} \subseteq W_n$ for all $n \in \omega$, so

$$\bigcap_{n \in \omega} U_n = U \cap \bigcap_{n \in \omega} U_{n+1} \subseteq U \cap \bigcap_{n \in \omega} W_n = \emptyset$$

and Player I wins.

We have shown when X is not a Baire space, Player I has a winning 1-tactic in G_X. Since the other implication of Theorem 1.1.6 says that if Player I has a winning strategy in G_X then X is not Baire, we have the following:

Observation 1.1.9. *If Player I has a winning strategy in G_X then Player I has a winning 1-tactic in G_X.*

The obvious question was asked. Need the same be true for Player II? That is, if X is a Choquet space, does Player II have a winning 1-tactic in G_X? Debs answered this in the negative, producing Choquet spaces where Player II could not have a winning 1-tactic, but did have a winning 2-tactic.

The simplest (and first) of these spaces failed to be regular, but traces of these in the literature are sparse, and at best, seem to only describe what the spaces were. We reconstruct some of the proofs for these later, within our own framework. A year after producing these initial spaces, Debs [10] produced a Tychonoff Choquet space X in which Player II has no winning 1-tactic, but does have a winning 2-tactic.

Such a space has a Hausdorff compactification Y, hence there is a winning 1-tactic for Player II in G_Y. Thus, X is a dense Choquet subspace of a Choquet space Y, where Player II has a winning 1-tactic in G_Y.

The next most obvious question to ask is whether there is a Choquet space X providing Player II with a winning 3-tactic, but no winning 2-tactic. More generally, one may ask for some $k \in \omega$, $k \geq 2$, whether there is a Choquet space X providing Player II with a winning $(k+1)$-tactic, but no winning k-tactic, or indeed, if there is a Choquet space X with no winning k-tactic for any $k \in \omega$. All of these remain open (see [26]). We concern ourselves predominantly with the first of these questions.

There are some known results about the amount of information that is sufficient for Player II to have a winning strategy. For instance, if X is Choquet, then Player II has a winning strategy that depends on only the previous move of Player I and the previous move of Player II. This was proved by Debs in [10] and independently by Galvin and Telgársky in [14]. One might wonder if it makes any difference if Player II is allowed to use the number of moves Player I has made so far in their figuring, in conjunction with the previous k moves of Player I, a so called Markov k-tactic. However, Player II has a winning k-tactic if and only if Player II has a winning Markov k-tactic; the case for $k = 1$ was established by Galvin and Telgársky in [14], while the $k \geq 2$ case is found in [1].

1.1.3 Overview

In section 1.2 we introduce the notion of a transfer map, a device used for analysing new strategies that we build from old ones. These are used extensively in the proofs that follow, and even naturally provide generalisations of known results. For instance, we prove that every dense G_δ subspace of a Choquet space is Choquet without any regularity assumptions, generalising a result of White. In the same section (in 1.2.2) we also provide some basic (folklore) results on k-tactics and strategies. The proofs are my own, and demonstrate typical use of transfer maps. We also introduce some new results, the highlight of which is a generalisation of the

fact that every pseudocompact topological space provides Player II with a winning 1-tactic.

Section 1.3 introduces the notion of a coding system, and we prove the coding lemma, that every Choquet space with a coding system provides Player II with a winning 2-tactic. Coding systems capture the possibility to encode a usable approximation of the entire history of a Choquet game using a relational structure between two open sets. Section 1.4 demonstrates some of the basic uses of the coding lemma, the tool which is central to the rest of the chapter. In particular we provide a proof of an unpublished result of Galvin, which provides a purely topological condition which is sufficient to give Player II a winning 2-tactic in a Choquet space.

Sections 1.5–1.7 explore Debs' counterexample spaces inside our new framework of coding systems, reconstructing unpublished proofs, and providing much simplified proofs for the published Tychonoff example. We also show that it is relatively consistent that coding systems are not necessary for Player II to have a winning 2-tactic (and no winning 1-tactic).

Section 1.8 studies the semi-regularisation of a Choquet space, an idea inspired by Debs' counterexample spaces. We prove in particular that the semi-regularisation of a Choquet space is Choquet, and that linearly ordered topological spaces admit coding systems, so that Player II has a winning 2-tactic in any Choquet linearly ordered topological space.

Having developed some more specialised language, section 1.9 explores more consequences of the coding lemma. In particular, we prove Player II has a winning 2-tactic in a Choquet space with π-weight not exceeding ω_1.

The final section of this chapter, section 1.10, introduces a generalisation of the Choquet game, where we no longer take a topological space, but create an abstract descending chain game with some of the properties of the Choquet game. The game is studied as a topological space in its own right, and the existence of winning strategies is characterised in terms of the topological properties of the set of winning plays. These tools indicate what one needs to do to create a Choquet-

like game where Player II has a winning strategy but no winning k-tactic, for some or all $k \in \omega$. This provides insight into the complexity of the problem. Furthermore, it should be easier to build such an abstract game, and then one might be able to realise it as a topological Choquet game by interpreting the abstract pieces in a suitable way.

1.2 k-tactics: basic properties

1.2.1 A method of building new strategies

Quite often we will be interested in questions of the form "given a winning strategy of a particular type, is there a winning strategy of some other type?" (for instance, given a winning strategy, is there a winning 2-tactic?). The general approach will be to take the first winning strategy σ and modify it in some way to create a new strategy which has some nice properties. However, we wish to be able to easily check that such a strategy is winning. In order to do this, we will typically build our new strategy by recursion, but simultaneously construct a transfer map which allows us to compare our new strategy with the old one. The transfer map is in each case designed to make analysis of the new strategy straightforward.

An additional benefit of introducing transfer maps to compare strategies is that it allows us to prove new theorems. This is because it provides new language that is easy to work with.

To avoid repetition later, and to improve clarity, we also abstract some properties we find in such transfer maps. For the following definition, view strategies as trees.

Definition 1.2.1. *Suppose σ and μ are strategies for Player II (Player I) in G_X. A transfer map (from μ to σ) is a map $\mathcal{T} \colon \mu \to \sigma$ such that*

1. $(s \subseteq t \to \mathcal{T}(s) \subseteq \mathcal{T}(t))$
2. *$\mathcal{T}$ preserves length, i.e. for all $s \in \mu$,* $\text{length}(s) = \text{length}(\mathcal{T}(s))$.

If $\mathcal{T}$ is a transfer map from μ to σ, then for $s \in [\mu]$ we may (abusing notation) define $\mathcal{T}(s)$ to be $\bigcup_{n\in\omega} \mathcal{T}(s \restriction n)$ (where n-tuples here are thought of as partial functions on the domain ω; this just gives us the obvious limit).

Note that the definition of a transfer map is just the natural definition for an embedding of one tree as a subtree of another.

Transfer maps are useful because of the following observations. These in application will be used to check that new strategies that we build are winning.

Observation 1.2.2. *Suppose $\mathcal{T}$ is a transfer map from μ to σ. If $s \in [\mu]$, then $\mathcal{T}(s) \in [\sigma]$. So if σ is a winning strategy for Player II then for all $s \in [\mu]$, $\bigcap \mathrm{ran}(\mathcal{T}(s)) \neq \emptyset$ (here we think of $\mathcal{T}(s)$ as a function on domain ω). Further, if $s \in [\mu]$, then for all $n \in \omega$ we have $\mathcal{T}(s \restriction n) = \mathcal{T}(s) \restriction n \subseteq \mathcal{T}(s)$.*

We will of course construct $\mathcal{T}$ so that $\mathrm{ran}(\mathcal{T}(s))$ allows us to say something about the sequence s. From now on the above properties of transfer maps will be used without reference.

1.2.2 Basic properties of k-tactics and strategies

Lemma 1.2.3. *Suppose X is a topological space and D is a dense subspace of X. Suppose Player II has a winning k-tactic (strategy) in G_D. Then Player II has a winning k-tactic (strategy) in G_X.*

Proof. Let σ be a winning k-tactic for Player II in G_D. We describe a winning k-tactic μ for Player II in G_X.

For each open non-empty U in D let $L(U)$ be open in X and satisfy $L(U) \cap D = U$. We build μ by recursion, alongside a transfer map $\mathcal{D}$ from μ to σ, with the properties that

1. if $(U_0, V_0, \ldots, V_n) \in \mu$ then $\mathcal{D}(U_0, V_0, \ldots, V_n) = t^\frown(V_n \cap D)$ for some t.

2. if $(U_0, V_0, \ldots, U_n) \in \mu$ then $\mathcal{D}(U_0, V_0, \ldots, U_n) = t^\frown(U_n \cap D)$, for some t.

Obviously, $\emptyset \in \mu$ and $\mathcal{D}(\emptyset) = \emptyset$, and if $(U_0, V_0, \ldots, V_n) \in \mu$ and U_{n+1} is non-empty and open in X with $U_{n+1} \subseteq V_n$ then $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \mu$, and

$$\mathcal{D}(U_0, V_0, \ldots, V_n, U_{n+1}) = \mathcal{D}(U_0, V_0, \ldots, V_n)^\frown (U_{n+1} \cap D).$$

Note that this is well-defined (i.e. $\mathcal{D}(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma$) if we have obeyed (1) and (2) in our recursion so far. This is because, in the notation of (1) and (2), if $U_{n+1} \subseteq V_n$, then $U_{n+1} \cap D \subseteq V_n \cap D$.

Now suppose $(U_0, V_0, \ldots, U_n) \in \mu$. Let $V_n' = \sigma(\mathcal{D}(U_0, V_0, \ldots, U_n))$, and put $V_n = L(V_n') \cap U_n$. Then declare $(U_0, V_0, \ldots, U_n, V_n) \in \mu$ and, as is necessary, $\mathcal{D}(U_0, V_0, \ldots, U_n, V_n) = \mathcal{D}(U_0, V_0, \ldots, U_n)^\frown (V_n')$. We demonstrate that $V_n \cap D = V_n'$, so that we maintain satisfaction of (1). It is enough to show that $V_n' \subseteq U_n$, but this is clear as, in the notation of (1) and (2), $V_n' \subseteq U_n \cap D$, because σ must prescribe this.

This defines μ and $\mathcal{D}$ with the properties we prescribed. Now if

$$s = \langle (U_i, V_i) \ : \ i \in \omega \rangle$$

is a μ-compatible play of G_X, then $\mathcal{D}(s)$ is σ-compatible, and $\bigcap \mathrm{ran}(\mathcal{D}(s)) \neq \emptyset$. But, property (1) which we prescribed on $\mathcal{D}$ ensures that

$$\bigcap \mathrm{ran}(\mathcal{D}(s)) \subseteq \bigcap_{n \in \omega} V_n,$$

so μ is in fact a winning strategy for Player II in G_X.

We now argue formally that μ is in fact a k-tactic. Suppose $s, t \in \mu$ both end on Player I's move, and s agrees with t on the previous k moves of Player I. Notice that by property (2) of $\mathcal{D}$, $\mathcal{D}(s)$ and $\mathcal{D}(t)$ agree on the previous k moves of Player I in the transferred Choquet game on D. By how we now define μ we see that $\mu(s) = \mu(t)$. □

Remark 1.2.4. *In the above proof, we decided that when $(U_0, V_0, \ldots, V_n) \in \mu$ and U_{n+1} is a non-empty open subset of V_n, that $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \mu$. It will be taken as implicit that this statement includes the possibility that no moves at all have been made, that is to say, this statement includes the statement that for any non-empty open*

set U_0 we declare $(U_0) \in \mu$. We will do this in all future proofs of the same type without mention.

Remark 1.2.5. *We cannot expect to prove a converse to 1.2.3. For instance $\mathbb{Q}$ is dense in $\mathbb{R}$, and Player II has a winning 1-tactic in $G_{\mathbb{R}}$, but $\mathbb{Q}$ is not even Baire.*

We may ask if we can do better when we restrict to the case when the dense subspace is Choquet, but this still is not good enough. In Debs' Tychonoff Choquet space X in [10], Player II has no winning 1-tactics in G_X. However, X is dense in a compact Hausdorff space Y, and Player II certainly has a winning 1-tactic in G_Y. See section 6 for details on Debs' Tychonoff counterexample space.

White in [34] has shown that if X is a weakly regular (i.e. for any non-empty open U there is a non-empty open V with $\overline{V} \subseteq U$) Choquet space, and Y is a dense G_δ subspace of X, then Y is Choquet. The following is a generalisation of White's result. The proof demonstrates how transfer maps provide a more effective language than ad hoc analysis. With the language of transfer maps we do not need the additional hypothesis of our space being weakly regular, and yet the proof procedes completely naturally.

Lemma 1.2.6. *Suppose X is a Choquet space and Y is a dense G_δ subspace of X. Then Y is Choquet.*

Proof. Suppose σ is a winning strategy for Player II in G_X. Write $Y = \bigcap_{n\in\omega} G_n$, where G_n is dense and open in X. For each U open in Y fix $L(U)$ open in X with $L(U) \cap Y = U$. Let us describe a winning strategy μ for Player II in G_Y; we build μ as a tree alongside a transfer map $\mathcal{L}$ from μ to σ, with the properties that

1. if $s \in \mu$ is of the form $(U_0, V_0, \ldots, U_{n+1})$ then $\bigcap \mathrm{ran}(\mathcal{L}(s)) \subseteq G_{n+1} \cap L(U_{n+1})$.
2. if $s \in \mu$ is of the form $(U_0, V_0, \ldots, U_n, V_n)$ and $\mathcal{L}(s) = t^\frown(V_n')$ for some t, then $V_n' \cap Y = V_n$.

Of course $\emptyset \in \mu$ and we put $\mathcal{L}(\emptyset) = \emptyset$, and if $s = (U_0, V_0, \ldots, V_n) \in \mu$ then for any U_{n+1} non-empty and open in Y, with $U_{n+1} \subseteq V_n$, put $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \mu$.

Meanwhile, set

$$\mathcal{L}(U_0, V_0, \ldots, V_n, U_{n+1}) = \mathcal{L}(s)^\frown \left(G_{n+1} \cap L(U_{n+1}) \cap \bigcap \mathrm{ran}(\mathcal{L}(s)) \right).$$

Now suppose $s = (U_0, V_0, \ldots, U_n) \in \mu$. Let $V_n' = \sigma(\mathcal{L}(s))$, and put $V_n = V_n' \cap Y$. Then put $s^\frown(V_n) \in \mu$ and set $\mathcal{L}(s^\frown(V_n)) = \mathcal{L}(s)^\frown(V_n')$. Observe that if we have satisfied (1) throughout this recursion, then $V_n' \subseteq L(U_n)$, so $V_n = V_n' \cap Y \subseteq L(U_n) \cap Y = U_n$, so we are justified in extending μ using V_n in this way.

We check μ is a winning strategy for Player II in G_Y. Suppose s is a μ-compatible sequence of plays in G_Y, and write $s = \langle (U_i, V_i) : i \in \omega \rangle$. Then $\mathcal{L}(s)$ is a σ-compatible sequence, so $\bigcap \mathrm{ran}(\mathcal{L}(s)) \neq \emptyset$. By property (1) of $\mathcal{L}$, $\bigcap \mathrm{ran}(\mathcal{L}(s)) \subseteq \bigcap_{n\in\omega} G_n = Y$, and also

$$\bigcap_{n\in\omega} V_n = \bigcap_{i\in\omega} ((\mathcal{L}(s))(2i) \cap Y) = \left(\bigcap \mathrm{ran}(\mathcal{L}(s)) \right) \cap Y = \bigcap \mathrm{ran}(\mathcal{L}(s)) \neq \emptyset$$

so μ is indeed a winning strategy for Player II in G_Y. □

The previous two lemmas give us the following.

Corollary 1.2.7. *Suppose X is a Baire space, and Y is comeagre in X. Then X is Choquet if and only if Y is Choquet.*

Proof. Y must contain a dense G_δ (in X), G say, and then note G is also dense in Y. For the converse, note Y is comeagre in the Baire space X and so dense. □

Another corollary to Lemma 1.2.6 (though we actually only need White's version of the result) is the following folklore result.

Corollary 1.2.8. *If X is Čech-complete then X is Choquet.*

The (fully general) analogous version of Lemma 1.2.6 for k-tactics remains open — our proof here relies on knowing the entire history of the game. It would be interesting to see if this can be avoided.

Question 1.2.9. *Is there a space X in which Player II has a winning k-tactic in G_X, but X has a dense G_δ subspace Y such that Player II does not have a winning k-tactic in G_Y?*

We obtain a result later in this section which provides a partial answer, eliminating spaces with a weak-base of regular open sets, and in particular, weakly regular spaces.

The above results are useful since it can be simpler to analyse subspaces or superspaces. Another approach is to reduce the amount of information about the topology that we use in the game.

In general, given a collection of non-empty subsets $\mathcal{S}$ of X, let $G_X^{\mathcal{S}}$ denote the Choquet game on X where both Players I and II are restricted to forming descending chains with only elements of $\mathcal{S}$ (instead of any non-empty open set), with the same winning condition. The following Lemma is very useful, and formalizes the above suggestion. First we make a definition.

Definition 1.2.10. *Let X be a topological space. $\mathcal{W}$ is said to be a weak-base for X if $\mathcal{W}$ is a set of non-empty open subsets of X such that if U is a non-empty open subset of X then there is a $W \in \mathcal{W}$ with $W \subseteq U$.*

Lemma 1.2.11. *Let $\mathcal{W}$ be a weak-base for X. Then Player II has a winning k-tactic (strategy) in G_X if and only if Player II has a winning k-tactic (strategy) in $G_X^{\mathcal{W}}$.*

Proof. For each non-empty open subset U of X, fix $[U]_{\mathcal{W}} \in \mathcal{W}$ with $[U]_{\mathcal{W}} \subseteq U$. First let σ be a winning k-tactic (strategy) for Player II in G_X. We define a k-tactic (strategy) $\sigma_{\mathcal{W}}$ for Player II in $G_X^{\mathcal{W}}$, and a transfer map $\mathcal{T}$ from $\sigma_{\mathcal{W}}$ to σ so that

1. if $(U_0, V_0, \ldots, U_n) \in \sigma_{\mathcal{W}}$ then $\mathcal{T}(U_0, V_0, \ldots, U_n) = t^\frown(U_n)$ for some t (let us refer to this property as "$\mathcal{T}$ does not change Player I's moves").

2. if $(U_0, V_0, \ldots, V_n) \in \sigma_{\mathcal{W}}$ then $\mathcal{T}(U_0, V_0, \ldots, V_n) = t^\frown(W_n)$ for some t, where $V_n = [W_n]_{\mathcal{W}}$.

First $\emptyset \in \sigma_{\mathcal{W}}$, with $\mathcal{T}(\emptyset) = \emptyset$, and for any $(U_0, V_0, \ldots, V_n) \in \sigma_{\mathcal{W}}$ and any $U_{n+1} \in \mathcal{W}$ with $U_{n+1} \subseteq V_n$ we declare $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma_{\mathcal{W}}$. In this case we also set $\mathcal{T}(U_0, V_0, \ldots, V_n, U_{n+1}) = \mathcal{T}(U_0, V_0, \ldots, V_n)^\frown(U_{n+1})$. Note that if we have enforced (1) and (2) in our recursion so far, then $\mathcal{T}(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma$, since in the notation of (2), $U_{n+1} \subseteq V_n \subseteq W_n$.

Now suppose $(U_0, V_0, \ldots, U_n) \in \sigma_{\mathcal{W}}$. Put $W_n = \sigma(\mathcal{T}(U_0, V_0, \ldots, U_n))$, and set $V_n = [W_n]_{\mathcal{W}}$. Declare $(U_0, V_0, \ldots, U_n, V_n) \in \sigma_{\mathcal{W}}$, and set $\mathcal{T}(U_0, V_0, \ldots, U_n, V_n) = \mathcal{T}(U_0, V_0, \ldots, U_n)^\frown(W_n)$. Note that, if so far in our recursion $\mathcal{T}$ obeys (1), then $W_n \subseteq U_n$, and hence $V_n \subseteq U_n$ also, so we are indeed justified in extending $\sigma_{\mathcal{W}}$ in this way.

Now if $s = \langle (U_i, V_i) : i \in \omega \rangle$ is $\sigma_{\mathcal{W}}$-compatible, then $\mathcal{T}(s)$ is σ-compatible, and hence, by property (1) of $\mathcal{T}$, $\bigcap_{i\in\omega} U_i \neq \emptyset$. So $\sigma_{\mathcal{W}}$ is a winning strategy for Player II in $G_X^{\mathcal{W}}$.

To argue that $\sigma_{\mathcal{W}}$ is a k-tactic, we do the same as in the proof of Lemma 1.2.3. Let $s, t \in \sigma_{\mathcal{W}}$ both end with Player I's move, with t agreeing with s on the previous k moves of Player I. Because $\mathcal{T}$ does not change Player I's moves (and because $(U \mapsto [U]_{\mathcal{W}})$ is a function), we see $\sigma_{\mathcal{W}}(s) = \sigma_{\mathcal{W}}(t)$ by how we defined $\sigma_{\mathcal{W}}$.

For the converse, suppose $\sigma_{\mathcal{W}}$ is a winning k-tactic (strategy) for Player II in $G_X^{\mathcal{W}}$. We define a k-tactic (strategy) σ for Player II in G_X and a transfer map $\mathcal{S}$ from σ to $\sigma_{\mathcal{W}}$, such that

1. if $(U_0, V_0, \ldots, U_n) \in \sigma$ then $\mathcal{S}(U_0, V_0, \ldots, U_n) = t^\frown([U_n]_{\mathcal{W}})$ for some t.

2. if $(U_0, V_0, \ldots, V_n) \in \sigma$ then $\mathcal{S}(U_0, V_0, \ldots, V_n) = t^\frown(V_n)$ for some t.

Starting with the obvious we put $\emptyset \in \sigma$ and set $\mathcal{S}(\emptyset) = \emptyset$. Further, if $(U_0, V_0, \ldots, V_n) \in \sigma$ then for any non-empty open U_{n+1} with $U_{n+1} \subseteq V_n$, we put $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma$ and set

$$\mathcal{S}(U_0, V_0, \ldots, V_n, U_{n+1}) = \mathcal{S}(U_0, V_0, \ldots, V_n)^\frown([U_{n+1}]_{\mathcal{W}}).$$

Note that, if we have enforced (2) in the recursion so far, then

$$\mathcal{S}(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma_{\mathcal{W}}$$

as we require.

Now if $(U_0, V_0, \ldots, U_n) \in \sigma$, then put $V_n = \sigma_{\mathcal{W}}(\mathcal{S}(U_0, V_0, \ldots, U_n))$, and note $V_n \subseteq [U_n]_{\mathcal{W}} \subseteq U_n$ by using (1) from our recursion so far. So we may declare $(U_0, V_0, \ldots, U_n, V_n) \in \sigma$ and $\mathcal{S}(U_0, V_0, \ldots, U_n, V_n) = \mathcal{S}(U_0, V_0, \ldots, U_n)^\frown(V_n)$.

σ is a winning strategy because if $s = \langle (U_i, V_i) \,:\, i \in \omega \rangle$ is σ-compatible, then $\bigcap \mathrm{ran}\,(\mathcal{S}\,(s)) \neq \emptyset$, so $\bigcap_{i\in\omega} V_i \neq \emptyset$ by property (2) of $\mathcal{S}$.

If $s, t \in \sigma$ both end on Player I's move, and Player I has the same previous k-moves in both s and t, then by property (1) of $\mathcal{S}$, $\mathcal{S}\,(s)$ and $\mathcal{S}\,(t)$ have the same previous k moves for Player I, and by definition of σ and the fact that $\sigma_{\mathcal{W}}$ is a k-tactic, we see $\sigma\,(s) = \sigma\,(t)$. So σ is a k-tactic when $\sigma_{\mathcal{W}}$ is. □

For topological spaces X_λ, for $\lambda \in \Lambda$, define the topological sum of the X_λ, as the disjoint union of the X_λ with the topology generated by the union of the topologies on the X_λ. Denote this space $\bigoplus_{\lambda\in\Lambda} X_\lambda$.

If there is, for each $k \in \omega$, some X_k which is Choquet, but so that Player II has no winning k-tactic in G_{X_k}, then the topological sum $\bigoplus_{k\in\omega} X_k$ is a Choquet space in which Player II has no winning k-tactic for any k. This provides a potential approach to one of the open problems mentioned above, in terms of the others. The details of the proof are in the following.

Lemma 1.2.12. *Suppose Player II has a winning k-tactic (strategy) in G_X, and U is a non-empty open subset of X. Then Player II has a winning k-tactic (strategy) in G_U.*

Proof. If σ is a winning k-tactic (strategy) for Player II in G_X, then σ is a winning k-tactic (strategy) for Player II in G_U also. □

Theorem 1.2.13. *Fix $k \in \omega$. Suppose X_λ is a topological space for each $\lambda \in \Lambda$. Write $X = \bigoplus_{\lambda\in\Lambda} X_\lambda$. Then Player II has a winning k-tactic (strategy) in G_X if and only if Player II has a winning k-tactic (strategy) in G_{X_λ} for each $\lambda \in \Lambda$.*

Proof. First suppose Player II has a winning k-tactic (strategy) in G_X. Note that for $\lambda \in \Lambda$, X_λ is an open subspace of X, so by the previous lemma, Player II has a winning k-tactic (strategy) in G_{X_λ}.

Now assume that for each X_λ, Player II has a winning k-tactic (strategy) in G_{X_λ}, σ_λ say. Let τ_λ denote the topology on X_λ. $\bigcup_{\lambda\in\Lambda} \tau_\lambda$ is a base for the topology on X, so it is enough to show that Player II has a winning k-tactic (strategy) in the game $G_X^{\left(\bigcup_{\lambda\in\Lambda}\tau_\lambda\right)}$ (by Lemma 1.2.11). But $\bigcup_{\lambda\in\Lambda} \sigma_\lambda$ is such a k-tactic (strategy). □

1.2.3 More on basic properties concerning G_δ sets

We return now to Question 1.2.9, and provide a partial answer. Recall that an open subset U of a space X is called regular open if and only if $\text{int}_X\left(\overline{U}\right) = U$.

Theorem 1.2.14. *Suppose the collection $\mathcal{W}$ of regular open sets in X is a weak-base, and suppose that Player II has a winning k-tactic in G_X. Let Y be a dense G_δ subspace of X. Then Player II has a winning k-tactic in G_Y.*

Proof. Let σ be a winning k-tactic for Player II in $G_X^{\mathcal{W}}$, and write $Y = \bigcap_{n\in\omega} G_n$, where G_n is open in X for each $n \in \omega$. Y has a weak-base given by

$$\{W \cap Y : W \in \mathcal{W}\};$$

let us call this weak-base $\mathcal{W}\#Y$. Let us also, for each non-empty open subset U of X, fix $[U]_{\mathcal{W}} \in \mathcal{W}$ with $[U]_{\mathcal{W}} \subseteq U$.

Suppose V is regular open in X. Then, in X, $V \cap Y$ is dense in V and $\overline{V}^X = \overline{V \cap Y}^X$. In particular the map L which takes open sets $U \in \mathcal{W}\#Y$ to $\text{int}_X\left(\overline{U}^X\right)$ (which is regular open in X) satisfies $L(U) \cap Y = U$ and $(U \subseteq V \to L(U) \subseteq L(V))$.

We construct a k-tactic μ for Player II in $G_Y^{\mathcal{W}\#Y}$, along with a transfer map $\mathcal{T}$ from μ to σ, with the following properties:

1. If $s = (U_1, V_1, \ldots, U_n) \in \mu$ then $\mathcal{T}(s) = t^\frown(W_n)$ for some tuple t, for each $n \in \omega$, where $W_n \subseteq G_n \cap L(U_n)$.

2. If $s = (U_1, V_1, \ldots, V_n) \in \mu$ then $\mathcal{T}(s) = t^\frown(L(V_n))$ for some tuple t, for each $n \in \omega$.

It is clear, given that we can construct μ and $\mathcal{T}$ in this way, that if $\langle (U_i, V_i) : i \in \omega\rangle$ is a μ-compatible play of $G_Y^{\mathcal{W}\#Y}$, then

$$\emptyset \neq \bigcap \text{ran}(\mathcal{T}(s)) \subseteq \bigcap_{n\in\omega} G_n = Y.$$

It follows that

$$\emptyset \neq \bigcap \text{ran}(\mathcal{T}(s)) = \bigcap \text{ran}(\mathcal{T}(s)) \cap Y = \bigcap_{n\in\omega} V_n,$$

so that μ is winning for Player II. So Player II has a winning strategy in G_Y.

So we only need convince ourselves that such a μ and $\mathcal{T}$ may be found. We build μ and $\mathcal{T}$ simultaneously by recursion. Of course, we begin with $\emptyset \in \mu$ and $\mathcal{T}(\emptyset) = \emptyset$. Suppose our recursion so far has satisfied (1) and (2).

If $s = (U_1, V_1, \ldots, V_n) \in \mu$, then for any $U_{n+1} \in \mathcal{W}\#Y$ which is a subset of V_n,

$$(U_1, V_1, \ldots, V_n, U_{n+1}) \in \mu.$$

Further, we may take k minimal so that $L(U_{n+1}) \not\subseteq G_k$ and set

$$W_n = [G_k \cap L(U_{n+1})]_{\mathcal{W}}$$

so that $W_n \subseteq G_k \cap L(U_{n+1})$, and then set

$$\mathcal{T}((U_1, V_1, \ldots, V_n, U_{n+1})) = \mathcal{T}(s)^\frown(W_n).$$

Note this is well-defined by (2) and monotonicity of L. Furthermore, by an easy induction, $W_n \subseteq G_n$.

Now suppose $s = (U_1, V_1, \ldots, U_n) \in \mu$. Let $V = \sigma(\mathcal{T}(s))$, and put $V_n = V \cap Y$. Now, $(U_1, V_1, \ldots, U_n, V_n) \in \mu$ and

$$\mathcal{T}(U_1, V_1, \ldots, U_n, V_n) = \mathcal{T}(s)^\frown(V).$$

Note that by injectivity of L, $L(V_n) = V$, so we have satisfied (2). It is clear from (1) and the fact that σ is a k-tactic, that defining μ in this way makes μ a k-tactic. □

Remark 1.2.15. *In the above, if X has a weak-base of regular open sets, then in particular the collection $\mathcal{W}$ of all regular open sets forms a weak-base for X.*

The observation below makes it clear why the above theorem is a strengthening of White's result mentioned before. We hence have two strengthening's of White's result: the above can be thought of as a strengthening of the conclusion, and Lemma 1.2.6 weakened the hypothesese.

Observation 1.2.16. *Suppose X is weakly regular (i.e. for each non-empty open U there is a non-empty open V with $\overline{V} \subseteq U$). Then X has a weak-base of regular open sets.*

Proof. For each non-empty open U, let V_U be non-empty and open with $\overline{V_U} \subseteq U$. Then

$$\left\{\text{int}_X\left(\overline{V_U}\right) : U \text{ non-empty and open}\right\}$$

is a weak-base of regular open sets. □

Corollary 1.2.17. *If X is Čech-complete then Player II has a winning 1-tactic in G_X.*

Proof. Note that βX is regular, Player II has a winning 1-tactic in $G_{\beta X}$, and X is a dense G_δ in βX. □

The condition that Y be a dense G_δ subset of X in the above allowed us to look at σ-compatible plays which got inside Y. We can weaken this idea slightly. The following is a well known fact.

Theorem 1.2.18 (Engelking [11], pg212, Exercise 3.10.F(a))**.** *Suppose X is a Tychonoff space. Then X is pseudocompact if and only if there are no non-empty G_δ subsets G of βX with $G \subseteq \beta X \setminus X$.*

Proof. We prove a generalisation later, see Remark 1.2.33. □

If σ is a strategy in G_X and $\langle (U_i, V_i) \ : \ i \in \omega \rangle$ is σ-compatible, then $\bigcap_{n\in\omega} U_n$ is G_δ in X. We use this to show that a winning 1-tactic is inherited in a pseudocompact space from its Stone-Čech compactification.

Theorem 1.2.19. *Suppose X is a pseudocompact (Tychonoff) space. Then Player II has a winning 1-tactic in G_X.*

Proof. Let $\mathcal{B} = \{f^{-1}(W) : f \in \mathsf{C}(X, [0,1]),\ W \text{ open in } [0,1]\}$ be the set of functionally open subsets of X. Because X is Tychonoff, $\mathcal{B}$ is a base for the space X. We will find a winning 1-tactic for Player II in $G_X^{\mathcal{B}}$.

We consider βX to be the set of ultrafilters on functionally closed subsets of X (a subset of X is functionally closed if and only if its complement is functionally open), with topology generated by the basic open sets $\mathcal{B}^*$, where

$$\mathcal{B}^* = \{U^* \colon U \in \mathcal{B}\}$$

and

$$U^* = \{\mathcal{F} \in \beta X : \exists C \in \mathcal{F}\ C \subseteq U\}.$$

Notice that if $V \subseteq U$ then $V^* \subseteq U^*$. The embedding of X into βX is given by

$$\beta : x \mapsto \mathcal{P}_x = \{F \subseteq X : F \text{ is functionally closed and } x \in F\},$$

and identifying X with its image under β in the usual way gives

$$U^* \cap X = U.$$

βX is a compact Hausdorff space, so Player II has a winning 1-tactic in $G_{\beta X}$, so Player II also has a winning 1-tactic σ in $G_{\beta X}^{\mathcal{B}^*}$.

We now construct a winning 1-tactic μ for Player II in $G_X^{\mathcal{B}}$ alongside a transfer map $\mathcal{T}$ from μ to σ, with the properties that

1. If $(U_1, V_1, \ldots, U_n) \in \mu$ then $\mathcal{T}(U_1, V_1, \ldots, U_n) = t^\frown(U_n^*)$ for some t.
2. If $(U_1, V_1, \ldots, V_n) \in \mu$ then $\mathcal{T}(U_1, V_1, \ldots, V_n) = t^\frown(V_n^*)$ for some t.

Of course, $\emptyset \in \mu$ and $\mathcal{T}(\emptyset) = \emptyset$. Suppose our construction so far has satisfied (1) and (2). If $s = (U_1, V_1, \ldots, U_n) \in \mu$ then let $W = \sigma(\mathcal{T}(s))$, and put $V_n = W \cap X$. Note $V_n^* = W$. Then prescribe $(U_1, V_1, \ldots, U_n, V_n) \in \mu$ and set

$$\mathcal{T}(U_1, V_1, \ldots, U_n, V_n) = s^\frown(V_n^*).$$

Now if $s = (U_1, V_1, \ldots, V_n) \in \mu$ then for any $U_{n+1} \in \mathcal{B}$ with $U_{n+1} \subseteq V_n$, put $(U_1, V_1, \ldots, V_n, U_{n+1}) \in \mu$ and

$$\mathcal{T}(U_1, V_1, \ldots, V_n, U_{n+1}) = \mathcal{T}(s)^\frown\left(U_{n+1}^*\right),$$

noting that this is well-defined.

By (1) and the way we have defined μ in terms of σ, it is clear that μ is a 1-tactic for Player II in $G_X^{\mathcal{B}}$. We now check that it is a winning strategy also. Suppose $s = \langle (U_i, V_i) \ : \ i \in \omega \rangle$ is a μ-compatible play of $G_X^{\mathcal{B}}$. Then $\mathcal{T}(s) = \langle (U_i^*, V_i^*) \ : \ i \in \omega \rangle$ is a σ-compatible play of $G_{\beta X}^{\mathcal{B}^*}$, so in particular

$$\bigcap_{n \in \omega} U_n^* \neq \emptyset.$$

But $\bigcap_{n\in\omega} U_n^*$ is then a non-empty G_δ subset of βX, so by psuedocompactness of X and the previous theorem

$$\emptyset \neq X \cap \bigcap_{n\in\omega} U_n^* = \bigcap_{n\in\omega} U_n,$$

and hence we see μ is a winning strategy for Player II in $G_X^{\mathcal{B}}$. □

1.2.4 Choquet initial topologies with winning 1-tactics

The proof of the previous result, that pseudocompact Tychonoff spaces give Player II a winning 1-tactic, is perhaps new, but the result is already known using an internal characterisation of pseudocompactness. The above proof is valuable for two reasons however — it shows that a winning 1-tactic is inherited from a winning 1-tactic in a compact Hausdorff space, and secondly, because we can generalise it to initial topologies generated by certain rings of real-valued functions.

Definition 1.2.20. *Let X be a set and $\mathcal{F} \subseteq \mathbb{R}^X$. We say $\mathcal{F}$ is a real functional subring (on X) if*

1. *$\mathcal{F}$ is closed under pointwise addition, multiplication and subtraction (so in particular $\mathcal{F}$ forms an algebraic ring), as well as $f \mapsto |f|$;*
2. *$\mathcal{F}$ separates the points of X (i.e. for $x, y \in X$ with $x \neq y$ there is an $f \in \mathcal{F}$ with $f(x) \neq f(y)$);*
3. *$\mathcal{F}$ contains all the constant maps.*
4. *$\mathcal{F}$ is closed under uniformly convergent sequences.*

Given a real functional subring $\mathcal{F}$ on a set X, we denote the initial topology on X generated by $\mathcal{F}$ as $\tau_{\mathcal{F}}$.

Closure under the absolute value operation, $f \mapsto |f|$, gives us closure under some other useful operations.

Lemma 1.2.21. *Suppose $\mathcal{F}$ is a real functional subring on X. Then $\mathcal{F}$ is closed under* max *and* min.

Proof. Observe that

$$\max(f,g) = \frac{1}{2}(f+g+|f-g|).$$

Also $\min(f,g) = -\max(-f,-g)$. The result then follows from the closure operations in the definition of a real functional subring. □

The following technical lemmas tell us some basic properties of the initial topology generated by a real functional subring $\mathcal{F}$. We use them to build a compactification of the initial topology. The proofs are intended to be similar to later proofs of technical results concerning this compactification.

Lemma 1.2.22. *Suppose $\mathcal{F}$ is a real functional subring on X. Suppose, $x \in X$, $f \in \mathcal{F}$ and U is open in $\mathbb{R}$ with $x \in f^{-1}(U)$. Then there is a $g \in \mathcal{F}$ with* $\operatorname{ran}(g) \subseteq [0,1]$, $g(x) = 1$ *and*

$$x \in g^{-1}\left(\left(\frac{1}{2},1\right]\right) \subseteq f^{-1}(U).$$

Proof. Without loss of generality, we may suppose

$$U = (f(x)-\epsilon, f(x)+\epsilon)$$

for some $\epsilon > 0$. Let $h = \frac{1}{2\epsilon}f$, and note $h \in \mathcal{F}$. Then

$$f^{-1}(U) = h^{-1}\left(\left(\frac{1}{2\epsilon}f(x)-\frac{1}{2}, \frac{1}{2\epsilon}f(x)+\frac{1}{2}\right)\right).$$

Now take $k = y \mapsto 1-|h(x)-h(y)|$, and observe $k \in \mathcal{F}$. Now $k(x) = 1$, and continuing the argument above

$$f^{-1}(U) = k^{-1}\left(\left(\frac{1}{2},1\right]\right).$$

We can now take $g = \max(k,0)$, and then $g \in \mathcal{F}$ (by the previous lemma) and will clearly have the properties we require. □

Lemma 1.2.23. *Suppose $\mathcal{F}$ is a real functional subring on X. Suppose U is open in $(X, \tau_{\mathcal{F}})$, and $x \in U$. Then there is an $f \in \mathcal{F}$ with* $\operatorname{ran}(f) \subseteq [0,1]$, $f(x) = 1$ *and*

$$x \in f^{-1}\left(\left(\frac{1}{2},1\right]\right) \subseteq U.$$

Proof. By the above lemma, we may suppose there are $g_1, \ldots, g_n$ with $\operatorname{ran}(g_i) \subseteq [0,1]$ and $g_i(x) = 1$ for each i, while

$$x \in \bigcap_{i=1}^{n} g_i^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \subseteq U.$$

Then we can take $f = \frac{1}{n}\sum_{i=1}^{n} g_i$ and observe that this choice of f works. □

These technical lemmas tell us some basic information about the topology, and allow us to construct an interesting compactification of the space.

Corollary 1.2.24. *Suppose $\mathcal{F}$ is a real functional subring on X. Then $(X, \tau_{\mathcal{F}})$ is Tychonoff. In particular, the set*

$$\mathcal{B} = \left\{ f^{-1}(U) : U \text{ open in } [0,1],\ f \in \mathcal{F} \cap [0,1]^X \right\}$$

forms a base for $(X, \tau_{\mathcal{F}})$.

Proof. It suffices to show that $\mathcal{B}$ is a base for the space $(X, \tau_{\mathcal{F}})$, and this follows from the previous lemma. □

Definition 1.2.25. *Given a real functional subring $\mathcal{F}$ on a set X, write $\mathcal{F}_b$ for $\mathcal{F} \cap [0,1]^X$. The above corollary says that the diagonal map*

$$\Delta_{f \in \mathcal{F}_b} f \colon X \to [0,1]^{\mathcal{F}_b}$$

is an embedding. This gives us a compactification of $(X, \tau_{\mathcal{F}})$, which we denote $\mathcal{F}X$, given by

$$\left(\Delta_{f \in \mathcal{F}_b} f, \overline{\left(\Delta_{f \in \mathcal{F}_b} f\right)(X)}^{[0,1]^{\mathcal{F}_b}} \right).$$

Observe also that the projection map π_f for $f \in \mathcal{F}_b$ is a continuous extension of f to $\mathcal{F}X$. When we write π_f we will mean $\pi_f \restriction \mathcal{F}X$, and not the map on the appropriate Tychonoff cube (unless noted otherwise).

We need some more technical results concerning the compactification $\mathcal{F}X$. We know the compactification is Tychonoff, but we want witnessing functions of the form π_f where $f \in \mathcal{F}_b$. We formalise this below.

Lemma 1.2.26. *Suppose U is open in $[0,1]$ and $f \in \mathcal{F}_b$ with $x \in \pi_f^{-1}(U)$. Then there is a $g \in \mathcal{F}_b$ with $\pi_g(x) = 1$ and*

$$x \in \pi_g^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \subseteq \pi_f^{-1}(U).$$

Proof. The proof here will look similar to the analogous result on X, but we need to consider the extensions of functions to the compactification $\mathcal{F}X$ also. We may suppose without loss of generality that

$$U = (\pi_f(x) - \epsilon, \pi_f(x) + \epsilon)$$

for some $\epsilon > 0$. As we build our function g we will track the preimage of U under f to help illustrate what is happening. Let $h = \frac{1}{2\epsilon} f$. Then $h \in \mathcal{F}$. Also

$$f^{-1}(U) = h^{-1}\left(\left(\frac{1}{2\epsilon}\pi_f(x) - \frac{1}{2}, \frac{1}{2\epsilon}\pi_f(x) - \frac{1}{2}\right)\right).$$

Now set $k = y \mapsto 1 - \left|\frac{1}{2\epsilon}\pi_f(x) - h(y)\right|$, so that $k \in \mathcal{F}$, and continuing the above

$$f^{-1}(U) = k^{-1}\left(\left(\frac{1}{2}, 1\right]\right).$$

As before, we set $g = \max(k, 0)$, and then $g \in \mathcal{F}_b$, and

$$f^{-1}(U) = g^{-1}\left(\left(\frac{1}{2}, 1\right]\right).$$

It remains to show $\pi_g(x) = 1$ and $\pi_g^{-1}\left(\left(\frac{1}{2}, 1\right]\right) = \pi_f^{-1}(U)$.

We have constructed g above in stages to illustrate what is happening. But if we write the formula for g out in full, we obtain

$$g = y \mapsto \max\left(1 - \left|\frac{1}{2\epsilon}\pi_f(x) - \frac{1}{2\epsilon}f(y)\right|, 0\right).$$

Observe that g is just the restriction of π_g to X, while

$$y \mapsto \max\left(1 - \left|\frac{1}{2\epsilon}\pi_f(x) - \frac{1}{2\epsilon}f(y)\right|, 0\right)$$

is just the restriction of

$$y \mapsto \max\left(1 - \left|\frac{1}{2\epsilon}\pi_f(x) - \frac{1}{2\epsilon}\pi_f(y)\right|, 0\right)$$

to X. Since X is dense in $\mathcal{F}X$ we must conclude

$$\pi_g = y \mapsto \max\left(1 - \left|\frac{1}{2\epsilon}\pi_f(x) - \frac{1}{2\epsilon}\pi_f(y)\right|, 0\right).$$

It is then easily seen that $\pi_g(x) = 1$ and $\pi_g^{-1}\left(\left(\frac{1}{2}, 1\right]\right) = \pi_f^{-1}(U)$, as required. □

As before, we now develop the previous lemma to obtain a basis for $\mathcal{F}X$ witnessed by single maps.

Lemma 1.2.27. *Suppose $x \in \mathcal{F}X$, and U is open in $\mathcal{F}X$ with $x \in U$. Then there is an $f \in \mathcal{F}_b$ with $\pi_f(x) = 1$ and*

$$x \in \pi_f^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \subseteq U.$$

Proof. By the above, we can find $g_1, \ldots, g_n \in \mathcal{F}_b$ with

$$x \in \bigcap_{i=1}^{n} \pi_{g_i}^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \subseteq U,$$

and $\pi_{g_i}(x) = 1$ for $i = 1, \ldots, n$. Let $f = \min_{i=1,\ldots,n} g_i$, so $f \in \mathcal{F}_b$. Then π_f and $\min_{i=1,\ldots,n} \pi_{g_i}$ agree on the dense set X, so

$$\pi_f = \min_{i=1,\ldots,n} \pi_{g_i}.$$

It follows that $\pi_f(x) = 1$. Furthermore,

$$x \in \pi_f^{-1}\left(\left(\frac{1}{2}, 1\right]\right) = \bigcap_{i=1}^{n} \pi_{g_i}^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \subseteq U,$$

which completes the proof. □

We make some minor modifications to the above lemma to serve our purposes more explicitly in the following theorem. We will use the modified version to build a uniformly convergent sequence of continuous functions in $\mathcal{F}$. The following lemma captures the Tychonoff separation property of $\mathcal{F}X$ in the ring $\mathcal{F}$.

Lemma 1.2.28. *Suppose $x \in \mathcal{F}X$ and C is closed in $\mathcal{F}X$ with $x \notin C$. Suppose $n \in \omega$. Then there is an $f \in \mathcal{F}_b$ with* $\operatorname{ran}(f) \subseteq \left[0, \frac{1}{2^n}\right]$*,*

$$\pi_f(C) \subseteq \{0\}$$

and

$$\pi_f(x) = \frac{1}{2^n}.$$

Proof. By the previous lemma, find $g \in \mathcal{F}_b$ with $\pi_g(x) = 1$ and

$$\pi_g^{-1}\left(\left(\frac{1}{2}, 1\right]\right) \subseteq \mathcal{F}X \setminus C.$$

Let

$$f = \max\left(\frac{1}{2^{n-1}}\left(g - \frac{1}{2}\right), 0\right).$$

Then $f \in \mathcal{F}_b$ and $\operatorname{ran}(f) \subseteq \left[0, \frac{1}{2^n}\right]$. Also, π_f and $\max\left(\frac{1}{2^{n-1}}\left(\pi_g - \frac{1}{2}\right), 0\right)$ agree on the dense subset X of $\mathcal{F}X$, so

$$\pi_f = \max\left(\frac{1}{2^{n-1}}\left(\pi_g - \frac{1}{2}\right), 0\right).$$

We see $\pi_f(x) = \frac{1}{2^n}$. Also, if $y \in C$, then $\pi_g(y) < \frac{1}{2}$, so $\pi_f(C) \subseteq \{0\}$. Hence f has the required properties. □

Remark 1.2.29. *None of the above required closure of $\mathcal{F}$ under uniformly convergent sequences.*

Definition 1.2.30. *Suppose Y is a topological space and X is a subspace of Y. We say that X is G_δ-dense in Y if and only if whenever G is a non-empty G_δ in Y we have $G \cap X \neq \emptyset$.*

Theorem 1.2.31. *Suppose $\mathcal{F}$ is a real functional subring on the set X. Then every $f \in \mathcal{F}_b$ has compact image if and only if X is G_δ-dense in $\mathcal{F}X$.*

Proof. Let us first observe that the condition of having a compact image is necessary, since if $f \in \mathcal{F}_b$ and $x \in \overline{\operatorname{ran}(f)} \setminus \operatorname{ran}(f)$, then $x \in \operatorname{ran}(\pi_f)$ and

$$\bigcap_{n\in\omega} \pi_f^{-1}\left(B\left(\pi_f(x), \frac{1}{n+1}\right)\right)$$

is a non-empty G_δ in $\mathcal{F}X$ which does not meet X.

Now we show that the condition is sufficient. Suppose G_n is open for each $n \in \omega$ and $G = \bigcap_{n\in\omega} G_n$ is a non-empty G_δ in $\mathcal{F}X$ which does not meet X. Let $x \in G$. Without loss of generality, let us suppose $G_{n+1} \subseteq G_n$ for all $n \in \omega$.

Claim 1.2.32. *For each $n \in \omega$ there is an $f_n \colon X \to \left[0, \frac{1}{2^{n+1}}\right]$ with $f_n(X \setminus G_n) \subseteq \{0\}$ and $\pi_{f_n}(x) = \frac{1}{2^{n+1}}$.*

Proof of Claim. The lemma preceeding this theorem guarantees the existence of such functions. □

Taking the f_n as in the claim above, we have that $f = \sum_{n\in\omega} f_n \in \mathcal{F}_b$, by closure under uniform convergence. Notice that $\pi_f = \sum_{n\in\omega} \pi_{f_n}$ (they agree on the dense subset X), and so $\pi_f(x) = 1$. Since X is dense in $\mathcal{F}X$,

$$\pi_f(\mathcal{F}X) = \pi_f\left(\overline{X}^{\mathcal{F}X}\right) \subseteq \overline{f(X)},$$

so $1 \in \overline{f(X)}$.

Now, if $y \in X$ and $y \notin G_n$, then $f(y) \leq \sum_{i\leq n} \frac{1}{2^{i+1}} < 1$. So $1 \in \overline{f(X)} \setminus f(X)$. This means f does not have closed image, and so does not have compact image. □

Remark 1.2.33. *If X is pseudocompact then we can take $\mathcal{F} = C(X, \mathbb{R})$. In this case $\mathcal{F}_b = C(X, [0,1])$. Then $\mathcal{F}X = \beta X$, and the above Theorem becomes a proof of the Hewitt theorem, Theorem 1.2.18.*

Theorem 1.2.34. *Suppose X is a topological space that is G_δ-dense in a Hausdorff compactification Y. Then X is Choquet and Player II has a winning 1-tactic in G_X.*

Proof. This is a generalisation of the pseudocompact (Tychonoff) case, where we took Y to be βX. The proof here needs to be slightly different because we don't have an explicit description of Y. Of course, the proof below will work when $Y = \beta X$.

Let $\mathcal{R}$ denote the set of regular open subsets of Y. Since Y is regular, we know that $\mathcal{R}$ forms a base for Y. Therefore $\mathcal{B} = \{U \cap X : U \in \mathcal{R}\}$ forms a base for X. We show that Player II has a winning 1-tactic in $G_X^{\mathcal{B}}$.

First, let us observe that since X is dense in Y, if $U \in \mathcal{R}$ then we have

$$U = \text{int}_Y\left(\overline{U}\right) = \text{int}_Y\left(\overline{U \cap X}\right)$$

so that the map $L : \mathcal{B} \to \mathcal{R}$ which sends V to $\text{int}_Y\left(\overline{V}^Y\right)$ is a bijection with inverse $U \mapsto U \cap X$. Furthermore, for $U, V \in \mathcal{B}$, $U \subseteq V$ if and only if $L(U) \subseteq L(V)$.

Since Y is compact and Hausdorff, we know that Player II has a winning 1-tactic in $G_Y^{\mathcal{R}}$; let σ be such a winning 1-tactic. We construct a winning 1-tactic μ in $G_Y^{\mathcal{B}}$ alongside a transfer map $\mathcal{T} : \mu \to \sigma$, such that

1. if $s = (U_1, V_1, \ldots, U_n) \in \mu$ then $\mathcal{T}(s) = t^\frown(L(U_n))$ for some t;

2. if $s = (U_1, V_1, \ldots, U_n, V_n) \in \mu$ then $\mathcal{T}(s) = t^\frown(L(V_n))$ for some t.

Of course $\emptyset \in \mu$ and $\mathcal{T}(\emptyset) = \emptyset$. If $s = (U_1, V_1, \ldots, U_n, V_n) \in \mu$, then for any $U_{n+1} \in \mathcal{B}$ with $U_{n+1} \subseteq V_n$, we put $s^\frown(U_{n+1}) \in \mu$ and define $\mathcal{T}(U_1, V_1, \ldots, V_n, U_{n+1}) = \mathcal{T}(s)^\frown(L(U_{n+1}))$. Observe that this is indeed a sequence from σ if we have obeyed (1) and (2) in all steps of the recursion so far, and this also ensures (1) is maintained.

Furthermore, if $s = (U_1, V_1, \ldots, U_n) \in \mu$, then let $V_n' = \sigma(\mathcal{T}(s))$, and put $V_n = V_n' \cap X$. We then prescribe $(U_1, V_1, \ldots, U_n, V_n) \in \mu$ and $\mathcal{T}(U_1, V_1, \ldots, U_n, V_n) = \mathcal{T}(s)^\frown(V_n')$. Note that this is well-defined, and by our above observations maintains (2).

It is easily seen that since σ is a 1-tactic, μ is a 1-tactic also. We must check that it is a winning strategy. Let $s = \langle (U_i, V_i) \,:\, i \in \omega \rangle$ be a μ-compatible play of $G_X^{\mathcal{B}}$. Then $\mathcal{T}(s) = \langle (L(U_i), L(V_i)) \,:\, i \in \omega \rangle$ is a σ-compatible play of $G_Y^{\mathcal{R}}$, and hence

$$\bigcap_{i \in \omega} L(V_i) \neq \emptyset.$$

But then $\bigcap_{i\in\omega} L(V_i)$ is a non-empty G_δ subset of Y, so $X \cap \bigcap_{i\in\omega} L(V_i) \neq \emptyset$. But

$$\emptyset \neq X \cap \bigcap_{i \in \omega} L(V_i) = \bigcap_{i \in \omega} (L(V_i) \cap X) = \bigcap_{i \in \omega} V_i$$

and hence we see that μ is a winning strategy, and hence a winning 1-tactic as required. □

Remark 1.2.35. *1. In the above Theorem, we do not need the full power of Y being a compact Hausdorff space. Observe that the above proof works as easily for the following generalised version of the theorem:*

Corollary 1.2.36. *Suppose Player II has a winning k-tactic in G_Y, and Y has a base of regular open sets. Suppose X is G_δ-dense in Y. Then X is Choquet and Player II has a winning k-tactic in G_X.*

Observe that open sets are G_δ, so if X is G_δ-dense in Y, then in particular, X is dense in Y.

2. *We already have a theorem for reducing k-tactics to dense G_δ ("large") subsets in regular spaces. The above theorem, and remarked generalised version, provide an analogous result for a different notion of "large". The property that X is G_δ-dense in Y says in some sense that X is so large that some class of simple subsets of Y cannot escape it. Note that this property is also equivalent to saying that X is not contained in a proper F_σ subset of Y.*

   Phrased in this way, we have that G_δ and F_σ subsets of Tychonoff cubes cannot fail to have winning 1-tactics. All Tychonoff spaces which are examples of spaces without winning k-tactics for some k, so in particular Debs' Tychonoff example, must therefore be quite complicated in the sense of Borel complexity — their Borel rank is greater than or equal to 2, if they are Borel at all.

3. *Note that being dense in a Choquet space is not sufficient for being Choquet. We can think of the property "X is G_δ-dense in Y" as a strengthening of density (noting it entails density). For spaces with a base of regular open sets this "countable closure of density" is enough to preserve Choquet-ness and the arity of tactics available.*

Corollary 1.2.37. *Suppose $\mathcal{F}$ is a real functional subring on the set X, consisting of functions with compact image. Then $(X, \tau_{\mathcal{F}})$ is Choquet, and Player II has a winning 1-tactic in $G_{(X,\tau_{\mathcal{F}})}$.*

Proof. Combine the two previous Theorems. □

The non-triviality of this generalisation was provided by the following example, which is joint work with Rolf Suabedissen.

Theorem 1.2.38 (joint work with Suabedissen)**.** *There is a set X and a real functional subring $\mathcal{F}$ on X such that every $f \in \mathcal{F}$ has compact image, but $(X, \tau_{\mathcal{F}})$ is not pseudocompact.*

Proof. Take $X = \omega_1 \oplus (0,1]$. Our real functional subring will generate the topology of the topological sum $\omega_1 \oplus (0,1]$, which is not pseudocompact (a function which is constant on ω_1 and $x \mapsto \frac{1}{x}$ on $(0,1]$ will witness this). However, every function in our ring will have compact image.

It is well known that every continuous function $f : \omega_1 \to \mathbb{R}$ is eventually constant (in the order topology) — for such an f let us write $\text{tail}(f)$ for this (eventually) constant value. We take

$$\mathcal{F} = \left\{ f \in C(\omega_1 \oplus (0,1]) : \text{tail}(f \restriction \omega_1) = \lim_{x \to 0} f \restriction (0,1] \right\}.$$

$\mathcal{F}$ is easily seen to form an algebraic ring (by standard algebra of limits arguments). Also, if $g : \mathbb{R} \to \mathbb{R}$ is continuous, and $f \in \mathcal{F}$ then $g \circ f \in \mathcal{F}$, because continuous functions preserve limits. In particular, if we take $g = x \mapsto |x|$ we see $\mathcal{F}$ is closed under taking absolute values.

Furthermore, it is easily seen that $\mathcal{F}$ contains all constant maps, and it is easily seen that $\mathcal{F}$ separates points. We show $\mathcal{F}$ is closed under uniform limits. So suppose $\langle f_n : n \in \omega \rangle$ is a uniformly convergent sequence, converging to some $f \in C(\omega_1 \oplus (0,1])$ (it is well known that $C(\omega_1 \oplus (0,1])$ is closed under uniform limits). We need to show $f \in \mathcal{F}$. For this it suffices to prove the following claim.

Claim 1.2.39. $\text{tail}(f \restriction \omega_1) = \lim_{x \to 0} f \restriction (0,1]$.

Proof of Claim. First observe that since ω_1 is a regular cardinal, there is an $\alpha < \omega_1$ such that for all $n \in \omega$, f_n is constant beyond α. Hence, since the uniform limit of functions is in particular a pointwise limit, we have

$$\text{tail}(f \restriction \omega_1) = \lim_{n \to \infty} \text{tail}(f_n \restriction \omega_1).$$

Now, for each $n \in \omega$, let $\hat{f}_n$ be the continous extension of $f_n \restriction (0,1]$ to $[0,1]$. Since taking a limit as $x \to 0$ preserves weak-inequalities, we can show that $\left\langle \hat{f}_n : n \in \omega \right\rangle$ is Cauchy and so convergent in $C([0,1])$, converging to some limit $\hat{f}$. Since $\hat{f}$ is in

particular the pointwise limit of the $\left\langle \hat{f}_n \,:\, n \in \omega \right\rangle$, we have

$$\begin{aligned}\hat{f}(0) &= \lim_{n\to\infty} \hat{f}_n(0) \\ &= \lim_{n\to\infty} \mathsf{tail}\left(f_n \restriction \omega_1\right) \\ &= \mathsf{tail}\left(f \restriction \omega_1\right).\end{aligned}$$

But if $\hat{f}$ is also a pointwise limit of $\left\langle \hat{f}_n \,:\, n \in \omega \right\rangle$ then

$$\hat{f} \restriction (0,1] = f \restriction (0,1].$$

Piecing this together we have

$$\lim_{x\to 0} f \restriction (0,1] = \lim_{x\to 0} \hat{f} \restriction (0,1] = \hat{f}(0) = \mathsf{tail}\left(f \restriction \omega_1\right),$$

as we require. □

So $\mathcal{F}$ forms a real functional subring. We now argue that every $f \in \mathcal{F}$ has compact image. To see this, let $f \in \mathcal{F}$, and let $\hat{f}$ be the continous extension of f to $\omega_1 \oplus [0,1]$ formed by defining

$$\hat{f}(0) = \mathsf{tail}\left(f \restriction \omega_1\right).$$

Observe that $\mathsf{ran}\,(f) = \mathsf{ran}\left(\hat{f}\right)$, and since $\omega_1 \oplus [0,1]$ is pseudocompact, $\hat{f}$ has compact image, and hence so does f.

It remains to show that $(X, \tau_{\mathcal{F}}) = \omega_1 \oplus (0,1]$. First note that for any clopen order interval (α, β) in ω_1, the indicator function $\chi_{(\alpha,\beta)} \colon \omega_1 \oplus (0,1] \to \mathbb{R}$ is continuous, in $\mathcal{F}$, and

$$(\alpha,\beta) = \chi^{-1}_{(\alpha,\beta)}\left((0,2)\right).$$

Since $\mathcal{F} \subseteq C\left(\omega_1 \oplus (0,1]\right)$, the topology induced on ω_1 by $\mathcal{F}$ is no finer than the order topology, and hence is just the order topology, and so ω_1 in the order topology occurs as an open subspace of $(X, \tau_{\mathcal{F}})$.

Now if $0 < q_1 < q_2$, where $q_1, q_2 \in \mathbb{Q}$, then on $[0,\infty)$ the map

$$g = x \mapsto \mathsf{max}\left(0, \mathsf{min}\left((x - q_1), (q_2 - x)\right)\right)$$

is continuous $[0, \infty) \to \mathbb{R}$ and non-zero precisely on the interval (q_1, q_2). If we now define f on X to be 0 on ω_1, and on $(0, 1]$ set $f = g \restriction (0, 1]$, then $f \in \mathcal{F}$ and

$$f^{-1}((0, \infty)) = (q_1, q_2).$$

Since $\mathcal{F} \subseteq C(\omega_1 \oplus (0, 1])$, $\mathcal{F}$ induces the usual Euclidean topology on $(0, 1]$ (again since the topology induced can be no finer than the Euclidean topology), and $(0, 1]$ occurs as an open subspace of $(X, \tau_{\mathcal{F}})$.

Putting these two pieces together we have $(X, \tau_{\mathcal{F}}) = \omega_1 \oplus (0, 1]$, as we required. So we have shown that $\mathcal{F}$ is a real functional subring on X consisting of functions with compact image, but that $\mathcal{F}$ generates the topology on X of the non-pseudocompact space $\omega_1 \oplus (0, 1]$. □

This example was originally of interest because of the following corollary,

Corollary 1.2.40. *There is a set X and a real functional subring $\mathcal{F}$ on X such that $C((X, \tau_{\mathcal{F}}), \mathbb{R}) \neq \mathcal{F}$.*

and also because examples like this provide interesting examples for the following, which shows that the Hewitt theorem characterising pseudocompactness (Theorem 1.2.18) does not generalise to arbitrary compactifications of X (not necessarily βX). This is in contrast with a property like Čech-completeness.

Corollary 1.2.41. *There is a space X which has a compactification $\mathcal{F}X$ such that X is G_δ-dense in $\mathcal{F}X$, but X is not G_δ-dense in βX.*

A simple example of this is to take X to be an uncountable discrete space, and to consider its one-point compactification. Our example is more obviously Stone-Čech-like, in the sense that we have a ring of functions which generate the topology and admit continuous extensions to the compactification. In fact, an uncountable discrete space can also be realised as an initial topology induced by a real functional subring of functions with compact image.

The advantage of our proof that Player II has a winning 1-tactic in the Choquet game on a pseudocompact Tychonoff space is that it generalises to deal with spaces of this kind.

1.3 Coding systems and coding tactics

In [10], Debs exhibited a Tychonoff Choquet space X where Player II does not have a winning 1-tactic in G_X, but does have a winning 2-tactic. In order to prove that Player II has a winning 2-tactic, Debs argues that, by cleverly picking responses to Player I's moves, Player II can deduce enough information about the history of the game in order to essentially play as if he has all the history in memory.

We abstract the idea of "coding and deducing approximate history" here, to exhibit a class of Choquet spaces which have winning 2-tactics. The objects which we will use for approximating the history of the game will be elements of some weak-base (see Definition 1.2.10).

To see the rough idea, suppose X is a topological space, and $\mathcal{W}$ is a weak-base for X. When Player I makes a move U, Player II can pretend that Player I has actually played some $W_U \in \mathcal{W}$ with $W_U \subseteq U$, and use this instead in deciding the next move. Formally we use Lemma 1.2.11. On any given turn, suppose Player II has access to the two previous "weak-basic approximations" of Player I's plays, (U_0, U_1) say. What we want is a device which given (U_0, U_1) provides a non-empty open subset of U_1, Z_1 say, which allows us, for any non-empty open $U_2 \subseteq Z_1$ to recover U_0 from (U_1, U_2). This device is hence a coding device, and is itself a 2-tactic.

Of course, this method of coding need not yield a winning 2-tactic, but such a 2-tactic can be refined using a winning strategy to create a winning 2-tactic. To do this we note the recovery of moves (for instance, of U_0 in the above) can be used inductively to recover the entire history of the game from the point of view of the weak-base. We then use our winning strategy with this recovered information.

There are some technical issues. For instance, Player II needs to track when the game begins. We could use the open set X as a marker for this, but to be more explicit, and to avoid confusion between the start of the game and the open set X, we introduce a start marker $\star$, where $\star \notin \tau$, and add this as a distinguished element to the weak-base. So, given a weak-base $\mathcal{W}$, we let $\mathcal{W}_\star = \mathcal{W} \cup \{\star\}$.

1.3.1 Coding systems and a coding lemma

A coding system is a device for remembering the approximate history of a Choquet game, and formalises the idea described above.

Definition 1.3.1 (Coding system)**.** *Suppose $\mathcal{W}$ is a weak-base for the topological space (X, τ). A coding system (with respect to $\mathcal{W}$) is a pair (μ, ν) where*

$$\mu \colon \mathcal{W}_\star \times \mathcal{W} \to \tau$$

$$\nu \colon \mathcal{W} \times \tau \to \mathcal{W}_\star$$

and μ and ν satisfy the following:

1. *$\mu(W_1, W_2) \subseteq W_2$ for $W_1 \in \mathcal{W}_\star$, $W_2 \in \mathcal{W}$;*
2. *if $W_1 \in \mathcal{W}_\star$ and $W_2 \in \mathcal{W}$, and $W_1 = \star$ or $(W_1 \neq \star \wedge W_2 \subseteq W_1)$, then for any non-empty open $V \subseteq \mu(W_1, W_2)$ we have*

$$W_1 = \nu(W_2, V)$$

We say $\mathcal{W}$ admits the coding system (μ, ν) in this case.

Intuitively, μ is a function which Player II will use to code the history of the game, while ν is a function Player II will use to decode it. Claim 1.3.3 below formalises how Player II decodes history, while Claim 1.3.4 captures how μ is used to do coding. The main result about coding systems is the following.

Theorem 1.3.2 (The coding lemma)**.** *Suppose (X, τ) is a topological space, $\mathcal{W}$ a weak-base for X and (μ, ν) is a coding system with respect to the weak-base $\mathcal{W}$. If X is Choquet, then Player II has a winning 2-tactic in G_X.*

Proof. The idea of the proof is to think of μ as an approximation to a winning 2-tactic. By decoding an approximate history of the game using ν, we may refine μ using a winning strategy for Player II, and obtain a winning 2-tactic.

First let us make some simplifications to improve readability. Note that we may as well assume that $\operatorname{ran}(\mu) \subseteq \mathcal{W}$, by assuming $\mu(U, V)$ actually takes the value W

where $W \in \mathcal{W}$ and $W \subseteq \mu(U, V)$. This still gives us a coding system for $\mathcal{W}$. One can explicitly construct the 2-tactic without making this simplification, by shrinking to elements of $\mathcal{W}$ where appropriate, but this makes the following harder to read. By Lemma 1.2.11, it is enough to show that Player II has a winning 2-tactic in $G_X^{\mathcal{W}}$. Since Player II has a winning strategy in G_X, Player II also has a winning strategy σ in $G_X^{\mathcal{W}}$, again by Lemma 1.2.11.

If Player II always uses μ while playing, then he can recover Player I's history of play. Suppose $\mathcal{C}$ is the set of descending finite sequences $(U_0, \ldots, U_n)$ of elements of $\mathcal{W}$, such that $U_1 \subseteq \mu(\star, U_0)$ and for each $i \in \omega$, $U_{i+2} \subseteq \mu(U_i, U_{i+1})$.

Claim 1.3.3. *The map λ with domain $\mathcal{C}$ defined by*

$$(U_0, \ldots, U_n) \mapsto (U_{n-1}, U_n)$$

when $n \geq 1$, and

$$(U_0) \mapsto (U_0)$$

is an injection.

Proof. We suppose the contrary and obtain a contradiction. Certainly if

$$\lambda(U_0, \ldots, U_n) = (U) = \lambda(U'_0, \ldots, U'_m),$$

then $n = m = 0$ and $(U_0) = (U'_0)$. Pick $(U_0, \ldots, U_n)$ with n minimal so that there exists a distinct $(U'_0, \ldots, U'_m)$ with $U'_{m-1} = U_{n-1}$ and $U'_m = U_n$, and fix such a tuple $(U'_0, \ldots, U'_m)$.

Now $n > 0$ by what we first noted, and from this we see also $m > 0$. For convenience, set $U_{-1} = \star$. Then $U_n = U'_m \subseteq \mu(U_{n-2}, U_{n-1}) \cap \mu\left(U'_{m-2}, U'_{m-1}\right)$ and hence

$$U_{n-2} = \nu(U_{n-1}, U_n) = \nu\left(U'_{m-1}, U'_m\right) = U'_{m-2}.$$

Now if $U_{n-2} = U'_{m-2} = \star$ then $n - 1 = m - 1 = 0$, and hence

$$(U_0, U_1) = \lambda((U_0, U_1)) = \lambda((U'_0, U'_1)) = (U'_0, U'_1)$$

a contradiction to our choices of tuples. But in the other case, $(U_0, \ldots, U_{n-1})$ is a shorter counterexample, which is also a contradiction. So λ is an injection. □

The above claim is the first step for turning a winning strategy into a winning 2-tactic. It says if we restrict where Player I plays in an appropriate way, then we might as well have access to all of Player I's history.

We continue our construction by building a slightly nicer winning strategy, namely one which respects the coding system.

Claim 1.3.4. *Player II has a winning strategy σ_μ in $G_X^{\mathcal{W}}$ such that if $\langle (U_i, V_i) \ : \ i \in \omega \rangle$ is σ_μ-compatible then $U_1 \subseteq \mu(\star, U_0)$ and for each $i \in \omega$, $U_{i+2} \subseteq \mu(U_i, U_{i+1})$.*

Proof. The idea here is that Player II essentially plays with σ, but instead replaces Player I's moves with open sets provided by μ. Throughout, take $U_{-1} = \star$.

We define σ_μ as a tree, and a transfer map $\mathcal{R}$ from σ_μ to σ simultaneously, by recursion. $\mathcal{R}$ will have the properties

1. if $(U_0, V_0, \ldots, U_n) \in \sigma_\mu$, then $\mathcal{R}(U_0, V_0, \ldots, U_n) = t^\frown(\mu(U_{n-1}, U_n))$ for some t.
2. if $(U_0, V_0, \ldots, V_n) \in \sigma_\mu$ then $\mathcal{R}(U_0, V_0, \ldots, V_n) = t^\frown(V_n)$ for some t.

Clearly if we build $\mathcal{R}$ with the above properties, then if $\langle (U_i, V_i) \ : \ i \in \omega \rangle$ is σ_μ-compatible, then for each i, there is a $t \in \sigma$ with

$$U_{i+1} \subseteq V_i = \sigma(t^\frown(\mu(U_{i-1}, U_i))) \subseteq \mu(U_{i-1}, U_i)$$

(recalling here $U_{-1} = \star$). Note that the equality in the middle is forced by the fact that $\mathcal{R}$ is a transfer map. This is exactly the property that we require. It is enough then to exhibit the construction of such a σ_μ and $\mathcal{R}$, and to check that such a σ_μ is winning.

First $\emptyset \in \sigma_\mu$, and $\mathcal{R}(\emptyset) = \emptyset$. For each $U_0 \in \mathcal{W}$, declare $(U_0) \in \sigma_\mu$ and $\mathcal{R}((U_0)) = (\mu(\star, U_0))$. If $s = (U_0, V_0, \ldots, V_n) \in \sigma_\mu$, for $n \geq 0$, then for any $U_{n+1} \in \mathcal{W}$ with $U_{n+1} \subseteq V_n$ we declare $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \sigma_\mu$ and put $\mathcal{R}((U_0, V_0, \ldots, V_n, U_{n+1})) = \mathcal{R}(s)^\frown(\mu(U_n, U_{n+1}))$. These last two parts of the construction ensure $\mathcal{R}$ satisfies property (1). Notice also that, provided we have satisfied (2) in our recursion so far, $\mathcal{R}$ is well-defined, that is $\mathcal{R}((U_0, V_0, \ldots, V_n, U_{n+1})) \in \sigma$, because $\mu(U_n, U_{n+1}) \subseteq U_{n+1} \subseteq V_n$.

If $s = (U_0, V_0, \ldots, U_n) \in \sigma_\mu$ and $\mathcal{R}(s)$ has been defined, then set $V_n = \sigma(\mathcal{R}(s))$, and declare $s^\frown(V_n) \in \sigma_\mu$, and $\mathcal{R}(s^\frown(V_n)) = \mathcal{R}(s)^\frown(V_n)$. This ensures $\mathcal{R}$ satisfies property (2). Note also if our recursion so far has satisfied (1), then $V_n \subseteq \mu(U_{n-1}, U_n) \subseteq U_n$, and so we may extend σ_μ in this way.

If $\langle (U_i, V_i) : i \in \omega \rangle$ is σ_μ-compatible, then $\mathcal{R}(s)$ is σ-compatible. By property (2) for $\mathcal{R}$ we have $\bigcap_{i\in\omega} V_i \neq \emptyset$. So σ_μ is a winning strategy for Player II in $G_X^{\mathcal{W}}$. □

It turns out that by respecting the coding system in this way, σ_μ is precisely the type of winning strategy we require.

Claim 1.3.5. *σ_μ is a 2-tactic.*

Proof. Suppose for a contradiction that σ_μ is not a 2-tactic. Then we may pick $s = (U_0, V_0, \ldots, U_{n+1}) \in \sigma_\mu$ with n minimal such that there is a $t = \left(U_0', V_0', \ldots, U_{m+1}'\right) \in \sigma_\mu$ with $U_n = U_m'$ and $U_{n+1} = U_{m+1}'$, but $\sigma_\mu(s) \neq \sigma_\mu(t)$; fix such a tuple $t \in \sigma_\mu$. Now, $(U_0, \ldots, U_{n+1}), \left(U_0', \ldots, U_{m+1}'\right) \in \mathcal{C}$ by Claim 1.3.4, and further we have $\lambda((U_0, \ldots, U_{n+1})) = \lambda\left(\left(U_0', \ldots, U_{m+1}'\right)\right)$ so $m = n$ and $U_i = U_i'$ for $i \leq n+1$ by Claim 1.3.3. But we also have

$$V_1 = \sigma_\mu(U_1) = V_1'$$

and by minimality of n that

$$V_i = \sigma_\mu(U_0, V_0, \ldots, U_{i-1}, V_{i-1}, U_i) = \sigma_\mu\left(U_0, V_0', \ldots, U_{i-1}, V_{i-1}', U_i\right) = V_i'$$

for $2 \leq i \leq n$. Hence $s = t$, a contradiction. □

The two previous claims give us that σ_μ is a winning 2-tactic for Player II in $G_X^{\mathcal{W}}$, and this completes the proof. □

1.4 Consequences of the coding lemma

We explore some of the consequences of Theorem 1.3.2. One such consequence is a translation of our coding lemma into a (more or less) equivalent theorem phrased in topological language. First let us make a definition.

Definition 1.4.1. *For a topological space X let the cellularity of X, denoted $\mathrm{c}(X)$, be the least cardinal κ such that there is no pairwise disjoint collection $\mathcal{D}$ of non-empty open subsets of X of cardinality κ. This is sometimes known as the Suslin number of X.*

Example 1.4.2. $\mathrm{c}(\mathbb{R}) = \omega_1$. *$X$ has the countable chain condition if and only if $\mathrm{c}(X) \leq \omega_1$.*

The following according to [1] has already been shown, but is unpublished. We show it follows from Theorem 1.3.2. I am unaware of how Galvin's proof procedes.

Theorem 1.4.3 (Galvin, unpublished)**.** *Suppose X is a Choquet space with a weak-base $\mathcal{W}$ with the property that for all $U \in \mathcal{W}$*

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathrm{c}(U).$$

Then Player II has a winning 2-tactic in G_X.

First we argue that we only need to consider parts of the space which have an infinite disjoint collection of open subsets. In fact, this is a useful observation in general for isolating the parts of a space of interest. We make use of the following proposition from [1].

Proposition 1.4.4 ([1], Proposition 41)**.** *Let X be a Choquet space such that every collection of pairwise disjoint open subsets is finite (i.e. $\mathrm{c}(X) \leq \aleph_0$). Then Player II has a winning 1-tactic in G_X.*

Let us state formally the procedure that we will use to isolate the part of a space of interest.

Lemma 1.4.5. *Suppose X is a topological space. There is an open subspace $X_{\mathcal{F}}$ of X such that*

1. *if X is Choquet, and if Player II has a winning k-tactic in $G_{X_{\mathcal{F}}}$ then Player II has a winning k-tactic in G_X;*

2. *if U is non-empty and open in $X_{\mathcal{F}}$ then* $\mathsf{c}(U) > \aleph_0$.

Note also that if X is Choquet, then so is $X_{\mathcal{F}}$.

Proof. Suppose (X, τ) is a topological space. Let $\mathcal{F}$ be maximal (using Zorn's Lemma) in the set

$$\{A \in \mathcal{P}(\tau \setminus \{\emptyset\}) : A \text{ is a disjoint collection of open sets } U \text{ with } \mathsf{c}(U) \leq \aleph_0\}$$

partially ordered by $\subseteq$.

Take $X_{\mathcal{F}} = X \setminus \overline{\bigcup \mathcal{F}}$. Note that if U is a non-empty open subset of $X_{\mathcal{F}}$, then $\mathsf{c}(U) > \aleph_0$ by maximality of $\mathcal{F}$, and hence we have (2).

Now suppose X is Choquet. (1) is clear if $\mathcal{F} = \emptyset$, so suppose otherwise. By the above Proposition, if $U \in \mathcal{F}$ then Player II has a winning 1-tactic in G_U. Hence by Lemma 1.2.13 and since $\bigcup \mathcal{F}$ is homeomorphic to $\bigoplus_{U \in \mathcal{F}} U$, Player II has a winning 1-tactic in $G_{(\bigcup \mathcal{F})}$.

Now suppose that Player II has a winning k-tactic in $G_{X_{\mathcal{F}}}$. Then Player II certainly has a winning k-tactic in $G_{((\bigcup \mathcal{F}) \oplus X_{\mathcal{F}})}$ by Lemma 1.2.13. But $(\bigcup \mathcal{F}) \oplus X_{\mathcal{F}}$ is homeomorphic to $(\bigcup \mathcal{F}) \cup X_{\mathcal{F}}$, which in turn is dense in X. So Player II has a winning k-tactic in G_X by Lemma 1.2.3. This gives us (1). □

Remark 1.4.6. *Notice in the above that if $\bigcup \mathcal{F}$ is dense in X then Player II has a winning 1-tactic in G_X by Lemma 1.2.3.*

With this in mind, we establish Theorem 1.4.3 using Theorem 1.3.2 and the following applied to $X_{\mathcal{F}}$ when $X_{\mathcal{F}} \neq \emptyset$. Note that if X has a weak-base $\mathcal{W}$ with the property that for all $U \in \mathcal{W}$, $|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U)$, then so does $X_{\mathcal{F}}$, namely the $X_{\mathcal{F}}$-weak-base $\{W \in \mathcal{W} \colon W \subseteq X_{\mathcal{F}}\}$.

Theorem 1.4.7. *Suppose X is a topological space with a weak-base $\mathcal{W}$ with the property that for all $U \in \mathcal{W}$,* $\mathsf{c}(U) \geq \aleph_0$ *and*

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U).$$

Then $\mathcal{W}$ admits a coding system (μ, ν).

Proof. For each $U \in \mathcal{W}$, choose a disjoint collection $\mathcal{C}_U$ of non-empty open subsets of U, with $|\mathcal{C}_U| = |\{W \in \mathcal{W} \colon U \subseteq W\} \cup \{\star\}|$ (here we use $\mathsf{c}(U) \geq \aleph_0$). Then, pick an injection

$$\phi_U \colon \{W \in \mathcal{W} \colon U \subseteq W\} \cup \{\star\} \to \mathcal{C}_U.$$

We may then define, for $U_1, U_2 \in \mathcal{W}$

$$\mu(\star, U_1) = \phi_{U_1}(\star)$$

$$\mu(U_1, U_2) = \begin{cases} \phi_{U_2}(U_1) & \text{when } U_2 \subseteq U_1 \\ X & \text{otherwise} \end{cases}$$

It does not matter how μ is defined on non-descending pairs, we use X here. Then for Z non-empty and open and $U \in \mathcal{W}$, let

$$\nu(U, Z) = \begin{cases} \phi_U^{-1}(Z') & \text{if } Z' \text{ is the (necessarily unique) element of } \mathcal{C}_U \text{ with } Z \subseteq Z' \\ U & \text{otherwise} \end{cases}$$

Then (μ, ν) is a coding system. To see this, suppose $U \in \mathcal{W}_\star$, $V \in \mathcal{W}$ and

$$(U \neq \star \to V \subseteq U).$$

Suppose Z is non-empty and open with $Z \subseteq \mu(U, V)$. Then $Z \subseteq \phi_V(U)$, and $\nu(V, Z) = U$. □

We now prove Theorem 1.4.3, using the above and previous results to restrict to the appropriate part of the space.

Proof of Theorem 1.4.3. Suppose $\mathcal{W}$ is a weak-base for X such that for all $U \in \mathcal{W}$

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U).$$

Let $X_{\mathcal{F}}$ be as in Lemma 1.4.5. By Lemma 1.4.5 it suffices to show Player II has a winning 2-tactic in $G_{X_{\mathcal{F}}}$. Let

$$\mathcal{V} = \{W \in \mathcal{W} \colon W \subseteq X_{\mathcal{F}}\}$$

so that $\mathcal{V}$ is a weak-base for $X_{\mathcal{F}}$. Furthermore, for $U \in \mathcal{V}$,

$$|\{V \in \mathcal{V} \colon U \subseteq V\}| \leq |\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U),$$

so by Theorem 1.4.7, $\mathcal{V}$ admits a coding system. Hence by Theorem 1.3.2, Player II has a winning 2-tactic in $G_{X_{\mathcal{F}}}$, and hence in G_X. □

Having proved Theorem 1.4.3, we prove a consequence which is analogous to a result of White. White ([34]) showed the following.

Theorem 1.4.8 (White, [34])**.** *If X is a Choquet space, and has a σ-disjoint weak-base, then Player II has a winning 1-tactic in G_X.*

We prove an analogous result for 2-tactics for higher cardinalities. First we make some definitions.

Definition 1.4.9. *Let X be a topological space. A collection $\mathcal{A}$ of sets is said to be locally-κ for a cardinal κ if and only if for every $x \in X$ there is an open U with $x \in U$ and*

$$|\{A \in \mathcal{A} \colon U \cap A \neq \emptyset\}| \leq \kappa.$$

By locally-countable we mean locally-$\aleph_0$.

Let $\mathcal{P}$ be a property of collections of sets. A collection $\mathcal{A}$ of sets is said to be σ_κ-$\mathcal{P}$ for a cardinal κ if and only if $\mathcal{A} = \bigcup_{\alpha<\kappa} \mathcal{A}_\alpha$ where $\mathcal{A}_\alpha$ has property $\mathcal{P}$. By σ-$\mathcal{P}$ we mean $\sigma_{\aleph_0}$-$\mathcal{P}$, as in the normal sense.

Theorem 1.4.10. *Suppose X is a Choquet space with the property that every non-empty open set U satisfies $\mathsf{c}(U) > \kappa$, where κ is an infinite cardinal. Suppose X has a σ_κ-disjoint weak-base $\mathcal{W}$. Then Player II has a winning 2-tactic in G_X.*

Proof. First write $\mathcal{W} = \bigcup_{\alpha<\kappa} \mathcal{W}_\alpha$, where each $\mathcal{W}_\alpha$ is a disjoint collection of open sets. Let U be an element of $\mathcal{W}$. U has at most one superset in each $\mathcal{W}_\alpha$ by disjointness. Hence

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| \leq \kappa < \mathsf{c}(U).$$

Now apply Theorem 1.4.3. □

It is natural to ask if weakening σ-disjoint to σ-locally-countable in White's result gives anything new. The following, a modified proof of [12] Corollary 2.4 with a statement to suit our own needs, says that we do not, even for larger cardinals.

Proposition 1.4.11 ([12], Corollary 2.4)**.** *Suppose X has a σ_κ-locally-κ weak-base. Then X has a σ_κ-disjoint weak-base.*

Proof. Suppose $\mathcal{W} = \bigcup_{\alpha<\kappa} \mathcal{W}_\alpha$ is a weak-base where each $\mathcal{W}_\alpha$ is a locally-κ collection. For each $\alpha < \kappa$ let $\mathcal{V}_\alpha$ be a maximal disjoint collection of non-empty open sets such that every $V \in \mathcal{V}_\alpha$ has non-empty intersection with at most κ-many elements of $\mathcal{W}_\alpha$, $\langle W_\beta^V : \beta < \kappa \rangle$ say. Now set

$$Z_\alpha^\beta = \{V \cap W_\beta^V : V \in \mathcal{V}_\alpha\}$$

which is a disjoint collection since $\mathcal{V}_\alpha$ is. Now suppose U is non-empty and open in X. Find $\alpha < \kappa$ and $W \in \mathcal{W}_\alpha$ with $W \subseteq U$. Then by maximality of $\mathcal{V}_\alpha$ there is a $V \in \mathcal{V}_\alpha$ with $V \cap W \neq \emptyset$. So $W = W_\beta^V$ for some $\beta < \kappa$. But $V \cap W_\beta^V \in Z_\alpha^\beta$ and $V \cap W_\beta^V \subseteq U$. So $\bigcup_{\alpha,\beta<\kappa} Z_\alpha^\beta$ is a σ_κ-disjoint weak-base for X. □

We have seen that Galvin's result follows from 1.3.2; we can also ask if 1.3.2 follows from Galvin's result. In fact the hypotheses to both Theorems are almost equivalent.

Proposition 1.4.12. *Suppose $\mathcal{W}$ is a weak-base for a topological space (X, τ) which admits a coding system (μ, ν). Then for each $U \in \mathcal{W}$,*

$$|\{W \in \mathcal{W} : U \subseteq W\}| < \mathfrak{c}(U).$$

Proof. Let $U \in \mathcal{W}$. Let $W_1, W_2 \in \{W \in \mathcal{W} : U \subseteq W\}$, and suppose $V = \mu(W_1, U) \cap \mu(W_2, U) \neq \emptyset$. Then $V \subseteq \mu(W_1, U), \mu(W_2, U)$, so

$$W_1 = \nu(U, V) = W_2.$$

Hence $\{\mu(W, U) : W \in \mathcal{W} \text{ and } U \subseteq W\}$ is a disjoint collection of open subsets of U of cardinality $|\{W \in \mathcal{W} : U \subseteq W\}|$. So

$$|\{W \in \mathcal{W} : U \subseteq W\}| < \mathfrak{c}(U)$$

as required. □

To summarise, we have a topological characterisation of coding systems on the parts of a space with large enough cellularity.

Theorem 1.4.13. *Suppose X is a topological space with a weak-base $\mathcal{W}$ such that for all $U \in \mathcal{W}$, $\mathsf{c}(U) \geq \aleph_0$. $\mathcal{W}$ admits a coding system if and only if for each $U \in \mathcal{W}$*

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U).$$

Proof. By the above Proposition and Theorem 1.4.7. □

In a topological space where all open sets have cellularity at least $\aleph_0$, we will consider having a coding system to be equivalent to having a weak-base $\mathcal{W}$ such that for all $U \in \mathcal{W}$, $|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U)$.

The following specialises the above results in more classical language. It is useful for particularly nice spaces, and in particular Debs' Tychonoff space in section 6. Recall that for a topological space X, $w(X)$ is the weight of X, the smallest cardinality of a base for the topology on X.

Theorem 1.4.14. *Suppose X is a topological space such that*

$$w(X) < \mathsf{c}(U)$$

for every non-empty open subset U of X (this is equivalent to saying $\mathsf{c}(U) = w(X)^+$). Then

1. *If $\aleph_0 \leq \mathsf{c}(U)$ for every non-empty open subset U of X, then X (has a weak-base which) admits a coding system.*

2. *If X is Choquet then Player II has a winning 2-tactic in G_X.*

Proof. Take a base $\mathcal{B}$ for X with $|\mathcal{B}| = w(X)$. Note that a base is in particular a weak-base.

1. This follows from Theorem 1.4.7.

2. Follows from (1) applied to $X_{\mathcal{F}}$, where $X_{\mathcal{F}}$ is as in Lemma 1.4.5, and the coding lemma Theorem 1.3.2. Alternatively, and more directly, apply Theorem 1.4.3. □

The following arises as a question about the nature of 2-tactics.

Question 1.4.15. *Suppose X is a topological space such that for all non-empty open sets U, $\mathsf{c}(U) \geq \aleph_0$. Is it the case that if Player II has a winning 2-tactic in G_X then either Player II has a winning 1-tactic in G_X, or X has a weak-base admitting a coding system?*

That is, do all 2-tactics arise from a 1-tactic or the ability to encode memory? We explore this further after reconstructing Debs' counterexample spaces.

1.5 Debs' counterexamples

Here we will reconstruct Debs' original counterexample spaces. The spaces themselves are easy enough to describe, and appear to have originally been described in a seminar given by Debs in 1984. However, the proofs appear to have never been published, and the literature only contains descriptions of these spaces referenced from the seminar. Debs published details of a Tychonoff counterexample space in 1985 ([10]), which, while having stronger separation properties than the original non-regular spaces, is less intuitive as a counterexample. First let us make some definitions.

Definition 1.5.1. *For a set A, let $\left([A]^{\leq\aleph_0}, \subseteq\right)$ be the σ-ideal of countable subsets of A. For a topological space X, let $(\mathrm{MGR}(X), \subseteq)$ be the σ-ideal of meagre subsets of X.*

Let $\tau_{\aleph_0}$ be the topology on $\mathbb{R}$ induced by the base

$$\left\{(a,b) \setminus C \colon C \in [\mathbb{R}]^{\leq\aleph_0}\right\}.$$

Let us call $(\mathbb{R}, \tau_{\aleph_0})$ "Debs' countable space". Our description of this space is taken from [29], pg235. Similarly, let τ_{MGR} be the topology on $\mathbb{R}$ induced by the base

$$\{(a,b) \setminus M \colon M \in \mathrm{MGR}(\mathbb{R})\}$$

and let $(\mathbb{R}, \tau_{\mathrm{MGR}})$ be referred to as "Debs' meagre space". Our description of this space is taken from [1].

Theorem 1.5.2. $(\mathbb{R}, \tau_{\aleph_0})$ *is Choquet.*

Proof. For each non-empty open U in $(\mathbb{R}, \tau_{\aleph_0})$, fix nonempty $(a_U, b_U) \setminus C_U$ so that C_U is countable and $(a_U, b_U) \setminus C_U \subseteq U$. For each countable subset C of $\mathbb{R}$, pick a surjection $\phi_C \colon \omega \to C$. We refer to open subsets of $\mathbb{R}$ in the Euclidean topology as Euclidean-open subsets of $\mathbb{R}$. Fix a complete Euclidean metric d on $\mathbb{R}$, so that we may speak of the diameter of a bounded subset of $\mathbb{R}$.

Suppose Player I has previously played $U_0, \ldots, U_n$. Let V_n' be a non-empty Euclidean-open subset of $\mathbb{R}$ with

$$\overline{V_n'}^d \subseteq (a_{U_n}, b_{U_n})$$

(where closure here is taken in the Euclidean topology; however, the $\tau_{\aleph_0}$-closure operator agrees with the d-closure on V_n'), and

$$\operatorname{diam}(V_n') < \frac{1}{2^n}$$

and so that

$$\left\{\phi_{C_{U_i}}(j) : i, j \leq n\right\} \cap V_n' = \emptyset.$$

Informally, V_n' avoids the first n points of each C_{U_i}, $i \leq n$. Then define

$$\sigma(U_0, V_0, \ldots, U_n) = V_n = V_n' \setminus C_{U_n}.$$

Whilst we have not constructed σ formally as a tree here, it is clear from this description how to go about doing so.

If $\langle (U_i, V_i) \ : \ i \in \omega \rangle$ is σ-compatible, pick for each $i \in \omega$ a point x_i in V_i to obtain a d-Cauchy sequence, which hence d-converges to some $x \in \mathbb{R}$. Now $x \in \overline{V_{i+1}}^d \subseteq (a_{U_i}, b_{U_i})$, so

$$x \in \bigcap_{i \in \omega} (a_{U_i}, b_{U_i}).$$

If $x \notin \bigcap_{i \in \omega} U_i$, then there is some $i \in \omega$ with $x \notin U_i$. As $x \in (a_{U_i}, b_{U_i})$, it must be the case that $x \in C_{U_i}$. Hence, there must be a $j \in \omega$ with $\phi_{C_{U_i}}(j) = x$. But then, letting $n = \max(i, j)$ in the notation above,

$$x \in \left(a_{U_{n+1}}, b_{U_{n+1}}\right) \subseteq V_n'$$

and

$$x = \phi_{C_{U_i}}(j) \notin V_n'$$

a contradiction. Hence $x \in \bigcap_{i\in\omega} U_i = \bigcap_{i\in\omega} V_i$, and so σ is a winning strategy for Player II in G_X. □

One can show $(\mathbb{R}, \tau_{\text{MGR}})$ is Choquet in a similar fashion.

The following is an adaptation of the proof in [10], where it is shown that Player II does not have a winning 1-tactic in the Choquet game on Debs' Tychonoff counterexample space. We borrow much of the notation.

Theorem 1.5.3. *Suppose $\mathcal{I}$ is a σ-ideal on $\mathcal{P}(\mathbb{R})$ which contains all singletons, and contains no non-empty open sets (so $[\mathbb{R}]^{\leq\aleph_0}$ and $\text{MGR}(\mathbb{R})$ are examples of such a σ-ideal). Let $\tau_{\mathcal{I}}$ be the topology generated by the base*

$$\{(a,b) \setminus I : a, b \in \mathbb{R},\ I \in \mathcal{I}\}$$

Then Player II has no winning 1-tactic in $G_{(\mathbb{R},\tau_{\mathcal{I}})}$.

Proof. The proof works by supposing Player II has a winning 1-tactic in $G_{(\mathbb{R},\tau_{\mathcal{I}})}$, and then constructing a sequence compatible with the 1-tactic which fails to win for Player II, to get a contradiction.

First however, let us introduce some notation. Let $\mathcal{J}$ denote the collection of non-empty bounded open intervals with rational endpoints. For $Z \in \mathcal{J}$ and $I \in \mathcal{I}$ let

$$V(Z, I) = Z \setminus I.$$

$\mathcal{B} = \{V(Z,I) : Z \in \mathcal{J},\ I \in \mathcal{I}\}$ is then a base of non-empty open sets for the topology $\tau_{\mathcal{I}}$. Let d be a complete metric which induces the Euclidean topology on $\mathbb{R}$. Let us partially order $\mathcal{J} \times \mathcal{I}$ via $(Z, I) \preceq (Z', I')$ if and only if $I' \subseteq I$ and $Z \subseteq Z'$ (which entails that $V(Z,I) \subseteq V(Z',I')$). For each $Z \in \mathcal{J}$, let $\frac{1}{2}Z$ be another element of $\mathcal{J}$ such that

$$\overline{\tfrac{1}{2}Z}^{d} \subseteq Z$$

and

$$\text{diam}\left(\frac{1}{2}Z\right) < \frac{1}{2}\text{diam}\,(Z)$$

where diameters here are taken with respect to d.

Suppose Player II has a winning 1-tactic in $G_{(\mathbb{R},\tau_{\mathcal{I}})}$; then Player II has a winning 1-tactic μ in $G^{\mathcal{B}}_{(\mathbb{R},\tau_{\mathcal{I}})}$. Define $g\colon \mathcal{J}\times\mathcal{I}\to\mathcal{J}\times\mathcal{I}$ so that if $g\,(Z,I)=(Z',I')$ then

$$\mu\left(V\left(\frac{1}{2}Z,I\right)\right)=V\,(Z',I')\,.$$

Let us say a sequence $\langle (Z_i,I_i)\ :\ i\in\omega\rangle$ in $\mathcal{J}\times\mathcal{I}$ is restricted by g if and only if $(Z_{i+1},I_{i+1})\preceq g\,(Z_i,I_i)$ for each $i\in\omega$.

Suppose $\langle (Z_i,I_i)\ :\ i\in\omega\rangle$ is restricted by g. Then since $Z_{i+1}\subseteq\frac{1}{2}Z_i$ for each $i\in\omega$, $\bigcap_{i\in\omega}Z_i$ is a singleton (using completeness of $(\mathbb{R},d)$). Further, since

$$\left\langle \left(V\left(\frac{1}{2}Z_i,I_i\right),\mu\left(V\left(\frac{1}{2}Z_i,I_i\right)\right)\right)\ :\ i\in\omega\right\rangle$$

is μ-compatible, $\bigcap_{i\in\omega}(Z_i\setminus I_i)\neq\emptyset$, and hence must be the singleton $\bigcap_{i\in\omega}Z_i$. Hence we have shown that if $\langle (Z_i,I_i)\ :\ i\in\omega\rangle$ is restricted by g, then

$$\left(\bigcap_{i\in\omega}Z_i\right)\cap\left(\bigcup_{i\in\omega}I_i\right)=\emptyset.$$

We now construct a sequence in $\mathcal{J}$ which our strategy resorts to using too often. One can think of the below as saying that Z_{i+1} occurs densely as a response to Z_i.

Claim 1.5.4. *There is a sequence $\langle Z_i\ :\ i\in\omega\rangle$ in $\mathcal{J}$ with the property that for any $I\in\mathcal{I}$ and $i\in\omega$ there is a $K\in\mathcal{I}$ with $I\subseteq K$ and so that*

$$(\pi_1\circ g)\,(Z_i,K)=Z_{i+1}$$

where $\pi_1\colon\mathcal{J}\times\mathcal{I}\to\mathcal{J}$ is the coordinate projection.

Proof. Enumerate $\mathcal{J}=\{A_i\colon i\in\omega\}$. Let Z_0 be any element of $\mathcal{J}$. We define our sequence recursively.

Suppose we have already defined $Z_0,\dots,Z_n$. Suppose for each A_i there is an $I_i\in\mathcal{I}$ such that for all $K\in\mathcal{I}$ with $I_i\subseteq K$, $(\pi_1\circ g)\,(Z_n,I_i)\neq A_i$. Let $I=\bigcup_{i\in\omega}I_i$, which is an element of $\mathcal{I}$ since $\mathcal{I}$ is a σ-ideal. Then for all $i\in\omega$, $I_i\subseteq I$ and

$$(\pi_1\circ g)\,(Z_n,I)\neq A_i,$$

a contradiction. Hence there must be some Z_{n+1} to extend our sequence in the way that we require. □

Let $\langle Z_i : i \in \omega \rangle$ be as in the claim above. Pick any $K \in \mathcal{I}$. By recursion, again using the claim above, we can find a sequence $\langle I_i : i \in \omega \rangle$ in $\mathcal{I}$ where $K \subseteq I_0$, and for each i, $(\pi_1 \circ g)(Z_i, I_i) = Z_{i+1}$ and $(\pi_2 \circ g)(Z_i, I_i) \subseteq I_{i+1}$ (where π_2 is the projection to the second coordinate). Then $\langle (Z_i, I_i) : i \in \omega \rangle$ is restricted by g, and hence $\bigcap_{i\in\omega} Z_i = \{a\}$ for some $a \in \mathbb{R}$. Note that a does not depend on K or our sequence $\langle I_i : i \in \omega \rangle$. In particular, we may take $K = \{a\}$ and then construct the I_i for $i \in \omega$. Then $a \in \bigcup_{i\in\omega} I_i$; but this contradicts $\left(\bigcap_{i\in\omega} Z_i\right) \cap \left(\bigcup_{i\in\omega} I_i\right) = \emptyset$. Hence Player II has no winning 1-tactic in $G_{(\mathbb{R},\tau_{\mathcal{I}})}$. □

The following summarises the consequences of interest from the above.

Corollary 1.5.5. *Player II has no winning* 1*-tactic in* $G_{\left(\mathbb{R},\tau_{\aleph_0}\right)}$*. Player II has no winning* 1*-tactic in* $G_{(\mathbb{R},\tau_{\text{MGR}})}$*.*

Let us now focus on showing that Player II has a winning 2-tactic in $G_{\left(\mathbb{R},\tau_{\aleph_0}\right)}$. For this we introduce a combinatorial device which allows us to talk about coding systems in $(\mathbb{R}, \tau_{\aleph_0})$ in terms of the ideal $[\mathbb{R}]^{\leq\aleph_0}$.

1.5.1 Pseudo-Lusin sets and 2-tactics in Debs' spaces

Definition 1.5.6. *Let* $(\mathbb{P}, \leq)$ *be a partial order. A subset* $\mathcal{C}$ *of* $\mathbb{P}$ *is said to be cofinal in* $(\mathbb{P}, \leq)$ *if and only if*

$$\forall p \in \mathbb{P}\ \exists q \in \mathcal{C}\ (p \leq q)\,.$$

The following combinatorial device allows us to isolate the information we need in an ideal to build coding systems in modified topologies on the reals, and in particular to build a coding system in Debs' countable space. Typically the following definition is of interest for directed partial orders, but it makes sense without this additional condition.

Definition 1.5.7. *Let $(\mathbb{P}, \leq)$ be a partial order and λ, κ cardinals. A subset L of $\mathbb{P}$ is said to be a (λ, κ) pseudo-Lusin set if and only if $|L| = \kappa$ and for all $p \in \mathbb{P}$*

$$|\{q \in L : q \leq p\}| < \lambda.$$

Obviously (λ, κ) pseudo-Lusin sets are only of interest when $\lambda \leq \kappa$. Pseudo-Lusin sets seem to arise naturally out of trying to find coding systems with respect to ideals; indeed the authors of [1] discuss them in their argument to show Player II has a winning 2-tactic in $G_{(\mathbb{R}, \tau_{\text{MGR}})}$.

We first make an easy observation that allows us to make liberal use of Theorem 1.4.13.

Observation 1.5.8. *Suppose $(\mathcal{I}, \subseteq)$ is a σ-ideal of subsets of $\mathbb{R}$ which contains no non-empty Euclidean open subsets of $\mathbb{R}$. Let $\tau_{\mathcal{I}}$ be the topology on $\mathbb{R}$ generated by the base $\{(a, b) \setminus I : a, b \in \mathbb{R},\ I \in \mathcal{I}\}$. If U is non-empty and open in $(\mathbb{R}, \tau_{\mathcal{I}})$ then* $\mathsf{c}(U) = \omega_1$.

Proof. Since $\mathcal{I}$ contains no non-empty Euclidean open subsets of $\mathbb{R}$,

$$((a, b) \setminus I) \cap ((a', b') \setminus I') = \emptyset \quad \text{if and only if} \quad (a, b) \cap (a', b') = \emptyset,$$

where $a, b, a', b' \in \mathbb{R}$ and $I, I' \in \mathcal{I}$. □

The following makes clear the link between cofinal pseudo-Lusin sets and coding systems.

Lemma 1.5.9. *Suppose $(\mathcal{I}, \subseteq)$ is a σ-ideal of subsets of $\mathbb{R}$ with a cofinal (ω_1, κ) pseudo-Lusin set L for some cardinal κ. Suppose further that $\mathcal{I}$ contains no non-empty Euclidean open subsets of $\mathbb{R}$. Then the topology $\tau_{\mathcal{I}}$ on $\mathbb{R}$ generated by the base*

$$\{(a, b) \setminus I : a, b \in \mathbb{R},\ I \in \mathcal{I}\}$$

admits a coding system.

Proof. Let $\mathcal{W}$ be the set

$$\{(a, b) \setminus J : a, b \in \mathbb{Q},\ a < b,\ J \in L\}$$

and note that since L is cofinal in $(\mathcal{I}, \subseteq)$, $\mathcal{W}$ is a weak-base for $(\mathbb{R}, \tau_{\mathcal{I}})$.

Now if $a, b \in \mathbb{R}$, $a < b$, and $I \in \mathcal{I}$ then

$$|\{W \in \mathcal{W} \colon (a,b) \setminus I \subseteq W\}| \leq |(\mathbb{Q} \times \mathbb{Q}) \times \{J \in L \colon J \subseteq I\}| \leq \aleph_0$$

since $\{J \in L \colon J \subseteq I\}$ is countable because L is an (ω_1, κ) pseudo-Lusin set. Hence by the above observation and Theorem 1.4.7, $(\mathbb{R}, \tau_{\mathcal{I}})$ admits a coding system. □

For $(\mathbb{R}, \tau_{\aleph_0})$ we must ask whether $[\mathbb{R}]^{\leq \aleph_0}$ has a cofinal $\left(\omega_1, 2^{\aleph_0}\right)$ pseudo-Lusin set. The following argument is well known, and is in fact just a specific case of the argument in [1] Proposition 1.

Proposition 1.5.10. *$\left([\mathbb{R}]^{\leq \aleph_0}, \subseteq\right)$ has a cofinal $\left(\omega_1, 2^{\aleph_0}\right)$ pseudo-Lusin set L.*

Proof. Let $\mathcal{Q}$ be a partition of $\mathbb{R}$ into $2^{\aleph_0}$ countable subsets of $\mathbb{R}$. Note $\left|[\mathbb{R}]^{\leq \aleph_0}\right| = 2^{\aleph_0}$. Enumerate

$$[\mathbb{R}]^{\leq \aleph_0} = \left\{p_\alpha \colon \alpha < 2^{\aleph_0}\right\}$$
$$\mathcal{Q} = \left\{q_\alpha \colon \alpha < 2^{\aleph_0}\right\}$$

and let $L = \left\{(p_\alpha \cup q_\alpha) : \alpha < 2^{\aleph_0}\right\}$. Clearly L has cardinality $2^{\aleph_0}$. L is cofinal in $[\mathbb{R}]^{\leq \aleph_0}$ because if $C \in [\mathbb{R}]^{\leq \aleph_0}$ then $C = p_\alpha$ for some α, in which case $C \subseteq (p_\alpha \cup q_\alpha)$. Finally, if $C \in [\mathbb{R}]^{\leq \aleph_0}$

$$|\{(p_\alpha \cup q_\alpha) : (p_\alpha \cup q_\alpha) \subseteq C\}| \leq |\{q \in \mathcal{Q} \colon q \subseteq C\}| = \aleph_0$$

since elements of $\mathcal{Q}$ are disjoint. Hence L is a cofinal $\left(\omega_1, 2^{\aleph_0}\right)$ pseudo-Lusin set for $\left([\mathbb{R}]^{\leq \aleph_0}, \subseteq\right)$. □

In particular we have the following.

Corollary 1.5.11. *Player II has a winning 2-tactic in $G_{\left(\mathbb{R}, \tau_{\aleph_0}\right)}$.*

Proof. By the previous Proposition and Lemma 1.5.9, $(\mathbb{R}, \tau_{\aleph_0})$ admits a coding system. Since $(\mathbb{R}, \tau_{\aleph_0})$ is Choquet, Player II has a winning 2-tactic in $G_{\left(\mathbb{R}, \tau_{\aleph_0}\right)}$. □

We may ask whether $(\mathrm{MGR}(\mathbb{R}), \subseteq)$ has a cofinal $\left(\omega_1, 2^{\aleph_0}\right)$ pseudo-Lusin set. The following is certainly true.

Proposition 1.5.12 (CH). $(\mathrm{MGR}(\mathbb{R}), \subseteq)$ *has a cofinal* $\left(\omega_1, 2^{\aleph_0}\right)$ *pseudo-Lusin set.*

Proof. If M is meagre in the Euclidean topology on $\mathbb{R}$, then we can write

$$M = \bigcup_{n \in \omega} F_n$$

where F_n is nowhere dense for each $n \in \omega$. Then $\overline{F_n}$ is nowhere dense and $\bigcup_{n\in\omega} \overline{F_n}$ is a meagre F_σ which contains M. So the set $\mathcal{F}$ of meagre F_σ subsets of $\mathbb{R}$ is cofinal in $(\mathrm{MGR}(\mathbb{R}), \subseteq)$. Since $\mathbb{R}$ has $2^{\aleph_0}$ F_σ subsets, under **CH** we have $|\mathcal{F}| = \omega_1$.

Let us enumerate $\mathcal{F}$, $\mathcal{F} = \{M_\alpha : \alpha < \omega_1\}$, and from this define

$$L = \left\{ \bigcup_{\alpha<\beta} M_\alpha : \beta < \omega_1 \right\}.$$

Then $L \subseteq \mathrm{MGR}(\mathbb{R})$ and for each $\beta < \omega_1$, $M_\beta \subseteq \bigcup_{\alpha<\beta+1} M_\alpha$. So L is cofinal in $(\mathrm{MGR}(\mathbb{R}), \subseteq)$, since $\mathcal{F}$ was. Further $|L| = \omega_1 = 2^{\aleph_0}$.

Finally, let us suppose that $M \in \mathrm{MGR}(\mathbb{R})$. We show M has only countably many subsets in L. $M \neq \mathbb{R}$, so pick $x \in \mathbb{R}$ with $x \notin M$. Then $\{x\}$ is a meagre F_σ subset of $\mathbb{R}$, so there is $\gamma < \omega_1$ with $M_\gamma = \{x\}$. Then for all $\beta < \omega_1$

$$\left(\beta > \gamma \to \bigcup_{\alpha<\beta} M_\alpha \not\subseteq M \right).$$

Hence if $\beta < \omega_1$ and $\bigcup_{\alpha<\beta} M_\alpha \subseteq M$ then $\beta \leq \gamma$. γ is a countable ordinal, so in other words we have

$$\{N \in L : N \subseteq M\}$$

is countable. Hence L is a cofinal $\left(\omega_1, 2^{\aleph_0}\right)$ pseudo-Lusin set for $(\mathrm{MGR}(\mathbb{R}), \subseteq)$. □

Corollary 1.5.13 (CH). $(\mathbb{R}, \tau_{\mathrm{MGR}})$ *admits a coding system. In particular, Player II has a winning* 2*-tactic in* $G_{(\mathbb{R},\tau_{\mathrm{MGR}})}$.

However, it is also consistent with **ZFC** that $(\mathrm{MGR}(\mathbb{R}), \subseteq)$ does not have a cofinal (ω_1, κ) pseudo-Lusin set for $\kappa > \aleph_0$ ($\kappa > \aleph_0$ is necessary because a countable union of meagre sets is meagre, hence no countable set can be cofinal in $\mathrm{MGR}(\mathbb{R})$). To see this, we begin with the following well known consequence of **MA** (see [18], Chapter 2, Section 2, in particular Theorem 2.20 pg58).

Theorem 1.5.14 (MA). *Suppose $\kappa < 2^{\aleph_0}$, and that $\langle M_\alpha : \alpha < \kappa \rangle$ is a sequence of meagre subsets of $\mathbb{R}$. Then $\bigcup_{\alpha<\kappa} M_\alpha$ is meagre.*

In particular, under $\mathbf{MA} + \neg\mathbf{CH}$ a union of ω_1 meagre subsets of $\mathbb{R}$ is meagre. In other words, $\operatorname{add}(\operatorname{MGR}(\mathbb{R}), \subseteq) > \omega_1$, where $\operatorname{add}(\mathbb{P}, \leq)$ is the additivity of $(\mathbb{P}, \leq)$, the least cardinality of a subset of $\mathbb{P}$ which is unbounded in $(\mathbb{P}, \leq)$.

The following is noted in [1].

Lemma 1.5.15. *Suppose $(\mathbb{P}, \leq)$ has a (λ, κ) pseudo-Lusin set L, where $\lambda \leq \kappa$. Then $\lambda \geq \operatorname{add}(\mathbb{P}, \leq)$.*

Proof. Pick $\mathcal{D} \subseteq L$ with $|\mathcal{D}| = \lambda$. If $\mathcal{D}$ were bounded by p in $(\mathbb{P}, \leq)$, then

$$\mathcal{D} \subseteq \{q \in L : q \leq p\}$$

a contradiction, since $|\{q \in L : q \leq p\}| < \lambda$. Hence $\mathcal{D}$ is unbounded in $(\mathbb{P}, \leq)$, and so $\lambda \geq \operatorname{add}(\mathbb{P}, \leq)$. □

Corollary 1.5.16 ($\mathbf{MA} + \neg\mathbf{CH}$, or, $\operatorname{add}(\operatorname{MGR}(\mathbb{R}), \subseteq) > \omega_1$). *For any $\kappa > \aleph_0$, $(\operatorname{MGR}(\mathbb{R}), \subseteq)$ has no (ω_1, κ) pseudo-Lusin set.*

This of course raises questions about coding systems in $(\mathbb{R}, \tau_{\text{MGR}})$, with respect to Question 1.4.15.

1.6 Debs' Tychonoff example

In [10], Debs provides an example of a Tychonoff Choquet space in which Player II does not have a winning 1-tactic, but does have a winning 2-tactic. The language of coding systems clarifies the proof that Player II has a winning 2-tactic, and makes the progression from Debs' countable space, $(\mathbb{R}, \tau_{\aleph_0})$, to a Tychonoff example more obvious. We present the more abstract and modular proof below. First we begin with a definition of the space.

Let $\mathcal{D} = \{C \subseteq \mathbb{R} \setminus \{0\} : C \neq \emptyset \text{ and } C \text{ is countable}\}$. Our space X will be given by putting a topology on the set of functions $x : \mathcal{D} \to \mathbb{R}$ with the property that there is a $D \in \mathcal{D}$ and $p \in \mathbb{R} \setminus (D \cup \{0\})$ such that

1. for each $C \subseteq D$, $x(C) = p$;

2. for each $C \not\subseteq D$, $x(C) = 0$.

We let $\mathcal{I}$ be the collection of bounded non-empty open intervals in $\mathbb{R} \setminus \{0\}$ with rational endpoints, and $\mathcal{S}$ be the collection of surjective partial functions $\mathbb{R} \setminus \{0\} \to \{0,1\}$ defined on a countable set. We define for $I \in \mathcal{I}$, $S \in \mathcal{S}$

$$V[S,I] = \left\{x \in X : x\left(S^{-1}(1)\right) = \{p\} \text{ where } p \in I \text{ and } \forall t \in S^{-1}(0),\ x(\{t\}) = 0\right\}.$$

We take $\mathcal{B} = \{V[S,I] : S \in \mathcal{S},\ I \in \mathcal{I}\}$ as a base generating a topology τ on X. Lemma 1.6.2 below provides the means to show $\mathcal{B}$ is indeed a base. (X,τ) is Debs' Tychonoff space.

Remark 1.6.1. *Debs also describes the topology as the uniform topology on countable subsets of $\mathcal{D}$.*

Lemma 1.6.2 (Proposition 7, [10])**.** *(X,τ) is Tychonoff and $V[S_1,I_1] \subseteq V[S_2,I_2]$ if and only if $S_2 \subseteq S_1$ and $I_1 \subseteq I_2$.*

Theorem 1.6.3. *(X,τ) is Choquet.*

Proof. It is enough to show that Player II has a winning strategy in $G_X^{\mathcal{B}}$.

For each countable $C \subseteq \mathbb{R} \setminus \{0\}$, fix an enumeration $\phi_C : \omega \to C$. Let d be the standard complete metric on $\mathbb{R}$, so we can speak of the diameter, diam, of a bounded set.

We build a winning strategy σ by recursion. σ will have the property that if

$$(V[S_1,I_1], V[T_1,J_1], \ldots, V[S_n,I_n], V[T_n,J_n]) \in \sigma$$

then

1. $T_n = S_n$;

2. $\operatorname{diam}(J_n) \leq 2^{-n}$;

3. $\overline{J_n} \subseteq I_n$;

4. $J_n \cap \left\{\phi_{S_i^{-1}(1)}(j) : i, j \leq n\right\} = \emptyset$.

5. $0 \notin \overline{J_1}^{\mathbb{R}}$.

(Condition (4) here amounts to saying that J_n misses the first n points of each $S_i^{-1}(1)$, $i \leq n$, but we need to be careful that we have a consistent enumeration of these points which is why the extra machinery is required).

Of course $\emptyset \in \sigma$, and if

$$(V[S_1, I_1], V[T_1, J_1], \ldots, V[S_n, I_n], V[T_n, J_n]) \in \sigma$$

then for any non-empty $V[S_{n+1}, I_{n+1}] \subseteq V[T_n, J_n]$, we demand

$$(V[S_1, I_1], V[T_1, J_1], \ldots, V[S_n, I_n], V[T_n, J_n], V[S_{n+1}, I_{n+1}]) \in \sigma.$$

So now suppose that $(V[S_1, I_1], V[T_1, J_1], \ldots, V[S_n, I_n]) \in \sigma$. Our conditions (1)-(4) tell us how to choose T_n and J_n. First take $T_n = S_n$. Now choose J_n so that (2)-(3) hold (which is clearly possible), and we can also easily obtain (5) when $n = 1$. We then demand

$$(V[S_1, I_1], V[T_1, J_1], \ldots, V[S_n, I_n], V[T_n, J_n]) \in \sigma.$$

This specifies our strategy σ. We must check that σ is indeed a winning strategy for Player II. Suppose $\langle (V[S_i, I_i], V[T_i, J_i]) : i \in \omega \rangle$ is a σ-compatible play of $G_X^{\mathcal{B}}$. Then $\bigcap_{i\in\omega} J_i = \{p\}$ for some $p \in \mathbb{R}$ by completeness of $\mathbb{R}$ with respect to d and the fact that $0 \notin \overline{J_1}^{\mathbb{R}}$ (note that $\mathcal{I}$ consists of intervals in $\mathbb{R} \setminus \{0\}$); indeed we must have $p \neq 0$. Observe that $S = \bigcup_{i\in\omega} S_i \in \mathcal{S}$, and if we have an $x \colon \mathcal{D} \to \mathbb{R}$ with $x(C) = p$ if and only if $C \subseteq S^{-1}(1)$ then

$$x \in \bigcap_{i\in\omega} V[S_i, I_i]$$

and we are done. But clearly, by the definition of our space, such an x exists if and only if $p \notin S^{-1}(1)$. But if $p \in S^{-1}(1)$ then $p \in S_i^{-1}(1)$ for some $i \in \omega$, and hence there is an $n \in \omega$ with $i \leq n$ and some $j \leq n$ with

$$\phi_{S_i^{-1}(1)}(j) = p.$$

But then $p \notin J_n$, a contradiction. Hence σ is indeed a winning strategy for Player II in $G_X^{\mathcal{B}}$. □

Lemma 1.6.4. *$w(X) \leq \mathfrak{c}$, where $w(X)$ denotes the weight of X, the least cardinality of a base for X.*

Proof. We bound the cardinality of $\mathcal{B}$. Observe that $|\mathcal{B}| \leq |\mathcal{S}| \times |\mathcal{I}|$. $\mathcal{I}$ clearly has cardinality $\aleph_0$, so we bound the cardinality of $\mathcal{S}$. There are $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ countable subsets of $\mathbb{R} \setminus \{0\}$, and for each of these countable subsets C, at most $2^{\aleph_0} = \mathfrak{c}$ functions $C \to \{0,1\}$. Hence

$$|\mathcal{S}| \leq \mathfrak{c} \times \mathfrak{c} = \mathfrak{c}.$$

But then $|\mathcal{B}| \leq \mathfrak{c} \times \aleph_0 = \mathfrak{c}$. Since $w(X) \leq |\mathcal{B}|$ we have the result. □

Lemma 1.6.5. *$\mathfrak{c}(V[S,I]) \geq \mathfrak{c}^+$ when $V[S,I]$ is non-empty.*

Proof. We construct $\mathfrak{c}$ many disjoint open subsets of $V[S,I]$. Fix a countably infinite subset A of $\mathbb{R} \setminus \{0\}$ with $\operatorname{dom}(S) \subseteq A$ with $A \setminus \operatorname{dom}(S)$ infinite, and consider characteristic functions $\chi_B \colon A \to \{0,1\}$ of subsets B of A. Observe that

$$\{\chi_B \colon B \subseteq A\} \subseteq \mathcal{S}$$

and for $B, C \subseteq A$, $\chi_B \subseteq \chi_C$ if and only if $B = C$. Hence for $B \neq C$, $V[\chi_B, I] \cap V[\chi_C, I] = \emptyset$ by Lemma 1.6.2. So we have a collection of size $\mathfrak{c}$ of disjoint open subsets of $V[S,I]$ given by

$$\left\{V[\chi_B, I] : B \subseteq A \text{ and } S^{-1}\{1\} \subseteq B\right\}$$

as we require. □

Corollary 1.6.6. *$w(X) = \mathfrak{c}$. If $V[S,I] \neq \emptyset$ then $\mathfrak{c}(V[S,I]) = \mathfrak{c}^+$.*

Proof. X has a disjoint collection of open subsets of size $\mathfrak{c}$, hence $w(X) \geq \mathfrak{c}$, and hence $w(X) = \mathfrak{c}$. But for $V[S,I] \neq \emptyset$, $\mathfrak{c}(V[S,I]) \leq w(X)^+ \leq \mathfrak{c}^+$, and hence in fact $\mathfrak{c}(V[S,I]) = \mathfrak{c}^+$. □

Theorem 1.6.7. *Player II has a winning 2-tactic in G_X.*

Proof. X is Choquet, and X admits a coding system by the previous Corollary and Theorem 1.4.14. □

1.7 Debs' meagre space and coding systems

We have already shown the following.

Theorem 1.7.1 (CH). *$(\mathbb{R}, \tau_{\text{MGR}})$ admits a coding system. In particular, Player II has a winning 2-tactic in $G_{(\mathbb{R},\tau_{\text{MGR}})}$.*

In this section, we will show

Theorem 1.7.2 ($\mathbf{MA} + \neg\mathbf{CH}$). *$(\mathbb{R}, \tau_{\text{MGR}})$ does not admit a coding system.*

So it is independent of **ZFC** whether $(\mathbb{R}, \tau_{\text{MGR}})$ admits a coding system. However, the following was shown in [1]:

Theorem 1.7.3 (ZFC). *Player II has a winning 2-tactic in $G_{(\mathbb{R},\tau_{\text{MGR}})}$.*

In particular, a positive answer to 1.4.15 cannot be obtained from **ZFC** alone.

Let us begin our proof of Theorem 1.7.2 with a general lemma about weak-bases with respect to coding systems.

Lemma 1.7.4. *Suppose (X, τ) is a topological space and has a weak-base $\mathcal{W}$ such that for all $U \in \mathcal{W}$*

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U).$$

Then if $\mathcal{V}$ is a weak-base for (X, τ) then there is a $\mathcal{V}' \subseteq \mathcal{V}$ which is a weak-base for (X, τ), with the property that for any $U \in \mathcal{V}'$

$$|\{W \in \mathcal{V}' \colon U \subseteq W\}| < \mathsf{c}(U).$$

Proof. For each $W \in \mathcal{W}$, let $W_{\mathcal{V}} \in \mathcal{V}$ be so that $W_{\mathcal{V}} \subseteq W$. Let

$$\mathcal{V}' = \{W_{\mathcal{V}} \colon W \in \mathcal{W}\}.$$

Clearly $\mathcal{V}'$ is a weak-base for (X, τ).

Let $U \in \mathcal{V}'$. Let $Z \in \mathcal{W}$ with $Z \subseteq U$. If $W \in \mathcal{W}$ and $U \subseteq W_{\mathcal{V}}$, then certainly $Z \subseteq W$. Hence

$$|\{V \in \mathcal{V}' \colon U \subseteq V\}| \leq |\{W \in \mathcal{W} \colon Z \subseteq W\}| < \mathsf{c}(Z) \leq \mathsf{c}(U)$$

and hence $\mathcal{V}'$ has the desired property. □

Proposition 1.7.5 ($\mathrm{add}(\mathrm{MGR}(\mathbb{R}), \subseteq) > \omega_1$)**.** *There is no weak-base $\mathcal{W}$ for $(\mathbb{R}, \tau_{\mathrm{MGR}})$ such that for every $U \in \mathcal{W}$,*

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U).$$

Proof. Suppose $\mathcal{W}$ is a weak-base for $(\mathbb{R}, \tau_{\mathrm{MGR}})$ and $\mathcal{W}$ is a subset of the weak-base

$$\{(a,b) \setminus M \colon a, b \in \mathbb{Q},\ M \in \mathrm{MGR}(\mathbb{R})\};$$

by the previous lemma it is enough to show that for any such $\mathcal{W}$ there is a $U \in \mathcal{W}$ with

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| \geq \mathsf{c}(U).$$

Note first that by Observation 1.5.8, for any non-empty open subset U of $(\mathbb{R}, \tau_{\mathrm{MGR}})$, $\mathsf{c}(U) = \omega_1$. Further, note that $\mathcal{W}$ cannot be countable, else, by Theorem 1.4.8, Player II would have a winning 1-tactic in $G_{(\mathbb{R}, \tau_{\mathrm{MGR}})}$. In particular then, there must be $a, b \in \mathbb{Q}$ with

$$\mathcal{M} = \{M \in \mathrm{MGR}(\mathbb{R}) : M \subseteq (a,b),\ (a,b) \setminus M \in \mathcal{W}\}$$

uncountable (note here if $M \in \mathrm{MGR}(\mathbb{R})$ then $M \cap (a,b) \in \mathrm{MGR}(\mathbb{R})$, and $(a,b) \setminus M = (a,b) \setminus (M \cap (a,b))$). Let M_α for $\alpha < \omega_1$ be distinct elements of $\mathcal{M}$. Then $M = \bigcup_{\alpha < \omega_1} M_\alpha$ is an element of $\mathrm{MGR}(\mathbb{R})$. So if $U \in \mathcal{W}$ with $U \subseteq (a,b) \setminus M$ then

$$\{(a,b) \setminus M_\alpha \colon \alpha < \omega_1\} \subseteq \{W \in \mathcal{W} \colon U \subseteq W\}$$

and hence $|\{W \in \mathcal{W} \colon U \subseteq W\}| \geq \omega_1 = \mathsf{c}(U)$. □

Theorem 1.7.2 follows from the following, since, as we have already noted, $\mathbf{MA} + \neg\mathbf{CH}$ implies $\mathrm{add}(\mathrm{MGR}(\mathbb{R}), \subseteq) > \omega_1$.

Corollary 1.7.6 ($\mathrm{add}(\mathrm{MGR}(\mathbb{R}), \subseteq) > \omega_1$)**.** $(\mathbb{R}, \tau_{\mathrm{MGR}})$ *does not admit a coding system.*

Proof. By Theorem 1.4.13. □

This raises the question of whether there is a **ZFC** example of a Choquet space X (in which if U is non-empty and open then $\mathsf{c}(U) \geq \aleph_0$) where Player II has no

winning 1-tactic in G_X, but such that X does not admit a coding system. If Player II also had a winning 2-tactic in G_X, then this gives a negative answer to Question 1.4.15. We may also ask whether a positive answer to Question 1.4.15 is consistent with **ZFC**.

1.8 The semi-regularisation of a topological space

Definition 1.8.1. *Let (X, τ) be a topological space. An open subset U of X is said to be regular open if and only if* $\text{int}_\tau\left(\overline{U}^\tau\right) = U$. *Let* $\text{RO}(X, \tau)$ *(or* $\text{RO}(X)$ *where the topology is clear) denote the set of regular open subsets of (X, τ). Observe that* $\text{RO}(X, \tau)$ *forms a base for a topology τ_s on X. We call (X, τ_s) the semi-regularisation of (X, τ).*

Clearly $\tau_s \subseteq \tau$. Observe also that if U is open in (X, τ) then $\text{int}_\tau\left(\overline{U}^\tau\right)$ is regular open, and contains U. We make some simple observations now to see $(X, (\tau_s)_s) = (X, \tau_s)$.

Observation 1.8.2. *Suppose U is open in (X, τ). Then $\overline{U}^\tau \setminus U$ is τ-nowhere dense.*

Proof. Clearly $\overline{U}^\tau \setminus U$ is τ-closed, and if V is open and $V \cap \left(\overline{U}^\tau \setminus U\right) \neq \emptyset$, then $V \cap U \neq \emptyset$. That is $V \not\subseteq \overline{U}^\tau \setminus U$, and from this we see $\overline{U}^\tau \setminus U$ is τ-codense. □

Observation 1.8.3. *Suppose U is open in (X, τ). Then $\overline{U}^\tau = \overline{U}^{\tau_s}$.*

Proof. If C is closed in (X, τ_s) then C is closed in (X, τ), so certainly $\overline{U}^\tau \subseteq \overline{U}^{\tau_s}$. For the reverse inclusion, suppose $x \in \overline{U}^{\tau_s}$, and let V be open in (X, τ) with $x \in V$. Then $\text{int}_\tau\left(\overline{V}^\tau\right)$ is τ-regular open and contains V and hence x, so $\text{int}_\tau\left(\overline{V}^\tau\right) \cap U \neq \emptyset$. Furthermore, $\text{int}_\tau\left(\overline{V}^\tau\right) \cap U$ is τ-open so is not τ-nowhere dense. But

$$\text{int}_\tau\left(\overline{V}^\tau\right) \cap U \subseteq (V \cap U) \cup \left(\overline{V}^\tau \setminus V\right)$$

and since $\overline{V}^\tau \setminus V$ is τ-nowhere dense, $V \cap U \neq \emptyset$. □

Observation 1.8.4. *Suppose C is closed in (X, τ). Then* $\text{int}_\tau(C) = \text{int}_{\tau_s}(C)$.

Proof. $\text{int}_\tau(C) = X \setminus \left(\overline{X \setminus C}^{\tau}\right) = X \setminus \left(\overline{X \setminus C}^{\tau_s}\right) = \text{int}_{\tau_s}(C)$. The middle equality here follows from the previous observation. □

Observation 1.8.5. *If U is open in (X, τ), then* $\text{int}_\tau\left(\overline{U}^{\tau}\right) = \text{int}_{\tau_s}\left(\overline{U}^{\tau_s}\right)$. *In particular* $\text{RO}(X, \tau) = \text{RO}(X, \tau_s)$, *so* $(\tau_s)_s = \tau_s$.

Proof. The first statement follows immediately from the previous two observations. The second set of observations follows from the first and $\tau_s \subseteq \tau$. □

The upshot of these observations is that nothing is gained from repeating the process of semi-regularisation more than once.

Semi-regularising a space allows us to consider the open sets which are generated by "fat" open sets. We can observe in Debs' meagre space that, under $\mathbf{MA} + \neg\mathbf{CH}$, there are too many meagre sets to code. But the meagre sets only account for small differences in open sets. More precisely, note that if M_1, M_2 are both meagre in $\mathbb{R}$, then

$$(a, b) = \text{int}_{\tau_{\text{MGR}}}\left(\overline{(a, b) \setminus M_i}^{\tau_{\text{MGR}}}\right); \quad i = 1, 2.$$

It is the meagre sets which we have trouble coding. Semi-regularising potentially removes the possibility of weak-bases becoming large in this way.

Debs meagre space is one of the motivations for studying the semi-regularisation of a Choquet space. Note that from

$$(a, b) = \text{int}_{\tau_{\text{MGR}}}\left(\overline{(a, b) \setminus M}^{\tau_{\text{MGR}}}\right)$$

(where M is meagre in $\mathbb{R}$) we see $(\mathbb{R}, (\tau_{\text{MGR}})_s) = (\mathbb{R}, \tau_{\text{metric}})$. Scheepers ([25], Theorem 22, pg1053) observed that one can think about $G_{(\mathbb{R}, \tau_{\text{MGR}})}$ as two different games; one the usual Choquet game on $(\mathbb{R}, \tau_{\text{metric}})$, and the other a suitable ideal game played with nowhere dense and meagre subsets of $(\mathbb{R}, \tau_{\text{metric}})$. Showing that Player II has a winning 2-tactic in $G_{(\mathbb{R}, \tau_{\text{MGR}})}$ was then reduced to showing Player II had a winning 2-tactic in the appropriate ideal game (the argument is completed in [1]). The description of $(\mathbb{R}, \tau_{\text{MGR}})$ is kind in the sense that it is built by modifying a very concrete space, $(\mathbb{R}, \tau_{\text{metric}})$, but we of course recover this space by semi-regularising.

In the abstract case, semi-regularising Choquet spaces might provide much kinder Choquet spaces, so that the business of finding a winning k-tactic could perhaps be reduced to a question of some suitable ideal game in a similar fashion. This is a promising line of inquiry, and gives rise to the following question, which motivates the material in this section.

Question 1.8.6. *Suppose (X, τ) is a topological space with a base of regular open sets, such that for any non-empty open subset U of X, $\mathsf{c}(U) \geq \aleph_0$. Is it the case that (X, τ) has a weak-base $\mathcal{W}$ which admits a coding system?*

Remark 1.8.7. *Suppose X is a topological space with a base of regular open sets, and $X_{\mathcal{F}}$ is as in Lemma 1.4.5. Suppose U is regular open in X with $U \subseteq X_{\mathcal{F}}$. If V is open in $X_{\mathcal{F}}$ (and so open in X) and $V \subseteq \overline{U}^{X_{\mathcal{F}}} = \overline{U}^{X} \cap X_{\mathcal{F}}$, then $V \subseteq \mathrm{int}_X\left(\overline{U}^{X}\right) = U$. Hence $\mathrm{int}_{X_{\mathcal{F}}}\left(\overline{U}^{X_{\mathcal{F}}}\right) = U$. So $X_{\mathcal{F}}$ has a base of $X_{\mathcal{F}}$-regular open sets.*

Question 1.8.6 is therefore really a question about all topological spaces with a base of regular open sets with respect to coding systems. The additional cellularity condition serves to capture the non-trivial parts of spaces where the combinatorics of coding systems is interesting.

An affirmative answer to the above question would be promising due to the following theorem, which says that the semi-regularisation of a Choquet space is Choquet. It follows that, if there is an affirmative answer to the above question, for any Choquet space X, the semi-regularisation of X admits a coding system, and hence a winning 2-tactic for Player II. In this case, if there is a Choquet space X in which Player II does not have a winning 2-tactic, the failure to have a winning 2-tactic must arise from having many open sets with small differences. An affirmative answer to the above question therefore tells us where to look for counterexamples. It also tells us that general spaces can be construed in a similar way to Debs meagre space, and to obtain winning 2-tactics in general one might only need to study appropriate ideal games.

Notice that the semi-regularisation of a regular space X is just the space X. Hence an affirmative answer to the above has as a corollary that Player II has a

winning 2-tactic in the Choquet game on any regular Choquet space X.

Theorem 1.8.8. *Suppose (X,τ) is Choquet. Then (X,τ_s) is Choquet.*

Proof. Suppose σ is a winning strategy for Player II in $G_{(X,\tau)}$. Let $\mathcal{W} = \mathrm{RO}(X,\tau) \setminus \{\emptyset\}$. We construct a winning strategy μ for Player II in $G^{\mathcal{W}}_{(X,\tau_s)}$ with a transfer map $\mathcal{T}$ from μ to σ by recursion such that

1. if $(U_0, V_0, \ldots, V_n) \in \mu$ then $\mathcal{T}(U_0, V_0, \ldots, V_n) = t^\frown(V_n')$, for some t, where $V_n = \mathrm{int}_\tau\left(\overline{V_n'}^\tau\right)$.

2. if $(U_0, V_0, \ldots, U_n) \in \mu$ then $\mathcal{T}(U_0, V_0, \ldots, U_n) = t^\frown(U_n')$, for some t, where $U_n' \subseteq U_n$.

Of course, $\emptyset \in \mu$ and $\mathcal{T}(\emptyset) = \emptyset$. If $s = (U_0, V_0, \ldots, V_n) \in \mu$ then for any $U_{n+1} \in \mathcal{W}$ with $U_{n+1} \subseteq V_n$, put $(U_0, V_0, \ldots, V_n, U_{n+1}) \in \mu$ and

$$\mathcal{T}(U_0, V_0, \ldots, V_n, U_{n+1}) = \mathcal{T}(s)^\frown\left(U_{n+1} \cap \bigcap \mathrm{ran}(\mathcal{T}(s))\right).$$

If $(U_0, V_0, \ldots, U_n) \in \mu$ then let $V_n' = \sigma(\mathcal{T}(U_0, V_0, \ldots, U_n))$. Provided we have satisfied property (2) so far in our recursion, $V_n' \subseteq U_n$, so $\mathrm{int}_\tau\left(\overline{V_n'}^\tau\right) \subseteq \mathrm{int}_\tau\left(\overline{U_n}^\tau\right) = U_n$, so we may put $(U_0, V_0, \ldots, U_n, V_n) \in \mu$, where $V_n = \mathrm{int}_\tau\left(\overline{V_n'}^\tau\right)$. Then set

$$\mathcal{T}(U_0, V_0, \ldots, U_n, V_n) = \mathcal{T}(U_0, V_0, \ldots, U_n)^\frown(V_n').$$

If $s = \langle (U_i, V_i) : i \in \omega \rangle$ is μ-compatible, then $\mathcal{T}(s)$ is σ-compatible, and so $\bigcap_{n\in\omega} U_n' \neq \emptyset$ (in the notation of property (2) of $\mathcal{T}$), and so $\bigcap_{n\in\omega} U_n \neq \emptyset$, by property (2) of $\mathcal{T}$. Hence μ is a winning strategy for Player II in $G^{\mathcal{W}}_{(X,\tau_s)}$. As $\mathcal{W}$ is a weak-base for (X,τ_s), (X,τ_s) is Choquet. □

Certainly the semi-regularisation has the property that the open sets cannot descend too "deeply", because of the following general fact.

Lemma 1.8.9. *Suppose X is a topological space. Suppose $\langle U_\alpha : \alpha < \gamma \rangle$ is a descending sequence of open subsets of X such that $\left(\alpha < \beta \to U_\alpha \setminus \overline{U_\beta} \neq \emptyset\right)$. Then $\gamma < \mathrm{c}(X)$.*

Proof. Otherwise $\left\{U_\alpha \setminus \overline{U_{\alpha+1}} \colon \alpha < \gamma\right\}$ is a set of disjoint open subsets of X of cardinality at least $\mathsf{c}(X)$. □

To simplify what needs to be shown to answer Question 1.8.6, we reduce the class of spaces we need to consider by building a weak-base with special properties.

Definition 1.8.10. *Let (X, τ) be a topological space. The deep cellularity of X,* $\mathrm{DCell}(X)$*, is the cardinal*

$$\min\left\{\mathsf{c}(U) : U \text{ is non-empty and open in } X\right\}.$$

In particular, there is a non-empty open subset U of X such that $\mathsf{c}(U) = \mathrm{DCell}(X)$.

The following is straightforward, but worth noting nonetheless.

Lemma 1.8.11. *Suppose X is a topological space with* $\mathrm{DCell}(X) = \mathsf{c}(X)$*. Then if U is non-empty and open in X then* $\mathrm{DCell}(U) = \mathsf{c}(U) = \mathsf{c}(X)$.

Proof. It is clear that $\mathsf{c}(U) \leq \mathsf{c}(X)$, so by definition of deep cellularity, $\mathsf{c}(U) = \mathsf{c}(X)$. Also we have $\mathrm{DCell}(U) \geq \mathrm{DCell}(X)$ as $U \subseteq X$, and that $\mathrm{DCell}(U) \leq \mathsf{c}(U)$. So in fact

$$\mathsf{c}(U) \geq \mathrm{DCell}(U) \geq \mathrm{DCell}(X) = \mathsf{c}(X) = \mathsf{c}(U),$$

and hence all the inequalities were in fact equalities. □

Lemma 1.8.12. *Suppose (X, τ) is a topological space. Then there is a weak-base $\mathcal{W}$ for X such that for all $W \in \mathcal{W}$,* $\mathrm{DCell}(W) = \mathsf{c}(W)$.

Proof. For each non-empty open subset U of X, let W_U be a non-empty open subset of U with $\mathrm{DCell}(U) = \mathsf{c}(W_U)$. Then by definition of deep cellularity, $\mathrm{DCell}(W_U) = \mathsf{c}(W_U)$. It therefore suffices to take

$$\mathcal{W} = \{W_U : U \text{ non-empty and open in } X\}.$$ □

Proposition 1.8.13. *Let us say here that a topological space X is "well-divisible" if and only if for every non-empty open subset U of X,* $\mathsf{c}(U) \geq \aleph_0$.

Every well-divisible topological space (X, τ) with a base of regular open sets admits a coding system if and only if every well-divisible topological space (Y, σ) with a base of regular open sets, and with $\mathsf{c}(Y) = \mathrm{DCell}(Y)$*, admits a coding system.*

Proof. By considering well-divisible spaces, we can freely make use of Theorem 1.4.13; we do so in this proof without further mention. Notice also that if $A \subseteq Y \subseteq X$ then the cellularity of A in X is the same as the cellularity of A in the subspace Y — in other words cellularity is absolute. This permits us to write $\mathrm{c}(A)$ for the cellularity of A when this quantity could be interpeted in more than one subspace. We do this freely below.

The implication from left to right is clear. So for the converse assume every well-divisible topological space (Y, σ) with a base of regular open sets and with $\mathrm{DCell}(Y) = \mathrm{c}(Y)$ admits a coding system. Let (X, τ) be any well-divisible topological space with a base of regular open sets, and let $\mathcal{W}$ be as in the previous lemma for X. We may also suppose $\mathcal{W}$ consists of regular open sets by Lemma 1.8.11, and since each element of $\mathcal{W}$ contains a non-empty regular open set.

Let $\mathcal{B}$ be a maximal disjoint collection of elements of $\mathcal{W}$ (which exists by Zorn's Lemma). If $B \in \mathcal{B}$ then $\mathrm{c}(B) = \mathrm{DCell}(B)$. Note also B is well-divisible because X is. Further B in the subspace topology has a base of regular open sets, for if U is regular open in X then $U \cap B$ is regular open in X, and hence

$$U \cap B \subseteq \mathrm{int}_B\left(\overline{U \cap B}^B\right) \subseteq \mathrm{int}_X\left(\overline{U \cap B}^X\right) = U \cap B$$

so $U \cap B$ is regular open in B. Hence B has a weak-base $\mathcal{W}_B$ which admits a coding system. Let

$$\mathcal{V} = \bigcup_{B \in \mathcal{B}} \mathcal{W}_B.$$

Then if $V \in \mathcal{V}$ then

$$\{W \in \mathcal{V} \colon V \subseteq W\} = \{W \in \mathcal{W}_B \colon V \subseteq W\}$$

where $B \in \mathcal{B}$ satisfies $V \in \mathcal{W}_B$. So, if $\mathcal{V}$ is a weak-base for X, then $\mathcal{V}$ admits a coding system.

To see $\mathcal{V}$ is a weak-base for X, note that $\bigcup \mathcal{B}$ is dense in X by maximality of $\mathcal{B}$. Then if U is non-empty and open in X, there is a $B \in \mathcal{B}$ with $B \cap U \neq \emptyset$, and since $B \cap U$ is open in B, there is a $V \in \mathcal{W}_B$ with $V \subseteq B \cap U \subseteq U$. □

Notice in the above proof we can refine $\mathcal{W}$ by taking

$$\mathcal{W}' = \{W \in \mathcal{W} \colon \exists B \in \mathcal{B}\,(W \subseteq B)\}$$

to get a weak-base $\mathcal{W}'$ satisfying the following.

Corollary 1.8.14 (to proof)**.** *Let (X, τ) be a topological space. Then there is a maximal disjoint collection of non-empty open subsets $\mathcal{B}$ of X such that every $B \in \mathcal{B}$ satisfies* $\mathrm{DCell}\,(B) = \mathsf{c}\,(B)$*, and there is a weak-base $\mathcal{W}$ of X such that for each $W \in \mathcal{W}$ there is a unique $B \in \mathcal{B}$ with $W \subseteq B$. Furthermore, if X has a base of regular open sets, then elements of $\mathcal{B}$ can be taken to be regular open in X, and elements W of $\mathcal{W}$ can be taken to be regular open in X and in B, where $B \in \mathcal{B}$ satisfies $W \subseteq B$.*

A space with a base of regular open sets informally restricts how "deep" nested open sets can go. The above proposition also restricts the "width" of spaces we need to consider. Further, we can prove the Erdős-Tarski theorem using the above language, which shows us that the cardinals representing these notions of "width" and "depth" are regular.

Theorem 1.8.15 (Erdős-Tarski)**.** *Let X be a topological space. Then* $\mathsf{c}\,(X)$ *is a regular cardinal.*

This is proved in two steps. The first is the following.

Lemma 1.8.16. *Suppose X is a topological space with* $\mathsf{c}\,(X) = \mathrm{DCell}\,(X)$*. Then* $\mathsf{c}\,(X)$ *is a regular cardinal.*

Proof. Suppose to the contrary that $\mathrm{cf}\,(\mathsf{c}\,(X)) < \mathsf{c}\,(X)$, and that $f \colon \mathrm{cf}\,(\mathsf{c}\,(X)) \to \mathsf{c}\,(X)$ is cofinal. Then we can find a collection $\mathcal{D}$ of non-empty disjoint open subsets of X with $|\mathcal{D}| = \mathrm{cf}\,(\mathsf{c}\,(X))$. As $\mathrm{DCell}\,(X) = \mathsf{c}\,(X)$, for each $D \in \mathcal{D}$, $\mathsf{c}\,(D) = \mathsf{c}\,(X)$. So enumerate

$$\mathcal{D} = \{D_\alpha \colon \alpha < \mathrm{cf}\,(\mathsf{c}\,(X))\}$$

and for each $\alpha < \mathrm{cf}\,(\mathsf{c}\,(X))$ let F_α be a disjoint collection of open subsets of D_α of cardinality $|f\,(\alpha)|$. Then $\bigcup_{\alpha < \mathrm{cf}(\mathsf{c}(X))} F_\alpha$ is a disjoint collection of open subsets of X of cardinality $\mathsf{c}\,(X)$, a contradiction. □

Proof of Theorem 1.8.15. Suppose for a contradiction that $\mathrm{c}(X)$ is singular; in particular then, $\mathrm{c}(X)$ is a limit cardinal. Let $\mathcal{B}$ be a maximal disjoint collection of non-empty open sets with $\mathrm{DCell}(B) = \mathrm{c}(B)$ for each $B \in \mathcal{B}$, and let $\mathcal{W}$ be a weak-base for X such that for each $W \in \mathcal{W}$ there is a (unique) $B \in \mathcal{B}$ with $W \subseteq B$. Corollary 1.8.14 guarantees the existence of such a $\mathcal{B}$ and $\mathcal{W}$. Then as $\mathcal{B}$ is a disjoint collection of open subsets of X, $|\mathcal{B}| < \mathrm{c}(X)$.

If $\mathcal{D}$ is a collection of non-empty disjoint open subsets of X, then by choosing for each $D \in \mathcal{D}$ a $W \in \mathcal{W}$ with $W \subseteq D$, we may suppose, without loss of generality from the point of view of cardinality considerations, that $\mathcal{D} \subseteq \mathcal{W}$. So $\mathrm{c}(X)$ is the least cardinal κ such that there is no disjoint subcollection of $\mathcal{W}$ of cardinality κ.

For $B \in \mathcal{B}$, $\mathrm{c}(B)$ is regular, by the previous lemma. Certainly $\mathrm{c}(B) \leq \mathrm{c}(X)$ so since $\mathrm{c}(X)$ is singular we must have $\mathrm{c}(B) < \mathrm{c}(X)$. Further,

$$\{\mathrm{c}(B) : B \in \mathcal{B}\}$$

is unbounded in $\mathrm{c}(X)$, else $\mathrm{c}(X)$ is no bigger than $\max(\sup\{\mathrm{c}(B) : B \in \mathcal{B}\}, |\mathcal{B}|)^+$, by what we have noted above. This is strictly less than $\mathrm{c}(X)$, as $\mathrm{c}(X)$ is a limit cardinal and $|\mathcal{B}| < \mathrm{c}(X)$.

So let $\langle B_\alpha : \alpha < \mathrm{cf}(\mathrm{c}(X))\rangle$ be a sequence in $\mathcal{B}$ with $\langle \mathrm{c}(B_\alpha) : \alpha < \mathrm{cf}(\mathrm{c}(X))\rangle$ a strictly increasing cofinal sequence in $\mathrm{c}(X)$. For each $\alpha < \mathrm{cf}(\mathrm{c}(X))$ let $\mathcal{D}_{\alpha+1}$ be a disjoint collection of open subsets of $B_{\alpha+1}$ of cardinality $\mathrm{c}(B_\alpha)$. But then $\bigcup_{\alpha<\mathrm{cf}(\mathrm{c}(X))} \mathcal{D}_{\alpha+1}$ is a disjoint collection of open subsets of X of cardinality $\mathrm{c}(X)$, a contradiction. □

In all of Debs' examples given, the semi-regularisations have a weak-base $\mathcal{W}$ of strictly smaller cardinality than the cellularity of the semi-regularisation. One might hope to improve on Question 1.8.6 by showing that if X has a base of regular open sets then X has a weak-base $\mathcal{W}$ with $|\mathcal{W}| < \mathrm{c}(X)$. However, the space 2^κ has the countable chain condition (so $\mathrm{c}(2^\kappa) = \omega_1$), and every weak-base for 2^κ has cardinality at least κ. However, the standard base for 2^κ is easily seen to admit a coding system, and in any case, 2^κ is compact and Hausdorff, so Player II has a winning 1-tactic in G_{2^κ}.

A more complicated example is a Suslin line (see [18], pg66, Definition 4.1), which has the countable chain condition (that is, cellularity ω_1), but is not separable, so cannot have a countable weak-base. Any linearly ordered topological space is regular, so in particular has a base of regular open sets. Suslin lines might then seem like candidates for counterexamples to Question 1.8.6. We will see now, however, that they cannot be.

Theorem 1.8.17. *Suppose X is a linearly ordered topological space (LOTS). Then X has a weak-base $\mathcal{W}$ such that for each $U \in \mathcal{W}$*

$$|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U).$$

Proof. First we use slightly modified versions of the above tools to reduce the cases we need to consider. First observe that an open interval of a LOTS is also a LOTS. Using that every non-empty open subset of a LOTS contains a non-empty open interval, along with Lemmas 1.8.11 and 1.8.12, let $\mathcal{W}$ be a weak-base of non-empty open intervals for X such that for each $W \in \mathcal{W}$ we have $\mathrm{DCell}(W) = \mathsf{c}(W)$. Let $\mathcal{B}$ be a maximal disjoint collection of elements of $\mathcal{W}$. If for each $B \in \mathcal{B}$, $\mathcal{W}_B$ is a weak-base for B such that for each $U \in \mathcal{W}_B$, $|\{W \in \mathcal{W}_B \colon U \subseteq W\}| < \mathsf{c}(U)$, then $\mathcal{W} = \bigcup_{B \in \mathcal{B}} \mathcal{W}_B$ is a weak-base for X such that for each $U \in \mathcal{W}$, $|\{W \in \mathcal{W} \colon U \subseteq W\}| < \mathsf{c}(U)$.

It is enough then to show the result for each $B \in \mathcal{B}$. That is, we may suppose X is a LOTS with $\mathrm{DCell}(X) = \mathsf{c}(X)$, so that for each non-empty open U in X, $\mathsf{c}(U) = \mathsf{c}(X)$.

We build the weak-base $\mathcal{W}$ in pieces by transfinite recursion. First let us introduce some (proof specific) terminology. Let us say that a collection $\mathcal{A}$ of sets refines a collection $\mathcal{B}$ of sets if and only if for all $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ with $A \subseteq B$ and $\big(A \neq B \to B \setminus \overline{A} \neq \emptyset\big)$. Let us say that a collection $\mathcal{C}$ of sets approximates a set D if and only if there is a $C \in \mathcal{C}$ with $C \subseteq D$.

Let $\mathcal{I}$ denote the set of non-empty order intervals of X; note these form a base for X. Well-order $\mathcal{I}$. Let $\mathcal{C}_0$ be a maximal disjoint collection of elements of $\mathcal{I}$ which contains the first element of $\mathcal{I}$.

We build disjoint collections of elements of $\mathcal{I}$, $\mathcal{C}_\alpha$, for ordinals α in some (yet to be determined) ordinal γ such that

1. If $\alpha \leq \beta < \gamma$ then $\mathcal{C}_\beta$ refines $\mathcal{C}_\alpha$.

2. $\mathcal{C}_{\alpha+1}$ approximates the least element of $\mathcal{I}$ not approximated by $\bigcup_{\beta\leq\alpha}\mathcal{C}_\beta$.

3. If $I \in \mathcal{I}$ is not approximated by $\bigcup_{\beta\leq\alpha}\mathcal{C}_\beta$, then there is a $J \in \mathcal{C}_\alpha$ with $I \cap J \neq \emptyset$.

4. Every $I \in \mathcal{I}$ is approximated by $\bigcup_{\alpha<\gamma}\mathcal{C}_\alpha$.

We then take $\mathcal{W} = \bigcup_{\alpha<\gamma}\mathcal{C}_\alpha$. $\mathcal{W}$ is then a weak-base for X by (4). If $J \in \mathcal{W}$, then J has at most one superset in each $\mathcal{C}_\alpha$ by disjointness, and by the same token there is a unique maximal, necessarily descending (by (1)), sequence $\langle W_\alpha : \alpha < \delta\rangle$ of supersets of J, where $\delta \leq \gamma$ and $W_\alpha \in \mathcal{C}_\alpha$. We need that $|\{W_\alpha \colon \alpha < \delta\}| < \mathfrak{c}(X)$, so that $|\{W \in \mathcal{W} \colon J \subseteq W\}| < \mathfrak{c}(X)$. But this follows from Lemma 1.8.9 and (1), because we may replace W_α by W_β where $\beta \geq \alpha$ is least such that $W_\beta \neq W_{\alpha'}$ for any $\alpha' < \alpha$, if it exists, and by terminating the sequence otherwise.

Let us provide the details for constructing the $\mathcal{C}_\alpha$. We have dealt with $\mathcal{C}_0$ and it is easy to see that $\mathcal{C}_0$ satisfies (1)-(3). Given $\mathcal{C}_\beta$ for $\beta \leq \alpha$, find the minimal element $I \in \mathcal{I}$ such that $\bigcup_{\beta\leq\alpha}\mathcal{C}_\beta$ does not approximate I, if it exists, else set $\gamma = \alpha + 1$ and finish the recursion. Then by (3) for α, there is a $J \in \mathcal{C}_\alpha$ with $J \cap I \neq \emptyset$. Since $\mathcal{I}$ is a base for X and X is regular, we can find $K \in \mathcal{I}$ so that $\overline{K} \subseteq J \cap I$ and hence $J \setminus \overline{K} \neq \emptyset$ (this is because $J \not\subseteq I$, so that $J \neq J \cap I$). Let $\mathcal{C}_{\alpha+1}$ be maximal extending $\{K\}$ in the set

$$\{A \in \mathcal{P}(\mathcal{I}) : A \text{ is a disjoint collection refining } \mathcal{C}_\alpha\}$$

partially ordered by $\subseteq$ (using Zorn's lemma). For $\alpha + 1$ we automatically have (1), and as $K \in \mathcal{C}_{\alpha+1}$, $\mathcal{C}_{\alpha+1}$ approximates I, so we also have (2). If $I \in \mathcal{I}$ is not approximated by $\bigcup_{\beta\leq\alpha+1}\mathcal{C}_\beta$ then there is a $J \in \mathcal{C}_\alpha$ with $J \cap I \neq \emptyset$. By maximality of $\mathcal{C}_{\alpha+1}$ there is a $J' \in \mathcal{C}_{\alpha+1}$ with $J' \cap (J \cap I) \neq \emptyset$, so in particular $J' \cap I \neq \emptyset$, and hence we have (3) for $\alpha + 1$.

Finally, let us suppose that λ is a limit ordinal, and that we have $\mathcal{C}_\alpha$ for $\alpha < \lambda$. If $\bigcup_{\alpha<\lambda} \mathcal{C}_\alpha$ approximates every $I \in \mathcal{I}$ then set $\gamma = \lambda$ and terminate the recursion. Otherwise, we define $\mathcal{C}_\lambda$ by

$$\mathcal{C}_\lambda = \left\{ \text{int}_X \left(\bigcap \text{ran}\,(f) \right) : f \in \left(\prod_{\alpha<\lambda} \mathcal{C}_\alpha \right) \text{ such that } f \text{ is decreasing} \right\} \setminus \{\emptyset\}.$$

$\mathcal{C}_\lambda$ is then a collection of elements of $\mathcal{I}$ and is disjoint because if $f, g \in \prod_{\alpha<\lambda} \mathcal{C}_\alpha$ are both decreasing and $f \neq g$ then choosing α such that $f(\alpha) \neq g(\alpha)$ we have $f(\alpha) \cap g(\alpha) = \emptyset$ by disjointness of $\mathcal{C}_\alpha$ and hence $\text{int}_X \left(\bigcap \text{ran}\,(f) \right) \cap \text{int}_X \left(\bigcap \text{ran}\,(g) \right) = \emptyset$. (1) is easily seen for λ and there is no need to check (2) here.

So we only need to check (3). Suppose $I \in \mathcal{I}$ is not approximated by $\bigcup_{\alpha\leq\gamma} \mathcal{C}_\alpha$.

Claim 1.8.18. *There is a descending $f \in \prod_{\alpha<\lambda} \mathcal{C}_\alpha$ such that for all $\alpha < \lambda$, $f(\alpha) \cap I \neq \emptyset$, and so that $(\alpha \mapsto f(\alpha) \cap I)$ is at most a two valued map; if it does take two values then $f(0) \cap I = I$.*

Proof. We construct f by transfinite recursion. Take $f(0)$ to be some $J_0 \in \mathcal{C}_0$ such that $J_0 \cap I \neq \emptyset$. Suppose now that we have defined $f(\alpha)$, and we wish to define $f(\alpha+1)$. $f(\alpha) \cap I$ has not been approximated by $\mathcal{C}_{\alpha+1}$ (else I would have been), so there is a $J_{\alpha+1} \in \mathcal{C}_{\alpha+1}$ such that $J_{\alpha+1} \cap (f(\alpha) \cap I) \neq \emptyset$. Moreover, since $\mathcal{C}_{\alpha+1}$ refines $\mathcal{C}_\alpha$, and $\mathcal{C}_\alpha$ is disjoint, we must have $J_{\alpha+1} \subseteq f(\alpha)$. So set $f(\alpha+1) = J_{\alpha+1}$.

It is enough to now show that if $I \not\subseteq f(\alpha)$, then $f(\alpha+1) \cap I = f(\alpha) \cap I$. To see this, suppose to the contrary. Write

$$f(\alpha) = (a_\alpha, b_\alpha)$$
$$f(\alpha+1) = (a_{\alpha+1}, b_{\alpha+1})$$
$$I = (c, d)$$

Suppose without loss of generality that $f(\alpha) \cap I = (c, b_\alpha)$ (noting $f(\alpha) \not\subseteq I$); the same argument works for (a_α, d). Now since $f(\alpha+1) \cap I \neq f(\alpha) \cap I$ and $f(\alpha+1) \not\subseteq I$, we must have $(b_{\alpha+1}, b_\alpha) \neq \emptyset$. $(b_{\alpha+1}, b_\alpha) \subseteq I$ so $\mathcal{C}_{\alpha+1}$ cannot approximate it, but there must be some $K \in \mathcal{C}_{\alpha+1}$ with $K \cap (b_{\alpha+1}, b_\alpha) \neq \emptyset$. $\mathcal{C}_{\alpha+1}$ refines the disjoint collection $\mathcal{C}_\alpha$, so necessarily $K \subseteq f(\alpha)$. But as $\mathcal{C}_{\alpha+1}$ is a disjoint collection, we must have

$f(\alpha+1) \cap K = \emptyset$. So, as K is an interval, $K \subseteq (b_{\alpha+1}, b_\alpha)$, a contradiction since $K \not\subseteq I$. So we must have $f(\alpha+1) \cap I = f(\alpha) \cap I$.

If ζ is a limit, we set

$$f(\zeta) = \text{int}_X \left(\bigcap \text{ran}\,(f \restriction \zeta) \right)$$

noting that this is well-defined. Suppose there is some $\beta < \zeta$ with $f(\alpha) \cap I = f(\beta) \cap I$ for all $\alpha \geq \beta$. In this case $f(\beta) \cap I \subseteq f(\zeta)$, and so $f(\beta) \cap I \subseteq f(\zeta) \cap I$, but as $f(\zeta) \subseteq f(\beta)$, we also have the opposite inclusion $f(\zeta) \cap I \subseteq f(\beta) \cap I$, and hence equality. Therefore, if $f \restriction \zeta$ is one (two) valued, then $f \restriction (\zeta+1)$ is one (two) valued. □

Given such an f, it is clear that $\text{int}_X (\bigcap \text{ran}\,(f)) \cap I \neq \emptyset$, and hence we have (3). This completes the construction of the collection $\mathcal{C}_\lambda$. Notice further that our condition for termination of the recursion (which must terminate because $\mathcal{I}$ is a set) ensures that we get (4) automatically. □

Remark 1.8.19. *Note that if X is a Suslin line, then in the above proof $\gamma \geq \omega_1$. This illustrates that generalising this method to other spaces is a non-trivial matter. Despite having a countable chain condition (cellularity ω_1) space, we still need a recursion of length at least ω_1 to build the weak-base. In the above proof, we know that for any element of $\mathcal{W}$ its supersets in $\mathcal{W}$ form a descending chain, and this is what allowed us to compute the number of supersets using the cellularity.*

It would be interesting, though likely difficult, to establish the above for regular spaces. The difficulty is in obtaining some abstract mechanism for limiting the number of supersets added to a weak-base as it is built. Examples such as Suslin lines, 2^κ and $[0,1]^\kappa$ illustrate that we can have large weak-bases and cellularity ω_1. All three of these examples admit coding systems.

Corollary 1.8.20. *Suppose X is a Choquet LOTS. Then Player II has a winning 2-tactic in G_X.*

Proof. By the above Theorem and Theorem 1.4.3. □

Proposition 1.8.13, Lemma 1.8.9, and Theorem 1.8.15 suggest a positive answer to Question 1.8.6. The above result on LOTS does not use all of these tools, but does rely on a more explicit description of a base for the space. Nonetheless, I conjecture the following.

Conjecture 1.8.21. *Suppose X has a base of regular open sets, and $\mathsf{c}(U) \geq \aleph_0$ for every non-empty open subset of X. Then X admits a coding system.*

The investigation into ideal games is work that has yet to be done. The following summarises the main points of interest.

Question 1.8.22. *Is it possible to capture the difference between a topological space and its semi-regularisation by some suitable (σ-)ideal? Is there a suitable game on this ideal from which a winning 2-tactic can be combined with a winning 2-tactic in the Choquet game on the semi-regularised space to build a winning 2-tactic in the Choquet game on the original space? Given a Choquet space, need there be such a winning 2-tactic in the appropriate ideal game?*

1.9 Some more basic properties of coding systems

The results in this section are similar to those in the previous section on consequences of the coding lemma, and could equally well be placed there. However, some of the language and results developed so far will help keep the proofs tidier. We begin by making a refinement to our previous point of view.

Theorem 1.9.1. *Suppose X is a topological space. Then $\mathsf{c}(X) \neq \aleph_0$.*

Proof. This is well known, and appears (for partial orders) as Exercise F1 in Chapter VII of Kunen, [18]. □

Given Lemma 1.4.5 and the above, we only consider spaces in this section whose open sets have cellularity at least ω_1, since these are the only Choquet spaces where Player II could possibly not have a winning 2-tactic. Furthermore, we may suppose for U non-empty and open in X, $\mathsf{c}(U) = \mathsf{c}(X)$; i.e. $\mathsf{DCell}(X) = \mathsf{c}(X)$.

To the notion of a base one associates a cardinal invariant, the weight of a space, which is the cardinality of the smallest base for a given space. We are more interested in weak-bases, so we define the appropriate cardinal invariant for this notion. The name here originates from the fact that weak-bases are also known as π-bases in the literature.

Definition 1.9.2. *For a topological space X we define the π-weight of X, $\pi w(X)$, to be the cardinality of a smallest weak-base for X.*

The following is obvious but worth noting.

Theorem 1.9.3. *Suppose $\mathcal{W}$ is a weak-base for X. Then there is a weak-base $\mathcal{V}$ for X with $\mathcal{V} \subseteq \mathcal{W}$ and $|\mathcal{V}| = \pi w(X)$.*

Proof. Let $\mathcal{U}$ be a weak-base for X with $|\mathcal{U}| = \pi w(X)$. For each $U \in \mathcal{U}$ choose $W_U \in \mathcal{W}$ with $W_U \subseteq U$. Then put $\mathcal{V} = \{W_U : U \in \mathcal{U}\}$. □

The following is a generalisation of a previous theorem, Theorem 1.4.14.

Theorem 1.9.4. *Suppose X is a topological space with $\mathsf{c}(X) \geq \omega_1$, with $\mathrm{DCell}(X) = \mathsf{c}(X)$, and with $\pi w(X) \leq \mathsf{c}(X)$. Then X admits a coding system.*

Proof. Let $\mathcal{W}$ be a weak-base for X with $|\mathcal{W}| = \pi w(X)$. Enumerate

$$\mathcal{W} = \{W_\alpha : \alpha < \pi w(X)\}.$$

We build a new weak-base $\mathcal{V}$ for X with the property that for every $V \in \mathcal{V}$

$$|\{U \in \mathcal{V} : V \subseteq U\}| < \mathsf{c}(X) = \mathsf{c}(V).$$

We build $\mathcal{V}$ by transfinite recursion on $\alpha < \pi w(X)$, constructing $\mathcal{V}_\alpha$ at stage α and setting $\mathcal{V} = \bigcup_{\alpha < \pi w(X)} \mathcal{V}_\alpha$. The $\mathcal{V}_\alpha$ will have the following properties:

1. $\mathcal{V}_{\alpha+1} \subseteq \{W_\beta : \beta \leq \alpha\}$ and, furthermore, $W_\alpha \in \mathcal{V}_{\alpha+1}$ if and only if for every $V \in \mathcal{V}_\alpha$ we have $V \not\subseteq W_\alpha$.

2. if $\alpha < \beta < \pi w(X)$ then $\mathcal{V}_\alpha \subseteq \mathcal{V}_\beta$;

Note that if our construction obeys (1) and (2) above, then we obtain the following property also, which says that later stages of our recursion do not add new supersets.

3. if $\alpha < \beta < \pi w(X)$, then for $U \in \mathcal{V}_\beta \setminus \mathcal{V}_\alpha$ we have $\{V \in \mathcal{V}_\alpha \colon V \subseteq U\} = \emptyset$.

Assuming the resulting $\mathcal{V}$ is indeed a weak-base for X, we check it has the required property. Let $U \in \mathcal{V}$, and let α be so that $U \in \mathcal{V}_\alpha$. Then properties (2) and (3) ensure

$$\{V \in \mathcal{V} \colon U \subseteq V\} \subseteq \mathcal{V}_\alpha,$$

and hence by property (1)

$$|\{V \in \mathcal{V} \colon U \subseteq V\}| < \pi w(X) \leq \mathsf{c}(X).$$

First set $\mathcal{V}_0 = \emptyset$. At limit stages λ, we put $\mathcal{V}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{V}_\alpha$. Given $\mathcal{V}_\alpha$, we describe how to build $\mathcal{V}_{\alpha+1}$. Consider W_α. If there is a $V \in \mathcal{V}_\alpha$ with $V \subseteq W_\alpha$, put $\mathcal{V}_{\alpha+1} = \mathcal{V}_\alpha$; otherwise, put $\mathcal{V}_{\alpha+1} = \mathcal{V}_\alpha \cup \{W_\alpha\}$. In this way we clearly satisfy property (1) and (2).

To see that this construction does indeed give us a weak-base, suppose U is a non-empty open subset of X and let α be so that $W_\alpha \subseteq U$. If $W_\alpha \in \mathcal{V}_{\alpha+1}$, then $W_\alpha \in \mathcal{V}$, otherwise, by property (1), there is some $V \in \mathcal{V}_\alpha$ with $V \subseteq W_\alpha$, in which case $V \in \mathcal{V}$ and $V \subseteq U$ also. □

Corollary 1.9.5. *Suppose X is a topological space with* $\mathrm{DCell}(X) = \mathsf{c}(X)$*, and* $\mathsf{c}(X) \geq \omega_1$*. Suppose* $\pi w(X) \leq \omega_1$*. Then X admits a coding system.*

Proof. Clearly $\pi w(X) \leq \mathsf{c}(X)$, so we only need apply the above Theorem. □

A corollary of White's work [34] is that Player II has a winning 1-tactic in G_X when X is Choquet and $\pi w(X) \leq \aleph_0$. The following are similar new results for 2-tactics.

Corollary 1.9.6. *Suppose X is a Choquet space with $\pi w(X) \leq \omega_1$. Then Player II has a winning 2-tactic in G_X.*

Proof. By Lemma 1.4.5, we may suppose $\mathsf{c}(X) \geq \omega_1$. By Corollary 1.8.14 and Theorem 1.9.3, we may assume $\mathrm{DCell}(X) = \mathsf{c}(X)$. The above Corollary then applies, and the results on coding systems ensure Player II has the required winning 2-tactic. □

Remark 1.9.7. *I was unable to generalise the above Corollary to larger π-weights, for instance to Choquet spaces X with $\pi w(X) = \omega_2$ — it likely requires new techniques.*

Remark 1.9.8. *If $\mathcal{D}$ is a collection of pairwise disjoint open subsets of a topological space X then we may without loss of generality suppose $\mathcal{D}$ consists of elements of some weak-base for X. Hence $|\mathcal{D}| \leq \pi w(X)$. So we must have that $\mathsf{c}(X) \leq \pi w(X)^+$. So the previous theorem only deals with the two largest values $\mathsf{c}(X)$ could possibly take.*

As an example, note that $\mathsf{c}(2^\kappa) = \omega_1$ and $\pi w(2^\kappa) = \kappa$, so the cellularity of a space can be very much smaller than the π-weight. Of course 2^κ is compact and Hausdorff, and hence Player II has a winning 1-tactic in G_{2^κ}. 2^κ nonetheless admits a coding system, since the set

$$\mathcal{B} = \left\{ \bigcap_{i=0}^{n} \pi_{\lambda_i}^{-1}\{k_i\} : n < \omega,\ k_i \in \{0,1\},\ \lambda_i \in \kappa \right\}$$

is a base for 2^κ, where the π_λ denote coordinate projections, noting every element of $\mathcal{B}$ has only finitely many supersets in $\mathcal{B}$.

Remark 1.9.9. *Given the previous Corollary, one might be inclined to collapse a large cardinal to $\aleph_2$ and hope to obtain a model in which Player II has a winning 2-tactic in the Choquet game on any Choquet space. [28] for instance, provides an example of this approach with respect to the Rothberger property.*

Question 1.9.10. *Is it consistent relative to the consistency of a supercompact cardinal that Player II has a winning 2-tactic in the Choquet game on any Choquet space?*

One approach is to try Lévy collapsing a supercompact cardinal κ to $\aleph_2$. The crux of the problem is in showing that if κ were collapsed to $\aleph_2$, then enough reflection is preserved from κ to reflect a 2-tactic from a generic extension of $M[G]$ (where M is a transitive subclass provided by λ-supercompactness for some suitable λ). However, the

difficulties of obtaining a 2*-tactic in* **ZFC** *from a winning strategy seem to be reflected in the combinatorial properties of the Lévy Collapse. This seemed to suggest that this approach wouldn't work, without more or less proving the result in* **ZFC**.

Since there is no particular reason to isolate supercompact cardinals as the preferred large cardinal axiom, one might try stronger large cardinal axioms to see if one obtains more control.

The following is a generalisation of a previous result, Theorem 1.4.10. It follows from the coding system lemma.

Theorem 1.9.11. *Suppose* X *is a topological space with a* σ_κ*-disjoint weak-base* $\mathcal{W}$*, and* $\mathsf{c}(X) = \mathrm{DCell}(X) \geq \kappa$*. Then* X *admits a coding system. In particular, if* X *is Choquet, then Player II has a winning* 2*-tactic in* G_X*.*

Proof. Write $\mathcal{W} = \bigcup_{\alpha<\kappa} \mathcal{W}_\alpha$, where each $\mathcal{W}_\alpha$ is a disjoint collection. For each $\alpha < \kappa$, define inductively disjoint collections

$$\mathcal{V}_\alpha = \{V \in \mathcal{W}_\alpha \colon \forall \beta < \alpha \, \forall W \in \mathcal{V}_\beta \; W \not\subseteq V\}$$

and put

$$\mathcal{V} = \bigcup_{\alpha<\kappa} \mathcal{V}_\alpha.$$

Now, $\mathcal{V}$ is a weak-base for X, since if U is open in X then there is some α and $W \in \mathcal{W}_\alpha$ with $W \subseteq U$. Then either $W \in \mathcal{V}_\alpha$ in which case we are done, or $W \notin \mathcal{V}_\alpha$, in which case there is some $\beta < \alpha$ and $V \in \mathcal{V}_\beta$ with $V \subseteq W$, in which case we simply observe $V \subseteq U$ also.

Clearly also, if $V \in \mathcal{V}_\alpha$, then

$$\{W \in \mathcal{V} \colon V \subseteq W\} = \left\{ W \in \bigcup_{\beta<\alpha} \mathcal{V}_\beta \colon V \subseteq W \right\}$$

and so by disjointness of $\mathcal{V}_\alpha$ this gives

$$|\{W \in \mathcal{V} \colon V \subseteq W\}| < \kappa = \mathsf{c}(V)$$

and so X admits a coding system. □

Corollary 1.9.12. *Suppose X is a Choquet space with a σ_{ω_1}-disjoint weak-base. Then Player II has winning 2-tactic in G_X.*

Proof. First note that the property of having a σ_{ω_1}-disjoint weak-base is open hereditary, and so by Lemma 1.4.5 we may suppose X has $\mathrm{DCell}(X) \geq \omega_1$. The above Theorem then applies. □

1.10 A characterisation of winning strategies in terms of the shape of collections of winning sequences

Here we obtain a complexity result for finding Choquet spaces where Player II does not have a winning k-tactic for some k. The general idea in what follows is to abstract the Choquet game to a general partial order, rather than a topological space, and specify a winning condition with similar properties to the topological Choquet game. The players in the abstract game will form descending chains of elements of a partial order, and wins for Player II will be prescribed by some subset of all possible descending chains. We characterize which subsets allow Player II to have a winning strategy, and a way in which we might prevent certain strategies from winning. One then hopes to prevent all k-tactics from winning in the abstract game while keeping a winning strategy for Player II. This would give a game with much of the character of the topological Choquet game, from which one might hope to reintroduce a topology, by pretending elements of the partial order are members of some weak-base, and placing points precisely where our winning set says they should be.

This turns out to be a hard problem, but from the characterization of when Player II has a winning strategy, and from the conditions required to prevent all k-tactics from winning, we are able to see why.

Definition 1.10.1. *Suppose $(\mathbb{P}, \leq)$ is a partial order, and*

$$W_{II} \subseteq \{z \in \mathbb{P}^\omega : z \textit{ is descending}\}.$$

The Choquet game on $\mathbb{P}$ with winning set W_{II}, $G_{(\mathbb{P},W_{II})}$, is defined as follows. Player I begins, and plays an element of $\mathbb{P}$. If on turn n Player i has played $p \in \mathbb{P}$, then on turn $n+1$ Player j, where $i \neq j$ is required to play some $q \in \mathbb{P}$ with $q \leq p$. A run of the game $G_{(\mathbb{P},W_{II})}$ can then be seen as an element of $\{z \in \mathbb{P}^\omega : z \text{ is descending}\}$. Player II is then said to win if and only if the run of the game $s \in \{z \in \mathbb{P}^\omega : z \text{ is descending}\}$ satisfies $s \in W_{II}$. The set $\{z \in \mathbb{P}^\omega : z \text{ is descending}\}$ is called the game space, consisting of all runs of the game.

Winning strategies etc. are defined in the obvious way, by making appropriate modifications to the definitions for the topological Choquet game.

Example 1.10.2. *Suppose (X, τ) is a topological space. Take $\mathbb{P} = \tau \setminus \{\emptyset\}$, $\leq = \subseteq$, and $W_{II} = \{z \in \mathbb{P}^\omega : z \text{ is descending and } \bigcap \operatorname{ran}(z) \neq \emptyset\}$. Then $G_{(\mathbb{P},W_{II})}$ is precisely the Choquet game on X, G_X.*

This justifies our notation; the game defined above can be seen as a generalisation of the (topological) Choquet game on X.

One of the properties of the set W_{II} in the topological Choquet game, is it does not depend on what happens in the first n moves for any $n \in \omega$. Whether a Player wins depends on the long term infinitistic behaviour of a run of the game. Though more can be said about closure properties of the set of winning sequences in the topological Choquet game, the following is a simple condition that must be true, and serves our purposes for the moment.

Definition 1.10.3. *If W_{II} is a winning set for Player II in a Choquet game $G_{(\mathbb{P},W_{II})}$, let us say that W_{II} is asymptotic if and only if, given any $x \in W_{II}$ and any $y \in \{z \in \mathbb{P}^\omega : z \text{ is descending}\}$ such that $\operatorname{ran}(x) \cap \operatorname{ran}(y)$ is infinite, we have $y \in W_{II}$.*

Intuitively, if W_{II} is asymptotic, then the game $G_{(\mathbb{P},W_{II})}$ is determined by the "long term behaviour" of the game. We obtain a characterisation of those asymptotic W_{II} for which Player II has a winning strategy, in terms of the category (in the sense of Baire) of W_{II}. First however, we introduce a different topological game, the Banach-Mazur game, and translate properties of our generalised Choquet game into properties of this new game.

Definition 1.10.4. *Let X be a topological space, and A a subset of X. The Banach-Mazur game on X with target set A, $BM(X, A)$, is an infinite length two player game, played as follows: Player I first plays a non-empty open subset of X; if Player i plays a non-empty open subset U_n of X on turn n, then on turn $n+1$, Player j, for $j \neq i$, must play a non-empty open subset U_{n+1} of X with $U_{n+1} \subseteq U_n$.*

On a run $\langle (U_i, V_i) \,:\, i \in \omega \rangle$ of $BM(X, A)$, Player II is said to win if and only if

$$\bigcap_{i\in\omega} U_i = \bigcap_{i\in\omega} V_i \subseteq A.$$

Winning strategies for each player are defined in the obvious way, by making the appropriate changes to the definition for the Choquet game.

Remark 1.10.5. *What we have labelled the (topological) Choquet game is often called the Banach-Mazur game in the literature. We have adopted the conventions of Kechris [17].*

Definition 1.10.6. *Suppose $\mathcal{S}$ is a collection of subsets of the space X. Define the Banach-Mazur game on X with winning set A with pieces from $\mathcal{S}$, $BM(X, A)^{\mathcal{S}}$, to be the modification of the Banach-Mazur game obtained by restricting Player I and II's moves to elements of $\mathcal{S}$, rather than non-empty open subsets of X. In this way we have $BM(X, A) = BM(X, A)^{\tau\setminus\{\emptyset\}}$.*

Observation 1.10.7. *Suppose $\mathcal{W}$ is a weak-base for the space X. Then Player II has a winning strategy in $BM(X, A)$ if and only if Player II has a winning strategy in $BM(X, A)^{\mathcal{W}}$.*

We translate the Choquet game on a partial ordering $\mathbb{P}$ with winning set W_{II} into a Banach-Mazur game on a topological space, when W_{II} is asymptotic. This gives us an external view on the game, by which we can identify when Player I or II has a winning strategy by looking at the shape of the set of winning plays as a whole. The requirement that W_{II} is asymptotic is used to compress finite sequences of play into single moves.

Proposition 1.10.8. *Suppose W_{II} is an asymptotic winning set for Player II in $G_{(\mathbb{P},W_{II})}$. Equip $\mathbb{P}$ with the discrete topology, and view W_{II} as a subset of the game space*

$G = \{z \in \mathbb{P}^\omega \colon z \text{ is descending}\}$, *where* G *is given the subspace topology of the Tychonoff product topology on* $\mathbb{P}^\omega$. *Then, Player II has a winning strategy in* $G_{(\mathbb{P},W_{II})}$ *if and only if Player II has a winning strategy in* $BM\left(G, W_{II}\right)$.

Proof. First let us note that $\mathcal{B} = \{N_s \colon s \in \mathbb{P}^{<\omega} \text{ such that } s \text{ is descending}\}$ forms a base for G, where $N_s = \bigcap_{i<\text{length}(s)} \pi_i^{-1}\left(s(i)\right)$. Hence Player II has a winning strategy in $BM\left(G, W_{II}\right)$ if and only if Player II has a winning strategy in $BM\left(G, W_{II}\right)^{\mathcal{B}}$, by the above observation. Observe also that $N_s \subseteq N_t$ if and only if $t \subseteq s$.

If $t \in \mathbb{P}^{<\omega}$ is decreasing, define $\text{last}\left(t\right) = t\left(\text{length}\left(t\right) - 1\right)$, the last term of t. Given a tuple of decreasing sequences in $\mathbb{P}^{<\omega}$, $\left(t_1, \ldots, t_n\right)$ define

$$\text{lasts}\left(t_1, \ldots, t_n\right) = \left(\text{last}\left(t_1\right), \ldots, \text{last}\left(t_n\right)\right).$$

We also extend lasts to infinite sequences of decreasing sequences in $\mathbb{P}^{<\omega}$ in the obvious way. lasts is used in a similar way to transfer maps in previous proofs.

Let us suppose first that Player II has a winning strategy σ in $G_{(\mathbb{P},W_{II})}$. We construct a winning strategy μ for Player II in the Banach-Mazur game $BM\left(G, W_{II}\right)^{\mathcal{B}}$. We construct μ recursively, and μ will have the property that if $\left(N_{r_1}, \ldots, N_{r_n}\right) \in \mu$ then $\text{lasts}\left(r_1, \ldots, r_n\right) \in \sigma$. Of course $\emptyset \in \mu$ and if $\left(N_{s_1}, N_{t_1}, \ldots, N_{s_n}, N_{t_n}\right) \in \mu$ then for any descending s_{n+1} with $t_n \subseteq s_{n+1}$, $\left(N_{s_1}, N_{t_1}, \ldots, N_{s_n}, N_{t_n}, N_{s_{n+1}}\right) \in \mu$. Suppose then that $\left(N_{s_1}, N_{t_1}, \ldots, N_{s_n}\right) \in \mu$. Put $s = \left(s_1, t_1, \ldots, s_n\right)$, so that $\text{lasts}\left(s\right) \in \sigma$. Let $p = \sigma\left(\text{lasts}\left(s\right)\right)$, and set $t_n = s_n{}^\frown\left(p\right)$. We demand that $\left(N_{s_1}, N_{t_1}, \ldots, N_{s_n}, N_{t_n}\right) \in \mu$. Note that μ defined in this way has the desired properties.

Now if $\langle\left(N_{s_i}, N_{t_i}\right) \ : \ i \in \omega\rangle$ is a μ-compatible play of $BM\left(G, W_{II}\right)^{\mathcal{B}}$, then

$$\text{lasts}\left(\langle\left(s_i, t_i\right) \ : \ i \in \omega\rangle\right)$$

is σ-compatible, so is an element of W_{II}. Hence, since W_{II} is asymptotic, $\bigcup_{i\in\omega} s_i \in W_{II}$. But $\bigcap_{i\in\omega} N_{s_i} = \left\{\bigcup_{i\in\omega} s_i\right\} \subseteq W_{II}$, so μ is a winning strategy for Player II in $BM\left(G, W_{II}\right)^{\mathcal{B}}$.

Now we establish the converse, and suppose that μ is a winning strategy for Player II in $BM\left(G, W_{II}\right)^{\mathcal{B}}$. We construct a strategy σ for Player II in $G_{(\mathbb{P},W_{II})}$, with a map $T \colon \sigma \to \mu$ with the properties

1. if $s \in \sigma$ then length $(T(s)) =$ length (s), and if $s \subseteq t$ then $T(s) \subseteq T(t)$;

2. if $T(s) = (N_{r_1}, \ldots, N_{r_n})$ then lasts $(r_1, \ldots, r_n) = s$.

(One might think of T as essentially being a transfer map which has been slightly adapted to suit present circumstances).

Of course, $\emptyset \in \sigma$ with $T(\emptyset) = \emptyset$, and if $(p_1, q_1, \ldots, p_n, q_n) \in \sigma$, then for any $p_{n+1} \leq q_n$, we must have $(p_1, q_1, \ldots, p_n, q_n, p_{n+1}) \in \sigma$. Furthermore, if $T(p_1, q_1, \ldots, p_n, q_n) = (N_{s_1}, N_{t_1}, \ldots, N_{t_n})$ then we define

$$T(p_1, q_1, \ldots, p_n, q_n, p_{n+1}) = \left(N_{s_1}, N_{t_1}, \ldots, N_{t_n}, N_{t_n{}^\frown(p_{n+1})}\right)$$

which clearly preserves conditions (1) and (2).

So suppose we have $s = (p_1, q_1, \ldots, p_n) \in \sigma$. Suppose $N_{t_n} = \mu(T(s))$. We declare $q_n =$ last (t_n), and put $s^\frown(q_n) \in \sigma$, and define $T(s^\frown(q_n)) = T(s)^\frown(N_{t_n})$, as is required by (2). This defines our strategy σ.

We argue that σ is a winning strategy. Suppose $s = \langle (p_i, q_i) : i \in \omega \rangle$ is a σ-compatible play of $G_{(\mathbb{P}, W_{II})}$. Then $T(s) = \bigcup_{n \in \omega} T(s \restriction n)$ is μ-compatible by condition (1). Hence, $\bigcap \text{ran}(T(s)) \subseteq W_{II}$. But $T(s) = \langle (N_{s_i}, N_{t_i}) : i \in \omega \rangle$, so $\bigcap \text{ran}(T(s)) = \left\{\bigcup_{i \in \omega} s_i\right\}$, so $\bigcup_{i \in \omega} s_i \in W_{II}$. Since W_{II} is asymptotic,

$$\text{lasts} \langle (s_i, t_i) : i \in \omega \rangle \in W_{II}$$

also, but by (2)

$$\text{lasts} \langle (s_i, t_i) : i \in \omega \rangle = \langle (p_i, q_i) : i \in \omega \rangle$$

and so we see that σ is a winning strategy for Player II in $G_{(\mathbb{P}, W_{II})}$. □

The previous theorem is useful, because the following characterises precisely when Player II has a winning strategy in $BM(X, A)$.

Theorem 1.10.9 ([17], Theorem 8.33). *Player II has a winning strategy in $BM(X, A)$ if and only if A is comeager in X.*

Combining this with what we have already, we obtain a theorem which says that Player II has a winning strategy in $G_{(\mathbb{P}, W_{II})}$ if and only if most plays lie in W_{II}.

Theorem 1.10.10. *Suppose W_{II} is an asymptotic winning set for Player II in $G_{(\mathbb{P},W_{II})}$. Equip $\mathbb{P}$ with the discrete topology, and view W_{II} as a subset of the game space $G = \{z \in \mathbb{P}^\omega \colon z \text{ is descending}\}$, where G is given the subspace topology of the Tychonoff product topology on $\mathbb{P}^\omega$. Then, Player II has a winning strategy in $G_{(\mathbb{P},W_{II})}$ if and only if W_{II} is comeager in G.*

Proof. Follows from the previous proposition and theorem. □

Remark 1.10.11. *G is a closed subspace of $\mathbb{P}^\omega$. Since $\mathbb{P}$ is completely metrizable (in the discrete topology), so is $\mathbb{P}^\omega$. Hence G is completely metrizable, and so a Baire space. Therefore, the property of being a comeager subset of G is a non-trivial notion.*

Remark 1.10.12. *We may do the analagous argument for Player I. We obtain for asymptotic W_{II}, that Player I has a winning strategy in $G_{(\mathbb{P},W_{II})}$ if and only if W_{II} is meagre in some non-empty open subset of the game space G. Since the game space G is Baire, every subset with the Baire Property (see Kechris [17], Chapter I, 8.F) is either comeagre or meagre in some open subset. So in particular, for an undetermined topological Choquet game, the winning set W_{II} cannot have the Baire Property, and in particular cannot be Borel.*

As mentioned previously, we can say more about the properties of W_{II} in the topological Choquet game.

Definition 1.10.13. *Consider a Choquet game $G_{(\mathbb{P},W_{II})}$, and a relation R on $\mathbb{P}$. Suppose G is the game space for $G_{(\mathbb{P},W_{II})}$, and $F \subseteq G$.*

1. *We define the squashed closure of F with respect to R,* $\text{Squash}_R(F)$ *to be the set*

$$\{z \in G \colon \exists\, f \in F\ \forall\, i \in \omega\ \exists\, j \in \omega\ z(j) R f(i)\}.$$

   We call a subset F of G, R-squash-closed if and only if $\text{Squash}_R(F) = F$. *We also abuse notation and write* $\text{Squash}_R(x)$ *for* $\text{Squash}_R(\{x\})$, *when $x \in G$.*

2. *We define the float closure of F with respect to R,* $\text{Float}_R(F)$ *to be the set*

$$\{z \in G \colon \exists\, f \in F\ \forall\, i \in \omega\ \exists\, j \in \omega\ f(j) R z(i)\}.$$

We call a subset F of G, R-float-closed if and only if $\text{Float}_R(F) = F$. *We also abuse notation and write* $\text{Float}_R(x)$ *for* $\text{Float}_R(\{x\})$, *when* $x \in G$.

Remark 1.10.14. *The naming here corresponds to some intuition.* $y \in \text{Squash}_R(F)$ *if and only if, with respect to* R *there is some* f *in* F *such that any point along* f *always ends up with some point along* y *below it. One can think of* y *as being "squashed by"* f. *Similarly,* $y \in \text{Float}_R(F)$ *if and only if (with respect to* R*) there is an* f *in* F *such that any point along* y *has some point along* f *underneath it. One can think of* y *as "floating on"* f *(with respect to* R*).*

We are primarily interested in the closure operators $F \mapsto \text{Squash}_{\leq}(F)$ and $F \mapsto \text{Float}_{\leq}(F)$ for our partial ordering $(\mathbb{P}, \leq)$.

Observation 1.10.15. *Suppose* W_{II} *is the winning set for a topological Choquet game* G_X, *and let* G *be the game space. Then*

1. $\text{Squash}_{\leq}(G \setminus W_{II}) = G \setminus W_{II}$ *(i.e.* $G \setminus W_{II}$ *is* $\leq$*-squash-closed).*
2. $\text{Float}_{\leq}(W_{II}) = W_{II}$ *(i.e.* W_{II} *is* $\leq$*-float-closed).*

The following says that any sequence that could lose can be made to lose for some appropriate choice of W_{II}.

Proposition 1.10.16. *Suppose* $(\mathbb{P}, \leq)$ *is a partial ordering, and* $x \in G$ *where* G *is the game space. Suppose for each* $i \in \omega$ *and* $t \leq x(i)$ *there is a* $j > i$ *and* $p \in \mathbb{P}$ *such that* $p \leq t$ *but there is no* $q \in \mathbb{P}$ *with* $q \leq x(j), p$ *(i.e. "*x *is strongly descending"), and furthermore, that there is no* $p \in \mathbb{P}$ *with* $p \leq x(i)$ *for all* $i \in \omega$. *Then* $\text{Squash}_{\leq}(x)$ *is nowhere dense in* G.

Proof. Suppose $N_s \cap \overline{\text{Squash}_{\leq}(x)}^G \neq \emptyset$, where s is a finite descending sequence of elements of $\mathbb{P}$ of length n say. Then $N_s \cap \text{Squash}_{\leq}(x) \neq \emptyset$. We show there is a finite descending sequence t of elements of $\mathbb{P}$ such that $s \subseteq t$ but $N_t \cap \text{Squash}_{\leq}(x) = \emptyset$. If this is the case, then $N_t \subseteq N_s$, and we see $N_t \cap \overline{\text{Squash}_{\leq}(x)}^G = \emptyset$, so in particular, $N_s \not\subseteq \overline{\text{Squash}_{\leq}(x)}^G$. Since s was arbitrary, this implies $\text{Squash}_{\leq}(x)$ is nowhere dense.

Claim 1.10.17. *There is a $p \leq s(n-1)$ such that for some $j \in \omega$ there is no $q \in \mathbb{P}$ with $q \leq p, x(j)$.*

Proof of Claim. If there is no $t \in \mathbb{P}$ with $t \leq s(n-1), x(0)$ then taking $p = s(n-1)$ will work. Otherwise, there is a $t \in \mathbb{P}$ with $t \leq s(n-1), x(0)$. Hence we may find a $p \leq t$ and $j \in \omega$ such that there is no $q \in \mathbb{P}$ with $q \leq p, x(j)$. Since $p \leq t \leq s(n-1)$, $p \leq s(n-1)$. □

So suppose $p \leq s(n-1)$ is such that for some $j \in \omega$ there is no $q \in \mathbb{P}$ with $q \leq p, x(j)$. Let $t = s^\frown(p)$. Suppose $y \in N_t$ and $k \in \omega$ with $y(k) \leq x(j)$. Since y is descending we can suppose $k \geq n$, so that $y(k) \leq x(j), p$. But this is a contradiction, so no such k exists, and we see $N_t \cap \text{Squash}_{\leq}(x) = \emptyset$. □

Proposition 1.10.18. *Suppose F is a subset of the game space G. Then*

$$\text{Float}_{\leq}\left(G \setminus \text{Squash}_{\leq}(F)\right) = G \setminus \text{Squash}_{\leq}(F).$$

Proof. It is clear that $G \setminus \text{Squash}_{\leq}(F) \subseteq \text{Float}_{\leq}\left(G \setminus \text{Squash}_{\leq}(F)\right)$, so we only need to show the reverse inclusion. So let $y \in \text{Float}_{\leq}\left(G \setminus \text{Squash}_{\leq}(F)\right)$. Then there is an $x \in G \setminus \text{Squash}_{\leq}(F)$ such that

$$\forall\, i \in \omega \exists\, j \in \omega\ x(j) \leq y(i).$$

Now let $f \in F$. We know that $x \notin \text{Squash}_{\leq}(F)$, so there is some $i \in \omega$ such that for any $j \in \omega$

$$\neg\, x(j) \leq f(i).$$

Now if $k \in \omega$ then we can find a $j \in \omega$ with $x(j) \leq y(k)$. In particular then, we cannot have $y(k) \leq f(i)$. Hence $y \notin \text{Squash}_{\leq}(F)$. So $y \in G \setminus \text{Squash}_{\leq}(F)$, obtaining the required reverse inclusion. □

Given a set of sequences F we can build a winning set with some of the flavour of the topological Choquet game by taking $W_{II} = G \setminus \text{Squash}_{\leq}(F)$. If F contains a sequence from the body of a strategy σ, then σ will not be a winning strategy in $G_{(\mathbb{P}, W_{II})}$. Note that the proposition preceeding the previous proposition tells us how

to select a sequence x from the body of such a strategy so that, in the case when $F = \{x\}$, we still ensure a winning strategy exists. But it is clear that

$$\text{Squash}_R(F) = \bigcup_{f \in F} \text{Squash}_R(f).$$

so in fact we can easily prevent countably many strategies from winning, and at the same time be sure a winning strategy exists.

However, in general, there are $2^{|\mathbb{P}|}$ many 2-tactics, and this perhaps indicates why the problem of making every 2-tactic fail to win is difficult — we require a meager set satisfying a large number of conditions. Of course, Debs' counterexamples show this is plausible; there are $2^{|\mathbb{P}|}$ many 1-tactics.

Chapter 2

ω^* and the Filter Dichotomy

2.1 Introduction to the Filter Dichotomy and ω^*

2.1.1 Introduction

$\beta\omega$ and $\omega^* = \beta\omega \setminus \omega$, the Stone-Čech compactification of ω and the remainder, are two of the most important spaces in general topology. A lot of what can be said about these spaces depends upon specific set-theoretic principles, tools which allow us to better analyse the inherent set-theoretic structure. In various group conversations at Oxford, an interest developed in the classical question of characterizing the closed subspaces of $\beta\omega$ and ω^*. Van Mill's article [32] provides a characterisation under **CH** for instance, which in part answered this curiosity — see Theorem $1.4.4$ of van Mill's article for details — but the group remained interested in what might be said without the continuum hypothesis, possibly in the presence of other set-theoretic principles.

David Fremlin then delivered a seminar on medial limits in Oxford, and introduced a new set-theoretic principle to the group, the Filter Dichotomy, **FD**. Although Fremlin's interests were centred on measure theory, the Filter Dichotomy appeared to say something of the structure of closed subsets of ω^*. This observation led to a collaborative investigation within the group, and a translation, via Stone Duality, of the Filter Dichotomy into a property of closed subsets of ω^*. My own

investigations developed into related questions about the Filter Dichotomy itself. Below is my contribution to this.

Indeed, the Filter Dichotomy is equivalent to an inequality between small cardinals, namely $\mathfrak{u} < \mathfrak{g}_f$. Small cardinals are quantities between $\aleph_1$ and $\mathfrak{c}$ which are of interest to mathematicians for a variety of reasons. In the above inequality for instance, $\mathfrak{u}$ is the ultrafilter number, the size of the smallest base for an ultrafilter on ω. Typically, when one establishes a result from **CH**, one often only requires that some small cardinal be equal to $\aleph_1$ or $\mathfrak{c}$. Similarly, if one establishes a result in some forcing extension, one often only needs some relationship to hold between two or more of these quantities. The relationships between small cardinals, and the non-existence (in **ZFC**) of other relationships has been a subject of extensive study. However, some relationships have been remarkably hard to establish or disprove. Perhaps the most well known of these problems was the question of whether $\mathfrak{p}$ and $\mathfrak{t}$ could be distinguished. To many, one of the difficulties here arose from the inability of countable support iterations to push $\mathfrak{c}$ beyond $\aleph_2$ — it was already known that if $\mathfrak{p} = \aleph_1$ then $\mathfrak{t} = \aleph_1$. This is an instance of a more general problem, called the continuum problem, of pushing $\mathfrak{c}$ beyond $\aleph_2$ while obtaining some properties of models typically obtained via countable support iterations of proper forcings. However, Shelah and Malliaris [20] have recently established in **ZFC** that $\mathfrak{p} = \mathfrak{t}$, exemplifying the deep relationships that occur between small cardinals. Another instance of the continuum problem is that of obtaining a model of the Filter Dichotomy with $\mathfrak{u} > \aleph_1$. We investigate this problem later.

We begin with an introduction to the basic concepts and results, most of which should be well known. The proofs in this section are my own unless otherwise indicated, though they are mostly extremely straightforward. Some basic results have been moved to appendices when they do not align with the bulk of the material.

After introducing the basic principles and results, the questions of interest in this chapter will be introduced and motivated, and an overview of the rest of the chapter provided. The work in subsequent sections is entirely my own unless indicated otherwise.

2.1.2 Notation

We recount some basic notation that is used almost universally. For a filter $\mathcal{F}$, we write $\chi(\mathcal{F})$ for the character of $\mathcal{F}$, the cardinality of the smallest base for $\mathcal{F}$. For subsets A and B of ω, we write $A \subseteq^* B$, and say A is almost a subset of B if and only if $B \setminus A$ is finite. We freely identify $\beta\omega$ and ω^* with the appropriate spaces of ultrafilters on ω. For a subset A of ω, we write A^* for the clopen subset of ω^* given by

$$A^* = \overline{A}^{\beta\omega} \cap \omega^* = \{\mathcal{U} \in \omega^* \colon A \in \mathcal{U}\}.$$

Recall that all clopen subsets of ω^* are of the form A^* for some $A \subseteq \omega$.

2.1.3 Some basic results

Definition 2.1.1. *Suppose $\mathcal{F}, \mathcal{G}$ are filters on ω. We say that $\mathcal{F} \leq_{RB} \mathcal{G}$ if and only if there is a finite-to-one map $\phi \colon \omega \to \omega$ such that $\phi(\mathcal{G}) = \mathcal{F}$, where here $\phi(\mathcal{G})$ is the filter generated by $\{\phi(G) : G \in \mathcal{G}\}$. We also say that $\mathcal{G}$ is almost $\mathcal{F}$ in this case. The ordering $\leq_{RB}$ on free filters is known as the Rudin-Blass ordering.*

The Rudin-Blass ordering was first named by Laflamme and Zhu in [19] in the context of ultrafilters. They discuss there its basic properties, and in particular compare it with the more classical Rudin-Keisler ordering, $\leq_{RK}$ (covered in some detail by Comfort and Negrepontis in [9]). The Rudin-Keisler ordering is defined in the same way as the Rudin-Blass ordering, but the finite-to-one restriction is removed. The two orderings behave quite differently. Van Mill had shown much earlier in [31] that there are free ultrafilters $\mathcal{U}, \mathcal{V}$ on ω with $\mathcal{U} \leq_{RK} \mathcal{V}$ but $\mathcal{U} \not\leq_{RB} \mathcal{V}$.

Interest in the Rudin-Blass ordering seems to have been motivated by Rudin's paper [24], where Rudin showed that if there are two ultrafilters $\mathcal{U}, \mathcal{V}$ such that for all finite-to-one maps $\phi \colon \omega \to \omega$, $\phi(\mathcal{U}) \neq \phi(\mathcal{V})$ (that is, **NCF** fails — see Appendix A), then $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$ has at least two composants. Here $\mathbb{H}$ is the half line $[0, \infty)$, and it is well known that $\mathbb{H}^*$ is a connected compact Hausdorff space (that is, a continuum) which cannot be written as a union of two proper subcontinua. A

composant of a point x in a continuum X (with at least two points) is the union of all proper subcontinua containing x. Clearly composants of X are connected and partition the space X. Like ω^*, $\mathbb{H}^*$ is an important space in set-theoretic topology.

The Rudin-Blass order can also be strengthened — the strong Rudin-Blass order is obtained by insisting in addition that our finite-to-one map also be increasing. Laflamme and Zhu show that the strong Rudin-Blass ordering captures some of the structure of "standard subcontinua" of $\mathbb{H}^*$ — see [19].

Definition 2.1.2. *The cofinite or Fréchet filter, Cof, is the filter*

$$\{A \subseteq \omega \colon \omega \setminus A \textit{ is finite}\}.$$

A filter $\mathcal{F}$ on ω is a free filter if and only if Cof $\subseteq \mathcal{F}$.

Since we are only interested in free filters, we assume from now on, unless stated otherwise, that all filters are free filters on ω.

Definition 2.1.3 (Filter Dichotomy, **FD**)**.** *The Filter Dichotomy,* **FD***, is the statement that for any free filter $\mathcal{F}$ on ω, one of the following is true.*

1. *Cof $\leq_{RB} \mathcal{F}$*
2. *$\mathcal{U} \leq_{RB} \mathcal{F}$, for some ultrafilter $\mathcal{U}$.*

That is, for any free filter $\mathcal{F}$ on ω, $\mathcal{F}$ is almost Cof or almost an ultrafilter.

By Stone Duality, any discussion of free filters on ω can be seen as a discussion of closed subsets of ω^*. Robert Leek observed that the Filter Dichotomy is equivalent to the statement that any closed subset C of ω^* can be mapped by some $\beta\phi$, where $\phi \colon \omega \to \omega$ is finite-to-one, to either a singleton or the entirety of ω^*. Note that the condition that ϕ is finite-to-one is equivalent to saying ϕ maps free filters to free filters, and in turn that $\beta\phi$ restricts to a map $\omega^* \to \omega^*$.

The free ultrafilters are in some sense the most complicated filters one can find on ω. Equally, *Cof* is the simplest free filter on ω; every other free filter contains it. So the next result tells us that the Filter Dichotomy is in fact the very best division

we might hope for of its kind — the almost ultrafilters and almost cofinite filters do not overlap. In other words, the Filter Dichotomy is indeed justified in being called a dichotomy, since (1) and (2) are mutually exclusive in the above definition of the principle. The following was proved in Oxford by Will Brian (and presumably many before him, although I was unable to find a reference), but neither of us have much recollection of his proof. Alternatively it follows from the fact that almost *Cof* filters are preserved under finite-to-one maps (shown for instance in [4], pg5), since clearly no ultrafilter is almost *Cof*. The following proof was a collaborative effort between myself and Rolf Suabedissen.

Lemma 2.1.4. *There is no free filter $\mathcal{F}$ which is both almost an ultrafilter and almost Cof.*

Proof. Let us suppose $\phi\colon \omega \to \omega$ and $\psi\colon \omega \to \omega$ are finite-to-one maps such that $\psi(\mathcal{F}) = \mathcal{U}$, where $\mathcal{U}$ is an ultrafilter. We construct a $B \in \mathcal{F}$ such that $\phi(B) \notin \mathit{Cof}$, and hence $\phi(\mathcal{F}) \neq \mathit{Cof}$. By composing with appropriate finite-to-one maps, we may as well suppose ϕ and ψ are surjective.

We construct, by recursion, distinct $\{p_i\colon i \in \omega\} \subseteq \omega$, and $B_i \subseteq \omega$, $i \in \omega$, such that $\phi^{-1}\{p_i\} \subseteq B_i$, $\psi^{-1}(\psi(B_i)) = B_i$, and $i \neq j$ implies $p_i \notin \phi(B_j)$. For the purposes of the construction, we also demand that B_i is finite for each $i \in \omega$ (this is simply so we can continue the recursion).

Given that we can construct these sets, we show how to obtain our set B. Let $C = \omega \setminus \bigcup_{i\in\omega} B_i$. Set $A_1 = \bigcup_{i\in\omega} B_{2i} \cup C$, and $A_2 = \bigcup_{i\in\omega} B_{2i+1}$. Then $\{A_1, A_2\}$ cover ω, hence so do $\{\psi(A_1), \psi(A_2)\}$. In particular either $\psi(A_1) \in \mathcal{U}$, in which case put $B = A_1$, or $\psi(A_2) \in \mathcal{U}$, in which case put $B = A_2$. Since $C = \psi^{-1}(\psi(C))$ (as $B_i = \psi^{-1}(\psi(B_i))$ for each $i \in \omega$), we have $B = \psi^{-1}(\psi(B))$. Since $\psi(B) \in \mathcal{U}$, this means $B \in \mathcal{F}$. But because for each $i, j \in \omega$, $\phi^{-1}\{p_i\} \cap B_j \neq \emptyset$ if and only if $i = j$, we have either $\phi(B) \cap \{p_{2i}\colon i \in \omega\} = \emptyset$ or $\phi(B) \cap \{p_{2i+1}\colon i \in \omega\} = \emptyset$, whence $\phi(B) \notin \mathit{Cof}$.

So we only need to justify the recursive construction. Suppose we have defined p_i and B_i for $i < j$ with the required properties. We define p_j and B_j. Now $\bigcup_{i<j} B_i$

is finite, so $\phi\left(\bigcup_{i<j} B_i\right)$ is finite. Pick $p_j \in \omega \setminus \phi\left(\bigcup_{i<j} B_i\right)$. We define B_j as the ψ-saturation of the ϕ-fibre of p_j, that is, put

$$B_j = \psi^{-1}\left(\psi\left(\phi^{-1}\left\{p_j\right\}\right)\right).$$

Note that B_j is finite because ϕ and ψ are finite-to-one. It is easily seen that p_j and B_j have the required properties. □

Corollary 2.1.5 (FD). *Suppose $\phi\colon \omega \to \omega$ is finite-to-one and $\mathcal{F}$ a free filter on ω. Then*

1. *If $\mathcal{F}$ is almost Cof, then $\phi(\mathcal{F})$ is almost Cof.*

2. *If $\mathcal{F}$ is almost an ultrafilter, then $\phi(\mathcal{F})$ is almost an ultrafilter.*

Proof. Observe that if ϕ and ψ are both finite-to-one maps $\omega \to \omega$, then $\phi \circ \psi$ is also a finite-to-one map $\omega \to \omega$. Hence, in the case of (1), if $\phi(\mathcal{F})$ were almost an ultrafilter, $\mathcal{F}$ would be also, in contradiction to the previous lemma. By **FD**, $\phi(\mathcal{F})$ must hence be almost *Cof*. A similar argument works for (2). □

In fact, **FD** is not needed for the above, there are proofs in **ZFC** — see the Appendix. The above proofs are retained for their simplicity.

We prove now some basic results concerning free filters in relation to some small cardinals. These motivate the questions we ask later. As is conventional, we write $\mathfrak{u}$ for the cardinality of the smallest base for a free ultrafilter on ω. The following is then clear:

Lemma 2.1.6. *If a free filter $\mathcal{F}$ is almost an ultrafilter, then $\chi(\mathcal{F}) \geq \mathfrak{u}$.*

Proof. Suppose $\phi\colon \omega \to \omega$ is finite-to-one and $\phi(\mathcal{F}) = \mathcal{U}$, where $\mathcal{U}$ is an ultrafilter. If $\mathcal{B}$ is a base for $\mathcal{F}$, then $\{\phi(B) : B \in \mathcal{B}\}$ is a base for $\mathcal{U}$, hence

$$\mathfrak{u} \leq \chi(\mathcal{U}) \leq |\mathcal{B}|.$$

It follows that $\mathfrak{u} \leq \chi(\mathcal{F})$. □

Corollary 2.1.7 (FD). *If $\mathcal{F}$ is a free filter on ω and $\chi(\mathcal{F}) < \mathfrak{u}$, then Cof $\leq_{RB} \mathcal{F}$.*

One naturally asks what values of $\mathfrak{u}$ are permissable alongside **FD**. In all known models of **FD** so far, $\mathfrak{u} = \aleph_1$. However, one easily sees that countably based free filters on ω are almost *Cof* — indeed this follows as a corollary to a result we prove shortly.

Recall that a collection of subsets $\mathcal{A}$ of ω has the strong finite intersection property if and only if for any finite subset $\mathcal{B}$ of $\mathcal{A}$, $\bigcap \mathcal{B}$ is infinite. A collection of subsets $\mathcal{A}$ of ω is said to have a pseudointersection if there is an infinite subset C of ω with

$$C \subseteq^* A$$

for each $A \in \mathcal{A}$. The smallest cardinality of a collection of subsets of ω with the strong finite intersection property but no pseudointersection is defined to be the cardinal $\mathfrak{p}$, the pseudointersection number.

Lemma 2.1.8. *If a free filter $\mathcal{F}$ on ω has a pseudointersection, then Cof $\leq_{RB} \mathcal{F}$.*

Proof. Suppose $\mathcal{F}$ has a pseudointersection, A say. We can extend any bijection $A \to \omega$ to a finite-to-one map $\phi\colon \omega \to \omega$. Then ϕ preserves $\subseteq^*$, so that for $F \in \mathcal{F}$ we have

$$\omega = \phi(A) \subseteq^* \phi(F)$$

and hence $\phi(F) \in \mathit{Cof}$. Since $\{\phi(F) : F \in \mathcal{F}\}$ is a base for the filter $\phi(\mathcal{F})$, we have $\phi(\mathcal{F}) = \mathit{Cof}$. □

Lemma 2.1.9. *If $\chi(\mathcal{F}) < \mathfrak{p}$ then Cof $\leq_{RB} \mathcal{F}$.*

Proof. If $\mathcal{F}$ is a free filter on ω, then $\mathcal{F}$ has the strong finite intersection property. Hence if $\chi(\mathcal{F}) < \mathfrak{p}$, $\mathcal{F}$ must have a pseudointersection. The above lemma then applies. □

As promised, we have as a corollary that countably based free filters on ω are almost *Cof*.

Corollary 2.1.10. *If $\mathcal{F}$ is a free filter on ω and $\chi(\mathcal{F}) = \aleph_0$, then Cof $\leq_{RB} \mathcal{F}$.*

Proof. Use the above lemma and the fact that $\mathfrak{p} \geq \aleph_1$. □

We also obtain a proof of a well known inequality between $\mathfrak{u}$ and $\mathfrak{p}$, an instance of a basic inequality between small cardinals which can be established in **ZFC**.

Corollary 2.1.11. $\mathfrak{p} \leq \mathfrak{u}$.

Proof. If $\mathfrak{u} < \mathfrak{p}$ then there would be an ultrafilter (which is in particular almost an ultrafilter) that is almost *Cof*. □

We also have the following. Martin's Axiom is well known for making all small cardinals equal to $\mathfrak{c}$.

Corollary 2.1.12 (**MA**). *If* $\chi(\mathcal{F}) < \mathfrak{c}$ *then Cof* $\leq_{RB} \mathcal{F}$.

Proof. Under **MA**, it is well-known that $\mathfrak{p} = \mathfrak{c}$. □

The above tells us it is possible to have models of **ZFC** where all free filters of character $\aleph_1$ are almost cofinite. This, in combination with our results concerning $\mathfrak{u}$ prompts the following question:

Question 2.1.13. *Is it consistent, relative to* **ZFC***, that the Filter Dichotomy holds and* $\mathfrak{u} > \aleph_1$*?*

We concern ourselves with this question later, and the more traditional instance of a continuum problem, namely the relative consistency of $\mathfrak{u} < \mathfrak{g}$ with $\mathfrak{u} > \aleph_1$ ($\mathfrak{g}$ is another instance of a small cardinal, and will be defined later). Analysis of these problems forms the bulk of the material in this part of the thesis.

One should note, of course, that $(\mathbf{MA} \rightarrow \neg\mathbf{FD})$, so under **MA** there is a free filter $\mathcal{F}$ which is not almost *Cof* and not almost an ultrafilter. Such a filter necessarily has character $\mathfrak{c}$.

2.1.4 Overview

We begin in section 2.2 with a brief study of the space of continuous functions $\omega^* \rightarrow \omega^*$. The finite-to-one maps $\omega \rightarrow \omega$ are precisely those maps whose Stone-Čech extensions restrict to maps $\omega^* \rightarrow \omega^*$. It seems interesting to ask precisely how

these maps appear inside the space of all continuous self-maps on ω^*, and we show that they are dense in the compact-open (and hence also pointwise) topology. The space of continuous maps $\omega^* \to \omega^*$ with the compact-open topology should be an interesting space to study, but it is not explored fully here, and is left as a future project.

Proofs of the existence of filters witnessing $\neg\mathbf{FD}$ tend to use indirect methods — either by comparing related small cardinals, or by considering medial limits. We construct in section 2.3, using $\mathbf{MA}$, a filter which is neither almost *Cof* nor almost an ultrafilter. The construction will make it explicit why the filter witnesses the failure of $\mathbf{FD}$.

The final section, section 2.4, examines the use of forcings to prove the relative consistency of $\mathfrak{u} < \mathfrak{g}$ (which is slightly stronger than $\mathbf{FD}$). We prove that "nice" linear iterations will not work. We then motivate and develop some tools for analysing forcings constructed along morasses. Adapting the arguments that precluded "nice" linear iterations to the morass forcing framework, we obtain bounds for $\mathfrak{u}$ and $\mathfrak{g}$, which in particular do not forbid $\mathfrak{u} < \mathfrak{g}$.

2.2 The finite-to-one maps in $C(\omega^*, \omega^*)$

$\mathbf{FD}$ is concerned with maps $\phi\colon \omega \to \omega$ whose Stone-Čech extension $\beta\phi$ restricts to a map $\omega^* \to \omega^*$. It is natural to ask what these $\beta\phi$ look like in the space $C(\omega^*, \omega^*)$. We equip $C(\omega^*, \omega^*)$ with the compact-open topology, that is the topology generated by subbasic open sets of the form

$$[K, U] = \{f \in C(\omega^*, \omega^*) : f(K) \subseteq U\}.$$

where K is a compact subset of ω^*, and U is an open subset of ω^*. We refer to this space as $C_k(\omega^*, \omega^*)$. We also make some observations about $C_p(\omega^*, \omega^*)$, that is $C(\omega^*, \omega^*)$ equipped with the pointwise topology, the topology that $C(\omega^*, \omega^*)$ inherits as a subspace of $(\omega^*)^{\omega^*}$ with the Tychonoff product topology.

Lemma 2.2.1. $|C(\omega^*, \omega^*)| = 2^{\mathfrak{c}}$.

Proof. There are $2^{\mathfrak{c}}$ constant functions. Also, ω^* is Hausdorff so continuous functions are determined completely by their action on a dense subset of ω^*. Given ω^* has density $\mathfrak{c}$ we have

$$2^{\mathfrak{c}} \leq |C(\omega^*, \omega^*)| \leq |(\omega^*)^{\mathfrak{c}}| = (2^{\mathfrak{c}})^{\mathfrak{c}} = 2^{\mathfrak{c}}. \qquad \square$$

One readily sees that $|\{\beta\phi \colon \phi \colon \omega \to \omega \text{ is finite-to-one}\}| \leq \omega^\omega = \mathfrak{c}$ (in fact we have equality), so, from a purely set-theoretic point of view, the Stone-Čech extensions of finite-to-one maps are small in the space of all continuous maps $\omega^* \to \omega^*$.

Lemma 2.2.2. *Suppose $\mathcal{W}$ is a base for ω^* of clopen sets, closed under finite unions. Then the set*

$$\{[A, B] : A, B \in \mathcal{W}\}$$

forms a subbase for $C_k(\omega^*, \omega^*)$.

The above lemma follows from the following more general theorem about spaces of self-maps on zero-dimensional spaces.

Theorem 2.2.3. *Suppose X is compact and zero-dimensional. Suppose $\mathcal{W}$ is a base of clopen sets for X closed under finite unions. Then*

$$\{[A, B] : A, B \in \mathcal{W}\}$$

forms a subbase for $C_k(X, X)$.

Proof. Let us consider a function $f \in C(X, X)$ and a subbasic open set $[C, U]$, with C compact and U open, with $f \in [C, U]$. Let us note that it is enough to show that there are $A, B \in \mathcal{W}$ with $f \in [A, B] \subseteq [C, U]$.

First, observe that $f(C)$ is compact and a subset of U, so using that $\mathcal{W}$ is a base for X closed under finite unions, find a $B \in \mathcal{W}$ with $f(C) \subseteq B \subseteq U$. Then $f \in [C, B] \subseteq [C, U]$.

Now, $f^{-1}(B)$ is open, and contains the compact set C, so using that $\mathcal{W}$ is a base for X closed under finite unions again, we may find an $A \in \mathcal{W}$ with $C \subseteq A \subseteq$

$f^{-1}(B)$. Note that since A is clopen it is compact. Furthermore $f(A) \subseteq B$, so $f \in [A, B]$. But clearly then

$$f \in [A, B] \subseteq [C, B] \subseteq [C, U]$$

which is precisely what we required. □

Corollary 2.2.4. $w(C_k(\omega^*, \omega^*)) \leq \mathfrak{c}$.

Proof. This follows immediately from the previous lemma. □

Lemma 2.2.5. $\mathrm{c}(C_k(\omega^*, \omega^*)) = \mathfrak{c}^+$. *In particular, there is a collection of size* $\mathfrak{c}$ *consisting of pairwise disjoint non-empty subsets of* $C_k(\omega^*, \omega^*)$.

Proof. We know that $C_k(\omega^*, \omega^*)$ cannot have weight exceeding $\mathfrak{c}$, and so

$$\mathrm{c}(C_k(\omega^*, \omega^*)) \leq \mathfrak{c}^+.$$

In ω^* however, one can find pairwise disjoint non-empty clopen sets $\{A_\alpha : \alpha < \mathfrak{c}\}$, which give rise to pairwise disjoint non-empty open sets $\{[\omega^*, A_\alpha] : \alpha < \mathfrak{c}\}$ in the space $C_k(\omega^*, \omega^*)$. Hence

$$\mathrm{c}(C_k(\omega^*, \omega^*)) \geq \mathfrak{c}^+.$$ □

Corollary 2.2.6. $w(C_k(\omega^*, \omega^*)) = \mathfrak{c}$.

Proof. From the above lemma (or more directly via its proof) we know there is a collection of size $\mathfrak{c}$ consisting of pairwise disjoint non-empty open subsets of $C_k(\omega^*, \omega^*)$. Accordingly, $w(C_k(\omega^*, \omega^*)) \geq \mathfrak{c}$. We already have the reverse inequality. □

Corollary 2.2.7. $C_k(\omega^*, \omega^*)$ *has density* $\mathfrak{c}$.

Proof. Because $w(C_k(\omega^*, \omega^*)) = \mathfrak{c}$, $C_k(\omega^*, \omega^*)$ has density at most $\mathfrak{c}$. But since there is a collection of disjoint open subsets of $C_k(\omega^*, \omega^*)$ of size $\mathfrak{c}$, $C_k(\omega^*, \omega^*)$ has density at least $\mathfrak{c}$. □

Corollary 2.2.8. $C_p(\omega^*, \omega^*)$ *has density* $\mathfrak{c}$.

Proof. The fact that the topology on $C_p(\omega^*, \omega^*)$ is coarser than that on $C_k(\omega^*, \omega^*)$ ensures the density of $C_p(\omega^*, \omega^*)$ is at most $\mathfrak{c}$. Again though, if we fix a point $p \in \omega^*$ and let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be a collection of pairwise disjoint non-empty clopen subsets of ω^*, then

$$\{[\{p\}, A_\alpha] : \alpha < \mathfrak{c}\}$$

is a collection of pairwise disjoint open subsets of $C_p(\omega^*, \omega^*)$. To see that these sets are in fact non-empty, observe that if A is clopen in ω^* then A and ω^* are homeomorphic. A homeomorphism $\omega^* \to A$ can be construed as a map $\omega^* \to \omega^*$, and this witnesses $[\{p\}, A] \neq \emptyset$. It follows that the density of $C_p(\omega^*, \omega^*)$ is at least $\mathfrak{c}$. □

We now prove that the set of Stone-Čech extensions of finite-to-one maps $\omega \to \omega$ (restricted to ω^*) is dense in $C_k(\omega^*, \omega^*)$. This seems reasonable on the basis that the Stone-Čech compactification is defined in terms of its ability to extend certain continuous functions on the base space, and so, intuitively, continuous maps on Stone-Čech remainders cannot be too far away from some continuous map extended from the base space.

We begin however with a simple observation. Corollary 2.2.18 is then best understood as essentially this observation with some additional combinatorics designed to deal with the additional complexity of a general basic open set.

Observation 2.2.9. *Suppose U and V are non-empty and clopen in ω^*. Then there is a $\phi : \omega \to \omega$ which is finite-to-one such that*

$$\beta\phi \restriction \omega^* \in [U, V].$$

Proof. First find $A, B \subseteq \omega$ with

$$A^* = \overline{A}^{\beta\omega} \cap \omega^* = U$$

and

$$B^* = \overline{B}^{\beta\omega} \cap \omega^* = V.$$

In order that U and V be non-empty it is necessary that A and B be infinite. Let $\phi\colon \omega \to \omega$ be any finite-to-one extension of a finite-to-one map $A \to B$. Then $\beta\phi\,(A^*) = B^*$ and so

$$\beta\phi \restriction \omega^* \in [U, V]\,. \qquad \square$$

One can directly generalise this observation to obtain a proof that the Stone-Čech extensions of finite-to-one maps are dense in $C_k\,(\omega^*, \omega^*)$, and a proof along these lines is provided in Appendix A. However, we refine our picture of basic open sets, and then weak-basic open sets below in order to make the result more obvious, and in fact we show that Stone-Čech extensions of injective maps are dense in $C_k\,(\omega^*, \omega^*)$.

Observation 2.2.10. *Suppose A_0, A_1, B_0 and B_1 are clopen in X, then, in $C_k\,(X, X)$,*

$$[A_0, B_0] \cap [A_1, B_1] = [A_0 \cap A_1, B_0 \cap B_1] \cap [A_0 \setminus A_1, B_0] \cap [A_1 \setminus A_0, B_1]\,.$$

Definition 2.2.11. *A collection of sets $\mathcal{A}$ is said to refine a collection of sets $\mathcal{B}$ if and only if for each $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ with $A \subseteq B$.*

Lemma 2.2.12. *Suppose A, B are clopen in X and A_i, B_i are clopen in X for $i \in \{0, \ldots, k\}$ with $\{A_0, \ldots, A_k\}$ pairwise disjoint. Then there are clopen U_i, V_i for $i \in \{0, \ldots, m\}$ with $\{U_0, \ldots, U_m\}$ pairwise disjoint and refining $\{A, A_0, \ldots, A_k\}$ and so that in $C_k\,(X, X)$,*

$$[A, B] \cap \bigcap_{i=0}^{k} [A_i, B_i] = \bigcap_{i=0}^{m} [U_i, V_i]\,.$$

Proof. We generalise the above observation to obtain

$$[A, B] \cap \bigcap_{i=0}^{k} [A_i, B_i] = \left[A \setminus \bigcup_{i=0}^{k} A_i, B\right] \cap \bigcap_{i=0}^{k} [A \cap A_i, B \cap B_i] \cap \bigcap_{i=0}^{k} [A_i \setminus A, B_i]\,,$$

from which the result follows. $\square$

Lemma 2.2.13. *Suppose A_i, B_i are clopen in X for $i \in \{0, \ldots, n\}$. Then there are clopen U_i, V_i for $i \in \{0, \ldots, m\}$ so that in $C_k\,(X, X)$,*

$$\bigcap_{i=0}^{n} [A_i, B_i] = \bigcap_{i=0}^{m} [U_i, V_i]\,,$$

and such that $\{U_0, \ldots, U_m\}$ is a pairwise disjoint refinement of $\{A_0, \ldots, A_n\}$.

Proof. We work by induction on n. The case $n = 0$ requires nothing to be proved, and the case $n = 1$ is the above Observation. So let us suppose $n \geq 2$, and the Lemma holds at $n - 1$. First, apply the inductive hypothesis to A_i, B_i for $i \in \{0, \ldots, n-1\}$, and obtain clopen U_i', V_i' for $i \in \{0, \ldots, m'\}$ with $\{U_0', \ldots, U_{m'}'\}$ a pairwise disjoint refinement of $\{A_0, \ldots, A_{n-1}\}$, and

$$\bigcap_{i=0}^{n-1} [A_i, B_i] = \bigcap_{i=0}^{m'} [U_i', V_i'] .$$

Then applying the above lemma with A_n, B_n and U_i', V_i' for $i \in \{0, \ldots, m'\}$ completes the induction. □

Corollary 2.2.14. *Suppose X is compact and zero-dimensional. Then $C_k(X, X)$ has a base of open sets of the shape*

$$\bigcap_{i=0}^{n} [A_i, B_i] ,$$

for A_i, B_i clopen in X, and $\{A_0, \ldots, A_n\}$ pairwise disjoint.

Proof. Combine Theorem 2.2.3 and the above Lemma. □

We now refine the picture a little further to obtain a nice weak-base for $C_k(\omega^*, \omega^*)$.

Lemma 2.2.15. *Suppose X is compact, zero-dimensional and perfect, and $A_0, \ldots, A_n$ and $B_0, \ldots, B_n$ are clopen sets in X with $\{A_0, \ldots, A_n\}$ a pairwise disjoint collection. Then there are non-empty clopen $C_0, \ldots, C_n$ such that $\{C_0, \ldots, C_n\}$ is a pairwise disjoint refinement of $\{B_0, \ldots, B_n\}$ and in $C_k(X, X)$,*

$$\emptyset \neq \bigcap_{i=0}^{n} [A_i, C_i] \subseteq \bigcap_{i=0}^{n} [A_i, B_i] .$$

Proof. Let $\mathcal{B}$ be the closure of $\{B_0, \ldots, B_n\}$ under non-empty (finite) intersections. Note that $\mathcal{B}$ is a finite set of clopen sets. Let $\mathcal{M}$ be the set of $\subseteq$-minimal elements of $\mathcal{B}$. For each $i \in \{0, \ldots, n\}$, let M_i be an element of $\mathcal{M}$ with $M_i \subseteq B_i$. Observe that if $M_i \cap M_j \neq \emptyset$, then $M_i = M_j$ by $\subseteq$-minimality and closure of $\mathcal{B}$ under non-empty intersections, for each pair i, j.

Now for each $M \in \mathcal{M}$, let $\{M^i \colon i = 0, \ldots, n\}$ be a partition of M into disjoint non-empty clopen sets. For each $i \in \{0, \ldots, n\}$, let $C_i = M_i^i$. Then each C_i is clopen, $C_i \subseteq M_i \subseteq B_i$, and

$$[A_i, C_i] \subseteq [A_i, M_i] \subseteq [A_i, B_i]$$

so

$$\bigcap_{i=0}^{n} [A_i, C_i] \subseteq \bigcap_{i=0}^{n} [A_i, B_i] \,.$$

Furthermore, if for $i, j \in \{0, \ldots, n\}$, $C_i \cap C_j \neq \emptyset$, then $M_i \cap M_j \neq \emptyset$, and hence $M_i = M_j$. But then $C_i = M_i^i$, and $C_j = M_i^j$, and since $\left\{M_i^k \colon k = 0, \ldots, n\right\}$ partitions M_i, we must have $i = j$. Hence $\{C_0, \ldots, C_n\}$ is a pairwise disjoint refinement of $\{B_0, \ldots, B_n\}$.

So the only thing left to show is that $\bigcap_{i=0}^{n} [A_i, C_i] \neq \emptyset$. For each $i \in \{0, \ldots, n\}$, let $p_i \in C_i$. Then let $f \colon \omega^* \to \omega^*$ be the map which sends points in A_i to p_i, and points in $\omega^* \setminus \bigcup_{i=0}^{n} A_i$ to p_0. Since $\{A_0, \ldots, A_n\}$ is a disjoint collection, f is well-defined, and it is easily seen from clopeness of the sets $A_0, \ldots, A_n$ that f is continuous, and $f \in \bigcap_{i=0}^{n} [A_i, C_i]$. □

Corollary 2.2.16. *Suppose X is compact, zero-dimensional and perfect. Then there is a weak-base for $C_k(X, X)$ consisting of sets of the shape*

$$\bigcap_{i=0}^{n} [A_i, C_i]$$

where $\{A_0, \ldots, A_n\}$ and $\{C_0, \ldots, C_n\}$ are pairwise disjoint collections of clopen subsets of X.

Proof. By Corollary 2.2.14 and the previous Lemma. □

Since when we are talking about Stone-Čech extensions of maps we are primarily interested in the restriction of said maps to ω^*, let us for notational convenience cease to distinguish between $\beta\phi$ and $\beta\phi \restriction \omega^*$.

Theorem 2.2.17. *The set $\{\beta\phi \colon \phi \colon \omega \to \omega$ is injective$\}$ is dense in $C_k(\omega^*, \omega^*)$.*

Proof. Let us consider the clopen base for ω^* given by $\{A^* \colon A \subseteq \omega\}$, and a non-empty weak-basic open set in $C_k(\omega^*, \omega^*)$ obtained from this base, i.e.

$$U = \bigcap_{i=1}^{n} [A_i^*, B_i^*]$$

with $\{A_1^*, \ldots, A_n^*\}$ and $\{B_1^*, \ldots, B_n^*\}$ pairwise disjoint (using the previous Corollary), and find an injective $\phi \colon \omega \to \omega$ with $\beta\phi$ in U. Since elements of $\{A_1^*, \ldots, A_n^*\}$ are pairwise disjoint, elements of $\{A_1, \ldots, A_n\}$ are almost disjoint, and since there are only finitely many elements in $\{A_1, \ldots, A_n\}$, we may suppose that $\{A_1, \ldots, A_n\}$ is a disjoint collection. Similarly, let us suppose $\{B_1, \ldots, B_n\}$ is a disjoint collection.

First let us suppose ψ_1 is an injection $A_1 \cup (\omega \setminus \bigcup_{i=1}^{n} A_i) \to B_1$, and ψ_i is an injection $A_i \to B_i$ for each i with $2 \leq i \leq n$. Then let $\phi = \bigcup_{i=1}^{n} \psi_i$, so $\phi \colon \omega \to \omega$. Then ϕ is well-defined by disjointness of $\{A_1, \ldots, A_n\}$, and injective by disjointness of $\{B_1, \ldots, B_n\}$.

Now for each $i \in \{1, \ldots, n\}$, if $\mathcal{U} \in A_i^*$, then $A_i \in \mathcal{U}$, so $\phi(A_i) \in \phi(\mathcal{U})$. But

$$\phi(A_i) \subseteq B_i,$$

so $B_i \in \phi(\mathcal{U})$, and hence $\phi(\mathcal{U}) \in B_i^*$. But that is to say, for each $i \in \{1, \ldots, n\}$,

$$\beta\phi(A_i^*) \subseteq B_i^*,$$

and hence

$$\beta\phi \in \bigcap_{i=1}^{n} [A_i^*, B_i^*] = U. \qquad \square$$

Corollary 2.2.18. *The set $\{\beta\phi \colon \phi \colon \omega \to \omega$ is finite-to-one$\}$ is dense in $C_k(\omega^*, \omega^*)$.*

Proof. Injective maps are finite-to-one. $\square$

Corollary 2.2.19. $|\{\beta\phi \colon \phi \colon \omega \to \omega$ *is injective*$\}| = \mathfrak{c}$.

Proof. Writing in the conventional manner $d(X)$ for the density of a space X, we have $\mathfrak{c} = d(C_k(\omega^*, \omega^*)) \leq |\{\beta\phi \colon \phi \colon \omega \to \omega$ is injective$\}| \leq \omega^\omega = \mathfrak{c}$. $\square$

Notice that the density of Stone-Čech extensions of injective or finite-to-one maps in the space $C_k(\omega^*, \omega^*)$ gives us density in the space $C_p(\omega^*, \omega^*)$, since the topology on $C_p(\omega^*, \omega^*)$ is coarser than that on $C_k(\omega^*, \omega^*)$.

2.3 An explicit proof of $(\mathbf{MA} \rightarrow \neg\mathbf{FD})$

It is well known that $\neg\mathbf{FD}$ follows from $\mathbf{MA}$. The usual proofs go via the fact that medial limits cannot exist under $\mathbf{FD}$ while they do under $\mathbf{MA}$ (seen in [13]), or via observing that $\mathbf{FD}$ is equivalent to the cardinal inequality $\mathfrak{u} < \mathfrak{g}_f$ (see Definition 2.4.26, and the discussion at the bottom of the second page of [21]), where $\mathfrak{g}_f \leq \mathfrak{c}$; of course $\mathbf{MA}$ entails $\mathfrak{u} = \mathfrak{c}$ and hence the Filter Dichotomy must fail.

However, neither of these proofs give us explicitly and directly a free filter which we can see is neither almost cofinite nor almost an ultrafilter. The medial limits approach uses additional and unnecessary measure theory, and the approach via small cardinals is not especially constructive. We use $\mathbf{MA}$ to directly build a filter which fails the Filter Dichotomy. This new proof has the advantage that it is relatively elementary, and the failure of $\mathbf{FD}$ is built directly into the construction of the filter. Furthermore, the proof is very natural — it is one of the first things one might try if one attempted to prove $\neg\mathbf{FD}$, but we use $\mathbf{MA}$ to overcome the insufficiencies of $\mathbf{ZFC}$. The equivalence of $\mathbf{FD}$ and $\mathfrak{u} < \mathfrak{g}_f$ is a more precise result, and provides a way of seeing that $(\mathbf{MA} \rightarrow \neg\mathbf{FD})$, but costs a loss of simplicity. Let us begin with a well known definition and theorem.

Definition 2.3.1. *A sequence of infinite subsets of ω, $\langle A_\alpha : \alpha < \lambda\rangle$, is a tower if and only if for $\alpha \leq \beta < \lambda$ we have $A_\beta \subseteq^* A_\alpha$. λ is the length of the tower. The cardinal number $\mathfrak{t}$ is the shortest length of a tower $\langle A_\alpha : \alpha < \mathfrak{t}\rangle$, such that $\{A_\alpha : \alpha < \mathfrak{t}\}$ has no infinite pseudointersection.*

Observation 2.3.2. $\mathfrak{p} \leq \mathfrak{t}$.

Theorem 2.3.3. $(\mathbf{MA} \rightarrow \mathfrak{t} = \mathfrak{c})$.

We show $(\mathfrak{t} = \mathfrak{c} \rightarrow \neg\mathbf{FD})$[1], by explicitly constructing a free filter on ω which is neither almost *Cof* nor almost an ultrafilter. We will build such a filter by building

[1]Shelah and Malliaris proved in [20] that $\mathfrak{p} = \mathfrak{t}$, so we could just as well state our result as $(\mathfrak{p} = \mathfrak{c} \rightarrow \neg\mathbf{FD})$. Our construction however makes use of towers, rather than arbitrary collections of reals with the strong finite intersection property, so we maintain the use of $\mathfrak{t}$ since it seems more appropriate.

a base by transfinite recursion, at each stage ensuring that a particular finite-to-one map cannot map the filter we are building to an ultrafilter or *Cof*. Ensuring that a finite-to-one image of a filter is not almost *Cof* is straightforward - we simply include a set with coinfinite image under the map.

To ensure that a finite-to-one image of a filter is not an ultrafilter, we need to ensure there is some subset B of ω such that neither B nor its complement $\omega \setminus B$ is in the image filter. This requires care, because we need to ensure at subsequent stages of our recursion we don't accidently add B in to the image filter while attempting to break a different map, despite having previously promised to exclude it.

To achieve both of these things simultaneously we will keep a list of promises in the form of two towers of infinite sets. At each stage we extend the towers with more refined infinite subsets of ω. The following two technical lemmas justify the refinements.

Lemma 2.3.4. *Suppose $\phi\colon \omega \to \omega$ is finite-to-one and A, B are infinite subsets of ω. Then there are infinite subsets $A' \subseteq A$ and $B' \subseteq B$ with $\phi(A' \cup B') \notin$ Cof.*

Proof. For S a subset of ω, define

$$\text{Fibres}(S) = \left\{n \in \omega\colon \phi^{-1}\{n\} \cap S \neq \emptyset\right\},$$

the set of ϕ-fibres that meet the set S. Because A and B are infinite, Fibres (A) and Fibres (B) are both infinite sets.

If Fibres $(A) \cap$ Fibres (B) is infinite, choose an infinite subset $S \subseteq$ Fibres $(A) \cap$ Fibres (B) with (Fibres $(A) \cap$ Fibres $(B)) \setminus S$ also infinite, and let $A' = A \cap \phi^{-1}(S)$ and $B' = B \cap \phi^{-1}(S)$. Then $\phi(A' \cup B') = S \notin$ *Cof*.

Otherwise, suppose Fibres $(A) \cap$ Fibres (B) is a finite set. Choose an infinite subset $S_A \subseteq$ Fibres (A) with Fibres $(A) \setminus S_A$ also infinite. Because Fibres $(A) \cap$ Fibres (B) is finite, $S_A \cup$ Fibres (B) is coinfinite, hence $S_A \cup$ Fibres $(B) \notin$ *Cof*. Now we set $A' = \phi^{-1}(S_A) \cap A$ and $B' = B$, and observe that $\phi(A') = S_A$ and $\phi(B') =$ Fibres (B). □

Lemma 2.3.5. *Suppose $\phi\colon \omega \to \omega$ is finite-to-one, and A, B are infinite subsets of ω. Then there is an infinite and coinfinite subset S of ω such that $\phi^{-1}(S) \cap A$ is infinite, and $\phi^{-1}(\omega \setminus S) \cap B$ is infinite.*

Proof. Because ϕ is finite-to-one, both $\phi(A)$ and $\phi(B)$ are infinite sets. If $\phi(A) \cap \phi(B)$ is finite, we can simply take $S = \phi(A)$. In the case when $\phi(A) \cap \phi(B)$ is infinite, we take an infinite subset S of $\phi(A) \cap \phi(B)$ with $(\phi(A) \cap \phi(B)) \setminus S$ also infinite. It is clear that this choice of S works. □

Theorem 2.3.6. *Suppose $\mathfrak{t} = \mathfrak{c}$. Then there is a free filter $\mathcal{F}$ on ω such that if $\phi\colon \omega \to \omega$ is finite-to-one then $\phi(\mathcal{F}) \neq$ Cof and $\phi(\mathcal{F})$ is not an ultrafilter.*

Proof. Suppose $\langle \phi_\alpha : \alpha < \mathfrak{c} \rangle$ enumerates the finite-to-one maps $\omega \to \omega$. We will build a $\subseteq^*$-decreasing sequence of infinite subsets of ω, $\langle A_\alpha : \alpha < \mathfrak{c} \rangle$ with some additional properties, and our filter $\mathcal{F}$ will be the filter generated by $\{A_\alpha\colon \alpha < \mathfrak{c}\} \cup$ *Cof*. We will choose A_α so that $\phi_\alpha(A_\alpha) \notin$ *Cof*. This will ensure $\phi_\alpha(\mathcal{F}) \neq$ *Cof*. We also want to ensure $\phi_\alpha(\mathcal{F})$ is not an ultrafilter. In order to achieve this, we create another sequence of sets $\langle B_\alpha : \alpha < \mathfrak{c} \rangle$ with the property that $\phi_\alpha(A_\beta)$ intersects both B_α and $\omega \setminus B_\alpha$ in an infinite set, for any $\beta < \mathfrak{c}$. This will ensure that $B_\alpha, \omega \setminus B_\alpha \notin \phi_\alpha(\mathcal{F})$, so that in particular $\phi_\alpha(\mathcal{F})$ is not an ultrafilter. Observe that if $\langle A_\alpha : \alpha < \mathfrak{c} \rangle$ is $\subseteq^*$-decreasing, if we have for $\beta \leq \alpha < \mathfrak{c}$ that $\phi_\beta(A_\alpha)$ intersects B_β and $\omega \setminus B_\beta$ in infinite sets, then the same is true for $\alpha < \beta$. Hence we only check this more restricted description in our recursion.

We build the $\langle B_\alpha : \alpha < \mathfrak{c} \rangle$ so that for any $\lambda < \mathfrak{c}$ both

- $\langle \phi_\alpha^{-1}(B_\alpha) \cap A_\alpha : \alpha < \lambda \rangle$ is a tower, and
- $\langle \phi_\alpha^{-1}(\omega \setminus B_\alpha) \cap A_\alpha : \alpha < \lambda \rangle$ is a tower.

Suppose, then, that we have defined A_β and B_β for $\beta < \alpha$. We show how to define A_α and B_α. First, since $\alpha < \mathfrak{t}$, let us take T_1 to be an infinite pseudointersection of $\{\phi_\beta^{-1}(B_\beta) \cap A_\beta\colon \beta < \alpha\}$, that is, for all $\beta < \alpha$ we have $T_1 \subseteq^* \phi_\beta^{-1}(B_\beta) \cap A_\beta$. Similarly, let T_2 be an infinite pseudointersection of $\{\phi_\beta^{-1}(\omega \setminus B_\beta) \cap A_\beta\colon \beta < \alpha\}$. In particular, $T_1 \cup T_2$ is an infinite pseudointersection of $\{A_\beta\colon \beta < \alpha\}$.

Since ϕ_α is finite-to-one, we may pick $B_\alpha \subseteq \omega$ so that B_α is infinite, coinfinite, and both $\phi_\alpha^{-1}(B_\alpha)$ intersects T_1 in an infinite set, and $\phi_\alpha^{-1}(\omega \setminus B_\alpha)$ intersects T_2 in an infinite set. The existence of B_α is justified in Lemma 2.3.5.

By Lemma 2.3.4 we can now find infinite subsets $A^1_\alpha \subseteq \phi_\alpha^{-1}(B_\alpha) \cap T_1$ and $A^2_\alpha \subseteq \phi_\alpha^{-1}(\omega \setminus B_\alpha) \cap T_2$ with $\phi_\alpha(A^1_\alpha \cup A^2_\alpha) \notin \mathit{Cof}$. We then set $A_\alpha = A^1_\alpha \cup A^2_\alpha$. Our construction obtains the following:

1. $\phi_\alpha(A_\alpha) \notin \mathit{Cof}$;
2. $A_\alpha \subseteq^* A_\beta$ for all $\beta \leq \alpha$ (since $A_\alpha \subseteq T_1 \cup T_2$, and $T_1 \cup T_2$ is a pseudointersection of $\{A_\beta : \beta < \alpha\}$);
3. for all $\beta \leq \alpha$, $\phi_\beta(A_\alpha) \cap B_\beta$ is infinite, and $\phi_\beta(A_\alpha) \cap (\omega \setminus B_\beta)$ is infinite.
4. $\left\langle \phi_\beta^{-1}(B_\beta) \cap A_\beta : \beta \leq \alpha \right\rangle$ is a tower;
5. $\left\langle \phi_\beta^{-1}(\omega \setminus B_\beta) \cap A_\beta : \beta \leq \alpha \right\rangle$ is a tower.

These are all the properties we require. Items (3)-(5) require a little justification.

(3) when $\beta = \alpha$ is clear, because A_α intersects both $\phi_\alpha^{-1}(B_\alpha)$ and $\phi_\alpha^{-1}(\omega \setminus B_\alpha)$ in an infinite set by how we chose A^1_α and A^2_α. For $\beta < \alpha$, note that A_α intersects T_1 in an infinite set, and $T_1 \subseteq^* \phi_\beta^{-1}(B_\beta)$, so that $\phi_\beta(A_\alpha) \cap B_\beta$ is infinite. A similar argument applied to T_2 shows $\phi_\beta(A_\alpha) \cap (\omega \setminus B_\beta)$ is an infinite set.

For (4), all we require is to show that $\phi_\alpha^{-1}(B_\alpha) \cap A_\alpha \subseteq^* \phi_\beta^{-1}(B_\beta) \cap A_\beta$, when $\beta < \alpha$. We know however, that $A^1_\alpha \subseteq \phi_\alpha^{-1}(B_\alpha)$ and $A^2_\alpha \subseteq \phi_\alpha^{-1}(\omega \setminus B_\alpha)$, so

$$\phi_\alpha^{-1}(B_\alpha) \cap A_\alpha = \phi_\alpha^{-1}(B_\alpha) \cap \left(A^1_\alpha \cup A^2_\alpha\right) = A^1_\alpha \subseteq T_1$$

and $T_1 \subseteq^* \phi_\beta^{-1}(B_\beta) \cap A_\beta$; (4) then follows. (5) is obtained similarly. This completes the construction. □

Corollary 2.3.7. $(\mathfrak{t} = \mathfrak{c} \to \neg\mathbf{FD})$.

Corollary 2.3.8. $(\mathbf{MA} \to \neg\mathbf{FD})$. *In particular* $(\mathbf{CH} \to \neg\mathbf{FD})$.

Proof. We noted that $\mathbf{MA}$ implies $\mathfrak{t} = \mathfrak{c}$. Also, $\mathbf{CH}$ implies $\mathbf{MA}$. □

We know that under $\mathbf{FD}$ we have two classes of free filters closed under the action of finite-to-one maps $\omega \to \omega$. We note in the appendices that under just $\mathbf{ZFC}$

the classes of almost *Cof* filters and almost ultrafilters remain closed under the action of finite-to-one maps. The results above demonstrate that there are possibly free filters which do not fall into either of these two classes. Therefore, it is natural to ask the following question.

Question 2.3.9. *What can be said of the structure of free filters that are not almost Cof and not almost ultrafilters?*

2.4 The consistency of $\mathfrak{u} < \mathfrak{g}$

2.4.1 Groupwise dense sets, $\mathfrak{g}$, and its relationship to the Filter Dichotomy

We begin by defining the small cardinal $\mathfrak{g}$, the groupwise density number. In order to do so, we need to define groupwise dense collections.

Definition 2.4.1. *A collection $\mathcal{G}$ of infinite subsets of ω is called groupwise dense if and only if*

1. *If $A \in \mathcal{G}$ and B is an infinite subset of ω with $B \subseteq^* A$ then $B \in \mathcal{G}$;*
2. *For any strictly increasing sequence $\langle i_j : j \in \omega \rangle$ of natural numbers, there is an infinite subset A of ω with*
$$\bigcup_{j \in A} [i_j, i_{j+1}) \in \mathcal{G}.$$

If, furthermore, $\mathcal{G}$ is closed under finite unions then we call $\mathcal{G}$ a groupwise dense ideal.

Definition 2.4.2. *The groupwise density number $\mathfrak{g}$, is defined to be the cardinality of the smallest collection of groupwise dense sets with empty intersection.*

Observation 2.4.3. $\aleph_1 \leq \mathfrak{g} \leq \mathfrak{c}$.

Theorem 2.4.4 ([3], Theorem 3)**.** *If $\mathfrak{u} < \mathfrak{g}$ then the Filter Dichotomy holds.*

Proof. Note that this also follows from Observation 2.4.27 and Theorem 2.4.28, which appear later.

We show $\mathfrak{u} < \mathfrak{g}$ implies **FD**. This proof appears as Theorem 3 in [3]. So suppose $\mathfrak{u} < \mathfrak{g}$. Let $\mathcal{U}$ be an ultrafilter with a base $\mathcal{B}$ of cardinality $\mathfrak{u}$. Let $\mathcal{F}$ be a free filter on ω.

For each $B \in \mathcal{B}$, let

$$\mathcal{G}_B = \{X \subseteq \omega \colon X \text{ is infinite and there is an } A \in \mathcal{F} \text{ such that whenever } x, y \in X \text{ and } x < y, \text{ and } [x, y) \cap A \neq \emptyset, \text{ then } [x, y) \cap B \neq \emptyset\}.$$

Observe that each $\mathcal{G}_B$ is closed under subsets. Also

Claim 2.4.5. *$\mathcal{G}_B$ is closed under adding finitely many points, that is if $X \in \mathcal{G}_B$ and $F \subseteq \omega$ is a finite set, then $X \cup F \in \mathcal{G}_B$.*

Proof of Claim. Suppose $X \in \mathcal{G}_B$ and $F \subseteq \omega$ is a finite set, $F \neq \emptyset$. We show $X \cup F \in \mathcal{G}_B$. Let $A \in \mathcal{F}$ be such that if $x, y \in X$ with $x < y$ and $[x, y) \cap A \neq \emptyset$ then $[x, y) \cap B \neq \emptyset$. Let

$$l = \max\left(\{t \in X \colon \forall p \in F \; t < p\} \cup \{0\}\right)$$
$$u = \min\left(\{t \in X \colon \forall p \in F \; t > p\} \cup \{0\}\right)$$

Note that $A \setminus [l, u) \in \mathcal{F}$, since $\mathcal{F}$ is free.

Now, suppose $x, y \in X \cup F$ and $[x, y) \cap (A \setminus [l, u)) \neq \emptyset$. Then one or both of x and y must be elements of X, and hence either $[x, l) \cap A \neq \emptyset$ or $[u, y) \cap A \neq \emptyset$. So, we must have either $[x, l) \cap B \neq \emptyset$ or $[u, y) \cap B \neq \emptyset$. But since

$$[x, l), [u, y) \subseteq [x, y)$$

we have $[x, y) \cap B \neq \emptyset$, and hence $X \cup F \in \mathcal{G}_B$. □

So $\mathcal{G}_B$ is closed under almost subsets, that is if $X \in \mathcal{G}_B$ and $Y \subseteq^* X$ then $Y \in \mathcal{G}_B$. The argument now divides into two cases. For the first case, suppose for some $B \in \mathcal{B}$, $\mathcal{G}_B$ is not groupwise dense. So let $\langle i_n : n \in \omega \rangle$ be an increasing sequence of natural numbers such that for all $A \subseteq \omega$, $\bigcup_{j \in A} [i_j, i_{j+1}) \notin \mathcal{G}_B$. By

taking an appropriate subsequence, we can, without loss of generality, suppose $[i_j, i_{j+1}) \cap B \neq \emptyset$ for each $j \in \omega$. Now let $f : \omega \to \omega$ be the finite-to-one map defined by $f^{-1}\{j\} = [i_j, i_{j+1})$. We show $f(\mathcal{F}) = \mathit{Cof}$. Suppose, to the contrary that $Y \subseteq \omega$ and $Y \notin \mathit{Cof}$ but $Y \in f(\mathcal{F})$. Then $f^{-1}(Y) \in \mathcal{F}$. We argue that this set witnesses that $f^{-1}(\omega \setminus Y) \in \mathcal{G}_B$. For if $x, y \in f^{-1}(\omega \setminus Y)$ with $x < y$ and $[x, y) \cap f^{-1}(Y) \neq \emptyset$, then we must have $[i_j, i_{j+1}) \subseteq [x, y)$ for some $j \in Y$, and hence, since $[i_j, i_{j+1}) \cap B \neq \emptyset$, $[x, y) \cap B \neq \emptyset$ also. So $f^{-1}(\omega \setminus Y) \in \mathcal{G}_B$. But

$$f^{-1}(\omega \setminus Y) = \bigcup_{j \in \omega \setminus Y} [i_j, i_{j+1}) \notin \mathcal{G}_B.$$

This is a contradiction, so we must have $f(\mathcal{F}) = \mathit{Cof}$.

The second case is when for each $B \in \mathcal{B}$, $\mathcal{G}_B$ is groupwise dense. Since $|\mathcal{B}| = \mathfrak{u} < \mathfrak{g}$, we can find an $X \in \bigcap_{B \in \mathcal{B}} \mathcal{G}_B$. Then define the finite-to-one map $f : \omega \to \omega$ by $f(n) = |X \cap [0, n]|$. For each $B \in \mathcal{B}$, because $X \in \mathcal{G}_B$, there is an $A \in \mathcal{F}$ such that whenever $x, y \in X$ with $x < y$ and $[x, y) \cap A \neq \emptyset$, we have $[x, y) \cap B \neq \emptyset$. By taking y as the immediate successor to x in X in this statement, we see $f(A) \subseteq f(B)$. It follows that $f(\mathcal{U}) \subseteq f(\mathcal{F})$, and hence $f(\mathcal{F})$ is just the ultrafilter $f(\mathcal{U})$. □

Remark 2.4.6. *It is relatively consistent that* $\mathfrak{g} \leq \mathfrak{u}$. *In a model of* **MA** *for instance* $\mathfrak{u} = \mathfrak{g}$, *while in a model obtained by adding* κ *Cohen reals, for a regular cardinal* κ, $\mathfrak{g} = \aleph_1$, *while* $\mathfrak{u} = \kappa$, *and so we can easily obtain* $\mathfrak{g} < \mathfrak{u}$.

The relative consistency of $\mathfrak{u} < \mathfrak{g}$ can be established by an $\aleph_2$-length countable support iteration of Miller or Matet forcing over a model of **CH** (see Blass [3]). In the resulting model, one has $\mathfrak{u} = \aleph_1$ and $\mathfrak{g} = \aleph_2$. These forcings have two important properties. First, they have the ability to preserve a p-point from the ground model, that is a p-point ultrafilter in the ground model will still generate an ultrafilter in the generic extension — in fact it remains a p-point. Secondly, a single stage of either forcing will diagonalise all groupwise dense collections in the ground model. More precisely, in the generic extension of a single stage of either forcing, there is a real $A \subseteq \omega$ such that if $\mathcal{G}$ is groupwise dense in the ground model, then there is a $B \in \mathcal{G}$ with $A \subseteq^* B$. Given these properties, in the generic extension it is clear

that $\mathfrak{u} = \aleph_1$, while $\mathfrak{g} = \aleph_2$ boils down to standard initial stage reflection arguments for iterated forcing extensions, so as to exploit the diagonalisation properties of the forcings.

2.4.2 Forcing notation and basics

We recount here some basic results from classical forcing. [18] and [16] are standard references on the subject. We note that for forcing partial orders we write $p \leq q$ to mean that p extends q, or p is a stronger condition than q.

We use $\Vdash$ to denote the forcing relation, V typically refers to the ground model, and $V[G]$ the extension of V by some filter G, where G is generic over V with respect to some specified partial order. Knowledge of the contents of [18] (single step forcing, basic iterations) and [16] (in particular countable support iteration of proper forcing) is taken as given. However, we provide here the definition for iterated forcing constructions in order to clarify notation (though it more or less follows [18]). All partial orders are (without loss of generality) assumed to have a maximal element $\mathbb{1}$. A subscript will be used when clarity of partial orders is required — for example $\mathbb{1}_{\mathbb{P}}$ is the maximal element of $\mathbb{P}$, and $\Vdash_{\mathbb{P}}$ is the forcing relation with respect to $\mathbb{P}$.

Furthermore, if S is a set in the generic extension, then $\dot{S}$ refers to a name for S in the ground model. If x is a set in the ground model, then $\check{x}$ refers to the canonical name for x in the generic extension.

Definition 2.4.7 ([18], Chapter VIII, Definition 5.8)**.** *Suppose α is an ordinal and $\mathcal{I}$ is an ideal of subsets of α containing all finite subsets of α. An α-stage iterated forcing construction with supports in $\mathcal{I}$ consists of two sequences in the ground model V; first we have a sequence of partial orders $\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle$, and secondly a sequence of names $\langle \pi_\xi : \xi < \alpha \rangle$, where each π_ξ is a $\mathbb{P}_\xi$-name for a partial order. Elements of $\mathbb{P}_\xi$ are sequences of names of length ξ, and if $p \in \mathbb{P}_\xi$ and $\eta < \xi$ then $p \restriction \eta \in \mathbb{P}_\eta$. If $\langle \rho_\alpha : \alpha < \xi \rangle \in \mathbb{P}_\xi$ then $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \rho_\alpha \in \pi_\alpha$ for each $\alpha < \xi$. $\mathbb{1}_{\mathbb{P}_\xi}$ is the sequence $\langle \mathbb{1}_{\pi_\alpha} : \alpha < \xi \rangle$.*

Let us define the support of a condition $\langle \rho_\alpha : \alpha < \xi \rangle$ *by*

$$\text{supp}\left(\langle \rho_\alpha : \alpha < \xi \rangle\right) = \{\alpha < \xi : \rho_\alpha \neq \mathbb{1}_{\pi_\alpha}\}.$$

We ask in addition the following:

1. $\mathbb{P}_0 = \{\emptyset\}$.
2. *If* $p = \langle \rho_\alpha : \alpha \leq \xi \rangle$ *then* $p \in \mathbb{P}_{\xi+1}$ *if an only if* $p \restriction \xi \in \mathbb{P}_\xi$ *and* $p \restriction \xi \Vdash_{\mathbb{P}_\xi} \rho_\xi \in \pi_\xi$. *Also if* $p = \langle \rho_\alpha : \alpha \leq \xi \rangle, p' = \langle \rho'_\alpha : \alpha \leq \xi \rangle \in \mathbb{P}_{\xi+1}$ *then* $p \leq p'$ *if and only if* $p \restriction \xi \leq p' \restriction \xi$ *and* $p \restriction \xi \Vdash \rho_\xi \leq \rho'_\xi$.
3. *Suppose* λ *is a limit ordinal and* $p = \langle \rho_\alpha : \alpha < \lambda \rangle$, *then* $p \in \mathbb{P}_\lambda$ *if and only if for each* $\xi < \lambda$, $p \restriction \xi \in \mathbb{P}_\xi$ *and* $\text{supp}\,(p) \in \mathcal{I}$. *For* $p, p' \in \mathbb{P}_\lambda$ *we say* $p \leq p'$ *if and only if for each* $\xi < \lambda$, $p \restriction \xi \leq p' \restriction \xi$.

Remark 2.4.8. *We remark on some of the details of the above definition, and how the definition will be used in practice.*

1. *We typically omit mention of the names* π_α *— it should be clear how they can be obtained. We may also call an* α*-stage iterated forcing construction an* α*-length iterated forcing construction or similar.*
2. *There are alternative ways to define the support of a condition. We used the above version for consistency with Kunen ([18]), but Jech ([16]) for instance defines the support of a condition* $\langle \rho_\alpha : \alpha < \xi \rangle$ *by*

   $$\text{supp}\left(\langle \rho_\alpha : \alpha < \xi \rangle\right) = \{\alpha < \xi : \neg\left(\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha} \rho_\alpha = \mathbb{1}_{\pi_\alpha}\right)\}.$$

   The required theory can be developed in the same way with either definition, and for the purposes of the following the precise definition does not matter.

There are specific ideals which we are often interested in, and iterations with respect to such ideals are given names of their own.

Definition 2.4.9. *The following are special kinds of iterated forcing constructions.*

1. *An α-length finite support iteration is an α-stage iterated forcing construction with supports in the ideal of finite subsets of α.*

2. *An α-length countable support iteration is an α-stage iterated forcing construction with supports in the ideal of V-countable subsets of α.*

The notion of complete embedding is used in the discussion and analysis of iterated forcings of various kinds.

Definition 2.4.10. *Let $\mathbb{P}, \mathbb{Q}$ be partial orders. We say that a map $i\colon \mathbb{P} \to \mathbb{Q}$ is an embedding if and only if*

1. *For $p, p' \in \mathbb{P}$, if $p \leq p'$ then $i(p) \leq i(p')$.*

2. *For $p, p' \in \mathbb{P}$, p and p' are incompatible if and only if $i(p)$ and $i(p')$ are incompatible.*

If i is the identity map, then we write $\mathbb{P} \subseteq_\perp \mathbb{Q}$. i is a complete embedding if it is both an embedding and satisfies the following condition:

3. *For each $q \in \mathbb{Q}$ there is a $p \in \mathbb{P}$ such that for any $p' \in \mathbb{P}$ if $p' \leq p$ then $i(p')$ and q are compatible in $\mathbb{Q}$. p is called a reduction of q to $\mathbb{P}$ with respect to the map i.*

If i is both the identity map and a complete embedding, we write $\mathbb{P} \subseteq_c \mathbb{Q}$.

Complete embeddings are used in iterated forcing arguments to pull back generic filters to initial stages of forcing constructions. More precisely, we have the following standard result.

Theorem 2.4.11 ([18], Theorem 7.5)**.** *Suppose $i\colon \mathbb{P} \to \mathbb{Q}$ is a complete embedding (in the ground model V). Suppose G is $\mathbb{Q}$-generic over V. Then $i^{-1}(G)$ is $\mathbb{P}$-generic over V and $V[i^{-1}(G)] \subseteq V[G]$.*

Finite support iterations are also characterised very nicely in terms of complete embeddings, as seen in the following.

Theorem 2.4.12 ([18], Chapter VIII, Exercise K5)**.** *Suppose* $\langle \mathbb{P}_\alpha : \alpha \leq \beta \rangle$ *is a sequence of partial orders with a commutative system of complete embeddings*

$$\langle i_{\xi\eta} \colon \mathbb{P}_\xi \to \mathbb{P}_\eta : \xi < \eta \leq \beta \rangle$$

such that for any limit ordinal $\lambda < \beta$

$$\mathbb{P}_\lambda = \bigcup \{ i_{\alpha\lambda} (\mathbb{P}_\alpha) : \alpha < \lambda \} .$$

(in fact, it is sufficient that $\bigcup \{ i_{\alpha\lambda} (\mathbb{P}_\alpha) : \alpha < \lambda \}$ *is dense in* $\mathbb{P}_\lambda$*). Then the forcing* $\langle \mathbb{P}_\alpha : \alpha \leq \beta \rangle$ *is equivalent to a* β*-length finite support iteration.*

Conversely, for any iterated forcing construction $(\langle \mathbb{P}_\xi : \xi \leq \alpha \rangle, \langle \pi_\xi : \xi < \alpha \rangle)$ we have complete embeddings $i_{\xi\eta} \colon \mathbb{P}_\xi \to \mathbb{P}_\eta$ for $\xi < \eta \leq \alpha$ given by

$$i_{\xi\eta} (p) = p^\frown \left\langle \mathbb{1}_{\pi_\beta} : \xi \leq \beta < \eta \right\rangle .$$

See [18], Chapter VIII, Definition 5.10 and Lemma 5.11 for details. Hence if G is $\mathbb{P}_\alpha$-generic over V, then $i_{\xi\alpha}^{-1} (G)$ is $\mathbb{P}_\xi$-generic over V. We write G_ξ for $i_{\xi\alpha}^{-1} (G)$.

The following is a standard reflection result used in iterated forcing arguments. It says that small sets in nice forcings must appear at initial stages.

Lemma 2.4.13 ([18], Chapter VIII, Lemma 5.14)**.** *Suppose* λ *is a limit ordinal and*

$$\langle \mathbb{P}_\alpha : \alpha \leq \lambda \rangle$$

is a λ*-stage iterated forcing construction with supports in* $\mathcal{I}$*, and that each element of* $\mathcal{I}$ *is bounded in* λ*. Suppose* $S \in V$*. Suppose* G *is* $\mathbb{P}_\lambda$*-generic over* V *and* $X \in V[G]$ *with*

$$V[G] \models X \subseteq S \wedge |S| < \mathrm{cf}(\lambda) .$$

Then for some $\alpha < \lambda$*,* $X \in V[G_\alpha]$*.*

2.4.3 Finite support iterations

We consider the possibility of using a finite support iteration to force $\mathfrak{u} < \mathfrak{g}$. We are only interested in forcings which are designed to do something, for which we provide the following definition.

Definition 2.4.14. *Let us say a notion of forcing $\mathbb{P}$ is non-trivial if*

1. $\mathbb{P} \neq \emptyset$*;*
2. $\mathbb{P}$ *is atomless, that is, for every* $p \in \mathbb{P}$*, there are elements* $q, r \in \mathbb{P}$ *with* $q, r \leq p$ *but* q *and* r *incompatible;*
3. $\mathbb{P}$ *is separative, that is, if* $p, q \in \mathbb{P}$ *satisfy* $p \not\leq q$ *then there is an* $r \in \mathbb{P}$ *such that* $r \leq p$ *but* r *and* q *are incompatible.*

We say an iteration is non-trivial when each stage is non-trivial over the previous stage.

Remark 2.4.15. *In the definition above, (2) says that a $\mathbb{P}$-generic filter cannot exist in the ground model. Also, condition (3) is without loss of generality — there is associated to every partial order a separative quotient, forcing over which is equivalent.*

We begin by making a straightforward observation which tells us there is no hope of making certain types of forcing work.

Proposition 2.4.16. *Suppose κ is an uncountable regular cardinal and $\mathbb{P}$ is a κ-length finite support iteration of (non-trivial) forcings, $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$, over a ground model V. If G is $\mathbb{P}$-generic over V, then $V[G] \models \mathfrak{u} \geq |\kappa|$.*

Proof. Suppose, in $V[G]$, that $\mathcal{F}$ is a free filter and $\chi(\mathcal{F}) < |\kappa|$. Then $\mathcal{F}$ must occur in some intermediate stage of the forcing (or more precisely, a base for $\mathcal{F}$ must occur in some intermediate stage), i.e. $\mathcal{F} \in V[G_\alpha]$ for some $\alpha < \kappa$, where G_α is $\mathbb{P}_\alpha$-generic over V (by Lemma 2.4.13). Now if λ is a limit ordinal of cofinality ω with $\alpha < \lambda < \kappa$, then the initial stage $V[G_\lambda]$ contains a real $A \subseteq \omega$ which is Cohen over $V[G_\alpha]$. It is well known that there can be no $B \in \mathcal{F}$ with either $B \subseteq A$ or $B \subseteq \omega \setminus A$ (because for each infinite real $B \subseteq \omega$ in the ground model, $\{p \in \mathrm{Fn}(\omega, 2) : \exists n \in B\, p(n) = 0\}$ is dense in $\mathrm{Fn}(\omega, 2)$), so since $A \in V[G]$ also, we see that $\mathcal{F}$ cannot be an ultrafilter in $V[G]$. Therefore, $\mathfrak{u} \geq |\kappa|$. □

The upshot of this proposition is that if one wishes to start with $\mathfrak{u}$ small in the ground model, and use a finite support iteration to increase $\mathfrak{g}$, then in the generic

extension $\mathfrak{u}$ will not have stayed small. In [22], Mildenberger and Shelah show how to increase $\mathfrak{g}$ with a finite support iteration of c.c.c. forcings (while keeping $\mathfrak{b}$ small), and note that it is open how to keep $\mathfrak{u}$ small. The above says that with respect to finite support iterations, one cannot.

There is an alternative option, however, that one might consider. One may start by first increasing $\mathfrak{g}$ without worrying about the value of $\mathfrak{u}$, for instance force a model of Martin's Axiom. One may then force a small ultrafilter into existence, and hope to keep $\mathfrak{g}$ large. However, we show that such a second stage forcing cannot have particularly nice properties.

Definition 2.4.17. *Suppose $\mathbb{P}$ is a κ-length iteration of forcings, $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$;*

1. *We say that forcing with $\mathbb{P}$ only adds reals at initial stages if for any G which is $\mathbb{P}$-generic over V, and any real $A \subseteq \omega$ in $V[G]$, there is an $\alpha < \kappa$ with $A \in V[G_\alpha]$.*

2. *We say that forcing with $\mathbb{P}$ regularly adds reals if and only if whenever G is $\mathbb{P}$-generic over V and $\alpha < \kappa$ then*

$$V[G] \models \mathcal{P}(\omega) \neq \mathcal{P}(\omega) \cap V[G_\alpha].$$

Example 2.4.18. *1. A κ-length iteration of Cohen reals, random reals, Matthias reals, Miller reals etc. regularly adds reals;*

2. *Any finite support iteration of non-trivial forcings of limit ordinal length regularly adds reals (since a Cohen real must be added at any stage of cofinality ω).*

3. *Any finite support iteration of length having uncountable cofinality (from the perspective of the generic extension) only adds reals at initial stages (by Lemma 2.4.13).*

The following generalises a result which appears in Blass' article [3]. Blass' article shows that adding uncountably many random or Cohen reals makes $\mathfrak{g} = \aleph_1$

in the generic extension (and hence, it is consistent that $\mathfrak{g} = \aleph_1$ while $\mathfrak{u} = \kappa$, for any regular cardinal κ). We show for a more general class of forcings, the length of the iteration is an upper bound for $\mathfrak{g}$ — our class of forcings need not have the symmetry inherent in Cohen and random forcings which allow us to view a κ-length iteration as a κ-length iteration followed by an $\aleph_1$-length iteration.

Theorem 2.4.19. *Suppose $\mathbb{P}$ is a κ-length iteration of forcings $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$ and suppose $\mathbb{P}$ only adds reals at initial stages, and $\mathbb{P}$ regularly adds reals. Suppose G is $\mathbb{P}$-generic over the ground model V. Then*

$$V[G] \models \mathfrak{g} \leq |\kappa| .$$

The proof is a generalisation of the proof in [3], and we quote the required tools from there.

Definition 2.4.20 ([3], pg24)**.** *A set $\mathcal{X}$ of infinite subsets of ω is said to be Turing cofinal if and only if the set of Turing equivalence classes*

$$\{[X]_{Tur} : X \in \mathcal{X}\}$$

is cofinal in the partial order of Turing degrees.

We say that a set $\mathcal{X}$ of infinite subsets of ω is almost Turing cofinal if and only if there is a real $A \subseteq \omega$ such that $\{X \vee_{rec} A : X \in \mathcal{X}\}$ is Turing cofinal, where $\vee_{rec}$ denotes the recursive join of two sets.

Theorem 2.4.21 ([3], Theorem 2, pg24)**.** *Suppose $\kappa < \mathfrak{g}$ and $\mathcal{X}$ is almost Turing cofinal. Suppose $\mathcal{X} = \bigcup_{\alpha<\kappa} \mathcal{X}_\alpha$. Then for some $\alpha < \kappa$, $\mathcal{X}_\alpha$ is almost Turing cofinal.*

We prove a corollary to this result, not because it is necessary, but because it tidies up the proof of Theorem 2.4.19, and relates the above to Turing cofinal sets.

Corollary 2.4.22. *Suppose $\mathcal{X}$ is Turing cofinal, and $\{[X]_{Tur} : X \in \mathcal{X}\}$ is upward directed with respect to the ordering of Turing degrees, $\leq_{Tur}$. Suppose $\{\mathcal{X}_\alpha : \alpha < \kappa\}$ is a directed collection of sets (more precisely, that for any $\alpha, \beta < \kappa$ there is a $\gamma < \kappa$ with $\mathcal{X}_\alpha, \mathcal{X}_\beta \subseteq \mathcal{X}_\gamma$) with $\kappa < \mathfrak{g}$ and $\mathcal{X} = \bigcup_{\alpha<\kappa} \mathcal{X}_\alpha$. Furthermore, let us suppose that for*

each $\alpha < \kappa$, $\{[X]_{Tur} : X \in \mathcal{X}_\alpha\}$ is upward directed with respect to $\leq_{Tur}$. Then for some $\alpha < \kappa$, $\mathcal{X}_\alpha$ is Turing cofinal.

Proof. By Theorem 2.4.21, there is some $\alpha < \kappa$ with $\mathcal{X}_\alpha$ almost Turing cofinal. Suppose $A \subseteq \omega$ is a real witnessing this, that is, that $\{X \vee_{\text{rec}} A \colon X \in \mathcal{X}_\alpha\}$ is Turing cofinal. Since $\mathcal{X}$ is Turing cofinal, we can find a $B \in \mathcal{X}$ with $[B]_{\text{Tur}} \geq_{\text{Tur}} [A]_{\text{Tur}}$. Since $\mathcal{X}$ is a union of a directed system, we can find β with $B \in \mathcal{X}_\beta$ and $\mathcal{X}_\alpha \subseteq \mathcal{X}_\beta$. We claim $\mathcal{X}_\beta$ is Turing cofinal. To see this, let $P \subseteq \omega$, and choose $X \in \mathcal{X}_\alpha$ with $[X \vee_{\text{rec}} A]_{\text{Tur}} \geq_{\text{Tur}} [P]_{\text{Tur}}$. Now $X \in \mathcal{X}_\beta$ and $B \in \mathcal{X}_\beta$, so we can find a $Y \in \mathcal{X}_\beta$ with

$$[Y]_{\text{Tur}} \geq_{\text{Tur}} [X]_{\text{Tur}} \vee_{\text{Tur}} [B]_{\text{Tur}} \geq_{\text{Tur}} [X]_{\text{Tur}} \vee_{\text{Tur}} [A]_{\text{Tur}} .$$

But

$$[X]_{\text{Tur}} \vee_{\text{Tur}} [A]_{\text{Tur}} = [X \vee_{\text{rec}} A]_{\text{Tur}} \geq_{\text{Tur}} [P]_{\text{Tur}} .$$

Hence we have $[Y]_{\text{Tur}} \geq_{\text{Tur}} [P]_{\text{Tur}}$, as required to demonstrate that $\mathcal{X}_\beta$ is Turing cofinal. □

We use the above corollary later, but the following specialisation better applies to our current situation.

Corollary 2.4.23. *Suppose $\mathcal{X}$ is Turing cofinal, and $\{[X]_{Tur} : X \in \mathcal{X}\}$ is upward directed with respect to the ordering of Turing degrees, $\leq_{Tur}$. Suppose $\langle \mathcal{X}_\alpha : \alpha < \kappa \rangle$ is an increasing chain of sets with $\kappa < \mathfrak{g}$ and $\mathcal{X} = \bigcup_{\alpha<\kappa} \mathcal{X}_\alpha$. Furthermore, let us suppose that for each $\alpha < \kappa$, $\{[X]_{Tur} : X \in \mathcal{X}_\alpha\}$ is upward directed with respect to $\leq_{Tur}$. Then for some $\alpha < \kappa$, $\mathcal{X}_\alpha$ is Turing cofinal.*

Proof. A chain is a directed system, so the above Corollary applies. □

Proof of Theorem 2.4.19. Suppose G is $\mathbb{P}$-generic over V. Let us write, for $\alpha < \kappa$, $\mathcal{X}_\alpha = V[G_\alpha] \cap \mathcal{P}(\omega)$. Observe that for $\alpha < \beta$, $\mathcal{X}_\alpha \subseteq \mathcal{X}_\beta$ and for each $\alpha < \kappa$, $\mathcal{X}_\alpha$ is closed under the Turing operations and is downward closed with respect to $\leq_{\text{Tur}}$ (since **ZFC** is more than enough for computational operations), and so in particular, for $X, Y \in \mathcal{X}_\alpha$, $X \vee_{\text{rec}} Y \in \mathcal{X}_\alpha$. It follows that $\{[X]_{\text{Tur}} : X \in \mathcal{X}_\alpha\}$ is upward directed with respect to $\leq_{\text{Tur}}$ (in fact it is closed under $\vee_{\text{Tur}}$).

Since $\mathbb{P}$ only adds reals at initial stages, $V[G] \models \mathcal{P}(\omega) = \bigcup_{\alpha<\kappa} \mathcal{X}_\alpha$, and trivially $V[G] \models (\mathcal{P}(\omega)$ is Turing cofinal). So in $V[G]$, if $\mathfrak{g} > \kappa$, then by the above Corollary, for some $\alpha < \kappa$, $\mathcal{X}_\alpha$ is Turing cofinal. But then

$$V[G] \models \mathcal{P}(\omega) = \mathcal{X}_\alpha = \mathcal{P}(\omega) \cap V[G_\alpha],$$

because $\mathcal{X}_\alpha$ is downward closed with respect to $\leq_{\text{Tur}}$. This is a contradiction, since $\mathbb{P}$ regularly adds reals. Hence $\mathfrak{g} \leq \kappa$, and since $\mathfrak{g}$ is a cardinal, $\mathfrak{g} \leq |\kappa|$. □

Corollary 2.4.24. *Suppose κ is an uncountable regular cardinal, and $\mathbb{P}$ is a κ-length finite support iteration of non-trivial c.c.c. forcings. If G is $\mathbb{P}$-generic over V, then $V[G] \models \mathfrak{g} \leq \kappa$. In particular, $V[G] \models \mathfrak{g} \leq \mathfrak{u}$.*

Proof. First observe that a κ-length finite support iteration of non-trivial forcings regularly adds reals, since it adds Cohen reals at limit stages of countable cofinality. Now, since $\mathbb{P}$ is a finite support iteration of c.c.c. forcings, it is itself c.c.c., so preserves cofinalities. In particular, Lemma 2.4.13 applies so that $\mathbb{P}$ only adds reals at initial stages. The above Theorem hence gives $\mathfrak{g} \leq |\kappa|$ in $V[G]$. Note that $\mathbb{P}$ preserves cardinals also, so $|\kappa| = \kappa$ in $V[G]$.

For the inequality between $\mathfrak{g}$ and $\mathfrak{u}$, use Proposition 2.4.16 to see that $V[G] \models \mathfrak{g} \leq \kappa \leq \mathfrak{u}$. □

Remark 2.4.25. *A few things are worth noting with respect to these results.*

1. *Note that we don't need full cofinality preservation in the proof above. We only need that $V[G] \models \text{cf}(\kappa) \geq \aleph_1$, so any finite support iteration with this property will also yield $\mathfrak{g} \leq \mathfrak{u}$.*

2. *One can also see from these results that if we iterate $\aleph_2$ many of the classical proper forcings for adding reals (Sacks forcing, Prikry-Silver forcing, Laver Forcing, Matthias forcing, Miller forcing etc.) over a model of* **GCH***, then we must add reals at limit stages of countable cofinality.*

   To see this, first observe that since these classical proper forcings all add reals, any such iteration regularly adds reals. Without loss of generality let us consider

an iteration of length a limit ordinal of countable cofinality. If such an iteration didn't add reals at the final stage, then the iteration would only add reals at initial stages. Using a cofinal sequence, we can factor the iteration as a countable length iteration of proper forcings (see Jech, [16], Part III, Theorem 5.17, The ω-Factor Lemma), and consequently, in the generic extension we would have $\mathfrak{g} \leq \aleph_0$. One of course asks what kind of reals must be added (or indeed that we can avoid adding), and these appear to be interesting questions.

3. *A key property of Miller and Matet forcing (those traditionally used to force $\aleph_1 = \mathfrak{u} < \mathfrak{g} = \aleph_2$) is that they preserve p-points from the ground model. In particular then, if the ground model contains p-points, as it will under* **CH** *for instance, then some ultrafilter must be preserved, and so at no stage of the forcing do we add a real which is Cohen over the ground model.*

   The values of small cardinals in generic extensions, and various preservation results (like the preservation of p-points), are therefore useful indicators of the kinds of reals we have avoided adding.

Altogether this means that if we wish to keep $\mathfrak{g}$ large and introduce a small ultrafilter (so as to make $\mathfrak{u}$ small), then we cannot hope to use finite support iterations of c.c.c. forcings. In particular, adding a small ultrafilter with the usual variants of Matthias forcing along various ultrafilters cannot simultaneously preserve $\mathfrak{g}$.

The Filter Dichotomy is slightly weaker than $\mathfrak{u} < \mathfrak{g}$, and it remains unclear whether the Filter Dichotomy can be shown to be consistent using a finite support iteration of c.c.c. forcings. We explore the non-trivial effect the Filter Dichotomy has on small cardinals below, since this is relevant for the following section and because it allows us to eliminate the possibility of using some simple forcings to obtain the Filter Dichotomy.

Definition 2.4.26. *The cardinal $\mathfrak{g}_f$ is the size of the smallest collection of groupwise dense ideals with empty intersection.*

Observation 2.4.27. $\mathfrak{g} \leq \mathfrak{g}_f$.

Theorem 2.4.28 (Noted in [21])**.** *The Filter Dichotomy* **FD** *is equivalent to the cardinal inequality* $\mathfrak{u} < \mathfrak{g}_f$.

Proof. The equivalence of **FD** and $\mathfrak{u} < \mathfrak{g}_f$ was noted in [21], with reference to papers with results in need of adaptation. We provide a little more detail here, namely the precise theorems which need adapting, the adaptations to be made, and sketches of why these adaptations work.

First we sketch how $\mathfrak{u} < \mathfrak{g}_f$ entails **FD**. This follows by taking Theorem 1 in [6] and replacing (0) there (which says $\mathfrak{u} < \mathfrak{g}$) with $\mathfrak{u} < \mathfrak{g}_f$. The collections $\mathscr{S}_\alpha$ in the "(0) implies (1)" part of the proof are shown to be groupwise dense. In fact, it is easily seen that $\mathscr{S}_\alpha$ is closed under finite unions (since the ideal $\mathscr{I}$ is closed under binary pointwise maxima), so $\mathscr{S}_\alpha$ is a groupwise dense ideal, and so the assumption $\mathfrak{u} < \mathfrak{g}_f$ suffices to obtain (1). (3) of the same theorem is just **FD**, and it is shown that (1) implies (3), so this together shows $\mathfrak{u} < \mathfrak{g}_f$ entails **FD**.

For the converse, we adapt Theorem 8 from [4]. This Theorem says that if $\mathfrak{u} < \mathfrak{g}$, then $\mathfrak{g} = \mathfrak{d}$. In fact, one can obtain, by more or less the same proof, that **FD** entails $\mathfrak{g}_f = \mathfrak{d}$. The main adaptation is to let $\mathscr{G}_\alpha$ for $\alpha < \lambda < \mathfrak{d}$ be arbitrary groupwise dense *ideals*, so that then, the sets $\mathscr{A}_\alpha$ are free filters. The Filter Dichotomy then provides for each α a finite-to-one $f_\alpha \colon \omega \to \omega$ with $f_\alpha(\mathscr{A}_\alpha)$ either the cofinite filter or an ultrafilter. The proof may then continue as in [4], with the exception that we may now conclude $\mathfrak{g}_f \geq \mathfrak{d}$. Note that all the results used in the proof from [2], which is referenced as [1] in [4], only depend on **NCF**, which is a consequence of the Filter Dichotomy. To finish, we use that $\mathfrak{g}_f \leq \mathfrak{c}$, and [4], Theorem 7 (or [5], Corollary 9.20), which says that the Filter Dichotomy implies $\mathfrak{d} = \mathfrak{c}$, so under **FD**[2] we have $\mathfrak{g}_f \leq \mathfrak{d}$, and hence $\mathfrak{g}_f = \mathfrak{d} = \mathfrak{c}$.

From this we obtain $\mathfrak{u} < \mathfrak{g}_f$, because **FD** entails **NCF** (see Appendix A — **NCF** says that any two free ultrafilters on ω can be identified by a finite-to-one map), and **NCF** entails $\mathfrak{u} < \mathfrak{d}$ (see Corollary 15, (a), [2]). Altogether we obtain from **FD** that $\mathfrak{u} < \mathfrak{d} = \mathfrak{g}_f$, as we require. □

[2]In fact in **ZFC**, $\mathfrak{g}_f \leq \mathfrak{d}$. In [8] the author notes that the proof of Theorem 3.1 in [7] shows this.

Addendum 2.4.29. *(to proof) If the Filter Dichotomy holds, then* $\mathfrak{g}_f = \mathfrak{d} = \mathfrak{c}$.

We also note the following remarkable result of Shelah, demonstrable of the deeper relationships between small cardinals. The proof uses the base tree theorem to explicitly construct groupwise dense ideals with empty intersection.

Theorem 2.4.30 (Shelah, [27])**.** $\mathfrak{g}_f \leq \mathfrak{b}^+$. *Hence,* $\mathfrak{g} \leq \mathfrak{b}^+$.

Corollary 2.4.31. *Suppose* $\mathfrak{u} < \mathfrak{g}_f$. *Then* $\mathfrak{b} = \mathfrak{u} < \mathfrak{u}^+ = \mathfrak{g}_f = \mathfrak{c}$.

Proof. [5], Theorem 9.10, says there is a filter $\mathcal{F}$ generated by $\mathfrak{b}$ sets which is not almost cofinite. Given **FD**, $\mathcal{F}$ must be almost an ultrafilter, and hence $\mathfrak{u} \leq \chi(\mathcal{F}) \leq \mathfrak{b}$. But it is well known that $\mathfrak{b} \leq \mathfrak{u}$, hence we have $\mathfrak{b} = \mathfrak{u}$. Then

$$\mathfrak{b} = \mathfrak{u} < \mathfrak{g}_f \leq \mathfrak{b}^+.$$

So $\mathfrak{g}_f = \mathfrak{b}^+ = \mathfrak{u}^+$. In addition to this, we have already seen that under **FD**, $\mathfrak{g}_f = \mathfrak{d} = \mathfrak{c}$. Hence altogether, if **FD** holds then

$$\mathfrak{b} = \mathfrak{u} < \mathfrak{g}_f = \mathfrak{b}^+ = \mathfrak{u}^+ = \mathfrak{d} = \mathfrak{c}. \qquad \square$$

Corollary 2.4.32. $\mathfrak{g} \leq \mathfrak{g}_f \leq \mathfrak{u}^+$.

Proof. Either $\mathfrak{g}_f \leq \mathfrak{u}$, or $\mathfrak{u} < \mathfrak{g}_f$, in which case the above applies. $\square$

We formalise the question concerning use of a c.c.c. (iterated) forcing to establish **FD**. It seems that the methods using Turing cofinal sets do not adapt for analysis of $\mathfrak{g}_f$, and it would be interesting to develop new machinary to do this analysis.

Question 2.4.33. *Is there a c.c.c. forcing* $\mathbb{P}$ *for which, whenever* G *is* $\mathbb{P}$*-generic over the ground model* V, $V[G] \models \mathbf{FD}$*?*

We already know that forcing over finite support iterations of non-trivial c.c.c. partial orders increases $\mathfrak{u}$ by the regular addition of Cohen reals, so one might look for an iterated forcing of c.c.c. partial orders amongst the type which first increase $\mathfrak{g}_f$, and then add a small ultrafilter. The addition of this ultrafilter is unlikely to

be simple (if possible at all). Observe that the usual way of adding an ultrafilter, a finite support iteration of Matthias forcings along ultrafilters, will not preserve large (that is, strictly greater than $\aleph_2$) values of $\mathfrak{g}_f$, because $\mathfrak{g}_f \leq \mathfrak{u}^+$ will entail the collapse of $\mathfrak{g}_f$ at limit stages of cofinality $\aleph_1$.

It is also worth noting that $\mathfrak{g} < \mathfrak{g}_f$ can be forced with a finite support iteration of c.c.c. forcings over a ground model with a suitable diamond principle — see [8].

2.4.4 gap-1 morasses

Throughout this section, κ is assumed to be an uncountable regular cardinal.

One of the consequences of Shelah's result, Theorem 2.4.30, that $\mathfrak{g} \leq \mathfrak{b}^+$, is that under $\mathfrak{u} < \mathfrak{g}$, $\mathfrak{g} = \mathfrak{u}^+$. This causes various technical difficulties for forcing arguments striving for $\aleph_1 < \mathfrak{u} < \mathfrak{g}$. One of these is that the standard method of preserving a p-point from the ground model will require a p-point of large character (the desired value of $\mathfrak{u}$) — in particular **CH** must fail in the ground model — and this means plenty of forcing theory will not apply, and large antichains appear in many of the traditional proper forcings that one might hope to adapt. Secondly, we have already seen that trying to chain two different iterations together causes difficulties. It is too easy to collapse $\mathfrak{g}$ and quite difficult to increase it without collapsing $\mathfrak{c}$ or failing to preserve $\mathfrak{u}$. Furthermore, we would need iterations of two different complexities, one of length κ for making $\mathfrak{u} = \kappa$, and one of length κ^+ to make $\mathfrak{g} = \kappa^+$. These won't interleave in a nice way, so it seems unreasonable to hope to force values for $\mathfrak{u}$ and $\mathfrak{g}$ simultaneously using a linear iteration.

We will here discuss various forcings constructed along gap-1 morasses. The idea of using a morass to construct a forcing originates from the capability of a gap-1 morass at κ to capture κ^+ in pieces of size less than κ. One hopes that this "jump" might be reflected in some suitably chosen forcing, so that on the one hand (using the κ-ness of the morass) one forces $\mathfrak{u} = \kappa$, while on the other (using the κ^+-ness) one forces $\mathfrak{g} = \kappa^+$. A morass might therefore be the correct tool for interleaving forcings of two different complexities. However, tools for forcing along morasses

have not yet been well developed. The aim of this section is to begin developing these tools.

We recall here the definition and basic properties of simplified gap-1 morasses at κ, in line with Velleman, [33]. Since we will have maps defined on ordinals it will be useful to be able to clearly distinguish between a map applied to an ordinal, and the image of the ordinal as a set. Hence we write $f[\beta] = \{f(\alpha) : \alpha < \beta\}$.

Definition 2.4.34 ([33], Definition 2.10)**.** *A sequence of ordinals $\langle \theta_\alpha : \alpha \leq \kappa \rangle$ along with a collection of sets of maps $\mathcal{F}_{\alpha,\beta}$ for $\alpha < \beta \leq \kappa$ is a (simplified) gap-1 morass at κ (or, more concisely, a $(\kappa, 1)$-morass) if and only if*

(M1) $\theta_\kappa = \kappa^+$ *and for each* $\alpha < \kappa$, $0 < \theta_\alpha < \kappa$;

(M2) $\mathcal{F}_{\alpha,\beta}$ *is a set of order preserving maps from* θ_α *to* θ_β;

(M3) For $\alpha < \beta < \kappa$, $|\mathcal{F}_{\alpha,\beta}| < \kappa$;

(M4) For $\alpha < \beta < \gamma \leq \kappa$, $\mathcal{F}_{\alpha,\gamma} = \{g \circ f : f \in \mathcal{F}_{\alpha,\beta},\ g \in \mathcal{F}_{\beta,\gamma}\}$;

(M5) For $\alpha < \kappa$, $\mathcal{F}_{\alpha,(\alpha+1)}$ *is either a singleton or a pair* $\{f, g\}$ *such that for some* $\beta < \theta_\alpha$, $f \restriction \beta = g \restriction \beta$ *and* $f[\theta_\alpha] \subseteq g(\beta)$. *The pair of functions* $\{f, g\}$ *is called an amalgamation pair and* β *the split point of the pair;*

(M6) Suppose $\lambda \leq \kappa$ *is a limit ordinal,* $\alpha_1, \alpha_2 < \lambda$ *and* $f_1 \in \mathcal{F}_{\alpha_1,\lambda}$, $f_2 \in \mathcal{F}_{\alpha_2,\lambda}$. *Then there is a* $\beta < \lambda$ *with* $\alpha_1, \alpha_2 < \beta$, *and* $g_1 \in \mathcal{F}_{\alpha_1,\beta}$, $g_2 \in \mathcal{F}_{\alpha_2,\beta}$ *and* $h \in \mathcal{F}_{\beta,\lambda}$ *so that* $f_1 = h \circ g_1$ *and* $f_2 = h \circ g_2$.

(M7) $\bigcup \{f[\theta_\alpha] : f \in \mathcal{F}_{\alpha,\kappa}, \alpha < \kappa\} = \kappa^+$.

Velleman also proves a generalisation of property (M6) in the case where $\lambda = \kappa$. This $(< \kappa)$-factorisation result is of use to us later when we analyse forcings constructed along a morass — it allows us to do initial stage style arguments without having a nice linear iteration of forcings.

Lemma 2.4.35 ([33], Corollary 3.5)**.** *Suppose* $(\langle \theta_\alpha : \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \kappa \rangle)$ *is a gap-1 morass at* κ, *and* $\mu < \kappa$. *Suppose for each* $\delta < \mu$ *we have an ordinal* $\beta_\delta < \kappa$

and a map $f_\delta \in \mathcal{F}_{\beta_\delta,\kappa}$. Then there is a $\gamma < \kappa$ with $\beta_\delta < \gamma$ for each $\delta < \mu$, and a map $g \in \mathcal{F}_{\gamma,\kappa}$ alongside maps $h_\delta \in \mathcal{F}_{\beta_\delta,\gamma}$ for each $\delta < \mu$, with the property that for each $\delta < \mu$, $f_\delta = g \circ h_\delta$.

Note that if there is a gap-1 morass at κ then there is a gap-1 morass at κ with the property that for all $\beta > 0$

$$\bigcup \{f\,[\theta_\alpha] : f \in \mathcal{F}_{\alpha,\beta}, \alpha < \beta\} = \theta_\beta \qquad (\text{M7}^+)$$

(see Velleman [33] for details). We assume all our gap-1 morasses at κ have this generalisation of property (M7), and refer to it as (M7$^+$) when we use it in the following.

We also suppose that the identity map occurs in $\mathcal{F}_{\alpha,(\alpha+1)}$ for each $\alpha < \kappa$, since if there is a gap-1 morass at κ, then there is a gap-1 morass at κ with this property (see [15] and [33]). In the notation of (M5) we then have $f = id$. This allows us to make use of the following result of Irrgang. It is useful to know that the identity is always in the set of maps between levels of the morass.

Lemma 2.4.36 (Irrgang, [15], Lemma 3.3)**.** *If $\mathcal{M}$ is a gap-1 morass at κ with a map system $\langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \kappa \rangle$, then for all $\alpha < \beta \leq \kappa$, $id \in \mathcal{F}_{\alpha,\beta}$.*

The structure of a morass naturally gives rise to a tree. We call this tree the morass tree (associated with a morass), and define it formally below.

Definition 2.4.37. *Suppose $\mathcal{M} = (\langle \theta_\alpha : \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \kappa \rangle)$ is a gap-1 morass at κ. Then this morass gives rise to a tree structure as follows. Define*

$$T = \{(\alpha, \nu) : \alpha \leq \kappa,\ \nu < \theta_\alpha\}$$

and an ordering $\prec$ on T by

$$(\alpha, \nu) \prec (\beta, \tau)$$

if and only if $\alpha < \beta$ and there is an $f \in \mathcal{F}_{\alpha,\beta}$ with $f(\nu) = \tau$. We call the structure $(T, \prec)$ the morass tree, or the tree associated to the morass $\mathcal{M}$.

For $s = (\alpha, \nu) \in T$ let us define $\nu(s) = \nu$ and $\alpha(s) = \alpha$. Furthermore, if $s \prec t$ in the morass tree, let us define $\pi_{st} = f \restriction \nu(s) + 1$ where f is a map witnessing $s \prec t$ (see [33], [15] or the lemma below to see this is well-defined).

See [33] or [15] for details on why this is in fact a tree. The tree structure captures the way in which the images of the morass maps cover κ^+. Observe that if we ignore the κ-th level of the tree, then we obtain a κ-Kurepa tree (see Kunen, [18]).

The following is a useful lemma to have, and it is used in proving that the tree associated with a morass is in fact a tree. It also shows us that π_{st} is well-defined for $s \prec t$ in the morass tree.

Lemma 2.4.38 (Lemma 3.1, [15] or Lemma 3.2, [33])**.** *Suppose $\mathcal{M}$ is a morass with maps $\langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \kappa \rangle$. Let $\alpha < \beta \leq \kappa$ and $f_1, f_2 \in \mathcal{F}_{\alpha,\beta}$. Suppose $\tau_1, \tau_2 < \theta_\alpha$ and $f_1(\tau_1) = f_2(\tau_2)$. Then $\tau_1 = \tau_2$ and $f_1 \restriction \tau_1 = f_2 \restriction \tau_2$.*

2.4.5 FS systems along gap-1 morasses

Here we are motivated by the idea of using a gap-1 morass to push $\mathfrak{u}$ and $\mathfrak{g}$ apart. Methods of forcing along morasses have been introduced in the literature already (for instance in [15] and [30]) but a body of tools and theory has not yet been developed. This section develops some of this theory, with a focus on iteration style arguments, and applies this theory to obtain bounds for $\mathfrak{u}$ and $\mathfrak{g}$ for certain classes of forcings constructed along gap-1 morasses. Interestingly, the arguments which preclude the use of finite support iterations of c.c.c. forcings carry across to this class of forcings, but it is from these arguments that we obtain bounds for our small cardinals. These bounds do not preclude the possibility of these forcings working to establish $\mathfrak{u} < \mathfrak{g}$.

FS systems along gap-1 morasses are introduced and defined by Irrgang in [15]. We provide the basic definition verbatim here. Tzimas' PhD thesis, [30], also outlines a framework for forcing along a morass, but instead of using Velleman's simplified morasses, he uses the older Jensen style morasses. His construction therefore also seems a little less natural, and his example comes with only a little, though by no means straightforward, analysis; he proves ω-distributivity and the ω_2-chain condition, and hence cardinal preservation, but does not present any additional

properties of the model obtained. Although Irrgang's FS systems seem to make more sense categorially — the ordinals and order maps of a morass being converted to partial orders and complete embeddings — Tzimas motivates his work as an attempt to deal with the continuum problem, that of generalising results we have only proved consistent in the $\aleph_1$-$\aleph_2$ case. $\aleph_1 < \mathfrak{u} < \mathfrak{g}$ is an instance of such a problem, so the existence and motivation for Tzimas' work lends credence to the idea of using a morass.

Definition 2.4.39 (FS system along a morass, Section 4, [15])**.** *Let*

$$\mathcal{M} = \left(\langle \theta_\alpha : \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \kappa \rangle\right)$$

be a gap-1 morass at κ, and $(T, \prec)$ the tree associated with the morass $\mathcal{M}$. We say that

$$\left(\left\langle \mathbb{P}_\eta : \eta \leq \kappa^+ \right\rangle, \langle \sigma_{st} : s \prec t \rangle, \langle e_\alpha : \alpha < \kappa \rangle\right)$$

is an FS system along $\mathcal{M}$ if and only if the following are true:

(FS1) *$\langle \mathbb{P}_\eta : \eta \leq \kappa^+ \rangle$ is a sequence of partial orders with the property that if $\eta \leq \nu$ then $\mathbb{P}_\eta \subseteq_\perp \mathbb{P}_\nu$, and for limit ordinals λ, $\mathbb{P}_\lambda = \bigcup_{\eta<\lambda} \mathbb{P}_\eta$.*

(FS2) *$\langle \sigma_{st} : s \prec t \rangle$ is a commutative system of injective embeddings*

$$\sigma_{st} \colon \mathbb{P}_{\nu(s)+1} \to \mathbb{P}_{\nu(t)+1}$$

such that if t is a limit point (that is, of limit height) in the tree $(T, \prec)$ then

$$\mathbb{P}_{\nu(t)+1} = \bigcup \left\{ \sigma_{st}\left[\mathbb{P}_{\nu(s)+1}\right] : s \prec t \right\}.$$

(FS3) *$e_\alpha \colon \mathbb{P}_{\theta_{\alpha+1}} \to \mathbb{P}_{\theta_\alpha}$ for each $\alpha < \kappa$.*

(FS4) *Suppose $s \prec t$. If $\pi_{st}(\nu') = \tau'$ and $s' = (\alpha(s), \nu')$, $t' = (\alpha(t), \tau')$, then σ_{st} extends $\sigma_{s't'}$, $\sigma_{s't'} \subseteq \sigma_{st}$. (Note $s' \prec t'$, see [33] or [15]).*

Note that now we can define, for $f \in \mathcal{F}_{\alpha,\beta}$,

$$\sigma_f = \bigcup \{ \sigma_{st} \colon s = (\alpha, \nu) \text{ and } t = (\beta, f(\nu)) \}.$$

(FS5) *If π_{st} is the identity, then so is σ_{st}. Hence, in the notation above, σ_{id} is just the identity map.*

Note that for $\alpha < \kappa$ there are at most two maps in $\mathcal{F}_{\alpha,(\alpha+1)}$, the identity (see Lemma 2.4.36) and f_α, say. We use this notation below.

(FS6) *Suppose $\alpha < \kappa$; we ask for the following two conditions. Firstly, $\mathbb{P}_{\theta_\alpha} \subseteq_c \mathbb{P}_{\theta_{\alpha+1}}$ and for $p \in \mathbb{P}_{\theta_{\alpha+1}}$, $e_\alpha(p)$ is a reduction of p to $\mathbb{P}_{\theta_\alpha}$ with respect to the identity map. Secondly, $\sigma_{f_\alpha} : \mathbb{P}_{\theta_\alpha} \to \mathbb{P}_{\theta_{\alpha+1}}$ is a complete embedding, and for $p \in \mathbb{P}_{\theta_{\alpha+1}}$, $e_\alpha(p)$ is a reduction of p to $\mathbb{P}_{\theta_\alpha}$ with respect to σ_{f_α}.*

(FS7) *Suppose $\alpha < \kappa$. If $p \in \mathbb{P}_{\theta_\alpha}$ then $e_\alpha(p) = p$. Also, if $p \in \text{ran}(\sigma_{f_\alpha})$, then $e_\alpha(p) = \sigma_{f_\alpha}^{-1}(p)$.*

The partial order $\mathbb{P}_{\kappa^+} = \mathbb{P}_{\theta_\kappa}$ will often be referred to as just $\mathbb{P}$ for forcings of this type.

The naming here is indicative of the currently poor understanding of how these forcings really look. While we have complete embeddings between successor levels, it is not clear in general what happens at limit stages, in particular, whether we get for limit ordinals λ and $f \in \mathcal{F}_{\alpha,\lambda}$ that σ_f is a complete embedding $\mathbb{P}_{\theta_\alpha} \to \mathbb{P}_{\theta_\lambda}$; indeed Irrgang himself, [15], questions whether his FS systems can be viewed as an iteration in some way. Our results below take this as an extra assumption, since this seems like a reasonable assumption to aid analysis, and makes categorial sense. That is, it makes sense to transfer all order preserving maps on a morass to complete embeddings, not just those at successor stages. One can then think of an FS system along a gap-1 morass at κ as κ^+-many κ-length iterations performed simultaneously. One of the reasons Irrgang may have ommitted this assumption was simply that he didn't need it for his analysis. Irrgang uses a gap-1 morass at ω_1 to build a c.c.c. forcing which adds an ω_2-Suslin tree. The initial stage analysis we seek here was not required.

In linear iterations it is easy to refer to initial stages by the ordinal level they occur at. A technical obstacle with FS systems along morasses is that $\mathbb{P}_{\theta_\alpha}$ is potentially

included in $\mathbb{P}_{\kappa^+}$ in many different ways (one way for each map in $\mathcal{F}_{\alpha,\kappa}$). Hence we introduce some new notation to refer to the pullback of generic filters onto these initial stages. Suppose G is $\mathbb{P}_{\kappa^+}$-generic over the ground model. For $\alpha < \kappa$ and $f \in \mathcal{F}_{\alpha,\kappa}$ we write G_f for the filter

$$G_f = \{p \in \mathbb{P}_{\theta_\alpha} : \sigma_f(p) \in G\} = \sigma_f^{-1}(G).$$

Note that, in the case when σ_f is a complete embedding, G_f is $\mathbb{P}_{\theta_\alpha}$-generic over the ground model.

Again, we will only be interested in forcings which are actually intended to do something. We define what we mean by a non-trivial FS system along a morass.

Definition 2.4.40. *We say that an FS system along a gap-1 morass at κ*

$$\left(\left\langle \mathbb{P}_\eta : \eta \leq \kappa^+ \right\rangle, \langle \sigma_{st} : s \prec t \rangle, \langle e_\alpha : \alpha < \kappa \rangle\right)$$

is non-trivial if and only if

1. *for each $\alpha \leq \kappa$, $\mathbb{P}_{\theta_\alpha} \neq \emptyset$;*
2. *for each $\alpha \leq \kappa$, $\mathbb{P}_{\theta_\alpha}$ is separative;*
3. *for each $\alpha < \kappa$, and $f \in \mathcal{F}_{\alpha,(\alpha+1)}$, if we factor $\mathbb{P}_{\theta_{\alpha+1}}$ along σ_f as $\mathbb{P}_{\theta_\alpha} \star \dot{Q}$, then for each $\mathbb{P}_{\theta_\alpha}$-name for an element of $\dot{Q}$, $\dot{q}$, there are $\mathbb{P}_{\theta_\alpha}$-names $\dot{r}, \dot{s}$ for elements of $\dot{Q}$ with*

$$\mathbb{1} \Vdash \dot{r}, \dot{s} \leq \dot{q}, \text{ and } \dot{r} \text{ and } \dot{s} \text{ are incompatible}.$$

Our first task is to develop tools to analyse the generic extension over such forcings. We then apply this to the cardinals of interest to obtain bounds. First we prove a reflection theorem for FS systems along a morass. Elements of the technique in the proof can be found in various results in Irrgang's paper [15].

Lemma 2.4.41. *Suppose $\left(\left\langle \mathbb{P}_\alpha : \alpha \leq \kappa^+ \right\rangle, \langle \sigma_{st} : s \prec t \rangle, \langle e_\alpha : \alpha < \kappa \rangle\right)$ is an FS system along a $(\kappa, 1)$-morass which preserves cardinals. Suppose further that all maps σ_f are complete embeddings. Suppose $X \in V$ and $S \in V[G]$, with $S \subseteq X$. Suppose $V[G] \models |X| < \kappa$. Then there is $\xi < \kappa$ and $g \in \mathcal{F}_{\xi,\kappa}$ with $S \in V[G_g]$.*

Proof. For each $x \in X$ we have $x \in S$ if and only if there exists a $p \in G$ with $p \Vdash \check{x} \in \dot{S}$. For $x \in S$ fix such a $p \in G$. Now $p \in \mathbb{P}_{\kappa^+}$, so by (FS2) we can find η minimal with $p \in \mathbb{P}_{\eta+1}$.

Now let $s = (\kappa, \eta)$, and find $s' \prec s$ minimal with $p \in \operatorname{ran}(\sigma_{s's})$. Suppose $\sigma_{s's}(p') = p$. Again, by (FS2) notice that $\alpha(s')$ must be a successor, and so in particular not κ. Write $\alpha_x = \alpha(s')$, so $\alpha_x < \kappa$. Furthermore, $\pi_{s's} \subseteq f_x$ for some $f_x \in \mathcal{F}_{\alpha_x,\kappa}$.

Now we consider all such f_x simultaneously. By Lemma 2.4.35, there is a $\gamma < \kappa$ and $g \in \mathcal{F}_{\gamma,\kappa}$, and for each $x \in S$ an $f'_x \in \mathcal{F}_{\alpha_x,\gamma}$ with $f_x = g \circ f'_x$.

Now σ_g is a complete embedding of $\mathbb{P}_{\theta_\gamma}$ into $\mathbb{P}_{\theta_\kappa} = \mathbb{P}_{\kappa^+}$, so G_g is a $\mathbb{P}_{\theta_\gamma}$-generic filter over V. For $x \in S$ we also have $\sigma_{f'_x}(p') \in G_g$ and

$$\sigma_g\left(\sigma_{f'_x}(p')\right) = \sigma_{g \circ f'_x}(p') = \sigma_{f_x}(p') = \sigma_{s's}(p') = p.$$

Hence we see that

$$S = \left\{x \in X : \exists p \in G_g \; \sigma_g(p) \Vdash \check{x} \in \dot{S}\right\} \in V[G_g],$$

noting that the forcing relation is definable over V, and when it is applied to atomic formulas is absolute between transitive models. □

Corollary 2.4.42. *Suppose $(\langle \mathbb{P}_\alpha : \alpha \leq \kappa^+ \rangle, \langle \sigma_{st} : s \prec t \rangle, \langle e_\alpha : \alpha < \kappa \rangle)$ is an FS system along a $(\kappa, 1)$-morass which preserves cardinals. Suppose further that all maps σ_f are complete embeddings. Then*

1. *if $A \subseteq \omega$ then there is a $\xi < \kappa$ and $g \in \mathcal{F}_{\xi,\kappa}$ with $A \in V[G_g]$;*
2. *if $V[G] \models (\mathcal{A} \subseteq \mathcal{P}(\omega)$ and $|\mathcal{A}| < \kappa)$ then there is a $\xi < \kappa$ and $g \in \mathcal{F}_{\xi,\kappa}$ with $\mathcal{A} \in V[G_g]$.*

In particular, small (size less than κ) filter bases appear at initial stages.

Proof. The above lemma applied directly. In the first case $X = \omega$, and in the second case $X = \lambda \times \omega$, where $\lambda = |\mathcal{A}|$. □

The following assumes a little more than complete embeddings between partial orders at morass levels. It would seem this extra strength is needed to make the following argument work. Note that all we really need to assume is that for $\alpha < \kappa$ if $\beta = \sup_{n\in\omega} \theta_{\alpha+n}$ then $\mathbb{P}_\beta$ is completely contained in $\mathbb{P}_{\theta_{\alpha+\omega}}$. It seems possible that we may have $\mathbb{P}_\beta$ looking like a finite support iteration (hence by the below argument, adding a Cohen real), while $\mathbb{P}_{\theta_{\alpha+\omega}}$ could be the countable support limit of the $\mathbb{P}_{\theta_{\alpha+n}}$s, which adds no reals, and hence cannot completely contain $\mathbb{P}_\beta$. Whether this can actually happen in a forcing along a morass remains unclear.

Theorem 2.4.43. *Suppose*

$$\left(\left\langle \mathbb{P}_\alpha \,:\, \alpha \leq \kappa^+ \right\rangle, \langle \sigma_{st} \,:\, s \prec t \rangle, \langle e_\alpha \,:\, \alpha < \kappa \rangle\right)$$

is a non-trivial FS system along a $(\kappa, 1)$-morass which preserves cardinals. Suppose further that all maps σ_f are complete embeddings, and also, that for $\xi < \eta < \kappa$, $\mathbb{P}_\xi$ is completely contained in $\mathbb{P}_\eta$ (via the identity map).

Now suppose $g \in \mathcal{F}_{\alpha,\kappa}$ for some $\alpha < \kappa$, and $h \in \mathcal{F}_{\alpha,(\alpha+\omega)}$, $f \in \mathcal{F}_{(\alpha+\omega),\kappa}$ with $g = f \circ h$. Then $V[G_f]$ contains a real which is Cohen over $V[G_g]$.

Proof. There are two cases we consider. For the first case, suppose g factors through the identity map $i\colon \theta_{\alpha+n} \to \theta_{\alpha+\omega}$ for some $n \in \omega$. Let $\lambda = \sup_{n\in\omega} \theta_{\alpha+n}$, so that $\lambda \leq \theta_{\alpha+\omega}$. Then $\mathbb{P}_\lambda$ is a finite support limit (in the normal linear sense) of the system $\left\langle \mathbb{P}_{\theta_{\alpha+n+k}} \,:\, k \in \omega \right\rangle$. Since $\mathbb{P}_\lambda$ is completely contained in $\mathbb{P}_{\theta_{\alpha+\omega}}$, G_f induces a generic filter on $\mathbb{P}_\lambda$. Hence since a generic filter on $\mathbb{P}_\lambda$ gives rise to a Cohen real over $V[G_{f\circ i}]$, and hence over $V[G_g]$, $V[G_f]$ must contain a real Cohen over $V[G_g]$.

The second case is therefore when for each $n \in \omega$ we cannot factor g through the identity map $\theta_{\alpha+n} \to \theta_{\alpha+\omega}$. We use g to build a new commutative system. We factor g through each stage to obtain maps $g_{ij}\colon \theta_{\alpha+i} \to \theta_{\alpha+j}$ for $i < j \leq \omega$, where $g_{ij} \in \mathcal{F}_{(\alpha+i),(\alpha+j)}$. We do this recursively. First, set $g_{0\omega} = g$. Given $g_{n\omega}$, we build $g_{n(n+1)}$ and $g_{(n+1)\omega}$ simply by factoring using morass property (M4), i.e. choosing $g_{n(n+1)} \in \mathcal{F}_{(\alpha+n),(\alpha+n+1)}$ and $g_{(n+1)\omega} \in \mathcal{F}_{(\alpha+n+1),(\alpha+\omega)}$ so that

$$g_{n\omega} = g_{(n+1)\omega} \circ g_{n(n+1)}.$$

To obtain the other maps g_{ij} we simply set, for $i < j < \omega$,

$$g_{ij} = g_{(j-1)j} \circ \ldots \circ g_{i(i+1)}.$$

Note that in no case do we have that $g_{n\omega}$ is the identity, so in particular

$$\langle \text{sup ran}\,(g_{n\omega}) \,:\, n \in \omega \rangle$$

is increasing. So let β be the limit ordinal given by

$$\beta = \bigcup_{n\in\omega} \text{sup ran}\,(g_{n\omega}).$$

We may construe each map $g_{n\omega}$ as a map $\theta_{\alpha+n} \to \beta$, and hence $\sigma_{g_{n\omega}}$ as a map $\mathbb{P}_{\theta_{\alpha+n}} \to \mathbb{P}_\beta$. By construction, $\langle g_{ij} \,:\, i < j \leq \omega \rangle$ forms a commutative system, hence so does $\left\langle \sigma_{g_{ij}} \,:\, i < j \leq \omega \right\rangle$.

Claim 2.4.44. *If, in the morass tree $s \prec r \prec t$, then*

$$\sigma_{st}\left[\mathbb{P}_{\nu(s)+1}\right] \subseteq \sigma_{rt}\left[\mathbb{P}_{\nu(r)+1}\right].$$

That is to say $\left\langle \sigma_{st}\left[\mathbb{P}_{\nu(s)+1}\right] \,:\, s \prec t \right\rangle$ is a (well-ordered) increasing chain.

Proof of Claim.

$$\sigma_{st}\left[\mathbb{P}_{\nu(s)+1}\right] = \sigma_{rt} \circ \sigma_{sr}\left[\mathbb{P}_{\nu(s)+1}\right] \subseteq \sigma_{rt}\left[\mathbb{P}_{\nu(r)+1}\right]. \qquad \square$$

Claim 2.4.45. $\mathbb{P}_\beta = \bigcup_{n\in\omega} \sigma_{g_{n\omega}}\left[\mathbb{P}_{\theta_{\alpha+n}}\right]$.

Proof of Claim. We have already remarked that $\bigcup_{n\in\omega} \sigma_{g_{n\omega}}\left[\mathbb{P}_{\theta_{\alpha+n}}\right] \subseteq \mathbb{P}_\beta$, so it remains to show the reverse inclusion. To this end, let $\tau' < \beta$ and pick $\tau > \tau'$ and $n \in \omega$ with $\tau \in \text{ran}\,(g_{n\omega})$. Since $\mathbb{P}_{\tau'} \subseteq \mathbb{P}_{\tau+1}$, if we show $\mathbb{P}_{\tau+1} \subseteq \bigcup_{n\in\omega} \sigma_{g_{n\omega}}\left[\mathbb{P}_{\theta_{\alpha+n}}\right]$ then the same is true for $\mathbb{P}_{\tau'}$.

Now $\tau \in \beta \subseteq \theta_{\alpha+\omega}$, so $t = (\alpha+\omega, \tau)$ is a point in the morass tree. But t also has limit height in the morass tree. This is because, by the additional property (M7$^+$) which we assumed for all our morasses, $\theta_{\alpha+\omega} = \bigcup \left\{ f\,[\theta_\xi] : \xi < \alpha+\omega,\ f \in \mathcal{F}_{\xi,(\alpha+\omega)} \right\}$, which ensures t has a predecessor in the tree, and the morass factoring property

(M4) which ensures every predecessor of t has a successor which still precedes t. Hence we have by (FS2) that

$$\mathbb{P}_{\tau+1} = \bigcup_{s \prec t} \sigma_{st} \left[\mathbb{P}_{\nu(s)+1} \right].$$

But the above claim allows us to simplify this union, more precisely, we have

$$\bigcup_{s \prec t} \sigma_{st} \left[\mathbb{P}_{\nu(s)+1} \right] = \bigcup_{s \prec t,\ \alpha(s)=\alpha+n} \sigma_{st} \left[\mathbb{P}_{\nu(s)+1} \right].$$

When $s \prec t$ and $\alpha(s) = \alpha + n$ however, then $\mathbb{P}_{\nu(s)+1} \subseteq \mathbb{P}_{\theta_{\alpha+n}}$ and σ_{st} is just an initial segment of $\sigma_{g_{n\omega}}$, by Lemma 2.4.38. These two facts together give us the inclusion

$$\bigcup_{s \prec t,\ \alpha(s)=\alpha+n} \sigma_{st} \left[\mathbb{P}_{\nu(s)+1} \right] \subseteq \bigcup_{n \in \omega} \sigma_{g_{n\omega}} \left[\mathbb{P}_{\theta_{\alpha+n}} \right].$$

Piecing the above three statements together, we have

$$\mathbb{P}_{\tau+1} \subseteq \bigcup_{n \in \omega} \sigma_{g_{n\omega}} \left[\mathbb{P}_{\theta_{\alpha+n}} \right].$$

So we have

$$\bigcup_{\tau < \beta} \mathbb{P}_\tau \subseteq \bigcup_{n \in \omega} \sigma_{g_{n\omega}} \left[\mathbb{P}_{\theta_{\alpha+n}} \right],$$

but since β is a limit ordinal, we have $\mathbb{P}_\beta = \bigcup_{\tau<\beta} \mathbb{P}_\tau$, and hence the reverse inclusion we required. □

Observation 2.4.46. *The partial orders $\left\{ \mathbb{P}_{\theta_{\alpha+n}} : n \in \omega \right\}$ and $\mathbb{P}_\beta$, along with maps $\left\{ \sigma_{g_{ij}} : i < j \leq \omega \right\}$, where $\sigma_{g_{n\omega}}$ is construed as a map $\mathbb{P}_{\theta_{\alpha+n}} \to \mathbb{P}_\beta$, form a commutative system of complete embeddings of partial orders, and furthermore*

$$\mathbb{P}_\beta = \bigcup_{n \in \omega} \sigma_{g_{n\omega}} \left[\mathbb{P}_{\theta_{\alpha+n}} \right].$$

Hence this system is a linear finite support iteration of length ω, by Theorem 2.4.12.

Now we can do the same argument as in the first case. Note that here $\mathbb{P}_\beta$ is completely contained in $\mathbb{P}_{\theta_{\alpha+\omega}}$, so the generic filter G_f induces a generic filter H on $\mathbb{P}_\beta$. Since $\mathbb{P}_\beta$ can be construed as a finite support iteration of length ω over $V[G_g]$, by the observation above, and by how we built our commutative system, $V[H]$ must contain a real Cohen over $V[G_g]$, and since $V[H] \subseteq V[G_f]$, so must $V[G_f]$. This completes the second case, and hence the proof. □

We now use these results to obtain some bounds for the small cardinals we are interested in. We begin with $\mathfrak{u}$.

Theorem 2.4.47. *Suppose*

$$\left(\left\langle \mathbb{P}_\alpha \,:\, \alpha \leq \kappa^+ \right\rangle, \langle \sigma_{st} \,:\, s \prec t \rangle, \langle e_\alpha \,:\, \alpha < \kappa \rangle\right)$$

is a non-trivial FS system along a $(\kappa, 1)$-morass which preserves cardinals. Suppose further that all maps σ_f are complete embeddings, and also, that for $\xi < \eta < \kappa$, $\mathbb{P}_\xi$ is completely contained in $\mathbb{P}_\eta$. Then

$$V[G] \models \mathfrak{u} \geq \kappa.$$

Proof. If $\mathcal{G}$ is a filter with character less than κ, then by Corollary 2.4.42, there is an $\alpha < \kappa$ and $g \in \mathcal{F}_{\alpha,\kappa}$ with $\mathcal{G} \in V[G_g]$ (or more precisely, so that a base for $\mathcal{G}$ is in $V[G_g]$). By factoring g through $\theta_{\alpha+\omega}$, find $f \in \mathcal{F}_{(\alpha+\omega),\kappa}$ and $g' \in \mathcal{F}_{\alpha,(\alpha+\omega)}$ with $g = f \circ g'$.

By the above Theorem, $V[G_f]$ contains a real which is Cohen over $V[G_g]$. Since $V[G_f] \subseteq V[G]$, $\mathcal{G}$ is not an ultrafilter in $V[G]$. □

Of course, one now asks about $\mathfrak{g}$ under these same hypotheses, and in fact we obtain an optimistic bound which in part reflects the "jump" in the morass we were trying to exploit.

Theorem 2.4.48. *Suppose*

$$\left(\left\langle \mathbb{P}_\alpha \,:\, \alpha \leq \kappa^+ \right\rangle, \langle \sigma_{st} \,:\, s \prec t \rangle, \langle e_\alpha \,:\, \alpha < \kappa \rangle\right)$$

is a non-trivial FS system along a $(\kappa, 1)$-morass which preserves cardinals. Suppose further that all maps σ_f are complete embeddings, and also, that for $\xi < \eta < \kappa$, $\mathbb{P}_\xi$ is completely contained in $\mathbb{P}_\eta$. Then

$$V[G] \models \mathfrak{g} \leq \left| \bigcup_{\alpha<\kappa} \mathcal{F}_{\alpha,\kappa} \right|.$$

Proof. Let $\mathcal{X} = \mathcal{P}(\omega)$ in $V[G]$. Let

$$\mathcal{D} = \left\{ \mathcal{P}(\omega)^{V[G_g]} : g \in \mathcal{F}_{\alpha,\kappa} \text{ for some } \alpha < \kappa \right\}.$$

Clearly $|\mathcal{D}| \leq \left|\bigcup_{\alpha<\kappa} \mathcal{F}_{\alpha,\kappa}\right|$, and $V[G] \models \bigcup \mathcal{D} = \mathcal{P}(\omega)$. Note that $\mathcal{D}$ is directed with respect to $\subseteq$. To see this, let $g \in \mathcal{F}_{\alpha,\kappa}$ and $h \in \mathcal{F}_{\beta,\kappa}$. Then there is a $\gamma < \kappa$ with $\alpha, \beta < \gamma$ and maps $g' \in \mathcal{F}_{\alpha,\gamma}$, $h' \in \mathcal{F}_{\beta,\gamma}$ and $k \in \mathcal{F}_{\gamma,\kappa}$ with $g = k \circ g'$ and $h = k \circ h'$. Then $\mathcal{P}(\omega)^{V[G_g]}, \mathcal{P}(\omega)^{V[G_h]} \subseteq \mathcal{P}(\omega)^{V[G_k]}$ and $\mathcal{P}(\omega)^{V[G_k]} \in \mathcal{D}$.

Now assume that in $V[G]$, $\mathfrak{g} > \left|\bigcup_{\alpha<\kappa} \mathcal{F}_{\alpha,\kappa}\right|$. By Corollary 2.4.22, in $V[G]$ some $\mathcal{Y} \in \mathcal{D}$ must be Turing cofinal.

Suppose $\mathcal{Y} = \mathcal{P}(\omega)^{V[G_g]}$, where $g \in \mathcal{F}_{\alpha,\kappa}$. Then $V[G] \models \mathcal{P}(\omega) = \mathcal{P}(\omega)^{V[G_g]}$. Suppose $h \in \mathcal{F}_{\alpha,(\alpha+\omega)}$ and $f \in \mathcal{F}_{(\alpha+\omega),\kappa}$ with $g = f \circ h$. By Theorem 2.4.43, we have that $V[G_f]$ contains a real which is Cohen over $V[G_g]$. But $V[G_f] \subseteq V[G]$, and we already have $\mathcal{P}(\omega)^{V[G]} = \mathcal{P}(\omega)^{V[G_g]}$. This is a contradiction. □

In the above scenario, note that elementary cardinal arithmetic yields bounds for $\left|\bigcup_{\alpha<\kappa} \mathcal{F}_{\alpha,\kappa}\right|$, namely

$$V[G] \models \kappa^+ \leq \left|\bigcup_{\alpha<\kappa} \mathcal{F}_{\alpha,\kappa}\right| \leq (\kappa.\kappa^\kappa)^V = (2^\kappa)^V.$$

In particular then, $V[G] \models \mathfrak{g} \leq (2^\kappa)^V \leq 2^\kappa$. Furthermore then, if $V \models 2^\kappa = \kappa^+$, then $V[G] \models \mathfrak{g} \leq \kappa^+$. Since assuming axioms like **GCH** in the ground model is often used to simplify the combinatorics involved in forcing arguments, this is a good bound under reasonable assumptions. However, we can do better, and establish the same bound, without additional assumptions in the ground model, using the morass theoretic lemma below, suggested by my supervisor, Robin Knight.

Lemma 2.4.49. *Suppose $\alpha < \beta \leq \kappa$ and $f, g \in \mathcal{F}_{\alpha,\beta}$ with $\sup f = \sup g$. Then $f = g$.*

Proof. We proceed by induction on β. When $\beta = \alpha + 1$ the result is clear from morass property (M5). Now suppose as an inductive hypothesis that $\alpha < \beta \leq \kappa$ with $\alpha + 1 < \beta$, and if $\alpha < \gamma < \beta$ and $f, g \in \mathcal{F}_{\alpha,\gamma}$ with $\sup f = \sup g$ then $f = g$. Now suppose $f, g \in \mathcal{F}_{\alpha,\beta}$ with $\sup f = \sup g$. We wish to show $f = g$.

First suppose β is a successor ordinal, say $\beta = \gamma + 1$. By morass property (M4), we can write

$$f = h \circ f'$$
$$g = k \circ g'$$

where $f', g' \in \mathcal{F}_{\alpha,\gamma}$ and $h, k \in \mathcal{F}_{\gamma,(\gamma+1)}$. If $h = k$ then we can conclude $\sup f' = \sup g'$ (by property (M2)), and hence by inductive hypothesis $f' = g'$, so that $f = g$. If $h \neq k$ then $\{h, k\}$ is an amalgamation pair (see property (M5)), with split point δ, say. But we have assumed that the identity map is in $\mathcal{F}_{\gamma,(\gamma+1)}$, so one of h and k is the identity map, so let us suppose, without loss of generality, that $h = id$, and so $\sup f = \sup f' < \theta_\gamma$. Now we must have $\mathrm{ran}\,(g') \subseteq \delta$, else $\sup g > \theta_\gamma > \sup f$, and so

$$g = (k \restriction \delta) \circ g' = id \circ g'$$

(see (M5)), and hence $\sup g = \sup g'$. Putting these steps together we have

$$\sup f' = \sup f = \sup g = \sup g'$$

so by the inductive hypothesis $f' = g'$, and hence

$$g = id \circ g' = id \circ f' = f.$$

The case when β is a limit ordinal is more straightforward. In this case, let us apply (M6) to obtain an ordinal γ with $\alpha < \gamma < \beta$ and a map $h \in \mathcal{F}_{\gamma,\beta}$ alongside maps $f', g' \in \mathcal{F}_{\alpha,\gamma}$ with

$$f = h \circ f'$$
$$g = h \circ g'.$$

By property (M2) we must have that $\sup f' = \sup g'$, so by inductive hypothesis $f' = g'$. It follows immediately that $f = g$. □

Corollary 2.4.50. *For $\alpha < \beta \leq \kappa$, $|\mathcal{F}_{\alpha,\beta}| \leq |\theta_\beta|$. In particular*

$$\left| \bigcup_{\alpha<\kappa} \mathcal{F}_{\alpha,\kappa} \right| \leq \kappa.\kappa^+ = \kappa^+.$$

Proof. The previous lemma says that $f \mapsto \sup f$ is an injection from $\mathcal{F}_{\alpha,\beta} \to \theta_\beta$. □

This combined with the previous theorem gives us a clear bound for $\mathfrak{g}$. We collect the combined results properly below.

Theorem 2.4.51. *Suppose*

$$\left(\left\langle \mathbb{P}_\alpha \,:\, \alpha \leq \kappa^+ \right\rangle, \langle \sigma_{st} \,:\, s \prec t \rangle, \langle e_\alpha \,:\, \alpha < \kappa \rangle\right)$$

is a non-trivial FS system along a $(\kappa, 1)$-morass which preserves cardinals. Suppose further that all maps σ_f are complete embeddings, and also, that for $\xi < \eta < \kappa$, $\mathbb{P}_\xi$ is completely contained in $\mathbb{P}_\eta$. Then

$$V[G] \models \mathfrak{u} \geq \kappa \text{ and } \mathfrak{g} \leq \kappa^+.$$

Proof. Combine the previous Corollary and previous two Theorems. □

These bounds for $\mathfrak{u}$ and $\mathfrak{g}$ look promising from the point of view of establishing $\aleph_1 < \mathfrak{u} < \mathfrak{g}$. In particular the "jump" in the morass has been mirrored in these bounds as we hoped. It is now natural to ask whether these bounds are sharp, which is equivalent to showing that one can establish $\mathfrak{u} < \mathfrak{g}$ using such a forcing construction. I am hopeful that in the future I might be able to construct such a forcing, and this is the most obvious direction in which to build on this work.

One of the difficulties in constructing such an FS system along a morass seems to be how to best exploit the κ^+-ness of the morass; the κ-flavour of the morass is the most prominent. One needs a way of effectively dealing with any groupwise dense collections in the generic extension at initial stages. The problem may require additional results for forcing along morasses, in particular more specialised reflection arguments, perhaps analogues of those used in the countable support iterations used to establish $\aleph_1 = \mathfrak{u} < \mathfrak{g} = \aleph_2$. Diamond style principles are also a possibility, and morasses can come equipped with diamond principles — see Appendix B. In any case, investigations in these directions will certainly be interesting.

Question 2.4.52. *Is there an FS system along a gap-1 morass at κ which yields a generic extension in which $\kappa = \mathfrak{u} < \mathfrak{g} = \kappa^+$? Is there an FS system along a gap-1 morass at κ which yields a generic extension satisfying $\mathbf{FD}$ and $\mathfrak{u} = \kappa$?*

Appendices

Appendix A

More basic results on FD

What follows is known, but provides some basic facts related to the Filter Dichotomy that might be instructive.

We use an extra piece of notation. For collections of subsets of ω, $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A}\#\mathcal{B}$, and say "$\mathcal{A}$ meshes with $\mathcal{B}$" if and only if for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$, $A \cap B \neq \emptyset$. If $C \subseteq \omega$, we write $C\#\mathcal{A}$ instead of $\{C\}\#\mathcal{A}$.

A.1 The action of finite-to-one maps $\omega \to \omega$

For the below, we identify $\beta\omega$ with the space of ultrafilters on ω.

Lemma A.1.1. *Suppose $\phi\colon \omega \to \omega$. Then we can consider $\beta\phi\colon \beta\omega \to \beta\omega$. Then,*

1. *For any ultrafilter $\mathcal{U}$, $\phi(\mathcal{U}) = \beta\phi(\mathcal{U})$.*
2. *ϕ is finite-to-one if and only if $\beta\phi$ restricts to a map $\omega^* \to \omega^*$.*

Proof. We provide a topological proof, as opposed to the usual via Stone-Duality. Suppose $\mathcal{U}$ is an ultrafilter, and let us recall the definition of the image filter

$$\phi(\mathcal{U}) = \left\{A \subseteq \omega\colon \phi^{-1}(A) \in \mathcal{U}\right\}.$$

Recall that since $\mathcal{U}$ is an ultrafilter, $\phi(\mathcal{U})$ is an ultrafilter. Then, working in $\beta\omega$, we have

$$\mathcal{U} \in \bigcap_{A \in \phi(\mathcal{U})} \overline{\phi^{-1}(A)}$$

Hence

$$\beta\phi\left(\mathcal{U}\right) \in \beta\phi\left(\bigcap_{A\in\phi(\mathcal{U})} \overline{\phi^{-1}\left(A\right)}\right) \subseteq \bigcap_{A\in\phi(\mathcal{U})} \beta\phi\left(\overline{\phi^{-1}\left(A\right)}\right).$$

Continuity of $\beta\phi$ however yields that

$$\beta\phi\left(\overline{\phi^{-1}\left(A\right)}\right) \subseteq \overline{\beta\phi\left(\phi^{-1}\left(A\right)\right)} \subseteq \overline{A},$$

so,

$$\beta\phi\left(\mathcal{U}\right) \in \bigcap_{A\in\phi(\mathcal{U})} \overline{A} = \left\{\phi\left(\mathcal{U}\right)\right\},$$

so that $\beta\phi\left(\mathcal{U}\right) = \phi\left(\mathcal{U}\right)$.

For assertion (2), let us first suppose ϕ is finite-to-one. Then for any $n \in \omega$, $\phi^{-1}\left\{n\right\}$ is finite, so $\phi^{-1}\left\{n\right\} \notin \mathcal{U}$ for any free ultrafilter $\mathcal{U}$. So for all $\mathcal{U} \in \omega^*$, $\left\{n\right\} \notin \phi\left(\mathcal{U}\right)$, so $\phi\left(\mathcal{U}\right)$ cannot be principal, and hence is an element of ω^*.

Conversely, let us suppose $\beta\phi$ restricts to a map $\omega^* \to \omega^*$. If for n in ω, $\phi^{-1}\left\{n\right\}$ were infinite, we could extend $\left\{\phi^{-1}\left\{n\right\}\right\}$ to a free ultrafilter $\mathcal{U}$ on ω, so that $\left\{n\right\} \in \phi\left(\mathcal{U}\right)$. But this means that $\phi\left(\mathcal{U}\right)$ is principal. Since this is a contradiction, $\phi^{-1}\left\{n\right\}$ must be finite. □

A.2 Filters on ω and subsets of ω^*

Given a filter $\mathcal{F}$ on ω, we may identify $\mathcal{F}$ with the subset

$$\mathcal{F}^* = \left\{\mathcal{U} \in \beta\omega \colon \mathcal{F} \subseteq \mathcal{U}\right\}$$

of ω^*, where here we are identifying $\beta\omega$ with the collection of ultrafilters on ω. $\mathcal{F}^*$ is the set of ultrafilter extensions of the filter $\mathcal{F}$. Observe that if $\mathcal{F}$ is a free filter, then $\mathcal{F}^* \subseteq \omega^*$. In particular, $Cof^* = \omega^*$, and if $\mathcal{U}$ is a free ultrafilter, then $\mathcal{U}^* = \left\{\mathcal{U}\right\}$.

Here we exhibit a few basic results describing how the "$*$" operator on filters interacts with maps $\omega \to \omega$.

Lemma A.2.1. *Suppose $\phi \colon \omega \to \omega$, $\mathcal{F}$ is a filter, and $\mathcal{G}$ is a filter with $\phi\left(\mathcal{F}\right) \subseteq \mathcal{G}$. Then there is a filter $\mathcal{V}$ with $\mathcal{F} \subseteq \mathcal{V}$ and $\phi\left(\mathcal{V}\right) = \mathcal{G}$.*

Proof. By Zorn's lemma, find a maximal filter $\mathcal{V}$ extending $\mathcal{F}$ with $\phi(\mathcal{V}) \subseteq \mathcal{G}$. Now if $A \in \mathcal{G}$, then $A \# \phi(\mathcal{V})$. In particular for every $B \in \mathcal{V}$, $\phi(B) \cap A \neq \emptyset$. But then $\mathcal{V} \# \phi^{-1}(A)$, so by maximality of $\mathcal{V}$, $\phi^{-1}(A) \in \mathcal{V}$, and hence $A \in \phi(\mathcal{V})$. Hence we may deduce $\phi(\mathcal{V}) = \mathcal{G}$. □

Remark A.2.2. *Notice in the above that if $\mathcal{G}$ is an ultrafilter, $\mathcal{V}$ can be taken to be an ultrafilter, since any ultrafilter $\mathcal{W}$ extending $\mathcal{V}$ must be mapped to an ultrafilter $\phi(\mathcal{W})$ by ϕ, which contains $\phi(\mathcal{V}) = \mathcal{G}$. Hence $\phi(\mathcal{W}) = \mathcal{G}$.*

Bearing this in mind, we see that the maximality of $\mathcal{V}$ in the proof entails that $\mathcal{V}$ is in fact an ultrafilter, whenever $\mathcal{G}$ is.

Corollary A.2.3. *Suppose $\phi\colon \omega \to \omega$, and $\mathcal{F}$ is a filter on ω. Then $\phi(\mathcal{F}^*) = \phi(\mathcal{F})^*$.*

Proof. If $\mathcal{U} \in \mathcal{F}^*$, then $\phi(\mathcal{F}) \subseteq \phi(\mathcal{U})$, and $\phi(\mathcal{U})$ is an ultrafilter, hence $\phi(\mathcal{U}) \in \phi(\mathcal{F})^*$. That is $\phi(\mathcal{F}^*) \subseteq \phi(\mathcal{F})^*$.

For the opposite inclusion, let us suppose $\mathcal{U} \in \phi(\mathcal{F})^*$. Using the previous lemma find a filter $\mathcal{V}$ extending $\mathcal{F}$ with $\phi(\mathcal{V}) = \mathcal{U}$. Now observe that $\mathcal{V}^* \neq \emptyset$, $\mathcal{V}^* \subseteq \mathcal{F}^*$, and $\phi(\mathcal{V}^*) = \{\mathcal{U}\}$. In particular, $\mathcal{U} \in \phi(\mathcal{F}^*)$, and hence we deduce the required equality. □

A.3 A direct translation of FD concerning ω^*

Robert Leek, via Stone Duality, provided the below translation of **FD** into properties of closed subsets of ω^*.

Theorem A.3.1 (Leek)**.** *Let $\mathcal{F}$ be a free filter on ω. Then*

1. *Cof $\leq_{RB} \mathcal{F}$ if and only if there is a finite-to-one $\phi\colon \omega \to \omega$ such that $\beta\phi(\mathcal{F}^*) = \omega^*$.*

2. *$\mathcal{U} \leq_{RB} \mathcal{F}$ for some ultrafilter $\mathcal{U}$, if and only if there is a finite-to-one $\phi\colon \omega \to \omega$ such that $\beta\phi(\mathcal{F}^*) = \{\mathcal{U}\}$.*

Corollary A.3.2 (Leek)**.** **FD** *is equivalent to the statement that for any non-empty closed subset* C *of* ω^**, there is a map* $\phi\colon \omega \to \omega$ *such that* $\beta\phi$ *restricts to a map* $\omega^* \to \omega^*$*, and such that either*

1. $\beta\phi\,(C) = \omega^*$*, or*
2. $\beta\phi\,(C)$ *is a singleton.*

It is hoped that something more can be said about the structure of ω^* under **FD**.

Remark A.3.3. *All of the previous results can be obtained via Stone-Duality.*

A.4 Weakenings of FD

Here we examine some of the weaker consequences of **FD**.

Definition A.4.1. *The principle of Near Coherence of Filters,* **NCF***, says that for any two free filters* $\mathcal{F}, \mathcal{G}$ *on* ω*, there is a finite-to-one map* $\phi\colon \omega \to \omega$ *such that* $\phi\,(\mathcal{F}) \# \phi\,(\mathcal{G})$.

By Stone-Duality we see:

Theorem A.4.2. **NCF** *is equivalent to the statement that for any two non-empty closed subsets* C, D *of* ω^**, there is a finite-to-one map* $\phi\colon \omega \to \omega$ *such that*

$$\beta\phi\,(C) \cap \beta\phi\,(D) \neq \emptyset.$$

Theorem A.4.3. **NCF** *is equivalent to the statement that for any two free ultrafilters* $\mathcal{U}, \mathcal{V}$ *on* ω*, there is a finite-to-one map* $\phi\colon \omega \to \omega$ *such that* $\phi\,(\mathcal{U}) = \phi\,(\mathcal{V})$.

Proof. Observe that in the definition of **NCF**, we may without loss of generality take $\mathcal{F}$ and $\mathcal{G}$ to be ultrafilters, and recall that the image of an ultrafilter is an ultrafilter. But two ultrafilters mesh if and only if they are equal. □

The above can be used to give a very quick and easy proof of $(\mathbf{FD} \to \mathbf{NCF})$ using the dual equivalent of **FD**.

Theorem A.4.4 (Proof due to Leek)**.** $(\mathbf{FD} \to \mathbf{NCF})$.

Proof. Suppose $\mathcal{U}, \mathcal{V}$ are free ultrafilters on ω. Then $\{\mathcal{U}, \mathcal{V}\}$ is a closed subset of ω^* of cardinality 2, hence cannot be mapped by any map onto ω^*. Hence there must be an ultrafilter $\mathcal{W}$ and a finite-to-one map $\phi\colon \omega \to \omega$ such that

$$\beta\phi\left(\{\mathcal{U}, \mathcal{V}\}\right) = \{\mathcal{W}\}$$

i.e. $\phi\left(\mathcal{U}\right) = \phi\left(\mathcal{V}\right) = \mathcal{W}$. Theorem A.4.3 then applies. □

However, Shelah and Mildenberger [23] have shown that $(\mathbf{NCF} \not\to \mathbf{FD})$. It is natural to wonder how much of **FD** can be recovered from **NCF**, i.e. what aspects of **FD** are used to deduce **NCF**.

The following lemma was provided by Suabedissen, although it is likely already known. The subsequent result previously relied on **FD**, but with the following can be done in **NCF**. It is part of a **ZFC** version of a result previously proved, Corollary 2.1.5.

Lemma A.4.5 (Suabedissen)**.** *Suppose $\mathcal{F}$ is almost an ultrafilter and $\phi\colon \omega \to \omega$ is finite-to-one. Then $\phi\left(\mathcal{F}\right)$ is almost an ultrafilter.*

Proof. Take a finite-to-one section λ of ϕ, i.e. a finite-to-one map $\lambda\colon \omega \to \omega$ with $\phi \circ \lambda = id_\omega$. Then for $F \in \mathcal{F}$, $\lambda\left(\phi\left(F\right)\right) \subseteq F$, so $\lambda\left(\phi\left(\mathcal{F}\right)\right)$ is a filter on ω which is finer than $\mathcal{F}$.

Now if ψ is finite-to-one and $\psi\left(\mathcal{F}\right)$ is an ultrafilter $\mathcal{U}$, then $\psi \circ \lambda\left(\phi\left(\mathcal{F}\right)\right)$ is a filter on ω containing $\mathcal{U}$, and so necessarily is equal to $\mathcal{U}$. □

From this we deduce the following.

Proposition A.4.6 (**NCF**)**.** *Suppose $\mathcal{F}$ and $\mathcal{G}$ are free filters which are both almost ultrafilters. Then there is a finite-to-one map $\phi\colon \omega \to \omega$ with*

$$\phi\left(\mathcal{F}\right) = \phi\left(\mathcal{G}\right).$$

Proof. Find a finite-to-one $\theta\colon \omega \to \omega$ with $\theta\left(\mathcal{F}\right)$ an ultrafilter. Then $\theta\left(\mathcal{G}\right)$ is almost an ultrafilter (by the above lemma), so we can find $\chi\colon \omega \to \omega$ with $\chi\left(\theta\left(\mathcal{G}\right)\right)$ an ultrafilter. But since images of ultrafilters are ultrafilters $\chi\left(\theta\left(\mathcal{F}\right)\right)$ is also an ultrafilter.

NCF is equivalent to the statement that any two ultrafilters are identified by a finite-to-one map on ω, so we can find a finite-to-one map ψ which identifies the ultrafilters $\chi(\theta(\mathcal{F}))$ and $\chi(\theta(\mathcal{G}))$. But then if we put $\phi = \psi \circ \chi \circ \theta$, we have that

$$\phi(\mathcal{F}) = \phi(\mathcal{G}). \qquad \square$$

Hence we have the following generalisation of the previous result A.4.3.

Corollary A.4.7. **NCF** *is equivalent to the statement: if $\mathcal{F}$ and $\mathcal{G}$ are almost ultrafilters, then there is a finite-to-one map $\phi\colon \omega \to \omega$ with $\phi(\mathcal{F}) = \phi(\mathcal{G})$.*

It is natural to ask whether two almost *Cof* filters may be identified under a finite-to-one map $\omega \to \omega$. In fact, this is true in **ZFC** alone. One easily has the following:

Proposition A.4.8. *Suppose $\phi\colon \omega \to \omega$ is finite-to-one. Then Cof $\leq_{RB} \phi(Cof)$.*

Proof. Let us first consider $\phi\colon \omega \to \mathrm{ran}(\phi)$, and observe that $\phi(Cof)$ is just *Cof* on $\mathrm{ran}(\phi)$. Hence in fact we think of elements of the filter on ω, $\phi(Cof)$, as just cofinite subsets of $\mathrm{ran}(\phi)$ unioned with subsets of $\omega \setminus \mathrm{ran}(\phi)$. This point of view prescribes our finite-to-one map.

Let $\psi\colon \mathrm{ran}(\phi) \to \omega$ be any bijection, and let $\chi\colon \omega \setminus \mathrm{ran}(\phi) \to \omega$ be any injection. Then $\psi \cup \chi$ is a finite-to-one (in fact 2-to-one) map on ω, and since ψ maps the cofinite filter on $\mathrm{ran}(\phi)$ to *Cof* on ω, we see $\psi \cup \chi$ maps $\phi(Cof)$ to *Cof*. $\square$

But this generalises to the following:

Lemma A.4.9 (Suabedissen (private communication), or, [4] pg5)**.** *Suppose Cof $\leq_{RB}$ $\mathcal{F}$ and $\phi\colon \omega \to \omega$ is finite-to-one. Then Cof $\leq_{RB} \phi(\mathcal{F})$.*

Hence we have the analogous theorem for almost *Cof* filters. Again this was initially seen under the assumption of **FD** using Corollary 2.1.5.

Proposition A.4.10. *Suppose $\mathcal{F}, \mathcal{G}$ are free filters which are both almost Cof. Then there is a finite-to-one map $\phi\colon \omega \to \omega$ with*

$$\phi(\mathcal{F}) = \phi(\mathcal{G}) = Cof.$$

Proof. Let $\psi\colon \omega \to \omega$ be finite-to-one with $\psi(\mathcal{F}) = \mathit{Cof}$. Then (by the previous lemma) $\psi(\mathcal{G})$ is almost *Cof*. So find a finite-to-one $\chi\colon \omega \to \omega$ with $\chi(\psi(\mathcal{G})) = \mathit{Cof}$. Now, $\chi(\mathit{Cof})$ is a free filter on ω, so

$$\mathit{Cof} \subseteq \chi(\mathit{Cof}).$$

But we also have that $\mathit{Cof} \subseteq \psi(\mathcal{G})$, so

$$\chi(\mathit{Cof}) \subseteq \chi(\psi(\mathcal{G})) = \mathit{Cof}$$

and hence $\chi(\mathit{Cof}) = \mathit{Cof}$. But that is to say, putting $\phi = \chi \circ \psi$, that

$$\phi(\mathcal{F}) = \phi(\mathcal{G}) = \mathit{Cof}. \qquad \square$$

Hence in **ZFC**, we can identify any two almost cofinite filters. This raises the following question:

Question A.4.11. *What is the cardinality of the smallest collection of almost cofinite filters which cannot be identified (without loss of generality to Cof) by a finite-to-one map?*

By the above, and a straigtforward induction, it is clearly at least $\aleph_0$.

A.5 Stone-Čech extensions of finite-to-one maps are dense in $C_k(\omega^*, \omega^*)$ (an alternative proof)

Theorem A.5.1. *The set* $\{\beta\phi\colon \phi\colon \omega \to \omega$ *finite-to-one*$\}$ *is dense in* $C_k(\omega^*, \omega^*)$.

Proof. Let us consider the clopen base for ω^* given by $\{A^*\colon A \subseteq \omega\}$, and a non-empty basic open set in $C_k(\omega^*, \omega^*)$ obtained from this base, i.e.

$$U = \bigcap_{i=1}^{n} [A_i^*, B_i^*]$$

and find a finite-to-one $\phi\colon \omega \to \omega$ with $\beta\phi$ in U.

Let us define $\mathcal{I} = \left\{F \subseteq \{1, \ldots, n\} : \bigcap_{i\in F} A_i \neq \emptyset\right\}$. We define our function on $\bigcup_{i=1}^{n} A_i$ by induction. First let us make an observation.

Claim A.5.2. *Suppose $F \in \mathcal{I}$, and further $\bigcap_{i\in F} A_i$ is infinite. Then $\bigcap_{i\in F} B_i$ is infinite.*

Proof of Claim. First note that if $\bigcap_{i\in F} A_i$ is infinite, then $\left(\bigcap_{i\in F} A_i\right)^* = \bigcap_{i\in F} A_i^*$ is non-empty. Now take $f \in U$. Then $f\left(\bigcap_{i\in F} A_i^*\right) \subseteq \bigcap_{i\in F} B_i^*$, so in particular, $\bigcap_{i\in F} B_i^* \neq \emptyset$. Since $\bigcap_{i\in F} B_i^* = \left(\bigcap_{i\in F} B_i\right)^*$, $\bigcap_{i\in F} B_i$ is infinite. □

We have seen already how to deal with the case $n = 1$ in Observation 2.2.9. Let us examine the potential complications in the case $n = 2$ and indicate how these complications can be resolved. The resolution motivates the construction below. Let us consider two subbasic open sets, $[A_1^*, B_1^*]$ and $[A_2^*, B_2^*]$ with $[A_1^*, B_1^*] \cap [A_2^*, B_2^*] \neq \emptyset$. If $A_1^* \cap A_2^* = \emptyset$, we can simply union finite-to-one maps $A_1 \to B_1$ and $A_2 \to B_2$ and extend to a finite-to-one map $\omega \to \omega$ to get our required ϕ, so the interesting case is when $A_1^* \cap A_2^* \neq \emptyset$. This means the subbasic components to our basic open set have some non-trivial interaction with each other. In this case, $A_1 \cap A_2$ must be infinite, and in constructing ϕ we must first ensure that $A_1 \cap A_2$ is mapped into $B_1 \cap B_2$ (which is also infinite by the above Claim), before making a finite-to-one extension to $A_1 \setminus (A_1 \cap A_2)$ and $A_2 \setminus (A_1 \cap A_2)$, and then finally extending to the entire of ω. Since there need not be any uniformity to the way these subbasic components interact when $n > 2$, we define a combinatorial device that allows us to leverage the power of induction to deal with these interactions in a more straightforward manner.

Let us call a subset $\mathcal{J}$ of $\mathcal{I}$ a crown for $\mathcal{I}$ if and only if, for any $F \in \mathcal{J}$ and any $G \in \mathcal{I}$, if $F \subseteq G$ then $G \in \mathcal{J}$ (i.e. $\mathcal{J}$ is closed under upsets). Observe, for example, that $\mathcal{I}$ is a crown for $\mathcal{I}$, $\emptyset$ is a crown for $\mathcal{I}$, and the set of $\subseteq$-maximal elements of $\mathcal{I}$ is a crown for $\mathcal{I}$. Our construction will be essentially a recursion on the crowns of $\mathcal{I}$.

Suppose $\mathcal{J}$ is a crown for $\mathcal{I}$ of minimal (necessarily finite) cardinality for which there is no finite-to-one function $\phi\colon \bigcup_{F\in\mathcal{J}} \bigcap_{j\in F} A_j \to \omega$ with the property that

$$\phi\left(\bigcap_{j\in F} A_j\right) \subseteq^* \bigcap_{j\in F} B_j$$

for all $F \in \mathcal{J}$. Note $\mathcal{J} \neq \emptyset$, vacuously.

Choose a $\subseteq$-minimal element of $\mathcal{J}$, G say. Then $\mathcal{J} \setminus \{G\}$ is a crown for $\mathcal{I}$, and is of smaller cardinality than $\mathcal{J}$, so we can find a finite-to-one function

$$\phi\colon \bigcup_{F \in \mathcal{J}\setminus\{G\}} \bigcap_{j \in F} A_j \to \omega$$

with the property that

$$\phi\left(\bigcap_{j\in F} A_j\right) \subseteq^* \bigcap_{j\in F} B_j$$

for $F \in \mathcal{J} \setminus \{G\}$. We show how to extend ϕ to $\bigcap_{j\in G} A_j$ with

$$\phi\left(\bigcap_{j\in G} A_j\right) \subseteq^* \bigcap_{j\in G} B_j$$

obtaining a contradiction to the existence of the crown $\mathcal{J}$.

First, note that we only need to extend ϕ to

$$C = \bigcap_{j\in G} A_j \setminus \bigcup_{F\in\mathcal{J}\setminus\{G\}, G\subseteq F} \left(\bigcap_{j\in F} A_j\right),$$

that is, the points upon which ϕ has not already been defined. This is clear if C is finite, since we can pick any injection $\psi\colon C \to \omega$, and extend ϕ to the finite-to-one map $\phi \cup \psi$. Then

$$\left\{ n \in \bigcap_{j\in G} A_j \colon \phi(n) \notin \bigcap_{j\in G} B_j \right\}$$

is a subset of

$$\bigcup_{F\in\mathcal{J}\setminus\{G\}, G\subseteq F} \left\{ n \in \bigcap_{j\in F} A_j \colon \phi(n) \notin \bigcap_{j\in F} B_j \right\} \cup C$$

which is finite.

We therefore turn our attention to the case where C is infinite. This case entails that $\bigcap_{j\in G} A_j$ is infinite, and hence by the above claim $\bigcap_{j\in G} B_j$ is infinite. We can therefore pick a finite-to-one $\psi\colon C \to \bigcap_{j\in G} B_j$, and extend ϕ to the finite-to-one map $\phi \cup \psi$. Now,

$$\left\{ n \in \bigcap_{j\in G} A_j \colon \phi(n) \notin \bigcap_{j\in G} B_j \right\}$$

is a subset of

$$\bigcup_{F\in\mathcal{J}\setminus\{G\}, G\subseteq F} \left\{ n \in \bigcap_{j\in F} A_j \colon \phi(n) \notin \bigcap_{j\in F} B_j \right\}$$

which is finite, and gives us the desired extension.

Therefore no such crown $\mathcal{J}$ can exist. In particular, there must be a finite-to-one map $\phi\colon \bigcup_{i=1}^{n} A_i \to \omega$ with

$$\phi(A_i) \subseteq^* B_i$$

for $i = 1, \ldots, n$, because $\mathcal{I}$ is a crown for $\mathcal{I}$. Now by taking a finite-to-one map $\psi\colon \omega \setminus \bigcup_{i=1}^{n} A_i \to \omega$ we can extend ϕ to a finite-to-one map $\phi \cup \psi\colon \omega \to \omega$, so let us just suppose ϕ is defined on all of ω.

Now, for $i \in \{1, \ldots, n\}$, if $\mathcal{U} \in A_i^*$, then $A_i \in \mathcal{U}$, so $\phi(A_i) \in \phi(\mathcal{U})$. But

$$\phi(A_i) \subseteq^* B_i$$

and $\phi(\mathcal{U})$ contains *Cof*, so $B_i \in \phi(\mathcal{U})$. But that is equivalent to saying that $\phi(\mathcal{U}) \in B_i^*$. So we have, for each $i \in \{1, \ldots, n\}$,

$$\beta\phi(A_i^*) \subseteq B_i^*$$

and hence

$$\beta\phi \in \bigcap_{i=1}^{n} [A_i^*, B_i^*] = U,$$

which completes the proof.

□

Appendix B

The diamond principle in terms of elementary submodels

B.1 Classical diamond principles in terms of elementary submodels

It is possible to have morasses which admit a diamond style principle. Typically this inbuilt principle is phrased in terms of elementary submodels. To see that this is a natural way to encode such a principle, it helps to see the classical diamond principle phrased in this way. The literature seems to contain no record of this — although it is likely widely known — so below we present an equivalent of the diamond principle in the language of elementary submodels. We then introduce morasses with inbuilt diamond principles and later establish a basic consequence. First however, let us recall the definition of the classical diamond principles.

Definition B.1.1. *The diamond principle, $\diamondsuit_\kappa$, for a regular cardinal κ, is the statement that there is a sequence $\langle A_\alpha : \alpha < \kappa \rangle$ of sets $A_\alpha \subseteq \alpha$ such that for any $A \subseteq \kappa$, the set*

$$\{\alpha < \kappa \colon A \cap \alpha = A_\alpha\}$$

is stationary in κ.

Lemma B.1.2. *If $\mathcal{U}$ is an $\mathcal{L}$-structure on κ, where κ is a regular cardinal and $\mathcal{L}$ is a first order language of cardinality less than κ, then*

$$C = \{\alpha < \kappa : \mathcal{U} \cap \alpha \preceq \mathcal{U}\}$$

is club in κ.

Proof. The standard Tarski-Vaught criterion for elementary submodels makes it easy to see C is closed. We show C is unbounded. Let $\alpha < \kappa$. Define $\beta_0 = \alpha + 1$. Given β_n, obtain β_{n+1} as follows. Using the downward Löwenheim-Skolem theorem, find an elementary submodel $\mathcal{M}_n$ of $\mathcal{U}$ with $\beta_n + 1 \subseteq \mathcal{M}_n$, and $|\mathcal{M}_n| < \kappa$. Let $\beta_{n+1} = \sup \mathrm{dom}\,(\mathcal{M}_n)$. The regularity of κ ensures $\beta_{n+1} < \kappa$ provided $\beta_n < \kappa$ also.

Now let β be the limit ordinal $\sup_{n\in\omega} \beta_n$. Clearly $\beta > \alpha$, and $\beta < \kappa$ by regularity of κ. Also, $\mathcal{U} \cap \beta = \bigcup_{n\in\omega} \mathcal{M}_n$ (this follows from the fact that $\beta_n \subseteq \mathcal{M}_{n+1} \subseteq \beta_{n+1}$), and so again, by the Tarski-Vaught criterion we see $\mathcal{U} \cap \beta$ is an elementary submodel of $\mathcal{U}$, and hence $\beta \in C$. □

Theorem B.1.3. *$\diamondsuit_\kappa$ is equivalent to the statement that for any countable first order language $\mathcal{L}$ (or more generally, first order language $\mathcal{L}$ with cardinality smaller than κ), there are $\mathcal{L}$-structures $\mathcal{U}_\alpha$ on α, for $\alpha < \kappa$, such that for any $\mathcal{L}$-structure $\mathcal{U}$ on κ, the set*

$$\{\alpha < \kappa : \mathcal{U}_\alpha \preceq \mathcal{U}\}$$

is stationary in κ.

Proof. Let $\langle A_\alpha : \alpha < \kappa\rangle$ be a $\diamondsuit_\kappa$-sequence. Without loss of generality, we may think of our language $\mathcal{L}$ as consisting of countably many (or less than κ many) predicate symbols, and hence construe an $\mathcal{L}$-structure $\mathcal{U}$ on any $\alpha \leq \kappa$ as a subset A of κ, via some appropriate coding (which does not depend on the structure). Given a code A, let us write $\langle A\rangle_{\mathcal{L}}$ for the partial-$\mathcal{L}$-structure which it codes.

We obtain our sequence of structures by attempting to decode the diamond sequence. If A_α is a code for a structure on α define $\mathcal{U}_\alpha$ to be $\langle A_\alpha\rangle_{\mathcal{L}}$, the $\mathcal{L}$-structure on α coded by A_α. If A_α does not code a structure on α, we just take $\mathcal{U}_\alpha$ to be the trivial structure on α (interpret all atomic predicates as always false). It is not clear

that here we have defined any non-trivial structures at all; our analysis later will show otherwise.

Let us now fix an $\mathcal{L}$-structure $\mathcal{U}$ on κ and a corresponding subset A of κ with $\langle A\rangle_{\mathcal{L}} = \mathcal{U}$. The $\diamondsuit_\kappa$ principle ensures that

$$S_1 = \{\alpha < \kappa \colon A_\alpha = A \cap \alpha\}$$

is stationary in κ.

Claim B.1.4. *$B = \{\alpha < \kappa \colon A \cap \alpha$ codes $\mathcal{U} \cap \alpha\}$ is club in κ.*

Proof of Claim. B is easily seen to be closed, so we concern ourselves with showing it is unbounded.

For each $\alpha < \kappa$, we define $f(\alpha) \in \kappa$ as the least ordinal β with the property that $\mathcal{U} \cap \beta$ contains the partial structure coded by $A \cap \alpha$. Note that such an ordinal exists, since $A \cap \alpha$ codes a structure which mentions less than κ many ordinals. Regularity of κ ensures we can find an upper bound $\gamma < \kappa$ for these ordinals. It is then easy to see β exists.

For each $\alpha < \kappa$, we also define $g(\alpha) \in \kappa$ to be the least ordinal $\beta < \kappa$ with the property that $\langle A \cap \beta\rangle_{\mathcal{L}}$ contains $\mathcal{U} \cap \alpha$. Note that this is possible, because $\mathcal{U} \cap \alpha$ has cardinality smaller than κ, and so is coded by some subset S of A of size less than κ. Then $S \subseteq \gamma$ for some $\gamma < \kappa$, and the existence of β follows.

Let D be the club $\{\alpha < \kappa \colon \alpha$ is closed under f and $g\}$ (see Kunen [18], Chapter 2, Lemma 6.13). Let D' denote the club of limit points of D. We argue that $D' \subseteq B$, hence in particular, B is unbounded. If $\lambda \in D'$, then $\langle A \cap \lambda\rangle_{\mathcal{L}} = \bigcup_{\alpha \in D \cap \lambda} \langle A \cap \alpha\rangle_{\mathcal{L}}$. Now, let $\alpha \in D \cap \lambda$. Then $f(\alpha) < \lambda$, and

$$\langle A \cap \alpha\rangle_{\mathcal{L}} \subseteq \mathcal{U} \cap f(\alpha) \subseteq \mathcal{U} \cap \lambda$$

so $\langle A \cap \lambda\rangle_{\mathcal{L}} \subseteq \mathcal{U} \cap \lambda$.

But we also have that $\mathcal{U} \cap \lambda = \bigcup_{\alpha \in D \cap \lambda} \mathcal{U} \cap \alpha$, and for $\alpha \in D \cap \lambda$ one has $g(\alpha) < \lambda$ and so

$$\mathcal{U} \cap \alpha \subseteq \langle A \cap g(\alpha)\rangle_{\mathcal{L}} \subseteq \langle A \cap \lambda\rangle_{\mathcal{L}}$$

and hence $\mathcal{U} \cap \lambda \subseteq \langle A \cap \lambda\rangle_{\mathcal{L}}$, and hence in fact $\mathcal{U} \cap \lambda = \langle A \cap \lambda\rangle_{\mathcal{L}}$, so that $D' \subseteq B$. □

So, $S = S_1 \cap (B \cap C)$ is stationary, where C is the club

$$\{\alpha < \kappa \colon \mathcal{U} \cap \alpha \preceq \mathcal{U}\}$$

given by the preceeding lemma. But for $\alpha \in S$ we have (since $\alpha \in B$ also) that $A \cap \alpha$ codes the structure $\mathcal{U} \cap \alpha$. But (since $\alpha \in S_1$) $A_\alpha = A \cap \alpha$, so A_α codes the structure $\mathcal{U} \cap \alpha$ also. But, since in particular, A_α codes a structure, we must have $\mathcal{U}_\alpha = \mathcal{U} \cap \alpha$. But the fact that $\alpha \in C$ tells us that $\mathcal{U} \cap \alpha \preceq \mathcal{U}$, so altogether we have for $\alpha \in S$

$$\mathcal{U}_\alpha \preceq \mathcal{U}.$$

Since any subset of κ containing S must be stationary, we have our desired sequence of structures $\langle \mathcal{U}_\alpha : \alpha < \kappa \rangle$.

The converse is more straightforward. Let $\mathcal{L}$ be the first order language $\{P\}$ consisting of a single unary predicate symbol. Let $\mathcal{U}_\alpha$ be $\mathcal{L}$-structures on α for $\alpha < \kappa$ with the properties hypothesised by the theorem. We obtain our $\diamondsuit_\kappa$-sequence as follows: for $\alpha < \kappa$ let $A_\alpha = \{\beta \in \alpha \colon \mathcal{U}_\alpha \models P(\beta)\}$. We show $\langle A_\alpha : \alpha < \kappa \rangle$ is a $\diamondsuit_\kappa$-sequence.

To this end, let $A \subseteq \kappa$. We define an $\mathcal{L}$-structure $\mathcal{U} = (\kappa; P_A)$ on κ by prescribing for $\beta < \kappa$,

$$\mathcal{U} \models P_A(\beta) \quad \text{ if and only if } \quad \beta \in A.$$

Then $S = \{\alpha < \kappa \colon \mathcal{U}_\alpha \preceq \mathcal{U}\}$ is stationary in κ. We show for $\alpha \in S$ that $A_\alpha = A \cap \alpha$. This is clear however, since for $\beta \in \alpha$, $\beta \in A_\alpha$ if and only if $\mathcal{U}_\alpha \models P(\beta)$, but

$$\mathcal{U}_\alpha \models P(\beta) \quad \text{ if and only if } \quad \mathcal{U} \models P(\beta)$$

since $\mathcal{U}_\alpha \preceq \mathcal{U}$. By definition of $\mathcal{U}$, $\mathcal{U} \models P(\beta)$ if and only if $\beta \in A$. This completes the proof. □

Given the above theorem, the definition of a morass with built in diamond seems much more natural.

Definition B.1.5. *We say that a gap-1 morass*

$$\mathcal{M} = (\langle \theta_\alpha : \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \kappa \rangle)$$

at κ has a built in diamond if and only if for any countable first order language $\mathcal{L}$ there is an $\mathcal{L}$-structure $\mathcal{U}_\alpha$ on θ_α for each $\alpha < \kappa$ such that for any $\mathcal{L}$-structure $\mathcal{U}$ on $\theta_\kappa = \kappa^+$ the set

$$\{\text{ran}\,(f) : f\,(\mathcal{U}_\alpha) \preceq \mathcal{U},\ f \in \mathcal{F}_{\alpha,\kappa},\ \alpha < \kappa\}$$

is stationary in $[\kappa^+]^{<\kappa}$. The sequence $\langle \mathcal{U}_\alpha : \alpha < \kappa\rangle$ is called a built in $\diamondsuit$-sequence for $\mathcal{L}$.

Remark B.1.6. *1. All countable first order languages are reducts of the countable first order language consisting of countably infinitely many symbols of each type. We could without loss of generality take the built in $\diamondsuit$-sequence for this language, and use this sequence of structures to generate the built in $\diamondsuit$-sequence for any other language by taking the appropriate reduct. Hence we may associate, before or without specifying a language, a built in $\diamondsuit$-sequence for a morass with built in diamond.*

2. It is sufficient (in the above definition) that our sequence $\langle \mathcal{U}_\alpha : \alpha < \kappa\rangle$ satisfy that for any first order $\mathcal{L}$-structure $\mathcal{U}$ on κ^+, there is an $\alpha < \kappa$ and $f \in \mathcal{F}_{\alpha,\kappa}$ with $f\,(\mathcal{U}_\alpha) \preceq \mathcal{U}$. That is to say, the full built in $\diamondsuit$-sequence property will follow from this.

B.2 A consequence of a morass with built in diamond

The following is an adaptation of the proof that $\diamondsuit_{\kappa^+}$ entails $2^\kappa = \kappa^+$.

Proposition B.2.1. *Suppose $\mathcal{M} = (\langle \theta_\alpha : \alpha \leq \kappa\rangle, \langle \mathcal{F}_{\alpha,\beta} : \alpha < \beta \leq \kappa\rangle)$ is a morass with built in diamond. Then*

$$\left|\left[\kappa^+\right]^{<\kappa}\right| = \kappa^+.$$

Proof. Let $\langle \mathcal{U}_\alpha : \alpha < \kappa\rangle$ be our built in diamond sequence for a morass with built in diamond with respect to the language $\mathcal{L}$ which consists of a single unary predicate P. Let $A \in [\kappa^+]^{<\kappa}$. Let $\mathcal{U}$ be the structure on κ^+ which interprets P so that

$$\mathcal{U} \models P\,(x) \quad \text{if and only if} \quad x \in A.$$

The diamond property tells us that

$$\{\text{ran}\,(f) : f\,(\mathcal{U}_\alpha) \preceq \mathcal{U},\ f \in \mathcal{F}_{\alpha,\kappa},\ \alpha < \kappa\}$$

is stationary in $[\kappa^+]^{<\kappa}$, so pick $\alpha < \kappa$ and $f \in \mathcal{F}_{\alpha,\kappa}$ with $A \subseteq \text{ran}\,(f)$ and $f\,(\mathcal{U}_\alpha) \preceq \mathcal{U}$. If $P_\alpha \subseteq \theta_\alpha$ is the interpretation of P in $\mathcal{U}_\alpha$, then $f\,(P_\alpha) = A$ because of how we chose f. Hence

$$A \in \{f\,(P_\alpha) : f \in \mathcal{F}_{\alpha,\kappa},\ \alpha < \kappa\}$$

so $[\kappa^+]^{<\kappa} \subseteq \{f\,(P_\alpha) : f \in \mathcal{F}_{\alpha,\kappa},\ \alpha < \kappa\}$, and $\{f\,(P_\alpha) : f \in \mathcal{F}_{\alpha,\kappa},\ \alpha < \kappa\}$ has cardinality no greater than the cardinality of $\bigcup_{\alpha<\kappa} \mathcal{F}_{\alpha,\kappa}$. Hence, by Corollary 2.4.50, we have

$$\kappa^+ \leq \left|\left[\kappa^+\right]^{<\kappa}\right| \leq \left|\bigcup_{\alpha<\kappa} \mathcal{F}_{\alpha,\kappa}\right| = \kappa^+,$$

and hence the result. □

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