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# A vibro-acoustic quality control approach for the elastic properties characterisation of thin orthotropic plates

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**Abstract.** Many engineering structures are built by assembly of plates, and in recent years there has been a widespread use of composite plates, and in particular of orthotropic plates, to improve dynamic structural performance and reduce structural weight. Depending on the manufacturing process used, orthotropic plates might be affected by variability of the mechanical properties, which in turn will affect their performance. The four elastic constants of orthotropic plates can be obtained via direct static or dynamic tests, or via inverse methods which combine dynamic experiments with mathematical models. However these approaches may not be appropriate for establishing manufacturing variability across a number of manufactured plates, since they might require expensive testing equipment and they may be time consuming. In this paper an inverse procedure based on the Chladni patterns is investigated to analyse thin orthotropic rectangular plates. A detailed mathematical derivation of the equations needed for estimating the elastic constants based on the knowledge of five natural frequencies is presented. The repeatability and reproducibility of the results obtained with the proposed procedure is first assessed, and the evaluation of the manufacturing variability of an ensemble of “nominally identical” plywood plates with the proposed approach is then investigated.

## 1. Introduction

In many areas of aeronautical and mechanical engineering there has been a widespread use of composite plates to improve dynamic structural performance and reduce structural weight of built-up structures. To investigate the dynamic performance of built-up structures at the design stage using virtual prototypes, it is necessary to characterise the material and geometrical properties, as well as damping, of the structural components. However, the manufacturing of composites generally involves the moulding of several layers of dry fabrics together with resin to form a laminate that is cured in moulds, which can lead to manufacturing variability. As a result, in recent years extensive research has been devoted to the development of efficient analysis methods which can take into account manufacturing variability and are suitable for use in the iterative design process. Nonetheless, little work has been carried out on the development of efficient strategies for quantifying the variability of the mechanical properties of the manufactured structural components. In this paper we focus our attention on the analysis of thin orthotropic plates.

The main direct testing methods that are used for the determination of elastic constants can be classified in three main groups: static, ultrasonic and Dynamic Mechanical Analysis (DMA) [1-4]. Static tests enable the evaluation of the elastic properties from the stress-strain curves. Nonetheless, these tests are destructive and can be time consuming. Moreover, as significant levels of creep (time increasing strains under constant stress) are found on the composite materials, the results of this type of tests cannot be directly used for the resolution of dynamic problems. Ultrasonic tests are based on measurements of the propagation of ultrasonic waves in the material tested. Composite materials often dissipate a



significant amount of the energy carried by the ultrasonic waves transmitted through them. Additionally, the wavelength that this test requires is often similar to the size of some structural elements part of the composite materials to be tested. For these two reasons the ultrasonic test results are also inappropriate for dynamic modelling of vibrating structures. DMA tests measure the strains of a material under sinusoidal stress and it provides values for the complex forms of the Young's Modulus and Shear Modulus of the materials. However, due to the small size of the specimen that can be tested and the high cost of the equipment involved, the industrial use of this method is limited.

Alternative approaches are based on inverse methods which combine dynamic experiments with mathematical models [3-6]. These approaches can be mainly grouped into (i) vibrational methods based on the optimization of the mathematical model to fit with certain tolerances the Frequency Response Function obtained via Experimental Modal Analysis (EMA); and (ii) Chladni approaches based on the optimization of a mathematical model to yield nodal lines (the so-called Chladni patterns) corresponding to the ones obtained with a Chladni setup. The latter approaches do not require expensive testing equipment and are characterised by a significantly reduced testing time, and is the focus of this paper.

The Chladni patterns have been used by Woodhouse and co-workers [3-5] to develop a simple yet effective procedure for evaluating the four elastic constants that affect the vibration thin orthotropic rectangular plate, namely: Young's moduli in x and y directions, in-plane shear modulus and Poisson's ratio. Their approach is based on evaluating with the Chladni pattern the mode shapes of 5 modes and their corresponding natural frequencies, and to use closed-form solutions relating these frequencies with the four elastic constants were provided [3-5]. Nonetheless, these papers do not investigate the possible industrialisation of the procedure for investigating manufacturing variability, neither include a detailed mathematical derivation of the equations needed for estimating the elastic constants.

In this paper an inverse procedure for the evaluation of the manufacturing variability of an ensemble of "nominally identical" orthotropic thin rectangular plates based on the Chladni patterns is investigated. A detailed mathematical derivation of the equations needed for estimating the elastic constants based on the knowledge of the natural frequencies of five modes is presented. A more accurate estimate of each natural frequency, as well as damping, is obtained by evaluating a single driving Frequency Response Function. The Chladni setup is then used to quickly visualise the mode shapes. The repeatability and reproducibility of the results obtained with the proposed procedure is first assessed, and the evaluation of the manufacturing variability of an ensemble of "nominally identical" plywood plates with the proposed approach is then investigated.

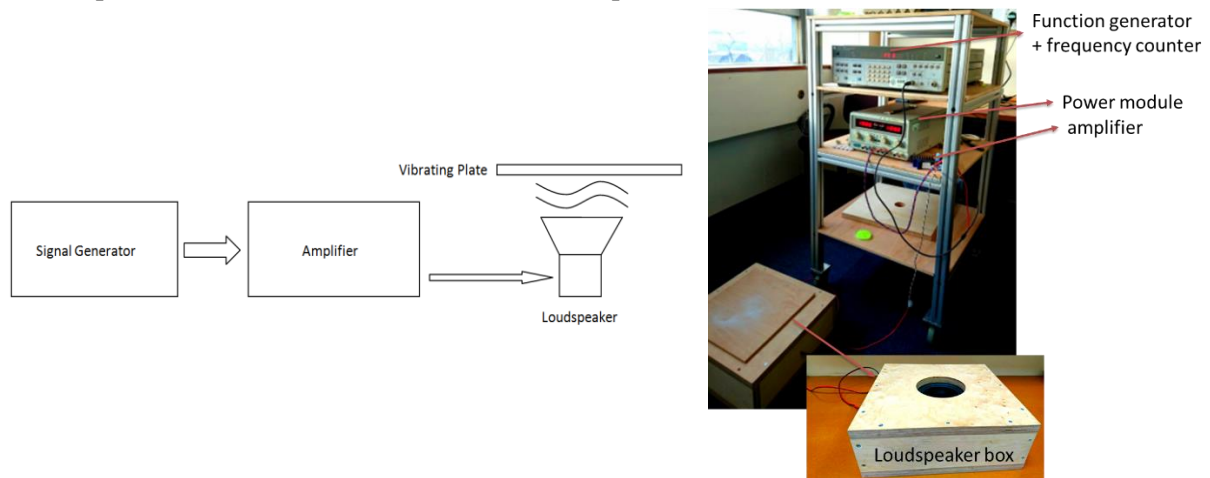
The Chladni experimental setup is described in section 2. Section 3 focuses on the evaluation of approximate expressions for the estimation of the elastic constants starting from the resonant frequencies of five modes. Section 4 introduces a procedure for obtaining refined resonant frequencies by combining EMA, Chladni and a mathematical model. Section 5 shows the results obtained for a plywood plate. Section 6 addresses the acceptability of the proposed approach for assessing the manufacturing variability evaluation of 30 plywood plates.

## 2. Chladni Experimental Setup

The Chladni setup, shown in figure 1, is composed by a flat surface equipped with a loudspeaker (Visaton WS 17E) driven by a sinewave generator (HP 3325B) with an audio power amplifier Prism LA50B with a power supply GW GPC-3030D. The plate is driven by a sinusoidal acoustic input which produces an out-of-plane displacement of the plate. Fine powder (tea leaves or salt) is spread on the plate resting on a flat surface and being supported by small blocks of foams (located under the nodal lines of the modes to reproduce free-free boundary conditions). A broad range of frequency is explored, and when the frequency of the acoustic input coincides with one of the plate natural frequencies, the powder moves from the plate's vibrating areas to the nodal lines where no movement occurs, forming the so-called Chladni patterns which enable the identification of the corresponding mode shape. Examples of three mode shapes are shown in figure 2.

This is a fast test that requires relative inexpensive equipment to evaluate the mode shapes (no sensor is attached to the plate, nor laser is used to measure the response) compared to that required standard Experimental Modal Analysis [6]. Knowing the frequency at which a set of mode shapes

occurs, it is possible to determine the elastic constant of the thin orthotropic plates by using some closed-form expressions [3-5]. In the next section, these expressions are derived.



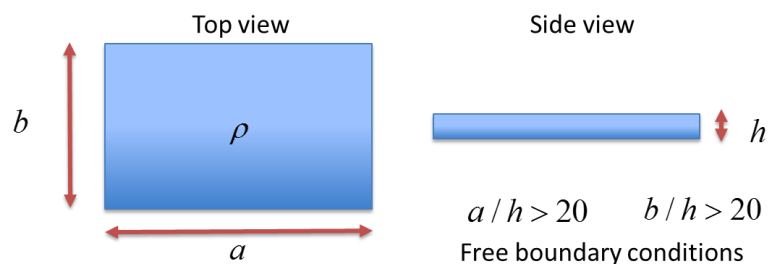
**Figure 1.** Chladni setup: schematic description (left figure) and portable setup used (right).



**Figure 2.** Chladni patterns example: plywood plate O-mode (left), FR4 plate +-mode (centre), FR4 plate X-mode (right).

### 3. Estimation of the Elastic Constants

Let us consider a thin flat orthotropic plate with free boundary conditions and dimensions  $a \times b \times h$  and density  $\rho$  (schematically shown in figure 3) characterised by the following elastic properties: Young's moduli in x and y directions ( $E_x$ ,  $E_y$ ), in-plane shear modulus  $G_{xy}$  and Poisson's ratios of a thin orthotropic plate  $\nu_{xy}$  and  $\nu_{yx}$ .



**Figure 3.** Schematic representation of a thin orthotropic plate with free boundary conditions

The equation describing the out-of-plane harmonic response  $w(x, y)$  of the plate at frequency  $\omega$  is:

$$\rho \omega^2 w = h^2 \left[ D_1 \frac{d^4 w}{dx^4} + (D_2 + D_4) \frac{d^4 w}{dx^2 dy^2} + D_3 \frac{d^4 w}{dy^4} \right] \quad (1)$$

Where the  $D$ s are elastic constants associated with [8]:

$$D_1 = \frac{E_x}{12(1-\nu_{xy}\nu_{yx})}; \quad D_3 = \frac{E_y}{12(1-\nu_{xy}\nu_{yx})}; \quad D_4 = \frac{G_{xy}}{3}; \quad D_2 = \frac{\nu_{xy}E_y}{6(1-\nu_{xy}\nu_{yx})} = \frac{\nu_{yx}E_x}{6(1-\nu_{xy}\nu_{yx})} \quad (2)$$

Only 4 independent constants need to be established because of the general reciprocal theorem [3]:

$$\nu_{yx}/E_y = \nu_{xy}/E_x \quad (3)$$

Considering free edges and that the plate is cut parallel to the symmetry axes of the material, the boundary conditions of the plate can be expressed as Torque and Shear force at the edges equal to zero, and also the condition that no external torques are applied at the corners of the plate must be enforced [3].

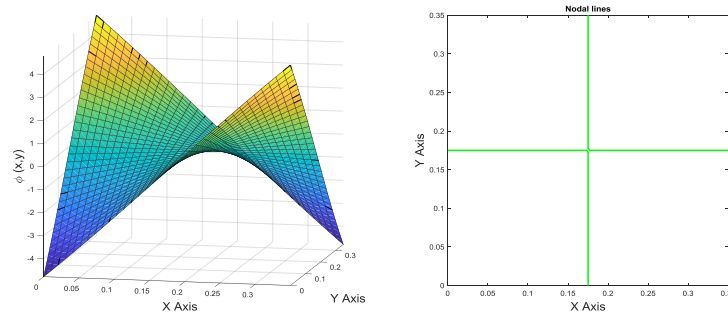
In order to derive the elastic constant expressions, we can use the Rayleigh's principle to derive an approximate estimate (always higher than the actual one) of the natural frequencies as the ratio of the potential energy functional  $V$  and the kinetic energy functional  $T$ :

$$\omega^2 \approx \frac{\frac{1}{2} \iint h^3 \left[ D_1 \frac{d^2 w}{dx^2} + D_2 \frac{d^2 w}{dx^2} \frac{d^2 w}{dy^2} + D_3 \frac{d^2 w}{dy^2} + D_4 \frac{d^2 w}{dxdy} \right] dx dy}{\frac{1}{2} \iint \rho h w^2 dx dy} \quad (4)$$

By using a trial function for the out-of-place function at a specific mode, it is possible to derive a closed-form expression for the natural frequencies as a function of the elastic constants.

### 3.1. Estimation of the Shear Modulus

A pure twisting mode is characterised by two nodal lines that run parallel to the  $x$  and  $y$  axes in the middle of the plate (the so-called cross mode), as shown in figure 4.



**Figure 4.** Cross (+) mode: mode shape and nodal lines.

This mode shape involves only one energy term  $G_{xy}$ , and the trial function  $w = xy$  can be used to approximate this mode shape. Substituting it in equation (4), and integrating from  $-b/2$  to  $b/2$  and  $-a/2$  to  $a/2$ , the following relationship is obtained [3]:

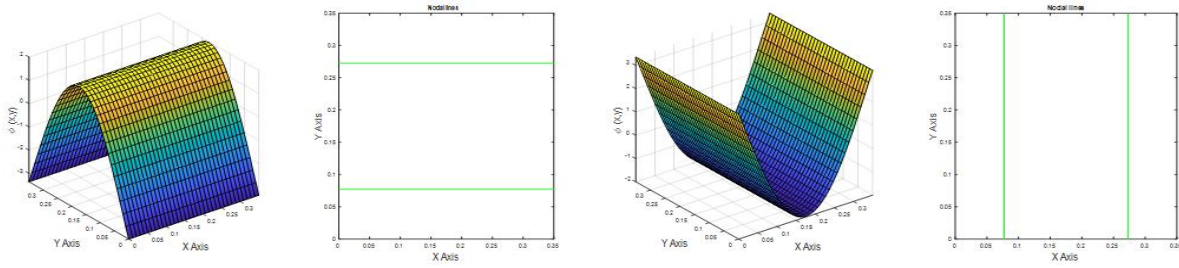
$$D_4 \approx 0.274 \frac{a^2 b^2 \rho}{h^2} f_+^2; \quad f_+ = \frac{\omega_+}{2\pi} \quad (5)$$

Therefore, by estimating with the Chladni the frequency  $f_+$  at which the cross mode occurs, it is possible to derive  $G_{xy}$  (using equation (2)) as:

$$G_{xy} \approx 0.822 \frac{a^2 b^2 \rho}{h^2} f_+^2 \quad (6)$$

### 3.2. Estimation of the Young's moduli in $x$ and $y$ directions

In order to evaluate the Young's moduli, we have to consider the two bending modes of the plates, which are shown in figure 5. These can be referred to as lower (L) and higher (H) bending modes, so that  $D_1 \geq D_3$  [3], and such that for the higher mode the nodal lines run parallel to the  $y$  axis.



**Figure 5.** Modal shapes and nodal lines of lower (left) and higher (right) bending modes.

Each plate bending mode shape can be approximated with the first bending mode of an equivalent free-free isotropic Euler-Bernoulli beam of length  $L$  and flexural wavenumber  $k_n = (\rho A / EI)^{1/4} \omega^{1/2}$  (being  $A$  the cross-sectional area and  $I$  the second moment of area), which can be found in closed-form (see [6]) as:

$$\cos(k_n L) \cosh(k_n L) = 1 \quad (7)$$

For the first mode  $n=1$  and the first non-zero solution of equation (7) is  $k_1 L = 4.73004$ . Since the equivalent beam of length  $L=a$  as a cross-sectional area of  $A=bh$  and  $I = bh^3 / 12$ , the Young modulus is obtained as:

$$E_x = \frac{12\rho a^4}{4.73004^4 h^2} f_H^2 (2\pi)^2 = 0.0789 \frac{12\rho a^4}{h^2} f_H^2 \approx 12D_1 \quad (8)$$

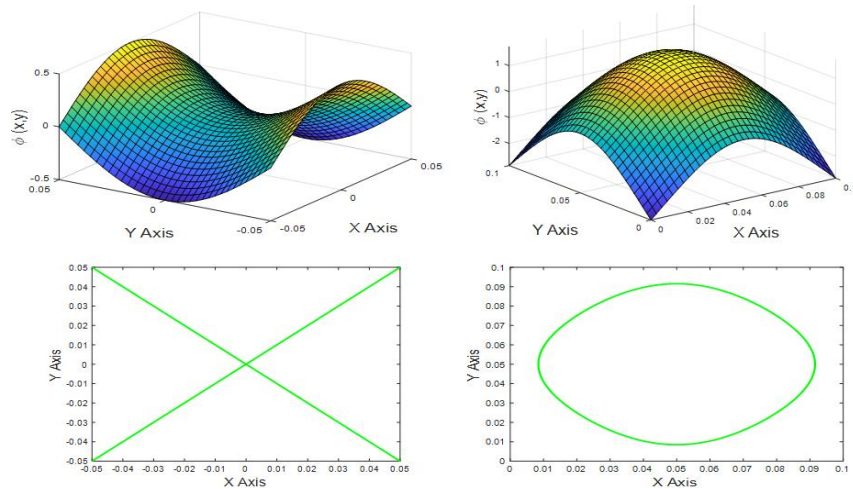
Being  $f_H$  the frequency (Hz) at which it is possible to observe the Chladni pattern of the higher bending mode of the plate. In a similar fashion, it can be found that:

$$E_y = \frac{12\rho b^4}{4.73004^4 h^2} f_L^2 (2\pi)^2 = 0.0789 \frac{12\rho b^4}{h^2} f_L^2 \approx 12D_3 \quad (9)$$

Being  $f_L$  the frequency (Hz) at which it is possible to observe the Chladni pattern of the lower bending mode of the plate.

### 3.3. Estimation of the Poisson's ratio

The Poisson's ratio can be derived from the difference of the so-called X and O mode of a thin plate which are shown in figure 6.



**Figure 6.** Left: X-mode, Right O-mode, shapes and nodal lines.



These modes are obtained from the coupling of the lower and higher bending modes and occur for specific aspect ratios of the plate. In particular, they will always occur in an isotropic square plate ( $a=b$ ). For orthotropic plates instead, the size of the plate needs to be adjusted to display those modes. The correct aspect ratio can be found by using equations (8) and (9) and setting  $f_H = f_L$  [3]:

$$\frac{a}{b} = \left( \frac{D_1}{D_3} \right)^{1/4} \quad (10)$$

To find a closed-form expression relating the frequency at which the X and O modes occur ( $f_x, f_o$  respectively) to the Poisson's ratio, the Rayleigh's principle (equation (4)) is applied again. The trial function to be used for the X mode shape can be expressed as [10]:

$$w = \sin \left[ \frac{\pi}{2} \left( \frac{y}{b} - \frac{x}{a} \right) \right] \sin \left[ \frac{\pi}{2} \left( \frac{y}{b} + \frac{x}{a} \right) \right] \quad (11)$$

The trial function for the O mode shape can be approximated by a linear combination of the free-free beam functions in the  $x$  and  $y$  directions [3]. This can be expressed as [11]:

$$w = \sin \left[ 4.73 \left( \frac{x}{a} - 0.5 \right) \right] - \frac{\sin(4.73/2)}{\sinh(4.73/2)} \cosh \left[ 4.73 \left( \frac{x}{a} - 0.5 \right) \right] + \cos \left[ 4.73 \left( \frac{y}{b} - 0.5 \right) \right] - \frac{\sin(4.73/2)}{\sinh(4.73/2)} \cosh \left[ 4.73 \left( \frac{y}{b} - 0.5 \right) \right] \quad (12)$$

By using equation (4) in combination with equations (12) and (13), and using also equations (8) and (9), the difference between  $f_x$  and  $f_o$  can be expressed as:

$$(f_o^2 - f_x^2) \approx \frac{D_2}{0.104 \frac{a^2 b^2 \rho}{h^2}} \quad (13)$$

Sato and Woodhouse [4] proposed a very similar expression, in which one coefficient has been corrected from comparison with numerical data and experimental data:

$$(f_o^2 - f_x^2) \approx \frac{D_2}{0.114 \frac{a^2 b^2 \rho}{h^2}} \quad (14)$$

Using the relationships in equation (2), it is therefore possible to evaluate the Poisson's ratio.

#### 4. Refined resonant frequencies using EMA, Chladni and a mathematical model – combined approach

The approximate expressions described in the previous section yield initial values of the elastic constants. These values can be refined by using a Rayleigh-Ritz model or a Finite Element Model of the plate by minimizing the difference between the predicted resonant frequencies and the measured ones. The Chladni procedure can be improved by integrating it with EMA and with mathematical models. The EMA can be used at the very beginning to establish only two average FRFs, and identify the resonances. This measurements can be performed with a relative cheap accelerometer. The Chladni setup can then be used to acoustically excite the plate only at these resonant frequencies and to identify which mode corresponds to which frequency, speeding up the process. A Rayleigh-Ritz *MATLAB* code is then used to refine the initially obtained elastic constants.

#### 5. Comparison of Experimental Modal Analysis, Chladni setup results

Experimental Modal Analysis (EMA) and the Chladni setup are used to evaluate the resonant frequencies, and consequently the material properties of the plywood plate using the approximate expressions derived in the previous section. The plywood plate has dimensions:  $a = 350\text{mm}$ ,  $b = 350\text{ mm}$ ,  $h = 5.50\text{ mm}$ , and density of  $568\text{ Kg / m}^3$ . Initially the Chladni test was

performed to evaluate the 5 mode shapes described in the previous section and the corresponding resonance frequencies. The mode shapes so obtained are shown in figure 7.



**Figure 7.** Modal shapes and nodal lines obtained with the Chladni setup

For obtained accurate results with EMA, the plates were vertically suspended using nylon threads to approximate the free boundary conditions that require no energy dissipation. The plates were excited using an instrumented hammer (PCB 086C03 with a rubber tip) and the velocity response was measured at the same point with a single pointer Laser Doppler Vibrometer (Polytec NLV-2500). The sampling rate was 10 kHz and the signal was recorded for 30 seconds. Three measurements were taken to evaluate an average FRF using the standard EMA procedures [6-7]. The mode shapes were not reconstructed by computing the FRFs over a grid of points [6-7], since they were already evaluated with the Chladni setup. The resonant frequencies obtained with the two approaches and with the refining approach described briefly in section 4 (referred to as “combined”) are compared in table 1.

**Table 1.** Resonant frequencies obtained with Chladni and EMA

Method	$f_+$ (Hz)	$f_L$ (Hz)	$f_H$ (Hz)	$f_x$ (Hz)	$f_o$ (Hz)
Chladni	54.4	123.6	173.6	171.0	181.2
EMA	54.8	128.3	175.8	172.4	182.0
Combined	54.8	128.3	175.8	172.8	183.8

It can be observed that there are small differences in the resonant frequencies estimated between the Chladni and EMA results. This is because there might be small mistakes from the operator in tuning the driving frequency of the Chladni setup to obtain precisely the peak response, and therefore the resonant frequency. Moreover, the Chladni results can be affected by the position of the foam blocks – care must be taken to move them under the nodal lines of each specific mode. It is possible to observe also that the combined approach will improve the prediction of  $f_x$  and  $f_o$ , which will affect the prediction of the Poisson’s ratio.

The corresponding elastic constants (in terms of  $D_s$ ) and the % differences obtained with the three approaches are shown in table 2, showing the improvements obtained with the combined approach.

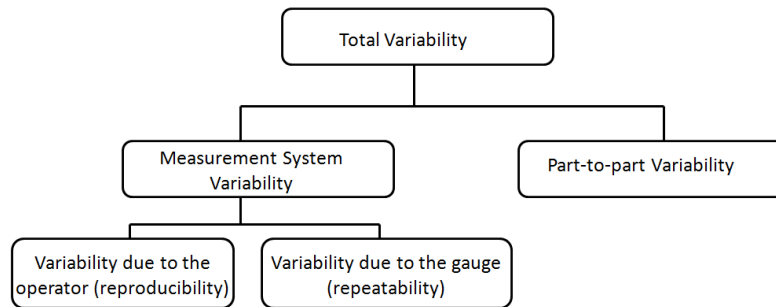
**Table 2.** Elastic constants based on Chladni and EMA results

	$D_1$ (MPa)	$D_2$ (MPa)	$D_3$ (MPa)	$D_4$ (MPa)
Chladni	729.9	89.2	369.6	248.9
EMA	748.2	84.5	398.4	252.1
Combined	747.0	84.5	401.6	267.1
% difference Chladni vs EMA	2.44	5.5	7.23	1.27
% difference EMA vs Combined	0.16	0	3.2	5.95
% difference Chladni vs Combined	2.32	5.5	8.65	7.31



## 6. Manufacturing Variability Analysis

In manufacturing most of the quality problems are caused by variability in the production process [12]. As a consequence, most of the quality improvement initiatives focus on reducing the causes of variation. Overall when performing a test, variability of the results can be divided into two categories: the variability due to the part-to-part differences and the variability due to the measurement system variation, as schematically shown in figure 8.



**Figure 8.** Sources of Variability.

For the measurement system to be acceptable, the measurement system variability must be within a certain limit, so that the part-to-part variability results can be trusted. In the next subsections, the acceptability of the proposed experimental setup as a quality control approach is assessed.

### 6.1. Measurement system variability

The adequacy of this Chladni setup as a measurement system for the elastic properties was tested. Measurement errors can be classified as bias and precision errors. The bias errors do not affect variability as they consistently shift the measurements in a fixed quantity. Precision errors add variability to the data, as they do not occur in all the readings and they may have different magnitude for different measurements [12,13]. There are two types of precision errors: Repeatability and Reproducibility (R&R) errors. Repeatability or gauge errors are found when the same person measures the same magnitude in several occasions. Reproducibility errors are referred to the variability that may be observed when the same magnitudes are measured by different appraisers.

The Gauge Repeatability and Reproducibility (GR&R) [12] quantifies the precision errors and assesses the ability of the measuring system to discriminate changes in the measured magnitude. To evaluate the GR&R, one plate was measured by three different appraisers. Each appraiser measured the plate five times. An ANOVA (Analysis of Variance) study was performed. To assess the acceptability of the GR&R result obtained, two ratios commonly used in industrial practises were considered [12]: (i) “GR&R” divided by the total variation found from experiments performed on an ensemble of plates; (ii) “GR&R” divided by the tolerance of the specification applicable (higher limit minus lower limit of the specification). The acceptability criteria are shown in table 3.

**Table 3.** Acceptability criteria for “GR&R” results commonly used during industrial practice [12].

Acceptability	Excellent	Good	Marginal
GR&R/Total Variation (%)	< 10 %	< 20 %	< 30 %
GR&R/Tolerance (%)	< 10 %	< 20 %	< 30 %

### 6.2 Assessment of the measurement system

Part-to-part variability of 30 plates was measured with the Chladni setup. The mean, standard deviation and total variation (6 sigma), and histograms were computed. The relative sizes of the GR&R to the sizes of the total variation of the distributions were first calculated. The results are shown in table 4. Acceptability of the measuring system for  $D_2$  are not shown since this would require to cut the 30 plates to satisfy the aspect ratio that would lead to the appearance of the X and O mode. While accurate estimates of  $D_1$  and  $D_3$  (which are related to the Young’s moduli) can be obtained with the proposed

setup,  $D_4$  measurements (which is related to the shear modulus) are unacceptable. It was found that acceptability of the measurement system can be improved if operators get appropriate training and rigorously follow operating procedures.

**Table 4.** Acceptability of the measurement system according to the ‘GR&R’ / total variation (%).

	GR&R f (Hz)	GR&R D's (MPa)	Total Variation D's (MPa)	‘GR&R’ / total variation (%)	Acceptability
$D_1$ - Mode h	1.6743	14.1	288.6	4.88%	Excellent
$D_3$ - Mode l	1.5036	9.0	106.2	8.47%	Excellent
$D_4$ - Mode +	1.5445	14.4	47.0	30.6%	Unacceptable

## 7. Conclusion

An inverse procedure based on the Chladni patterns has been investigated to evaluate the manufacturing variability of the elastic properties of thin orthotropic rectangular plates. A detailed mathematical derivation of the equations needed for estimating the elastic constants based on the knowledge of five natural frequencies has been presented, and a combined approach using the Chladni setup with EMA and a mathematical model has been described and applied to a plywood plate, showing very good results. The manufacturing variability of an ensemble of 30 “nominally identical” plywood plates was investigated, showing that the variability of Young’s moduli can be estimated accurately, while care must be taken by the operator when assessing the shear modulus.

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