Affinoid Enveloping Algebras and their Representations

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Abstract

We develop the basic theory of Picard algebroids and twisted differential operators on a smooth, reduced, locally of finite type scheme over a commutative ring. We also give a new geometric proof of the classical Duflo's theorem. We next move to the study the affinoid enveloping algebra of a semisimple Lie algebra defined over a discrete valuation ring. We prove that there exists a one-to-one correspondence between the lattice of submodules of an affinoid Verma module of a given weight and the corresponding classical Verma module. Finally, we classify all the primitive ideals in the affinoid enveloping algebra and prove that a large class of two-sided ideals in the affinoid enveloping algebra is controlled by two-sided ideals in the classical enveloping algebra.

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Chapter 1

Introduction

Representation theory is the subfield of algebra which aims to understand an abstract algebraic object (groups, rings, etc) by representing its elements as linear transformations of vector spaces, known as a representation or module. One can think of a representation as showing some 'face' of the algebraic object: some of them are trivial and provide no information, some of them are *faithful* when no information is lost. We are mostly interested in representations that lie in between of these two and in my research I am particularly interested in representations of groups and rings.

The field of real numbers $\mathbb{R}$ can be described as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the Cauchy norm. This allows for the development of a very rich and fruitful theory of classical analysis; however, there are limited applications of $\mathbb{R}$ in number theory.

Let us fix a prime number p. We may endow $\mathbb{Q}$ with another norm, called the *p-adic* norm whose completion we denote $\mathbb{Q}_p$, the field of p-adic rational integers. To allow for a more general theory, we work not only over $\mathbb{Q}_p$, but over fields K that are finite extensions of $\mathbb{Q}_p$. Similar to how $\mathbb{Z}$ is the ring of integers of $\mathbb{Q}$, we have associated with K a ring of integers denoted $\mathcal{O}_K$; in the case $K = \mathbb{Q}_p$, we denote $\mathbb{Z}_p = \mathcal{O}_{\mathbb{Q}_p}$ the ring of p-adic integers. We also let π be the uniformiser of $R = \mathcal{O}_K$ and $k = R/\pi R$ the residue field.

We call a group, a *linear algebraic group* if it is a closed subgroup of a matrix group in the Zariski topology. We let G be a linear algebraic group defined over R and we let $\mathfrak{G}$ be a compact open subgroup of $G(K)$; the group $\mathfrak{G}$ has a more rigid structure than G which allows for a richer theory. We are interested to study the group $\mathfrak{G}$ and its representation theory when the p-adic Lie group $\mathfrak{G}$ is compact. Classically, to study the representation theory of a finite group H, we associate to H a ring $K[H]$ called the group ring, such that there is a one-to-one correspondence between H-representations and $K[H]$-representations. This correspondence allows using techniques from non-commutative ring theory to study the representation theory of H. In our case, the ring $K[\mathfrak{G}]$ is not rich enough to allow for a nice theory as it ignores the underlying topology of $\mathfrak{G}$. Therefore, we need to look for a new algebra to replace the group ring.

The R-Iwasawa algebra of $\mathfrak{G}$ is defined to be:

$$R\mathfrak{G} := \varprojlim_{\mathfrak{N} \trianglelefteq_o \mathfrak{G}} R[\mathfrak{G}/\mathfrak{N}],$$

where the inverse limit is taken over all open normal subgroups. We remark that since $\mathfrak{G}$ is compact, we have that all normal open subgroups have finite index in $\mathfrak{G}$. We call $K\mathfrak{G} := R\mathfrak{G} \underset{R}{\otimes} K$ the Iwasawa algebra of $\mathfrak{G}$. The main properties of $K\mathfrak{G}$ were established in Lazard's seminal paper [40]; in particular, the ring $K\mathfrak{G}$ is Noetherian. We can also view $K\mathfrak{G}$ as the continuous dual of the algebra of continuous K-valued functions on $\mathfrak{G}$. The importance of $K\mathfrak{G}$ in the representation theory of $\mathfrak{G}$ is given by the following theorem:

Theorem 1.1.1. *[52, Theorem 3.5] There is an anti-equivalence of categories between K-Banach admissible $\mathfrak{G}$-modules and finitely generated $K\mathfrak{G}$-modules.*

We make a further restriction on G and assume that G is semisimple and split over K. This restriction has a strong number theory justification: the representation theory of G forms one side of the celebrated *Local Langlands* programme. Specifically, we are interested in *simple* representations, the ones which do not have any subrepresentations. From an algebraic point of view, Jacobson suggested that we should study the representation theory of a ring by studying the *primitive ideals*; an ideal is primitive if it is the annihilator of a non-zero *simple* module. This allows us to group simple representations by their annihilators. The motivating question regarding the primitive spectrum of the Iwasawa algebras is:

Question 1.1.2. *Can we characterise all the primitive ideals in $K\mathfrak{G}$?*

Understanding the structure of primitive ideals of $K\mathfrak{G}$ seems to be a hard problem; even in the simplest case, there is no direct approach to computing these ideals.

There are two completions of $K\mathfrak{G}$ that may provide some light on its primitive ideals: the distribution algebra $D(\mathfrak{G}, K)$ and the affinoid enveloping algebra $\widehat{U(\mathfrak{g})_K}$, where $\mathfrak{g}$ is a certain Lie algebra associated to $\mathfrak{G}$. Understanding the relation between the representation theory of these 3 rings is a topic of considerable current interest and our project follows this direction. We should mention that the distribution algebra $D(\mathfrak{G}, K)$ can be defined as the continuous dual of K-valued locally analytic functions on $\mathfrak{G}$. From the perspective of number theory, specifically the local Langlands programme, we should mention a theorem of Schneider and Teitalbum that states that there is an anti-equivalence of categories between *admissible* $\mathfrak{G}$-representations and *coadmissible* $D(\mathfrak{G}, K)$-modules. Algebraically, the $D(\mathfrak{G}, K)$ is *faithfully flat* as a $K\mathfrak{G}$-module, so understanding the structure of the distribution algebra should provide some information about the Iwasawa algebra.

We let $\mathfrak{g}$ be the R-Lie algebra associated with $\mathfrak{G}$, where we assume that $\mathfrak{G}$ is uniform, further denote $\mathfrak{g}_K := \mathfrak{g} \underset{R}{\otimes} K$. Recall that π denotes the uniformiser of K. The affinoid enveloping algebra of $\mathfrak{g}$ is defined to be:

$$\widehat{U(\mathfrak{g})_K} := (\varprojlim U(\mathfrak{g})/\pi^i U(\mathfrak{g})) \underset{R}{\otimes} K.$$

We also let

$$\widehat{U(\mathfrak{g})_{n,K}} = (\varprojlim U(\pi^n \mathfrak{g})/\pi^i U(\pi^n \mathfrak{g})) \underset{R}{\otimes} K.$$

Let $u_1, u_2, \dots u_d$ be a free R-basis of $\mathfrak{g}$. Then as a K-vector space we have

$$\widehat{U(\mathfrak{g})_{n,K}} = \{\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha u^\alpha : \quad \lambda_\alpha \in K, p^{-n|\alpha|}\lambda_\alpha \to 0 \text{ as } |\alpha| \to \infty\}.$$

Here for a d-tuple $\alpha = (\alpha_1, \alpha_2, \dots \alpha_d)$, $|\alpha| = \sum_{i=1}^d \alpha_i$ and $u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2} \dots u_d^{\alpha_d}$.

We should note that the affinoid enveloping algebra can be defined for any Lie algebra $\mathfrak{g}$ which is free as an R-module. For example, if $\mathfrak{g}$ is the Abelian Lie algebra of dimension d, then $\widehat{U(\mathfrak{g})_K}$ is isomorphic with the K-Tate algebra in d-variables. Informally, we may think about the affinoid enveloping algebras as noncommutative restricted power series instead of noncommutative polynomial rings (which informally describe classical enveloping algebras) with the same commutation relations.

For $n = 0$, there is a natural embedding $K\mathfrak{G} \to \widehat{U(\mathfrak{g})_K}$ that makes $\widehat{U(\mathfrak{g})_K}$ a flat $K\mathfrak{G}$-module. The ring $\widehat{U(\mathfrak{g})_K}$ is easier to understand and work with, as it can be described as a certain completion of the classical enveloping algebra $U(\mathfrak{g} \underset{R}{\otimes} K) \cong U(\mathfrak{g}) \underset{R}{\otimes} K$.

The study of the universal enveloping algebra of a semisimple Lie algebra has been a topic of considerable interest since the 1960s. Among the mathematicians that have worked on this topic, we should mention Anthony Joseph, Michel Duflo, Jacques Dixmier, Alexandre Beilinson, Joseph Bernstein, Masaki Kashiwara, David Kazhdan, George Lusztig, David Vogan, Walter Borho, Jean-Luc Brylinsky. In a seminal paper [25], Duflo obtained a characterisation of all the primitive ideals in the affinoid enveloping algebra of a semisimple Lie algebra defined over the complex numbers. Beilinson-Bernstein [8] related the modules over the enveloping algebra to modules over twisted differential operators on the flag variety. Via the Riemann-Hilbert correspondence this allows for a proof of the celebrated Kazhdan-Lusztig conjectures.

The goal of my D.Phil. project is to answer the following question posed by Ardakov and Wadsley in [6]:

Question 1.1.3. *Is it the case that every primitive ideal of $\widehat{U(\mathfrak{g})_{n,K}}$ with K-rational infinitesimal central character is the annihilator of a simple affinoid highest weight module?*

A positive answer to Question 1.1.3 would be a step towards answering Question 1.1.2. Specifically, Ardakov and Wadsley claim that if one can prove that any infinite dimensional affinoid highest weight $K\mathfrak{G}$-module is *faithful*, one could use the faithful flatness of $D(\mathfrak{G}, K)$ over $K\mathfrak{G}$ together with the affinoid Quillen's Lemma [5] to prove that every non-zero prime ideal of $K\mathfrak{G}$ is the annihilator of a finite dimensional simple module.

Let us formulate the main results of the thesis; we let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}^*$ denote the set of linear maps from $\mathfrak{h} \to R$.

Theorem A. *Let $\lambda \in \mathfrak{h}^*$ and consider the affinoid Verma module $\widehat{M(\lambda)}$. There exists a one to one correspondence between the lattice of submodules of $\widehat{M(\lambda)}$ and the lattice of submodules of $M(\lambda)$. In particular, $\widehat{M(\lambda)}$ has a unique simple quotient denoted $\widehat{L(\lambda)}$.*

Theorem B. *Let R be a mixed characteristic $(0, p)$ complete discrete valuation ring and let G be a connected, simply-connected, split semisimple, smooth affine algebraic group scheme over* $\operatorname{Spec} R$. *Denote $\mathfrak{g} := \operatorname{Lie}(G)$ the Lie algebra of G.*

Any primitive ideal in the affinoid enveloping algebra $\widehat{U(\mathfrak{g})_{n,K}}$ with K-rational infinitesimal central character is the annihilator of some $\widehat{L(\lambda)}$.

In particular, we obtain a positive answer to Question 1.1.3. As corollary we obtain in Proposition 5.7.6:

Proposition 1.1.4. *Any ideal in* $\widehat{U(\mathfrak{g})_{n,K}}$ *with* K*-rational central character induced by a regular weight* $\lambda \in \mathfrak{h}^*$ *is controlled by its intersection with* $U(\mathfrak{g})_K$*.*

To prove Theorem B, we use an affinoid version of the Beilinson-Bernstein localisation theorem, together with an affinoid version of the Borho-Brylinski equivalence between equivariant modules on the double flag variety and equivariant modules on the flag variety.

We also prove an affinoid version of Quillen's Lemma improving the result proved by Ardakov and Wadsley [5, Theorem 9.4].

Theorem C. *Let* M *be a finitely generated* $\widehat{U(\mathfrak{g})_{n,K}}$ *module and* φ *a simple endomorphism of* M*. Then* φ *is algebraic over* K*.*

In fact, by combining Theorem B and Theorem C, we obtain in Theorem 6.5.2 a full characterisation of the primitive spectrum of $\widehat{U(\mathfrak{g})_{n,K}}$. Furthermore, we are able to prove that a large class of two-sided ideals $\widehat{U(\mathfrak{g})_{n,K}}$ are controlled by ideals in the classical enveloping algebra.

Theorem D. *Let* I *be a two-sided ideal in* $\widehat{U(\mathfrak{g})_{n,K}}$ *and assume it has a central character* $\chi : Z(\widehat{U(\mathfrak{g})_{n,K}}) \to \overline{K}$ *generated by a dominant regular weight. Then* I *is controlled by* $I \cap U(\mathfrak{g})_K$*.*

Finally, let us fix a central character $\chi = \chi_\lambda$ generated by a dominant regular weight. Let m_λ and $\widehat{m_\lambda}$ be two-sided ideals in $U(\mathfrak{g})_K$ and $\widehat{U(\mathfrak{g})_{n,K}}$ generated by $\ker \chi_\lambda$, and also let $U(\mathfrak{g})_K^\lambda$ and $\widehat{U(\mathfrak{g})_{n,K}^\lambda}$ be the corresponding quotients.

Theorem E. *The map* $\overline{\psi} : \operatorname{Spec}(\widehat{U(\mathfrak{g})_{n,K}^\lambda}) \to \operatorname{Spec}(U(\mathfrak{g})_K^\lambda)$*,* $\overline{\psi}(P+\widehat{m_\lambda}) = P \cap U(\mathfrak{g})_K + m_\lambda$ *is a homeomorphism with inverse* $\overline{\psi}^{-1}(Q + m_\lambda) = \widehat{U(\mathfrak{g})_{n,K}} Q \widehat{U(\mathfrak{g})_{n,K}} + \widehat{m_\lambda}$*.*

This thesis is organised as follows: in Chapter 2, we introduce the basic noncommutative algebra and geometry tools that we use in this thesis. Next, we develop in Chapter 3 the theory of twisted differential operators on smooth, reduced and locally of finite type schemes defined over a commutative ring. We then give in Chapter 4 a geometric proof of the classical Duflo's theorem based on the theory developed in 3. In Chapter 5, we complete the proof of Theorem A and Theorem B using the techniques from the previous 2 chapters. Lastly, in Chapter 6, we generalise the affinoid Quillen's Lemma in [5] (Theorem C) and prove Theorem D.

Chapter 2

Preliminaries

This chapter aims to describe the non-commutative rings and Lie theory tools that are used in this thesis.

2.1 Filtrations, gradations, and completions

Throughout this document, all rings will be unital. By a module over a ring, we will mean a left module unless otherwise stated. By a Noetherian ring, we mean a ring that is both left and right Noetherian.

2.1.1 Filtered rings and modules

Definition 2.1.1. *We call R a filtered ring with filtration FR if there is an ascending chain of additive subgroups F_nR, $n \in \mathbb{Z}$ satisfying $1 \in F_0R$, $F_nR \subset F_{n+1}R$ and $F_iR.F_jR \subset F_{i+j}R$ for all $n, i, j \in \mathbb{Z}$.*

Given a filtered ring, we may also define the notion of a filtered module.

Definition 2.1.2. *Let R be a ring with filtration FR and M an R-module. A filtration of M consists of an ascending chain of additive subgroups F_nM, $n \in \mathbb{Z}$ satisfying $F_nM \subset F_{n+1}M$ and $F_iR.F_jM \subset F_{i+j}M$ for all $n, i, j \in \mathbb{Z}$.*

Given R-modules M and N with filtrations FM and FN, we call $f : M \to N$ a filtered morphism if f is a module homomorphism and $f(F_iM) \subset F_iN$ for all $i \in \mathbb{Z}$.

Lastly, we say that f if strict *if $f(F_iM) = f(M) \cap F_iN$.*

Let us now introduce some terminology for filtered modules. This will apply to filtered rings as well by considering a ring as a module over itself via left multiplication. We call a filtration *positive* if $F_iM = 0$ for $i < 0$. We say that it is *exhaustive* if $M = \cup_{i \in \mathbb{Z}} F_iM$. Finally, we say that it is *separated* if $\cap_{i \in \mathbb{Z}} F_iM = 0$.

From now on, we will assume that all the filtrations that appear are exhaustive. Let us now define the filtration topology on M.

Definition 2.1.3. *Let M be a filtered R-module with filtration FM. Define a topology Ω on M by saying that $U \subset M$ is open if for all $u \in U$, there exists $i \in \mathbb{Z}$ such that $u + F_iM \subset U$. We call this the filtration topology.*

It is easy to check that the notation above indeed defines a topology. Furthermore, the condition that the filtration is separated is equivalent to the topology being Hausdorff. Let us finish this subsection by stating some basic properties of the filtration topology.

Proposition 2.1.4. *[33, I.3.1] Let R be a filtered ring with filtration FR and M a filtered module with filtration FM.*

i) The sets $m + F_iM$, $m \in M$ and $i \in \mathbb{Z}$ form a basis for the filtration topology.

ii) Let N be a submodule of M. Then N is open in M if and only $F_iM \subset N$ for some $i \in \mathbb{Z}$. Furthermore, if N is open, it is also closed.

iii) Let S be a subset of M. The closure of S, denoted by $\overline{S}$, is defined to be:

$$\overline{S} = \bigcap_{i \in \mathbb{Z}} (S + F_iM).$$

2.1.2 Completions of modules

Throughout this subsection, R will denote a filtered ring with filtration FR. Most of this subsection is based on [33, I.3]. Let us begin by defining the notion of Cauchy sequences with respect to the filtration topology.

Definition 2.1.5. *Let M be a filtered R-module with filtration FM. A sequence (x_n) of elements of M is said to be Cauchy if for all $n' > 0$, there exists an integer $N > 0$ such that $x_i - x_j \in F_{-n'}M$ for all $i, j > N$. The sequence converges to $x \in M$ if for every integer $n' \geq 0$, there exists N such that $x - x_i \in F_{-n'}M$ for $i \geq N$.*

Definition 2.1.6. *We say that a filtered module M with filtration topology FM is complete with respect to the filtration topology if FM is a separated filtration and every Cauchy sequence in M converges to an element of M.*

We may now define the completion of a filtered module.

Definition 2.1.7. *Let M be a filtered R-module with filtration FM. A completion of M with respect to the filtration topology is a triple $(\hat{M}, F\hat{M}, f_M)$, where $\hat{M}$ is a filtered R-module with filtration $F\hat{M}$ and $f_M : M \to \hat{M}$ a filtered ring morphism such that:*

1. *$\hat{M}$ is a complete module with respect to $F\hat{M}$.*
2. *The morphism $f_M : M \to \hat{M}$ is strict.*
3. *$f_M(M)$ is everywhere dense in $\hat{M}$.*
4. *$\ker f_M = \cap_{i\in\mathbb{Z}} F_iM$.*

It is clear that if such a completion exists, then it is unique. Let us now prove its existence.

Let M be a filtered R-module with filtration FM. Consider the projective system M/F_iM with the corresponding transition maps $\psi_{i,j}$ and let $\hat{M} = \varprojlim M/F_iM$. Let $\pi_i : \hat{M} \to M/F_iM$ and $u_i : M \to M/F_iM$ denote the canonical morphisms. Define a filtration on $\hat{M}$, by $F_i\hat{M} = \ker \pi_i$ and let $f_M : M \to \hat{M}$ be given by $f_M(m) = (u_i(m))$.

Proposition 2.1.8. *[33, I.3.5] The triple $(\hat{M}, F\hat{M}, f_M)$ is the completion of M with respect to the filtration topology.*

We also denote $\hat{R}$ the completion of R with respect to its filtration. It turns out that $\hat{M}$ is a filtered $\hat{R}$-module with respect to the filtrations $F\hat{R}$ and $F\hat{M}$.

2.1.3 Associated graded rings and modules

Definition 2.1.9. *A $\mathbb{Z}$-grading of a ring R consists of a family of subgroups G_iR, $i \in \mathbb{Z}$ such that $G_iR.G_jR = G_{i+j}R$ and $R = \bigoplus_{i\in\mathbb{Z}} G_iR$.*

A $\mathbb{Z}$-grading of an R-module M consists of a family of subgroups G_iM, $i \in \mathbb{Z}$ such that $G_iR.G_jM = G_{i+j}M$ and $M = \bigoplus_{i\in\mathbb{Z}} G_iM$.

We may now define the associated graded ring/module of a filtered ring/module.

Definition 2.1.10. *Let R be a filtered ring with filtration FR and M a filtered R-module with filtration FM.*

The associated graded ring of R is defined to be

$$\operatorname{gr}(R) = \bigoplus_{i \in \mathbb{Z}} F_{i+1}R/F_iR.$$

The associated graded module of M is defined to be

$$\operatorname{gr}(M) = \bigoplus_{i \in \mathbb{Z}} F_{i+1}M/F_iM.$$

It then follows that $\operatorname{gr}(R)$ is a $\mathbb{Z}$-graded ring with $G_iM = F_{i+1}R/F_iR$ and $\operatorname{gr}(M)$ is a graded $\operatorname{gr}(R)$-module with $G_iM = F_{i+1}M/F_iM$. There are many properties that we can deduce about a ring/module from its associated graded ring/module such as being Noetherian or non-zero. The following proposition shows that completions interact well with taking the associated graded.

Proposition 2.1.11. *[33, I.4.2.2] Let R be a filtered ring with filtration FR and M a filtered R-module with filtration FM. Then $\operatorname{gr} R \cong \operatorname{gr} \hat{R}$ as rings and $\operatorname{gr} M \cong \operatorname{gr} \hat{M}$ as $\operatorname{gr} R$-modules.*

We finish the subsection by defining the Rees ring/module and show how it connects to the associated graded ring/module.

Definition 2.1.12. *Let R be a filtered ring with filtration FR and M a filtered R-module with filtration FM. The Rees ring of R is defined to be the subring of $R[t, t^{-1}]$ given by*

$$\tilde{R} := \bigoplus_{i \in \mathbb{Z}} t^i F_iR.$$

The Rees module of M is the submodule of $M[t, t^{-1}]$ given by

$$\tilde{M} := \bigoplus_{i \in \mathbb{Z}} t^i F_iM.$$

Proposition 2.1.13. *We have the following isomorphisms:*

- *$\tilde{R}/(t)\tilde{R} \cong \mathrm{gr}(R)$ and $\tilde{R}/(t-1)\tilde{R} \cong R$ as rings.*
- *$\tilde{M}/(t)\tilde{M} \cong \mathrm{gr}(M)$ as $\mathrm{gr}(R)$-modules and $\tilde{M}/(t-1)\tilde{M} \cong M$ as R-modules.*

Definition 2.1.14. *Let R be a filtered ring with filtration FR. We say that a filtration on a R-module M is* good *if $\tilde{M}$ is finitely generated as a $\tilde{R}$-module or equivalently, there exist $m_1, m_2, \dots m_s \in M$ and $k_1, k_2, \dots k_s \in \mathbb{Z}$ such that:*

$$F_iM = \sum_{j=1}^{s} F_{n-k_j} R m_j.$$

We should note that any finitely generated R-module admits a good filtration. Furthermore, it follows from [33, Theorem I.5.7] that if R is complete with respect to its filtration, FM is a good filtration if and only if $\mathrm{gr}(M)$ is finitely generated over $\mathrm{gr}(R)$. In general, only the if part holds.

2.1.4 Artin-Rees property and Zariski rings

Definition 2.1.15. *Let R be a filtered ring with filtration FR and M a filtered R-module. We say that M has the Artin-Rees property if for every finitely generated R-module $N = \sum_{i=1}^{s} Rn_i$, there exists $c \in \mathbb{Z}$ such that for all $j \in \mathbb{Z}$,*

$$F_jM \cap N \subseteq \sum_{i=1}^{s} F_{j+c} R n_i.$$

We finish this section by providing the definition and characterisation of Zariski rings.

Definition 2.1.16. *Let R be a filtered ring with filtration FR. We say that R is a left Zariski ring if $\tilde{R}$ is Noetherian and $F_{-1}R \subset J(F_0R)$, where $J(\bullet)$ denotes the Jacobson radical.*

Theorem 2.1.17. *[33, Theorem II.2.2] The following are equivalent:*

1. *R is a left Zariski ring.*
2. *FR is separated, $\mathrm{gr}(R)$ is left Noetherian, $F_{-1}R \subset J(F_0R)$ and every good filtration on an R-module has the Artin-Rees property.*

3. *FR is separated,* $\operatorname{gr}(R)$ *is left Noetherian,* $F_{-1}R \subset J(F_0R)$ *and* FR *has the Artin-Rees property.*

4. $\operatorname{gr}(R)$ *is left Noetherian and* $\hat{R}$ *is a faithfully flat* R*-module.*

5. $\operatorname{gr}(R)$ *is left Noetherian,* $F_{-1}R \subset J(F_0R)$ *and for every* I *left ideal of* R *with good filtration* FI*, we have*

$$F_iI = \cap_{j\in\mathbb{Z}}(F_iI + F_jI).$$

2.1.5 Algebraic microlocalisation

Throughout this subsection, we let R be a Zariski ring with filtration FR, T an Ore set in $\operatorname{gr} R$ consisting of homogeneous elements and M a finitely generated R-module with good filtration FM.

Lemma 2.1.18. *[5, Lemma 2.4]*

Let $S := \{s \in R | \operatorname{gr} s \in T\}$*. Then:*

a) *S is an Ore set in R.*

b) *The induced filtration $F'R_S$ on the localisation R_S is Zariskian and furthermore* $\operatorname{gr} R_S \cong (\operatorname{gr} R)_T$*.*

c) *The induced filtration $F'M_S$ is good and there is an isomorphism of* $\operatorname{gr} R_S$ *modules* $\operatorname{gr} M_S \cong (\operatorname{gr} M)_T$*.*

Definition 2.1.19. *Let the notation be as in the previous lemma. We define the microlocalisation of R at S, $Q_T(R)$, to be the completion of R_S with respect to $F'R_S$. Similarly, we define the microlocalisation of M at S, $Q_T(M)$ to be the completion of M_S with respect to $F'M_S$.*

Corollary 2.1.20. *[5, Corollary 2.4]*

The ring $Q_T(R)$ is flat over R and as $Q_T(R)$-modules, we have $Q_T(R) \underset{R}{\otimes} M \cong Q_T(M)$*. Moreover,* $Q_T(M) = 0$ *if and only* $M_S = 0$*.*

2.2 Affinoid and Tate algebras

Throughout this section, we let R be a commutative ring.

We call a map $| \bullet | : R \to \mathbb{R}$ a valuation of R if for all $x, y \in R$:

1. $|xy| = |x||y|$.
2. $|x + y| \leq \max(|x|, |y|)$.
3. $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$.

Given a valuation of a ring, we have a topology induced by the corresponding metric space given by $d(x, y) = |x - y|$. We call the pair $(R, | \bullet |)$ a valued ring.

Let $(F, | \bullet |)$ be a valued field. We call the set $R = \{r \in F||r| \leq 1\}$ the valuation ring of F. In general, we call $(R, | \bullet |)$ a valuation ring if it is the valuation ring of a valued field. If R is Noetherian, it is automatically a principal ideal domain. The units of R are elements of norm 1. In general, R contains a unique maximal ideal $\mathfrak{m}$ given by elements of norm strictly less than 1. In particular, R is a local ring and we call $R/\mathfrak{m}$ the residue field of R.

Definition 2.2.1. *We call R a discrete valuation ring if the value of $R \backslash 0$ is a discrete subset of $\mathbb{R}_{>0}$. This happens if and only if its value group G is a discrete, hence cyclic subgroup of $\mathbb{R}_{>0}$. We let ϵ be such a generator. We call any element π such that $|\pi| = \epsilon$ a uniformiser of R.*

It follows from Krull's intersection theorem that every Noetherian valuation ring R is a discrete valuation ring. Furthermore, given a discrete valuation ring with uniformiser π, all the ideals of R are given by $\mathfrak{m}^n = (\pi^n)$ for $n \in \mathbb{N}$. This allows to define a negative filtration on R by $F_{-n}R = \mathfrak{m}^n$ for $n \in \mathbb{N}$, which can be extended to the field of fractions of R. It then follows that the completion of R with respect to the $| \bullet |$-norm of $\operatorname{Frac}(R)$ coincides with the completion with respect to the filtration topology.

We let $| \bullet |_p$ denote the p-adic valuation on $\mathbb{Q}$ for p prime. It turns out that all the non-trivial non-archimedean valuations are given by $| \bullet |_p$ for some p. We denote $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to $| \bullet |_p$.

For the rest of this section, we let K be a field equipped with a non-archimedean norm $| \bullet |$.

Definition 2.2.2. *The Tate algebra in n variables is defined to be the subring of formal power series $K[[x_1, x_2, \dots x_n]]$ given by:*

$$T_n = K\langle x_1, x_2 \dots x_n\rangle := \{\sum_{J\in\mathbb{N}^d} a_J x^J | \quad |a_J| \to 0 \text{ as } |J| \to \infty\}.$$

We may endow the Tate algebra with the Gauss norm: for $f \in T_n$, define $|f(x_1, x_2 \dots x_n)| := \max_{J\in\mathbb{N}^d} |a_J|$. This norm is non-archimidean and one may check that the Tate algebra is a Banach K-algebra with respect to it.

We may also describe the units in T_n; let R be the valuation ring of K and k the residue field of R. Let $T_n^0 = \{f \in T_n| \quad |f| \leq 1\}$ and consider the natural surjection $q : T_n^0 \to k[x_1, x_2, \dots x_n]$.

Proposition 2.2.3. *An element $f \in T_n$ is a unit if and only if $f \in T_n^0$ and $q(f)$ is a constant polynomial.*

The following proposition describes the main ring theoretic properties of Tate algebras:

Proposition 2.2.4. *The ring T_n is Noetherian, Jacobson, factorial, regular of equidimension n, and the Nullstellensatz holds: for each maximal ideal I, the field T_n/I is a finite extension of K.*

Definition 2.2.5. *We say that a ring A is a K-affinoid algebra if there is $n \in \mathbb{N}$ and I an ideal in T_n such that $A \cong T_n/I$.*

We may endow such a K-affinoid algebra with the residue norm and with respect to this norm A becomes a K-Banach algebra. It follows from the definition that A inherits many properties of T_n; in particular, we obtain from Proposition 2.2.4 that for any maximal ideal J in A, the field A/J is a finite extension of K.

Tate algebras and affinoid algebras form the starting point of the beautiful theory of rigid geometry. The classical spectrum of the polynomial ring is replaced by the maximal spectrum of the Tate algebra and affinoid algebras correspond to rigid varieties. An excellent book that introduces this theory is [18].

2.2.1 Doubly-filtered rings

Throughout this subsection, we let R be a discrete valuation ring with field of fractions K, uniformiser π, and residue field k.

Definition 2.2.6. *Let V be a K-vector space. We say that an R-submodule L of V is a R-lattice if $V = K.L$ and $\cap_{i=0}^{\infty}\pi^i L = 0$.*

We call the vector space $\operatorname{gr}_0 V := L/\pi L$ the slice of V.

Definition 2.2.7. *We call A a doubly-filtered K-algebra if it has an R-subalgebra F_0A which is an R-lattice in A and if the slice $\operatorname{gr}_0 A$ of A is a $\mathbb{Z}$-filtered ring. We say that A is a complete doubly filtered K-algebra if F_0A is complete with respect to its π-adic filtration, and the filtration on $\operatorname{gr}_0 A$ is also complete. A morphism of doubly filtered K-algebras is a K-linear ring homomorphism $\varphi : A \to B$ which preserves the lattices in A and B and which induces a filtered k-linear homomorphism $\operatorname{gr}_0 \varphi : \operatorname{gr}_0 A \to \operatorname{gr}_0 B$ between the slices.*

Lemma 2.2.8. *Let A be a doubly-filtered K-algebra and consider the natural π-adic filtration on A. Then*

$$\operatorname{gr} A \cong \operatorname{gr}_0 A[x, x^{-1}].$$

Proof. This is [5, Lemma 3.1]. □

We will use the notation $\operatorname{Gr}(A)$ to denote the associated graded ring of the slice of A. Then Gr is a functor from the category of doubly-filtered K-algebras to the category of graded k-algebras.

2.3 Dimension theory

Let us begin by defining the general version of a dimension function on a Noetherian ring.

Definition 2.3.1. *Let A be a Noetherian ring. A dimension function is map δ that associates to each finitely generated A-module M, a value $\delta(M) \in \mathbb{Z} \cup \{-\infty\}$ satisfying the conditions:*

i) $\delta(0) = -\infty$

ii) If $0 \to M' \to M \to M" \to 0$ *is a short exact sequence of finitely generated* A*-modules, then* $\delta(M) \geq \max(\delta(M'), \delta(M''))$*; the equality always holds if the sequence is split.*

iii) If P *is a prime ideal such that* $PM=0$ *and* M *is a torsion* A/P*-module, then* $\delta(M) + 1 \leq \delta(R/P)$*.*

If the equality in ii) is always true, δ *is called an exact dimension.*

A module M will be called homogeneous if $\delta(M) = \delta(M')$ for all non-zero submodules M'. In fact, given a dimension function δ, we may always construct an exact dimension function, δ^* by setting $\delta^*(M) = \max\{\delta(N) | N \text{ is a subfactor of } M\}$.

Definition 2.3.2. *Let* A *be a Noetherian ring. A dimension function will be called finitely partitive if for every finitely generated* A*-module, there is an integer* n *such that whenever* $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots M_i \supseteq \dots$ *is a descending chain of* A*-submodules, we have* $\delta(M_i/M_{i+1}) < \delta(M)$ *for all* $i \geq n$*.*

2.3.1 Krull dimension

We first introduce the Krull dimension; this is a dimension that measures how close a module or a ring is to being Artinian. In commutative algebra, we define the Krull dimension as the maximum length of a chain of prime ideals. This is not a good measure in non-commutative ring theory, as there are simple Noetherian rings which are not Artinian. As an example, one may consider the family of Weyl Algebras defined over a field of characteristic 0.

Let I be a partially ordered set. If $a, b \in I$ with $a \geq b$, we define $a/b = \{x \in I | a \geq x \geq b\}$ to be the factor of a by b. By a descending chain $\{a_n\}$ of elements in I, we mean a sequence $a_1 \geq a_2 \geq \dots$ and the factors a_i/a_{i+1} are the factors of the chain. We say that I satisfies the descending chain condition if every descending chain is eventually constant.

Definition 2.3.3. *Let* I *be a partially ordered set. If* I *is trivial, then* I *has deviation* $-\infty$*. If* I *satisfies the descending chain condition, then* $\operatorname{dev} I = 0$*. If* α *is a general ordinal, we say that* $\operatorname{dev} I = \alpha$ *if* $\operatorname{dev} I \neq \beta < \alpha$ *and in any descending chain of* I *all but finitely many factors have deviation less than* α*.*

Proposition 2.3.4. *[43, Proposition 1.8] Let* I *be a partially ordered set that satisfies the ascending chain condition. Then* I *has a deviation.*

We may now define the Krull dimension for finitely generated modules over Noetherian rings. Given a R-module M, we may view its lattice of submodules $\mathcal{L}(M)$ as a partially ordered set.

Definition 2.3.5. *Let A be a Noetherian ring and M a left A-module, we define its Krull dimension* $\mathrm{Kdim}_A(M)$ *to be the deviation of $\mathcal{L}(M)$. We define the Krull dimension of A to be* $\mathrm{Kdim}_A(A)$*, where we view A as a left A-module.*

Lemma 2.3.6. *Let M be a finitely generated A-module. Then* $\mathrm{Kdim}_A(M)$ *is defined.*

Proof. This follows from Proposition 2.3.4. □

Given a finitely generated A-module $M \neq 0$, we deduce immediately that M is Artinian if and only if $\mathrm{Kdim}_A(M) = 0$.

2.3.2 Auslander-Gorenstein rings

Definition 2.3.7. *Let A be a Noetherian ring. We say that a finitely generated A-module M satisfies the Auslander's condition if for every $i \geq 0$ and for every submodule N of* $\mathrm{Ext}^i_A(M, A)$*, we have* $\mathrm{Ext}^j_A(N, A) = 0$ *for all $j < i$.*

We say that A is Auslander-Gorenstein if the left and right self-injective dimension of A is finite and every finitely generated (left or right) A-module satisfies the Auslander's condition.

We should also note that commutative Auslander-Gorenstein rings are exactly the Gorenstein rings.

Definition 2.3.8. *Let A be a Noetherian ring and M a finitely generated A-module. We define the grade of M to be*

$$j_A(M) = \inf\{i | \mathrm{Ext}^i_A(M, A) \neq 0\}.$$

If A is an Auslander-Gorenstein ring we define the canonical dimension of M to be

$$d_A(M) = inj.dim_A(A) - j_A(M).$$

We say that a module M is pure if it is homogeneous with respect to the canonical dimension.

Proposition 2.3.9. *Let A be an Auslander-Gorenstein ring. Then the canonical dimension, d_A, is a finitely partitive exact dimension function.*

Proof. This follows from [41, Proposition 4.5]. ☐

2.4 Algebraic groups

Throughout this section, R denotes a commutative ring.

Definition 2.4.1. *We call a representable functor $G : \{R\text{-algebras}\} \to \{\text{groups}\}$ an affine group scheme. We use the notation $R[G]$ to denote the R-algebra representing G.*

By viewing an affine scheme as a functor from R-algebras to sets, we may view G as a scheme via the forgetful functor from groups to sets.

Example 2.4.2. *The functor G_a defined by $G_a(A) = (A, +)$ is represented by the polynomial ring $R[X]$.*

The functor G_m defined by $G_m(A) = (A^*, \cdot)$ is represented by the $R[X,Y]/(XY - 1)$.

The functor GL_n defined by $GL_n(A) = \{$ invertible $n \times n$ matrices with entries in the ring $A\}$ is representable.

It turns out that the algebra $R[G]$ has a natural structure of a Hopf Algebra. In fact, it follows from [60, Theorem 1.4] that there is a one-to-one correspondence between R-Hopf Algebras and group schemes over R.

Definition 2.4.3. *An affine algebraic group scheme G is a subfunctor of GL_n, i.e. for any R-algebra A, $G(A)$ is a subgroup of $GL_n(A)$.*

Most of this thesis will be about studying a special class of affine algebraic group schemes called semisimple. Before we define semisimple group schemes, let us introduce some language.

Definition 2.4.4. *We say that an algebraic group scheme G is connected if it has no proper open subgroup scheme.*

We call G smooth if the corresponding morphism $G \to \operatorname{Spec} R$ is smooth.

A homomorphism of connected smooth algebraic group schemes $G' \to G$ is called an isogeny if it is surjective with finite kernel.

We call an algebraic group scheme G simply-connected if G is connected smooth and any isogeny $G' \to G$ is an isomorphism.

Definition 2.4.5. *We call a group scheme G semisimple if its centre $Z(G)$ is a finite group scheme.*

We say that G is split semisimple if it has a torus T inside a Borel subgroup $B \subset G$ such that $B = T[B, B]$, where $[B, B]$ denotes the unipotent radical of B.

There is a wonderful theory of split-semisimple algebraic groups. In particular, these groups have a root datum. It turns out that a connected semisimple algebraic group defined over an algebraically closed field is completed determined by its root datum. This is explained in detail in [36, Chapter II.1].

2.4.1 Representations of algebraic groups

We call a functor $F : \{R\text{-algebras}\} \to \{\text{sets}\}$ an R-functor. For an R-module M, we denote by M_a the R-functor $M_a(A) = A \underset{R}{\otimes} M$ for all R-algebras A.

Definition 2.4.6. *Let G be an affine algebraic group scheme and M an R-module. We call M a G-module (or G-representation) if $G(A)$ acts on $M_a(A) = A \underset{R}{\otimes} M$ through A-linear maps for any R-algebra A.*

There is a natural notion of a comodule over a Hopf Algebra [46, Chapter I]. It turns out that G-modules correspond to $R[G]$-comodules:

Proposition 2.4.7. *[36, Section I.2.8]*

The category of G-modules is equivalent to the category of $R[G]$-comodules.

We should remark that when R is an algebraically closed field and G is a reduced affine group scheme, there is a natural notion of a rational $G(R)$-representation. It turns out that this notion coincides with the one in Definition 2.4.6.

2.4.2 The Lie algebra of an algebraic group

Definition 2.4.8. *Let G be an affine algebraic group scheme and define the ring of dual numbers $R[\epsilon] := R[X]/(X^2)$. The Lie algebra of G, denoted by* $\mathrm{Lie}(G)$ *is defined to be:*

$$\mathrm{Lie}(G) = \ker(G(R[\epsilon]) \to G(R)).$$

We also provide an alternatively description of the Lie algebra of G. For an R-algebra A and an A-module M, we call an additive map $D : A \to M$ an R-derivation if it satisfies $D(ab) = aD(b) + bD(a)$ for all $a, b \in A$ and $D(R) = 0$. When $M = A$, we simply call D an A-derivation.

Assume that A is a Hopf Algebra with comodule map Δ. We say that an A-derivation D is left-invariant if $\Delta D = (\mathrm{id} \otimes D)\Delta$.

Proposition 2.4.9. *[60, Theorem 12.2] The following are isomorphic as Lie algebras:*

- $\mathrm{Lie}(G)$.
- *The left-invariant derivations on* $R[G]$.
- *The space of derivations of* $R[G]$ *into* R.

We call a Lie algebra semisimple if it has no non-trivial Abelian ideals. When R is an algebraically closed field of characteristic 0, there is a one-to-one correspondence between semisimple Lie algebras over R and connected, split-semisimple group schemes over R.

A very important tool in studying Lie algebras is the universal enveloping algebra $\mathfrak{g}$.

Definition 2.4.10. *Let* $\mathfrak{g}$ *be a Lie algebra and let* $T(\mathfrak{g})$ *be the tensor algebra of* $\mathfrak{g}$, *i.e.,* $T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} T^n(\mathfrak{g})$, *where* $T^n(\mathfrak{g}) = \bigotimes^n \mathfrak{g}$. *Further, let* I *be a two-sided ideal of* $T(\mathfrak{g})$ *generated by* $[x, y] - x \otimes y - y \otimes x$. *The universal enveloping algebra of* $\mathfrak{g}$, $U(\mathfrak{g})$, *is defined to be:*

$$U(\mathfrak{g}) = T(\mathfrak{g})/I.$$

The universal enveloping algebra satisfies the following universal property, there exists a unique Lie Algebra homomorphism $\epsilon : \mathfrak{g} \to U(\mathfrak{g})$ such that for all associative R-algebras A and all Lie algebra homomorphism $\alpha : \mathfrak{g} \to A$, there exists a unique ring homomorphism $f : U(\mathfrak{g}) \to A$ such that the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\epsilon"] \arrow[d, "\alpha"] & U(\mathfrak{g}) \arrow[dl, "f"] \\
A &
\end{tikzcd}$$

commutes. The importance of the enveloping algebra follows from the following simple lemma: the identity functor provides a one-to-one correspondence between Lie algebra representations of $\mathfrak{g}$ and $U(\mathfrak{g})$-modules. Therefore, we can use tools from non-commutative ring theory to study the representation theory of $U(\mathfrak{g})$.

The ring $U(\mathfrak{g})$ can also be defined via generators and relations. Assume that $\mathfrak{g}$ is a free R-module of rank d with basis $u_1, u_2, \dots u_d$. Then the enveloping algebra can be described as:

$$U(\mathfrak{g}) = R\langle u_1, u_2 \dots u_d \rangle / I,$$

where $R\langle u_1, u_2, \dots u_d \rangle$ denotes the free algebra and I is the two-sided ideal generated by $u_i u_j - u_j u_i - [u_i, u_j]$ for $1 \leq i, j \leq d$.

Let us finish this section by stating the famous PBW theorem.

Theorem 2.4.11 (PBW theorem)**.** *Assume that* $\mathfrak{g}$ *is a free* R*-module of finite rank with basis* $u_1, u_2, \dots u_d$*. Then* $U(\mathfrak{g})$ *is a free* R*-module with basis*

$$u_1^{n_1} u_2^{n_2} \dots u_d^{n_d}, \text{ for } n_1, n_2 \dots n_d \in \mathbb{N}.$$

Using the PBW theorem, we define a filtration on $U(\mathfrak{g})$ by $F_0 U(\mathfrak{g}) = R$, $F_1 U(\mathfrak{g}) = \mathfrak{g}$ and $F_i U(\mathfrak{g}) = u_1^{n_1} u_2^{n_2} \dots u_d^{n_d}$, with $n_1 + n_2 \dots + n_d \leq i$. Therefore, we may form the graded ring

$$\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} F_{i+1} U(\mathfrak{g}) / F_i U(\mathfrak{g}).$$

Proposition 2.4.12. *We have an isomorphism* $\operatorname{gr} U(\mathfrak{g}) \cong R[u_1, u_2 \dots, u_d]$*. In particular,* $\operatorname{gr} U(\mathfrak{g})$ *is Noetherian if and only if* R *is Noetherian.*

Chapter 3

Lie algebroids and twisted differential operators

3.1 Introduction

This chapter is based on [58]; the paper has been submitted for publication. The aim of this chapter is to establish a good framework to use geometric representation theory to obtain in Chapter 4 a new proof of the classical Duflo's theorem [25]:

Theorem 3.1.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra defined over a field K of characteristic 0. Then any primitive ideal in $U(\mathfrak{g})$ with K-rational infinitesimal central character is the annihilator of the simple quotient of some Verma module. In case $K = \mathbb{C}$, the theorem gives a classification of all primitive ideals.*

Assume for now that our ground field is $\mathbb{C}$ and let $\mathfrak{g}$ be a semisimple $\mathbb{C}$-Lie algebra. For a complex variety X, we will denote $\mathcal{D}_X$ the sheaf of differential operators on X. Let G be the semisimple connected affine algebraic group associated with $\mathfrak{g}$ and fix B a Borel subgroup. Let $X = G/B$ be the flag variety associated to $\mathfrak{g}$. In [17, Proposition 3.6], the authors prove an equivalence of categories between G-equivariant coherent $\mathcal{D}_{X\times X}$-modules and B-equivariant coherent $\mathcal{D}_X$-modules. Further, the Beilinson-Bernstein theorem [8] establishes an equivalence of categories between coherent $\mathcal{D}_X$-modules and finitely generated $U(\mathfrak{g})$-modules with trivial central character. Combining these two results, one obtains a geometric proof of Duflo's theorem for ideals with *trivial central character*. In the next chapter, we remove some restrictions from [17, Proposition 3.6]: we prove that the results hold over a general commutative Noetherian ring and that the equivalence holds between coherent modules over certain *homogeneous sheaves of twisted differential operators*, which can be regarded as *equivariant* twisted differential operators.

There is a well established theory of twisted differential operators and homogeneous twisted differential operators over a *complex* variety introduced in [9] and treated in more detail in [45]. The authors also explore the connections between twisted differential operators(tdo's) and Picard algebroids and the following question is answered:

Question 3.1.2. *Let $f : Y \to X$ be a map of smooth complex varieties and let $\mathcal{D}$ be a tdo on X. How should we define the pullback of $\mathcal{D}$, call it $f^{\bullet}\mathcal{D}$, such that $f^{\bullet}\mathcal{D}$ is a tdo on Y?*

The solution proposed in [9] was to define $f^{\bullet}\mathcal{D}$ using the Dif functor introduced by Grothendieck:

$$f^{\bullet}\mathcal{D} := \mathrm{Dif}_{f^{-1}\mathcal{D}}(f^{*}\mathcal{D}, f^{*}\mathcal{D})$$

is the sheaf of differential operators from $f^{*}\mathcal{D}$ to itself that commute with the right $f^{-1}\mathcal{D}$-action. In particular when $\mathcal{D} = \mathcal{D}_X$, we obtain $f^{\bullet}\mathcal{D}_X = \mathcal{D}_Y$.

Now, let R be a commutative base ring and X be an R-scheme that is smooth, separated and locally of finite type. In order to build a good theory of twisted differential operators over a commutative ring, there are two basic questions we need to answer:

Question What constitutes a good definition of a tdo on X?

Question Given $f : Y \to X$ a map of smooth, separated and locally of finite type R-schemes and $\mathcal{D}$ a tdo on X, how should we define the pullback of $\mathcal{D}$ such that it is also a tdo on Y?

There are two possible candidates of sheaves of differential operators that one can define over X: $\mathcal{D}_X$-the sheaf of crystalline differential operators and $\mathscr{D}_X$-the sheaf of Grothendieck's differential operators.

One of the key properties satisfied by twisted differential operators over complex varieties is that they come equipped with a filtration such that the associated graded is isomorphic with the symmetric algebra of the tangent sheaf over the ring of functions. The sheaf of Grothendieck's differential operators $\mathscr{D}_X$ has a natural filtration given by the order of differential operators, but the associated grading ring does not satisfy the desired property. Therefore, we choose to work with the sheaf of crystalline differential operators.

Attempting working with the classical definition of the pullback of tdo's we immediately encounter a problem. Assume that $Y = \mathbb{A}^1$ is the affine line over R, let $X = \operatorname{Spec} R$ be the base of Y and $f : Y \to X$ be the natural projection. Let $\mathcal{D}_X \cong R$ be the sheaf of crystalline differential operators on X. Then using the classical definition we obtain

$$f^{\bullet}\mathcal{D} \cong \mathscr{D}_{\mathbb{A}^1},$$

which is the sheaf of *Grothendieck's differential operators*. In particular, if we work with the classical definition we obtain that the pullback of a twisted differential operator does not satisfy the desired property.

To resolve the problem with the definition, we will explore the correspondence between twisted differential operators (shortened tdo's frow now on) and *Picard algebroids* and define the pullback of tdo's by first defining the pullback of Picard algebroids.

Specifically, we will be able to go from tdo's to Picard algebroids and vice-versa using two functors Lie and $\mathscr{T}$. For a Picard algebroid $\mathcal{L}$ on X, we can consider the pullback $f^{\#}\mathcal{L}$ as a Lie algebroid on Y and for a general tdo on X we define

$$f^{\#}\mathcal{D} := \mathscr{T}(f^{\#}(\mathrm{Lie}(\mathcal{D}))).$$

In the case when $R = \mathbb{C}$, our definition coincides with the definition found in [9] and [45].

Statement of the main results of the chapter

Let G be a smooth affine algebraic group of finite type over $\operatorname{Spec} R$ and let $f : Y \to X$ be a locally trivial G-torsor. Then for a tdo $\mathcal{D}$ equipped with a suitable G-action (we call this a G-htdo), we may define its descent $f_{\#}\mathcal{D}^G$, which is a tdo on X.

Proposition 3.1.3 (Corollary 3.10.12)**.** *The functors $f_{\#}(-)^G$ and $f^{\#}(-)$ induce quasi-inverse equivalences between G-htdo's on Y and tdo's on X.*

For a G-htdo $\mathcal{D}$ we may define the $\mathrm{Coh}(\mathcal{D}, G)$ the category of G-equivariant coherent $\mathcal{D}$-modules. Further, let $\mathcal{A}$ be a tdo on X and $\mathcal{M}$ a $\mathcal{A}$-module. Then we may endow the $\mathcal{O}$-module pullback $f^{*}\mathcal{M}$ with the structure of a $f^{\#}\mathcal{A}$-module and we call this module $f^{\#}\mathcal{M}$.

Theorem 3.1.4 (Theorem 3.11.8)**.** *Assume that R is a Noetherian ring. Let G be a smooth affine algebraic group of finite type. Let X, Y be smooth separated and locally of finite type R-schemes and let $f : Y \to X$ be a locally trivial G-torsor. Further, let $\mathcal{D}$ be a sheaf of G-homogeneous twisted differential operators on Y. The functors:*

$$\begin{aligned} f_*(-)^G &: \mathrm{Coh}(\mathcal{D}, G) \to \mathrm{Coh}(f_\#\mathcal{D}^G) \\ f^\#(-) &: \mathrm{Coh}(f_\#\mathcal{D}^G) \to \mathrm{Coh}(\mathcal{D}, G). \end{aligned} \tag{3.1}$$

are quasi-inverse equivalences of categories between coherent G-equivariant $\mathcal{D}$-modules and coherent $(f_\#\mathcal{D})^G$-modules.

In fact, we generalise the proposition and the theorem in two directions. First, for $r \in R$ a regular element, we define the notion of r-deformed Picard algebroids and r-deformed G-htdo's. Then we may prove that the descent of an r-deformed G-htdo is an r-deformed tdo and similarly the theorem holds for coherent modules over r-deformed G-htdo's.

Secondly, for another smooth affine algebraic group of finite type B acting on X and Y, we prove that for a $G \times B$-htdo on Y, its descent under a B-equivariant G-torsor is a B-htdo. A similar result also holds for the corresponding coherent $G \times B$-modules. This generality will be needed in future applications: in [56] we use this framework to give a geometric proof of 3.1.1 and in [57] we prove an affinoid version of the same theorem. The proofs of these theorems will be the highlights of the next chapters.

Structure of the chapter

In Sections 3.2 and 3.3, we review the theory of equivariant $\mathcal{O}$-modules and equivariant descent for a locally trivial torsor. Next, we define in Section 3.4 the sheaf of crystalline differential operators and the notion of r-deformed twisted differential operators on a smooth, separated and locally of finite type R-scheme. In the next two sections, we establish correspondences between r-deformed Picard algebroids/equivariant Picard algebroids and r-deformed twisted differential operators/homogeneous twisted differential operators. We then define the pullback of r-deformed Picard algebroids and r-deformed tdo's in Section 3.7. In the next two sections, we explore the connections between modules over r-deformed Picard algebroids and r-deformed twisted differential operators.

Finally, in Section 3.10, we prove equivariant descent for r-deformed homogeneous twisted differential operators (Proposition 3.1.3) and in Section 3.11, we prove equivariant descent for modules over r-deformed homogeneous twisted differential operators (Theorem 3.1.4).

Conventions

Throughout this chapter, R will denote a commutative ring of arbitrary characteristic and all the schemes will be R-schemes. For a map $f : Y \to X$ of R-schemes, we will denote f^* the pullback in the category of $\mathcal{O}$-modules and f_* the pushforward sheaf. Unadorned tensor products will be assumed to be taken over R. An element $r \in R$ is called regular if it is not a zero divisor.

3.2 Equivariant $\mathcal{O}$-modules

Let G be an affine algebraic group scheme acting on a scheme X; denote the action by $\sigma_X : G \times X \to X$. Furthermore, we denote $p_X : G \times X \to X$ and $p_{2X} : G \times G \times X \to X$ the projections onto the X factor, $p_{23X} : G \times G \times X \to G \times X$ the projection onto the second and third factor and $m : G \times G \to G$ the multiplication of the group G.

Definition 3.2.1. *Let G an algebraic group scheme acting on a scheme X. A G-equivariant $\mathcal{O}_X$-module is a pair $(\mathcal{M}, \alpha)$, where $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module and $\alpha : \sigma_X^*\mathcal{M} \to p_X^*\mathcal{M}$ is an isomorphism of $\mathcal{O}_{G\times X}$-modules such that the diagram*

$$\begin{tikzcd}
(1_G \times \sigma_X)^* p_X^* \mathcal{M} \arrow[r, "p_{23X}^*\alpha"] & p_{2X}^* \mathcal{M} \\
(1_G \times \sigma_X)^* \sigma_X^* \mathcal{M} \arrow[u, "(1_G\times\sigma_X)^*\alpha"] \arrow[r, leftrightarrow, "id"] & (m \times 1_X)^* \sigma_X^* \mathcal{M} \arrow[u, "(m\times 1_X)^*\alpha"]
\end{tikzcd}$$

of $\mathcal{O}_{G\times G\times X}$-modules commutes (the cocycle condition) and the pullback

$$(e \times 1_X)^*\alpha : \mathcal{M} \to \mathcal{M}$$

is the identity map.

We prove a crucial lemma that will be used in the future; it is stated on the stack project, but the proof is omitted.

Lemma 3.2.2. *[54, 03LG]*

Let G be an affine algebraic group acting on schemes X and Y and let $f : Y \to X$ be a G-equivariant morphism. Then the pullback functor f^* given by

$$(\mathcal{M}, \alpha) \mapsto (f^*\mathcal{M}, (1_G \times f)^*\alpha)$$

defines a functor from G-equivariant $\mathcal{O}_X$-modules to G-equivariant $\mathcal{O}_Y$-modules.

Proof. Let $\mathcal{M}$ be a G-equivariant $\mathcal{O}_X$-module. Since $\alpha : \sigma_X^*\mathcal{M} \to p_X^*\mathcal{M}$ is an isomorphism, we get that $(1_G \times f)^*\alpha : (1_G \times f)^*\sigma_X^*\mathcal{M} \to (1_G \times f)^*p_X^*\mathcal{M}$ is also an isomorphism. We have

$$\begin{gathered}
\sigma_X \circ (1_G \times f)(g, y) = gf(y) = f(gy) = f \circ \sigma_Y(g, y), \text{ so } (1_G \times f)^*\sigma_X^*\mathcal{M} = \sigma_Y^* f^*\mathcal{M}. \\
p_X \circ (1_G \times f)(g, y) = f(y) = f \circ p_Y, \text{ so } (1_G \times f)^*p_X^*\mathcal{M} = p_Y^* f^*\mathcal{M}.
\end{gathered} \tag{3.2}$$

Thus, $(1_G \times f)^*\alpha : \sigma_Y^*(f^*\mathcal{M}) \to p_Y^*(f^*\mathcal{M})$ is an isomorphism. Next, we need to prove that the morphism $(1_G \times f)^*\alpha$ satisfies the cocycle condition; that is we need to show that the diagram

$$\begin{tikzcd}
(1_G \times \sigma_Y)^* p_Y^* f^*\mathcal{M} \arrow[r, "p_{23Y}^*(1_G \times f)^*\alpha"] & p_{2Y}^* f^*\mathcal{M} \\
(1_G \times \sigma_Y)^* \sigma_Y^* f^*\mathcal{M} \arrow[u, "(1_G \times \sigma_Y)^*(1_G \times f)^*\alpha"] \arrow[r, leftrightarrow, "\mathrm{id}"] & (m \times 1_Y)^* \sigma_Y^* f^*\mathcal{M} \arrow[u, "(m \times 1_Y)^*(1_G \times f)^*\alpha"]
\end{tikzcd} \tag{3.3}$$

of $\mathcal{O}_{G \times G \times Y}$-modules commutes given that the diagram

$$\begin{tikzcd}
(1_G \times \sigma_X)^* p_X^*\mathcal{M} \arrow[r, "p_{23X}^*\alpha"] & p_{2X}^*\mathcal{M} \\
(1_G \times \sigma_X)^* \sigma_X^*\mathcal{M} \arrow[u, "(1_G \times \sigma_X)^*\alpha"] \arrow[r, leftrightarrow, "\mathrm{id}"] & (m \times 1_X)^* \sigma_X^*\mathcal{M} \arrow[u, "(m \times 1_X)^*\alpha"]
\end{tikzcd} \tag{3.4}$$

of $\mathcal{O}_{G \times G \times X}$-modules commutes.

We shall prove that the diagram 3.3 is the pullback of the diagram 3.4 under the morphism $(1_G \times 1_G \times f)$. We have that

$$\begin{aligned}
&f \circ p_Y \circ (1_G \times \sigma_Y)(g_1, g_2, y) = f(g_2 y) = p_X \circ (1_G \times \sigma_X) \circ (1_G \times 1_G \times f)(g_1, g_2, y).\\
&f \circ p_{2Y}(g_1, g_2, y) = y = p_{2X} \circ (1_G \times 1_G \times f)(g_1, g_2, y).\\
&f \circ \sigma_Y \circ (1_G \times \sigma_Y)(g_1, g_2, y) = f(g_1 g_2 y) = \sigma_X \circ (1_G \times \sigma_X) \circ (1_G \times 1_G \times f)(g_1, g_2, y).\\
&f \circ \sigma_Y \circ (m \times 1_Y)(g_1, g_2, y) = g_1 g_2 f(y) = \sigma_X \circ (m \times 1_X) \circ (1_G \times 1_G \times f)(g_1, g_2, y).
\end{aligned}\tag{3.5}$$

Therefore,

$$\begin{aligned}
((1_G \times \sigma_Y)^* p_Y^*) f^* \mathcal{M} &= (1_G \times 1_G \times f)^* ((1_G \times \sigma_X)^* p_X^* \mathcal{M}).\\
p_{2Y}^* (f^* \mathcal{M}) &= (1_G \times 1_G \times f)^* (p_{2X}^* \mathcal{M}).\\
((1_G \times \sigma_Y)^* \sigma^*) f^* \mathcal{M} &= (1_G \times 1_G \times f)^* ((1_G \times \sigma_X)^* \sigma_X^* \mathcal{M}).\\
((m \times 1_Y)^* \sigma^*) f^* \mathcal{M} &= (1_G \times 1_G \times f)^* ((m \times 1_X)^* \sigma_X^* \mathcal{M}).
\end{aligned}\tag{3.6}$$

Similarly, we get

$$\begin{aligned}
&(1_G \times f) \circ p_{23Y}(g_1, g_2, y) = (g_2, f(y)) = p_{23X} \circ (1_G \times 1_G \times f)(g_1, g_2, y).\\
&(1_G \times f) \circ (1_G \times \sigma_Y)(g_1, g_2, y) = (g_1, f(g_2 y)) = (1_G \times \sigma_X) \circ (1_G \times 1_G \times f)(g_1, g_2, y).\\
&(1_G \times f) \circ (m \times 1_Y)(g_1, g_2, y) = (g_1 g_2, f(y) = (m \times 1_X) \circ (1_G \times 1_G \times f)(g_1, g_2, y).
\end{aligned}\tag{3.7}$$

Thus,

$$\begin{aligned}
p_{23Y}^* (1_G \times f)^* \alpha &= (1_G \times 1_G \times f)^* p_{23X}^* \alpha.\\
(1_G \times \sigma)^* (1_G \times f)^* \alpha &= (1_G \times 1_G \times f)^* (1_G \times \sigma_X)^* \alpha.\\
(m \times 1_X)^* (1_G \times f)^* \alpha &= (1_G \times 1_G \times f)^* (m \times 1_X)^* \alpha.
\end{aligned}\tag{3.8}$$

Combining equations (3.6) and (3.8), we get that the diagram (3.3) is indeed the pullback of the diagram (3.4) under the morphism $1_G \times 1_G \times f$, so $(1_G \times f)^* \alpha$ satisfies the cocycle condition.

Finally, we need to prove that the map $(e \times 1_Y)^* (1_G \times f)^* \alpha : f^* \mathcal{M} \to f^* \mathcal{M}$ is the identity map using $(e \times 1_X)^* \alpha : \mathcal{M} \to \mathcal{M}$ is the identity map.

We have that $(1_G \times f) \circ (e \times 1_Y)(g, y) = (e, f(y)) = (e \times 1_X)(1_G \times f)$, so

$$(e \times 1_Y)^*(1_G \times f)^*\alpha = (1_G \times f)^*(e \times 1_X)^*\alpha = (1_G \times f)^*(\mathrm{id}) = \mathrm{id},$$

since the identity map is preserved by any functor. □

Definition 3.2.3. *Let G an affine algebraic group acting on a scheme X via σ_X. We define the category of G-equivariant quasi-coherent $\mathcal{O}_X$-modules. Objects are given by G-equivariant $\mathcal{O}_X$-modules.*

A morphism of G-equivariant $\mathcal{O}_X$-modules $(\mathcal{M}, \alpha_M)$ and $(\mathcal{N}, \alpha_N)$ is a map $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ such that the following diagram commutes:

$$\begin{tikzcd}
\sigma_X^*\mathcal{M} \arrow[r, "\alpha_M"] \arrow[d, "\sigma_X^*\phi"] & p_X^*\mathcal{M} \arrow[d, "p_X^*\phi"] \\
\sigma_X^*\mathcal{N} \arrow[r, "\alpha_N"] & p_X^*\mathcal{N}.
\end{tikzcd}$$

We call such a morphism G-equivariant and denote the category of G-equivariant $\mathcal{O}_X$-modules together with G-equivariant morphisms by $\mathrm{QCoh}(\mathcal{O}_X, G)$.

Proposition 3.2.4. *Let G an affine algebraic group acting on a scheme X via σ_X. Then the category $\mathrm{QCoh}(\mathcal{O}_X, G)$ is Abelian.*

Proof. By construction, we have that $\mathrm{QCoh}(\mathcal{O}_X, G)$ is additive. Let $(\mathcal{M}, \alpha_M)$ and $(\mathcal{N}, \alpha_N) \in \mathrm{QCoh}(\mathcal{O}_X, G)$ and let $\phi \in \mathrm{Hom}_{\mathrm{QCoh}(\mathcal{O}_X,G)}(\mathcal{M}, \mathcal{N})$. Consider the exact sequence:

$$0 \to \ker(\phi) \to \mathcal{M} \to \mathcal{N} \to \mathrm{coker}(\phi) \to 0.$$

We aim to prove that $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are in $\mathrm{QCoh}(\mathcal{O}_X, G)$. Since σ_X and p_X are smooth morphisms, the pullback functors σ_X^* and p_X^* are exact, so we get two short exact sequences:

$$\begin{aligned}
&0 \to \sigma_X^* \ker(\phi) \to \sigma_X^*\mathcal{M} \to \sigma_X^*\mathcal{N} \to \sigma_X^* \mathrm{coker}(\phi) \to 0, \\
&0 \to p_X^* \ker(\phi) \to p_X^*\mathcal{M} \to p_X^*\mathcal{N} \to p_X^* \mathrm{coker}(\phi) \to 0.
\end{aligned} \tag{3.9}$$

Consider now the diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \sigma_X^* \ker(\phi) \arrow[r] & \sigma_X^*\mathcal{M} \arrow[r] \arrow[d, "\alpha_{\mathcal{M}}"] & \sigma_X^*\mathcal{N} \arrow[d, "\alpha_{\mathcal{N}}"] \\
0 \arrow[r] & 0 \arrow[r] & p_X^* \ker(\phi) \arrow[r] & p_X^*\mathcal{M} \arrow[r] & p_X^*\mathcal{N}.
\end{tikzcd}$$

By construction, we have that $\alpha_{\mathcal{M}}$ and $\alpha_{\mathcal{N}}$ are isomorphisms, so by the Five Lemma we obtain an isomorphism $\beta : \sigma_X^* \ker(\phi) \to p_X^* \ker(\phi)$. Furthermore, since $\alpha_{\mathcal{M}}$ and $\alpha_{\mathcal{N}}$ satisfy the cocycle condition, so does β. Thus, we have proven that $\ker(\phi) \in \mathrm{QCoh}(\mathcal{O}_X, G)$. A similar argument applying the Five Lemma shows that $\mathrm{coker}(\phi) \in \mathrm{QCoh}(\mathcal{O}_X, G)$. Finally, by construction we have that any monorphism and epimorphism in $\mathrm{QCoh}(\mathcal{O}_X, G)$ is normal, so $\mathrm{QCoh}(\mathcal{O}_X, G)$ is indeed Abelian category. $\square$

From now on, when we use the notion of morphism of G-equivariant $\mathcal{O}_X$-modules, we always view it as a morphism in the category $\mathrm{QCoh}(\mathcal{O}_X, G)$.

A reformulation of equivariance

We wish to reformulate the notion of an equivariant $\mathcal{O}$-module. Until the end of the section, we fix X a scheme defined over R acted on by an affine algebraic group G. For any R-algebra A, we define $X_A := \operatorname{Spec} A \times_{\operatorname{Spec} R} X$. We start with a very simple observation: viewing $\mathcal{O}_X$ as a left $\mathcal{O}_X$-module, $(\mathcal{O}_X, \mathrm{id})$ is a G-equivariant $\mathcal{O}_X$-module. We may reformulate this following ideas in [11]: for each R-algebra A inducing a map $s : \operatorname{Spec} A \to \operatorname{Spec} R$ and for each geometric point $i_g : \operatorname{Spec} A \to G$ which induces an automorphism $g : X_A \to X_A$ there exists an isomorphism

$$q_g : s^*\mathcal{O} \to (g^{-1})^* s^*\mathcal{O}, \text{ satisfying}$$

$$q_e = \mathrm{id} \text{ and } q_{gh} = (g^{-1})^*(q_h) q_g \tag{3.10}$$

in such a way that (q_g)'s are compatible with base change. Let $r_g = g^* \circ q_g$. For each $U \subset X_A$ affine open, r_g induces a map $\mathcal{O}_{X_A}(U) \to \mathcal{O}_{X_A}(g^{-1}U)$. The equation (3.10) translates as $r_e = \mathrm{id}$ and $r_{gh} = r_h r_g$. Furthermore, the $\mathcal{O}$-module compatibility requires that for any $f_1, f_2 \in \mathcal{O}_{X_A}(U)$, we have $r_g(f_1 f_2) = r_g(f_1) r_g(f_2)$.

We define r_g via $r_g(f)(x) = f(g^{-1}x)$ for all R-algebras A, $U \subset X_A$ affine open, $x \in U$, $f \in \mathcal{O}_{X_A}(U)$, $g : X_A \to X_A$ and it is easy to see that r_g's make $\mathcal{O}_X$ a G-equivariant $\mathcal{O}_X$-module according to equation (3.10). We may now make an abuse of notation: for each $i_g : \operatorname{Spec} A \to G$ and each $f \in \mathcal{O}_{X_A}(U)$, we denote $g.f = r_{g^{-1}}(f)$ and we translate the equivariance structure as

$$e.f_1 = f, \quad g.(h.f_1) = (gh).f_1, \quad g.(f_1 f_2) = (g.f_1)(g.f_2) \text{ for all } g, h \in G, f_1, f_2 \in \mathcal{O}_X.$$

Lemma 3.2.5. *A $\mathcal{O}_X$-module $\mathcal{M}$ is G-equivariant if and only if for each R-algebra A, for each $s : \operatorname{Spec} A \to \operatorname{Spec} R$ and for each geometric point $i_g : \operatorname{Spec} A \to G$ which induces an automorphism $g : X_A \to X_A$ there exists an isomorphism of $\mathcal{O}_A$-modules*

$$q_g : s^*\mathcal{M} \to (g^{-1})^* s^*\mathcal{M}$$

satisfying

$$q_e = \mathrm{id} \ \textit{ and } q_{gh} = (g^{-1})^*(q_h)q_g \tag{3.11}$$

in such a way that (q_g)'s are compatible with base change.

Proof. The proof repeats the argument in [11, Proposition 2.2/Proposition 1.3.1] working over a commutative ring rather than a field and using the structure sheaf $\mathcal{O}$ instead of the sheaf of differential operators $\mathcal{D}$. □

Again, by setting $s_g = g^* \circ q_g$, we may reformulate equation (3.11) as: for each R-algebra A and for each $i_g : \operatorname{Spec} A \to G$, we have an isomorphism of $\mathcal{O}$-modules $s_g : \mathcal{M}_{X_A} \to \mathcal{M}_{X_A}$ such that for each $U \subset X_A$ affine open:

$$\begin{aligned} & s_e = \mathrm{id}, \\ & s_{gh} = s_h s_g, \\ & s_g's \text{ are compatible with base change}, \\ & r_g(f.m) = r_g(f).s_g(m), \text{ for all } f \in \mathcal{O}_{X_A}(U), m \in \mathcal{M}(U). \end{aligned} \tag{3.12}$$

Again, we make an abuse of notation: for each $i_g : \operatorname{Spec} A \to G$ and each $m \in \mathcal{M}_A(U)$, we denote $g.m = s_{g^{-1}}(m)$ and we translate the equivariance structure as:

$$\begin{aligned} & e.m = m, \\ & gh.m = g.(h.m), \\ & g.(f.m) = (g.f).(g.m), \end{aligned} \tag{3.13}$$

for all $g, h \in G$, $m \in \mathcal{M}$, $f \in \mathcal{O}_X$. Here, we used an abuse of notation: by $m \in \mathcal{M}$ we mean $m \in \mathcal{M}(U)$ for some open U.

Using the definition above, we reformulate the notion of G-equivariance of a morphism of G-equivariant $\mathcal{O}_X$-modules, $\phi : \mathcal{M} \to \mathcal{N}$ as:

$$g.\phi(m) = \phi(g.m) \text{ for all } g \in G, m \in \mathcal{M} \tag{3.14}$$

3.3 Equivariant descent for $\mathcal{O}$-modules

Definition 3.3.1. *[5, Section 4.3]*

Let G be a smooth affine algebraic group of finite type, Y a scheme equipped with an action $G \times Y \to Y$ and lastly, let X be a scheme. We say that a morphism $\xi : Y \to X$ is a G-torsor if ξ is faithfully flat and locally of finite type, ξ is a G-equivariant morphism, and the map

$$G \times Y \to Y \times_X Y, \qquad (g, y) \mapsto (gy, y)$$

is an isomorphism.

An open subscheme U of X is said to trivialise the torsor ξ if there is a G-invariant isomorphism

$$G \times U \to \xi^{-1}(U),$$

where G acts on $G \times U$ by left multiplication on the first factor.

Finally, let $\mathcal{S}_X$ be the set of affine open subschemes $U \subset X$ such that U trivialises ξ and $\mathcal{O}(U)$ is a finitely generated R-algebra. We say that ξ is a locally trivial torsor if it can be covered by opens in $\mathcal{S}_X$.

Definition 3.3.2 (definition-proposition)**.** *Let $\xi : Y \to X$ be a locally trivial G-torsor and let $(\mathcal{M}, \alpha_M)$ be a quasi-coherent G-equivariant $\mathcal{O}_Y$-module. Then the presheaf $(\xi_*\mathcal{M})^G$ acquires the structure of a quasi-coherent $\mathcal{O}_X$-module. Furthermore, if we are given $\psi : (\mathcal{M}, \alpha_M) \to (\mathcal{N}, \alpha_N)$ a map of G-equivariant $\mathcal{O}_Y$-modules there is a canonical induced map $(\xi_*)^G\psi : (\xi_*\mathcal{M})^G \to (\xi_*\mathcal{N})^G$.*

Proof. The question is local, so we may assume that X is affine, $\xi : Y \to X$ is $p_X : G \times X \to X$ and G acts on $G \times X$ via left multiplication on the first factor. Since G and X are affine, the category of G-equivariant $\mathcal{O}_{G\times X}$-modules is equivalent to the category of $(\mathcal{O}(G \times X), G)$-modules by the same arguments as in the proof of [11, Proposition 1.4.1]. Modules in this category are modules equipped with compatible actions of the ring $\mathcal{O}(G \times X)$ and of the group G.

Let $\mathcal{M}$ be a G-equivariant $\mathcal{O}_{G\times X}$-module and let $M = \Gamma(G \times X, \mathcal{M})$ be its global sections. Since M is a $(\mathcal{O}(G \times X), G)$-module it acquires a comodule structure $\rho_M : M \to \mathcal{O}(G) \underset{R}{\otimes} M$. Furthermore, let $\rho_G : \mathcal{O}(G \times X) \to \mathcal{O}(G) \underset{R}{\otimes} \mathcal{O}(G \times X)$ denote the comodule structure on $\mathcal{O}(G \times X)$ induced from the G-action on $G \times X$ by left multiplication on the first factor.

As X is affine, to prove the first statement, it is enough to prove that the Abelian group $(p_*\mathcal{M})^G(X) = M^G$ has the structure of an $\mathcal{O}(X)$-module.

Let $f \in \mathcal{O}(X)$ and define $\phi \in \mathcal{O}(G \times X)$ by $\phi(h, x) := f(x)$ for all $h \in G, x \in X$, and for any $m \in M^G$ define $f.m := \phi.m$

We need to prove that $f.m \in M^G$. By construction, it is clear that $\rho_G(\phi) = 1 \otimes \phi$. Since M is a $(\mathcal{O}(G \times X), G)$-module, we have

$$\rho_M(\phi.m) = \rho_G(\phi)\rho_M(m) = (1 \otimes \phi)(1 \otimes m) = 1 \otimes \phi.m,$$

so $\phi.m \in M^G$, thus the action of f is well-defined.

For the second statement, notice that if $\varphi : (M, \alpha_M) \to (N, \alpha_N)$ is a morphism of $(\mathcal{O}(G \times X), G)$-modules, it is G-equivariant, so φ restricts to a map $M^G \to N^G$.

In general if $\xi : \mathcal{M} \to \mathcal{N}$ is a map of G-equivariant $\mathcal{O}_Y$ modules and we let U be affine open in X, we have canonical maps $(\xi_*)^G\psi : (\xi_*\mathcal{M})^G(U) \to (\xi_*\mathcal{N})^G(U)$ compatible with restrictions given by restricting the map ψ. Therefore, glueing together the local morphisms we get a map of sheaves. □

In particular, we have proven that if $\xi : Y \to X$ is a locally trivial G-torsor, we obtain a functor ξ_*^G from G-equivariant $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules. We would like to prove that this is an equivalence of categories.

Recall that for an R-Hopf algebra H, a Hopf module M is a left H-module, together with a comodule map $\rho : M \to H \underset{R}{\otimes} M$ such that ρ is a map of H-modules; here we view H as a module over itself via left multiplication. For a Hopf module M, denote $M^{\mathrm{co}\,H}$ the coinvariants of M. Similarly, one may define the notion of a right Hopf module.

Lemma 3.3.3. *Let H be an R-Hopf algebra and let M be an R-module. Then $H \underset{R}{\otimes} M$ is a left H-module and $(H \underset{R}{\otimes} M)^{\mathrm{co}\,H} \cong M$ and R-modules.*

Proof. We will prove the dual version of this statement, that is for a right R-module M, we have $M \underset{R}{\otimes} H$ is a right Hopf comodule and $M \cong (M \underset{R}{\otimes} H)^{\mathrm{co}H}$. The first statement follows from [62, 12.7(1)]. Furthermore, we have by [62, 12.12] that $(M \underset{R}{\otimes} H)^H \cong \mathrm{Hom}^H(R, M \underset{R}{\otimes} H)$, where the morphism is considered in the category of right H-comodules and we view R as a H-comodule. Finally, it follows from [62, 7.9] that $\mathrm{Hom}^H(R, M \otimes_R H) \cong \mathrm{Hom}_R(R, M)$. The claim follows since $\mathrm{Hom}_R(R, M) \cong M$.

We should remark that the compositions of isomorphisms $M \cong \mathrm{Hom}_R(R, M) \cong \mathrm{Hom}^H(R, M \underset{R}{\otimes} H) \cong (M \underset{R}{\otimes} H)^{\mathrm{co}H}$ maps m to $m \otimes 1$. ☐

We will also need the the Fundamental Theorem of Hopf modules:

Theorem 3.3.4. *[31, Theorem 4.13] Let H be a Hopf algebra over a commutative ring R and M a Hopf module. Then the map*

$$\mu : H \underset{R}{\otimes} M^{\mathrm{co}H} \to M, \quad \mu(h \otimes m) = h.m$$

is an isomorphism.

We may now prove the main result of this section. This is presumably known, but we record it for the sake of completeness.

Proposition 3.3.5. *[Equivariant descent for $\mathcal{O}$-modules] Let G be a smooth affine algebraic group of finite type and let $\xi : Y \to X$ be a locally trivial G-torsor. Then the functors $\xi_*(-)^G$ and $\xi^*(-)$ induce quasi-inverse equivalences of categories between G-equivariant quasi-coherent $\mathcal{O}_Y$-modules and quasi-coherent $\mathcal{O}_X$-modules.*

Proof. For $\mathcal{M} \in \mathrm{QCoh}(\mathcal{O}_Y, G)$ and $\mathcal{N} \in \mathrm{QCoh}(\mathcal{O}_X)$ we obtain by functoriality maps $\mathcal{M} \to \xi^*(\xi_*\mathcal{M})^G$ and $(\xi_*(\xi^*\mathcal{N}))^G \to \mathcal{N}$, respectively. Thus we only need to prove the statement locally. We may then assume that $Y = G \times X$, $p : G \times X \to X$ is the projection onto the second factor and G acts on $G \times X$ via left multiplication on the first factor.

We start by constructing a natural isomorphism $\eta : \mathrm{id} \to (p_*)^G p^*$. Let $\mathcal{M}$ be a $\mathcal{O}_X$-module. We aim to define a map $\eta_M : \mathcal{M} \to p_*^G p^* \mathcal{M}$. For any open affine $U \subset X$, we have

$$(p_*^G p^* \mathcal{M})(U) = (\mathcal{O}(G) \underset{R}{\otimes} \mathcal{M}(U))^G.$$

Let $\eta_{\mathcal{M}U} : \mathcal{M}(U) \to (\mathcal{O}(G) \underset{R}{\otimes} \mathcal{M}(U))^G$ be defined by $m \mapsto 1 \otimes m$. We have by Lemma 3.3.3 that $\eta_{\mathcal{M}U}$ is an isomorphism.

Let $V \subset U$ be open affine and let $\mathrm{res}_{UV} : \mathcal{M}(U) \to \mathcal{M}(V)$ be the restriction map. It is easy to see that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{M}(U) \arrow[r, "\eta_{\mathcal{M}U}"] \arrow[d, "\mathrm{res}_{UV}"] & (\mathcal{O}(G) \underset{R}{\otimes} \mathcal{M}(U))^G \arrow[d, "\mathrm{id} \underset{R}{\otimes} \mathrm{res}_{UV}"] \\
\mathcal{M}(V) \arrow[r, "\eta_{\mathcal{M}V}"] & (\mathcal{O}(G) \underset{R}{\otimes} \mathcal{M}(V))^G.
\end{tikzcd}$$

Thus, η is a map of sheaves. Next, we prove that the isomorphism η is natural. Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a morphism of $\mathcal{O}_X$-modules. It is enough to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\eta_M"] \arrow[d, "\varphi"] & (p_*)^G p^* \mathcal{M} \arrow[d, "(p_*)^G p^* \varphi"] \\
\mathcal{N} \arrow[r, "\eta_N"] & (p_*)^G p^* \mathcal{N}.
\end{tikzcd}$$

We can work locally. Let $U \subset X$ be affine open and let $m \in \mathcal{M}(U)$. Then $\eta_N(\varphi(m)) = 1 \otimes \varphi(m)$. On the other hand we have $p^*\varphi : p^*\mathcal{M}(G \times U) \to p^*\mathcal{N}(G \times U)$ defined by $p^*\varphi(F \otimes m) = F \otimes \varphi(m)$ for any $F \in \mathcal{O}(G), m \in \mathcal{M}(U)$.

Thus, we have that $(p_*)^G p^* \varphi : (p_*)^G p^* \mathcal{M}(U) \to (p_*)^G p^* \mathcal{M}(U)$ is defined by $(p_*)^G p^* \varphi(1 \otimes m) = 1 \otimes \varphi(m)$, for all $m \in \mathcal{M}(U)$. In particular, we get that

$$(p_*)^G p^* \varphi(\eta_M(m)) = (p_*)^G p^* \varphi(1 \otimes m) = 1 \otimes \varphi(m) = \eta_N(\varphi(m)),$$

which shows that the diagram is indeed commutative.

Now, let $(\mathcal{M}, \alpha)$ be a G-equivariant $\mathcal{O}_{G \times X}$-module. By construction $(p_*)^G \mathcal{M}$ is a subsheaf of $p_* \mathcal{M}$ and since there is a canonical sheaf map $p^*(p_*)\mathcal{M} \to \mathcal{M}$, we get by functoriality that there is a map $\nu_{\mathcal{M}} : p^*(p_*)^G \mathcal{M} \to \mathcal{M}$.

Let $U \subset X$ open affine. Then we have the induced map

$$\nu_{\mathcal{M}_{G \times U}} : \mathcal{O}(G) \underset{R}{\otimes} \mathcal{M}(G \times U)^G = p^*(p_*)^G \mathcal{M}(G \times U) \to \mathcal{M}(G \times U)$$

given by $\nu_{\mathcal{M}_{G \times U}}(f \otimes m) = f.m$. We aim to prove that this map is an isomorphism. Since $\mathcal{M}$ is G-equivariant the isomorphism α induces an automorphism of $\mathcal{O}_{G \times G \times X}$-modules on $p^* \mathcal{M}$, so in particular we obtain an automorphism on $\mathcal{O}(G) \otimes \mathcal{M}(G \times U)$

of $\mathcal{O}(G \times G \times U)$-modules. This induces a Hopf module structure on $\mathcal{M}(G \times U)$ for the Hopf algebra $\mathcal{O}(G)$. Thus, by Theorem 3.3.4 we get that $\nu_{\mathcal{M}_{G\times U}}$ is indeed an isomorphism.

Let $\{U_i\}_{i\in I}$ be an affine open cover of X and let $\{G \times U_i\}_{i\in I}$ be the corresponding affine cover of $G \times X$. As $\nu_{\mathcal{M}_{G\times U_i}}$ is an isomorphism, the sheaves $p^*(p_*)^G\mathcal{M}$ and $\mathcal{M}$ agree on an affine open cover and there is a sheaf map between the two, ν is an isomorphism.

To finish the proof, notice that by construction, we have $\nu : p^*(p_*)^G \to \mathrm{id}$ is a natural isomorphism. This concludes the proof of the proposition. ☐

We will also consider a slightly more general setting. Let Y be a variety acted on by two smooth affine algebraic groups of finite type, G and B. Let us denote the two actions by . and $\star$.

Observation 3.3.6. *Let $\mathcal{M}$ be a $G \times B$-equivariant $\mathcal{O}_Y$-module. Then $\mathcal{M}^G$ is a B-equivariant submodule of $\mathcal{M}$. Let $\phi : \mathcal{M} \to \mathcal{M}'$ be a $G \times B$ equivariant map of $G\times B$-equivariant modules. Then ϕ restricts to a B-equivariant map $\phi : \mathcal{M}^G \to \mathcal{M}'^G$.*

Proof. We view the equivariance structure via the equations (3.13). We have that for $g \in G, b \in B$ and $m \in \mathcal{M}$ that

$$g.(b \star m) = b \star (g.m).$$

If $m \in \mathcal{M}^G$, then $g.m = m$, so by the equation above $g.(b \star m) = b \star m$, so $b \star m \in \mathcal{M}^G$. Thus, the B-equivariance on $\mathcal{M}$ induces B-equivariance on $\mathcal{M}^G$.

Similarly, using the equivariance of morphisms in equation (3.14), we have since ϕ is particular G-equivariant that for $m \in \mathcal{M}^G$, $g.\phi(m) = \phi(g.m) = \phi(m)$, so ϕ restricts to a map $\mathcal{M}^G \to \mathcal{M}'^G$.

Finally, since ϕ is in particular B-equivariant, we have that for $m \in \mathcal{M}^G$

$$b \star \phi(m) = \phi(b \star m),$$

concluding the proof. ☐

Lemma 3.3.7. *Let G and B be smooth affine algebraic groups of finite type acting on R-schemes X and Y such that the action of B and G on Y commute. Let $\xi : Y \to X$ be a locally trivial G-torsor that is B-equivariant.*

- *Let $\mathcal{M}$ be a $G \times B$-equivariant $\mathcal{O}_Y$-module. Then $(\xi_*\mathcal{M})^G$ is a B-equivariant $\mathcal{O}_X$-module.*
- *Let $\mathcal{N}$ be a B-equivariant $\mathcal{O}_X$-module. Then $\xi^*\mathcal{N}$ is a $G \times B$-equivariant $\mathcal{O}_Y$-module.*

Proof. Since ξ is B-equivariant, the B-action on $\mathcal{O}_X \cong (\xi_*\mathcal{O}_Y)^G$ is induced from the B-action on $\mathcal{O}_Y$. Further, using the observation above we may define a B-action on $(\xi_*\mathcal{M})^G$ which is compatible with the B-action on $\mathcal{O}_Y$ since $\mathcal{M}$ is B-equivariant, so the first claim is proven.

For the second claim, we let G act on $\mathcal{N}$ via $g.n = n$ for all $g \in G$ and $n \in \mathcal{N}$, so that $\mathcal{N}$ is $G \times B$-equivariant. The claim follows from Lemma 3.2.2. ☐

Corollary 3.3.8. *Let G and B be smooth affine algebraic groups of finite type acting on Y and X such that the action of B and G on Y commute. Let $\xi : Y \to X$ be a locally trivial G-torsor that is B-equivariant. The functors $\xi_*(-)^G$ and $\xi^*(-)$ induce quasi-inverse equivalences of categories between $G \times B$-equivariant quasi-coherent $\mathcal{O}_Y$-modules and quasi-coherent B-equivariant $\mathcal{O}_X$-modules.*

Proof. This follows from Proposition 3.3.5, Lemma 3.3.7 and Observation 3.3.6. ☐

3.4 Deformed twisted differential operators

Definition 3.4.1. *We call an R-scheme X that is smooth, separated and locally of finite type an* R-variety*.*

We write $\mathcal{T}_X$ for the sheaf of sections of the tangent bundle TX.

Definition 3.4.2. *[5, Definition 4.2]*

Let X be an R-variety. The sheaf of crystalline differential operators is defined to be the enveloping algebra $\mathcal{D}_X$ of the Lie algebroid $\mathcal{T}_X$.

We can view $\mathcal{D}_X$ as a sheaf of ring generated by $\mathcal{O}_X$ and $\mathcal{T}_X$ modulo the relations:

- $f\partial = f \cdot \partial$;
- $\partial f - f\partial = \partial(f)$;

- $\partial\partial' - \partial'\partial = [\partial, \partial']$,

for all $f \in \mathcal{O}_X$ and $\partial, \partial' \in \mathcal{T}_X$. The sheaf $\mathcal{D}_X$ comes equipped with a natural PBW filtration:

$$0 \subset F_0(\mathcal{D}_X) \subset F_1(\mathcal{D}_X) \subset \dots$$

consisting of coherent $\mathcal{O}_X$-modules such that

$$F_0(\mathcal{D}_X) = \mathcal{O}_X, \quad F_1(\mathcal{D}_X) = \mathcal{O}_X \oplus \mathcal{T}_X, \quad F_m(\mathcal{D}_X) = F_1(\mathcal{D}_X) \cdot F_{m-1}(\mathcal{D}_X) \text{ for } m > 1.$$

Since X is smooth, the tangent sheaf $\mathcal{T}_X$ is locally free and the associated graded sheaf of algebras of $\mathcal{D}_X$ is isomorphic to the symmetric algebra of $\mathcal{T}_X$:

$$\operatorname{gr}(\mathcal{D}_X) = \bigoplus_{m=0}^{\infty} \frac{F_m(\mathcal{D}_X)}{F_{m-1}(\mathcal{D}_X)} \cong \operatorname{Sym}_{\mathcal{O}_X} \mathcal{T}_X. \tag{3.15}$$

If $q : T^*X \to X$ is the cotangent bundle of X defined by the locally free sheaf $\mathcal{T}_X$, then we can also identify $\operatorname{gr}(\mathcal{D}_X)$ with $q_*\mathcal{O}_{T^*X}$.

Let X be an R-variety and let $U = \operatorname{Spec}(A) \subset X$ be open affine. Further, we consider $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-bimodules quasi-coherent with respect to the left action. We define a filtration on $M = \mathcal{M}(U)$ given by $F_\bullet M$:

- $F_{-1}(M) = 0$,
- $F_n(M) = \{m \in M \,|\, \operatorname{ad}(a_0)\operatorname{ad}(a_1)...\operatorname{ad}(a_n)(m) = 0, \text{ for any } a_0, a_1, \dots a_n \in A\}$, for $n \geq 0$.

We say that M is differential if $M = \cup_{n \in \mathbb{N}} F_n(M)$ and we call $\mathcal{M}$ a differential $\mathcal{O}_X$-bimodule if there is an affine open cover $(U_i)_{i \in I}$ such that $\mathcal{M}(U_i)$ is a differential bimodule for all $i \in I$.

Let $\mathcal{M}, \mathcal{N}$ be two quasi-coherent $\mathcal{O}_X$-modules. Then for any affine open U in X, the set $\operatorname{Hom}_R(\mathcal{M}(U), \mathcal{N}(U))$ has the structure of a $\mathcal{O}_X(U)$-bimodule. Let $\mathcal{F} \in \operatorname{Hom}_R(\mathcal{M}, \mathcal{N})$; we say that $\mathcal{F}$ is a differential operator of degree $\leq n$ if for any affine open U, $\mathcal{F}(U) \in F_n(\operatorname{Hom}_R(\mathcal{M}(U), \mathcal{N}(U))$. We denote $\operatorname{Dif}^n(\mathcal{M}, \mathcal{N})$ the subsheaf of differential operators of degree $\leq n$ and $\mathrm{Dif}(\mathcal{M},\mathcal{N}) = \cup_{n\in\mathbb{N}} \mathrm{Dif}^n(\mathcal{M},\mathcal{N})$ the subsheaf of differential operators of finite degree.

We may construct differential $\mathcal{O}_X$-modules using the following proposition:

Proposition 3.4.3. *Let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module and let $\mathcal{N}$ be a $\mathcal{O}_X$-module. Then $\mathrm{Dif}(\mathcal{M},\mathcal{N})$ is differential $\mathcal{O}_X$-bimodule.*

Proof. The proof follows by repeating the argument in [59, Proposition 2.1.3]. □

Definition 3.4.4. *Let $\mathcal{B}$ be a $\mathcal{O}_X$-algebra. We say that $\mathcal{B}$ is a differential algebra if $\mathcal{B}$ is a flat R-module and multiplication makes $\mathcal{B}$ a differential $\mathcal{O}_X$-bimodule. The filtration $F_\bullet(\mathcal{B})$ becomes a ring filtration and with respect to this filtration $\mathrm{gr}^F(\mathcal{B})$ is commutative.*

Definition 3.4.5. *Let $r \in R$ be a regular element. An algebra of r-deformed twisted differential operators(tdo) is an $\mathcal{O}_X$-differential algebra $\mathcal{D}$ such that:*

i) The natural map $\mathcal{O}_X \to F_0(\mathcal{D})$ is an isomorphism.

ii) The morphism $\mathrm{gr}_1^F \mathcal{D} \to \mathcal{T}_X = \mathrm{Der}_R(\mathcal{O}_X, \mathcal{O}_X)$ defined by $\psi \mapsto \mathrm{ad}_\psi$ for $\psi \in F_1(\mathcal{D})$ induces an isomorphism $\mathrm{gr}_1^F \mathcal{D} \cong r\mathcal{T}_X$.

iii) The morphism of $\mathcal{O}_X$-algebras $\mathrm{Sym}_{\mathcal{O}_X}(\mathrm{gr}_1^F \mathcal{D}) \to \mathrm{gr}^F \mathcal{D}$ is an isomorphism.

A morphism of tdo's is a morphism of $\mathcal{O}_X$ algebras compatible with the $\mathcal{O}_X$-bimodule structure and the maps in i) $\to$ iii).

We should make some remarks about this definition: when $r = 1$ we call $\mathcal{D}$ a sheaf of twisted differential operators. Classically, working with twisted differential operators over a complex variety the condition *iii*) is implied by *i*) and *ii*). This is no longer true in our case. Further, the sheaf of *Grothendieck*'s differential operators does not satisfy condition *iii*) for a general ring R. This is the main reason why we develop the theory of twisted differential operators using the connection with Lie algebroids rather than using the classical Dif definition.

Lemma 3.4.6. *Assume that X is locally Noetherian R-variety and let $\mathcal{D}$ be an r-deformed tdo on X. Then $\mathcal{D}$ is locally Noetherian.*

Proof. We have by conditions *ii*) and *iii*) that $\mathrm{gr}^F \mathcal{D} \cong \mathrm{Sym}_{\mathcal{O}_X}(r\mathcal{T}_X)$ and because r is regular $\mathrm{Sym}_{\mathcal{O}_X}(r\mathcal{T}_X) \cong \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X)$. Since $\mathcal{T}_X$ is a free $\mathcal{O}_X$-module and X is locally Noetherian, we obtain that $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X)$ is locally Noetherian. Therefore, we have $\mathrm{gr}^F \mathcal{D}$ is locally Noetherian, which implies the same for $\mathcal{D}$. □

3.5 Connections between deformed Lie algebroids and deformed tdo's

Throughout this section, we let X denote an R-variety and $r \in R$ a regular element.

Definition 3.5.1. *A Lie algebroid $\mathcal{L}$ on X is a quasi-coherent $\mathcal{O}_X$-module equipped with a morphism of $\mathcal{O}_X$-modules $\rho : \mathcal{L} \to \mathcal{T}_X$ (the anchor map) and an R-linear pairing $[\bullet, \bullet] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that:*

- *$[\bullet, \bullet]$ defines the structure of a Lie algebra on $\mathcal{L}$ and ρ is a morphism of Lie algebras.*
- *$[l_1, fl_2] = f[l_1, l_2] + \rho(l_1)(f)l_2$ for $l_i \in \mathcal{L}, f \in \mathcal{O}_X$.*

A morphism of Lie algebroids is a morphism of $\mathcal{O}_X$-modules compatible with the anchor maps and bracket.

In particular, locally we obtain that for any $U \subset X$ affine open that $\mathcal{L}(U)$ is an $(R, \mathcal{O}_X(U))$-Lie Rinehart algebra, see [49] for definition and basic properties of Lie Rinehart algebras. We may think of $\mathcal{L}$ as a sheaf of $(R, \mathcal{O}_X)$-Lie Rinehart algebras; we will use this local description soon.

Definition 3.5.2. *The universal enveloping algebra of $\mathcal{L}$, denoted $U(\mathcal{L})$, is the sheaf of R-algebras generated by $\mathcal{O}_X$ and $\mathcal{L}$ modulo the relations:*

- *$i : \mathcal{O}_X \to U(\mathcal{L})$ is a morphism of R-algebras,*
- *$j : \mathcal{L} \to U(\mathcal{L})$ is a morphism of Lie algebras,*
- *$j(fl) = i(f)j(l)$ and $[j(l), i(f)] = i(\rho(l)(f))$.*

Locally, $U(\mathcal{L})$ is just the enveloping algebra of the corresponding $(R, \mathcal{O}_X(U))$-algebra.

We want to establish a correspondence between Lie algebroids and r-deformed tdo's on an R- variety X. For an $\mathcal{O}_X$-differential algebra $\mathcal{D}$ we define $\mathrm{Lie}(\mathcal{D}) := F_1(\mathcal{D})$; one may prove that when $\mathcal{D}$ is a tdo, $\mathrm{Lie}(\mathcal{D})$ is a Lie algebroid, see [9, 1.2.5]; unfortunately not all Lie algebroids induce tdo's, so we need a more specific notion.

Definition 3.5.3. *We call a Lie algebroid $\mathcal{L}$ an r-deformed Picard algebroid if there exists a short exact sequence of Lie algebras and $\mathcal{O}_X$-modules:*

$$0 \to \mathcal{O}_X \to \mathcal{L} \to r\mathcal{T}_X \to 0.$$

One should notice that the Lie algebra structure imposed on $\mathcal{O}_X$ is the trivial one. We should also denote $1_{\mathcal{L}}$ the image of $1 \in \mathcal{O}_X$ under the inclusion map.

A morphism of r-deformed Picard algebroids is a morphism of Lie algebroids compatible with the maps in the short exact sequence defining r-deformed Picard structure.

Proposition 3.5.4. *Let $\mathcal{L}$ be an r-deformed Picard algebroid on X. Then the sheaf of rings $\mathcal{D} := U(\mathcal{L})/U(\mathcal{L})(i(1) - j(1))$ is an r-deformed tdo with* $\mathrm{Lie}(\mathcal{D}) = \mathcal{L}$.

Proof. The question is local so we may assume that X is affine; $A = \mathcal{O}_X(X)$, $T = \mathcal{T}_X(X)$, $L = \mathcal{L}(X)$ and let $i : A \to L$ denote the injection induced by the short exact sequence defining $\mathcal{L}$.

Consider the enveloping algebra $U(L)$ of the (R, A)-Lie algebra L. We can think of it as being generated by A and the universal enveloping algebra of the Lie algebra L subject to the relations: $fl = f \cdot l$ (the module action), $fl - lf = \rho(l)(f)$ for $f \in A$ and $l \in L$, $l_1 l_2 - l_2 l_1 = [l_1, l_2]$ for $l_1, l_2 \in L$. The natural filtration on $U(L)$ is given by:

- $F_0(U(L)) = A$,
- For $n \geq 1$, $F_n(U(L)) = A + l_1 l_2 \dots l_m$, where $m \leq n$ and $l_1, l_2, \dots l_m \in L$.

Since $fl - lf = \rho(l)(f)$ for $f \in A$ and $l \in L$, it easy to see that $U(L)$ becomes a differential A algebra with respect to this filtration.

Let $I = \langle i(1) - 1_L \rangle$ be the central two sided ideal and $D := \mathcal{D}(X) = U(L)/IU(L)$. We give D the quotient natural quotient filtration induced from $U(L)$. By construction, we have that D is also a differential A-algebra since $U(L)$ is. Furthermore, we have $F_0(D) \cong F_0(U(L)) \cong A$.

Let $g : F_1(U(L)) \to L$ given by $g(a + l) = i(a) + l$ for $a \in A, l \in L$. Then it is clear that g is surjective and $\ker(g) = a - i(a) = F_1(I)$. Thus, we obtain $F_1(D) \cong L$, so $\mathrm{Lie}(D) \cong L$.

Since L is an r-deformed Picard algebroid, we obtain immediately that $\operatorname{gr}_1 D \cong L/A \cong rT$. Finally, because X is an R-variety, T is projective as an A-module. Therefore, L is also projective as A-module, so we have by [49, Theorem 3.1] that $\operatorname{gr} D \cong \operatorname{Sym}_A(\operatorname{gr}_1 D)$. Thus, we have proven all the required properties to make D an r-deformed tdo on X. ☐

Using the Lemma above we make the following definition:

Definition 3.5.5. *Let X be an R-variety. Define a functor $\mathscr{T}$: category of r-deformed Picard algebroids on X → category of r-deformed twisted differential operators on X by*

$$\mathscr{T}(\mathcal{L}) := U(\mathcal{L})/U(\mathcal{L})(i(1) - j(1)).$$

Lemma 3.5.6. *Let $\mathcal{D}$ be an r-deformed tdo on X. The sheaf* $\operatorname{Lie}(\mathcal{D}) := F_1(\mathcal{D})$ *is an r-deformed Picard algebroid and furthermore*

$$\mathscr{T}(\operatorname{Lie}\mathcal{D}) = U(\operatorname{Lie}(\mathcal{D}))/U(\operatorname{Lie}(\mathcal{D}))(i(1) - j(1)) \cong \mathcal{D}.$$

Proof. Let $\mathcal{L} := \operatorname{Lie}(\mathcal{D})$. Since $\mathcal{D}$ is a differential algebra, $\mathcal{L}$ is a Lie algebroid by [9, 1.2.5], with the anchor map $\rho : \mathcal{L} \to \mathcal{T}_X$ induced by axiom ii) of Definition 3.4.5. Further by axioms i) and ii) of 3.4.5 we observe that $\ker(\rho) = \mathcal{O}_X$ and $\operatorname{im}(\rho) = r\mathcal{T}_X$, so $\mathcal{L}$ is indeed an r-deformed Picard algebroid.

Let $\mathcal{A} := U(\mathcal{L})/U(\mathcal{L})(i(1) - j(1))$. By Proposition 3.5.4, $\mathcal{A}$ is an r-deformed tdo. Further by construction we have that there is morphism of filtered algebras $\mathcal{A} \to \mathcal{D}$ and $\operatorname{gr}_1(\mathcal{A}) \cong \operatorname{gr}_1(\mathcal{D})$. Since $\mathcal{A}$ and $\mathcal{D}$ are r-deformed tdo's, we have by axiom iii) of Definition 3.4.5 that $\operatorname{gr}(\mathcal{A}) \cong \operatorname{Sym}_{\mathcal{O}_X}(\operatorname{gr}_1 \mathcal{A})$ and $\operatorname{gr}(\mathcal{D}) \cong \operatorname{Sym}_{\mathcal{O}_X}(\operatorname{gr}_1 \mathcal{D})$. Therefore, we get $\operatorname{gr}(\mathcal{A}) \cong \operatorname{gr}(\mathcal{D})$, so $\mathcal{A} \cong \mathcal{D}$ since there is a filtered morphism between them.

☐

Corollary 3.5.7. *Let X be an R-variety. The functors $\mathscr{T}$ and* Lie *induce quasi-inverse equivalences of categories between the category of r-deformed Picard algebroids on X and the category of r-deformed tdo's on X.*

Proof. This follows from Proposition 3.5.4 and Lemma 3.5.6. ☐

3.6 Equivariant deformed Picard algebroids and deformed homogeneous twisted differential operators

Throughout this section, we fix X an R-variety, G a smooth affine algebraic group of finite type acting on X and $r \in R$ a regular element. Recall from 3.13 that for a G-equivariant $\mathcal{O}_X$-module $\mathcal{M}$ we denoted by abuse of notation by $\cdot$ the group action giving the equivariance.

Definition 3.6.1. *Let $(\mathcal{L}, \rho)$ be a Lie algebroid. We say that $\mathcal{L}$ is r-deformed G-equivariant if $\mathcal{L}$ is a G-equivariant as a $\mathcal{O}_X$-module and it is equipped with a Lie algebra morphism $i_{\mathfrak{g}} : r\mathfrak{g} \to \mathcal{L}$ such that:*

i) $g \cdot [x, y] = [g \cdot x, g \cdot y]$, *for* $g \in G$, $x, y \in \mathcal{L}$.

ii) $g.\rho(l)(f) = \rho(g.l)(g.f)$, *for* $g \in G, l \in \mathcal{L}, f \in \mathcal{O}_X$. *This is equivalent to ρ being G-equivariant.*

iii) $i_{\mathfrak{g}}(g.\psi) = g.i_{\mathfrak{g}}(\psi)$, *for $g \in G$ and $\psi \in r\mathfrak{g}$. Here G acts on $r\mathfrak{g} \subset \mathfrak{g}$ via the Adjoint action.*

Similarly, we may define the notion of equivariant differential algebra.

Definition 3.6.2. *Let $\mathcal{D}$ be a differential $\mathcal{O}_X$-algebra. We call $\mathcal{D}$ an r-deformed G-*equivariant *differential algebra if it is G-equivariant as a left $\mathcal{O}_X$-module and it is equipped with a Lie algebra map $i_{\mathfrak{g}} : r\mathfrak{g} \to \mathcal{D}$ such that:*

1. $g.1 = 1$ *and* $g.(d_1 d_2) = (g.d_1)(g.d_2)$, *for $g \in G$ and $d_1, d_2 \in \mathcal{D}$.*
2. $g.(fd) = (g.f)(g.d)$, *for $f \in \mathcal{O}_X$ and $d \in \mathcal{D}$.*
3. $i_{\mathfrak{g}}(g.\psi) = g.i_{\mathfrak{g}}(\psi)$, *for $g \in G$ and $\psi \in r\mathfrak{g}$.*

Lemma 3.6.3. *Let $(\mathcal{L}, \rho, i_{\mathfrak{g}})$ be an r-deformed G-equivariant Lie algebroid. Then $U(\mathcal{L})$ is an r-deformed G-equivariant differential algebra.*

Proof. Since $\mathcal{L}$ is quasi-coherent as a $\mathcal{O}_X$-module, so is $U(\mathcal{L})$. We define G-action on $U(\mathcal{L})$ by defining $g.(l_1 l_2 \dots g.j) = (g.l_1)(g.l_2)\dots(g.l_j)$ for $l_1, l_2, \dots l_j \in \mathcal{L}$ and $g.(f l_1 l_2 \dots l_j) = (g.f)(g.l_1 l_2 \dots l_j)$ for $f \in \mathcal{O}_X$ and $l_1, l_2, \dots l_j \in \mathcal{L}$. We have

$$\begin{aligned} g.(l_1l_2 - l_2l_1) &= (g.l_1)(g.l_2) - (g.l_2)(g.l_1) \\ &= [g.l_1, g.l_2] \\ &= g.[l_1, l_2] \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} g.(fl - lf) &= (g.f)(g.l) - (g.l)(g.f) \\ &= [g.f, g.l] \\ &= \rho(g.l)(g.f) \\ &= g.\rho(l)(f). \end{aligned} \tag{3.17}$$

Since $U(\mathcal{L})$ is generated by $\mathcal{O}_X$ and the enveloping algebra of $\mathcal{L}$, subject to the relations $(fl - lf) = \rho(l)(f)$ and $fl = f.l$, it follows from the equations and definition that $U(\mathcal{L})$ is G-equivariant as $\mathcal{O}_X$-module and furthermore, axioms $i)$ and $ii)$ of Definition 3.6.2 are satisfied. Further it easy to check that G-action preserves the filtration on $U(\mathcal{L})$.

The morphism $i_{\mathfrak{g}} : r\mathfrak{g} \to \mathcal{L}$ can be extended to a morphism $i_{\mathfrak{g}} : r\mathfrak{g} \to U(\mathcal{L})$ via the natural map $\mathcal{L} \to U(\mathcal{L})$ and it is clear by construction that under G-action and the map $i_{\mathfrak{g}}$ defined above that $U(\mathcal{L})$ becomes an r-deformed G-equivariant differential algebra. □

As we are interested in deformed Picard algebroids, we define the notion of an r-deformed G-equivariant Picard algebroid. The G action on $\mathcal{L}$ induces by differentiation a $\mathfrak{g} := \mathrm{Lie}(G)$ action via a map $\beta_{\mathcal{M}} : \mathfrak{g} \to \mathrm{End}(\mathcal{L})$. We also let $\eta : \mathfrak{g} \to \mathcal{T}_X$ denote the infinitesimal action of $\mathfrak{g}$ on X.

Definition 3.6.4. *Let $\mathcal{L}$ be an r-deformed Picard algebroid. We say that $\mathcal{L}$ is an r-deformed G-equivariant Picard algebroid if $\mathcal{L}$ is an r-deformed G-equivariant algebroid, in the short exact sequence*

$$0 \to \mathcal{O}_X \to \mathcal{L} \to r\mathcal{T}_X \to 0$$

all the morphisms are G-equivariant and

- *The derivative of the G-action induces a $\mathfrak{g}$ action and thus a $r\mathfrak{g}$ action on $\mathcal{L}$. This must coincide with the action $l \mapsto [i_{\mathfrak{g}}(\psi), l]$ for $\psi \in r\mathfrak{g}$ and $l \in \mathcal{L}$.*

- $\eta_{|r\mathfrak{g}} = \rho \circ i_{\mathfrak{g}}$.

A morphism of r-deformed G-equivariant Picard algebroids is a morphism of r-deformed Picard algebroids compatible with the equivariance structures.

Definition 3.6.5. *Let $\mathcal{D}$ be a $\mathcal{O}_X$-algebra. We say that $\mathcal{D}$ is a sheaf of r-deformed G-homogeneous twisted differential operators (r-deformed G-htdo) if $(\mathcal{D}, i_{\mathfrak{g}})$ is an r-deformed G-equivariant differential $\mathcal{O}_X$-algebra and a sheaf of r-deformed twisted differential operators, and furthermore:*

- *The image of $i_{\mathfrak{g}}$ lies in $F_1\mathcal{D}$.*
- *The derivative of the G-action induces a $\mathfrak{g}$ action and thus a $r\mathfrak{g}$-action on $\mathcal{D}$. This must coincide with the action $d \mapsto [i_{\mathfrak{g}}(\psi), d]$ for $\psi \in r\mathfrak{g}$ and $d \in \mathcal{D}$.*
- *$\eta_{|r\mathfrak{g}} = \rho \circ i_{\mathfrak{g}}$, where $\rho : F_1\mathcal{D} = \mathrm{Lie}(\mathcal{D}) \to \mathcal{T}_X$ is the natural anchor map.*

A morphism of r-deformed G-htdo's is a morphism of r-deformed tdo's compatible with the equivariance structures.

One should notice that since $g.1 = 1$, the morphism $\mathcal{O}_X \to F_0(\mathcal{D})$ is automatically G-equivariant.

Lemma 3.6.6. *Let $(\mathcal{L}, \rho, i_{\mathfrak{g}})$ be an r-deformed G-equivariant Picard algebroid. Then $\mathscr{T}(\mathcal{L})$ is an r-deformed G-htdo.*

Proof. We have by Lemma 3.6.3 that $U(\mathcal{L})$ is a G-equivariant differential algebra. Now, since $\mathcal{L}$ is G-equivariant, the action of G stabilises the ideal generated by $i(1) - j(1)$, so the G action descends on $U(\mathcal{L})/U(\mathcal{L})(i(1) - j(1))$. Similarly, composing the map $i_{\mathfrak{g}} : r\mathfrak{g} \to U(\mathcal{L})$ with the natural projection, we obtain a map $r\mathfrak{g} \to \mathscr{T}(\mathcal{L})$. Further, by Proposition 3.5.4, $U(\mathcal{L})/U(\mathcal{L})(i(1) - j(1))$ is an r-deformed tdo. Finally, the axioms of Definition 3.6.4 imply that $\mathscr{T}(\mathcal{L})$ is an r-deformed G-htdo. □

Lemma 3.6.7. *Let $\mathcal{D}$ be an r-deformed G-htdo. Then $\mathcal{L} := F_1(\mathcal{D})$ is an r-deformed G-equivariant Picard algebroid.*

Proof. We have by Lemma 3.5.6 that $\mathcal{L}$ is an r-deformed Picard algebroid and the axioms for $\mathcal{D}$ imply that $\mathcal{L}$ is a G-equivariant. Since $\mathcal{D}$ is a G-equivariant differential algebra the map $\mathcal{L} \to \mathcal{T}_X$ is G-equivariant. Lastly, the morphism $\mathcal{O}_X \to F_0(\mathcal{D})$ is G-equivariant and the other axioms in definition Definition 3.6.4 follow from the corresponding axioms in 3.6.5. □

Corollary 3.6.8. *Let X be an R-variety and $r \in R$ a regular element. The maps $\mathscr{T}$ and* Lie *induce quasi-inverse equivalences of categories between the category of r-deformed G-equivariant Picard algebroids on X and the category of r-deformed G-htdo's on X.*

Proof. This follows by combining Lemmas 3.6.6 and 3.6.7 and Corollary 3.5.7. ☐

3.7 Pullback of deformed Picard algebroids

Throughout this section, we fix $f : Y \to X$ a morphism of R-varieties and $r \in R$ a regular element. The map f induces a morphism $f^*\Omega^1_X \to \Omega^1_Y$ and by dualising we obtain $\alpha : \mathcal{T}_Y \to f^*\mathcal{T}_X$. Here Ω^1_X and Ω^1_Y denote the sheaf of differential 1-forms.

Definition 3.7.1. *Let $(\mathcal{L}, \rho_X)$ be an r-deformed Picard algebroid on X and let $\beta = f^*(\rho_X)$. Then we let*

$$f^{\#}\mathcal{L} := r\mathcal{T}_Y \times_{rf^*\mathcal{T}_X} f^*\mathcal{L} = \{(d,l) | d \in r\mathcal{T}_Y, l \in f^*\mathcal{L}, \alpha(d) = \beta(l)\}.$$

We give $f^{\#}\mathcal{L}$ the structure of a Lie algebroid by setting $\rho_Y(d,l) = d$ and the Lie bracket be induced by

$$[(\psi, f \otimes P), (\eta, g \otimes Q)] := ([\psi, \eta], fg \otimes [P,Q] + \psi(g) \otimes P - \eta(f) \otimes Q),$$

for $\psi, \eta \in r\mathcal{T}_Y, f, g \in \mathcal{O}_Y, P, Q \in f^{-1}\mathcal{L}$. We call $f^{\#}\mathcal{L}$ the pullback of $\mathcal{L}$.

Lemma 3.7.2. *Let $(\mathcal{L}, \rho_X)$ be an r-deformed Picard algebroid. Then $(f^{\#}\mathcal{L}, \rho_Y)$ is an r-deformed Picard algebroid.*

Proof. Since $\mathcal{L}$ is an r-deformed Picard algebroid, it fits into a short exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{L} \to r\mathcal{T}_X \to 0.$$

By assumption, X is a smooth variety, thus $\mathcal{T}_X$ is a free $\mathcal{O}_X$-module, so in particular is flat. Since $r\mathcal{T}_X \cong \mathcal{T}_X$ as $\mathcal{O}_X$-modules, by pulling back along f we obtain a short exact sequence:

$$0 \to \mathcal{O}_Y \to f^*\mathcal{L} \to rf^*\mathcal{T}_X \to 0. \tag{3.18}$$

Considering the pullback diagram

$$\begin{tikzcd}
f^{\#}\mathcal{L} \arrow[r] \arrow[d] & f^*\mathcal{L} \arrow[d] \\
r\mathcal{T}_Y \arrow[r] & rf^*\mathcal{T}_X,
\end{tikzcd}$$

we obtain $\ker(\rho_Y) = \ker(f^{\#}\mathcal{L} \to r\mathcal{T}_Y) \cong \ker(f^*\mathcal{L} \to rf^*\mathcal{T}_X) \cong \mathcal{O}_Y$. Finally, since $\operatorname{im}(f^*\mathcal{L} \to rf^*\mathcal{T}_X) = rf^*\mathcal{T}_X$, we obtain that $\operatorname{im}(\rho_Y) = r\mathcal{T}_Y$. □

We would like to describe how the pullback interacts with composition of morphisms. In general, one would like to prove that for $u : Z \to Y, f : Y \to X$ maps of R-varieties and $\mathcal{L}$ an r-deformed Picard algebroid on X, we have $u^{\#}f^{\#}\mathcal{L} \cong (f \circ u)^{\#}\mathcal{L}$. Unfortunately, we can not prove that it is always true; there is some evidence in [9, 1.4.4] that this may fail in a slightly different context. In the following, we give sufficient conditions.

Lemma 3.7.3. *Let $u : Z \to Y, f : Y \to X$ be maps of R-varieties and $\mathcal{L}$ be an r-deformed Picard algebroid on X. Assume that u is flat or f is etale. Then*

$$u^{\#}f^{\#}\mathcal{L} \cong (f \circ u)^{\#}\mathcal{L}.$$

Proof. First assume u is flat. Then:

$$\begin{aligned}
u^{\#}f^{\#}\mathcal{L} &= u^{\#}(r\mathcal{T}_Y \times_{rf^*\mathcal{T}_X} f^*\mathcal{L}) \\
&= r\mathcal{T}_Z \times_{ru^*\mathcal{T}_Y} u^*(r\mathcal{T}_Y \times_{rf^*\mathcal{T}_X} f^*\mathcal{L}) \\
&= r\mathcal{T}_Z \times_{ru^*\mathcal{T}_Y} ru^*\mathcal{T}_Y \times_{ru^*f^*\mathcal{T}_X} u^*f^*\mathcal{L} \text{ (u is flat, so commutes with limits)} \\
&\cong r\mathcal{T}_Z \times_{ru^*f^*\mathcal{T}_X} u^*f^*\mathcal{L} \\
&\cong (f \circ u)^{\#}\mathcal{L}.
\end{aligned} \tag{3.19}$$

Now suppose f is etale, thus $f^*\mathcal{T}_X \cong \mathcal{T}_Y$. Then:

$$\begin{aligned}
u^{\#}f^{\#}\mathcal{L} &= u^{\#}(r\mathcal{T}_Y \times_{rf^*\mathcal{T}_X} f^*\mathcal{L}) \\
&\cong u^{\#}f^*\mathcal{L} \\
&\cong r\mathcal{T}_Z \times_{ru^*\mathcal{T}_Y} u^*f^*\mathcal{L} \\
&\cong r\mathcal{T}_Z \times_{ru^*f^*\mathcal{T}_X} u^*f^*\mathcal{L} \\
&\cong (f \circ u)^{\#}\mathcal{L}.
\end{aligned}$$

□

Lemma 3.7.4. *Let $(\mathcal{L}, \rho_X, i_{\mathfrak{g}})$ be an r-deformed G-equivariant Picard algebroid on X and assume further $f : Y \to X$ is a G-equivariant. Then $(f^{\#}\mathcal{L}, \rho_Y)$ is an r-deformed G-equivariant Picard algebroid.*

Proof. By the Lemma above, $f^{\#}\mathcal{L}$ is an r-deformed Picard algebroid. Since $f : Y \to X$ is G-equivariant, we obtain by Lemma 3.2.2 that $f^*\mathcal{L}$ is a G-equivariant $\mathcal{O}_Y$-module and that the maps α and β are G-equivariant. We define a G action on $f^{\#}\mathcal{L}$ via

$$g.(d, l) = (g.d, g.l) \text{ for } d \in r\mathcal{T}_Y, l \in f^*\mathcal{L}.$$

We need to check that this action is well-defined and ρ_Y is G-equivariant. The second statement is easy, we have $g.\rho_Y(a, b) = \rho_Y(g.(a, b))$. For the other statement, let $d \in r\mathcal{T}_Y, l \in f^*\mathcal{L}$ such that $\alpha(d) = \beta(l)$. Then, we have

$$\alpha(g.d) = g.\alpha(d) = g.\beta(l) = \beta(g.l).$$

So, we are left to prove that G-action interacts correctly with the Lie bracket. Let $\psi, \eta \in r\mathcal{T}_Y, f, h \in \mathcal{O}_Y, P, Q \in f^{-1}\mathcal{L}$. Then

$$\begin{aligned} g.[(\psi, f \otimes P), (\eta, h \otimes Q)] &= g.([\psi, \eta], fh \otimes [P, Q] + \psi(h) \otimes P - \eta(f) \otimes Q) \\ &= (g.[\psi, \eta], g.fh \otimes g.[P, Q] \\ &\quad + g.\psi(h) \otimes g.P - g.\eta(f) \otimes g.Q) \\ &= ([g.\psi, g.\eta], (g.f)(g.h) \otimes [g.P, g.Q] \\ &\quad + \psi(g.h) \otimes g.P - \eta(g.f) \otimes g.Q) \\ &= [g.(\psi, f \otimes P), g.(\eta, h \otimes Q)]. \end{aligned} \tag{3.20}$$

Since G acts on Y we obtain the infinitesimal map $\eta : \mathfrak{g} \to \mathcal{T}_Y$. The map $i_{\mathfrak{g}} : r\mathfrak{g} \to \mathcal{L}$ can be extended to a map $i_{\mathfrak{g}} : \mathcal{O}_X \otimes r\mathfrak{g} \to \mathcal{L}$ and by pulling back we obtain a map $i_{\mathfrak{g}}^* : \mathcal{O}_Y \otimes r\mathfrak{g} \to f^*\mathcal{L}$. We let $i : r\mathfrak{g} \to f^*\mathcal{L}$ be the restriction of $i_{\mathfrak{g}}^*$ to $r\mathfrak{g}$; by construction, we have that the $r\mathfrak{g}$ action induced by i coincides with the one induced by the G action. Therefore we obtain a map $(\eta_{|r\mathfrak{g}}, i) : r\mathfrak{g} \to f^{\#}\mathcal{L}$ and it follows by the construction that the $r\mathfrak{g}$-action induced by the derivative of the G-action coincides with the $r\mathfrak{g}$-action induced by $(\eta_{|r\mathfrak{g}}, i)$ and further that $\eta_{|r\mathfrak{g}} = \rho_Y \circ (\eta_{|r\mathfrak{g}}, i)$. Thus $(f^{\#}\mathcal{L}, \rho_Y, (\eta_{|r\mathfrak{g}}, i))$ is an r-deformed G-equivariant Picard algebroid. □

Definition 3.7.5. *Let $\mathcal{D}$ be an r-deformed tdo/G-htdo on X and let $f : Y \to X$ be a morphism/G-equivariant morphism of R-varieties. We call*

$$f^\#\mathcal{D} := \mathscr{T}(f^\# \operatorname{Lie}(\mathcal{D}))$$

the pullback of $\mathcal{D}$ along f.

Corollary 3.7.6. *Let the notation as above. Then $f^\#\mathcal{D}$ is well defined and furthermore it is an r-deformed tdo/htdo.*

Proof. This follows from Lemmas 3.7.2, 3.7.4 and Corollaries 3.5.7, 3.6.8. □

Corollary 3.7.7. *Let $u : Z \to Y, f : Y \to X$ be maps of R-varieties and $\mathcal{D}$ be an r-deformed tdo on X. Assume that u is flat or f is etale. Then*

$$u^\# f^\#\mathcal{D} \cong (f \circ u)^\#\mathcal{D}.$$

Proof. This follows from Lemma 3.7.3 and Corollary 3.5.7. □

Let us explain how our condition for an r-deformed G-htdo fits into a diagram satisfying the cocycle condition. Denote the G-action by $\sigma_X : G \times X \to X$. Furthermore, we denote $p_X : G \times X \to X$ and $p_{2X} : G \times G \times X \to X$ the projections on the X factor, $p_{23X} : G \times G \times X \to G \times X$ the projection onto the second and third factor and $m : G \times G \to G$ the multiplication of the group G.

Lemma 3.7.8. *Let $\mathcal{D}$ be an r-deformed G-htdo on X. Then there exists $\alpha : \sigma_X^\#\mathcal{D} \to p_X^\#\mathcal{D}$ an isomorphism of $\mathcal{O}_{G\times X}$-algebras such that the diagram:*

$$\begin{tikzcd}
(1_G \times \sigma_X)^\# p_X^\#\mathcal{D} \arrow[r, "p_{23X}^\#\alpha"] & p_{2X}^\#\mathcal{D} \\
(1_G \times \sigma_X)^\# \sigma_X^\#\mathcal{D} \arrow[u, "(1_G\times\sigma_X)^\#\alpha"] \arrow[r, leftrightarrow, "id"] & (m \times 1_X)^\# \sigma_X^\#\mathcal{D} \arrow[u, "(m\times 1_X)^\#\alpha"']
\end{tikzcd} \tag{3.21}$$

of $\mathcal{O}_{G\times G\times X}$-algebras commutes (the cocycle condition) and the pullback

$$(e \times 1_X)^\#\alpha : \mathcal{D} \to \mathcal{D}$$

is the identity map. We note that this is the same condition as in [59, Section 5.2].

Proof. Let $\mathcal{L} = \mathrm{Lie}(D)$; we have by Lemma 3.6.7 that $\mathcal{L}$ is a G-equivariant r-deformed Picard algebroid. In particular, we obtain an isomorphism $\beta : \sigma_X^* \mathcal{L} \to p_X^* \mathcal{L}$, which can be extended to an isomorphism of r-deformed Picard algebroids $\sigma_X^{\#}\mathcal{L} \to p_X^{\#}\mathcal{L}$ and thus to an isomorphsim of $\mathcal{O}_{G\times X}$-algebras $\alpha : \sigma_X^{\#}\mathcal{D} = \mathscr{T}(\sigma_X^{\#}\mathcal{L}) \to \mathscr{T}(p_X^{\#}\mathcal{L}) = p_X^{\#}\mathcal{D}$. Further, since the map β satisfies the cocycle condition, so does α. $\square$

3.8 Representations of Lie algebroids

Throughout this section we fix X an R-variety, $r \in R$ a regular element and $(\mathcal{L}, \rho_X)$ a Lie algebroid on X.

Definition 3.8.1. *Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. We say that $\mathcal{M}$ is a $\mathcal{L}$-*module *if $\mathcal{M}$ is a sheaf of modules over the sheaf of Lie algebras $\mathcal{L}$ and for all $f \in \mathcal{O}_X$, $l \in \mathcal{L}$, $m \in \mathcal{M}$, we have*

$$\begin{aligned} f.(l.m) &= l.(f.m) - \rho_X(l)(f).m, \\ (f.l).m &= f.(l.m). \end{aligned} \tag{3.22}$$

We define a morphism of $\mathcal{L}$-modules to be a morphism of $\mathcal{O}_X$-modules compatible with the $\mathcal{L}$-action.

Definition 3.8.2. *Assume that $(\mathcal{L}, \rho_X)$ is an r-deformed Picard algebroid and let $f : Y \to X$ be a map of R-varieties and $\mathcal{M}$ a $\mathcal{L}$-module. Then we define the pullback of $\mathcal{M}$ along f, via $f^{\#}\mathcal{M} = f^*\mathcal{M}$ as an $\mathcal{O}_Y$-module and*

$$(\psi, P \otimes l).(Q \otimes m) := \psi(Q) \otimes m + PQ \otimes l.m,$$

for $\psi \in r\mathcal{T}_Y$, $P, Q \in \mathcal{O}_Y, l \in f^{-1}\mathcal{L}, m \in f^{-1}\mathcal{M}$.

Lemma 3.8.3. *The action defined above makes $f^{\#}\mathcal{M}$ a $f^{\#}\mathcal{L}$-module.*

Proof. First, we check the bracket action. We have for $\psi, \eta \in r\mathcal{T}_Y$, $P, Q, R \in \mathcal{O}_Y$, $a, b \in f^{-1}\mathcal{L}$ and $m \in f^{-1}\mathcal{M}$ that

$$\begin{aligned} (\psi, P \otimes a).((\eta, R \otimes b).Q \otimes m) &= (\psi, P \otimes a).(\eta(Q) \otimes m + RQ \otimes b.m) \\ &= \psi(\eta(Q)) \otimes m + P\eta(Q) \otimes a.m \\ &+ \psi(RQ) \otimes b.m + PQR \otimes a.(b.m) \end{aligned} \tag{3.23}$$

and

$$\begin{aligned}(\eta, R\otimes b).((\psi, P\otimes a).Q\otimes m) = \eta(\psi(Q))\otimes m + R\psi(Q)\otimes b.m \\ + \eta(PQ)\otimes a.m + PQR\otimes b.(a.m).\end{aligned} \tag{3.24}$$

Thus, combining the equations above, we obtain

$$\begin{aligned}(\psi, P\otimes a).((\eta, R\otimes b).Q\otimes m) - (\eta, R\otimes b).((\psi, P\otimes a).Q\otimes m)& \\ = \psi(\eta(Q)) - \eta(\psi(Q))\otimes m& \\ + PQR\otimes a.(b.m) - b.(a.m)& \\ + P\eta(Q) - \eta(PQ)\otimes a.m + \psi(RQ) - R\psi(Q)\otimes b.m& \\ = [\psi,\eta](Q)\otimes m + PQR\otimes [a,b].m& \\ - Q\eta(P)\otimes a.m + Q\psi(R)\otimes b.m& \\ = [(\psi, P\otimes a), (\eta, R\otimes b)].(Q\otimes m).&\end{aligned} \tag{3.25}$$

To check the first axiom, we have

$$\begin{aligned}R.((\psi, P\otimes l).Q\otimes m) &= R.(\psi(Q)\otimes m + PQ\otimes l.m) \\ &= R\psi(Q)\otimes m + PQR\otimes l.m \\ &= \psi(RQ)\otimes m - Q\psi(R)\otimes m + PQR\otimes l.m \\ &= (\psi, P\otimes l).R.(Q\otimes m) - \rho_Y(\psi, P\otimes l)(R).(Q\otimes m).\end{aligned} \tag{3.26}$$

Finally, we have

$$\begin{aligned}R.((\psi, P\otimes l).Q\otimes m) &= R.(\psi(Q)\otimes m + PQ\otimes l.m) \\ &= R\psi(Q)\otimes m + PQR\otimes l.m \\ &= (R.(\psi, P\otimes l)).Q\otimes m. \qquad \square\end{aligned}$$

We now define equivariant representations. Let G be a smooth affine algebraic group of finite type acting on an R-variety X.

Definition 3.8.4. *Let $(\mathcal{L}, \rho, i_{\mathfrak{g}})$ be an r-deformed G-equivariant Lie algebroid on X. We say that $\mathcal{M}$ is a G-equivariant $\mathcal{L}$-module if:*

i) $\mathcal{M}$ *is a* G*-equivariant* $\mathcal{O}_X$*-module and* $\mathcal{M}$ *is a* $\mathcal{L}$*-module.*

ii) $g.(l.m) = (g.l).(g.m)$ *for any* $g \in G$*,* $l \in \mathcal{L}$ *and* $m \in \mathcal{M}$*.*

iii) The $r\mathfrak{g}$ *action induced by restricting the* $\mathfrak{g}$ *action induced from the derivative of the* G*-action on* $\mathcal{M}$ *coincides with the* $r\mathfrak{g}$*-action induced from* $i_{\mathfrak{g}}$*.*

A morphism of G*-equivariant* $\mathcal{L}$*-modules is a morphism of* G*-equivariant* $\mathcal{O}_X$*-modules compatible with the* $\mathcal{L}$*-action.*

Lemma 3.8.5. *Let* $\mathcal{L}$ *be an* r*-deformed* G*-equivariant Lie algebroid on* X *and* $\mathcal{M}$ *a* G*-equivariant* $\mathcal{L}$*-module. Further, let* Y *be a variety and* $f : Y \to X$ *a* G*-equivariant morphism. Then* $f^{\#}\mathcal{M}$ *is a* G*-equivariant* $f^{\#}\mathcal{L}$*-module.*

Proof. We have the G action on $f^{\#}\mathcal{M}$ induced by the action on the simple tensors $g.(Q \otimes m) = g.Q \otimes g.m$ for $g \in G$, $Q \in \mathcal{O}_Y$, $m \in f^{-1}\mathcal{M}$. Since $f^{\#}\mathcal{M} = f^{*}\mathcal{M}$ as a $\mathcal{O}_Y$-module, the first axiom of Definition 3.8.4 follows from Lemma 3.2.2.

Next, we have for $g \in G$, $\psi \in \mathcal{T}_Y, Q \in \mathcal{O}_Y, l \in f^{-1}\mathcal{L}, m \in f^{-1}\mathcal{M}$:

$$\begin{aligned} g.((\psi, P \otimes l).Q \otimes m) &= g.(\psi(Q) \otimes m + PQ \otimes l.m) \\ &= g.\psi(Q) \otimes g.m + g.(PQ) \otimes g.(l.m) \\ &= (g.\psi)(g.Q) \otimes g.m + g.Pg.Q \otimes (g.l).(g.m) \\ &= (g.\psi, g.(P \otimes l).(g.Q \otimes g.m) \\ &= (g.(\psi, P \otimes l)).(g.(Q \otimes m)). \end{aligned} \tag{3.27}$$

Finally, the third axiom follows easily from the definition of G-action on $f^{\#}\mathcal{L}$ and on $f^{\#}\mathcal{M}$. □

We should remark that one could define a more general notion of equivariance over an r-deformed G-equivariant Lie algebroid $(\mathcal{L}, \rho, i_g)$. Let L be a closed subgroup of G, then one may relax condition $i)$ to L-equivariance, impose condition $ii)$ to hold just for $g \in L$ and change condition $iii)$ to: the $r\mathfrak{l} = \mathrm{Lie}(L)$ action induced by derivative of the L-action on $\mathcal{M}$ coincides with the $r\mathfrak{l}$-action induced by the restriction of $i_{\mathfrak{g}}$ to $r\mathfrak{l}$.

3.9 Modules over twisted differential operators

We keep the notation from the previous section. As our main interest is in modules over deformed htdo's, we need a definition of a representation of a deformed Picard algebroid. Recall that $(\mathcal{L}, \rho)$ is an r-deformed Picard algebroid if $\mathcal{L}$ fits into the following short exact sequence $0 \to \mathcal{O}_X \to \mathcal{L} \to r\mathcal{T}_X \to 0$, Thus, for a $\mathcal{L}$-module, we get two actions of the structure sheaf: one since $\mathcal{M}$ is an $\mathcal{O}_X$-module by assumption and one induced by the short exact sequence.

We say that $\mathcal{M}$ is a *Picard module* if the two actions defined above coincide.

Lemma 3.9.1. *Let $(\mathcal{L}, \rho)$ be an r-deformed Picard algebroid and let $\mathcal{M}$ be a Picard $\mathcal{L}$-module. Then $\mathcal{M}$ is a module over $\mathcal{D} := \mathscr{T}(\mathcal{L})$.*

Proof. Since $\mathcal{M}$ is a $\mathcal{L}$-module, we obtain that $\mathcal{M}$ is also a $U(\mathcal{L})$-module in a similar fashion as Lie algebra representations correspond to enveloping algebra modules. Further, the condition that $\mathcal{M}$ is a Picard is exactly the condition that allows us to descend to a $\mathcal{D}$-module. □

As a corollary of the proof, we obtain immediately:

Corollary 3.9.2. *Let $(\mathcal{L}, \rho)$ be an r-deformed Picard algebroid on X and $\mathcal{D} = \mathscr{T}(\mathcal{L})$. Then the identity map provides a one-to-one correspondence between Picard $\mathcal{L}$-modules and $\mathcal{D}$-modules.*

For an r-deformed Picard algebroid $\mathcal{L}$ we will denote $\mathrm{Mod}(\mathcal{L})$ the category of Picard $\mathcal{L}$-modules. Similarly, for an r-deformed tdo $\mathcal{D}$ we denote $\mathrm{Mod}(\mathcal{D})$ the category of quasi-coherent $\mathcal{D}$-modules and $\mathrm{Coh}(\mathcal{D})$ its full subcategory consisting of coherent modules.

Equivariant representations

We now move to the equivariant setting. Recall that G is an algebraic group acting on X with Lie algebra $\mathfrak{g}$.

Definition 3.9.3. *Let $(\mathcal{D}, i_{\mathfrak{g}})$ be an r-deformed G-htdo. We say that a $\mathcal{D}$-module $\mathcal{M}$ is a G-equivariant module over $\mathcal{D}$ if:*

i) $\mathcal{M}$ is a G-equivariant $\mathcal{O}_X$-module.

ii) $g.(d.m) = (g.d).(g.m)$, for all $g \in G, d \in \mathcal{D}, m \in \mathcal{M}$.

iii) The $r\mathfrak{g}$-action induced by the derivative of the G-action on $\mathcal{M}$ coincides with the $r\mathfrak{g}$-action induced by $i_{\mathfrak{g}}$.

Lemma 3.9.4. *Let $(\mathcal{L}, \rho, i_{\mathfrak{g}})$ be an r-deformed G-equivariant Picard algebroid and $\mathcal{M}$ be a G-equivariant Picard $\mathcal{L}$-module. Further, let $\mathcal{D} = \mathscr{T}(\mathcal{L})$ the r-deformed G-htdo corresponding to $\mathcal{L}$. Then $\mathcal{M}$ is a G-equivariant $\mathcal{D}$-module.*

Proof. We have by Corollary 3.9.2 that $\mathcal{M}$ is a $\mathcal{D}$-module, so we only have to prove equivariance. Axioms i) and iii) follow from the corresponding axioms in Definition 3.8.4, while axiom ii) follows by an easy induction argument by using the definition of the G-action on $U(\mathcal{L})$. We only check the first step: we have that for $g \in G$, $l_1, l_2 \in \mathcal{L}$ and $m \in \mathcal{M}$ that

$$\begin{aligned} g.(l_1 l_2.m) &= g.(l_1.(l_2.m)) \\ &= (g.l_1).(g.(l_2.m)) \\ &= [(g.l_1)(g.l_2)].(g.m) \\ &= (g.l_1 l_2).(g.m). \end{aligned}$$

□

Similarly to the non-equivariant case, we obtain

Corollary 3.9.5. *Let X be an R-variety and $(\mathcal{L}, \rho, i_{\mathfrak{g}})$ an r-deformed G-equivariant Picard algebroid and let $\mathcal{D} = \mathscr{T}(\mathcal{L})$ be the corresponding r-deformed G-htdo. The identity map provides a one-to-one correspondence between G-equivariant Picard $\mathcal{L}$-modules and G-equivariant $\mathcal{D}$-modules.*

For an r-deformed G-equivariant Picard algebroid $\mathcal{L}$ we will denote $\mathrm{Mod}(\mathcal{L}, G)$ the category of G-equivariant Picard $\mathcal{L}$-modules. Similarly, for a G-htdo $\mathcal{D}$ we denote $\mathrm{Mod}(\mathcal{D}, G)$ the category of quasi-coherent G-equivariant $\mathcal{D}$-modules and $\mathrm{Coh}(\mathcal{D}, G)$ its full subcategory consisting of coherent modules. A similar argument to the one in Proposition 3.2.4 proves that these categories are Abelian.

Pullback of modules over twisted differential operators

We conclude the section by defining of the pullback of a module over a sheaf of twisted differential operators.

Definition 3.9.6. *Let $f : Y \to X$ be a morphism of R-varieties, $\mathcal{D}$ an r-deformed tdo on X and $\mathcal{M}$ a $\mathcal{D}$-module. We define the pullback of $\mathcal{M}$ under the map f to be $f^{\#}\mathcal{M}$.*

We should remark that the sheaf $f^{\#}\mathcal{M}$ has the structure of a $f^{\#}\mathcal{D}$-module by Corollary 3.5.7, Lemma 3.8.3 and Corollary 3.9.2.

Lemma 3.9.7. *Let $\mathcal{D}$ be an r-deformed G-htdo, $\mathcal{M}$ a G-equivariant $\mathcal{D}$-module and $f : Y \to X$ a G-equivariant morphism of R-varieties. Then $f^{\#}\mathcal{M}$ is a G-equivariant $f^{\#}\mathcal{D}$-module.*

Proof. We have by the definition above that $f^{\#}\mathcal{M}$ is a $f^{\#}\mathcal{D}$ module and by Corollary 3.9.2, $f^{\#}\mathcal{D}$ is an r-deformed G-htdo, so the statement makes sense. The claim now follows by combining Lemma 3.8.5 and Corollary 3.9.5. □

As for the r-deformed G-htdo's, we may prove that a G-equivariant module over an r-deformed G-htdo module satisfies a cocycle condition. Denote the G-action on X by $\sigma_X : G \times X \to X$. Further, we denote $p_X : G \times X \to X$ and $p_{2X} : G \times G \times X \to X$ the projections on the X factor, $p_{23X} : G \times G \times X \to G \times X$ the projection onto the second and third factor and $m : G \times G \to G$ the multiplication of the group G. Let $\mathcal{D}$ be an r-deformed G-htdo and $\mathcal{M}$ a G-equivariant $\mathcal{D}$-module. Then there exists $\alpha : \sigma_X^{\#}\mathcal{M} \to p_X^{\#}\mathcal{M}$ an isomorphism of $p_X^{\#}\mathcal{D}$-modules such that the diagram 3.21 commutes, where we replace $\mathcal{D}$ by $\mathcal{M}$. Again, we note that the condition is similar to the one in [59, Section 5.2.9].

3.10 Equivariant descent for Lie algebroids and homogeneous twisted differential operators

Throughout this section, we let X, Y be R-varieties, G a smooth affine algebraic group of finite type acting *freely* on Y. Further, we fix $f : Y \to X$ a locally trivial G-torsor and $(\mathcal{L}, \rho, i_{\mathfrak{g}})$ an r-deformed G-equivariant Picard algebroid on Y. Let $\alpha : \mathfrak{g} \to \mathcal{T}_Y$ be the derivative of the G-action on Y. Recall that by Definition 3.6.4, $\alpha = \rho \circ i_{\mathfrak{g}}$ as maps from $r\mathfrak{g}$ to $\mathcal{T}_X$. Throughout this section, we will use without further specifying that all the R-modules appearing have no r-torsion.

Let $\widetilde{\mathcal{T}_X} := (f_*\mathcal{T}_Y)^G$ and let $\sigma : \widetilde{\mathcal{T}_X} \to \mathcal{T}_X$ denote the anchor map. Further, we denote $\widetilde{\mathfrak{g}_X} := (f_*\mathcal{O}_Y \otimes \mathfrak{g})^G$, where G acts on $\mathfrak{g}$ via the Adjoint action. We let $\tilde{\alpha}$ the induced map $\widetilde{\mathfrak{g}_X} \to \widetilde{\mathcal{T}_X}$. Since $f_*\mathcal{O}_Y \otimes \mathfrak{g}$ has no r-torsion, it is easy to see that $r\widetilde{\mathfrak{g}_X} \cong (f_*\mathcal{O}_Y \otimes r\mathfrak{g})^G$.

Lemma 3.10.1. *The maps $\tilde{\alpha}$ and σ induce a short exact sequence*

$$0 \to \widetilde{\mathfrak{g}_X} \xrightarrow{\tilde{\alpha}} \widetilde{\mathcal{T}_X} \xrightarrow{\sigma} \mathcal{T}_X \to 0.$$

Proof. The question is local, so we may assume that X is affine $Y = G \times X$, G acts on Y via left multiplication on the first factor and f is the projection on the second factor. In that case, we have

$$\begin{aligned}
\widetilde{\mathfrak{g}_X}(X) :&= (f_*\mathcal{O}_Y \otimes \mathfrak{g})^G(X) \\
&= (\mathcal{O}_Y(Y) \otimes \mathfrak{g})^G \\
&\cong (\mathcal{O}_X(X) \otimes \mathcal{O}_G(G) \otimes \mathfrak{g})^G \\
&\cong \mathcal{O}_X(X) \otimes (\mathcal{O}_G(G) \otimes \mathfrak{g})^G \\
&\cong \mathcal{O}_X(X) \otimes (\mathcal{T}(G))^G.
\end{aligned} \tag{3.28}$$

Further, we have by the proof of [5, Lemma 4.4] that $\widetilde{\mathcal{T}_X}(X) \cong \mathcal{O}_X(X) \otimes \mathcal{T}(G)^G \oplus \mathcal{T}_X(X)$, so the conclusion follows. ☐

By abuse of notation we denote $i_{\mathfrak{g}}$ the map $i_{\mathfrak{g}} : \mathcal{O}_Y \otimes r\mathfrak{g} \to \mathcal{L}$ and $\tilde{i}_{\mathfrak{g}} : r\widetilde{\mathfrak{g}_X} \to (f_*\mathcal{L})^G$ the induced map.

Definition 3.10.2. *Let $f_\#\mathcal{L}^G$ be the $\mathcal{O}_X$-module $(f_*\mathcal{L})^G / \tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X})$. We call this the descent of $\mathcal{L}$.*

Lemma 3.10.3. *The $\mathcal{O}_X$-module $f_\#\mathcal{L}^G$ has the structure of an r-deformed Picard algebroid.*

Proof. The bracket structure on $\widetilde{\mathcal{L}} := (f_*\mathcal{L})^G$ is induced from the bracket structure on $\mathcal{L}$; furthermore this descends to a bracket structure on $f_\#\mathcal{L}^G$ by setting $[a + \tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X}), b + \tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X})] = [a, b] + \tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X})$ for $a, b \in (f_*\mathcal{L})^G$. Since the image of $i_{\mathfrak{g}}$ is an ideal in $\mathcal{L}$, we obtain that the image of $\tilde{i}_{\mathfrak{g}}$ is an ideal in $(f_*\mathcal{L})^G$, thus the bracket is well defined. Therefore, we are left to construct an anchor map.

Consider the short exact sequence $0 \to \mathcal{O}_Y \to \mathcal{L} \xrightarrow{\rho} r\mathcal{T}_Y \to 0$. Applying Proposition 3.3.5, we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to \widetilde{\mathcal{L}} \xrightarrow{\tilde{\rho}} r\widetilde{\mathcal{T}_X} \to 0.$$

By construction, we have $\tilde{\alpha} = \tilde{\rho} \circ \tilde{i}_{\mathfrak{g}}$, so the map

$$\bar{\tilde{\rho}} : \widetilde{\mathcal{L}} / \tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X}) \to \widetilde{\mathcal{T}_X} / \tilde{\alpha}(r\widetilde{\mathfrak{g}_X})$$

is well defined. Furthermore, since there is no r-torsion, we have by Lemma 3.10.1 an induced isomorphism $\bar{\sigma} : r\widetilde{\mathcal{T}}_X/r\tilde{\alpha}(\widetilde{\mathfrak{g}_X}) \to r\mathcal{T}_X$. We define the anchor map on $\widetilde{\mathcal{L}}/\,\tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X})$ to be $\rho_X := \bar{\sigma}\circ\bar{\tilde{\rho}}$ and it is clear that together with the bracket $\widetilde{\mathcal{L}}/\,\tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X})$ becomes a Lie algebroid, so it remains to show that it is an r-deformed Picard algebroid. It is easy to see that $\operatorname{im}\rho_X = r\mathcal{T}_X$, so it is enough to prove that $\ker(\rho_X) \cong \ker\widetilde{\rho} \cong \mathcal{O}_X$.

Since $\bar{\sigma}$ is injective we have that

$$\ker\rho_X = \ker(\bar{\tilde{\rho}}) = (\ker\tilde{\rho} + \tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X}))/\,\tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X}) \cong \ker\tilde{\rho}/\ker\tilde{\rho}\cap\tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X})$$

by the second isomorphism theorem. Thus, it is enough to prove $\ker\tilde{\rho}\cap\tilde{i}_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X}) = 0$ and since $\tilde{\alpha} = \tilde{\rho}\circ\tilde{i}_{\mathfrak{g}}$, this reduces to proving $\ker\tilde{\alpha} = 0$. By assumptions, the action of G on Y is free; thus the map $\alpha : \mathfrak{g} \to \mathcal{T}_Y$ is injective and since $\mathcal{O}_Y$ is a faithfully flat R-module, the induced map $\alpha : \mathcal{O}_Y \otimes r\mathfrak{g} \to r\mathcal{T}_Y$ injective and thus so is $\tilde{\alpha}$. □

Lemma 3.10.4. *Let $\mathcal{L}$ as before. Then $f^{\#}(f_{\#}\mathcal{L}^G) \cong \mathcal{L}$.*

Proof. Recall that by abuse of notation, we denote $i_{\mathfrak{g}} : \mathcal{O}_Y \otimes r\mathfrak{g} \to \mathcal{L}$ and $\alpha : \mathcal{O}_Y \otimes r\mathfrak{g} \to \mathcal{T}_Y$ the induced maps; we still have $\alpha = \rho\circ i_{\mathfrak{g}}$. Let $\widetilde{\mathcal{L}} := (f_*\mathcal{L})^G$ so that there is a short exact sequence

$$0 \to (f_*\mathcal{O}_Y \otimes r\mathfrak{g})^G \to \widetilde{\mathcal{L}} \to f_{\#}\mathcal{L}^G \to 0$$

Pulling back under the torsor f and applying Proposition 3.3.5, we obtain a short exact sequence

$$0 \to \mathcal{O}_Y \otimes r\mathfrak{g} \xrightarrow{i_{\mathfrak{g}}} \mathcal{L} \to f^*(f_{\#}\mathcal{L}^G) \to 0,$$

so, $f^*(f_{\#}\mathcal{L}^G) \cong \mathcal{L}/i_{\mathfrak{g}}(\mathcal{O}_Y \otimes r\mathfrak{g})$.

Similarly, by pulling back under f the short exact sequence in Lemma 3.10.1 and taking into account there is no r-torsion we obtain a short exact sequence

$$0 \to \mathcal{O}_Y \otimes r\mathfrak{g} \xrightarrow{\alpha} r\mathcal{T}_Y \to rf^*\mathcal{T}_X \to 0, \tag{3.29}$$

so $rf^*\mathcal{T}_X \cong r\mathcal{T}_Y/\alpha(\mathcal{O}_Y \otimes r\mathfrak{g})$.

Therefore, we obtain

$$f^{\#}(f_{\#}\mathcal{L}^G) \cong f^*(f_{\#}\mathcal{L}^G) \times_{rf^*\mathcal{T}_X} r\mathcal{T}_Y \cong \mathcal{L}/i_{\mathfrak{g}}(\mathcal{O}_Y \otimes r\mathfrak{g}) \times_{r\mathcal{T}_Y/\alpha(\mathcal{O}_Y\otimes r\mathfrak{g})} r\mathcal{T}_Y.$$

Define $\varphi : \mathcal{L} \to f^{\#}(f_{\#}\mathcal{L}^G)$ by $\varphi(l) = (l + i_{\mathfrak{g}}(\mathcal{O}_Y \otimes r\mathfrak{g}), \rho(l))$ for any $l \in \mathcal{L}$.

To see that φ is injective, we use that α is an injective map, thus so is $i_{\mathfrak{g}}$, therefore $i_{\mathfrak{g}}(\mathcal{O}_Y \otimes r\mathfrak{g}) \cap \ker(\rho) = 0$.

Finally, let $(l + i_{\mathfrak{g}}(\mathcal{O}_Y \otimes r\mathfrak{g}), x) \in \mathcal{L}/i_{\mathfrak{g}}(\mathcal{O}_Y \otimes r\mathfrak{g}) \times_{r\mathcal{T}_Y/\alpha(\mathcal{O}_Y \otimes r\mathfrak{g})} r\mathcal{T}_Y$, so that $\rho(l) + \alpha(\mathcal{O}_Y \otimes r\mathfrak{g}) = x + \alpha(\mathcal{O}_Y \otimes r\mathfrak{g})$, thus $x = \rho(l) + i$ for some $i \in \alpha(\mathcal{O}_Y \otimes r\mathfrak{g})$. Since α is injective there exist a unique $j \in \mathcal{O}_Y \otimes r\mathfrak{g}$, so that $\alpha(j) = i$. Thus $\varphi(l + i_{\mathfrak{g}}(j)) = (l + i_{\mathfrak{g}}(\mathcal{O}_Y \otimes r\mathfrak{g}), x)$, so φ is surjective. ☐

Lemma 3.10.5. *Let $\mathcal{P}$ be an r-deformed Picard algebroid on X. Then $f_{\#}(f^{\#}\mathcal{P})^G \cong \mathcal{P}$.*

Proof. We may view $\mathcal{P}$ as an r-deformed G-equivariant Picard algebroid by letting $g.p = p$ for all $g \in G, p \in P$ and the map $r\mathfrak{g} \to \mathcal{P}$ to be the 0 map. Then by Lemma 3.7.4, $f^{\#}\mathcal{P}$ is an r-deformed G-equivariant Picard algebroid. The induced map $i_{\mathfrak{g}} : r\mathfrak{g} \to f^*\mathcal{P} \times_{rf^*\mathcal{T}_X} r\mathcal{T}_Y$ is defined as $i_{\mathfrak{g}}(\psi) = (0, \alpha(\psi))$.

Using Proposition 3.3.5 for $\mathcal{O}$-modules together with the fact that $(f_*)^G$ commutes with limits we get:

$$\begin{aligned}(f_*f^{\#}\mathcal{P})^G &= (f_*(f^*\mathcal{P} \times_{rf^*\mathcal{T}_X} r\mathcal{T}_Y))^G \\ &\cong (f_*f^*\mathcal{P})^G \times_{r(f_*f^*\mathcal{T}_X)^G} (rf_*\mathcal{T}_Y)^G \\ &\cong \mathcal{P} \times_{r\mathcal{T}_X} r\widetilde{\mathcal{T}_X}.\end{aligned} \tag{3.30}$$

Therefore, we obtain:

$$\begin{aligned}f_{\#}(f^{\#}\mathcal{P})^G &\cong (f_*f^{\#}\mathcal{P})^G / i_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X}) \\ &\cong (\mathcal{P} \times_{r\mathcal{T}_X} r\widetilde{\mathcal{T}_X}) / i_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X}) \\ &\cong \mathcal{P} \times_{r\mathcal{T}_X} r\mathcal{T}_X \text{ (by Lemma 3.10.1).} \\ &\cong \mathcal{P}.\end{aligned}$$

☐

Corollary 3.10.6. *The functors $f_{\#}(-)^G$ and $f^{\#}(-)$ induce quasi-inverse equivalences of categories between the category of r-deformed G-equivariant Picard algebroids on Y and the category of r-deformed Picard algebroids on X.*

Proof. The proof follows from Lemmas 3.10.4 and 3.10.5. ☐

We consider again a more general setting: let B another smooth affine algebraic group acting on Y and X such that $f : Y \to X$ is B-equivariant. Recall that we are assuming that f is a locally trivial G-torsor.

Lemma 3.10.7. *With the assumption as above we have:*

- *let* $(\mathcal{L}, \rho, i_{\mathfrak{g}\times\mathfrak{b}})$ *be an* r*-deformed* $G \times B$*-equivariant Picard algebroid on* Y*. Then* $f_{\#}\mathcal{L}^G$ *may be given the structure of an* r*-deformed* B*-equivariant Picard algebroid on* X*.*
- *let* $(\mathcal{K}, \rho_X, j_{\mathfrak{b}})$ *be an* r*-deformed* B*-equivariant Picard algebroid on* X*. Then* $f^{\#}\mathcal{K}$ *may be given the structure of an* r*-deformed* $G \times B$*-equivariant Picard algebroid on* Y*.*

Proof. For the first claim, we start by proving that $\widetilde{\mathcal{L}} = (f_*\mathcal{L})^G$ is a B-equivariant Lie algebroid. First, we get a B-action on $\widetilde{\mathcal{L}}$ from Lemma 3.3.7, so that $\widetilde{\mathcal{L}}$ becomes a B-equivariant $\mathcal{O}_X$-module. Further, since $\mathcal{L}$ is B-equivariant, axiom $i)$ of definition 3.6.1 is also satisfied.

Next, since the actions of G and B on Y commute the anchor map $\sigma : \widetilde{\mathcal{T}_X} \to \mathcal{T}_X$ is B-equivariant. Composing with the B-equivariant map $\tilde{\rho} : \widetilde{\mathcal{L}} \to \widetilde{\mathcal{T}_X}$ we obtain a B-equivariant map $\sigma \circ \tilde{\rho} : \widetilde{\mathcal{L}} \to \mathcal{T}_X$, so axiom $ii)$ of 3.6.1 is satisfied.

We let $i_{\mathfrak{g}}$ and $i_{\mathfrak{b}}$ denote the restriction of $i_{\mathfrak{g}\times\mathfrak{b}}$ to $r\mathfrak{g} \cong r\mathfrak{g} \times 0 \subset r(\mathfrak{g} \times \mathfrak{b})$ and $r\mathfrak{b} \cong 0 \times r\mathfrak{b} \subset r(\mathfrak{g} \times \mathfrak{b})$, respectively. Let $\beta : \mathfrak{b} :\to \mathcal{T}_Y$, $\gamma : \mathfrak{b} \to \mathcal{T}_X$ denote the infinitesimal action B on Y and X; since $\mathcal{L}$ is in particular B-equivariant, we have $\beta = \rho \circ i_{\mathfrak{b}}$. By descending we obtain maps $\tilde{\beta} : r\mathfrak{b} \to \widetilde{\mathcal{T}_X}$ and $\tilde{i_{\mathfrak{b}}} : r\mathfrak{b} \to \widetilde{\mathcal{L}}$ such that $\tilde{\beta} = \tilde{\rho} \circ \tilde{i_{\mathfrak{b}}}$. Since the actions of G and B commute, we also get $\gamma = \sigma \circ \tilde{\beta}$ so $\gamma = \sigma \circ \tilde{\beta}$. Therefore combining this two we get:

$$\gamma = (\sigma \circ \tilde{\rho}) \circ \tilde{i_{\mathfrak{b}}}.$$

We get since $\mathcal{L}$ is B-equivariant, that the Lie algebroid $(\widetilde{\mathcal{L}}, \sigma \circ \tilde{\rho}, \tilde{i_{\mathfrak{b}}})$ is also B-equivariant. By the proof of Lemma 3.10.3, to show that $f_{\#}\mathcal{L}^G$ is B-equivariant it suffices to prove that $i_{\mathfrak{g}}(\widetilde{r\mathfrak{g}_X})$ is B-equivariant as an $\mathcal{O}_X$-module. Further, by using Lemma 3.3.7 this is equivalent to proving $i_{\mathfrak{g}}(\mathcal{O}_Y \otimes r\mathfrak{g})$ is a B-equivariant $\mathcal{O}_Y$-module. This is true since $i_{\mathfrak{g}\times\mathfrak{b}}$ is in particular B-equivariant, so $i_{\mathfrak{g}}$ sends a B-equivariant module to a B-equivariant module.

Now, we prove the second claim. We may endow $\mathcal{K}$ with a trivial G-action $g.k = k$ for all $k \in \mathcal{K}$ and with the zero map $j_{\mathfrak{g}} \to \mathcal{K}$ so that $(\mathcal{K}, \rho_X, j_{\mathfrak{g}} \times j_{\mathfrak{b}})$ becomes $G \times B$-equivariant. The claim follows from Lemma 3.7.4. ☐

Corollary 3.10.8. *The functors $f_\#(-)^G$ and $f^\#(-)$ induce quasi-inverse equivalences of categories between the category of r-deformed $G \times B$-equivariant Picard algebroids on Y and the category of r-deformed B-equivariant Picard algebroids on X.*

Proof. This follows from Corollary 3.10.6 and Lemma 3.10.7. ☐

Descent of twisted differential operators

We keep the notations from the start of the section. Further, we let $\mathcal{D}$ be a sheaf of r-deformed G-homogeneous twisted differential operators on Y.

Definition 3.10.9. *We define the descent of $\mathcal{D}$ under the torsor $f : Y \to X$ to be the sheaf*

$$f_\#\mathcal{D}^G := \mathscr{T}(f_\#(\operatorname{Lie}\mathcal{D})^G).$$

Lemma 3.10.10. *Let the notations be as above. Then:*

i) $f_\#\mathcal{D}^G$ *is an r-deformed tdo on X.*

ii) $f^\#(f_\#\mathcal{D}^G) \cong \mathcal{D}$.

Proof. The first claim follows from Lemma 3.10.3 and Corollary 3.5.7. The second claim follows from Lemma 3.10.4 and Corollary 3.6.8. ☐

Lemma 3.10.11. *Let $\mathcal{A}$ be an r-deformed tdo on X. Then $(f_\# f^\# \mathcal{A})^G \cong \mathcal{A}$.*

Proof. We view $\mathcal{A}$ as an r-deformed G-equivariant htdo on X with the trivial G-action, so that $\operatorname{Lie}(\mathcal{A})$ is an r-deformed G-equivariant Picard algebroid with the trivial G-action. The claim follows from Lemma 3.10.5 and Corollary 3.5.7. ☐

Similarly to the deformed Picard algebroids case we obtain:

Corollary 3.10.12. *The functors $f_\#(-)^G$ and $f^\#(-)$ induce quasi-inverse equivalences between the category of r-deformed G-equivariant htdo's on Y and the category of r-deformed tdo's on X.*

Proof. This follows from the Lemmas 3.10.10 and 3.10.11. □

Corollary 3.10.13. *Assume that f is also B-equivariant. The functors $f_{\#}(-)^G$ and $f^{\#}(-)$ induce quasi-inverse equivalences between the category r-deformed $G \times B$-equivariant htdo's on Y and the category r-deformed B-equivariant htdo's on X.*

Proof. This follows from the previous corollary, along with Lemma 3.10.7 and Corollary 3.6.8. □

3.11 Equivariant descent for equivariant Picard algebroids and htdo's modules

We keep the notations from the previous section: recall that G is a smooth affine algebraic group of finite type, X and Y are R-varieties, and $f : Y \to X$ is a locally trivial G-torsor. Further, we assume that $(\mathcal{L}, \rho, i_{\mathfrak{g}})$ is an r-deformed G-equivariant Picard algebroid on Y.

Lemma 3.11.1. *Let $\mathcal{M}$ be a G-equivariant Picard $\mathcal{L}$-module. Then $(f_*\mathcal{M})^G$ is a Picard $f_{\#}\mathcal{L}^G$-module. We call $(f_*\mathcal{M})^G$ the* descent *of $\mathcal{M}$.*

Proof. Let $\widetilde{\mathcal{L}} = (f_*\mathcal{L})^G$. Then $(f_*\mathcal{M})^G$ is a Picard $\widetilde{\mathcal{L}}$-module, so it remains to prove that the action of $i_{\mathfrak{g}}(r\widetilde{\mathfrak{g}_X})$ kills $(f_*\mathcal{M})^G$.

Since the G action on $(f_*\mathcal{M})^G$ is constant, by differentiating this action we obtain a trivial $\mathfrak{g}$ action, so a trivial $r\mathfrak{g}$ action. But by our assumption on $\mathcal{M}$ this coincides with the $r\mathfrak{g}$ action induced from $i_{\mathfrak{g}} : r\mathfrak{g} \to \mathcal{L}$; the conclusion follows. □

Proposition 3.11.2. *Let $\mathcal{L}$ be an r-deformed G-equivariant Picard algebroid on Y. The functors*

$$\begin{aligned} f_*(-)^G &: \operatorname{Mod}(\mathcal{L}, G) \to \operatorname{Mod}(f_{\#}\mathcal{L}^G), \\ f^{\#}(-) &: \operatorname{Mod}(f_{\#}\mathcal{L}^G) \to \operatorname{Mod}(\mathcal{L}, G), \end{aligned} \tag{3.31}$$

are quasi-inverse equivalences of categories.

Proof. This follows from the Lemma above, Lemma 3.10.4 and Proposition 3.3.5. □

Corollary 3.11.3. *Let $\mathcal{D}$ be an r-deformed G-htdo on Y. The functors*

$$\begin{aligned} f_*(-)^G &: \mathrm{Mod}(\mathcal{D}, G) \to \mathrm{Mod}(f_\# \mathcal{D}^G), \\ f^\#(-) &: \mathrm{Mod}(f_\# \mathcal{D}^G) \to \mathrm{Mod}(\mathcal{D}, G), \end{aligned} \tag{3.32}$$

are quasi-inverse equivalences of categories.

Proof. This follows from the Proposition above and Corollaries 3.9.2 and 3.9.5. □

Let B is another smooth affine algebraic group acting on Y such that the actions of G and B commute and $f : Y \to X$ is B-equivariant. Let $\mathcal{D}$ be an r-deformed $G \times B$-htdo. Similar to the $\mathcal{O}$-module case (Corollary 3.3.8), we obtain using Corollary 3.11.3:

Corollary 3.11.4. *Let $\mathcal{D}$ be an r-deformed $G \times B$-htdo on Y and assume f is B-equivariant. The functors*

$$\begin{aligned} f_*(-)^G &: \mathrm{Mod}(\mathcal{D}, G \times B) \to \mathrm{Mod}(f_\# \mathcal{D}^G, B), \\ f^\#(-) &: \mathrm{Mod}(f_\# \mathcal{D}^G, B) \to \mathrm{Mod}(\mathcal{D}, G \times B), \end{aligned} \tag{3.33}$$

are quasi-inverse equivalences of categories.

To prove the main theorem of the chapter, we need two easy results. Recall that for an R-variety Z we denote $\mathcal{D}_Z$ the sheaf of crystalline differential operators on Z. For $r \in R$ regular element, we call $\mathcal{D}_{Z,r} = \mathscr{T}(\mathcal{O}_Z \oplus r\mathcal{T}_Z)$ the sheaf of r-deformed differential operators and $\mathcal{D}(Z)_r = \Gamma(Z, \mathcal{D}_{Z,r})$. It is clear by construction that $\mathcal{D}_{Z,r}$ is an r-deformed tdo. We also make the following assumption:

Assumption 3.11.5. *The base ring R is Noetherian.*

Lemma 3.11.6. *Let $\mathcal{M}$ be a $\mathcal{D}_{X,r}$-module. Then $f^*\mathcal{M}$ is a coherent $\mathcal{D}_{Y,r}$-module if and only $\mathcal{M}$ is a coherent $\mathcal{D}_{X,r}$-module.*

Proof. Since coherence is a local property, we may assume that X is affine, $Y = G \times X$, f is the projection onto the second factor and G acts on Y via left multiplication on the first factor. Let M be a $\mathcal{D}(X)_r$ module, so $f^*M \cong \mathcal{O}(G) \otimes M$.

We may view $D(Y)_r$ as an r-deformation of the ring $D(Y)$. Since $D(Y) \cong D(G) \otimes D(X)$ and deformations commutes with tensor product, we obtain that the ring $\mathcal{D}(Y)_r \cong \mathcal{D}(G)_r \otimes \mathcal{D}(X)_r$ acts naturally on f^*M. Further, since our base ring R is Noetherian we have by Lemma 3.4.6 that $D(Y)_r$ and $D(X)_r$ are Noetherian rings, so coherence is equivalent to local finite generation. Thus it is enough to prove the following claim:

Claim. M is a finitely generated $\mathcal{D}(X)_r$ module if and only if $\mathcal{O}(G) \otimes M$ is a finitely generated $\mathcal{D}(G)_r \otimes \mathcal{D}(X)_r$ module.

The direct implication is trivial. For the reverse implication, we will prove the contrapositive: if M is not Noetherian, neither is $\mathcal{O}(G)\otimes M$. Let $M_1 \subsetneq M_2 \subsetneq M_3 \ldots \subsetneq$ be an infinite ascending of M-submodules. Since G is a smooth group, $\mathcal{O}(G)$ is a faithfully flat R-module, thus tensoring with $\mathcal{O}(G)$ preserves injections. In particular, we obtain an infinite chain

$$\mathcal{O}(G) \otimes M_1 \subsetneq \mathcal{O}(G) \otimes M_2 \subsetneq \mathcal{O}(G) \otimes M_3 \ldots \subsetneq$$

of submodules of $\mathcal{O}(G) \otimes M$. Therefore, the claim is proven. □

Corollary 3.11.7. *Let $\mathcal{D}$ be an r-deformed G-htdo on Y and let $\mathcal{M}$ be a $(f_\#\mathcal{D})^G$-module. Then $f^\#\mathcal{M}$ is a coherent $\mathcal{D}$-module if and only if $\mathcal{M}$ is coherent.*

Proof. The claim is local, thus we may assume that $\mathcal{D} = \mathcal{D}_{Y,r}$ so that $(f_\#\mathcal{D})^G = \mathcal{D}_{X,r}$. The claim follows by the Lemma above and Lemma 3.10.10. □

We may now prove the main theorem.

Theorem 3.11.8. *Assume the base ring R is Noetherian. Let G be a smooth affine algebraic group of finite type. Let X, Y be R-varieties and let $f : Y \to X$ be a locally trivial G-torsor. Further, let $\mathcal{D}$ be a sheaf of r-deformed G-homogeneous twisted differential operators on Y. The functors:*

$$\begin{aligned} f_*(-)^G &: \mathrm{Coh}(\mathcal{D}, G) \to \mathrm{Coh}(f_\#\mathcal{D}^G), \\ f^\#(-) &: \mathrm{Coh}(f_\#\mathcal{D}^G) \to \mathrm{Coh}(\mathcal{D}, G), \end{aligned} \tag{3.34}$$

are quasi-inverse equivalences of categories between coherent G-equivariant $\mathcal{D}$-modules and coherent $(f_\#\mathcal{D})^G$-modules.

Proof. This follows from Corollaries 3.11.3 and 3.11.7. □

As a corollary, we obtain using Corollary 3.11.4:

Corollary 3.11.9. *Let the assumptions be as above. Further, assume that B is a smooth affine algebraic group of finite type such that $f : Y \to X$ is B-equivariant. Further assume that $\mathcal{D}$ is an r-deformed $G \times B$-htdo on Y. The functors:*

$$\begin{aligned} &f_*(-)^G : \mathrm{Coh}(\mathcal{D}, G \times B) \to \mathrm{Coh}(f_\# \mathcal{D}^G, B), \\ &f^\#(-) : \mathrm{Coh}(f_\# \mathcal{D}^G, B) \to \mathrm{Coh}(\mathcal{D}, G \times B), \end{aligned} \tag{3.35}$$

are quasi-inverse equivalences of categories.

Chapter 4

A geometric proof of classical Duflo's theorem

4.1 Introduction

This chapter is based on the second paper of the series ([58], [56], [57]). The goal of this chapter is to explain the well-known Beilinson-Bernstein localisation mechanism in a more general setting. In particular, we will work over a commutative ring R of arbitrary characteristic rather than over an algebraically closed field of characteristic 0. Together with the framework developed in the previous chapter, we give a new proof of Duflo's theorem classifying primitive ideals in enveloping algebras of semisimple Lie algebras defined over a field of characteristic 0.

Let K be a field of characteristic 0 and G be a connected, simply-connected, split semisimple, smooth affine algebraic group over K with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Fix $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ a Cartan decomposition. In a seminal paper [12], the authors define the category $\mathcal{O}$ of representations for the algebra $U(\mathfrak{g})$. The building blocks are given by Verma modules $M(\lambda) = U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} K$ for each $\lambda \in \mathfrak{h}^*$; $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. These are highest weight modules with unique maximal submodule $N(\lambda)$ and unique simple quotient $L(\lambda)$. Moreover, this category $\mathcal{O}$ is Artinian and the set of simple objects is characterised exactly by $L(\lambda)$ for $\lambda \in \mathfrak{h}^*$. An excellent exposition of category $\mathcal{O}$ can be found in [34].

The importance of category $\mathcal{O}$ in the representation theory of the ring $U(\mathfrak{g})$ can be seen in the following theorem:

Theorem 4.1.1 (Duflo's Theorem). *[25, Theorem 4.3]*

Let I be a primitive/prime ideal in $U(\mathfrak{g})$ with K-rational infinitesimal central character. Then

$$I = \operatorname{Ann}(L(\lambda)) \text{ for some } \lambda \in \mathfrak{h}^*.$$

Duflo's original statement requires the ground field to be $\mathbb{C}$. In their paper [13], Bernstein and Gelfand extend this result for algebraically closed fields of characteristic 0 which in turn can be extended to the generality stated by a base change argument. Another purely algebraic proof of Duflo's theorem can be found in [37] and for a categorical proof, see [28]. One should note that if K is algebraically closed, all primitive ideals have K-rational central character, so the theorem gives a full classification of the primitive spectrum.

Fix B a Borel subgroup of G and let $X = G/B$ denote the flag variety of the group G. For $\lambda \in \mathfrak{h}^*$, let χ_λ denote the corresponding central character. Furthermore, let $U(\mathfrak{g})^\lambda$ be the quotient of $U(\mathfrak{g})$ by the two-sided ideal generated by $\ker(\chi_\lambda)$ and $\mathcal{D}_\lambda$ denote the sheaf of λ-twisted differential operators on X as defined in [8].

For a Lie algebra $\mathfrak{g}$, we let ρ denote the half sum of positive roots. Throughout this document, we call a weight $\lambda \in \mathfrak{h}^*$ dominant if it is ρ-dominant, i.e. for any coroot $\alpha^\vee$ corresponding to a positive root α we have $(\lambda+\rho)(\alpha^\vee) \notin \{-1, -2, -3, \ldots\}$.

Theorem 4.1.2 (Beilinson-Bernstein localisation, [8])**.** *Let $\lambda : \mathfrak{h} \to K$ be a dominant weight. Consider the functors:*

$$\begin{aligned} &\operatorname{Loc} : \operatorname{Mod}(U(\mathfrak{g})^\lambda) \to \operatorname{QCoh}(\mathcal{D}_\lambda), &&\operatorname{Loc}(M) := \mathcal{D}_\lambda \underset{U(\mathfrak{g})^\lambda}{\otimes} M, \\ &\Gamma : \operatorname{QCoh}(\mathcal{D}_\lambda) \to \operatorname{Mod}(U(\mathfrak{g})^\lambda), &&\Gamma(\mathcal{M}) := \mathcal{M}(X). \end{aligned} \tag{4.1}$$

The functors Loc *and* Γ *induce quasi-inverse equivalences of categories between* $\operatorname{Mod}(U(\mathfrak{g})^\lambda)$ *and the quotient category* $\operatorname{QCoh}(\mathcal{D}_\lambda)/\ker\Gamma$. *In the case when λ is also regular, one has* $\ker\Gamma = 0$.

Furthermore, this equivalence restricts to an equivalence of categories between finitely generated $U(\mathfrak{g})^\lambda$-modules and coherent $\mathcal{D}_\lambda$-modules.

Let L be a closed subgroup of G. In [8], the authors define the notion of L-equivariant $U(\mathfrak{g})$ and $\mathcal{D}$-modules. A more detailed definition can be found in [32, Section 11.5] . The equivariant localisation theorem states:

Theorem 4.1.3 (Equivariant Beilinson-Bernstein localisation, [8]). *Let $\lambda : \mathfrak{h} \to K$ be a dominant weight. Consider the functors:*

$$\begin{aligned} &\mathrm{Loc} : \mathrm{Mod}(U(\mathfrak{g})^\lambda, L) \to \mathrm{QCoh}(\mathcal{D}_\lambda, L), \quad && \mathrm{Loc}(M) := \mathcal{D}_\lambda \underset{U(\mathfrak{g})^\lambda}{\otimes} M, \\ &\Gamma : \mathrm{QCoh}(\mathcal{D}_\lambda, L) \to \mathrm{Mod}(U(\mathfrak{g})^\lambda, L), && \Gamma(\mathcal{M}) := \mathcal{M}(X). \end{aligned} \tag{4.2}$$

The functors Loc *and* Γ *induce quasi-inverse equivalences of categories between* $\mathrm{Mod}(U(\mathfrak{g})^\lambda, L)$ *and the quotient category* $\mathrm{QCoh}(\mathcal{D}_\lambda, L)/\ker \Gamma$.

Using the equivariant Beilinson-Bernstein localisation, the authors prove in [17] a version of Duflo's theorem for ideals with trivial central character for Lie algebras defined over the complex numbers.

Statement of the main results of the chapter

Let R be a commutative base ring and let G be a connected, simply-connected, split semisimple, smooth affine algebraic group defined over R and let $r \in R$ be a regular element. Fix B a Borel subgroup of G and let $X = G/B$ be the flag scheme.

Let $\mathcal{D}$ be a sheaf of r*-deformed* G*-homogeneous twisted differential operators (*G*-htdo)* on the double flag variety $X \times X$, see Section 3.6 for the definition of an r-deformed G-htdo. Let $x_0 = eB$ be a base point of X and let $i_r : X \to X \times X$, $i_r(x) = (x_0, x)$ be the inclusion of X into the right copy. One can define the pullback of $\mathcal{D}$, $i_r^{\#}\mathcal{D}$ as a sheaf of rings on X, see Section 3.7 for the definition of the pullback. We should remark that i_r does not depend on the deformation parameter, we use the index r here to specify that it is the inclusion into the *right* copy.

Theorem 4.1.4 (Theorem 4.3.5). *The pullback $i_r^{\#}\mathcal{D}$ is an r-deformed B-htdo on X and there is an equivalence of categories between G-equivariant coherent $\mathcal{D}$-modules and B-equivariant coherent $i_r^{\#}\mathcal{D}$-modules.*

In our applications, $\mathcal{D}$ will be a sheaf of r-deformed λ-*twisted* differential operators on $X \times X$ for some $\lambda \in (r\mathfrak{h} \times r\mathfrak{h})^*$. Here $\mathfrak{h} = \mathrm{Lie}(H)$ is the Cartan subalgebra correspoding to a Cartan subgroup of G. In particular, when $r = 1$, $R = \mathbb{C}$ and $\mathcal{D} = \mathcal{D}_{X\times X}$ we obtain the Borho-Brylinski equivalence [17, Proposition 3.6].

Next, we consider the r-deformation of the enveloping algebra of $\mathfrak{g} := \mathrm{Lie}(G)$. This is isomorphic with $U(r\mathfrak{g})$; further we let L be a closed subgroup of G and $\mathcal{A}$ be a sheaf of r-deformed L-htdo on X.

Proposition 4.1.5 (Propositions 4.5.1 and 4.5.4)**.** *Let $\mathcal{A}$ be as above and let M an L-equivariant $U(r\mathfrak{g})$-module. Then* $\operatorname{Loc}(M) := \mathcal{A} \underset{U(r\mathfrak{g})}{\otimes} M$ *is an L-equivariant $\mathcal{A}$-module.*

Let $\mathcal{M}$ be an L-equivariant $\mathcal{A}$-module. Then $\Gamma(X, \mathcal{M})$ *is an L-equivariant $U(r\mathfrak{g})$-module.*

In particular when $r = 1$, $R = \mathbb{C}$, $\mathcal{A} = \mathcal{D}_\lambda$ with $\lambda \in \mathfrak{h}^*$ dominant, this follows directly from the Beilinson-Bernstein equivalence Theorem 4.1.3.

As an application of theory developed, we give a geometric proof of classical Duflo's theorem in Theorem 4.7.1. As mentioned before, this approach was studied in [17], where the authors use similar geometric methods to prove the theorem in the case of primitive ideals with trivial central character. Our new approach deals with all the rational central characters and the corresponding sheaves of twisted differential operators building on the framework developed in the previous chapter.

In the next chapter, we will use Theorem 4.1.4 and Proposition 4.1.5, to obtain an affinoid version of Duflo's theorem.

Structure of the chapter

This chapter is organised as follows: in Section 4.3, we prove a more general version of Borho-Brylinski equivalence [17, Proposition 3.6]. Next, in 4.4, we introduce the notion of G-equivariant modules over deformed enveloping algebras and prove that certain two-sided ideals are G-equivariant as bimodules over deformed enveloping algebras. In Section 4.5, we explain the localisation mechanism connecting equivariant modules over deformed enveloping algebras with equivariant modules over deformed homogeneous twisted differential operators. Next, in Section 4.6 we combine the results in Sections 4.3 and 4.5 to compute the pullback of certain equivariant modules over deformed homogeneous twisted differential operators on the double flag variety under the left/right inclusion of the flag variety. Finally, in Section 4.7, we complete the proof of classical Duflo's theorem.

Conventions. Throughout this chapter R will denote a commutative Noetherian base ring of arbitrary characteristic. All the varieties/algebraic groups will be considered over R unless otherwise specified. Unadorned tensor products and scheme products will be assumed to be taken over R and over $\operatorname{Spec}(R)$, respectively.

All the algebraic groups appearing in this chapter will be assumed to be connected and locally of finite type unless otherwise stated. All the modules will be regarded as left modules unless explicitly stated otherwise.

For a ring R, we will use R^{op} to denote the opposite ring. Given an R-algebra A and any R-Lie algebra/module $\mathfrak{g}/M$, we define $\mathfrak{g}_A = \mathfrak{g} \underset{R}{\otimes} A$ and $M_A = M \underset{R}{\otimes} A$. Further, we will assume that all the ring filtrations appearing in this document are positive, exhaustive and separated.

For a scheme X and $\mathcal{D}$ a sheaf of rings on X, a $\mathcal{D}$-module is called quasi-coherent if it is quasi-coherent as an $\mathcal{O}_X$-module.

Lastly, given $f : X \to Y$ a map of schemes, we use f^* to denote the pullback in the category of $\mathcal{O}/\mathcal{D}$-modules and f^{-1}/f_* to denote the inverse/direct image sheaf.

4.2 Deformations

Definition 4.2.1. *Let A be a positively $\mathbb{Z}$-filtered R-algebra such that F_0A is an R-subalgebra of A. We call A a* deformable *R-algebra if $\operatorname{gr} A$ is a flat R-module. A morphism of deformable R-algebras is an R-linear filtered ring homomorphism.*

Definition 4.2.2. *Let A be a deformable R-algebra and let $r \in R$ be a regular element. The r-th deformation of A is the following R-submodule of A:*

$$A_r := \sum_{i=0}^{\infty} r^i F_i A.$$

By construction A_r is an R-subalgebra of R. Further, the definition is functorial, and the following lemma states that we have a family of endofunctors $A \mapsto A_r$.

Lemma 4.2.3. *Let A be a deformable R-algebra and $r \in R$ a regular element. Then A_r is also a deformable R-algebra and there is a natural isomorphism $\operatorname{gr} A \cong \operatorname{gr} A_r$.*

Proof. We give A_r the subspace filtration $F_iA_r := F_iA \cap A_r$. As $\operatorname{gr} A$ is flat over R we have $F_iA_r = \sum_{j=0}^{i} r^j F_j A$. For $i \geq 1$ define a R-linear map

$$f : F_iA/F_{i-1}A \to F_iA_r/F_{i-1}A_r, \quad f(x + F_{i-1}A) = r^i x + F_{i-1}A_r.$$

To finish the proof it is enough to check that f is bijective. First, we prove that f is injective. Assume that $r^i x \in F_{i-1}A_r$, so $r^i x \in F_{i-1}A$ which implies that $x \in F_{i-1}A$ since $\operatorname{gr} A$ is flat, so in particular R-torsion free. It is straightforward to see that f is also surjective. □

Recall Definition 3.4.1; we call an R-scheme X that is smooth, separated and locally of finite type an *R-variety.* For the rest of the section, we let $r \in R$ be a regular element.

Further in the chapter, we will use the following lemma:

Lemma 4.2.4. *Let $\mathcal{D}$ be a tdo(1-tdo). Let $\mathcal{D}_r$ be the sheafification obtained by composing $\mathcal{D}$ with the deformation functor. Then $\mathcal{D}_r$ is an r-deformed tdo.*

Proof. The claim is local, so we may assume that X is affine and $D = \mathcal{D}(X)$ is the sheaf of crystalline differential operators on X. Clearly, D_r is a differential algebra. Next, we have $F_1 D = \mathcal{O}(X) \oplus \mathcal{T}(X)$. By construction we have $F_0 D_r = \mathcal{O}(X)$ and $F_1 D_r = \mathcal{O}(X) \oplus r\mathcal{T}(X)$, so the first two axioms of Definition 3.4.5 are satisfied. Since r is regular, $r \operatorname{gr}_1 D_r \cong \operatorname{gr}_1 D$ as $\mathcal{O}(X)$-modules, so the last claim follows from Lemma 4.2.3. $\square$

We recall the definition of a G-htdo and rephrase it for later use in this chapter:

Definition 4.2.5 (Definition 3.6.5). *Let $\mathcal{D}$ be a differential $\mathcal{O}_X$-algebra. We call $\mathcal{D}$ a sheaf r-deformed G-homogeneous twisted differential operators(G-htdo) if it is G-equivariant as a left $\mathcal{O}_X$-module, $\mathcal{D}$ is an r-deformed tdo and $\mathcal{D}$ is equipped with a Lie algebra map $i_{\mathfrak{g}} : r\mathfrak{g} \to \mathcal{D}$ such that:*

i) $g.1 = 1$ *and* $g.(d_1 d_2) = (g.d_1)(g.d_2)$ *for* $g \in G$ *and* $d_1, d_2 \in \mathcal{D}$.

ii) $g.(fd) = (g.f)(g.d)$ *for* $f \in \mathcal{O}_X$ *and* $d \in \mathcal{D}$.

iii) $i_{\mathfrak{g}}(g.\psi) = g.i_{\mathfrak{g}}(\psi)$ *for* $g \in G, \psi \in r\mathfrak{g}$.

iv) The derivative of the G-action induces a $\mathfrak{g}$-action and so a $r\mathfrak{g} \subset \mathfrak{g}$-action. This must coincide with the action $d \to [i_{\mathfrak{g}}(\psi), d]$ for $\psi \in r\mathfrak{g}$ and $d \in \mathcal{D}$.

v) $i_{\mathfrak{g}}(r\mathfrak{g}) \subset F_1\mathcal{D}$.

vi) $\eta = \rho \circ i_{\mathfrak{g}}$ as maps from $r\mathfrak{g}$ to $r\mathcal{T}_X$ where $\eta : \mathfrak{g} \to \mathcal{T}_X$ is the infinitesimal map and $\rho : F_1\mathcal{D} \to \mathcal{T}_X$ is the natural anchor map.

Definition 4.2.6. *Let $(\mathcal{D}, i_{\mathfrak{g}})$ be a r-deformed G-htdo and L be a closed subgroup of G, with Lie algebra $\mathfrak{l}$. A $\mathcal{D}$-module $\mathcal{M}$ is weakly L-equivariant if:*

i) $\mathcal{M}$ is an L-equivariant $\mathcal{O}_X$-module.

ii) *$g.(D.m) = (g.D).(g.m)$ for any $g \in L, d \in \mathcal{D}, m \in \mathcal{M}$.*

We call $\mathcal{M}$ L-equivariant if in addition:

iii) *The $r\mathfrak{l}$-action induced by the derivative of the L-action on $\mathcal{M}$ coincides with the $r\mathfrak{l}$-action induced by the restriction of $i_{\mathfrak{g}}$ to $r\mathfrak{l}$.*

A morphism of (weakly) equivariant $\mathcal{D}$-modules is a morphism of L-equivariant $\mathcal{O}_X$-modules that is $\mathcal{D}$-linear.

In case $L = G$, we recover the Definition 3.9.3, but we will need this more general definition for explaining the localisation mechanism. We denote $\mathrm{Coh}(\mathcal{D}, G)$ the category of coherent G-equivariant coherent $\mathcal{D}$-modules.

4.3 An equivalence à la Borho-Brylinski

Throughout this section, we let G be a connected, smooth, affine algebraic group over $\operatorname{Spec} R$, B a closed subgroup of G and we make the following assumption:

Assumption 4.3.1. *The quotient scheme $X = G/B$ is an R-variety and the quotient map $d_B : G \to X$ given by $d_B(g) = gB$ is a locally trivial B-torsor with respect to the action $\diamond$ given by $b \diamond g = gb^{-1}$.*

This is satisfied in particular when B is a Borel subgroup of G and $X = G/B$ becomes the flag scheme.

4.3.1 Notation and preliminaries

First, we introduce some notation: we will denote the standard action (left multiplication) of G or B on X by gx for all $g \in G$ and $x \in X$. Furthermore, the standard B action on X (the restriction of the G-action to B) will also be denoted act_B.

Define:

- $\mathrm{act}_{1,G,B}(\square) : (G \times B) \times (G \times X) \to (G \times X) : (g, b)\square(h, x) := (ghb^{-1}, bx)$.
- $\mathrm{act}_{2,G,B}(\triangle) : (G \times B) \times (G \times X) \to (G \times X) : (g, b)\triangle(h, x) := (ghb^{-1}, gx)$.
- $p : G \times X \to X, \quad p(g, x) := x$.
- $\delta : G \times X \to X, \quad \delta(g, x) := g^{-1}x$.

- $\alpha : G \times X \to G \times X, \quad \alpha(g,x) := (g, g^{-1}x)$.

Lemma 4.3.2. *The following statements are easy to see:*

- $p \circ \alpha = \delta$.
- α *is a bijective* $G \times B$*-equivariant map with respect to the actions* $\triangle$ *and* $\square$.

Lemma 4.3.3. *The map* δ *is a* B*-equivariant locally trivial* G*-torsor. Further, the* $\triangle$ *actions of* G *and* B *on* $G \times X$ *commute.*

Proof. By construction, we have that p is a locally trivial G-torsor. The $\square$ actions of G and B actions on $G \times X$ commute. The claim follows by Lemma 4.3.2, by viewing α as a B-equivariant locally trivial G-torsor. $\square$

We introduce further notation.

- Let $\mathrm{act}_G : G \times (X \times X) \to X \times X, \mathrm{act}_G(g,x,y) := (gx, gy)$ be the diagonal action .
- Let $d : G \times X \to X \times X, d(h,x) := (hB, x)$ be the quotient map.

Lemma 4.3.4. *By abuse of notation, we will denote* $\triangle$ *the* B *action on* $G \times X$ *given by* $b\triangle(g,x) = (gb^{-1}, x)$*. Then the descent map* d *is a* G*-equivariant locally trivial* B*-torsor with respect to the* B*-action given by* $\triangle$*.*

Proof. We have

$$d(g\triangle(h,x)) = d(ghb^{-1}, gx) = (ghB, gx) = \mathrm{act}_G(g, d(h,x)),$$

so d is indeed G-equivariant.

We have that $d = d_B \times \mathrm{id}_X$ and since d_B is faithfully flat and locally of finite type so is d. Since d_B is a B-torsor, we have that the map

$$G \times B \to G \times_X G$$

sending $(g,b) \to (g, gb^{-1})$ is an isomorphism, so the map

$$(G \times X) \times B \to (G \times X) \times_{X \times X} (G \times X)$$

sending $((g, x), b) \to ((g, x), (gb^{-1}, x))$ is also an isomorphism. Therefore, d is indeed a B-torsor.

To conclude, one needs to show that d is locally trivial. Let $S = \{U_i\}_{i \in I}$ be an affine open cover of X trivialising the torsor d_B and let $\phi_i : B \times U_i \to d_B^{-1}(U_i)$ the corresponding B-invariant isomorphisms. Define $\{V_{ij} := U_i \times U_j | U_i, U_j \in S\}$. Then $\{V_{ij}\}$ is an affine open cover of $X \times X$. We have that

$$\mathcal{O}_{X \times X}(V_{ij}) = \mathcal{O}_{X \times X}(U_i \times U_j) \cong \mathcal{O}_X(U_i) \underset{R}{\otimes} \mathcal{O}_X(U_j) \text{ for any } i, j \in I.$$

By assumption, for any $k \in I, \mathcal{O}_X(U_k)$ is a finitely generated R-algebra; therefore the ring $\mathcal{O}_X(U_i) \underset{R}{\otimes} \mathcal{O}_X(U_j)$ is also a finitely generated R-algebra.

We claim that any $\{V_{ij}\}$ trivialises the torsor d. We have by definition $d^{-1}(V_{ij}) = (d_B^{-1} U_i, U_j)$.

Let $\varphi_{ij} : B \times U_i \times U_j \to (d_B^{-1} U_i, U_j)$ be defined by $\varphi(b, x, y) := (\phi_i(b, x), y)$.

Then it is trivial to check that this is an isomorphism compatible with the B-action given that ϕ_i is. Therefore, we have checked all the conditions to make d a locally trivial B-torsor. □

4.3.2 Pullback of representations of homogeneous twisted differential operators on $X \times X$

We keep the notation from the previous subsection. Let $i_r : X \to X \times X, i_r(x) = (eB, x)$ denote the inclusion of X into the right copy of $X \times X$. We also fix $(\mathcal{D}, i_{\mathfrak{g}})$ an r-deformed G-htdo on $X \times X$ with respect to the G-action defined by act_G. The goal of this subsection is to prove the following theorem:

Theorem 4.3.5. *The pullback $i_r^{\#}\mathcal{D}$ is an r-deformed B-htdo and the functor*

$$i_r^{\#} : \mathrm{Coh}(\mathcal{D}, G) \to \mathrm{Coh}(i_r^{\#}\mathcal{D}, B)$$

is an equivalence of categories.

To prove this theorem, we will need some additional results:

Lemma 4.3.6. *The pullback $d^{\#}\mathcal{D}$ is an r-deformed $G \times B$-htdo and the functors*

$$\begin{aligned} d^{\#}(-) &: \mathrm{Coh}(\mathcal{D}, G) \to \mathrm{Coh}(d^{\#}\mathcal{D}, G \times B), \\ d_*(-)^B &: \mathrm{Coh}(d^{\#}\mathcal{D}, G \times B) \to \mathrm{Coh}(\mathcal{D}, G) \end{aligned} \tag{4.3}$$

are quasi-inverses equivalences of categories.

Proof. This follows from Lemma 4.3.4, Corollary 3.10.13 and Corollary 3.11.9. □

Lemma 4.3.7. *Denote $\mathcal{A} := \delta_{\#}(d^{\#}\mathcal{D})^G$. Then $\mathcal{A}$ is an r-deformed B-htdo and the functors:*

$$\begin{aligned} \delta^{\#}(-) &: \mathrm{Coh}(\mathcal{A}, B) \to \mathrm{Coh}(d^{\#}\mathcal{D}, G \times B), \\ \delta_*(-)^G &: \mathrm{Coh}(d^{\#}\mathcal{D}, G \times B) \to \mathrm{Coh}(\mathcal{A}, B) \end{aligned} \tag{4.4}$$

are quasi-inverses equivalences of categories.

Proof. This follows from Lemma 4.3.3, Corollary 3.10.13 and Corollary 3.11.9. □

As a corollary, we obtain from the previous two lemmas an equivalence between $\mathrm{Coh}(\mathcal{D}, G)$ and $\mathrm{Coh}(\mathcal{A}, B)$.

Lemma 4.3.8. *The sheaves of r-deformed twisted differential operators $\mathcal{A}$ and $i_r^{\#}\mathcal{D}$ are isomorphic. In particular, $i_r^{\#}\mathcal{D}$ is an r-deformed B-htdo.*

Proof. Define the maps $s : G \times X \to G \times X \times X, s(g, x) := (g^{-1}, gB, x)$ and $q : G \times X \times X \to X \times X, q(g, x, y) := (x, y)$.

By construction, $d = q \circ s$, so $d^{\#}\mathcal{D} \cong (q \circ s)^{\#}\mathcal{D}$. Since d and q are locally trivial B respectively G-torsors, s is flat, so by Corollary 3.7.7, we get $(q \circ s)^{\#}\mathcal{D} \cong s^{\#}q^{\#}\mathcal{D}$. Further, since $\mathcal{D}$ is a deformed G-htdo, we have $q^{\#}\mathcal{D} \cong \mathrm{act}_G^{\#}\mathcal{D}$. Therefore, using again that s is flat, we have by Corollary 3.7.7 $s^{\#}q^{\#}\mathcal{D} \cong (\mathrm{act}_G \circ s)^{\#}\mathcal{D}$.

By construction, one has $\mathrm{act}_G \circ s = i_r \circ \delta$, so $(\mathrm{act}_G \circ s)^{\#}\mathcal{D} \cong (i_r \circ \delta)^{\#}\mathcal{D}$. Since δ is smooth, it is in particular flat, so by Corollary 3.7.7, $(i_r \circ \delta)^{\#}\mathcal{D} \cong \delta^{\#}i_r^{\#}\mathcal{D}$. Therefore, we have obtained that $d^{\#}\mathcal{D} \cong \delta^{\#}i_r^{\#}\mathcal{D}$. To conclude we use Corollary 3.10.13 to obtain:

$$\mathcal{A} = \delta_{\#}(d^{\#}\mathcal{D})^G \cong \delta_{\#}(\delta^{\#}i_r^{\#}\mathcal{D})^G \cong i_r^{\#}\mathcal{D}.$$ □

Lemma 4.3.9. *Let* $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}, G)$. *Then*

$$(d_*^B \circ \delta^\#) \circ i_r^\# \mathcal{M} \cong \mathcal{M},$$

i.e. the functor $d_*^B \circ \delta^\#$ *is a left quasi-inverse to* $i_r^\#$.

Proof. By repeating the argument in the previous lemma, we get $d^\# \mathcal{M} \cong \delta^\# \circ i_r^\# \mathcal{M}$. The claim then follows by Lemma 4.3.6. □

To complete the proof of the equivalence theorem, we need one more categorical lemma:

Lemma 4.3.10. *Let* C *and* D *be two categories and let* $H : C \to D$ *and* $F_1, F_2 : D \to C$ *be functors such that:*

$$HF_1 \cong 1_D, \quad F_1H \cong 1_C, \quad HF_2 \cong 1_D.$$

Then $F_2H \cong 1_C$.

Proof. We have

$$H(F_2H)F_1 \cong (HF_2)(HF_1) \cong 1_D, \text{ so}$$

$$F_1(HF_2HF_1)H \cong F_1H \cong 1_C. \tag{4.5}$$

On the other hand we have:

$$F_1(HF_2HF_1)H \cong (F_1H)(F_2H)(F_1H) \cong 1_C(F_2H)1_C = F_2H. \tag{4.6}$$

Thus, by combining the equations (4.5) and (4.6) we obtain $F_2H \cong 1_C$. □

We can now prove Theorem 4.3.5:

Proof. We have by Lemmas 4.3.6, 4.3.7 and 4.3.8 that $d_*^B \circ \delta^\#$ and $\delta_*^G \circ d^\#$ provide quasi-inverse equivalences of the categories mentioned in the statement. By Lemma 4.3.9, $d_*^B \circ \delta^\#$ is a left quasi-inverse to $i_r^\#$, so by applying Lemma 4.3.10 it is also a right quasi-inverse. Thus, for $\mathcal{M}$ a G-equivariant coherent $\mathcal{D}$-module, we have $i_r^\# \mathcal{M} \cong \delta_*^G \circ d^\# \mathcal{M}$. □

By setting $R = \mathbb{C}$, $r = 1$ and $\mathcal{D} = \mathcal{D}_{X\times X}$, we recover the classical Borho-Brylinski equivalence [17, Proposition 3.6d)].

Let $i_l : X \to X \times X$, $i_l(x) := (x, eB)$ to be the inclusion of X into the left copy. Then by duality, we get:

Corollary 4.3.11. *The pullback* $i_l^{\#}\mathcal{D}$ *is a* B*-htdo and the functor*

$$i_l^{\#} : \mathrm{Coh}(\mathcal{D}, G) \to \mathrm{Coh}(i_l^{\#}\mathcal{D}, B)$$

is an equivalence of categories.

4.3.3 Global sections of $\mathcal{O}$-modules

We keep the notation from the previous subsection. We aim to prove the following proposition:

Proposition 4.3.12. *Let* $\mathcal{N} \in \mathrm{QCoh}(\mathcal{O}_X)$ *such that* $\Gamma(X, \mathcal{N}) = 0$*. Then*

$$\Gamma(X \times X, (d_*)^B \delta^* \mathcal{N}) = 0.$$

We will need two additional lemmas:

Lemma 4.3.13. *Let* $\mathcal{N}$ *be a quasi-coherent* $\mathcal{O}_X$*-module such that* $\Gamma(X, \mathcal{N}) = 0$*. Then*

$$\Gamma(G \times X, p^*\mathcal{N}) = 0.$$

Proof. Let $(U_i)_{i\in I}$ be an affine finite open cover of X. Then $(G \times U_i)_{i\in I}$ is an affine finite open cover of $G \times X$ stable under the map p. Let $U_{ij} := U_i \cap U_j$.

Consider the Cech complex

$$0 \to \mathcal{N}(X) \to \prod_{i\in I} \mathcal{N}(U_i) \to \prod_{i,j\in I} \mathcal{N}(U_{ij}).$$

The assumption tells us that the last map is injective.

Consider now the Cech Complex

$$0 \to p^*\mathcal{N}(G \times X) \to \prod_{i\in I} p^*(\mathcal{N})(G \times U_i) \to \prod_{i,j\in I} p^*(\mathcal{N})(G \times U_{ij}). \tag{4.7}$$

We have $p^*(\mathcal{N})(G \times U_i) = \mathcal{O}(G) \underset{R}{\otimes} \mathcal{N}(U_i)$ and since tensor product commutes finite products we get

$$\prod_{i \in I} p^*(\mathcal{N})(G \times U_i) \cong \mathcal{O}(G) \underset{R}{\otimes} \prod_{i \in I} \mathcal{N}(U_i),$$

and similarly

$$\prod_{i,j \in I} p^*(\mathcal{N})(G \times U_{ij}) \cong \mathcal{O}(G) \underset{R}{\otimes} \prod_{i,j \in I} \mathcal{N}(U_{ij}).$$

Since $\mathcal{O}(G)$ is flat over R we get that the tensor product preserves injections, so the last map in equation (4.7) is injective; the claim follows. □

Lemma 4.3.14. *Let Y be a smooth scheme let $\beta : Y \to Y$ be an automorphism and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_Y$-module such that $\Gamma(Y, \mathcal{M}) = 0$. Then $\Gamma(Y, \beta^* \mathcal{M}) = 0$.*

Proof. Since β is an automorphism of a smooth scheme, it is smooth, so in particular it is faithfully flat. The same is true for β^{-1}. Therefore, we get an injection

$$\alpha : \Gamma(Y, \beta^* \mathcal{M}) \to \Gamma(Y, \beta^{-1^*} \beta^* \mathcal{M}) \cong \Gamma(Y, \mathcal{M}) = 0,$$

so the claim is proven. □

Proof of Proposition 4.3.12. Since $\delta^* \mathcal{N} = \alpha^* p^* \mathcal{N}$, we have by Lemma 4.3.13 and Lemma 4.3.14 that $\Gamma(G \times X, \delta^* \mathcal{N}) = 0$. The claim follows from the definition of pushforward sheaf. □

As a corollary we obtain:

Corollary 4.3.15. *Let $\mathcal{N} \in \mathrm{Coh}(\mathcal{D}, G)$ with $\Gamma(X, i_r^{\#} \mathcal{N}) = 0$. Then $\Gamma(X \times X, \mathcal{N}) = 0$.*

Proof. We have that at the level of $\mathcal{O}$-modules that $i_r^{\#} = i_r^*$ and $\delta^{\#} = \delta^*$. The claim then follows from Theorem 4.3.5 and Proposition 4.3.12. □

By duality we obtain:

Corollary 4.3.16. *Let $\mathcal{N} \in \mathrm{Coh}(\mathcal{D}, G)$ with $\Gamma(X, i_l^{\#} \mathcal{N}) = 0$. Then $\Gamma(X \times X, \mathcal{N}) = 0$.*

4.4 Representations of deformed enveloping algebras

4.4.1 Notation

Let G be a connected, simply connected, split semisimple, smooth affine algebraic group over a commutative ring R and $\mathfrak{g}$ denote its Lie algebra which is free as an R-module. Fix $\mathfrak{h}$ a Cartan subalgebra and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ a triangular decomposition. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra. We make the following notation:

Notation 4.4.1.

- $\mathfrak{h}^*$ *the dual space of* $\mathfrak{h}$*- elements of* $\mathfrak{h}^*$ *are called weights.*
- ϕ *- set of roots in* $\mathfrak{g}$*.*
- ϕ^+*- set of positive roots.*
- $\triangle$*- set of simple roots.*
- *For a root* α*, we denote* $\alpha^\vee$ *the corresponding coroot.*
- ρ*- half sum of positive roots.*
- W*- the Weyl group associated with* $\mathfrak{g}$*.*
- w_o*- the longest element of* W*.*

Definition 4.4.2. *We define a shifted dot action of* W *on* $\mathfrak{h}^*$ *by*

$$w \cdot \lambda = w(\lambda + \rho) - \rho \text{ for } w \in W, \lambda \in \mathfrak{h}^*.$$

We say that a weight is dominant *if* $(\lambda + \rho)(\alpha^\vee) \notin \mathbb{Z}^{\leq -1}$ *for all* $\alpha \in \phi^+$*. Any* W*-orbit contains a dominant weight.*

We say that a weight is regular *if* $(\lambda + \rho)(\alpha^\vee) \neq 0$ *for all* $\alpha \in \phi^+$*. This is equivalent to the stabiliser of* λ *under the shifted dot action being trivial.*

Let $Z(\mathfrak{g}) = Z(U(\mathfrak{g}))$ denote the center of the universal enveloping algebra. For any $\lambda \in \mathfrak{h}^*$ there is an associated central character $\chi_\lambda : Z(\mathfrak{g}) \to R$. Furthermore, we have that for all $w \in W$, $\chi_\lambda = \chi_{w \cdot \lambda}$.

4.4.2 Equivariant modules over deformed enveloping algebras

Fix $r \in R$ a regular element and consider the r-th deformation of $U(\mathfrak{g})$ denoted $U(\mathfrak{g})_r$. Using the PBW theorem we obtain that $U(\mathfrak{g})_r \cong U(r\mathfrak{g})$. The enveloping algebra $U(\mathfrak{g})$ is a G-representation via the Adjoint action, so by the module-comodule duality we obtain a map $\rho : U(\mathfrak{g}) \to \mathcal{O}(G) \underset{R}{\otimes} U(\mathfrak{g})$ making $U(\mathfrak{g})$ a comodule for the Hopf algebra $\mathcal{O}(G)$. Furthermore, since the G action commutes with the R action, the map ρ restricts to a map $\rho : U(r\mathfrak{g}) \to \mathcal{O}(G) \underset{R}{\otimes} U(r\mathfrak{g})$.

Let L be a closed subgroup of G. Then L also acts on $U(r\mathfrak{g})$ via the restriction to L of the Adjoint action of G. Again, by duality we obtain a comodule map $\rho_{r\mathfrak{g},L} : U(r\mathfrak{g}) \to \mathcal{O}(L) \underset{R}{\otimes} U(r\mathfrak{g})$.

Let M be a $U(r\mathfrak{g})$-module that is also an $\mathcal{O}(L)$-comodule. The comodule structure induces an action of L; the derivative of the L-action induces an action of the Lie algebra $\mathfrak{l} = \mathrm{Lie}(L)$, and so of $r\mathfrak{l}$, on M. Furthermore, since $U(r\mathfrak{g})$ and M are $\mathcal{O}(L)$-comodules, so is $U(r\mathfrak{g}) \underset{R}{\otimes} M$, see [46, Section 1.8] for details.

Definition 4.4.3. *A weakly L-equivariant $U(r\mathfrak{g})$ module is a triple (M, α, ρ), where M is a R-module, $\alpha : U(r\mathfrak{g}) \underset{R}{\otimes} M \to M$ is a left $U(r\mathfrak{g})$-action, $\rho : M \to \mathcal{O}(L) \underset{R}{\otimes} M$ is a $\mathcal{O}(L)$ co-action such that α is a morphism of $\mathcal{O}(L)$-comodules.*

Furthermore, if the action of $r\mathfrak{l} \subset \mathfrak{l} = \mathrm{Lie}(L)$ induced by ρ by derivating coincides with the restriction of the $r\mathfrak{g}$ action to $r\mathfrak{l}$, we say that (M, α, ρ) is L-equivariant. As for equivariant $\mathcal{D}$-modules, we will omit the equivariance structure when it is understood from the context.

A morphism of (weakly) L-equivariant $U(r\mathfrak{g})$-modules (M, α, ρ_1) and (N, β, ρ_2) is a map $f : M \to N$ of Abelian groups that is $U(r\mathfrak{g})$-linear with respect to actions α, β and $\mathcal{O}(L)$-co-linear with respect to ρ_1 and ρ_2. We call such a morphism L-equivariant.

Denote $\mathrm{Mod}(U(r\mathfrak{g}), L)$ the category of consisting of L-equivariant $U(r\mathfrak{g})$-modules together with L-equivariant morphisms.

We can reformulate the weakly equivariant condition in the following way: by the module-comodule correspondence M can be viewed as a representation of the algebraic group L. Since $U(r\mathfrak{g})$ is also an L-representation we may rewrite the condition that the map $\alpha : U(r\mathfrak{g}) \underset{R}{\otimes} M \to M$ is a morphism of $\mathcal{O}(L)$-comodules as:

$$l.(\psi.m) = (l.\psi).(l.m),$$

for all R-algebras A, $l \in L(A)$, $\psi \in U(r\mathfrak{g})_A$ and $m \in M_A$. By abuse of language we define an equivalent notion of a weakly L-equivariant $U(r\mathfrak{g})$-module by:

$$M \text{ is a representation of } L.$$

$$l.(\psi.m) = (l.\psi).(l.m) \text{ for all } l \in L, \psi \in U(r\mathfrak{g}), m \in M. \tag{4.8}$$

We will also need the following notion: let $\phi : U(r\mathfrak{g}) \to S$ be a map of rings. We say that an S-module is (weakly) L-equivariant if it (weakly) L-equivariant as $U(r\mathfrak{g})$-module. We denote $\operatorname{Mod}(S, L)$ the category of L-equivariant S-modules.

4.4.3 Equivariance of two-sided ideals in classical enveloping algebras

In this subsection we prove that any two-sided ideal in $U(\mathfrak{g})$ is a G-equivariant $U(\mathfrak{g} \times \mathfrak{g})$-module ($U(\mathfrak{g}) - U(\mathfrak{g})$-bimodule) when the base ring is a field of characteristic 0. Here we view G via its isomorphism with the diagonal subgroup of $G \times G$. The group G acts on the enveloping algebra $U(\mathfrak{g})$ via the Adjoint action-denoted Ad.

Let τ denote the principal anti-automorphism of $U(\mathfrak{g})$ induced by $x \mapsto -x$ for all $x \in \mathfrak{g}$. To simplify the notation, we will use x^τ to denote $\tau(x)$. We have an action of $U(\mathfrak{g} \times \mathfrak{g}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g})$ (by [53, III.2.2)]) on $U(\mathfrak{g})$ via

$$(x \otimes y) \cdot a = yax^\tau \text{ for all } x, a, y \in U(\mathfrak{g}).$$

Proposition 4.4.4. *Assume that R is a field of characteristic 0. Let I be a two-sided ideal in $U(\mathfrak{g})$. Then $I \in \operatorname{Mod}(U(\mathfrak{g} \times \mathfrak{g}), G)$.*

Proof. First, since R is a field of characteristic 0, we have by [24, Proposition 2.4.17] that I is invariant under the Adjoint action, so I is a G-module. By construction, it is clear I is a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$-module under the action defined above. We then have that for all $g \in G$, $x, y \in U(\mathfrak{g}), u \in I$ that

$$\begin{aligned}
\operatorname{Ad}(g) \cdot ((x \otimes y)u) &= \operatorname{Ad}(g)(xuy^\tau) \\
&= \operatorname{Ad}(g)y \operatorname{Ad}(g)u \operatorname{Ad}(g)x^\tau \\
&= \operatorname{Ad}(g)y \operatorname{Ad}(g)u(\operatorname{Ad}(g)x)^\tau \\
&= (\operatorname{Ad}(g)x \otimes \operatorname{Ad}(g)y) \cdot \operatorname{Ad}(g)u.
\end{aligned} \tag{4.9}$$

The derivative of the Ad action is the ad-action of the Lie Algebra $\mathfrak{g}$. Since $\mathrm{Lie}(G)$ embeds into $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ via $x \to x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$, the two actions coincide. Therefore, we have proven all the conditions for I to be a G-equivariant $U(\mathfrak{g} \times \mathfrak{g})$-module. $\square$

We now specialise to two-sided ideals in $U(\mathfrak{g})$ with a given central character. For $\lambda \in \mathfrak{h}^*$, we let $\chi_\lambda : Z(\mathfrak{g}) \to R$ the associated central character. Let $m_\lambda = \ker(\chi_\lambda)$ and $U(\mathfrak{g})^\lambda = U(\mathfrak{g})/m_\lambda U(\mathfrak{g})$.

Consider the centre $C := Z(U(\mathfrak{g} \times \mathfrak{g})) = Z(\mathfrak{g} \times \mathfrak{g})$. Since, $U(\mathfrak{g} \times \mathfrak{g}) \cong U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and since $U(\mathfrak{g})$ is a free R-module one obtains $C \cong Z(\mathfrak{g}) \otimes Z(\mathfrak{g})$. Any central character of C is determined by a pair θ_1, θ_2, where $\theta_i : Z(\mathfrak{g}) \to R$. Let $\lambda, \mu \in \mathfrak{h}^*$ and let $\theta_1 = \ker_{\chi_\lambda}$ and $\theta_2 = \ker_{\chi_\mu}$, so that we obtain:

$$\ker(\chi_\lambda, \chi_\mu) = m_\lambda \otimes Z(\mathfrak{g}) + Z(\mathfrak{g}) \otimes m_\mu.$$

Therefore we obtain:

$$\begin{aligned} U(\mathfrak{g} \times \mathfrak{g})^{\lambda,\mu} &\cong U(\mathfrak{g}) \otimes U(\mathfrak{g}) / U(\mathfrak{g}) \otimes U(\mathfrak{g})(m_\lambda \otimes Z(\mathfrak{g}) + Z(\mathfrak{g}) \otimes m_\mu) \\ &\cong U(\mathfrak{g})/U(\mathfrak{g})m_\lambda \otimes U(\mathfrak{g})/U(\mathfrak{g})m_\mu \\ &\cong U(\mathfrak{g})^\lambda \otimes U(\mathfrak{g})^\mu. \end{aligned} \tag{4.10}$$

Recall that if I is a two-sided ideal in $U(\mathfrak{g})$ and R a field of characteristic 0, we proved in Proposition 4.4.4 that $I \in (U(\mathfrak{g} \times \mathfrak{g}), G)$. Furthermore, if I is a two-sided ideal in $U(\mathfrak{g})^\lambda$, we can view it as a $U(\mathfrak{g})^\lambda$-bimodule, so a module over the ring $U(\mathfrak{g})^{\lambda^{\mathrm{op}}} \otimes U(\mathfrak{g})^\lambda$. Further, we have by [10, Lemma 5.4-Equation 5.5] that τ induces an isomorphism $U(\mathfrak{g})^{\lambda^{\mathrm{op}}} \cong U(\mathfrak{g})^{-w_o\lambda}$; recall that w_o denotes the longest element of W. Therefore, using equation (4.10), we deduce that a $U(\mathfrak{g})^\lambda$-bimodule is the same as a $U(\mathfrak{g} \times \mathfrak{g})^{-w_o\lambda,\lambda}$-module. In particular, we obtain:

Corollary 4.4.5. *Assume R is a field of characteristic 0. Let I be a two-sided ideal in $U(\mathfrak{g})^\lambda$. Then $I \in \mathrm{Mod}(U(\mathfrak{g} \times \mathfrak{g})^{-w_o\lambda,\lambda}, G)$.*

4.4.4 Equivariance of two-sided ideals in deformed enveloping algebras

Throughout this subsection only, we will assume that R is a Noetherian local ring of characteristic 0.

Let $r \in R$ a regular element and consider the r-th deformation $U(\mathfrak{g})_r \cong U(r\mathfrak{g})$. Unfortunately, we cannot prove that any two-sided ideal in $U(r\mathfrak{g})$ is G-equivariant with respect to the Adjoint action using the method below, so we restrict to a special class of ideals. We call a two-sided ideal I in $U(r\mathfrak{g})$ an *r-ideal* if $r^n i \in I$ for some $n \in \mathbb{N}$ and $i \in U(\mathfrak{g})$ implies that $i \in I$. This is equivalent to $U(r\mathfrak{g})/I$ having no r-torsion.

Lemma 4.4.6. *Let I be an r-ideal in $U(r\mathfrak{g})$ and $x \in U(\mathfrak{g})$. Then $\operatorname{ad}_x(I) \subset I$. In other words, I is closed under the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$.*

Proof. Since $x \in U(\mathfrak{g})$, there exists $n \in \mathbb{N}$ such that $r^n x \in U(r\mathfrak{g})$. Since I is an ideal in $U(r\mathfrak{g})$, we have $\operatorname{ad}_{r^n x}(I) \subset I$. The claim follows since I is an r-ideal. □

For $\alpha \in \phi$, we let $x_\alpha : G_a \to G$ and $e_\alpha = (dx_\alpha)(1) \in \mathfrak{g}$ be the root homomorphism and root vector corresponding to α, respectively.

Corollary 4.4.7. *Let I be an r-ideal in $U(r\mathfrak{g})$. Then $\operatorname{Ad}(G(R))(I) \subset I$; in other words I is closed under the Adjoint action.*

Proof. We have by [6, Lemma 4.1b,c)] that for all $\alpha \in \phi$ and $s \in R$:

$$x_\alpha(s) \cdot a = \sum_{i=0}^{\infty} \frac{\operatorname{ad}(se_\alpha)^m}{m!}(a), \tag{4.11}$$

and there exists $n \in \mathbb{N}$ such that $\frac{\operatorname{ad}(se_\alpha)^n}{n!}(a) = 0$ for all $a \in U(\mathfrak{g})$. In particular, combining Lemma 4.4.6 with equation (4.11), we obtain $x_\alpha(s) \cdot I \subset I$. To finish, we have that R is a local ring, so by [1, Proposition 1.6] the Chevalley group $G(R)$ is generated by elements of the form $x_\alpha(s)$. □

Recall that ${}^\tau$ denote the principal anti-automorphism of $U(\mathfrak{g})$ induced by $x \mapsto -x$ for all $x \in \mathfrak{g}$. We have by combining [4, Lemma 3.3] and the PBW theorem that $U(\mathfrak{g} \times \mathfrak{g})_r \cong U(r\mathfrak{g}) \otimes U(r\mathfrak{g})$. We get an action of $U(r\mathfrak{g}) \otimes U(r\mathfrak{g})$ on $U(r\mathfrak{g})$ via

$$(x \otimes y) \cdot a = yax^\tau \text{ for all } x, a, y \in U(r\mathfrak{g}).$$

Proposition 4.4.8. *Let I be an r-ideal in $U(r\mathfrak{g})$. Then $I \in \operatorname{Mod}(U(\mathfrak{g} \times \mathfrak{g})_r, G)$.*

Proof. We have by the Corollary above that I is a G-module. By construction, it is clear I is a $U(r\mathfrak{g}) \otimes U(r\mathfrak{g})$-module under the action defined above. We then have by the same arguments as in Proposition 4.4.4 that for all $g \in G$, $x, y \in U(r\mathfrak{g}), u \in I$

$$\mathrm{Ad}(g) \cdot ((x \otimes y)u) = (\mathrm{Ad}(g)x \otimes \mathrm{Ad}(g)y) \cdot \mathrm{Ad}(g)u.$$

The derivative of the Ad is the ad-action. Since $r\,\mathrm{Lie}(G)$ embeds into $U(r\mathfrak{g}) \otimes U(r\mathfrak{g})$ via $x \to x \otimes 1 + 1 \otimes x$ for $x \in r\mathfrak{g}$, the derivative of the G-action coincides with the Lie algebra action. This concludes the proof. $\square$

4.4.5 Verma modules

An important tool in studying the representation theory of semisimple Lie Algebras are the Verma modules.

Definition 4.4.9. *Let $\lambda \in r\mathfrak{h}^*$. We extend λ to a map $r\mathfrak{n}^- \oplus r\mathfrak{h} \to R$ and to a ring morphism $\lambda : U(r\mathfrak{b}) \to R$ and denote R_λ the corresponding $U(r\mathfrak{b})$-module. The Verma module of weight λ is defined to be*

$$M(\lambda) = U(r\mathfrak{g}) \underset{U(r\mathfrak{b})}{\otimes} R_\lambda.$$

For the rest of this subsection assume that $R = K$ is *a field of characteristic* 0. Given an $U(\mathfrak{g})$-module M and a weight $\lambda \in \mathfrak{h}^*$, we denote $M_\lambda = \{m \in M | hm = \lambda(h)m\}$ for all $h \in \mathfrak{h}$. Consider the BGG category $\mathcal{O}$ of finitely generated $U(\mathfrak{g})$-modules M such that $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ and the action of $\mathfrak{n}$ on M is locally finite.

We recall some basic facts about objects in category $\mathcal{O}$ that will be useful in the later sections.

Proposition 4.4.10.

- *The Verma module $M(\lambda)$ has a unique maximal submodule denoted $N(\lambda)$ and a unique simply quotient denoted $L(\lambda)$.*
- *The annihilator of $M(\lambda)$ is given by* $\ker(\chi_\lambda)U(\mathfrak{g})$.
- *Any module in category $\mathcal{O}$ has finite length.*
- *The composition factors of objects in $\mathcal{O}$ are of the form $L(\mu)$ for $\mu \in \mathfrak{h}^*$.*

4.5 The localisation mechanism

Throughout this section G will denote a connected, simply-connected, split semisimple, smooth affine algebraic group over R, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra, X will denote an R-variety with a G-action and $r \in R$ a regular element.

4.5.1 Equivariant localisation theory

We fix $(\mathcal{D}, i_{\mathfrak{g}})$ an r-deformed G-htdo on X. We aim to prove that even though Beilinson-Bernstein equivalence theorem does not work over an arbitrary commutative ring, the equivariance structure is preserved under localisation and taking global sections. Let L be a closed subgroup of G; since $\mathcal{D}$ is a G-htdo, it is in particular an L-htdo and we denote by $\star$ the L-action on $\mathcal{D}$. The Lie algebra map $i_{\mathfrak{g}} : r\mathfrak{g} \to \mathcal{D}$ can be extended to a ring homomorphism $i_{\mathfrak{g}} : U(r\mathfrak{g}) \to \mathcal{D}$.

Proposition 4.5.1. *Let L be a closed subgroup of G and M an L-equivariant $U(r\mathfrak{g})$-module. Then $\mathcal{D} \underset{U(r\mathfrak{g})}{\otimes} M$ is an L-equivariant quasi-coherent $\mathcal{D}$-module.*

To prove the proposition we will use two additional lemmas:

Lemma 4.5.2. *Let M be weakly L-equivariant $U(r\mathfrak{g})$-module. Then $\mathcal{D} \underset{U(r\mathfrak{g})}{\otimes} M$ is a weakly L-equivariant quasi-coherent $\mathcal{D}$-module.*

Proof. We follow the idea in [26, Section 10.4]. Consider the induced G-actions on $\mathcal{D}$ and M induced by equivariance condition. We define a twisted G-action on $\mathcal{D} \underset{R}{\otimes} M$ by viewing M as a constant sheaf on X and defining:

$$l \cdot (D \otimes m) = l \star D \otimes l.m,$$

for any $D \in \mathcal{D}$, $m \in M$ and $l \in L$.

We begin by proving that the $\mathcal{D}$-module $\mathcal{D} \underset{R}{\otimes} M$ is weakly L-equivariant for the action defined above. We have:

$$\begin{aligned} l \cdot D_1(D_2 \otimes m) &= l \cdot (D_1 D_2 \otimes m) \\ &= l \star (D_1 D_2) \otimes l.m \\ &= (l \star D_1)(l \star D_2) \otimes l.m \text{ (by 4.2.5 i))} \\ &= (l \star D_1)(l \cdot (D_2 \otimes m)), \end{aligned} \tag{4.12}$$

for all $l \in L$, $D_1, D_2 \in \mathcal{D}, m \in M$.

To prove that the L-action is well-defined on $\mathcal{D} \underset{U(r\mathfrak{g})}{\otimes} M$ (and thus $\mathcal{D} \underset{U(r\mathfrak{g})}{\otimes} M$ is also weakly L-equivariant) it remains to prove that for any $\psi \in U(r\mathfrak{g})$, $m \in M$, $l \in L$ and $D \in \mathcal{D}$ a local section that

$$l \cdot (Di_{\mathfrak{g}}(\psi) \otimes m) = l \cdot (D \otimes \psi.m).$$

We have

$$\begin{aligned} l \cdot (D \otimes \psi.m) &= l \star D \otimes l.(\psi.m) \\ &= l \star D \otimes (\mathrm{Ad}(l)\psi).(l.m) \text{ (by equation (4.8))} \\ &= (l \star D) i_{\mathfrak{g}}(\mathrm{Ad}(l)\psi) \otimes l.m \\ &= (l \star D)(l \star i_{\mathfrak{g}}(\psi)) \otimes l.m \text{ (by 4.2.5 iii))} \\ &= l \star (Di_{\mathfrak{g}}\psi) \otimes l.m \text{ (by 4.2.5 i))} \\ &= l \cdot (Di_{\mathfrak{g}}(\psi) \otimes m). \end{aligned}$$

□

Lemma 4.5.3. *Let L a closed subgroup of G and let M a $U(r\mathfrak{g})$-module that is also is an $\mathcal{O}(L)$-comodule and assume that the action of $r\mathfrak{l} \subset \mathfrak{l} = \mathrm{Lie}(L)$ induced by derivating the L-action coincides with the restriction of the $r\mathfrak{g}$ action to $r\mathfrak{l}$.*

Consider the restriction to $r\mathfrak{l}$ of the $r\mathfrak{g}$-action on $\mathcal{D} \underset{U(\mathfrak{g})}{\otimes} M$ induced by $i_{\mathfrak{g}} : r\mathfrak{g} \to \mathcal{D}$ and the action of $r\mathfrak{l}$ induced by the derivative of the L action on $\mathcal{D} \underset{U(\mathfrak{g})}{\otimes} M$. Then these two actions coincide.

Proof. Fix $\psi \in r\mathfrak{l}$, $D \in \mathcal{D}_X$ a local section and $m \in M$.

The first action is given by $\psi \cdot (D \otimes m) = i_{\mathfrak{g}}(\psi) D \otimes m$.

Let $\star_1$ be the derivative of the L-action on $\mathcal{D}$ and $\star_2$ the derivative of the L-action on M. Then the second action (denote it $\star$) is given applying the chain rule by

$$\psi \star (D \otimes m) = \psi \star_1 D \otimes m + D \otimes \psi \star_2 m.$$

We have by Definition 4.2.5 iv) that $\psi \star_1 D = i_{\mathfrak{g}}(\psi) D - Di_{\mathfrak{g}}(\psi)$ and because of the assumption on M, $\psi \star_2 m = \psi m$. Therefore, we get

$$\begin{aligned}\psi \star (D \otimes m) &= \psi \star_1 D \otimes m + D \otimes \psi \star_2 m \\ &= [i_{\mathfrak{g}}(\psi)D - Di_{\mathfrak{g}}(\psi)] \otimes m + D \otimes \psi m \\ &= i_{\mathfrak{g}}(\psi)D \otimes m - Di_{\mathfrak{g}}(\psi) \otimes m + D \otimes \psi m \\ &= i_{\mathfrak{g}}(\psi)D \otimes m - D \otimes \psi m + D \otimes \psi m \\ &= i_{\mathfrak{g}}(\psi)D \otimes m \\ &= \psi \cdot (D \otimes m).\end{aligned} \tag{4.13}$$

Thus the lemma is proved. □

Proposition 4.5.1 now follows from Lemmas 4.5.2 and 4.5.3.

Proposition 4.5.4. *Let L be a closed subgroup of G and $\mathcal{M}$ an L-equivariant quasi-coherent $\mathcal{D}$-module. Then $M := \Gamma(X, \mathcal{M})$ is an L-equivariant $U(r\mathfrak{g})$-module.*

We will do this in two steps:

Lemma 4.5.5. *Let L be a closed subgroup of G and $\mathcal{M}$ a weakly L-equivariant quasi-coherent $\mathcal{D}$-module. Then $M := \Gamma(X, \mathcal{M})$ is a weakly L-equivariant $U(\mathfrak{g})$-module.*

Proof. First, notice that the $U(r\mathfrak{g})$-module structure on M is given by

$$\psi.m = i_{\mathfrak{g}}(\psi).m, \qquad \text{for } \psi \in U(r\mathfrak{g}), m \in M.$$

We have that for $l \in L$, $\psi \in U(r\mathfrak{g})$ and $m \in M$

$$\begin{aligned} l.(\psi.m) &= l.(i_{\mathfrak{g}}(\psi).m) \\ &= (l.i_{\mathfrak{g}}(\psi)).(l.m) \text{ (by 4.2.6 ii))} \\ &= i_{\mathfrak{g}}(l.\psi).(l.m) \text{ (by 4.2.5 iii))} \\ &= (l.\psi).(l.m). \end{aligned}$$

□

Lemma 4.5.6. *Let $\mathcal{M}$ be weakly L-equivariant $\mathcal{D}$-module and assume that the $r\mathfrak{l} \subset \mathfrak{l} = \mathrm{Lie}(L)$ action induced by the derivative of the L-action coincides with the restriction to $r\mathfrak{l}$ of the $r\mathfrak{g}$ action on $\mathcal{M}$ induced by $i_{\mathfrak{g}}$. Then the same holds on $M := \Gamma(X, \mathcal{M})$.*

Proof. The lemma follows from the fact that $\mathcal{D}$ is G-htdo and definition 4.2.6, notice it is much easier to prove than the corresponding Lemma 4.5.3. □

Proposition 4.5.4 now follows from Lemmas 4.5.5, 4.5.6.

4.5.2 Homogeneous twisted differential operators on the flag variety

In this subsection, we aim to prove that the r-th deformation of the sheaf of λ-twisted differential operators on the flag variety is an r-deformed G-htdo. We begin by reviewing the constructions in [5, Section 4, Section 6.4] and [3, Section 5.1]. For now, we keep the notation from the previous section. Recall that by [3, Definition 5.1.1] there is an infinitesimal action of $\mathfrak{g}$ on X given by a R-linear Lie algebra map $\varphi' : \mathfrak{g} \to \mathcal{T}_X(X)$. This morphism is G-equivariant by [3, Lemma 5.1.3] and can be extended to a G-equivariant ring homomorphism $\alpha : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X)$.

Lemma 4.5.7. *The sheaf* $(\mathcal{D}_X, \alpha)$ *is a* G*-htdo.*

Proof. Recall by Definition 3.4.2 that $\mathcal{D}_X$ is a differential algebra with $F_0\mathcal{D}_X \cong \mathcal{O}_X$ and $\operatorname{gr}_1 \mathcal{D}_X \cong \mathcal{T}_X$. Further by equation (3.15), we have $\operatorname{gr} \mathcal{D}_X \cong \operatorname{Sym}_{\mathcal{O}_X} \mathcal{T}_X$, so $\mathcal{D}_X$ is a tdo on X.

We endow $\mathcal{D}_X$ with a G action by setting:

$$\begin{aligned} &g.f(x) = f(gx) \text{ for } g \in G, f \in \mathcal{O}, x \in X, \\ &g.\tau(f) = g.\tau(g^{-1}f) \text{ for } g \in G, \tau \in \mathcal{T}, f \in \mathcal{O}. \end{aligned} \tag{4.14}$$

With this action, $\mathcal{D}_X$ satisfies axioms i) and ii) from Definition 4.2.5. Since α is G-equivariant, axiom iii) is also satisfied. Further, it is clear that the map α satisfies axioms v) and vi). Finally, axiom iv) follows from the proof of [11, Proposition 2.2]. ☐

Lemma 4.5.8. *Let H be another smooth affine algebraic group acting on X such that the actions of G and H commute. Then* $\operatorname{im}(\alpha) \subset \Gamma(X, \mathcal{D}_X)^H$.

Proof. This follows from [5, Section 4.8]. ☐

Recall that we assume that G is a connected, simply connected, split semisimple, smooth affine algebraic group scheme over R. Let B be a closed and flat Borel R-subgroup scheme, N its unipotent radical and $H = B/N$ the abstract Cartan group. Let $\widetilde{X} = G/N$ denote the basic affine space and $X = G/B$ denote the flag scheme. Define an action of H on $\widetilde{X}$ via

$$bN \cdot gN := gbN, \qquad b \in B, g \in G.$$

Lemma 4.5.9 ([5, Lemma 4.7]). *i) The action of H commutes with the natural action of G.*

ii) $\widetilde{X}$ and X are smooth separated schemes over R, locally of finite type.

iii) The natural projection $\xi : \widetilde{X} \to X$ is a locally trivial H-torsor.

Definition 4.5.10. *The relative enveloping algebra is the sheaf of H-invariants of $\xi_* \mathcal{D}_{\widetilde{X}}$:*

$$\widetilde{\mathcal{D}}_X := (\xi_* \mathcal{D}_{\widetilde{X}})^H.$$

The sheaf comes with a natural filtration $F_i \widetilde{\mathcal{D}}_X := (\xi_* F_i \mathcal{D}_{\widetilde{X}})^H$ induced by the natural filtration on $\mathcal{D}_{\widetilde{X}}$.

Proposition 4.5.11. *Let $\widetilde{\mathcal{T}}_X := (\xi_* \mathcal{T}_{\widetilde{X}})^H$, $\mathfrak{h} = \operatorname{Lie}(H)$, U be an affine open set of X trivialising ξ. Then:*

i) $\mathcal{D}_{\widetilde{X}}(U) \underset{R}{\otimes} U(\mathfrak{h}) \cong \widetilde{\mathcal{D}}_X(U)$.

ii) $\operatorname{Sym}_{\mathcal{O}_X} \widetilde{\mathcal{T}}_X \cong \operatorname{gr} \widetilde{\mathcal{D}}_X$.

iii) The derivative of the H action on $\widetilde{X}$ induces a central embedding $j : \mathfrak{h} \to \widetilde{\mathcal{D}}_X$.

Proof. The first two claims follow by [5, Proposition 4.6] and the third follows by [5, Section 4.10] along with Lemma 4.5.9 ii) and iii). ☐

The sheaf $\widetilde{\mathcal{D}}_X$ will be used to construct the sheaf of λ-twisted differential operators. Proposition 4.5.11 i) shows that $\widetilde{\mathcal{D}}_X$ can not be a htdo as it is not a tdo on X.

Lemma 4.5.12. *The sheaf $(\widetilde{\mathcal{D}}_X, \alpha)$ is a differential algebra, G-equivariant as an $\mathcal{O}_X$-module and furthermore it satisfies axioms i) to vi) of Definition 4.2.5.*

Proof. The filtration on $\widetilde{\mathcal{D}}_X$ makes it a differential algebra. Since the actions of G and H commute we have by [58, Lemma 3.6] that the G action on $\mathcal{D}_{\widetilde{X}}$ descends to a G-action on $\widetilde{\mathcal{D}}_X$. Thus, axioms $i), ii)$ of Definition 4.2.5 are satisfied. Axioms $iii) - vi)$ follow from construction and Lemma 4.5.8 since α and ξ are G-equivariant. ☐

Definition 4.5.13. *Let $\widetilde{\mathcal{D}}_r$ be the sheafification of the presheaf obtained by postcomposing $\widetilde{\mathcal{D}}_X$ with the deformation functor $A \to A_r$. Since the G-action preserves the filtration on $\widetilde{\mathcal{D}}_X$ induced by deformation, it restricts to a G-action on $\widetilde{\mathcal{D}}_r$.*

Now, let $\lambda : r\mathfrak{h} \to R$; the map λ can be extended to a ring homomorphism $\lambda : U(r\mathfrak{h}) \to R$ and we denote R_λ the resulting $U(r\mathfrak{h})$-module. Furthermore, recall that by Proposition 4.5.11 *iii*), there exists a central embedding $j : \mathfrak{h} \to \widetilde{\mathcal{D}}_X$ which can be extended to a ring morphism $j : U(\mathfrak{h}) \to \widetilde{\mathcal{D}}_X$. By applying the deformation functor we obtain a homomorphism $j : U(r\mathfrak{h}) \to \widetilde{\mathcal{D}}_r$.

Definition 4.5.14. *The sheaf of r-deformed λ-twisted differential operators on the flag variety X is the central reduction*

$$\mathcal{D}_{\lambda,r} := \widetilde{\mathcal{D}}_r \underset{U(r\mathfrak{h})}{\otimes} R_\lambda.$$

We give R_λ the trivial filtration and we view $\mathcal{D}_{\lambda,r}$ as a sheaf of filtered R-algebras with the tensor filtration.

Corollary 4.5.15. *The sheaf $(\mathcal{D}_{\lambda,r}, \alpha)$ is an r-deformed G-htdo on the flag variety.*

Proof. We have by [5, Lemma 6.4] that $\mathcal{D}_\lambda$ is a tdo on X. Since $\mathcal{D}_{\lambda,r}$ can be viewed as the r-th deformation of $\mathcal{D}_\lambda$, we have by Lemma 4.2.4 that $\mathcal{D}_{\lambda,r}$ is an r-deformed tdo. Let $\mathcal{I} = \langle j(h) - \lambda(h) | h \in r\mathfrak{h}\rangle$ be two-sided ideal in $\widetilde{\mathcal{D}}_r$ such that $\mathcal{D}_{\lambda,r} = \widetilde{\mathcal{D}}_r/\mathcal{I}$. We have by Lemma 4.5.8 (note that we swap G and H) that $g.j(h) = h$. Therefore, the ideal $\mathcal{I}$ is stable under the G-action, so the G-action on $\widetilde{\mathcal{D}}_r$ descends to $\mathcal{D}_{\lambda,r}$. By abuse of notation we denote α the composition $\alpha : U(r\mathfrak{g}) \to \widetilde{\mathcal{D}}_r$ with the projection $\widetilde{\mathcal{D}}_r \to \mathcal{D}_{\lambda,r}$. Then $(\mathcal{D}_{\lambda,r}, \alpha)$ is an r-deformed G-htdo. $\square$

We finish the subsection by computing the geometric fibre of $\mathcal{D}_\lambda = \mathcal{D}_{\lambda,1}$ at the identity. This is also sketched is [45, I.2.4], but we use different conventions to construct the sheaf $\mathcal{D}_\lambda$. Since the group G is split we can find a Cartan group T of G complementary to N in B. Using the natural isomorphism $H \to T$, we view $\mathfrak{h}$ as a Cartan subalgebra of $\mathfrak{g}$. We let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ to be the corresponding decompositions for the Lie algebra and Borel subalgebra respectively. The adjoint action of H on $\mathfrak{g}$ induces a decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where we regard $\mathfrak{n}^- = \mathrm{Lie}(N)$ as being spanned by *negative* roots.

Lemma 4.5.16. *Let m be the ideal sheaf of functions on X vanishing on eB and let $i : [eB] \to X$ denote the natural inclusion. Then:*

$$U(\mathfrak{g})/\mathfrak{n}^- U(\mathfrak{g}) \cong \Gamma(X, \widetilde{\mathcal{D}}_X / m\widetilde{\mathcal{D}}_X).$$

Proof. Let $Y = G/N$ denote the basic affine space, let $x_0 = eN$ denote the base point and m_N denote the ideal sheaf of functions vanishing on x_0.

Let $\rho : \mathfrak{g} \to G/N$ encode the infinitesimal action of $\mathfrak{g}$ on G/N induce by the action of G on G/N by left multiplication. Since the stabiliser of x_0 with respect to this action is N we have that:

$$\rho(\mathfrak{n}^-) \subset m_N \mathcal{D}_{G/N}.$$

Applying the descent functor $(\xi_*)^H$ we obtain a map $\tilde{\rho} : \mathfrak{n}^- \to (m_N \mathcal{D}_{G/N})^H \cong (m_N)^H \widetilde{\mathcal{D}}_X$. By construction $\tilde{\rho}$ is just the restriction of the infinitesimal action of $\mathfrak{g}$ on G/B to $\mathfrak{n}^-$. By considering the following diagram

$$\begin{tikzcd}
G/N \arrow[d] & x_0 = N/N \arrow[l] \arrow[d] \\
G/B & eB \arrow[l]
\end{tikzcd}$$

we observe that $(m_N)^H \widetilde{\mathcal{D}}_X \subset m\widetilde{\mathcal{D}}_X$, so $\tilde{\rho}(\mathfrak{n}^-) \subset m\widetilde{\mathcal{D}}_X$. Therefore we obtain a surjective map

$$U(\mathfrak{g})/\mathfrak{n}^- U(\mathfrak{g}) \to \Gamma(X, \widetilde{\mathcal{D}}_X / m\widetilde{\mathcal{D}}_X).$$

By passing to the associated grading rings and applying Proposition 4.5.11 *ii*) we obtain an isomorphism $\mathrm{gr}(U(\mathfrak{g})/\mathfrak{n}^- U(\mathfrak{g})) \to \Gamma(X, \mathrm{gr}(\widetilde{\mathcal{D}}_X / m\widetilde{\mathcal{D}}_X))$, so the claim follows.

□

Corollary 4.5.17. *Let $\lambda \in \mathfrak{h}^*$ and let m the ideal sheaf of functions on X vanishing on eB. Further, let R_λ be the $U(\mathfrak{h})$-module induced by the $\mathfrak{h}$ module on which $\mathfrak{h}$ acts by λ. Then*

$$R_\lambda \underset{U(\mathfrak{h})}{\otimes} (U(\mathfrak{g})/\mathfrak{n}^- U(\mathfrak{g})) \cong \Gamma(X, \mathcal{D}_\lambda / m\mathcal{D}_\lambda).$$

Proof. We have:

$$\begin{aligned}\Gamma(X, \mathcal{D}_\lambda/m\mathcal{D}_\lambda) &= \Gamma(X, (i_*)\mathcal{O}_{eB} \underset{\mathcal{O}_X}{\otimes} \widetilde{\mathcal{D}}_X) \underset{U(\mathfrak{h})}{\otimes} R_\lambda \\ &\cong \Gamma(X, \widetilde{\mathcal{D}}_X)/m\widetilde{\mathcal{D}}_X) \underset{U(\mathfrak{h})}{\otimes} R_\lambda) \\ &\cong \Gamma(X, R_\lambda \underset{U(\mathfrak{h})}{\otimes} \widetilde{\mathcal{D}}_X)/m\widetilde{\mathcal{D}}_X) \text{ (by Proposition 4.5.11)} \\ &\cong R_\lambda \underset{U(\mathfrak{h})}{\otimes} \Gamma(X, \widetilde{\mathcal{D}}_X)/m\widetilde{\mathcal{D}}_X) \\ &\cong R_\lambda \underset{U(\mathfrak{h})}{\otimes} U(\mathfrak{g})/\mathfrak{n}^- U(\mathfrak{g}) \text{ (by Lemma 4.5.16).}\end{aligned} \tag{4.15}$$

□

4.5.3 Applications of the localisation mechanism

We retain the notation from the previous subsection. Recall that for any weight $\lambda : \mathfrak{h} \to R$, we defined χ_λ the corresponding central character and we let $U(\mathfrak{g})^\lambda = U(\mathfrak{g})/\ker(\chi_\lambda)U(\mathfrak{g})$.

By [5, Section 6.10], we have a map $\varphi : U(\mathfrak{g})^\lambda \to \mathcal{D}_\lambda$; by functoriality of deformation, we obtain a map $\varphi : U(\mathfrak{g})^\lambda_r \to \mathcal{D}_{\lambda,r}$. Define the localisation functor $\mathrm{Loc}^{\lambda,r} : \mathrm{Mod}(U(\mathfrak{g})^\lambda_r) \to \mathrm{Mod}(\mathcal{D}_{\lambda,r})$. This functor is adjoint to the global sections functor $\Gamma(X,-) : \mathrm{Mod}(\mathcal{D}_{\lambda,r}) \to \mathrm{Mod}(U(\mathfrak{g})^\lambda_r)$.

Proposition 4.5.18. *Let L be a closed subgroup of G. The functors $\mathrm{Loc}^{\lambda,r}$ and $\Gamma(X,-)$ induce a pair of adjoint functors between $\mathrm{Mod}(U(\mathfrak{g})^\lambda_r, L)$ and $\mathrm{Mod}(\mathcal{D}_{\lambda,r}, L)$.*

Proof. This follows by combining Corollary 4.5.15 and Propositions 4.5.1 and 4.5.4. □

In general, the functors above do not provide an equivalence of categories. However, when R is a field of characteristic 0 we get the well-known Beilinson-Bernstein localisation. We will also need the following proposition:

Proposition 4.5.19. *Assume that R is a field of characteristic 0. Then the map $\varphi : U(\mathfrak{g})^\lambda \to \Gamma(X, \mathcal{D}_\lambda)$ is an isomorphism.*

Theorem 4.5.20. *([8], [38, Section 9.2]) Assume that R is a field of characteristic 0. Let λ be a dominant weight and L a closed subgroup of G. Then the functor Loc^λ is an equivalence of categories between $\mathrm{Mod}(U(\mathfrak{g})^\lambda, L)$ and the quotient category $\mathrm{Mod}(\mathcal{D}_\lambda, L)/\ker\Gamma$. A quasi-inverse is given by $\Gamma(X,-)$.*

Furthermore, under this equivalence finitely generated $U(\mathfrak{g})^\lambda$*-modules correspond to coherent* $\mathcal{D}_\lambda$*-modules and the global sections functor is exact.*

If λ *is also regular,* $\ker \Gamma = 0$*, so one obtains an equivalence of categories between* $\operatorname{Mod}(U(\mathfrak{g})^\lambda, L)$ *and* $\operatorname{Mod}(\mathcal{D}_\lambda, L)$*.*

The same statements hold if we remove the L*-equivariance from both sides, i.e. setting* $L = e$*.*

As an easy consequence we obtain:

Corollary 4.5.21. *Assume that* R *is a field of characteristic 0 and let* λ *be a dominant weight. Consider* $M \in \operatorname{Mod}(U(\mathfrak{g})^\lambda, L)$*. Then*

$$\Gamma(X, \operatorname{Loc}^\lambda M) \cong M.$$

In the next sections, we will apply the theorem in two particular cases: B is a Borel subgroup of G acting on $X = G/B$ and $G = \{(g, g) | g \in G\} \subset G \times G$ acting diagonally on the flag variety of $G \times G$, which is $X \times X$. Notice, we are using a slight abuse notation and identify G with the diagonal subgroup.

4.6 Pullback of modules over deformed homogeneous twisted differential operators

We retain the notation from the beginning of the previous section: recall that G denotes a connected, simply-connected, split semisimple, smooth affine algebraic group scheme over R, $\mathfrak{g} = \operatorname{Lie}(G)$, B a closed and flat Borel subscheme of G and $X = G/B$ the flag scheme. We consider the triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, we let $\mathfrak{b}^- := \mathfrak{n}^- \oplus \mathfrak{h}$, $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$.

By construction, X is a G-homogeneous space; furthermore we have by [36, II,1.10(2)] and by Lemma 4.5.9 *ii*) that Assumption 4.3.1 is satisfied. Therefore, we may apply the machinery developed in Sections 4.3 and 4.5.

We consider homogeneous twisted differential operators on the double flag variety $X \times X$ of the group $G \times G$. The Cartan algebra of $r\mathfrak{g} \times r\mathfrak{g}$ is given by $r\mathfrak{h} \times r\mathfrak{h}$, so picking a weight of the deformed Cartan Lie subalgebra is equivalent to picking a pair of weights $\lambda, \mu \in r\mathfrak{h}^*$. We have by Corollary 4.5.15 that $\mathcal{D}_{\lambda,\mu,r}$ is an r-deformed $G \times G$-htdo on $X \times X$. In particular, it is an r-deformed G-htdo with respect to the diagonal G action defined in Section 4.3. Recall also that $i_l : X \to X \times X$ denotes the inclusion into the left copy. We have by Corollary 4.3.11 that $\mathcal{D}_{\lambda,r} \cong i_l^{\#}\mathcal{D}_{\lambda,\mu,r}$ is an r-deformed B-htdo and the functor $i_l^{\#} : \operatorname{Coh}(\mathcal{D}_{\lambda,\mu,r}, G) \to \operatorname{Coh}(\mathcal{D}_{\lambda,r}, B)$ is an equivalence of categories. From now on, until the end of this section, we will assume that $\lambda, \mu : r\mathfrak{h} \to R$ are dominant weights.

Definition 4.6.1. *Let $i : eB \to X$ be the natural inclusion and let R to be trivial $\mathcal{O}_{eB}$-module. We let $\mathcal{M}_{\mu,r}$ be the right $\mathcal{D}_{\mu,r}$-module defined by:*

$$\mathcal{M}_{\mu,r} := i_*R \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{\mu,r}.$$

Lemma 4.6.2.

i) $\mathcal{M}_{\mu,r}$ is coherent as a right $\mathcal{D}_{\mu,r}$-module.

ii) Let $\mathcal{M}_\mu = \mathcal{M}_{\mu,1}$. Then $\Gamma(X, \mathcal{M}_\mu) = R_\mu \underset{U(\mathfrak{b}^-)}{\otimes} U(\mathfrak{g})$; we will use $T(\mu)$ to denote this module.

Proof. Since $\mathcal{D}_{\mu,r}$ is in particular an r-deformed tdo, we have by Definition 3.4.5 that $\operatorname{gr}\mathcal{D}_{\mu,r} \cong \operatorname{Sym}_{\mathcal{O}_X}(r\mathcal{T}_X)$ is a sheaf of Noetherian rings, so $\mathcal{D}_{\mu,r}$ is a sheaf of Noetherian rings. Therefore, coherence is equivalent to locally finite generation. Let $U \subset X$ be affine open; if $eB \notin U$, then $\mathcal{M}_{\mu,r}(U) = 0$. If $eB \in U$, then $1 \otimes 1$ is a generator for $\mathcal{M}_{\mu,r}(U)$.

For the second part, we have by construction that $\mathcal{M}_\mu \cong i_*(i^*\mathcal{D}_\mu) \cong T_{eB}(\mathcal{D}_\mu)$ (the geometric fibre at the identity). Let m be the ideal sheaf of functions on X vanishing on eB. We have:

$$\begin{aligned} T_{eB}(\mathcal{D}_\mu) &= \Gamma(X, \mathcal{D}_\mu/m\mathcal{D}_\mu) \\ &= R_\mu \underset{U(\mathfrak{h})}{\otimes} U(\mathfrak{g})/\mathfrak{n}^-U(\mathfrak{g}) \text{ (by Corollary 4.5.17)} \\ &= R_\mu \underset{U(\mathfrak{b}^-)}{\otimes} U(\mathfrak{g}). \end{aligned}$$ ☐

Lemma 4.6.3. *Let $p_r : X \times X \to X$ denote the projection onto the right factor. Then*

$$p_r^{-1}(\mathcal{M}_{\mu,r}) \underset{p_r^{-1}\mathcal{D}_{\mu,r}}{\otimes} \mathcal{D}_{\lambda,\mu,r} \cong i_{l_*}i_l^*\mathcal{D}_{\lambda,\mu,r}.$$

Proof. We have

$$\begin{aligned} i_{l_*} i_l^* \mathcal{D}_{\lambda,\mu,r} &= i_{l_*}(\mathcal{O}_X \underset{i_l^{-1}\mathcal{O}_{X\times X}}{\otimes} i_l^{-1}\mathcal{D}_{\lambda,\mu,r}) \\ &\cong i_{l_*}\mathcal{O}_X \underset{\mathcal{O}_{X\times X}}{\otimes} \mathcal{D}_{\lambda,\mu,r} \\ &\cong i_{l_*}\mathcal{O}_X \underset{\mathcal{O}_{X\times X}}{\otimes} p_r^{-1}\mathcal{D}_{\mu,r} \underset{p_r^{-1}\mathcal{D}_{\mu,r}}{\otimes} \mathcal{D}_{\lambda,\mu,r}. \end{aligned} \tag{4.16}$$

Therefore, it is enough to prove that

$$p_r^{-1}\mathcal{M}_{\mu,r} \cong i_{l_*}\mathcal{O}_X \underset{\mathcal{O}_{X\times X}}{\otimes} p_r^{-1}\mathcal{D}_{\mu,r}.$$

Consider the following Cartesian square:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "i_l"] & eB \arrow[d, "i"] \\ X \times X \arrow[r, "p_r"] & X, \end{tikzcd}$$

where i and p denote the natural inclusion and projection. Consider the constant sheaf R on the trivial scheme eB. We then have by [54, 02KG]

$$p_r^*(i_*R) \cong i_{l_*}(p^*R).$$

Here the pullback is taken in the category of right modules instead of left modules. Since $p^*R \cong \mathcal{O}_X$ as right $\mathcal{O}_X$-modules, we obtain that as right $\mathcal{O}_{X\times X}$-modules

$$i_{l_*}\mathcal{O}_X \cong p_r^*(i_*R).$$

Therefore, we obtain

$$\begin{aligned} i_{l_*}\mathcal{O}_X \underset{\mathcal{O}_{X\times X}}{\otimes} p_r^{-1}\mathcal{D}_{\mu,r} &\cong p_r^*(i_*R) \underset{\mathcal{O}_{X\times X}}{\otimes} p_r^{-1}\mathcal{D}_{\mu,r} \\ &\cong p_r^{-1}(i_*R) \underset{p_r^{-1}\mathcal{O}_X}{\otimes} \mathcal{O}_{X\times X} \underset{\mathcal{O}_{X\times X}}{\otimes} p_r^{-1}\mathcal{D}_{\mu,r} \\ &\cong p_r^{-1}(i_*R) \underset{p_r^{-1}\mathcal{O}_X}{\otimes} p_r^{-1}\mathcal{D}_{\mu,r} \\ &\cong p_r^{-1}(i_*R \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{\mu,r}) \\ &\cong p_r^{-1}\mathcal{M}_{\mu,r}. \end{aligned}$$

□

Corollary 4.6.4. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_{\lambda,\mu,r}$-module. Then*

$$i_{l_*} i_l^{\#} \mathcal{M} \cong p_r^{-1}(\mathcal{M}_{\mu,r}) \underset{p_r^{-1}\mathcal{D}_{\mu,r}}{\otimes} \mathcal{M}.$$

Proof. We have

$$\begin{aligned}
i_{l_*} i_l^{\#} \mathcal{M} &\cong i_{l_*} i_l^{*} \mathcal{M} \\
&\cong i_{l_*}(\mathcal{O}_X \underset{i_l^{-1}\mathcal{O}_{X\times X}}{\otimes} i_l^{-1}\mathcal{M}) \\
&\cong i_{l_*}(\mathcal{O}_X \underset{i_l^{-1}\mathcal{O}_{X\times X}}{\otimes} i_l^{-1}\mathcal{D}_{\lambda,\mu,r} \underset{i_l^{-1}\mathcal{D}_{\lambda,\mu,r}}{\otimes} i_l^{-1}\mathcal{O}_{X\times X} \underset{i_l^{-1}\mathcal{O}_{X\times X}}{\otimes} i_l^{-1}\mathcal{M}) \\
&\cong i_{l_*}(i_l^{*}\mathcal{D}_{\lambda,\mu,r} \underset{i_l^{-1}\mathcal{D}_{\lambda,\mu,r}}{\otimes} i_l^{-1}\mathcal{M}) \\
&\cong i_{l_*} i_l^{*}\mathcal{D}_{\lambda,\mu,r} \underset{\mathcal{D}_{\lambda,\mu,r}}{\otimes} \mathcal{M} \text{ (Since } i_l \text{ is closed)} \\
&\cong p_r^{-1}(\mathcal{M}_{\mu,r}) \underset{p_r^{-1}\mathcal{D}_{\mu,r}}{\otimes} \mathcal{D}_{\lambda,\mu,r} \underset{\mathcal{D}_{\lambda,\mu,r}}{\otimes} \mathcal{M} \text{ (By Lemma 4.6.3)} \\
&\cong p_r^{-1}(\mathcal{M}_{\mu,r}) \underset{p_r^{-1}\mathcal{D}_{\mu,r}}{\otimes} \mathcal{M}.
\end{aligned}$$

□

4.6.1 Global sections under the pullback

Throughout this subsection, we will assume that $R = K$ is a field of characteristic 0.

Recall that by equation (4.10), we have $U(\mathfrak{g}\times\mathfrak{g})^{\lambda,\mu} \cong U(\mathfrak{g})^{\lambda} \otimes U(\mathfrak{g})^{\mu}$. We aim to prove the following theorem:

Theorem 4.6.5. *Let λ, μ be dominant weights and let $\mathcal{M}$ be a coherent G-equivariant $\mathcal{D}_{\lambda,\mu}$-module. Then we obtain an isomorphism of left $U(\mathfrak{g})^{\lambda}$-modules*

$$\Gamma(X, i_l^{\#}\mathcal{M}) \cong T(\mu) \underset{U(\mathfrak{g})^{\mu}}{\otimes} \Gamma(X\times X, \mathcal{M}).$$

We should make several remarks before we proceed to prove this. We view $\Gamma(X\times X, \mathcal{M})$ as a left $U(\mathfrak{g})^{\mu}$-module by defining $y.m = (1\otimes y).m$ for $y \in U(\mathfrak{g})^{\mu}$ and $m \in \Gamma(X\times X, \mathcal{M})$. Further, we note that the space $T(\mu) \underset{U(\mathfrak{g})^{\mu}}{\otimes} \Gamma(X\times X, \mathcal{M})$ has the structure of a left $U(\mathfrak{g})^{\lambda}$-module via $x.(m\otimes m') = m \otimes (x\otimes 1).m'$ for $x \in U(\mathfrak{g})^{\lambda}$, $m \in T(\mu)$ and $m' \in \Gamma(X\times X, \mathcal{M})$.

In order to prove this theorem, we need to introduce additional notation. Let Y, Z be K-schemes, $\mathcal{M}$ an $\mathcal{O}_Y$-module and $\mathcal{N}$ an $\mathcal{O}_Z$-module. We will use the notation:

$$\mathcal{M} \boxtimes \mathcal{N} := \mathcal{O}_{Y\times Z} \underset{q_Y^{-1}\mathcal{O}_Y \otimes q_Z^{-1}\mathcal{O}_Z}{\otimes} q_Y^{-1}\mathcal{M} \otimes q_Z^{-1}\mathcal{N} \cong q_Y^{*}\mathcal{M} \underset{\mathcal{O}_{Y\times Z}}{\otimes} q_Z^{*}\mathcal{N}, \tag{4.17}$$

where $q_Y : Y \times Z \to Y$ and $q_Z : Y \times Z \to Z$ denote the natural projections. We refer to [32, Section I.1.5] for the basic properties of box tensor product. We will also identify X with $X \times eB$ and the map i_l with $\mathrm{id} \times i$; recall $i : eB \to X$ denotes the natural inclusion. We have that:

$$\begin{aligned}
i_l^{*}\mathcal{D}_{\lambda,\mu} &\cong (\mathrm{id} \times i)\mathcal{D}_{\lambda,\mu} \\
&\cong (\mathrm{id} \times i)(\mathcal{D}_{\lambda} \boxtimes \mathcal{D}_{\mu}) \text{ (by [29, Section 5.4])} \\
&\cong \mathcal{D}_{\lambda} \boxtimes i^{*}\mathcal{D}_{\mu} \\
&\cong \mathcal{D}_{\lambda} \underset{K}{\otimes} \Gamma(eB, i^{*}\mathcal{D}_{\mu}) \\
&\cong \mathcal{D}_{\lambda} \underset{K}{\otimes} \Gamma(X, i_{*}i^{*}\mathcal{D}_{\mu}) \\
&\cong \mathcal{D}_{\lambda} \underset{K}{\otimes} \Gamma(X, \mathcal{M}_{\mu}) \\
&\cong \mathcal{D}_{\lambda} \underset{K}{\otimes} T(\mu) \text{ (by Lemma 4.6.2).}
\end{aligned} \tag{4.18}$$

Therefore we obtain that as left $\mathcal{D}_{\lambda}$-modules $i_l^{\#}\mathcal{D}_{\lambda,\mu} \cong \mathcal{D}_{\lambda} \underset{K}{\otimes} T(\mu)$, so by Theorem 4.1.2, we have that as left $U(\mathfrak{g})^{\lambda}$-modules:

$$\Gamma(X, i_l^{\#}\mathcal{D}_{\lambda,\mu}) \cong U(\mathfrak{g})^{\lambda} \underset{K}{\otimes} T(\mu). \tag{4.19}$$

Proof of theorem 4.6.5. Let $\mathscr{F}, \mathscr{J} : \mathrm{Coh}(\mathcal{D}_{\lambda,\mu}, G) \to \mathrm{Mod}(U(\mathfrak{g})^{\lambda}, B)$ defined by

$$\begin{aligned}
\mathscr{F}(\mathcal{M}) &: = \Gamma(X, i_l^{\#}\mathcal{M}), \\
\mathscr{J}(\mathcal{M}) &: = T(\mu) \underset{U(\mathfrak{g})^{\mu}}{\otimes} \Gamma(X \times X, \mathcal{M}).
\end{aligned} \tag{4.20}$$

The theorem follows if we can prove that $\mathscr{F}(\mathcal{M}) \cong \mathscr{J}(\mathcal{M})$. First, we extend the functors $\mathscr{F}, \mathscr{J}$ to the category $\mathrm{WCoh}(\mathcal{D}_{\lambda,\mu}, G)$; this is the category of coherent $\mathcal{D}_{\lambda,\mu}$-modules that are equivariant $\mathcal{O}$-modules together with G-equivariant morphisms. The reason to do this is that $\mathcal{D}_{\lambda,\mu} \in \mathrm{WCoh}(\mathcal{D}_{\lambda,\mu}, G)$, but $\mathcal{D}_{\lambda,\mu} \notin \mathrm{Coh}(\mathcal{D}_{\lambda,\mu}, G)$.

Since $i_l^{\#} \cong i_l^{*}$ as $\mathcal{O}$-modules, $i_l^{\#}$ is an equivalence of Abelian categories and λ is dominant, we obtain by Theorem 4.1.2 that $\mathscr{F}$ is right exact. Furthermore since λ

and μ are dominant weights, by the same theorem applied on $\mathcal{D}_{\lambda,\mu}$-modules, $\mathscr{J}$ is also right exact.

Next, let us construct a map from $\mathscr{J}(\mathcal{M}) \to \mathscr{F}(\mathcal{M})$. Let $\mathcal{N} = i_{l_*} i_l^* \mathcal{D}_{\lambda,\mu}$ and consider the space $\mathcal{N} \underset{\mathcal{D}_{\lambda,\mu}}{\otimes} \mathcal{M}$. By construction, this is isomorphic to $i_{l_*} i_l^* \mathcal{M}$. Therefore, we have

$$\begin{aligned}
\mathscr{F}(\mathcal{M}) &= \Gamma(X, i_l^{\#}\mathcal{M}) \\
&= \Gamma(X, i_l^{*}\mathcal{M}) \\
&\cong \Gamma(X \times X, i_{l_*} i_l^{*}\mathcal{M}) \\
&\cong \Gamma(X \times X, \mathcal{N} \underset{\mathcal{D}_{\lambda,\mu}}{\otimes} \mathcal{M}).
\end{aligned} \tag{4.21}$$

Therefore, it is enough to construct a map from $\mathscr{J}(\mathcal{M})$ to $\Gamma(X \times X, \mathcal{N} \underset{\mathcal{D}_{\lambda,\mu}}{\otimes} \mathcal{M})$. This reduces to proving that $\mathscr{J}(\mathcal{M}) \cong \Gamma(X \times X, \mathcal{N}) \underset{\Gamma(X\times X, \mathcal{D}_{\lambda,\mu})}{\otimes} \Gamma(X \times X, \mathcal{M})$. Let $\mathcal{B} := \Gamma(X \times X, \mathcal{N}) \underset{\Gamma(X\times X, \mathcal{D}_{\lambda,\mu})}{\otimes} \Gamma(X \times X, \mathcal{M})$.

We have

$$\begin{aligned}
\mathcal{B} &= \Gamma(X \times X, \mathcal{N}) \underset{\Gamma(X\times X, \mathcal{D}_{\lambda,\mu})}{\otimes} \Gamma(X \times X, \mathcal{M}) \\
&= \Gamma(X \times X, i_{l_*} i_l^{*}\mathcal{D}_{\lambda,\mu}) \underset{\Gamma(X\times X, \mathcal{D}_{\lambda,\mu})}{\otimes} \Gamma(X \times X, \mathcal{M}) \\
&\cong U(\mathfrak{g})^{\lambda} \underset{K}{\otimes} T(\mu) \underset{U(\mathfrak{g})^{\lambda} \underset{K}{\otimes} U(\mathfrak{g})^{\mu}}{\otimes} \Gamma(X \times X, \mathcal{M}) \\
&\cong T(\mu) \underset{U(\mathfrak{g})^{\mu}}{\otimes} \Gamma(X \times X, \mathcal{M}) \\
&\cong \mathscr{J}(\mathcal{M}).
\end{aligned} \tag{4.22}$$

Finally, we have that

$$\begin{aligned}
\mathscr{J}(\mathcal{D}_{\lambda,\mu}) &= T(\mu) \underset{U(\mathfrak{g})^{\mu}}{\otimes} \Gamma(X \times X, \mathcal{D}_{\lambda,\mu}) \\
&\cong T(\mu) \underset{U(\mathfrak{g})^{\mu}}{\otimes} U(\mathfrak{g})^{\lambda} \underset{K}{\otimes} U(\mathfrak{g})^{\mu} \\
&\cong U(\mathfrak{g})^{\lambda} \underset{K}{\otimes} T(\mu) \\
&\cong \Gamma(X, i_l^{\#}\mathcal{D}_{\lambda,\mu}) \text{ (by equation (4.19))} \\
&\cong \mathscr{F}(\mathcal{D}_{\lambda,\mu}).
\end{aligned} \tag{4.23}$$

The claim follows by picking a presentation $(\mathcal{D}_{\lambda,\mu})^n \to (\mathcal{D}_{\lambda,\mu})^m \to \mathcal{M} \to 0$ and applying the Five Lemma. $\square$

As a corollary, we obtain using Corollary 4.6.4:

Corollary 4.6.6. *Let λ, μ be dominant weights and let $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_{\lambda,\mu}, G)$. Then:*

$$\Gamma(X \times X, p_r^{-1}(\mathcal{M}_\mu) \underset{p_r^{-1}\mathcal{D}_\mu}{\otimes} \mathcal{M}) \cong T(\mu) \underset{U(\mathfrak{g})^\mu}{\otimes} \Gamma(X \times X, \mathcal{M}).$$

4.7 Primitive/prime ideals in the universal enveloping algebra of a semisimple Lie algebra

We retain the notations from 4.6.1, recall that $R = K$ is a field of characteristic 0. In this section, we aim to finish the proof of Duflo's Theorem, that is we aim to prove:

Theorem 4.7.1. *Let $\lambda : \mathfrak{h} \to K$ be a dominant weight and let I be a primitive/prime ideal in $U(\mathfrak{g})^\lambda$. Then*

$$I = \mathrm{Ann}(L(\mu)) \text{ for some } \mu \in \mathfrak{h}^*.$$

For the rest of this section, fix $\lambda \in \mathfrak{h}^*$ a dominant weight and let $\lambda^* = -w_o\lambda$; if λ is dominant, λ^* will also be dominant. Recall further that τ induces an isomorphism $U(\mathfrak{g})^{\lambda^{\mathrm{op}}} \cong U(\mathfrak{g})^{\lambda^*}$.

Consider the functor $\mathscr{F} : \mathrm{Mod}_{\mathrm{fg}}(U(\mathfrak{g} \times \mathfrak{g})^{\lambda^*,\lambda}, G) \to \mathrm{Mod}_{\mathrm{fg}}(U(\mathfrak{g})^{\lambda^*}, B)$ defined by

$$\mathscr{F}(M) := \Gamma(X, i_l^{\#} \circ \mathrm{Loc}^{\lambda^*,\lambda}(M)).$$

Proposition 4.7.2. *The functor $\mathscr{F}$ is exact and $\mathscr{F}(M) \cong T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} M$ as $U(\mathfrak{g})^{\lambda^*}$-modules.*

Proof. Let $\mathcal{M} := \mathrm{Loc}^{\lambda^*,\lambda}(M)$; we have by Corollary 4.5.21 that $\Gamma(X \times X, \mathcal{M}) \cong M$ and by Theorem 4.5.20 that $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_{\lambda^*,\lambda}, G)$, so the second claim follows from Theorem 4.6.5. Consider a short exact sequence of finitely generated G-equivariant $U(\mathfrak{g} \times \mathfrak{g})^{\lambda^*,\lambda}$-modules:

$$0 \to N \to M \to P \to 0.$$

We let $\mathcal{N}, \mathcal{M}, \mathcal{P}$ be the localisations of N, M and P respectively and let $\mathcal{K} = \ker(\mathcal{N} \to \mathcal{M})$. Since the functor $\mathrm{Loc}^{\lambda^*,\lambda}$ is right exact we obtain an exact sequence

$$0 \to \mathcal{K} \to \mathcal{N} \to \mathcal{M} \to \mathcal{P} \to 0.$$

Since $i_l^{\#}$ is an equivalence of Abelian categories by Theorem 4.3.5, $i_l^{\#}$ is in particular exact. Furthermore, by Theorem 4.5.20 the global sections functor is also exact, so we obtain an exact sequence

$$0 \to \Gamma(X, i_l^{\#}\mathcal{K}) \to \Gamma(X, i_l^{\#}\mathcal{N}) \to \Gamma(X, i_l^{\#}\mathcal{M}) \to \Gamma(X, i_l^{\#}\mathcal{P}) \to 0.$$

Applying Theorem 4.6.5 we obtain an exact sequence

$$\begin{aligned} 0 \to T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} \Gamma(X \times X, \mathcal{K}) \to T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} N \to \\ \to T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} M \to T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} P \to 0. \end{aligned} \tag{4.24}$$

The claim follows since $\Gamma(X \times X, \mathcal{K}) = 0$ by definition of $\mathcal{K}$, Theorem 4.5.20 and Corollary 4.5.21. □

Lemma 4.7.3. *Let $M \in \mathrm{Mod}_{\mathrm{fg}}(U(\mathfrak{g} \times \mathfrak{g})^{\lambda^*,\lambda}, G)$ and assume that $\mathscr{F}(M) = 0$. Then $M = 0$.*

Proof. Let $\mathcal{M} = \mathrm{Loc}^{\lambda^*,\lambda}(M)$. Then, by assumption, we have that $\Gamma(X, i_l^{\#}\mathcal{M}) = 0$. Applying Corollary 4.3.16, we obtain $\Gamma(X \times X, \mathcal{M}) = 0$, so by Corollary 4.5.21, $M = 0$. □

We now specialise to two sided ideals in $U(\mathfrak{g})^\lambda$; recall that a two-sided ideal I can be viewed as a module over $U(\mathfrak{g})^{\lambda^*} \otimes U(\mathfrak{g})^\lambda$ via $(x \otimes y).i = yi\tau(x)$ for $x \in U(\mathfrak{g})^{\lambda^*}, y \in U(\mathfrak{g})^\lambda$ and $i \in I$. Further, by Corollary 4.4.5, $I \in \mathrm{Mod}_{\mathrm{fg}}(U(\mathfrak{g} \times \mathfrak{g})^{\lambda^*,\lambda}, G)$, so $\mathscr{F}(I)$ is well-defined. As a corollary, we obtain immediately:

Corollary 4.7.4. *Let I, J be two-sided ideals in $U(\mathfrak{g})^\lambda$ such that $I \subseteq J$. Assume that $\mathscr{F}(I) \cong \mathscr{F}(J)$. Then $I = J$.*

Proof. Consider the short exact sequence:

$$0 \to I \to J \to J/I \to 0.$$

By Proposition 4.7.2, $\mathscr{F}$ is an exact functor, so we obtain an exact sequence

$$0 \to \mathscr{F}(I) \to \mathscr{F}(J) \to \mathscr{F}(J/I) \to 0.$$

By assumption, we have $\mathscr{F}(I) \cong \mathscr{F}(J)$, so $\mathscr{F}(J/I) = 0$. The claim follows by Lemma 4.7.3. ☐

Lemma 4.7.5. *Let I a two-sided ideal in $U(\mathfrak{g})^\lambda$. Then as $U(\mathfrak{g})^{\lambda^*}$-modules we have:*

$$\mathscr{F}(I) \cong T(\lambda)I.$$

We should remark that $U(\mathfrak{g})^{\lambda^*}$ acts on $T(\lambda)I$ via $x.(ti) = t(x.i) = ti\tau(x)$ for $x \in U(\mathfrak{g})^{\lambda^*}, t \in T(\lambda), i \in I$ and the isomorphism is natural in I.

Proof. Consider the following short exact sequence:

$$0 \to I \to U(\mathfrak{g})^\lambda \to U(\mathfrak{g})^\lambda/I \to 0.$$

Applying the exact functor $\mathscr{F}$ and using Proposition 4.7.2, we obtain an exact sequence:

$$0 \to T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} I \to T(\lambda) \to T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} U(\mathfrak{g})^\lambda/I \to 0.$$

This exact sequence fits into the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} I \arrow[r] \arrow[d] & T(\lambda) \arrow[r] \arrow[d] & T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} U(\mathfrak{g})^\lambda/I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T(\lambda)I \arrow[r] & T(\lambda) \arrow[r] & T(\lambda) \underset{U(\mathfrak{g})^\lambda}{\otimes} U(\mathfrak{g})^\lambda/I \arrow[r] & 0.
\end{tikzcd}$$

It is easy to see that the first map is a surjection and the second and the third maps are isomorphisms. Furthermore, by the Five Lemma, the first map is injective, so indeed we get $\mathscr{F}(I) \cong T(\lambda)I$. ☐

Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ denote the Chevalley involution that swaps $\mathfrak{n}^+$ and $\mathfrak{n}^-$ and fixes $\mathfrak{h}$. Then σ extends to an anti-automorphism of $U(\mathfrak{g})$ that fixes the center by [34, Exercise 1.10], so it descends to an anti-automorphism $\sigma : U(\mathfrak{g})^\lambda \to U(\mathfrak{g})^\lambda$. Recall that by construction we have $T(\lambda) = K_\lambda \underset{U(\mathfrak{b}^-)}{\otimes} U(\mathfrak{g})$.

Lemma 4.7.6. *The map*

$$\phi : T(\lambda) \to M(\lambda), \qquad \phi(k \otimes x) = \sigma(x) \otimes k, \quad k \in K_\lambda, x \in U(\mathfrak{g})$$

is a K-linear isomorphism of vector spaces satisfying $\phi(tu) = \sigma(u)\phi(t)$ for all $u \in U(\mathfrak{g})$ and $t \in T(\lambda)$.

Proof. Since σ swaps $\mathfrak{n}^+$ and $\mathfrak{n}^-$ and fixes $\mathfrak{h}$, it is easy to check that the map ϕ is well-defined. Furthermore, as σ is a K-linear anti-automorphism, ϕ is a K-linear isomorphism of vector spaces.

Finally, we have for $k \otimes x \in T(\lambda)$ and $u \in U(\mathfrak{g})$:

$$\begin{aligned} \phi((k \otimes x)u) &= \phi(k \otimes xu) \\ &= \sigma(xu) \otimes k \\ &= \sigma(u)\sigma(x) \otimes k \\ &= \sigma(u)\phi(k \otimes x). \end{aligned}$$ □

In particular, we obtain that if I is a two-sided ideal in $U(\mathfrak{g})^\lambda$, $\phi(T(\lambda)I) = \sigma(I)M(\lambda)$.

Corollary 4.7.7. *Let I a two-sided ideal in $U(\mathfrak{g})^\lambda$. Then*

$$I = \operatorname{Ann}(M(\lambda)/IM(\lambda)).$$

Proof. Let $J := \operatorname{Ann}(M(\lambda)/IM(\lambda))$. Since $I(M(\lambda)/IM(\lambda)) = 0$, we obtain that $I \subseteq J$, so $\sigma(I) \subseteq \sigma(J)$. We remark that since σ is anti-automorphism of $U(\mathfrak{g})^\lambda$, $\sigma(I)$ and $\sigma(J)$ are two-sided ideals in $U(\mathfrak{g})^\lambda$. Since $\mathscr{F}$ preserves injections, we obtain $\mathscr{F}(\sigma(I)) \subseteq \mathscr{F}(\sigma(J))$.

Consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(\sigma(I)) \arrow[r] \arrow[d] & \mathscr{F}(\sigma(J)) \arrow[d] \\
T(\lambda)\sigma(I) \arrow[r] \arrow[d, "\phi"] & T(\lambda)\sigma(J) \arrow[d, "\phi"] \\
IM(\lambda) \arrow[r] & JM(\lambda)
\end{tikzcd}$$

By construction, the bottom diagram commutes and by the definition of J, we have $JM(\lambda) = IM(\lambda)$. Using Lemma 4.7.6, we get $T(\lambda)\sigma(I) = T(\lambda)\sigma(J)$. Furthermore, since the isomorphism in Lemma 4.7.5 is natural on ideals, we obtain that the top diagram also commutes. Therefore, $\mathscr{F}(\sigma(I)) \cong \mathscr{F}(\sigma(J))$. The claim now follows by Corollary 4.7.4 since σ is an anti-automorphism. □

We now have all the ingredients to prove Duflo's Theorem; the idea of the proof was first given by Dixmier in [24, Problem 30].

Proof of Theorem 4.7.1. Let I be a prime ideal in $U(\mathfrak{g})^\lambda$. Then, by Corollary 4.7.7, we have $I = \operatorname{Ann}(M(\lambda)/IM(\lambda))$. Since $M(\lambda)/IM(\lambda)$ is quotient of $M(\lambda)$, we have by Proposition 4.4.10 that there exists a finite composition series

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = M(\lambda)/IM(\lambda).$$

Let $I_i := \operatorname{Ann}(M_i/M_{i-1})$ for $1 \leq i \leq n$. Then

$$\begin{aligned} I_1 I_2 \ldots I_n(M(\lambda)/IM(\lambda)) &= I_1 I_2 \ldots I_n M_n \\ &= I_1 I_2 \ldots I_{n-1} M_{n-1} \\ &= \ldots \\ &= 0, \end{aligned} \tag{4.25}$$

so $I_1 I_2 \ldots I_n \subset I$. Since the ideal I is prime, there exists $1 \leq j \leq n$ such that $I_j \subset I$. On the other hand, by construction $I \subset I_j$. Thus, we obtain that $I = I_j = \operatorname{Ann}(M_j/M_{j-1})$. The claim follows since by Proposition 4.4.10, $M_j/M_{j-1} \cong L(\mu)$ for some $\mu \in \mathfrak{h}^*$.

Since any primitive ideal is in particular prime, we obtain that any primitive ideal in $U(\mathfrak{g})^\lambda$ is the annihilator of some $L(\mu)$. □

In the case when K is algebraically closed, for example, $K = \mathbb{C}$, we may characterise all the primitive ideals in $U(\mathfrak{g})$.

Corollary 4.7.8. *Assume that K is an algebraically closed field of characteristic 0 and let I be a primitive ideal in $U(\mathfrak{g})$. Then*

$$I = \operatorname{Ann}(L(\mu)) \text{ for some } \mu : \mathfrak{h} \to K.$$

Proof. This follows by combining Proposition 4.4.10, Theorem 4.7.1 and Theorems [24, 8.4.3-8.4.4 d)]. □

Chapter 5

Primitive ideals in affinoid enveloping algebras

5.1 Introduction

This chapter is based on the final paper series [58], [56], [57]. The goal of the chapter is to prove Theorem A and Theorem B mentioned in the introduction. Recall that for a Lie algebra $\mathfrak{g}$ defined over a discrete valuation ring R, with uniformiser π and field of fractions K, we have defined a family of affinoid algebras

$$\widehat{U(\mathfrak{g})_{n,K}} = (\varprojlim U(\pi^n\mathfrak{g})/\pi^i U(\pi^n\mathfrak{g})) \underset{R}{\otimes} K.$$

Our strategy for proving Theorem A and Theorem B is to enhance the affinoid Beilinson-Bernstein localisation [5, Theorem C] developed by Ardakov and Wadsley to the equivariant setting to obtain an affinoid version of Theorem 4.5.20. Further, we prove an affinoid versions of Theorem 4.3.5, Proposition 4.7.2 and Corollary 4.7.7.

We should also remark that our initial approach was to try to adapt one of the classical proofs in [25], [12], [28], [37] to the affinoid setting. Unfortunately, these approaches failed to produce results for $\mathfrak{g} \neq \mathfrak{sl}_2$. It boils down to the fact that the weight spaces of the ad-action of the Cartan subalgebra on the affinoid enveloping algebra are not finite dimensional. This is in contrast to what happens in Theorem 5.2.25, where we can adapt classical machinery to obtain a correspondence between the lattices of submodules of $\widehat{M(\lambda)}$ and $M(\lambda)$, respectively.

Structure of the chapter

The chapter is organised as follows: in Section 5.2, we introduce affinoid enveloping algebras and affinoid Verma modules. We prove that for any weight λ of the Cartan subalgebra, there is an explicit one-to-one correspondence between submodules of affinoid Verma module of weight λ and the classical Verma module of weight λ.

In Chapter 4, we have proven that there is an equivalence of categories between G-equivariant (λ, μ)-twisted $\mathcal{D}$-modules on the double flag variety and B-equivariant λ-twisted $\mathcal{D}$-modules on the flag variety for any all weights λ, μ. In Section 5.3, we prove an affinoid version of this equivalence.

Next, in Section 5.4, we enhance the affinoid Beilinson-Bernstein equivalence proven by Ardakov and Wadsley in [5] to the equivariant setting. We further prove that any two-sided ideal in the affinoid enveloping algebra is G-equivariant when viewed as a bimodule over the affinoid enveloping algebra.

In Section 5.5, we compute global sections under the affinoid pullback functor defined in Section 5.3. Finally, in Section 5.6, we prove an affinoid version of Duflo's theorem.

Conventions

Throughout this chapter, except otherwise stated, R will denote a mixed characteristic $(0, p)$ complete discrete valuation ring with uniformiser π and field of fractions K. We use $||\cdot||$ to denote the norm of an element in R or K.

Given an R-module M, we define $M_K := M \underset{R}{\otimes} K$. For any R-algebra A and for $\mathfrak{g}$ a R-Lie algebra, we define $\mathfrak{g}_A := \mathfrak{g} \underset{R}{\otimes} A$; if M is an R-module, we denote $M_A := M \underset{R}{\otimes} A$. If $\mathcal{M}$ is a sheaf of R-modules on a topological space Y, we define a sheaf of K vector spaces on Y, $\mathcal{M}_K$, by $\mathcal{M}_K(U) := \mathcal{M}(U) \underset{R}{\otimes} K$ for any $U \subset Y$ open.

Following [5, definition 2.7], an R-module/sheaf of R-modules $M/\mathcal{M}$ of a K-vector space/sheaf of K-vector spaces $V/\mathcal{V}$ will be called a *lattice* if

$$M \underset{R}{\otimes} K \cong V / \mathcal{M} \underset{R}{\otimes} K \cong \mathcal{V} \text{ and } \cap_{n \in \mathbb{N}^*} \pi^n M = 0 / \cap_{n \in \mathbb{N}^*} \pi^n \mathcal{M} = 0.$$

We will use $\hat{\otimes}$ to denote the completed tensor product, see [54, 0AMU] for definition. We will assume that all the filtrations appearing are exhaustive. Given a filtered A with filtration $F_iA, i \in \mathbb{Z}$, we will use $\operatorname{gr} A$ to denote the associated graded ring with respect to the filtration. Further, for any ring A, $Z(A)$ will denote its centre. We will use the notation (V_i) to denote a set of objects indexed by the non-negative natural numbers.

Lastly, given $f : X \to Y$ a map of schemes, we will use f^* to denote the pullback in the category of $\mathcal{O}$ and $\mathcal{D}$-modules and f^{-1}, f_* to denote the inverse/direct image sheaf.

5.2 Affinoid enveloping algebras and Verma modules

From now on, till the end of the chapter, we will assume that R is a complete mixed characteristic $(0, p)$ discrete valuation ring with field of fractions K, uniformiser π and residue field k.

For a deformable R-algebra A and $n \in \mathbb{N}^*$, we denote $A_n := A_{\pi^n}$ the π^n-th deformation of A.

5.2.1 Background on affinoid enveloping algebras

In this subsection, we recall the main construction and results concerning affinoid enveloping algebras.

Let G be a connected, simply connected, split semisimple, smooth affine algebraic group scheme over $\operatorname{Spec} R$. Denote $\mathfrak{g}$ the Lie algebra of G. The Lie algebra $\mathfrak{g}$ is a linear G representation via the Adjoint action; see [36, II.1.12] for details. In particular the functor of points $G(R)$ acts on $\mathfrak{g}$. Using the functoriality one may extend this action to the enveloping algebra $U(\mathfrak{g})$. For example, if we consider a monomial $x_1 x_2 \ldots x_n \in U(\mathfrak{g})$, with $x_i \in \mathfrak{g}$, we get that for each $g \in G(R)$ we have

$$g \cdot x_1 x_2 \ldots x_n = (g \cdot x_1)(g \cdot x_2) \ldots (g \cdot x_n).$$

It follows that the action of $G(R)$ preserves the standard PBW filtration on $U(\mathfrak{g})$. Consider the corresponding comodule structure on $\mathcal{O}(G)$ induced by the action of G and let $\rho : U(\mathfrak{g}) \to \mathcal{O}(G) \otimes U(\mathfrak{g})$ be the defining map. It follows from the definition of the $G(R)$ action that the comodule map satisfies $\rho(ab) = \rho(a)\rho(b)$ for any $a, b \in U(\mathfrak{g})$.

Let H be a fixed maximal torus for G and $\mathbf{\Phi}$ the corresponding root system, and $x_\alpha : G_a \to G$ and $e_\alpha := (dx_\alpha)(1) \in \mathfrak{g}$ be the root homomorphism and root vector corresponding to a root $\alpha \in \mathbf{\Phi}$.

Lemma 5.2.1 ([6], Lemma 4.1)**.** *Let $r \in R$ and $a \in \mathbf{\Phi}$. Then the following hold:*

1. *For every G module M, the action of $\frac{e_\alpha^m}{m!}$ preserves M.*

2. For all $b \in U(\mathfrak{g})$, there exists $i \geq 1$ such that $\frac{\mathrm{ad}(re_\alpha)^i}{i!} \cdot b = 0$.

3. $x_\alpha(r) \cdot a = \sum_{m=0}^{\infty} \frac{\mathrm{ad}(re_\alpha)^m}{m!}(a)$ for all $a \in U(\mathfrak{g})$.

Definition 5.2.2. *Let A be an R-algebra. The π-adic completion of the R-algebra A is defined to be $\widehat{A} = \varprojlim A/\pi^i A$.*

Let $u_1, u_2, \dots u_d$ be a free R-basis of $\mathfrak{g}$. Then as a vector space we have

$$\widehat{U(\mathfrak{g})_{n,K}} = \{\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha u^\alpha : \quad \lambda_\alpha \in K, \pi^{-n|\alpha|}\lambda_\alpha \to 0 \text{ as } |\alpha| \to \infty\}. \tag{5.1}$$

Here for a d-tuple $\alpha = (\alpha_1, \alpha_2, \dots \alpha_d)$, we define $|\alpha| = \sum_{i=1}^{d} \alpha_i$ and $u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2} \dots u_d^{\alpha_d}$.

By functoriality, the Adjoint action of G on $U(\mathfrak{g})$ extends to a G-action on $\widehat{U(\mathfrak{g})_{n,K}}$. The following proposition extends the classical results for enveloping algebras defined over a field of characteristic 0.

Lemma 5.2.3. *[6, Corollary 4.3]*

i) Every two sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$ is preserved by $G(R)$.

ii) For any $z \in Z(\widehat{U(\mathfrak{g})_{n,K}})$ and for any $g \in G(R)$, we have $g \cdot z = z$.

One may wonder if the converse of Lemma 5.2.3 *ii*) also holds. Classically, we have $Z(U(\mathfrak{g}_K)) \cong U(\mathfrak{g}_K)^G$. The following theorem states that the result carries in the affinoid setting:

Theorem 5.2.4. *[6, Theorem 4.4] We have $Z(\widehat{U(\mathfrak{g})_{n,K}}) \cong \widehat{U(\mathfrak{g})_{n,K}^G}$.*

Recall that $H \subset B^-$ is a split maximal torus in G contained in B^-. The unipotent radical N^- of B^- will be considered as generated by negative roots corresponding to the adjoint action of H on G. Furthermore, let N^+ be the unipotent radical of the opposite Borel group B^+ containing H. Let $\mathfrak{h}, \mathfrak{b}^-, \mathfrak{n}^-, \mathfrak{n}^+, \mathfrak{b}^+$ be the Lie algebras corresponding to the algebraic groups so that we have a decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Let $\lambda : \pi^n\mathfrak{h} \to R$ be an R-linear character; extend this to an R-linear map $\pi^n\mathfrak{b}^+ \to R$ by pulling along the projection map $\pi^n\mathfrak{b}^+ \to \pi^n\mathfrak{h}$. Similar to the classical case, we denote K_λ the corresponding one dimensional module over $\widehat{U(\mathfrak{b}^+)_{n,K}}$; the Lie algebra $\mathfrak{b}^+$ acts on K_λ via the corresponding map $\pi^n \mathfrak{b}^+ \to R$ and we may extend this action to the whole algebra $\widehat{U(\mathfrak{b}^+)_{n,K}}$.

Definition 5.2.5. *The affinoid Verma module with highest weight λ is defined to be*

$$\widehat{M(\lambda)} := \widehat{U(\mathfrak{g})_{n,K}} \underset{\widehat{U(\mathfrak{b}^+)_{n,K}}}{\otimes} K_\lambda.$$

Notice that affinoid Verma modules are non-trivial for $\mathfrak{h}_K$-weights induced by weights of $\pi^n\mathfrak{h}$; for a general $\mathfrak{h}_K$-weight, a unit in $\widehat{U(\mathfrak{g})_{n,K}}$ may annihilate the affinoid Verma module. It is clear by construction that, similarly to the classical case, the affinoid Verma modules are cyclic: $\widehat{M(\lambda)}$ is generated by $v_\lambda = 1 \underset{\widehat{U(\mathfrak{b}^+)_{n,K}}}{\otimes} 1$.

The centre $Z(\mathfrak{g}_K)$ of $U(\mathfrak{g}_K)$ acts on the classical Verma module defined by $M(\lambda) := U(\mathfrak{g}_K) \underset{U(\mathfrak{b}_K^+)}{\otimes} K_\lambda$ by a character $\chi_\lambda : Z(\mathfrak{g}_K) \to K_\lambda$. As $\widehat{M(\lambda)}$ contains $M(\lambda)$ as a dense subset the action of $Z(\mathfrak{g}_K)$ on $\widehat{M(\lambda)}$ also factors through χ_λ. In [6], the authors compute the annihilator of the affinoid Verma module $\widehat{M(\lambda)}$.

Theorem 5.2.6. *[6, Theorem 4.6] If p is a very good prime for G then the annihilator of the affinoid Verma module $\widehat{M(\lambda)}$ inside $\widehat{U(\mathfrak{g})_{n,K}}$ is*

$$\widehat{I_\lambda} := \ker \chi_\lambda \widehat{U(\mathfrak{g})_{n,K}}.$$

For the rest of section we fix a R-linear map $\lambda : \pi^n\mathfrak{h} \to R$ and let $\widehat{M(\lambda)}$ and $M(\lambda)$ be the affinoid respectively classical Verma module of weight λ. In the next subsections, we prove there is an explicit one-to-one correspondence between submodules of $\widehat{M(\lambda)}$ and submodules of $M(\lambda)$.

5.2.2 The height function

For the semisimple Lie algebra $\mathfrak{g}$, let Δ denote the set of simple positive roots and $\mathbf{\Phi}^+$ the set of positive roots. For any root α, we will denote $\alpha^\vee$ the corresponding coroot. In the Killing form identification of $\mathfrak{h}$ and $\mathfrak{h}^*$ the coroot $\alpha^\vee$ corresponds to $h_\alpha \in \mathfrak{h}$.

Definition 5.2.7. *Let $\beta \in \mathbf{\Phi}^+$ be a positive root. Then $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha$, with $c_\alpha \in \mathbb{Z}^+$ determined uniquely, see [34, section 0.2] for details. We define the* height *of β to be*

$$\mathrm{ht}(\beta) := \sum_{\alpha\in\Delta} c_\alpha.$$

We now extend this definition to monomials in the universal enveloping algebra using the correspondence between roots and root vectors. Fix an order between the positive roots and let $e_1, e_2 \dots, e_m$ the corresponding order between root vectors. For a root vector e_i, we define the height, $\mathrm{ht}(e_i)$, to be the height of the root corresponding to e_i.

Definition 5.2.8. *For $a_1, a_2 \dots a_m \in \mathbb{N}$, let $e^A := e_1^{a_1} e_2^{a_2} \dots e_m^{a_m} \in U(\mathfrak{n}^+)$ be such that $e_i \in \mathfrak{n}^+$. Then we define the* height *of e^A to be*

$$\mathrm{ht}(e^A) := \sum_{i=1}^{m} a_i \, \mathrm{ht}(e_i).$$

Let $f^B = f_1^{b_1} f_2^{b_2} \dots f_m^{b_m} \in U(\mathfrak{n}^-)$ such that $f_i \in \mathfrak{n}^-$. Then we define the height *of f^B to be*

$$\mathrm{ht}(f^B) := \sum_{i=1}^{m} b_i \, \mathrm{ht}(e_i),$$

where e_i is the positive root vector corresponding to f_i.

Let ρ be the half sum of positive roots and $\delta = \rho^\vee \in \mathfrak{h}$ the corresponding coroot. Let α be a positive root; then by the roots-coroots duality we have $\alpha(\delta) = \rho(\alpha^\vee)$; furthermore by [34, section 0.6], we have $\rho(\alpha^\vee) = \mathrm{ht}(\alpha)$, therefore we obtain $\alpha(\delta) = \mathrm{ht}(\alpha)$.

Lemma 5.2.9. *[6, Section 4.7] Let $f^B = f_1^{b_1} f_2^{b_2} \cdots f_m^{b_m} \in U(\mathfrak{n}^-)$ for $\beta \in \mathbb{N}^m$. Then for any $h \in \mathfrak{h}$, we have:*

$$h \cdot f^B v_\lambda = (\lambda - \sum_{j=1}^{m} b_j \alpha_j)(h) f^B v_\lambda.$$

For the ease of notation, denote $\Lambda := \lambda(\delta)$. Setting $h = \delta$ in the equation above we get:

$$\delta f^B v_\lambda = (\lambda - \sum_{i=1}^{m} b_i \alpha_i)(\delta) f^B v_\lambda = (\Lambda - \sum_{i=1}^{m} b_i \, \mathrm{ht}(\alpha_i)) f^B v_\lambda = (\Lambda - \mathrm{ht}(f^B)) f^B v_\lambda. \quad (5.2)$$

As an easy corollary we get:

Corollary 5.2.10. *Let $a \in \mathbb{N}, f^B \in U(\mathfrak{n}^-)$. Then:*

$$(\delta - \Lambda + a)(f^B)v_\lambda = (a - \mathrm{ht}(f^B))f^B v_\lambda.$$

Definition 5.2.11. *Let M be a $U(\mathfrak{h})$-module and $\mu \in \mathfrak{h}_K^*$. We say that $m \in M$ has weight μ if $hm = \mu(h)m$ for all $h \in \mathfrak{h}_K$. The set of vectors of weight μ is denoted M_μ.*

The following lemma follows easily from the construction of affinoid Verma modules:

Lemma 5.2.12. *Let N a submodule of $\widehat{M(\lambda)}$. Then N_μ is a finite dimensional vector space for any $\mu \in \mathfrak{h}_K^*$.*

5.2.3 Submodules of affinoid Verma modules

Throughout this subsection, we will make free use of the following well known facts:

- $U(\mathfrak{g}_K) \cong U(\mathfrak{g})_K = U(\mathfrak{g}) \underset{R}{\otimes} K$.
- $\widehat{U(\mathfrak{g})_n}$ is flat over $U(\mathfrak{g})_n$ and $\widehat{U(\mathfrak{g})_{n,K}}$ is flat over $U(\mathfrak{g})_K$.
- Given N a submodule of $M(\lambda)$, we may view N as a subset of the topological module $\widehat{M(\lambda)}$. The closure of N inside $\widehat{M(\lambda)}$ is given by
$$\hat{N} := \overline{N} = \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g})_K}{\otimes} N.$$
- The affinoid Verma module $\widehat{M(\lambda)}$ has a K-topological basis given by $f^B v_\lambda$, where $f^B \in U(\mathfrak{n}^-)$; recall that $v_\lambda = 1 \otimes 1$.

We begin by extending our definition of the height function to homogeneous polynomials, homogeneity being given by height. We say that a polynomial in $U(\mathfrak{n}_K^-)$ has height n if all the monomials appearing in its expansion have height n. We also let $M = \max(\mathrm{ht}(e_i))$, so that we have the inequality

$$M|B| \geq \mathrm{ht}(f^B) \geq |B|. \tag{5.3}$$

By construction, we know that as a vector space

$$\widehat{U(\mathfrak{n}^-)_{n,K}} = \{ \sum_{B\in\mathbb{N}^m} a_B f^B, \quad p^{-n|B|}||a_B|| \to 0 \text{ as } |B| \to \infty \}.$$

We can reformulate this in terms of height function using equation (5.3):

$$\widehat{U(\mathfrak{n}^-)_{n,K}} = \{ \sum_{B\in\mathbb{N}^m} a_B f^B, \quad p^{-n|B|}||a_B|| \to 0 \text{ as } \operatorname{ht}(f^B) \to \infty \}.$$

Let N a closed submodule of $\widehat{M(\lambda)}$ and let $\nu \in \mathfrak{h}^*$. We know by equation 5.2 that $\delta f^B v_\lambda = (\Lambda - \operatorname{ht}(f^B)) f^B v_\lambda$ so any element of N_ν must be of the form Pv_λ, where $P \in U(\mathfrak{n}_K^-)$ is a homogeneous polynomial of height $\Lambda - \nu(\delta)$.

Proposition 5.2.13. *Let N be a closed submodule of $\widehat{M(\lambda)}$. Then*

$$N = \overline{\bigoplus_{\mu\in\mathfrak{h}_K^*} N_\mu} = \overline{N \cap M(\lambda)},$$

where $M(\lambda)$ is the Verma module of weight λ.

Fix the closed submodule N and an element $u \in N$, which we write as

$$u = \sum_{B\in\mathbb{N}^m} a_B f^B v_\lambda,$$

with $p^{-n|B|}||a_B|| \to 0$ as $\operatorname{ht}(f^B) \to \infty$. Furthermore, fix $Pv_\lambda \in N_\mu$, with $P \neq 0$ of height L appearing in the expansion of u. To prove Proposition 5.2.13, it is enough to prove that $Pv_\lambda \in N$. To do this, we begin by eliminating all the other terms of height L appearing in the expansion of u. We write u as

$$u = \sum_{B\in\mathbb{N}^m, \operatorname{ht}(f^B)\neq L} a_B f^B v_\lambda + Pv_\lambda + \sum_{s\in S} Q_s v_\lambda,$$

where S is a set such that Q_s has height L and $Q_s v_\lambda \in N_{\nu_s}$ for some weight $\nu_s \neq \mu$. By Lemma 5.2.12, the set S is finite.

Let $s \in S$. As $\nu_s \neq \mu$, there exists $h_s \in \mathfrak{h}_K$ such that $\mu(h_s) \neq \nu_s(h_s)$. So we can define an operator $H_S := \prod_{s\in S}(h_s - \nu_s(h_s)) \in U(\mathfrak{h}_K)$. Since $Q_s \in N_{\nu_s}$, we get $H_S \cdot Q_s v_\lambda = 0$. Therefore applying the operator H_S to u we obtain a new element $u' \in N$, which can be written as

$$u' = \sum_{B\in\mathbb{N}^m, \operatorname{ht}(f^B)\neq L} b_B f^B v_\lambda + \prod_{s\in S}(\mu(h_s) - \nu_s(h_s)) P v_\lambda,$$

with $p^{-n|B|}||b_B|| \to 0$ as $\mathrm{ht}(f^B) \to \infty$. By our construction $\prod_{s\in S}(\mu(h_s) - \nu_s(h_s)) \neq 0$, so for the ease of notation we set $P = \prod_{s\in S}(\mu(h_s) - \nu_s(h_s))P \neq 0$, so that

$$N \ni u' = \sum_{B\in\mathbb{N}^m, \mathrm{ht}(f^B)\neq L} b_B f^B v_\lambda + P v_\lambda.$$

To complete the proof we need to define a new set of operators in $U(\mathfrak{h}_K)$. We use the convention that for $x \in U(\mathfrak{h}_K)$ and $i \in \mathbb{N}$ the symbol $\binom{x}{i}$ will denote

$$\prod_{l=0}^{i-1} \frac{1}{i!}(x - l) \in U(\mathfrak{h}_K).$$

Definition 5.2.14. *For $i, j \in \mathbb{N}$ define $\epsilon_{i,j} := \binom{\delta-\Lambda+i+j}{i} \in U(\mathfrak{h}_K)$.*

Lemma 5.2.15. *For $B \in \mathbb{N}^m$, we have that $\epsilon_{i,j} f^B v_\lambda = \binom{i+j-\mathrm{ht}(f^B)}{i} f^B v_\lambda$.*

Proof. We have

$$\begin{aligned}\epsilon_{i,j} f^B v_\lambda &= \frac{1}{i!}\prod_{l=0}^{i-1}(\delta - \Lambda + i + j - l) f^B v_\lambda \\ &= \frac{1}{i!}\prod_{l=0}^{i-1}(i + j - l - \mathrm{ht}(f^B)) f^B v_\lambda \text{ (by Corollary 5.2.10)} \\ &= \binom{i + j - \mathrm{ht}(f^B)}{i} f^B v_\lambda. \end{aligned}$$ □

We are now ready to prove Proposition 5.2.13; recall that it is enough to prove that for $u' = \sum_{B\in\mathbb{N}^m, \mathrm{ht}(f^B)\neq L} b_B f^B v_\lambda + P v_\lambda$, we have $P v_\lambda \in N$.

Proof of Proposition 5.2.13. Write u' as

$$u' = \sum_{B\in\mathbb{N}^m, \mathrm{ht}(f^B)<L} b_B f^B v_\lambda + P v_\lambda + \sum_{B\in\mathbb{N}^m, \mathrm{ht}(f^B)>L} b_B f^B v_\lambda.$$

Consider the operator $\epsilon_{L-1,0}$ acting on u'. We get

$$
\begin{aligned}
N \ni u'' :&= \epsilon_{L-1,0} \cdot u' \\
&= \sum_{B\in\mathbb{N}^m,\mathrm{ht}(f^B)<L} \binom{L-1-\mathrm{ht}(f^B)}{L-1} b_B f^B v_\lambda + \binom{L-1-L}{L-1} P v_\lambda + \\
&+ \sum_{B\in\mathbb{N}^m,\mathrm{ht}(f^B)>L} \binom{L-1-\mathrm{ht}(f^B)}{L-1} b_B f^B v_\lambda \\
&= b_0 v_\lambda + (-1)^{L-1} P v_\lambda + \sum_{B\in\mathbb{N}^m,\mathrm{ht}(f^B)>L} \binom{L-1-\mathrm{ht}(f^B)}{L-1} b_B f^B v_\lambda .
\end{aligned}
\tag{5.4}
$$

Next, we apply the operator $\frac{(-1)^L(\delta-\Lambda)}{L}$. We have

$$
\begin{aligned}
N \ni u^{(3)} :&= \frac{(-1)^n(\delta-\Lambda)}{L} \cdot u'' \\
&= \frac{(-1)^L}{L}[(\Lambda-\Lambda)b_0 v_\lambda + (\Lambda - L - \Lambda)(-1)^{L-1} P v_\lambda + \\
&+ ^{(\text{by } 5.2.10)} \sum_{B\in\mathbb{N}^m,\mathrm{ht}(f^B)>L} (\Lambda - \mathrm{ht}(f^B) - \Lambda) \binom{(L-1-\mathrm{ht}(f^B))}{L-1} b_B f^B v_\lambda] \\
&= P v_\lambda + \sum_{B\in\mathbb{N}^m,\mathrm{ht}(f^B)>L} c_B f^B v_\lambda ,
\end{aligned}
\tag{5.5}
$$

for some $c_B \in K$ with $p^{-n|B|}||c_B|| \to 0$ as $\mathrm{ht}(f^B) \to \infty$. Finally, we consider the family of operators $\epsilon_{i,L}$ where $i \in \mathbb{N}$ varies. By Lemma 5.2.15, we have that

$$
\epsilon_{i,L} f^B v_\lambda = \binom{i+L-\mathrm{ht}(f^B)}{i} f^B v_\lambda .
$$

In particular, $\epsilon_{i,L} P v_\lambda = P v_\lambda$ and $\epsilon_{i,L} f^B v_\lambda = 0$, for $L < \mathrm{ht}(f^B) \le i + L$. Therefore, for any $i \in \mathbb{N}$ one has

$$
N \ni u_i = \epsilon_{i,L} \cdot u^{(3)} = P v_\lambda + \sum_{B\in\mathbb{N}^m,\mathrm{ht}(f^B)>i+L} \binom{i+L-\mathrm{ht}(f^B)}{i} c_B f^B v_\lambda .
$$

Since $p^{-n|B|}||c_B|| \to 0$ as $\mathrm{ht}(f^B) \to \infty$ and $||\binom{i+L-\mathrm{ht}(f^B)}{i}|| \le 1$, we get $\lim_{i\to\infty} u_i = P v_\lambda$. Since we assumed that N is closed submodule of $\widehat{M(\lambda)}$, we may conclude that $P v_\lambda \in N$, which is the desired result. $\square$

Recall that we aim to prove that there is a one-to-one correspondence between submodules of affinoid Verma module $\widehat{M(\lambda)}$ and the corresponding classical Verma module $M(\lambda)$ where $\lambda : \pi^n\mathfrak{h} \to R$ is a R-linear map. Define a function $\mathscr{F}$ from submodules of $\widehat{M(\lambda)}$ to submodules of $M(\lambda)$ sending the submodule N to $N \cap M(\lambda)$.

Lemma 5.2.16. *The function $\mathscr{F}$ is injective.*

Proof. Let N_1, N_2 be submodules of $\widehat{M(\lambda)}$ such that

$$\mathscr{F}(N_1) = N_1 \cap M(\lambda) = N_2 \cap M(\lambda) = \mathscr{F}(N_2).$$

As the induced metric topology on $\widehat{M(\lambda)}$ is complete, any submodule of $\widehat{M(\lambda)}$ is closed by [33, I.5.5]. Thus, applying Proposition 5.2.13, we obtain $N_1 = \overline{N_1 \cap M(\lambda)} = \overline{N_2 \cap M(\lambda)} = N_2$, so $N_1 = N_2$. Therefore, the function $\mathscr{F}$ is injective. □

We aim to prove that $\mathscr{F}$ is also surjective. For any ring A, S a subset of A and M an A-module, we say that M is not S-torsion if there exists $m \in M$ such that for all $s \in S$, $sm \neq 0$.

Proposition 5.2.17. *Let $\lambda : \pi^n\mathfrak{h} \to R$ be an R-linear map and let M be a $U(\mathfrak{g}_K)$ subquotient of $M(\lambda)$ that is not $1 + \pi U(\mathfrak{g})$-torsion. Then*

$$\widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} M \neq 0.$$

To prove this proposition we will need a few additional results. We say that a right ideal I in a right Noetherian ring A has the Artin-Rees property if for any right ideal J, there exists $n \in \mathbb{N}^*$ such that $J \cap I^n \subset JI$.

Proposition 5.2.18. *[43, Proposition 4.2.9]*

Let A be a right Noetherian ring and I an ideal with the right Artin-Rees property. Then:

1. *$1 - I$ is a right Ore set, so a right denominator set.*
2. *Writing S for $1 - I$, we have $I_S \subset J(A_S)$, where $J(\bullet)$ denotes the Jacobson radical of a ring and I_S and A_S denote the sets $S^{-1}I$ and $S^{-1}A$, respectively.*

Corollary 5.2.19. *Consider the ideal $I = \pi U(\mathfrak{g})_n$ of $U(\mathfrak{g})_n$. Then $U(\mathfrak{g})_{n_{1+I}}$ exists and is non-zero; furthermore $\pi U(\mathfrak{g})_{n_{1+I}} \subset J(U(\mathfrak{g})_{n_{1+I}})$.*

Proof. By [43, Proposition 4.2.6], any ideal generated by normal elements in a right Noetherian ring has the Artin-Rees property, so in particular we get that the ideal $\pi U(\mathfrak{g})_n$ in $U(\mathfrak{g})_n$ has the Artin-Rees property. The claim follows from Proposition 5.2.18. □

Remark 5.2.20. *Notice that as the ring $U(\mathfrak{g})_n$ is both left and right Noetherian we get the same results for the left localization. Thus, ${}_{1+\pi U(\mathfrak{g})_n}U(\mathfrak{g})_n = U(\mathfrak{g})_{n_{1+\pi U(\mathfrak{g})}}$ by [43, Corollary 2.1.4].*

From now on until the end of the section, S will denote the set $1 + \pi U(\mathfrak{g})_n$. The ring $U(\mathfrak{g})_{n_S}$ has a π-adically negative filtration $F_\bullet$ given by

$$F_i U(\mathfrak{g})_{n_S} = \pi^{-i} U(\mathfrak{g})_{n_S}, \text{ for } i \leq 0.$$

Denote $\widehat{U(\mathfrak{g})_{n_S}}$ the π-adically completion of $U(\mathfrak{g})_{n_S}$, i.e. the completion induced by the filtration $F_\bullet$.

Proposition 5.2.21. *$\widehat{U(\mathfrak{g})_{n_S}}$ is a faithfully flat right $U(\mathfrak{g})_{n_S}$-module.*

Proof. Since $U(\mathfrak{g})_n$ is left and right Noetherian, and S is an Ore set, we get that the ring $U(\mathfrak{g})_{n_S}$ is also left and right Noetherian. Furthermore, we have that the ideal inducing the π-adic filtration on $U(\mathfrak{g})_S$ is generated by a central element, so by [47, Proposition 3.12], the Rees ring $\widetilde{U(\mathfrak{g})_{n_S}}$ is also left and right Noetherian. By Corollary 5.2.19, we have $F_{-1}U(\mathfrak{g})_{n_S} \subset J(F_0 U(\mathfrak{g})_{n_S})$, therefore by combining the two statements, we get that $U(\mathfrak{g})_S$ is a left *Zariski* ring as defined in Definition 2.1.16. The claim follows from Theorem 2.1.17. □

Corollary 5.2.22. *Let M be a non-zero $U(\mathfrak{g})_n$-module that is not S-torsion. Then*

$$\widehat{U(\mathfrak{g})_n} \underset{U(\mathfrak{g})_n}{\otimes} M \neq 0.$$

Proof. Since M is not S-torsion, we have by localising

$$0 \neq S^{-1}M = S^{-1}U(\mathfrak{g})_n \underset{U(\mathfrak{g})_n}{\otimes} M = U(\mathfrak{g})_{n_S} \underset{U(\mathfrak{g})_n}{\otimes} M.$$

By Proposition 5.2.21, $\widehat{U(\mathfrak{g})_{n_S}}$ is faithfully flat over $U(\mathfrak{g})_{n_S}$, so

$$\widehat{U(\mathfrak{g})_{n_S}} \underset{U(\mathfrak{g})_{n_S}}{\otimes} S^{-1}M \neq 0.$$

Recall that all the elements of S are of the form $1+\pi x$, with $x \in U(\mathfrak{g})_n$, but when we π-adically complete $U(\mathfrak{g})_{n_S}$ everything in S becomes a unit, so $\widehat{U(\mathfrak{g})_{n_S}} \cong \widehat{U(\mathfrak{g})_n}$. We then get:

$$\begin{aligned} 0 \neq \widehat{U(\mathfrak{g})_{n_S}} \underset{U(\mathfrak{g})_{n_S}}{\otimes} S^{-1}M &\cong \widehat{U(\mathfrak{g})_n} \underset{U(\mathfrak{g})_{n_S}}{\otimes} S^{-1}M \\ &\cong \widehat{U(\mathfrak{g})_n} \underset{U(\mathfrak{g})_{n_S}}{\otimes} U(\mathfrak{g})_{n_S} \underset{U(\mathfrak{g})_n}{\otimes} M \\ &\cong \widehat{U(\mathfrak{g})_n} \underset{U(\mathfrak{g})_n}{\otimes} M. \qquad \square \end{aligned}$$

We now apply the results for objects in category $\mathcal{O}$ for the enveloping algebra $U(\mathfrak{g}_K)$. Let $\lambda : \pi^n\mathfrak{h} \to R$ be an R-linear map and consider the simple $U(\mathfrak{g}_K)$ module $L(\lambda)$-the unique simple quotient of $M(\lambda)$- and view it as a $U(\mathfrak{g})$-module.

Lemma 5.2.23. *Let the notations be as above. The module $L(\lambda)$ is not $1+\pi U(\mathfrak{g})$-torsion.*

Proof. The module $L(\lambda)$ is cyclic being generated by $v_\lambda + N(\lambda)$, where $N(\lambda)$ is the unique maximal submodule of $M(\lambda)$. It is enough to prove that $v_\lambda + N(\lambda)$ is $1+\pi U(\mathfrak{g})$ torsion-free.

Consider the Cartan Lie subalgebra $\mathfrak{h}$. We extend the character $\lambda : \mathfrak{h} \to R$ to an R algebra homomorphism $\lambda : U(\mathfrak{h}) \to R$. We use the decomposition of $U(\mathfrak{g})$ given by $U(\mathfrak{g}) = (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+) \oplus U(\mathfrak{h})$. Notice that if $x \in U(\mathfrak{g})\mathfrak{n}^+$, then $xv_\lambda = 0$. Furthermore, if $x \in \mathfrak{n}^- U(\mathfrak{g})$, then

$$xv_\lambda \in \mathfrak{n}_K^- U(\mathfrak{n}_K^-)v_\lambda. \tag{5.6}$$

In fact, one may prove that xv_λ is in $\mathfrak{n}^- U(\mathfrak{n}^-)v_\lambda$, but that requires a messy computation and we do not need this in our argument. Next, for $y \in U(\mathfrak{h})$, we have

$$yv_\lambda = \lambda(y)v_\lambda, \text{ where } \lambda(y) \in R. \tag{5.7}$$

Now, let $U(\mathfrak{g}) \ni z = x + y$ be such that $x \in \mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+$ and $y \in U(\mathfrak{h})$. By equation (5.6) there exists $s \in \mathfrak{n}_K^- U(\mathfrak{n}_K^-)$ such that $xv_\lambda = sv_\lambda$ and by equation (5.7), $yv_\lambda = \lambda(y)v_\lambda$. Therefore, we get

$$(1 + \pi z) \cdot (v_\lambda + N(\lambda)) = (1 + \pi s + \pi\lambda(y))v_\lambda + N(\lambda).$$

Proving that $(1+\pi z)v_\lambda + N(\lambda) \neq 0$ is equivalent to $(1+\pi z)v_\lambda \notin N(\lambda)$. Assume for a contradiction that $(1+\pi z)v_\lambda \in N(\lambda)$. Then $(1+\pi\lambda(y))v_\lambda + \pi s v_\lambda \in N(\lambda)$. View $(1+\pi\lambda(y))v_\lambda + \pi s v_\lambda$ as an element in $M(\lambda)$. Consider the decomposition of $M(\lambda)$ given by

$$M(\lambda) = M(\lambda)_\lambda \oplus M(\lambda)_{<\lambda} = Kv_\lambda \oplus M(\lambda)_{<\lambda},$$

where $M(\lambda)_{<\lambda}$ denotes the K-span of all $v_\mu \in M(\lambda)_\mu$ with $\mu < \lambda$. Notice that since $s \in \mathfrak{n}_K^- U(\mathfrak{n}_K^-)$, we have $\pi s v_\lambda \in M(\lambda)_{<\lambda}$; furthermore, $1 + \pi\lambda(y)v_\lambda \in M(\lambda)_\lambda$. Now since the module $N(\lambda)$ is itself $\mathfrak{h}_K$-semisimple, we have

$$(1+\pi\lambda(y))v_\lambda \in N(\lambda).$$

By construction, $\lambda(y) \in R$, so $||\pi\lambda(y)|| < 1$, thus $1+\pi\lambda(y)$ is a unit in R. Therefore, multiplying by its inverse we conclude that $v_\lambda \in N(\lambda)$, so $N(\lambda) = M(\lambda)$ which is the desired contradiction. We conclude that $L(\lambda)$ is indeed not $1+\pi U(\mathfrak{g})$-torsion. □

As an easy corollary of the Lemma above, using the fact $1+\pi U(\mathfrak{g})_n$ is a subset of $1+\pi U(\mathfrak{g})$, we obtain:

Corollary 5.2.24. *Let the notations as in the previous lemma. View $L(\lambda)$ as $U(\mathfrak{g})_n$-module. Then $L(\lambda)$ is not $1+\pi U(\mathfrak{g})_n$-torsion.*

We may now prove Proposition 5.2.17:

Proof. Let M be a subquotient of $M(\lambda)$ and view it as a $U(\mathfrak{g})_n$-module. Any subquotient of the Verma module $M(\lambda)$ has finite length and can be viewed as extension of modules of the form $L(\mu)$. Each $L(\mu)$ is not $1+\pi U(\mathfrak{g})_n$-torsion by Corollary 5.2.24. As finite extension of modules that are not $1+\pi U(\mathfrak{g})_n$-torsion is not $1+\pi U(\mathfrak{g})_n$-torsion, M is not $1+\pi U(\mathfrak{g})_n$-torsion. Therefore, by Corollary 5.2.22, we have

$$\widehat{U(\mathfrak{g})_n} \underset{U(\mathfrak{g})_n}{\otimes} M \neq 0.$$

As this space has no π-torsion we get

$$\begin{aligned} 0 \neq (\widehat{U(\mathfrak{g})_n} \underset{U(\mathfrak{g})_n}{\otimes} M) \underset{R}{\otimes} K &= (\widehat{U(\mathfrak{g})_n} \underset{R}{\otimes} K) \underset{(U(\mathfrak{g})_n \underset{R}{\otimes} K)}{\otimes} (M \underset{R}{\otimes} K) \\ &= \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} M. \end{aligned}$$

□

We are now able to prove Theorem A:

Theorem 5.2.25. *Let λ be a weight in $\pi^n\mathfrak{h}^*$ and extend this to a weight $\lambda \in \mathfrak{h}_K^*$. There is a one to one correspondence between submodules of $\widehat{M(\lambda)}$ and submodules of $M(\lambda)$.*

Proof. Recall the function $\mathscr{F}$ going from submodules of $\widehat{M(\lambda)}$ to submodules of $M(\lambda)$ sending a submodule N to $N \cap M(\lambda)$. We have already proven in Lemma 5.2.16 that $\mathscr{F}$ is injective, so we only need prove that $\mathscr{F}$ is surjective.

Let N be a submodule of $M(\lambda)$ and let $\overline{N} = \widehat{N} = \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N$. Furthermore, let $N' = \widehat{N} \cap M(\lambda)$. We aim to prove that $N = N' = \mathscr{F}(\widehat{N})$. By construction we have that $N \subset N'$ and by Proposition 5.2.13, $\widehat{N} = \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N'$. Assume for a contradiction that N is strictly included in N'. Consider the short exact sequence

$$0 \to N \to N' \to N'/N \to 0.$$

As $\widehat{U(\mathfrak{g})_{n,K}}$ is flat over $U(\mathfrak{g}_K)$ we get a short exact sequence

$$0 \to \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N \to \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N' \to \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N/N' \to 0, \text{ so}$$

$$0 \to \widehat{N} \to \widehat{N} \to \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N/N' \to 0$$

is a short exact sequence, which implies that $\widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N/N' = 0$. Finally, by Proposition 5.2.17, we have $\widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N/N' \neq 0$, which is the desired contradiction. ☐

One might also try to prove the theorem above using [23, Korollar 1.3.12]; we were not aware of the existence of this paper at the time of the proof.

Using the theorem above, we obtain immediately:

Proposition 5.2.26. *Let $\lambda : \pi^n\mathfrak{h} \to R$ be an R-linear map. The affinoid Verma module $\widehat{M(\lambda)}$ has finite length equal to the length of classical Verma module $M(\lambda)$.*

Proof. By Theorem 5.2.25, there is a one to one correspondence between submodules of $\widehat{M(\lambda)}$ and submodules of $M(\lambda)$. As the module $M(\lambda)$ has finite length by

[34, Theorem 1.11], it follows that $\widehat{M(\lambda)}$ also has finite length. Furthermore, the correspondence is 1-1, so the lengths must be the same. □

For λ as in Theorem 5.2.25 we get the following corollaries:

Corollary 5.2.27. *An affinoid Verma module $\widehat{M(\lambda)}$ is simple if and only if the corresponding classical Verma module $M(\lambda)$ is simple.*

Corollary 5.2.28. *Any affinoid Verma module has a unique maximal submodule and a unique simple quotient. The unique simple quotient $\widehat{L(\lambda)}$ of $\widehat{M(\lambda)}$ is given by*

$$\widehat{L(\lambda)} := \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} L(\lambda),$$

where $L(\lambda)$ denotes the unique simple quotient of $M(\lambda)$.

Proof. Let $N(\lambda)$ denote the unique maximal submodule of $M(\lambda)$. Consider the short exact sequence

$$0 \to N(\lambda) \to M(\lambda) \to L(\lambda) \to 0.$$

Since $\widehat{U(\mathfrak{g})_{n,K}}$ is flat over $U(\mathfrak{g}_K)$ we obtain a short exact sequence

$$0 \to \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N(\lambda) \to \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} M(\lambda) \to \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} L(\lambda) \to 0. \tag{5.8}$$

By construction, $\widehat{M(\lambda)} \cong \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} M(\lambda)$ and by Theorem 5.2.25, one obtains $\widehat{N(\lambda)} := \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} N(\lambda)$ is the unique maximal submodule of $\widehat{M(\lambda)}$. The claim now follows from equation (5.8). □

Proposition 5.2.29. *Let $\hat{M}$ be a subquotient of $\widehat{M(\lambda)}$. Then $\hat{M}$ has a finite composition series and all the simple quotients are of the form $\widehat{L(\mu)}$ for some $\mu \in \pi^n\mathfrak{h}^*$.*

Proof. The first statement follows directly from Proposition 5.2.26. It is enough to prove the second statement in the case $\hat{M} = \widehat{M(\lambda)}$. Let

$$0 = \hat{M}_0 \subset \hat{M}_1 \subset \hat{M}_2 \subset \dots \hat{M}_n = \widehat{M(\lambda)},$$

be a composition series for $\widehat{M(\lambda)}$. By Theorem 5.2.25, there exists a composition series of $M(\lambda)$

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots M_n = M(\lambda),$$

such that $\hat{M}_i = \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} M_i$ for $0 \leq i \leq n$.

Fix $1 \leq j \leq n$; it is enough to prove that $\hat{M}_j/\hat{M}_{j-1} \cong \widehat{L(\mu)}$ for some $\mu \in \mathfrak{h}^*$. Consider the short exact sequence:

$$0 \to M_{j-1} \to M_j \to M_j/M_{j-1}.$$

Since $\widehat{U(\mathfrak{g})_{n,K}}$ is flat over $U(\mathfrak{g}_K)$ we obtain by tensoring on the left a short exact sequence

$$0 \to \hat{M}_{j-1} \to \hat{M}_j \to \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} M_j/M_{j-1},$$

so $\hat{M}_j/\hat{M}_{j-1} \cong \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} M_j/M_{j-1}$. Since M_j/M_{j-1} is a simple subquotient of $M(\lambda)$, we have $M_j/M_{j-1} \cong L(\mu)$ for some $\mu \in \mathfrak{h}_K^*$ by [34, Section 1.11]. This is induced by some R-linear map $\mu : \pi^n\mathfrak{h} \to R$. The conclusion follows from Corollary 5.2.28. □

5.3 An affinoid equivalence of categories à la Borho-Brylinski

Recall that G is a connected, simply connected smooth affine algebraic group scheme defined over $\operatorname{Spec} R$ with Lie algebra $\mathfrak{g}$. We also let B be a closed subgroup of G. Throughout this section, we retain Assumption 4.3.1; that is we assume that the quotient scheme $X = G/B$ is an R-variety and the quotient map $d_B : G \to X$ given by $d_B(g) = gB$ is a locally trivial B-torsor with respect to the action $\diamond$ given by $b \diamond g = gb^{-1}$.

5.3.1 Introduction to $\widehat{\mathcal{D}}$-modules

We use the following convention, for a sheaf of R-modules $\mathcal{M}$, we define its π-adic completion $\widehat{\mathcal{M}} := \varprojlim \mathcal{M}/\pi^i\mathcal{M}$.

Let Y be an R-variety and $\mathcal{D}$ be a sheaf of Noetherian rings on Y. Since π-adic completion preserves Noetherianity we obtain that $\widehat{\mathcal{D}}$ is a sheaf of Noetherian rings. Thus, a module $\mathcal{M}$ over $\widehat{\mathcal{D}}$ is coherent if and only if it is locally finitely generated. Furthermore, we will use without further comments that if $\mathcal{M}$ is a coherent $\widehat{\mathcal{D}}$-module, then $\mathcal{M} \cong \varprojlim \mathcal{M}/\pi^i\mathcal{M}$; this follows from [5, Lemma 5.4]. We also use that for any $i \in \mathbb{N}^*$, we have $\widehat{\mathcal{D}}/\pi^i\widehat{\mathcal{D}} \cong \mathcal{D}/\pi^i\mathcal{D}$. For more background on $\widehat{\mathcal{D}}$-modules, the reader is advised to consult [5, Section 5] and [14, Section 3].

In general, it is hard to determine whether a $\widehat{\mathcal{D}}$-module $\mathcal{M}$ is coherent. This is true for example if $\mathcal{M} = \widehat{\mathcal{N}}$ for some coherent $\mathcal{D}$-module $\mathcal{N}$. In the following, we give a more general set of sufficient conditions.

Proposition 5.3.1. *[14, Lemme 3.2.2] Let D be a ring and I an ideal generated by finitely many central elements, and let $D_i = D/I^iD$, $i \in \mathbb{N}^*$. Furthermore, suppose there exists (M_i) an inverse system of D_i-modules such that for $j \geq 2$ the canonical morphisms $M_j/\pi^{j-1}M_j \to M_{j-1}$ are isomorphisms. We let $M = \varprojlim M_i$. Then:*

1. *For $i \geq 1$ the canonical morphisms*
$$M/I^iM \to M_i$$
are isomorphisms.
2. *If M_1 is finitely generated over D_1, then M is finitely generated over $\widehat{D} := \varprojlim D_i$. Furthermore, a generating set for M can be obtained by lifting a generating set for M_1.*

Corollary 5.3.2. *Let Y be an R-variety and $\mathcal{D}$ a sheaf of Noetherian rings on Y. Further, let $(\mathcal{M}_i)$ be an inverse system of coherent modules over $\mathcal{D}/\pi^i\mathcal{D}$ and suppose that the connecting maps induce isomorphisms $\mathcal{M}_i/\pi^{i-1}\mathcal{M}_i \cong \mathcal{M}_{i-1}$ for all $i \geq 2$. Define*
$$\mathcal{M} := \varprojlim \mathcal{M}_i.$$
Then $\mathcal{M}$ is a coherent $\widehat{\mathcal{D}}$-module and $\mathcal{M}_i \cong \mathcal{M}/\pi^i\mathcal{M}$ for all $i \geq 1$.

Proof. The question is local; as $\widehat{\mathcal{D}}$ is a sheaf of Noetherian rings, a module is coherent if and only if it is locally finitely generated. Let $U \subset Y$ be open affine and let $M_{iU} := \mathcal{M}_i(U)$ and $D_{iU} := \mathcal{D}(U)/\pi^i\mathcal{D}(U)$, so that $\mathcal{M}(U) = \varprojlim M_{iU}$.

Since $\mathcal{M}_i$ is coherent as a $\mathcal{D}/\pi^i\mathcal{D}$-module, we get that M_{iU} is a finitely generated D_{iU}-module. By definition, we have $M_{iU}/\pi^{i-1}M \cong M_{i-1U}$, so by the second part of Proposition 5.3.1, we get that $\mathcal{M}(U)$ is finitely generated as a $\widehat{\mathcal{D}}(U)$-module, so $\mathcal{M}$ is indeed a coherent $\widehat{\mathcal{D}}$-module. For the second part of the statement we have by the first part of Proposition 5.3.1 that

$$M_{iU} \cong \mathcal{M}(U)/\pi^i\mathcal{M}(U).$$

As this is true for any open affine and there is a map $\mathcal{M}/\pi^i\mathcal{M} \to \mathcal{M}_i$, we get the desired conclusion. □

5.3.2 Pullback of $\widehat{\mathcal{D}}$-modules

Lemma 5.3.3. *Let $f : Z \to Y$ be a map of smooth R-varieties and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_Y$-module. Then as $\mathcal{O}_Z$-modules we have*

$$f^*(\mathcal{M}/\pi^j\mathcal{M}) \cong f^*(\mathcal{M})/\pi^j f^*(\mathcal{M}), \text{ for any } j \geq 1.$$

Proof. Consider the short exact sequence:

$$\mathcal{M} \xrightarrow{\cdot\pi^j} \mathcal{M} \to \mathcal{M}/\pi^j\mathcal{M} \to 0.$$

The functor f^* is right exact, so applying this to the short exact sequence above we get:

$$f^*\mathcal{M} \xrightarrow{f^*(\cdot\pi^j)} f^*\mathcal{M} \to f^*(\mathcal{M}/\pi^j\mathcal{M}) \to 0.$$

Finally, notice that $f^*(\cdot\pi^j) = \cdot\pi^j$, so we get that indeed

$$f^*\mathcal{M}/\pi^j f^*\mathcal{M} \cong f^*(\mathcal{M}/\pi^j\mathcal{M}).$$ □

For the rest of this section, we fix n a deformation parameter. Let $\mathcal{D}$ be a π^n-deformed tdo on an R-variety Y.

Definition 5.3.4. *Let $f : Z \to Y$ be a map of smooth R-varieties and let $\mathcal{M}$ be a coherent $\widehat{\mathcal{D}}$-module on Y. Then we define the π-adic pullback of $\mathcal{M}$ to be*

$$\hat{f}^{\#}(\mathcal{M}) := \varprojlim f^{\#}(\mathcal{M}/\pi^i\mathcal{M}).$$

Remark 5.3.5. *The inverse limit is considered in the category of presheaves over* Z*. By construction, we have that* $\mathcal{M}_i := \mathcal{M}/\pi^i\mathcal{M}$ *is in particular a* $\mathcal{D}$*-module, so* $f^\#(\mathcal{M}_i)$ *is a* $f^\#\mathcal{D}$*-module. Since* $\pi^i\mathcal{M}_i = 0$*, we obtain* $\pi^i f^\#(\mathcal{M}_i) = 0$*, thus* $f^\#(\mathcal{M}_i)$ *is a* $f^\#\mathcal{D}/\pi^i f^\#\mathcal{D}$*-module. Therefore, we obtain that* $\hat{f}^\#(\mathcal{M})$ *has the structure of a* $\widehat{f^\#\mathcal{D}}$*-module.*

Let L be a smooth affine algebraic group locally of finite type defined over $\operatorname{Spec} R$ acting on Y and let $\mathcal{D}$ be a π^n-deformed L-htdo on Y. We define the notion of $\hat{L}$-equivariant $\widehat{\mathcal{D}}$-modules.

Definition 5.3.6. *A* $\hat{L}$*-equivariant coherent* $\widehat{\mathcal{D}}$*-module is a triple* $(\mathcal{M}, (\mathcal{M}_i), (\alpha_i))$ *such that:*

1. $(\mathcal{M}_i)$ *is an inverse system of* $\mathcal{D}$*-modules and* $\pi^i\mathcal{M}_i = 0$*.*
2. *For* $i \in \mathbb{N}^*$*,* $(\mathcal{M}_i, \alpha_i) \in \operatorname{Coh}(\mathcal{D}, L)$*.*
3. *For* $i \geq 2$*, the connecting map in the inverse system induces an isomorphism* $\mathcal{M}_i/\pi^{i-1}\mathcal{M}_i \cong \mathcal{M}_{i-1}$ *of* L*-equivariant* $\mathcal{D}$*-modules.*
4. $\mathcal{M} \cong \varprojlim \mathcal{M}_i$ *as* $\widehat{\mathcal{D}}$*-modules.*

A $\hat{L}$*-equivariant morphism between* $\hat{L}$*-equivariant* $\widehat{\mathcal{D}}$*-modules* $(\mathcal{M}, (\mathcal{M}_i), (\alpha_i))$ *and* $(\mathcal{N}, (\mathcal{N}_i), (\beta_i))$ *is a* $\widehat{\mathcal{D}}$*-linear morphism* $\phi : \mathcal{M} \to \mathcal{N}$ *such that there exist compatible maps* $\phi_i \in \operatorname{Hom}_{\operatorname{Coh}(\mathcal{D},L)}(\mathcal{M}_i, \mathcal{N}_i)$ *with* $\phi = \varprojlim \phi_i$*.*

We define the category of $\hat{L}$*-equivariant* $\widehat{\mathcal{D}}$*-modules to consist of* $\hat{L}$*-equivariant objects and* $\hat{L}$*-equivariant morphisms. As before, we will omit the equivariance structure when it is understood from the context. We denote* $\operatorname{Coh}(\widehat{\mathcal{D}}, L)$ *the category of* $\hat{L}$*-equivariant coherent* $\widehat{\mathcal{D}}$*-modules.*

Proposition 5.3.7. *Let the notation be as above. The category* $\operatorname{Coh}(\widehat{\mathcal{D}}, L)$ *is Abelian.*

To prove this proposition, we will need the following lemma:

Lemma 5.3.8. *Let* A *be a* π*-adically complete Noetherian* R*-algebra. Let* $(M_i)_{i\in\mathbb{N}^*}$ *and* $(N_i)_{i\in\mathbb{N}^*}$ *be inverse systems of* A*-modules such that* $\pi^i M_i = \pi^i N_i = 0$ *for all* $i \in \mathbb{N}^*$ *and assume that transition maps induce isomorphisms* $M_i/\pi^{i-1}M_i \cong M_{i-1}$ *and* $N_i/\pi^{i-1}N_i \cong N_{i-1}$*. Let* $(f_i) : (M_i) \to (N_i)$ *be a map of inverse systems and* $(K_i) = \ker(f_i)$*. Then* $K_i/\pi^{i-1}K_i \cong K_{i-1}$*.*

Proof. We follow the idea in [54, 087X]. Let $M := \varprojlim M_i$, $N := \varprojlim N_i$ and $f : M \to N$ the induced map; further let $K = \ker(f)$. We have by Proposition 5.3.1 that for any $j \in \mathbb{N}^*$, $M_j \cong M/\pi^j M$ and $N_j \cong N/\pi^j N$, so we may assume that the map $f_j : M/\pi^j M \to N/\pi^j N$ is given by $f_j(m + \pi^j M) = f(m) + \pi^j N$ for all $m \in M$.

Next, we know by [14, 3.2.3i)] that there exists $c \in \mathbb{N}$ such that for $n \geq c$, we have $\pi^n N \cap f(M) \subset \pi^{n-c} f(M)$. In particular, we obtain:

$$f^{-1}(\pi^n N) \subset K + \pi^{n-c} M. \tag{5.9}$$

For $s, t \in \mathbb{N}, s \geq t$, we let $K'_{s,t} := \operatorname{im}(\ker(f_s) \to M_t)$. We claim that for a fixed t, $K'_{s,t}$ is eventually constant and we denote K'_t this value. We have that for $s \geq t + c$

$$\begin{aligned} K'_{s,t} &= f^{-1}(\pi^s N) + \pi^t M/\pi^t M \\ &= K + \pi^t M/\pi^t M \text{ (by equation (5.9))} \\ &\cong K/K \cap \pi^t M. \end{aligned} \tag{5.10}$$

Therefore $K'_t = K/K \cap \pi^t M$ is the constant value we seek. We claim that for any $n \in \mathbb{N}$ the system $(K'_t/\pi^n K'_t)_{t \geq n}$ is eventually constant with value $K/\pi^n K$. Again, we have by [14, 3.2.3i)] that there exists $d \in \mathbb{N}$ such that

$$K \cap \pi^u M \subset \pi^{u-d} \text{ for any } u \geq d. \tag{5.11}$$

Therefore we obtain that for $t \geq n + d$

$$\begin{aligned} K'_t/\pi^n K'_t &\cong K/K \cap \pi^t M/(\pi^n K/K \cap \pi^t M) \\ &\cong K/(K \cap \pi^t M + \pi^n K) \\ &\cong K/\pi^n K \text{ (by equation (5.11))}. \end{aligned} \tag{5.12}$$

Finally, to prove that $K/\pi^n K \cong K_n$ for all $n \in \mathbb{N}$, we repeat the argument in [54, 087X] to prove that the inverse system $(K/\pi^i K)$ is indeed the the kernel of (f_i). □

Proof of Proposition 5.3.7. We have by Section 3.9 that the category $\operatorname{Coh}(\mathcal{D}, L)$ is Abelian. We view $\operatorname{Coh}(\widehat{\mathcal{D}}, L)$ as a full subcategory of the Abelian category of towers consisting of objects in $\operatorname{Coh}(\mathcal{D}, L)$. It is easy to see that $0 \in \operatorname{Coh}(\widehat{\mathcal{D}}, L)$ and the category is closed under direct sums. Therefore, we only need to prove that $\operatorname{Coh}(\widehat{\mathcal{D}}, L)$ is closed under kernels and cokernels.

Let $\phi : (\mathcal{M}, (\mathcal{M}_i), (\alpha_i)) \to (\mathcal{N}, (\mathcal{N}_i), (\beta_i))$ be a map of objects in $\mathrm{Coh}(\widehat{\mathcal{D}}, L)$. For $i \in \mathbb{N}^*$, let $\phi_i : \mathcal{M}_i \to \mathcal{N}_i$ be the corresponding map and $\mathcal{K}_i = \ker(\phi_i)$. Since $\mathrm{Coh}(\mathcal{D}, L)$ is Abelian, we have $\mathcal{K}_i \in \mathrm{Coh}(\mathcal{D}, L)$; further by construction we have $\pi^i \mathcal{K}_i = 0$ and that $(\mathcal{K}_i)$ forms an inverse system of $\mathcal{D}$-modules. Finally, by working locally and using Lemma 5.3.8, we obtain that for any $i \in \mathbb{N}^*$, $\mathcal{K}_i/\pi^{i-1}\mathcal{K}_i \cong \mathcal{K}_{i-1}$, so $\mathcal{K} = \ker \phi = \varprojlim \mathcal{K}_i \in \mathrm{Coh}(\widehat{\mathcal{D}}, L)$; the coherence of $\mathcal{K}$ follows form Corollary 5.3.2.

A similar argument proves that $\mathrm{Coh}(\widehat{\mathcal{D}}, L)$ is closed under cokernels. □

Recall that $i_l : X \to X \times X$ denotes the inclusion of X into the left copy of $X \times X$. Further, recall from Theorem 4.3.5 that for a π^n-deformed G-equivariant htdo on $X \times X$, the functor $i_l^\#$ induces an equivalence of categories between $\mathrm{Coh}(\mathcal{D}, G)$ and $\mathrm{Coh}(i_l^\# \mathcal{D}, B)$. We denote $\mathscr{H}_l$ the quasi-inverse of $i_l^\#$.

Proposition 5.3.9. *Let $\mathcal{D}$ be a π^n-deformed G-equivariant htdo on $X \times X$. The functor $\hat{i_l}^\#$ induces an equivalence of categories between $\mathrm{Coh}(\widehat{\mathcal{D}}, G)$ and $\mathrm{Coh}(\widehat{i_l^\# \mathcal{D}}, B)$. A quasi-inverse is given $\widehat{\mathscr{H}_l}$ defined by $\widehat{\mathscr{H}_l}(\mathcal{N}) := \varprojlim \mathscr{H}_l(\mathcal{N}/\pi^i\mathcal{N})$ for $\mathcal{N} \in \mathrm{Coh}(\widehat{i_l^\# \mathcal{D}}, B)$.*

Proof. Let $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}}, G)$ and $\mathcal{M}_i := \mathcal{M}/\pi^i\mathcal{M}$ for $i \geq 1$. By construction $\mathcal{M}_i \in \mathrm{Coh}(\mathcal{D}, G)$, so applying Theorem 4.3.5, we obtain $\mathcal{N}_i := i_l^\# \mathcal{M}_i \in \mathrm{Coh}(i_l^\# \mathcal{D}, B)$. Further, we have $\pi^i \mathcal{N}_i = 0$ since $\pi^i \mathcal{M}_i = 0$. By Lemma 5.3.3, $\mathcal{N}_i/\pi^{i-1}\mathcal{N}_i \cong \mathcal{N}_{i-1}$. Therefore, we obtain by Corollary 5.3.2 that

$$\widehat{\mathcal{N}} := \hat{i_l}^\# \mathcal{M} = \varprojlim \mathcal{N}_i \in \mathrm{Coh}(\widehat{i_l^\# \mathcal{D}}, B).$$

We have by Corollary 5.3.2 that $\mathcal{N}_i \cong \mathcal{N}/\pi^i\mathcal{N}$, so we get:

$$\begin{aligned} \widehat{\mathscr{H}_l} \circ \hat{i_l}^\#(\mathcal{M}) &= \widehat{\mathscr{H}_l}(\mathcal{N}) \\ &\cong \varprojlim \mathscr{H}_l(\mathcal{N}_i) \\ &\cong \varprojlim \mathcal{M}_i \text{ (by Theorem 4.3.5)} \\ &\cong \mathcal{M}. \end{aligned} \tag{5.13}$$

Thus $\widehat{\mathscr{H}_l}$ is a left quasi-inverse for $\hat{i_l}^\#$. A similar argument shows that $\widehat{\mathscr{H}_l}$ is also a right quasi-inverse. □

5.3.3 Some category theory lemmas

To prove an affinoid version of the Borho-Brylinski theorem, we need some lemmas for R-linear Abelian categories.

Throughout this subsection we fix $\mathcal{A}$ an R-linear small Abelian category and let $\mathcal{B}$ be the full $\mathcal{A}$-subcategory of *π-torsion elements*, i.e. $\mathrm{ob}(\mathcal{B}) = \{A \in \mathcal{A} | \quad \pi^n \mathrm{id}_A = 0 \text{ for some } n \in \mathbb{N}\}$ (here id_A denotes the identity morphism going from A to A). We also call a morphism $f \in \mathrm{Hom}(A, B)$ *π-torsion* if there exists $n \in \mathbb{N}$ such that $\pi^n f = 0$.

Throughout this subsection we use that in an R-linear category, we have for $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$ and $r \in R$

$$r(g \circ f) = (rg) \circ f = g \circ (rf).$$

Define a new category $\mathcal{A}_K$, where $\mathrm{ob}(\mathcal{A}_K) = \mathrm{ob}(\mathcal{A})$ and Hom sets given by $\mathrm{Hom}_{\mathcal{A}_K}(M, N) := \mathrm{Hom}_{\mathcal{A}}(M, N) \underset{R}{\otimes} K$, for all $M, N \in \mathrm{ob}(\mathcal{A})$. Furthermore, denote $\mathcal{F}$ the natural functor $\mathcal{A} \to \mathcal{A}_K$.

The aim of this subsection is to establish the following theorem:

Theorem 5.3.10. *There exists an equivalence of categories between the quotient category $\mathcal{A}/\mathcal{B}$ and the category $\mathcal{A}_K$.*

One should notice that apriori it is not clear why the quotient category $\mathcal{A}/\mathcal{B}$ is well-defined, so we should begin by proving that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$. We start by proving a very useful lemma:

Lemma 5.3.11. *Let $B \in \mathcal{B}$, $C \in \mathcal{A}$ and consider morphisms $f \in \mathrm{Hom}(B, C)$ and $g \in \mathrm{Hom}(C, B)$. Then f and g are π-torsion.*

Proof. Since $B \in \mathcal{B}$, there exists $n \in \mathbb{N}$ such that $\pi^n \mathrm{id}_B = 0$. We have

$$\pi^n f = \pi^n(\mathrm{id}_B \circ f) = (\pi^n \mathrm{id}_B) \circ f = 0,$$

so f is indeed π-torsion. A similar argument shows that g is also π-torsion. □

Proposition 5.3.12. *The category $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$.*

Proof. Consider a short exact sequence:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

One needs to prove that $B \in \mathcal{B}$ if and only if $A, C \in \mathcal{B}$.

First assume that $B \in \mathcal{B}$. By Lemma 5.3.11, f is π-torsion so there exists $n \in \mathbb{N}$ such that $\pi^n f = 0$, so that

$$0 = \pi^n f = \pi^n (f \circ \mathrm{id}_A) = f \circ \pi^n \mathrm{id}_A .$$

Since f is a monomorphism, we can left cancel to get $\pi^n \mathrm{id}_A = 0$, so $A \in \mathcal{B}$.

By Lemma 5.3.11, g is π-torsion, so there exists $n \in \mathbb{N}$ such that $\pi^n g = 0$, so that

$$0 = \pi^n g = \pi^n (\mathrm{id}_C \circ g) = \pi^n \mathrm{id}_C \circ g.$$

As g is an epimorphism, we can right cancel to obtain $\pi^n \mathrm{id}_C = 0$, so $C \in \mathcal{B}$.

Now assume that $A, C \in \mathcal{B}$. By Lemma 5.3.11, f, g are π-torsion so there exist $n_1, n_2 \in \mathbb{N}$ such that $\pi^{n_1} f = \pi^{n_2} g = 0$. Let $n = \max(n_1, n_2)$ and $h := \pi^n \mathrm{id}_B$. We have

$$\begin{aligned} 0 &= \pi^n f = \pi^n (\mathrm{id}_B \circ f) = (\pi^n \mathrm{id}_B) \circ f = h \circ f. \\ 0 &= \pi^n g = \pi^n (g \circ \mathrm{id}_B) = g \circ (\pi^n \mathrm{id}_B) = g \circ h. \end{aligned} \tag{5.14}$$

Since $h \in \mathrm{Hom}(B, B)$, we have by Lemma 5.3.13 below that $h^2 = 0$, so $\pi^{2n} \mathrm{id}_B = 0$. Thus $B \in \mathcal{B}$. ☐

Lemma 5.3.13. *Let $\mathcal{C}$ be a small Abelian category and let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence. Let $h \in \mathrm{Hom}(B, B)$ such that $h \circ f = g \circ h = 0$. Then $h^2 = 0$.

Proof. By the Freyd-Mitchell embedding we may assume that $\mathcal{C} = S$-mod for some ring S. In particular, we may assume that A, B and C are Abelian groups. Let $b \in B$; then $g(h(b)) = 0$, so $h(b) \in \ker(g) = \mathrm{im}(f)$. Thus, there exists $a \in A$ with $f(a) = h(b)$. Then

$$0 = h(f(a)) = h(h(b)),$$

proving that $h^2 = 0$. ☐

Let S be the collection of $\mathcal{B}$-isomorphisms, i.e. morphisms f in $\mathcal{A}$ such that $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{B}$. Then S is a multiplicative system in the sense defined in [61, Appendix II]. Furthermore, by [61, Example A.1.2], the quotient category $\mathcal{A}/\mathcal{B}$ is equivalent to the localised category $\mathcal{A}_S$. Denote $\mathrm{loc} : \mathcal{A} \to \mathcal{A}_S$ the localisation functor.

Proof of Theorem 5.3.10. By the discussion above, it is enough to prove that there exists an equivalence of categories between $\mathcal{A}_S$ and $\mathcal{A}_K$. By construction, we have that for any $s \in S$, $\mathcal{F}(s)$ is an isomorphism, so by the universal property of localisation there exists a unique functor $\mathcal{G} : \mathcal{A}_S \to \mathcal{A}_K$ defined by $\mathcal{G}(s^{-1}f) = \mathcal{F}(s)^{-1}F(f)$ for any $s^{-1}f$ in $\mathrm{Hom}_{\mathcal{A}_S}(X, Y)$.

We claim that $\mathcal{G}$ is an equivalence of categories. It is clear that $\mathcal{G}$ is essentially surjective, so we need to prove that it is fully faithful.

Let $\phi \in \mathrm{Hom}_{\mathcal{A}_K}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B) \underset{R}{\otimes} K$. Then there exists $n \in \mathbb{N}$ such that $\phi = f \otimes \pi^{-n}$ for some $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$. By construction, we have that $\pi^n \mathrm{id}_B \in S$, so we get that

$$\begin{aligned} \mathcal{G}((\pi^n \mathrm{id}_B)^{-1}f) &= \mathcal{F}(\pi^n \mathrm{id}_B)^{-1} \circ \mathcal{F}(f) \\ &= (\mathrm{id}_B \otimes \pi^{-n}) \circ (f \otimes 1) \\ &= f \otimes \pi^{-n} \\ &= \phi. \end{aligned} \tag{5.15}$$

Thus, $\mathcal{G}$ is indeed full. Lastly, we need to prove that $\mathcal{G}$ is faithful. As all the categories involved are Abelian it is enough to prove that for $s^{-1}f \in \mathrm{Hom}_{\mathcal{A}_S}(X, Y)$, if $\mathcal{G}(s^{-1}f) = 0$, then $s^{-1}f = 0$. Here we assume $s \in \mathrm{Hom}_{\mathcal{A}}(X', X)$, $s \in S$ and $f \in \mathrm{Hom}_{\mathcal{A}}(X', Y)$. We have $0 = \mathcal{G}(s^{-1}f) = \mathcal{F}(s)^{-1} \circ \mathcal{F}(f)$, so $F(f) = 0$. Therefore, we get that f is π-torsion, so there exists $n \in \mathbb{N}$ such that $\pi^n f = 0$. Then:

$$f \circ \pi^n \mathrm{id}_Y = \pi^n f \circ \mathrm{id}_Y = 0,$$

and since $\pi^n \mathrm{id}_Y \in S$, we obtain by [44, Lemma 2.1.5] that $s^{-1}f = 0$. Thus, $\mathcal{G}$ is indeed faithful. ☐

We finish the subsection by proving a categorical proposition that we will need in the next subsection.

Proposition 5.3.14. *Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be an equivalence of Abelian categories. Let $\mathcal{C}$ and $\mathcal{D}$ be Serre subcategories of $\mathcal{A}$ and $\mathcal{B}$, respectively such that $\mathcal{F}$ restricts to an equivalence $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. Then $\mathcal{F}$ induce an equivalence between the quotient categories $\mathcal{A}/\mathcal{C}$ and $\mathcal{B}/\mathcal{D}$.*

Proof. Let $q_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ and $q_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}/\mathcal{D}$ denote the localisation functors and let $\mathcal{H} := q_{\mathcal{B}} \circ \mathcal{F}$. By assumptions, we have $\ker \mathcal{H} = \mathcal{C}$, so by [48, Exercise 5, Section 4.4], there exists a faithful and exact functor $\overline{\mathcal{H}} : \mathcal{A}/\mathcal{C}$ such that $\overline{\mathcal{H}} \circ q_{\mathcal{A}} = \mathcal{H} = q_{\mathcal{B}} \circ \mathcal{F}$. Since $q_{\mathcal{B}} \circ \mathcal{F}$ is essentially surjective, we obtain that $\overline{\mathcal{H}}$ is also essentially surjective. Finally, for any morphism f in $\mathcal{B}/\mathcal{D}$, there is a morphism g in $\mathcal{A}/\mathcal{C}$, such that $\overline{\mathcal{H}}(g) = f$, so $\overline{\mathcal{H}}$ is also full. □

5.3.4 Affinoid equivariant equivalence a la Borho-Brylinski

Let Y be an R-variety and L a smooth affine algebraic group locally of finite type defined over R and $\mathcal{D}$ a sheaf of π^n-deformed L-equivariant htdo on Y. Recall that by Proposition 5.3.7 the category of $\hat{L}$-equivariant coherent $\widehat{\mathcal{D}}$-modules, $\operatorname{Coh}(\widehat{\mathcal{D}}, L)$, is Abelian.

Definition 5.3.15. *Let Y be a quasi-compact R-variety and L an algebraic group acting on Y. Let $\operatorname{Coh}(\widehat{\mathcal{D}}, L)^{\pi}$ be the full subcategory of $\operatorname{Coh}(\widehat{\mathcal{D}_Y}, L)$ consisting of π-torsion objects. As Y is quasi-compact this is equivalent to the full subcategory of $\operatorname{Coh}(\widehat{\mathcal{D}_Y}, L)$ such that all the sections are π-torsion.*

Proposition 5.3.16. *There is an equivalence of categories between the quotient category*
$\operatorname{Coh}(\widehat{\mathcal{D}}, L)/\operatorname{Coh}(\widehat{\mathcal{D}}, L)^{\pi}$ *and the category* $\operatorname{Coh}(\widehat{\mathcal{D}}, L)_K$*.*

Proof. This follows directly from Theorem 5.3.10. □

Definition 5.3.17. *Let Y be a R-variety; recall that $\widehat{\mathcal{D}_K} = \widehat{\mathcal{D}} \underset{R}{\otimes} K$. A coherent $\hat{L}_K$-equivariant $\widehat{\mathcal{D}_K}$-module is quadruple $(\mathcal{M}, \mathcal{M}_0, (\mathcal{M}_i), (\alpha_i))$ such that $\mathcal{M}_0$ is a lattice of $\mathcal{M}$ and $(\mathcal{M}_0, (\mathcal{M}_i), (\alpha_i)) \in \operatorname{Coh}(\widehat{\mathcal{D}}, L)$.*

Let $(\mathcal{M}, \mathcal{M}_0, (\mathcal{M}_i), (\alpha_i))$ and $(\mathcal{N}, \mathcal{N}_0, (\mathcal{N}_i), (\beta_i))$ be $\hat{L}_K$-equivariant $\widehat{\mathcal{D}_K}$-modules and let $\phi : \mathcal{M} \to \mathcal{N}$ be a $\widehat{\mathcal{D}_K}$-linear morphism. We have by the proof of [14, Proposition 3.4.5] that

$$\mathrm{Hom}_{\widehat{\mathcal{D}}}(\mathcal{M}_0, \mathcal{N}_0) \underset{R}{\otimes} K \cong \mathrm{Hom}_{\widehat{\mathcal{D}_K}}(\mathcal{M}, \mathcal{N}),$$

so there exists a pair (ϕ_0, x), $\phi_0 \in \mathrm{Hom}_{\widehat{\mathcal{D}}}(\mathcal{M}_0, \mathcal{N}_0)$ *and* $x \in K$ *such that* $\phi = \phi_0 \otimes x$. *We say that* ϕ *is* $\hat{L}_K$*-equivariant if* ϕ_0 *is* $\hat{L}$*-equivariant.*

We denote $\mathrm{Coh}(\widehat{\mathcal{D}_K}, L)$ *the category of coherent* $\widehat{\mathcal{D}_K}$*-modules consisting of* $\hat{L}_K$*-equivariant objects together with* $\hat{L}_K$*-equivariant morphisms.*

We will ignore the equivariance structure when it is well understood from the context and just call $\mathcal{M}$ an $\hat{L}_K$-equivariant $\widehat{\mathcal{D}_K}$-module.

Lemma 5.3.18. *Assume that* Y *is quasi-compact. Then there exists an explicit equivalence of categories between* $\mathrm{Coh}(\widehat{\mathcal{D}}, L)_K$ *and* $\mathrm{Coh}(\widehat{\mathcal{D}_K}, L)$.

Proof. Define $\mathcal{F} : \mathrm{Coh}(\widehat{\mathcal{D}}, L)_K \to \mathrm{Coh}(\widehat{\mathcal{D}_K}, L)$ by $F(\mathcal{M}) = \mathcal{M} \underset{R}{\otimes} K$ for any object $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}}, L)_K$ and

$$\mathcal{F}(f \otimes x) = f \otimes x, \text{ for all } f \otimes x \in \mathrm{Hom}(\mathcal{M}, \mathcal{N}) \underset{R}{\otimes} K.$$

By construction, it is clear that F is essentially surjective and since the tensors in $\mathrm{Hom}(M, N) \underset{R}{\otimes} K$ are all pure, $\mathcal{F}$ is also faithful. Furthermore, it follows by definition of the morphisms in $\mathrm{Coh}(\widehat{\mathcal{D}_K}, H)$ that $\mathcal{F}$ is also full. □

Until the end of the section, we assume that $\mathcal{D}$ is a π^n-deformed G-equivariant htdo on $X \times X$.

Lemma 5.3.19. *The functor* $\hat{i_l}^{\#}$ *in Proposition 5.3.9 restricts to an equivalence between* $\mathrm{Coh}(\widehat{\mathcal{D}}, G)^{\pi}$ *and* $\mathrm{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)^{\pi}$. *A quasi-inverse is given* $\widehat{\mathscr{H}_l}$.

Proof. Let $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}}, G)^{\pi}$ and define $\mathcal{M}_i := \mathcal{M}/\pi^i\mathcal{M}$. By definition, there exists $m \in \mathbb{N}^*$ such that for $j \geq m$, $\mathcal{M}_j = \mathcal{M}$. Let $\mathcal{N}_i = i_l^{\#}\mathcal{M}_i$; we have $\hat{i_l}^{\#}\mathcal{M} = \varprojlim \mathcal{N}_i$ and by Corollary 5.3.2, $\mathcal{N}_i = \hat{i_l}^{\#}\mathcal{M}/\pi^i\hat{i_l}^{\#}\mathcal{M}$. Further by construction, we have that for $j \geq m$, $\mathcal{N}_j = i_l^{\#}\mathcal{M}$, therefore $\hat{i_l}^{\#}\mathcal{M} \in \mathrm{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)^{\pi}$.

An analogous argument proves that for $\mathcal{N} \in \mathrm{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)^{\pi}$, we have $\widehat{\mathscr{H}_l}(\mathcal{N}) \in \mathrm{Coh}(\widehat{\mathcal{D}}, G)^{\pi}$. The conclusion follows from Proposition 5.3.9. □

We can now prove an equivariant version of Corollary 4.3.11:

Theorem 5.3.20. *There is an equivalence of categories between* $\operatorname{Coh}(\widehat{\mathcal{D}_K}, G)$ *and* $\operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}_K}, B)$*.*

Proof. To simplify the proof, we use $\cong$ to denote an equivalence of categories. Since G is affine and the quotient map $G \to G/B$ is surjective, we obtain that X is quasi-compact, thus so is $X \times X$. We have by Lemma 5.3.18:

$$\operatorname{Coh}(\widehat{\mathcal{D}_K}, G) \cong \operatorname{Coh}(\widehat{\mathcal{D}}, G)_K \text{ and } \operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}_K}, B) \cong \operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)_K. \tag{5.16}$$

Furthermore, we have by Proposition 5.3.16 that

$$\begin{aligned} \operatorname{Coh}(\widehat{\mathcal{D}}, G)_K &\cong \operatorname{Coh}(\widehat{\mathcal{D}}, G)/\operatorname{Coh}(\widehat{\mathcal{D}}, G)^{\pi}, \\ \operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)_K &\cong \operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)/\operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)^{\pi}. \end{aligned} \tag{5.17}$$

Next, we have by Proposition 5.3.9 that there is an equivalence of categories $\hat{i}_l^{\#} : \operatorname{Coh}(\widehat{\mathcal{D}}, G) \cong \operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)$ and by Lemma 5.3.19 this restricts to an equivalence $\operatorname{Coh}(\widehat{\mathcal{D}}, G)^{\pi} \cong \operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)^{\pi}$, so applying Proposition 5.3.14, we obtain an equivalence between the quotient categories:

$$\operatorname{Coh}(\widehat{\mathcal{D}}, G)/\operatorname{Coh}(\widehat{\mathcal{D}}, G)^{\pi} \cong \operatorname{Coh}\left(\widehat{i_l^{\#}\mathcal{D}}, B\right)/\operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}}, B)^{\pi}. \tag{5.18}$$

Therefore, by combining equations (5.16), (5.17) and (5.18), we get

$$\operatorname{Coh}(\widehat{\mathcal{D}_K}, G) \cong \operatorname{Coh}(\widehat{i_l^{\#}\mathcal{D}_K}, B). \qquad \square$$

Remark 5.3.21. *Denote* $\hat{i}^{\#}_{l,K}$ *the equivalence functor from the category* $\operatorname{Coh}(\widehat{\mathcal{D}_K}, G)$ *to the category* $\operatorname{Coh}(i_l^{\#}\widehat{\mathcal{D}_K}, B)$*. Let* $\mathcal{M} \in \operatorname{Coh}(\widehat{\mathcal{D}_K}, G)$ *and let* $\mathcal{M}_0$ *be the corresponding lattice of* $\mathcal{M}$*. Then under the equivalence of categories above we have that*

$$\hat{i}^{\#}_{l,K}\mathcal{M} = (\hat{i}_l^{\#}\mathcal{M}_0) \underset{R}{\otimes} K.$$

Let us finish the section by proving an affinoid version of Corollary 4.3.16.

Corollary 5.3.22. *Let* $\mathcal{M} \in \operatorname{Coh}(\widehat{\mathcal{D}_K}, G)$ *and assume that* $\Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{M}) = 0$*. Then* $\Gamma(X \times X, \mathcal{M}) = 0$*.*

Proof. Let $\mathcal{M}_0$ be the corresponding lattice of $\mathcal{M}$ and define $\mathcal{M}_i := \mathcal{M}_0/\pi^i\mathcal{M}_0$ and $\mathcal{N}_i := i_l^{\#}\mathcal{M}_i$. By construction, we have $\mathcal{N} := \hat{i}_l^{\#}\mathcal{M} = \varprojlim \mathcal{N}_i$ and by Corollary 5.3.2, $\mathcal{N}_i = \mathcal{N}/\pi^i\mathcal{N}$.

By assumption, we know that $\Gamma(X, \hat{i}_{l,K}^{\#}\mathcal{M}) = \Gamma(X, \mathcal{N}) \underset{R}{\otimes} K = 0$. Since $\mathcal{N} \in \mathrm{Coh}(\widehat{\mathcal{D}}, L)$, the sections of $\mathcal{N}$ are finitely generated over $\widehat{\mathcal{D}}$; in particular, there exists $m \in \mathbb{N}$ such that $\pi^m\Gamma(X, \mathcal{N}) = 0$, so $\Gamma(X, \pi^m\mathcal{N}) = 0$.. Since $\Gamma(X, \mathcal{N}) = \varprojlim \Gamma(X, \mathcal{N}_i)$, we obtain that for $j \geq m$, $\Gamma(X, \mathcal{N}_j) = \Gamma(X, \mathcal{N})$, so $\Gamma(X, \pi^m\mathcal{N}_j) = 0$. Therefore, by applying Corollary 4.3.16, we obtain $\Gamma(X \times X, \pi^m\mathcal{M}_j) = 0$ for $j \geq m$, so $\pi^m\Gamma(X \times X, \mathcal{M}_j) = 0$. Since $\Gamma(X \times X, \mathcal{M}_0) = \varprojlim \Gamma(X \times X, \mathcal{M}_j)$, we conclude that $\pi^m\Gamma(X \times X, \mathcal{M}_0) = 0$, so $\Gamma(X \times X, \mathcal{M}) = \Gamma(X \times X, \mathcal{M}_0) \underset{R}{\otimes} K = 0$. ☐

5.4 Affinoid equivariant Beilinson-Bernstein localisation

Throughout this section we let G be a connected, simply connected, smooth affine algebraic group scheme locally of finite type defined over $\mathrm{Spec}\, R$ and we let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. We also let X be a quasi-compact R-variety on which G acts.

5.4.1 Affinoid localisation mechanism

We fix L a closed subgroup of G and n a deformation parameter. We denote $U(\mathfrak{g})_n$ the π^n-th deformation of $U(\mathfrak{g})$ and we let $(\mathcal{D}, i_{\mathfrak{g}})$ be a π^n-deformed L-htdo on X. Throughout this section we also make the following assumption:

Assumption 5.4.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module. Then $\Gamma(X, \mathcal{M})$ is a finitely generated $U(\mathfrak{g})_n$-module.*

This is a restriction that is always satisfied when $\mathcal{D}$ is a π^n-deformed tdo on the flag scheme X (the equivariance does not play any role). This is the case we are mainly interested in, but we choose the write theory in this more general setting.

Recall that $\widehat{U(\mathfrak{g})_n}$ denotes the π-adic completion of $U(\mathfrak{g})_n$; further we denoted $\widehat{U(\mathfrak{g})_{n,K}} := \widehat{U(\mathfrak{g})_n} \underset{R}{\otimes} K$. Similar to Definition 5.3.6, we define the notion $\hat{L}$-equivariant $\widehat{U(\mathfrak{g})_n}$-modules by extending Definition 4.4.3.

Definition 5.4.2. *A $\hat{L}$-equivariant $\widehat{U(\mathfrak{g})_n}$-module is quadruple $(M,(M_i),(\alpha_i),(\rho_i))$ such that M is a finitely generated $\widehat{U(\mathfrak{g})_n}$-module, (M_i) is an inverse system of $U(\mathfrak{g})_n$-modules and*

- *(M_i,α_i,ρ_i) is a finitely generated L-equivariant $U(\mathfrak{g})_n$-module and $\pi^i M_i = 0$.*
- *The transition maps induce isomorphisms $M_i/\pi^{i-1}M_i \cong M_{i-1}$ of L-equivariant $U(\mathfrak{g})_n$-modules.*
- *$M \cong \varprojlim M_i$ as $\widehat{U(\mathfrak{g})_n}$-modules.*

A morphism between two $\hat{L}$-equivariant $\widehat{U(\mathfrak{g})_n}$-modules $(M,(M_i),(\alpha_i),(\rho_{Mi}))$ and $(N,(N_i),(\beta_i),(\rho_{Ni}))$ is a map of $f : M \to N$ of $\widehat{U(\mathfrak{g})_n}$-modules such that there is a family of compatible L-equivariant morphisms $f_i : M_i \to N_i$ such that $f = \varprojlim f_i$. We call such a morphism $\hat{L}$-equivariant and denote $\mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})_n}, L)$ the subcategory of $\mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})_n})$ consisting of $\hat{L}$-equivariant modules and morphisms. As for equivariant $\widehat{\mathcal{D}}$-modules, we will omit the equivariance structure when it is clear in the context and just call M a $\hat{L}$-equivariant $\widehat{U(\mathfrak{g})_n}$-module.

We also define the notion of equivariant modules for the ring $\widehat{U(\mathfrak{g})_{n,K}}$.

Definition 5.4.3. *A $\hat{L}_K$-equivariant $\widehat{U(\mathfrak{g})_{n,K}}$-module is a quintuple $(M,M_0,(M_i),(\alpha_i),(\rho_{Mi}))$ such that M_0 is a lattice for M and $(M_0,(M_i),(\alpha_i),(\rho_{Mi})) \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})_n}, L)$.*

Next, let $(M,M_0,(M_i),(\alpha_i),(\rho_{Mi}))$ and $(N,N_0,(N_i),(\beta_i),(\rho_{Ni}))$ be $\hat{L}_K$-equivariant $\widehat{U(\mathfrak{g})_{n,K}}$-modules, and let $f : M \to N$ be a $\widehat{U(\mathfrak{g})_{n,K}}$ linear morphism. As M and N are finitely generated, we have

$$\mathrm{Hom}_{\widehat{U(\mathfrak{g})_n}}(M_0,N_0) \underset{R}{\otimes} K \cong \mathrm{Hom}_{\widehat{U(\mathfrak{g})_{n,K}}}(M,N),$$

so there exists $f_0 : M_0 \to N_0$ and $x \in K$ such that $f = f_0 \otimes x$. We say that f is $\hat{L}_K$-equivariant if f_0 is $\hat{L}$-equivariant. Denote $\mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})_{n,K}}, L)$ the subcategory of finitely generated $\widehat{U(\mathfrak{g})_{n,K}}$ modules consisting of $\hat{L}_K$-equivariant objects along with $\hat{L}_K$-equivariant morphisms. We will ignore the equivariance structure when it is well understood from the context.

Before stating the affinoid localisation mechanism, we need one more lemma:

Lemma 5.4.4. *Let B be a Noetherian R-algebra and A a finitely generated B-module. Let $A_i = A/\pi^i A$, $B_i = B/\pi^i B$, $\hat{A} = \varprojlim A_i$, $\hat{B} = \varprojlim B_i$. Further, let C_i be a inverse system of B_i-modules and $C = \varprojlim C_i$. Assume that $C_i = C/\pi^i C$. Then:*

$$\varprojlim (A_i \underset{B_i}{\otimes} C_i) \cong \hat{A} \underset{\hat{B}}{\otimes} C.$$

Proof. Since B is a Noetherian R-algebra, we have $\hat{A} \cong A \underset{B}{\otimes} \hat{B}$, so

$$\hat{A} \underset{\hat{B}}{\otimes} C \cong A \underset{B}{\otimes} \hat{B} \underset{\hat{B}}{\otimes} C \cong A \underset{B}{\otimes} C.$$

Consider the exact sequence

$$C \xrightarrow{\cdot \pi} C \to C/\pi^i C \to 0.$$

Since tensor product is right exact, one obtains:

$$A \underset{B}{\otimes} C \xrightarrow{\cdot \pi} A \underset{B}{\otimes} C \to A \underset{B}{\otimes} C_i \to 0.$$

Therefore, we get

$$[A \underset{B}{\otimes} C/\pi^i (A \underset{B}{\otimes} C)] \cong A \underset{B}{\otimes} C_i \cong A_i \underset{B_i}{\otimes} C_i.$$

The claim follows since $A \underset{B}{\otimes} C$ is π-adically complete. □

Recall by Definition 4.2.5 that there exists a map $i_{\mathfrak{g}} : U(\mathfrak{g})_n \to \mathcal{D}$. By functoriality, the map $i_{\mathfrak{g}} : U(\mathfrak{g})_n \to \mathcal{D}$ induces a map $\widehat{i_{\mathfrak{g}}} : \widehat{U(\mathfrak{g})_n} \to \widehat{\mathcal{D}}$ and thus a map $\widehat{i_{\mathfrak{g}}} : \widehat{U(\mathfrak{g})_{n,K}} \to \widehat{\mathcal{D}_K}$.

Definition 5.4.5. *We define two functors:*

$$\begin{aligned} &\mathrm{Loc} : \mathrm{Mod}(\widehat{U(\mathfrak{g})_{n,K}}) \to \mathrm{Mod}(\widehat{\mathcal{D}_K}), && \mathrm{Loc}(M) = \widehat{\mathcal{D}_K} \underset{\widehat{U(\mathfrak{g})_{n,K}}}{\otimes} M, \\ &\Gamma : \mathrm{Mod}(\widehat{\mathcal{D}_K}) \to \mathrm{Mod}(\widehat{U(\mathfrak{g})_{n,K}}), && \Gamma(\mathcal{M}) = \Gamma(X, \mathcal{M}). \end{aligned} \tag{5.19}$$

Proposition 5.4.6.

i) Let $M \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})_{n,K}}, L)$. Then $\mathrm{Loc}(M) \in \mathrm{Coh}(\widehat{\mathcal{D}_K}, L)$.

ii) Let $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}_K}, L)$*. Then* $\Gamma(X, \mathcal{M}) \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})_{n,K}}, L)$*.*

Proof. Let M_0 be the lattice of M such that $M_0 \in \mathrm{Mod}(\widehat{U(\mathfrak{g})_n}, L)$. Then

$$(\widehat{\mathcal{D}} \underset{\widehat{U(\mathfrak{g})_n}}{\otimes} M_0) \underset{R}{\otimes} K \cong \widehat{\mathcal{D}_K} \underset{\widehat{U(\mathfrak{g})_{n,K}}}{\otimes} M \cong \mathrm{Loc}(M),$$

so $\mathcal{M} := \widehat{\mathcal{D}} \underset{\widehat{U(\mathfrak{g})_n}}{\otimes} M_0$ is a lattice for $\mathrm{Loc}(M)$, so we need to prove $\mathcal{M}$ is L-equivariant.

Let $M_i = M_0/\pi^i M_0$ and $\mathcal{M}_i := \mathcal{D} \underset{U(\mathfrak{g})_n}{\otimes} M_i$. Then, we have by applying Lemma 5.4.4 that $\mathcal{M} \cong \varprojlim \mathcal{M}_i$. Fix $i \in \mathbb{N}^*$; by construction we have $\pi^i \mathcal{M}_i = 0$; next, by definition we have that M_i is a L-equivariant finitely generated $U(\mathfrak{g})_n$-module, so $\mathcal{M}_i$ is a quasi-coherent L-equivariant $\mathcal{D}$-module by Proposition 4.5.1. Since M_i is finitely generated as a $U(\mathfrak{g})_n$-module, by picking a presentation of M_i we obtain that $\mathcal{M}_i$ is also coherent.

Finally, consider the short exact sequence:

$$M_i \xrightarrow{\cdot \pi} M_i \to M_{i-1} \to 0.$$

Since tensor product is right exact, we get a short exact sequence:

$$\mathcal{D} \underset{U(\mathfrak{g})_n}{\otimes} M_i \xrightarrow{\cdot \pi} \mathcal{D} \underset{U(\mathfrak{g})_n}{\otimes} M_i \to \mathcal{D} \underset{U(\mathfrak{g})_n}{\otimes} M_{i-1} \to 0,$$

so $\mathcal{M}_i/\pi^{i-1}\mathcal{M}_i \cong \mathcal{M}_{i-1}$. Thus, we proved that $\mathcal{M}$ is indeed L-equivariant, so $\mathrm{Loc}(M)$ is also L-equivariant. This proves the first statement.

On the other hand, consider $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}_K}, L)$ and let $M = \Gamma(X, \mathcal{M})$. Further, let $\mathcal{M}_0 \in \mathrm{Coh}(\widehat{\mathcal{D}}, L)$ be the corresponding lattice of $\mathcal{M}$ and $M_0 = \Gamma(X, \mathcal{M}_0)$. By construction $M_0 \underset{R}{\otimes} K \cong M$, so it is enough to prove that M_0 is L-equivariant. Let $\mathcal{M}_i := \mathcal{M}/\pi^i\mathcal{M}$, $M_i := \Gamma(X, \mathcal{M}_i)$. Since $\mathcal{M}$ is coherent, we have $\mathcal{M} \cong \varprojlim \mathcal{M}_i$, so $M_0 = \varprojlim M_i$. Further, $M_i/\pi^{i-1}M_i \cong M_{i-1}$ and $\pi^i M_i = 0$ for all $i \in \mathbb{N}$, so we are left to prove that M_i's are L-equivariant finitely generated $U(\mathfrak{g})_n$-modules.

Since $\mathcal{M}_0 \in \mathrm{Coh}(\widehat{\mathcal{D}}, L)$, we obtain by construction and Corollary 5.3.2 that for all $i \in \mathbb{N}^*$, $\mathcal{M}_i$ is a L-equivariant coherent $\mathcal{D}$-module. Then by Proposition 4.5.4 and Assumption 5.4.1 we obtain that for all $i \in \mathbb{N}^*$, M_i is a L-equivariant finitely generated $U(\mathfrak{g})_n$-module. This concludes the proof. □

5.4.2 Applications of the localisation mechanism

Throughout this subsection, we assume that G is a connected, simply connected, split semisimple, smooth affine algebraic group scheme over $\operatorname{Spec} R$. We also let $X = G/B$ denote the flag scheme which is a quasi-compact R-variety. Fix n a deformation parameter and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ a Cartan decomposition of $\mathfrak{g} = \operatorname{Lie}(G)$. Further, we fix $\lambda : \pi^n\mathfrak{h} \to R$ an R-linear map and denote R_λ the corresponding $U(\mathfrak{h})_n$-module. By [5, Section 6.10] we have an induced map $(U(\mathfrak{g})^G)_n \to U(\mathfrak{h})_n$ and we view R_λ as a $(U(\mathfrak{g})^G)_n$-module via this map. We also let $K_\lambda := R_\lambda \underset{R}{\otimes} K$ the corresponding $\widehat{U(\mathfrak{g})}^G_{n,K}$-module. We make the following definitions:

- $U(\mathfrak{g})^\lambda_n := U(\mathfrak{g})_n \underset{(U(\mathfrak{g})^G)_n}{\otimes} R_\lambda$,
- $\widehat{U(\mathfrak{g})}^\lambda_n := \varprojlim U(\mathfrak{g})^\lambda_n / \pi^i U(\mathfrak{g})^\lambda_n$ and
- $\widehat{U(\mathfrak{g})}^\lambda_{n,K} := \widehat{U(\mathfrak{g})}^\lambda_n \underset{R}{\otimes} K$.

We should remark that by [5, Theorem 6.10a)], $\widehat{U(\mathfrak{g})}^\lambda_{n,K} \cong \widehat{U(\mathfrak{g})}_{n,K} \underset{\widehat{U(\mathfrak{g})}^G_{n,K}}{\otimes} K_\lambda$, so in particular $\widehat{U(\mathfrak{g})}^\lambda_{n,K}$ is a quotient of $\widehat{U(\mathfrak{g})}_{n,K}$.

We also let $\mathcal{D}^\lambda_n$ be as in [5, Section 6.4]. This coincides with $\mathcal{D}_{\lambda,\pi^n}$ as defined in Definition 4.5.14. By [5, Theorem 6.10b)] one has $\Gamma(X, \widehat{\mathcal{D}}^\lambda_{n,K}) \cong \widehat{U(\mathfrak{g})}^\lambda_{n,K}$. We define a localisation functor

$$\operatorname{Loc}^\lambda : \operatorname{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})}^\lambda_{n,K}) \to \operatorname{Coh}(\widehat{\mathcal{D}}^\lambda_{n,K}) \qquad \operatorname{Loc}^\lambda(M) := \widehat{\mathcal{D}}^\lambda_{n,K} \underset{\widehat{U(\mathfrak{g})}^\lambda_{n,K}}{\otimes} M.$$

We say that $\lambda \in \mathfrak{h}^*_K$ is *dominant* if $(\lambda + \rho)(h) \geq 0$ for any positive coroot $h \in \mathfrak{h}$. Given $\lambda : \pi^n\mathfrak{h} \to R$, we say that λ is *dominant* if the corresponding root $\lambda \in \mathfrak{h}^*_K$ is dominant. We say that $\lambda : \pi^n\mathfrak{h} \to R$ is *regular* if the corresponding $\lambda \in \mathfrak{h}^*_K$ is regular, i.e. the stabiliser of the Weyl group action on λ is trivial.

In [5], the authors prove an affinoid version of Beilinson-Bernstein localisation:

Theorem 5.4.7. *[5, Theorem C] [3, Theorem 5.3.13] Let $\lambda : \pi^n\mathfrak{h} \to R$ be a dominant weight. The functor Γ is exact and the functors $\operatorname{Loc}^\lambda$ and Γ induce quasi-inverse equivalences of categories between $\operatorname{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})}^\lambda_{n,K})$ and the quotient category*

$\mathrm{Coh}(\widehat{\mathcal{D}^{\lambda}_{n,K}})/\ker\Gamma$. *In case* λ *is also regular, then* $\ker\Gamma = 0$ *whenever* $n > 0$ *or* p *is a very good prime for* G.

We should remark that the part of the proof where λ is dominant does not require p to be a very good prime for G. The restriction on p has been removed in [3, Theorem 5.3.13] provided that $n > 0$.

We may prove an equivariant version of the affinoid localisation theorem.

Theorem 5.4.8. *Let* L *be a closed subgroup of* G *and let* $\lambda : \pi^n\mathfrak{h} \to R$ *be a dominant weight. The functors* Loc^{λ} *and* Γ *induce quasi-inverse equivalences of categories between* $\mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}, L)$ *and the quotient category* $\mathrm{Coh}(\widehat{\mathcal{D}^{\lambda}_{n,K}}, L)/\ker\Gamma$. *In case* λ *is also regular, then* $\ker\Gamma = 0$ *whenever* $n > 0$ *or* p *is a very good prime for* G.

Proof. By Theorem 5.4.7, it is enough to prove that Loc^{λ} and Γ preserve the L-equivariance. We have by Corollary 4.5.15 that $\mathcal{D}^{\lambda}_{n}$ is a π^n-deformed G-htdo, so in particular it is π^n-deformed L-htdo. Further, we have by [5, Proposition 5.15] that $\mathcal{D}^{\lambda}_{n}$ satisfies Assumption 5.4.1. The claim follows from Proposition 5.4.6 since $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$ is a quotient of $\widehat{U(\mathfrak{g})_{n,K}}$. □

As a corollary we obtain:

Corollary 5.4.9. *Let* $M \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})^{\lambda}_{n,K}})$. *Then*

$$\Gamma(X, \mathrm{Loc}^{\lambda}(M)) \cong M.$$

In the next section, we will apply Theorem 5.4.8 in two cases: B is a Borel subgroup of a G and $G \cong G_d = \{(g,g) | g \in G\}$ is the diagonal subgroup of $G \times G$.

5.4.3 Equivariance of two-sided ideals

We keep the notation from the previous section. The Lie algebra of the algebraic group $G \times G$ is given by $\mathrm{Lie}(G\times G) = \mathrm{Lie}(G) \times \mathrm{Lie}(G) = \mathfrak{g}\times\mathfrak{g}$. We aim to prove that any two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$ is G-equivariant when viewed as $\widehat{U(\mathfrak{g}\times\mathfrak{g})_{n,K}}$-module. Here we view G as the diagonal subgroup of $G \times G$. To avoid confusion, we denote this group G_d.

Recall that the enveloping algebra $U(\mathfrak{g})$ is a G-representation. In particular, the group $G(R)$ acts on $U(\mathfrak{g})$ via the Adjoint action inducing a comodule map

$$\rho : U(\mathfrak{g}) \to \mathcal{O}(G) \otimes U(\mathfrak{g}).$$

Further, since the G-action preserves $U(\mathfrak{g})_n$, the map ρ restricts to a comodule map $\rho : U(\mathfrak{g})_n \to \mathcal{O}(G) \otimes U(\mathfrak{g})_n$. Let $\widehat{\mathcal{O}(G)} := \varprojlim \mathcal{O}(G)/\pi^i\mathcal{O}(G)$ denote the π-adic completion of the Hopf algebra $\mathcal{O}(G)$ corresponding to the group G. Using the fact that the π-adic completion is a functor, we obtain a map

$$\hat{\rho} : \widehat{U(\mathfrak{g})_n} \to \widehat{\mathcal{O}(G)} \hat{\otimes} \widehat{U(\mathfrak{g})_n},$$

where $\hat{\otimes}$ denotes the completed tensor product.

Definition 5.4.10. *We say that a two-sided ideal I in $\widehat{U(\mathfrak{g})_n}$ is π-closed if the quotient $\widehat{U(\mathfrak{g})_n}/I$ is π-torsion-free.*

For the rest of this subsection, we let I a π-closed two-sided ideal in $\widehat{U(\mathfrak{g})_n}$.

By construction, we have that $g \cdot x = \hat{\rho}(x)(g)$, for all $g \in G(R), x \in \widehat{U(\mathfrak{g})_n}$, so applying [6, Corollary 4.3], we obtain

$$\hat{\rho}(x)(g) = g \cdot x \in I, \text{ for all } g \in G(R), x \in I. \tag{5.20}$$

For each $g \in G(R)$ consider the map $\epsilon_g : \widehat{\mathcal{O}(G)} \to R$, $\epsilon_g(f) := f(g)$ and let $q : \widehat{U(\mathfrak{g})_n} \to \widehat{U(\mathfrak{g})_n}/I$ denote the natural projection. Consider the following commutative diagram:

$$\begin{tikzcd}
\widehat{\mathcal{O}(G)} \hat{\otimes} \widehat{U(\mathfrak{g})_n} \arrow[r, "\mathrm{id}\,\hat{\otimes} q"] \arrow[d, "\epsilon_g \hat{\otimes}\,\mathrm{id}"] & \widehat{\mathcal{O}(G)} \hat{\otimes} \widehat{U(\mathfrak{g})_n}/I \arrow[d, "\epsilon_g \hat{\otimes}\,\mathrm{id}"] \\
R \hat{\otimes} \widehat{U(\mathfrak{g})_n} \arrow[r, "\mathrm{id}\,\hat{\otimes} q"] & R \hat{\otimes} \widehat{U(\mathfrak{g})_n}/I.
\end{tikzcd} \tag{5.21}$$

By equation (5.20), we have that $(\mathrm{id}\,\hat{\otimes} q) \circ (\epsilon_g \hat{\otimes}\,\mathrm{id}) \circ \hat{\rho}(i) = 0$, for all $i \in I$ and $g \in G(R)$, therefore we obtain

$$(\epsilon_g \hat{\otimes}\,\mathrm{id}) \circ (\mathrm{id}\,\hat{\otimes} q) \circ \hat{\rho}(i) = 0 \text{ for all } g \in G(R), i \in I. \tag{5.22}$$

Let $\widehat{K(G)} := \widehat{\mathcal{O}(G)} \underset{R}{\otimes} K$. We wish to prove that the Jacobson radical of $\widehat{K(G)}$ is 0, and if $f \in \widehat{\mathcal{O}(G)}$, viewed as an element of $\widehat{K(G)}$, is such that $\epsilon_g(f) = 0$ fo all

$g \in G(\mathcal{O}_L)$ and all L/K finite extensions, then f is in the intersection of all maximals ideals of $\widehat{K(G)}$. By combining the results, we obtain $f = 0$.

Proposition 5.4.11. *The Jacobson radical of $\widehat{K(G)}$, $J(\widehat{K(G)})$, is 0.*

Proof. Any free Tate algebra over a non-archimidean field K is a Jacobson ring by [18, Proposition 3.1.3]; in particular as $\widehat{K(G)} = K\langle x_1, x_2, \dots x_n\rangle/J$ is a quotient of a free Tate algebra by some closed ideal J, we have $J(\widehat{K(G)})$ =nilradical($\widehat{K(G)}$), so it suffices to prove that nilradical($\widehat{K(G)}$)=0.

As G is a reductive connected group scheme, we have by [36, II.1.9 (4)] that $\mathcal{O}(G)$ is an integral domain, therefore $k(G) := \mathcal{O}(G) \underset{R}{\otimes} k$ is also an integral domain. Consider the π-adic filtrations on $\mathcal{O}(G)$ and $\widehat{\mathcal{O}(G)}$; we have by the properties of π-adic completions

$$\mathrm{gr}(\widehat{\mathcal{O}(G)}) = \mathrm{gr}(\mathcal{O}(G)) \cong (\mathrm{gr}\ R)(G).$$

As (gr R) is a polynomial ring over k and $k(G)$ is an integral domain, we obtain that $\mathrm{gr}(\widehat{\mathcal{O}(G)})$ is an integral domain, so $\widehat{\mathcal{O}(G)}$ is an integral domain. Therefore, $\widehat{K(G)} = \widehat{\mathcal{O}(G)} \underset{R}{\otimes} K$ is an integral domain, so in particular $\widehat{K(G)}$ has trivial nilradical. □

Proposition 5.4.12. *Let $f \in \widehat{\mathcal{O}(G)}$ such that $\epsilon_g(f) = 0$ for all $g \in G(\mathcal{O}_L)$ and all L finite extensions of K. Then $f = 0$.*

Proof. View f as an element of $\widehat{K(G)}$. Further, let $K\langle x_1, x_2 \dots, x_n\rangle$ be a free Tate algebra projecting onto $\widehat{K(G)}$ via a map denoted ϕ; let $J = \ker \phi$. Finally, let $\mathfrak{m} \subset \widehat{K(G)}$ be a maximal ideal of $\widehat{K(G)}$; we aim to prove that $f \in \mathfrak{m}$.

As $\mathfrak{m} \subset \widehat{K(G)}$ is maximal, $\phi^{-1}(\mathfrak{m})$ is a maximal ideal in $K\langle x_1, x_2 \dots, x_n\rangle$, so we get an induced map $\zeta : \widehat{K(G)}/\mathfrak{m} \to K\langle x_1 x_2 \dots x_n\rangle/\phi^{-1}(\mathfrak{m})$. Further, we have by [18, Corollary 2.2.12], $K\langle x_1, x_2, \dots, x_n\rangle/\phi^{-1}(\mathfrak{m}) \cong L$, where L is a finite extension of K. The image of f under the composition of the maps (call this composition $\eta : \widehat{\mathcal{O}(G)} \to L$) lies into the ring of integers of L, $\mathcal{O}_L$.

By [20, Example 1.8 ii)], there is a correspondence between maps from $\widehat{K(G)}$ to L and the zero locus of a system of generators for the ideal defining J (recall $\widehat{K(G)} = K\langle x_1, x_2 \dots, x_n\rangle/J$) inside $\mathcal{O}_L^n$. Therefore, as $\epsilon_g(f) = 0$ for all $g \in G(\mathcal{O}_L)$, we obtain $\eta(f) = 0$. Consider the composition defining η:

$$\widehat{K(G)} \to \widehat{K(G)}/\mathfrak{m} \to K\langle x_1, x_2 \dots x_n\rangle/\phi^{-1}(\mathfrak{m}) \cong L.$$

As $\mathfrak{m}$ is a maximal ideal, $\widehat{K(G)}/\mathfrak{m}$ is a field, so the map $\widehat{K(G)}/\mathfrak{m} \to L$ is an injection. Thus, as $\eta(f) = 0$, one obtains that $f \in \mathfrak{m}$. In conclusion, f lies in all the maximal ideals of $\widehat{K(G)}$, i.e. $f \in J(\widehat{K(G)})$; applying Proposition 5.4.11, we get $f = 0$. □

Theorem 5.4.13. *Let I be a π-closed two-sided ideal in $\widehat{U(\mathfrak{g})_n}$. Then $\hat{\rho}(I) \subset \widehat{\mathcal{O}(G)}\hat{\otimes} I$.*

Proof. Consider the composition map $(\epsilon_g \hat{\otimes} \operatorname{id}) \circ (\operatorname{id} \hat{\otimes} q) \circ \hat{\rho} : I \to R\hat{\otimes}\widehat{U(\mathfrak{g})_n}/I$. By equation (5.22), we know that for all $i \in I$, $(\epsilon_g \hat{\otimes} \operatorname{id}) \circ (\operatorname{id} \hat{\otimes} q) \circ \hat{\rho}(i) = 0$.

Let $I_K = I \underset{R}{\otimes} K$ and notice that I_K is a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$. As I is a π-closed ideal, the space $\widehat{U(\mathfrak{g})_n}/I$ has no π-torsion, so we obtain

$$\widehat{U(\mathfrak{g})_n}/I \underset{R}{\otimes} K \cong \widehat{U(\mathfrak{g})_{n,K}}/I_K.$$

The space $\widehat{U(\mathfrak{g})_{n,K}}/I_K$ is a K-Banach space that has a countably dimensional dense subspace consisting of elements of the form $x + I_K$, $x \in U(\mathfrak{g}_K)$. Therefore, applying [51, Proposition 10.4], we get that $\widehat{U(\mathfrak{g})_{n,K}}/I_K$ has a countable topological K-basis, so $\widehat{U(\mathfrak{g})_n}/I$ has a countable topological R-basis; denote this basis $\{y_i | i \in \mathbb{N}\}$. Another way to see the existence of this basis is that the space $\widehat{U(\mathfrak{g})_{n,K}}/I_K$ is a separable K-Banach space, so it has a Schauder basis.

Consider an element $a = \sum_{i=1}^{\infty} f_i \hat{\otimes} y_i \in \widehat{\mathcal{O}(G)}\hat{\otimes}\widehat{U(\mathfrak{g})_n}$. Then we have for all $g \in G(R)$,

$$0 = (\epsilon_g \hat{\otimes} \operatorname{id})(\sum_{i=1}^{\infty} f_i \hat{\otimes} y_i) = \sum_{i=1}^{\infty} f_i(g) y_i.$$

As y_i's form a topological basis of $\widehat{U(\mathfrak{g})_n}$, we obtain

$$\epsilon_g(f_i) = 0 \text{ for all } g \in G(R), i \in \mathbb{N}. \tag{5.23}$$

Now, let A be π-adically complete commutative R-algebra finitely generated as an R-module. For $g \in G(A)$ let $\epsilon_g : \widehat{A(G)} \to A$ denote the evaluation map by abusing notation.

Recall that $\widehat{U(\mathfrak{g})_n}$ is a G-representation by extending the Adjoint action of G on $\mathfrak{g}$. Consider the set $I \underset{R}{\otimes} A$ inside $\widehat{U(\mathfrak{g})_{n,A}} := \widehat{U(\mathfrak{g})_n} \underset{R}{\otimes} A$. Notice that since A is finitely generated as an R-module, we only need to take the standard tensor product, not the completed one. Let $x_1 \otimes y_1$ be a simple tensor in $\widehat{U(\mathfrak{g})_{n,A}}$ and $x_2 \otimes y_2$ be a simple tensor in $I \underset{R}{\otimes} A$. Then

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2.$$

As $x_2 \in I$ and I is a two-sided ideal $x_1 x_2 \in I$, so $(x_1 \otimes y_1)(x_2 \otimes y_2) \in I \underset{R}{\otimes} A$. Extending this to non-simple tensors, taking in account all the possible ways to represent elements in $I \underset{R}{\otimes} A$ and $\widehat{U(\mathfrak{g})_{n,A}}$ as sums of simple tensors, we get that $I \underset{R}{\otimes} A$ is a left ideal in $\widehat{U(\mathfrak{g})_{n,A}}$. By symmetry it is also a right ideal, so $I \underset{R}{\otimes} A$ is indeed a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,A}}$. As $\widehat{U(\mathfrak{g})_n}$ is a G-representation and $G(R)$ preserves I, G is a flat group scheme, we deduce that I is a G-subrepresentation of $\widehat{U(\mathfrak{g})_n}$, so $G(A) \cdot (I \underset{R}{\otimes} A) \subset I \underset{R}{\otimes} A$. Therefore, by base changing equation (5.23) to A we get

$$\epsilon_g(f_i) = 0 \text{ for all } g \in G(A), i \in \mathbb{N}. \tag{5.24}$$

In particular we get that the result is true for any $\mathcal{O}_L$, where L is a finite extension of K. Applying Proposition 5.4.12, we obtain $f_i = 0$ for all $i \in \mathbb{N}$. Thus, we have obtained that $a = 0$, so $(\mathrm{id} \hat{\otimes} q) \circ \hat{\rho}(i) = 0$, which implies

$$\hat{\rho}(i) \in \ker(\mathrm{id} \hat{\otimes} q) = \widehat{\mathcal{O}(G)} \hat{\otimes} I.$$

Therefore, $\hat{\rho}(I) \subset \widehat{\mathcal{O}(G)} \hat{\otimes} I$. ☐

Let τ be the principal anti-automorphism of $U(\mathfrak{g})$ induced by $x \to -x$ for all $x \in \mathfrak{g}$. We use x^τ to denote $\tau(x)$. For all $x_1, x_2 \dots x_n \in \mathfrak{g}$, we have

$$(x_1 x_2 \dots x_n)^\tau = (-1)^n x_n x_{n-1} \dots x_2 x_1.$$

We define the action of the ring $\widehat{U(\mathfrak{g} \times \mathfrak{g})_n} \cong \widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n}$ on $\widehat{U(\mathfrak{g})_n}$ via

$$(a \otimes b)x = bxa^\tau, \text{ for all } a, b, x \in \widehat{U(\mathfrak{g})_n}.$$

Let $m : (\widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n}) \hat{\otimes} \widehat{U(\mathfrak{g})_n} \to \widehat{U(\mathfrak{g})_n}$ denote the action map. The set of submodules of $\widehat{U(\mathfrak{g})_n}$ under this action coincide with the set of two-sided ideals. The group $G \times G$ acts on $\widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n}$ via the adjoint action:

$$(g_1, g_2) \cdot (x \hat{\otimes} y) = (\operatorname{Ad}(g_1)x \hat{\otimes} \operatorname{Ad}(g_2)y).$$

In particular we get an action of the group $G_d \cong G$. Let

$$\hat{\rho}_{\text{bimod}} : \widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n} \to \widehat{\mathcal{O}(G)} \hat{\otimes} \widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n}$$

be the corresponding comodule map.

Finally, let $\mathfrak{g}_d = \operatorname{Lie}(G_d)$. It embeds into $\widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n}$ via $x \mapsto x \hat{\otimes} 1 + 1 \hat{\otimes} x$ for all $x \in \mathfrak{g}_d$.

Proposition 5.4.14. *Let I be a π-closed two-sided ideal in $\widehat{U(\mathfrak{g})_n}$. Then, $I \in \operatorname{Mod}(\widehat{U(\mathfrak{g} \times \mathfrak{g})_n}, G_d)$.*

Proof. By abuse of notation let $\hat{\rho} : I \to \widehat{\mathcal{O}(G)} \hat{\otimes} I$ be the restriction of $\hat{\rho}$ to I induced by the Ad action; by Theorem 5.4.13 this map is well defined. Furthermore, since the ring $\widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n}$ is Noetherian, I is also finitely generated. Let

$$\hat{\rho}_{\text{tensor}} : \widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n} \hat{\otimes} I \to \widehat{\mathcal{O}(G)} \hat{\otimes} \widehat{U(\mathfrak{g})_n} \hat{\otimes} \widehat{U(\mathfrak{g})_n} \hat{\otimes} I$$

be the comodule map induced by $\hat{\rho}$ and $\hat{\rho}_{\text{bimod}}$. To prove that the multiplication m is a morphism of comodules it is enough to prove that for all $g \in G, x, y \in \widehat{U(\mathfrak{g})_n}, u \in I$.

$$\operatorname{Ad}(g) \cdot ((x \hat{\otimes} y) \cdot u) = (\operatorname{Ad}(g)x \hat{\otimes} \operatorname{Ad}(g)y) \cdot (\operatorname{Ad}(g)u)$$

We have:

$$\begin{aligned} \operatorname{Ad}(g) \cdot ((x \hat{\otimes} y) \cdot u) &= \operatorname{Ad}(g)(yux^{\tau}) \\ &= \operatorname{Ad}(g)y \operatorname{Ad}(g)u \operatorname{Ad}(g)x^{\tau} \\ &= \operatorname{Ad}(g)y \operatorname{Ad}(g)u(\operatorname{Ad}(g)x)^{\tau} \\ &= (\operatorname{Ad}(g)x \hat{\otimes} \operatorname{Ad}(g)y) \cdot (\operatorname{Ad}(g)u). \end{aligned} \tag{5.25}$$

Next, the differentiation of the Ad action is the ad action which coincides with the action of the Lie Algebra $\mathfrak{g}_d$. (*)

Now, consider $I_i = I/\pi^i I$. Then it is easy to see that I_i is finitely generated as $U(\mathfrak{g} \times \mathfrak{g})_n \cong U(\mathfrak{g})_n \otimes U(\mathfrak{g})_n$-module (here $U(\mathfrak{g})_n \otimes U(\mathfrak{g})_n$ acts on I_i via $(x \otimes y) \cdot (u + \pi^i I) =$

$xuy^\tau + \pi^i I)$, $\pi^i I_i = 0$ and $I = \varprojlim I_i$. The map $\hat{\rho} : I \to \widehat{\mathcal{O}(G)}\hat{\otimes} I$ descends to a map $\rho_i : I_i \to \mathcal{O}(G) \otimes I_i$ which is compatible with the action map since $\hat{\rho}_{\text{tensor}}$ is a comodule homomorphism. Finally, by (*) the differentiation of the Ad action descend to I_i, so I_i is indeed a G_d-equivariant $U(\mathfrak{g})_n \otimes U(\mathfrak{g})_n$-module. Thus, we have proven all the conditions required to make I a $\hat{G}_d$-equivariant $\widehat{U(\mathfrak{g} \times \mathfrak{g})_n}$-module. □

We may now prove an affinoid version of Proposition 4.4.4.

Corollary 5.4.15. *Let J be a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$. Then*

$$J \in \operatorname{Mod}_{\text{fg}}(\widehat{U(\mathfrak{g} \times \mathfrak{g})_{n,K}}, G_d).$$

Proof. Clearly, J is finitely generated since $\widehat{U(\mathfrak{g} \times \mathfrak{g})_{n,K}}$ is a Noetherian ring. Let $I = J \cap \widehat{U(\mathfrak{g})_n}$. It is easy to see that I is a two sided ideal in $\widehat{U(\mathfrak{g})_n}$; we claim it is π-closed. Suppose there exists $x \in \widehat{U(\mathfrak{g})_n}$ and $n \in \mathbb{N}^*$ such that $\pi^n(x + I) = 0 + I$. Then we obtain $\pi^n x \in I \subset J$. Since J is a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$, we have $x \in J$. By the construction of I, we obtain $x \in I$, i.e. $x + I = 0 + I$, so I is indeed π-closed. Therefore, by applying Proposition 5.4.14, we obtain $I \in \operatorname{Mod}(\widehat{U(\mathfrak{g} \times \mathfrak{g})_n}, G_d)$.

To finish the proof, we need to prove that I is a lattice for J. Notice that $I \underset{R}{\otimes} K \subset J$. Let $x \in J$; there exists $n \in \mathbb{N}$ such that $\pi^n x \in \widehat{U(\mathfrak{g})_n}$, so $\pi^n x \in I$. Thus $x = \pi^n x \underset{R}{\otimes} \pi^{-n} \in I \underset{R}{\otimes} K$, so $J = I \underset{R}{\otimes} K$. Finally, since $\cap_{i=1}^\infty \widehat{U(\mathfrak{g})_n}/\pi^i \widehat{U(\mathfrak{g})_n} = 0$ we obtain that $\cap_{i=1}^\infty I/\pi^i I = 0$, so I is indeed a lattice for J. □

5.4.4 Ideals with a given central character

We now specialise to ideals in $\widehat{U(\mathfrak{g})_{n,K}}$ with a given central character. The Cartan subalgebra of $\mathfrak{g} \times \mathfrak{g}$ is given by $\mathfrak{h} \times \mathfrak{h}$, so picking a weight $\nu : \pi^n(\mathfrak{h} \times \mathfrak{h}) \to R$ is the same as picking a pair of weights (λ, μ) where $\lambda, \mu : \pi^n\mathfrak{h} \to R$.

Proposition 5.4.16. *We have $\widehat{U(\mathfrak{g} \times \mathfrak{g})}^{\lambda,\mu}_{n,K} \cong \widehat{U(\mathfrak{g})}^{\lambda}_{n,K} \underset{K}{\hat{\otimes}} \widehat{U(\mathfrak{g})}^{\mu}_{n,K}$.*

We need the following Lemma:

Lemma 5.4.17. *Let A and B be complete normed K-algebras and I, J be closed two-sided ideals in A and B, respectively. Then*

$$A \underset{K}{\hat{\otimes}} B / (I \underset{K}{\hat{\otimes}} B + A \underset{K}{\hat{\otimes}} J) \cong A/I \underset{K}{\hat{\otimes}} B/J.$$

Proof. We call a continuous morphism $\phi : M \to N$ between two semi-normed K-vector spaces *strict* if the natural morphism $\operatorname{coim}\phi \to \operatorname{im}\phi$ is a homeomorphism. If M and N are Banach spaces, then by [16, Lemma 2.6], ϕ is strict if and only if the image of ϕ is closed in N.

Let $\phi_1 : A \to A/I$, $\phi_2 : B \to B/J$ be the natural projections. Since A and B are Banach spaces and I and J are closed ideals, we get by the discussion above that ϕ_1 and ϕ_2 are strict morphisms. Further, R is mixed characteristic $(0, p)$, so the valuation on $\mathbb{Q} \subset K = \operatorname{Frac}(R)$ is non-trivial. Thus, by [16, Theorem 2.8] the morphism $\phi_1 \underset{K}{\otimes} \phi_2 : A \underset{K}{\otimes} B \to A/I \underset{K}{\otimes} B/J$ is also strict. Furthermore, as I, J are closed in A and B respectively, we get that the natural inclusion $I \underset{K}{\otimes} B + A \underset{K}{\otimes} J \to A \underset{K}{\otimes} B$ is also strict. Therefore, we obtain a *strict* short exact sequence

$$0 \to I \underset{K}{\otimes} B + A \underset{K}{\otimes} J \to A \underset{K}{\otimes} B \to A/I \underset{K}{\otimes} B/J \to 0.$$

Applying [19, Corollary 1.1.9/6], we get a strict exact sequence

$$0 \to \overline{I \underset{K}{\otimes} B + A \underset{K}{\otimes} J} \to A \hat{\underset{K}{\otimes}} B \to A/I \hat{\underset{K}{\otimes}} B/J \to 0. \tag{5.26}$$

Since I and J are closed ideals, we have $\overline{I \underset{K}{\otimes} B + A \underset{K}{\otimes} J} = I \hat{\underset{K}{\otimes}} B + A \hat{\underset{K}{\otimes}} J$. The lemma follows from equation (5.26). ☐

We can now prove Proposition 5.4.16:

Proof. For $\lambda : \pi^n \mathfrak{h} \to R$ denote $\chi_\lambda : U(\mathfrak{g})_n^G \to R$ the map obtained by composing the map λ with the map $U(\mathfrak{g})_n^G \to U(\mathfrak{h})_n$. Recall by Theorem 5.2.4, we have $Z(\widehat{U(\mathfrak{g})_{n,K}}) \cong \widehat{U(\mathfrak{g})^G_{n,K}}$, so χ_λ determines a central character of $\widehat{U(\mathfrak{g})_{n,K}}$ which we denote χ_λ by abuse of language. Let $m_\lambda = \ker \chi_\lambda$, so that $\widehat{U(\mathfrak{g})^\lambda_{n,K}} = \widehat{U(\mathfrak{g})_{n,K}} / \widehat{U(\mathfrak{g})_{n,K}} m_\lambda$. Further, consider $\chi_{\lambda,\mu} : U(\mathfrak{g} \times \mathfrak{g})_n^{G \times G} \to R$ and let $m_{\lambda,\mu} = \ker \chi_{\lambda,\mu}$ so that $\widehat{U(\mathfrak{g} \times \mathfrak{g})^{\lambda,\mu}_{n,K}} = \widehat{U(\mathfrak{g} \times \mathfrak{g})_{n,K}} / m_{\lambda,\mu} \widehat{U(\mathfrak{g} \times \mathfrak{g})_{n,K}}$.

We have by definition that $m_{\lambda,\mu} = m_\lambda \otimes (U(\mathfrak{g})^G)_n + (U(\mathfrak{g})^G)_n \otimes m_\mu$, so

$$m_{\lambda,\mu} \widehat{U(\mathfrak{g} \times \mathfrak{g})_{n,K}} = m_\lambda \hat{\underset{K}{\otimes}} \widehat{U(\mathfrak{g})_{n,K}} + \widehat{U(\mathfrak{g})_{n,K}} \hat{\otimes} m_\mu. \tag{5.27}$$

The claim now follows by applying Lemma 5.4.17. ☐

Recall that τ denotes the principal anti-automorphism of $U(\mathfrak{g})$. It can be extended to an anti-automorphism of $\widehat{U(\mathfrak{g})_{n,K}}$, which we will also call τ. We have by [10, Lemma 5.4-Equation 5.5] that τ induces an isomorphism $U(\mathfrak{g})^{\lambda^{\mathrm{op}}} \cong U(\mathfrak{g})^{-w_o\lambda}$; here w_o denotes the longest element of W. The map τ extends to an isomorphism $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}^{\mathrm{op}} \cong \widehat{U(\mathfrak{g})^{-w_o\lambda}_{n,K}}$. From now on, until the end of the document we will use λ^* to denote $-w_o\lambda$.

Recall that if I is a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$, we have shown in Corollary 5.4.15 that $I \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g}\times\mathfrak{g})_{n,K}}, G_d)$. Furthermore, if I has central character χ_λ, i.e. $m_\lambda \subset I$, we view I as a two-sided ideal in $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$. We have by Proposition 5.4.16 that I is a module over the ring $\widehat{U(\mathfrak{g}\times\mathfrak{g})^{\lambda^*,\lambda}_{n,K}} \cong \widehat{U(\mathfrak{g})^{\lambda^*}_{n,K}} \underset{K}{\hat{\otimes}} \widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$, so we have:

Corollary 5.4.18. *Let I be a two-sided ideal in $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$. Then*

$$I \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g}\times\mathfrak{g})^{\lambda^*,\lambda}_{n,K}}, G).$$

.

We should remark that we have dropped the index d as this should not cause any confusion for the rest of the document. Recall that we view G via its diagonal embedding into $G \times G$. The corollary above is an affinoid version of Corollary 4.4.5.

5.5 Global sections under affinoid pullback

Throughout this section, we aim to compute global sections under the affinoid pullback defined in Remark 5.3.21. We proceed by developing some machinery.

5.5.1 Preliminary lemmas

Recall that for any sheaf R-modules $\mathcal{F}$ on a topological space Y, we denote $\hat{\mathcal{F}} := \varprojlim \mathcal{F}/\pi^i\mathcal{F}$ its π-adic completion and $\mathcal{F}_K$ the sheaf defined by $\mathcal{F}_K(U) := F(U) \underset{R}{\otimes} K$ for any $U \subset Y$ open.

Throughout this section we will freely make use of the following easy result: Let $f : Y \to W$ be a map of R-schemes and $\mathcal{F}$ be a sheaf of R-modules on Y, then $(f_*\mathcal{F})_K \cong f_*(\mathcal{F}_K)$.

Lemma 5.5.1. *Let $j : Y \to W$ be a closed embedding of R-varieties and let $(\mathcal{F}_i)$ be an inverse system of sheaves of R-modules such that $\mathcal{F}_i/\pi^{i-1}\mathcal{F}_i \cong \mathcal{F}_{i-1}$ for $i \geq 1$. Let $\mathcal{F} := \varprojlim \mathcal{F}_i$. Then:*

$$j_*\mathcal{F} \cong (\varprojlim j_*\mathcal{F}_i).$$

Proof. Notice that apriori it is not clear that the right-hand side is well defined. However, since j is a closed embedding, it is in particular right exact. Therefore, one can prove in a similar fashion to Lemma 5.3.3 that

$$j_*\mathcal{F}_i/\pi^{i-1}j_*\mathcal{F}_i \cong j_*\mathcal{F}_{i-1}.$$

Let $\mathcal{G} := \varprojlim j_*\mathcal{F}_i$. Since j_* is a right adjoint functor to the inverse image functor, j_* commutes with inverse limits, thus

$$j_*\mathcal{F} = j_*(\varprojlim \mathcal{F}_i) \cong \varprojlim(j_*\mathcal{F}_i) = \mathcal{G}. \qquad \square$$

Lemma 5.5.2. *Let Y be an R-variety and let $p_r : Y \times Y \to Y$ be the projection on the right factor, and $\mathcal{F}$ a sheaf of R-modules. Then:*

$$\widehat{p_r^{-1}\mathcal{F}} \cong p_r^{-1}\hat{\mathcal{F}}.$$

Proof. First, notice that there is a map from the right hand side to the left-hand side via the maps $\hat{\mathcal{F}} \to \hat{\mathcal{F}}/\pi^i\hat{\mathcal{F}} \cong \mathcal{F}/\pi^i\mathcal{F}$. Let $U, V \subset Y$ be affine open, so that $U \times V \subset Y \times Y$ is affine open. Then, we have:

$$\begin{aligned}
\widehat{p_r^{-1}\mathcal{F}}(U \times V) &= \varprojlim(p_r^{-1}\mathcal{F}/\pi^i p_r^{-1}\mathcal{F})(U \times V) \\
&\cong \varprojlim p_r^{-1}\mathcal{F}(U \times V)/\pi^i p_r^{-1}\mathcal{F}(U \times V) \\
&\cong \varprojlim \mathcal{F}(V)/\pi^i\mathcal{F}(V) \\
&\cong \hat{\mathcal{F}}(V) \\
&\cong p_r^{-1}\hat{\mathcal{F}}(U \times V).
\end{aligned} \tag{5.28}$$

Now let $(U_i)_{i\in I}$ be an affine open covering of Y. Then $\{U_j \times U_k | j, k \in I\}$ is an affine open covering of $Y \times Y$. By equation (5.28), we have that the sheaves $\widehat{p_r^{-1}\mathcal{F}}$ and $p_r^{-1}\hat{\mathcal{F}}$ agree on affine open cover and since there exists a map between the two, they are isomorphic. $\square$

5.5.2 Simplyfing the pullback functor

We retain the notation from the previous section. Further, for the rest of the section we assume $\lambda, \mu : \pi^n\mathfrak{h} \to R$ are R-linear *dominant weights*. Since $\mathcal{D}_n^{\lambda,\mu}$ is a sheaf of π^n-deformed G-htdo by Corollary 4.5.15 and the double flag variety $X \times X$ is quasi-compact we have by Theorem 4.3.5, Proposition 5.3.9 and Theorem 5.3.20 equivalences of categories:

$$\begin{aligned} i_l^{\#} &: \mathrm{Coh}(\mathcal{D}_n^{\lambda,\mu}, G) \to \mathrm{Coh}(\mathcal{D}_n^{\lambda}, B), \\ \hat{i}_l^{\#} &: \mathrm{Coh}(\widehat{\mathcal{D}_n^{\lambda,\mu}}, G) \to \mathrm{Coh}(\widehat{\mathcal{D}_n^{\lambda}}, B), \\ \hat{i}_{l,K}^{\#} &: \mathrm{Coh}(\widehat{\mathcal{D}_{n,K}^{\lambda,\mu}}, G) \to \mathrm{Coh}(\widehat{\mathcal{D}_{n,K}^{\lambda}}, B). \end{aligned} \tag{5.29}$$

We let $i : eB \to X$ and $p_r : X \times X \to X$ denote the natural inclusion and projection onto the right factor, respectively. We also define $\mathcal{M}_n^{\mu} := i_* R \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_n^{\mu}$.

We can now give a description of the pullback of G-equivariant coherent $\widehat{\mathcal{D}_n^{\lambda,\mu}}$-modules and $\widehat{\mathcal{D}_{n,K}^{\lambda,\mu}}$-modules. We start by proving π-adic and affinoid versions of Corollary 4.6.4.

Proposition 5.5.3. *Let* $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}_n^{\lambda,\mu}}, G)$*. Then:*

$$i_{l_*}(\hat{i_l}^{\#}\mathcal{M}) \cong p_r^{-1}\widehat{\mathcal{M}_n^{\mu}} \underset{p_r^{-1}\widehat{\mathcal{D}_n^{\mu}}}{\otimes} \mathcal{M}.$$

Proof. We know by construction that $\mathcal{M}/\pi^i\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_n^{\lambda,\mu}, G)$. We have

$$\begin{aligned} i_{l_*}(\hat{i_l}^{\#}\mathcal{M}) &= i_{l_*}(\varprojlim i_l^{\#}(\mathcal{M}/\pi^i\mathcal{M})) \\ &\cong \varprojlim(i_{l_*}(i_l^{\#}(\mathcal{M}/\pi^i\mathcal{M}))) \text{ (by Lemma 5.5.1)} \\ &\cong \varprojlim(p_r^{-1}\mathcal{M}_n^{\mu} \underset{p_r^{-1}\mathcal{D}_n^{\mu}}{\otimes} \mathcal{M}/\pi^i\mathcal{M}) \text{ (by Corollary 4.6.4)} \\ &\cong \varprojlim(p_r^{-1}\mathcal{M}_n^{\mu}/\pi^i p_r^{-1}\mathcal{M}_n^{\mu} \underset{p_r^{-1}\mathcal{D}_n^{\mu}/\pi^i p_r^{-1}\mathcal{D}_n^{\mu}}{\otimes} \mathcal{M}/\pi^i\mathcal{M}) \\ &\cong \widehat{p_r^{-1}\mathcal{M}_n^{\mu}} \underset{\widehat{p_r^{-1}\mathcal{D}_n^{\mu}}}{\otimes} \mathcal{M} \text{ (by Lemma 5.4.4)} \\ &\cong p_r^{-1}\widehat{\mathcal{M}_n^{\mu}} \underset{p_r^{-1}\widehat{\mathcal{D}_n^{\mu}}}{\otimes} \mathcal{M} \text{ (by Lemma 5.5.2).} \end{aligned}$$

□

Corollary 5.5.4. *Let* $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}^{\lambda,\mu}_{n,K}}, G)$*. Then:*

$$i_{l_*}(\hat{i}^{\#}_{l,K}\mathcal{M}) \cong (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}}}{\otimes} \mathcal{M}.$$

Proof. Let $\mathcal{M}_0$ be the correspoding lattice for $\mathcal{M}$ such that $\mathcal{M}_0 \in \mathrm{Coh}(\widehat{\mathcal{D}^{\lambda,\mu}_{n}}, G)$. By definition $(\hat{i}^{\#}_{l,K}\mathcal{M}) = (\hat{i_l}^{\#}\mathcal{M}_0)_K$. We have

$$\begin{aligned}
i_{l_*}(\hat{i}^{\#}_{l,K}\mathcal{M}) &= i_{l_*}((\hat{i_l}^{\#}\mathcal{M}_0)_K) \\
&\cong (i_{l_*} i_l^{\#}\mathcal{M}_0)_K \\
&\cong (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}} \underset{p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n}}}{\otimes} \mathcal{M}_0)_K \text{ (by Proposition 5.5.3)} \\
&\cong (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{(p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n}})_K}{\otimes} (\mathcal{M}_0)_K \\
&\cong (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{(p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n}})_K}{\otimes} \mathcal{M} \\
&\cong (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}}}{\otimes} \mathcal{M}.
\end{aligned}$$

$\square$

We should remark that the argument above proves that the functor $\hat{i}^{\#}_{l,K}$ is well defined, i.e. it does not depend on the lattice of $\mathcal{M}$.

5.5.3 Computation of global sections

We aim to compute $\Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{M})$ for $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}^{\lambda,\mu}_{n,K}}, G)$. Recall that $Z = X \times X$ denotes the double flag variety. We start by making some notations.

To simplify the notation, we denote $\mathcal{A} := \Gamma(X, \widehat{\mathcal{D}^{\mu}_{n}})$, so that $\mathcal{A}_K = \mathcal{A} \underset{R}{\otimes} K \cong \Gamma(X, \widehat{\mathcal{D}^{\mu}_{n,K}})$. Further, we let $\mathcal{B} := \Gamma(X, \widehat{\mathcal{D}^{\lambda,\mu}_{n}})$. We have by combining Proposition 5.4.16 and [5, Theorem 6.10b)] that $\widehat{U(\mathfrak{g} \times \mathfrak{g})^{\lambda,\mu}_{n,K}} \cong \mathcal{B}_K \cong \widehat{U(\mathfrak{g})^{\lambda}_{n,K}} \underset{K}{\hat{\otimes}} \mathcal{A}_K$. Given a $\mathcal{B}_K$-module M, we may view it as $\mathcal{A}_K$-module via $x.m = (1 \otimes x).m$ for $x \in \mathcal{A}_K$ and $m \in M$.

Proposition 5.5.5. *Let* $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}^{\lambda,\mu}_{n,K}}, G)$*. Then as* $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$*-modules we have an isomorphism:*

$$\Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{M}) \cong \Gamma(X, \widehat{\mathcal{M}^{\mu}_{n,K}}) \underset{\mathcal{A}_K}{\otimes} \Gamma(Z, \mathcal{M}).$$

Proof. We have $\Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{M}) \cong \Gamma(Z, i_{l_*}(\hat{i}^{\#}_{l,K}\mathcal{M}))$, so using Corollary 5.5.4, it is enough to compute $\Gamma(Z, (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}}}{\otimes} \mathcal{M})$.

We have by the first part of Lemma 4.6.2 that there is an exact sequence $(\mathcal{D}^{\mu}_{n})^a \to (\mathcal{D}^{\mu}_{n})^b \to \mathcal{M}^{\mu}_{n} \to 0$ for some $a, b \in \mathbb{N}^*$. Since π-adic completion is exact on coherent modules, we obtain an exact sequence $(\widehat{\mathcal{D}^{\mu}_{n}})^a \to (\widehat{\mathcal{D}^{\mu}_{n}})^b \to \widehat{\mathcal{M}^{\mu}_{n}} \to 0$, so an exact sequence

$$(\widehat{\mathcal{D}^{\mu}_{n,K}})^a \to (\widehat{\mathcal{D}^{\mu}_{n,K}})^b \to \widehat{\mathcal{M}^{\mu}_{n,K}} \to 0.$$

Since $p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n,K}} \cong (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K$ and tensor product is right exact we obtain an exact sequence:

$$(p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}})^a \underset{p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}}}{\otimes} \mathcal{M} \to (p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}})^b \underset{p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}}}{\otimes} \mathcal{M} \to (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}}}{\otimes} \mathcal{M} \to 0.$$

To simplify the notation, we let $\mathcal{E} = p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}}$ and $M = \Gamma(Z, \mathcal{M})$. The above short sequence fits into the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^a \underset{\mathcal{E}}{\otimes} \mathcal{M} \arrow[r] \arrow[d] & \mathcal{E}^b \underset{\mathcal{E}}{\otimes} \mathcal{M} \arrow[r] \arrow[d] & (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{\mathcal{E}}{\otimes} \mathcal{M} \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{M}^a \arrow[r] & \mathcal{M}^b \arrow[r] & (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{p_r^{-1}\widehat{\mathcal{D}^{\mu}_{n,K}}}{\otimes} \mathcal{M} \arrow[r] & 0.
\end{tikzcd}$$

Since $\Gamma(Z, -)$ is exact on coherent modules by Theorem 5.4.7 we obtain a commutative diagram:

$$\begin{tikzcd}
\Gamma(Z, \mathcal{E}^a \underset{\mathcal{E}}{\otimes} \mathcal{M}) \arrow[r] \arrow[d] & \Gamma(Z, \mathcal{E}^b \underset{\mathcal{E}}{\otimes} \mathcal{M}) \arrow[r] \arrow[d] & \Gamma(Z, (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{\mathcal{E}}{\otimes} \mathcal{M}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Gamma(Z, \mathcal{M}^a) \arrow[r] \arrow[d] & \Gamma(Z, \mathcal{M}^b) \arrow[r] \arrow[d] & \Gamma(Z, (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{\mathcal{E}}{\otimes} \mathcal{M}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
M^a \arrow[r] \arrow[d] & M^b \arrow[r] \arrow[d] & \Gamma(Z, (p_r^{-1}\widehat{\mathcal{M}^{\mu}_{n}})_K \underset{\mathcal{E}}{\otimes} \mathcal{M}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\Gamma(X, \widehat{\mathcal{D}^{\mu}_{n,K}})^a \underset{\mathcal{A}_K}{\otimes} M \arrow[r] & \Gamma(X, \widehat{\mathcal{D}^{\mu}_{n,K}})^b \underset{\mathcal{A}_K}{\otimes} M \arrow[r] & \Gamma(X, \widehat{\mathcal{M}^{\mu}_{n,K}}) \underset{\mathcal{A}_K}{\otimes} M \arrow[r] & 0.
\end{tikzcd}$$

By construction, we have that the vertical arrows on the first, second and fourth columns are isomorphisms. Considering the first and fourth row and applying the Five Lemma, we get the desired isomorphism. $\square$

5.5.4 Global sections of $\widehat{\mathcal{M}^{\lambda}_{n,K}}$

Recall that $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$ denotes the negative Borel subalgebra of $\mathfrak{g}$. Let $T(\mu)_0 = R_\mu \underset{U(\mathfrak{b}^-)_n}{\otimes} U(\mathfrak{g})_n$ denote the right $U(\mathfrak{g})_n$-module such that $U(\mathfrak{b}^-)_n$ acts on R via μ. Further, we let $T(\mu) := T(\mu)_0 \underset{R}{\otimes} K \cong K_\mu \underset{U(\mathfrak{b}^-)_K}{\otimes} U(\mathfrak{g})_K$ and $\widehat{T(\mu)} := \widehat{T(\mu)_0} \underset{R}{\otimes} K \cong T(\mu) \underset{U(\mathfrak{g})_K}{\otimes} \widehat{U(\mathfrak{g})_{n,K}}$. We should remark that $T(\mu)$ is the same as defined in Lemma 4.6.2.

Proposition 5.5.6. *We have* $\Gamma(X, \widehat{\mathcal{M}^{\mu}_{n,K}}) \cong \widehat{T(\mu)}$ *as right* $\widehat{U(\mathfrak{g})_{n,K}}$*-modules.*

Proof. We have by construction that $\Gamma(X, \mathcal{M}^{\mu}_{n}) \underset{R}{\otimes} K \cong \Gamma(X_K, \mathcal{M}^{\mu}_{n,K|X_K})$ and by Lemma 4.6.2 that $\Gamma(X_K, \mathcal{M}^{\mu}_{n,K|X_K}) \cong T(\mu)$. Therefore

$$\Gamma(X, \mathcal{M}^{\mu}_{n}) \underset{R}{\otimes} K \cong T(\mu)_0 \underset{R}{\otimes} K \cong T(\mu).$$

By [5, Proposition 5.15b)], $\Gamma(X, \mathcal{M}^{\mu}_{n})$ is finitely generated as a $U(\mathfrak{g})_n$-module, so $\Gamma(X, \mathcal{M}^{\mu}_{n})$ and $T(\mu)_0$ are both lattices for $T(\mu)$. Therefore, they agree modulo bounded π-torsion, i.e. there exists an exact sequence

$$T(\mu)_0 \to \Gamma(X, \mathcal{M}^{\mu}_{n}) \to C \to 0,$$

such that $\pi^m C = 0$ for some $m \in \mathbb{N}$ and C is a finitely generated $U(\mathfrak{g})_n$-module. Since, π-adic completion is exact on finitely generated $U(\mathfrak{g})_n$-modules we get an exact sequence

$$\widehat{T(\mu)_0} \to \widehat{\Gamma(X, \mathcal{M}^{\mu}_{n})} \to \widehat{C} \to 0,$$

with $\pi^m \widehat{C} = 0$. So by tensoring with K we obtain

$$\widehat{T(\mu)} = \widehat{T(\mu)_0} \underset{R}{\otimes} K \cong \widehat{\Gamma(X, \mathcal{M}^{\mu}_{n})} \underset{R}{\otimes} K \cong \Gamma(X, \widehat{\mathcal{M}^{\mu}_{n,K}}).$$

$\square$

We may now prove the main theorem of this section, an affinoid version of Theorem 4.6.5.

Theorem 5.5.7. *Let $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}_{n,K}^{\lambda,\mu}}, G)$. Then as $\widehat{U(\mathfrak{g})_{n,K}^{\lambda}}$-modules we have:*

$$\Gamma(X, \hat{i}_{l,K}^{\#}\mathcal{M}) \cong \widehat{T(\mu)} \underset{\widehat{U(\mathfrak{g})_{n,K}^{\mu}}}{\otimes} \Gamma(Z, \mathcal{M}).$$

Proof. This follows by combining Corollary 5.4.8, Proposition 5.5.6 and [5, Theorem 6.10b)]. □

5.6 Affinoid Duflo's theorem

Throughout this section $\lambda : \pi^n\mathfrak{h} \to R$ denotes an R-linear dominant weight. Recall that we use λ^* to denote the weight $-w_o\lambda$, where w_o is the longest element of the Weyl group. If λ is a dominant weight, so is λ^*. To prove an affinoid version of Duflo's theorem, we follow the same lines as in Section 4.7 at the affinoid level. We consider the functor

$$\begin{aligned}
&\mathscr{F} : \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g}\times\mathfrak{g})_{n,K}^{\lambda^*,\lambda}}, G) \to \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g})_{n,K}^{\lambda^*}}, B),\\
&\mathscr{F}(M) := \Gamma(X, \hat{i}_{l,K}^{\#}\,\mathrm{Loc}^{\lambda}(M)).
\end{aligned}$$

Proposition 5.6.1. *The functor $\mathscr{F}$ is exact and $\mathscr{F}(M) \cong \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})_{n,K}^{\lambda}}}{\otimes} M$ as $\widehat{U(\mathfrak{g})_{n,K}^{\lambda^*}}$-modules for $M \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g}\times\mathfrak{g})_{n,K}^{\lambda^*,\lambda}}, G)$.*

Proof. Let $M \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g}\times\mathfrak{g})_{n,K}^{\lambda^*,\lambda}}, G)$ and $\mathcal{M} := \mathrm{Loc}^{\lambda^*,\lambda}(M)$. We have by Corollary 5.4.9 that $\Gamma(Z, \mathcal{M}) \cong M$ and by Theorem 5.4.8 that $\mathcal{M} \in \mathrm{Coh}(\widehat{\mathcal{D}_{n,K}^{\lambda^*,\lambda}}, G)$, so the second claim follows from Theorem 5.5.7. Consider a short exact sequence in $\mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g}\times\mathfrak{g})_{n,K}^{\lambda^*,\lambda}}, G)$:

$$0 \to N \to M \to P \to 0.$$

We let $\mathcal{N}, \mathcal{M}, \mathcal{P}$ denote the localisation of N, M and P respectively. Further, we denote $\mathcal{L} := \ker(\mathcal{N} \to \mathcal{M})$. By construction, $\mathrm{Loc}^{\lambda^*,\lambda}$ is right exact, so we obtain an exact sequence:

$$0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{M} \to \mathcal{P} \to 0.$$

Since $\hat{i}^{\#}_{l,K}$ is an equivalence of Abelian categories, it is exact. Furthermore, by Theorem 5.4.8 the global sections functor is also exact, so we obtain an exact sequence

$$0 \to \Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{L}) \to \Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{N}) \to \Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{M}) \to \Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{P}) \to 0.$$

Combining Theorem 5.5.7 and Corollary 5.4.9 we obtain an exact sequence

$$\begin{aligned} 0 \to \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}}{\otimes} \Gamma(Z, \mathcal{L}) \to \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}}{\otimes} N \\ \to \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}}{\otimes} M \to \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}}{\otimes} P \to 0. \end{aligned} \tag{5.30}$$

The claim follows since $\Gamma(Z, \mathcal{L}) = 0$ by definition of $\mathcal{L}$ and Corollary 5.4.9. ☐

Lemma 5.6.2. *Let* $M \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g}\times\mathfrak{g})^{\lambda^*,\lambda}_{n,K}}, G)$ *and assume* $\mathscr{F}(M) = 0$. *Then* $M = 0$.

Proof. Let $\mathcal{M} := \mathrm{Loc}^{\lambda^*,\lambda}(M)$. Then, by assumption, we have that $\Gamma(X, \hat{i}^{\#}_{l,K}\mathcal{M}) = 0$. By applying Corollary 5.4.9 and Corollary 5.3.22, we obtain $M = 0$. ☐

We now specialise to two sided ideals in $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$; recall that a two-sided ideal I can be viewed as a module over $\widehat{U(\mathfrak{g})^{\lambda^*}_{n,K}} \otimes \widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$ via $(x \otimes y).i = yi\tau(x)$ for $x \in \widehat{U(\mathfrak{g})^{\lambda^*}_{n,K}}, y \in \widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$ and $i \in I$. Further, by Corollary 5.4.15, we have $I \in \mathrm{Mod}_{\mathrm{fg}}(\widehat{U(\mathfrak{g}\times\mathfrak{g})^{\lambda^*,\lambda}_{n,K}}, G)$, so $\mathscr{F}(I)$ is well-defined. As a corollary, we obtain immediately:

Corollary 5.6.3. *Let* I, J *be two-sided ideals in* $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$ *such that* $I \subseteq J$. *Assume that* $\mathscr{F}(I) \cong \mathscr{F}(J)$. *Then* $I = J$.

Proof. Consider the short exact sequence:

$$0 \to I \to J \to J/I \to 0.$$

By Proposition 5.6.1 the functor $\mathscr{F}$ is exact, so we obtain an exact sequence

$$0 \to \mathscr{F}(I) \to \mathscr{F}(J) \to \mathscr{F}(J/I) \to 0.$$

Using the assumption, we obtain $\mathscr{F}(J/I) = 0$. The claim follows by Lemma 5.6.2. ☐

Corollary 5.6.4. *Let I be a two-sided ideal in $\widehat{U(\mathfrak{g})}_{n,K}^{\lambda}$. Then as left $\widehat{U(\mathfrak{g})}_{n,K}^{\lambda^*}$-modules we have*

$$\mathscr{F}(I) \cong \widehat{T(\lambda)}I.$$

We should remark that $\widehat{U(\mathfrak{g})}_{n,K}^{\lambda^*}$ acts on $\widehat{T(\lambda)}I$ via $x.(ti) = t(x.i) = ti\tau(x)$ for $x \in \widehat{U(\mathfrak{g})}_{n,K}^{\lambda^*}, t \in \widehat{T(\lambda)}$ and $i \in I$. Further, the isomorphism is natural in I.

Proof. Consider the following exact sequence:

$$0 \to I \to \widehat{U(\mathfrak{g})}_{n,K}^{\lambda} \to \widehat{U(\mathfrak{g})}_{n,K}^{\lambda}/I \to 0.$$

Applying Proposition 5.6.1 we obtain a short exact sequence:

$$0 \to \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})}_{n,K}^{\lambda}}{\otimes} I \to \widehat{T(\lambda)} \to \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})}_{n,K}^{\lambda}}{\otimes} \widehat{U(\mathfrak{g})}_{n,K}^{\lambda}/I \to 0.$$

This short exact sequence fits in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})}_{n,K}^{\lambda}}{\otimes} I \arrow[r] \arrow[d] & \widehat{T(\lambda)} \arrow[r] \arrow[d] & \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})}_{n,K}^{\lambda}}{\otimes} \widehat{U(\mathfrak{g})}_{n,K}^{\lambda}/I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{T(\lambda)}I \arrow[r] & \widehat{T(\lambda)} \arrow[r] & \widehat{T(\lambda)} \underset{\widehat{U(\mathfrak{g})}_{n,K}^{\lambda}}{\otimes} \widehat{U(\mathfrak{g})}_{n,K}^{\lambda}/I \arrow[r] & 0.
\end{tikzcd}$$

It is easy to see that the first map is a surjection and the second and the third maps are isomorphisms. Furthermore, by the diagram above, the first map is also injective, so indeed we get $\mathscr{F}(I) \cong \widehat{T(\lambda)}I$. ☐

Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ denote the Chevalley involution that swaps $\mathfrak{n}^+$ and $\mathfrak{n}^-$ and fixes $\mathfrak{h}$. Then σ extends to an anti-automorphism of $U(\mathfrak{g})_K$ that fixes the center by [34, Exercise 1.10]. Therefore, we obtain that an anti-automorphism $\hat{\sigma} : \widehat{U(\mathfrak{g})}_{n,K}^{\lambda} \to \widehat{U(\mathfrak{g})}_{n,K}^{\lambda}$. Recall that $T(\lambda)_0 = R_\lambda \underset{U(\mathfrak{b}^-)_n}{\otimes} U(\mathfrak{g})$, so that $\widehat{T(\lambda)} \cong K_\lambda \underset{\widehat{U(\mathfrak{b}^-)_{n,K}}}{\otimes} \widehat{U(\mathfrak{g})}_{n,K}$.

Lemma 5.6.5. *The map*

$$\hat{\phi} : \widehat{T(\lambda)} \to \widehat{M(\lambda)}, \qquad \hat{\phi}(k \otimes x) = \hat{\sigma}(x) \otimes k, \qquad k \in K_\lambda, x \in \widehat{U(\mathfrak{g})_{n,K}}$$

is a K-linear isomorphism of vector spaces satisfying $\hat{\phi}(tu) = \hat{\sigma}(u)\hat{\phi}(t)$ for all $u \in \widehat{U(\mathfrak{g})_{n,K}}$ and $t \in \widehat{T(\lambda)}$. In particular, if I is a two-sided ideal in $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$, then $\hat{\phi}(\widehat{T(\lambda)}I) = \hat{\sigma}(I)\widehat{M(\lambda)}$.

Proof. We have by Lemma 4.7.6 that the map $\phi : T(\lambda) \to M(\lambda)$, $\phi(k \otimes x) = \sigma(x) \otimes K$ is a K-linear isomorphism of vector spaces satisfying $\phi(tu) = \sigma(u)\phi(t)$ for all $t \in T(\lambda)$ and $u \in U(\mathfrak{g})_K$. Therefore, the claims follow from the construction of $\hat{\phi}$ and $\hat{\sigma}$. □

To prove the main theorem, we will need the following corollary:

Corollary 5.6.6. *Let I be a two-sided ideal in $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$. Then:*

$$I = \mathrm{Ann}(\widehat{M(\lambda)}/I\widehat{M(\lambda)}).$$

Proof. Let $J := \mathrm{Ann}(\widehat{M(\lambda)}/I\widehat{M(\lambda)})$. Since $I(\widehat{M(\lambda)}/I\widehat{M(\lambda)}) = 0$, we obtain that $I \subseteq J$, so $\hat{\sigma}(I) \subseteq \hat{\sigma}(J)$. We remark that since $\hat{\sigma}$ is an anti-automorphism of $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$, $\hat{\sigma}(I)$ and $\hat{\sigma}(J)$ are also two-sided ideals in $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$. Since $\mathscr{F}$ is exact, in particular left exact, we obtain $\mathscr{F}(\hat{\sigma}(I)) \subseteq \mathscr{F}(\hat{\sigma}(J))$. Consider the following diagram:

$$\begin{tikzcd}
\mathscr{F}(\hat{\sigma}(I)) \arrow[r] \arrow[d] & \mathscr{F}(\hat{\sigma}(J)) \arrow[d] \\
\widehat{T(\lambda)}\hat{\sigma}(I) \arrow[r] \arrow[d, "\hat{\phi}"] & \widehat{T(\lambda)}\hat{\sigma}(J) \arrow[d, "\hat{\phi}"] \\
I\widehat{M(\lambda)} \arrow[r] & J\widehat{M(\lambda)}
\end{tikzcd}$$

By construction, the bottom diagram commutes and by the definition of J, we have $J\widehat{M(\lambda)} = I\widehat{M(\lambda)}$. Using Lemma 5.6.5, we get $\widehat{T(\lambda)}\hat{\sigma}(I) = \widehat{T(\lambda)}\hat{\sigma}(J)$. Furthermore, since the isomorphism is Lemma 5.6.4 is natural on ideals, the top diagram is also commutative. Therefore, $\mathscr{F}(\hat{\sigma}(I)) \cong \mathscr{F}(\hat{\sigma}(J))$. The claim follows from Corollary 5.6.3 since $\hat{\sigma}$ is an anti-automorphism. □

Theorem 5.6.7. *Let I be a prime ideal in $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$. Then*

$$I = \mathrm{Ann}(\widehat{L(\mu)}) \textit{ for some } \mu : \pi^n\mathfrak{h} \to R.$$

Proof. Since $\widehat{M(\lambda)}/I\widehat{M(\lambda)}$ is a quotient of $\widehat{M(\lambda)}$, we have by Proposition 5.2.29 that there exists a finite composition series:

$$0 = M_0 \subset M_1 \subset \dots M_l = \widehat{M(\lambda)}/I\widehat{M(\lambda)}.$$

Let $I_i = \operatorname{Ann}(M_i/M_{i-1})$ for $1 \leq i \leq l$. We have

$$I_1 I_2 \dots I_l M_l = I_1 I_2 \dots I_{l-1} M_{l-1} = \dots = 0,$$

so $I_1 I_2 \dots I_l \subset \operatorname{Ann}(\widehat{M(\lambda)}/I\widehat{M(\lambda)}) = I$ by Corollary 5.6.6. Since I is prime, there exists $1 \leq j \leq l$ such that $I_j \subset I$. On the other hand, we have by construction $I \subset I_j$, so $I = I_j = \operatorname{Ann}(M_j/M_{j-1})$. Finally, we have by Proposition 5.2.29 that $M_j/M_{j-1} \cong \widehat{L(\mu)}$ for some $\mu : \pi^n \mathfrak{h} \to R$. ☐

As an easy corollary, we obtain a positive answer to [6, Question A], thus completing the proof of Theorem B.

Corollary 5.6.8. *Every primitive ideal of $\widehat{U(\mathfrak{g})_{n,K}}$ with K-rational infinitesimal central character is the annihilator of a simple affinoid highest weight module.*

Proof. Any primitive ideal in $\widehat{U(\mathfrak{g})_{n,K}}$ with K-rational infinitesimal central character intersects $Z(\widehat{U(\mathfrak{g})_{n,K}})$ in a maximal ideal of the form $\ker \chi_\lambda$; here we view $\ker \chi_\lambda$ as a central character of $\widehat{U(\mathfrak{g})_{n,K}}$ via Theorem 5.2.4. Therefore, classifying these ideals reduces to classifying the ideals in $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}}$ for all $\lambda \in \pi^n \mathfrak{h}^*$. There is an action of the Weyl group W on the set of weights such that for two weights λ and μ, $\widehat{U(\mathfrak{g})^{\lambda}_{n,K}} = \widehat{U(\mathfrak{g})^{\mu}_{n,K}}$ if and only if λ and μ are W-conjugate. Further, every W-conjugacy class contains at least one dominant weight. The claim follows by Theorem 5.6.7 since every primitive ideal is prime. ☐

We should remark that in the case p is a very good prime for G, we have by Theorem 5.2.6 that the ideals $\{\hat{I}_\lambda = \operatorname{Ann}(\widehat{M(\lambda)}) | \lambda \in \pi^n \mathfrak{h}^*\}$ form the set of minimal primitive ideals with K-rational central character.

5.7 A controller theorem

We keep the notations and assumptions from the previous section; we further assume that $\lambda : \pi^n \mathfrak{h} \to R$ is also *regular*. We will also make use of the fact that two-sided ideals in $\widehat{U(\mathfrak{g})_{n,K}}$ that contain $\widehat{U(\mathfrak{g})_{n,K}} \ker \chi_\lambda$ correspond to two-sided ideals in $\widehat{U(\mathfrak{g})^\lambda_{n,K}}$.

Lemma 5.7.1. *Let I be a two-sided ideal in $U(\mathfrak{g}_K)$. Then $\widehat{U(\mathfrak{g})_{n,K}}I$ is a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}I$ and furthermore, $\widehat{U(\mathfrak{g})_{n,K}}I = \widehat{U(\mathfrak{g})_{n,K}}I\widehat{U(\mathfrak{g})_{n,K}}$.*

Proof. Clearly, it is enough to prove that $I\widehat{U(\mathfrak{g})_{n,K}} \subset \widehat{U(\mathfrak{g})_{n,K}}I$.

Viewing $\widehat{U(\mathfrak{g})_{n,K}}$ as a left $\widehat{U(\mathfrak{g})_{n,K}}$-module via left multiplication, we have that $\widehat{U(\mathfrak{g})_{n,K}}I$ is a $\widehat{U(\mathfrak{g})_{n,K}}$-submodule. Since the topology $\widehat{U(\mathfrak{g})_{n,K}}$ is complete, we have by [33, I.5.5] that $\widehat{U(\mathfrak{g})_{n,K}}I$ is a closed subset.

Let $i \in I$ and $x \in \widehat{U(\mathfrak{g})_{n,K}}$. Recall that assuming that $u_1, u_2 \dots u_d$ is a free R-basis for $\mathfrak{g}$, we may write as $x = \sum_{\alpha \in \mathbb{N}^d} c_\alpha u^\alpha$, with $||p^{-n} c_\alpha|| \to 0$ as $|\alpha| \to \infty$. For $k \in \mathbb{N}$, let $x_k = \sum_{\alpha \in \mathbb{N}^d, |\alpha| \leq k} c_\alpha u^\alpha$. Since I is a two-sided ideal, we obtain that $ix_k \in I \subset \widehat{U(\mathfrak{g})_{n,K}}I$.

Finally, we have $ix = i \lim_{k\to\infty} x_k = \lim_{k\to\infty} ix_k$. Since $\widehat{U(\mathfrak{g})_{n,K}}I$ is closed, we obtain $ix \in \widehat{U(\mathfrak{g})_{n,K}}I$ finishing the proof. ☐

Proposition 5.7.2. *Let I be a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$ such that $\widehat{U(\mathfrak{g})_{n,K}} \ker \chi_\lambda \subset I$. Then there exists an unique two-sided ideal J in $U(\mathfrak{g}_K)$ such that $I = \widehat{U(\mathfrak{g})_{n,K}}J$.*

Proof. Let $\hat{M} = I\widehat{M(\lambda)}$. We have by Theorem 5.2.25 that there exists M a submodule of $M(\lambda)$ such that $\hat{M} = \widehat{U(\mathfrak{g})_{n,K}}.M$. Further, we have by [13, Theorem 4.3] that there exists an unique two-sided ideal J in $U(\mathfrak{g}_K)$ such that $JM(\lambda) = M$. By applying Lemma 5.7.1 we obtain:

$$\begin{aligned}(\widehat{U(\mathfrak{g})_{n,K}}J).\widehat{M(\lambda)} &= (\widehat{U(\mathfrak{g})_{n,K}}J)\widehat{U(\mathfrak{g})_{n,K}}.M(\lambda) \\ &= (\widehat{U(\mathfrak{g})_{n,K}}J).M(\lambda) = \widehat{U(\mathfrak{g})_{n,K}}.M = \hat{M},\end{aligned} \tag{5.31}$$

so $(\widehat{U(\mathfrak{g})_{n,K}}J)\widehat{M(\lambda)} = I\widehat{M(\lambda)}$. Let $J' = \widehat{U(\mathfrak{g})_{n,K}}J + I$, so that $J'\widehat{M(\lambda)} = I\widehat{M(\lambda)}$ and $I \subset J'$. We have by Lemma 5.7.1 that J' is also a two-sided ideal. Further, by combining Corollary 5.6.3, Corollary 5.6.4 and Lemma 5.6.5 we get $J' = I$, so $\widehat{U(\mathfrak{g})_{n,K}}J \subset I$. Applying the same strategy again, we obtain $\widehat{U(\mathfrak{g})_{n,K}}J = I$. ☐

To finish the proof, we need one more lemma. This is probably well-known among the experts, but we have not been able to locate a reference.

Lemma 5.7.3. *Let $S \subset T$ two rings and let I be a left ideal of T generated by $X \subset S$. Then $I = T(I \cap S)$.*

Proof. Let $J = I \cap S$. Obviously, we have $TJ \subset I$. On the other hand, we have $X \subset S$, $X \subset I$, so $X \subset J$. Therefore, $I = T.X \subset TJ$. The claim follows. $\square$

Theorem 5.7.4. *Let $\lambda : \pi^n \mathfrak{h} \to R$ be a R-linear dominant regular weight. Let I be a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$ with χ_λ-central character. Then:*

$$I = \widehat{U(\mathfrak{g})_{n,K}}(I \cap U(\mathfrak{g}_K)).$$

Proof. This follows immediately from Proposition 5.7.2 and Lemma 5.7.3. $\square$

As a corollary, we obtain immediately:

Corollary 5.7.5. *Let $\lambda : \pi^n \mathfrak{h} \to R$ be a R-linear dominant regular weight. The maps:*

$$\begin{aligned} I &\mapsto I \cap U(\mathfrak{g}_K) \\ J &\mapsto \widehat{U(\mathfrak{g})_{n,K}} J \end{aligned} \tag{5.32}$$

induce inverse bijections between the set of two sided ideals in $\widehat{U(\mathfrak{g})_{n,K}}$ with χ_λ central character and the set of two sided ideals in $\widehat{U(\mathfrak{g})_{n,K}}$ with χ_λ central character.

We may also prove which ideal controls the annihilator of the simple affinoid module $\widehat{L(\mu)}$.

Proposition 5.7.6. *Let $\mu : \pi^n \mathfrak{h} \to R$ and assume that μ is W-linked to λ. Then:*

$$\operatorname{Ann}(\widehat{L(\mu)}) = \widehat{U(\mathfrak{g})_{n,K}} \operatorname{Ann} L(\mu).$$

Proof. Let $\hat{P} := \operatorname{Ann}(\widehat{L(\mu)})$, $P = \operatorname{Ann}(L(\mu))$ and $J = \hat{P} \cap U(\mathfrak{g}_K)$. Then, we have by construction $P \subset J$.

We claim that $\widehat{U(\mathfrak{g})_{n,K}} P$ contains the annihilator of $\widehat{L(\mu)}$. We have by the proof of Lemma 5.7.1 that for all $p \in P$ and $x \in \widehat{U(\mathfrak{g})_{n,K}}$, there exist $q \in P$ and $y \in \widehat{U(\mathfrak{g})_{n,K}}$

such that $px = yq$. Therefore, we have for all $z \in \widehat{U(\mathfrak{g})_{n,K}}$ and $x \otimes l \in \widehat{U(\mathfrak{g})_{n,K}} \underset{U(\mathfrak{g}_K)}{\otimes} L$ that

$$zp.(x \otimes l) = z(px \otimes l) = zy(q \otimes l) = zy(1 \otimes q.l) = 0,$$

so the claim is proven. Therefore, we obtain using Corollary 5.7.5 that $P = J$ and so, $\hat{P} = \widehat{U(\mathfrak{g})_{n,K}} P$. □

Let $\widehat{m_\lambda}$ and m_λ be the corresponding ideals generated by $\ker \chi_\lambda$ in $\widehat{U(\mathfrak{g})_{n,K}}$ and $U(\mathfrak{g})_K$, respectively. As a corollary, we obtain

Corollary 5.7.7. *Let $\lambda : \pi^n \mathfrak{h} \to R$ be a R-linear dominant regular weight. The map $\psi : \operatorname{Spec}(\widehat{U(\mathfrak{g})^\lambda_{n,K}}) \to \operatorname{Spec}(U(\mathfrak{g})^\lambda_K)$, $\psi(P + \widehat{m_\lambda})) = (P \cap U(\mathfrak{g})_K) + m_\lambda$ is a homeomorphism with inverse $\psi^{-1} : \operatorname{Spec}(U(\mathfrak{g})^\lambda_K) \to \operatorname{Spec}(\widehat{U(\mathfrak{g})^\lambda_{n,K}})$, $\psi^{-1}(Q + m_\lambda) = (\widehat{U(\mathfrak{g})^\lambda_{n,K}} Q) + \widehat{m_\lambda}$.*

Proof. This follows by combining Theorem 5.6.7, Corollary 5.7.5 and Proposition 5.7.6.

□

Question 5.7.8. *Does the result of Theorem 5.7.4 hold if we drop the assumption that λ is regular? What about if we restrict ourselves to just prime ideals?*

Chapter 6

Affinoid Quillen's Lemma

The goal of this chapter is to prove a more general version of affinoid Quillen's Lemma [5, Theorem C]. The original proof works only for the case $n > 0$, while our version will also work in the case $n = 0$. The proof will follow the same lines as the one by Ardakov and Wadsley; we also introduce new techniques to deal with the fact that the slice of $\widehat{U(\mathfrak{g})_K}$, which is isomorphic to $U(\mathfrak{g}_k)$, is not commutative. Using this version of affinoid Quillen's Lemma we are able to characterise all the primitive ideals in $\widehat{U(\mathfrak{g})_{n,K}}$ for all $n \in \mathbb{N}$.

Throughout this chapter, we fix a ring A such that A is a completely filtered K-algebra, F_0A is an R-lattice in A and the slice $\operatorname{gr}_0 A = F_0A/\pi F_0A$ is an almost commutative Noetherian k-algebra (we call such algebras almost commutative affinoid algebra) such that $\operatorname{gr}_0 A$ is finitely generated as a module over a central subalgebra Z.

6.1 Introduction

Let M be a finitely generated A module and $\varphi \in \operatorname{End}_A(M)$. We say that φ is simple if every non-zero element of $K[\varphi]$ acts invertibly on M. We also let $K(\varphi)$ be the subfield of $\operatorname{End}_A(M)$ generated by K and φ.

From now on, we fix M and φ and assume that φ is simple.

Definition 6.1.1. *We say that $N \subset M$ is a* regular $F_0A - \varphi$*-lattice if N is a lattice of M, N is a finitely generated F_0A-module and the subring*

$$B := \{\phi \in K(\varphi) | \phi(N) \subset N\}$$

is a discrete valuation ring and a lattice in $K(\varphi)$.

Definition 6.1.2. *Let S be an integral domain and T an S-algebra. For $f \neq 0$, we denote S_f the localisation of S at powers of f and $T_f := S_f \underset{S}{\otimes} T$.*

We say that T is generically flat over S if for every finitely generated S-module P, there exists $0 \neq f \in S$ such that the module $P_f = T_f \underset{T}{\otimes} P$ is free over S_f.

Lemma 6.1.3. *The ring $\operatorname{gr}_0 A[x]$ is generically flat over $k[x]$.*

Proof. Consider the filtration on $\operatorname{gr}_0 A[x]$ induced by the filtration of $\operatorname{gr}_0 A$. In respect to this filtration, $\operatorname{gr}(\operatorname{gr}_0 A[x]) \cong \operatorname{gr}(\operatorname{gr}_0(A))[x]$, and by our assumption on A, we have $\operatorname{gr}(\operatorname{gr}_0(A))$ is finitely generated k-algebra. Therefore, by [54, Lemma 10.117.2], we get that $\operatorname{gr}(\operatorname{gr}_0 A[x])$ is generically flat over $k[x]$. The claim follows from [43, Proposition 4.9] ☐

Lemma 6.1.4. *Suppose that M has a regular $F_0A - \varphi$-lattice N. Then the residue field of the discrete valuation ring B, k', is an algebraic extension of k.*

Proof. Let τ be a uniformiser of B and note that the residue field $k' := B/\tau B$ of B acts faithfully on $N/\tau N$ by the definition of B. In particular, $N/\tau N$ is non-zero. Since B is an R-lattice in $K(\varphi)$, π is not a unit in B. Hence, $\pi \in \tau B$ and $N/\tau N$ is a finitely generated $\operatorname{gr}_0 A$-module.

Now, let $s \in k'$; then s acts on $N/\tau N$, so we may view $N/\tau N$ as a finitely generated $\operatorname{gr}_0 A[s]$-module. We have by Lemma 6.1.3 that there exists $f \in k[s]$, $f \neq 0$, such that $(N/\tau N)_f$ is a free $k[s]_f$-module. Since $k[s]_f$ is a subring of k' and k' acts invertibly on $N/\tau N$, $k[s]_f$ will also act invertibly on the free $k[s]_f$-module $(N/\tau N)_f$, forcing $k[s]_f$ to be a field. The claim now follows from the Nullstellansatz. ☐

Proposition 6.1.5. *Let the notations and assumptions be as in Lemma 6.1.4. Then k' is a finite extension of k.*

Proof. Since k' acts invertibly on $N/\tau N$ we may identify k' as a subfield of the endomorphism ring $\operatorname{End}_{\operatorname{gr}_0 A}(N/\tau N) \subset \operatorname{End}_Z(N/\tau N)$; recall that Z is a central subring of $\operatorname{gr}_0 A$ such that $\operatorname{gr}_0 A$ is finitely generated as a Z-module..

Let $P \in \operatorname{Spec}(Z)$ be a minimal prime over $\operatorname{Ann}_Z(N/\tau N)$ and let $k(P)$ be the residue field of Z_P. Letting $X := k(P) \underset{Z}{\otimes} N/\tau N$, we have a natural k-linear homomorphism $\operatorname{End}_Z(N/\tau N) \to \operatorname{End}_{k(P)}(X)$ given by $\phi \mapsto 1 \otimes \phi$. Since $N/\tau N$ is a finitely generated $\operatorname{gr}_0 A$-module and $\operatorname{gr}_0 A$ is finitely generated as a Z-module by our assumption, we get that $N/\tau N$ is a finitely generated Z-module. Therefore, X is a finite dimensional $k(P)$-vector space and we view k' as a subfield of the matrix ring $\mathrm{End}_{k(P)}(X)$.

Finally, since Z is a finitely generated k-algebra, $k(P)$ is a finitely generated field extension of k. Since the image of $k(P)$ lies in the centre of $\mathrm{End}_{k(P)}(X)$, the algebra $k'k(P)$ is finite dimensional over $k(P)$. Thus, if $Q \in \mathrm{Spec}(k'k(P))$, then $L = k'k(P)/Q$ is an integral domain and further it is a finite dimensional $k(P)$-vector space. In particular, we get that L is a finitely generated field extension of k. It follows from [39, Propostion III.6] that every subextension, and in particular, the image of k' in L, is a finitely generated field extension of k. The claim follows since the image of k' in L is non-trivial. □

Corollary 6.1.6. *Assume that M has a regular $F_0A - \varphi$-lattice. Then φ is algebraic over K.*

Proof. Let N be a regular $F_0A - \varphi$-lattice of M. Then we have by Proposition 6.1.5 that the residue field of $B := \{\phi \in K(\varphi) | \phi(N) \subset N\}$ is a finite extension of k. Since the τ-adic filtration on B is separated we have by [33, Theorem I.5.7] that B is a finitely generated R-module, so $K(\varphi) = \mathrm{Frac}(B)$ is a finite dimensional $K = \mathrm{Frac}(R)$ vector space. □

6.2 Microlocalisation

Throughout this section, we fix F_0M an F_0A- lattice of M. We also let $P_1, P_2, \dots P_r$ be the distinct minimal primes in Z above $\mathrm{Ann}_Z(\mathrm{gr}_0 M)$ and $T := Z \setminus \bigcup_{i=1}^r P_i$.

We have by Lemma 2.2.8 that $\mathrm{gr}\, A := \mathrm{gr}_0\, A[x, x^{-1}]$ and since $T \subset \mathrm{gr}_0\, A$ consists of homogeneous elements of degree 0, we have by Lemma 2.1.18 that we may consider the microlocalisations of A and M at T as in Definition 2.1.19; denote the microlocalisations by $Q := Q_T(A)$ and $Q_T(M)$, respectively.

Lemma 6.2.1. *There are ring/module filtrations on Q and $Q_T(M)$ such that $\mathrm{gr}(Q) \cong (\mathrm{gr}_0\, A)_T[s, s^{-1}]$ and $\mathrm{gr}(Q_T(M)) \cong (\mathrm{gr}_0\, M)_T[s, s^{-1}]$. In particular, the ring Q is Noetherian.*

Proof. We have by [7, Proposition 3.10] that

$$\operatorname{gr}(Q) \cong T^{-1} \operatorname{gr} A \cong T^{-1}(\operatorname{gr}_0 A[x, x^{-1}]) \cong (\operatorname{gr}_0 A)_T[x, x^{-1}]. \tag{6.1}$$

A similar argument computes $\operatorname{gr}(Q_T(M))$. The last claim follows from the equation above since $(\operatorname{gr}_0 A)$ is Noetherian, the localisation of a Noetherian ring is also Noetherian and Hilbert's basis theorem. □

Lemma 6.2.2. *The module $Q_T(M)$ is Artinian as a Q-module. In particular, the slice $\operatorname{gr}_0(Q_T(M))$ is an Artinian $\operatorname{gr}_0 Q$ module.*

Proof. Since some product of $P_i's$ annihilates $\operatorname{gr}_0 M$, we may find a finite chain of Z_T-submodules of $(\operatorname{gr}_0 M)_T$ such that each quotient is isomorphic to $(Z/P_i)_T$ for some i. On the other hand, the ring $(Z/P_i)_T$ is the residue field of the Z_{P_i}, so, in particular, $(\operatorname{gr}_0 M)_T$ has finite length.

Since $(\operatorname{gr}_0 M)_T$ has finite length as a $(\operatorname{gr}_0 A)_T$-module, we have by Lemma 6.2.1 that $\operatorname{gr} M$ is an Artinian $\operatorname{gr} Q$-module. Since the filtration on M is separated and Q is complete, the claim follows from [33, Proposition I.7.1.2]. □

Since microlocalisation is a functor, we may extend any A-linear endomorphism of M to a Q-linear endomorphism of $Q_T(M)$. Therefore, we may view $Q_T(M)$ as a $Q - K(\varphi)$-bimodule.

Lemma 6.2.3. *Let V be a simple $Q - K(\varphi)$ quotient of $Q_T(M)$. The module V has at least one F_0Q lattice. Furthermore, for any such F_0Q lattice L, $L/\pi L$ has finite length and $\operatorname{Kdim}_{F_0Q}(L) = 1$.*

Proof. Let L_0 be the image of $F_0Q_T(M)$ in V. By Lemma 6.2.1, F_0Q is Noetherian and since it is π-adically complete we have by [5, Lemma 3.2a)] that L_0 is a F_0Q-lattice in V. Furthermore, since $L_0/\pi L_0$ is a quotient of $\operatorname{gr}_0(Q_T(M))$, we get by Lemma 6.2.2, that $L_0/\pi L_0$ has finite length.

Now, let L be another F_0Q-lattice in V. We have by [27, Proposition 1.1.2] that the class of $L/\pi L$ is equal to the class of $L_0/\pi L_0$ in the Grothendieck group of $\operatorname{gr}_0 Q$ modules, so, in particular, $L/\pi L$ has finite length.

Next, by applying [33, Proposition I.7.1.2], we obtain

$$\operatorname{Kdim}_{F_0Q}(L) \leq \operatorname{Kdim}_{\operatorname{gr}(F_0Q)}(\operatorname{gr} L) = \operatorname{Kdim}_{\operatorname{gr}_0 Q[x]}(L/\pi L[x]).$$

Since $L/\pi L$ is a quotient of $\mathrm{gr}_0(Q_T(M))$, we have by Lemma 6.2.2 that $\mathrm{Kdim}_{\mathrm{gr}_0 Q}(L/\pi L) = 0$, so by [30, Theorem 15.19], $\mathrm{Kdim}_{\mathrm{gr}_0 Q[x]}(L/\pi L[x]) = 1$.

Since L has an infinite descending chain of F_0Q-submodules given by $\pi^n L$, it cannot be Artinian, so $\mathrm{Kdim}_{F_0Q}(L) = 1$. □

Definition 6.2.4. *We say that a subring B of $K(\varphi)$ is a lattice preserver if there is an F_0Q lattice L of V such that $BL \subset L$.*

We fix L_0 a F_0Q lattice in V and we denote $\mathcal{L}$ the set of F_0Q lattices contained in L_0, but not contained in πL_0. We also denote $\mathcal{P}$ the set of lattice preservers in $K(\varphi)$.

Lemma 6.2.5. *The set $\mathcal{P}$ has a maximal element.*

Proof. This repeats the argument in [5, Proposition 8.3] word for word. □

Theorem 6.2.6. *Every maximal lattice preserver is a discrete valuation ring.*

Proof. We repeat the argument in [5, Theorem 8.4] word for word. □

6.3 Finding the lattice using Auslander-Gorenstein theory

Throughout this section, we assume that $\mathrm{gr}_0 A$ is an Auslander-Gorenstein ring.

Lemma 6.3.1. *i) The ring F_0A is Auslander-Gorenstein.*

ii) For every finitely generated π-torsion-free F_0A-module N, we have

$$j_{F_0A}(N) = j_{\mathrm{gr}_0 A}(N) + 1.$$

Proof. We have by [21, Theorem 2.1] that $\mathrm{gr}_0 A[x]$ is also Auslander-Gorenstein. Since $\mathrm{gr}(F_0A) \cong \mathrm{gr}_0 A[x]$ and the π-adic filtration on F_0A is Zariskian, the first claim follows from [15, Theorem 3.9].

We have by Rees Lemma [2, Lemma 1.1] an isomorphism of Abelian groups

$$\mathrm{Ext}^i_{\mathrm{gr}_0 A}(N, \mathrm{gr}_0 A) \cong \mathrm{Ext}^{i+1}_A(N, A) \text{ for } i \geq 0.$$

The second claim follows from the definition of grade. □

Definition 6.3.2. *We say that a pair* (B, C) *has property* $\boldsymbol{P}$ *if* B *is a Noetherian ring which is finitely generated over a central subring* C*, and for all finitely generated* B*-modules* $N_1 \subset N_2$ *such that* $P_1, P_2, \ldots, P_r$ *denote the minimal primes above* $\operatorname{Ann}_C(N_2)$*, if there exists* $t \in C \setminus \bigcup_{i=1}^r P_i$ *with* $tN_1 = 0$*, then*

$$j_B(N_1) > j_B(N_2).$$

We say that a Noetherian ring B *is centrally grade preserved on modules if it has a central subring* C *such that* (B, C) *has property* $\boldsymbol{P}$*.*

From now on, we further assume that $\operatorname{gr}_0 A$ is *centrally grade preserved on modules* and we let Z be a central subring such that $(\operatorname{gr}_0 A, Z)$ has property $\mathbf{P}$.

Lemma 6.3.3. *Let* N *be a finitely generated* A*-module and let* L_1, L_2 *be two* F_0A *lattices of* N*. Then* $\operatorname{Supp}_Z(L_1/\pi L_1) = \operatorname{Supp}_Z(L_2/\pi L_2)$*.*

Proof. Let $I_1 = \operatorname{Ann}_Z(L_1/\pi L_1)$, $I_2 = \operatorname{Ann}_Z(L_2/\pi L_2)$ and let $z \in I_1$. Let $z' \in F_0A$ be a lift of z. Since L_1, L_2 are F_0A lattices, there exists $c, d \in \mathbb{N}$ such that $\pi^c L_2 \subset L_1$ and $\pi^d L_1 \subset \pi^{c+1} L_2$. Therefore, for $l_2 \in \pi^c L_2$ we have $z'^d l_2 \in z'^d L_1 \subset \pi^d L_1 \subset \pi^{c+1} L_2$. Thus, $z^d \in \operatorname{Ann}_Z(\pi^c L_2/\pi^{c+1} L_2) = \operatorname{Ann}_Z(L_2/\pi L_2)$.

In particular, we proved that $z \in \sqrt{I_2}$, so $I_1 \subset \sqrt{I_2}$ and it follows that $\sqrt{I_1} \subset \sqrt{I_2}$. A similar argument shows $\sqrt{I_2} \subset \sqrt{I_1}$. The claim follows since for any ideal I that is the annihilator of a finitely generated Z-module M, we have $\operatorname{Supp}(M) = V(I) = V(\sqrt{I})$. □

Proposition 6.3.4. *Let* N *be a finitely generated* F_0A*-module.*

i) If N *is* π*-torsion-free, then* $j_{F_0A}(N) = j_A(N \otimes_R K)$*.*

ii) If L *is an* F_0A *lattice in* M *and* N *a submodule of* M/L *such that* $Q_T(N) = 0$*, then* $j_{F_0A}(N) \geq j_{F_0A}(L) + 2$*.*

Proof. The first part repeats the argument in [5, Lemma 8.5] word for word.

Since $N \subset M/L$ is finitely generated, it is contained in $\pi^{-m}L/L$ for some $m \geq 1$. By considering the filtration N induced by $L \subsetneq \pi^{-1}L \ldots \subsetneq \pi^{-m}L$, we may assume that $m = 1$, so $\pi N = 0$. By our assumptions, N and $(L/\pi L)$ are finitely generated Z-modules, so $\operatorname{Supp}_Z(N) \subset \operatorname{Supp}_Z(\pi^{-1}L/L) = \operatorname{Supp}_Z(L/\pi L)$. Further, we have by

Lemma 6.3.3 that $\mathrm{Supp}_Z(L/\pi L) = \mathrm{Supp}_Z(\mathrm{gr}_0 M)$, so $L/\pi L$ is annihilated as a Z-module by some product of the minimal primes $P_1, P_2, \dots, P_r$ above $\mathrm{Ann}_Z(\mathrm{gr}_0 M)$. Since $\pi N = 0$ and $Q_T(N) = 0$, we have by Corollary 2.1.20 that N is T-torsion. Further, as N is finitely generated as a Z-module, we get $tN = 0$ for some $t \in T$.

Therefore, viewing N as a submodule of $L/\pi L$ and using the assumptions in Definition 6.3.2, we obtain $j_{\mathrm{gr}_0 A}(L/\pi L) + 1 \leq j_{\mathrm{gr}_0 A}(N)$. Thus, applying part *ii*) of Lemma 6.3.1 twice we obtain:

$$j_{F_0 A}(N) = j_{\mathrm{gr}_0 A}(N) + 1 \geq j_{\mathrm{gr}_0 A}(L/\pi L) + 2 = j_{F_0 A}(L/\pi L) + 1.$$

To finish, we use part *i*) of Lemma 6.3.1 and Proposition 2.3.9 to obtain that $j_{F_0 A}(L/\pi L) \geq j_{F_0 A}(L) + 1$. □

Proposition 6.3.5. *Suppose that M is a simple $A - K(\varphi)$-bimodule. Then M has a regular φ-lattice.*

Proof. The proof repeats the argument in [5, Proposition 8.6]; we remark that since $1 \in T$, if we let S be the preimage of T in $F_0 A$, it still follows that π is in the Jacobson radical of $F_0 A_S$. □

Corollary 6.3.6. *Let A be an almost commutative affinoid K-algebra such that its slice is an almost commutative Auslander-Gorenstein k-algebra and centrally grade preserved on modules. Then every simple endomorphism of every finitely generated A-module is algebraic over K.*

Proof. This repeats the argument in [5, Corollary 8.6]. □

6.4 Affinoid Quillen's Lemma for $\widehat{U(\mathfrak{g})_K}$

Let us provide sufficient conditions for property **P** to be satisfied.

Definition 6.4.1. *Let B a Noetherian ring and δ a finite partitive function on B. We call B δ-Cohen-Macaulay if for every finitely generated B-module M, we have*

$$\delta(B) = \delta(M) + j_B(M).$$

Lemma 6.4.2. *Let B be a Noetherian ring and C a central subring such that B is finitely generated as a C-module,* $\operatorname{Kdim}(B) < \infty$ *and B is Krull-Cohen-Macaulay. Then the pair (B, C) has property $\boldsymbol{P}$.*

Proof. Let $N_1 \subset N_2$ be two finitely generated B-modules, $P_1, P_2 \ldots, P_r$ the minimal primes above $\operatorname{Ann}_C(N_2)$ and $t \in C \setminus \cup_{i=1}^r P_i$ such that $tN_1 = 0$. Following definition 6.3.2, we need to prove $j_B(N_1) > j_B(N_2)$.

By our assumptions, we have $\operatorname{Kdim}_C(N_1) < \operatorname{Kdim}_C(N_2)$. Further, we have by [43, Corollary 10.1.10] that $\operatorname{Kdim}_B(N_i) = \operatorname{Kdim}_C(N_i)$ for $i = 1, 2$. The claim follows since $\operatorname{Kdim}_B(N_i) = \operatorname{Kdim}_B(B) - j_B(N_i)$ for $i = 1, 2$. □

Lemma 6.4.3. *Let B be a ring and C a central subring such that C is Gorenstein and* $\operatorname{Hom}_C(B, C) \cong B$ *as B-modules. Then the pair (B, C) has property $\boldsymbol{P}$.*

Proof. Let N_1, N_2, t be as in the proof of the previous lemma. Since C is Gorenstein, our assumption gives us $j_C(N_1) > j_C(N_2)$.

We claim that for any finitely generated B-module M, we have $j_B(M) = j_C(M)$. First, notice that

$$\operatorname{Hom}_C(M, C) = \operatorname{Hom}_B(M, \operatorname{Hom}_C(B, C)) \cong \operatorname{Hom}_B(M, B)$$

for any finitely generated B-module M. Therefore, the functors $\operatorname{Hom}_C(-, C)$ and $\operatorname{Hom}_B(-, B)$ coincide, so viewing Ext as the right derived functor of Hom, we obtain $\operatorname{Ext}^i_B(M, B) \cong \operatorname{Ext}^i_C(M, C)$ for $i \geq 0$. □

As a corollary, we obtain immediately:

Corollary 6.4.4. *Let $C \subset B$ a Frobenius extension and assume that C is Gorenstein. Then (B, C) has property $\boldsymbol{P}$.*

Let us now prove that the ring $\widehat{U(\mathfrak{g})_K}$ satisfies the assumptions of Corollary 6.3.6. It is clear that $\widehat{U(\mathfrak{g})_K}$ is an almost commutative affinoid K-algebra, so we only need to prove that $U(\mathfrak{g}_k) = \operatorname{gr}_0(\widehat{U(\mathfrak{g})_K})$ is an almost commutative Auslander-Gorenstein k-algebra that is centrally grade preserved on modules.

Lemma 6.4.5. *The ring $U(\mathfrak{g}_k)$ is an almost commutative Auslander-Gorenstein ring that is Cohen-Macaulay with respect to the GK-dimension.*

Proof. This follows from the PBW filtration and [55, Corollary 4.5] since $S(\mathfrak{g}_k)$ is a polynomial identity ring of finite injective dimension. □

Proposition 6.4.6. *The pair* $(U(\mathfrak{g}_k), Z(U(\mathfrak{g}_k)))$ *has property* $\boldsymbol{P}$.

Proof. We have by [55, Lemma 4.3] that $\mathrm{Kdim}_{U(\mathfrak{g}_k)}(M) = \mathrm{GKdim}_{U(\mathfrak{g}_k)}(M)$ and by Lemma 6.4.5 that $\mathrm{GKdim}_{U(\mathfrak{g}_k)}((M) + j_{U(\mathfrak{g}_k)}(M) = \mathrm{GKdim}(U(\mathfrak{g}_k))$ for every finitely generated $U(\mathfrak{g}_k)$-module M. The claim follows by combining [35, Proposition 2] and Lemma 6.4.2. □

We can now prove Theorem C from the introduction, the case $n > 0$ is proven in [5, Theorem 9.4].

Theorem 6.4.7. *Let* M *be a finitely generated* $\widehat{U(\mathfrak{g})_K}$ *module and* φ *a simple endomorphism of* M. *Then* φ *is algebraic over* K.

Proof. This follows by combining Proposition 6.4.6 and Corollary 6.3.6. □

6.5 Applications of Affinoid Quillen's Lemma

Let us now mention two important applications of Theorem 6.4.7.

Proposition 6.5.1. *Assume that* p *is a very good prime for* G *and let* B *be a Borel subgroup of* G. *Then the Krull dimension of* $\widehat{U(\mathfrak{g})_K}$, $\mathrm{Kdim}(\widehat{U(\mathfrak{g})_K})$, *satisfies the inequality:*

$$\mathrm{Kdim}(\widehat{U(\mathfrak{g})_K}) \leq \dim(B).$$

Proof. As remarked in [4, Theorem 4.3] the restriction to the case $n > 0$ is only required to apply the affinoid Quillen's lemma in [5] which only works for $n > 0$. This restriction is removed by Theorem 6.4.7. □

We are now able to characterise all the primitive ideals in $\widehat{U(\mathfrak{g})_{n,K}}$ for any $n \in \mathbb{N}$.

Theorem 6.5.2. *Let* I *be a primitive ideal in* $\widehat{U(\mathfrak{g})_{n,K}}$. *Then there exists a finite extension* L/K *and a primitive ideal* $J \in \widehat{U(\mathfrak{g})_{n,K}} \underset{K}{\otimes} L$ *with* L*-rational central character such that:*

$$I = J \cap \widehat{U(\mathfrak{g})_{n,K}}.$$

Further, this ideal J is of the form $\mathrm{Ann}(\widehat{L(\lambda)})$ *for some suitable λ.*

Proof. We have by Theorem 5.2.4 that $Z(\widehat{U(\mathfrak{g})_{n,K}})$ is isomorphic with a Tate algebra. Further, we have by Theorem 6.4.7 that I has some central character. Therefore, there exists L/K finite extension such that $I \underset{K}{\otimes} L \subset \widehat{U(\mathfrak{g})_{n,K}} \underset{R}{\otimes} K$ has L-rational central character.

Let e be the ramification index of L/K, $\mathcal{O}_L$ the ring of integers of L, π' the uniformiser of $\mathcal{O}_L$ and $\mathfrak{g}' = \mathfrak{g} \underset{R}{\otimes} \mathcal{O}_L$. We have by [5, Lemma 3.9 c)] that $\widehat{U(\mathfrak{g})_{n,K}} \otimes L \cong \widehat{U(\mathfrak{g}')_{en,L}}$. Finally, we have by [43, Theorem 10.2.9] that there exists a prime ideal $J \in \widehat{U(\mathfrak{g}')_{en,L}}$ such that $I = J \cap \widehat{U(\mathfrak{g})_{n,K}}$. Thus J contains $I \underset{K}{\otimes} L$, so in particular it has an L-rational central character. The claim follows from Theorem 5.6.7. □

Let us end this section by proving Theorem D. First, we need to prove an auxilary proposition.

Proposition 6.5.3. *Let I be a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$ and assume it has a central character $\chi : Z(\widehat{U(\mathfrak{g})_{n,K}}) \to \overline{K}$ generated by a dominant regular weight. Then I is controlled by a two-sided ideal in $U(\mathfrak{g})_K$.*

Proof. We have by the proof of Theorem 6.5.2 that there exists a finite extension L/K such that $I \underset{K}{\otimes} L$ has a L-rational central character. By passing to the Galois closure, if necessary, we may assume that L/K is a Galois extension; let $\mathcal{G} = \mathrm{Gal}(L/K)$ denote the corresponding Galois group. Again, let e be the ramification index of L/K, $\mathcal{O}_L$ the ring of integers of L, π' the uniformiser of $\mathcal{O}_L$ and $\mathfrak{g}' = \mathfrak{g} \underset{R}{\otimes} \mathcal{O}_L$. As before, it follows from [5, Lemma 3.9 c)] that $\widehat{U(\mathfrak{g})_{n,K}} \underset{K}{\otimes} L \cong \widehat{U(\mathfrak{g}')_{en,L}}$.

Since I is a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}}$ and L/K is finite, we get that $I \underset{K}{\otimes} L$ is a two-sided ideal in $\widehat{U(\mathfrak{g})_{n,K}} \underset{K}{\otimes} L \cong \widehat{U(\mathfrak{g}')_{en,L}}$. Therefore, we have by Proposition 5.7.2 that there exists J, a two-sided ideal in $U(\mathfrak{g}')_L$, such that $I \underset{K}{\otimes} L = \widehat{U(\mathfrak{g}')_{en,L}} J$.

By construction, $I \underset{K}{\otimes} L$ is closed under the natural action of $\mathcal{G}$, therefore J is also closed under the corresponding $\mathcal{G}$-action on $U(\mathfrak{g}')_L$. We have

$$U(\mathfrak{g}')_L \cong U(\mathfrak{g} \underset{R}{\otimes} \mathcal{O}_L) \underset{\mathcal{O}_L}{\otimes} L \cong U(\mathfrak{g}) \underset{R}{\otimes} \mathcal{O}_L \underset{\mathcal{O}_L}{\otimes} L \cong U(\mathfrak{g}) \underset{R}{\otimes} L \cong U(\mathfrak{g})_K \underset{K}{\otimes} L. \tag{6.2}$$

Since L/K is a finite Galois extension, we have by Galois descent and the same arguments as in [22, Theorem 4.2] that $J = J_0 U(\mathfrak{g}')_L$ for some two-sided ideal J_0 in $U(\mathfrak{g})_K$. We have

$$\begin{aligned} I\widehat{U(\mathfrak{g}')_{en,L}} &= I \underset{K}{\otimes} L = J\widehat{U(\mathfrak{g}')_{en,L}} = (J_0 U(\mathfrak{g}')_L)\widehat{U(\mathfrak{g}')_{en,L}} \\ &= J_0\widehat{U(\mathfrak{g}')_{en,L}} = (J_0\widehat{U(\mathfrak{g})_{n,K}}) \underset{K}{\otimes} L. \end{aligned} \tag{6.3}$$

Therefore, by taking $\mathcal{G}$-invariants and taking into account that I and $J_0\widehat{U(\mathfrak{g})_{n,K}}$ are fixed by the $\mathcal{G}$-action, we obtain

$$I = (I \underset{K}{\otimes} L)^{\mathcal{G}} = ((J_0\widehat{U(\mathfrak{g})_{n,K}}) \underset{K}{\otimes} L)^{\mathcal{G}} = J_0\widehat{U(\mathfrak{g})_{n,K}}. \tag{6.4}$$

In conclusion, I is controlled by J_0 and the claim is proven. □

Proof of Theorem D. This follows by combining Proposition 6.5.3 and Lemma 5.7.3. □

For the rest of this section, we fix $\chi = \chi_\lambda : Z(\widehat{U(\mathfrak{g})_{n,K}}) \to \overline{K}$ a central character generated by λ dominant.

Proposition 6.5.4. *The map* $\psi : \{$ *two-sided ideals in* $\widehat{U(\mathfrak{g})_{n,K}}$ *with character* χ $\}$ $\to \{$ *two-sided ideals in* $U(\mathfrak{g})_K$ *with character* χ $\}$

$$\psi(I) = I \cap U(\mathfrak{g})_K$$

is bijective, with inverse $\psi^{-1}(I_0) = \widehat{U(\mathfrak{g})_{n,K}} I_0 \widehat{U(\mathfrak{g})_{n,K}}$.

Proof. First, we prove that ψ is injective. Let $I, J \trianglelefteq \widehat{U(\mathfrak{g})_{n,K}}$ and assume that $\psi(I) = \psi(J)$. Then $\widehat{U(\mathfrak{g})_{n,K}}(I \cap U(\mathfrak{g})_K)\widehat{U(\mathfrak{g})_{n,K}} = \widehat{U(\mathfrak{g})_{n,K}}(J \cap U(\mathfrak{g})_K)\widehat{U(\mathfrak{g})_{n,K}}$ and the claim follows from Theorem D.

To prove ψ is surjective, it is sufficient to prove that if $I_0, J_0 \trianglelefteq U(\mathfrak{g})_K$ and $\widehat{U(\mathfrak{g})_{n,K}} I_0 \widehat{U(\mathfrak{g})_{n,K}} = \widehat{U(\mathfrak{g})_{n,K}} J_0 \widehat{U(\mathfrak{g})_{n,K}}$, then $I_0 = J_0$.

Again, we choose L/K a Galois extension such that $(\widehat{U(\mathfrak{g})_{n,K}} I_0 \widehat{U(\mathfrak{g})_{n,K}}) \underset{K}{\otimes} L$ has L-rational central character; let $\mathcal{G} = \operatorname{Gal}(L/K)$ the Galois group.

By construction, we have that $(\widehat{U(\mathfrak{g})_{n,K}} I_0 \widehat{U(\mathfrak{g})_{n,K}}) \underset{K}{\otimes} L$ is controlled by $I_0 \underset{K}{\otimes} L$ and $J_0 \underset{K}{\otimes} L$. Thus, applying Proposition 5.7.2, we obtain that $I_0 \underset{K}{\otimes} L = J_0 \underset{K}{\otimes} L$. Therefore, we get by Galois descent

$$I_0 = (I_0 \underset{K}{\otimes} L)^{\mathcal{G}} = (J_0 \underset{K}{\otimes} L)^{\mathcal{G}} = J.$$

□

Lemma 6.5.5. *Let ψ be as in the previous proposition. Then $\psi(I)$ is a prime ideal if and only if I is a prime ideal.*

Proof. First assume that I is prime. Let $A, B \trianglelefteq U(\mathfrak{g})_K$ be two-sided ideals such that $AB \subset \psi(I) = I \cap U(\mathfrak{g})_K$. Then

$$(\widehat{U(\mathfrak{g})_{n,K}} A \widehat{U(\mathfrak{g})_{n,K}})(\widehat{U(\mathfrak{g})_{n,K}} B \widehat{U(\mathfrak{g})_{n,K}}) \subset \widehat{U(\mathfrak{g})_{n,K}} (I \cap U(\mathfrak{g})_K) \widehat{U(\mathfrak{g})_{n,K}} = I,$$

and since I is prime we obtain that $\widehat{U(\mathfrak{g})_{n,K}} A \widehat{U(\mathfrak{g})_{n,K}} \subseteq I$ or $\widehat{U(\mathfrak{g})_{n,K}} B \widehat{U(\mathfrak{g})_{n,K}} \subseteq I$. Therefore, by Proposition 6.5.4, we obtain that $A \subseteq I \cap U(\mathfrak{g})_K$ or $B \subseteq I \cap U(\mathfrak{g})_K$, so $\psi(I)$ is prime.

For the other direction, let $I_0 = \psi(I)$. Let $A, B \trianglelefteq \widehat{U(\mathfrak{g})_{n,K}}$ such that $AB \subset I$. We have by Proposition 6.5.4 that $A = \widehat{U(\mathfrak{g})_{n,K}} A_0 \widehat{U(\mathfrak{g})_{n,K}}$ and $B = \widehat{U(\mathfrak{g})_{n,K}} B_0 \widehat{U(\mathfrak{g})_{n,K}}$ for some $A_0, B_0 \trianglelefteq U(\mathfrak{g})_K$. Therefore, we obtain

$$\widehat{U(\mathfrak{g})_{n,K}} (A_0 B_0) \widehat{U(\mathfrak{g})_{n,K}} \subseteq \widehat{U(\mathfrak{g})_{n,K}} (I_0) \widehat{U(\mathfrak{g})_{n,K}}.$$

Thus, applying again Proposition 6.5.4, we get $A_0 B_0 \subseteq I_0$. Since I_0 is prime, either $A_0 \subseteq I_0$ or $B_0 \subseteq I_0$. In conclusion, we obtain that either $A \subseteq I$ or $B \subseteq I$. □

Now, let us finish this section by proving Theorem E; this can be seen as a more general version of Corollary 5.7.7. Let m_λ and $\widehat{m_\lambda}$ be two-sided ideals in $U(\mathfrak{g})_K$ and $\widehat{U(\mathfrak{g})_{n,K}}$ generated by $\ker \chi_\lambda$, and also let $U(\mathfrak{g})_K^\lambda$ and $\widehat{U(\mathfrak{g})_{n,K}^\lambda}$ be the corresponding quotients.

Corollary 6.5.6 (Theorem E)**.** *The map* $\overline{\psi} : \operatorname{Spec}(\widehat{U(\mathfrak{g})_{n,K}^\lambda}) \to \operatorname{Spec}(U(\mathfrak{g})_K^\lambda)$, $\overline{\psi}(P + \widehat{m_\lambda}) = \psi(P) + m_\lambda$ *is a homeomorphism with inverse* $\overline{\psi}^{-1}(Q + m_\lambda) = \widehat{U(\mathfrak{g})_{n,K}} Q \widehat{U(\mathfrak{g})_{n,K}} + \widehat{m_\lambda}$.

Proof. This follows from Proposition 6.5.4 and Lemma 6.5.5. □

6.6 An algebraic proof of Duflo's theorem for $\mathfrak{g} = \mathfrak{sl}_2$

Let us now give an algebraic proof of Duflo's theorem for $\mathfrak{g} = \mathfrak{sl}_2$ using the bound on the Krull dimension of $\widehat{U(\mathfrak{g})_K}$. We will only present the proof in the case of trivial central characters and $n = 0$, but this can easily be extended to all rational central characters and to all $n \in \mathbb{N}^*$. The proof only works in the case $\mathfrak{g} = \mathfrak{sl}_2$ as this is the only Lie Algebra in which the centre of the affinoid enveloping algebra is a PID, in this case, $Z(\widehat{U(\mathfrak{g})_K})$ is isomorphic to a Tate algebra in one variable.

6.6.1 ad-Action on the Affinoid Enveloping Algebra

Frow now on, we will assume that $\mathfrak{g} = \mathfrak{sl}_2$ and we denote e, f, h the standard $\mathfrak{sl}_2$-basis, and let c denote the Casimir operator. Consider the $\mathrm{ad}_h = [h, -]$ action on $U(\mathfrak{g}_K)$.

Lemma 6.6.1. *[42, Lemma 2.29]*

For $i, j, k \in \mathbb{N}$, the operator $[h, -]$ acts on the monomial $e^i h^j f^k$ with eigenvalue $2(k - i)$. In particular, the action ad_h on $U(\mathfrak{g}_K)$ is diagonalisable.

We need a more general definition of a diagonalisable action to apply for $\widehat{U(\mathfrak{g})_K}$ in order to replace the direct sum decomposition with a convergent sum decomposition as in [50]. A module M will be called $\mathfrak{h}^*$-diagonalisable if there is a set of weights $\prod(M) \subset \mathfrak{h}^*$ with the property: for every $m \in M$ there exists a family $\{m_\lambda \in M_\lambda\}_{\lambda \in \prod(M)}$ converging cofinite to zero in M and satisfying

$$m = \sum_{\lambda \in \prod(M)} m_\lambda.$$

In particular, if we let $M = \widehat{U(\mathfrak{g})_K}$ and consider the ad_h action on it we get by Lemma 6.6.1 that $\widehat{U(\mathfrak{g}_K)}$ is diagonalisable with the set of weights given by $2s$, $s \in \mathbb{Z}$.

We would like to compute the space $\widehat{U(\mathfrak{g})_K}_0$, i.e, the centralizer of h in the affinoid enveloping algebra. For this we need an auxiliary result; recall that c denotes the Casimir operator.

Lemma 6.6.2. *[42, Proposition 2.30]*

$$U(\mathfrak{g}_K)_0 = K[h, c],$$

As a corollary we obtain:

Corollary 6.6.3. *The centralizer of h inside the affinoid enveloping algebra of $\mathfrak{sl}_2$ is given by*

$$\widehat{U(\mathfrak{g})_K}_0 = K\langle h, c\rangle.$$

We now use the decomposition of $\widehat{U(\mathfrak{g})_K}$ into $\mathfrak{h}^*$ subspaces to deduce a crucial property about the $\mathfrak{h}^*$ subspace decomposition of two sided ideals in $\widehat{U(\mathfrak{g})_K}$.

Proposition 6.6.4. *Let I be a two sided ideal in $\widehat{U(\mathfrak{g})_K}$ and let $i \in I$. Write*

$$i = a_0 i_0 + \sum_{s\in\mathbb{Z}, s\neq 0} a_s i_s,$$

with $i_0 \in \widehat{U(\mathfrak{g})_K}_0$, $i_s \in \widehat{U(\mathfrak{g})_K}_{2s}$ and such that $||a_s|| \to 0$ as $|s| \to \infty$. Then $a_0 i_0 \in I$.

To prove the proposition we define a new family of operators H_j for $j \in \mathbb{N}$ that act on the affinoid enveloping algebra. Let

$$H_j := (-1)^j \frac{1}{2^{2j}(j!)^2} \prod_{l=1}^{j} (\mathrm{ad}_h - 2l)(\mathrm{ad}_h + 2l).$$

Lemma 6.6.5. *Let $m_s \in \widehat{U(\mathfrak{g})_K}_{2s}$. Then:*

$$H_j m_s = \begin{cases} m_s \text{ if } s = 0. \\ 0 \text{ if } |s| \leq j. \\ b_s m_s \text{ if } |s| > j, \text{ with } b_s \in K, ||b_s|| \leq 1. \end{cases} \tag{6.5}$$

Proof. We have that

$$\begin{aligned} H_j m_s &= (-1)^j \frac{1}{2^{2j}(j!)^2} \prod_{l=1}^{j} (\mathrm{ad}_h - 2l)(\mathrm{ad}_h + 2l) m_s \\ &= (-1)^j \frac{1}{2^{2j}(j!)^2} \prod_{l=1}^{j} (2s - 2l)(2s + 2l) m_s \\ &= (-1)^j \frac{1}{2^{2j}(j!)^2} 4^j \prod_{l=1}^{j} (s - l)(s + l) m_s \\ &= (-1)^j \frac{1}{(j!)^2} \prod_{l=1}^{j} (s - l)(s + l) m_s. \end{aligned} \tag{6.6}$$

For $s = 0$, we obtain

$$H_j m_0 = (-1)^j \frac{1}{(j!)^2} \prod_{l=1}^{j} -l^2 m_0 = m_0.$$

Next, if $|s| \leq j$, there exist $l \leq j$ such that $s + l = 0$ or $s - l = 0$, so $H_j m_s = 0$.

Finally, assume $|s| > j$ and without loss of generality that $s > 0$, so that $s > j$. Then

$$\frac{1}{j!} \prod_{l=1}^{j} (s + l) = \binom{s+j}{s} \text{ and } \frac{1}{j!} \prod_{l=1}^{j} (s - l) = \binom{s-j}{s}.$$

Thus, we get $H_j m_s = (-1)^j \binom{s+j}{s}\binom{s-j}{s} m_s$, so $b_s = (-1)^j \binom{s+j}{s}\binom{s-j}{s} \in \mathbb{N}$, which immediately implies $||b_s|| \leq 1$. A similar result holds for $s < 0$. This concludes the proof of the lemma. ☐

Proof of proposition 6.6.4. Define a new sequence $i^{(j)} := H_j.i$. Since I is a two sided ideal, we have that $i^{(j)} \in I$. Furthermore, by applying Lemma 6.6.5 one gets

$$i^{(j)} = a_0 i_0 + \sum_{s \in \mathbb{Z}, |s| > j} a_s b_{js} i_s,$$

with $||b_{js}|| \leq 1$. Since $||a_s|| \to 0$ as $s \to \infty$ and $||b_s|| \leq 1$ the sequence $i^{(j)}$ has limit $\hat{i}$ given by

$$\begin{aligned} \hat{i} &= \lim_{j \to \infty} i^{(j)} \\ &= \lim_{j \to \infty} a_0 i_0 + \sum_{s \in \mathbb{Z}, |s| > j} a_s b_{sj} i_s \qquad (6.7) \\ &= a_0 i_0 \end{aligned}$$

By [33, Corollary I.5.5], I is a closed ideal, so $a_0 i_0 = \hat{i} \in I$. ☐

6.6.2 Ideals in the Tate algebra in one variable

Let us now prove some basic results about the ideals in the Tate algebra in one variable. Let $K\langle t \rangle$ denote this Tate algebra. Recall from Section 2.2 the Gauss norm on $K\langle t \rangle$ given by $||\sum_{j=0}^{\infty} a_j t^j|| = \max_{j \in \mathbb{N}} ||a_j||$.

Definition 6.6.6. *We call $f = \sum_{j=0}^{\infty} a_j t^j \in K\langle t\rangle$ distinguished of degree s if $||a_s|| \geq 1$, $||f|| = ||a_s||$ and $||a_t|| < ||a_s||$ for $t > s$.*

We call an element of $w \in K[t]$ a Weierstrass polynomial if w is monic and $||w|| = 1$. Denote by W the set of Weierstrass polynomials in $K[t]$.

Proposition 6.6.7 (Weierstrass Preparation Theorem)**.** *Let $f \in K\langle t\rangle$ be distinguished of degree s. Then there exists unique $w \in W$ and $g \in K\langle t\rangle^*$ such that $f = wg$.*

Corollary 6.6.8. *Let $f = \sum_{j=0}^{\infty} a_j t^j \in K\langle t\rangle$. Assume that $a_0 \neq 0$. Let $I = (f)$ be the ideal generated by f. Then there exists $f' = \sum_{j=0}^{N} b_j t^j \in K[t] \cap I$ with $b_0 \neq 0$. In particular, one may assume that $b_0 = 1$.*

Proof. By scaling with a suitable power of π, we may assume that f is distinguished of degree s. Therefore, by Proposition 6.6.7, we have $f = we$ for $w \in W$ and $e \in K\langle t\rangle^*$. Write $w = \sum_{j=0}^{N} b_j t^j$. Since $a_0 \neq 0$, $b_0 \neq 0$ as well. Therefore, we have $w = wee^{-1} = fe^{-1} \in (f)$ and the claim is proved. □

Remark 6.6.9. *It follows from Proposition 6.6.7 that any ideal in $K\langle t\rangle$ intersects $K[t]$. This only holds for the Tate algebra in one variable. Indeed, one may construct an ideal J in the Tate algebra $K\langle t_1, t_2\rangle$ such that $J \cap K[t_1, t_2] = \emptyset$.*

6.6.3 Primitive ideals with trivial central character

Let us now describe the primitive ideals in $\widehat{U(\mathfrak{g})_K}$ with trivial central character. We let $U^0 = \widehat{U(\mathfrak{g})_K}/(c)$ and we will identify the ideals in U^0 with ideals in $\widehat{U(\mathfrak{g})_K}$ with trivial central characters. We aim to prove that U^0 contains a unique proper prime ideal corresponding to the augmentation ideal, $\widehat{U(\mathfrak{g})_K}^0$, in $\widehat{U(\mathfrak{g})_K}$.

Proposition 6.6.10. *Let I be a prime ideal in U^0. Assume that I is proper. Then $I = \widehat{U(\mathfrak{g})_K}^0/(c)$.*

Proof. Let $\widehat{M(0)}$ denote the Verma module of weight 0 and let $M = I\widehat{M(0)}$. We have by Theorem 5.2.25 that $M \in \{0, \widehat{N(0)}, \widehat{M(0)}\}$, where $\widehat{N(0)}$ denotes the maximal submodule of $\widehat{M(0)}$.

Case I. $M = 0$. Then $I \subset \mathrm{Ann}(\widehat{M(0)}) + (c) = 0 + (c)$, so $I = 0 + (c)$, which is not a proper ideal.

Case II. $M = \widehat{N(0)}$. View I as a two-sided ideal in $\widehat{U(\mathfrak{g})_K}$ containing (c). We have $0 \subsetneq (c) \subsetneq I \subset \mathrm{Ann}(\widehat{M(0)}/\widehat{N(0)}) = \widehat{U(\mathfrak{g})_{n,K}}^0$. Since I is a prime ideal it follows from combining [43, Lemma 4.5] and Proposition 6.5.1 that $I = \widehat{U(\mathfrak{g})_{n,K}}^0$.

Case III. $M = \widehat{M(0)}$. Let $v_0 = 1 \otimes 1 \in \widehat{M(0)}$. Since v_0 generates $\widehat{M(0)}$, there exists $i \in I$ such that $iv_0 = v_0$, so i is of the form

$$1 + \sum_{x,y,z \in \mathbb{N}, x+y+z>0} \alpha_{x,y,z} f^x h^y e^z + (c).$$

Next, write i as $i_0 + i'$, where

$$i_0 = 1 + \sum_{x,y,z \in \mathbb{N}, x+y+z>0, x=z} f^x h^y e^z.$$

We have by combining Lemma 6.6.1 and Proposition 6.6.4 that $i_0 + (c) \in I$. Further, we have by Corollary 6.6.3 that $i_0 \in K\langle h, c\rangle$. Therefore, there exist $(a_j)_{j \in \mathbb{N}^*} \in K$ such that $1 + \sum_{j=1}^{\infty} a_j h^j + (c) \in I$, where $||a_j|| \to 0$ as $j \to \infty$. Therefore, by Corollary 6.6.8, there exist $(b_j)_{j \in \mathbb{N}^*} \in K$ such that $x = 1 + \sum_{j=1}^{N} b_j h^j + (c) \in I$.

Finally, view I as an ideal containing (c) and let $I' = I \cap U(\mathfrak{g}_K)$. Then $x \in I'$ and it follows that $I'M(0) = M(0)$. Therefore, by applying [13, Theorem 4.3], we obtain $I' = U(\mathfrak{g}_K)$, so $I = \widehat{U(\mathfrak{g})_K}$. □

As a corollary, we obtain:

Corollary 6.6.11. *The only prime ideals in $\widehat{U(\mathfrak{sl}_2)_K}$ with trivial central character are the ideal generated by the Casimir and the augmentation ideal.*

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