Default Forecasting in KMV

Yuqian (Steven) Lu

June 19, 2008

Dissertation for MSc Mathematical and Computational Finance

Oriel College

University of Oxford

Abstract

In this dissertation, we present the basic ideals and structrues of the KMV in the framework of both Merton and Vasicek and Kealhofer Models, and also explain some conditions before implementing these two models. Moreover, we extend the Merton's model to a special case in KMV. We use the real data to examine the default probability of the several firms which have different financial conditions in three industries, and find out some implications among the parameters we input and derive.

Keywords: KMV, Merton, Distance-to-Default (DD), Expected Default Frequency (EDF), Implied Default Probability (IDP).

1 Introduction

Credit risk is the risk that an obligor does not meet its repayments on time. These repayments takes a wide range of forms such as the debt and the principle. Failure to meet repayment can experience several consequences. Here, we mainly focus on the credit risk of the firm. The credit risk of the firm is often referred as the default risk of the firm, and indeed both terms are interchangeable in this project. Default of the firm usually associated with the bankruptcy of the firm. However, this is just one among several credit events[1]. We are interested in the credit event that the firm fail to meet its repayment of the debt. Although the default of the firm is a rare event, once it happens, it will have significant losses, and indeed there is no way to discriminate unambiguously between that will default and those that will not prior to the default event.Consequently, modeling of credit risk to forecast the time is paid closed attention by many individuals and firms. Many credit rating agencies such as Standard and Poor, Fitch and Moody's were born in such a case. The main functions of these rating agencies are similar, evaluating the credit risk outlook for individual companies and assign credit ratings.

The quantitative modeling of credit risk has become a extensive topic today, because of the innovation of the credit derivatives and firm debt products. Since then, many academics and practitioners have shown great interest in models that forecast the credit risk of the firm. One major application has been widely used is the one initialed by Merton (see [1]). Later on, Merton's model was developed by the firm called KMV Corporation[2], a firm specialized in credit risk analysis. The model is sometimes called Merton's KMV. KMV deployed the framework of Merton, in which the equity value of the firm is a call option on the underlying value of the firm's asset with a strike price equal to the face value of the firm's debt, basing on some simplifying assumptions about the structure of the typical firm's finances. In Merton's KMV, the methodology uses the value of the equity, the volatility of equity and several other observable to obtain the value of the firm's asset and

[1] For more details of the credit events, please refer to the International Securities and Derivatives Association(ISDA) website.

[2] In 2002, Moody's Corporation Completes Acquisition of KMV. KMV Corporation is now renamed as Moody's KMV.

volatility, in which are both non-observable. After obtaining these two inferred quantities, it applies the assumption the value of the firm follows a geometric Brownian motion to specify the default probability of the firm.

Merton's Model is a foundation in modeling credit risk. KMV smartly uses this application in forecasting the credit risk of the firm. However, how well it performs substantially relies on the simplifying assumptions facilitated its implementation. In practice, these simplifying assumptions are not realistic. KMV Corporation does not rely solely on these assumptions. Indeed, the founders of KMV, Oldrich Vasicek and Stephen Kealhofer, developed a new model called Vasicek-Kealhofer (VK) (see [2]) to estimate the "Distance-to-default" of an individual firm and then to use a proprietary database of US firms to map into an "Expected Default Frequency" .

Many practitioners are doing the research on KMV, to examine the accuracy of the model and seek some methods to improve it. Perhaps most of them are from Moody's KMV website. Crosbie and Bohn (see [3]) summarized KMV's default probability model after making some modifications on the assumptions. a applied the variant of the Merton model to calculate the market value and volatility of the firm's asset from equity value to improve the accuracy in obtaining the "distance-to-default". Kealhofer and Kurbat (see[4]) replicated Moody's research results to argue that Moody's model captured more information and react more quick compared to those traditional rating agency. Beside these practitioners, many scholars also are interested in KMV methodology. Bharath and Shumway (see [5]) examine the accuracy and the contribution of the KMV-Merton default forecasting model by constructing its naive alternative "probability". Therefore, we wish to find out how the "Distance-to-default" link to some observable factors, and how the methodology can be improved.

The paper is organized as follows. Section 2 briefly reviews the literatures on the derivations of the Merton's equation, and the equity based models of firm default in the frameworks of Merton and VK. Section 3 shows how the original Merton model can be extended to a special case. Section 4 outlines how the model may be tested, and describes the data on US firms in the estimations. Section 5 concludes.

2 Default Forecasting Models

Merton extended the work of Black and Scholes (see [6]) on option pricing theory in the default prediction of the firm, along with certain strong assumptions. In late 1980s, the application of Merton's model to forecast default of the firm was developed by KMV Corporation, and we call this application the KMV-Merton Model. This model relies on the idea that a firm's equity could be viewed as an option on the underlying value of the firm's assets in a certain time horizon. Later on, Oldrich Vasicek and Stephen Kealhofer have extended the Black-Scholes-Merton framework to produce a model of default probability known as the Vasicek-Kealhofer (VK) model. This model assumes a firm's equity could be viewed as a perpetual barrier option on the underlying value of the firm's asset in a time horizon. Once the asset value of the firm drops below some threshold level, which is also called the default point (DD), at or before the time horizon, the firm would immediately default. Since this model has proved its better behavior with respect to default prediction in the market, it has been taken over by rating agency Moody; it is called Moody's KMV today. Moody's KMV uses its large historical database to estimate the empirical distribution of changes in distance to default, and calculates default probabilities based on that distribution (see [5]). This default probability is known as EDF credit measure (see [7]), which is firm-specific. Due to Moody's commercial characteristic, the modeling choices made by Moody's KMV become the commercial secret. Simultaneously, the modeling choice will trigger the accuracy concern of forecasting the default risk.

2.1 Merton's (1974) Model

In 1974, Merton proposed a model, which based on the option pricing theory of the Black-Scholes (see [6]) due to the "observable variables" of the final function, to assess the credit risk of a firm. The model links the credit risk to the capital structure of the company. This model is perhaps the most significant contribution to the area of the qualitative credit risk research. Relying on the some implicit assumption, the model assumes that equity is a call option on the value of assets of the firm. From this insight, the value of debt can be derived from the equity value. A description of this model is presented in following section.

2.1.1 Assumptions

The Merton model made some assumptions to develop the Black-Scholes equation (see [6]). I categorized these assumptions in the four sections.

- Debt

The firm has issued just a single, homogeneous class of bond maturing in Tperiods. The firm promise to pay the bond to the bondholder at maturity T.

- Capital Structure

In the Merton model, it is simply assumed that the public traded firm[3] is funded using debt and equity; its balance sheet looks like:

	Asset	Liabilities
	Firm Value:$F(t)$	Debt: $C(F,t)$
		Equity: $E(F,t)$
Total	$F(t)$	$F(t)$

Figure 1: Balance sheet of Merton's Firm

Naturally, following the accounting identity, we would have the equation

$$F(t) = E(F,t) + C(F,t). \tag{1}$$

Remark: Prior to the maturity of the debt, the firm can't issue any new senior claims or repurchase on shares on the firms.

- The dynamic of the value of the firm's asset

It assumes that the firm's assets are tradable assets, and they follow a geometric Brownian motion on the probability space (Ω, F, P) such as

$$dF = \mu_F F dt + \sigma_F F dW, \tag{2}$$

[3]It refers to the company that is permitted to offer its registered securities for sale to general public.

where μ_F and σ_F are the instaneous expected rate of the return and the volatility of the firm respectively, dW is a standard Weiner process and $W_t \sim N(0,t)$. $F(t)$ is log-normal distributed with expected value at time t, such that

$$F(t) = F(0)\exp\left\{(r - \frac{1}{2}\sigma_F^2)t + \sigma_F\sqrt{t}W_t\right\}. \tag{3}$$

- Market Perfection

In this assumption, it assumes that coupon and dividend payments, taxes have been ignored. There is no penalty to short sales. Market is fully liquid; investors can purchase or sell any assets at the desirable market price. Borrowing and lending are at the same risk free interest rate, and this interest rate is constant through the horizon. These assumptions do not violate the formulations of the model; they are illustrated only for expositional convenience.

2.1.2 Setup

Following the Black-Scholes derivation, we assume that $Y_1 = V_1(F(t),t)$ and $Y_2 = V_2(F(t),t)$ are two functions of the value of the term and time, by Ito's Lemma (see [8]), we have

$$dY_i = \frac{\partial V_i}{\partial F}dF + \frac{\partial V_i}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V_i}{\partial F^2}(dF)^2 \cdots, \tag{4}$$

where $i = 1,2$ and $F = F(t)$. Since the firm's asset follows a geometric Brownian motion, we have $(dF)^2 = \sigma_F F^2 dt$, rearrange equation (4), then

$$dY_i = \frac{\partial V_i}{\partial F}dF + \frac{\partial V_i}{\partial t}dt + \frac{1}{2}\sigma_F^2 F^2\frac{\partial^2 V_i}{\partial F^2}dt. \tag{5}$$

By choosing a portfolio of Y_1 and Y_2, the appropriate portfolio is long an amount of value Y_1 and short an amount of ΔY_2. Define Π as the value of portfolio such that

$$\Pi = Y_1 - \Delta Y_2. \tag{6}$$

Substituting the equation (2) and (5) into equation (6), we have

$$\begin{aligned} d\Pi &= (\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V_1}{\partial F^2} + \mu_F F \frac{\partial V_1}{\partial F})dt - \Delta(\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V_2}{\partial F^2} + \mu_F F \frac{\partial V_2}{\partial F})dt \\ &+ \left(\sigma_F F \frac{\partial V_1}{\partial F} - \Delta \frac{\partial V_2}{\partial F}\right) dW. \end{aligned} \tag{7}$$

To ensure the portfolio is risk-less during time dt, we take $\Delta = \frac{\partial V_1/\partial F}{\partial V_2/\partial F}$. In the absence of arbitrage, it follows that

$$d\Pi = r\Pi dt, \tag{8}$$

where r is the risk-free interest rate.

By equaling the equation (7) and (8), we get

$$(\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V_1}{\partial F^2})dt - \Delta(\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V_2}{\partial F^2})dt = r\Pi dt = r(V_1 - \triangle V_2 dt). \tag{9}$$

Rearranging,

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V_1}{\partial F^2} - rV_1}{\frac{\partial V_1}{\partial F}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V_2}{\partial F^2} - rV_2}{\frac{\partial V_2}{\partial F}}. \tag{10}$$

Both side are equal to an arbitrage function of F and t, since this holds for any two function $V_1(F,t)$ and $V_2(F,t)$. Say this arbitrage function is $\alpha(F,t)$.

For any function $Y = V(F,t)$, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV - \alpha(F,t)\frac{\partial V}{\partial F} = 0. \tag{11}$$

We then choose $\alpha(F,t) = (\sigma_F \lambda - \mu_F)F$, where $\lambda = \lambda(F,t)$ is the market price of risk.

We can rewrite equation (11) as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV - (\sigma_F \lambda - \mu_F) F \frac{\partial V}{\partial F} = 0. \tag{12}$$

Since the firm's asset is tradable, the firm's asset is a solution to equation (12), so we have $(\mu_F - \sigma_F \lambda)V - rV = 0$. Hence, $\lambda = \frac{\mu - r}{\sigma}$ is the market price of risk. By simplifying terms, we can reduce equation (12) to

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV + rF \frac{\partial V}{\partial F} = 0. \tag{13}$$

This is the so called Black-Scholes-Merton equation. It must be satisfied by any equities whose value is a function of the value of the firm and the time.

2.1.3 Option nature of the equity

According to the company law, at the maturity of debt obligation, the bondholders will receive their debts in full; the equity-holders will get rest. But in the event that the payment of the debt is not met, the bondholders will take control of the remaining asset of the firm. Hence, the equity-holders will receive nothing.

Therefore, we would consider that the value of the equity is equivalent to a call option on the underlying value of the firm with a strike price equal to the face value D of the debt at maturity of debt obligation T, it can be written as

$$Equity\,Value\ E(F,t) = \max\left[F(T) - D, 0\right]. \tag{14}$$

At the maturity time T, if the value of the firm's asset is greater than the debt, we exercise the option to gain the payoff of $F(T) - D$; otherwise, we have nothing. By inspection of the capital structure, the value of debt is the minimum value between the firm's asset and debt, and is equivalent to the value of a debt minus the value of put option on the underlying value of the firm's asset with a strike price equal to the face value D of the firm's debt at the maturity of the debt obligation. Again, it can be written as

$$Debt\,Value\ C(F,t) = \min\left[F(T), D\right] = D - \max\left[D - F(T)\right]. \tag{15}$$

In the case that both equity and debt are tradable, we can rewrite equation (13) for both case by writing $V_1 = C(F,t)$ and $V_2 = E(F,t)$.

2.1.4 Merton's result

Due to the option nature of the equity and debt, Merton claims that we can extend the Black-Scholes option pricing model to the both equity and debt cases, and write down the solution to (14) and (15) directly.

In accordance with the option pricing theory, we then have

$$\begin{aligned} E(F,t) &= F(t)N(d_1) - e^{-r(T-t)}DN(d_2) \\ d_{1,2} &= \frac{\log(D/F(t)) \pm (r - \frac{1}{2}\sigma_F^2)(T-t)}{\sigma_{F\sqrt{T-t}}}, \end{aligned} \tag{16}$$

where $N(\cdot)$ is the cumulative standard normal distribution.

From equation (15) and the capital structure $C(F,t) = F(t) - E(F,t)$, we then have

$$C(F,t) = F(t)N(-d_1) + De^{-r(T-t)}N(d_2). \tag{17}$$

2.2 Merton's KMV

The KMV relies on the Merton model applied to the value of the firm's assets, and regards the equity as a call option on the assets in the framework of the Black-Scholes-Merton equation to generate the default probability for each firm in the sample at any given point in time. In this model, we need to estimate the firm's asset quantities—the current value and the volatility—from the market value of the firm's equity and the equity's instantaneous volatility, along with knowing the outstanding and maturity of debt. The debt's maturity is chosen and the book value of the debt is set to equal the face value of the debt.The firm is default when the value of firm's asset falling below the default point (DD), which is the face value of the debt. To calculate the default probability, the new parameter is introduced, called the distance to default, which is the distance between the expected value of the firm's assets and the default point and then then divides this difference by an estimate of the volatility of the firm in a time horizon. And then, the distance to default is substituted into a cumulative density function to calculate the probability that the value of the firm will be less than the face value of debt at the maturity of the debt.

2.2.1 Estimation of the value of firm's assets V and volatility of the firm return σ_F

Under Merton's assumption, equity is a call option on the value of the firm's assets and the time, and it follows the following stochastic differential equation

$$dE = \mu_E E dt + \sigma_E E dW, \tag{18}$$

μ_E and σ_E are the instaneous expected rate of return on this equity and its volatility.

By using Ito's Lemma, we can write the dynamics of the equity as

$$\begin{aligned} dE &= \frac{\partial E}{\partial F} dF + \frac{\partial E}{\partial t} dt + \frac{1}{2}\sigma_E^2 F^2 \frac{\partial^2 E}{\partial F^2} (dF)^2 \cdots \\ &= (\frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 E}{\partial F^2} + \mu_F F \frac{\partial E}{\partial F} + \frac{\partial E}{\partial t}) dt + \sigma_F F \frac{\partial E}{\partial F} dW. \end{aligned} \tag{19}$$

Comparing diffusion terms in equations (18) and (19), we can retrieve the relationship such that

$$E\sigma_E = F\sigma_F \frac{\partial E}{\partial F}. \tag{20}$$

In addition, we can derive *Equity Delta* $\Delta^E = \frac{\partial E}{\partial F} = N(d_1) > 0$ from equation (16).

Hence, the new relation between the volatility of the firm and that of the equity is

$$E\sigma_E = F\sigma_F N(d_1). \tag{21}$$

Similarly, comparing drift terms in equations (18) and (19), we have

$$\mu_E E = \frac{1}{2}\sigma_F^2 F^2 \frac{\partial^2 E}{\partial F^2} + \mu_F F \frac{\partial E}{\partial F} + \frac{\partial E}{\partial t}. \tag{22}$$

In addition, we can derive *Equity Gamma* $\Gamma^E = \frac{\partial^2 E}{\partial F^2} = \frac{n(d_1)}{F\sigma_F\sqrt{T-t}} > 0$ and *Equity Theta* $\theta^E = \frac{\partial E}{\partial t} = -\frac{Fn(d_1)\sigma_F}{2\sqrt{T-t}} - rDe^{-r(T-t)}N(d_2)$ from equation (16).

Hence, we have

$$\mu_F = \frac{\mu_E E - \theta^E - \frac{1}{2}\sigma_F^2 F^2 \Gamma^E}{F\Delta^E}. \tag{23}$$

In practice, value of equity for the public firms can be directly observed from stock exchange market. It directly implies that the value of the option written on the underlying value of the firm's asset can be observed. In addition, the volatility's of the equity can be obtained by estimating the implied volatility from an observed option price or by using a historical stock returns data. Once we have risk-free interest rate and the time horizon of the debt, the only unknown quantities are the value of the firm's assets $F(t)$ and the volatility of the firm σ_F. Thus, we can solve the two nonlinear simultaneous equation (16) and (20) to determine $F(t)$ and σ_F by the equity value, volatility value and capital structure.

2.2.2 Calculation of Distance-to-Default (DD)

Merton's assumption regards that the firm' asset are tradable is violated by KMV. KMV is aware of this point. Instead of this point, KMV only uses the Black-Scholes and Merton setups as motivation to calculate an intermediate phase called "distance-to-default" (DD) before computing the probability of default.

To derive the default probability of a particular firm, beside results of the values of the firm's asset and firm's volatility, we must to calculate the distance to default. The default event happens when the value of firm's asset is below the default point. The face value of the debt is regarded as the default point in Merton's Model. By using the volatility of the firm's asset to measure, we can calculate the Distance-to-default. The larger the number is in the Distance-to-default, the less chance the company will default. Hence, we can express DD under the some risk-neutral probability measure as the following equation

$$Distance - to - Default(DD) = \frac{In(^{F(t)}/_{D}) + (r - \frac{1}{2}\sigma_F)(T-t)}{\sigma_F\sqrt{T-t}}, \tag{24}$$

where r is the risk-free rate of the return of the firm's asset, $F(t)$ is the current value of the firm's asset and D is the face value of the debt.

2.2.3 Derivation of the probabilities of default

When we obtain $F(T)$ and σ_F, we can immediately derive the probabilities of default. According to the definition of default that the value of firm's asset is below the value of debt, we can express the probabilities of default under risk-neutral measure at time t as

$$\begin{aligned} P_t &= Pr\left[F(T) < D\right] \\ &= Pr\left[F(t)exp\left\{(r-\frac{\sigma_F^2}{2})(T-t)+\sigma_F W_{T-t}\right\} < D\right] \\ &= Pr\left[W_{T-t} < \frac{In(D/F(t))-(r-\frac{\sigma_F^2}{2})(T-t)}{\sigma_F}\right] \\ &= Pr\left[Z < \frac{In(F(t)/D)-(r-\frac{\sigma_F^2}{2})(T-t)}{\sigma_{F\sqrt{T-t}}}\right] \\ &= Pr\left[Z < -DD\right] \\ &= N(-D), \end{aligned} \tag{25}$$

where $Z \backsim N(0,1)$ and $DD = \frac{In(F(t)/D)+(r-\frac{\sigma_F^2}{2})(T-t)}{\sigma_{F\sqrt{T-t}}}$.

The shaded area in Figure 2 below the default point is equal to $N(-DD)$.

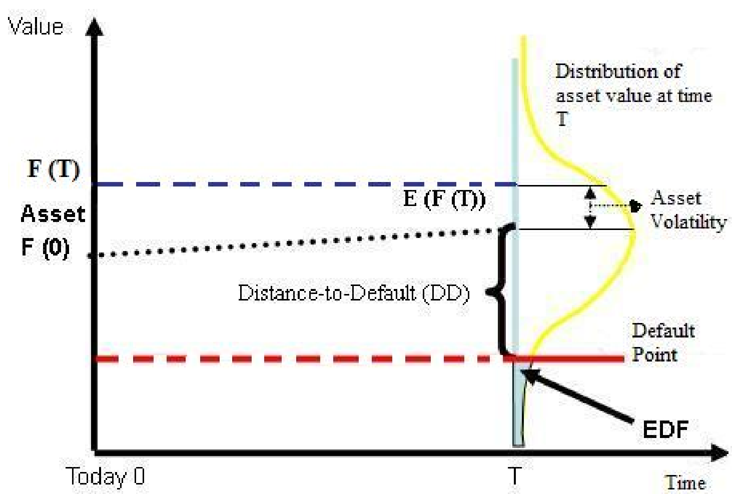


Figure 2 Distribution of the firm's asset value at maturity of the debt

This default probability does not represent the actual default probability of a firm. Since the underlying asset is risky, the firm value is not drift at the risk-free interest rate r. In order to get an objective default probability, we need to replace the risk-free interest rate r by the expected return on the firm's asset μ_F. Then, we have

$$N(-\widehat{DD}) = N(-\frac{\ln(\frac{F(t)}{D}) + \left(\mu_F - \frac{\sigma_F^2}{2}\right)T}{\sigma_F\sqrt{T}}). \tag{26}$$

To evaluate the expected return on the firm's asset μ_F, we can use equation (20). In reality, the drift is higher than the risk-free rate on the return of the firm's asset, in despite of the same diffusion terms in both the objective and risk-neutral distribution of the value of the firm's asset. In fact, risk-neutral probability serves as an upper bound to objective default probabilities (see [9]).

2.3 Vasicek-Kealhofer (VK)'s Model

We have addressed that how KMV works in the framework of Merton's assumption so far. However, these assumptions may be not the same as the way how Moody operates. Moody's KMV uses a proprietary model called the VK model (see [2]) which is an generalization of the Merton's Model. In fact, Moody adopts the new concept to measure the default probability of the firm, called Expected Default Frequency (EDF) (see [7]). The EDF modifies the some assumptions in Merton's Model, and uses its large historical database to estimate the empirical distribution of distances to default. Basing on that distribution calculated, Moody's KMV calculates Expected Default Probability. An EDF credit measure is a physical probability of default for a given firm. EDF credit measures can be estimated by a software product called Credit Monitor (CM) by empirical mapping based on actual default rate to get the default probabilities from year 1 through 5. The newest version is EDF 8.0 (see [7]), which refines the mapping of the DD to the EDF using a much larger default database observed over a longer time period and updates the risk-free interest rate every month for conceptual consistency.

Since the modeling choices made by Moody are the proprietary information, we are only informed some fundamental properties from their few research papers so far.

Merton proposed that the company was funded by a single class of debt and a single class of equity without any coupon or dividend payments. This is not the case employed in the Moody's KMV. Moody incorporates the capital structure of the company are composed of five types of claims on the firm's cash flow: short term and long term liabilities, common and preference equities and convertible equity, not just the single class of equity and debt. In addition, the coupons and dividends are paid in continuous time. The firm can issue any new senior claims or repurchase on shares on the firms. Moreover, we need to concern the market value of the firm's assets[4] and the market value of the liabilities. Both two quantities can not simply equal to the corresponding face value, because the market value of firm's assets are not traded and can not observed directly, and the market value of liabilities is not a constant quantity, which fluctuated with its credit quality. In fact, Moody makes proprietary adjustments to the accounting information that they use to calculate the face value of debt.

The option nature of equity is the foundation of the Merton's model. In the Merton-KMV case, the option has the characteristic of "European call" at maturity fixed time T, but in Moody's KMV, it regards the option as "Perpetual down-out" option that would never expire and can exercise at any time.

According to observations in dozens of firms, Moody find out that when the default of the firm happens, the market value of the firm's assets lie between some point between total face value of liabilities and the short-term liabilities. Hence, the definition of the default of the firm that the value of the firm is less than the value of debt can not accurately be implied in calculating the actual default probabilities.

Basing on the empirical research of Moody, default point (DP) approximately equals to the sum of the short-term liabilities and half of the long-term liabilities. Hence, we can represent the distance-to-default as the number of standard deviations the asset value is away from default(See [3]).

$$DD = \frac{E(F(T)) - DP}{E(F(T))\sigma_F}, \tag{27}$$

D_S: Short-term debt

[4]The market value of the firm's asset is the net present value of firm's future cash flow.

D_L: Long-term debt

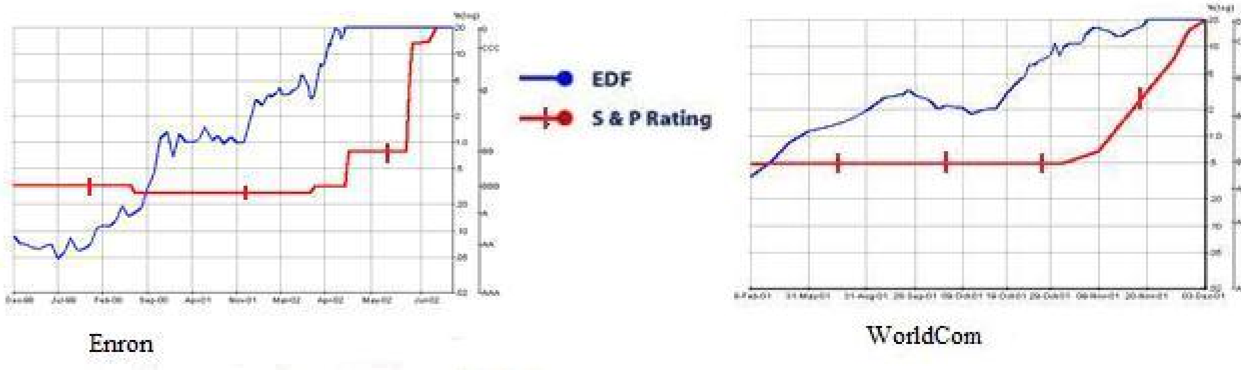
DP: Default point=$D_S + \frac{1}{2}D_L$

DD Distance-to-default: The distance between the market value of the firm's asset and the default point.

Finally, departing from the ideal of normal distribution in defining the default probability, Moody's KMV uses its large historical database to estimate the expected default frequency. Thus, we can use the database to determine a one-to-one relationship between DD and EDF. Due to this one-to-one relationship, two firms with the same DD will have the same EDF-even if they have different size, industry, and geography.

The most important implication in Merton approach is that Merton-KMV uses the normal distribution to define the probability default. In fact, using the normal distribution is very poor choice to define the probability default (see [3]). Firstly, let's go back to the default point. In Merton approach, the default point is a constant, and equals to the debt. However, in Moody's approach, the default point is a variable; it somehow links to the repurchase or issue of debts. In particular, the firm often adjusts their liabilities as they near default. Secondly, the default time is not necessary equal to the maturity time of the debt obligation; it could be any time before or at the time horizon. Indeed, market data can be updated daily because of changes in default point. Finally, the asset returns are wider tails than the normal distribution.

In Moody's case studies, KMV picks up WorldCom (see [10]) and Enron (see [11]) cases as examples of how its method takes advantages over its competitors, such as Standard & Poor. In both cases, when the equity price of both companies fell, the distance to default immediately decreased, following by the jump in EDF. The traditional rating agents took several days to corporate with this change. Clearly, EDF provides early warning power than those competitors. From Figure 3, we also observe that the EDF credit measure lead the traditional ratings in some sense. While traditional ratings are adjusted in discrete steps, EDF reacts any change in default risk dynamically and continuously. These results present that using equity values to infer default probabilities reflect information faster and more comprehensive than the traditional ones.



Source: Moody's KMV
Figure 3 Enron and WorldCom EDF and S&P Rating

3 Extension to two classes of debts

In Merton's assumption, the capital structure of the firm only has single class of debt and equity. Now we extend this assumption to the one that consists of two classes of debts and one equity. The debt claims on the firm are differentiated by their maturity times—the long-term and short-term debts. The short-time debt has a maturity time of T_S with the face value of D_S, and the long-term debt has a maturity time of T_L with the face value of D_L. All debts are paid without financing new investments. Now, let's discuss the value of the equity and the default probability of the firm under this new assumption in the firm's capital structure.

Suppose the current time is $t = 0$, and the firm need to pay its debt at the time $t = T_S$. If the firm can not meet the payment of the debt as defined in event of default previously, it defaults immediately. Otherwise, the firm survives, but the value of the firm at time T_S reduced by the payment of the debt D_S, as shown in Figure 4. When it reaches the time T_L, the firm needs to pay its due debt—namely the long-term debt D_L. Again, if the firm can't meet its obligation, it defaults immediately. To discuss the value of the equity and the default probability of the firm from time $t = 0$ to time $t = T_S$, we need split up the time period into two independent parts. First part is time period $0 \to T_S$, and the other is $T_S \to T_L$.

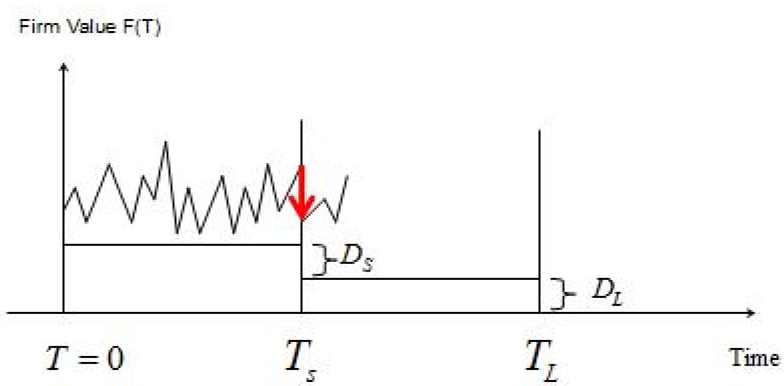


Figure 4 two classes of debts

- The value of the equity

To work out the value of the equity, we need to caculate backward. First, we consider the value of the equity at time period between T_S and T_L. At the maturity of the long-term debt T_L, the debt is worth the $\mathrm{Min}(F(T_L), D_L)$, and the equity is worth the $\mathrm{Max}(F(T_S) - D_S, 0)$. The bondholder gets the face value of the debt, D_L, and the equityholder gets the difference between the value of the firm and the face value of the debt, $F(T_S) - D_S$, whenever the value of firm at T_L is greater than the value of the debt D_L.

Inferring from the Black-Scholes-Merton equation (13) and equation (16), we could write the value of the equity during the period between T_S and T_L as the function

$$\begin{aligned} E(F(t), t) &= F(t)N(d_1^{T_S}) - D_L e^{-r(T_L - t)} N(d_2^{T_S}) \\ d_{1,2}^{T_S} &= \frac{\log(F(T_S)/D_L) \pm (r - \frac{1}{2}\sigma_F^2)(T_L - t)}{\sigma_F \sqrt{T_L - T_S}}, \end{aligned} \quad (28)$$

where $N(\cdot)$ is cumulative normal distribution.

Since the firm can't refinance new investment to cover its debt payment, the value of the firm has a discrete jump at the time T_S. But, the equity value would be continuous if the firm is not default at this point, otherwise there is an arbitrage oppotunity. We need to discuss the value of the firm before and after this discrete jump, which represented the time before and after as T_S^- and T_S^+. Thus we have the following relation at time T_S:

$$\begin{cases} F(T_S^-) & = F(T_S^+) + D_S \\ E(T_S^-, F(T_S^-)) & = E(T_S^+, F(T_S^+)) \end{cases} \quad (29)$$

At time of T_S^-, the debt is worth $\mathrm{Min}(D_S, F(T_S^-))$, and the equity is worth $\mathrm{Max}(E(T_S^-, F(T_S^-)), 0)$. Since the equity is viewed as an option on the value of the firm, this call option can be regarded as an option on option, or a compound option. Then we could write the value of the equity, along with equation (28) and (29), as following:

$$E(F(t), t) = \begin{cases} 0 & 0 < F(t) < D_S \\ C_{BS}(F(t) - D_S, t; D_L, T_L) & F(t) > D_S \end{cases} \quad (30)$$

We could write equation (30) as

$$\begin{aligned} E(F(t),t) &= e^{-r(T_S-t)} E(C_{BS}(F(t)-D_S,t;D_L,T_L)1_{(F(t)\geq D_S)}) \\ &= e^{-r(T_S-t)} \int_{D_S}^{\infty} C_{BS}(F(t)^{'}-D_S,t;D_L,T_L)p(F(t),t;F(t)^{'},T_S)dF^{'} \\ &= e^{-r(T_S-t)} \int_{0}^{\infty} C_{BS}(F(t)^{''},t;D_L,T_L)p(F(t),t;F(t)^{''}+D_S,T_S)dF^{''} \\ &= e^{-r(T_S-t)} \int_{0}^{\infty} f(F(t)^{''})dF^{''}, \end{aligned} \tag{31}$$

where p is the transitional probability density function and
$f(F(t)^{''}) = C_{BS}(F(t)^{''},t;D_L,T_L)p(F(t),t;F(t)^{''}+D_S,T_S)$.

The equation (31) can not be solved analytically. We need use composite trapezium rule to solve it numerically. The basic idea is to fix the infinite in the integral to some value to make $f(F(t)^{''})$ samll enough, and using the equation of the composite trapezium rule to write down the approximated solution.

- the default probability

To caculate the default probability of the firm under two classes of the firm's debts, we need to calculate the survival probabilities for both parts. By multiplying two survival probabilities, we have the default probability in this period.

We start with the period between 0 and T_S. To calculate the survival probability at the time T_S, we would simply adopt the probability of default for the single class of debt D_S such that

$$\begin{aligned} S_{T_S} &= 1 - \Pr\left[F(T_S) < D_S\right] \\ &= 1 - N(-DD_S) \\ &= N(DD_S), \end{aligned} \tag{32}$$

where $DD_S = \frac{In(\frac{F(0)}{D_S})+(\mu_F-\frac{1}{2}\sigma_F^2)T_S}{\sigma_F\sqrt{T_S}}$.

Similarly, the survive probability in the period between T_S and T_L is

$$\begin{aligned} S_{T_L} &= 1 - \Pr\left[F(T_S) - D_S < D_{T_L}\right] \\ &= 1 - N(-DD_L) \\ &= N(DD_L), \end{aligned} \tag{33}$$

where $DD_L = \frac{In(\frac{F(T_S)-D_S}{D_L}) + (\mu_F - \frac{1}{2}\sigma_F^2)(T_S - T_L)}{\sigma_F\sqrt{T_S - T_L}}$.

With the results above, we would have the default probability of the firm during the period between $t = 0$ and T_S as following

$$\begin{aligned} Default\,Probability &= 1 - S_{T_L} \times S_{T_S} \\ &= 1 - N(DD_S) \cdot N(DD_L) \end{aligned} \tag{34}$$

Once again, we can't just simplely work out the default probability due to the unknown quantities $F(t)$ and σ_F. However, we can solve two nonlinear simultaneous equation (20) and (31) to determine these two unknown quantities. Within knowing the other "observable" quantities, we would easily compute the default probability for the case of two classes of the debts.

4 Testing the model

4.1 Data Selection

To use the equations derived in section three, I choose the eighteen firms in total to observe their default probabilities from year 2004 to year 2006. Each six firms are in the same industry which are Energy, Electrical & Electronical (EE) and Wholesales. In each industry, the firms are categorized as the good, normal and poor performance firms[5] according to their size, revenue and earning per share. We call the poor, normal and good performance as Sample one, two and three respectively.

4.2 Parameter's Estimation

When we calculate the parameters in KMV, we use both Excel and Matlab to implement the corresponding data's and solve the two nonlinear simultaneously equations (16) and (21) to work out the value and volatility of the firm's assets. Within knowing all parameters, we start to solve the Distance-to-Default (DD) by equation (24). Relying the assumption that the value of the firm's assets follows a geometric Brownian motion, we would work out the implied default probability in equation (25) or the actual default probability through the equation (26) by calculating the Equity Delta, Gamma and Theta. The calculation steps are in the following sequence.

- The volatility of the equity

The volatility of the equity is calculated by the historical equity return data. Since in the assumption that the stock price follows the geometric Brownian motion, we would assume that μ_i is the log return at the i_{th} day, S_i and S_{i-1} are the closing price of the stock at the i_{th} and $i-1_{th}$ day respectively. Then we have

$$\mu_i = \ln \frac{S_i}{S_{i-1}}. \tag{35}$$

By using the historical data to predict the volatility introduced by Hull (see [12]), we can work out the volatility of the equity in the following year.

[5] All these company have the positive net income, we define their categories in the comparable senses.

$$\sigma_E = \frac{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\mu_i^2 - \frac{1}{(n-1)n}\left(\sum_{i=1}^{n}\mu_i\right)^2}}{\sqrt{\frac{1}{n}}} \tag{36}$$

where n is the trading day, which is approximately equal to 253.

- The market value of the equity

The value of the equity value is directly extracted from the *Wharton Research Data Service*(See [13]) at the beginning of the each year in which we start to forecast.

- Risk-free interest rate

In this case, I use the interest rate of the One-Year Treasury Bill as the risk-free interest rate. The data are from *Econstats* (see [14]). But since this rate is fluctuated from month to month in last few years, I take the average of the 12 month's interest rates in the forecasting year in order to produce more accurate results. The risk-free interest rate for the year 2004, 2005 and 2006 are 1.87%, 3.62% and 4.93%.

- Time

In general, the firm has a complex liability structure, and also we can't gain the access to the details of the maturity time of this structure. Here, we assume the firm's liabilities will be matured in the time of one year. So literally, the time $\tau = T - t = 1$.

- Liability of the firm

From the Moody's Research, the value of the firm's liability is roughly equal to the short-term one plus half of the long-term one. The data of the short-term and long-term are from the *Wharton Research Data Service* (see [13]).

- The value and volatility of the firm's asset

Once we have derived the above five parameters, we start to work out the value and volatility of the firm's asset. The two nonlinear simultaneous equations (16) and (21) are complicated, we use matlab to solve the solutions of the system of the equations, and also modify this system of the equation into the following set:

$$\begin{cases} f(F) &= FN(d_1) - e^{-r\tau} DN(d_2) - E \\ f(\sigma_E) &= \frac{F\sigma_F N(d_1)}{E} - \sigma_E \end{cases} \tag{37}$$

The basic ideals of the calculations are listed step by step below.

1. The initial volatility of the firm's asset is replaced by the volatility of the equity. Substituting this new value in first function in equation (37), we derive the corresponding value of the firm's asset.

2. Substituting the value of the firm's asset calculated in step 1 into the second function in equation (37) to get the corresponding volatility of the equity.

3. If the volatility of the equity calculated in step 2 is equal to the real volatility of the equity[6], the program stops, see Figure 5. Otherwise, we need to readjust the volatility of the firm's asset, and iterate the step 1 and 2 till the condition in step 3 is reached.

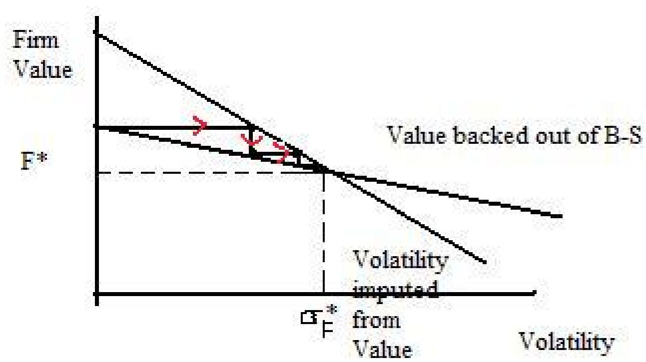


Figure 5 Iteration between value and volatility of firm's asset

For the equation (37), when we find the solution set of the volatility and value of the firm's asset, we need to concern if this solution set is unique. The answer is YES. We could simply differentiate the firt function in equation (37) with respect to firm's value F to get

[6]The real volatility is equal to the one we computed by the historical data from the daily stock price of the last year.

$\frac{\partial f(F)}{\partial F} = N(d_1) + \frac{FN'(d_1) - De^{-r\tau}N'(d_2)}{F\sigma_F\sqrt{\tau}}$, where $N'(\cdot)$ is the normal probability density function. Rearrange the above equation, we then have $\frac{\partial f(F)}{\partial F} = N(d_1)$. Because $N(d_1)$ is greater than zero, $\frac{\partial f(F)}{\partial F}$ is greater than zero. $f(F)$ is the increasing function of F, then $f(F)$ has the unique solution. Consequently, σ_E is the unique solution to the equation (30). Actually, we could interpret above explanations as the confirmation that the solutions of the volatility and value of the firm's asset are the unique solutions.

- Distance-to-default (DD) and Implied Default Probability (IDP)

Once we have all the parameters, we could start to calculate the Distance-to-default, and then the default probability by using normal distribution mapping in the assumption that the value of the firm's asset follows a geometric Brownian motion. For the implied default probability, we could simply substitute the parameters in equation (25). However, for the actual default probability, we need to calculate the delta, theta and gamma for the equity, and then substitute these parameters, along with the others, into the equation (21) to work out the expected return of the firm's asset. Associating with equation (26), we would have actual default probability of each firm. In this case, we only talk about the the implied default probability.

4.3 Data Analysis

In Appendix 1, 2 and 3, we have recorded and derived all the parameters mentioned above. In all three appendixes, we could easily see that volatility of the equity is always higher than that of the firm's asset. This results are due to the firm's capital structure that the value of the firm's asset involve the value of the equity and the liability, which is always greater than zero. We also see that the IDP varies from one to another.Now we are analyzing the result by listing them by their performance and industry.

4.3.1 DD and IDP

First, we consider their performances. We categorized these 18 firms into the three sample (see Figure 6). According to their performance, we name the the poor, normal and good performance as Sample one, two and three respectively. We discuss the 18-pairs data in 2004 in order to make some observations or comments.

Sample1		Sample2		Sample3	
DD	IDP	DD	IDP	DD	IDP
1.7381	4.11E-02	4.2815	9.28E-06	4.31	8.16E-06
2.8204	2.40E-03	2.7317	3.15E-03	3.5336	2.05E-04
4.9214	4.30E-07	4.6497	1.66E-06	5.9993	9.91E-10
3.6318	1.41E-04	5.7588	4.23E-09	5.6735	7.00E-09
2.7146	3.32E-03	2.5556	5.30E-03	6.1362	4.23E-10
2.5556	5.30E-03	3.1956	6.98E-04	3.9512	3.89E-05

Figure 6 DD and IDP in three samples of YEAR 2004

In general, the good performance firms have less chance to default, and the bad performance firms have more chance to default. The normal performance firms has the possibility to default lying between good and bad performance firms. Here, DD presents as a ordinal number in indicating the default probability. In Figure 7, it obviously reflects the difference in DD by their corresponding performances. Even viewing their DD in terms of the average number, we find out that DD in good performance firms is 4.93, 3.86 in normal performance firms, and 3.06 in bad performance firms. These figures also reflected the difference in DD by their performance. We can interpret these observations as credit reliability among these public firms.

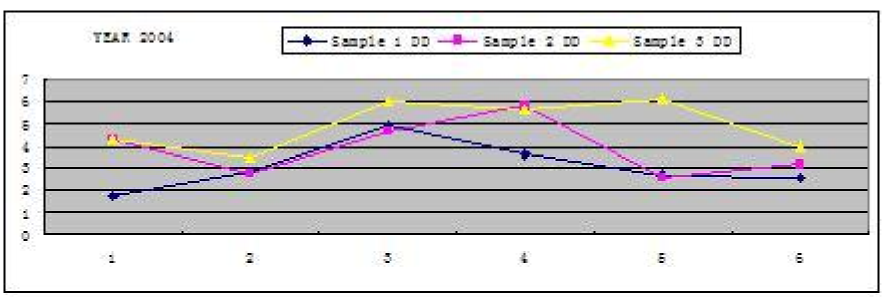


Figure 7 DD comparisons in three samples

From Figure 8, IDP has proved its function to indicate the difference in default probability, but it only plays a role as a ordinal number. Figure 8 presents the variations of IDF of the different performance firms. Although the difference has been showed, the difference between the normal and good performance firms is tiny, it may not satisfy the phenomenon in the reality. In addition, the numerical values of IDF in all three samples are obviously small.

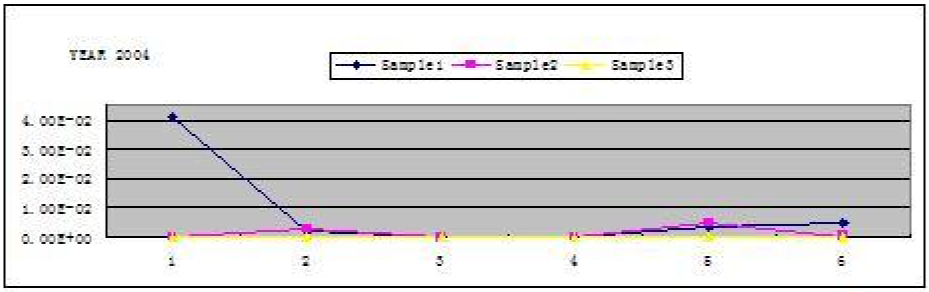


Figure 8 IDP comparisons in three samples

In Figure 9, it presents the relationship between DD and IDP among 18 firms, and roughly shows the inverse one-to-one relationship between DD and IDP, confirming Moody's research results (see [3]). Moreover, we observe that the inverse relation is more obvious before DD reaches 3.5. After this critical point, the relationship almost becomes a straight line, which is not sensitive in projecting DD. This fact explains that when DD reach some points, DD is vulnerable to project IDF, consequently, IDF is lack of accuracy in distinguishing the creditability of the firm.

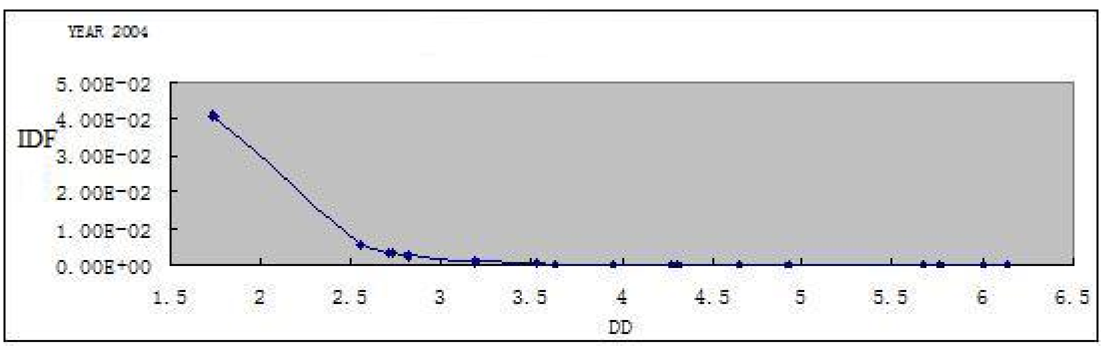


Figure 9 Relationship between DD and IDP

4.3.2 DD and Volatility of the firm

Secondly, we now are focused on 18 firms in different industries. Figure 9 and Figure 10 presents the average of DD and IDP in each year. No surprisingly, DD and IDP varies in each industry. This is due to the volatility of the firm's asset in each industry, which possesses the different economic characteristics. From the caculation, we observe that the the volatility of the equity is inverse proportional to the DD.

Figure 9 Average DD comparisons

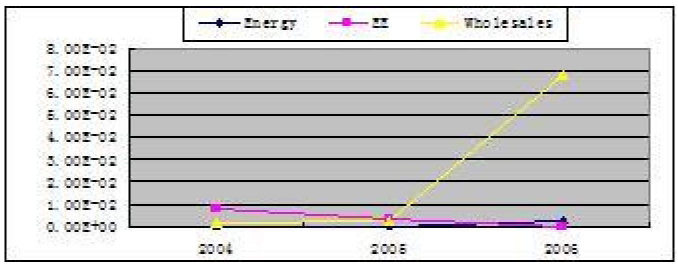


Figure 10 Average IDP comparisons

4.3.3 DD and Value of firm

In Figure 11, 12, 13, we illustrate the relationship between the DD and the value of the firm's asset in three industries. To make the easy comparisons, $Log(F)$ is used, which represents the logarithm of the value of the firm's asset. We would observe that no matter which industry is, the larger the value of the firm is , the greater Distance-to-Default is. In the other word, the larger the value of the firm is, the less chance the firm defaults.

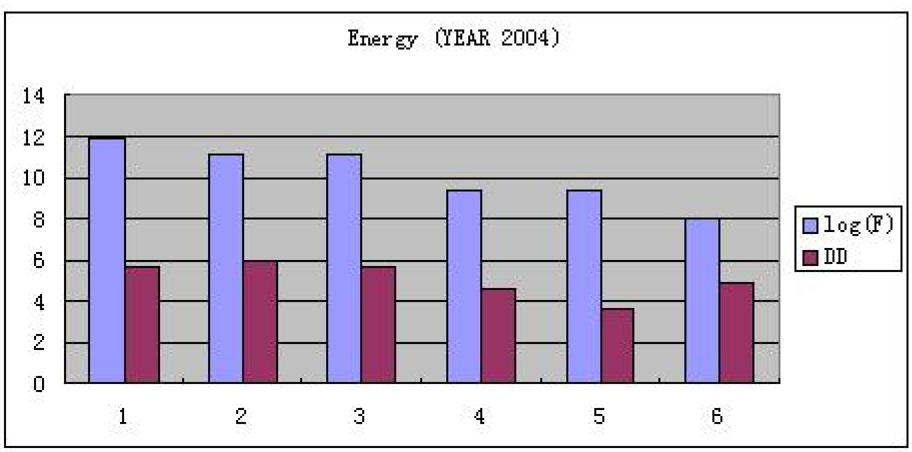


Figure 11 Log(F) and DD in Energy

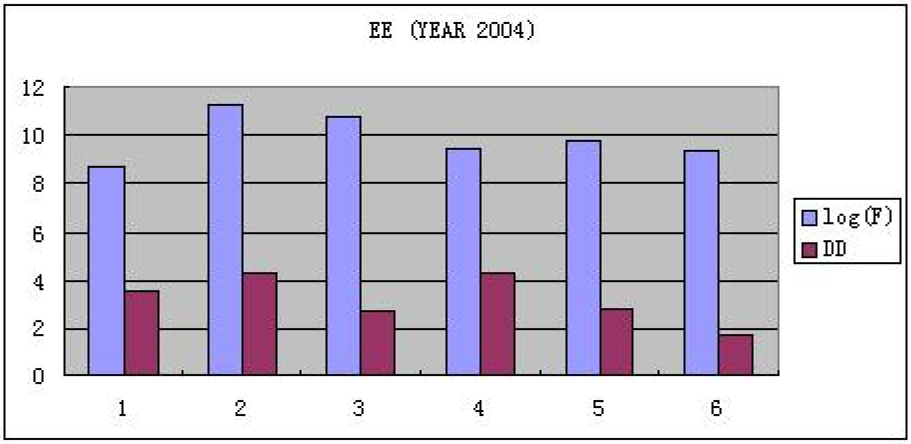


Figure 12 Log(F) and DD in EE

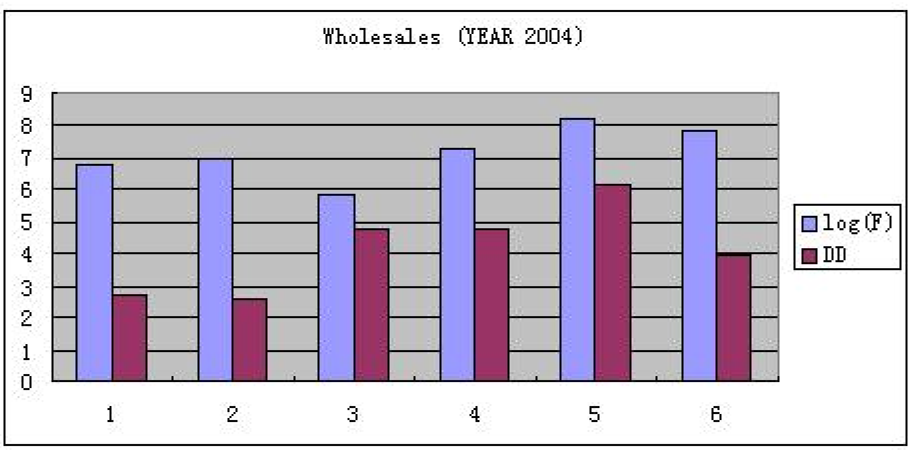


Figure 13 Log(F) and DD in Wholesales

4.4 Discussion

The KMV is the structural model, which relies on the basic ideal of the Merton's Model. It has the concrete theory to support it and also uses the market value of equity price and financial data as inputs. In some sense, it offers the reliable results to assess the credit risk of the firm. The above examples have well explained their functions. However, it may not be precise as we expect.

When we derived the implied default probability of the firm, we relied on the assumption that the value of the firm's asset follows the geometric Brownian motion. But, the normal distribution is a very poor choice to define the probability of default (see [3]). The Moody's uses large historical default data of the US firms to find the relationship between Distance-to-default and probability of default. Nonetheless, we need ask if the trend these large database inferred is sufficient to apply in the different market. And, the default point can not be regarded as a certain number, it is a random variable (see [3]) in fact. Beside these subjective reasons, there are some other objective reasons. They also contribute to the accuracy in assessing the credit risk of the firm. Although DD is gained by calculating the equity value that is fully decided by the market and includes backgrounds on financial situation, we can't guarantee the equity value indicates the true value of the firm. There are several reasons for this but the most important is the case that the unreliable accounting data mislead the effectiveness of the exchange market. For some industries and public traded firms, the model may not work smoothly. For example, the recent example is that sub-prime crisis causes the troubles of Northern Rock. The story of this mortgage bank was finally ended by the interference of the government. This kind of incidents happen in almost every industry with those firm having great social and political impacts. In these situations, the default of the firm would be saved by "outside" forces. The most critical part is that we can not just use simple risk-free interest rate as the expected return of the firm's asset. Instead, we need use the real expected return of the firm's asset by some methodologies.

5 Conclusion

This article describes the basic idea of the KMV under two different models, and extend Merton's KMV to fit into the firm, which has two classes of the debts. Moreover, we applied Merton's KMV to a number of US firms in three different industries over the period from 2004 to 2006.

By examining the real data, KMV appears to have the ability to forecast the default of the firm, and also the result confirms the KMV's claims that the default probability is inverse proportional to the distance-to-default. However, our analysis suggests that the model are useful in ranking companies rather than in identifying their default probabilities. This result is caused partly by the assumption that the firm value follows a geometric Brownian motion. Nonetheless, the model we implement is successful in finding the relation among the Distance-to-default (DD), the volatility of the firm's asset and the value of the firm's asset.

We acknowledge that our implementation of the Merton's KMV is not the same as the one that the Moody's implements, and therefore VK's Model would produce better result than the one tested in this paper.

Acknowledgment

I thank my supervisor Professor Sam Howison for his continuous support in this paper.

References

[1] Merton, R.C.: "On the pricing of corporate debt: The risk structure of interest rates", *The Journal of Finance*, Vol. 29, No. 2, pp. 449-470, 1974.

[2] Arora,N., Bohn,J.R. and Zhu, F.: "Reduced Form vs. Structural Models of Credit Risk: A Case Study of Three Models", Research Paper, *Moody's KMV*, 2005.

[3] Crosbie, P.J. and Bohn, J.R.: "Modeling Default Risk", Research Paper, *Moody's KMV*, 2002.

[4] Kealhofer,S. and Kurbat,M.: "Benchmarking Quantitative Default Risk Models: A Validation Methodology", Research Paper, *Moody's KMV*, 2000.

[5] Bharath, S.T. and Shumway, T.: "Forecasting Default with the KMV-Merton Model", Working Paper, *The University of Michigan*, 2004.

[6] Black,F. and Scholes: "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, Vol. 81, pp. 637-659, 1973.

[7] Dwyer, D. and Qu, S.: "EDF^{TM}8.0 Model Enhancements, Research Paper", *Moody's KMV*, 2007.

[8] McKean, H.P., 1969, Stochastic Integrals, New York, Academic Press.

[9] Delianedis, G. and Geske, R.: "Credit Risk And Risk Neutral Default Probabilities - Information About Rating Migrations And Defaults", *EFA 2003 Annual Conference Paper*, No. 962, 2003.

[10] EDF Case Study: Worldcom, *Moody's KMV*, 2003.

[11] EDF Case Study:Enron, *Moody's KMV*, 2002.

[12] Hull, J., 2002, Options, Futures, and Other Derivatives, Fifth Edition, International Edition, Prentice Hall.

[13] Wharton Research Data Service, http://wrds.wharton.upenn.edu.

[14] EconStats, http://econstats.com/r/r_em1.htm.

	Firm	Year	σ_E	E(Million)	DP(Million)	σ_F	F(Million)	DD	IDP
		2004	0.5531	6491	5311.5	0.3067	11657.1	1.7381	4.11E-02
	Sun Microsystems	2005	0.4634	6438	6589	0.2332	12792.7	2.0992	1.79E-02
Sample		2006	0.3223	6674	6141	0.1718	12519.6	3.0620	1.10E-03
One		2004	0.3501	13372	5424.5	0.2504	18696.0	2.8204	2.40E-03
	Sprint	2005	0.2636	13521	17227.5	0.1183	30136.0	3.7607	8.47E-05
		2006	0.2167	51937	32223	0.1362	82609.9	4.5903	2.21E-06
		2004	0.2320	6460	6075.5	0.1206	12422.9	4.2815	9.28E-06
	Emerson Electric	2005	0.1768	7238	6731	0.0932	13729.7	5.6325	8.88E-09
Sample		2006	0.1687	7400	7379	0.0865	14424.0	5.9079	1.73E-09
Two		2004	0.3625	37846	8088	0.2997	45784.2	2.7317	3.15E-03
	Intel	2005	0.3054	28579	8785	0.2356	37051.7	3.2470	5.83E-04
		2006	0.2058	36182	10683	0.1606	46351.1	4.8420	6.43E-07
		2004	0.2306	27864	57246.5	0.0765	84049.9	4.3100	8.16E-06
	IBM	2005	0.1472	29747	59617	0.0502	87244.5	6.7789	6.06E-12
Sample		2006	0.1787	33098	53901	0.0701	84406.1	5.5759	1.23E-08
Three		2004	0.2811	1301	4824.5	0.0606	6036.1	3.5336	2.05E-04
	Whirlpool	2005	0.2329	1606	5280	0.0558	6698.3	4.2731	9.64E-06
		2006	0.2715	1745	5402	0.0688	6887.1	3.6573	1.27E-04

Appendix 1. EE Sector

	Firm	Year	σ_E	E(Million)	DP(Million)	σ_F	F(Million)	DD	IDP
Sample One	Sunoco	2004	0.2023	965.4	2035.0	0.0659	2962.7	4.9214	4.30E-07
		2005	0.2670	1327.1	11301.0	0.0290	12226.3	3.7318	9.50E-05
		2006	0.3439	1887.0	16625.0	0.0366	17712.3	2.8913	1.92E-03
	Valero Engery	2004	0.2703	5534.7	6496.7	0.1256	11911.0	3.6318	1.41E-04
		2005	0.3082	7589.9	8063.6	0.1522	15366.8	3.1662	7.72E-04
		2006	0.4189	14982.0	12491.5	0.2335	26872.6	2.2813	1.13E-02
Sample Two	Tesoro	2004	0.2140	4140.5	8642.8	0.0702	12623.2	4.6497	1.66E-06
		2005	0.1508	4726.9	9877.5	0.0500	14253.2	6.6133	1.88E-11
		2006	0.2337	4915.5	14056.5	0.0628	18295.8	4.2566	1.04E-05
	ConocoPhillips	2004	0.1730	34366.0	38055.5	0.0829	71716.5	5.7588	4.23E-09
		2005	0.2062	42723.0	40655.0	0.1075	81932.6	4.8232	7.06E-07
		2006	0.3006	52731.0	48493.0	0.1603	98891.3	3.2894	5.02E-04
Sample Three	Chevron	2004	0.1656	36295.0	30643.0	0.0905	66370.3	5.9993	9.91E-10
		2005	0.1693	45230.0	33386.5	0.0989	77429.5	5.8649	2.25E-09
		2006	0.2240	62676.0	44084.0	0.1342	104639.4	4.4098	5.17E-06
	Exxon Mobile	2004	0.1750	89915.0	61374.5	0.1048	150152.5	5.6735	6.996E-09
		2005	0.1551	101756.0	68240.5	0.0942	167570.4	6.4101	7.273E-11
		2006	0.2315	111186.0	71728.0	0.1434	179463.6	4.2649	9.997E-06

Appendix 2. Energy Sector

	Firm	Year	σ_E	E(Million)	DP(Million)	σ_F	F(Million)	DD	IDP
Sample One	WESCO International	2004	0.3642	167.7	736.6	0.0686	890.6	2.7146	3.318E-03
		2005	0.4007	353.6	783.8	0.1278	1109.4	2.4489	7.165E-03
		2006	0.5701	491.5	939.8	0.2104	969.3	0.2685	3.942E-01
	Reliance Stl & Almn	2004	0.3842	647.6	462.3	0.2259	1101.4	2.5556	5.301E-03
		2005	0.3929	822.6	514.8	0.2450	1319.0	2.4981	6.244E-03
		2006	0.3421	1029.9	536.5	0.2287	1540.6	2.8851	1.956E-03
Sample Two	World Fuel Services	2004	0.2090	148.4	207.0	0.0882	351.6	4.7592	9.720E-07
		2005	0.3154	188.5	495.3	0.0892	666.2	3.1382	8.498E-04
		2006	0.4308	353.3	648.1	0.1571	970.2	2.2668	1.170E-02
	Airgas	2004	0.3092	691.9	740.8	0.1508	1419.0	3.1956	6.977E-04
		2005	0.2567	814.2	905.5	0.1239	1687.5	3.8631	5.597E-05
		2006	0.2663	947.2	1001.5	0.1327	1900.4	3.7223	9.872E-05
Sample Three	Genuine Parts	2004	0.1625	2312.3	1410.6	0.1016	3696.7	6.1362	4.225E-10
		2005	0.1717	2544.4	1521.8	0.1089	4012.1	5.8057	3.205E-09
		2006	0.1466	2694.0	1663.3	0.0924	4277.3	6.8024	5.146E-12
	Grainger	2004	0.2515	1845.1	743.1	0.1802	2574.5	3.9512	3.889E-05
		2005	0.2089	2068.0	702.0	0.1574	2745.0	4.7670	9.351E-07
		2006	0.2137	2289.0	773.0	0.1617	3024.7	4.6595	1.585E-06

Appendix 3. Wholesales Sector