

Investigations in Calabi-Yau Modularity and Mirror Symmetry



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and

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Dedicated to my
sister and brother,
Charlotte and Stewart.

Abstract

This thesis lays out a number of different projects, linked by the common thread of Calabi-Yau manifolds.

We produce tables of instanton numbers for various multiparameter Calabi-Yau manifolds. Studying those tables reveals, in some cases, a Coxeter group of symmetries that act on sets of instanton numbers. Instanton numbers are constant over the orbits of group actions on the curve classes, and these groups can be infinite.

We investigate to what extent this symmetry can be used to constrain the holomorphic ambiguity that arises in higher genus topological string free energy computations. In the example we study, at genus 2, combining this Coxeter constraint with the set of vanishing numbers fixes the holomorphic ambiguity.

A new class of solutions is provided to the supersymmetric flux vacuum equations, which have been conjectured elsewhere to give weight-two modular manifolds.

Another search, complementary to those that have already been carried out, is made for rank-two attractors in the AESZ list. Two novel examples are found, both of which belong to moduli spaces with two points of maximal unipotent monodromy each. For one of these operators, the additional MUM point corresponds to another derived equivalent geometry. For the other operator, we compute *nonintegral* values of the triple intersection number and genus 0 instanton numbers, and so a geometric interpretation is lacking for the second MUM point.

We provide several instances of summation identities that express a ratio of critical L-values as an infinite sum, whose terms contain the Gromov-Witten invariants of a Calabi-Yau manifold Y . In one example there is no manifold Y , only an operator, but a set of invariants can nonetheless be computed for this and a summation identity is found. The L-functions are associated to a modular manifold X which is mirror to Y . These sums can be divergent, but Padé resummation cures this.

In addition to these results, we briefly review informing aspects of Calabi-Yau geometry, black holes in 4d $\mathcal{N} = 2$ supergravity, flux compactifications, topological string theory, and number theory.

The contents of the tables of this thesis are available to download at [1].

Statement of Originality

Parts of this thesis are based on joint work with Philip Candelas, Xenia de la Ossa, and Pyry Kuusela.

Chapters One and Two serve as an introduction and review of relevant background material, neither contains new results.

Chapter Three describes ongoing work, building on ideas that appeared in [2] where the appearance of the Coxeter groups was noted.

Chapter Four also describes ongoing work. Parts of the discussion of flux vacua appear in [3]. The series solution method for the IIA attractor equations was first given in [4].

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Chapter 1

Introduction

I must go down to the seas again, for the call of the running tide
Is a wild call and a clear call that may not be denied;

John Masefield, *Sea Fever*

1.1 Motivation

It is difficult to overstate the wealth of interesting mathematics attached to Calabi-Yau manifolds as they appear in physics. This thesis documents an attempt to look into these geometries, specifically compact threefolds, to see what can be found. This approach is informed by two broad ideas, both of which have seen major recent advancement on which we rely. These are *modularity* and *mirror symmetry*.

Modularity has a good modern physics pedigree, appearing for instance as a highly constraining symmetry in CFT partition functions. The modular group also appears as a symmetry of type IIB string theory, which is vital for the geometric uplift to F-theory by way of elliptic fibrations. The “modular” here refers to the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$.

This group Γ has a deep significance to the arithmetic of elliptic curves. Arithmetic, by which we mean the study of varieties over finite fields, is perhaps less familiar to physicists

of some schools than the aforementioned role played by Γ . It is a fact that the set of point counts for an elliptic curve, considered as a variety over a finite field, are collected into a generating function that transforms in a highly constrained way under a subgroup of Γ . Namely, this function is a weight-two modular newform for a congruence subgroup of Γ . This is the modularity theorem for elliptic curves [5–7]. We will better explain that claim in §2.4, but for now press on to make the point that this can, rarely, happen for Calabi-Yau threefolds [8].

Whether or not this happens can change as the Calabi-Yau is subject to a complex structure deformation, so that the question of “Which Calabi-Yau threefolds are modular?” becomes the question “Which families contain modular members, and where in the moduli space are they?”. The reason that the physicist should care about this question lies in two putative answers, each reliant on some conjectures [9–12]:

- Rank-two attractors, in the context of four dimensional $\mathcal{N} = 2$ supergravity, are weight four modular.
- Supersymmetric vacua in IIB flux compactifications are weight-two modular.

At the most extreme end of optimism then, one could hope that a better understanding of arithmetic modularity could shed light on black hole physics and string model building. This would see a well-developed area of mathematics being brought to bear on some persistent problems in physics. We come far short of that in this thesis, but these aspirations, and other recent works realising these possibilities, serve as the ultimate motivation. For example, the works [13, 14] were able to express D-brane masses and semiclassical black hole entropies using Mellin transforms of the modular forms associated to modular geometries.

The other *raison d’être* of this thesis is mirror symmetry, specifically two-dimensional mirror symmetry. This is a duality between certain quantum field theories, specifically twisted $\mathcal{N} = (2, 2)$ nonlinear sigma models with Calabi-Yau target space. Any duality between two quantum field theories should be squeezed for all that can be learnt from it, and this duality

comes with a particularly nice geometric picture. It is possible to obtain, suitably defined, counts of holomorphic embeddings of genus- g curves of various degrees into a Calabi-Yau threefolds Y by studying the complex-structure periods of a mirror geometry X .

There is a captivating problem in carrying out this computation at higher genera. There is a procedure for doing this, BCOV recursion [15],[16], which at each genus suffers from a holomorphic ambiguity. This is a rational function that must be fixed, up to which the complete B-model free energy at genus g is determined by the lower genus results. This ambiguity can be fixed by a number of considerations, for instance [17], but these do not suffice to reach arbitrarily high genus. A full solution to this stubborn problem could involve a strong handle on patterns in the sets of instanton numbers, which motivates our work along these lines.

Additionally, the mirror symmetry computation is understood to give higher derivative stringy correction terms to the supergravity action [18, 19]. This duality then, is offering one way to approach the problem of formulating string theory nonperturbatively. Topological string theory is a simplified version of string theory, and so it is a natural expectation that a full understanding of the latter will be helped by first understanding the topological theory, as discussed recently in [20].

Mirror symmetry gives a duality between supergravity compactifications. These are low energy limits of string theories, with there conjecturally being a duality between two full string compactifications that exchanges the nonperturbative contents of both theories. This supergravity duality gives one a means of studying the interplay between mirror symmetry and the modularity that can be present in supergravity compactifications. One can consider extremal black holes from the IIB or IIA perspectives. It is in IIB that we will first consider rank two attractors. In the IIA description the enumerative invariants, computed by mirror symmetry techniques, enter as quantum corrections to the supergravity action. The upshot is that the mirror map provides identities relating the enumerative geometry of one space to the modularity of another, and these are naturally encountered in the course of investigating

black hole solutions.

1.2 Thesis overview

We have throughout this thesis strived to denote Calabi-Yau manifolds by X and Y , and always have it so that B-model computations are performed on X while A-model computations are reserved for Y . We will consider IIB on X and IIA on Y . Where Y is a CICY, X is a CICY mirror. We will always refer to instanton numbers of Y .

Chapter §2

This is a brief collection of some preliminary results and terminology. We define Calabi-Yaus, and introduce the favourable CICY set. These will recur throughout the thesis as a sufficiently varied set of examples for our purposes. We go on to set up the period computation that we will use frequently. We explain how, for CICY mirrors X , the CICY matrix defines the topological quantities and periods of X . We also review the splitting process for CICYs, and all of these will bear relation to discussion in §3.

We go on to give a very basic account of arithmetic modularity, sufficient for the purposes of searching for rational rank-two attractors later on in §4. We will meet the zeta function, which provides the means of testing for modularity. The zeta function is first defined by suitably collecting point counts into a power series, but remarkably this is resummed into a rational function. For modular manifolds, the numerator of this function factorises. It is from these factors that the relevant modular form is read off.

Chapter §3

Here we detail our studies in mirror symmetry. We compute genus 0 numbers for manifolds with $h^{1,1}$ as high as 7. We compute genus one numbers for some geometries with lower $h^{1,1}$, but cannot do this for the 6 and 7 parameter models. The genus 1 computations requires a piece of data not strictly necessary at genus 0, the manifold's discriminant locus. We

compute a number of examples of these, and comment on some patterns in this data set, in the hopes of eventually returning to the 6 and 7 parameter models.

It is already known that the process of splitting a Calabi-Yau partitions its instanton numbers, so obtaining the instanton numbers of the new manifold. We showcase some examples of this, for a wider class of models than simply CICYs.

The data from our tables reveals a surprising fact, that sometimes sets of instanton numbers can be symmetric under the action of a Coxeter group on the manifold's second homology. These are groups that are generated by reflections, and are encoded by diagrams similar to Dynkin diagrams. The Weyl group associated to a Dynkin diagram is also a Coxeter group, but Coxeter groups are a much larger set with a different classification theory. We go on to tabulate some Coxeter groups that appear in our data, giving their diagrams and their representation on the curve classes of the Calabi-Yaus to which they are associated.

Armed with Coxeter symmetry, we address the higher genus computation. We review the construction of the propagators out of which higher genus B-model free energies can be built, and cover BCOV recursion. We work an example at genus 2, presented in such a way that we hope is valuable to anybody hoping to carry out this computation. When it comes time to fix the genus-2 holomorphic ambiguity, we apply the Coxeter symmetry and observe that this does not fully fix the ambiguity. We discuss what other information is required.

Chapter §4

This chapter opens by recalling some textbook information on Calabi-Yau supergravity compactifications. Then, we undertake a substantial review of the attractor mechanism's derivation, including the recasting of the attractor equations into Strominger's form for the fixed points of the flow. With the attractor mechanism explained for general 4d $\mathcal{N} = 2$ supergravities, we then discuss its details where Calabi-Yau compactifications are concerned.

Next, we depart from black holes and instead look at flux compactifications. We review both the IIB construction and its lift into F-theory à la Sen, to see how precisely the IIB

axiodilaton profile is realised as the modulus for the elliptic fibre in a Calabi-Yau fourfold. At this point in the thesis, we make a small digression. We discuss possible sources of corrections to the potential in a Calabi-Yau flux compactification, and explain how the no-scale relation is violated by a nonzero Euler characteristic χ , or by worldsheet instantons if $\chi = 0$.

Then, we return to flux compactifications and explain our solution method for the equations defining supersymmetric vacua. We provide a number of examples, the crux of these being their axiodilaton's dependence on the complex structure moduli. This allows for the F-theory curve to be identified.

Next we turn our sights back to black holes, and use zeta function computations to identify two new rank-two attractors. There exists a way, given a rank-two attractor, to construct a summation identity relating L value ratios to Gromov-Witten invariants. We display a number of these. Not all of these sums converge, but the divergent ones can be treated with Padé resummation.

The identities so given bring together critical L-values and instanton numbers, in some examples distinct sets of instanton numbers, for different spaces that are derived equivalent. We discuss some of the geometric aspects of these derived equivalent partners.

Chapter 2

Some essentials

This is where the fun begins.

Anakin Skywalker

A manifold is *Calabi-Yau* if it is complex, Kähler, and has vanishing first Chern class. In this thesis we will be concerned with compact Calabi-Yau manifolds. They possess a twofold utility in string theory. One side of this is that the two-dimensional $\mathcal{N} = (2, 2)$ nonlinear sigma model is conformal exactly when the target space is Calabi-Yau. The other side is that an equivalent characterisation of a Calabi-Yau n -fold is that the manifold has $SU(n)$ special holonomy, and so possesses a covariantly constant spinor. This allows for supersymmetry to be preserved in compactifications of supergravity or string theory on these manifolds. Specifically, Calabi-Yau compactifications preserve one half of the supercharges found in the ten-dimensional theory. They saw introduction to the theoretical physics literature in [21].

In complex dimension 1, the Calabi-Yau condition is met only by elliptic curves. These can be realised as quotients of the complex plane $\mathbb{C}$ by a lattice Λ_τ , with complex structure parameter the lattice parameter τ . The choices τ and $-\tau$ give isomorphic elliptic curves. Two lattices $\Lambda_\tau, \Lambda_{\tau'}$ are equal if τ and τ' are related by an $SL(2, \mathbb{Z})$ transformation, and so the elliptic curves obtained as quotients $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda_{\tau'}$ are also equal. Therefore, the moduli space of one-dimensional Calabi-Yau manifolds is given by the upper half plane

modulo $SL(2, \mathbb{Z})$ transformations.

In complex dimension two, the set of complex Kähler manifolds with vanishing first Chern class consists of products of elliptic curves, Abelian surfaces, and K3 surfaces (see [22, 23]). Only K3 surfaces have holonomy $SU(2)$, and they all lie in one complex analytic, but not algebraic, family.

The situation in three dimensions and higher is starkly different, and does not currently possess a full characterisation. There are distinct families, unrelated by complex structure deformation, and it is not known if the number of such families in any given dimension is finite. Our studies will concern the three-dimensional manifolds. It was shown in [24] that the moduli space of a Calabi-Yau manifold X factorises locally, into a product of the space of Kähler structures on X , $\mathcal{M}_K(X)$, with the space of complex structures on X , $\mathcal{M}_{CS}(X)$. The first of these spaces can be thought of as the coefficients of X 's Kähler form when expanded in a basis of $H^2(X, \mathbb{Z})$ while the second parametrises complex structure deformations of X , which in the simplest cases are variations of the coefficients of the polynomials defining X as a complete intersection. Recall that the spaces of the n^{th} cohomology of a complex manifold admit a Hodge decomposition into (p, q) -forms, with $p + q = n$, with each of these spaces having dimension equal to the Hodge number $h^{p,q}$.

We will only consider examples with full $SU(3)$ holonomy and not a subgroup. This is the case for manifolds with $h^{1,0} = h^{2,0} = 0$, because it can be shown that the existence of a holomorphic $(1, 0)$ or $(2, 0)$ form reduces the holonomy from $SU(3)$ to a subgroup [25]. In so doing we exclude the product of three elliptic curves, and the product of an elliptic curve with a K3 surface. All Hodge numbers of X are fixed once $h^{1,1}$ and $h^{1,2}$ are specified. This is because complex conjugation equates $h^{p,q}$ to $h^{q,p}$, while Hodge duality equates $h^{p,q}$ to $h^{3-q, 3-p}$. We have stated that $\text{Dim } \mathcal{M}_K(X) = \text{Dim } H^2(X, \mathbb{Z})$, which in turn equals $h^{1,1}$ because $h^{2,0} = h^{0,2} = 0$. It can also be shown that $\text{Dim } \mathcal{M}_{CS}(X) = h^{2,1}(X)$ [24].

2.1 Periods and the triple intersection form

Up to scale, Calabi-Yau manifolds possess a unique holomorphic $(3,0)$ form denoted Ω . After expanding Ω in a basis of $H^3(X, \mathbb{Z})$, one obtains the period vector of X . The $b_3 = 2h^{1,2} + 2$ components of this vector vary as the complex structure of X is modified, and satisfy the Picard-Fuchs system of X . For manifolds with a one-dimensional space of complex structures (so $h^{2,1} = 1$), the Picard-Fuchs equation is a fourth order Fuchsian differential equation (that is, a fourth order ODE, with coefficients that are rational functions in the dependent variable, with regular singular points). There are a number of choices of the basis in which one expands Ω , but the two that we will most frequently see are the Frobenius basis and integral symplectic basis. The Frobenius period vector, denoted ϖ , arises as the first set of solutions obtained by the method of Frobenius. The integral symplectic basis sees the most use in physical applications. Recall that linear systems of differential equations possess monodromies, whereby continuing a vector of solutions analytically around a singularity effects a linear transformation on the solution vector. Importantly, the moduli space $\mathcal{M}_{CS}$ possesses a point of maximal unipotent monodromy, or MUM point, around which the monodromy matrix M obeys $(M - \mathbb{1})^{b_3} = 0$ but $(M - \mathbb{1})^{b_3-1} \neq 0$.

We shall write φ^i , $i = 1, \dots, h^{1,2}$ for the complex structure moduli coordinates of X , or φ to denote the list of all coordinates. The Picard-Fuchs system admits as a solution a power series ϖ_0 about a MUM point, which we shall take to be at $\varphi = \mathbf{0}$, with radius of convergence given by the distance from the MUM point to the nearest singularity of the system. By normalising this function ϖ_0 to equal 1 at the MUM point we uniquely specify it. Once an analytic expression for the coefficients in the power series defining ϖ_0 in an expansion about $\varphi = \mathbf{0}$ is known (as a ratio of factorials), along with some topological data that we will explain, it is possible to obtain the other periods using the method of [26, 27] as follows.

First, introduce dependence on formal parameters ϵ_i into ϖ_0 as follows:

$$\varpi_0(\varphi) = \sum_{\mathbf{m} \geq 0} c(\mathbf{m}) \varphi^{\mathbf{m}} \quad \rightarrow \quad \varpi^\epsilon = \sum_{\mathbf{m} \geq 0} \frac{c(\mathbf{m} + \boldsymbol{\epsilon})}{c(\boldsymbol{\epsilon})} \varphi^{\mathbf{m} + \boldsymbol{\epsilon}}. \quad (2.1)$$

Our choice of coordinates φ is discussed in Appendix §A. The function $c(\mathbf{m})$ is extended to a function of non-integer argument above by replacing factorials with Gamma functions, $n! \mapsto \Gamma(1 + n)$. $h^{2,1}$ periods with logarithmic singularities at the MUM point can then be found by taking ϵ -derivatives:

$$\varpi_{1,i} = \left. \partial_{\epsilon_i} \varpi^\epsilon \right|_{\boldsymbol{\epsilon}=0}. \quad (2.2)$$

We will shortly introduce and define a set of constants Y_{ijk} . This can be used to construct a further $h^{2,1}$ solutions with logarithm-squared singularities at the MUM point like so:

$$\varpi_{2,i} = \left. \frac{1}{2} Y_{ijk} \partial_{\epsilon_j} \partial_{\epsilon_k} \varpi^\epsilon \right|_{\boldsymbol{\epsilon}=0}. \quad (2.3)$$

The last period, which has logarithm-cubed behaviour near $\varphi = 0$, is obtained via

$$\varpi_3 = \left. \frac{1}{6} Y_{ijk} \partial_{\epsilon_i} \partial_{\epsilon_j} \partial_{\epsilon_k} \varpi^\epsilon \right|_{\boldsymbol{\epsilon}=0}. \quad (2.4)$$

These functions are then collected into the Frobenius period vector,

$$\varpi = \begin{pmatrix} \varpi_0 \\ \varpi_{1,i} \\ \varpi_{2,i} \\ \varpi_3 \end{pmatrix}. \quad (2.5)$$

Let us draw attention to a notational point. Where one-parameter manifolds have been studied in the literature, it has sometimes been common to use the symbols ϖ_2 and ϖ_3 to denote the above functions divided by Y_{111} , i.e. it has been common to have ϖ_2 and ϖ_3 with expansions beginning respectively with $\log(\varphi)^2$ and $\log(\varphi)^3$. In this thesis we will consider

one-parameter manifolds on par with the multiparameter geometries, and so will always use the above contraction with Y_{ijk} .

The quantities Y_{ijk} are topological quantities not on X , but on a *mirror manifold* to X . If Y denotes this mirror family, then Y_{ijk} gives the triple intersection form of two-cycles on Y . That is, if e_i form a basis of the second cohomology of Y ,

$$Y_{ijk} = \int_Y e_i \wedge e_j \wedge e_k . \quad (2.6)$$

This quantity is an integer, equalling the intersection number of the four cycles that are Poincaré dual to each e_i .

Having explained how to form a Frobenius basis of solutions from the data of ϖ_0 and Y_{ijk} , it remains to determine analytic expressions for the coefficients in ϖ_0 's series expansion. This can be done by analysing the Picard-Fuchs equations directly, but then one must first write these down, which is not always straightforward. ϖ_0 can be determined by certain residue integrals, as in [28]. The method that we will adopt (outside of the one-parameter cases where we have the Picard-Fuchs operator), is to use Liebgober's formula that relates the sought coefficient $c(\mathbf{m})$ to the total Chern class of Y , when X and Y are intersections in toric varieties [29]. The total Chern class can be formally written as a rational function of generators of the second cohomology e_i , with a numerator and denominator that factorise into linear factors. By effecting for each such factor the replacement

$$\left(1 + \sum_{i=1}^{h^{2,1}(X)} a_i e_i \right) \rightarrow \Gamma \left(1 + \sum_{i=1}^{h^{2,1}(X)} a_i m_i \right)^{-1} , \quad (2.7)$$

we obtain $c(\mathbf{m})$ as a ratio of Gamma functions.

When we turn to studying string-theory compactifications, we shall find a need to express period vectors in an integral symplectic basis, as we now explain. Consider a path in moduli space that begins and ends at some value φ_* , and goes around a singularity. Encircling

singularities in the complex structure moduli space effects monodromy transformations of the period vector. Since such an encircling returns us to the same point φ_* in moduli space that we started at, these monodromy transformations must be a symmetry of the theory. By choosing to expand the period vector in a basis such that monodromy transformations are integral symplectic matrices, we can pair the monodromy transformations of the period vector with symplectic transformations of the charge vector, to have a manifest monodromy invariance of quantities like D-brane masses or black hole central charges (these are the symplectic inner products of charge vectors with period vectors). Indeed, $\mathcal{N} = 2$ Maxwell theory possesses a symplectic duality symmetry. We explain here how to make the change of basis from Frobenius to integral symplectic.

First, one divides each Frobenius period ϖ by an appropriate power of $2\pi i$, since upon circling 0 anticlockwise we have $\frac{1}{2\pi i} \log(\varphi) \mapsto \frac{1}{2\pi i} \log(\varphi) + 1$:

$$\widehat{\varpi}_0 = \varpi_0 , \quad \widehat{\varpi}_{1,i} = \frac{1}{2\pi i} \varpi_{1,i}, \quad \widehat{\varpi}_{2,i} = \frac{1}{(2\pi i)^2} \varpi_{2,i} , \quad \widehat{\varpi}_3 = \frac{1}{(2\pi i)^3} \varpi_3 . \quad (2.8)$$

The integral symplectic period vector Π is then computed from

$$\Pi = \rho \widehat{\varpi} , \quad \rho = \begin{pmatrix} -\frac{1}{3}Y_{000} & -\frac{1}{2}\mathbf{Y}_{00}^T & \mathbf{0}^T & 1 \\ -\frac{1}{2}\mathbf{Y}_{00} & -Y_0 & -\mathbb{1} & \mathbf{0} \\ 1 & \mathbf{0}^T & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{1} & \mathbf{0} & \mathbf{0} \end{pmatrix} . \quad (2.9)$$

We have already seen the triple intersection numbers Y_{ijk} , with i running from 1 to $h^{2,1}(X)$. it will turn out to be natural to consider these as members of a larger set Y_{abc} with indices running from 0 to $h^{2,1}$. The vector $\mathbf{Y}_{00}$ has components Y_{00i} , which are defined in terms of the mirror manifold Y 's second Chern class c_2 as

$$Y_{00i} = -\frac{1}{12} \int_Y c_2 \wedge e_i . \quad (2.10)$$

The quantity Y_{000} is

$$Y_{000} = \frac{3\zeta(3)}{(2\pi i)^3} \chi(X) = -\frac{3\zeta(3)}{(2\pi i)^3} \chi(Y) , \quad (2.11)$$

where in the last equality we have used that the mirror pairing exchanges the complex structure and complexified Kähler structure of X and Y , and so their Hodge numbers $h^{1,1}$ and $h^{2,1}$ are interchanged, and thereby X and Y have Euler characteristics that differ by a sign.

The quantities Y_{0ij} , which we collect into the matrix $\mathbb{Y}_0$, can be taken to have values 0 or $1/2$. Symplectic transformations allow for shifts by integer values, so we make the choice to take values in $\{0, 1/2\}$. As was explained in [27], exactly which number each Y_{0ij} equals is determined by demanding that the monodromy matrix that acts on Π after encircling the MUM point at $\varphi = 0$ is integral and symplectic. This condition relates the Y_{0ij} to the Y_{ijk} , and one can compute in the 1, 2, 3 modulus cases that

$$Y_{0ij} = \frac{Y_{ijj} \bmod 2}{2} . \quad (2.12)$$

2.2 CICYs

A set that admits a particularly simple characterisation are the *Complete Intersection Calabi-Yaus*, or CICYs [30]. These are the intersection of hypersurfaces in products of projective space. Such a manifold Y can be represented by a configuration matrix, a rectangular array where for each projective space factor in the ambient space there is a row. Each column represents one of the defining equations. The (i, j) -entry of the array gives the degree of the j^{th} equation in the i^{th} projective space. Three illustrative examples are

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} , \quad \mathbb{P}^5 \begin{bmatrix} 3 & 3 \end{bmatrix} , \quad \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} . \quad (2.13)$$

The first of these represents a hypersurface (the vanishing set of a single equation) in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ defined by a polynomial with degrees $(2, 2, 3)$ in the homogeneous coordinates on each projective space. The second represents the intersection of two degree-3 hypersurfaces in $\mathbb{P}^5$. The third represents the intersection of two hypersurfaces in the space $\mathbb{P}^1 \times \mathbb{P}^4$. The general configuration matrix, which we display to fix notation in this section, is

$$\begin{matrix} \mathbb{P}^{n_1} \\ \vdots \\ \mathbb{P}^{n_K} \end{matrix} \begin{bmatrix} d_1^{(1)} & \dots & d_C^{(1)} \\ \vdots & \dots & \vdots \\ d_1^{(K)} & \dots & d_C^{(K)} \end{bmatrix} . \quad (2.14)$$

For the purposes of this section, K will be the number of projective space factors in the ambient space and C will be the number of equations defining the intersection.

Not any configuration matrix yields a Calabi-Yau. One must verify that the first Chern class $c_1(TY)$ vanishes. This is an exercise in classical algebraic geometry: one can form the total Chern class from the entries of the configuration matrix, which is Taylor expanded to get the Chern classes c_1 , c_2 and c_3 . The first of these needs to vanish, while the third can be used to compute the Euler characteristic $\chi(Y)$. The space Y is an intersection in an ambient space $\mathcal{A} = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_K}$, and there is a split exact sequence

$$0 \rightarrow TY \rightarrow T\mathcal{A} \rightarrow NY \rightarrow 0 , \quad (2.15)$$

where the tangent bundle TY of Y injects into the tangent bundle of $\mathcal{A}$, which in turn projects onto the normal bundle NY of Y . Stated in more pedestrian terms, the following decomposition (which is non-canonical, depending on the choice of the embedding of NY as a subbundle of $T\mathcal{A}$) holds at points on Y :

$$T\mathcal{A} = TY \oplus NY . \quad (2.16)$$

The Chern class of the vector sum of two spaces is the product of their individual Chern

classes, and so

$$\text{Ch}(T\mathcal{A}) = \text{Ch}(TY) \cdot \text{Ch}(NY) , \quad \text{whereby} \quad \text{Ch}(TY) = \frac{\text{Ch}(T\mathcal{A})}{\text{Ch}(NY)} . \quad (2.17)$$

The total Chern class of $\mathcal{A}$ is the product of the Chern classes of the factors $\mathbb{P}^{n_i}$. Since $\text{Ch}(\mathbb{P}^{n_i}) = (1 + E_i)^{n_i+1}$, where E_i generates the second cohomology of $\mathbb{P}^{n_i}$, we get

$$\text{Ch}(T\mathcal{A}) = \prod_{i=1}^K (1 + E_i)^{n_i+1} . \quad (2.18)$$

It is through the total Chern class of NY that the entries of the configuration matrix enter.

One has that

$$\text{Ch}(NY) = \prod_{j=1}^C \left(1 + \sum_{i=1}^K d_j^{(i)} E_i \right) . \quad (2.19)$$

The total Chern class then is

$$\text{Ch}(TY) = \frac{\prod_{i=1}^K (1 + E_i)^{n_i+1}}{\prod_{j=1}^C \left(1 + \sum_{i=1}^K d_j^{(i)} E_i \right)} , \quad (2.20)$$

which we expand to linear order in the cohomology generators E_i to find the first Chern class c_1 . This gives

$$c_1(TY) = \sum_{i=1}^K \left(1 + n_i - \sum_{j=1}^C d_j^{(i)} \right) E_i , \quad (2.21)$$

whereupon we learn that the Calabi-Yau condition is met precisely when the sum of the configuration matrix's i^{th} row equals $n_i + 1$. Since we are interested in threefolds, we must have $c + 3 = \sum_{i=1}^K n_i$.

There is a computational problem in enumerating all possible rectangular matrices with nonnegative entries that satisfy the above two conditions. However, there are a number of possible redundancies which mean that a straightforward listing of all possible configuration matrices overcounts the number of CICY threefolds. For example, for m -vectors $\mathbf{a}$, $\mathbf{b}$, an

$n \times m$ matrix M , and a ‘vector’ P of n projective spaces, there is the identity

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ P \end{array} \begin{bmatrix} 1 & \mathbf{a}^T \\ 1 & \mathbf{b}^T \\ \mathbf{0} & M \end{bmatrix} = \begin{array}{c} \mathbb{P}^1 \\ P \end{array} \begin{bmatrix} \mathbf{a}^T + \mathbf{b}^T \\ M \end{bmatrix}, \quad \text{for example} \quad \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}. \quad (2.22)$$

This follows from the fact that

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbb{P}^1. \quad (2.23)$$

Upon removing such redundant descriptions as above, one arrives at a naive number of 7890 CICY threefolds. This is still not the full story, as additional nontrivial redundancies exist beyond those used to find this number. This list is held in many places, we have in this work made use of Lukas’s website [31], from which one can for instance find the redundancy

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (2.24)$$

This is a family to which we shall give some attention in later sections, preferring to use the more symmetric description on the left-hand side.

More generally, one could consider intersections in toric varieties (of which CICYs are but a special case). Hypersurfaces in toric varieties are in bijection with reflexive four-dimensional polytopes, which were tabulated in [32]. The enumeration of more general intersections of hypersurfaces in toric varieties of dimension greater than 4 has not been completed. Using gauge theory constructions, intersections in toric varieties are realised as phases of an abelian gauged linear sigma model. By dropping this abelian condition, one could reach a wider-still set of manifolds. Examples of this construction include the Rødland manifold [33] and the two-parameter model of Knapp and Hori [34].

As we turn to the study of particular problems, we shall lean on the CICY threefolds as our

examples. However, we stress that a number of the properties that we will observe can be expected to hold in more general examples.

There is an important partition of the CICY manifolds into two sets: those that are *favourable* and those that are not. A complete intersection Y is favourable if its second cohomology is spanned by the pullbacks to Y of the generators for each projective space's second cohomology. For example,

$$\mathbb{P}^5[3, 3] \quad \text{and} \quad \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \quad (2.25)$$

respectively have one- and two-dimensional second cohomologies. In the first example this is spanned by the pullback of the Kähler form for the ambient $\mathbb{P}^5$, while in the second the pullbacks of the Kähler forms for the ambient $\mathbb{P}^1$ and $\mathbb{P}^4$ span the manifold's second cohomology. Note that a favourable manifold necessarily has $h^{1,1} = K$. This is to be contrasted with non-favourable CICYs, for which $h^{1,1} > K$ (we will always take K to be the number of projective space factors in the representative among redundant configurations with the smallest number of rows). An example of a CICY with this feature is

$$\begin{matrix} \mathbb{P}^4 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 0 & 0 & 2 & 2 & 1 \\ 2 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad (2.26)$$

which has $h^{1,1} = 12$. The pullbacks of the Kähler form for each $\mathbb{P}^4$ factor each appear in the manifold's second cohomology, but there are ten other generators. To see where these come from, we should recall that the (non-Calabi-Yau) intersection $\mathbb{P}^4[2 \ 2]$ is the Segre surface, isomorphic to the projective plane blown up in 5 points. In addition to the hyperplane class, this space has an additional 5 generators in the second cohomology dual to cycles wrapping each blown up point. So by viewing the above CICY manifold as an intersection in the product of two Segre surfaces, we can account for $h^{1,1} = 12$ by pulling back to X the six independent generators in each Segre surface's cohomology. More generally, the $h^{1,1}$ of a non-favourable manifold can be explained in terms of the cohomology classes on del Pezzo surfaces that arise from the configuration matrix in a similar manner as above.

To compute $h^{2,1}$, it suffices to compute $h^{1,1}$ as in the previous paragraph and then use the total Chern class to compute $\chi = \int_Y c_3$. Recall that the Euler characteristic of Y is the alternating sum of Betti numbers, $\chi(Y) = \sum_{i=0}^6 (-1)^i b_i$. Now, $b_0 = b_6 = 1$ as our manifolds each have a single connected component. Further, $b_1 = b_5 = 0$ (in light of full $SU(3)$ holonomy). We have that $b_2 = b_4 = h^{1,1}$ and $b_3 = 2 + 2h^{1,2}$, so that

$$\chi = 2(h^{1,1} - h^{1,2}) . \quad (2.27)$$

We see that $h^{2,1}$ can be determined via $h^{2,1} = h^{1,1} - \frac{1}{2}\chi$, with χ determined from the configuration matrix via the total Chern class formula. Note that for a CICY Y , $\chi(Y) \leq 0$. Following [30], the triple intersection numbers of a favourable CICY Y can be computed by noting that each two-form e_i on Y is the pullback, with respect to the inclusion map $\iota : Y \mapsto \mathcal{A}$, of the Kähler forms E_i of each $\mathbb{P}^{n_i}$. That is,

$$e_i = \iota^*(E_i). \quad (2.28)$$

As a result of this

$$Y_{ijk} = \int_Y e_i e_j e_k = \int_{\mathcal{A}} E_i E_j E_k \prod_{j=1}^C \left(1 + \sum_{i=1}^K d_j^{(i)} E_i \right) , \quad (2.29)$$

where the product inserted in the right-hand integral is the Poincaré dual to Y in the cohomology of $\mathcal{A}$. The above integral is performed by singling out the coefficient of the volume form on $\mathcal{A}$, which is the product $\prod_{i=1}^K E_i^{n_i}$.

We have now provided the total Chern class and triple intersection numbers of a favourable CICY Y . We can now use (2.20) and (2.29) to compute the periods of mirror manifolds X as in (2.1).

2.3 Splittings and the conifold transition

The data of a CICY matrix gives only the degrees of the polynomials defining the Calabi-Yau variety, and says nothing about the coefficients of these polynomials. Generic choices of these polynomials, related by variation of complex structure, will be smooth and be of the same topological type (in particular they will have the same Hodge numbers and triple intersection form). Nonetheless, for nongeneric choices of these coefficients the geometry will acquire singularities. Deformation (varying the polynomial coefficients again) will return a smooth manifold in the generic family, but one could also resolve these singularities via a blowup to obtain some smooth member of a different family of manifolds, with different topological properties. The latter family obtained from the singular space by resolution is said to be a ‘split’ of the former family obtained from the singular space via deformation.

Let us be more concrete, and explain the splittings via operations on CICY matrices. Let us begin with the family Y , with configuration matrix

$$Y : \quad \mathbf{P} \left[\mathbf{v} \quad \mathbf{M} \right] , \quad (2.30)$$

with $\mathbf{P}$ denoting a list of projective spaces, $\mathbf{v}$ being a vector of nonnegative integers, and $\mathbf{M}$ being a matrix. Now it may be the case that for some nongeneric choice of coefficients, the first equation of the above configuration can be expressed as a determinant. That is to say, the equation read off from $\mathbf{P}[\mathbf{v}]$ is

$$\text{Det}(C^a_b) = 0 , \quad (2.31)$$

where the square matrix C , of some dimension $(n+1) \times (n+1)$, has entries that are polynomials in the coordinates of $\mathbf{P}$. The set of such equations as the polynomials are varied may have members defining singular spaces, in which case the splitting we go on to describe is said to be *effective* and yields a manifold of a new topological type. Alternatively we

may be describing an *ineffective* splitting if the family of vanishing loci of $\det(C)$ has no singularities, so that we are not changing the topology of the family considered, instead just giving a different redundant description.

Vanishing of C 's determinant implies the existence of a nullvector for C , a nonzero $n + 1$ vector x^b such that

$$C^a{}_b x^b = 0 . \quad (2.32)$$

There are $n + 1$ such equations for each choice of the index a . Promoting x^b to homogeneous coordinates on $\mathbb{P}^n$, The above equation then is some element of the set of polynomials read off from

$$\mathbb{P}^n \begin{bmatrix} 1 & 1 & \dots & 1 \\ \mathbf{P} & \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_{n+1} \end{bmatrix} , \quad (2.33)$$

where $\mathbf{v} = \sum_{i=1}^{n+1} \mathbf{u}_i$. We learn that the family X can be split to a family X' with configuration

$$Y' : \quad \mathbb{P}^n \begin{bmatrix} 1 & 1 & \dots & 1 & \mathbf{0}^T \\ \mathbf{P} & \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_{n+1} & \mathbf{M} \end{bmatrix} , \quad (2.34)$$

which is guaranteed to be Calabi-Yau by the fact that Y was Calabi-Yau (one can check that the rows sum to appropriate values). Some examples of pairs (A, B) where B splits A are

$$\left(\mathbb{P}^4[5] , \mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \right) , \quad \left(\mathbb{P}^4[5] , \mathbb{P}^4 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \right) , \quad \left(\begin{array}{c} \mathbb{P}^1 \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \\ \mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \right) . \quad (2.35)$$

Note that a manifold can have distinct splits, as exemplified by the Quintic $\mathbb{P}^4[5]$ above.

2.4 Calabi-Yau modularity

We turn now to explaining what is meant by a *modular* Calabi-Yau manifold. Having done this, we will go on to see how Calabi-Yau modularity is related to supersymmetric physics.

The conventional physicist's view of a Calabi-Yau manifold is that it is an algebraic variety over the field of complex numbers $\mathbb{C}$. From a wider perspective, one is free to consider Calabi-Yau manifolds over any field. Of particular relevance to modularity are the finite fields $\mathbb{F}_{p^n}$, where in the subscript p is a prime and n a positive integer. This number p^n is the order of the finite field, which is necessarily the power of a prime.

Given a polynomial with rational coefficients, we can multiply through by the least common multiple of the coefficient's denominators in order to get a polynomial with integer coefficients. This in turn can be reduced modulo a prime.

Alternatively, suppose now that a Calabi-Yau X 's complex structure moduli take rational values m/n . X is then an algebraic variety over $\mathbb{Q}$, for which one writes $X/\mathbb{Q}$. The integers m, n have representatives in $\mathbb{F}_p$, simply found as $m \bmod p$ and $n \bmod p$. Now, $\mathbb{F}_p$ is a field and so for each integer n such that $p \nmid n$, there is an element $1/n$. So the product $m \cdot \frac{1}{n}$ can be found in $\mathbb{F}_p$. This allows for the equations defining X as an algebraic variety over $\mathbb{C}$, assumed to have rational coefficients, to be replaced with equations with coefficients in $\mathbb{F}_p$. All of that is to say, we can consider the variety X over the field $\mathbb{F}_p$, denoted $X/\mathbb{F}_p$.

The variety $X/\mathbb{F}_p$ is the set of solutions to these equations, where the indeterminates (coordinates on the ambient space containing X) take values in $\mathbb{F}_p$. So, this variety $X/\mathbb{F}_p$ is the union of a finite number of points in the ambient space.

The finite fields $\mathbb{F}_{p^n}$ can be realised as the field extension of $\mathbb{F}_p$ by the root of an irreducible polynomial with order n . That is to say, we can realise $\mathbb{F}_{p^n}$ by taking

$$\mathbb{F}_{p^n} \cong \mathbb{F}_p(\alpha) , \quad \text{with } \alpha \text{ a root of an irreducible polynomial } Q \text{ with coefficients in } \mathbb{F}_p. \quad (2.36)$$

There is a standard field embedding

$$\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^n} \quad (2.37)$$

which simply sends the field element $r \in \mathbb{F}_p$ to $r \in \mathbb{F}_p(\alpha) \cong \mathbb{F}_{p^n}$.

Since $\mathbb{F}_p$ embeds this way into $\mathbb{F}_{p^n}$, we can also take the equations defining the variety $X/\mathbb{F}_p$ to define another variety $X/\mathbb{F}_{p^n}$, which consists of another union of a finite number of coordinate vectors with elements in $\mathbb{F}_{p^n}$ so that these equations are solved.

Crucially, the information contained in each of these finite sets ‘fits together’ in a way that we shall now describe.

2.4.1 The local zeta function

Fix a prime p , and a set of algebraic moduli φ for the variety X_φ , which we shall subscript with φ in this subsection so as to make dependencies clear. We will write $N_{p^n}(\varphi)$ for the number of points constituting the variety $X_\varphi/\mathbb{F}_{p^n}$.

Now we introduce a formal variable T and collect these point counts into the following generating function:

$$\zeta_p(X_\varphi; T) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} N_{p^n}(\varphi) T^n \right). \quad (2.38)$$

This is the local zeta function of $X_\varphi/\mathbb{F}_p$. The three Weil conjectures [35], proved by Dwork [36], Grothendieck [37], and Deligne [38], describe the behaviour of this function. We shall state them on the assumption that $X_\varphi/\mathbb{C}$ is a d -complex dimensional projective variety, and not necessarily a Calabi-Yau threefold.

(1) Rationality

The zeta function is a rational function of T , with the numerator and denominator factorising over $\mathbb{Z}$ into the form

$$\zeta_p(X_\varphi; T) = \frac{P_1(X_\varphi; T) P_3(X_\varphi; T) \dots P_{2d-1}(X_\varphi; T)}{P_0(X_\varphi; T) P_2(X_\varphi; T) \dots P_{2d}(X_\varphi; T)}, \quad (2.39)$$

where each polynomial $P_i(X_\varphi; T)$ has degree equal to the Betti number b_i . In addition,

$$P_0 = 1 - T \quad \text{and} \quad P_{2d} = 1 - p^d T. \quad (2.40)$$

(2) Functional Equation

This function obeys the following transformation of argument property:

$$\zeta_p \left(X_\varphi; \frac{1}{p^d T} \right) = \pm p^{\frac{d\chi}{2}} T^\chi \zeta_p(X_\varphi; T), \quad (2.41)$$

with χ the Euler characteristic of X_φ .

(3) The Riemann hypothesis

Each polynomial P_i factorises over $\mathbb{C}$,

$$P_i(X_\varphi; T) = \prod_{j=1}^{b_i} (1 - \lambda_{ij}(X_\varphi)T), \quad (2.42)$$

and each λ_{ij} is an algebraic integer. That is, each λ_{ij} is a root in $\mathbb{C}$ of a monic polynomial with integer coefficients. Further, each λ_{ij} has absolute value $p^{i/2}$.

Note that the rational function (2.39) is further simplified when X_φ is a Calabi-Yau threefold. Then $d = 3$, and the Betti numbers are $b_0 = b_6 = 1$, $b_2 = b_4 = h^{1,1}$, $b_1 = b_5 = 0$, and $b_3 = 2h^{2,1} + 2$. The form of (2.39) is further constrained by the functional equation (2.41), so that for a Calabi-Yau threefold we get

$$\zeta_p(X_\varphi; T) = \frac{R_p(X_\varphi; T)}{(1 - T)(1 - pT)^{h^{1,1}}(1 - p^2T)^{h^{1,1}}(1 - p^3T)}, \quad (2.43)$$

with all φ dependence captured in the numerator $R_p(X_\varphi; T)$, which is a degree $b_3 = 2h^{2,1} + 2$ polynomial in T .

Let us stress the following point: for generic moduli φ we obtain some typically irreducible

polynomial $R_p(X_\varphi; T)$, but for particular choices of moduli $R_p(X_\varphi; T)$ can factorise for each p . When this happens, the coefficients of quadratic factors are conjecturally related to modular forms. We shall now explain this in greater detail.

2.4.2 Modular forms and modularity of elliptic curves

For elliptic curves E_φ (Calabi-Yau one-folds) over $\mathbb{Q}$, the zeta function numerator Q_p is a quadratic function of T of the form

$$Q_p(X_\varphi; T) = 1 - c_p(\varphi)pT + p^3T^2 . \quad (2.44)$$

We remark that the Weil conjectures for elliptic curves were originally proved by Hasse [39]. The modularity theorem for elliptic curves over $\mathbb{Q}$, proved in work by Breuil, Conrad, Diamond, Taylor, and Wiles [5–7] guarantees that the c_p so defined are the coefficients of q^p in the Fourier expansion of a weight-two modular form (in fact, a newform) for a congruence subgroup $\Gamma_0(N)$, where N is the conductor of the elliptic curve E_φ . This number N is divisible by each prime p that has bad reduction, which means that the elliptic curve $E_\varphi/\mathbb{F}_p$ is singular.

The modular group $\Gamma \cong \mathrm{PSL}(2, \mathbb{Z})$ of 2×2 matrices with integer components and determinant ± 1 acts on the upper half plane $\mathbb{H}$ (complex numbers with positive imaginary part) by the Möbius action:

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d} , \quad \text{where the matrix } \gamma \text{ has components } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \quad (2.45)$$

We shall always take an elements γ of Γ to have matrix components a, b, c, d as above. A modular form of weight $k \geq 0$ is a holomorphic function f on the upper half plane with the following transformation properties under the modular group:

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) . \quad (2.46)$$

There are functions that satisfy the above property not for Γ , but for certain subgroups. Specifically, *congruence subgroups*, which we now define. The principal congruence subgroup of level N is the group

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : a, d = 1 \text{ and } b, c = 0 \bmod N \right\}. \quad (2.47)$$

A congruence subgroup is any subgroup of Γ that contains a $\Gamma(N)$ as a subgroup. The examples that we shall make frequent return to, appearing in the statement of the modularity theorem for elliptic curves that we gave at the start of this subsection, are the groups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c = 0 \bmod N \right\}. \quad (2.48)$$

A modular form of level $N \in \mathbb{N}$ and weight $k \geq 0$ is a holomorphic function obeying (2.46) for all $\gamma \in \Gamma_0(N)$. These functions form a finite-dimensional vector space over $\mathbb{C}$, and newforms are particular basis elements of this space. These newforms are tabulated in the L-Functions and Modular Forms Database, LMFDB, [40]. Newforms are in particular cusp forms, which have 0 constant term in their Fourier expansions. We shall make frequent use of the LMFDB labels when discussing examples in later sections.

L functions

The Mellin transform of a modular form f yields the associated L -function:

$$L(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy f(iy) y^{s-1}. \quad (2.49)$$

Naturally, $\Gamma(s)$ is the Gamma function and not a group. In practice, one usually has the first few hundred (or more) terms in a Fourier expansion of f , which can be inserted into the above integral so as to compute values $L(s)$. The accuracy improves with more terms in f 's Fourier expansion, but a manipulation of the integrand using transformation properties of f allows for a quicker evaluation to a given accuracy. To do this, first break up the integral

like so:

$$L(s) = \frac{(2\pi)^s}{\Gamma(s)} \left(\int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy) y^{s-1} dy + \int_0^{\frac{1}{\sqrt{N}}} f(iy) y^{s-1} dy \right) . \quad (2.50)$$

Because the lower range of integration on the left-hand integral is nonzero, the approximations to this term obtained by truncating f 's Fourier expansion are much better than for the right-hand integral. To better approximate the right-hand integral we will make use of the Fricke involution, a property of cusp forms for $\Gamma_0(N)$. Such functions, with weight k , admit the following transformation (Fricke involution):

$$f\left(-\frac{1}{N\tau}\right) = \epsilon N^{k/2} \tau^k f(\tau) . \quad (2.51)$$

The quantity ϵ is the Fricke sign of the form f , which is equal to plus or minus one. We now effect a change of variables $\tau \mapsto -\frac{1}{N\tau}$ in the right-hand integral of (2.50), and then apply the formula (2.51), to obtain

$$L(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_{\frac{1}{\sqrt{N}}}^{\infty} dy f(iy) (y^{s-1} + (-1)^{k/2} \epsilon N^{k/2-s} y^{k-1-s}) . \quad (2.52)$$

This expression is well-approximated upon replacing f by a truncation of its Fourier expansion.

2.4.3 Modularity of Calabi-Yau threefolds

Whereas the modularity of elliptic curves over $\mathbb{Q}$ is a proven fact for all such curves, the same is not true of threefolds. The topic of Calabi-Yau modularity was reviewed in [8].

A Calabi-Yau is rigid if it has a pointlike space of complex structures, i.e. $h^{2,1} = 0$. Then $b_3 = 2$ and the zeta function numerators $R_p(X; T)$ are, by the Weil conjectures, automatically quadratic. It was proven by Gouvea and Yui [41] that such threefolds are indeed modular,

of weight-four. For these Rigid threefolds

$$R_p(X; T) = 1 - \beta_p T + p^3 T^2 \quad (2.53)$$

and the integers β_p are the coefficients of the p^{th} terms in the Fourier expansion of a weight-four modular form.

This is in contrast to manifolds X_φ with $h^{2,1} \neq 0$. The most general approach here, which we do not take, is to recognise in the zeta function of X_φ the coefficients of an automorphic form for a symplectic group's Langlands dual, in the style of the Langlands program [42–44]. The problem that we turn to is, for which moduli φ is X_φ modular? This means that the zeta function numerator $R_p(X_\varphi; T)$ possesses for all good primes p a quadratic factor from which we can read off a modular form.

There are a number of threefolds proven to be weight-four modular. Certain members of a family associated to the A_4 lattice were shown to be so by Hulek and Verrill [45], Schoen proved modularity of singular quintic threefolds in [46], and in [13] a number of conjecturally modular manifolds were displayed, and among them it was proven that the mirror of $\mathbb{P}^7 \left[\begin{smallmatrix} 2 & 2 & 2 & 2 \end{smallmatrix} \right]$ was modular at the conifold. In addition to these proven cases, a number of manifolds are conjectured to be modular with evidence given by an extensive computation of the zeta function. Such cases are included in the works [14, 47].

There are two problems in compactifications of type IIB superstring theory on X_φ , solved by restrictions on the moduli φ to values φ_* , so that conjecturally the manifolds X_{φ_*} are modular. The first of these is the identification of *attractor points of rank-two*, a program instigated by Moore in [9, 10]. The relevant manifolds here are conjecturally weight-four modular. The second such set of moduli are those that give *supersymmetric flux vacua*. The manifolds here should be weight-two, as conjectured by Kachru, Nally, and Yang [11, 12]. We will in later sections exposit both of these, and display new results in both directions. Here, we discuss some general features of modular Calabi-Yau threefolds that bears relevance to both physical setups.

The Frobenius map

We review this background material following [14, 48–50]. Fermat’s little theorem gives, for integer c ,

$$c^p = c \quad \text{Mod } p . \quad (2.54)$$

Further, when working modulo p we have a simplified form of the binomial theorem for c and d in $\mathbb{F}_{p^k}$:

$$(c + d)^p = c^p + d^p \quad \text{Mod } p . \quad (2.55)$$

When considering varieties over finite fields, we refer to the vanishing set of some polynomials. Note that in light of the above two identities, we get a result for polynomials Q that have coefficients in $\mathbb{F}_p$ and indeterminates in any $\mathbb{F}_{p^k}$ with $k \in \mathbb{N}$. This is

$$Q(x) = 0 \implies Q(x)^p = 0 \implies Q(x^p) = 0 . \quad (2.56)$$

So, if we act on the coordinates x of the ambient space containing X_φ by the Frobenius map

$$\text{Frob}_p : x \mapsto x^p , \quad (2.57)$$

we leave invariant the vanishing locus X_φ . In light of this, the Frobenius map is an automorphism of any variety $X_\varphi/\mathbb{F}_{p^k}$ with coefficients in $\mathbb{F}_p$.

The fixed points of this automorphism obey $x^p = x$, which means that x is defined in $\mathbb{F}_p \subset \mathbb{F}_{p^k}$. Fixed points of $(\text{Frob}_p)^2$ have $x^{p^2} = x$, and so are defined in $\mathbb{F}_{p^2} \subset \mathbb{F}_{p^k}$, and so on for each higher field.

So one has that the zeroes of the defining polynomials of X_φ , which we aim to count, are fixed points of a continuous automorphism of X_φ . By providence, the number of fixed points

of such an automorphism appears in Lefschetz's trace formula:

$$\sum_{i=0}^6 (-1)^i \text{Tr} \left((\text{Frob}_p)^k | H^i(X_\varphi) \right) = N_{p^k}(\varphi) . \quad (2.58)$$

Here, one uses the p -adic cohomology $H^i(X_\varphi)$ (we do not discuss this point further, instead referring to [50]). The 6 in the above formula is the real dimension of the threefolds we study, for a complex n -fold one would replace 6 by $2n$.

Using this formula, working from the exponential form (2.38) one can show that the polynomials P_k appearing in the rational form of the zeta function (2.39) are in fact determinants:

$$P_k(X_\varphi; T) = \det \left(1 - T \text{Frob}_p^{-1} | H^k(X_\varphi) \right) . \quad (2.59)$$

It should be noted that our notation for P_k suppresses their dependence on the prime p .

Counting the $\mathbb{F}_{p^k}$ -points of the varieties X_φ , as done for example in [51], is a laborious process. In [50] a method was developed, using the above determinant formula, to more efficiently compute zeta function numerators.

Chapter 3

Reflections in the mirror

And I saw how the stars of Heaven come out, and counted the Gates out of which they come, and wrote down all their outlets, for each one, individually, according to their number. And their names, according to their constellations, their positions, their times, and their months, as the Angel Uriel, who was with me, showed me.

Enoch 33.3

Given a complex manifold M , a quantity of foremost interest in classical algebraic geometry is the number of holomorphic embeddings of a Riemann surface with given genus and degree in M . This statement requires some delicacy, for such curves can lie in continuous families and so a more refined notion of ‘counting’ is required. A major success of string theory is the means it provides of obtaining these curve counts when M is Calabi-Yau. This is done by studying the partition function of two distinct topological string theories, the A and B models. The A-model bears a direct relation to these curve counts, the partition function is their generating function. A direct computation of the A-model partition function is infeasible, but the B-model partition function can be computed from the period functions of M (there are however complications with increasing genus). The two models are related by a string duality, *mirror symmetry*, which means that once either partition function is computed, both are known. And so the ‘easy’ B-model computation yields the sought curve

counts.

To elaborate, as covered in the textbook [52] the A and B models are topological twists of the two-dimensional $\mathcal{N} = (2, 2)$ nonlinear sigma model with target space M . The R-symmetry group of a 2d $\mathcal{N} = (2, 2)$ theory is $U(1)_A \times U(1)_V$, the product of axial and vector $U(1)$ symmetries. The vector $U(1)_V$ symmetry is never anomalous, while the axial $U(1)_A$ is anomalous unless M is Calabi-Yau. The A model is obtained by twisting the theory with respect to the $U(1)_V$ R-symmetry, while the B-model comes from twisting with respect to $U(1)_A$. The bosonic field content of the nonlinear sigma model includes the map ϕ that gives the coordinates of an embedding of the worldsheet Σ into the target space M ,

$$\phi : \Sigma \mapsto M . \tag{3.1}$$

The path integral of a supersymmetric quantum field theory can be computed from the field configurations where the supersymmetry variations of the fermions vanish. This is the principle of supersymmetric localisation.

The characterisation of such field configurations is different in the A and B twists, and is covered in the textbook [52]. The upshot is that in the A model the fermionic variation vanishes for field configurations where the map ϕ is holomorphic, i.e.

$$\partial_{\bar{z}}\phi = 0 \tag{3.2}$$

where $(z, \bar{z})$ are complex coordinates on the Riemann surface Σ . For such a field configuration, ϕ embeds the string *holomorphically* into the target space. Since the closed string worldsheet is a genus- g Riemann surface, the partition function will serve as a generating function for counts of holomorphic embeddings, of various degrees, of genus- g Riemann surfaces in the target space.

On the other hand, in the B-model the path integral localises onto *constant* maps. Very roughly speaking, the space of constant maps to X is X , and so the B-model partition

function is computed by integrals over the target space, using Hodge theory. We will now give a computationally-oriented account of these identities.

3.1 The genus 0 prepotential

We will denote $m = h^{2,1}(X) = h^{1,1}(Y)$. In terms of the Frobenius periods ϖ as detailed in (2.2), the B-model prepotential at genus 0 is

$$\mathcal{F}^{(0)}(\varphi) = \frac{1}{2} \left(-\varpi_0 \varpi_3 + \sum_{i=1}^m \varpi_{1,i} \varpi_{2,i} \right) , \quad (3.3)$$

with each of the periods a function of the complex structure variables φ .

The A-model prepotential at genus 0 admits the expansion

$$F^{(0)} = \frac{1}{6} Y_{ijk} t^i t^j t^k + \frac{1}{(2\pi i)^3} \sum_{\mathbf{b} \geq 0} n_{\mathbf{b}}^{(0)} \text{Li}_3 \left(e^{2\pi i \mathbf{b} \cdot \mathbf{t}} \right) , \quad (3.4)$$

with the Y_{ijk} being the triple-intersection numbers on the manifold Y . The quantity $\mathbf{b} \cdot \mathbf{t}$ should be understood as the integral of the Kähler form over a two-cycle which, in the basis dual to canonical generators of the Kähler cone, has nonnegative components $\mathbf{b}$. The summation over $\mathbf{b} \in \mathbb{Z}_{\geq 0}^{h^{1,1}(Y)}$ gives a sum over all possible degree vectors that a rational curve embedded in Y can have (neglecting cases with torsion classes), and $\mathbf{b} \cdot \mathbf{t}$ is the area of such a curve.

Since $\mathcal{F}^{(0)}$ is given in terms of solutions to differential equations (which we can find as explicit power series), the above pair of formulae provide the means of determining the genus 0 instanton numbers (integers also nomenclated genus 0 Gopakumar-Vafa invariants) $n_{\beta}^{(0)}$ of Y . It only remains to find a suitable transformation of coordinates

$$\varphi \rightarrow \mathbf{t} \quad (3.5)$$

and make a compatible choice of gauge.

The complex structure moduli space is itself Kähler, with Kähler potential $K = -\log \left(i \bar{\Pi}^T \Sigma \Pi \right)$ and metric $G_{ij} = \partial_{\varphi^i} \partial_{\varphi^j} K$. The necessary choice of coordinates is the flat one, in which the Christoffel symbols for the metric G vanish. This is

$$t^i = \frac{1}{2\pi i} \frac{\varpi_{1,i}(\varphi)}{\varpi_0(\varphi)} . \quad (3.6)$$

If φ_i is moved in a circle about the MUM point 0, we get a monodromy transformation $t^i \mapsto t^i + 1$. This is a symmetry of the theory, and leaves the free energy $F^{(0)}$ invariant when the symplectic transformations of the Y_{ijk} are taken into account (this amounts to a change of basis for $H_2(Y, \mathbb{Z})$). A convenient repackaging of the $\mathbf{t}$ in a new coordinate $\mathbf{q}$ makes this manifest:

$$q^i = \exp \left(2\pi i t^i \right) \equiv \exp \left(\frac{\varpi_{1,i}(\varphi)}{\varpi_0(\varphi)} \right) . \quad (3.7)$$

The appropriate change of scale is to multiply (3.3) by $\frac{1}{(2\pi i)^3 \varpi_0^2}$, and so the numbers $n_{\mathbf{b}}^{(0)}$ are fixed by

$$\begin{aligned} (2\pi i)^3 F^{(0)}(\mathbf{q}) &= \frac{1}{2\varpi_0^2} \left(-\varpi_0 \varpi_3 + \sum_{i=1}^m \varpi_{1,i} \varpi_{2,i} \right) \Big|_{\varphi=\varphi(\mathbf{q})} \\ &= \frac{1}{6} Y_{ijk} \log(q^i) \log(q^j) \log(q^k) + \sum_{\mathbf{b} \geq 0} n_{\mathbf{b}}^{(0)} \text{Li}_3(\mathbf{q}^{\mathbf{b}}) . \end{aligned} \quad (3.8)$$

By $\mathbf{q}^{\mathbf{b}}$ we mean the product $\prod_i (q^i)^{b_i}$. This form of $F^{(0)}(\mathbf{q})$ holds for manifolds whose homology groups lack torsion. Manifolds with torsion homology classes were considered in [53–55]. The problem of efficiently performing this computation was addressed first in the package INSTANTON [56], and more recently in the library CYtools [57, 58].

Performing the computation efficiently

Let us make three points on the efficient computation of the genus 0 invariants.

One is that instead of replacing φ with the q -series $\varphi(\mathbf{q})$ in the first line of (3.8), the

appearances of q^i in the second line can be replaced with φ -series. This means that, if only genus 0 numbers are sought, the inverse mirror map $\varphi(\mathbf{q})$ does not need to be computed.

Secondly, if $\mathbf{q}$ is replaced with $\mathbf{q}(\varphi)$ in the bottom line of (3.8), the $\log(q^i)$ terms can be expanded and one finds (as necessitated) that the terms proportional to $\log(\varphi^i)$ in the top line (coming from the logarithmic parts of the Frobenius periods) can be subtracted from both sides, so we need only work with the power series that appear in each Frobenius period. Said another way, we can set up the computation so that the logarithmic terms in each ϖ are removed from the start.

Finally, we draw attention to the use of symmetric polynomials in suitable examples. Computing the instanton numbers up to degree vectors $\mathbf{b}$ with total n involves finding the n^{th} order Taylor series of the prepotential, which can be found from the n^{th} order series parts of each Frobenius period. For one-parameter manifolds the number of terms in this series is n . If the number of parameters is greater than 1 ($h^{1,1}(Y) \geq 2$), then these truncated Taylor series become very large much faster as n is increased, with a number of new terms at order n equal to the number of partitions of n with length $h^{1,1}(Y)$. The process of multiplying the large intermediate polynomials is computationally expensive, and prevents one from straightforwardly computing instanton numbers to very high degrees for multiparameter manifolds.

This latter problem, of an exponential growth in the number of terms in each power series, can to some extent be mitigated in examples with a symmetry in their parameters. Suppose three (or more) of the parameters enter on symmetric footing, for example the case where Y is a CICY with three identical rows. Then ϖ_0 and $F^{(0)}$ are symmetric functions of the three parameters.

A symmetric polynomial in n variables a_i can be uniquely expressed in terms of the elementary symmetric polynomials

$$A_i = \sum_{\substack{L \text{ a length-} i \\ \text{subset of } \{1, \dots, n\}}} \prod_{k \in L} a_k, \quad (3.9)$$

with a convention $A_0 = 0$.

The utility here is that the number of distinct new monomials in the polynomials A_i at each degree is smaller than the number of new monomials in the a_i , because A_i has degree i . The asymptotic growth is still exponential, but with a smaller base.

Now, a number of the functions involved in the genus 0 computation will not be symmetric in these variables. However, if the coordinates φ^j, φ^k enter the problem symmetrically then $\varpi_{1,k}, \varpi_{2,k}$ can be obtained from $\varpi_{1,j}, \varpi_{2,j}$ by effecting a swap $\varphi^j \leftrightarrow \varphi^k$. So, the number of computations to perform has decreased.

Moreover, identities between the A_i and a_i allow for further simplification of the less symmetric functions like $\varpi_{1,i}$. We have that, for each k ,

$$\sum_{i=1}^m (-a_k)^i A_{m-i} = 0 . \quad (3.10)$$

This means that, for the power series of a function like $\varpi_{1,k}$ where a_k breaks the symmetry, we can uniquely express a truncation to a finite degree as a polynomial in the A_i and a_k with degree less than m in a_k . This decreases the number of terms in the expression.

Using these ideas, and the ARC supercomputing resources [59], it was possible in [2] to compute genus 0 instanton numbers up to degree 29 for the five-parameter Mirror Hulek-Verrill model

$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} , \quad (3.11)$$

which possesses an S_5 symmetry corresponding to permuting rows of this matrix.

To illustrate our claims anew here, we present in table 3.1 genus 0 numbers for the family¹

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{array}{c} (6,36) \\ \\ \\ \\ \\ -60 \end{array}, \quad (3.12)$$

which similarly possesses an S_5 symmetry originating in the freedom to permute the first five rows. These numbers were computed on a 6-CPU desktop machine with 16GB of RAM, and so we do not reach so high a degree as was done for the five parameter model (3.11) in [2].

More can be done, and we also present in table 3.2 the genus 0 numbers up to degree 13 for the family

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^4 \\ \mathbb{P}^4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{c} (7,27) \\ \\ \\ \\ \\ \\ -40 \end{array}. \quad (3.13)$$

Another nice example is found in the family of tetraquadrics

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \begin{array}{c} (4,68) \\ \\ \\ -128 \end{array}. \quad (3.14)$$

We present the genus 0 numbers up to degree 20 for this model in table 3.3.

We conclude this run of examples with a model previously studied by Hosono and Takagi,

¹We will, as and when useful in this section but not throughout the thesis, write CICY matrices with a superscript displaying the Hodge numbers $(h^{1,1}, h^{2,1})$ and a subscript displaying the Euler characteristic.

and for which we will have more to say in our later discussions on higher genus invariants and modularity. This is the 2-parameter family

$$\mathbb{P}^4 \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]_{-100}^{(2,52)}. \quad (3.15)$$

Genus 0 invariants up to total degree 18 were already given in [60]. We provide them to degree 37 in table 3.4.

One motivation for studying these families is their presence in [61], along with the mirror Hulek-Verrill manifold. These geometries are related by splittings, and all possess certain symmetries that allow for quotient constructions with small Hodge numbers.

The content of the following tables is available in electronic form [1].

$\mathbf{p}$	$n_{\mathbf{p}}^{(0)}$	$\mathbf{p}$	$n_{\mathbf{p}}^{(0)}$	$\mathbf{p}$	$n_{\mathbf{p}}^{(0)}$	$\mathbf{p}$	$n_{\mathbf{p}}^{(0)}$
0,0,0,0,0,1	10	0,1,2,2,2,5	540	1,1,1,1,2,3	27492	1,1,2,2,4,5	964648
0,0,0,0,1,0	12	0,1,2,2,3,2	22	1,1,1,1,2,4	8892	1,1,2,2,5,2	36
0,0,0,0,1,1	12	0,1,2,2,3,3	2796	1,1,1,1,2,5	66	1,1,2,2,5,3	-400
0,0,0,1,1,0	1	0,1,2,2,3,4	8860	1,1,1,1,3,1	-24	1,1,2,2,5,4	2560
0,0,0,1,1,1	22	0,1,2,2,3,5	2796	1,1,1,1,3,2	180	1,1,2,2,6,2	-4
0,0,0,1,1,2	1	0,1,2,2,3,6	22	1,1,1,1,3,3	9444	1,1,2,2,6,3	30
0,0,1,1,1,1	56	0,1,2,2,4,3	56	1,1,1,1,3,4	9444	1,1,2,3,3,2	2
0,0,1,1,1,2	56	0,1,2,2,4,4	1344	1,1,1,1,3,5	180	1,1,2,3,3,3	139918
0,0,1,1,2,1	1	0,1,2,2,4,5	1344	1,1,1,1,3,6	-24	1,1,2,3,3,4	2449206
0,0,1,1,2,2	22	0,1,2,2,4,6	56	1,1,1,1,4,1	3	1,1,2,3,3,5	5705460
0,0,1,1,2,3	1	0,1,2,2,5,4	1	1,1,1,1,4,2	-22	1,1,2,3,4,3	2676
0,0,1,2,2,2	12	0,1,2,2,5,5	22	1,1,1,1,4,3	59	1,1,2,3,4,4	766084
0,0,1,2,2,3	12	0,1,2,3,3,3	1344	1,1,1,1,4,4	1024	1,1,2,3,5,3	-376
0,0,2,2,2,2	10	0,1,2,3,3,4	13968	1,1,1,1,4,5	59	1,1,2,4,4,3	20
0,0,2,2,2,3	60	0,1,2,3,3,5	13968	1,1,1,1,4,6	-22	1,1,3,3,3,3	112196
0,0,2,2,2,4	10	0,1,2,3,3,6	1344	1,1,1,1,4,7	3	1,1,3,3,3,4	4878540
0,0,2,2,3,3	12	0,1,2,3,4,3	22	1,1,1,2,2,2	10792	1,1,3,3,4,3	2775
0,0,2,2,3,4	12	0,1,2,3,4,4	2796	1,1,1,2,2,3	95648	1,2,2,2,2,2	27600
0,0,2,3,3,3	1	0,1,2,3,4,5	8860	1,1,1,2,2,4	95648	1,2,2,2,2,3	1585020
0,0,2,3,3,4	22	0,1,2,3,5,4	12	1,1,1,2,2,5	10792	1,2,2,2,2,4	8537376
0,0,2,3,3,5	1	0,1,2,4,4,4	540	1,1,1,2,3,2	276	1,2,2,2,2,5	8537376
0,0,3,3,3,4	56	0,1,3,3,3,3	798	1,1,1,2,3,3	43326	1,2,2,2,2,6	1585020
0,0,3,3,3,5	56	0,1,3,3,3,4	25662	1,1,1,2,3,4	125676	1,2,2,2,3,2	1344
0,0,3,3,4,4	1	0,1,3,3,3,5	69528	1,1,1,2,3,5	43326	1,2,2,2,3,3	1097052
0,0,3,3,4,5	22	0,1,3,3,4,3	12	1,1,1,2,3,6	276	1,2,2,2,3,4	14667648
0,1,1,1,1,1	174	0,1,3,3,4,4	6708	1,1,1,2,4,2	-48	1,2,2,2,3,5	32044488
0,1,1,1,1,2	756	0,2,2,2,2,2	756	1,1,1,2,4,3	432	1,2,2,2,4,2	-552
0,1,1,1,1,3	174	0,2,2,2,2,3	25662	1,1,1,2,4,4	22320	1,2,2,2,4,3	26364
0,1,1,1,2,1	12	0,2,2,2,2,4	69516	1,1,1,2,4,5	22320	1,2,2,2,4,4	4907940
0,1,1,1,2,2	540	0,2,2,2,2,5	25662	1,1,1,2,4,6	432	1,2,2,2,5,2	198
0,1,1,1,2,3	540	0,2,2,2,2,6	756	1,1,1,2,5,2	3	1,2,2,2,5,3	-3960
0,1,1,1,2,4	12	0,2,2,2,3,2	56	1,1,1,2,5,3	-22	1,2,2,2,6,2	-48
0,1,1,1,3,1	-2	0,2,2,2,3,3	13968	1,1,1,2,5,4	59	1,2,2,3,3,2	24
0,1,1,1,3,2	10	0,2,2,2,3,4	103000	1,1,1,2,5,5	1024	1,2,2,3,3,3	878696
0,1,1,1,3,3	96	0,2,2,2,3,5	103000	1,1,1,3,3,3	22680	1,2,2,3,3,4	28577520
0,1,1,1,3,4	10	0,2,2,2,3,6	13968	1,1,1,3,3,4	194664	1,2,2,3,4,2	-2
0,1,1,1,3,5	-2	0,2,2,2,4,3	756	1,1,1,3,3,5	194664	1,2,2,3,4,3	24120
0,1,1,2,2,2	474	0,2,2,2,4,4	25662	1,1,1,3,3,6	22680	1,2,3,3,3,3	823008
0,1,1,2,2,3	1852	0,2,2,2,4,5	69516	1,1,1,3,4,3	276	2,2,2,2,2,2	52740
0,1,1,2,2,4	474	0,2,2,2,5,4	540	1,1,1,3,4,4	43326	2,2,2,2,2,3	7545520
0,1,1,2,3,2	12	0,2,2,3,3,2	1	1,1,1,3,4,5	125676	2,2,2,2,2,4	80109420
0,1,1,2,3,3	540	0,2,2,3,3,3	8860	1,1,1,3,5,3	-24	2,2,2,2,2,5	166265920
0,1,1,2,3,4	540	0,2,2,3,3,4	176251	1,1,1,3,5,4	180	2,2,2,2,3,2	2928
0,1,1,2,3,5	12	0,2,2,3,3,5	425712	1,1,1,4,4,4	10792	2,2,2,2,3,3	6157524
0,1,1,2,4,3	1	0,2,2,3,4,3	540	1,1,2,2,2,2	15902	2,2,2,2,3,4	154063572
0,1,1,2,4,4	22	0,2,2,3,4,4	54504	1,1,2,2,2,3	369646	2,2,2,2,4,2	-1338
0,1,1,2,4,5	1	0,2,2,4,4,3	10	1,1,2,2,2,4	920760	2,2,2,2,4,3	178908
0,1,1,3,3,3	174	0,2,3,3,3,3	6708	1,1,2,2,2,5	369646	2,2,2,2,5,2	660
0,1,1,3,3,4	756	0,2,3,3,3,4	342924	1,1,2,2,2,6	15902	2,2,2,3,3,2	142
0,1,1,3,3,5	174	0,2,3,3,4,3	474	1,1,2,2,3,2	596	2,2,2,3,3,3	5734779
0,1,1,3,4,4	56	0,3,3,3,3,3	6204	1,1,2,2,3,3	210748	2,2,2,3,4,2	-24
0,1,1,3,4,5	56	1,1,1,1,1,1	700	1,1,2,2,3,4	1399712		
0,1,1,4,4,4	1	1,1,1,1,1,2	8900	1,1,2,2,3,5	1399712		
0,1,1,4,4,5	22	1,1,1,1,1,3	8900	1,1,2,2,3,6	210748		
0,1,2,2,2,2	540	1,1,1,1,1,4	700	1,1,2,2,4,2	-176		
0,1,2,2,2,3	6708	1,1,1,1,2,1	66	1,1,2,2,4,3	3558		
0,1,2,2,2,4	6708	1,1,1,1,2,2	8892	1,1,2,2,4,4	360222		

Table 3.1: The genus 0 instanton numbers of total degree ≤ 15 for the family (3.12). The numbers not in this list are either zero, or given by those in the table after a permutation of the first five indices. The sixth index **cannot** be exchanged with the others.

$\mathbf{p}$	$n_{\mathbf{p}}^{(0)}$	$\mathbf{p}$	$n_{\mathbf{p}}^{(0)}$	$\mathbf{p}$	$n_{\mathbf{p}}^{(0)}$
0,0,0,0,0,0,1	10	0,1,1,1,2,2,2	264	1,1,1,1,1,2,3	4120
0,0,0,0,1,0,0	6	0,1,1,1,2,2,3	264	1,1,1,1,1,2,4	330
0,0,0,0,1,0,1	6	0,1,1,1,2,2,4	6	1,1,1,1,1,3,3	4120
0,0,0,0,1,1,1	6	0,1,1,1,2,3,3	264	1,1,1,1,1,3,4	330
0,0,0,1,1,0,1	1	0,1,1,1,2,3,4	6	1,1,1,1,1,4,4	20
0,0,0,1,1,1,1	20	0,1,1,1,3,1,3	-2	1,1,1,1,2,1,2	15
0,0,0,1,1,1,2	1	0,1,1,1,3,2,3	10	1,1,1,1,2,1,3	36
0,0,1,1,1,1,1	28	0,1,1,1,3,3,3	80	1,1,1,1,2,1,4	15
0,0,1,1,1,1,2	28	0,1,1,1,3,3,4	10	1,1,1,1,2,2,2	1692
0,0,1,1,1,2,2	28	0,1,1,2,2,2,2	72	1,1,1,1,2,2,3	5478
0,0,1,1,2,1,2	1	0,1,1,2,2,2,3	330	1,1,1,1,2,2,4	1692
0,0,1,1,2,2,2	20	0,1,1,2,2,2,4	72	1,1,1,1,2,2,5	15
0,0,1,1,2,2,3	1	0,1,1,2,2,3,3	1192	1,1,1,1,2,3,3	16464
0,0,1,2,2,2,2	6	0,1,1,2,2,3,4	330	1,1,1,1,2,3,4	5478
0,0,1,2,2,2,3	6	0,1,1,2,3,2,3	6	1,1,1,1,3,1,3	-12
0,0,1,2,2,3,3	6	0,1,1,2,3,2,4	6	1,1,1,1,3,1,4	-12
0,0,2,2,2,2,3	10	0,1,1,2,3,3,3	264	1,1,1,1,3,2,3	90
0,0,2,2,2,3,3	40	0,1,2,2,2,2,2	6	1,1,1,1,3,2,4	90
0,0,2,2,2,3,4	10	0,1,2,2,2,2,3	264	1,1,1,1,3,3,3	4644
0,0,2,2,3,3,3	6	0,1,2,2,2,2,4	264	1,1,1,1,4,1,4	3
0,1,1,1,1,1,1	24	0,1,2,2,2,3,3	3090	1,1,1,2,2,2,2	480
0,1,1,1,1,1,2	126	0,1,2,2,3,2,3	1	1,1,1,2,2,2,3	4916
0,1,1,1,1,1,3	24	0,2,2,2,2,2,3	126	1,1,1,2,2,2,4	4916
0,1,1,1,1,2,2	504	1,1,1,1,1,1,1	20	1,1,1,2,2,3,3	42908
0,1,1,1,1,2,3	126	1,1,1,1,1,1,2	330	1,1,1,2,3,2,3	54
0,1,1,1,1,3,3	24	1,1,1,1,1,1,3	330	1,1,2,2,2,2,2	80
0,1,1,1,2,1,2	6	1,1,1,1,1,1,4	20	1,1,2,2,2,2,3	3293
0,1,1,1,2,1,3	6	1,1,1,1,1,2,2	4120	1,2,2,2,2,2,2	6

Table 3.2: The genus 0 instanton numbers of total degree ≤ 13 for the family (3.13). The numbers not in this list are either zero, or given by those in the table after permutations of the first five and last two indices. The sixth and seventh indices **cannot** be exchanged with the first five.

p	$n_p^{(0)}$	p	$n_p^{(0)}$	p	$n_p^{(0)}$
1,0,0,0	48	5,4,0,0	48	6,5,4,3	70283944963932592401152
1,1,0,0	160	5,4,1,0	3265984	6,5,4,4	5129348718881933358532960
1,1,1,0	2432	5,4,1,1	14129065088	6,5,5,0	1335543626995200
1,1,1,1	86016	5,4,2,0	1017784976	6,5,5,1	15470217373819587584
2,1,0,0	48	5,4,2,1	3874428076928	6,5,5,2	11284911987863253245440
2,1,1,0	5056	5,4,2,2	1074177329475184	6,5,5,3	2276112606495732006202112
2,1,1,1	518784	5,4,3,0	59821118208	6,5,5,4	195588673772290431107673344
2,2,0,0	128	5,4,3,1	260287936814720	6,6,0,0	128
2,2,1,0	34640	5,4,3,2	82385600163793920	6,6,1,0	57521568
2,2,1,1	7037184	5,4,3,3	7471070109015318656	6,6,1,1	845112783616
2,2,2,0	507904	5,4,4,0	1268723299376	6,6,2,0	59821080576
2,2,2,1	171165840	5,4,4,1	6894039650704512	6,6,2,1	645375311627376
2,2,2,2	6547900416	5,4,4,2	2643331064630855200	6,6,2,2	435566317842866176
3,1,1,0	2432	5,4,4,3	291051848701788699648	6,6,3,0	9911375239392
3,1,1,1	899072	5,4,4,4	13862614815470765167056	6,6,3,1	108118212077621248
3,2,0,0	48	5,5,0,0	160	6,6,3,2	74937314945510009952
3,2,1,0	61824	5,5,1,0	11746432	6,6,3,3	13961655375523769282816
3,2,1,1	28639616	5,5,1,1	73320878080	6,6,4,0	549828820220928
3,2,2,0	2089008	5,5,2,0	5243776704	6,6,4,1	6736033668350881952
3,2,2,1	1297013760	5,5,2,1	27965750549248	6,6,4,2	5160552306598346379264
3,2,2,2	80605022416	5,5,2,2	10380731579151872	6,6,4,3	1080443741007758375555472
3,3,0,0	160	5,5,3,0	432161466624	6,6,4,4	95469681182710628821921792
3,3,1,0	308352	5,5,3,1	2545321829650432	6,6,5,0	13962620228584512
3,3,1,1	241754112	5,5,3,2	1042877238127097984	6,6,5,1	201987366061445211648
3,3,2,0	17677056	5,5,3,3	120304302520611954688	6,6,5,2	178213464042928851755872
3,3,2,1	17200647552	5,5,4,0	12709250594240	6,6,5,3	42979893686208656793686016
3,3,2,2	1557963029504	5,5,4,1	90222224188616064	6,6,6,0	191827050005069824
3,3,3,0	252810752	5,5,4,2	43400958203031979648	6,6,6,1	3375664486259351631456
3,3,3,1	352652451840	5,5,4,3	5917889244182186813696	6,6,6,2	3521325683147770386726912
3,3,3,2	43431181213824	5,5,4,4	347399539693965379619648	7,2,1,1	518784
3,3,3,3	1646607181615104	5,5,5,0	176223512332928	7,2,2,0	34640
4,1,1,0	160	5,5,5,1	1575672398623395840	7,2,2,1	1297013760
4,1,1,1	518784	5,5,5,2	922075476759511908864	7,2,2,2	1108729141920
4,2,1,0	34640	5,5,5,3	151692725887218685190144	7,3,1,0	2432
4,2,1,1	44662400	5,5,5,4	10726096697730587601447552	7,3,1,1	241754112
4,2,2,0	3265280	5,5,5,5	399456970261757309912334336	7,3,2,0	17677056
4,2,2,1	4042603552	6,1,1,1	2432	7,3,2,1	249714336000
4,2,2,2	426305384448	6,2,1,0	48	7,3,2,2	165296580806656
4,3,0,0	48	6,2,1,1	7037184	7,3,3,0	3808483584
4,3,1,0	508160	6,2,2,0	507904	7,3,3,1	39529406119936
4,3,1,1	789382400	6,2,2,1	4042603552	7,3,3,2	24136954745543808
4,3,2,0	57521568	6,2,2,2	1514352078848	7,3,3,3	3672469607846903808
4,3,2,1	94371041920	6,3,1,0	61824	7,4,1,0	508160
4,3,2,2	13002686858864	6,3,1,1	789382400	7,4,1,1	14129065088
4,3,3,0	1424625024	6,3,2,0	57521568	7,4,2,0	1017784976
4,3,3,1	3031512413312	6,3,2,1	342823329792	7,4,2,1	12163074768640
4,3,3,2	527889850255360	6,3,2,2	125771915722896	7,4,2,2	8102781267426048
4,3,3,3	27526058324060160	6,3,3,0	5243776704	7,4,3,0	188051992832
4,4,0,0	128	6,3,3,1	30008993843584	7,4,3,1	1984799528343552
4,4,1,0	2089008	6,3,3,2	11575415612576768	7,4,3,2	1294128535920058368
4,4,1,1	4985888864	6,3,3,3	1203234568961692800	7,4,3,3	219681085367035373952
4,4,2,0	360824832	6,4,1,0	2089008	7,4,4,0	9911375239392
4,4,2,1	871452197440	6,4,1,1	19804670208	7,4,4,1	112338620588205056
4,4,2,2	166647022068736	6,4,2,0	1424616960	7,4,4,2	79594696896349282688
4,4,3,0	13391051328	6,4,2,1	9173232641616	7,4,4,3	15110868372965862621696
4,4,3,1	39844405855232	6,4,2,2	3846287763468288	7,4,4,4	1189045428745766422260768
4,4,3,2	9199812356321968	6,4,3,0	141793830384	7,5,1,0	11746432
4,4,3,3	627205272329861504	6,4,3,1	938779827042688	7,5,1,1	255547666432
4,4,4,0	188051928064	6,4,3,2	419700792333851408	7,5,2,0	18178255616
4,4,4,1	741058165634496	6,4,3,3	51546803704385918976	7,5,2,1	233839970568832
4,4,4,2	217057048330727424	6,4,4,0	4650991239168	7,5,2,2	176514847485662208
4,4,4,3	18647281489625577504	6,4,4,1	35938232545114592	7,5,3,0	3600925105536
4,4,4,4	699794636853614635008	6,4,4,2	18502170216819978240	7,5,3,1	43715576259518464
5,1,1,1	86016	6,4,4,3	2653048822718080612368	7,5,3,2	32805238780261668736
5,2,1,0	5056	6,4,4,4	161629518930438998396928	7,5,3,3	6480417642541472432128
5,2,1,1	28639616	6,5,0,0	48	7,5,4,0	221844802863872
5,2,2,0	2089008	6,5,1,0	17677056	7,5,4,1	2932079676057142912
5,2,2,1	5834714624	6,5,1,1	188011416064	7,5,4,2	2387966888748874810112
5,2,2,2	1108729141920	6,5,2,0	13391051328	7,5,4,3	523912487781610422477568
5,3,1,0	308352	6,5,2,1	107087122937856	7,5,4,4	47981652609161486365088768
5,3,1,1	1154723840	6,5,2,2	55530466229981712	7,5,5,0	6036801603750144
5,3,2,0	84056832	6,5,3,0	1651763971584	7,5,5,1	92621472022310780928
5,3,2,1	249714336000	6,5,3,1	13704006681994624	7,5,5,2	85897234681261670146944
5,3,2,2	54663940002816	6,5,3,2	7493044028521115648	7,5,5,3	21540883712581909578244096
5,3,3,0	3808483584	6,5,3,3	1120675249090550046720	7,6,0,0	48
5,3,3,1	12944311033856	6,5,4,0	69188204803344	7,6,1,0	84056832
5,3,3,2	3295227059205504	6,5,4,1	661446106005076224	7,6,1,1	2010040031104
5,3,3,3	240139807447941120	6,5,4,2	409473371380697240000	7,6,2,0	141793830384

Table 3.3: The genus 0 instanton numbers of total degree ≤ 20 for the family (3.14). The numbers not in this list are either zero, or given by those in the table after permuting indices.

p	$n_p^{(0)}$	p	$n_p^{(0)}$	p	$n_p^{(0)}$
7,6,2,1	2208595900084224	8,6,3,1	1462074329960965376	9,7,1,1	23624438407168
7,6,2,2	2021884379122016048	8,6,3,2	1771264367116721723008	9,7,2,0	1651763971584
7,6,3,0	33789217322496	8,6,3,3	542307422558027394152960	9,7,2,1	70930323016569984
7,6,3,1	503495581721955456	8,6,4,0	7456149925576704	9,7,2,2	150750588373893541888
7,6,3,2	455218569908331757568	8,6,4,1	162132923614385349360	9,7,3,0	1071736381056384
7,6,3,3	107719074363142059583360	8,6,4,2	203054957441346394726912	9,7,3,1	37743466871928029184
7,6,4,0	2566483940906640	8,6,5,0	3439455622331791376	9,7,4,0	191827050036601856
7,6,4,1	41379145350412949504	8,6,5,1	8263402711793164324608	9,8,0,0	48
7,6,4,2	40123559676936131346720	8,6,6,0	8277365987776740864	9,8,1,0	1424625024
7,6,4,3	10410193126310745548984320	8,7,0,0	48	9,8,1,1	142582803775232
7,6,5,0	87046894784712960	8,7,1,0	360828928	9,8,2,0	9911375239392
7,6,5,1	1607540052866407577728	8,7,1,1	18129723322752	9,8,2,1	497106580728095936
7,6,5,2	1749693070862210564431872	8,7,2,0	1268723299376	9,8,3,0	7456149932740608
7,6,6,0	1576912001012723760	8,7,2,1	36298079877646592	9,9,0,0	160
7,6,6,1	34462954830661856456704	8,7,2,2	55987725237356145792	9,9,1,0	3808483584
7,7,0,0	160	8,7,3,0	549828822442752	9,9,1,1	487860526727168
7,7,1,0	252810752	8,7,3,1	14005384303898305024	9,9,2,0	33789217322496
7,7,1,1	8087692763136	8,7,3,2	20171041734451334944000	10,2,2,1	34640
7,7,2,0	567589415680	8,7,4,0	71290241748070896	10,2,2,2	6547900416
7,7,2,1	11584959011321216	8,7,4,1	1864208769268632504064	10,3,1,1	2432
7,7,2,2	13491384176760961024	8,7,5,0	3988612138839842816	10,3,2,0	48
7,7,3,0	176223512332928	8,8,0,0	128	10,3,2,1	1297013760
7,7,3,1	3369477175185817600	8,8,1,0	1017784976	10,3,2,2	13002686858864
7,7,3,2	3792683314958877584384	8,8,1,1	66748924807328	10,3,3,0	17677056
7,7,3,3	1099277650161884440166400	8,8,2,0	4650991239168	10,3,3,1	3031512413312
7,7,4,0	17175761255142144	8,8,2,1	170418886235419696	10,3,3,2	11575415612576768
7,7,4,1	348295401563531685632	8,8,2,2	328173467451404489600	10,3,3,3	7079306295276204288
7,7,4,2	412395106599944043177984	8,8,3,0	2566483940906640	10,4,1,1	44662400
7,7,5,0	740932627227834624	8,8,3,1	82211057906371385344	10,4,2,0	3265280
7,7,5,1	16875996286670980751360	8,8,4,0	417176124089319424	10,4,2,1	871452197440
7,7,6,0	16999688846773790208	9,2,2,1	7037184	10,4,2,2	3846287763468288
8,2,1,1	5056	9,2,2,2	80605022416	10,4,3,0	13391051328
8,2,2,0	128	9,3,1,1	899072	10,4,3,1	938779827042688
8,2,2,1	171165840	9,3,2,0	61824	10,4,3,2	2508525839463392816
8,2,2,2	426305384448	9,3,2,1	17200647552	10,4,3,3	1332539766738727936000
8,3,1,1	28639616	9,3,2,2	54663940002816	10,4,4,0	4650991239168
8,3,2,0	2089008	9,3,3,0	252810752	10,4,4,1	219342311907269104
8,3,2,1	94371041920	9,3,3,1	12944311033856	10,4,4,2	487713798985291233280
8,3,2,2	125771915722896	9,3,3,2	24136954745543808	10,5,1,0	5056
8,3,3,0	1424625024	9,3,3,3	8791828001566988288	10,5,1,1	14129065088
8,3,3,1	30008993843584	9,4,1,0	160	10,5,2,0	1017784976
8,3,3,2	30743146145765888	9,4,1,1	789382400	10,5,2,1	107087122937856
8,3,3,3	7079306295276204288	9,4,2,0	57521568	10,5,2,2	347926332689829040
8,4,1,0	34640	9,4,2,1	3874428076928	10,5,3,0	1651763971584
8,4,1,1	4985888864	9,4,2,2	8102781267426048	10,5,3,1	86320414207198848
8,4,2,0	360824832	9,4,3,0	59821118208	10,5,3,2	204999947699695388672
8,4,2,1	9173232641616	9,4,3,1	1984799528343552	10,5,4,0	439063136203008
8,4,2,2	10353839703716352	9,4,3,2	3120841582270747392	10,5,4,1	18557339208009894656
8,4,3,0	141793830384	9,4,3,3	1094190143621388343680	10,5,5,0	38952597058723520
8,4,3,1	2538964516666880	9,4,4,0	9911375239392	10,6,1,0	2089008
8,4,3,2	2508525839463392816	9,4,4,1	273504607073902144	10,6,1,1	845112783616
8,4,3,3	602927912105481193728	9,4,4,2	400065629964453039984	10,6,2,0	59821080576
8,4,4,0	12709250181888	9,4,4,3	141106319957374906048512	10,6,2,1	4533354437642480
8,4,4,1	219342311907269104	9,5,1,0	308352	10,6,2,2	13486794785676976128
8,4,4,2	219737543096535650304	9,5,1,1	73320878080	10,6,3,0	69188204803344
8,4,4,3	56255611905947864806512	9,5,2,0	5243776704	10,6,3,1	3368380766765580288
8,4,4,4	5785403639953201842686976	9,5,2,1	233839970568832	10,6,4,0	17175761244601344
8,5,1,0	3265984	9,5,2,2	435172594319284224	10,7,1,0	84056832
8,5,1,1	188011416064	9,5,3,0	3600925105536	10,7,1,1	18129723322752
8,5,2,0	13391051328	9,5,3,1	108024514779267072	10,7,2,0	1268723299376
8,5,2,1	302207140853120	9,5,3,2	167816512674800089472	10,7,2,1	88462669038163968
8,5,2,2	347926332689829040	9,5,3,3	61564588462323994361856	10,7,3,0	1335543626995200
8,5,3,0	4650991497216	9,5,4,0	549828822442752	10,8,1,0	1017784976
8,5,3,1	86320414207198848	9,5,4,1	15172240325024688384	10,8,1,1	182970871579264
8,5,3,2	91592941667768938752	9,5,4,2	22894439058514947334656	10,8,2,0	12709250181888
8,5,3,3	24378159414883707592448	9,5,5,0	31786723861681536	10,9,0,0	48
8,5,4,0	439063136203008	9,5,5,1	913426712060194627584	10,9,1,0	5243776704
8,5,4,1	8247835975441925056	9,6,1,0	17677056	10,10,0,0	128
8,5,4,2	9033581369124138593840	9,6,1,1	2010040031104	11,2,2,2	171165840
8,5,4,3	2573204027887680476759296	9,6,2,0	141793830384	11,3,2,1	28639616
8,5,5,0	17175761255142144	9,6,2,1	5744002412760576	11,3,2,2	1557963029504
8,5,5,1	356496506323192211328	9,6,2,2	10966073503132221968	11,3,3,0	308352
8,5,5,2	428090722447387987896384	9,6,3,0	87594475420800	11,3,3,1	352652451840
8,6,1,0	57521568	9,6,3,1	2738079662657568640	11,3,3,2	3295227059205504
8,6,1,1	2668638725632	9,6,3,2	4572692498129982156800	11,3,3,3	3672469607846903808
8,6,2,0	188051928064	9,6,4,0	13962620228584512	11,4,1,1	518784
8,6,2,1	4533354437642480	9,6,4,1	420332107728557286912	11,4,2,0	34640
8,6,2,2	5860852677002366976	9,6,5,0	895919580397273344	11,4,2,1	94371041920
8,6,3,0	69188204803344	9,7,1,0	252810752	11,4,2,2	1074177329475184

table 3.3 continued.

p	$n_p^{(0)}$	p	$n_p^{(0)}$	p	$n_p^{(0)}$	p	$n_p^{(0)}$
11,4,3,0	1424625024	11,6,2,0	13391051328	12,4,2,1	4042603552	13,3,3,1	241754112
11,4,3,1	260287936814720	11,6,2,1	2208595900084224	12,4,2,2	166647022068736	13,4,2,1	44662400
11,4,3,2	1294128535920058368	11,6,3,0	33789217322496	12,4,3,0	57521568	13,4,3,0	508160
11,4,4,0	1268723299376	11,7,1,0	11746432	12,4,3,1	39844405855232	13,5,1,1	86016
11,4,4,1	112338620588205056	11,7,1,1	8087692763136	12,4,4,0	188051928064	13,5,2,0	5056
11,5,1,1	1154723840	11,7,2,0	567589415680	12,5,1,1	28639616		
11,5,2,0	84056832	11,8,1,0	360828928	12,5,2,0	2089008		
11,5,2,1	27965750549248	12,2,2,2	507904	12,5,2,1	3874428076928		
11,5,2,2	176514847485662208	12,3,2,1	61824	12,5,3,0	59821118208		
11,5,3,0	432161466624	12,3,2,2	80605022416	12,6,1,0	48		
11,5,3,1	43715576259518464	12,3,3,0	160	12,6,1,1	19804670208		
11,5,4,0	221844802863872	12,3,3,1	17200647552	12,6,2,0	1424616960		
11,6,1,0	61824	12,3,3,2	527889850255360	12,7,1,0	508160		
11,6,1,1	188011416064	12,4,1,1	160	13,3,2,2	1297013760		

table 3.3 continued.

p	$n_p^{(0)}$	p	$n_p^{(0)}$	p	$n_p^{(0)}$
1,0	50	8,8	2218998811196105750	17,5	75885200
1,1	650	9,7	782114760930236000	12,11	1169273852253720661047855850
2,1	1475	10,6	28126794522576400	13,10	261422628471008778452895000
2,2	29350	11,5	50034381769600	14,9	11425471345666372778573625
3,1	650	12,4	545403950	15,8	71553701937328489430500
3,2	148525	9,8	30429684503634827875	16,7	34680269311023701250
4,1	50	10,7	4075722566708421875	17,6	334030085380350
3,3	3270050	11,6	51642034298930775	18,5	148525
4,2	250550	12,5	24352783493100	12,12	26334146932509192721297606250
4,3	24162125	13,4	24162125	13,11	12925735995730366743674988500
5,2	148525	9,9	688579463588598857500	14,10	1443695931340763707228964750
4,4	545403950	10,8	270605922599775866950	15,9	30455096636986392995454400
5,3	75885200	11,7	14322131205924119500	16,8	8345187338400446556500
6,2	29350	12,6	63157566038079800	17,7	14322131205924119500
5,4	5048036025	13,5	7175800860250	18,6	27995704239850
6,3	110273275	14,4	250550	13,12	418749190393922926454264339775
7,2	1475	10,9	9910287533252141060075	14,11	105192221331381719953056030200
5,5	114678709000	11,8	1632070561204989561850	15,10	5983442985255839039237107500
6,4	22945154050	12,7	34680269311023701250	16,9	60960783194314781836252175
7,3	75885200	13,6	51642034298930775	17,8	71553701937328489430500
6,5	1231494256550	14,5	1231494256550	18,7	4075722566708421875
7,4	55531376500	15,4	50	19,6	1231494256550
8,3	24162125	10,10	223872593965525056524000	13,13	9418685010993246523213147309050
6,6	27995704239850	11,9	96088214450066089180650	14,12	4871005171529900672353774747800
7,5	7175800860250	12,8	6879784715166845894000	15,11	64422511633993038725675093800
8,4	74278763500	13,7	58660895139129344250	16,10	18858521597882017598430587450
9,3	3270050	14,6	28126794522576400	17,9	92220396289894438953276250
7,6	334030085380350	15,5	114678709000	18,8	45029496161343522802000
8,5	24352783493100	11,10	3351508924925685769008400	19,7	782114760930236000
9,4	55531376500	12,9	652460665889917943662000	20,6	22945154050
10,3	148525	13,8	20691735554891324819375	14,13	153346515556322207530993642061375
7,7	7584889119913750	14,7	69837157468295256300	15,12	42551462569251226858792953023050
8,6	2329042266808650	15,6	10084936612321850	16,11	3018482957837620078514600231550
9,5	50034381769600	16,5	5048036025	17,10	45641749489863534673387285075
10,4	22945154050	11,11	75590773394298108275641400	18,9	105822944845470819411145650
11,3	650	12,10	34897260129076709170702250	19,8	20691735554891324819375
8,7	97887416945961075	13,9	3187856349624109686861750	20,7	97887416945961075
9,6	10084936612321850	14,8	45029496161343522802000	21,6	110273275
10,5	63477362571125	15,7	58660895139129344250	14,14	344506366410127667138401567917450
11,4	5048036025	16,6	2329042266808650	15,13	1863829015771384379547064210671650

Table 3.4: The genus 0 instanton numbers of total degree ≤ 37 for the family (3.15). The numbers not in this list are either zero, or given by those in the table after permuting indices.

p	$n_p^{(0)}$	p	$n_p^{(0)}$
16,12	284748690462403703849327898236400	22,11	194151779259881472757738870486275
17,11	10956684127545224855242712323800	23,10	45641749489863534673387285075
18,10	85415237718946312165876907100	24,9	652460665889917943662000
19,9	92220396289894438953276250	25,8	97887416945961075
20,8	6879784715166845894000	26,7	1475
21,7	7584889119913750	17,17	187534310501258236886420065561355215672850
22,6	29350	18,16	112636401191272489352275415887745870490500
15,14	57245804146829441141855075686421600	19,15	23921213512060867824603015906296302482550
16,13	17311291067627079301898110530922200	20,14	1687381561533043508236480530888841957000
17,12	1482101901841193380170774085390975	21,13	35280439004752280205968167087919443350
18,11	31105645711336079302777239697100	22,12	182005705950504851032041523111610550
19,10	124182064589288217124451139225	23,11	173303334724056406174320632632850
20,9	60960783194314781836252175	24,10	18858521597882017598430587450
21,8	1632070561204989561850	25,9	96088214450066089180650
22,7	334030085380350	26,8	2329042266808650
15,15	1284701496853760180092631899781959250	18,17	3267643622168623942896097738545068717322400
16,14	722903234302220208838481980961898100	19,16	1206865467127098649476297800122829258555125
17,13	125101067315642934987038992528141500	20,15	158613328551370248346752880868799119011250
18,12	6071452882998806212881647096382400	21,14	6853883162475990624130911021611665107450
19,11	69561055478912921075310779919000	22,13	85346765578856941178005038357338030125
20,10	140638929443068626672454410250	23,12	249140684221048318682166766371696025
21,9	30455096636986392995454400	24,11	123171917767680954127111199469800
22,8	270605922599775866950	25,10	5983442985255839039237107500
23,7	7175800860250	26,9	9910287533252141060075
16,15	21731435735419411130551512688193970350	27,8	24352783493100
17,14	7084069788870929694102349973792379125	18,18	73136802934944577315316758502397717744613800
18,13	713360358445986060308899284173421250	19,17	45145186660990785137737197513727989291648750
19,12	19755724290719957639765719857823000	20,16	10440390902521863107237190843251830811038400
20,11	123171917767680954127111199469800	21,15	858330748648887482867022141394382548080200
21,10	124182064589288217124451139225	22,14	22822795902093633748803493742442397239000
22,9	11425471345666372778573625	23,13	168970246230273909199356496016178585000
23,8	30429684503634827875	24,12	276580864121735152873483395276585800
24,7	55531376500	25,11	69561055478912921075310779919000
16,16	487219827311876979652523699533587321150	26,10	1443695931340763707228964750
17,15	283802089997587090105839672112883654150	27,9	688579463588598857500
18,14	54760100433813765734980222782145239000	28,8	74278763500
19,13	3245572721974319554949303862415944250	19,18	1290315264071203979827100216377332535448538000
20,12	51423738528601615934359979687767300	20,17	502246809291235370794554217924188280591690325
21,11	173303334724056406174320632632850	21,16	73738372223409643492419300864401743615326225
22,10	85415237718946312165876907100	22,15	3821053219977680771955490543838867117725025
23,9	3187856349624109686861750	23,14	62641363244422442148059754716581069033750
24,8	2218998811196105750	24,13	274635658050542709324085728636859818125
25,7	75885200	25,12	249140684221048318682166766371696025
17,16	837198589815033172483098467977619591325	26,11	31105645711336079302777239697100
18,15	291574357716209286756587154545088037975	27,10	261422628471008778452895000
19,14	338273641516001759928541421158949393625	28,9	30429684503634827875
20,13	11885728626129063303448250398108564900	29,8	24162125
21,12	107659493684116349217356173601610375		

table 3.4 continued.

3.2 Genus 1 mirror symmetry: Counting elliptic curves

The B-model prepotential was determined in [16] and is given in the topological limit $\bar{t} \rightarrow -i\infty$ by

$$\mathcal{F}^{(1)} = \log \left[\varpi_0^{-\left(3+h^{2,1}(X)+\chi(X)/12\right)/2} \det \left(\frac{\partial \varphi}{\partial \mathbf{t}} \right)^{1/2} f(\varphi) \right] , \quad (3.16)$$

with the function f giving the correct behaviour at singularities of the moduli space, to be fixed after imposing a boundary condition that imposes consistency with the A-model expansion. The large complex structure expansion of the A-model genus 1 prepotential is

$$F^{(1)} = \frac{1}{2} Y_{00i} t^i + \sum_{\mathbf{b} \geq 0} \left(n_{\mathbf{b}}^{(1)} + \frac{n_{\mathbf{b}}^{(0)}}{12} \right) \text{Li}_1(\mathbf{q}^{\mathbf{b}}) . \quad (3.17)$$

The function f should have a factor that gives zeroes at the roots of the discriminant locus Δ , the set of moduli φ for which the manifold X becomes singular. Additionally, f should vanish at the large complex structure² (LCS) point $\varphi = \mathbf{0}$, with the order of vanishing such that the leading behaviour of $F^{(1)}$ is recovered. These considerations fix f to be of the form

$$f = \frac{1}{\Delta^c \prod_{i=1}^{h^{2,1}(X)} (\varphi^i)^{(1-Y_{00i})/2}} . \quad (3.18)$$

We remark that the determinant in (3.16) goes like $(\varphi^i)^{1/2}$ as $\varphi^i \rightarrow 0$, hence the exponents $\frac{1-Y_{00i}}{2}$ above. The exponent c is specified by the kind of singularities encoded by Δ . For a conifold singularity, c equals $1/12$. More complicated behaviours are possible at orbifold singularities, see for instance the computation of genus 1 numbers on quotient manifolds in [14], informed by [62]. For our purposes, Δ is always a conifold locus and we work with $c = 1/12$. For clarity, we remark that while f encapsulates a number of singularities, there is only one $\mathcal{F}^{(1)}$, and this behaves appropriately at each different singularity through f 's appearance in (3.16).

The polynomial Δ is in the variables φ^i , and we explain this choice of complex structure

²so named because this point is taken by the mirror map to the large volume point $t = i\infty$.

coordinates in Appendix §A. Δ encodes singularities not of the CICYs that we display, but of their mirrors X . Appendix §A discusses the construction of those mirror families X , and their parametrisation in terms of φ^i .

We shall display some genus 1 numbers for two families, the Tetraquadric (3.14) and the maximally split Quintic (3.15), whose mirrors respectively have discriminants

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \begin{array}{c} (4,68) \\ \\ \\ -128 \end{array}, \quad \Delta = \prod_{\epsilon_i \in \{1, -1\}} \left(1 + 2\epsilon_1 \sqrt{\varphi^1} + 2\epsilon_2 \sqrt{\varphi^2} + 2\epsilon_3 \sqrt{\varphi^3} + 2\epsilon_4 \sqrt{\varphi^4} \right) ;$$

$$\begin{array}{c} \mathbb{P}^4 \\ \mathbb{P}^4 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{c} (2,52) \\ -100 \end{array}, \quad \Delta = \prod_{1 \leq n_1, n_2 \leq 5} \left(1 - e^{\frac{2\pi i}{5} n_1} (\varphi^1)^{1/5} - e^{\frac{2\pi i}{5} n_2} (\varphi^2)^{1/5} \right) .$$
(3.19)

The q -series that we have used for obtaining genus-1 curve counts $n_{\mathbf{b}}^{(1)}$, formula (3.17), is that given by the Gopakumar-Vafa prescription [18, 19]. In earlier work by Bershadsky, Cecotti, Ooguri and Vafa [16], the prescription differed slightly (counting what they called “primitive elliptic curves”) and so some care is needed when comparing curve counts across the literature. In either case the B-model prepotential is the same, and the two sets of data are equivalent.

The content of the following tables is available in electronic form [1].

p	$n_p^{(1)}$	p	$n_p^{(1)}$	p	$n_p^{(1)}$
2,2,0,0	4	5,5,1,0	-631296	6,6,4,0	-267545068581728
2,2,1,0	-96	5,5,1,1	-193064960	6,6,4,1	98216197349510400
2,2,1,1	1984	5,5,2,0	-831856256	6,6,4,2	1487879821243945966432
2,2,2,0	-10272	5,5,2,1	156514373632	6,6,4,3	546260879195635250051616
2,2,2,1	269280	5,5,2,2	951193768367744	6,6,4,4	66049985769182760347092224
2,2,2,2	93388992	5,5,3,0	-114273286144	6,6,5,0	-8478485684300928
3,2,1,0	-320	5,5,3,1	23771145744384	6,6,5,1	3465741933227797120
3,2,1,1	10496	5,5,3,2	165492284557610240	6,6,5,2	63508834166133397153792
3,2,2,0	-69568	5,5,3,3	33247249326979840000	6,6,5,3	26872551390971959525026816
3,2,2,1	2765056	5,5,4,0	-4598298755008	6,6,6,0	-136960390214771328
3,2,2,2	1812768672	5,5,4,1	1075385950542336	6,6,6,1	64863757395907531392
3,3,1,0	-4864	5,5,4,2	9348192801960947328	6,6,6,2	1466437377855662330826240
3,3,1,1	112640	5,5,4,3	2224378492069768184832	7,2,2,0	-96
3,3,2,0	-1046656	5,5,4,4	177840753331297192191808	7,2,2,1	2765056
3,3,2,1	54265088	5,5,5,0	-78791963521792	7,2,2,2	35189119872
3,3,2,2	57695471872	5,5,5,1	21862949555982336	7,3,1,1	112640
3,3,3,0	-25362432	5,5,5,2	243574478608897658880	7,3,2,0	-1046656
3,3,3,1	1683640320	5,5,5,3	70047199737530626793472	7,3,2,1	1021728256
3,3,3,2	2698701579264	5,5,5,4	6756842515156306030742016	7,3,2,2	10223759052288
3,3,3,3	173800712052736	5,5,5,5	310274010609799385697632256	7,3,3,0	-578600960
4,2,1,0	-96	6,2,1,1	1984	7,3,3,1	276199923712
4,2,1,1	20096	6,2,2,0	-10272	7,3,3,2	2739373720985856
4,2,2,0	-124616	6,2,2,1	9983872	7,3,3,3	748137868720717824
4,2,2,1	9983872	6,2,2,2	49952569216	7,4,1,0	-10112
4,2,2,2	12011885472	6,3,1,0	-320	7,4,1,1	-14325760
4,3,1,0	-10112	6,3,1,1	241408	7,4,2,0	-127854976
4,3,1,1	241408	6,3,2,0	-4386240	7,4,2,1	65476152960
4,3,2,0	-4386240	6,3,2,1	1445166080	7,4,2,2	723323393538432
4,3,2,1	353758784	6,3,2,2	7576252775296	7,4,3,0	-45623185536
4,3,2,2	621519800480	6,3,3,0	-831856256	7,4,3,1	18148629097216
4,3,3,0	-188206400	6,3,3,1	205607705344	7,4,3,2	206208348590980608
4,3,3,1	17552569088	6,3,3,2	1238547915076352	7,4,3,3	62079704012640705280
4,3,3,2	43026117967872	6,3,3,3	226786087463462912	7,4,4,0	-3510327932032
4,3,3,3	3851778847706880	6,4,1,0	-69568	7,4,4,1	1335576781170688
4,4,0,0	4	6,4,1,1	-25295744	7,4,4,2	17562256414556246528
4,4,1,0	-69568	6,4,2,0	-188206960	7,4,4,3	5884952965650513045248
4,4,1,1	-1580992	6,4,2,1	48618551008	7,4,4,4	636300785891465292898752
4,4,2,0	-38445312	6,4,2,2	323138860266848	7,5,1,0	-631296
4,4,2,1	3987616000	6,4,3,0	-33378648000	7,5,1,1	-1115140096
4,4,2,2	10558546637760	6,4,3,1	8232411529408	7,5,2,0	-3380586368
4,4,3,0	-2397896960	6,4,3,2	61783631376376704	7,5,2,1	1393481688832
4,4,3,1	286009826304	6,4,3,3	13300730517821622784	7,5,2,2	20022577101423360
4,4,3,2	999142624314816	6,4,4,0	-1541331810048	7,5,3,0	-1166173618944
4,4,3,3	117575052436736704	6,4,4,1	404052229218240	7,5,3,1	469840676702208
4,4,4,0	-45623190048	6,4,4,2	3723977589477458176	7,5,3,2	6554880932704430336
4,4,4,1	6666610324992	6,4,4,3	935179511020340807456	7,5,3,3	2281737394795361083392
4,4,4,2	31633436508163392	6,4,4,4	77803599946908880718016	7,5,4,0	-100934478032512
4,4,4,3	4709891230826927232	6,5,1,0	-1046656	7,5,4,1	41094723073773568
4,4,4,4	239082475099257093312	6,5,1,1	-734128384	7,5,4,2	656136930137156080768
5,2,1,1	10496	6,5,2,0	-2397896960	7,5,4,3	253382111085115792710400
5,2,2,0	-69568	6,5,2,1	626843119104	7,5,4,4	31839291301539012129070080
5,2,2,1	15218816	6,5,2,2	5792732588373728	7,5,5,0	-3469762473325056
5,2,2,2	35189119872	6,5,3,0	-497787993344	7,5,5,1	1531922193657643008
5,3,1,0	-4864	6,5,3,1	139578486511360	7,5,5,2	29300021039056213398016
5,3,1,1	286720	6,5,3,2	1362193196667713536	7,5,5,3	12921015476002583834605568
5,3,2,0	-6904192	6,5,3,3	356453634018444596992	7,6,1,0	-6904192
5,3,2,1	1021728256	6,5,4,0	-28770544038624	7,6,1,1	-16517906944
5,3,2,2	3034052551424	6,5,4,1	8683903357985664	7,6,2,0	-33378648000
5,3,3,0	-578600960	6,5,4,2	101579932691307744640	7,6,2,1	13324886863872
5,3,3,1	83573229568	6,5,4,3	30534454347465167308416	7,6,2,2	271114676514738784
5,3,3,2	317438354195968	6,5,4,4	3044826118116035108486016	7,6,3,0	-13264263786752
5,3,3,3	40174633448570880	6,5,5,0	-692429114488064	7,6,3,1	5997956867302144
5,4,1,0	-124608	6,5,5,1	237977798080726528	7,6,3,2	106978781273942999040
5,4,1,1	-14325760	6,5,5,2	3452295445836449045632	7,6,3,3	44494307569970341498112
5,4,2,0	-127854976	6,5,5,3	1220175570920202576920832	7,6,4,0	-1392349606298016
5,4,2,1	19566085376	6,5,5,4	143356009888676753465771648	7,6,4,1	649011301399128064
5,4,2,2	80961860336992	6,6,0,0	4	7,6,4,2	12931687111454709126208
5,4,3,0	-12801118336	6,6,1,0	-4386240	7,6,4,3	5900151322919540445706752
5,4,3,1	2116485979904	6,6,1,1	-5444761792	7,6,5,0	-59273706883766400
5,4,3,2	10738139481866240	6,6,2,0	-12801129824	7,6,5,1	29832962488756485888
5,4,3,3	1691948070016716544	6,6,2,1	3899397189472	7,6,5,2	699633783726226960540672
5,4,4,0	-372907450720	6,6,2,2	52718694526710016	7,6,6,0	-1270394426831547360
5,4,4,1	71025289697408	6,6,3,0	-3510327932032	7,6,6,1	719306357328985736192
5,4,4,2	466267539878814528	6,6,3,1	1211618774875136	7,7,1,0	-25362432
5,4,4,3	89381762755780427776	6,6,3,2	15814322874224859712	7,7,1,1	-93294921728
5,4,4,4	5783834578826599864896	6,6,3,3	5160195188934834782144	7,7,2,0	-154245226112

Table 3.5: The genus 1 instanton numbers of total degree ≤ 20 for the family (3.14). The numbers not in this list are either zero, or given by those in the table after permuting indices.

p	$n_p^{(1)}$	p	$n_p^{(1)}$	p	$n_p^{(1)}$
7,7,2,1	67556863891200	8,7,1,1	-249012852224	9,7,2,2	26201449192835298304
7,7,2,2	2038282247914091008	8,7,2,0	-372907450720	9,7,3,0	-547093483682304
7,7,3,0	-78791963521792	8,7,2,1	201277836622720	9,7,3,1	513891279029764096
7,7,3,1	42886489670017024	8,7,2,2	9200312079392193664	9,7,4,0	-136960390218830336
7,7,3,2	1004665124573369920512	8,7,3,0	-267545068649344	9,8,1,0	-188206400
7,7,3,3	512053569223265100480512	8,7,3,1	185891068420360448	9,8,1,1	-2888853456640
7,7,4,0	-10569489713182080	8,7,3,2	5838261073702086594688	9,8,2,0	-3510327932032
7,7,4,1	5915250606113727488	8,7,4,0	-47957368939626656	9,8,2,1	2188585915040384
7,7,4,2	149886488946568628294656	8,7,4,1	33506015096031958400	9,8,3,0	-4345925673853440
7,7,5,0	-572125762297261056	8,7,5,0	-3380856350018782208	9,9,1,0	-578600960
7,7,5,1	340591525223538673664	8,8,0,0	4	9,9,1,1	-12088456933376
7,7,6,0	-15558927658117177088	8,8,1,0	-127854976	9,9,2,0	-13264263786752
8,2,2,0	4	8,8,1,1	-1181782901504	10,2,2,1	-96
8,2,2,1	269280	8,8,2,0	-1541331810048	10,2,2,2	93388992
8,2,2,2	12011885472	8,8,2,1	844741431788416	10,3,2,1	2765056
8,3,1,1	10496	8,8,2,2	59548327861798208892	10,3,2,2	621519800480
8,3,2,0	-69568	8,8,3,0	-1392349606298016	10,3,3,0	-1046656
8,3,2,1	353758784	8,8,3,1	1140095263765735424	10,3,3,1	17552569088
8,3,2,2	7576252775296	8,8,4,0	-311703356855240960	10,3,3,2	1238547915076352
8,3,3,0	-188206400	9,2,2,1	1984	10,3,3,3	1507518654558092544
8,3,3,1	205607705344	9,2,2,2	1812768672	10,4,1,1	20096
8,3,3,2	3556448533895424	9,3,2,0	-320	10,4,2,0	-124616
8,3,3,3	1507518654558092544	9,3,2,1	54265088	10,4,2,1	3987616000
8,4,1,0	-96	9,3,2,2	3034052551424	10,4,2,2	323138860266848
8,4,1,1	-1580992	9,3,3,0	-25362432	10,4,3,0	-2397896960
8,4,2,0	-38445312	9,3,3,1	83573229568	10,4,3,1	82324111529408
8,4,2,1	48618551008	9,3,3,2	2739373720985856	10,4,3,2	418143162721593312
8,4,2,2	942605235833872	9,3,3,3	1899437094349107200	10,4,3,3	419022629321006302208
8,4,3,0	-33378648000	9,4,1,1	241408	10,4,4,0	-1541331810048
8,4,3,1	23534563926272	9,4,2,0	-4386240	10,4,4,1	2691518925004800
8,4,3,2	418143162721593312	9,4,2,1	19566085376	10,4,4,2	119867722557338716800
8,4,3,3	181018160115934668416	9,4,2,2	723323393538432	10,5,1,1	-14325760
8,4,4,0	-4598298768072	9,4,3,0	-12801118336	10,5,2,0	-127854976
8,4,4,1	2691518925004800	9,4,3,1	18148629097216	10,5,2,1	626843119104
8,4,4,2	51542075860983487936	9,4,3,2	527901584611872640	10,5,2,2	41383783201391104
8,4,4,3	23528568383214020784160	9,4,3,3	340169678300685140992	10,5,3,0	-497787993344
8,4,4,4	3349905544323212030802336	9,4,4,0	-3510327932032	10,5,3,1	955800484656896
8,5,1,0	-124608	9,4,4,1	3390956750776448	10,5,3,2	45752405840566558208
8,5,1,1	-734128384	9,4,4,2	97201242671174343424	10,5,4,0	-210165664436480
8,5,2,0	-2397896960	9,4,4,3	61931896048581461894144	10,5,4,1	280378657159159808
8,5,2,1	1809944501504	9,5,1,0	-4864	10,5,5,0	-25243927618418816
8,5,2,2	41383783201391104	9,5,1,1	-193064960	10,6,1,0	-69568
8,5,3,0	-1541331760640	9,5,2,0	-831856256	10,6,1,1	-5444761792
8,5,3,1	955800484656896	9,5,2,1	1393481688832	10,6,2,0	-12801129824
8,5,3,2	19486899779598178176	9,5,2,2	52565098296996352	10,6,2,1	27081415941696
8,5,3,3	9230145138666014389504	9,5,3,0	-1166173618944	10,6,2,2	2036173541408295680
8,5,4,0	-210165664436480	9,5,3,1	1207709952303104	10,6,3,0	-28770544038624
8,5,4,1	120646194440285120	9,5,3,2	37014997608595211776	10,6,3,1	42837663308834816
8,5,4,2	2670176886160388673088	9,5,3,3	24481664831474093735936	10,6,4,0	-10569489712041568
8,5,4,3	1347917974198473929771648	9,5,4,0	-267545068649344	10,7,1,0	-6904192
8,5,5,0	-10569489713182080	9,5,4,1	227415315078834432	10,7,1,1	-249012852224
8,5,5,1	6203840748464569344	9,5,4,2	7109119409066563817856	10,7,2,0	-372907450720
8,5,5,2	158349506217985346295936	9,5,5,0	-20339595410138112	10,7,2,1	463223659993600
8,6,1,0	-4386240	9,5,5,1	16444600976022953984	10,7,3,0	-692429114488064
8,6,1,1	-23596592896	9,6,1,0	-1046656	10,8,1,0	-127854976
8,6,2,0	-45623190048	9,6,1,1	-16517906944	10,8,1,1	-3867646677632
8,6,2,1	27081415941696	9,6,2,0	-33378648000	10,8,2,0	-4598298768072
8,6,2,2	840745818190013632	9,6,2,1	34157599278208	10,9,1,0	-831856256
8,6,3,0	-28770544038624	9,6,2,2	1634935031609724864	10,10,0,0	4
8,6,3,1	18089515225187456	9,6,3,0	-37107163518144	11,2,2,2	269280
8,6,3,2	449402422813104411520	9,6,3,1	34589813440859136	11,3,2,1	10496
8,6,3,3	243302732065835955711744	9,6,3,2	1221567939348425063424	11,3,2,2	57695471872
8,6,4,0	-4345925673694464	9,6,4,0	-8478485684300928	11,3,3,0	-4864
8,6,4,1	2674825644762726656	9,6,4,1	7170055316812400128	11,3,3,1	1683640320
8,6,4,2	71095410940695469670768	9,6,5,0	-699282768851250048	11,3,3,2	317438354195968
8,6,5,0	-254128031458899136	9,7,1,0	-25362432	11,3,3,3	748137868720717824
8,6,5,1	162347641909980632448	9,7,1,1	-342488981504	11,4,2,0	-96
8,6,6,0	-7295475375082553200	9,7,2,0	-497787993344		
8,7,1,0	-38443520	9,7,2,1	377316002640896		

table 3.5 continued.

$\mathbf{p}$	$n_{\mathbf{p}}^{(1)}$	$\mathbf{p}$	$n_{\mathbf{p}}^{(1)}$	$\mathbf{p}$	$n_{\mathbf{p}}^{(1)}$
11,4,2,1	353758784	11,6,2,0	-2397896960	12,4,4,0	-45623190048
11,4,2,2	80961860336992	11,6,2,1	13324886863872	12,5,1,1	10496
11,4,3,0	-188206400	11,6,3,0	-13264263786752	12,5,2,0	-69568
11,4,3,1	2116485979904	11,7,1,0	-631296	12,5,2,1	19566085376
11,4,3,2	206208348590980608	11,7,1,1	-93294921728	12,5,3,0	-12801118336
11,4,4,0	-372907450720	11,7,2,0	-154245226112	12,6,1,1	-25295744
11,4,4,1	1335576781170688	11,8,1,0	-38443520	12,6,2,0	-188206960
11,5,1,1	286720	12,2,2,2	-10272	12,7,1,0	-10112
11,5,2,0	-6904192	12,3,2,1	-320	13,3,2,2	2765056
11,5,2,1	156514373632	12,3,2,2	1812768672	13,3,3,1	112640
11,5,2,2	20022577101423360	12,3,3,1	54265088	13,4,2,1	20096
11,5,3,0	-114273286144	12,3,3,2	43026117967872	13,4,3,0	-10112
11,5,3,1	469840676702208	12,4,2,1	9983872		
11,5,4,0	-100934478032512	12,4,2,2	10558546637760		
11,6,1,0	-320	12,4,3,0	-4386240		
11,6,1,1	-734128384	12,4,3,1	286009826304		

table 3.5 continued.

$\mathbf{p}$	$n_{\mathbf{p}}^{(1)}$	$\mathbf{p}$	$n_{\mathbf{p}}^{(1)}$
3,3	1475	10,5	3573290410020
4,3	29350	11,4	46911250
4,4	2669500	8,8	428216622053327300
5,3	148525	9,7	135767820281303350
5,4	46911250	10,6	3417167213249325
6,3	250550	11,5	2731112702750
5,5	2311178040	12,4	2669500
6,4	303610050	9,8	7360276988409757150
7,3	148525	10,7	817523002761866550
6,5	38756326500	11,6	6658383337394000
7,4	882636150	12,5	1207298050100
8,3	29350	13,4	29350
6,6	1477879258975	9,9	213796802016132371125
7,5	298784327925	10,8	77723709111160034550
8,4	1249719025	11,7	3186381984770132650
9,3	1475	12,6	8301844531611000
7,6	24724246516200	13,5	298784327925
8,5	1207298050100	10,9	3733143718641168532250
9,4	882636150	11,8	534623718661493240750
7,7	824125289385950	12,7	8273823575633968400
8,6	217335663077200	13,6	6658383337394000
9,5	2731112702750	14,5	38756326500
10,4	303610050	10,10	104214421442680518762070
8,7	13948250904141600	11,9	42137416928528774899500
9,6	1103600201154950	12,8	2489528873573792774625

Table 3.6: The genus 1 instanton numbers of total degree ≤ 37 for the family (3.15). The numbers not in this list are either zero, or given by those in the table after permuting indices.

p	$n_p^{(1)}$	p	$n_p^{(1)}$
13,7	14571606313456936800	18,10	80361250809900507464623047700
14,6	3417167213249325	19,9	60438328652210496631088500
15,5	2311178040	20,8	2489528873573792774625
11,10	1846950755735426212125500	21,7	824125289385950
12,9	322658009930052499145200	15,14	98295022688621627634296681347227000
13,8	8054854261423242104500	16,13	28288279315284624001518199057961000
14,7	17578605828858033550	17,12	2182612077022792907314617911510950
15,6	1103600201154950	18,11	38633710437048284516888316634750
16,5	46911250	19,10	118872243137920021974764986250
11,11	50041665253951501461197500	20,9	39078728637782455132498000
12,10	22037047145009664053322650	21,8	534623718661493240750
13,9	1732761485763012342489325	22,7	24724246516200
14,8	18428114211388322399550	15,15	2483451908782029978483441284953085840
15,7	14571606313456936800	16,14	1366181074996754720323336949015132850
16,6	217335663077200	17,13	220457095613704667909688795083550075
17,5	148525	18,12	9451168057601953137717939435446150
12,11	899012021570846648502276300	19,11	89342683428580923415751095264250
13,10	184364061855133820169125300	20,10	135397354308622811768958599380
14,9	6680798279094720126093800	21,9	18802673258937852338243175
15,8	30150777878498691717250	22,8	77723709111160034550
16,7	8273823575633968400	23,7	298784327925
17,6	24724246516200	16,15	46546285854998069136664551148160029750
12,12	23797576472047430629503926275	17,14	14543614745639013135983680186883590850
13,11	11242357248165502750651190625	18,13	1340566237649146039899848536120642500
14,10	1113322592174963231485326725	19,12	32177823874685392221678086500438300
15,9	18802673258937852338243175	20,11	161956900412845480278959039133500
16,8	35503691126837007672225	21,10	118872243137920021974764986250
17,7	3186381984770132650	22,9	6680798279094720126093800
18,6	1477879258975	23,8	7360276988409757150
13,12	432820127858166659059675434050	24,7	882636150
14,11	101324272018566859752134278000	16,16	1163420080671892184401603599755019719900
15,10	4956305470261096852879492520	17,15	664598046187475942195256766297105189925
16,9	39078728637782455132498000	18,14	120763194531086741850020681429354372600
17,8	30150777878498691717250	19,13	6439973193384613124797926096685847700
18,7	817523002761866550	20,12	86828419567453882491964244521715475
19,6	38756326500	21,11	231063040271162785144114394861625
13,13	11245532977494243857131235680275	22,10	80361250809900507464623047700
14,12	5634805973327046211368971793400	23,9	1732761485763012342489325
15,11	675189575619716190356450822375	24,8	428216622053327300
16,10	16520552027852727161441204150	25,7	148525
17,9	60438328652210496631088500	17,16	21983863245711864916146007548819920783200
18,8	18428114211388322399550	18,15	7380688961068662640476640993756532397450
19,7	135767820281303350	19,14	793510861046493220781342058077180934050
20,6	303610050	20,13	24681216007215096706670234574240357500
14,13	206807254359453225476163905375300	21,12	186826323241756428747689323102774900
15,12	54117569679383890033269008256800	22,11	260054031226217864291042886267450
16,11	3390671418197231819920137904000	23,10	41701179917420632714678892850
17,10	41701179917420632714678892850	24,9	322658009930052499145200
18,9	69860383953641753591175350	25,8	13948250904141600
19,8	8054854261423242104500	17,17	544441278609756260678075733228688830971900
20,7	13948250904141600	18,16	321496661014907630133282804314901041773300
21,6	250550	19,15	64811223550225163381933657270162177991375
14,14	5291910264323169346514519871886400	20,14	4173727947710375348427686143589666021850
15,13	2787959348321923974606513689960775	21,13	76049417732402225585766755889537693250
16,12	392220183802149653816905782829075	22,12	321974061113618057538097390567433500
17,11	13018317138588633706128782358825	23,11	231063040271162785144114394861625

table 3.6 continued.

p	$n_p^{(1)}$
24,10	16520552027852727161441204150
25,9	42137416928528774899500
26,8	217335663077200
18,17	10364068253575077523538680193754837859801100
19,16	3707131738581515278187095373395684544431750
20,15	455990757375323038689152386845291780997980
21,14	17736215361969271597308350645891203085550
22,13	189541398071564094191656020501685749700
23,12	445795478920394854284300481376874700
24,11	161956900412845480278959039133500
25,10	4956305470261096852879492520
26,9	3733143718641168532250
27,8	1207298050100
18,18	254628881812169884863144737907071363850384275
19,17	154848648177340961815627818425036135998755750
20,16	34212563460249368657994755599372244965731625
21,15	2598107539139344864070481843220029434149400
22,14	61346528057542695552753204264092617938350
23,13	383896440392157077525881068439189823375
24,12	496770523785734165693381637859440175
25,11	89342683428580923415751095264250
26,10	1113322592174963231485326725
27,9	213796802016132371125
28,8	1249719025
19,18	4879913180567429516866248000617525315434124200
20,17	1846594387973758522093973511561276425586720500
21,16	255797150291719302931481295802475608460495500
22,15	12092961029767718697960992221193131180072800
23,14	173738502202568390450523304606890631973000
24,13	634051294393795149147968219347562400500
25,12	445795478920394854284300481376874700
26,11	38633710437048284516888316634750
27,10	184364061855133820169125300
28,9	7360276988409757150
29,8	29350

table 3.6 continued.

3.3 Discriminant loci and Yukawa couplings

We do not give genus one numbers for six and seven parameter families (3.12), (3.13). Performing this computation would be possible if we knew the relevant discriminants Δ , which we do not. To make progress on this, we compute a number of discriminants for other models, in the hopes of recognising a formula that reproduces the known examples. This can be done for a CICY matrix with identical columns, as in this case Δ takes a factored form as for the two families above.

To compute a model's discriminant, we compute the Yukawa couplings. These are rational functions (in the coordinates defined in Appendix §A), and the discriminant can be identified in the denominators. This process can be run over all two-parameter CICYs, and we also do this for a number of three-parameter CICYs. However, this has not yet led to any good candidates for the models (3.12), (3.13).

The Yukawa couplings C_{ijk} can be computed from the formula

$$C_{ijk} = \frac{\partial}{\partial \varphi^i} \frac{\partial}{\partial \varphi^j} \frac{\partial}{\partial \varphi^k} \mathcal{F}^{(0)} = -\Pi^T \cdot \Sigma \cdot \frac{\partial}{\partial \varphi^i} \frac{\partial}{\partial \varphi^j} \frac{\partial}{\partial \varphi^k} \Pi = -\varpi^T \cdot \sigma \cdot \frac{\partial}{\partial \varphi^i} \frac{\partial}{\partial \varphi^j} \frac{\partial}{\partial \varphi^k} \varpi . \quad (3.20)$$

We give again here for ease of reading the matrix Σ , and also the matrix σ :

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1}_{m+1} \\ -\mathbb{1}_{m+1} & 0 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \mathbb{1}_m & 0 \\ 0 & -\mathbb{1}_m & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad m = h^{2,1}(X) . \quad (3.21)$$

These functions C_{ijk} are rational function of φ^i , and their denominators all contain a factor of Δ , which equals the polynomial least common multiple of the denominators of C_{ijk} . In practice formula (3.20) serves to compute C_{ijk} as power series in ϖ^i , and one can fit a rational function to this. If the discriminant Δ is known, then one can form the power series of $\varphi^i \varphi^j \varphi^k \cdot \Delta \cdot C_{ijk}$ to find a polynomial, the numerator of C_{ijk} . If Δ is not known then

one must fit a rational function to C_{ijk} , which becomes prohibitively complicated at large numbers of parameters. A more detailed discussion of the Yukawa couplings can be found in [63, 64].

We remark that a rule holds for CICYs with identical columns $(d^{(1)}, d^{(2)}, \dots, d^{(k)})^T$, so of the form

$$\begin{matrix} \mathbb{P}^{n_1} \\ \vdots \\ \mathbb{P}^{n_k} \end{matrix} \begin{bmatrix} d^{(1)} & \dots & d^{(1)} \\ \vdots & \dots & \vdots \\ d^{(k)} & \dots & d^{(k)} \end{bmatrix}. \quad (3.22)$$

This rule is

$$\Delta = \prod_{j_1=1}^{n_1+1} \cdots \prod_{j_k=1}^{n_k+1} \left(1 - \sum_{i=1}^k d^{(i)} \exp \left(\frac{2\pi i j_i}{n_i + 1} \right) \varphi_i^{\frac{1}{n_i+1}} \right). \quad (3.23)$$

This does not give the discriminant of the families (3.22), but of their mirrors. We discuss the parametrisation of the mirror manifolds that we are employing in Appendix §A. We verify that this holds for all CICY configurations of the form (3.22), so the following 11 families:

$$\begin{aligned} & \mathbb{P}^4 \begin{bmatrix} 5 \end{bmatrix}, \quad \mathbb{P}^5 \begin{bmatrix} 3 & 3 \end{bmatrix}, \quad \mathbb{P}^7 \begin{bmatrix} 2 & 2 & 2 & 2 \end{bmatrix}, \\ & \begin{matrix} \mathbb{P}^4 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \\ & \begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned} \quad (3.24)$$

For the last model above, with five parameters, this discriminant is

$$\Delta = \prod_{\epsilon_i \in \{1, -1\}} \left(1 + \epsilon_1 \sqrt{\varphi^1} + \epsilon_2 \sqrt{\varphi^2} + \epsilon_3 \sqrt{\varphi^3} + \epsilon_4 \sqrt{\varphi^4} + \epsilon_5 \sqrt{\varphi^5} \right). \quad (3.25)$$

We proceed to tabulate all CICYs with $h^{1,1} = 2$ [31], together with Δ , which we compute by forming the polynomial least common multiple of the Yukawa coupling denominators [27].

CICYs with $h^{1,1} = 2$, with the discriminant polynomial Δ of the mirror. [1]	
$\mathbb{P}^2 \begin{bmatrix} 0 & 0 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}$ $(1 - 4\varphi_1)^2 (1 - 8\varphi_1 + 16\varphi_1^2 - 48\varphi_2 - 320\varphi_1\varphi_2 + 768\varphi_2^2 + 256\varphi_1\varphi_2^2 - 4096\varphi_2^3)$	
$\mathbb{P}^4 \begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \end{bmatrix}$ $(1 - 4\varphi_1) (1 - 4\varphi_2) (1 - 12\varphi_1 + 48\varphi_1^2 - 64\varphi_1^3 - 12\varphi_2 - 336\varphi_1\varphi_2 - 192\varphi_1^2\varphi_2 + 48\varphi_2^2 - 192\varphi_1\varphi_2^2 - 64\varphi_2^3)$	
$\mathbb{P}^2 \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ $(1 - 4\varphi_1)^2 (1 - 8\varphi_1 + 16\varphi_1^2 - 81\varphi_2 - 540\varphi_1\varphi_2 + 2187\varphi_2^2 + 729\varphi_1\varphi_2^2 - 19683\varphi_2^3)$	
$\mathbb{P}^2 \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 6 & 2 & 2 & 1 & 1 \end{bmatrix}$ $(1 - \varphi_1)^2 (1 - 3\varphi_1 + 3\varphi_1^2 - \varphi_1^3 - 48\varphi_2 - 336\varphi_1\varphi_2 - 48\varphi_1^2\varphi_2 + 768\varphi_2^2 - 768\varphi_1\varphi_2^2 - 4096\varphi_2^3)$	
$\mathbb{P}^3 \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}$ $(1 - \varphi_1) (1 - 4\varphi_1 + 6\varphi_1^2 - 4\varphi_1^3 + \varphi_1^4 - 16\varphi_2 - 496\varphi_1\varphi_2 - 496\varphi_1^2\varphi_2 - 16\varphi_1^3\varphi_2 + 96\varphi_2^2 - 1984\varphi_1\varphi_2^2 + 96\varphi_1^2\varphi_2^2 - 256\varphi_2^3 - 256\varphi_1\varphi_2^3 + 256\varphi_2^4)$	
$\mathbb{P}^2 \begin{bmatrix} 0 & 2 & 1 \\ 2 & 2 & 1 & 2 \end{bmatrix}$ $(1 - 4\varphi_1) (1 - 12\varphi_1 + 48\varphi_1^2 - 64\varphi_1^3 - 48\varphi_2 - 696\varphi_1\varphi_2 + 96\varphi_1^2\varphi_2 + 768\varphi_2^2 + 384\varphi_1\varphi_2^2 - 432\varphi_1^2\varphi_2^2 - 4096\varphi_2^3)$	
$\mathbb{P}^3 \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$ $(1 - 4\varphi_1) (1 - 12\varphi_1 + 48\varphi_1^2 - 16\varphi_2 - 664\varphi_1\varphi_2 - 544\varphi_1^2\varphi_2 + 96\varphi_2^2 - 1280\varphi_1\varphi_2^2 + 16\varphi_1^2\varphi_2^2 - 256\varphi_2^3 + 128\varphi_1\varphi_2^3 + 256\varphi_2^4)$	
$\mathbb{P}^4 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ $1 - 5\varphi_1 + 10\varphi_1^2 - 10\varphi_1^3 + 5\varphi_1^4 - \varphi_1^5 - 5\varphi_2 - 605\varphi_1\varphi_2 - 1905\varphi_1^2\varphi_2 - 605\varphi_1^3\varphi_2 - 5\varphi_1^4\varphi_2 + 10\varphi_2^2 - 1905\varphi_1\varphi_2^2 - 10\varphi_1^2\varphi_2^2 - 10\varphi_1^3\varphi_2^2 - 10\varphi_2^3 - 605\varphi_1\varphi_2^3 - 10\varphi_1^2\varphi_2^3 + 5\varphi_2^4 - 5\varphi_1\varphi_2^4 - \varphi_2^5$	
$\mathbb{P}^3 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ $1 - 16\varphi_1 + 96\varphi_1^2 - 256\varphi_1^3 + 256\varphi_1^4 - 16\varphi_2 - 1261\varphi_1\varphi_2 - 2968\varphi_1^2\varphi_2 + 304\varphi_1^3\varphi_2 + 96\varphi_2^2 - 2968\varphi_1\varphi_2^2 + 69\varphi_1^2\varphi_2^2 - 27\varphi_1^3\varphi_2^2 - 256\varphi_2^3 + 304\varphi_1\varphi_2^3 - 27\varphi_1^2\varphi_2^3 + 256\varphi_2^4$	

CICYs with $h^{1,1} = 2$, with the discriminant polynomial Δ of the mirror. [1]	
$\mathbb{P}^1 \begin{bmatrix} 0 & 2 \\ 3 & 2 \end{bmatrix}$	
$(1 - 4\varphi_1)^2 (1 - 8\varphi_1 + 16\varphi_1^2 - 216\varphi_2 - 864\varphi_1\varphi_2 + 11664\varphi_2^2)$	
$\mathbb{P}^2 \begin{bmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$	
$(1 - \varphi_1) (1 - 4\varphi_1 + 6\varphi_1^2 - 4\varphi_1^3 + \varphi_1^4 - 48\varphi_2 - 664\varphi_1\varphi_2 - 320\varphi_1^2\varphi_2 + 8\varphi_1^3\varphi_2 + 768\varphi_2^2 - 2176\varphi_1\varphi_2^2 + 16\varphi_1^2\varphi_2^2 - 4096\varphi_2^3)$	
$\mathbb{P}^2 \begin{bmatrix} 0 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$	
$(1 - \varphi_1)^2 (1 - 3\varphi_1 + 3\varphi_1^2 - \varphi_1^3 - 81\varphi_2 - 567\varphi_1\varphi_2 - 81\varphi_1^2\varphi_2 + 2187\varphi_2^2 - 2187\varphi_1\varphi_2^2 - 19683\varphi_2^3)$	
$\mathbb{P}^3 \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$	
$1 - 5\varphi_1 + 10\varphi_1^2 - 10\varphi_1^3 + 5\varphi_1^4 - \varphi_1^5 - 16\varphi_2 - 905\varphi_1\varphi_2 - 1889\varphi_1^2\varphi_2 - 318\varphi_1^3\varphi_2 + 3\varphi_1^4\varphi_2 + 96\varphi_2^2 - 4296\varphi_1\varphi_2^2 + 814\varphi_1^2\varphi_2^2 - 3\varphi_1^3\varphi_2^2 - 256\varphi_2^3 - 784\varphi_1\varphi_2^3 + \varphi_1^2\varphi_2^3 + 256\varphi_2^4$	
$\mathbb{P}^1 \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}$	
$(1 - \varphi_1) (1 - 4\varphi_1 + 6\varphi_1^2 - 4\varphi_1^3 + \varphi_1^4 - 128\varphi_2 - 768\varphi_1\varphi_2 - 128\varphi_1^2\varphi_2 + 4096\varphi_2^2)$	
$\mathbb{P}^1 \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$	
$(1 - \varphi_1)^2 (1 - 3\varphi_1 + 3\varphi_1^2 - \varphi_1^3 - 128\varphi_2 - 320\varphi_1\varphi_2 + 16\varphi_1^2\varphi_2 + 4096\varphi_2^2)$	
$\mathbb{P}^1 \begin{bmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}$	
$(1 - \varphi_1)^3 (1 - 2\varphi_1 + \varphi_1^2 - 128\varphi_2 - 128\varphi_1\varphi_2 + 4096\varphi_2^2)$	
$\mathbb{P}^2 \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$	
$1 - 5\varphi_1 + 10\varphi_1^2 - 10\varphi_1^3 + 5\varphi_1^4 - \varphi_1^5 - 48\varphi_2 - 1280\varphi_1\varphi_2 - 1672\varphi_1^2\varphi_2 - 124\varphi_1^3\varphi_2 - \varphi_1^4\varphi_2 + 768\varphi_2^2 - 5888\varphi_1\varphi_2^2 - 128\varphi_1^2\varphi_2^2 - 4096\varphi_2^3$	
$\mathbb{P}^1 \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$	
$(1 - 4\varphi_1)^2 (1 - 8\varphi_1 + 16\varphi_1^2 - 128\varphi_2 - 512\varphi_1\varphi_2 + 4096\varphi_2^2)$	

CICYs with $h^{1,1} = 2$, with the discriminant polynomial Δ of the mirror. [1]
$\mathbb{P}^1 \begin{bmatrix} 0 & 0 & 0 & 2 \\ 2 & 2 & 2 & 1 \end{bmatrix}$ $\mathbb{P}^6 \begin{bmatrix} 2 & 2 & 2 & 1 \end{bmatrix}$ $(1 - 4\varphi_1)^3 (1 - 4\varphi_1 - 128\varphi_2 + 4096\varphi_2^2)$
$\mathbb{P}^2 \begin{bmatrix} 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ $1 - 16\varphi_1 + 96\varphi_1^2 - 256\varphi_1^3 + 256\varphi_1^4 - 81\varphi_2 - 1809\varphi_1\varphi_2 + 440\varphi_1^2\varphi_2 - 400\varphi_1^3\varphi_2 + 2187\varphi_2^2 + 1458\varphi_1\varphi_2^2 - 3375\varphi_1^3\varphi_2^2 + 3125\varphi_1^3\varphi_2^3 - 19683\varphi_2^3$
$\mathbb{P}^1 \begin{bmatrix} 0 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ $\mathbb{P}^5 \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ $(1 - \varphi_1)^2 (1 - 3\varphi_1 + 3\varphi_1^2 - \varphi_1^3 - 216\varphi_2 - 540\varphi_1\varphi_2 + 27\varphi_1^2\varphi_2 + 11664\varphi_2^2)$
$\mathbb{P}^2 \begin{bmatrix} 2 & 1 \\ 3 & 2 & 2 \end{bmatrix}$ $1 - 16\varphi_1 + 96\varphi_1^2 - 256\varphi_1^3 + 256\varphi_1^4 - 48\varphi_2 - 2656\varphi_1\varphi_2 - 5120\varphi_1^2\varphi_2 + 512\varphi_1^3\varphi_2 + 768\varphi_2^2 - 8704\varphi_1\varphi_2^2 + 256\varphi_1^2\varphi_2^2 - 4096\varphi_2^3$
$\mathbb{P}^2 \begin{bmatrix} 0 & 2 & 1 \\ 4 & 2 & 2 & 1 \end{bmatrix}$ $\mathbb{P}^4 \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}$ $(1 - 4\varphi_1) (1 - 12\varphi_1 + 48\varphi_1^2 - 64\varphi_1^3 - 48\varphi_2 - 1344\varphi_1\varphi_2 - 768\varphi_1^2\varphi_2 + 768\varphi_2^2 - 3072\varphi_1\varphi_2^2 - 4096\varphi_2^3)$
$\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 4 & 3 & 2 \end{bmatrix}$ $1 - 5\varphi_1 + 10\varphi_1^2 - 10\varphi_1^3 + 5\varphi_1^4 - \varphi_1^5 - 216\varphi_2 - 2052\varphi_1\varphi_2 - 873\varphi_1^2\varphi_2 + 16\varphi_1^3\varphi_2 + 11664\varphi_2^2$
$\mathbb{P}^3 \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$ $1 - 16\varphi_1 + 96\varphi_1^2 - 256\varphi_1^3 + 256\varphi_1^4 - 16\varphi_2 - 1984\varphi_1\varphi_2 - 7936\varphi_1^2\varphi_2 - 1024\varphi_1^3\varphi_2 + 96\varphi_2^2 - 7936\varphi_1\varphi_2^2 + 1536\varphi_1^2\varphi_2^2 - 256\varphi_2^3 - 1024\varphi_1\varphi_2^3 + 256\varphi_2^4$
$\mathbb{P}^1 \begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$ $\mathbb{P}^6 \begin{bmatrix} 3 & 2 & 1 & 1 \end{bmatrix}$ $(1 - \varphi_1)^3 (1 - 2\varphi_1 + \varphi_1^2 - 216\varphi_2 - 216\varphi_1\varphi_2 + 11664\varphi_2^2)$
$\mathbb{P}^2 \begin{bmatrix} 1 & 1 \\ 4 & 3 & 1 & 1 \end{bmatrix}$ $1 - 5\varphi_1 + 10\varphi_1^2 - 10\varphi_1^3 + 5\varphi_1^4 - \varphi_1^5 - 81\varphi_2 - 1674\varphi_1\varphi_2 - 1417\varphi_1^2\varphi_2 + 55\varphi_1^3\varphi_2 - 8\varphi_1^4\varphi_2 + 2187\varphi_2^2 - 10206\varphi_1\varphi_2^2 + 108\varphi_1^2\varphi_2^2 - 16\varphi_1^3\varphi_2^2 - 19683\varphi_2^3$

CICYs with $h^{1,1} = 2$, with the discriminant polynomial Δ of the mirror. [1]	
$\mathbb{P}^1 \begin{bmatrix} 0 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ $\mathbb{P}^5 \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$ $(1 - 4\varphi_1)^3 (1 - 4\varphi_1 - 216\varphi_2 + 11664\varphi_2^2)$	
$\mathbb{P}^1 \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ $\mathbb{P}^5 \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}$ $(1 - \varphi_1) (1 - 4\varphi_1 + 6\varphi_1^2 - 4\varphi_1^3 + \varphi_1^4 - 216\varphi_2 - 864\varphi_1\varphi_2 + 72\varphi_1^2\varphi_2 - 16\varphi_1^3\varphi_2 + 11664\varphi_2^2)$	
$\mathbb{P}^1 \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$ $\mathbb{P}^4 \begin{bmatrix} 2 & 3 \end{bmatrix}$ $(1 - 4\varphi_1) (1 - 12\varphi_1 + 48\varphi_1^2 - 64\varphi_1^3 - 216\varphi_2 - 2592\varphi_1\varphi_2 + 11664\varphi_2^2)$	
$\mathbb{P}^2 \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ $\mathbb{P}^3 \begin{bmatrix} 3 & 1 \end{bmatrix}$ $1 - 16\varphi_1 + 96\varphi_1^2 - 256\varphi_1^3 + 256\varphi_1^4 - 81\varphi_2 - 4968\varphi_1\varphi_2 - 11664\varphi_1^2\varphi_2 + 256\varphi_1^3\varphi_2 + 2187\varphi_2^2 - 31347\varphi_1\varphi_2^2 - 19683\varphi_2^3$	
$\mathbb{P}^2 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $\mathbb{P}^2 \begin{bmatrix} 3 \end{bmatrix}$ $1 - 81\varphi_1 + 2187\varphi_1^2 - 19683\varphi_1^3 - 81\varphi_2 - 15309\varphi_1\varphi_2 - 59049\varphi_1^2\varphi_2 + 2187\varphi_2^2 - 59049\varphi_1\varphi_2^2 - 19683\varphi_2^3$	
$\mathbb{P}^1 \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\mathbb{P}^4 \begin{bmatrix} 4 & 1 \end{bmatrix}$ $1 - 5\varphi_1 + 10\varphi_1^2 - 10\varphi_1^3 + 5\varphi_1^4 - \varphi_1^5 - 512\varphi_2 - 2816\varphi_1\varphi_2 + 320\varphi_1^2\varphi_2 - 144\varphi_1^3\varphi_2 + 27\varphi_1^4\varphi_2 + 65536\varphi_2^2$	
$\mathbb{P}^1 \begin{bmatrix} 0 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}$ $\mathbb{P}^5 \begin{bmatrix} 4 & 1 & 1 \end{bmatrix}$ $(1 - \varphi_1)^3 (1 - 2\varphi_1 + \varphi_1^2 - 512\varphi_2 - 512\varphi_1\varphi_2 + 65536\varphi_2^2)$	
$\mathbb{P}^1 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $\mathbb{P}^3 \begin{bmatrix} 4 \end{bmatrix}$ $1 - 16\varphi_1 + 96\varphi_1^2 - 256\varphi_1^3 + 256\varphi_1^4 - 512\varphi_2 - 12288\varphi_1\varphi_2 - 8192\varphi_1^2\varphi_2 + 65536\varphi_2^2$	
$\mathbb{P}^1 \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix}$ $\mathbb{P}^4 \begin{bmatrix} 4 & 1 \end{bmatrix}$ $(1 - 4\varphi_1)^3 (1 - 4\varphi_1 - 512\varphi_2 + 65536\varphi_2^2)$	

3.4 Contractions and instanton summations

Upon inspecting the CICY matrices, one can see that the (7,27) model³ is a split of the (6,36) model, which is itself a split of the (5,45) Mirror Hulek Verrill family. Moreover, this (5,45) family is a split of the Tetraquadric with Hodge numbers (4,68).

Further still, a sequence of five splits from the (2,52) family⁴ yields the (7,27) family. Finally, the (2,52) family is a split of the Quintic $\mathbb{P}^4 \begin{bmatrix} 5 \\ -200 \end{bmatrix}^{(1,101)}$.

To pass from a manifold Y to its split $\tilde{Y}$ involves blowing up some number of exceptional divisors, which are degree-1 rational curves (so genus 0). Now consider a rational curve $\tilde{C}$ on the split manifold. Corresponding to this curve is a degree vector $\tilde{\mathbf{b}}$, giving the curve's homology class. In blowing down, we project out one of the homology classes on $\tilde{Y}$. The image of $\tilde{C}$ in Y is a rational curve C with degree vector $\mathbf{b}$ obtained from $\tilde{\mathbf{b}}$ by deleting the entries corresponding to the homology class that was projected out. Let ρ denote the projection operator on degree vectors that deletes the entry corresponding to the projected-out homology class. We expect to be able to recover the instanton numbers $n_{\mathbf{b}}^{(0)}$ for Y from the numbers $n_{\tilde{\mathbf{b}}}^{(0)}$ for $\tilde{Y}$ via a sum rule

$$n_{\mathbf{b}}^{(0)} = \sum_{\tilde{\mathbf{b}} | \rho(\tilde{\mathbf{b}}) = \mathbf{b}} n_{\tilde{\mathbf{b}}}^{(0)} . \quad (3.26)$$

This can also be seen at the level of the Calabi-Yau genus prepotential F , which is closely related to the topological string free energy F_0 , but incorporates the full set of topological quantities Y_{abc} as follows:

$$F = -\frac{1}{6} Y_{abc} t^a t^b t^c - \frac{1}{(2\pi i)^3} \sum_{\mathbf{b} \geq 0} n_{\mathbf{b}}^{(0)} \text{Li}_3(\mathbf{q}^{\mathbf{b}}) . \quad (3.27)$$

³It is clearest here for us to refer to these manifolds by their Hodge numbers $(h^{1,1}, h^{2,1})$. The configurations for the families referred to in this paragraph can be found in formulae (3.13), (3.12), (3.11), (3.14).

⁴See the configuration (3.15).

Reiterating, a, b, c run from 0 to $h^{1,1}(Y)$ while i, j, k run from 1 to $h^{1,1}(Y)$. The t^i give coefficients in the expansion of Y 's complexified Kähler form in a basis of $H^2(Y, \mathbb{Z})$. We have said this again in order to set the stage for shrinking two-cycles, a birational map between threefolds realised by sending one of the t^i to zero.

Let Y have $h^{1,1} = n$. Take $\tilde{Y}$ with $h^{1,1} = n + 1$ to be a split of Y . The coordinates t^i , $i = 1, \dots, n + 1$ give the complexified Kähler structure parameters on $\tilde{Y}$, which equal the integral of the complexified Kähler form⁵ $B + iJ$ of two-cycles in a basis of $H_2(\tilde{Y}, \mathbb{Z})$. Let t^{n+1} be the integral of $B + iJ$ over the two-cycle wrapped by the collapsing two-spheres. Then we should have

$$F_Y(t^1, \dots, t^n) = F_{\tilde{Y}}(t^1, \dots, t^n, t^{n+1}) \Big|_{t^{n+1}=0} . \quad (3.28)$$

Inspecting (3.4), setting $t_{n+1} = 0$, and comparing the q -expansions of both sides, we recover the sum rule for all degree vectors $\mathbf{b}$ except the zero vector. Equating the terms linear, quadratic, and cubic in t^i we get equalities

$$\tilde{Y}_{abc} = Y_{abc} , \quad 0 \leq a, b, c \leq n \text{ and } (a, b, c) \neq (0, 0, 0) , \quad (3.29)$$

with the left hand side giving topological numbers $\tilde{Y}_{abc}$ on $\tilde{Y}$ and the right hand side giving the numbers Y_{abc} for Y .

The exception to the sum rule is related to the change in Euler characteristic. In light of $\text{Li}_3(1) = \zeta(3)$, equating constant terms in (3.28) yields

$$\sum_k n_{0, \dots, 0, k}^{(0)} = -\frac{1}{2} \left[\chi(\tilde{Y}) - \chi(Y) \right] . \quad (3.30)$$

This formula could have been anticipated more geometrically, as the left hand side counts the curves that are blown down in the birational map $\tilde{Y} \mapsto Y$. Since $\chi(\mathbb{P}^1) = 2$, minus twice the above left hand side gives the change in Euler characteristic.

⁵ J is the Kähler form and B is the Kalb-Ramond B-field.

This can be verified for each pair of CICYs where one splits the other, but other examples can be found for suitable $\tilde{Y}$. For instance, the genus 0 invariants for the families

$$\frac{\mathbb{P}^3 \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}}{\mathbb{P}^3 \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}}, \quad \frac{\mathbb{P}^3 \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}{\mathbb{P}^3 \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}}, \quad \frac{\mathbb{P}^1 \begin{bmatrix} 2 \\ 4 \end{bmatrix}}{\mathbb{P}^3 \begin{bmatrix} 2 \\ 4 \end{bmatrix}}, \quad (3.31)$$

can be summed in this manner to recover invariants for the degree 8 hypersurface in the weighted projective space $\mathbb{W}\mathbb{P}_{(1,1,1,1,4)}^4$. This is mirror to a manifold with Picard-Fuchs operator labelled AESZ7 in the database [65], a hypergeometric model with indices $(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8})$. This family has also seen recent attention in [66], wherein three-loop “wheel” Feynman diagrams were evaluated in terms of Calabi-Yau periods.

In [34], a two-parameter non-Abelian gauged linear sigma model (GLSM) [67] with gauge group

$$G = \frac{U(1) \times U(1) \times O(2)}{\{\pm 1\} \times \{\pm 1\} \times \{\pm \mathbb{1}_2\}} \quad (3.32)$$

was studied. This model has six phases, and of relevance to our discussion is their phase I_+ and phase IV. These were both geometric, in the sense that the GLSM flowed to a nonlinear sigma model on Calabi-Yau spaces with Hodge numbers $(h^{1,1}, h^{2,1}) = (2, 24)$. The phase I_+ model was a quotient of a complete intersection in a toric variety, while the phase IV geometry was a determinantal hypersurface in another toric variety. We remark that the phase I_+ geometry can be contracted in either of two ways, and the instanton sums recover the genus-0 invariants and Euler characteristic of either the Reye congruence [60, 68] or a $\mathbb{Z}_2$ quotient of the quadric in $\mathbb{P}^7$:

$$\frac{\mathbb{P}^4 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}{\mathbb{P}^4 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{/\mathbb{Z}_2}, \quad \mathbb{P}^7[2, 2, 2, 2]_{/\mathbb{Z}_2}. \quad (3.33)$$

On the other hand, instanton number summations suggest that the phase IV geometry can be contracted to the intersection of two degree four hypersurfaces in $\mathbb{W}\mathbb{P}_{(1,1,1,1,2,2)}^5$. This manifold is mirror to a geometry with Picard-Fuchs operator AESZ10, a hypergeometric operator with indices $(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4})$.

We will briefly mention, but say no more about, the question of connecting all Calabi-Yau threefolds by such transitions (including generalisations, such as where instead of two-spheres, orbifolds shrink), which in the mathematics literature relates to a conjecture of Reid [69], see also [70] . The conifold transition in string theory was studied in [71–73]. It was shown in [74] that these transitions could be explained by black hole condensation.

Studying the prepotential F makes clear that the genus 0 instanton numbers on the split manifold can be summed into the genus 0 numbers of the contracted manifold, with the topological quantities and change in Euler character accounted for. At higher genera, setting a t^k to zero reproduces the same summation rules for higher genus instanton numbers. So the rule (3.26) also holds at higher genus:

$$n_{\mathbf{b}}^{(g)} = \sum_{\tilde{\mathbf{b}} | \rho(\tilde{\mathbf{b}}) = \mathbf{b}} n_{\tilde{\mathbf{b}}}^{(g)} . \quad (3.34)$$

Such data can serve as useful boundary data in higher genus free energy computations, as was noted in [56].

3.5 Coxeter groups

When the tables of genus 0 and 1 instanton numbers in [2] were produced, a number of repetitions of values were noted. A similar phenomenon can be observed in our tables table 3.3 and table 3.5 for the Tetraquadric. For example, we have equalities

$$\begin{aligned} n_{(1,1,1,0)}^{(0)} &= n_{(6,1,1,0)}^{(0)} = n_{(10,3,1,1)}^{(0)} = 2432 \\ n_{(3,3,1,1)}^{(0)} &= n_{(7,3,1,1)}^{(0)} = n_{(13,3,3,1)}^{(0)} = 241754112 \\ n_{(4,2,1,0)}^{(1)} &= n_{(7,2,2,0)}^{(1)} = n_{(8,4,1,0)}^{(1)} = -96 \\ n_{(3,2,2,2)}^{(1)} &= n_{(9,2,2,2)}^{(1)} = n_{(12,3,2,2)}^{(1)} = 1812768672 . \end{aligned} \quad (3.35)$$

It should be noted that when genus 0 numbers $n_{\mathbf{p}}^{(0)}$ and $n_{\mathbf{q}}^{(0)}$ are equal, then the genus 1 numbers $n_{\mathbf{p}}^{(1)}$ and $n_{\mathbf{q}}^{(1)}$ are equal.

In fact, instanton numbers are equal for degree vectors related by a certain linear transformation:

$$n_{(i,j,k,l)}^{(g)} = n_{-i+2j+2k+2l,j,k,l}^{(g)} . \quad (3.36)$$

Our tables support this claim for genera 0 and 1, but this is expected to hold at every genus. It is true to the full extent of the tables of genus one numbers, but at genus 0 there is one exception, which is that this operation takes $(1, 0, 0, 0)$ to $(-1, 0, 0, 0)$, yet the instanton number $n_{(1,0,0,0)}^{(0)} = 48$ is nonzero. For this particular Tetraquadric family, the above identity (including the mentioned exception) can be proven at every genus, either by inheriting an action from the splitting $(5,45)$ family (which itself has a proven Coxeter symmetry [2]), or by the arguments from flop transitions that appear in [75]. As we go on to give more examples of such identities, they will for the purposes of this thesis strictly be conjectures supported by the instanton number computations. Any example where these reflections can be traced to a flop transition (such as CICYs that split another family) would see a rigorous proof along the same lines, and indeed a number of our examples appear in [75] with proofs.

We will denote this operation by

$$g_1 : (i, j, k, l) \mapsto (2d - 3i, j, k, l) , \quad d = i + j + k + l . \quad (3.37)$$

This operation is an involution, with $g_1^2 = 1$. Additionally, there are the obvious permutation operations that leave instanton numbers invariant, because the mirror Tetraquadric's complex structure moduli φ_i can be exchanged by an S_4 symmetry. The group S_4 has three generators, whose actions on $\mathbb{Z}^4$ are

$$s_1 : (i, j, k, l) \mapsto (j, i, k, l) , \quad s_2 : (i, j, k, l) \mapsto (i, k, j, l) , \quad s_3 : (i, j, k, l) \mapsto (i, j, l, k) . \quad (3.38)$$

We should be clear that there are a few distinct group orbits with the same instanton number,

a point that demands further study. Collecting these together, we find the following group presentation

$$\langle g_1, s_1, s_2, s_3 \mid d_1^2 = s_1^2 = s_2^2 = s_3^2 = 1, (d_1 s_2)^2 = (d_1 s_3)^2 = 1, (s_1 s_2)^3 = (s_1 s_3)^3 = (s_2 s_3)^3 = 1 \rangle. \quad (3.39)$$

Note that the product $d_1 s_1$ is a group element of infinite order. This is a group generated by reflections r_i , and a symmetric matrix m_{ij} (with two formally infinite entries) gives the relations that products of reflections obey:

$$(r_i r_j)^{m_{ij}} = 1. \quad (3.40)$$

Groups with such a presentation are termed *Coxeter groups*. For our group, the Coxeter matrix m_{ij} reads

$$m_{ij} = \begin{pmatrix} 1 & \infty & 2 & 2 \\ \infty & 1 & 3 & 2 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{pmatrix}. \quad (3.41)$$

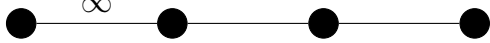
The number of generators r_i is termed the *rank* of the Coxeter group. The data of a Coxeter matrix can be neatly encoded in a Coxeter diagram. This is a graph with a number of nodes equal to the group's rank. If $m_{ij} = 2$, then no edges connect the i, j nodes. If $m_{ij} = 3$ then an unlabelled edge connects nodes i, j . If $m_{ij} \geq 4$ then nodes i, j are connected by an edge labelled with the entry m_{ij} . The Tetraquadric's Coxeter group has the following diagram:



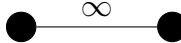
The appearance of Coxeter group actions as symmetries of sets of instanton numbers was also discussed in [75, 76], and were explained by flop transitions between manifolds in the same complex deformation family. We incorporate permutation symmetries in our discussion, so our presented Coxeter groups differ to those of [75]. The paper [77] explained how an infinite symmetry group could arise from infinite sequences of birational transformations.

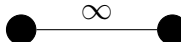
We list some CICY matrices for which a Coxeter group symmetry acts on the instanton numbers. Each of these geometries possesses an $S_{h^{1,1}}$ symmetry, which is extended by an operation g for which we give the action on the second homology, to get the group specified by the diagram that we give. When we give the operation g , the symbol d refers to the sum of a vector's elements. For instance in the first example $d = i + j + k + l + m$, while in the third $d = i + j + k$.

$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad g : (i, j, k, l, m) \mapsto (d-2i, j, k, l, m),$$

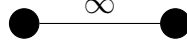

$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad g : (i, j, k, l) \mapsto (2d-3i, j, k, l),$$


$$\begin{matrix} \mathbb{P}^2 \\ \mathbb{P}^2 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad g : (i, j, k) \mapsto (2d-3i, j, k),$$


$$\begin{matrix} \mathbb{P}^4 \\ \mathbb{P}^4 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad g : (i, j) \mapsto (4d-5i, j),$$


$$\begin{matrix} \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad g : (i, j) \mapsto (6d-7i, j),$$


$$\begin{matrix} \mathbb{P}^3 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} , \quad g : (i, j) \mapsto (7d - 8i, j) ,$$



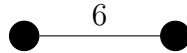
In the latter three two-parameter examples we see the appearance of the infinite dihedral group. In §3.7 we will discuss the application of this infinite symmetry to the computation of higher genus invariants. All of the above examples possess an $S_{h^{1,1}}$ permutation symmetry, but Coxeter group symmetries can also be found for manifolds that do not have this permutation symmetry. Some examples, for which we now give a full set of generators, are as follows:

$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} , \quad g : (i, j) \mapsto (4j - i, j) ,$$



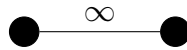
$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} , \quad g : (i, j, k) \mapsto (2k + j - i, j, k) ,$$

$$s : (i, j, k) \mapsto (j, i, k)$$



$$\begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \end{matrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} , \quad g : (i, j, k) \mapsto (3k + 2j - i, j, k) ,$$

$$s : (i, j, k) \mapsto (j, i, k)$$



In the first of the above two examples, we have a $\mathbb{Z}_2$ action but no permutation symmetry. This should be contrasted with manifolds that possess a permutation symmetry but have no extra $\mathbb{Z}_2$ symmetry action that changes the sum $i + j$, for example the $h^{1,1} = 2$ bicubic.

The second of the above three examples is interesting insofar as the Coxeter group is finite, being the dihedral group of order 12. This is the Weyl group of G_2 . It would be interesting

in future work to address the question of whether a Kac-Moody symmetry can be realised in string compactifications on manifolds with Coxeter symmetries of the forms that we describe, and were this the case then this example could be the simplest place to start.

While the instanton numbers are constant across orbits of the Coxeter groups we have given, it is not the case that each such orbit has a distinct associated instanton number. There is additional work to be done in explaining the additional repetitions. For example, the Coxeter group actions that we have given above for the families

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^2 \end{array} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \begin{array}{c} (3,75) \\ \\ -144 \end{array}, \quad \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^3 \end{array} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{array}{c} (3,55) \\ \\ -104 \end{array}, \quad (3.42)$$

do not change the third component k of the index vector (i, j, k) . Nonetheless, the tables of instanton numbers reveal some repetitions of instanton numbers for different values of k . We produce those tables to a low order here. Note for instance that for the $\chi = -144$ family

$$n_{(1,0,13)}^{(0)} = n_{(2,0,7)}^{(0)} = 5221882080 \quad \text{and} \quad n_{(1,0,13)}^{(0)} = n_{(5,0,4)}^{(0)} = 2115255492, \quad (3.43)$$

while for the $\chi = -104$ family we observe

$$n_{(1,0,5)}^{(0)} = n_{(3,0,3)}^{(0)} = 125824 \quad \text{and} \quad n_{(3,0,9)}^{(0)} = n_{(5,0,7)}^{(0)} = 130181768448 \quad (3.44)$$

Equalities like this are not explained by the flop operations appearing in [75], and for the time being we cannot explain them. It may be of note that the above index vectors have components that are 0.

In another attempt to generalise the results of [75], we can see manifolds with Coxeter symmetries that are not CICYs. We defer discussion of examples with groups larger than $\mathbb{Z}_2$ to future work, and here remark that nontrivial $\mathbb{Z}_2$ symmetries can be found for the following manifolds, which appeared in the tables of [78]. Details of Calabi-Yau intersections

in weighted projective space that we avail of are found in [79]. Since the instanton numbers that we compute for their $\chi = -112$ families agree, we conjecture that these two families are identical. We also fix a typo in the configuration matrix for the $\chi = -98$ family.

$$\mathbb{P}^1 \quad \mathbb{W}\mathbb{P}_{(1,1,1,1,2,2)}^5 \begin{bmatrix} 0 & 1 & 1 \\ 4 & 2 & 2 \end{bmatrix}_{\chi=-112} \cong \mathbb{P}^1 \quad \mathbb{W}\mathbb{P}_{(1,1,1,1,2)}^4 \begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix}_{\chi=-112}, \quad g : (i, j) \mapsto (2j - i, j) . \quad (3.45)$$

$$\mathbb{P}^2 \quad \mathbb{W}\mathbb{P}_{(1,1,1,1,2)}^4 \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}_{\chi=-98}, \quad g : (i, j) \mapsto (4j - i, j) . \quad (3.46)$$

Finally, we remark that the instanton numbers for the two-parameter geometry given in Phase IV of the Knapp-Hori model [34] possess a $\mathbb{Z}_2$ symmetry given by the action $g : (i, j) \mapsto (i, 5i - j)$, but this only holds for vectors with $i > 0$ at genus 0. Such $\mathbb{Z}_2$ symmetries and their utility in higher genus computations were identified in [56].

The content of the following tables is available in electronic form [1].

p	$n_p^{(0)}$	p	$n_p^{(0)}$	p	$n_p^{(0)}$
0,0,1	128	3,2,3	598081536	2,1,8	1320335902720
1,0,0	32	3,3,2	11713408	2,2,7	22815978525184
0,0,2	120	4,0,4	4264256	3,0,8	21118162688
1,0,1	384	4,1,3	27253248	3,1,7	5536125819904
1,1,0	32	4,2,2	2005824	3,2,6	39458161495040
0,0,3	128	4,3,1	384	3,3,5	26335226884096
1,0,2	2368	5,0,3	10496	4,0,7	28827891712
1,1,1	3072	5,1,2	2368	4,1,6	3144631756416
2,0,1	128	0,0,9	128	4,2,5	8535762597376
0,0,4	104	1,0,8	2692704	4,3,4	1874830230016
1,0,3	10496	1,1,7	1625063424	4,4,3	31657668096
1,1,2	75648	2,0,7	119377920	5,0,6	5865465088
2,0,2	5056	2,1,6	23048243712	5,1,5	224531369984
2,1,1	3072	2,2,5	111830923392	5,2,4	186198155264
0,0,5	128	3,0,6	368140672	5,3,3	8915561216
1,0,4	38624	3,1,5	25668767488	5,4,2	11713408
1,1,3	958464	3,2,4	40106733568	6,0,5	119377920
2,0,3	70656	3,3,3	3737054208	6,1,4	1082082304
2,1,2	293376	4,0,5	119377920	6,2,3	142328576
2,2,1	8000	4,1,4	2487784448	6,3,2	293376
3,0,2	2368	4,2,3	945297408	7,0,4	38624
3,1,1	384	4,3,2	11713408	7,1,3	10496
0,0,6	120	4,4,1	128	0,0,12	104
1,0,5	125824	5,0,4	2692704	1,0,11	36442112
1,1,4	8368448	5,1,3	9744384	1,1,10	113160255872
2,0,4	626432	5,2,2	293376	2,0,10	8125777088
2,1,3	9744384	6,0,3	128	2,1,9	8186269737984
2,2,2	2005824	0,0,10	120	2,2,8	238488938059776
3,0,3	125824	1,0,9	6679296	3,0,9	130181768448
3,1,2	293376	1,1,8	7268894432	3,1,8	58621972428800
3,2,1	3072	2,0,8	529534464	3,2,7	755784523825152
4,0,2	120	2,1,7	188236812288	3,3,6	1028951153011840
0,0,7	128	2,2,6	1803958100480	4,0,8	310514571520
1,0,6	373952	3,0,7	3018823680	4,1,7	63737659456000
1,1,5	57106432	3,1,6	428709670912	4,2,6	351572541694080
2,0,5	4264960	3,2,5	1533012331520	4,3,5	186761410553984
2,1,4	180973568	3,3,4	430848150336	4,4,4	11079782988096
2,2,3	142328576	4,0,6	2151740096	5,0,7	130181768448
3,0,4	2692704	4,1,5	111475558400	5,1,6	10983601432704
3,1,3	27253248	4,2,4	128007090432	5,2,5	22728981286912
3,2,2	3662848	4,3,3	8915561216	5,3,4	3805839345664
3,3,1	3072	4,4,2	20508560	5,4,3	47775759360
4,0,3	70656	5,0,5	174847616	5,5,2	11713408
4,1,2	75648	5,1,4	2487784448	6,0,6	8125777088
4,2,1	128	5,2,3	598081536	6,1,5	224531369984
0,0,8	104	5,3,2	3662848	6,2,4	128007090432
1,0,7	1033472	6,0,4	626432	6,3,3	3737054208
1,1,6	326085888	6,1,3	958464	6,4,2	2005824
2,0,6	24162944	6,2,2	5056	7,0,5	36442112
2,1,5	2324460544	0,0,11	128	7,1,4	180973568
2,2,4	5014078976	1,0,10	15887680	7,2,3	9744384
3,0,5	36442112	1,1,9	29762052096	7,3,2	2368
3,1,4	1082082304	2,0,9	2151748608	8,0,4	104

Table 3.7: The genus 0 instanton numbers of total degree ≤ 12 for the $\chi = -104$ family (3.42). The numbers not in this list are either zero, or given by those in the table after permuting the first two indices.

p	$n_p^{(0)}$	p	$n_p^{(0)}$	p	$n_p^{(0)}$	p	$n_p^{(0)}$
0,0,1	168	5,0,4	2115255492	5,4,3	6055793581127544	6,4,4	19702612873693234944
1,0,0	54	5,1,3	30179989584	5,5,2	29817003490128	6,5,3	400441073987635488
0,0,2	168	5,2,2	6738481008	6,0,6	29850028039080	6,6,2	678024552756840
1,0,1	1080	5,3,1	35294184	6,1,5	2540902631155632	7,0,7	6624537453484920
1,1,0	180	5,4,0	54	6,2,4	5011092898162560	7,1,6	1009832275947370032
0,0,3	144	6,0,3	5686200	6,3,3	801503420918760	7,2,5	3836114182573117632
1,0,2	9504	6,1,2	9589752	6,4,2	9842930030808	7,3,4	1448583132341564928
1,1,1	22968	6,2,1	84240	6,5,1	3363048504	7,4,3	65054184230486808
2,0,1	1080	0,0,10	168	6,6,0	144	7,5,2	246714051981816
2,1,0	54	1,0,9	112746384	7,0,5	685227318336	7,6,1	24516763128
0,0,4	168	1,1,8	408220124400	7,1,4	18624092277168	7,7,0	180
1,0,3	55080	2,0,8	29153182176	7,2,3	9866075528304	8,0,6	170870441516784
1,1,2	801720	2,1,7	26462705388768	7,3,2	275408356176	8,1,5	9942236934310944
2,0,2	55080	2,2,6	565228067371704	7,4,1	179638056	8,2,4	12999327967495584
2,1,1	84240	3,0,7	414019483488	8,0,4	828397800	8,3,3	1333022580438624
2,2,0	144	3,1,6	137077369593336	8,1,3	3897248904	8,4,2	9842930030808
3,0,1	168	3,2,5	1091242103367168	8,2,2	212527800	8,5,1	1672396776
0,0,5	168	3,3,4	771574529680320	8,3,1	84240	9,0,5	414019483488
1,0,4	258876	4,0,6	685227318336	9,0,3	144	9,1,4	6324878723688
1,1,3	14272344	4,1,5	86145995276352	0,0,13	168	9,2,3	1709274209400
2,0,3	1045440	4,2,4	249147504104832	1,0,12	2115255492	9,3,2	19569181320
2,1,2	9589752	4,3,3	56646795125808	1,1,11	42320995599600	9,4,1	2286360
2,2,1	823968	4,4,2	950998199904	2,0,11	2966972060160	10,0,4	12531888
3,0,2	94248	5,0,5	147357745992	2,1,10	13783408869528072	10,1,3	14272344
3,1,1	84240	5,1,4	6324878723688	2,2,9	1570644007964714736	10,2,2	55080
3,2,0	54	5,2,3	5553133901424	3,0,10	211313193184296	0,0,15	144
0,0,6	144	5,3,2	275408356176	3,1,9	391767299135571456	1,0,14	12507646968
1,0,5	1045440	5,4,1	387427104	3,2,8	19261188790077746538	1,1,13	669793850973648
1,1,4	169945416	5,5,0	180	3,3,7	108334675692791766768	2,0,13	46536192247248
2,0,4	12531888	6,0,4	2868991776	4,0,9	2030806663104960	2,1,12	532443125472289380
2,1,3	422121240	6,1,3	30179989584	4,1,8	1737261009324236286	2,2,11	148032484883296635024
2,2,2	212527800	6,2,2	4691149344	4,2,7	39401939588604883920	3,0,12	8041290548966712
3,0,3	5686200	6,3,1	14832456	4,3,6	102365041178451406320	3,1,11	37113307255824419664
3,1,2	37017000	7,0,3	1045440	4,4,5	42886986818729501952	3,2,10	4635692472845342699712
3,2,1	2286360	7,1,2	801720	5,0,8	3683509791835230	3,3,9	71228168955194892241824
3,3,0	180	7,2,1	1080	5,1,7	1475899281351731232	4,0,11	19097819261948528
4,0,2	55080	0,0,11	168	5,2,6	15437538722622496320	4,1,10	428898270234051873648
4,1,1	22968	1,0,10	312318288	5,3,5	17666622173504257920	4,2,9	266355777378628399257880
0,0,7	168	1,1,9	2077856570952	5,4,4	2952225900540710424	4,3,8	208542584860846725573792
1,0,6	3781080	2,0,9	147357745992	5,5,3	65441894934804480	4,4,7	310728018286366928659728
1,1,5	1538714160	2,1,8	241891770932622	6,0,7	1358732492843328	5,0,10	936802041472321344
2,0,5	112746384	2,2,7	9578647470994416	6,1,6	243556290929859120	5,1,9	1084155094548776246256
2,1,4	10651393728	3,0,8	3764269848150	6,2,5	1092271314577190688	5,2,8	34504030323455683365540
2,2,3	18704746728	3,1,7	2356453861300944	6,3,4	485201908448389176	5,3,7	137334444454235316481488
3,0,4	159172380	3,2,6	37164898364815152	6,4,3	25578337459800960	5,4,6	100771753799790046258992
3,1,3	3897248904	3,3,5	58494821385825792	6,5,2	113389478053344	5,5,5	15002509184093317799448
3,2,2	1536760944	4,0,7	12074918985360	6,6,1	12857494104	6,0,9	1114166808793427904
3,3,1	14832456	4,1,6	3130393529188872	7,0,6	89683487215200	6,1,8	662747101613708918166
4,0,3	12531888	4,2,5	20023868970613584	7,1,5	6340790927783952	6,2,7	10648710770871871691424
4,1,2	57195792	4,3,4	12074786392584528	7,2,4	10262891970293004	6,3,6	20684312507372786347752
4,2,1	2286360	4,4,3	815165175453336	7,3,3	1333022580438624	6,4,5	6916476884469525094032
4,3,0	54	5,0,6	6028970554656	7,4,2	13029988164048	6,5,4	411546921208916198364
5,0,2	9504	5,1,5	618734398390992	7,5,1	3363048504	6,6,3	3494470176162937224
5,1,1	1080	5,2,4	1475296893039852	7,6,0	54	7,0,8	326486183204225142
0,0,8	168	5,3,3	283970290298616	8,0,5	685227318336	7,1,7	95346746262039029904
1,0,7	12531888	5,4,2	4180709760048	8,1,4	14260130266464	7,2,6	729068917506240634848
1,1,6	11407448232	5,5,1	1672396776	8,2,3	5553133901424	7,3,5	629003708472330915264
2,0,6	828397800	6,0,5	414019483488	8,3,2	105124396536	7,4,4	82046840116560882948
2,1,5	185136252912	6,1,4	14260130266464	8,4,1	35294184	7,5,3	1485487116716515272
2,2,4	688185209088	6,2,3	98660755283076	9,0,4	159172380	7,6,2	2309426676472032
3,0,5	2868991776	6,3,2	377099230176	9,1,3	422121240	7,7,1	85286277432
3,1,4	196654202136	6,4,1	387427104	9,2,2	9589752	8,0,7	20915462494951344
3,2,3	268467230952	6,5,0	54	9,3,1	168	8,1,6	2715306672487631616
3,3,2	19569181320	7,0,4	2115255492	0,0,14	168	8,2,5	8705702714908296384
4,0,4	828397800	7,1,3	15496835472	1,0,13	5221882080	8,3,4	2759145061379596596
4,1,3	15496835472	7,2,2	1536760944	1,1,12	173059142952312	8,4,3	102967117070019696
4,2,2	4691149344	7,3,1	2286360	2,0,12	12074918985360	8,5,2	318500079686208
4,3,1	35294184	8,0,3	55080	2,1,11	89309343886076376	8,6,1	24516763128
4,4,0	144	8,1,2	9504	2,2,10	16165319885559734832	8,7,0	54
5,0,3	12531888	0,0,12	144	3,0,11	1358732492843328	9,0,6	211313193170328
5,1,2	37017000	1,0,11	828397800	3,1,10	4045017572706202992	9,1,5	9942236934310944
5,2,1	823968	1,1,10	9721605877056	3,2,9	323487707741738306640	9,2,4	10262891970293004
6,0,2	168	2,0,10	685227318336	3,3,8	3085305331283577953136	9,3,3	801503420918760
0,0,9	144	2,1,9	1932024378377232	4,0,10	20915462494951344	9,4,2	4180709760048
1,0,8	38713950	2,2,8	133082696708836560	4,1,9	29595710277925790904	9,5,1	387427104
1,1,7	72542163168	3,0,9	29850028039080	4,2,8	1140647229711563099904	10,0,5	147357745992
2,0,7	5221882080	3,1,8	33021971994940200	4,3,7	5313981987997190937072	10,1,4	1560583187460
2,1,6	2455050545136	3,2,7	949255452430119360	4,4,6	4394846243085819779592	10,2,3	268467230952
2,2,5	26018243190288	3,3,6	2949179390777334672	5,0,9	64083374604252864	10,3,2	1536760944
3,0,6	38437207344	4,0,8	170870441516784	5,1,8	44892419985923962284	10,4,1	22968
3,1,5	6157144423728	4,1,7	83229565047206400	5,2,7	846447272733537148032	11,0,4	258876
3,2,4	22069473542568	4,2,6	1047154408278044472	5,3,6	1887461261292502202904	11,1,3	55080
3,3,3	5853218557032	4,3,5	1392090222866615136	5,4,5	706177757611379483424		
4,0,5	29153182176	4,4,4	259340008844756376	5,5,4	45586829807821571112		
4,1,4	1560583187460	5,0,7	170870441516784	6,0,8	44262594615526560		
4,2,3	1709274209400	5,1,6	36397140905520432	6,1,7	15022136654860343184		
4,3,2	105124396536	5,2,5	193464393164553024	6,2,6	133596413514797157576		
4,4,1	179638056	5,3,4	100247137259690592	6,3,5	132683253333213798864		

Table 3.8: The genus 0 instanton numbers of total degree ≤ 15 for the $\chi = -144$ family (3.42). The numbers not in this list are either zero, or given by those in the table after permuting the first two indices.

3.6 Higher genus mirror symmetry

The higher genus B-model prepotentials can be computed via a recursive procedure, originally due to Bershadsky, Cecotti, Ooguri, and Vafa [15, 16]. This was refined in work by Yamaguchi and Yau [80] who demonstrated that the genus- g prepotential was a polynomial in a set of propagators. This was further refined in work by Alim, Laenge, and Scheidegger [81, 82], and in [83] Huang, Katz, and Klemm phrased the recursion as a set of PDEs. This has also been reviewed by Elmi [84] and in the notes [85]. Work by Klemm, Huang, and Quackenbush [17] drove the computation to genus 51 for the Quintic, and this has recently been driven higher still by incorporating the modularity of D4-D2-D0 bound states [86, 87]. Additional higher genus results and other developments are found in [83, 88] .

The point of departure for this computation is in realising the complex structure moduli space of X as a complex, Kähler metric space, with Kähler potential

$$K(\varphi, \bar{\varphi}) = -\log(-i\Pi^\dagger \Sigma \Pi) \ . \quad (3.47)$$

The metric then is

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \ . \quad (3.48)$$

This metric can be used to raise and lower indices, and doing so twice on the complex conjugate of the holomorphic Yukawa coupling gives a quantity

$$\bar{C}_{\bar{i}}{}^{jk} = G^{j\bar{j}} G^{k\bar{k}} \overline{C_{ijk}} \ . \quad (3.49)$$

From this, we define non-holomorphic propagators S^{ij} , S^i , S as potentials:

$$\partial_i S^{ij} = \bar{C}_{\bar{i}}{}^{ij} \ , \quad \partial_i S^j = G_{i\bar{i}} S^{ij} \ , \quad \partial_i S = G_{i\bar{i}} S^i \ . \quad (3.50)$$

Note that S^{ij} is a symmetric tensor, $S^{ij} = S^{ji}$. These functions transform as tensors

under coordinate transformations of the moduli space. Under a Kähler transformation $K \mapsto K + f + \bar{f}$ for some holomorphic f , each propagator transforms as $S \mapsto e^{2f} S$, $S^i \mapsto e^{2f} S^i$, $S^{ij} \mapsto e^{2f} S^{ij}$ (one says that they have weight (2,0) under Kähler transformations). We display their covariant, with respect to coordinate and Kähler transformations, derivatives below, and also do so for $K_i = \partial_{\varphi^i} K$:

$$\begin{aligned}
D_i S^{jk} &= \partial_i S^{jk} + \Gamma_{il}^j S^{lk} + \Gamma_{il}^k S^{jl} - 2K_i S^{jk} , \\
D_i S^j &= \partial_i S^j + \Gamma_{il}^j S^l - 2K_i S^j , \\
D_i S &= \partial_i S - 2K_i S , \\
D_i K_j &= \partial_i K_j - \Gamma_{ij}^l K_l .
\end{aligned} \tag{3.51}$$

The Christoffel symbols Γ_{jk}^i give the Levi-Civita connection for the metric G . We also have provided the covariant derivative $D_i K_j$ to avoid any confusion in what follows.

For $g \geq 2$, the genus- g B-model free energy is a polynomial of degree $3g - 3$ in the functions S^{ij} (degree 1), S^i (degree 2), S (degree 3), and K_i (degree 1). The coefficients of this polynomial are rational functions of the moduli φ , to be determined soon. The Kähler potential K can be computed from the periods as in (3.47), and we now explain how to obtain the functions S^{ij} , S^i , S from the periods. The key insight is the following special geometry relation, which originates in the tt^* equations as explained in [85] (which we avoid discussing):

$$\begin{aligned}
R_{i\bar{i}j}^{l} &= [\partial_{\bar{i}}, D_i]^l_{j} \\
&= \partial_{\bar{i}} \Gamma_{ij}^l = \delta_i^l G_{j\bar{i}} + \delta_j^l G_{i\bar{i}} - C_{ijk} \bar{C}_{\bar{i}}^{kl} .
\end{aligned} \tag{3.52}$$

This equation is a total antiholomorphic derivative, with the left and right hand side explicitly given as $\partial_{\bar{i}}$ of quantities that we can either already compute, or of $C_{ijk} S^{kl}$. So we integrate this equation, to get the *integrated special geometry relation* below that we use to fix the first propagator S^{ij} :

$$C_{ijk} S^{kl} = -\Gamma_{ij}^l + \delta_i^l K_j + \delta_j^l K_i + s_{ij}^l . \tag{3.53}$$

The quantity s_{ij}^l is the holomorphic ‘constant of integration’ which lives in the kernel of $\partial_{\bar{i}}$. It is the first of several ‘propagator ambiguities’ that we will meet. There is some freedom in the choice of s_{ij}^l . Note that to solve the above equation for S^{kl} we must clear the C_{ijk} from the left hand side. We shall make comments on how to go about this later, and for now press on with reviewing the BCOV procedure.

We now seek to express covariant derivatives of the propagators in terms of the propagators themselves. This involves some algebra. Consider for example $D_i S^{jk}$. One hits this with $\partial_{\bar{i}}$, and then expands the expression to get $\partial_{\bar{i}} (\partial_i S^{jk} + \Gamma_{i\bar{l}}^j S^{lk} + \Gamma_{i\bar{l}}^k S^{jl} - 2K_i S^{jk})$. We expand this by the Leibniz rule, with antiholomorphic derivatives of the Christoffel symbols replaced by the special geometry relation (3.52). After some manipulation, each term on the right hand side can be written as $\partial_{\bar{i}}$ of something, and this is integrated to get the following:

$$\begin{aligned}
D_i S^{jk} &= \delta_i^j S^k + \delta_i^k S^j - C_{imn} S^{mj} S^{nk} + h_i^{jk} , \\
D_i S^j &= 2\delta_i^j S - C_{imn} S^m S^{nj} + h_i^{jk} K_k + h_i^j , \\
D_i S &= -\frac{1}{2} C_{imn} S^m S^n + \frac{1}{2} h_i^{mn} K_m K_n + h_i^j K_j + h_i , \\
D_i K_j &= -K_i K_j - C_{ijk} S^k + C_{ijk} S^{kl} K_l + h_{ij} .
\end{aligned} \tag{3.54}$$

h_i^{jk} , h_i^j , h_i , h_{ij} are all further propagator ambiguities, rational functions of φ over which there is a degree of choice. It is the first and second of the above equations that are used in practice to compute the propagators S^i and S .

The BCOV recursion relation takes its simplest form after a change of variables. One writes the tilded propagators

$$\tilde{S}^{ij} = S^{ij} , \quad \tilde{S}^i = S^i - S^{ij} K_j , \quad \tilde{S} = S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j , \quad \tilde{K}_i = K_i , \tag{3.55}$$

which, we parenthetically remark, obey

$$\partial_i \tilde{S}^{ij} = \bar{C}_i^{ij} , \quad \partial_i \tilde{S}^i = -\bar{C}_i^{ij} K_j , \quad \partial_i \tilde{S} = \frac{1}{2} \bar{C}_i^{ij} K_i K_j . \quad (3.56)$$

This change of variables allows for a significant simplification: higher genus prepotentials $\mathcal{F}^{(g \geq 2)}$ are degree $3g - 3$ polynomials in $\tilde{S}^{ij}$, $\tilde{S}^i$, and $\tilde{S}$ but *do not explicitly depend on $\tilde{K}$* .

For $g \geq 2$, the BCOV recursion relation takes the form [83]

$$\begin{aligned} \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}^{ij}} &= \frac{1}{2} \partial_i (\partial'_j \mathcal{F}^{(g-1)}) + \frac{1}{2} (C_{ijl} S^{lk} - s_{ij}^k) \partial'_k \mathcal{F}^{(g-1)} + \frac{1}{2} (C_{ijk} S^k - h_{ij}) c_{g-1} \\ &\quad + \frac{1}{2} \sum_{h=1}^{g-1} (\partial'_i \mathcal{F}^{(h)}) (\partial'_j \mathcal{F}^{(g-h)}) , \\ \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}^i} &= (2g - 3) \partial'_i \mathcal{F}^{(g-1)} + \sum_{h=1}^{g-1} c_h \partial'_i \mathcal{F}^{(g-h)} , \\ \frac{\partial \mathcal{F}^{(g)}}{\partial \tilde{S}} &= (2g - 3) c_{g-1} + \sum_{h=1}^{g-1} c_h c_{g-h} . \end{aligned} \quad (3.57)$$

In the above equations we have used

$$\begin{aligned} c_g &= \begin{cases} \frac{\chi(Y)}{24} - 1 , & g = 1 , \\ (2g - 2) \mathcal{F}^{(g)} , & g > 1 , \end{cases} \\ \partial'_i \mathcal{F}^{(g)} &= \begin{cases} \frac{1}{2} C_{ijk} S^{jk} + \partial_i f^{(1)} , & g = 1 , \\ \partial_i \mathcal{F}^{(g)} , & g > 1 . \end{cases} \end{aligned} \quad (3.58)$$

The above symbol $f^{(1)}$ denotes the genus-1 holomorphic ambiguity, but is *not* the function f of (3.18) that appeared inside the logarithm of the genus 1 prepotential. Instead,

$$f^{(1)} = -\log \left(\Delta^c \prod_{i=1}^{h^{2,1}(X)} (\varphi^i)^{(1-Y_{00i})/2} \overline{\Delta^c \prod_{i=1}^{h^{2,1}(X)} (\varphi^i)^{(1-Y_{00i})/2}} \right) , \quad (3.59)$$

with $c = 1/12$ in the CICY examples that concern us. We stress that $f^{(1)}$ is a function of φ and $\bar{\varphi}$. We labour this point so as to avoid confusion when we take the topological limit.

The recursive procedure involves working out each $\mathcal{F}^{(g)}$ by first taking as ansatz the most general polynomial in the tilded propagators of degree $3g-3$, and then comparing the partial derivatives (with respect to the propagators) thereof to the right hand side of (3.57) (which only involves lower-genus prepotentials, guaranteeing a recursion). That right hand side can be written as a polynomial in the tilded propagators by replacing every derivative of a propagator produced by (3.58) through the BCOV closure relations (3.54). For simplicity, we rewrite those closure relations here for the tilded propagators, and expanding the covariant derivatives:

$$\begin{aligned}
\partial_i \tilde{S}^{jk} &= C_{imn} \tilde{S}^{mj} \tilde{S}^{nk} + \delta_i^j \tilde{S}^k + \delta_i^k \tilde{S}^j - s_{im}^j \tilde{S}^{mk} - s_{im}^k \tilde{S}^{mj} + h_i^{jk} , \\
\partial_i \tilde{S}^j &= C_{imn} \tilde{S}^{mj} \tilde{S}^n + 2\delta_i^j \tilde{S} - s_{im}^j \tilde{S}^m - h_{ik} \tilde{S}^{kj} + h_i^j , \\
\partial_i \tilde{S} &= \frac{1}{2} C_{imn} \tilde{S}^m \tilde{S}^n - h_{ij} \tilde{S}^j + h_i , \\
\partial_i K_j &= K_i K_j - C_{ijn} \tilde{S}^{mn} K_m + s_{ij}^m K_m - C_{ijk} \tilde{S}^k + h_{ij} .
\end{aligned} \tag{3.60}$$

This procedure fixes the polynomial form of $\mathcal{F}^{(g)}(\tilde{S}^{ij}, \tilde{S}^i, \tilde{S})$ up to the ‘constant’ term $f^{(g)}$, which is a rational function of φ not fixed by (3.57). That this function is holomorphic is buried in the derivation of (3.57), which is a rearrangement of antiholomorphic derivatives that annihilate this *holomorphic ambiguity* $f^{(g)}$.

Determination of this ambiguity $f^{(g)}$ is the major remaining conceptual problem in topological string theory. It is a rational function, with poles at the zeroes of Δ and any singularities of the propagators. That the propagators can have singularities outside of $\Delta = 0$ is an artefact of how the equations (3.53) and (3.54) are solved for the propagators. $\mathcal{F}^{(g)}$ should be nonsingular away from $\Delta = 0$, so $f^{(g)}$ must have a residue at these spurious poles so that $\mathcal{F}^{(g)}$ is regular.

The singular behaviour of $\mathcal{F}^{(g)}$ at the conifold locus $\Delta = 0$ is fixed by the conifold gap

condition [17], which shows that as the conifold locus is approached $\mathcal{F}^{(g)}$ should go like

$$\mathcal{F}^{(g)} \sim \frac{1}{\Delta^{2g-2}} + \text{regular} \quad (3.61)$$

where “regular” denotes terms that are nonsingular at $\Delta = 0$. The degree of $f^{(g)}$ ’s numerator should not be too high, so that $\mathcal{F}^{(g)}$ is regular at ∞ .

In this thesis we shall only make it as high as genus 2. To this end we display an expression for the genus 2 prepotential that we will make use of, first derived in [15]. The authors thereof were able to compute higher-still prepotentials but these do not fit on a page. Note that we now revert to using untilded propagators.

$$\begin{aligned} \mathcal{F}^{(2)} = & \frac{1}{2}C_{ij}^{(1)} + \frac{1}{2}C_i^{(1)}S^{ij}C_j^{(1)} - \frac{1}{8}S^{jk}S^{mn}C_{jkmn} - \frac{1}{2}S^{ij}C_{ijm}S^{mn}C_n^{(1)} + \frac{\chi(Y)}{24}S^iC_i^{(1)} \\ & + \frac{1}{8}S^{ij}C_{ijp}S^{pq}C_{qmn}S^{mn} + \frac{1}{12}S^{ij}S^{pq}S^{mn}C_{ipm}C_{jqn} - \frac{\chi(Y)}{48}S^iC_{ijk}S^{jk} \\ & + \frac{\chi(Y)}{24} \left(\frac{\chi(Y)}{24} - 1 \right) S + f^{(2)}(\varphi) . \end{aligned} \quad (3.62)$$

We now fix the above missing pieces of notation:

$$\begin{aligned} C_i^{(1)} &= D_i\mathcal{F}^{(1)} = \partial_i\mathcal{F}^{(1)} , \quad C_{ij}^{(1)} = D_iD_j\mathcal{F}^{(1)} = \partial_i\partial_j\mathcal{F}^{(1)} - \Gamma_{ij}^l\partial_l\mathcal{F}^{(1)} , \\ C_{jklm} &= D_jC_{klm} = \partial_jC_{klm} - \Gamma_{jk}^nC_{nlm} - \Gamma_{jl}^nC_{knm} - \Gamma_{jm}^nC_{kln} + 2K_jC_{klm} . \end{aligned} \quad (3.63)$$

Note that C_{jklm} is symmetric in its four indices.

In order to obtain curve counts, a change of variables to A-model quantities must be made.

The genus- g A model free energy dependence on higher genus instanton numbers $n_{\mathbf{b}}^{(g)}$ is given

by the Gopakumar-Vafa formula [18, 19]:

$$\begin{aligned}
F^{\text{All Genus}}(\lambda, \mathbf{t}) &= \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) \\
&= \frac{c(\mathbf{t})}{\lambda} + l(\mathbf{t}) + \sum_{g=0}^{\infty} \sum_{\beta \in H^2(Y, IZ)} n_{\beta}^{(g)} \sum_{k=1}^{\infty} \frac{1}{k} \left(2 \sin \frac{k\lambda}{2} \right)^{2g-2} \exp[-2\pi k \beta \cdot \mathbf{t}] .
\end{aligned} \tag{3.64}$$

λ is the string coupling. $c(\mathbf{t})$ and $l(\mathbf{t})$ are respectively cubic and linear polynomials in t . This formula fixes $F_g(\mathbf{t})$ at each genus up to a constant term, which we neglect to discuss. This was studied at different genera in the series of papers [15, 16, 89, 90] and nicely presented in the thesis [84]. We give the genus 2 expansion below ahead of making use of it in the next section.

$$F_2(\mathbf{t}) = \frac{\chi(Y)}{5760} + \sum_{\mathbf{b}} \left(\frac{n_{\mathbf{b}}^{(0)}}{240} + n_{\mathbf{b}}^{(2)} \right) \text{Li}_{-1}(\mathbf{q}^{\mathbf{b}}) . \tag{3.65}$$

One can note that genus 1 numbers $n_{\mathbf{b}}^{(1)}$ do not appear in this formula. Their absence was noted in [15], where it was explained as the lack of “toroidal bubbling” at genus 2.

Quantities in the holomorphic limit

Up until now the propagators have had a dependence on φ and $\bar{\varphi}$. To compute enumerative invariants, it suffices to work in the topological limit $\bar{t} \mapsto -i\infty$, or $\bar{\varphi} \mapsto \infty$. The Kähler potential can be massaged,

$$K(\varphi, \bar{\varphi}) = -\log(-i\Pi^\dagger \Sigma \Pi) = \log(k(\bar{\varphi})) - \log(\varpi_0(\varphi) + p(\varphi, \bar{\varphi})) , \tag{3.66}$$

where the function $p(\varphi, \bar{\varphi})$ vanishes when $\bar{\varphi} \mapsto \infty$. Since it is derivatives of the Kähler potential that play a role in computing the propagators, we can remove the piece $\log(k(\bar{\varphi}))$ and retain in this limit

$$K(\varphi) = -\log(\varpi_0(\varphi)) . \tag{3.67}$$

Similarly, the Christoffel symbols become in this limit

$$\Gamma_{jk}^i = \frac{\partial \varphi^i}{\partial t^l} \frac{\partial^2 t^l}{\partial a^j \partial a^k} . \quad (3.68)$$

3.7 A worked example at genus 2

We will set about computing genus 2 numbers on the maximally split quintic (3.15) . These were already computed to degree 18 in [60], where the holomorphic ambiguity was fixed with an involved implementation of the conifold gap condition. This involves expanding the periods about the conifold locus and imposing the gap condition on the prepotential, expressed in terms of these conifold-adapted periods.

We instead make the following claim: The genus 2 ambiguity, in this example, can be fixed from two sources of information. These are

- The known vanishing instanton numbers .
- An infinite Coxeter symmetry .

This example will involve several appearances of the coordinates φ^i raised to powers. In the name of sanity, we will temporarily write these coordinates with lower indices so that superscripts can be read as exponents. For instance, φ_2^4 is φ^2 to the power of four. The only exception to this rule will be when the superscript is an abstract index, as in φ^i , so that tensor contractions behave as they should.

The three-point functions, or Yukawa couplings, are

$$\begin{aligned}
C_{111} &= \frac{5(1 + 37\varphi_1 + 73\varphi_1^2 + 14\varphi_1^3 - 3\varphi_2 + 176\varphi_1\varphi_2 - 73\varphi_1^2\varphi_2 + 3\varphi_2^2 + 37\varphi_1\varphi_2^2 - \varphi_2^3)}{\varphi_1^3\Delta}, \\
C_{112} &= \frac{5(2 + 14\varphi_1 - 9\varphi_1^2 - 7\varphi_1^3 - \varphi_2 - 88\varphi_1\varphi_2 + 64\varphi_1^2\varphi_2 - 4\varphi_2^2 - 51\varphi_1\varphi_2^2 + 3\varphi_2^3)}{\varphi_1^2\varphi_2\Delta}, \\
C_{122} &= \frac{5(2 - \varphi_1 - 4\varphi_1^2 + 3\varphi_1^3 + 14\varphi_2 - 88\varphi_1\varphi_2 - 51\varphi_1^2\varphi_2 - 9\varphi_2^2 + 64\varphi_1\varphi_2^2 - 7\varphi_2^3)}{\varphi_1\varphi_2^2\Delta}, \\
C_{222} &= \frac{5(1 - 3\varphi_1 + 3\varphi_1^2 - \varphi_1^3 + 37\varphi_2 + 176\varphi_1\varphi_2 + 37\varphi_1^2\varphi_2 + 73\varphi_2^2 - 73\varphi_1\varphi_2^2 + 14\varphi_2^3)}{\varphi_2^3\Delta}.
\end{aligned} \tag{3.69}$$

The discriminant Δ is

$$\begin{aligned}
\Delta &= 1 - 5\varphi_1 + 10\varphi_1^2 - 10\varphi_1^3 + 5\varphi_1^4 - \varphi_1^5 - 5\varphi_2 - 605\varphi_1\varphi_2 - 1905\varphi_1^2\varphi_2 - 605\varphi_1^3\varphi_2 \\
&\quad - 5\varphi_1^4\varphi_2 + 10\varphi_2^2 - 1905\varphi_1\varphi_2^2 + 1905\varphi_1^2\varphi_2^2 - 10\varphi_1^3\varphi_2^2 - 10\varphi_2^3 - 605\varphi_1\varphi_2^3 \\
&\quad - 10\varphi_1^2\varphi_2^3 + 5\varphi_2^4 - 5\varphi_1\varphi_2^4 - \varphi_2^5.
\end{aligned} \tag{3.70}$$

The undisplayed C_{ijk} can be obtained from those given by symmetry, e.g. $C_{211} = C_{112}$.

A digression on matrix inversion

Note that there is a $\mathbb{Z}_2$ symmetry to the example, so that for instance C_{222} can be obtained from C_{111} by effecting a swap $\varphi_1 \leftrightarrow \varphi_2$. This symmetry means that in our final result, the instanton numbers should possess a symmetry $n_{(i,j)}^{(g)} = n_{(j,i)}^{(g)}$. The A and B model prepotentials should both be symmetric functions.

This means that, if we set up the problem correctly, we can have the ambiguity $f^{(2)}$ be a symmetric function of φ_1, φ_2 . Let us explain this point. We begin with the determination of S^{ij} from (3.53). In the one-parameter setting, one is free to simply divide both sides of this equation by the quantity C_{111} . In multiparameter examples we must invert C_{ijk} . The traditional way to do this is to pick a direction in moduli space, say 1, and then write out

the $i = 1$ component of (3.53):

$$C_{1jk}S^{kl} = -\Gamma_{1j}^l + \delta_1^l K_j + \delta_j^l K_1 + s_{1j}^l . \quad (3.71)$$

Now although covariance has been thrown out the window, C_{1jk} is a legitimate matrix that generically has full rank, so that we can find an inverse of it, so that we obtain

$$S^{kl} = -(C_1^{-1})^{kj} \Gamma_{1j}^l + \delta_1^l (C_1^{-1})^{kj} K_j + (C_1^{-1})^{kl} K_1 + (C_1^{-1})^{kj} s_{1j}^l . \quad (3.72)$$

Naively we might say that $s_{1j}^l = 0$ is the simplest choice, so that we read off S^{kl} directly from the above. This would be wrong, because the S^{kl} so defined does not equal S^{lk} . The way to proceed is to write

$$(C_1^{-1})^{kj} s_{1j}^l = A^{kl} + E^{kl} , \quad (3.73)$$

where A^{kl} and E^{kl} are respectively antisymmetric and symmetric in their indices. Our freedom is to redefine the symmetric part E^{kl} , and we take this to vanish. Then we get

$$S^{kl} = \frac{1}{2} \left[\left(-(C_1^{-1})^{kj} \Gamma_{1j}^l + \delta_1^l (C_1^{-1})^{kj} K_j + (C_1^{-1})^{kl} K_1 \right) + (k \leftrightarrow l) \right] . \quad (3.74)$$

This choice would be correct, and one could proceed from here to compute all of the other propagators and propagator ambiguities using the $i = 1$ components of (3.54) . For completeness, we give the components of the symmetric matrix $(C_1^{-1})^{ij}$:

$$\begin{aligned} (C_1^{-1})^{11} &= \frac{-\varphi_1^3 (2 - \varphi_1 - 4\varphi_1^2 + 3\varphi_1^3 + 14\varphi_2 - 88\varphi_1\varphi_2 - 51\varphi_1^2\varphi_2 - 9\varphi_2^2 + 64\varphi_1\varphi_2^2 - 7\varphi_2^3)}{5(2 - 7\varphi_1 - 2\varphi_2)} , \\ (C_1^{-1})^{12} &= \frac{\varphi_1^2\varphi_2 (2 + 14\varphi_1 - 9\varphi_1^2 - 7\varphi_1^3 - \varphi_2 - 88\varphi_1\varphi_2 + 64\varphi_1^2\varphi_2 - 4\varphi_2^2 - 51\varphi_1\varphi_2^2 + 3\varphi_2^3)}{5(2 - 7\varphi_1 - 2\varphi_2)} , \\ (C_1^{-1})^{22} &= \frac{-\varphi_1\varphi_2^2 (1 + 37\varphi_1 + 73\varphi_1^2 + 14\varphi_1^3 - 3\varphi_2 + 176\varphi_1\varphi_2 - 73\varphi_1^2\varphi_2 + 3\varphi_2^2 + 37\varphi_1\varphi_2^2 - \varphi_2^3)}{5(2 - 7\varphi_1 - 2\varphi_2)} . \end{aligned} \quad (3.75)$$

A problem with this approach is that this breaks the $(1 \leftrightarrow 2)$ symmetry. For instance, S^{11}

does not give S^{22} upon swapping $(\varphi_1 \leftrightarrow \varphi_2)$. At the end of the process, $f^{(2)}$ will not be a symmetric function.

This is not such a problem for computations with two-parameter manifolds. However, we are motivated by the power of symmetry evidenced in the genera 0 and 1 computations for manifolds with ≥ 3 parameters to seek another way of determining the propagators, so that we maintain any symmetries that we begin with. The reader who does not care for maintaining this symmetry can use the previous expressions of this subsection for S^{ij} .

Resuming the computation

In what follows, we forget the last five equations and compute S^{ij} anew. To this end, we contract the integrated special geometry relation (3.53) with φ^i to obtain

$$\varphi^i C_{ijk} S^{kl} = -\varphi^i \Gamma_{ij}^l + \varphi^l K_j + \delta_j^l \varphi^i K_i + \varphi^i s_{ij}^l . \quad (3.76)$$

The matrix $\mathfrak{C}_{jk} = \varphi^i C_{ijk}$ respects the $(1 \leftrightarrow 2)$ symmetry, as does its inverse. We introduce more notation,

$$\mathfrak{C}_{jk} = \varphi^i C_{ijk} , \quad \Upsilon_j^l = \varphi^i \Gamma_{ij}^l , \quad \mathfrak{K} = \varphi^i K_i , \quad \mathfrak{s}_j^l = \varphi^i s_{ij}^l . \quad (3.77)$$

We shall also use another abuse of notation, whereby we denote the inverse of $\mathfrak{C}_{ij}$ with raised components:

$$\mathfrak{C}^{ij} \mathfrak{C}_{jk} = \delta_k^i . \quad (3.78)$$

The components of $\mathfrak{C}_{ij}$ are

$$\begin{aligned}\mathfrak{C}_{11} &= \frac{5(3 + 51\varphi_1 + 64\varphi_1^2 + 7\varphi_1^3 - 4\varphi_2 + 88\varphi_1\varphi_2 - 9\varphi_1^2\varphi_2 - \varphi_2^2 - 14\varphi_1\varphi_2^2 + 2\varphi_2^3)}{\varphi_1^2\Delta}, \\ \mathfrak{C}_{12} &= \frac{5(4 + 13\varphi_1 - 13\varphi_1^2 - 4\varphi_1^3 + 13\varphi_2 - 176\varphi_1\varphi_2 + 13\varphi_1^2\varphi_2 - 13\varphi_2^2 + 13\varphi_1\varphi_2^2 - 4\varphi_2^3)}{\varphi_1\varphi_2\Delta}, \\ \mathfrak{C}_{22} &= \frac{5(3 - 4\varphi_1 - \varphi_1^2 + 2\varphi_1^3 + 51\varphi_2 + 88\varphi_1\varphi_2 - 14\varphi_1^2\varphi_2 + 64\varphi_2^2 - 9\varphi_1\varphi_2^2 + 7\varphi_2^3)}{\varphi_2^2\Delta}.\end{aligned}\tag{3.79}$$

The inverse has components

$$\begin{aligned}\mathfrak{C}^{11} &= \frac{-\varphi_1^2(3 - 4\varphi_1 - \varphi_1^2 + 2\varphi_1^3 + 51\varphi_2 + 88\varphi_1\varphi_2 - 14\varphi_1^2\varphi_2 + 64\varphi_2^2 - 9\varphi_1\varphi_2^2 + 7\varphi_2^3)}{5(7 - 2\varphi_1 - 2\varphi_2)}, \\ \mathfrak{C}^{12} &= \frac{\varphi_1\varphi_2(4 + 13\varphi_1 - 13\varphi_1^2 - 4\varphi_1^3 + 13\varphi_2 - 176\varphi_1\varphi_2 + 13\varphi_1^2\varphi_2 - 13\varphi_2^2 + 13\varphi_1\varphi_2^2 - 4\varphi_2^3)}{5(7 - 2\varphi_1 - 2\varphi_2)}, \\ \mathfrak{C}^{22} &= \frac{-\varphi_2^2(3 + 51\varphi_1 + 64\varphi_1^2 + 7\varphi_1^3 - 4\varphi_2 + 88\varphi_1\varphi_2 - 9\varphi_1^2\varphi_2 - \varphi_2^2 - 14\varphi_1\varphi_2^2 + 2\varphi_2^3)}{5(7 - 2\varphi_1 - 2\varphi_2)}.\end{aligned}\tag{3.80}$$

The denominator appearing above will recur, and so we give it a name:

$$Z = 7 - 2\varphi^1 - 2\varphi^2.\tag{3.81}$$

Analogously to what we did with (3.73), we have a freedom to change the symmetric part of $\mathfrak{C}^{kj}\mathfrak{s}_j^l$, and we set this to zero. Then from (3.76) we get

$$S^{kl} = \frac{1}{2} [(-\mathfrak{C}^{kj}\Upsilon_j^l + \varphi^l\mathfrak{C}^{kj}K_j + \mathfrak{K}\mathfrak{C}^{kl}) + (k \leftrightarrow l)].\tag{3.82}$$

The first few terms in S^{ij} 's Taylor expansions are

$$\begin{aligned}S^{11} &= -\frac{3\varphi_1^2}{35} + \frac{64\varphi_1^3}{245} - \frac{102}{49}\varphi_1^2\varphi_2 + \frac{79\varphi_1^4}{1715} - \frac{7654\varphi_1^3\varphi_2}{1715} - \frac{7488\varphi_1^2\varphi_2^2}{1715} + \dots, \\ S^{12} &= \frac{4\varphi_1\varphi_2}{35} + \frac{24}{49}\varphi_1^2\varphi_2 + \frac{24}{49}\varphi_1\varphi_2^2 - \frac{1426\varphi_1^3\varphi_2}{1715} - \frac{11182\varphi_1^2\varphi_2^2}{1715} - \frac{1426\varphi_1\varphi_2^3}{1715} + \dots, \\ S^{22} &= -\frac{3\varphi_2^2}{35} - \frac{102}{49}\varphi_1\varphi_2^2 + \frac{64\varphi_2^3}{245} - \frac{7488\varphi_1^2\varphi_2^2}{1715} - \frac{7654\varphi_1\varphi_2^3}{1715} + \frac{79\varphi_2^4}{1715} + \dots.\end{aligned}\tag{3.83}$$

Now we turn to computing S^i , which in the interests of symmetry we do by considering $\varphi^i D_i S^{jk}$ as in (3.54). There is another new notation,

$$\mathfrak{h}^{jk} = \varphi^i h_i^{jk} . \quad (3.84)$$

We have the freedom to set $\mathfrak{h}^{1,1} = \mathfrak{h}^{2,2} = 0$, and then we can find S^1 and S^2 from the $(j, k) = (1, 1)$ and $(j, k) = (2, 2)$ components of $\varphi^i D_i S^{jk}$ as below:

$$\varphi^i D_i S^{jk} = \varphi^j S^k + \varphi^k S^j - \mathfrak{C}_{mn} S^{mj} S^{nk} + \mathfrak{h}^{jk} . \quad (3.85)$$

The first few terms in the expansions of the S^i so obtained are

$$\begin{aligned} S^1 &= -\frac{3\varphi_1}{70} + \frac{64\varphi_1^2}{245} - \frac{929\varphi_1\varphi_2}{490} + \frac{1389\varphi_1^3}{1715} - \frac{13539\varphi_1^2\varphi_2}{1715} - \frac{7641\varphi_1\varphi_2^2}{1715} + \dots , \\ S^2 &= -\frac{3\varphi_2}{70} - \frac{929\varphi_1\varphi_2}{490} + \frac{64\varphi_2^2}{245} - \frac{7641\varphi_1^2\varphi_2}{1715} - \frac{13539\varphi_1\varphi_2^2}{1715} + \frac{1389\varphi_2^3}{1715} + \dots . \end{aligned} \quad (3.86)$$

We will use the $(j, k) = 1, 2$ component of (3.85) in order to compute $\mathfrak{h}^{12}$, over which we have no freedom having fixed $\mathfrak{h}^{11} = \mathfrak{h}^{22} = 0$. We will need all components of $\mathfrak{h}^{ij}$ when we

consider $\varphi^i D_i S^j$, which we will use to compute the remaining propagator S . We find

$$\begin{aligned} \mathfrak{h}^{12} = \frac{\varphi_1 \varphi_2}{10Z^2 \Delta} \bigg(& 98 + 679\varphi_1 - 5669\varphi_1^2 - 6701\varphi_1^3 - 29717\varphi_1^4 - 29363\varphi_1^5 - 4824\varphi_1^6 - 2656\varphi_1^7 \\ & + 21\varphi_1^8 + 7\varphi_1^9 + 679\varphi_2 - 72532\varphi_1\varphi_2 - 848410\varphi_1^2\varphi_2 - 2858677\varphi_1^3\varphi_2 \\ & - 2218750\varphi_1^4\varphi_2 - 307684\varphi_1^5\varphi_2 + 97632\varphi_1^6\varphi_2 + 4550\varphi_1^7\varphi_2 + 67\varphi_1^8\varphi_2 \\ & - 5669\varphi_2^2 - 848410\varphi_1\varphi_2^2 - 1272670\varphi_1^2\varphi_2^2 + 8757690\varphi_1^3\varphi_2^2 + 3502930\varphi_1^4\varphi_2^2 \\ & - 588156\varphi_1^5\varphi_2^2 + 880\varphi_1^6\varphi_2^2 + 280\varphi_1^7\varphi_2^2 - 6701\varphi_2^3 - 2858677\varphi_1\varphi_2^3 \\ & + 8757690\varphi_1^2\varphi_2^3 - 13388420\varphi_1^3\varphi_2^3 + 1228780\varphi_1^4\varphi_2^3 - 37094\varphi_1^5\varphi_2^3 + 672\varphi_1^6\varphi_2^3 \\ & - 29717\varphi_2^4 - 2218750\varphi_1\varphi_2^4 + 3502930\varphi_1^2\varphi_2^4 + 1228780\varphi_1^3\varphi_2^4 - 66890\varphi_1^4\varphi_2^4 \\ & + 1022\varphi_1^5\varphi_2^4 - 29363\varphi_2^5 - 307684\varphi_1\varphi_2^5 - 588156\varphi_1^2\varphi_2^5 - 37094\varphi_1^3\varphi_2^5 \\ & + 1022\varphi_1^4\varphi_2^5 - 4824\varphi_2^6 + 97632\varphi_1\varphi_2^6 + 880\varphi_1^2\varphi_2^6 + 672\varphi_1^3\varphi_2^6 - 2656\varphi_2^7 \\ & + 4550\varphi_1\varphi_2^7 + 280\varphi_1^2\varphi_2^7 + 21\varphi_2^8 + 67\varphi_1\varphi_2^8 + 7\varphi_2^9 \bigg) . \end{aligned} \quad (3.87)$$

We continue to contract propagator ambiguities with φ^i , and write

$$\mathfrak{h}^j = \varphi^i h_i^j \quad (3.88)$$

so that $\varphi^i D_i S^j$ reads

$$\varphi^i D_i S^j = 2\varphi^j S - \mathfrak{C}_{mn} S^m S^{nj} + \mathfrak{h}^{jk} K_k + \mathfrak{h}^j . \quad (3.89)$$

We now make another choice, $\frac{1}{\varphi_1} \mathfrak{h}^1 + \frac{1}{\varphi_2} \mathfrak{h}^2 = 0$. Then we can get a symmetric S by taking

$$S = \frac{1}{2} \left[\frac{1}{2\varphi_1} (\varphi^i D_i S^1 + \mathfrak{C}_{mn} S^m S^{nj} - \mathfrak{h}^{1k} K_k) + (\varphi_1 \leftrightarrow \varphi_2) \right] . \quad (3.90)$$

The first few terms of S 's Taylor expansion are

$$S = -\frac{3}{140} - \frac{188\varphi_1}{245} - \frac{188\varphi_2}{245} - \frac{5659\varphi_1^2}{6860} - \frac{34163\varphi_1\varphi_2}{3430} - \frac{5659\varphi_2^2}{6860} + \dots \quad (3.91)$$

Having computed each of the propagators, we can work back through the relations (3.54) to obtain all of the propagator ambiguities, which would be necessary to go beyond genus 2. We decline to show those functions here, but remark that all of their denominators are products of powers of Z and Δ . The most complicated propagator ambiguity is h_i , which has as denominator $40\varphi_i Z^4 \Delta^3$, and a numerator of degree 21.

We now plug our propagators into (3.62). It remains to fix the holomorphic ambiguity, which is of the form

$$f^{(2)} = \frac{P_{15}}{Z^3 \Delta^2}, \quad (3.92)$$

where P_{15} is a degree-15 symmetric polynomial in φ_1, φ_2 .

We consider the combined set of unknowns, coefficients of P_{15} and the genus 2 instanton numbers $n_{(i,j)}^{(2)}$. We have 73 unknown coefficients of P_{15} . Increasing the order to which we work at provides more unknowns $n_{(i,j)}^{(2)}$, and as many equations for these. Some of these new unknowns we can fix in terms of other unknowns by Coxeter symmetry, until with a high enough order we exhaust the independent constraints that Coxeter symmetry offers.

Demanding that the q -expansion of $\mathcal{F}^{(2)}$ possesses infinite dihedral Coxeter symmetry reduces the number of unknowns, but unfortunately these constraints are not all independent and we cannot get by solely by working to a higher order and imposing Coxeter symmetry. Fortunately, for this example, we know ahead of time which instanton numbers should vanish from the tables of Hosono and Takagi [60]. These are the numbers $n_{(i,j)}^{(2)}$ with i or j less than 4, and also $n_{(4,4)}^{(2)}$. If we include these zeroes as known values, then we get an integer set of solutions for the instanton numbers that agrees with Hosono and Takagi's tables where they overlap, as well as fixing the ambiguity $f^{(2)}$. Curiously, we need *all* of these zeroes. If we leave out $n_{(4,4)}^{(2)} = 0$ then we have a solution with one degree of freedom, eliminated by imposing $n_{(4,4)}^{(2)} = 0$.

We have taken a barbaric way to know these zeroes, from Hosono and Takagi's computation. Since they used the conifold gap condition, we have not escaped its necessity. Future investigation will determine whether there is a means of fixing all zeroes from another principle,

like a Castelnuovo bound (vital for the computation of [17], with a possible generalisation discussed in [87]), and then we could hope to check how high of a genus Coxeter symmetry will take us. The genus 2 numbers that we compute are tabulated in table 3.9 .

In [60], Hosono and Takagi display numbers $n_{(i,j)}^{(g)}$ for $0 \leq g \leq 2$, $0 \leq i \leq 12$, $0 \leq j \leq 6$. We have in this thesis displayed numbers for this model at the same genera, but for degrees $i + j \leq 37$, which includes numbers not displayed before (even when the Coxeter action is taken into account). It would be interesting to return to this problem at genus 3.

The contents of the following table is available in electronic form [1].

p	$n_p^{(2)}$	p	$n_p^{(2)}$
5,4	2500	12,8	307911899250593057650
5,5	2238300	13,7	1083681563433259200
6,4	36800	14,6	91091698085900
6,5	101188225	15,5	2238300
7,4	132150	11,10	389901597341682586245250
6,6	10885677450	12,9	59381136406580839191450
7,5	1276419800	13,8	1088683595004165702950
8,4	191850	14,7	1328923645662952325
7,6	321011805475	15,6	25512018106050
8,5	6764994000	16,5	2500
9,4	132150	11,11	13181153378953434491429200
7,7	19785981206800	12,10	5475299266824539894041150
8,6	4007017841650	13,9	356988331876350933661250
9,5	17585700200	14,8	2645148270492863092650
10,4	36800	15,7	1083681563433259200
8,7	497495418446900	16,6	4007017841650
9,6	25512018106050	12,11	282712962376014816940856175
10,5	24017901850	13,10	52176320562765050835877250
11,4	2500	14,9	1500298428800313660065325
8,8	23487690756165150	15,8	4481489156958762453000
9,7	6354579702758450	16,7	585085137096464550
10,6	91091698085900	17,6	321011805475
11,5	17585700200	12,12	9047714570978243256779338500
9,8	546653742258204975	13,11	4082629478671789440045910650
10,7	46325408551373425	14,10	349654611337758233664220900
11,6	192086807308450	15,9	4499834280177683754617100
12,5	6764994000	16,8	5337842575936922957650
9,9	22058659953217981800	17,7	206305163005291900
10,8	7207510049560719850	18,6	10885677450
11,7	206305163005291900	13,12	192252132113764705987353896200
12,6	245649059538250	14,11	41394311546286990570767724825
13,5	1276419800	15,10	1690595367815132163307836750
10,9	492220346020866314150	16,9	9770462296518547305273800
11,8	58488168886336533925	17,8	4481489156958762453000
12,7	585085137096464550	18,7	46325408551373425
13,6	192086807308450	19,6	101188225
14,5	101188225	13,13	5896815924361936623044444626800
10,10	17908353146570145820800	14,12	2847327956358643190359156085300
11,9	6703938825093094075300	15,11	303781402592961107944725255700

Table 3.9: The genus 2 instanton numbers of total degree ≤ 37 for the family (3.15). The numbers not in this list are either zero, or given by those in the table after permuting indices.

p	$n_p^{(2)}$
16,10	6008644011089373537248656800
17,9	15502680926528679507634750
18,8	2645148270492863092650
19,7	6354579702758450
20,6	36800
14,13	124565089119075700247816681507000
15,12	30450650946759882533741285631525
16,11	1650396599182174240918908567600
17,10	15912299153776396184830406150
18,9	18071193444493417015126625
19,8	1088683595004165702950
20,7	497495418446900
14,14	3693414921590391334562655080548700
15,13	1887312824415684091294137374297850
16,12	241434752646453338262009308998450
17,11	6750850050453875649938033430650
18,10	31701961212371099761950706550
19,9	15502680926528679507634750
20,8	307911899250593057650
21,7	19785981206800
15,14	77734854231576430735155964501483500
16,13	21141578696054339753623844206366900
17,12	1447892872574562822004056745806925
18,11	21056470435268391362146639838000
19,10	47822198309268358643758769375
20,9	9770462296518547305273800
21,8	58488168886336533925
22,7	321011805475
15,15	2242005633475379243138889611530224350
16,14	1202166271275554852096319284514781050
17,13	179193911305392936323515910617613500

table 3.9 continued.

p	$n_p^{(2)}$
18,12	6669236758606676544482977924485150
19,11	50556113264139731816624429986400
20,10	54823814338848321169711111550
21,9	4499834280177683754617100
22,8	7207510049560719850
23,7	1276419800
16,15	47083170110743580006872424368633531875
17,14	14025908356241673430452097295836041700
18,13	1169983477395354290630422427608175550
19,12	23876076766141789732446125447010225
20,11	94078435233076347832828692471750
21,10	47822198309268358643758769375
22,9	1500298428800313660065325
23,8	546653742258204975
24,7	132150
16,16	1327080986654885164312931683366667358200
17,15	741736053327841335761519208560504382350
18,14	126013267019536739983934798259312856100
19,13	5967526679788804873035522660386672150
20,12	67042622214148592803250933902047750
21,11	136315316135643017078981825252200
22,10	31701961212371099761950706550
23,9	356988331876350933661250
24,8	23487690756165150
17,16	27835064011278355479795425089182885779850
18,15	8971331196558149334377988222442191858000
19,14	885994523265940239167051685255212381600
20,13	24041369335716235988574456740045846350
21,12	148673916874105664720736408737803900
22,11	154206354444180862857899859688400
23,10	15912299153776396184830406150
24,9	59381136406580839191450
25,8	497495418446900
17,17	769459996977013484532529807380612024950900
18,16	445912967342824716142914500096902010818850

table 3.9 continued.

p	$n_p^{(2)}$
19,15	84847587251331673881762239411974072983050
20,14	4938016593847719974314383381477983741100
21,13	77169654314981775489114851835598126800
22,12	261702486079440770739117430365216750
23,11	136315316135643017078981825252200
24,10	6008644011089373537248656800
25,9	6703938825093094075300
26,8	4007017841650
18,17	16130063331032815277955386421606900094355875
19,16	5569304833367577335205278477402592908516350
20,15	636726757004598625316020644295171791557250
21,14	22041710721238453533303229084611701286825
22,13	198702763601020024146439712249527136275
23,12	366918571341072517525517535457060450
24,11	94078435233076347832828692471750
25,10	1690595367815132163307836750
26,9	492220346020866314150
27,8	6764994000
18,18	438539109875379528721919654529846561401409750
19,17	262362104844021123045694638864495986589287000
20,16	55131178495812759321176609589875814140037750
21,15	3836414759887507545665007708487359916157800
22,14	79449270275648960296194539460339370694050
23,13	412564879312424855910387890012240484400
24,12	410576929861745683208869036387758800
25,11	50556113264139731816624429986400
26,10	349654611337758233664220900
27,9	22058659953217981800
28,8	191850
19,18	9192063299791665753644261196926491990414773625
20,17	3372577227429580157843234196650003432550205075
21,16	438359755343431485741657365115289687329201500
22,15	18737856667070391183320951837536512233506700
23,14	232781528281126665497995449029490453266300
24,13	693400092277019907121874794244775804900
25,12	366918571341072517525517535457060450
26,11	21056470435268391362146639838000
27,10	52176320562765050835877250
28,9	546653742258204975

Chapter 4

Supergravity Compactifications

Rose : I can't even touch it. Seems to be in a state of flux.

Donna : What does that mean?

Rose : I don't know. Sort of thing the Doctor would say.

Russell T Davies, *Turn Left*

Our discussion of background material in this section follows the textbooks [91, 92]. If massive degrees of freedom are neglected, then any of the five ten-dimensional superstring theories reduce to ten-dimensional supergravity theories. Of foremost concern to us are the two type II superstring theories (type IIA and IIB), whose low energy effective theories are the two maximal $\mathcal{N} = 2$ supergravities (of type IIA and IIB respectively). Either of the maximal supergravities can be compactified on a Calabi-Yau threefold¹ to obtain a four-dimensional $\mathcal{N} = 2$ matter coupled supergravity, and at the level of supergravity, mirror symmetry gives a duality between each of the supergravity theories obtained by the compactifications

$$\begin{array}{ccc} \text{Type IIB supergravity} & \cong & \text{Type IIA supergravity} \\ \text{on CY threefold X} & & \text{on mirror Y} \end{array} . \quad (4.1)$$

We will review briefly the field content of these 4d theories, in particular recalling that the

¹Complex dimension three, so real dimension six.

scalar fields in the matter content include geometric moduli of the threefolds. By comparing supersymmetric black hole solutions to both of the above theories, we can in some cases obtain formulae that relate number-theoretic quantities computed on X to the enumerative invariants of Y .

Moreover, one approach to realising realistic phenomenological models is *flux compactifications*, which we shall always view from the IIB perspective. Here, in addition to compactifying IIB supergravity (or indeed the full string theory) on a Calabi-Yau threefold X , one gives nonzero vacuum expectation values to form fields supported on the cohomology of X . This breaks supersymmetry further, and the four-dimensional massless theory has $\mathcal{N} = 1$ supersymmetry. We will study supersymmetric vacuum configurations, and provide new examples in support of the *flux modularity conjecture* [11, 12]. Such IIB setups can be related to F-theory on an elliptically fibred Calabi-Yau *fourfold* $\tilde{X}$, and we will see that (in line with previous conjectures) the elliptic fibre in a supersymmetric vacuum configuration has a surprising relation to the modularity of the threefold X .

4.1 4d $\mathcal{N} = 2$ matter coupled supergravities

The $\mathcal{N} = 2$ gravity multiplet consists of the spin-2 spacetime metric $g_{\mu\nu}$, two spin-3/2 gravitini, and a spin-1 vector field dubbed the graviphoton. The CPT conjugates to all of these must be included in the theory as well, so as to have CPT symmetry. Additional vector fields can be found in the matter content, residing in 4d $\mathcal{N} = 2$ vector multiplets which each consist of a spin-1 vector field, two spin-1/2 gaugini, and two real scalars. These scalars are packaged into one complex scalar. Again, this content is not CPT self-conjugate and so fields with opposite helicities must also be included. Finally, there can be CPT self-conjugate hypermultiplets, which contain two spin-1/2 fields, their CPT conjugate spin-1/2 fields, and four real scalars.

The attractor mechanism governs the values of the vector multiplet scalars near a stationary

black hole solution, so we pause to comment on the possible manifolds in which the vector multiplet scalars of a $\mathcal{N} = 2$ theory can live in. With there being n vector multiplets, we have $n + 1$ $U(1)$ vector fields when the graviphoton is incorporated. We will only be concerned with gauge fields for Abelian gauge groups $U(1)^{n+1}$, i.e. extended Maxwell theory. Just as classical electromagnetism (albeit with electric and magnetic charges) possesses an electromagnetic duality group $SL(2, \mathbb{Z})$ under which the charge vector $(q_{\text{electric}}, q_{\text{magnetic}})^T$ transforms as a doublet, extended Maxwell theory enjoys a symplectic electromagnetic duality group $Sp(2n + 2, \mathbb{Z})$ under which the charge vector $(\mathbf{q}_{\text{electric}}, \mathbf{q}_{\text{magnetic}})$ transforms in the vector representation.

If the 4d theory is to both have these duality transformations and be supersymmetric, then the vector multiplet scalars must be coordinates on a Kähler manifold with an $SP(2n + 2, \mathbb{R}) \otimes GL(1, \mathbb{C})$ bundle that possesses a holomorphic symplectic section² $\mathcal{S}$ [93]. The n vector multiplet scalars can be projectivised into $n + 1$ scalar fields living on a Kähler manifold $\mathcal{M}$. A symplectic duality transformation on the $2n + 2$ -component charge vector is paired with a symplectic transformation of the section

$$\mathcal{S} = \begin{pmatrix} X^I \\ \mathcal{F}_I \end{pmatrix} \quad (4.2)$$

on the scalar manifold $\mathcal{M}$. We are assuming the existence of such a section $\mathcal{S}$ in line with [93], and denoting components of $\mathcal{S}$ by X_I and $\mathcal{F}_I$. In this equation and the following, I, J, K run from 0 to n . With $\langle \circ, \circ \rangle$ giving the symplectic form, the Kähler potential of $\mathcal{M}$ is

$$K = i \langle \mathcal{S}, \bar{\mathcal{S}} \rangle . \quad (4.3)$$

There is an additional requirement, that

$$\langle \partial_I \mathcal{S}, \partial_J \mathcal{S} \rangle = 0 . \quad (4.4)$$

²When we turn to studying IIB compactifications on a CY threefold X , the bundle will simply be $H^3(X, \mathbb{C})$ and the section will be the holomorphic three-form Ω .

Kähler manifolds possessing such a section $\mathcal{S}$ are termed *special Kähler*. So long as the matrix $\partial_I X^J$ is invertible, then there must locally exist a holomorphic function $\mathcal{F}(X)$, homogeneous of degree 2, such that

$$\mathcal{F}_I = \frac{\partial \mathcal{F}(X)}{\partial X^I}. \quad (4.5)$$

In the absence of fluxes, the Kaluza-Klein reduction (as recounted for instance in the textbook [91]) of either maximal 10d supergravity on a Calabi-Yau is a 4d $\mathcal{N} = 2$ supergravity with some number of the aforementioned two kinds of matter multiplets. We turn now to describing these theories, with particular mind to the prepotential for their vector multiplet scalars.

Type IIA compactifications

If type IIA supergravity is compactified on a Calabi-Yau Y , the resulting theory possesses $h^{1,1}$ vector multiplets and $h^{2,1} + 1$ hypermultiplets. The $h^{1,1}$ vector multiplet's complex scalars parametrise the space of complexified Kähler classes on Y . The hypermultiplet moduli space contains the space of complex structures on Y , in addition to further scalars coming from reduction of the type IIA form fields on Y .

The first half of our symplectic section $\mathcal{S}$, which we denoted X^I in the previous subsection, will be taken to be the projective coordinates z^I .

The prepotential is closely related to the genus-0 topological A-model free energy,

$$\begin{aligned} \mathcal{F} &= -\frac{1}{6z^0} Y_{IJK} z^I z^J z^K - \frac{(z^0)^2}{(2\pi i)^3} \sum_{\mathbf{k} \geq 0} n_{\mathbf{k}}^{(0)} \text{Li}_3 \left(\exp \left(2\pi i k_i \frac{z^i}{z^0} \right) \right) \\ &\equiv -\frac{1}{6z^0} Y_{IJK} z^I z^J z^K - (z^0)^2 \mathcal{I} \left(\frac{\mathbf{z}}{z^0} \right). \end{aligned} \quad (4.6)$$

We have collected the z^i into the vector quantity $\mathbf{z}$. Here i runs from 1 to $h^{1,1}(Y)$ while I, J, K run from 0 to $h^{1,1}(Y)$. The Y_{ijk} are the topological numbers that we encountered in (2.6), (2.10), (2.12) and (2.11).

The projective coordinates z^I relate to the usual complexified Kähler coordinates t^i via

$$t^i = \frac{z^i}{z^0} . \quad (4.7)$$

Note that in (4.6) we have given the prepotential $\mathcal{F}$ together with its nonperturbative instanton corrections $(z^0)^2 \mathcal{I}(z^i/z^0)$ and perturbative quantum corrections, those being the terms in $\frac{1}{6z^0} Y_{IJK} z^I z^J z^K$ where any index takes the value 0. We will understand the purely classical value of the prepotential to be

$$\mathcal{F}_{\text{Classical}} = -\frac{(z^0)^2}{6} Y_{ijk} t^i t^j t^k . \quad (4.8)$$

Type IIB compactifications

Compactifying the type IIB supergravity on a CY threefold X leads to a 4d $\mathcal{N} = 2$ theory with $h^{2,1}(X)$ vector multiplets and $h^{1,1}(X) + 1$ hypermultiplets. The $h^{2,1}$ vector multiplet scalars parametrise complex structures on X , and within the hypermultiplet moduli are $h^{1,1}(X)$ complex scalars parametrisng the complexified Kähler structures on X . When X is mirror to Y , the field content of IIB on X and IIA on Y are identical. Both theories are actually identical, but this is a more involved duality than simply identifying dimensions by swapping the Hodge numbers, as the hypermultiplet moduli space of either compactification must contain the vector multiplet moduli space of the other. In both compactifications there is a universal hypermultiplet containing the axiodilaton field.

By using the mirror map, so that the t^i parametrisng Y 's Kähler structures are coordinates on the complex structure moduli space of X , we can find the same symplectic section as in the previous subsection on IIA compactifications,

$$\mathcal{S} = \begin{pmatrix} z^I \\ \frac{\partial \mathcal{F}}{\partial z^I} \end{pmatrix} , \quad (4.9)$$

with the same $\mathcal{F}$ as in (4.6). There is an important point to be made on classical versus nonclassical physics. In the IIA compactification, we have seen that the prepotential $\mathcal{F}$ is the sum of a classical part and quantum corrections. However, in the IIB frame this quantity is purely classical. Only after changing duality frames does the instanton sum $\mathcal{I}$ concealed in $\mathcal{F}$ above take on a nonperturbative interpretation.

The symplectic section $\mathcal{S}$ has a natural relation to the holomorphic three-form Ω on X . After introducing a symplectic basis α_I, β^I of $H^3(X, \mathbb{Z})$ we can write

$$\Omega = z^I \alpha_I - \frac{\partial \mathcal{F}}{\partial z^I} \beta^I . \quad (4.10)$$

Or in other words, the symplectic section is the integral period vector

$$\mathcal{S} = \Pi . \quad (4.11)$$

4.2 The attractor mechanism

This subsection serves to review some background material. We follow [92] and display a derivation of the attractor equations, including the necessary manipulations to obtain Strominger's form of the equations [94]. We adopt a set of notations in this section that make the supergravity analysis tractable, but this section is mostly self-contained and included to provide better background to the work on rank-two attractors.

In words, the attractor mechanism fixes the values of the vector multiplet scalars at the horizon of a supersymmetric black hole configuration in terms of the charges of said black hole. We shall at the end of this section explain how in Calabi-Yau compactifications these attractor equations can be interpreted in terms of the internal geometry's cohomology.

We begin with a more concrete account of electromagnetic duality, which was important in arguing for the existence of a holomorphic symplectic section on the scalar manifold. Then

we set about describing dilatation-gauge fixing, which is necessary in order to fix the form of the action that we will consider black hole solutions for, and the kinetic terms for the gauge fields.

Subsequently, we will take a spherically symmetric solution ansatz and derive the equations of motion. The scalar field's equations of motions will be recast as a gradient flow equation. Finally we will follow Ferrara and Kallosh [95] to find algebraic expressions for the fixed points of these gradient flows. Note that although we will study attractor points, we never work with the gradient flow equations, so do not find attractors by solving differential equations. We instead will in later sections deal with solutions to the algebraic equations given at the end of this section.

Electromagnetic duality

We shall write the Maxwell fields as A_μ^I , with μ a spacetime index and I running from 0 to n (one of these is the graviphoton and the remaining n are fields in the vector multiplets). The field strengths will then be

$$F_{\mu\nu}{}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I. \quad (4.12)$$

These field strengths form the first $n + 1$ components of a symplectic vector with $2n+2$ components. With the action S given below, the quantities

$$G_{\mu\nu I} = \kappa^2 \epsilon_{\mu\nu\rho\sigma} \frac{\delta S}{\delta F_{\rho\sigma}{}^I} \quad (4.13)$$

form the remaining $n + 1$ components. If $F_{\mu\nu}{}^I$ and $G_{\mu\nu I}$ solve the gauge field equations of motion then so do $\widehat{F}_{\mu\nu}{}^I$ and $\widehat{G}_{\mu\nu I}$ defined by

$$\begin{pmatrix} \widehat{F}_{\mu\nu}{}^I \\ \widehat{G}_{\mu\nu I} \end{pmatrix} = \mathbf{W} \begin{pmatrix} F_{\mu\nu}{}^I \\ G_{\mu\nu I} \end{pmatrix}, \quad (4.14)$$

where $\mathbf{W}$ is a $(2n+2) \times (2n+2)$ symplectic matrix. There is a corresponding transformation law for the gauge kinetic matrix (and hence the scalars, upon which this matrix depends). This is the statement of electromagnetic duality for Maxwell fields.

The electric and magnetic charges q_I and p^I in a volume V are given by

$$q_I = -\frac{1}{2} \int_V d^3x \epsilon^{ijk} \partial_i G_{jkI} , \quad p^I = -\frac{1}{2} \int_V d^3x \epsilon^{ijk} \partial_i F_{jk}{}^I . \quad (4.15)$$

Under electromagnetic duality the charge vector $(p^I, q_I)^T$ is mapped to $\mathbf{W} (p^I, q_I)^T$. Quantization forces the charges to lie on a lattice:

$$(p^I, q_I) \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{1}_{n \times n} \\ -\mathbf{1}_{n \times n} & \mathbf{0}_{n \times n} \end{pmatrix} \begin{pmatrix} p^I \\ q_I \end{pmatrix} = 2\pi n, \quad n \in \mathbb{Z}, \quad (4.16)$$

and so $\mathbf{W}$ must have integer entries: $\mathbf{W} \in \text{Sp}(2n+2, \mathbb{Z})$.

Dilatation gauge fixing and the gauge field kinetic term

We have already argued that this electromagnetic duality, plus supersymmetry, leads to the requirement that the vector multiplet scalars take values in a special Kähler manifold with symplectic section $\mathcal{S}$ and Kähler potential $K = i\langle \mathcal{S}, \bar{\mathcal{S}} \rangle$.

With the symplectic section $\mathcal{S}$ defined in terms of a prepotential $\mathcal{F}$ as in (4.2), the gauge field kinetic matrix is

$$\kappa^2 \mathcal{N}_{IJ}(z, \bar{z}) = \bar{F}_{IJ} + i \frac{K_{IK} X^K K_{JL} X^L}{K_{MN} X^M X^N}, \quad (4.17)$$

in which all subscripts denote partial derivatives. This couples the scalars to the gauge fields.

We write

$$\mathbf{P}_{IJ} = \text{Re} [\mathcal{N}_{IJ}], \quad \mathbf{I}_{IJ} = \text{Im} [\mathcal{N}_{IJ}]. \quad (4.18)$$

The dilatation operator's action on the scalars is to flow them along a Killing vector of this

manifold. To gauge fix, one identifies all points in the same orbits of this action. Using the Frobenius theorem one can choose coordinates y, z^α so that ∂_y is parallel to this Killing vector. We take the quantities X^I to be projective by introducing functions $Z^I(z)$, that do not depend on y , such that $X^I = yZ^I(z)$. One can then write

$$\mathcal{S} = \begin{pmatrix} X^I \\ F_I(X) \end{pmatrix} = y \begin{pmatrix} Z^I(z) \\ F_I(Z(z)) \end{pmatrix}. \quad (4.19)$$

The dilatation gauge is fixed by choosing a specific value for y . This value is chosen to be real and to keep the Kähler potential K at a fixed constant value of $-\kappa^{-2}$.

Performing this fixing we are left with a *projective special Kähler* manifold. This has coordinates z^α , which are the scalars that will appear in the gauge-fixed action. The Kähler potential of this projective manifold is $\mathcal{K}(z, \bar{z})$, where

$$e^{-\kappa^2 \mathcal{K}} = -\kappa^2 \frac{\partial K}{\partial X^I \partial \bar{X}^{\bar{J}}} Z^I \bar{Z}^{\bar{J}} = \frac{1}{|y|^2}. \quad (4.20)$$

There is some freedom in our choice of scale for Z^I and y . Had we instead opted for

$$\hat{y} = ye^{\kappa^2 f(z)}, \quad \hat{Z}^I = Z^I e^{-\kappa^2 f(z)} \quad (4.21)$$

with f an arbitrary holomorphic function, then we would have arrived at a projective potential

$$\hat{\mathcal{K}} = \mathcal{K} + f + \bar{f}. \quad (4.22)$$

A connection is introduced to give derivatives covariant under the transformations (4.21):

$$\begin{aligned}
\nabla_\alpha Z^I &= \partial_\alpha Z^I + \kappa^2 (\partial_\alpha \mathcal{K}) Z^I, \\
\bar{\nabla}_{\bar{\alpha}} \bar{Z}^{\bar{I}} &= \partial_{\bar{\alpha}} \bar{Z}^{\bar{I}} + \kappa^2 (\partial_{\bar{\alpha}} \mathcal{K}) \bar{Z}^{\bar{I}}, \\
\nabla_\alpha \bar{Z}^{\bar{I}} &= \partial_\alpha \bar{Z}^{\bar{I}} = 0, \\
\bar{\nabla}_{\bar{\alpha}} Z^I &= \partial_{\bar{\alpha}} Z^I = 0.
\end{aligned} \tag{4.23}$$

The metric on the projective manifold can be written as

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K} = i \langle \nabla_\alpha \mathcal{S}, \bar{\nabla}_{\bar{\beta}} \bar{\mathcal{S}} \rangle. \tag{4.24}$$

We end this subsection discussing the scalars by stating the bilinear form $\mathcal{N}_{IJ}$ used in the kinetic term for the gauge fields when a prepotential does not exist. This is

$$\kappa^2 \bar{\mathcal{N}}_{IJ} = (\bar{F}_I \nabla_\alpha F_I) (\bar{X}^J \nabla_\alpha X^J)^{-1}. \tag{4.25}$$

We will still in this case use the definitions (4.18).

Equations of motion

The bosonic part of the action for this theory is

$$S = \frac{1}{2\kappa^2} \int d^4x \left[\sqrt{-g} \left(R - 2g^{\mu\nu} g_{\alpha\bar{\beta}} \partial_\mu z^\alpha \partial_\nu \bar{z}^{\bar{\beta}} + \frac{1}{2} \mathcal{H}_{IJ} F_{\mu\nu}^I F^{\mu\nu J} \right) - \frac{1}{4} \mathcal{P}_{IJ} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^I F_{\rho\sigma}^J \right]. \tag{4.26}$$

One takes the following ansatz for a spherically symmetric, static metric:

$$ds^2 = -e^{2U(\tau)} dt^2 + e^{-2U(\tau)} \left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2} (d\theta^2 + \sin^2(\theta) d\phi^2) \right]. \tag{4.27}$$

The coordinates θ and ϕ are the familiar 2-sphere coordinates. The extremal Reissner-Nordstrom coordinate can be brought to this form, and in this case τ is $1/r$, where the

shifted radial variable r is 0 at the horizon.

In a static, spherically symmetric solution the three-dimensional electric and magnetic fields will only have radial components. Thus our field strengths are of the form

$$F^I = F_{t\tau}{}^I dt \wedge d\tau + F_{\theta\phi}{}^I d\theta \wedge d\phi. \quad (4.28)$$

Staticity and spherical symmetry gives

$$\nabla_t F_{\mu\nu}{}^I = \nabla_\theta F_{\mu\nu}{}^I = \nabla_\phi F_{\mu\nu}{}^I = 0. \quad (4.29)$$

The latter two of these enter into a Bianchi identity from which one obtains the equation

$$\nabla_\tau F_{\theta\phi}{}^I = 0. \quad (4.30)$$

These equations are solved by

$$F_{t\tau}{}^I = f^I(\tau), \quad F_{\theta\phi}{}^I = -\frac{p^I}{4\pi} \sin(\theta), \quad (4.31)$$

with the constant of integration $-p^I/4\pi$ fixed by the magnetic charge integral in (4.15).

The equation of motion $\delta S/\delta A_t^I = 0$ following from the action (4.26) is

$$\sin(\theta) \partial_\tau \left[e^{-2U} \mathbb{I}_{IJ} f^J(\tau) - P_{IJ} \frac{p^I}{4\pi} \right] = 0. \quad (4.32)$$

Integration gives $f^I(\tau)$.

$$f^I(\tau) = e^{2U} (\mathbb{I}^{-1})^{IJ} \left[P_{JK} \frac{p^K}{4\pi} - \frac{q_J}{4\pi} \right]. \quad (4.33)$$

Here the constant of integration is obtained by comparing the expression in brackets in (4.32) with that of $G_{\mu\nu I} = \kappa^2 \epsilon_{\mu\nu\rho\sigma} \frac{\delta S}{\delta F_{\rho\sigma}^I}$ and then using the electric charge integral in (4.15).

The stress energy tensor $T_{\mu\nu}$ for the above field content is

$$\kappa^2 T_{\mu\nu} = -\mathbb{I}_{IJ} \left(F_{\mu\rho}{}^I F_{\nu}{}^{\rho J} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}{}^I F^{\rho\sigma J} \right) + g_{\alpha\bar{\beta}} \left(\partial_\mu z^\alpha \partial_\nu \bar{z}^{\bar{\beta}} + \partial_\nu z^\alpha \partial_\mu \bar{z}^{\bar{\beta}} - g_{\mu\nu} \partial_\rho z^\alpha \partial^\rho \bar{z}^{\bar{\beta}} \right). \quad (4.34)$$

It is most convenient to work with the following form of the Einstein equations:

$$R_{\mu\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_\rho{}^\rho \right). \quad (4.35)$$

Expanding the right hand side using (4.34) leads to

$$R_{\mu\nu} = -\mathbb{I}_{IJ} \left(F_{\mu\rho}{}^I F_{\nu}{}^{\rho J} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}{}^I F^{\rho\sigma J} \right) + g_{\alpha\bar{\beta}} \left(\partial_\mu z^\alpha \partial_\nu \bar{z}^{\bar{\beta}} + \partial_\nu z^\alpha \partial_\mu \bar{z}^{\bar{\beta}} \right). \quad (4.36)$$

A significant simplification is met by introducing the black hole potential V_{BH} .

$$V_{BH} = \frac{1}{32\pi^2} \begin{pmatrix} p^I & q_I \end{pmatrix} \begin{pmatrix} -(\mathbb{I} + \mathbf{P}\mathbb{I}^{-1}\mathbf{P})_{IJ} & (\mathbf{P}\mathbb{I}^{-1})_I{}^J \\ (\mathbb{I}^{-1}\mathbf{P})^I{}_J & -(\mathbb{I}^{-1})^{IJ} \end{pmatrix} \begin{pmatrix} p^J \\ q_J \end{pmatrix}. \quad (4.37)$$

There are only two independent Einstein equations.

$$R_{tt} = e^{4U} \tau^4 \ddot{U} = e^{6U} \tau^4 V_{BH}, \quad (4.38)$$

$$R_{\tau\tau} = \ddot{U} - 2\dot{U}^2 = -e^{2U} V_{BH} + 2g_{\alpha\bar{\beta}} \dot{z}^\alpha \dot{\bar{z}}^{\bar{\beta}}.$$

The dot represents differentiation with respect to τ . Neatening these gives

$$\ddot{U} = e^{2U} V_{BH}, \quad (4.39)$$

$$\dot{U}^2 = e^{2U} V_{BH} - g_{\alpha\bar{\beta}} \dot{z}^\alpha \dot{\bar{z}}^{\bar{\beta}}.$$

The equations of motion of the scalars are

$$\frac{d}{d\tau} \left(g_{\alpha\bar{\beta}} \dot{z}^{\bar{\beta}} \right) - (\partial_{\alpha} g_{\gamma\bar{\delta}}) \dot{z}^{\gamma} \dot{z}^{\bar{\delta}} - e^{2U} V_{BH} = 0, \quad (4.40)$$

plus the complex conjugate of these equations.

The black hole potential and the central charge

The central charge $\mathcal{Z}$ is related to the electromagnetic charges and the scalar Kähler geometry by

$$\mathcal{Z} = 2\kappa^{-2} e^{\frac{\kappa^2}{2}\mathcal{K}} Z^I (q_I - \mathcal{N}_{IJ} p^J) = 2\kappa^{-2} (X^I q_I - F_I p^I). \quad (4.41)$$

After some unwinding, one can relate the central charge and the black hole potential.

$$(4\pi)^2 V_{BH} = \frac{\kappa^4}{4} |\mathcal{Z}|^2 + \frac{\kappa^2}{4} (\nabla_{\alpha} \mathcal{Z}) g^{\alpha\bar{\beta}} (\nabla_{\bar{\beta}} \bar{\mathcal{Z}}). \quad (4.42)$$

Some formulae from special geometry give further simplification. $\nabla_{\alpha} \bar{\mathcal{Z}} = 0$ and $\nabla_{\alpha} |\mathcal{Z}|^2 = \partial_{\alpha} |\mathcal{Z}|^2$ lead to

$$\nabla_{\alpha} \mathcal{Z} = 2\sqrt{\frac{\mathcal{Z}}{\bar{\mathcal{Z}}}} \partial_{\alpha} |\mathcal{Z}|. \quad (4.43)$$

This allows for

$$\frac{V_{BH}}{G^2} = |\mathcal{Z}|^2 + 4g^{\alpha\bar{\beta}} \partial_{\alpha} |\mathcal{Z}| \partial_{\bar{\beta}} |\mathcal{Z}|, \quad G = \frac{\kappa}{8\pi} \frac{64\pi^2}{\kappa^2} \quad (4.44)$$

The one-dimensional effective action

The large amount of symmetry has greatly reduced the number of equations. Moreover, the equations (4.40) and the first of (4.39) extremise a single one-dimensional effective action:

$$S_{1D} [U, z, \bar{z}] = \int_0^{\infty} d\tau \left(\dot{U}^2 + g_{\alpha\bar{\beta}} \dot{z}^{\alpha} \dot{\bar{z}}^{\bar{\beta}} + e^{2U} V_{BH} \right). \quad (4.45)$$

Since the above Lagrangian is independent of τ there is a conserved quantity:

$$\mathcal{E} = \dot{U}^2 + g_{\alpha\bar{\beta}} \dot{z}^{\alpha} \dot{\bar{z}}^{\bar{\beta}} - e^{2U} V_{BH}. \quad (4.46)$$

By fixing $\mathcal{E} = 0$ the second equation of (4.39) is satisfied. With this constant of integration fixed, this one-dimensional action captures the metric and scalar dynamics.

One can go further, by inserting the central charge expression (4.44) into the action (4.45) the attractor behaviour can be made clear. By completing squares and recognising a total derivative, one can write

$$S_{1D}[U, z, \bar{z}] = \int_0^\infty d\tau \left[\left(\dot{U} + Ge^U |\mathcal{Z}| \right)^2 + g_{\alpha\bar{\beta}} \left(\dot{z}^\alpha + 2Ge^U g^{\alpha\bar{\delta}} \partial_{\bar{\delta}} |\mathcal{Z}| \right) \left(\dot{\bar{z}}^{\bar{\beta}} + 2Ge^U g^{\gamma\bar{\beta}} \partial_\gamma |\mathcal{Z}| \right) \right] - 2Ge^U |\mathcal{Z}| \Big|_{\tau=0}^{\tau=\infty}. \quad (4.47)$$

The attractor equations

The integral in this action has an integrand that is the sum of two squares. It is thus minimised when both of these squares vanish. This requires that the following attractor equations are satisfied:

$$\dot{U} = -Ge^U |\mathcal{Z}|, \quad \dot{z}^\alpha = -2Ge^U g^{\alpha\bar{\beta}} \partial_{\bar{\beta}} |\mathcal{Z}|, \quad (4.48)$$

in addition to the complex conjugate of this last equation.

Note that these minimising equations together directly give an expression for $\dot{U}^2 + g_{\alpha\bar{\beta}} \dot{z}^\alpha \dot{\bar{z}}^{\bar{\beta}}$. This is

$$\dot{U}^2 + g_{\alpha\bar{\beta}} \dot{z}^\alpha \dot{\bar{z}}^{\bar{\beta}} = G^2 e^{2U} \left(|\mathcal{Z}|^2 + 4g^{\alpha\bar{\beta}} \partial_\alpha |\mathcal{Z}| \partial_{\bar{\beta}} |\mathcal{Z}| \right) = e^{2U} V_{BH}. \quad (4.49)$$

Equation (4.44) has been used to make the rightmost transition. This implies that $\mathcal{E} = 0$ for any solutions to (4.48) and so the second equation in (4.39) is satisfied.

The formulae (4.48) describe the evolution of U and motion of the scalars on their target space as τ increases. The second formula implies that under this evolution

$$\frac{d}{d\tau}|\mathcal{Z}| = -4Ge^U g^{\alpha\bar{\beta}} \partial_\alpha |\mathcal{Z}| \partial_{\bar{\beta}} |\mathcal{Z}|. \quad (4.50)$$

Since $g_{\alpha\bar{\beta}}$ is positive definite this equation informs us that $|\mathcal{Z}|$ is monotonically decreasing along attractor flows, and also that the flows end where $|\mathcal{Z}|$ takes a critical value. Since $|\mathcal{Z}|$ is bounded below (by virtue of being nonnegative), there must be a limiting value for $|\mathcal{Z}|$ as $\tau \rightarrow \infty$. The scalars z^α thus flow to the values such that $|\mathcal{Z}|$ takes a critical value. These are the *attractor points* of the scalar target manifold.

Saturation of the BPS bound

One can read off the mass of the black hole from the leading correction to g_{tt} at large r . It will be shown that this mass equals the modulus of the central charge at infinity, which is the BPS condition. We seek to find M as per $g_{tt} = -e^{2U} = -1 + 2MG\tau + \dots$, and obtain

$$M = \frac{-1}{2G} \frac{d}{d\tau} e^{2U} \Big|_{\tau=0}. \quad (4.51)$$

The flow equations (4.48) facilitate such a computation. Rearranging the equation involving $\dot{U}$, one obtains

$$\frac{d}{d\tau} e^{-U} = G|\mathcal{Z}|. \quad (4.52)$$

One can proceed by

$$M = \frac{-1}{2G} \frac{d}{d\tau} e^{2U} \Big|_{\tau=0} = \frac{-1}{2G} \frac{d}{d\tau} (e^{-U})^{-2} \Big|_{\tau=0} = \frac{1}{G} e^{3U} \frac{d}{d\tau} e^{-U} \Big|_{\tau=0} = |\mathcal{Z}|_\infty. \quad (4.53)$$

Stabilisation equations

We have seen that the scalars flow to attractor points where $|\mathcal{Z}|$ is minimised. A set of equations give the locations of these attractor points as a function of the black hole charges

p^I and q_I . These were initially derived by Strominger in 1996 [94]. Later that year they were derived using a slightly different approach by Ferrara and Kallosh [95].

Strominger considered solutions to the field equations with constant scalars (which must necessarily have an attractor point value throughout spacetime). The spacetime geometry in such setups is Reissner-Nordstrom. As these black holes are BPS and have vanishing Fermi fields, the bosonic parts of the fermionic supersymmetry variations necessarily vanish. The gaugino variations give the sought equations for the scalars in terms of the charges.

Ferrara and Kallosh's approach, which we detail here, involves directly extremising the central charge's modulus. The starting point is formula (4.41), reproduced here:

$$\mathcal{Z} = 2\kappa^{-2} (X^I q_I - F_I p^I). \quad (4.54)$$

By formula (4.43), $\partial_\alpha |\mathcal{Z}|$ is a nonzero multiple of $\nabla_\alpha \mathcal{Z}$. Thus the scalars that minimise $|\mathcal{Z}|$ satisfy

$$\frac{\kappa^2}{2} \nabla_\alpha \mathcal{Z} = (\nabla_\alpha X^I) q_I - (\nabla_\alpha F_I) p^I = 0. \quad (4.55)$$

The argument will make use of some formulae from special geometry. There are the raising and lowering identities

$$F_I = \mathcal{N}_{IJ} X^J, \quad \bar{F}_I = \bar{\mathcal{N}}_{IJ} \bar{X}^J, \quad \nabla_\alpha F_I = \bar{\mathcal{N}}_{IJ} \nabla_\alpha X^J, \quad \bar{\nabla}_{\bar{\alpha}} \bar{F}_I = \mathcal{N}_{IJ} \bar{\nabla}_{\bar{\alpha}} \bar{X}^J. \quad (4.56)$$

Using these one arrives at the following identities involving the vector $\mathcal{S}$ of (4.19):

$$\begin{aligned}
\langle \mathcal{S}, \bar{\mathcal{S}} \rangle &= X^I \bar{F}_I - \bar{X}^I F_I = X^I \bar{X}^J (\bar{\mathcal{N}}_{IJ} - \mathcal{N}_{IJ}) = -2i X^I \bar{X}^J \mathbb{I}_{IJ}, \\
\langle \nabla_\alpha \mathcal{S}, \bar{\nabla}_{\bar{\beta}} \bar{\mathcal{S}} \rangle &= (\nabla_\alpha X^I) (\bar{\nabla}_{\bar{\beta}} \bar{F}_I) - (\bar{\nabla}_{\bar{\beta}} \bar{X}^I) (\nabla_\alpha F_I) = 2i (\nabla_\alpha X^I) (\bar{\nabla}_{\bar{\beta}} \bar{X}^J) \mathbb{I}_{IJ}, \\
\langle \mathcal{S}, \bar{\nabla}_{\bar{\alpha}} \bar{\mathcal{S}} \rangle &= X^I \bar{\nabla}_{\bar{\alpha}} \bar{F}^J - F_I \bar{\nabla}_{\bar{\alpha}} \bar{X}^I = X^I \mathcal{N}_{IJ} \bar{\nabla}_{\bar{\alpha}} \bar{X}^J - X^J \mathcal{N}_{JI} \bar{\nabla}_{\bar{\alpha}} \bar{X}^I = 0, \\
\langle \bar{\mathcal{S}}, \nabla_\alpha \mathcal{S} \rangle &= \bar{X}^I \nabla_\alpha F_I - \bar{F}_I \nabla_\alpha X^I = \bar{X}^I \bar{\mathcal{N}}_{IJ} \nabla_\alpha X^J - \bar{X}^J \bar{\mathcal{N}}_{JI} \nabla_\alpha X^I = 0.
\end{aligned} \tag{4.57}$$

The second of these gives a formula for the metric component $g_{\alpha\bar{\beta}} = i\langle \nabla_\alpha \mathcal{S}, \bar{\nabla}_{\bar{\beta}} \bar{\mathcal{S}} \rangle$. The first formula above is an equation for the Kähler potential of the rigid manifold inside which our projective manifold is embedded³, $K = i\langle \mathcal{S}, \bar{\mathcal{S}} \rangle$. In choosing a dilatation gauge we chose values for the coordinate y so that K had a constant value of $-\kappa^{-2}$, but so as to find a general set of stabilisation equations not depending on gauge we do not fix this value here. One should bear in mind that since $\mathcal{S}$ is holomorphic, K is real.

Noting that by construction $\nabla_\alpha y = \nabla_\alpha \bar{y}$, all of the equations in (4.57) can be packaged into a single $(n+1) \times (n+1)$ matrix equation:

$$\begin{pmatrix} -K & 0 \\ 0 & g_{\alpha\bar{\beta}} \end{pmatrix} = -2y\bar{y} \begin{pmatrix} \bar{Z}^I \\ \nabla_\alpha Z^I \end{pmatrix} \mathbb{I} (Z^I \bar{\nabla}_{\bar{\beta}} \bar{Z}^I z). \tag{4.58}$$

A comment on the layout of this equation is in order. The matrix on the LHS is read in the obvious way. In the leftmost matrix on the RHS $\bar{Z}^I$ is understood as a row vector and $\nabla_\alpha Z^I$ is an $n \times (n+1)$ matrix. $\mathbb{I} = \text{Im}[\mathcal{N}]$ is read in the obvious way. In the rightmost matrix on the LHS Z^I is a column vector and $\bar{\nabla}_{\bar{\beta}} \bar{Z}^I$ is an $(n+1) \times n$ matrix.

The point of this involved digression is to arrive at a useful formula in the derivation of the stabilisation equations. One can rearrange (4.58) to reach

³This embedding is one stage in the construction of supergravity actions by gauge-fixing conformal supergravity, as discussed in the textbook [92].

$$\mathbb{I}^{-1} = -2y\bar{y} \begin{pmatrix} Z^I & \bar{\nabla}_{\bar{\beta}} \bar{Z}^I \end{pmatrix} \begin{pmatrix} -K^{-1} & 0 \\ 0 & g^{\alpha\bar{\beta}} \end{pmatrix} \begin{pmatrix} \bar{Z}^I \\ \nabla_{\alpha} Z^I \end{pmatrix}. \quad (4.59)$$

The matrix multiplication can be carried out to get an equation for the components of I^{-1} .

$$(\mathbb{I}^{-1})^{IJ} = \frac{2}{K} \bar{X}^I X^J - 2g^{\alpha\bar{\beta}} (\nabla_{\alpha} X^I) (\bar{\nabla}_{\bar{\beta}} \bar{X}^J). \quad (4.60)$$

It is now possible to extremise the central charge. First, substitute the third identity from (4.56) in (4.55) and use the fact that $\mathcal{N}_{IJ}$ is symmetric to obtain

$$(\nabla_{\alpha} X^I) q_I - \bar{\mathcal{N}}_{IJ} (\nabla_{\alpha} X^I) p^J = 0. \quad (4.61)$$

Next contract with $g^{\alpha\bar{\beta}} (\bar{\nabla}_{\bar{\beta}} \bar{X}^K)$ and use formula (4.60) to replace $g^{\alpha\bar{\beta}} (\nabla_{\alpha} X^I) (\bar{\nabla}_{\bar{\beta}} \bar{X}^K)$.

$$\left(\frac{1}{K} \bar{X}^I X^K - \frac{1}{2} (\mathbb{I}^{-1})^{IK} \right) (q_I - \bar{\mathcal{N}}_{IJ} p^J) = 0. \quad (4.62)$$

One then rearranges this equation,

$$\begin{aligned} 0 &= \frac{1}{K} (\bar{X}^I q_I - \bar{\mathcal{N}}_{IJ} \bar{X}^I p^J) X^K - \frac{1}{2} (\mathbb{I}^{-1})^{IK} q_I + \frac{1}{2} (\mathbb{I}^{-1})^{IK} \bar{\mathcal{N}}_{IJ} p^J \\ &= \frac{1}{K} (\bar{X}^I q_I - \bar{F}_J p^J) X^K - \frac{1}{2} (\mathbb{I}^{-1})^{IK} q_I + \frac{1}{2} (\mathbb{I}^{-1})^{IK} (\mathbf{P}_{IJ} - \mathbf{i}\mathbb{I}_{IJ}) p^J. \\ &= \frac{\kappa^2}{2K} \bar{\mathcal{Z}} X^K - \frac{1}{2} (\mathbb{I}^{-1})^{IK} (q_I - \mathbf{P}_{IJ} p^J) - \frac{\mathbf{i}}{2} p^K. \end{aligned} \quad (4.63)$$

Passing from (4.62) to the first line of (4.63) involves only moving terms around. The passage from the first to the second line makes use of the second lowering identity in (4.56) and breaks $\bar{\mathcal{N}}$ into its real and imaginary parts. Passing from the second to the third line necessitates formula (4.54) and multiplying some matrices together.

Since κ^2 , $\mathbb{I}$, $\mathbf{P}$, K , p^I and q_I are all real, one can extract the imaginary part of the last equation:

$$p^K = \text{Im} \left[\frac{\kappa^2}{K} \bar{\mathcal{Z}} X^K \right]. \quad (4.64)$$

The real part of (4.63), contracted with $\mathbb{I}_{JK}$, is

$$\begin{aligned} q_J &= P_{JK} p^K + \text{Re} \left[\frac{\kappa^2}{K} \bar{\mathcal{Z}} \mathbb{I}_{JK} X^K \right] \\ &= \text{Im} \left[\frac{\kappa^2}{K} \bar{\mathcal{Z}} P_{JK} X^K \right] + \text{Im} \left[\frac{\kappa^2}{K} \bar{\mathcal{Z}} i \mathbb{I}_{JK} X^K \right] \\ &= \text{Im} \left[\frac{\kappa^2}{K} \bar{\mathcal{Z}} \mathcal{N}_{JK} X^K \right] \\ &= \text{Im} \left[\frac{\kappa^2}{K} \bar{\mathcal{Z}} F_J \right]. \end{aligned} \quad (4.65)$$

The quantity $\frac{\kappa^2}{K} \bar{\mathcal{Z}}$ depends on the dilatation gauge and the charges. Denoting this quantity's value at an attractor point as C , we have reached the stabilisation equations that relate the scalar values at an attractor point to the electromagnetic charges when the central charge does not vanish:

$$\begin{pmatrix} p^I \\ q_I \end{pmatrix} = \text{Im} \left[C \begin{pmatrix} X^I \\ F_I \end{pmatrix} \right]. \quad (4.66)$$

These are $2n + 2$ real equations for $2n + 4$ real variables — the real and imaginary parts of C and the $2n + 2$ attractor coordinates on the rigid manifold. Fixing a choice of gauge constrains two of these variables, and so the number of equations equals the number of physical variables to solve for.

Multi-centred solutions

In [96] the attractor mechanism was extended to multi-centered black holes in $\mathcal{N} = 2$ supergravity. It was shown that in the presence of N sources located at $\vec{x}_i$, $i = 1, \dots, N$, each of charge Γ_i , the vector multiplet scalars take attractor values associated to the charge vector Γ_i as $\vec{x}_i$ is approached.

4.3 Calabi-Yau attractors in IIB

In a IIB compactification on a Calabi-Yau threefold X , a charged black hole has a string-theoretic description as a bound state of D3-branes wrapping special lagrangian cycles in the middle cohomology of X . The charge vector Q then is associated to a three-cycle γ in $H_3(X, \mathbb{Z})$. This is dual in cohomology to a threeform Γ , which we will expand in the same integral symplectic basis as used in (4.10), whose components we collect in the vector Q .

$$\Gamma = p^I \alpha_I - q_I \beta^I, \quad Q = \begin{pmatrix} q_I \\ p^I \end{pmatrix}. \quad (4.67)$$

The stabilisation equations (4.66) give the following constraint on the holomorphic three-form Ω :

$$\Gamma = \text{Im} [C \Omega] , \quad \text{or in components } Q = \text{Im} [C \Pi] . \quad (4.68)$$

Note that the ordering of the electric and magnetic charges is reversed compared to that in §4.2, which is purely a matter of changing convention and not conceptually significant. If we fix the charge vector, then the above equation is a constraint on the complex structure moduli of X . A Calabi-Yau threefold whose moduli satisfy the above equation for some integral charge vector is said to be an attractor variety.

If we let the complex constant C have real part a and imaginary part b , then the above equation reads

$$Q = a \text{Im} [\Pi] + b \text{Re} [\Pi] . \quad (4.69)$$

In the Dolbeault decomposition of $H^3(X, \mathbb{Z})$, the holomorphic three form generates the $(3, 0)$ part. The real and imaginary part of Ω both belong to $H^{(3,0)} \oplus H^{(0,3)}$, and so the above equation tells us that inside $H^{(3,0)} \oplus H^{(0,3)}$ lives the span of the integral vector Q . The statement is that the $(3, 0) + (0, 3)$ part of X 's Dolbeault cohomology contains a 1-dimensional integral lattice.

In some cases, there may be a point φ_* in the complex structure moduli space of X so that,

at this point, the equation (4.68) is solved for two independent charge vectors Q_1, Q_2 (with different values of C). “Independence” here means that the symplectic inner product $Q_1 \Sigma Q_2$ does not vanish, and so Q_1 and Q_2 are necessarily linearly independent.

For such moduli values φ_* , X ’s cohomology has the property that $H^{(3,0)} \oplus H^{(0,3)}$ is the complexification of a rank-two integral lattice in $H^3(X, \mathbb{Z})$ [14]. This gives a splitting of the Hodge structure. We see this at the level of deRham cohomology, however conjecturally this persists into a suitable étale cohomology and so restricts the form of the zeta function. We shall look at this more closely in §4.10.

4.4 One-parameter Calabi-Yau attractors in IIA

We will only be concerned with IIA compactifications on manifolds Y with $h^{1,1} = 1$. If the period vector Π solves the attractor equations for two independent charge vectors Q_1, Q_2 , then there must exist independent integral vectors Q_3, Q_4 such that

$$\Pi \Sigma Q_3 = \Pi \Sigma Q_4 = 0 . \quad (4.70)$$

This must be so, because the attractor equations imply

$$\text{Span}(\text{Re}[\Pi] , \text{Im}[\Pi]) = \text{Span}(Q_1 , Q_2) . \quad (4.71)$$

The symplectic complement of $\text{Span}(Q_1 , Q_2) \subset \mathbb{Z}^4$ is two-dimensional, with generators some Q_3 , Q_4 , which are by the above relation symplectic-orthogonal to $\text{Re}[\Pi]$ and $\text{Im}[\Pi]$. We shall in what follows make reference to the *orthogonality equation*

$$Q \Sigma \Pi = 0 , \quad (4.72)$$

in contrast to the attractor equations (4.68) .

This equivalence will be useful when we discuss a set of summation identities based on solutions to the IIA attractor equations, which we now set up. In a one-parameter compactification the IIA prepotential (4.6) reads

$$\begin{aligned}\mathcal{F} &= - (z^0)^2 \left[\frac{1}{6} Y_{111} t^3 + \frac{1}{2} Y_{110} t^2 + \frac{1}{2} Y_{100} t + \frac{1}{6} Y_{000} + \mathcal{I}(t) \right] , \\ \mathcal{I} &= \frac{1}{(2\pi i)^3} \sum_{k=1}^{\infty} n_k^{(0)} \text{Li}_3(e^{2\pi i k t}) = \frac{1}{(2\pi i)^3} \sum_{k=1}^{\infty} \frac{N_k}{k^3} e^{2\pi i k t} .\end{aligned}\tag{4.73}$$

We have written $t = \frac{z^1}{z^0}$, and in the above expression for the instanton sum $\mathcal{I}$ we have repackaged the genus-0 instanton numbers into *scaled Gromov-Witten invariants* N_k . These are computed from the $n_k^{(0)}$ and related to the usual Gromov-Witten invariants N_k^{GW} [97] by

$$N_k = \sum_{d|k} d^3 n_d^{(0)} , \quad N_k = k^3 N_k^{GW} .\tag{4.74}$$

The four-component period vector Π is

$$\Pi(t) = \begin{pmatrix} \frac{\partial}{\partial z^0} \mathcal{F} \\ \frac{\partial}{\partial z^1} \mathcal{F} \\ z^0 \\ z^1 \end{pmatrix} = z^0 \begin{pmatrix} \frac{1}{6} Y t^3 - \frac{1}{2} Y_{100} t - \frac{1}{3} Y_{000} - 2\mathcal{I}(t) + t\mathcal{I}'(t) \\ -\frac{1}{2} Y t^2 - Y_{110} t - \frac{1}{2} Y_{100} - \mathcal{I}'(t) \\ 1 \\ t \end{pmatrix} , \quad Y = Y_{111} .\tag{4.75}$$

We will now fix a gauge⁴, $z^0 = 1$. The period vector is seen to be a sum of a quantity Π_0 that is corrected by instantons:

$$\begin{aligned}\Pi &= \Pi_0 + \Delta_{\mathcal{I}} \Pi , \\ \Pi_0 &= \begin{pmatrix} \frac{1}{6} Y t^3 - \frac{1}{2} Y_{100} t - \frac{1}{3} Y_{000} \\ -\frac{1}{2} Y t^2 - Y_{110} t - \frac{1}{2} Y_{100} \\ 1 \\ t \end{pmatrix} , \quad \Delta_{\mathcal{I}} \Pi = \begin{pmatrix} -2\mathcal{I}(t) + t\mathcal{I}'(t) \\ -\mathcal{I}'(t) \\ 0 \\ 0 \end{pmatrix} .\end{aligned}\tag{4.76}$$

In [4] a method of solving the attractor and orthogonality equations for the above form of

⁴Note that this is done only after taking the above derivatives $\frac{\partial}{\partial z^0} \mathcal{F}$

period vector was displayed. We provide a brief summary of the solution method. First, the uncorrected equations

$$Q = \text{Im}[C\Pi_0] \quad \text{or} \quad Q\Sigma\Pi_0 = 0 \quad (4.77)$$

are solved, to find a ‘perturbative’ solution t_0 , with real and imaginary part x_0 and y_0 , so

$$t_0 = x_0 + i y_0 . \quad (4.78)$$

Notice that either of (4.77) gives algebraic equations for x_0 and y_0 . In [98] the attractor equations for such uncorrected prepotentials were considered. A full ‘instanton-corrected’ solution t is found by performing perturbation theory in the small parameter

$$e^{-2\pi y_0} = |e^{2\pi i t_0}| . \quad (4.79)$$

The precise form of the solution depends on the charge vector Q . The most general form, with electric charges q_0, q and magnetic charges p^0, p , is

$$Q = \begin{pmatrix} q_0 \\ q \\ p^0 \\ p \end{pmatrix} = \begin{pmatrix} q_{D0} \\ q_{D2} \\ q_{D6} \\ q_{D4} \end{pmatrix} , \quad (4.80)$$

where in the second equality we interpret the integral charges as giving the wrapping number of even-dimensional D-branes. We adopt a convention that does not include charges induced on each brane by world-volume curvature coupling.

We will consider only charge vectors that take one of the two following forms:

$$Q_{D4} = \kappa \begin{pmatrix} \Lambda \\ \Upsilon \\ 0 \\ 1 \end{pmatrix} , \quad Q_{D6} = \kappa \begin{pmatrix} \Lambda \\ \Upsilon \\ 1 \\ 0 \end{pmatrix} . \quad (4.81)$$

The integer κ gives the total D4 or D6 charge, while Λ and Υ are integer multiples of $\frac{1}{\kappa}$.

Our attention will be fixed to solutions of the ‘D4-D2-D0 orthogonality equations’:

$$Q_{D4}\Sigma\Pi = 0 . \quad (4.82)$$

To explain this choice, we note first that if t is a rank-two attractor, then an integral vector of the form Q_{D4} can be found so that the above equation (4.82) is satisfied. We work with the equations (4.82) and not the attractor equations themselves because, at present, the solution to (4.82) takes a much simpler form where the terms in the perturbative series can be given in closed form (rather than solely by recurrences).

Neglecting instantons, the equation $Q_{D4}\Sigma\Pi_0 = 0$ is a simple quadratic,

$$Y_{111}t_0^2 + 2(Y_{110} + \Upsilon)t_0 + Y_{100} + 2\Lambda = 0 . \quad (4.83)$$

We will assume that $(Y_{110} + \Upsilon)^2 - Y_{111}(Y_{100} + 2\Lambda) < 0$. In that case, the solution to (4.83) is $t_0 = x_0 + iy_0$ with real and imaginary parts

$$x_0 = \frac{-Y_{110} - \Upsilon}{Y_{111}} , \quad y_0 = \frac{\sqrt{Y_{111}(Y_{100} + 2\Lambda) - (Y_{110} + \Upsilon)^2}}{Y_{111}} . \quad (4.84)$$

To reiterate, this algebraic number $x_0 + iy_0$ does not solve the attractor or orthogonality equations, but is termed the *perturbative* solution of the orthogonality equation as it solves the second of the equations (4.77). Then, the full solution of (4.82), as conjectured⁵ in [4], is

$$t = t_0 - \frac{i}{\sqrt{2\pi^3 Y_{111}}} \sum_{j=1}^{\infty} e^{2\pi i x_0 j} \sum_{\mathbf{p} \in \text{pt}(j)} a_{\mathbf{p}} N_{\mathbf{p}} \left(\frac{j}{2\pi y_0 Y_{111}} \right)^{\ell(\mathbf{p})-1/2} K_{\ell(\mathbf{p})-1/2}(2\pi j y_0) . \quad (4.85)$$

$K_{\nu}(z)$ is the modified Bessel function of the second kind. $\text{pt}(j)$ is the set of partitions of

⁵The conjectural part of this analysis is in identifying the combinatoric functions $a_{\mathbf{p}}$ and modified Bessel functions in the series coefficients. It is a theorem that some series solution with recursively defined coefficients exists when y_0 is sufficiently small, see [4].

the integer j . For such a partition $\mathbf{p}$, j is partitioned into a set of integers as $j = \sum_{k=0}^{\infty} \mu_k k$. That is to say, μ_k is the multiplicity of the integer k in the partition $\mathbf{p}$ of j .

That said, $a_{\mathbf{p}}$ is a combinatorial factor given by

$$a_{\mathbf{p}} = \prod_{k=1}^{\infty} \frac{1}{k^{2\mu_k} \mu_k!} \quad (4.86)$$

and $N_{\mathbf{p}}$ is the following product of the enumerative invariants N_k :

$$N_{\mathbf{p}} = \prod_{k=1}^{\infty} N_k^{\mu_k} . \quad (4.87)$$

Finally, $\ell(\mathbf{p})$ is the length of the partition $\mathbf{p}$,

$$\ell(\mathbf{p}) = \sum_{k=0}^{\infty} \mu_k . \quad (4.88)$$

4.5 Flux compactifications

Flux compactifications are string theory compactifications with nontrivial background values for the $(p+1)$ -form field strengths. Our account of this topic follows [99] and [12]. The first supersymmetric configurations were given in [100]. The prospect of realising deSitter vacua was addressed in [101].

Type IIB supergravity's massless bosonic field content includes the even p -form Ramond-Ramond fields C_0 , C_2 , and C_4 which have field strengths

$$F_i = dC_{i-1} . \quad (4.89)$$

Additionally there is the Kalb-Ramond two-form B_2 which has field strength

$$H_3 = dB_2 . \quad (4.90)$$

There is also the dilaton ϕ , which is packaged along with the axion C_0 into a single complex scalar, the axiodilaton

$$\tau = C_0 + i e^{-\phi} . \quad (4.91)$$

Another notational change introduces the three-form field G_3 , defined as

$$G_3 = F_3 - \tau H_3 . \quad (4.92)$$

The utility of these two redefinitions is that it simplifies the action of the $\text{SL}(2, \mathbb{Z})$ symmetry of the IIB theory. The axiodilaton τ and complexified field strength G_3 transform via

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} , \quad G_3 \mapsto \frac{1}{c\tau + d} G_3 , \quad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) . \quad (4.93)$$

In a flux compactification, we dimensionally reduce on a Calabi-Yau threefold X , or an orientifold thereof, but do so with nonzero values given to F_3 and H_3 supported on the cohomology of X . Expanding in an integral symplectic basis, we write the components of F_3 and H_3 as

$$F_3 = (2\pi)^2 \alpha' (f^a \alpha_a - f_b \beta^b) , \quad H_3 = (2\pi)^2 \alpha' (h^a \alpha_a - h_b \beta^b) , \quad (4.94)$$

where α' is the string coupling and f_a, f^b, h_a, h^b are all integers. We collect these into the component vectors

$$F = \begin{pmatrix} f^a \\ f_b \end{pmatrix} , \quad H = \begin{pmatrix} h^a \\ h_b \end{pmatrix} . \quad (4.95)$$

When compactifying with such fluxes turned on, more supersymmetry is broken and the resulting four-dimensional supergravity has $\mathcal{N} = 1$ supersymmetry. The geometric moduli of X (both complex structure and Kähler) and the axiodilaton are scalars in $\mathcal{N} = 1$ chiral multiplets [99]. These scalars are coupled together by a potential term V in the action, which in accordance with $\mathcal{N} = 1$ supersymmetry is constructed in a standard manner from a superpotential W . Incorporating or neglecting nonperturbative string theory contributions,

like D-brane instantons, leads to different superpotentials W . We shall work with the uncorrected, classical superpotential which only depends on the complex structure moduli of X and the axiodilaton, reading

$$W = \int_X G_3 \wedge \Omega = (2\pi)^2 \alpha' (F - \tau H)^T \Sigma \Pi, \quad \Sigma = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \quad (4.96)$$

There are two constraints on the choices of flux vectors F and H , which bound $F^T \Sigma H$ from above and below.

$$0 < F^T \Sigma H \leq \frac{\chi(X_4)}{24}. \quad (4.97)$$

The lower bound is necessary to get nontrivial supersymmetric solutions to the equations of motion, while the upper bound is the D3 tadpole condition, which is a consistency condition coming from F-theory. Namely, the different sources of D3 charge must locally cancel out one another. Flux compactifications on an orientifold of X can be lifted to F-theory on an elliptically fibred Calabi-Yau fourfold X_4 [102], and there is a condition

$$\frac{1}{2\kappa_{10}^2 T_3} \int_X H_3 \wedge F_3 + Q^{\text{D3}} = \frac{\chi(X_4)}{24}. \quad (4.98)$$

Q^{D3} is the total charge of any present D3 branes. We can add D3 branes (but not anti-D3 branes) without breaking supersymmetry, from which the upper bound follows.

4.6 F-theory lifts

Sen demonstrated in [102] that F-theory on an elliptically fibred Calabi-Yau fourfold reduces at weak coupling to an Orientifold compactification of type IIB string theory. We shall briefly review his construction, informed also by [11, 12].

The modulus τ of the elliptic fibration is identified with the axiodilaton of the IIB theory. This gives a geometric interpretation to the $\text{SL}(2, \mathbb{Z})$ symmetry of IIB. The base of the fibration admits a double covering by a Calabi-Yau threefold, as depicted in figure 4.1.

$$\begin{array}{ccc}
\mathcal{E}_\tau & \hookrightarrow & X_4 \\
& & \downarrow \\
X & \xrightarrow{2:1} \twoheadrightarrow & \mathcal{B}
\end{array}$$

Figure 4.1: *The F-theory fourfold is an elliptic fibration over the base $\mathcal{B}$. This base is in turn doubly covered by the Calabi-Yau threefold X .*

A method for constructing these fourfolds as intersections in toric varieties, given X as such an intersection, was described in [103, 104] .

Sen's description gave an explicit relation between the elliptic fibration and the locations of D7 branes and O7 planes in the IIB limit. The most general elliptic fibration over a base $\mathcal{B}$ has a Weierstrass form

$$y^2 = x^3 + f(u)x + g(u) , \quad (4.99)$$

with u being coordinates on the base $\mathcal{B}$. As u varies so does the above elliptic curve. For certain values of u , given by the vanishing of the discriminant

$$\Delta(u) = 4f^3 + 27g(u)^2 , \quad (4.100)$$

the elliptic curve becomes singular. Sen computed monodromies of τ about this singular locus, and recognised the same monodromies as one expects upon circling D7 branes or O7 planes. In so doing, the singular locus $\Delta(u) = 0$ on the base $\mathcal{B}$ is recognised as giving the intersections of spacetime filling 7-dimensional extended objects with the internal orientifold geometry.

To delineate between the D7 and O7 objects, it is convenient to rewrite f and g in terms of functions h , η , and χ , and a constant C (which is not the C of the previous section):

$$f(u) = C\eta(u) - 3h(u)^2 , \quad g(u) = h(u) [C\eta(u) - 2h(u)^2] + C^2\chi(u) . \quad (4.101)$$

Having done this, the discriminant is

$$\Delta(u) = C^2 [\eta^2 (4C\eta - 9h^2) + 54h (C\eta - 2h^2) \chi + 27C^2 \chi^2] . \quad (4.102)$$

The j -invariant $j_{\mathcal{E}}$ of an elliptic curve is, in terms of the Weierstrass data f, g ,

$$j_{\mathcal{E}}(u) = \frac{6912f^3}{4f^3 + 27g^2} . \quad (4.103)$$

Our prefactor 6912 differs to Sen's convention. Since the elliptic fibration's complex structure parameter τ (itself some function of the base coordinate u) is to be identified with the axiodilaton τ , we should have

$$j_{\mathcal{E}}(u) = j(\tau(u)) , \quad (4.104)$$

where on the right hand side we have Klein's j -invariant

$$j(\tau) = e^{-2\pi i \tau} + 744 + 196884e^{2\pi i \tau} + 21493760e^{4\pi i \tau} + 864299970e^{6\pi i \tau} + \dots . \quad (4.105)$$

Note that this differs from the function $J(\tau) = \frac{1}{1728}j$, the former being implemented in Mathematica as `KleinInvariantJ`.

In terms of the functions f, h, χ , this j -invariant is

$$j(\tau(u)) = \frac{6912 (C\eta - 3h)^2}{C^2 [\eta^2 (4C\eta - 9h^2) + 54h (C\eta - 2h^2) \chi + 27C^2 \chi^2]} . \quad (4.106)$$

In the limit $C \mapsto 0$ the above j -invariant diverges like C^{-2} . As discussed in [102] (wherein their λ is our τ), up to $\text{SL}(2, \mathbb{Z})$ transformations this limit corresponds to sending the axiodilaton τ to $i\infty$ via $\tau = \frac{1}{4\pi i} \log(C) + O(C)$.

In this large τ limit, taken with constant h, η, χ and $C \mapsto 0$, the discriminant (4.102) becomes

$$\Delta(u) = -C^2 h^2 (\eta^2 + 12h\chi) . \quad (4.107)$$

This vanishes on the loci

$$h = 0 , \quad \text{and} \quad \eta = \pm \sqrt{-12h\chi} . \quad (4.108)$$

Sen computed the monodromies of the axiodilaton around these loci, and interpreted these as being due to charged objects sourcing the axiodilaton being present. Seven-dimensional extended objects fill the macroscopic four-dimensional spacetime and intersect the base $\mathcal{B}$ on the loci $\Delta(u) = 0$. The locus $h = 0$ gives the position of an $O7$ plane, while the loci $\eta = \pm \sqrt{-12h\chi}$ give D-brane positions.

In F-theory, a supersymmetric configuration will be one where the D7 and O7s coincide. By making a choice $\chi = 0$ and $h^2 = \eta$, both objects will be positioned at $\eta = 0$.

At the IIB level a maximally supersymmetric configuration should have a spatially constant axiodilaton profile. The sought profile is obtained from Sen's fibration, and the choices made for η , h , χ give the following f and g as in (4.101):

$$f(u) = (C - 3)h(u)^2 , \quad g(u) = (C - 2)h(u)^3 . \quad (4.109)$$

This gives an elliptic fibration over $\mathcal{B}$ with Weierstrass model

$$y^2 = x^3 + (C - 3)h(u)^2 + (C - 2)h(u)^3 . \quad (4.110)$$

The singular locus of this curve is $C^2(4C - 9)h(u)^6$. Away from the locus $h(u) = 0$ we can make a rescaling of coordinates $y \rightarrow h^{3/2}y$, $x \rightarrow hx$ so that our model is constant over the base, away from the charged objects.

$$y^2 = x^3 + (C - 3)x + C - 2 . \quad (4.111)$$

The j -invariant of this curve is

$$j_C = \frac{6912(C - 3)^3}{C^2(4C - 9)} . \quad (4.112)$$

From this follows a cubic equation for C , which can be solved in terms of $j(\tau)$. This choice of a C then gives the correct elliptic fibre for the F-theory uplift of the IIB supersymmetric flux vacuum with axiodilaton τ .

4.7 The scalar potential

Let us temporarily forget the fact that we have an internal Calabi-Yau manifold, and discuss some general features of these supergravity theories following [92]. In any four-dimensional $\mathcal{N} = 1$ supergravity coupled to N chiral multiplets, the scalars in those multiplets are coordinates on a projective Kähler manifold. The theory is specified by the choice of two functions of the moduli; the Kähler potential $\mathcal{K}$ for the scalar manifold and a superpotential W . From these two functions, one calculates the following potential:

$$V \stackrel{\text{def}}{=} e^{\mathcal{K}} \left(G^{\alpha\bar{\beta}} D_{\alpha} W \overline{D_{\beta} W} - 3|W|^2 \right). \quad (4.113)$$

The sum in the above expression runs over all moduli, so in the case of a Calabi-Yau compactification on X , $\alpha, \beta = 1, \dots, h^{1,1}(X) + h^{1,2}(X) + 1$. The metric $G_{\alpha\bar{\beta}}$ is the metric on the scalar manifold, and, by virtue of the Kähler condition, $G_{\alpha\bar{\beta}} = \partial_{\alpha} \partial_{\bar{\beta}} \mathcal{K}$. As in [24, 93], the quantity $D_{\alpha} W$ is the Kähler covariant derivative of W , which has Kähler weight (1,0), so

$$D_{\alpha} W = \partial_{\alpha} W + (\partial_{\alpha} \mathcal{K}) W.$$

In the case of a type IIB Calabi-Yau compactification, the total scalar manifold is the product of the upper half plane (in which the axiodilaton is valued) and the moduli space of the Calabi-Yau manifold X , which factorises, at least locally, into the moduli spaces of complex structures and complexified Kähler structures of X . The Kähler potential for the total moduli space is then a sum of the Kähler potentials for the axiodilaton, complex structure,

and Kähler structure factors of the moduli space:

$$\mathcal{K} = K^{\text{AD}} + K^{\text{CS}} + K^{\text{CK}} .$$

These depend on the moduli as follows:

$$K^{\text{AD}} = -\log(-i(\tau - \bar{\tau})) , \quad K^{\text{CS}} = -\log(-i\Pi^\dagger \Sigma \Pi) , \quad K^{\text{CK}} = -\log(-i\Pi^\dagger \Sigma \Pi) , \quad (4.114)$$

where Π is the period vector for the mirror manifold, so a function of the Kähler moduli of X . The Kähler potentials for the axiodilaton and complex structure moduli are exact at tree level, while the potential for complexified Kähler structures, K^{CK} , is corrected by fundamental string instantons and α' corrections (this corrected K^{CK} is the uncorrected $\tilde{K}^{\text{CS}}$ of the mirror manifold). Additionally, nonperturbative effects can modify W in such a way as to give it a dependence on the Kähler moduli of X . If these latter corrections are neglected, then derivatives of W with respect to Kähler moduli vanish and the potential (4.113) reduces to

$$V_{(W \text{ classical})} = e^{\mathcal{K}} \left(\text{Im}[\tau]^2 |D_\tau W|^2 + g^{i\bar{j}} D_i W \overline{D_{\bar{j}} W} + (\tilde{g}^{r\bar{s}} K_r^{\text{CK}} K_{\bar{s}}^{\text{CK}} - 3) |W|^2 \right) . \quad (4.115)$$

In this expression $g_{i\bar{j}}$ is the metric on the space of complex structures and $\tilde{g}_{r\bar{s}}$ is the metric on the space of complexified Kähler structures. The indices i, j run from 1 to $h^{2,1}(X)$, and r, s run from 1 to $h^{1,1}(X)$.

In supersymmetric configurations the superpotential W vanishes, which greatly simplifies the above expression for the scalar potential V , which must vanish⁶ in a vacuum configuration.

$$V = e^{\mathcal{K}} \left(\text{Im}[\tau]^2 |\partial_\tau W|^2 + g^{i\bar{j}} \partial_{\varphi^i} W \overline{\partial_{\varphi^{\bar{j}}} W} \right) . \quad (4.116)$$

⁶Note that V does not automatically vanish on a locus in moduli space where W vanishes, because V depends on derivatives of W in each moduli coordinate.

This potential is manifestly positive semidefinite, and thus the equations defining a supersymmetric flux vacuum, requiring that both W and V vanish, read:

$$W = 0, \quad \partial_\tau W = 0, \quad \partial_{\varphi^i} W = 0. \quad (4.117)$$

Depending on the vacuum expectation values of the field strengths F_3 and H_3 , these equations might be solved for the complex structure moduli φ^i as well as the value of the axiodilaton field. The Kähler moduli are unconstrained, although this ceases to be true when instanton corrections to the action are incorporated [105].

4.8 Corrections to the potential and a digression on moduli stabilisation

In what follows, we only consider the case where the superpotential is given by the classical formula (4.96), that is, we work in the approximation where we can safely ignore the non-perturbative stringy corrections. Incorporating these corrections to W could alter the space of supersymmetric flux vacua. In particular, the condition of vanishing W , which has a cohomological interpretation central to the modularity that we will discuss, only holds when the classical expression (4.96) for W is used. Additionally, these corrections give W a dependence on the Kähler moduli, and so additional derivative terms will appear in the potential.

The nonperturbative corrections to W that we are referring to are the same as those considered in [105], where it was argued that precisely these corrections allowed for a realisation of metastable de Sitter vacua. One possible source of these corrections is Euclidean D3-instantons, first discussed in [106]. Additionally, in some setups gluino condensation could occur on spacetime-filling D7-branes which gives a contribution to W . Neither of these will be discussed here.

In attempting to build realistic cosmological models from flux compactifications, one can expect to meet the moduli problem. Namely, string theory constructions typically come with a large number of massless scalars. This is phenomenologically undesirable because there is no evidence for the existence such massless scalars. In fact, since these scalars will be coupled to gravity, they would give rise to unobserved fifth-force effects [107, 108] that would, among other things, affect planetary orbits. Hopes of getting around this problem lie in finding suitable mechanisms of moduli stabilisation. This involves recognising nonperturbative corrections to the scalar potential, so that it acquires a steep local minimum in which the scalars will settle. When the scalars have taken this minimising value, the scalar potential term in the action becomes an effective cosmological constant. The observational fact that the universe is deSitter necessitates that this minimum be positive. There then remains the problem of giving a natural explanation for the cosmological constant's value being of the order 10^{-122} in Planck units.

We will not have anything to say about this problem. However, we will make a short digression now to discuss how the topology of the internal geometry X can affect the large volume behaviour of the potential for the Kähler moduli. We proceed from the scalar potential (4.115) that follows from the superpotential (4.96), which crucially does not depend on the Kähler moduli.

The square-bracketed quantities in the first line of (4.115) are positive definite, so V can be minimised by setting them to zero. This gives $h^{2,1} + 1$ equations for the $h^{2,1} + 1$ quantities of the axiodilaton τ and the full set of complex structure moduli φ . Let us assume that these equations all provide independent constraints, so that all complex structure moduli and the axiodilaton are fixed. Let these respectively be fixed to values τ_* and φ_* . What remains is

$$V(\tau_*, \varphi_*) = \frac{e^{C(\varphi_*)}}{\text{Im}[\tau_*]} |W_*|^2 \cdot e^K \left(K^{i\bar{j}} K_i K_{\bar{j}} - 3 \right) , \quad (4.118)$$

which depends on the as-yet-unfixed Kähler moduli $\mathbf{t}$ through $K(\mathbf{t})$.

We want to study the contraction

$$\mathcal{C} = K^{i\bar{j}} K_i K_j . \quad (4.119)$$

Recall that K can be computed from the prepotential $\mathcal{F}$ via

$$e^{-K} = -i \left(z^a \bar{\mathcal{F}}_a - \bar{z}^a \mathcal{F}_a \right) . \quad (4.120)$$

$\mathcal{F}$ depends on the projectivised Kähler moduli by

$$\mathcal{F} = -\frac{1}{3!} Y_{abc} \frac{z^a z^b z^c}{z^0} - (z^0)^2 \mathcal{I} \left(\frac{\mathbf{z}}{z^0} \right) , \quad \mathcal{I}(\mathbf{t}) = \frac{1}{(2\pi i)^3} \sum_{\mathbf{k} \in H_2(X, \mathbb{Z})} n_{\mathbf{k}} \text{Li}_3 \left(e^{2\pi i \mathbf{k} \cdot \mathbf{t}} \right) . \quad (4.121)$$

with a, b, c running from 0 to $h^{1,1}$, the above sum includes the perturbative worldsheet corrections to X 's quantum volume, in the terms with a zero index. Including or excluding the Y_{0ij} and Y_{00i} terms do not lead to any change in $\mathcal{C}$, a fact already well-appreciated in the supergravity literature (ultimately because Y_{0ij} and Y_{00i} are real). However the Y_{000} term does play heavily in our discussion, so we take a moment to fix a notation.

$$Y_{000} \equiv -3 \frac{\zeta(3)}{(2\pi i)^3} \chi(X) = 3 \frac{\zeta(3)i}{8\pi^3} \chi = 6i\zeta , \quad \text{where} \quad \zeta = \frac{\zeta(3)}{16\pi^3} \chi . \quad (4.122)$$

A salient point, ζ has the same sign as the Euler characteristic χ .

We shall write the real and imaginary parts of $\mathbf{t}$ as $\boldsymbol{\mu}$ and $\boldsymbol{\rho}$,

$$\mathbf{t} = \boldsymbol{\mu} + i\boldsymbol{\rho} . \quad (4.123)$$

We will denote $\partial_{t^i} \mathcal{I}$ and $\partial_{t^i} \partial_{t^j} \mathcal{I}$ by $\mathcal{I}_i$ and $\mathcal{I}_{ij}$. A computation gives

$$e^{-K} = 4 |z^0|^2 \left[\frac{1}{3} Y_{ijk} \rho^i \rho^j \rho^k - \rho^i \text{Re} [\mathcal{I}_i] + \text{Im} [\mathcal{I}] + \zeta \right] . \quad (4.124)$$

Differentiating this equation yields

$$\begin{aligned}
e^{-K} K_i &= 2 |z^0|^2 \left[i Y_{imn} \rho^m \rho^n + \rho^m \mathcal{I}_{im} - \text{Im} [\mathcal{I}_i] \right] , \\
e^{-K} K_{\bar{j}} &= -2 |z^0|^2 \left[i Y_{imn} \rho^m \rho^n - \rho^m \mathcal{I}_{im} + \text{Im} [\mathcal{I}_i] \right] , \\
e^{-K} K_{i\bar{j}} - e^{-K} K_i K_{\bar{j}} &= -2 |z^0|^2 \left[Y_{ijn} \rho^n + \text{Im} [\mathcal{I}_{ij}] \right] .
\end{aligned} \tag{4.125}$$

By introducing the matrix M with components

$$M_{ij} = Y_{ijn} \rho^n - i \mathcal{I}_{ij} \tag{4.126}$$

we can write

$$K_{i\bar{j}} = -e^K (M + \bar{M})_{ij} + 4e^{2K} \left[M_{in} \rho^n + i \text{Im} [\mathcal{I}_i] \right] \left[\bar{M}_{jm} \rho^m - i \text{Im} [\mathcal{I}_j] \right] . \tag{4.127}$$

Computing $\mathcal{C}$ necessitates inverting this matrix, so now is a good time to note an elementary formula. For an invertible symmetric matrix A and vectors $\mathbf{a}$, $\mathbf{b}$, the matrix B with components

$$B_{ij} = A_{ij} + a_i b_j \tag{4.128}$$

has inverse

$$B^{ij} = A^{ij} - \frac{1}{1 + \mathbf{a}^T A^{-1} \mathbf{b}} A^{im} A^{jn} a_m b_n . \tag{4.129}$$

Let us introduce yet more notation,

$$P_{ij} = \text{Re} [M_{ij}] , \quad v_i = M_{in} \rho^n + i \text{Im} [\mathcal{I}_i] . \tag{4.130}$$

We find

$$\mathcal{C} = \frac{-8 \mathbf{v}^T P^{-1} \bar{\mathbf{v}} + 64 e^K \left([\mathbf{v}^T P^{-1} \bar{\mathbf{v}}]^2 - [\mathbf{v}^T P^{-1} \mathbf{v}] [\bar{\mathbf{v}}^T P^{-1} \bar{\mathbf{v}}] \right)}{e^{-K} - 8 \mathbf{v}^T P^{-1} \bar{\mathbf{v}}} . \tag{4.131}$$

If all terms in the instanton expansion are set to zero, then this reduces to

$$\mathcal{C} = \frac{3Y_{ijk}\rho^i\rho^j\rho^k}{Y_{ijk}\rho^i\rho^j\rho^k - 6\zeta} . \quad (4.132)$$

If the Euler characteristic of the manifold is 0, then the above expression takes the constant value 3. Then (4.118) vanishes, recovering the standard no-scale supergravity potential, for instance appearing in [105].

However, if the Euler characteristic is not zero, then $\mathcal{C}$ is not constant and V is less trivial. In fact, V exhibits typical Dine-Seiberg behaviour [109]. The classical value of the manifold's volume is

$$v = Y_{ijk}\rho^i\rho^j\rho^k , \quad (4.133)$$

and for large v , $\mathcal{C}$ asymptotes to the value 3. The difference $\mathcal{C} - 3$ falls off like $\frac{1}{v}$. The asymptote is from above if $\chi > 0$, and from below if $\chi < 0$. In the first case, $\chi > 0$, a barrier prevents $\mathcal{C}$ from becoming smaller than v (in dimensionless variables) and the potential runs away to infinity. For $\chi < 0$, there is a runaway towards $v = 0$. Since this latter point is at small volume, we can expect that mechanisms not incorporated in our analysis become important and so we cannot say much at all about $\mathcal{C}$ near $v = 0$.

The denouement of this section is that equation (4.131) allows for one to study how instantons affect the previous paragraph's content. For $\chi \neq 0$ there is no substantial change, but the $\chi = 0$ scenario is altered.

If $\chi = 0$ but instantons are not neglected, then at large ρ the contraction $\mathcal{C}$ will asymptote to 3 from above or below depending on the sign of the instanton corrections. What is more, the asymptotic approach will be exponential, rather than the powerlike decay discussed before. This is because, if $\chi = 0$, the first nonclassical term in (4.121) that contributes to (4.131) is the first instanton contribution, which falls off exponentially as the Kähler parameters are increased.

None of that has direct relevance to any useful string models, because (among other prob-

lems) we have not stabilised any moduli. We have demonstrated how the values of χ and the leading instanton number can affect the large volume behaviour of this unrealistic model.

4.9 Modularity of supersymmetric vacua in flux compactifications

The central equations of this section are those giving the simultaneous vanishing of the potential V and superpotential W in a IIB flux compactification on a CY threefold X . We work with the classical superpotential, which has no dependence on the Kähler moduli. The potential V then has the form (4.116) which we repeat here.

$$W = \int_X G_3 \wedge \Omega = (2\pi)^2 \alpha' (F - \tau H) \Sigma \Pi \quad , \quad V = e^K \left(\text{Im}[\tau]^2 |\partial_\tau W|^2 + g^{i\bar{j}} \partial_{\varphi^i} W \overline{\partial_{\varphi^j} W} \right) . \quad (4.134)$$

The equations (4.117) satisfied by supersymmetric vacuum configurations, together with the consistency condition $F^T \Sigma H \neq 0$, can be recast as the SFV equations

$$F^T \Sigma \Pi = 0 \quad , \quad H^T \Sigma \Pi = 0 \quad , \quad F^T \Sigma H \neq 0 \quad , \quad (F - \tau H) \Sigma (\partial_{\varphi^i} \Pi) = 0 . \quad (4.135)$$

It is unknown if, for a given manifold X , there is always a pair of flux vectors F and H so that within X 's moduli space exists a region where the above equations are satisfied. In [12] it was conjectured by Kachru, Nally, and Yang that such manifolds, with their moduli on the SFV locus and valued in an algebraic number field, are weight-two modular. They were able to test this conjecture by appealing to two sources of information. One of these was a solution method produced by deWolfe [110] for solving the SFV equations, valid for X the mirror of a hypersurface in weighted projective space. The other key source of information was the thesis [51] of Kadir, in which tables of zeta functions were given, with various choices of moduli, for the mirror of the octic hypersurface in $\mathbb{WP}_{1,1,2,2,2}$. Kachru, Nally, and Yang supported their conjecture by reporting that on deWolfe's solution locus, the tables of Kadir

demonstrated weight-two modularity.

We seek to make further tests of this conjecture. This requires more solutions to the SFV equations and more computations of zeta functions. Concerning the first requirement, we determine a method of solving the SFV equations that works on any manifold X whose period vector possesses a $\mathbb{Z}_2$ symmetry. In the thesis [111], the method of [50] was extended to multiparameter models which, using the solution method that we now describe, allowed for the flux modularity conjecture to be extensively tested, with additional tests carried out in [3].

4.9.1 Solving the SFV equations via permutation symmetry

It is required that the period vector Π has two pairs of entries such that each pair is permuted when two of the moduli are swapped. That is to say, we should be able to identify a pair of moduli φ^I, φ^J so that

$$\begin{aligned}\Pi^{1+a}(\varphi^I, \varphi^J) &= \Pi^{1+b}(\varphi^J, \varphi^I) , \\ \Pi^{2+h^{2,1}+a}(\varphi^I, \varphi^J) &= \Pi^{2+h^{2,1}+b}(\varphi^J, \varphi^I) ,\end{aligned}\tag{4.136}$$

where we have suppressed Π 's dependence on the other moduli. The superscripts indicate components of the vector Π , with $1 \leq a, b \leq h^{2,1}$.

The dependence of the other components of Π on any remaining moduli is unimportant. If Π has this property, then on the locus $\varphi^I = \varphi^J$ flux vectors F and H satisfying the first three equations in (4.135) are

$$F = e_{2+h^{2,1}+I} - e_{2+h^{2,1}+J} , \quad H = e_{1+I} - e_{1+J} ,\tag{4.137}$$

where the e_k are standard orthonormal basis vectors for $\mathbb{R}^{2+2h^{2,1}}$, with all components 0 except for the k^{th} component being 1.

With the flux vectors and this restriction on the moduli space $\varphi^I = \varphi^J$ specified, all but two of the SFV equations are satisfied for any value of the axiodilaton. The remaining two

equations are equivalent and fix the axiodilaton:

$$\begin{aligned}
\tau(\varphi) &= - \frac{F^T \Sigma (\partial_{\varphi^I} - \partial_{\varphi^J}) \Pi}{H^T \Sigma (\partial_{\varphi^I} - \partial_{\varphi^J}) \Pi} \Big|_{\varphi^I = \varphi^J} \\
&= \frac{i}{2\pi} \cdot \frac{\partial_{\varphi^I} (\varpi_{2,I} - \varpi_{2,J})}{\partial_{\varphi^I} (\varpi_{1,I} - \varpi_{1,J})} \Big|_{\varphi^I = \varphi^J} + Y_{0IJ} - Y_{0II} .
\end{aligned} \tag{4.138}$$

It is perhaps interesting to note that for real moduli in the large complex structure region, the real part of the axiodilaton is 0 or 1/2 depending on the value of $Y_{0IJ} - Y_{0II}$.

We will now give a select few worked examples. Many more are possible to find than those given here.

4.9.2 Examples of supersymmetric flux vacua

The Hulek-Verrill Manifold

The first example of such a suitable symmetric manifold is the Hulek-Verrill manifold, which is mirror to the CICY

$$\begin{array}{c}
\mathbb{P}^1 \left[\begin{array}{cc} 1 & 1 \end{array} \right]^{(5,45)} \\
\mathbb{P}^1 \left[\begin{array}{cc} 1 & 1 \end{array} \right] \\
\mathbb{P}^1 \left[\begin{array}{cc} 1 & 1 \end{array} \right] \\
\mathbb{P}^1 \left[\begin{array}{cc} 1 & 1 \end{array} \right] \\
\mathbb{P}^1 \left[\begin{array}{cc} 1 & 1 \end{array} \right]_{\chi=-80}
\end{array} \tag{4.139}$$

and has the following topological quantities:

$$Y_{ijk} = \begin{cases} 2 & i, j, k \text{ distinct.} \\ 0 & \text{otherwise,} \end{cases} \quad Y_{ij0} = 0, \quad Y_{i00} = -2, \quad Y_{000} = 240 \frac{\zeta(3)}{(2\pi i)^3} . \tag{4.140}$$

Analytic expressions for the periods are available in terms of integrals of Bessel functions, as detailed in [2]. Subject to the condition

$$\operatorname{Re} \left[\sum_{i=1}^5 \sqrt{\varphi^i} \right] < 1 , \quad (4.141)$$

we can give the relevant periods as

$$\begin{aligned} \varpi_0(\varphi) &= \int_0^\infty dz \, z K_0(z) \prod_{i=1}^5 I_0 \left(\sqrt{\varphi^i} z \right) , \\ \varpi_{1,j}(\varphi) &= -2 \int_0^\infty dz \, z K_0(z) K_0 \left(\sqrt{\varphi^j} z \right) \prod_{i \neq j} I_0 \left(\sqrt{\varphi^i} z \right) , \\ \varpi_{2,j}(\varphi) &= 8 \sum_{\substack{m < n \\ m, n \neq j}} \int_0^\infty dz \, z K_0(z) K_0 \left(\sqrt{\varphi^m} z \right) K_0 \left(\sqrt{\varphi^n} z \right) \prod_{i \neq m, n} I_0 \left(\sqrt{\varphi^i} z \right) - 4\pi^2 \varpi_0(\varphi) . \end{aligned} \quad (4.142)$$

The five-parameter Hulek-Verrill manifolds support supersymmetric flux vacua on the loci where any two complex structure moduli φ^i are equal. For definiteness, let us set two of the φ^1 and φ^2 equal to a value φ , and relabel the three remaining φ^i as ψ^1, ψ^2, ψ^3 . Use of standard Bessel function identities reveals that in the case of Hulek-Verrill manifolds, the axiodilaton given in (4.138) is

$$\tau(\psi^1, \psi^2, \psi^3) = \frac{2i}{\pi} \frac{\int_0^\infty dz \, z K_0(z) \left[K_0 \left(\sqrt{\psi^1} z \right) I_0 \left(\sqrt{\psi^2} z \right) I_0 \left(\sqrt{\psi^3} z \right) + \text{cyclic} \right]}{\int_0^\infty dz \, z K_0(z) I_0 \left(\sqrt{\psi^1} z \right) I_0 \left(\sqrt{\psi^2} z \right) I_0 \left(\sqrt{\psi^3} z \right)} . \quad (4.143)$$

The dependence of the j -invariant can be found numerically by computing the value of $j(\tau)$ on numerous points on the moduli space, and fitting the points on a rational function. It turns out that the j -function takes a remarkably simple form

$$j(\tau(\psi^1, \psi^2, \psi^3)) = \frac{(\Delta_F + 16\psi^1\psi^2\psi^3)^3}{\Delta_F(\psi^1\psi^2\psi^3)^2} , \quad (4.144)$$

where the polynomial Δ_F , which is related to the discriminant (3.25), is given by

$$\begin{aligned}\Delta_F &= \prod_{\epsilon_i=\pm 1} \left(1 + \epsilon_1 \sqrt{\psi^1} + \epsilon_2 \sqrt{\psi^2} + \epsilon_3 \sqrt{\psi^3}\right) \\ &= \left((1 - \psi^1 - \psi^2 - \psi^3)^2 - 4 (\psi^1 \psi^2 + \psi^2 \psi^3 + \psi^3 \psi^1) \right)^2 - 64 \psi^1 \psi^2 \psi^3 .\end{aligned}\tag{4.145}$$

Note that to derive (4.143) and then subsequently arrive at (4.144), we have used the expressions (4.142) for the periods, which are valid only in the region (4.141). However, since the expression (4.144) for the j -function is well-defined everywhere outside of the discriminant locus, this is the unique analytic continuation of the left-hand side into this region and the expression (4.144) is correct throughout moduli space.

It is interesting to note that (4.143) and (4.144) do not involve the complex structure coordinate φ , and only depend on the three coordinates ψ^1, ψ^2, ψ^3 . In other examples, as we will see in imminent subsections, τ does depend on the value to which we set two of the coordinates in our solution method. For this particular example at hand, the elimination of φ from these formula was explained geometrically in [3] in terms of the fibred product structure of the Hulek-Verrill manifold. Understanding why this φ independence does or does not occur for other examples is an open problem.

The mirror bicubic

This manifold is mirror to the CICY

$$\mathbb{P}^2 \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{\chi=-162}, \tag{4.146}$$

with the constants Y_{abc} given by

$$Y_{ijk} = \begin{cases} 0 & i = j = k , \\ 3 & \text{otherwise,} \end{cases} \quad Y_{ij0} = \begin{cases} 0 & i = j , \\ \frac{1}{2} & i \neq j , \end{cases} \quad Y_{i00} = -3 , \quad Y_{000} = 486 \frac{\zeta(3)}{(2\pi i)^3} . \tag{4.147}$$

The periods can be written in terms of integrals of hypergeometric and Meijer G-functions:

$$\begin{aligned}
\varpi_0 &= \int_0^\infty du e^{-u} {}_0F_2(1, 1; u^3\varphi^1) {}_0F_2(1, 1; u^3\varphi^2) , \\
\varpi_{1,i} &= - \int_0^\infty du e^{-u} G_{0,3}^{2,0}(0, 0, 0|u^3\varphi^i) {}_0F_2(1, 1; u^3\varphi^j) , \quad j \neq i , \\
\varpi_{2,i} &= 3 \int_0^\infty du e^{-u} \left[G_{0,3}^{2,0}(0, 0, 0|u^3\varphi^i) G_{0,3}^{2,0}(0, 0, 0|u^3\varphi^j) \right. \\
&\quad \left. + {}_0F_2(1, 1; u^3\varphi^i) (G_{0,3}^{3,0}(0, 0, 0|-u^3\varphi^j) + i\pi G_{0,3}^{2,0}(0, 0, 0|u^3\varphi^j)) \right] \\
&\quad - 2\pi^2 \varpi_0 , \quad j \neq i ,
\end{aligned} \tag{4.148}$$

whence it follows that the axiodilaton profile on the $\mathbb{Z}_2$ symmetric locus $\varphi^1 = \varphi^2$ is given by

$$\tau(\varphi) = \frac{3i}{2\pi} \frac{\int_0^\infty du e^{-u} [{}_0F_2(1, 1; u^3\varphi) G_{0,3}^{3,0}(0, 0, 1|-u^3\varphi) - {}_0F_2(2, 2; u^3\varphi) G_{0,3}^{3,0}(1, 1, 1|-u^3\varphi)]}{\int_0^\infty du e^{-u} [{}_0F_2(1, 1; u^3\varphi) G_{0,3}^{2,0}(0, 1, 0|u^3\varphi) + {}_0F_2(2, 2; u^3\varphi) G_{0,3}^{2,0}(1, 1, 1|u^3\varphi)]} - 1 .$$

Even though it is not immediately obvious, this expression has, for real φ , a real part equal to $\frac{1}{2}$. One should bear in mind that we have further simplified the ratio of integrals to arrive at the above expression.

Numerical methods strongly suggest that the j -invariant $j(\tau(\varphi))$ is given by the following rational function of the moduli:

$$j(\tau(\varphi)) = -\frac{(1 + 24\varphi)^3}{\varphi^3(1 + 27\varphi)} . \tag{4.149}$$

The mirror maximally split quintic

This manifold is the mirror of the maximal split of the quintic hypersurface in $\mathbb{P}^5$, given by the configuration described by the CICY matrix

$$\begin{array}{c} \mathbb{P}^4 \\ \mathbb{P}^4 \end{array} \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]_{\chi=-100} . \tag{4.150}$$

The quantities Y_{abc} of the split quintic are

$$Y_{ijk} = \begin{cases} 5 & i = j = k, \\ 10 & \text{otherwise,} \end{cases} \quad Y_{0ij} = \begin{cases} \frac{1}{2} & i = j, \\ 0 & i \neq j, \end{cases} \quad Y_{i00} = -\frac{25}{6}, \quad Y_{000} = \frac{300\zeta(3)}{(2\pi i)^3}. \quad (4.151)$$

Denoting the Meijer G-functions as $G_{c,d}^{a,b}(0,0,0,0,0|x) = G_{c,d}^{a,b}(x)$, and the hypergeometric functions as ${}_aF_b(1,1,1,1;x) = {}_aF_b(x)$, the periods relevant to our analysis can be written as

$$\begin{aligned} \varpi_0 &= \int_0^\infty du \, G_{0,5}^{5,0}(u) {}_0F_4(u\varphi^1) {}_0F_4(u\varphi^2), \\ \varpi_{1,i} &= - \int_0^\infty du \, G_{0,5}^{5,0}(u) G_{0,5}^{2,0}(u\varphi^i) {}_0F_4(u\varphi^j), \quad i \neq j, \\ \varpi_{2,i} &= 5 \int_0^\infty du \, G_{0,5}^{5,0}(u) [2G_{0,5}^{2,0}(u\varphi^i) G_{0,5}^{2,0}(u\varphi^j) + (G_{0,5}^{3,0}(-u\varphi^i) + i\pi G_{0,5}^{2,0}(u\varphi^i)) {}_0F_4(u\varphi^2) \\ &\quad + 2 (G_{0,5}^{3,0}(-u\varphi^j) + i\pi G_{0,5}^{2,0}(u\varphi^j)) {}_0F_4(u\varphi^i) - \frac{5\pi^2}{3} {}_0F_4(u\varphi^i) {}_0F_4(u\varphi^j)] , \quad i \neq j, \end{aligned} \quad (4.152)$$

and the axiodilaton is given on the $\mathbb{Z}_2$ symmetric locus $\varphi^1 = \varphi^2 = \varphi$ by

$$\begin{aligned} \tau(\varphi) &= \frac{5i}{2\pi} \frac{\int_0^\infty du \, G_{0,5}^{5,0}(u) [G_{0,5}^{3,0}(0,0,1,0,0|-u\varphi) {}_0F_4(u\varphi) - G_{0,5}^{3,0}(1,1,1,1,1|-u\varphi) {}_0F_4(2,2,2,2;u\varphi)]}{\int_0^\infty du \, G_{0,5}^{5,0}(u) [G_{0,5}^{2,0}(0,1,0,0,0|u\varphi) {}_0F_4(u\varphi) - G_{0,5}^{2,0}(1,1,1,1,1|u\varphi) {}_0F_4(2,2,2,2;u\varphi)]} \\ &\quad + 2. \end{aligned} \quad (4.153)$$

For real φ , this expression has real part equal to $-\frac{1}{2}$. Attempts to numerically integrate the above combinations of Meijer G functions are met with problems in Mathematica, so instead it is best to expand each of the hypergeometric functions ${}_0F_4$ as a power series in u up to some large order (we used 360). After interchanging the order of summation and integration, each integral in the resulting sum can be evaluated in Mathematica exactly as a single Meijer G function, yielding a series expression amenable to fast evaluation.

The numerical evidence strongly suggests that the j -invariant $j(\tau)$ is again a rational function

of the complex structure parameter φ ,

$$j(\tau(\varphi)) = -\frac{(1 + 12\varphi + 14\varphi^2 - 12\varphi^3 + \varphi^4)^3}{\varphi^5(1 + 11\varphi - \varphi^2)} . \quad (4.154)$$

4.9.3 Speculation on elliptic curves inside the F-theory fourfold

Consider again the F-theory fourfold, as discussed in §4.6. This is an elliptic fibration, and for the supersymmetric vacuum configurations which we have described, the fibre is the elliptic curve E_τ with τ the axiodilaton (4.138) of our solution to the supersymmetric flux vacuum equations.

The base of this fibration, $\mathcal{B}$, is doubly covered by the threefold X , which we have argued is modular of weight two. In line with the conjectures of Kachru, Nally, and Yang in [11, 12], the weight-two modular form associated to X is itself associated to the elliptic curve E_τ in the sense of the modularity theorem for elliptic curves.

For the case of the Hulek-Verrill family, it was explained in [3] that the threefold X contained a ruled surface $E_\tau \times \mathbb{P}^1$, which was nontrivial in homology. This gave an account of why the threefold's zeta function numerator contained a factor corresponding to the elliptic curve E_τ .

For this Hulek-Verrill example then, the F-theory fourfold that appears in the uplift of the supersymmetric flux vacuum that we describe contains E_τ in two different ways: once as the fibre and again in (a double cover) of the base. It could be interesting to understand whether this happens for every example, and more speculatively, to see if any string duality explains or utilises this twofold appearance of E_τ in the F-theory fourfold.

4.10 New weight four modular manifolds

We have seen in §4.2 that the attractor mechanism fixes the values taken by vector multiplet scalar fields on the horizon of an extremal black hole in $\mathcal{N} = 2$ supergravity. In IIB Calabi-Yau compactifications those scalars give the complex structure moduli of the Calabi-Yau X , and the fixed points of attractor flows are the solutions of Strominger’s equations

$$Q = \text{Im} [C\Pi] \quad . \quad (4.155)$$

In this section we will describe a method, first worked through in [14] and later fully developed in [50], for finding rank-two attractors. These are complex structure moduli φ_* that solve the above equation (4.155) for two independent charge vectors Q_1, Q_2 . Tables that display the main piece of information that we use, factorisation counts, are collected in §B. Our search returns a number of known examples, but also some new weight-four manifolds.

We will search among a subset of the one-parameter manifolds. These possess fourth-order Picard Fuchs operators which are of Calabi-Yau type [112, 113]. Such operators are tabulated in the database [65], which is attached to the paper [112]. We shall refer to geometries by the AESZ label of their operator. Note that in the one-parameter setting, rank-two attractors are also solutions to the supersymmetric flux vacuum equations, and indeed all examples of weight-four modular one-parameter manifolds are also weight-two modular.

The search process

If a rational number φ_* is such that X_{φ_*} is weight-four modular, then the numerator of the zeta function $\zeta_p(X_{\varphi_*}; T)$ should factorise for each prime p . Turning this on its head, we will search for rank-two attractors by first tabulating the zeta function for $\varphi^p \in \{0, \dots, p-1\}$, for a number of primes p . Fixing a prime p , we will count the number of times the zeta function factorises for $\varphi^p \in \{0, \dots, p-1\}$. If this number is ‘usually’ greater than zero (with

exceptions expected to correspond to a small number of bad primes), then we inspect the zeta function numerators for the φ_*^p that have factorised numerators and search the LMFDB database [40] for a weight-four modular form whose p^{th} Fourier coefficient is that read off of the zeta function numerator, to the extent of our tables.

If such a modular form can be found, then we seek a rational φ_* that reduces modulo p to φ_*^p . Having found such a value φ_* , we can numerically check to see if a pair of rational charge vectors Q_1, Q_2 can be found so that (4.155) is solved by $\Pi(\varphi_*)$.

We perform this check for 61 of the operators on the AESZ list. In Appendix §B we give bar charts showing the number of factorisations for each prime $5 \leq p \leq 131$, for a total of 30 primes. To spot a rational rank-two attractor, we scroll down this list and stop where the bars all have heights greater than 0. In this way we recover a number of known examples: AESZ34 [14], and the pair AESZ4 and AESZ11 [13], as well as a number of examples identified previously in [47]. We will highlight those examples in Appendix §B, and also comment on this choice of 61 operators.

Moreover, we are also able to identify two hitherto-undiscovered attractors. These are in the moduli spaces of AESZ17 and AESZ22, and both at $\varphi_* = -1$.

The operators AESZ17 and AESZ22 possess a property not shared by any other operators for which rank-two attractors are known: 17 and 22 both possess two MUM points, at 0 and infinity. By a change of variables that exchanges the MUM points, the operators AESZ17 and AESZ22 can be respectively transformed to AESZ118 and AESZ290. Indeed, the charts in §B for the pair (22,118) are identical. We give the associated modular forms in the following table.

AESZ no.	Attractor	weight 2 form	weight 4 form
17	-1	14.2.a.a	14.4.a.b
22	-1	11.2.a.a	33.4.a.b

Table 4.1: New rank two attractors.

One should bear in mind that these are only conjectured to be modular varieties, and that conjecture is supported by tabulating the zeta function for finitely many primes.

We will say more about these geometries in §4.11, and wrap up this section with the first few terms of the above modular form's q -expansions, with $q = e^{2\pi i\tau}$:

$$\begin{aligned} f_{14.2.a.a}(\tau) &= q - q^2 - 2q^3 + q^4 + 2q^6 + q^7 - q^8 + q^9 + \dots, \\ f_{14.4.a.b}(\tau) &= q + 2q^2 - 2q^3 + 4q^4 - 12q^5 - 4q^6 + 7q^7 + 8q^8 - 23q^9 + \dots, \end{aligned} \tag{4.156}$$

$$\begin{aligned} f_{11.2.a.a}(\tau) &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + \dots, \\ f_{33.4.a.b}(\tau) &= q - q^2 - 3q^3 - 7q^4 - 4q^5 + 3q^6 - 26q^7 + 15q^8 + 9q^9 + \dots. \end{aligned}$$

LMFDB [40] provides the following eta-function expressions for the weight-two forms:

$$f_{14.2.a.a}(\tau) = \eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau), \quad f_{11.2.a.a}(\tau) = \eta(\tau)^2\eta(11\tau)^2, \tag{4.157}$$

$$\text{with} \quad \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) .$$

4.11 Summation identities: L-value ratios from GW invariants

Note that all summation identities in this section are at the time of writing conjectures, supported to the various degrees of numerical precision that we give.

In this section, we will specialise the series solution (4.85) to the orthogonality equations to a selection of rank-two attractors. Following [13, 14], we can express the Frobenius periods as linear combinations of the two critical L-values associated to a weight-four modular form. Therefore, the mirror coordinate t is a rational function of those L-values. This value t is a solution to the instanton-corrected IIA attractor equations, and we explained in §4.4 how

such a t necessarily solved the orthogonality equations. By expressing t via the solution (4.85), we arrive at a series identity. Certain combinations of L-values can be expressed as infinite sums whose terms are built out of Gromov-Witten invariants.

In practice, we evaluate no more than the first 75 terms of the series, because the j^{th} term contains a sum over partitions of j . Beyond $j = 75$ there are too many partitions to work with on a laptop.

These sums converge for small enough values of y_0 , and in those cases we get an honest numerical equality. Shanks transformations [114] can be applied to the series in order to improve the number of figures to which our identities hold. However, for a number of rank-two attractors the appropriate value of y_0 is too large and the sums formally diverge.

Building on the work of [4], where such divergences were not further addressed, we identify Padé resummation as the appropriate scheme for summing these series for larger values of y_0 . We shall write

$$\text{LHS} \stackrel{\text{P}}{=} \sum_{j=1}^{\infty} e^{2\pi i x_0 j} c(j, y_0) \quad (4.158)$$

to indicate that while the series on the right hand side diverges, the Padé approximants about 0 in the variable X to the series $\sum_{j=1}^{\infty} X^j c(j, y_0)$ approach (as the order of the approximant is increased) the value on the LHS when the substitution $X = e^{2\pi i x_0}$ is made.

Our examples will feature rank two attractors found in [13, 14] for the operators AESZ34, AESZ4, and AESZ11, as well as the new attractors found in §4.10 for AESZ22 and AESZ17.

Note that when we give a χ among a set of topological data at the start of a subsection, we do so for the B-model geometry.

AESZ4, The mirror of $\mathbb{P}^5[3, 3]$

The Picard-Fuchs equation for this geometry is the following hypergeometric equation:

$$\left[\theta^4 - 729 z \left(\theta + \frac{1}{3} \right)^2 \left(\theta + \frac{2}{3} \right)^2 \right] F(z) = 0 . \quad (4.159)$$

The topological data is

$$Y_{111} = 9 , \quad Y_{110} = \frac{1}{2} , \quad Y_{100} = -\frac{9}{2} , \quad \chi = 144 . \quad (4.160)$$

It was shown in [13] that this geometry has a rank two attractor at $z = -2^{-3} \cdot 3^{-6}$. The Frobenius periods were shown to be given by the following combinations of critical L-values associated to the weight-4 modular form with LMFDB label 54.4.a.c :

$$\begin{aligned} \widehat{\omega}_0 &= -18 \frac{L(2)}{(2\pi i)^2} , \\ \widehat{\omega}_1 &= -9 \frac{L(2)}{(2\pi i)^2} - \frac{9}{4} \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_2 &= 135 \frac{L(2)}{(2\pi i)^2} - \frac{81}{8} \frac{L(1)}{(2\pi i)} , \\ \widehat{\omega}_3 &= \frac{297}{4} \frac{L(2)}{(2\pi i)^2} + \frac{81}{16} \frac{L(1)}{2\pi i} + \frac{\chi \zeta(3)}{(2\pi i)^3} \widehat{\omega}_0 . \end{aligned} \quad (4.161)$$

These critical L-values have decimal expansions

$$\begin{aligned} L(1) &= 3.84405587339769322173023 \dots , \\ L(2) &= 2.18020008917513916885513 \dots . \end{aligned} \quad (4.162)$$

In the t -plane, the attractor is

$$t = \frac{1}{2} + \frac{i\pi}{4} \frac{L(1)}{L(2)} . \quad (4.163)$$

The attractor equations are solved by $z = -2^{-3} \cdot 3^{-6}$ for the following charge vectors:

$$Q_1 = (0, -5, 0, 1)^T, \quad Q_2 = (-3, 12, 1, 0)^T. \quad (4.164)$$

It follows that the integral basis period vector Π is symplectic-orthogonal to the charge vectors

$$Q_3 = (-3, 0, 1, 0)^T, \quad Q_4 = (12, -5, 0, 1)^T. \quad (4.165)$$

Q_4 is of ‘D4-type’, as the entry corresponding to the D6 charge is zero. This means that we can form an identity as in [4]. The perturbative solution of the orthogonality equation is given by

$$t_0 = \frac{1}{2} + \frac{i\sqrt{69}}{6}. \quad (4.166)$$

We arrive at

$$\frac{3\pi}{2} \cdot \frac{L(1)}{L(2)} = \sqrt{69} - \sqrt{\frac{2}{\pi^3}} \sum_{j=1}^{\infty} (-1)^j \sum_{\mathfrak{p} \in \text{pt}(j)} a_{\mathfrak{p}} N^{\mathfrak{p}} \left(\frac{j}{3\pi\sqrt{69}} \right)^{l(\mathfrak{p})-1/2} K_{l(\mathfrak{p})-1/2} \left(\frac{j\pi\sqrt{69}}{3} \right). \quad (4.167)$$

This sum converges, and there is no need for Padé resummation. The equality should be understood as usual. Using 60 terms of the series, we observe agreement to 73 figures. By performing a Shanks transformation up to 15 times, we observe improvement to 105 figures. Applying further Shanks transformations does not yield further improvement.

AESZ11, The mirror of $\mathbb{WP}_{1,1,1,1,1,2}^5[4, 3]$

This geometry also has a hypergeometric Picard-Fuchs equation:

$$\left[\theta^4 - 1728z \left(\theta + \frac{1}{4} \right) \left(\theta + \frac{1}{3} \right) \left(\theta + \frac{2}{3} \right) \left(\theta + \frac{3}{4} \right) \right] F(z) = 0. \quad (4.168)$$

The topological data is

$$Y_{111} = 6 , \quad Y_{110} = 0 , \quad Y_{100} = -4 , \quad \chi = 156 . \quad (4.169)$$

It was shown in [13] that this geometry has a rank two attractor at $z = -2^{-4} \cdot 3^{-3}$. The Frobenius periods were again shown to be given by combinations of critical L-values , this time associated to the weight-4 modular form with LMFDB label 180.4.a.e :

$$\begin{aligned} \widehat{\omega}_0 &= -18 \frac{L(2)}{(2\pi i)^2} , \\ \widehat{\omega}_1 &= -9 \frac{L(2)}{(2\pi i)^2} - \frac{6}{10} \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_2 &= 45 \frac{L(2)}{(2\pi i)^2} - \frac{9}{5} \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_3 &= 27 \frac{L(2)}{(2\pi i)^2} + \frac{3}{5} \frac{L(1)}{2\pi i} + \frac{\chi \zeta(3)}{(2\pi i)^3} \widehat{\omega}_0 . \end{aligned} \quad (4.170)$$

These critical L-values have decimal expansions

$$\begin{aligned} L(1) &= 10.01558944150908412493925 \dots , \\ L(2) &= 1.99897335900796011552100 \dots . \end{aligned} \quad (4.171)$$

In the t -plane, the attractor this time is

$$t = \frac{1}{2} + \frac{i\pi}{15} \cdot \frac{L(1)}{L(2)} . \quad (4.172)$$

The value $z = -2^{-4} \cdot 3^{-3}$ solves the attractor equations for the following charge vectors:

$$Q_1 = (-1, 6, 1, 0)^T, \quad Q_2 = (1, -3, 0, 1)^T . \quad (4.173)$$

One computes that the integral basis period vector Π is symplectic-orthogonal to the charge

vectors

$$Q_3 = (-1, 1, 1, 0)^T, \quad Q_4 = (6, -3, 0, 1)^T. \quad (4.174)$$

Q_4 is of D4-type, so we can form an identity. The perturbative solution is given by

$$t_0 = \frac{1}{2} + \frac{i\sqrt{39}}{6}. \quad (4.175)$$

We thereby find

$$\frac{2\pi}{5} \cdot \frac{L(1)}{L(2)} \stackrel{\mathbb{P}}{=} \sqrt{39} - \sqrt{\frac{3}{\pi^3}} \sum_{j=1}^{\infty} (-1)^j \sum_{\mathfrak{p} \in \text{pt}(j)} a_{\mathfrak{p}} N^{\mathfrak{p}} \left(\frac{j}{2\pi\sqrt{39}} \right)^{l(\mathfrak{p})-1/2} K_{l(\mathfrak{p})-1/2} \left(\frac{j\pi\sqrt{39}}{3} \right). \quad (4.176)$$

This sum does not converge. As discussed in the overview, the identity is supported numerically by Padé resummation. A diagonal Padé approximant of order 30 gives agreement to 38 figures.

AESZ34, A quotient of the Hulek-Verrill manifold

Three rank two attractor points were found for this geometry in [14]. One of these, at $z = 33 - 8\sqrt{17}$, lies inside the LCS region and was a central example in [4], where the first sum of the form considered here was given. We discuss now the other two attractor points, at $z = -7^{-1}$ and $z = 33 + 8\sqrt{17}$, which lie outside of the LCS region and so require Padé resummation.

The Picard-Fuchs equation is not hypergeometric, and is

$$\begin{aligned} & [(1-z)(1-9z)(1-25z)\theta^4 + 2z(675z^2 - 518z + 35)\theta^3 + z(2925z^2 - 1580z + 63)\theta^2 \\ & + 4z(675z^2 - 272z + 7)\theta + 5z(180z^2 - 57z + 1)] F(z) = 0. \end{aligned} \quad (4.177)$$

The topological data is

$$Y_{111} = 24 , \quad Y_{110} = 0 , \quad Y_{100} = -2 , \quad \chi = 16 . \quad (4.178)$$

We will divide the remainder of this section into subsections discussing each rank two attractor separately. We choose to work with the ‘ $\kappa = 2$ ’ geometry of [14], which has triple intersection number $Y_{111} = 24$. The choice does not affect the summation identities, as all factors of κ drop out.

$$z = -\frac{1}{7}$$

The periods were conjectured (with 1000 digits of numerical precision supporting) to be given in terms of critical L-values for the modular form 14.4.a.a .

$$\begin{aligned} \widehat{\omega}_0 &= -28 \frac{L(2)}{(2\pi i)^2} , \\ \widehat{\omega}_1 &= -14 \frac{L(2)}{(2\pi i)^2} - \frac{5}{2} \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_2 &= -28 \frac{L(2)}{(2\pi i)^2} - 30 \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_3 &= 14 \frac{L(2)}{(2\pi i)^2} - \frac{11}{2} \frac{L(1)}{2\pi i} + \frac{\chi \zeta(3)}{(2\pi i)^3} \widehat{\omega}_0 . \end{aligned} \quad (4.179)$$

The decimal expansions of these L-values begin

$$\begin{aligned} L(1) &= 0.67496319716994177129270 \dots , \\ L(2) &= 0.91930674266912115653914 \dots . \end{aligned} \quad (4.180)$$

The attractor is, in the t -plane,

$$t = \frac{1}{2} + \frac{5\pi i}{28} \cdot \frac{L(1)}{L(2)} . \quad (4.181)$$

Independent charge vectors for which the attractor equations are solved are

$$Q_1 = \left(-\frac{8}{5}, 6, 1, 0\right)^T, \quad Q_2 = \left(\frac{16}{5}, -12, 0, 1\right)^T. \quad (4.182)$$

The integral symplectic period vector is then orthogonal to

$$Q_3 = \left(-\frac{8}{5}, \frac{16}{5}, 1, 0\right)^T, \quad Q_4 = (6, -12, 0, 1)^T. \quad (4.183)$$

We shall form our identity using Q_4 . The perturbative solution is

$$t_0 = \frac{1}{2} + \frac{i}{\sqrt{6}}. \quad (4.184)$$

From these considerations, we can find the identity

$$\frac{15\pi}{14} \cdot \frac{L(1)}{L(2)} \stackrel{P}{=} \sqrt{6} - \frac{1}{2} \cdot \sqrt{\frac{3}{\pi^3}} \sum_{j=1}^{\infty} (-1)^j \sum_{\mathfrak{p} \in \text{pt}(j)} a_{\mathfrak{p}} N^{\mathfrak{p}} \left(\frac{j}{8\pi\sqrt{6}} \right)^{l(\mathfrak{p})-1/2} K_{l(\mathfrak{p})-1/2} \left(\frac{\pi j \sqrt{6}}{3} \right). \quad (4.185)$$

A diagonal Padé approximant of order 30 gives agreement to 34 figures.

$$z = 33 + 8\sqrt{17}$$

This constitutes perhaps our strangest example. The real parts of the critical L-values associated to 34.4.b.a have decimal expansions beginning

$$\begin{aligned} \lambda(1) &= 0.61300748403501690756896 \dots, \\ \lambda(2) &= 0.72053904959503349611019 \dots. \end{aligned} \quad (4.186)$$

The complete L-values are

$$\begin{aligned} L(1) &= \lambda(1) \left(1 + i \left(\frac{1 + \sqrt{17}}{4} \right)^3 \right) , \\ L(2) &= \lambda(2) \left(1 - i \frac{1 - \sqrt{17}}{4} \right) . \end{aligned} \quad (4.187)$$

Introduce symbols for the following numbers:

$$\epsilon_- = 4 - \sqrt{17} , \quad \delta_+ = \frac{3 + \sqrt{17}}{2} , \quad \delta_- = \frac{3 - \sqrt{17}}{2} . \quad (4.188)$$

In terms of the real parts $\lambda(1)$, $\lambda(2)$ of the L-values, the periods as computed in [14] are (again, conjecturally with 1000 digits of precision found numerically by those authors)

$$\begin{aligned} \widehat{\omega}_0 &= -\frac{7\sqrt{17}\epsilon_-^2\delta_+}{2^3\pi^2}\lambda(2) - i\frac{3 \cdot 5\delta_-^3}{2^4\sqrt{17}\pi}\lambda(1) , \\ \widehat{\omega}_1 &= -\frac{\sqrt{17}\epsilon_-^2\delta_+}{2\pi^2}\lambda(2) - i\frac{5\delta_-^3}{2^5\sqrt{17}\pi}\lambda(1) , \\ \widehat{\omega}_2 &= -\frac{5^2\sqrt{17}\epsilon_-^2\delta_+}{2^3\pi^2}\lambda(2) + i\frac{3 \cdot 5\delta_-^3}{2^4\sqrt{17}\pi}\lambda(1) , \\ \widehat{\omega}_3 &= -\frac{\sqrt{17}\epsilon_-^2\delta_+}{2\pi^2}\lambda(2) + i\frac{13\delta_-^3}{2^5\sqrt{17}\pi}\lambda(1) + \frac{\chi\zeta(3)}{(2\pi i)^3}\widehat{\omega}_0 . \end{aligned} \quad (4.189)$$

In the t -plane, the attractor is

$$t = \frac{1}{6} + \frac{1156L(2)}{2856L(2) - 45\pi i (9 + \sqrt{17}) L(1)} . \quad (4.190)$$

The attractor equations are solved for the charge vectors

$$Q_1 = \left(-\frac{72}{85}, \frac{66}{17}, 1, 0\right)^T , \quad Q_2 = \left(\frac{296}{85}, -\frac{192}{17}, 0, 1\right)^T . \quad (4.191)$$

The period vector is orthogonal to the charge vectors

$$Q_3 = \left(-\frac{72}{85}, \frac{296}{85}, 1, 0\right)^T, \quad Q_4 = \left(\frac{66}{17}, -\frac{192}{17}, 0, 1\right)^T. \quad (4.192)$$

Q_4 shall be used to obtain an identity. The perturbative solution is

$$t_0 = \frac{8}{17} + \frac{i}{34} \sqrt{\frac{65}{3}}. \quad (4.193)$$

We arrive at

$$\begin{aligned} \frac{2^3 17^3 \lambda(2)}{2856 \lambda(2) - 45 \pi i (9 + \sqrt{17}) \lambda(1)} &\stackrel{P}{=} \frac{31}{3} + i \sqrt{\frac{65}{3}} \\ &- \frac{17i}{2\sqrt{3}\pi^3} \sum_{j=1}^{\infty} e^{\frac{16\pi i}{17} \cdot j} \sum_{\mathfrak{p} \in \text{pt}(j)} a_{\mathfrak{p}} N^{\mathfrak{p}} \left(\frac{17j}{8\pi\sqrt{195}} \right)^{l(\mathfrak{p})-1/2} K_{l(\mathfrak{p})-1/2} \left(\frac{\pi j}{17} \sqrt{\frac{65}{3}} \right). \end{aligned} \quad (4.194)$$

This identity is much trickier to verify numerically. Using a diagonal Padé approximant of order 30, agreement can be found to six figures. Our conjectural summation identity for this example then is not very well supported. In fact, agreement to six figures can be found using diagonal Padé approximants of orders 23 through 30.

In case our series (4.194) does not hold, and in case the other (better evidenced) conjectural series identities from this section should prove true, it would be interesting to understand what sets this example apart.

The pair AESZ22 and AESZ118

The following two operators are mapped to one another by a change of variables and scaling transformation

$$z = \frac{1}{32x}, \quad F_{22}(z) = \frac{1}{32x} F_{118}(x). \quad (4.195)$$

$$\begin{aligned}
\text{AESZ22: } & [(7-4z)^2 (1+11z-z^2) \theta^4 \\
& -2z(7-4z) (143+4942z-2084z^2+256z^3) \theta^3 \\
& -z (1638+102261z-72568z^2+23024z^3-3072z^4) \theta^2 \\
& -z (637+66094z-30072z^2+12896z^3-2048z^4) \theta \\
& -2z (49+7868z-1904z^2+1472z^3-256z^4)] F_{22}(z) = 0 , \tag{4.196}
\end{aligned}$$

$$\begin{aligned}
\text{AESZ118: } & [(1-x)(1-56x)^2 (1-352x-1024x^2) \theta^4 \\
& -2x(1-56x)(9-64x) (33+384x-1792x^2) \theta^3 \\
& -x (431+15136x-335424x^2+4386816x^3-19267584x^4) \theta^2 \\
& -2x (67+7072x-41088x^2+996532x^3-6422528x^4) \theta \\
& -16x (1+176x-144x^2+23296x^3-20070x)] F_{118}(x) = 0 .
\end{aligned}$$

In the first of the above equations, θ denotes the operator $z \frac{d}{dz}$, while in the second θ denotes $x \frac{d}{dx}$.

Every operator in the AESZ list is ‘adapted’ to a MUM point, in that the point $0 \in \mathbb{C}$ is a point of maximal unipotent monodromy for any operator and the exponents for a basis of solutions about this point are $\{0, 0, 0, 0\}$.

However, as demonstrated by these examples, Calabi-Yau operators can have multiple MUM points. AESZ22 has, in addition to the origin, a MUM point at infinity with exponents $\{1, 1, 1, 1\}$. Under the above change of variables, this point is mapped to 0 while 0 is mapped to AESZ118’s secondary MUM point at infinity. The scaling transformation $F(z) \mapsto xF(x)$ is performed so that the exponents about the origin are $\{0, 0, 0, 0\}$.

The q -expansion of either the Yukawa coupling or the prepotential about $z = 0$ returns integer invariants n_d^{RC} that are interpreted as counts of genus 0 curves with degree d on a

mirror geometry, the *Reye Congruence*. This is a $\mathbb{Z}_2$ quotient of the complete intersection

$$\begin{array}{c} \mathbb{P}^4 \\ \mathbb{P}^4 \end{array} \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{array} \right]_{\chi=-100}. \quad (4.197)$$

by the freely acting $\mathbb{Z}_2$ quotient that exchanges the two $\mathbb{P}^4$ factors. The first few n_d^{RC} are

$$n_1^{\text{RC}} = 50, n_2^{\text{RC}} = 325, n_3^{\text{RC}} = 1475, n_4^{\text{RC}} = 15325, n_5^{\text{RC}} = 148575, n_6^{\text{RC}} = 1885575. \quad (4.198)$$

On the other hand, q -expanding about $x = 0$ returns integers n_d^{QS} that count genus 0 degree d curves on the orthogonal linear section of the double quintic symmetroid [68].

The first few of these numbers are

$$n_1^{\text{QS}} = 550, n_2^{\text{QS}} = 19150, n_3^{\text{QS}} = 1165550, n_4^{\text{QS}} = 106612400, n_5^{\text{QS}} = 12279982850. \quad (4.199)$$

The formal statement, around which we avoid working, is that these two spaces RC and QS are derived equivalent, having the same derived category of coherent sheaves [115]. They are not birational, possessing different topological invariants (such as their nonequal triple intersection numbers), but have the same Hodge numbers and the mirror symmetry computation of their curve counts takes place in a common complex structure moduli space for which the Picard-Fuchs operator can be taken to be AESZ22 or AESZ118 by suitable choices of coordinates.

A rational rank-two attractor

We have identified the point $z = -1$ as a rank-two attractor. Equivalently, this is $x = -1/32$. The zeta function numerator factorises and the ensuing weight-four modular form has LMFDB label 33.4.a.b .

Upon a Mellin transformation, this modular form gives an L-function with special values

$$\begin{aligned} L(1) &= -1.0538249565444346000240 \dots , \\ L(2) &= 0 . \end{aligned} \tag{4.200}$$

About either $z = 0$ or $x = 0$ a Frobenius basis of periods can be computed. It is possible to express the imaginary parts of the period vectors $\widehat{\omega}$ as rational multiples of $\frac{L(1)}{2\pi}$ (except that $\widehat{\omega}_3$ involves $\frac{\zeta(3)}{(2\pi i)^3}$). The real parts involve another nonzero number $\frac{\Phi}{(2\pi)^2}$, with

$$\Phi = 16.671069275646809663 \dots . \tag{4.201}$$

Since $L(2)$ is zero, it is not useful to this end. It would be interesting to approach this issue from the perspective of Beilinson's conjecture and give Φ in terms of the nonzero value $L'(2)$, a problem that would involve calculating a suitable Beilinson regulator [116]. However, it is possible to express Φ in terms of the L-values for a different modular form related to 33.4.a.b by a twist. In particular, Φ can be expressed in terms of a critical value of the Mellin transform of the form with LMFDB label 528.4.a.h:

$$\Phi = \frac{\pi}{6} L_{528.a.h}(1) , \tag{4.202}$$

The Reye Congruence

Pertaining to the mirror geometry RC, the topological invariants are

$$Y_{111} = 35 , \quad Y_{110} = \frac{1}{2} , \quad Y_{100} = -\frac{25}{6} , \quad \chi = 50 . \tag{4.203}$$

at $z = -1$, the Frobenius periods (about $z = 0$) evaluate as follows:

$$\begin{aligned}
\widehat{\omega}_0 &= -\frac{\Phi}{(2\pi i)^2} - \frac{5}{3} \frac{L(1)}{2\pi i} , \\
\widehat{\omega}_1 &= -\frac{1}{2} \frac{\Phi}{(2\pi i)^2} , \\
\widehat{\omega}_2 &= -\frac{10}{3} \frac{\Phi}{(2\pi i)^2} + \frac{175}{36} \frac{L(1)}{2\pi i} , \\
\widehat{\omega}_3 - \frac{\chi \zeta(3)}{(2\pi i)^3} \widehat{\omega}_0 &= -\frac{5}{24} \frac{\Phi}{(2\pi i)^2} + \frac{5}{6} \frac{L(1)}{2\pi i} .
\end{aligned} \tag{4.204}$$

The attractor equations are solved for charge vectors given in the integral basis by

$$Q_1 = \left(-\frac{1}{2}, 5, 1, 0\right)^T , \quad Q_2 = \left(\frac{7}{2}, -13, 0, 1\right) . \tag{4.205}$$

The orthogonality equations are solved by

$$Q_3 = (5, -13, 0, 1)^T , \quad Q_4 = \left(-\frac{1}{2}, \frac{7}{2}, 1, 0\right) . \tag{4.206}$$

Q_3 has zero $D6$ charge. One can determine the perturbative solution to the orthogonality equation to be

$$t_0 = \frac{15}{42} + \frac{\sqrt{69}}{42} i . \tag{4.207}$$

These considerations lead to a sum

$$-\frac{3 - 25 \frac{\pi L(1)}{\Phi} i}{3 + 10 \frac{\pi L(1)}{\Phi} i} \stackrel{P}{=} \frac{\sqrt{69}}{6} - \sqrt{\frac{7}{10\pi^3}} \sum_{j=1}^{\infty} e^{\frac{15\pi i}{21} j} \sum_{\mathfrak{p} \in \text{pt}(j)} a_{\mathfrak{p}} N^{\mathfrak{p}} \left(\frac{3j}{5\pi\sqrt{69}} \right)^{l(\mathfrak{p})-1/2} K_{l(\mathfrak{p})-1/2} \left(\frac{\pi j \sqrt{69}}{21} \right) , \tag{4.208}$$

in which the N_k are the scaled Gromov-Witten invariants constructed from the invariants n_d^{RC} . We use a diagonal Padé approximant of order 30, and find agreement to six figures. This example is also only weakly supported, and our conjectural sum here is not well-evidenced.

The Quintic Symmetroid

This time, for the mirror geometry QS, the topological invariants are

$$Y_{111} = 10 , \quad Y_{110} = 0 , \quad Y_{100} = -\frac{10}{3} , \quad \chi = 50 . \quad (4.209)$$

at $x = -1/32$, the Frobenius periods (about $x = 0$) evaluate to:

$$\begin{aligned} \widehat{\omega}_0 &= -2 \frac{\Phi}{(2\pi i)^2} , \\ \widehat{\omega}_1 &= -\frac{\Phi}{(2\pi i)^2} + \frac{10}{3} \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_2 &= \frac{5}{3} \frac{\Phi}{(2\pi i)^2} + \frac{50}{3} \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_3 - \frac{\chi \zeta(3)}{(2\pi i)^3} \widehat{\omega}_0 &= \frac{5}{3} \frac{\Phi}{(2\pi i)^2} + \frac{10}{9} \frac{L(1)}{2\pi i} . \end{aligned} \quad (4.210)$$

The attractor equations are solved for the charge vectors

$$Q_1 = (-1, 5, 1, 0)^T , \quad Q_2 = (2, -5, 0, 1)^T . \quad (4.211)$$

The orthogonality equations are in turn solved by

$$Q_3 = (-1, 2, 1, 0)^T , \quad Q_4 = (5, -5, 0, 1)^T . \quad (4.212)$$

From Q_4 , one can determine the perturbative solution

$$t_0 = \frac{1}{2} + \frac{i\sqrt{15}}{6} . \quad (4.213)$$

Collecting the various pieces, one can reach the sum

$$-20 \frac{\pi L(1)}{\Phi} \stackrel{\text{P}}{=} \sqrt{15} - \frac{1}{\sqrt{10\pi^3}} \sum_{j=1}^{\infty} (-1)^j \sum_{\mathfrak{p} \in \text{pt}(j)} a_{\mathfrak{p}} N^{\mathfrak{p}} \left(\frac{j}{60\pi\sqrt{15}} \right)^{l(\mathfrak{p})-1/2} K_{l(\mathfrak{p})-1/2} \left(\pi j \sqrt{\frac{5}{3}} \right) , \quad (4.214)$$

in which the scaled Gromov-Witten invariants are constructed from the instanton numbers n_d^{QS} . We use a diagonal Padé approximant of order 36, and find agreement to 29 figures.

The pair AESZ17 and AESZ290

These are another pair of operators with two MUM points that are exchanged by a coordinate transformation and scaling

$$z = z \mapsto -\frac{1}{3^6 x} , \quad F_{17}(z) = -\frac{1}{3^6 x} F_{290}(x) . \quad (4.215)$$

$$\begin{aligned} \text{AESZ17: } & [(5-9z)^2(1-27z)(1+27z^2)\theta^4 \\ & - 36z(5-9z)(7-15z+621z^2-729z^3)\theta^3 \\ & - 6z(59049z^4-91935z^3+39591z^2-541z+180)\theta^2 \\ & - 6z(39366z^4-64233z^3+34155z^2-155z+75)\theta \\ & - 3z(19683z^4-32562z^3+21060z^2-30z+25)]F_{17}(z) = 0 , \end{aligned}$$

$$\begin{aligned} \text{AESZ290: } & [(1+27x)(1+405x)^2(1+19683x^2)\theta^4 \\ & - 108x(1+405x)(7-729x-177147x^2-7971615x^3)\theta^3 \\ & - 6x(80-37017x-8155323x^2-1506635235x^3-87169610025x^4)\theta^2 \\ & - 6x(17-17415x-3720087x^2-789189885x^3-58113073350x^4)\theta \\ & - 9x(1-1998x-454896x^2-111602610x^3-9685512225x^4)]F_{290}(x) = 0 . \end{aligned} \quad (4.216)$$

The geometric picture is markedly different to the pair AESZ22/118, in that q -expansions about $x = 0$ do not return the BPS invariants of a manifold because there is no mirror manifold. Indeed, we will see that any putative mirror should have $Y_{111} = -30/13$, which defies an interpretation as a triple intersection number.

The MUM point $z = 0$ is mirror the the large volume point of a one-parameter family of

threefolds constructed as the $\mathbb{Z}_3$ quotient of the split bicubic:

$$\mathbb{P}^2 \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right]_{/\mathbb{Z}_3} \quad (4.217)$$

The above quotient geometry has Euler characteristic $\chi = -30$. Since $h^{1,1} = 1$, we have that $h^{2,1} = 16$ for the above space. AESZ17 is the Picard Fuchs operator for a mirror X to (4.217). From the Hodge diamond reflection, we have that $h^{1,1}(X) = 16$ and $h^{2,1}(X) = 1$. If the mum point $x = 0$ (which is $z = \infty$) in the complex structure moduli space of X has a mirror Y , then Y 's Hodge numbers must read $h^{1,1}(Y) = 1$, $h^{2,1}(Y) = 16$. That is to say, Y should have the same Hodge numbers as (4.217). As a consequence, Y and (4.217) must have the same Euler characteristic.

Now we consider monodromies [117] for the operator AESZ290. This has conifold singularities at $x = -1/27$, $x = \pm \frac{\sqrt{3}i}{343}$. As usual, since the operator's local exponents at $x = 0$ are $(0, 0, 0, 0)$ we can construct a Frobenius basis

$$\begin{aligned} \varpi_0(x) &= f_0(x) , \\ \varpi_1(x) &= \frac{1}{2\pi i} (f_0(x) \log(x) + f_1(x)) , \\ \varpi_2(x) &= \frac{Y_{111}}{2(2\pi i)^2} (f_0(x) \log(x)^2 + 2f_1(x) \log(x) + f_2(x)) , \\ \varpi_3(x) &= \frac{Y_{111}}{6(2\pi i)^3} (f_0(x) \log(x)^3 + 3f_1(x) \log(x)^2 + 3f_2(x) \log(x) + f_3(x)) , \end{aligned} \quad (4.218)$$

where all of the f_i are power series, with $f_0(0) = 1$ and $f_1(0) = f_2(0) = f_3(0) = 0$. Now we attempt to construct an integral symplectic period vector

$$\Pi = \begin{pmatrix} -\frac{\chi(Y)\zeta(3)}{(2\pi i)^3} & -\frac{1}{2}Y_{100} & 0 & 1 \\ -\frac{1}{2}Y_{100} & -Y_{110} & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \varpi . \quad (4.219)$$

The task at hand then is to find real values of Y_{111} , Y_{100} and Y_{110} so that the monodromy

matrices about each conifold are integral. Note that we have fixed the value $\chi(Y) = 30$ by our earlier argument. In practice we compute monodromy matrices for a modified version of the Frobenius vector that does not include the factors of Y_{111} , and then restore these symbolically with a diagonal change of basis.

It is instructive to check the monodromy matrix that we compute at $x = \frac{i\sqrt{3}}{343}$ for the above Π . The $(1, 1)$ component is

$$-\frac{7}{8} + \frac{9Y_{100}}{Y_{111}} + \frac{9i(30 + 13Y_{111})\zeta(3)}{4Y_{111}\pi^3} . \quad (4.220)$$

In order for this to be real, we must choose Y_{111} so that the imaginary part vanishes. So we learn that the triple intersection number of our manifold Y should be

$$Y_{111} = -\frac{30}{13} . \quad (4.221)$$

On these grounds, we dispense with the idea that there is a mirror manifold at $x = 0$. We cannot by any choice of Y_{100} and Y_{110} render the conifold matrix integral, so we press on without fixing any value for these.

A rational rank-two attractor

Here we are able to identify a rank-two attractor at $z = -1$, which is $x = \frac{1}{36}$, with the corresponding weight-four modular form having LMFDB label 14.4.a.b . Mellin transforming this gives the critical L-values

$$\begin{aligned} L(1) &= 0.81476235013261396706625 \dots , \\ L(2) &= 1.13620338571911095621858 \dots . \end{aligned} \quad (4.222)$$

AESZ17

The topological invariants are

$$Y_{111} = 30 , \quad Y_{110} = 0 , \quad Y_{100} = -3 , \quad \chi = 30 . \quad (4.223)$$

The Frobenius periods about $z = 0$ can be evaluated at $z = -1$, giving

$$\begin{aligned} \widehat{\omega}_0 &= -14 \frac{L(2)}{(2\pi i)^2} , \\ \widehat{\omega}_1 &= -7 \frac{L(2)}{(2\pi i)^2} - \frac{3}{4} \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_2 &= -42 \frac{L(2)}{(2\pi i)^2} - \frac{45}{4} \frac{L(1)}{2\pi i} , \\ \widehat{\omega}_3 - \frac{\chi \zeta(3)}{(2\pi i)^3} \widehat{\omega}_0 &= -\frac{7}{2} \frac{L(2)}{(2\pi i)^2} - \frac{19}{8} \frac{L(1)}{2\pi i} . \end{aligned} \quad (4.224)$$

Independent charge vectors for which the attractor equations are solved are

$$Q_1 = \left(-\frac{4}{3}, 6, 1, 0 \right)^T , \quad Q_2 = \left(\frac{14}{3}, -15, 0, 1 \right)^T , \quad (4.225)$$

while the orthogonality equations are solved for

$$Q_3 = (6, -15, 0, 1)^T , \quad Q_4 = \left(-\frac{4}{3}, \frac{14}{3}, 1, 0 \right)^T . \quad (4.226)$$

Using Q_3 one can find the perturbative solution

$$t_0 = \frac{1}{2} + \frac{i\sqrt{5}}{10} . \quad (4.227)$$

We obtain the sum

$$\frac{3\pi}{14} \frac{L(1)}{L(2)} \stackrel{P}{=} \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{15}\pi^3} \sum_{j=1}^{\infty} (-1)^j \sum_{\mathfrak{p} \in \text{pt}(j)} a_{\mathfrak{p}} N^{\mathfrak{p}} \left(\frac{j}{6\pi\sqrt{5}} \right)^{l(\mathfrak{p})-1/2} K_{l(\mathfrak{p})-1/2} \left(\frac{\pi j}{\sqrt{5}} \right) . \quad (4.228)$$

A diagonal Padé approximant of order 36 provides agreement to 10 figures.

AESZ290

We lack a geometric interpretation for a mirror at $x = 0$, but proceed with the computation using the topological invariants

$$Y_{111} = -\frac{30}{13}, \quad Y_{110} = Y_{110}, \quad Y_{100} = Y_{100}, \quad \chi = 30. \quad (4.229)$$

The rank two attractor $z = -1$ maps to $x = 3^{-6}$, and we find that at this point the Frobenius periods for AESZ290 evaluate to

$$\begin{aligned} \widehat{\omega}_0 &= \frac{9\sqrt{3}}{2} \frac{L(1)}{(2\pi)}, \\ \widehat{\omega}_1 &= 21i\sqrt{3} \frac{L(2)}{(2\pi)^2}, \\ \widehat{\omega}_2 &= \frac{45\sqrt{3}}{8} \frac{L(1)}{2\pi}, \\ \widehat{\omega}_3 - \frac{\chi \zeta(3)}{(2\pi i)^3} \widehat{\omega}_0 &= \frac{315i\sqrt{3}}{52} \frac{L(2)}{(2\pi)^2}. \end{aligned} \quad (4.230)$$

The attractor equations are solved for the charge vectors

$$Q_1 = \left(\frac{15}{52} - \frac{Y_{100}}{2}, -Y_{110}, 0, 1 \right)^T, \quad Q_2 = \left(0, -\frac{5}{4} - \frac{Y_{100}}{2}, 1, 0 \right)^T. \quad (4.231)$$

The orthogonality equations are solved for charges

$$Q_3 = \left(0, \frac{15}{52} - \frac{Y_{100}}{2}, 1, 0 \right)^T, \quad Q_4 = \left(-\frac{5}{4} - \frac{Y_{100}}{2}, -Y_{110}, 0, 1 \right)^T. \quad (4.232)$$

The orthogonality equation for the charge vector Q_4 is solved at the perturbative level by

$$t_0 = \frac{i}{2} \sqrt{\frac{13}{3}}. \quad (4.233)$$

From this information, we construct the sum

$$\frac{14}{3\pi} \frac{L(2)}{L(1)} = \sqrt{\frac{13}{3}} - \sqrt{\frac{13}{15\pi^3}} \sum_{j=1}^{\infty} \sum_{\mathfrak{p} \in \text{pt}(j)} a_{\mathfrak{p}} N^{\mathfrak{p}} (-1)^{l(\mathfrak{p})} \left(\frac{\sqrt{\frac{13}{3}} j}{10\pi} \right)^{l(\mathfrak{p})-1/2} K_{l(\mathfrak{p})-1/2} \left(\pi j \sqrt{\frac{13}{3}} \right) . \quad (4.234)$$

There is no Padé resummation here, as this sum converges (by the same arguments as appeared in Appendix F of [4]). By computing 60 terms in the series, we find agreement to 50 figures. By taking two Shanks transformations [114], this agreement can be improved to 55 figures. Three Shanks transformations also yields 55 figures of agreement, and further Shanks transformations worsen the agreement.

Appendix A

Coordinates on complex structure moduli space

Smith : Because of you, I'm no longer an Agent of this system. Because of you, I've changed. I'm unplugged. A new man, so to speak. Like you, apparently, free.

Neo : Congratulations.

Smith : Thank you. But, as you well know, appearances can be deceiving, which brings me back to the reason why we're here.

Lana and Lilly Wachowski, *The Matrix Reloaded*

In this appendix, we explain our choice of coordinates for the complex structure moduli spaces of the mirrors of complete intersections. We follow the presentations of [27, 118], whose choice of coordinates we use. This involves a combinatoric problem in toric geometry, a subject that we do not go into detail on, instead referring to the textbook [119]. The construction of mirror manifolds via operations on polyhedrons that encode intersections in toric varieties goes back to work of Batyrev, Borisov, and Nill [120–124].

The presentation that we offer is solely given to define a choice of coordinates. We will not discuss the proofs underlying these methods.

As in the main body of the thesis, Y will be a family of complete intersections. For this

appendix we shall allow the ambient space to be a product of weighted projective spaces, which is a mild generalisation of the CICYs discussed in §2.2. So the data of Y is

$$\begin{array}{c} \mathbb{WP}^{n_1}_{(w_1^{(1)}, \dots, w_{n_1+1}^{(1)})} \\ \vdots \\ \mathbb{WP}^{n_K}_{(w_1^{(K)}, \dots, w_{n_K+1}^{(K)})} \end{array} \begin{bmatrix} d_1^{(1)}, \dots, d_C^{(1)} \\ \vdots \\ d_1^{(K)}, \dots, d_C^{(K)} \end{bmatrix}. \quad (\text{A.1})$$

We assume, following [27], that the intersection Y does not intersect with the singularities of the ambient variety $\mathcal{A} = \mathbb{WP}^{n_1}_{(w_1^{(1)}, \dots, w_{n_1+1}^{(1)})} \times \dots \times \mathbb{WP}^{n_K}_{(w_1^{(K)}, \dots, w_{n_K+1}^{(K)})}$, and without loss of generality take $w_{n_i+1}^{(i)} = 1$ for each i . We will also assume a favourability condition: $h^{1,1} = K$. We will display here formulae relevant for such intersections in products of weighted projective spaces. Results for CICYs in products of (unweighted) projective spaces, as in all our main lines of discussion in this thesis, can be obtained by setting all $w_i^{(k)}$ equal to 1.

Each weighted projective space $\mathbb{WP}^{n_i}_{\mathbf{w}^{(i)}}$ is a toric variety, associated to a reflexive simplicial polyhedron Δ_i . Δ_i has integer vertices in $\mathbb{R}^{n_i}$. The ambient space $\mathcal{A}$ is also a toric variety, and the associated polyhedron is

$$\Delta = \Delta_1 \times \dots \times \Delta_K \subset \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_K}. \quad (\text{A.2})$$

The Batyrev-Borisov procedure produces a mirror X to an intersection in a toric variety, like Y above, from the data of the polyhedron Δ^* , which is the polar dual of Δ . We intentionally gloss over the specifics of Δ and its association to $\mathcal{A}$, and skip to the presentation of the vertices $\nu_{i,j}^*$ of Δ^* . For each fixed i , there are vertices $\nu_{i,j}^*$ that have components 0 in each

factor $\mathbb{R}^{n_j}$ of $\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_K}$ with $j \neq i$. The components of $\nu_{i,j}^*$ in the factor $\mathbb{R}^{n_i}$ are

$$\begin{aligned} \nu_{i,1}^* &= (1, 0, \dots, 0) , & \dots , & & \nu_{i,n_i}^* &= (0, \dots, 0, 1) , \\ \nu_{i,n_i+1}^* &= \left(-w_1^{(1)} , \dots , -w_{n_i}^{(i)} \right) . \end{aligned} \quad (\text{A.3})$$

Denote the set of all the vertices by E , and partition E into distinct sets E_m , $1 \leq m \leq C$. For each fixed i , each set E_m contains $d_m^{(i)}$ vertices from the set $\nu_{i,j}^*$.

Now for each fixed m , each vertex $\nu_{i,j}^* \in E_m$ is extended to

$$\bar{\nu}_{i,j}^* = (e^{(m)}, \nu_{i,j}^*) \in \mathbb{R}^C \times \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_K} . \quad (\text{A.4})$$

By $e^{(m)}$ we mean the unit vector in the m^{th} direction of $\mathbb{R}^C$. We then form additional vectors

$$\bar{\nu}_{0,p}^* = (e^{(p)}, \mathbf{0}) \in \mathbb{R}^C \times \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_K} . \quad (\text{A.5})$$

Importantly, it turns out that there are K independent linear relations between the vectors $\bar{\nu}_{i,j}^*$,

$$\sum_{i,j} l_{i,j}^{(s)} \bar{\nu}_{i,j}^* = \mathbf{0} , \quad 1 \leq s \leq K . \quad (\text{A.6})$$

The components of $l_{i,j}^{(s)}$ are

$$\begin{aligned} l_{0,j}^{(s)} &= -d_j^{(s)} , & 1 \leq j \leq C , \\ l_{i,j}^{(s)} &= \delta_i^{(s)} w_j^{(s)} , & 1 \leq i \leq K , \quad 1 \leq j \leq n_{n_s+1}^{(s)} , \\ \delta_i^{(s)} &= \begin{cases} 0 & i \neq s , \\ 1 & i = s . \end{cases} \end{aligned} \quad (\text{A.7})$$

The choice of overall sign, with minus signs in the components of $l_{0,j}^{(s)}$, is one of convention. We fix this choice in our work, so that MUM points are located at the origin. We remark that for the more general case of intersections in toric varieties, relations of the form (A.6)

hold. It is only in our simplified case of complete intersections in weighted projective spaces that the particular form (A.7), with the Kronecker delta, hold.

Let Δ_i^* denote the convex hull of E_i and the origin. The mirror family of Y consists of varieties X birational to the vanishing loci of C equations $P_r = 0$ in the toric variety $\mathbb{P}_{\Delta_1^* + \dots + \Delta_C^*}$, where $+$ denotes the Minkowski sum of polyhedra. The ambient space coordinates are $X_{m,n}$, with $1 \leq m \leq K$ and $1 \leq n \leq n_m$. Those equations read

$$P_r \equiv a_{0r} - \sum_{\nu_{i,j}^* \in E_r} a_{i,j} X^{\nu_{i,j}^*} = 0 . \quad (\text{A.8})$$

Let us clarify the notation here: $X^{\nu_{i,j}^*}$ means $\prod_{j=1}^{n_i} X_{i,j}^{\nu_{i,j}^*}$.

The set of a_{ij} that appear in (A.8) outnumber the set of complex structure moduli of X . Coordinate redefinitions remove the redundancies, and these are encoded in the relations $l_{i,j}^s$.

We work with coordinates

$$\varphi^s = \frac{\prod_{j=1}^{n_s+1} a_{s,j}}{\prod_{j=1}^C a_{0,j}^{d_j^{(s)}}} \equiv a^{l^{(s)}} . \quad (\text{A.9})$$

Inspecting this process, one sees that there is some superficial ambiguity in the choice of sets E_m which can lead to different polynomials P_m . The set of coordinates obtained from (A.9) does not depend on this choice. The apparent ambiguity in the polynomials is removed when these varieties are desingularised, but we do not discuss that in this appendix.

Appendix B

Zeta function factorisation counts

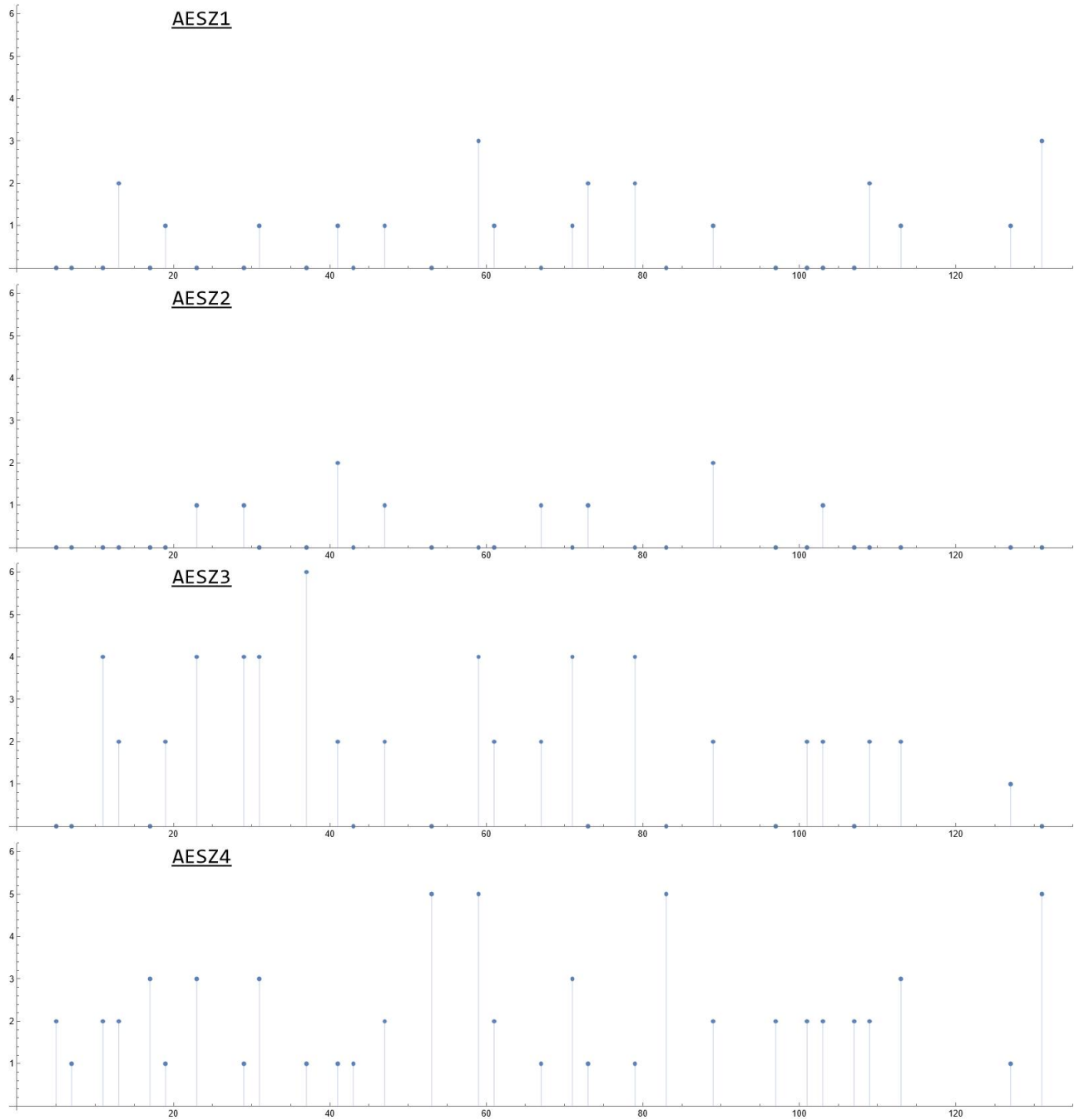
We're not banging rocks together here.

Cave Johnson

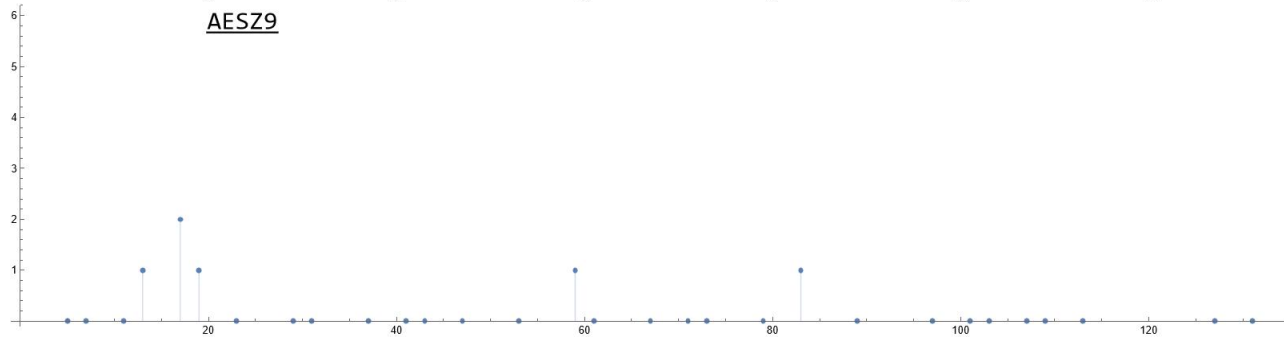
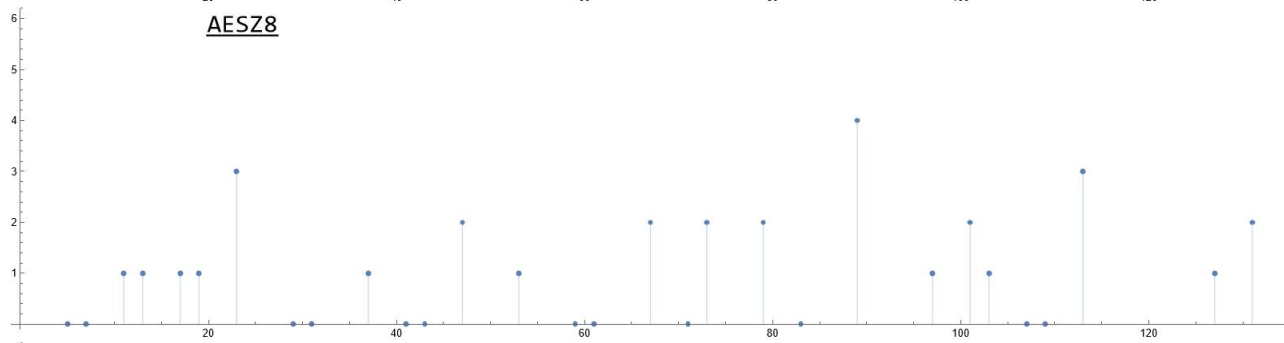
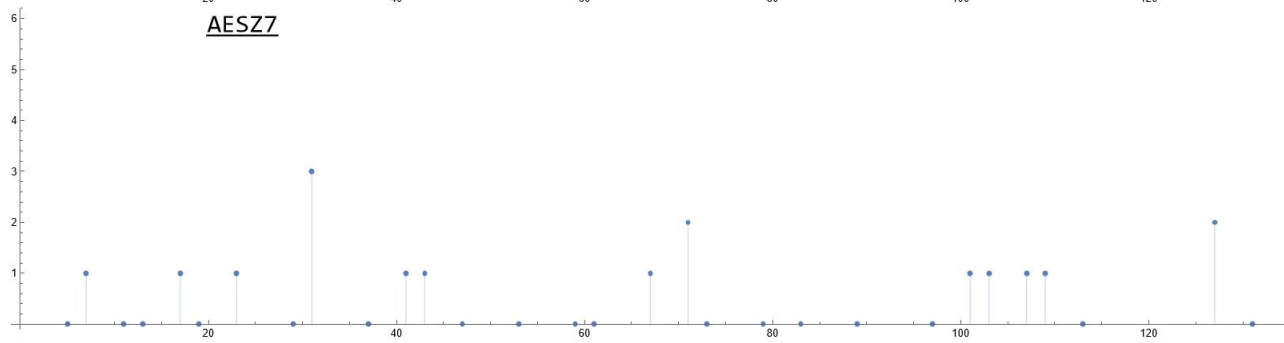
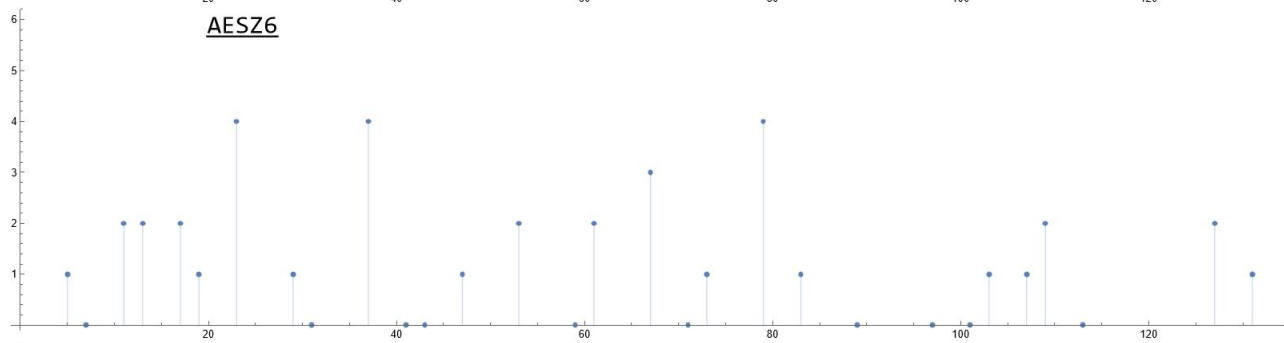
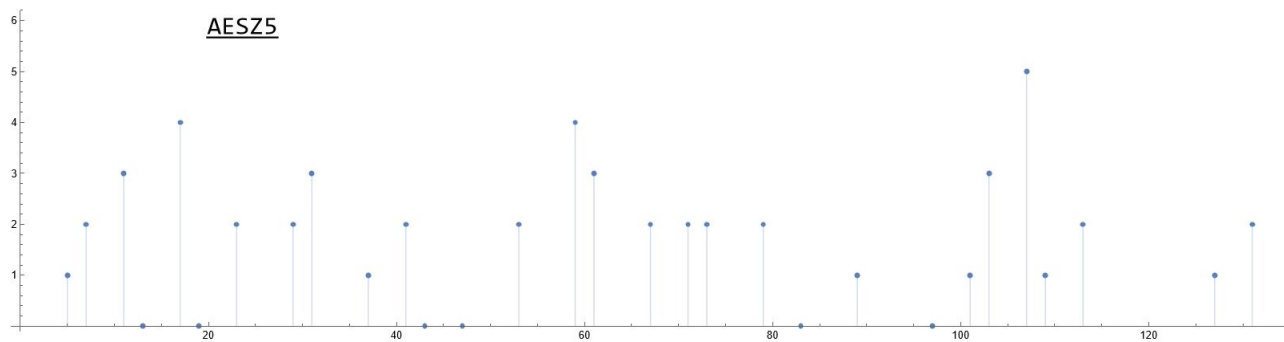
This appendix contains bar charts that display for each labelled operator, with p on the x -axis, the number of times that the zeta function $\zeta_p(\varphi; T)$ factorises as φ runs over $\mathbb{F}_p$. Our tables are for primes $5 \leq p \leq 131$.

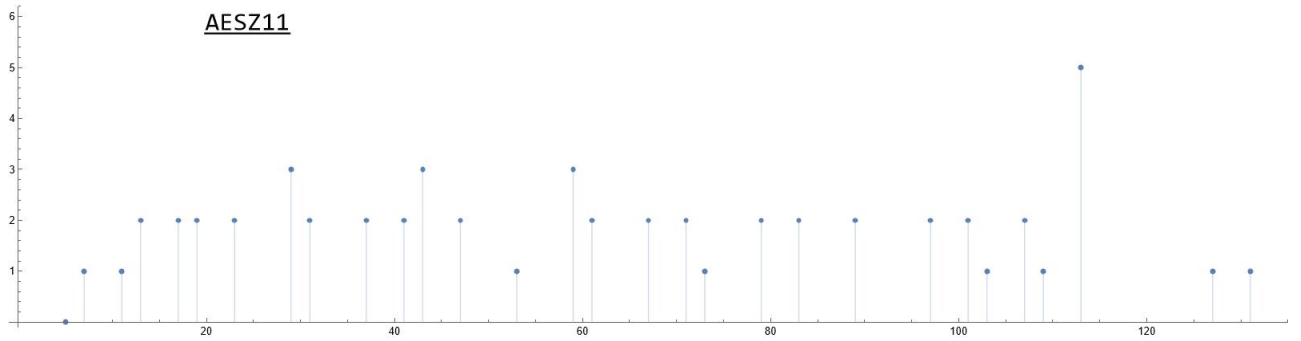
It should be understood that this list is not exhaustive: other operators have been studied elsewhere (see [13, 14, 47]), and many more remain uninvestigated. We depend on the digital database [65]. The computation that obtains these factorisation counts needs the topological numbers Y_{111} and χ , which are not given for every operator in [65]. Further, a number of the operators listed in [65] are typeset in such a way that posed a problem for our code that recovered the operators from the website. Rather than fix these surmountable problems, we omit those examples. We also did not make an effort to work through the whole set for which there was sufficient data available for our purposes. This choice of 61 is a matter of simplistic practicality and does not belie anything interesting.

These tables were generated using the methods of [50], and are available in electronic form [1].

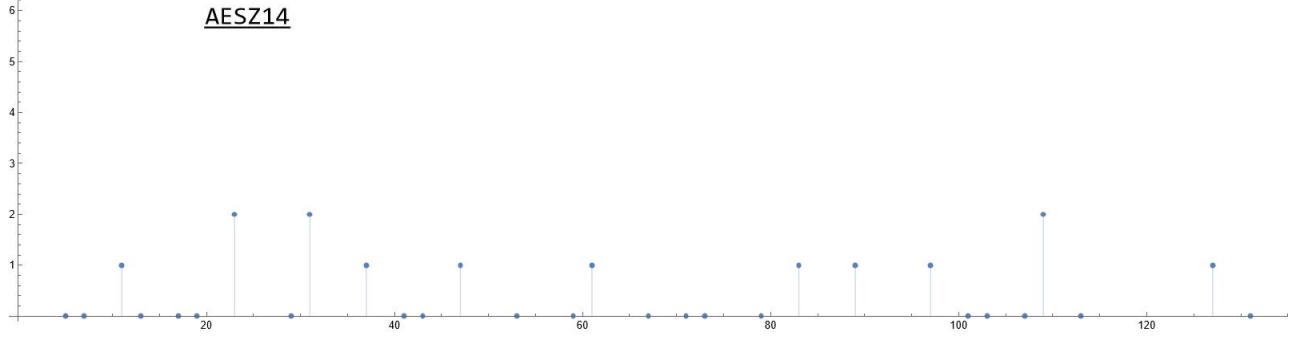
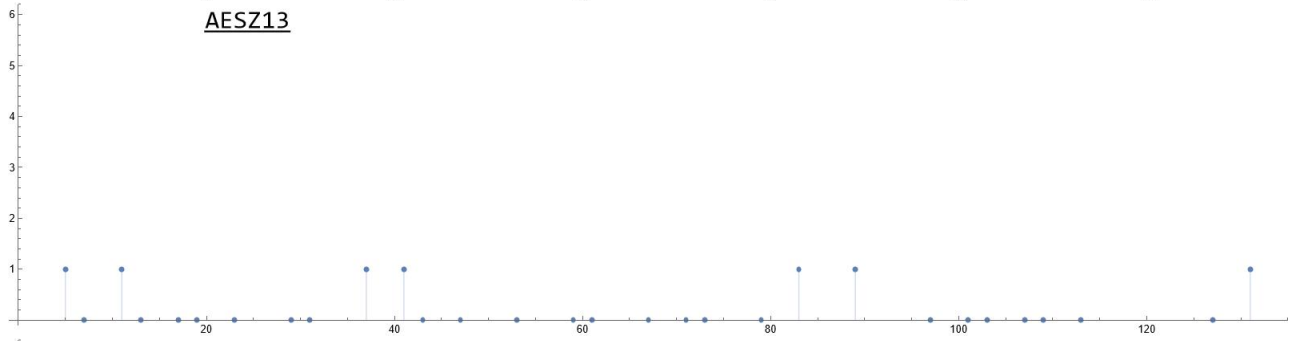
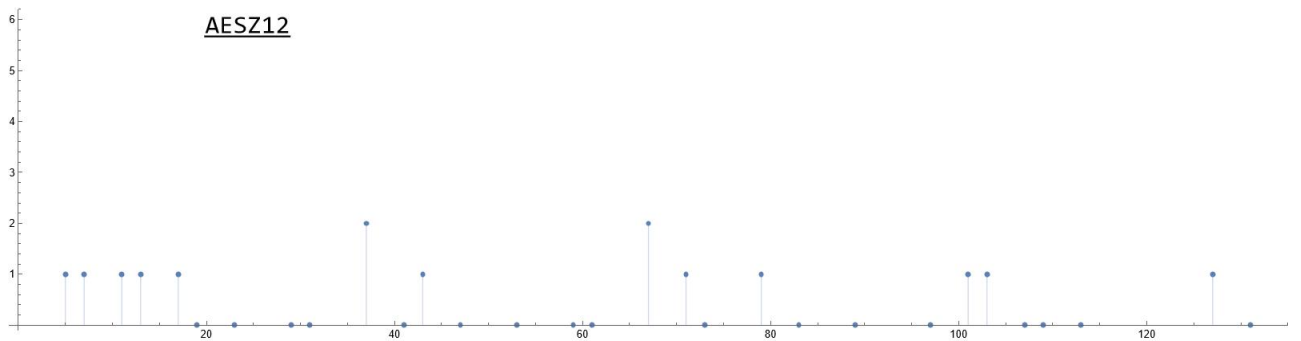


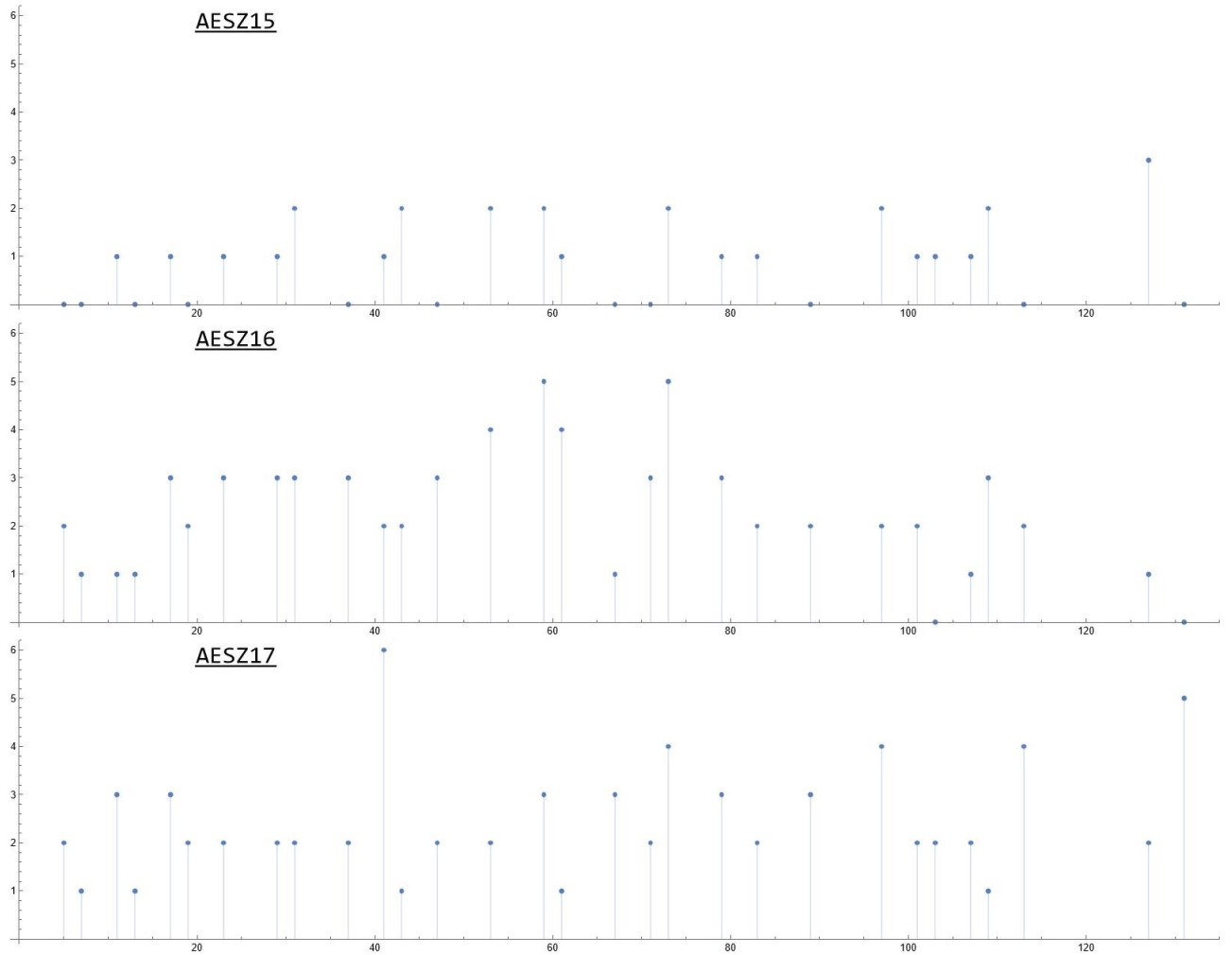
AESZ4 was studied in [13, 47] and is known to possess a rational rank two attractor.



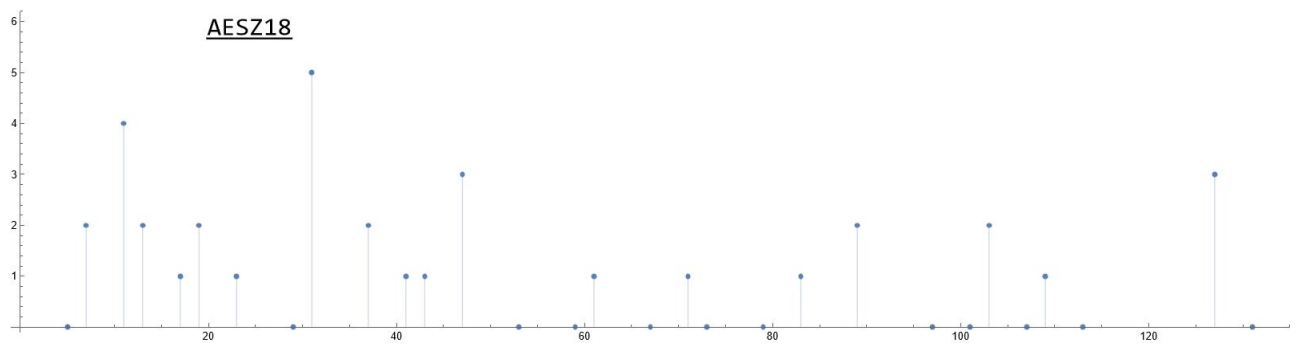


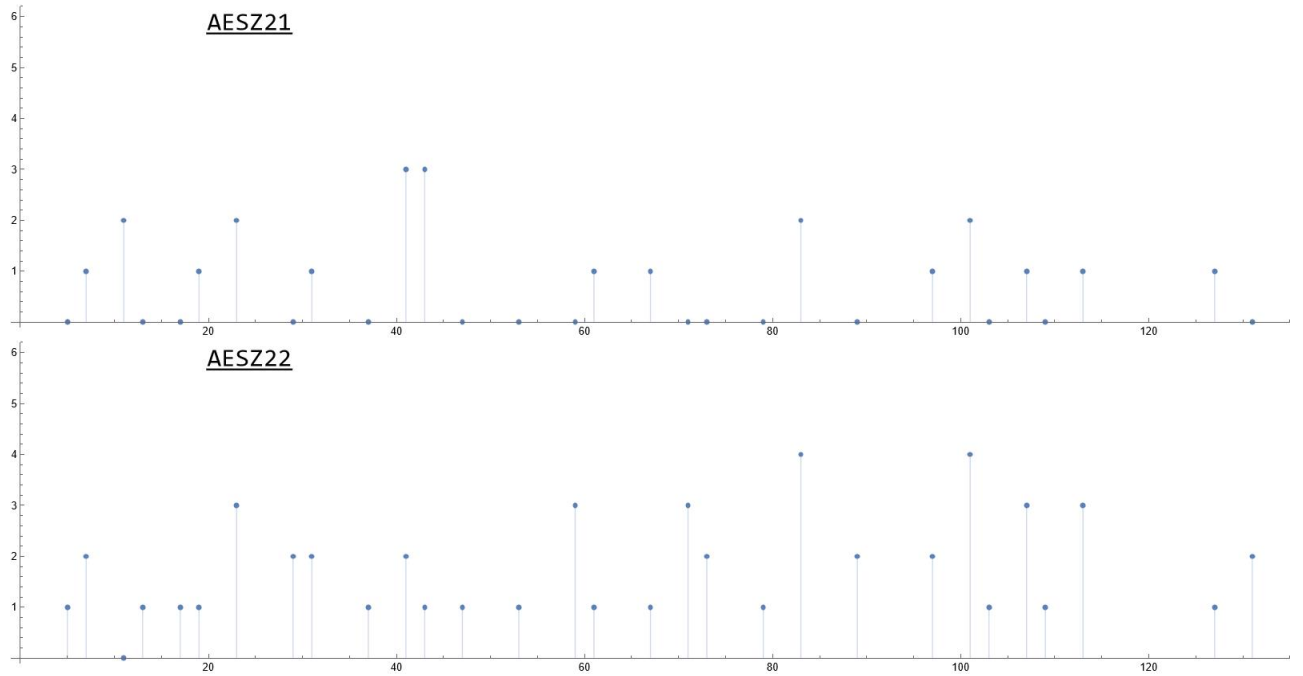
AESZ11 was studied in [13, 47] and is known to possess a rational rank two attractor.



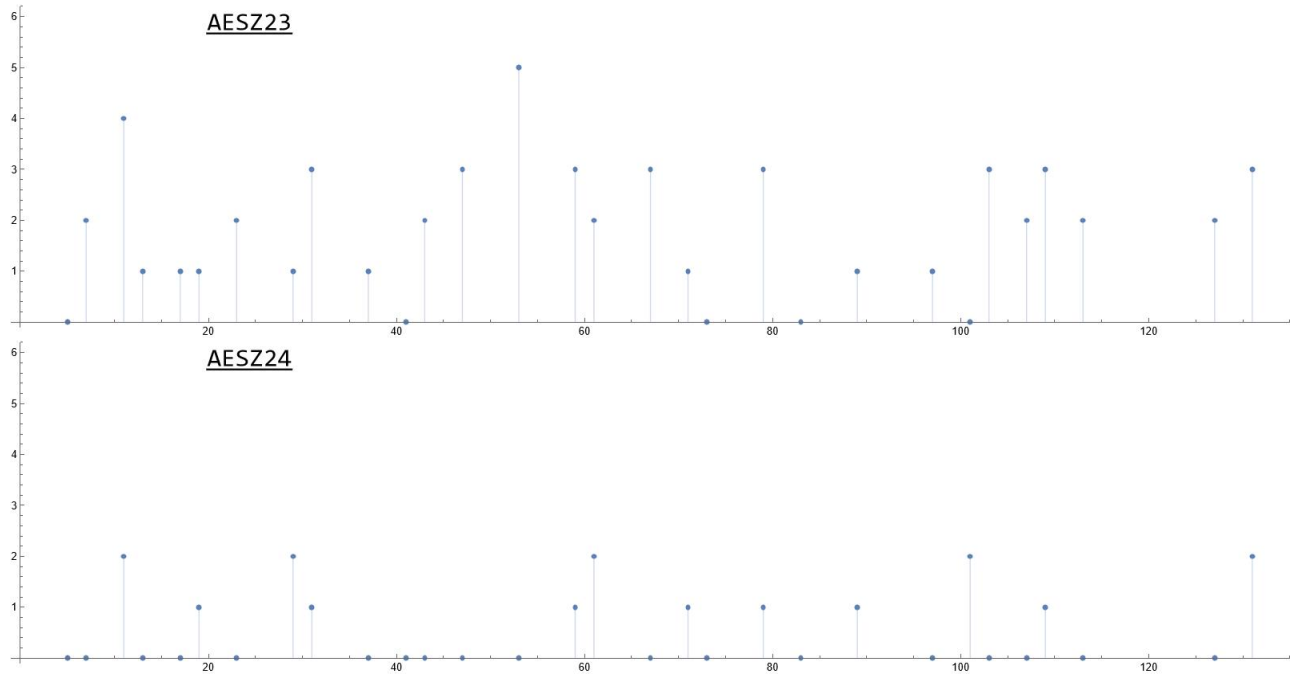


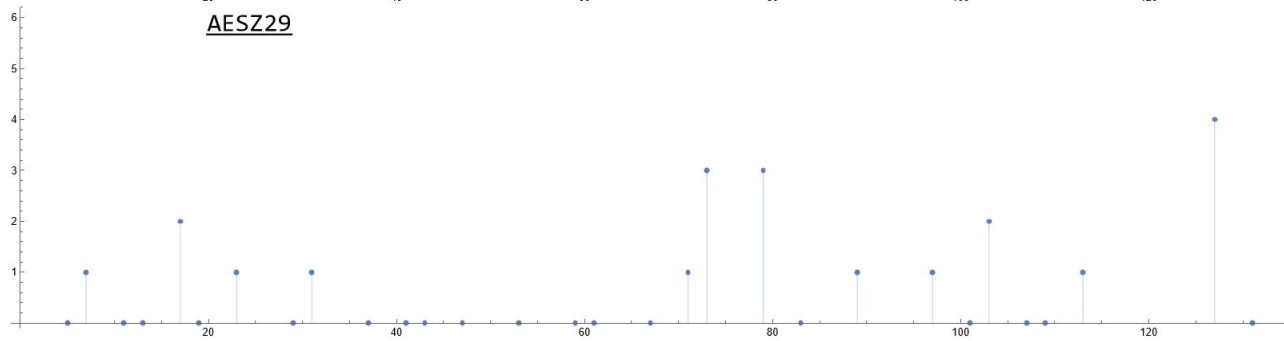
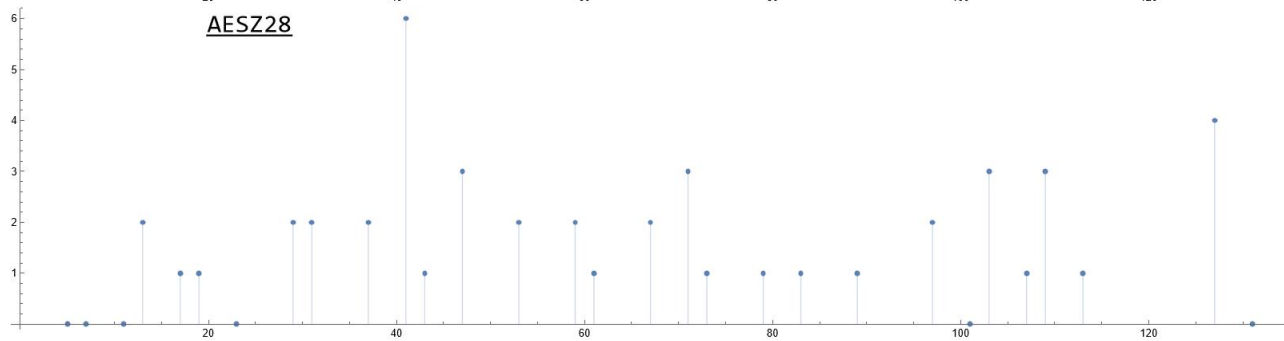
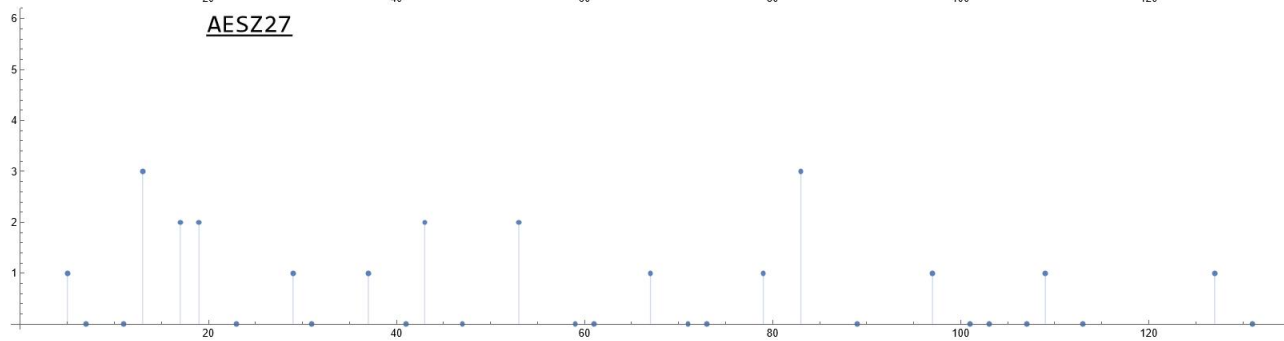
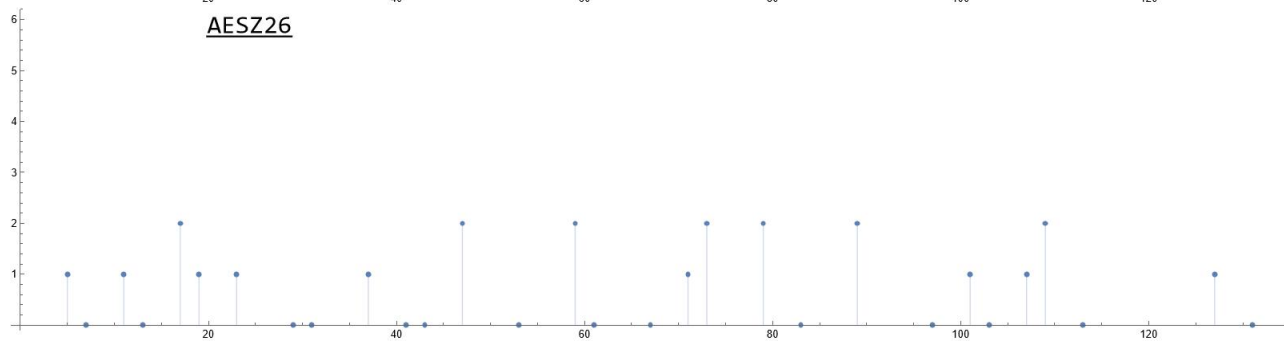
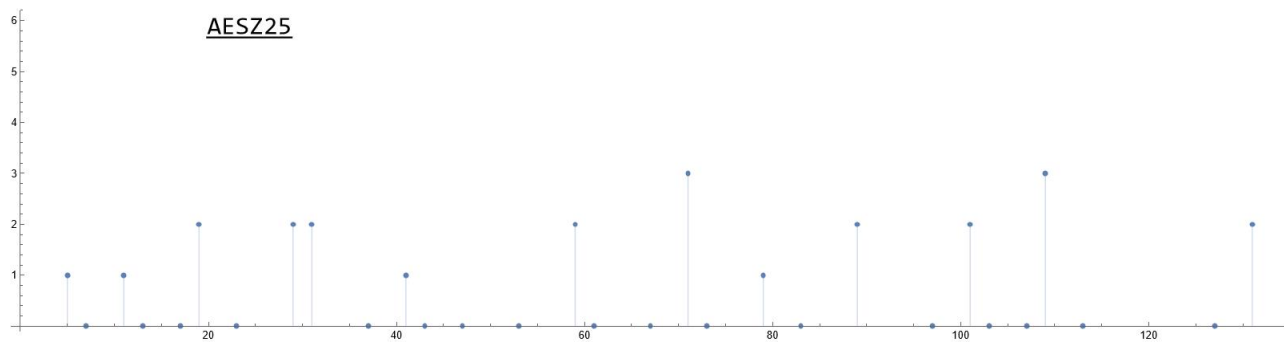
The existence of a rational rank two attractor for AESZ17 is a new result of this thesis.

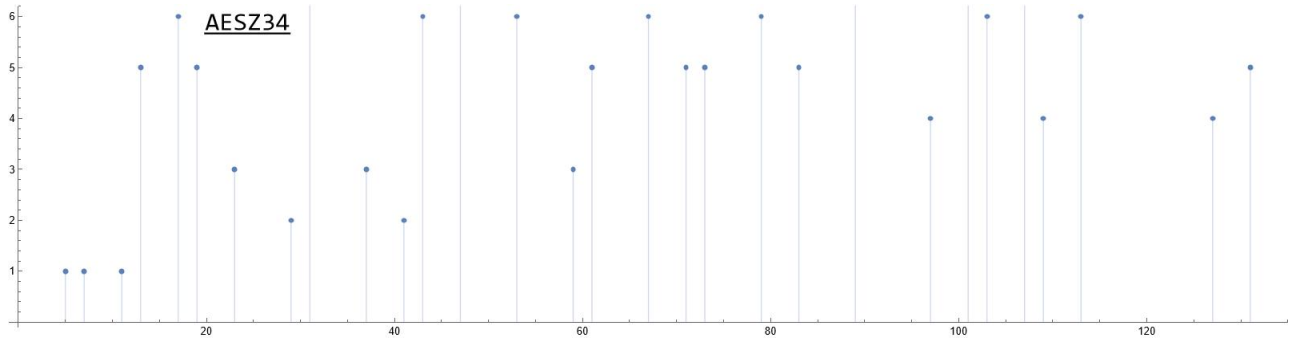




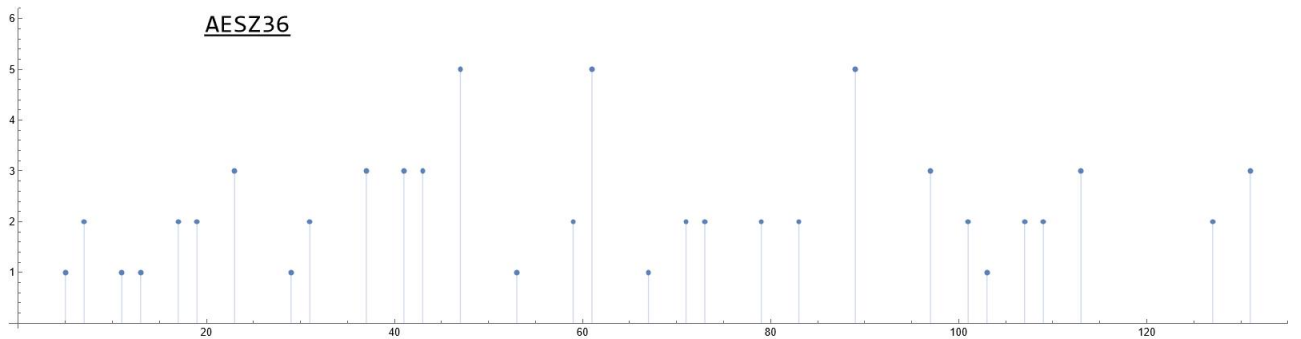
The existence of a rational rank two attractor for AESZ22 is a new result of this thesis.



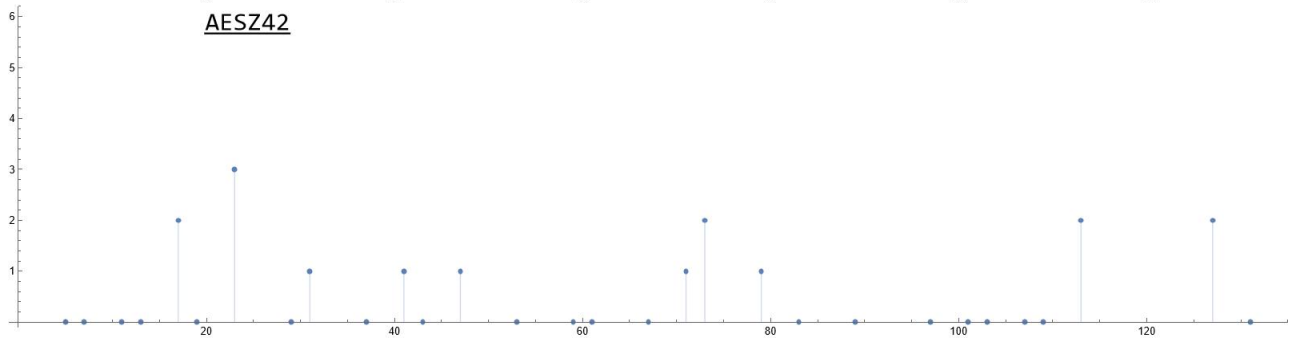
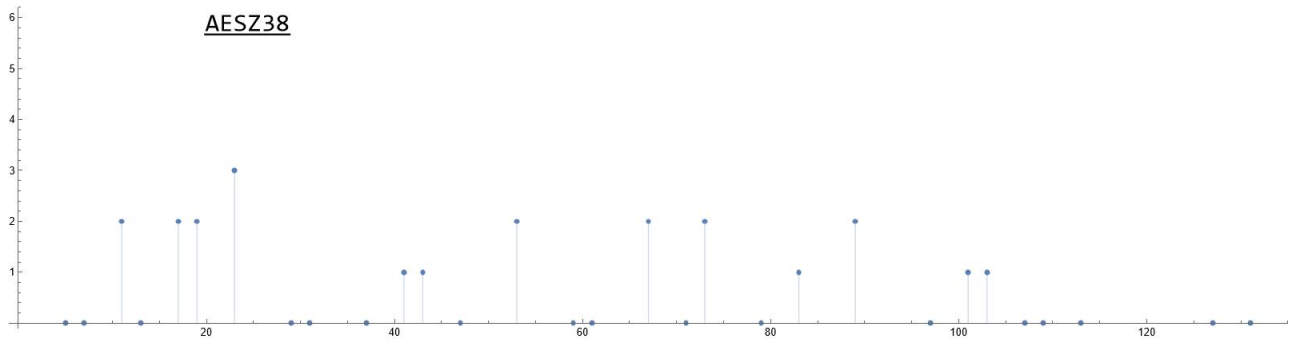


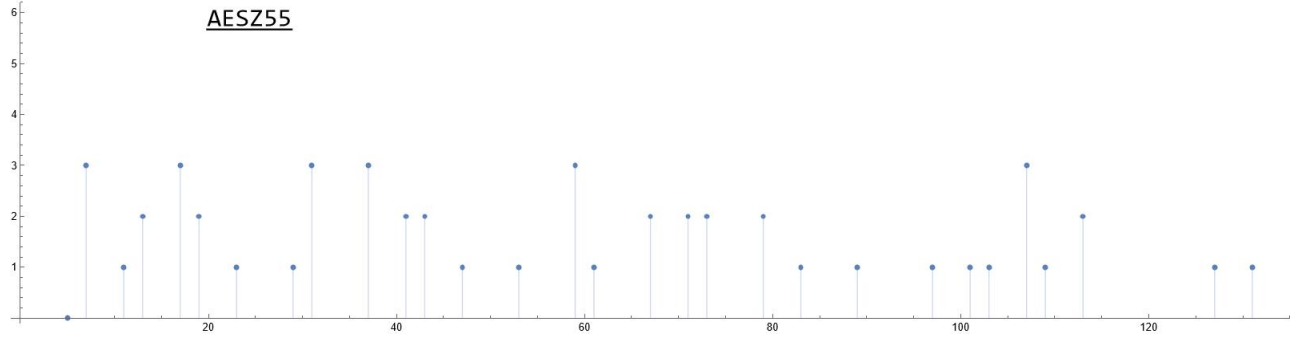
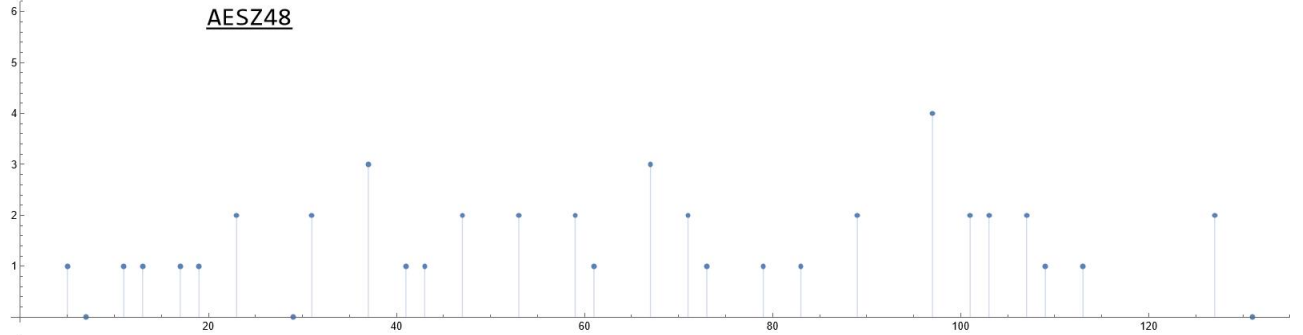
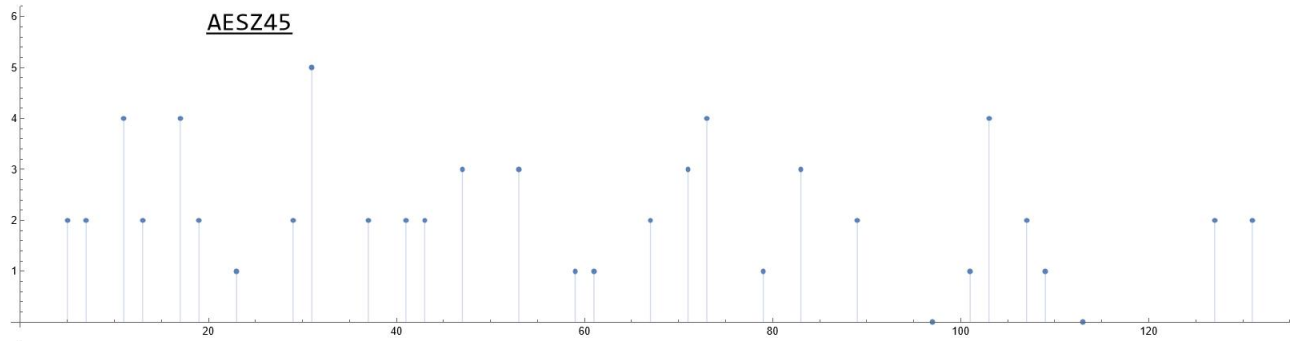


AESZ34 was studied in [14]. It is known to possess one rational and two quadratic rank two attractors.

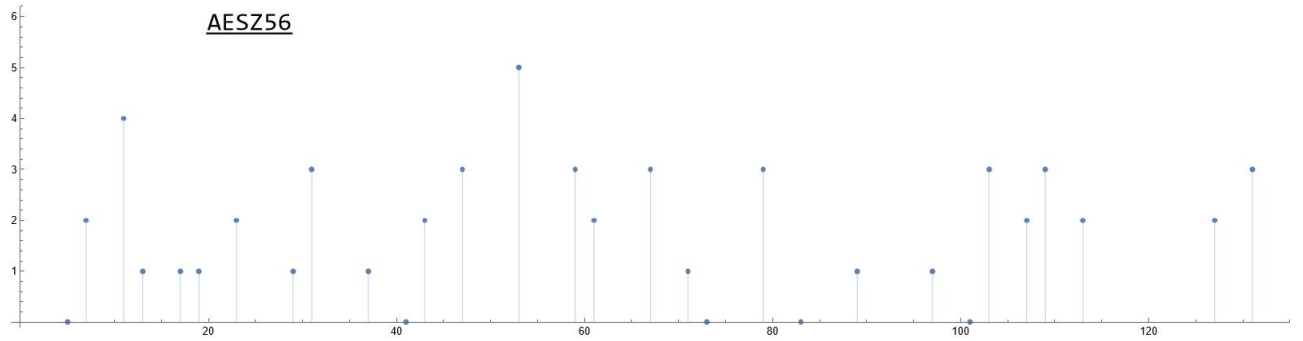


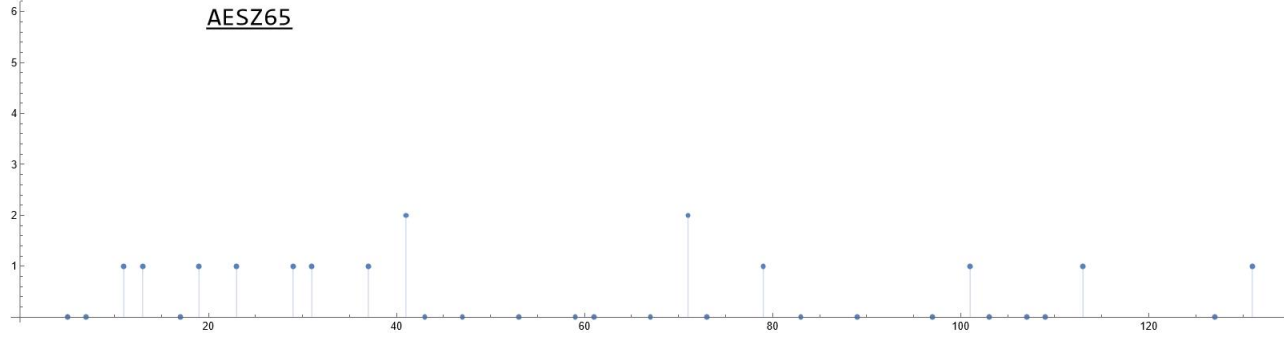
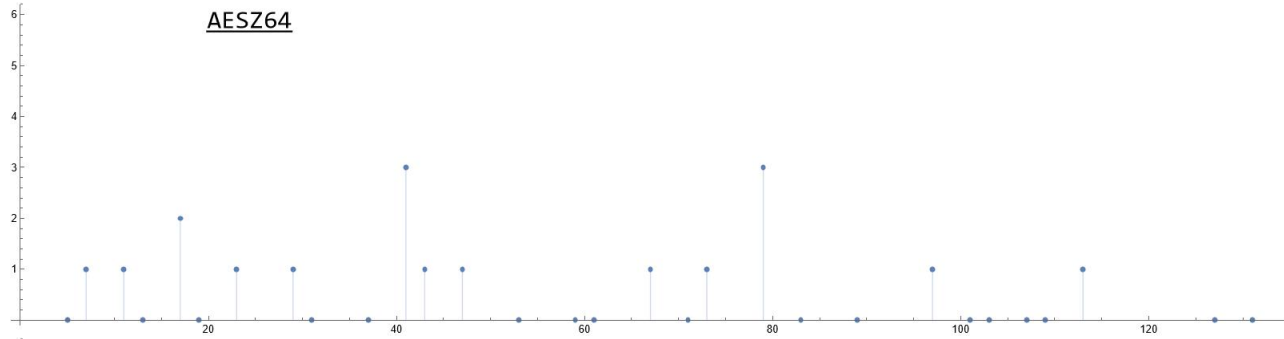
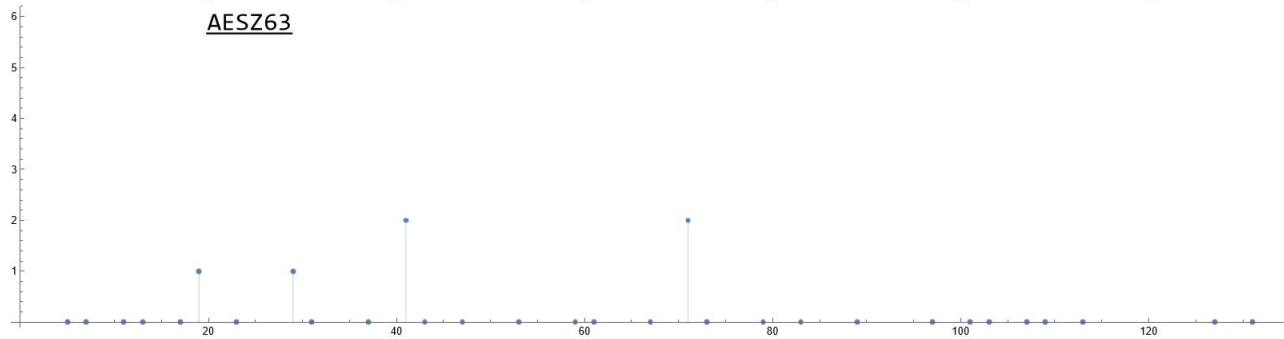
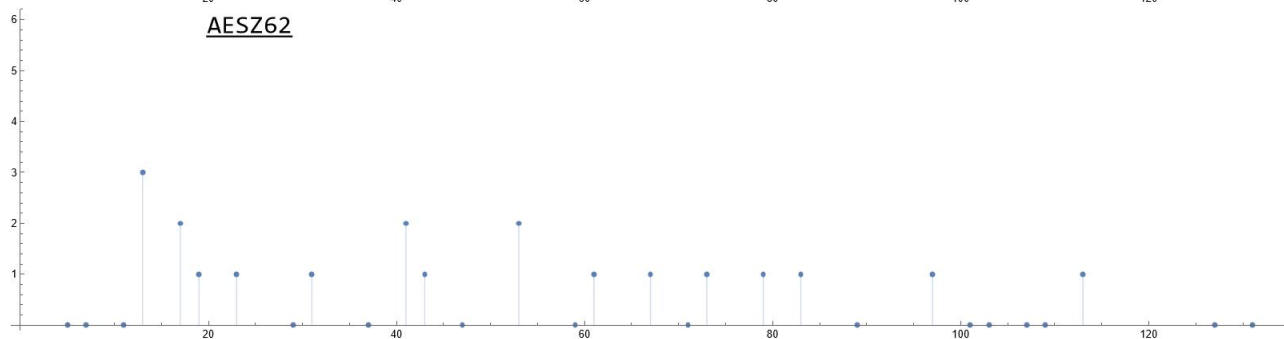
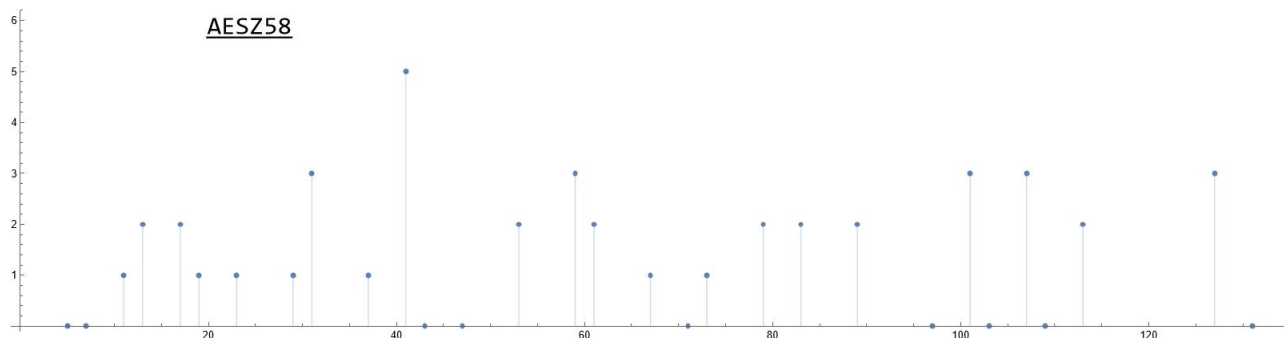
AESZ36 was studied in [47] and is known to possess a rational rank two attractor.

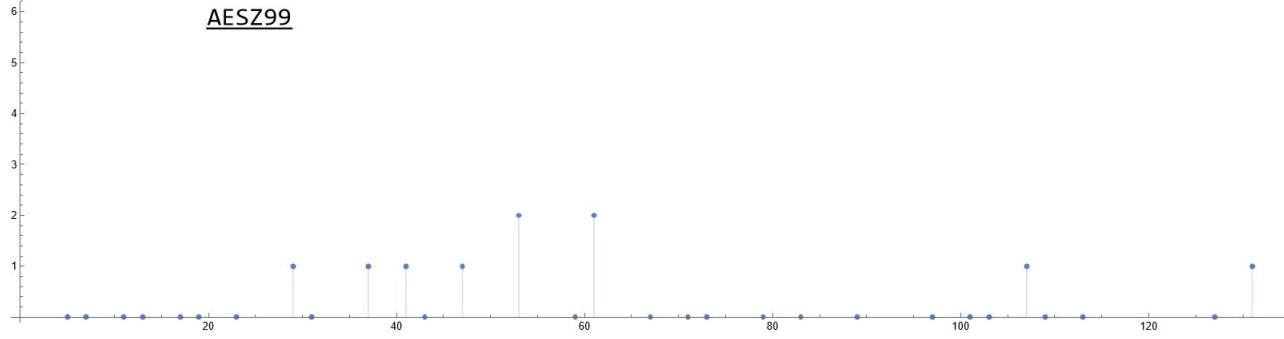
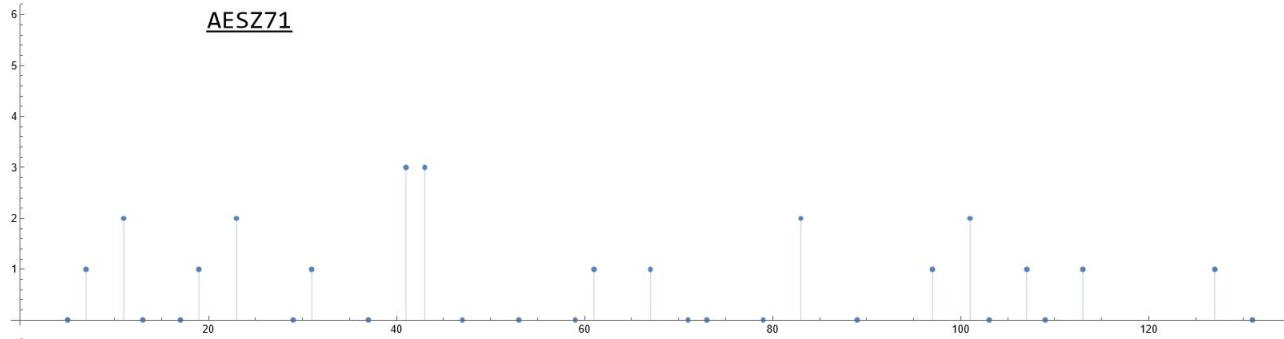
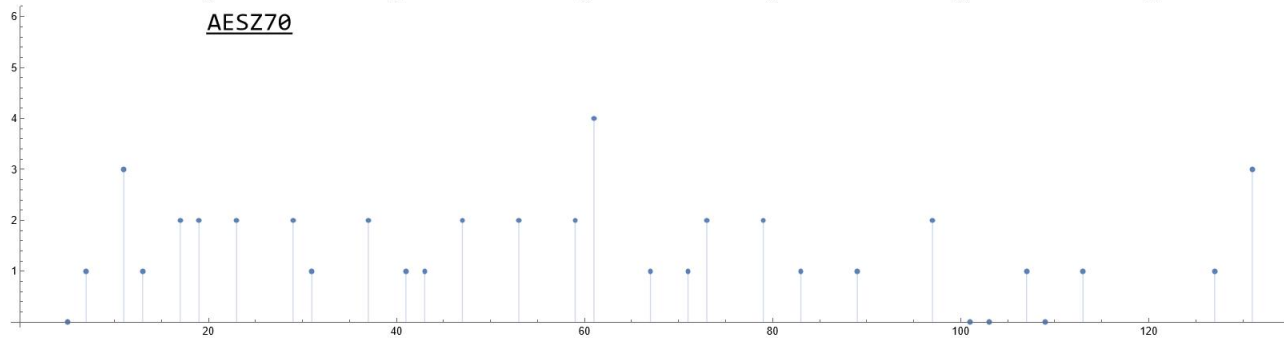
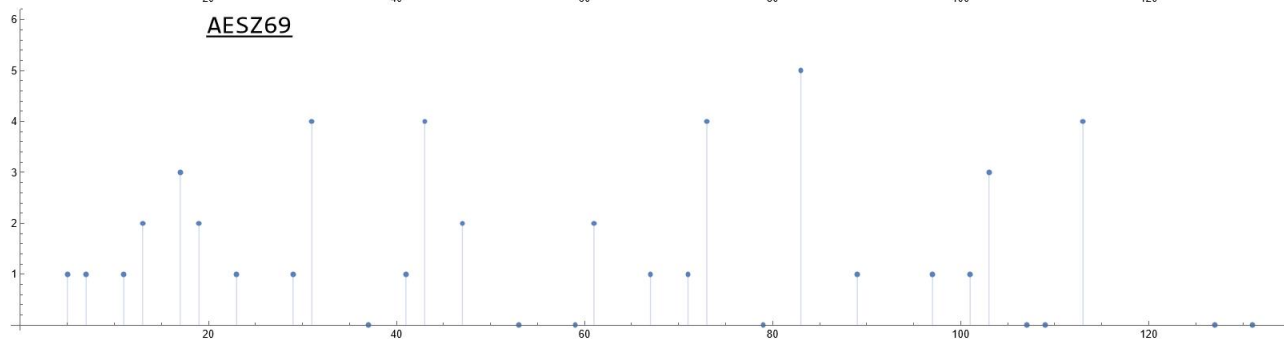
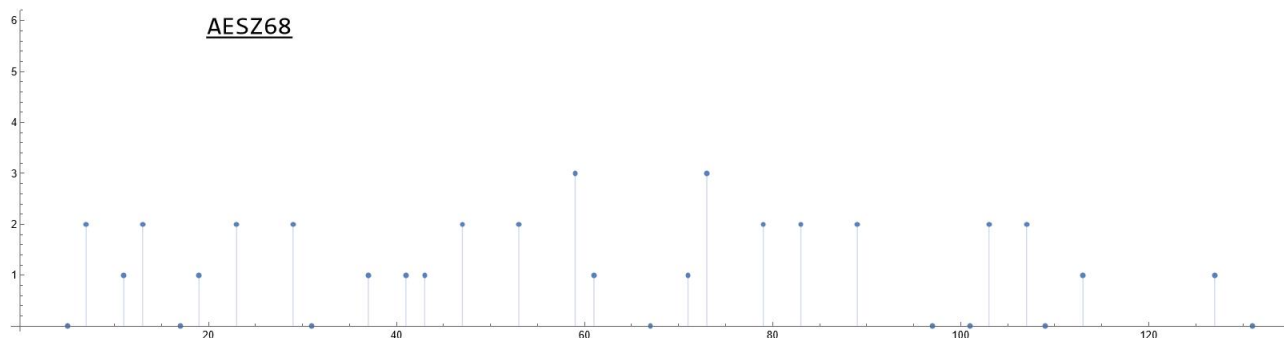


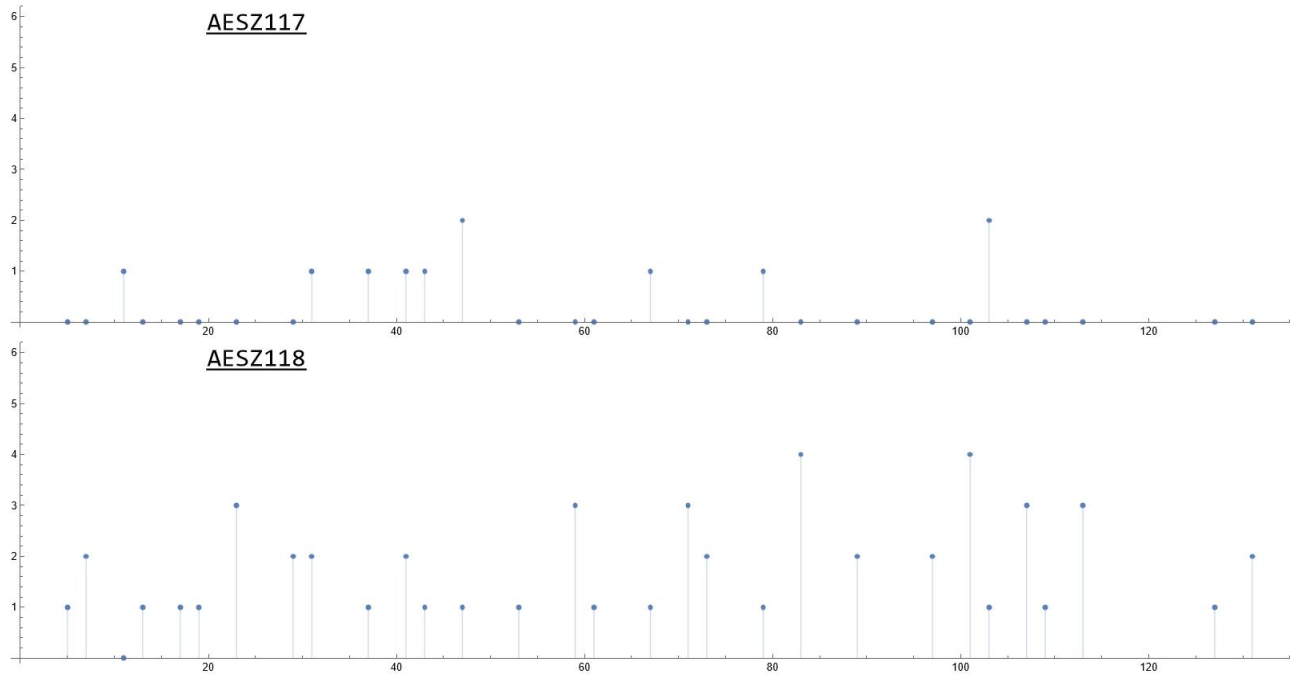


AESZ55 was studied in [47] and is known to possess a rational rank two attractor.

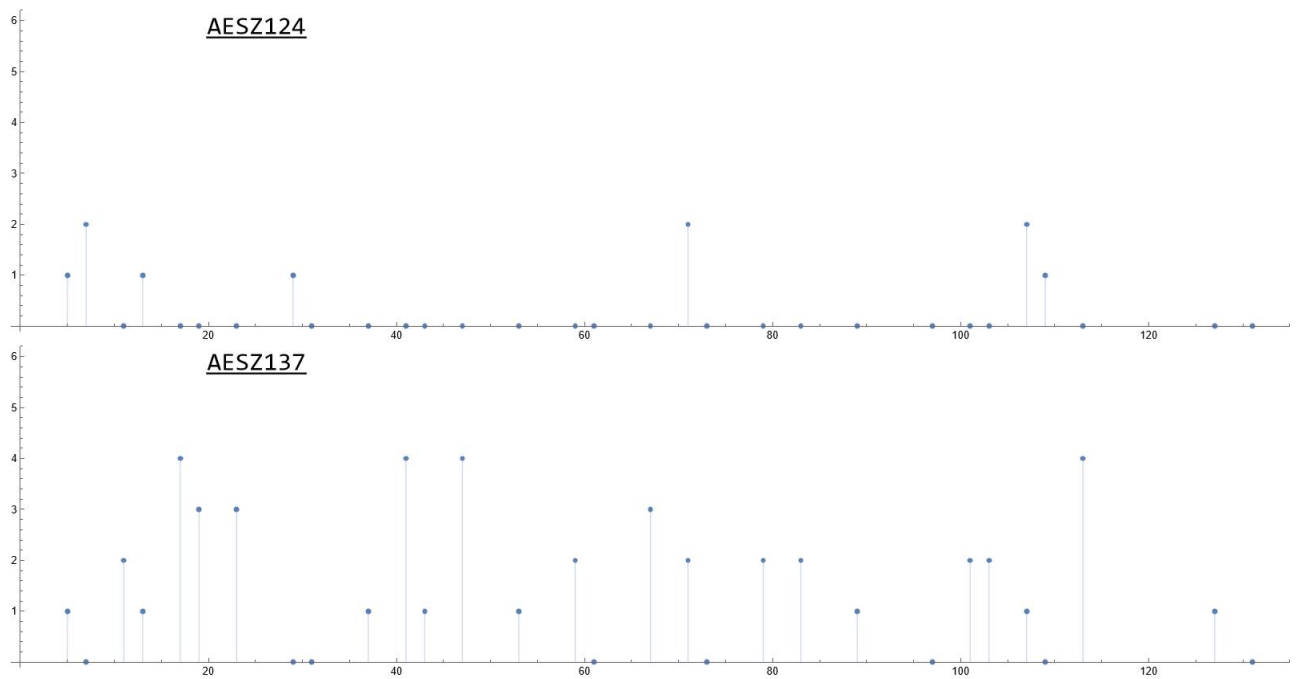


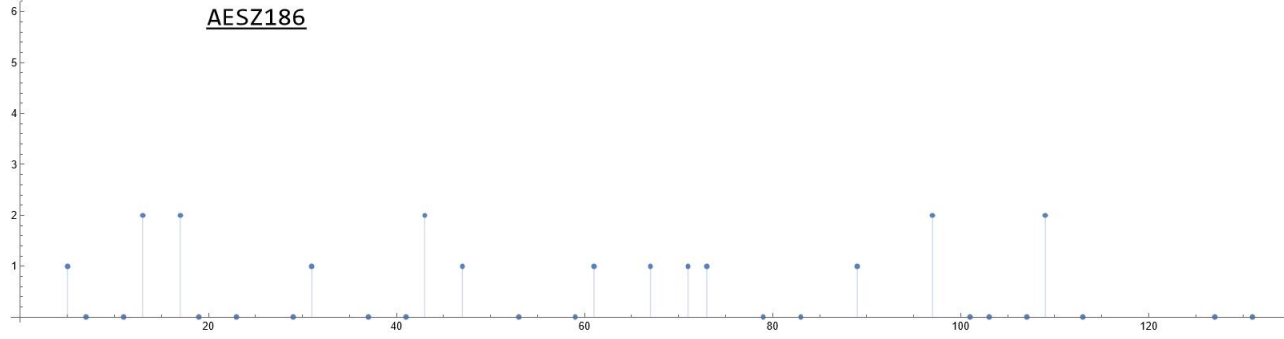
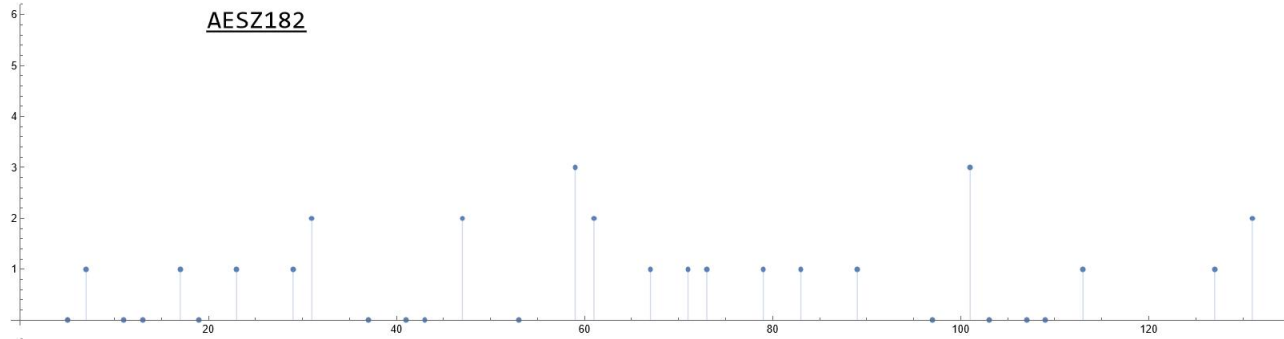
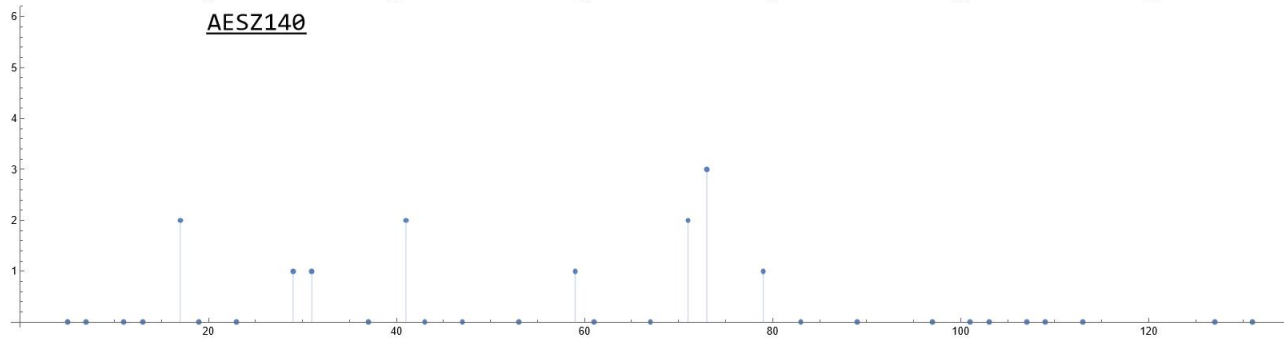
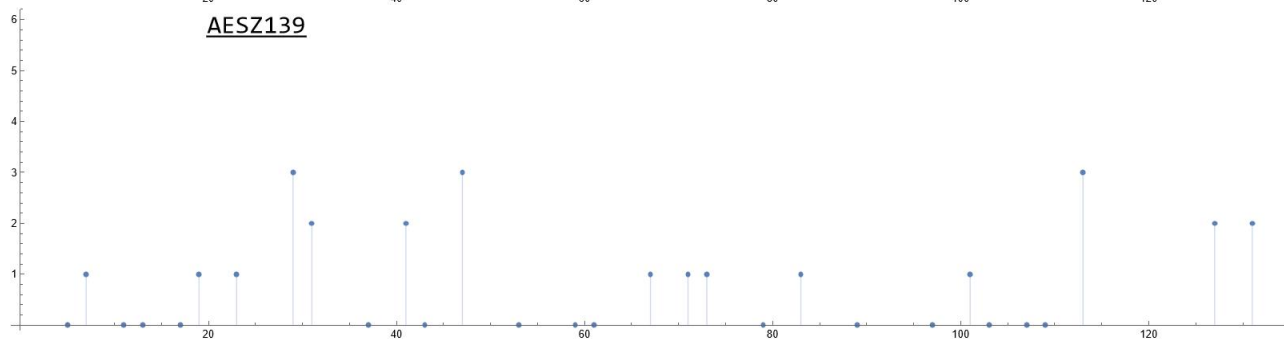
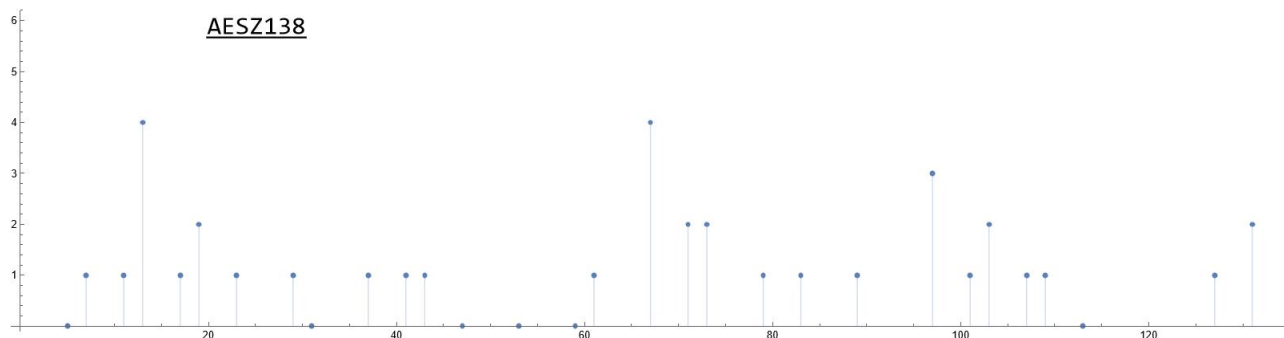


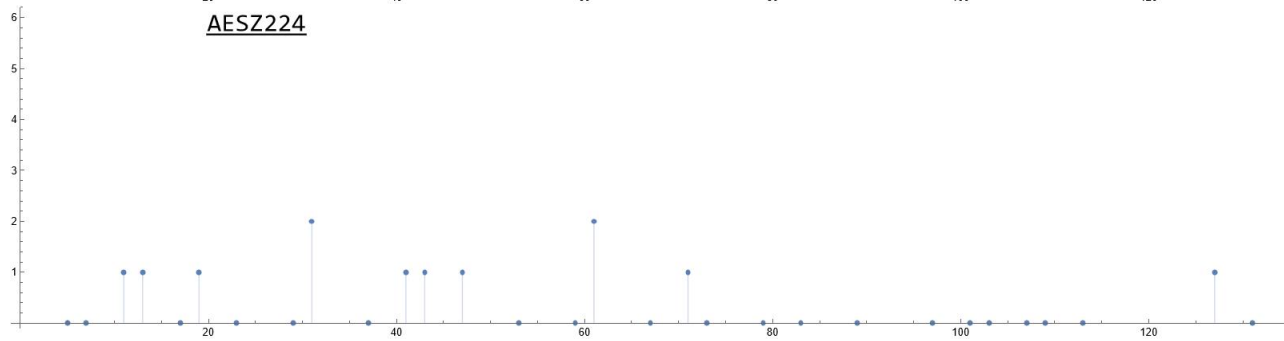
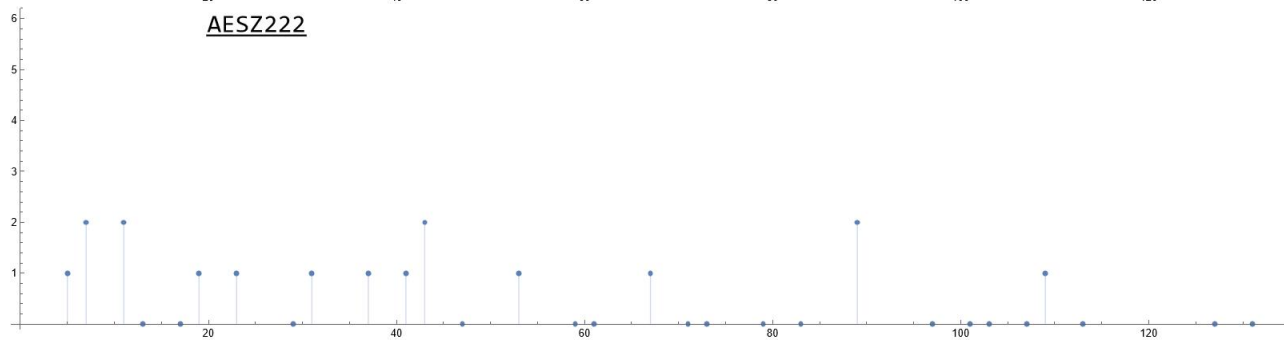
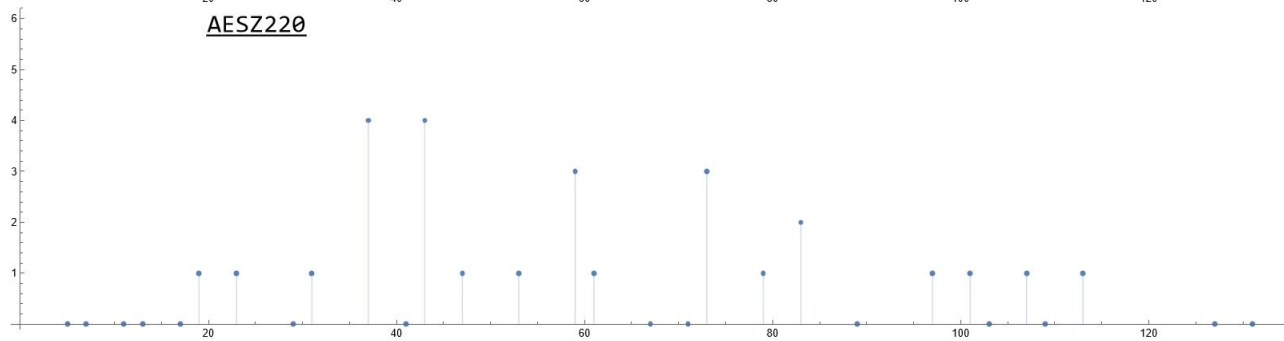
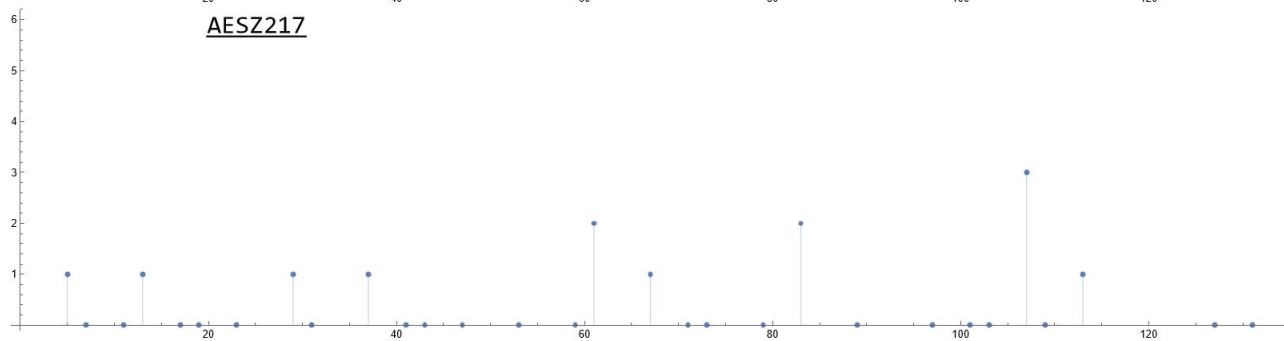
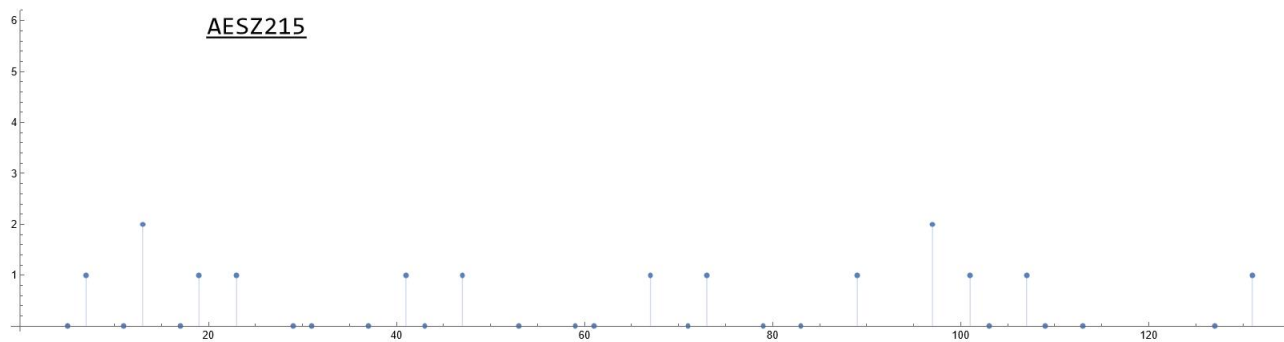


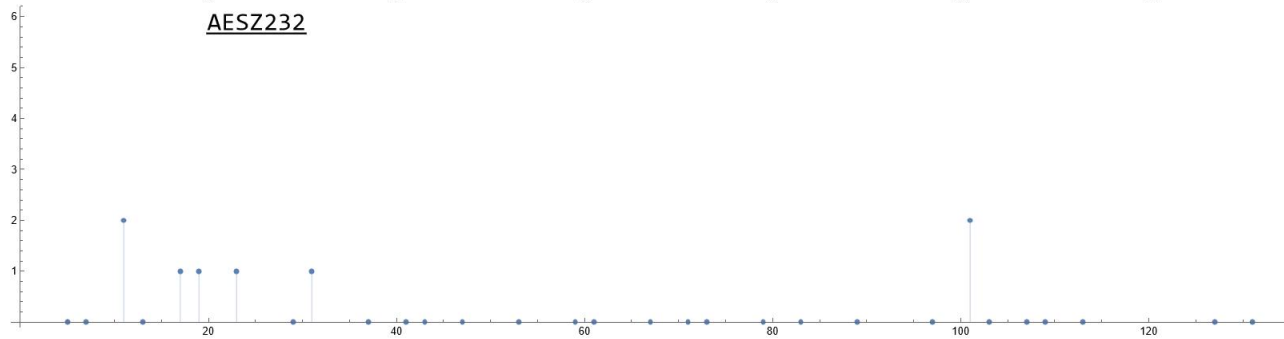
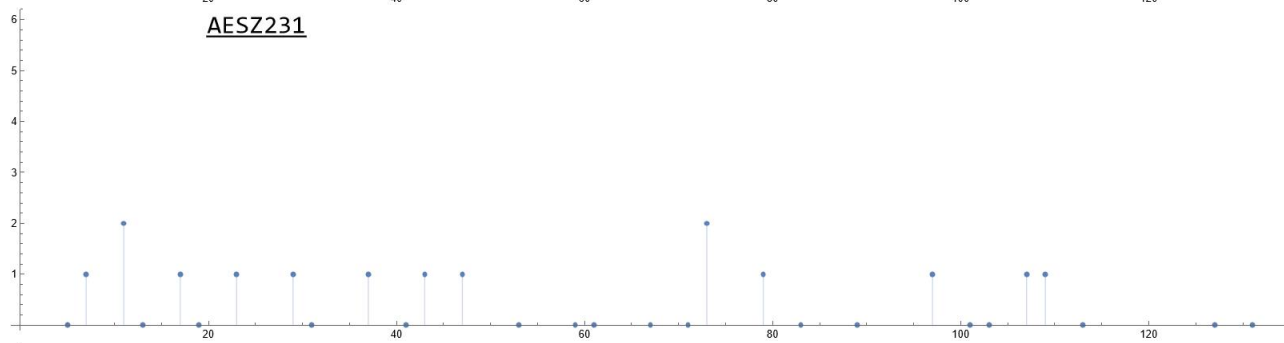
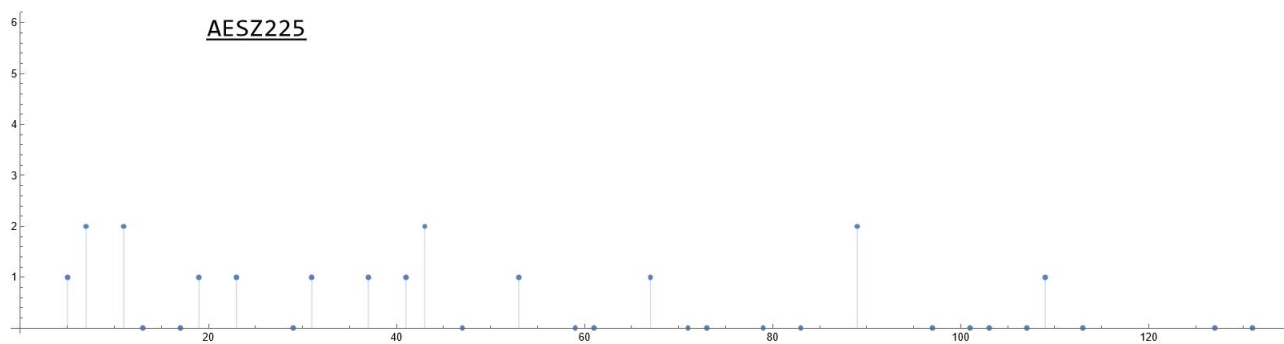


AESZ118 is equivalent to AESZ22 after a change of variables (see the AESZ22 plot).









Bibliography

- [1] J. McGovern and P. Candelas, “Joseph McGovern DPhil Thesis Data.”
<http://dx.doi.org/10.5287/ora-5r2n780jk>, 2023. DOI: 10.5287/ora-5r2n780jk.
- [2] P. Candelas, X. de la Ossa, P. Kuusela and J. McGovern, *Mirror Symmetry for Five-Parameter Hulek-Verrill Manifolds*, 2111.02440.
- [3] P. Candelas, X. de la Ossa, P. Kuusela and J. McGovern, *Flux Vacua and Modularity for $\mathbb{Z}_2$ Symmetric Calabi-Yau Manifolds*, 2302.03047.
- [4] P. Candelas, P. Kuusela and J. McGovern, *Attractors with large complex structure for one-parameter families of Calabi-Yau manifolds*, *JHEP* **11** (2021) 032 [2104.02718].
- [5] A. Wiles, *Modular elliptic curves and Fermat’s last theorem*, *Ann. of Math. (2)* **141** (1995) 443.
- [6] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, *Ann. of Math. (2)* **141** (1995) 553.
- [7] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises*, *J. Amer. Math. Soc.* **14** (2001) 843.
- [8] N. Yui, *Modularity of Calabi–Yau Varieties: 2011 and Beyond*, pp. 101–139. Springer New York, New York, NY, 2013. 10.1007/978-1-4614-6403-7_4.
- [9] G. W. Moore, *Arithmetic and attractors*, [hep-th/9807087](#).
- [10] G. W. Moore, *Attractors and arithmetic*, [hep-th/9807056](#).

- [11] S. Kachru, R. Nally and W. Yang, *Flux Modularity, F-Theory, and Rational Models*, 2010.07285.
- [12] S. Kachru, R. Nally and W. Yang, *Supersymmetric Flux Compactifications and Calabi-Yau Modularity*, 2001.06022.
- [13] K. Bönisch, A. Klemm, E. Scheidegger and D. Zagier, *D-brane masses at special fibres of hypergeometric families of Calabi-Yau threefolds, modular forms, and periods*, 2203.09426.
- [14] P. Candelas, X. de la Ossa, M. Elmi and D. Van Straten, *A One Parameter Family of Calabi-Yau Manifolds with Attractor Points of Rank Two*, *JHEP* **10** (2020) 202 [1912.06146].
- [15] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].
- [16] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories*, *Nucl. Phys. B* **405** (1993) 279 [hep-th/9302103].
- [17] M.-x. Huang, A. Klemm and S. Quackenbush, *Topological string theory on compact Calabi-Yau: Modularity and boundary conditions*, *Lect. Notes Phys.* **757** (2009) 45 [hep-th/0612125].
- [18] R. Gopakumar and C. Vafa, *M theory and topological strings. 1.*, hep-th/9809187.
- [19] R. Gopakumar and C. Vafa, *M theory and topological strings. 2.*, hep-th/9812127.
- [20] I. Bah, D. S. Freed, G. W. Moore, N. Nekrasov, S. S. Razamat and S. Schafer-Nameki, *A Panorama Of Physical Mathematics c. 2022*, 2211.04467.
- [21] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, *Nucl. Phys. B* **258** (1985) 46.

- [22] P. S. Aspinwall, *K3 surfaces and string duality*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 421–540, 11, 1996, [hep-th/9611137](#).
- [23] S. Filip, *An introduction to K3 surfaces and their dynamics*, in *Teichmüller theory and dynamics*, vol. 58 of *Panor. Synthèses*, pp. 1–46. Soc. Math. France, Paris, [2022] ©2022.
- [24] P. Candelas and X. de la Ossa, *Moduli Space of Calabi-Yau Manifolds*, *Nucl. Phys. B* **355** (1991) 455.
- [25] S.-T. Yau, *A survey of Calabi-Yau manifolds*, in *Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry*, vol. 13 of *Surv. Differ. Geom.*, pp. 277–318. Int. Press, Somerville, MA, 2009. DOI.
- [26] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, *Commun. Math. Phys.* **167** (1995) 301 [[hep-th/9308122](#)].
- [27] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, *Nucl. Phys. B* **433** (1995) 501 [[hep-th/9406055](#)].
- [28] P. Berglund, P. Candelas, X. De La Ossa, A. Font, T. Hubsch, D. Jancic et al., *Periods for Calabi-Yau and Landau-Ginzburg vacua*, *Nucl. Phys. B* **419** (1994) 352 [[hep-th/9308005](#)].
- [29] A. Libgober, *Chern classes and the periods of mirrors*, *arXiv preprint math/9803119* (1998) .
- [30] P. Candelas, A. M. Dale, C. A. Lutken and R. Schimmrigk, *Complete Intersection Calabi-Yau Manifolds*, *Nucl. Phys. B* **298** (1988) 493.

- [31] Andre Lukas, “Andre Lukas’s Webpage.”
<https://www-thphys.physics.ox.ac.uk/people/AndreLukas/Site/Andre.Lukas.html>,
 2023.
- [32] M. Kreuzer and H. Skarke, *Complete classification of reflexive polyhedra in four-dimensions*, *Adv. Theor. Math. Phys.* **4** (2000) 1209 [[hep-th/0002240](#)].
- [33] E. A. Rødland, *The pfaffian calabi–yau, its mirror, and their link to the grassmannian $g(2, 7)$* , *Compositio Mathematica* **122** (2000) 135.
- [34] K. Hori and J. Knapp, *A pair of Calabi-Yau manifolds from a two parameter non-Abelian gauged linear sigma model*, [1612.06214](#).
- [35] A. Weil, *Numbers of solutions of equations in finite fields*, *Bull. Amer. Math. Soc.* **55** (1949) 497.
- [36] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, *American Journal of Mathematics* **82** (1960) 631.
- [37] A. Grothendieck, *Formule de Lefschetz et rationalité des fonctions L* , in *Séminaire Bourbaki, Vol. 9*, pp. Exp. No. 279, 41–55. Soc. Math. France, Paris, 1995.
- [38] P. Deligne, *La conjecture de Weil. I*, *Inst. Hautes Études Sci. Publ. Math.* (1974) 273.
- [39] H. Hasse, *Zur Theorie der abstrakten elliptischen Funktionenkörper. I: Die Struktur der Gruppe der Divisorenklassen endlicher Ordnung*, *J. Reine Angew. Math.* **175** (1936) 55.
- [40] The LMFDB Collaboration, “The L-functions and Modular Forms Database.”
<https://www.lmfdb.org/>, 2023.
- [41] F. Q. Gouvêa and N. Yui, *Rigid calabi–yau threefolds over q are modular*, *Expositiones Mathematicae* **29** (2011) 142.

- [42] E. Frenkel, *Lectures on the Langlands program and conformal field theory*, in *Les Houches School of Physics: Frontiers in Number Theory, Physics and Geometry*, pp. 387–533, 2007, [hep-th/0512172](#), DOI.
- [43] S. Gelbart, *An elementary introduction to the Langlands program*, *Bull. Amer. Math. Soc. (N.S.)* **10** (1984) 177.
- [44] M. Goresky, *Langlands’ conjectures for physicists*, March, 2004.
- [45] K. Hulek and H. Verrill, *On modularity of rigid and nonrigid Calabi-Yau varieties associated to the root lattice A_4* , *Nagoya Mathematical Journal* **179** (2005) 103 .
- [46] C. Schoen, *On the geometry of a special determinantal hypersurface associated to the mumford-horrocks vector bundle.*, *Journal für die reine und angewandte Mathematik* **364** (1986) 85.
- [47] K. Bönisch, M. Elmi, A.-K. Kashani-Poor and A. Klemm, *Time reversal and CP invariance in Calabi-Yau compactifications*, *JHEP* **09** (2022) 019 [[2204.06506](#)].
- [48] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, *Calabi-Yau manifolds over finite fields. 1.*, [hep-th/0012233](#).
- [49] P. Candelas, X. de la Ossa and F. Rodriguez Villegas, *Calabi-Yau manifolds over finite fields. 2.*, *Fields Inst. Commun.* **38** (2013) 121 [[hep-th/0402133](#)].
- [50] P. Candelas, X. De La Ossa and D. Van Straten, *Local Zeta Functions From Calabi-Yau Differential Equations*, [2104.07816](#).
- [51] S. N. Kadir, *The Arithmetic of Calabi-Yau manifolds and mirror symmetry*, dphil thesis, 9, 2004. [hep-th/0409202](#).
- [52] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa et al., *Mirror symmetry*, vol. 1 of *Clay mathematics monographs*. AMS, Providence, USA, 2003.

- [53] V. Braun, M. Kreuzer, B. A. Ovrut and E. Scheidegger, *Worldsheet Instantons and Torsion Curves*, *AMS/IP Stud. Adv. Math.* **44** (2008) 231 [0801.4154].
- [54] V. Braun, M. Kreuzer, B. A. Ovrut and E. Scheidegger, *Worldsheet instantons and torsion curves, part A: Direct computation*, *JHEP* **10** (2007) 022 [hep-th/0703182].
- [55] V. Braun, M. Kreuzer, B. A. Ovrut and E. Scheidegger, *Worldsheet Instantons and Torsion Curves, Part B: Mirror Symmetry*, *JHEP* **10** (2007) 023 [0704.0449].
- [56] A. Klemm, M. Kreuzer, E. Riegler and E. Scheidegger, *Topological string amplitudes, complete intersection Calabi-Yau spaces and threshold corrections*, *JHEP* **05** (2005) 023 [hep-th/0410018].
- [57] M. Demirtas, A. Rios-Tascon and L. McAllister, *CYTools: A Software Package for Analyzing Calabi-Yau Manifolds*, 2211.03823.
- [58] M. Demirtas, M. Kim, L. McAllister, J. Moritz and A. Rios-Tascon, *Computational Mirror Symmetry*, 2303.00757.
- [59] A. Richards, *University of oxford advanced research computing*, Aug., 2015. 10.5281/zenodo.22558.
- [60] S. Hosono and H. Takagi, *Mirror symmetry and projective geometry of reye congruences i*, *Journal of Algebraic Geometry* **23** (2014) 279.
- [61] P. Candelas and R. Davies, *New Calabi-Yau Manifolds with Small Hodge Numbers*, *Fortsch. Phys.* **58** (2010) 383 [0809.4681].
- [62] R. Gopakumar and C. Vafa, *Branes and fundamental groups*, *Adv. Theor. Math. Phys.* **2** (1998) 399 [hep-th/9712048].
- [63] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, vol. 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999, 10.1090/surv/068.

- [64] P. Candelas, *Yukawa Couplings Between (2,1) Forms*, *Nucl. Phys. B* **298** (1988) 458.
- [65] D. van Straten and P. Metelitsyn, “Calabi-Yau differential operator database v.3.”
<https://cydb.mathematik.uni-mainz.de>, 2023.
- [66] J. L. Bourjaily, A. J. McLeod, C. Vergu, M. Volk, M. Von Hippel and M. Wilhelm,
Embedding Feynman Integral (Calabi-Yau) Geometries in Weighted Projective Space,
JHEP **01** (2020) 078 [[1910.01534](#)].
- [67] E. Witten, *Phases of $N=2$ theories in two-dimensions*, *Nucl. Phys. B* **403** (1993) 159
[[hep-th/9301042](#)].
- [68] S. Hosono and H. Takagi, *Double quintic symmetroids, reye congruences, and their
derived equivalence*, *Journal of Differential Geometry* **104** (2016) 443.
- [69] M. Reid, *The moduli space of 3-folds with $k = 0$ may nevertheless be irreducible.*,
Mathematische Annalen **278** (1987) 329.
- [70] M. Gross, *Primitive calabi-yau threefolds*, *Journal of Differential Geometry* **45** (1997)
288.
- [71] P. Candelas, P. S. Green and T. Hubsch, *Rolling Among Calabi-Yau Vacua*, *Nucl.
Phys. B* **330** (1990) 49.
- [72] P. Candelas and X. C. de la Ossa, *Comments on Conifolds*, *Nucl. Phys. B* **342**
(1990) 246.
- [73] P. Candelas, P. S. Green and T. Hubsch, *Finite Distances Between Distinct
Calabi-yau Vacua: (Other Worlds Are Just Around the Corner)*, *Phys. Rev. Lett.* **62**
(1989) 1956.
- [74] B. R. Greene, D. R. Morrison and A. Strominger, *Black hole condensation and the
unification of string vacua*, *Nucl. Phys. B* **451** (1995) 109 [[hep-th/9504145](#)].

- [75] A. Lukas and F. Ruehle, *Symmetries of Calabi-Yau prepotentials with isomorphic flops*, *JHEP* **02** (2023) 175 [2210.09369].
- [76] C. Brodie, A. Constantin, A. Lukas and F. Ruehle, *Flops for complete intersection Calabi-Yau threefolds*, *J. Geom. Phys.* **186** (2023) 104767 [2112.12106].
- [77] S. Hosono, H. Takagi et al., *Movable vs monodromy nilpotent cones of calabi-yau manifolds*, *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* **14** (2018) 039.
- [78] J. K. Kim, C. Park and Y. Yoon, *Calabi-yau manifold from complete intersections in products of weighted complex projective spaces*, *Physics Letters B* **224** (1989) 108.
- [79] B. R. Greene, *Superconformal compactifications in weighted projective space*, *Commun. Math. Phys.* **130** (1990) 335.
- [80] S. Yamaguchi and S.-T. Yau, *Topological string partition functions as polynomials*, *JHEP* **07** (2004) 047 [hep-th/0406078].
- [81] M. Alim and J. D. Lange, *Polynomial Structure of the (Open) Topological String Partition Function*, *JHEP* **10** (2007) 045 [0708.2886].
- [82] M. Alim and E. Scheidegger, *Topological Strings on Elliptic Fibrations*, *Commun. Num. Theor. Phys.* **08** (2014) 729 [1205.1784].
- [83] M.-x. Huang, S. Katz and A. Klemm, *Topological String on elliptic CY 3-folds and the ring of Jacobi forms*, *JHEP* **10** (2015) 125 [1501.04891].
- [84] M. A. J. Elmi, *Modular Calabi-Yau threefolds in string compactifications*, Ph.D. thesis, Oxford U., 2020.
- [85] M. Alim, *Lectures on Mirror Symmetry and Topological String Theory*, 1207.0496.
- [86] S. Alexandrov, S. Feyzbakhsh, A. Klemm, B. Pioline and T. Schimannek, *Quantum geometry, stability and modularity*, 2301.08066.

- [87] S. Alexandrov, N. Gaddam, J. Manschot and B. Pioline, *Modular bootstrap for $D4$ - $D2$ - $D0$ indices on compact Calabi-Yau threefolds*, 2204.02207.
- [88] C. F. Cota, A. Klemm and T. Schimannek, *Topological strings on genus one fibered Calabi-Yau 3-folds and string dualities*, *JHEP* **11** (2019) 170 [1910.01988].
- [89] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, *Nucl. Phys. B* **359** (1991) 21.
- [90] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, *Inventiones mathematicae* **139** (2000) 173 [math/9810173].
- [91] K. Becker, M. Becker and J. H. Schwarz, *String theory and M-theory: A modern introduction*. Cambridge University Press, 12, 2006, 10.1017/CBO9780511816086.
- [92] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 5, 2012.
- [93] A. Strominger, *Special Geometry*, *Commun. Math. Phys.* **133** (1990) 163.
- [94] A. Strominger, *Macroscopic entropy of $N=2$ extremal black holes*, *Phys. Lett. B* **383** (1996) 39 [hep-th/9602111].
- [95] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev. D* **54** (1996) 1514 [hep-th/9602136].
- [96] F. Denef, *Supergravity flows and D-brane stability*, *JHEP* **08** (2000) 050 [hep-th/0005049].
- [97] M. Kontsevich, *Enumeration of rational curves via Torus actions*, hep-th/9405035.
- [98] M. Shmakova, *Calabi-Yau black holes*, *Phys. Rev. D* **56** (1997) 540 [hep-th/9612076].
- [99] M. R. Douglas and S. Kachru, *Flux compactification*, *Rev. Mod. Phys.* **79** (2007) 733 [hep-th/0610102].

- [100] K. Dasgupta, G. Rajesh and S. Sethi, *M theory, orientifolds and G - flux*, *JHEP* **08** (1999) 023 [[hep-th/9908088](#)].
- [101] S. Kachru, R. Kallosh, A. D. Linde and S. P. Trivedi, *De Sitter vacua in string theory*, *Phys. Rev. D* **68** (2003) 046005 [[hep-th/0301240](#)].
- [102] A. Sen, *Orientifold limit of F theory vacua*, *Phys. Rev. D* **55** (1997) R7345 [[hep-th/9702165](#)].
- [103] A. Collinucci, *New F-theory lifts*, *JHEP* **08** (2009) 076 [[0812.0175](#)].
- [104] A. Collinucci, *New F-theory lifts. II. Permutation orientifolds and enhanced singularities*, *JHEP* **04** (2010) 076 [[0906.0003](#)].
- [105] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, *de sitter vacua in string theory*, *Physical Review D* **68** (2003) .
- [106] E. Witten, *Nonperturbative superpotentials in string theory*, *Nucl. Phys. B* **474** (1996) 343 [[hep-th/9604030](#)].
- [107] M. Cicoli, J. P. Conlon, A. Maharana, S. Parameswaran, F. Quevedo and I. Zavala, *String Cosmology: from the Early Universe to Today*, [2303.04819](#).
- [108] J. P. Conlon, *Moduli Stabilisation and Applications in IIB String Theory*, *Fortsch. Phys.* **55** (2007) 287 [[hep-th/0611039](#)].
- [109] M. Dine and N. Seiberg, *Is the Superstring Weakly Coupled?*, *Phys. Lett. B* **162** (1985) 299.
- [110] O. DeWolfe, *Enhanced symmetries in multiparameter flux vacua*, *JHEP* **10** (2005) 066 [[hep-th/0506245](#)].
- [111] P. Kuusela, *Modular Calabi-Yau manifolds, attractor points, and flux vacua*, Ph.D. thesis, Oxford University, 2022.

- [112] G. Almkvist, C. van Enckevort, D. van Straten and W. Zudilin, *Tables of calabi–yau equations*, 2010.
- [113] D. van Straten, *Calabi–yau operators*, 2017.
- [114] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers*. McGraw-Hill Book Co., New York,, 1978.
- [115] M. Kontsevich, *Homological Algebra of Mirror Symmetry*, [alg-geom/9411018](#).
- [116] A. A. Beilinson, *Higher regulators and values of L-functions*, in *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pp. 181–238. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.
- [117] C. van Enckevort and D. van Straten, *Monodromy calculations of fourth order equations of Calabi-Yau type*, in *Mirror symmetry. V*, vol. 38 of *AMS/IP Stud. Adv. Math.*, pp. 539–559. Amer. Math. Soc., Providence, RI, 2006. DOI.
- [118] S. Hosono, B. H. Lian and S.-T. Yau, *GKZ generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces*, *Commun. Math. Phys.* **182** (1996) 535 [[alg-geom/9511001](#)].
- [119] D. A. Cox, J. B. Little and H. K. Schenck, *Toric varieties*, vol. 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011, 10.1090/gsm/124.
- [120] L. Borisov, *Towards the mirror symmetry for calabi-yau complete intersections in gorenstein toric fano varieties*, 1993.
- [121] V. V. Batyrev and L. A. Borisov, *On Calabi-Yau complete intersections in toric varieties*, [alg-geom/9412017](#).
- [122] V. V. Batyrev, *Hodge Theory of Hypersurfaces in Toric Varieties and Recent Developments in Quantum Physics*, 2308.15187.

- [123] V. Batyrev and B. Nill, *Combinatorial aspects of mirror symmetry*, in *Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics*, vol. 452 of *Contemp. Math.*, pp. 35–66. Amer. Math. Soc., Providence, RI, 2008. DOI.
- [124] V. Batyrev and B. Nill, *Multiples of lattice polytopes without interior lattice points*, *Mosc. Math. J.* **7** (2007) 195.