# Estimates and regularity for some fully nonlinear PDEs in conformal geometry

Jonah A. J. Duncan

Keble College

University of Oxford

Supervised by Prof. Luc Nguyen

A thesis submitted for the degree of

*Doctor of Philosophy*

Trinity 2021

To my parents and grandparents

# Acknowledgements

I would first like to thank my supervisor, Prof. Luc Nguyen, for his support, guidance and collaboration during these last few years. Thank you Luc for being so generous with your time and for providing so many interesting problems and insightful comments. Your mentorship, comprehensive expertise and incisive ability to see through a problem and pose thought-provoking questions have been pivotal to my development in this new world, and are qualities that I deeply aspire to develop in myself.

My time at Oxford would not have been nearly as enjoyable or productive without the vibrant community that is OxPDE; it has been a pleasure to learn from both established faculty and my PDE CDT colleagues alike. I am particularly grateful to Prof. Melanie Rupflin and Prof. Endre Süli, two friendly faces who have always been willing to give up their time to read my work, provide valuable feedback and help with my career development. A special mention also to my officemates and friends: John, James and Asad. I hope we will be back distracting each other in the office again soon. And thank you to James and Francis for helping to preserve some sense of (socially-distanced) normalcy during the pandemic.

I have always been very fortunate to be surrounded by a strong support network of family and friends. Thank you to my parents, Pippa and Alex, who have provided me with so many opportunities and have always remained unconditionally loving and supportive. Thank you to my friends from Oxford, Warwick and back home in London, for giving me a life outside of mathematics (although I know they like to think I don't have one). And thank you to Molly, not least for putting up with me every day of this pandemic. I am so lucky to have you by my side to lighten up stressful times, and to make the good times so much more special.

# Statement of Originality

I declare that the contents of this thesis are, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known, nor has any part of this thesis been submitted for a degree at another university. The results in §2.1–2.6 are the product of joint work with Luc Nguyen, and may also be found in the preprint [**DN20**]. The results in Chapter 4 are also the product of joint work with Luc Nguyen.

# Abstract

In this thesis we obtain new estimates and regularity results for some fully nonlinear elliptic equations arising in conformal geometry.

Our first set of new results concern local pointwise second derivative estimates for elliptic solutions to the $\sigma_k$-Yamabe equation. In Chapter 2 we obtain such estimates for $W^{2,p}$-strong solutions on Euclidean domains, addressing both the so-called positive and negative cases. We explore two methods for obtaining these estimates: an integrability improvement argument coupled with Moser iteration, and a method using the Alexandrov-Bakelman-Pucci estimate. Our estimates are obtained in the more general context of augmented Hessian equations. In Chapter 3 we obtain similar estimates for smooth solutions on manifolds when $k = 2$. Our work here contributes to a growing literature on the regularity theory for the $\sigma_k$-Yamabe equation and, from a broader perspective, the regularity theory for fully nonlinear, non-uniformly elliptic equations.

In Chapter 4 we study the existence of conformal metrics satisfying $g^{-1}A_g^\tau \in \Gamma_2^+$, where $A_g^\tau$ is the trace-modified Schouten tensor. When $\tau = 1$, this is an important question in the context of the $\sigma_2$-Yamabe problem, and it is also of independent geometric and topological interest when $\tau \leq 1$. Our focus will be on three dimensions; we prove a new existence result when $\tau < 1$, and obtain a new proof of a result of Ge, Lin & Wang [**GLW10**] when $\tau = 1$, with an eye towards tackling some related problems.

In Chapter 5 we obtain integral estimates for a fourth order perturbation of the (trace-modified) $\sigma_2$-Yamabe equation in three dimensions, in the spirit of Chang, Gursky & Yang [**CGY02b**]. Our study of this equation is partly motivated by the existence problems considered in Chapter 4, but also from the analytic viewpoint of using fourth order regularisations to study non-uniformly elliptic PDEs of second order.

# Contents

# Chapter 1

# Introduction

Fully nonlinear elliptic partial differential equations (PDEs) involving the eigenvalues of the Hessian of an unknown function have been the subject of intensive study over the last few decades, since the seminal work of Caffarelli, Kohn, Nirenberg & Spruck in the series of papers [**CNS84, CKNS85, CNS85, CNS86, CNS88**]. Perhaps the most well-known examples of these PDEs are formulated using the *symmetric elementary polynomials*:

**Definition 1.0.1.** Let $\mathrm{Sym}_n(\mathbb{R})$ denote the space of real symmetric $n \times n$ matrices and let $1 \leq k \leq n$ be an integer. For a matrix $A \in \mathrm{Sym}_n(\mathbb{R})$ with eigenvalues $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, we define the *k'th symmetric elementary polynomial of A* by

$$\sigma_k(A) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k}.$$

In particular, $\sigma_1(A) = \mathrm{tr}(A)$, $\sigma_2(A) = \frac{1}{2}(\mathrm{tr}(A)^2 - |A|^2)$ and $\sigma_n(A) = \det(A)$.

In this thesis we will be concerned with elliptic solutions to equations of the form

$$\sigma_k^{1/k}\Big(\nabla^2 u(x) - H[u](x)\Big) = f\big(x, u(x), \nabla u(x)\big) > 0, \tag{1.0.1}$$

where $H[u](x) = H(x, u(x), \nabla u(x)) \in \mathrm{Sym}_n(\mathbb{R})$. We will also study solutions to some variants of (1.0.1), such as a quotient equation involving the $\sigma_1$ and $\sigma_2$ operators, and solutions to a fourth order perturbation of (1.0.1) when $k = 2$.

## 1.1 An overview of the equations considered in this thesis

We begin this introduction with an overview of some of the equations considered in this thesis. The focus will be on familiarising the reader with (1.0.1); we will explain the ellipticity and concavity properties of (1.0.1), and explain how equations of this form are to be interpreted on Riemannian manifolds. We will also introduce a special case of (1.0.1) known as the $\sigma_k$-*Yamabe equation*, which serves as our primary motivation throughout this thesis. We postpone any discussion on the aforementioned variants of (1.0.1) until later.

### 1.1.1 Augmented Hessian equations

In this thesis we will restrict our attention to the case $k \geq 2$ in (1.0.1), which ensures that these equations are fully nonlinear; we will not consider the case $k = 1$, in which (1.0.1) is semilinear.

Let us start by considering (1.0.1) on a domain $\Omega \subset \mathbb{R}^n$. As mentioned above, we will be concerned with elliptic solutions to (1.0.1). It is shown in [**CNS85**] that (1.0.1) is elliptic (non-uniformly, *a priori*) when restricted to solutions $u$ for which the eigenvalues of

$$A_H[u](x) = \nabla^2 u(x) - H[u](x)$$

belong to the positive cone

$$\Gamma_k^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \sigma_j(\lambda_1, \dots, \lambda_n) > 0 \text{ for } 1 \leq j \leq k\}$$

for all $x \in \Omega$. Note that when $k = n$ and $H \equiv 0$, we recover the well-known fact that the Monge-Ampère equation is elliptic when restricted to either convex or concave solutions; for a detailed discussion on the general notion of ellipticity for fully nonlinear equations, we refer the reader to [**Kry95**].

From now on, we will therefore consider in place of (1.0.1) the elliptic equation

$$\sigma_k^{1/k}\big(A_H[u](x)\big) = f(x, u(x), \nabla u(x)) > 0, \quad A_H[u](x) \in \Gamma_k^+ \quad \text{for } x \in \Omega. \tag{1.1.1}$$

**Notation:** For $A \in \mathrm{Sym}_n(\mathbb{R})$, we write $A \in \Gamma_k^+$ as shorthand for $\lambda(A) \in \Gamma_k^+$, where $\lambda(A) \in \mathbb{R}^n$ denotes the vector of eigenvalues of $A$.

A further useful feature is that $\sigma_k^{1/k}$ is concave when restricted to matrices with eigenvalues in $\Gamma_k^+$ [**Gar59**]. Moreover, *a priori* $C^2$ estimates on solutions to (1.1.1) yield uniform ellipticity. Therefore, once *a priori* $C^2$ estimates are known, one may apply the regularity theory of Evans-Krylov [**Eva82,Kry82**] (see also [**CC95**]) for fully nonlinear, uniformly elliptic concave equations to obtain an *a priori* $C^{2,\alpha}$ estimate. Classical Schauder theory then allows one to bootstrap according to the regularity of $f$. Therefore, in the context of existence and regularity theory for (1.1.1), $C^2$ estimates are key.

We now explain how (1.1.1) may be interpreted as an equation on a Riemannian manifold $(\mathcal{M}, g)$. Let $\nabla^2 u \in \Gamma(\mathrm{Sym}(T^*\mathcal{M} \otimes T^*\mathcal{M}))$ denote the Hessian of $u$ with respect to the metric $g$, suppose $H[u] \in \Gamma(\mathrm{Sym}(T^*\mathcal{M} \otimes T^*\mathcal{M}))$ is also a symmetric $(0,2)$-tensor, and denote as before $A_H[u] = \nabla^2 u - H[u]$. Raising an index via the inverse metric $g^{-1} = (g^{ij})$, we obtain the $(1,1)$-tensor $g^{-1}A_H[u]$, with $(i,j)$-component in local coordinates given by $g^{ik}(A_H[u])_{kj}$. By the canonical identification of $T^*\mathcal{M} \otimes T\mathcal{M}$ with the endomorphism bundle $\mathrm{End}(T\mathcal{M})$, $g^{-1}A_H[u]$ can be viewed at each point $x \in \mathcal{M}$ as a linear map on the tangent space $T_x\mathcal{M}$ which, by symmetry of the $(0,2)$-tensor $A_H[u]$, is self-adjoint. In particular, the eigenvalues of $g^{-1}A_H[u](x)$ are well-defined and real for each $x \in \mathcal{M}$, and it makes sense to consider on $(\mathcal{M}, g)$ the elliptic equation

$$\sigma_k^{1/k}\big(g^{-1}A_H[u](x)\big) = f\big(x, u(x), \nabla u(x)\big) > 0, \quad g^{-1}A_H[u](x) \in \Gamma_k^+ \quad \text{for } x \in \mathcal{M}. \tag{1.1.2}$$

Both the equations (1.1.1) on Euclidean domains and (1.1.2) on Riemannian manifolds will feature heavily in this thesis. Following the recent work of Jiang & Trudinger

[**JT17, JT18, JT19**], we refer to these collectively as *augmented Hessian equations*.

The augmented Hessian equations include some well-known PDEs as special cases, such as the $k$-Hessian equation, which is obtained by taking $H \equiv 0$, and the $\sigma_k$-Yamabe equation, which is our main focus in this thesis and will be described in the following subsection. We note that the $k$-Hessian equation includes the standard Poisson equation (by taking $k = 1$ and $f = f(x)$) and, for convex or concave $u$, the elliptic Monge-Ampère equation (by taking $k = n$ and $f = f(x)$). For additional detail on the $k$-Hessian equation, we refer the reader to [**TW97, TW99, TW02**], where these equations are considered in a very general sense. See also [**Wan09**] for a detailed survey.

### 1.1.2 The $\sigma_k$-Yamabe equation

Our main motivation for studying the augmented Hessian equations introduced above comes from conformal geometry, namely in relation to the so-called $\sigma_k$-*Yamabe problem*, for which the corresponding $\sigma_k$-Yamabe equations fall into the framework of (1.1.2).

Suppose $(\mathcal{M}^n, g_0)$ is a closed manifold of dimension $n \geq 3$, and denote by $A_0$ the *Schouten tensor* of $g_0$:

$$A_0 = \frac{1}{n-2}\bigg(\operatorname{Ric}_0 - \frac{R_0}{2(n-1)}g_0\bigg). \tag{1.1.3}$$

Here, $\operatorname{Ric}_0$ is the Ricci tensor of $g_0$ and $R_0$ is the scalar curvature of $g_0$. In light of the Ricci decomposition (to be explained in more detail in §2.1.2), the conformal transformation properties of the full Riemann curvature tensor are completely determined by those of the Schouten tensor, and thus the Schouten tensor is a natural quantity to consider in the context of conformal geometry.

The (elliptic) $\sigma_k$-Yamabe problem, first studied by Viaclovsky in [**Via00a**], is to establish the existence of a conformal metric $g \in [g_0] = \{fg_0 : 0 < f \in C^\infty(\mathcal{M}^n)\}$ satisfying

$$\sigma_k^{1/k}(g^{-1}A_g) = \text{constant} > 0, \quad g^{-1}A_g \in \Gamma_k^+ \quad \text{on } \mathcal{M}^n, \tag{1.1.4}$$

where $A_g$ is the Schouten tensor of $g$. More precisely, we refer to (1.1.4) as the $\sigma_k$-Yamabe problem *in the positive case*; the $\sigma_k$-Yamabe problem *in the negative case* is to establish the existence of a conformal metric $g \in [g_0]$ satisfying

$$\sigma_k^{1/k}(-g^{-1}A_g) = \text{constant} > 0, \quad -g^{-1}A_g \in \Gamma_k^+ \quad \text{on } \mathcal{M}^n. \tag{1.1.5}$$

The $\sigma_k$-Yamabe problem derives its name from the fact that the trace of the Schouten tensor is a positive multiple of the scalar curvature, and thus when $k = 1$, (1.1.4) (resp. (1.1.5)) reduces to the original semilinear Yamabe problem in the case of positive (resp. negative) Yamabe invariant.

When $k \geq 2$, both (1.1.4) and (1.1.5) are fully nonlinear elliptic equations; *a priori*, the ellipticity is non-uniform. Moreover, the $\sigma_k$-Yamabe problem in the positive (resp. negative) case is usually posed under a natural admissibility condition on the background metric $g_0$, namely $g_0^{-1}A_0 \in \Gamma_k^+$ (resp. $-g_0^{-1}A_0 \in \Gamma_k^+$). Under these respective assumptions, a simple continuity argument using the connectedness of $\Gamma_k^+$ (see [**Via02, Prop. 2**]) demonstrates that any metric $g \in [g_0]$ with $\sigma_k^{1/k}(g^{-1}A_g) > 0$ (resp. $\sigma_k^{1/k}(-g^{-1}A_g) > 0$) automatically satisfies $g^{-1}A_g \in \Gamma_k^+$ (resp. $-g^{-1}A_g \in \Gamma_k^+$).

The Yamabe problem was solved through the combined works of Yamabe [**Yam60**], Trudinger [**Tru68**], Aubin [**Aub76**] and Schoen [**Sch84**] – see also [**LP87**]. Under the admissibility condition $g_0^{-1}A_0 \in \Gamma_k^+$, the $\sigma_k$-Yamabe problem in the positive case has also been solved when $k = 2$ [**CGY02a, GW06, STW07**] and when $(\mathcal{M}^n, g_0)$ is locally conformally flat (LCF) [**LL03, GW03a, BV04, STW07**]. Largely relevant here is the fact that (1.1.4) is variational when $k = 1$ [**Yam60**], $k = 2$ or $(\mathcal{M}^n, g_0)$ is LCF [**Via00a, BV04**]. Relevant also to the LCF case is the work of Schoen & Yau [**SY88**] on the developing map of a LCF manifold. The $\sigma_k$-Yamabe problem in the positive case has also been solved when $k \geq \frac{n}{2}$ [**GV07, LN14**]; here, positivity of the Ricci curvature (discussed in more detail in §1.2.2) plays an important role. On the other hand, the existence problem is open in the positive case for $2 < k < \frac{n}{2}$ when

$(\mathcal{M}^n, g_0)$ is not LCF, and there are no general existence results for smooth solutions in the negative case when $k \geq 2$.

Now, if we write our conformal metrics in the form $g = e^{-2u}g_0$, where $u \in C^\infty(\mathcal{M}^n)$, then the Schouten tensors $A_0$ and $A_g$ are related by the formula

$$A_g = \nabla_0^2 u + du \otimes du - \frac{|\nabla_0 u|_0^2}{2} g_0 + A_0 \tag{1.1.6}$$

(see [**Bes87**]), where we have denoted using subscript 0s those metric quantities that are defined with respect to $g_0$. Therefore, after writing $g^{-1} = e^{2u}g_0^{-1}$ and multiplying both sides of the equation in (1.1.4) by $e^{-2u}$, we see that (1.1.4) is equivalent to

$$\sigma_k^{1/k}\left(g_0^{-1}\left(\nabla_0^2 u + du \otimes du - \frac{|\nabla_0 u|_0^2}{2} g_0 + A_0\right)\right) = ce^{-2u}, \quad g_0^{-1}A_g \in \Gamma_k^+. \tag{1.1.7}$$

This is precisely of the form seen in (1.1.2). Similarly, by writing conformal metrics as $g = e^{2u}g_0$, (1.1.5) can be written in a form which falls into the framework of (1.1.2).

To encompass more general prescribed $\sigma_k$-curvature problems, in this thesis we will refer to equations of the form

$$\sigma_k^{1/k}(g^{-1}A_g) = f(x, u(x), \nabla u(x)) > 0, \quad g^{-1}A_g \in \Gamma_k^+ \tag{1.1.8}$$

as $\sigma_k$-*Yamabe equations in the positive case*, and equations of the form

$$\sigma_k^{1/k}(-g^{-1}A_g) = f(x, u(x), \nabla u(x)) > 0, \quad -g^{-1}A_g \in \Gamma_k^+ \tag{1.1.9}$$

as $\sigma_k$-*Yamabe equations in the negative case.* Note that in either case, we drop the prefix 'elliptic' for brevity, and unless otherwise specified, we will always be assuming that $k \geq 2$ when referring to either the '$\sigma_k$-Yamabe problem' or the '$\sigma_k$-Yamabe equation'.

## 1.2 Summary of new results

In this section we will briefly motivate and state some of our new results. To keep the exposition concise, more general statements will appear in later chapters, alongside more detailed literature reviews.

### 1.2.1 Local second derivative estimates for the $\sigma_k$-Yamabe equation

Our first set of new results concern local pointwise second derivative estimates for $W^{2,p}$-strong solutions to (1.1.8) and (1.1.9) on Euclidean domains (for any $2 \leq k \leq n$) and smooth solutions to (1.1.8) and (1.1.9) on Riemannian manifolds (for $k = 2$). In particular, these estimates apply to (1.1.4) and (1.1.5). In fact, our estimates on Euclidean domains will be obtained for more general augmented Hessian equations of the form (1.1.1), although we postpone the statements of these more general results until Chapter 2.

As we mentioned in §1.1, $C^2$ estimates on solutions to (1.1.4) and (1.1.5) are key in the context of existence and regularity for the $\sigma_k$-Yamabe problem. Let us begin with an overview of the known estimates in the positive case (1.1.4). It has been known since the work of Viaclovsky [**Via02**] that global *a priori* $C^2$ estimates for (1.1.4) follow from $C^0$ estimates. Local *a priori* $C^2$ estimates depending only on one-sided $C^0$ bounds have also been established, see e.g. [**LL03, GW03b, Che05, Wan06, JLL07, Li09**]. The desired $C^0$ estimates (assuming that $(\mathcal{M}^n, g_0)$ is not conformally equivalent to the round $n$-sphere) have also been established in certain settings, see e.g. [**CGY02a, LL03, GV07, LN14**], leading to some of the aforementioned existence results for (1.1.4). We note that most of these estimates apply more generally to (1.1.8), at least when $f = f(x)$.

We emphasise that the estimates mentioned above are *a priori* estimates, in that they are obtained for smooth solutions. On the other hand, the regularity theory for (1.1.8) has received less attention. Our first new result is a local pointwise second derivative estimate for $W^{2,p}$-strong solutions to the $\sigma_k$-Yamabe equation (1.1.8) on Euclidean domains, and is the product of joint work with Luc Nguyen (see the preprint [**DN20**]):

**Theorem A** ([**DN20**]). *Let* $\Omega$ *be a domain in* $\mathbb{R}^n$ *(*$n \geq 3$*) and let* $f \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ *be a positive function. Suppose that* $2 \leq k \leq n$, $p > \frac{kn}{2}$ *and* $g = u^{-2}|dx|^2$ *with* $0 < u \in W^{2,p}_{\rm loc}(\Omega)$ *is a solution to*

$$\sigma_k^{1/k}\big(g^{-1}A_g(x)\big) = f(x, u(x), \nabla u(x)) > 0, \quad g^{-1}A_g(x) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega.$$

*Then* $u \in C^{1,1}_{\rm loc}(\Omega)$*, and for any concentric balls* $B_R \subset B_{2R} \Subset \Omega$ *we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where* $C$ *is a constant depending only on* $n, p, R, f$ *and an upper bound for* $\|\ln u\|_{W^{2,p}(B_{2R})}$.

For the negative case (1.1.5) of the $\sigma_k$-Yamabe problem, the situation is quite different. Here, *a priori* $C^1$ estimates on solutions have been established by Gursky & Viaclovsky in [**GV03b**], but it is an open problem as to whether solutions satisfy an *a priori* second derivative estimate. Consequently, there are no general existence results for smooth solutions to (1.1.5) when $k \geq 2$. Our second new result is an analogue of Theorem A for strong solutions to the $\sigma_k$-Yamabe equation in the negative case (1.1.9), with a slightly stronger integrability assumption on $\nabla^2 u$:

**Theorem B** ([**DN20**]). *Let* $\Omega$ *be a domain in* $\mathbb{R}^n$ *(*$n \geq 3$*) and let* $f \in C^{1,1}_{\rm loc}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ *be a positive function. Suppose that* $2 \leq k \leq n$, $p > \frac{(k+1)n}{2}$ *and* $g = u^{-2}|dx|^2$ *with* $0 < u \in W^{2,p}_{\rm loc}(\Omega)$ *is a solution to*

$$\sigma_k^{1/k}\big(-g^{-1}A_g(x)\big) = f(x, u(x), \nabla u(x)) > 0, \quad -g^{-1}A_g(x) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega.$$

*Then* $u \in C^{1,1}_{\rm loc}(\Omega)$*, and for any concentric balls* $B_R \subset B_{2R} \Subset \Omega$ *we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where* $C$ *is a constant depending only on* $n, p, R, f$ *and an upper bound for* $\|\ln u\|_{W^{2,p}(B_{2R})}$.

The proof of Theorems A and B is based on an integrability improvement argument followed by an application of Moser iteration; see also §2.7 for partial improvements using the Alexandrov-Bakelman-Pucci estimate. We note that Moser iteration has previously been utilised in the context of the $\sigma_k$-Yamabe equation to establish local boundedness of solutions, see for instance [**Han04, Gon05, Gon06**]. We refer the reader to Theorem F in Chapter 2 for the more general result that includes Theorems A and B as special cases.

As far as the author is aware, Theorem B currently provides the only available pointwise second derivative estimate for solutions to the $\sigma_k$-Yamabe problem in the negative case. It also has the added strength of being a regularity result, rather than just an *a priori* estimate. If one is concerned only with *a priori* second derivative estimates for smooth solutions on manifolds which, as we have discussed, is still an open problem in the negative case, then in fact our methods for Theorems A and B can be adjusted to this setting when $k = 2$. Our result in the negative case is as follows:

**Theorem C.** *Let $(\mathcal{M}^n, g_0)$ be a manifold of dimension $n \geq 3$, let $f \in C^2_{\mathrm{loc}}(\mathcal{M}^n \times \mathbb{R} \times T\mathcal{M}^n)$ be a positive function and suppose $p > \frac{3n}{2}$. Let $B_{2R}$ be a geodesic ball in $(\mathcal{M}^n, g_0)$ of radius $2R < i_0$, where $i_0$ is the injectivity radius of $(\mathcal{M}^n, g_0)$ at the centre of the ball. If $g = u^{-2}g_0$, $0 < u \in C^4(B_{2R})$, is a solution to*

$$\sigma_2^{1/2}\big(-g^{-1}A_g(x)\big) = f(x, u(x), \nabla u(x)) > 0, \quad -g^{-1}A_g(x) \in \Gamma_2^+ \quad \text{on } B_{2R}, \qquad (1.2.1)$$

*then there exists a constant $C$ depending only on $n, p, R, f, g_0$ and an upper bound for $\|\ln u\|_{W^{2,p}(B_{2R})}$ such that*

$$\sup_{B_R} |\nabla^2 u| \leq C.$$

We refer the reader to Theorem J in Chapter 3 for a slightly more general version of Theorem C which encompasses the positive counterpart to (1.2.1).

### 1.2.2 The existence of conformal metrics with $g^{-1}A_g^\tau \in \Gamma_2^+$

For the remainder of the introduction, we will only need to consider the $\sigma_k$-Yamabe equation in the positive case, so we drop any mention of 'the positive case'.

As we mentioned in §1.1.2, it is standard to assume the admissibility condition $g_0^{-1}A_0 \in \Gamma_k^+$ when considering the $\sigma_k$-Yamabe problem on $[g_0]$. When $k = 1$, this is equivalent to imposing positivity of the Yamabe invariant of $[g_0]$, although this condition is less well-understood for $k \geq 2$. Therefore, it is of interest to ask – from both the perspective of the $\sigma_k$-Yamabe problem, and more generally – whether certain conformally invariant conditions on a manifold $(\mathcal{M}^n, g_0)$ imply the existence of a conformal metric $g \in [g_0]$ with $g^{-1}A_g \in \Gamma_k^+$.

One of the first significant results in this direction is due to Chang, Gursky & Yang [**CGY02b**], who showed that if $(\mathcal{M}^4, g_0)$ is a closed 4-manifold with $R_0 > 0$ and $\int_{\mathcal{M}^4} \sigma_2(g_0^{-1}A_0)\, dv_0 > 0$ (the latter being a conformally invariant condition in four dimensions – see §4.1.1), then there exists a conformal metric $g \in [g_0]$ satisfying $g^{-1}A_g \in \Gamma_2^+$, necessarily with positive Ricci curvature. We note that the conclusion $\mathrm{Ric}_g > 0$ is a special case of a more general result due to Guan, Viaclovsky & Wang [**GVW03**], who showed that if $g^{-1}A_g \in \Gamma_k^+$ for some $k \geq \frac{n}{2}$, then $\mathrm{Ric}_g > 0$. If this is the case, it follows as a well-known consequence of Myers' theorem that the fundamental group of $\mathcal{M}^n$ is finite, thus providing a topological obstruction to the existence of metrics with $g^{-1}A_g \in \Gamma_k^+$ for $k \geq \frac{n}{2}$.

It is of particular interest to generalise the result of [**CGY02b**] to three dimensions, where one retains the implication of positive Ricci curvature if $g^{-1}A_g \in \Gamma_2^+$. In this direction, Ge, Lin & Wang [**GLW10**] showed that if $(\mathcal{M}^3, g_0)$ is a closed 3-manifold with $R_0 > 0$ and $\int_{\mathcal{M}^3} \sigma_2(g_0^{-1}A_0)\, dv_0 > 0$, then there exists a conformal metric $g \in [g_0]$ satisfying $g^{-1}A_g \in \Gamma_2^+$. The authors also point out that $\int_{\mathcal{M}^3} \sigma_2(g^{-1}A_g)\, dv_g$ is no longer a conformal invariant in three dimensions. Moreover, a simple algebraic argument (see Appendix 5.A) shows that in three dimensions, one only needs to show $g^{-1}A_g^\tau \in \Gamma_2^+$

for some $\tau \geq \frac{2}{3}$ in order to conclude that $\mathrm{Ric}_g > 0$, where

$$A_g^\tau := A_g + (1-\tau)\sigma_1(g^{-1}A_g)g = \frac{1}{n-2}\bigg(\mathrm{Ric}_g - \frac{\tau R_g}{2(n-1)}g\bigg)$$

is the so-called *trace-modified Schouten tensor*. We therefore ask:

**Question 1.** *Fix $\tau \in [\frac{2}{3}, 1)$ and suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold with $R_0 > 0$ and $\int_{\mathcal{M}^3} \sigma_2(g_0^{-1}A_0^\tau)\, dv_0 > 0$. Does there exist a metric $g \in [g_0]$ with $g^{-1}A_g^\tau \in \Gamma_2^+$ (necessarily with positive Ricci curvature)?*

The trace-modified Schouten tensor was introduced independently in [**GV03a**] and [**LL03**], where the parameter $\tau \leq 1$ is used to construct a family of equations for a continuity method and degree argument, respectively. In particular, each of these methods culminates in an existence result for $\tau = 1$. On the other hand, it is also of independent interest to study the quantities $\sigma_k(\pm g^{-1}A_g^\tau)$ for $\tau < 1$, e.g. in relation to the existence of conformal metrics of positive Ricci curvature, as discussed above. Indeed, existence results for $\tau < 1$ have been studied by various authors – see e.g. [**GV03b**] for an existence result in the negative case that is still open for $\tau = 1$, and the following discussion on a result of [**CD10**] in the positive case.

A first step towards answering Question 1 is to see whether the conclusion $\mathrm{Ric}_g > 0$ in three dimensions can be obtained under *any* weakening of the hypothesis $\int_{\mathcal{M}^3} \sigma_2(g_0^{-1}A_0)\, dv_0 > 0$ of [**GLW10**]. Indeed, Catino & Djadli [**CD10**] showed that if a closed 3-manifold $(\mathcal{M}^3, g_0)$ satisfies $R_0 > 0$ and $\int_{\mathcal{M}^3} \sigma_2(g_0^{-1}A_0)\, dv_0 \geq 0$ (note the non-strict inequality here), then there exists a conformal metric $g \in [g_0]$ satisfying $g^{-1}A_g^\tau \in \Gamma_2^+$ for $\tau \approx \frac{7}{10}$, necessarily with positive Ricci curvature. It is natural to ask whether, under the same hypotheses as [**CD10**] and for *any* $\tau < 1$, one can obtain a conformal metric satisfying $g^{-1}A_g^\tau \in \Gamma_2^+$; this would bridge the gap between the results of [**GLW10**] and [**CD10**]. In fact, we are able to prove a slightly stronger result than this; the following result is the product of joint work with Luc Nguyen:

**Theorem D.** *Let* $(\mathcal{M}^3, g_0)$ *be a closed 3-manifold with positive Yamabe invariant and suppose*

$$\sup_{g=e^{-2u}g_0,\ R_g>0} \frac{\int_{\mathcal{M}^3} \sigma_2(g^{-1}A_g)\, dv_g}{\int_{\mathcal{M}^3} e^{4u}\, dv_g} \geq 0.$$

*Then for all* $\tau < 1$ *there exists a metric* $g \in [g_0]$ *satisfying* $g^{-1}A_g^\tau \in \Gamma_2^+$ *(necessarily with positive Ricci curvature if* $\tau \geq \frac{2}{3}$*).*

For the proof of Theorem D, see Chapter 4. Returning more directly to Question 1, one may ask whether the methods of [**GLW10**] (which gives a positive answer to Question 1 when $\tau = 1$) could be adjusted to attack the case $\tau < 1$. However, as we will explain in more detail in Chapter 4, the flow-based approach of [**GLW10**] (which involves the quotient equation alluded to at the start of the introduction – see (4.1.4)) uses the variational properties of $\sigma_2(g^{-1}A_g^\tau)$ when $\tau = 1$. We therefore ask whether there is an elliptic alternative to this approach that is independent of any variational structure. Our progress on this problem is partial, and so far consists of reproving the result of [**GLW10**] using an elliptic method, although still with some dependence on the variational structure. We refer the reader to Chapter 4 for the details.

### 1.2.3 Integral estimates for a perturbed $\sigma_2$-Yamabe equation

In a similar vein, one may also ask whether the methods of [**CGY02b**] in four dimensions could be modified to address Question 1. Therein, the authors consider a fourth order perturbation of a $\sigma_2$-Yamabe equation given by

$$\sigma_2(g^{-1}A_g) = \frac{\delta}{4}\Delta_g R_g + f, \tag{1.2.2}$$

where $f$ is a carefully chosen positive function. A large part of the proof in [**CGY02b**] consists of obtaining *a priori* integral estimates on solutions to (1.2.2) that are independent of $\delta$, and it is interesting to ask – both in relation to Question 1, and from a more general PDE perspective – whether similar estimates can be obtained for such a vanishing viscosity-type problem in three dimensions. The following result generalises the higher order integral estimates of [**CGY02b**] to three dimensions, and also addresses the trace-modified case:

**Theorem E.** *Suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold with positive Yamabe invariant. Then for given $s < 6$, $\tau \in [\frac{2}{3}, 1]$, $C_1 > 0$ and positive $f \in C^1(\mathcal{M}^3 \times \mathbb{R})$, there exist constants $\delta_0 > 0$ and $C > 0$ (depending only on $g_0$, $s$, $\tau$, $C_1$, $\sup_{\mathcal{M}^3\times[-C_1,C_1]}(f + |\nabla_x f|) < \infty$ and $\inf_{\mathcal{M}^3\times[-C_1,C_1]} f > 0$) such that for every $C^4$ solution $g = e^{2w}g_0$ to*

$$\sigma_2(g^{-1}A_g^\tau) = \frac{\delta}{4}\Delta_g R_g + f(x,w)$$

*with $0 \leq \delta < \delta_0$ satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,6}(\mathcal{M}^3,g_0)} \leq C_1 \quad \text{and} \quad R_g \geq 0, \tag{1.2.3}$$

*one has*

$$\|w\|_{W^{2,s}(\mathcal{M}^3,g_0)} \leq C. \tag{1.2.4}$$

*If $\tau \in [\frac{2}{3}, 1)$, we may replace the $W^{1,6}$ norm in (1.2.3) with the $W^{1,4}$ norm, and the same conclusion holds for all $s < 12$.*

We refer the reader to Chapter 5 for more background context to Theorem E, including a discussion of the work of Chang, Gursky & Yang [**CGY02b**].

## 1.3 An outline of the thesis and a list of theorems

We conclude the introduction with an outline of the thesis. In Chapter 2 we consider local pointwise second derivative estimates for $W^{2,p}$-strong solutions to the $\sigma_k$-Yamabe equation on Euclidean domains. We prove Theorems A and B in the more general context of augmented Hessian equations. More precisely, we state and prove Theorem F, which contains Theorems A and B as special cases. Various extensions and partial improvements to Theorem F will also be addressed (see Theorems G, H and I).

In Chapter 3 we continue our study of local pointwise second derivative estimates for the $\sigma_k$-Yamabe equation, shifting our focus to smooth solutions on manifolds when $k = 2$. We will state and prove Theorem J, which supersedes Theorem C.

In Chapter 4 we study the existence of conformal metrics with $g^{-1}A_g^\tau \in \Gamma_2^+$ in three dimensions. We will prove Theorem D and give an alternative proof of the aforementioned result of [**GLW10**], in which we consider a quotient equation involving the $\sigma_1$ and $\sigma_2$ operators.

In Chapter 5 we prove Theorem E in the spirit of [**CGY02b**], obtaining *a priori* integral estimates for solutions to a fourth order perturbation of the (trace-modified) $\sigma_2$-Yamabe equation. We also give an application of our work in Chapter 2.

Finally, in Chapter 6 we summarise our work and list some further problems.

The following list of main results may serve as a useful reference for the reader:

| **Theorem** | **Brief description** | **Superseded by** | **First stated** |
|---|---|---|---|
| Theorem A | Local pointwise second derivative estimates for strong solutions to the $\sigma_k$-Yamabe equation in the positive case on Euclidean domains | Theorem F | Introduction |
| Theorem B | Local pointwise second derivative estimates for strong solutions to the $\sigma_k$-Yamabe equation in the negative case on Euclidean domains | Theorem F | Introduction |
| Theorem C | Local pointwise second derivative estimates for smooth solutions to the $\sigma_2$-Yamabe equation in the negative case on manifolds | Theorem J | Introduction |
| Theorem D | Existence of conformal metrics with $g^{-1}A_g^\tau \in \Gamma_2^+$ on closed 3-manifolds, $\tau < 1$ | – | Introduction |
| Theorem E | *A priori* $W^{2,p}$ estimates for smooth solutions to a perturbed $\sigma_2$-Yamabe equation on closed 3-manifolds | – | Introduction |
| Theorem F | Local pointwise second derivative estimates for strong solutions to augmented Hessian equations on Euclidean domains | – | Chapter 2 |
| Theorem G | Extension of Theorem F | – | Chapter 2 |
| Theorem H | Extension of Theorems F and G | – | Chapter 2 |
| Theorem I | Partial improvement of Theorem F | – | Chapter 2 |
| Theorem J | Local pointwise second derivative estimates for smooth solutions to the $\sigma_2$-Yamabe equation in the positive and negative cases on manifolds | – | Chapter 3 |

# Chapter 2

# Local pointwise second derivative estimates for strong solutions to the $\sigma_k$-Yamabe equation on Euclidean domains

In this chapter we consider the problem of obtaining local pointwise second derivative estimates for elliptic $W^{2,p}$-strong solutions to the $\sigma_k$-Yamabe equation on Euclidean domains. We will address simultaneously both the positive and negative cases (see Theorems A and B in the introduction) by working in the more general context of augmented Hessian equations. That is, we consider more general equations of the form

$$\sigma_k^{1/k}\big(A_H[u](x)\big) = f(x,u(x),\nabla u(x)) > 0, \quad A_H[u](x) \in \Gamma_k^+ \tag{2.0.1}$$

where $A_H[u](x) = \nabla^2 u(x) - H[u](x)$ and $H[u](x) = H(x,u(x),\nabla u(x)) \in \mathrm{Sym}_n(\mathbb{R})$. For the more general result that includes Theorems A and B as special cases, we refer to Theorem F below. We also prove, concurrently with Theorem F, a more general result which applies when $k = 2$ – see Theorem G below. The plan of this chapter is as follows:

1. We begin in §2.1 with a literature review that will serve to motivate Theorems F and G independently of conformal geometry. After stating Theorems F and G, we will then return to the setting of the $\sigma_k$-Yamabe equation, and provide a more extensive literature review of the $\sigma_k$-Yamabe problem. We will explain how Theorems A and B follow as special cases of Theorem F.

2. In §2.2 we provide a detailed outline of the proof of Theorems F and G. This will include a comparison with previous work of Urbas [**Urb00**], who considered the $k$-Hessian case of (2.0.1) where $H \equiv 0$ and $f = f(x)$.

3. The outline given in §2.2 will prompt us to consider the divergence structure of the linearised operator, which we address in §2.3.

4. In §2.4, we carry out the main body of our integral estimates, as outlined in §2.2.

5. In §2.5, we use our integral estimates and the Moser iteration technique to obtain the desired $C^{1,1}_{\mathrm{loc}}$ estimates, completing the proofs of Theorems F and G (and therefore Theorems A and B).

After proving Theorems F and G, we will then consider an extension of Theorems F and G and a partial improvement of Theorem F. More precisely:

6. In §2.6, we give an extension of Theorems F and G to the case $k \geq 3$.

7. In §2.7, we strengthen Theorem F for a certain class of augmented Hessian equations. More precisely, we are able to weaken our integrability assumption on $\nabla^2 u$ for certain values of $k$ by appealing to an alternative method which does not require any divergence structure or Moser iteration. The class of equations considered includes the $\sigma_k$-Yamabe equation in the positive and negative cases.

The results in §2.1–2.6 of this chapter are the product of joint work with Luc Nguyen and may also be found in the preprint [**DN20**].

## 2.1 Augmented Hessian equations and the $\sigma_k$-Yamabe equation

In this section we begin with a brief discussion on some of the known estimates for augmented Hessian equations. This will motivate (independently of conformal geometry) our new results Theorems F and G. After stating these new results in §2.1.1, we will return to the case of the $\sigma_k$-Yamabe equation in §2.1.2, giving a detailed account of the current status of the $\sigma_k$-Yamabe problem and known estimates. We will explain how Theorems A and B follow as special cases of Theorem F.

### 2.1.1 Background and new results for augmented Hessian equations

Let us begin by stating a known *a priori* second derivative estimate for augmented Hessian equations, which will help to motivate our main result. The following theorem was stated in [**Tru06**], and a proof due to Jiang & Trudinger can be found in [**JT18**] (see also [**MTW05**] for the case $k = n$ in the context of optimal transport):

**Theorem 2.1.1** ([**MTW05, Tru06, JT18**]). *Let $\Omega$ be a domain in $\mathbb{R}^n$ $(n \geq 3)$, $f \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ a positive function and $H \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n; \mathrm{Sym}_n(\mathbb{R}))$. Suppose $u \in C^4(\Omega)$ is a solution to (2.0.1) in $\Omega$, and that there exists a constant $a_0 > 0$ such that on the set $\mathcal{U} = \{(x, z, \xi) : z = u(x), \xi = \nabla u(x)\}$,*

$$\sum_{i,j,k,l} \partial^2_{\xi_k \xi_l} H_{ij}(x, z, \xi)\zeta_i\zeta_j\eta_k\eta_l \geq a_0|\zeta|^2|\eta|^2 \quad \text{for all } \zeta, \eta \in \mathbb{R}^n \text{ s.t. } \zeta \cdot \eta = 0. \tag{2.1.1}$$

*Then for all $\Omega' \Subset \Omega$, we have*

$$\sup_{\Omega'} |\nabla^2 u| \leq C,$$

*where $C$ is a constant depending only on $n, \Omega', \Omega, f, H$ and $\|u\|_{C^1(\Omega)}$.*

**Remark 2.1.2.** In [**JT18**], the authors are primarily concerned with boundary value problems for augmented Hessian equations, and in fact the main contributions of [**JT18**] are *global* estimates, rather than the local estimates of Theorem 2.1.1 above. However, we do not consider boundary value problems in this thesis, and state only the local version of Jiang & Trudinger's estimates.

The condition (2.1.1) is variably referred to in the literature as *strict co-dimension one convexity of $H$*, *strict regularity of $H$*, or the *strict MTW condition*; if we allow the inequality (2.1.1) to hold for $a_0 = 0$, then we drop the prefix 'strict'. It has been well-known since the work of Ma, Trudinger & Wang [**MTW05**] that such structural properties of the augmenting matrix $H$ play a role in obtaining *a priori* estimates for (2.0.1). Moreover, Loeper [**Loe09**] has constructed counterexamples to $C^1$ regularity when co-dimension one convexity is violated. Theorem 2.1.1 therefore gives rise to two natural questions: first, to what extent is a convexity condition such as (2.1.1) necessary to obtain *a priori* second derivative estimates, and second, can such estimates be obtained for less regular solutions?

Regarding the first question, we note that global *a priori* estimates have been obtained in [**JT17**] allowing for $a_0 = 0$ in (2.1.1), under certain conditions on $\partial\Omega$ and the boundary data. Regarding interior estimates, the situation is less well-understood: even for $H \equiv 0$, it is still unknown whether smooth solutions to (2.0.1) with $f \equiv 1$, $k = 2$ and $n \geq 4$ admit an *a priori* local second derivative estimate in terms of $\|u\|_{C^1}$ (for $n = 3$, such an estimate does hold - see [**WY09**]).

The answer to the second question depends on one's notion of a generalised solution to (2.0.1). We again mention Loeper's counterexample to $C^1$ regularity in the absence of co-dimension one convexity, obtained in the context of optimal transport [**Loe09**]. In [**Pog78**], Pogorelov constructed a counterexample to interior $C^2$ regularity for generalised solutions (in the sense of Alexandrov) to the Monge-Ampère equation $\det \nabla^2 u = 1$ in three dimensions. This counterexample was extended by Urbas in [**Urb90**] to viscosity solutions of the $k$-Hessian equations for $k \geq 3$. On the other hand, Urbas showed in [**Urb00**] that for $k \geq 2$ and $p > \frac{kn}{2}$, $W^{2,p}$-strong solutions to the $k$-Hessian equation $\sigma_k^{1/k}(\nabla^2 u(x)) = f(x)$ with $0 < f \in C^{1,1}_{\text{loc}}(\Omega)$ satisfy a $C^{1,1}_{\text{loc}}(\Omega)$ estimate depending on the $L^p$ norm of $\nabla^2 u$.

Our main result of this section is an extension of the interior second derivative estimates of Urbas [**Urb00**] to $W^{2,p}$-strong solutions of (2.0.1), assuming that $H$ is a multiple of the identity matrix. We note that when $H$ is of the form $H(x, z, \xi) =$

$H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$, our constraint $p > \frac{kn}{2}$ coincides with that of [**Urb00**]:

**Theorem F** ([**DN20**]). *Let $\Omega$ be a domain in $\mathbb{R}^n$ ($n \geq 3$), $f \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ a positive function and $H \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\, \text{Sym}_n(\mathbb{R}))$. Suppose $2 \leq k \leq n$, $p \geq 1$ and $u \in W^{2,p}_{\text{loc}}(\Omega)$ is a solution to*

$$\sigma_k^{1/k}\big(A_H[u](x)\big) = f(x,u(x),\nabla u(x)) > 0, \quad A_H[u](x) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega, \tag{2.1.2}$$

*and that one of the following conditions holds:*

1. *$H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$ and $p > \frac{kn}{2}$,*
2. *$H(x,z,\xi) = H_2(x,z,\xi)I$ and $p > \frac{(k+1)n}{2}$.*

*Then $u \in C^{1,1}_{\text{loc}}(\Omega)$, and for any concentric balls $B_R \subset B_{2R} \Subset \Omega$ we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C, \tag{2.1.3}$$

*where $C$ is a constant depending only on $n, p, R, f, H$ and an upper bound for $\|u\|_{W^{2,p}(B_{2R})}$.*

**Remark 2.1.3.** More precisely, the constant $C$ in (2.1.3) depends only on $n, p, R$ and upper bounds for $\|u\|_{W^{2,p}(B_{2R})}$, $\|H\|_{C^{1,1}(\Sigma)}$ and $\|\ln f\|_{C^{1,1}(\Sigma)}$, where $\Sigma = \overline{B}_{2R} \times [-M,M] \times \overline{B}_M(0) \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $M \geq \|u\|_{C^1(\overline{B}_{2R})}$. Note that since $p > n$ in Theorem F, an upper bound for $\|u\|_{W^{2,p}(B_{2R})}$ implies an upper bound for $\|u\|_{C^1(\overline{B}_{2R})}$, in light of the Morrey embedding theorem.

The proof of Theorem F uses an integrability improvement argument followed by an application of Moser iteration, and is inspired by the methods of Urbas [**Urb00**], who proved Theorem F in the case $H \equiv 0$, $f = f(x)$. Other lower bounds on $p$ leading to $C^{1,1}_{\text{loc}}$ regularity for $k$-Hessian equations have been considered in [**CM17, LB05, Urb88, Urb01, Urb07**], for instance. As we will see, new difficulties arise when $H \not\equiv 0$. The estimates of Theorem F are also closely related to certain analytical aspects in the work of Chang, Gursky & Yang [**CGY02b**], although we postpone the discussion of this work until later in the thesis (see Chapter 5).

Besides providing a regularity result for a large class of augmented Hessian equations, one of the points to take away from Theorem F is that no convexity assumption on $H$ is required to obtain interior second derivative estimates, once sufficiently strong integral estimates are known. In essence, we are able to trade a convexity condition (see the discussion after Remark 2.1.2) for an integrability condition. This is of interest even for smooth solutions (in which case the proof of Theorem F is largely simplified).

One of the main difficulties compared with the $k$-Hessian case considered by Urbas is that the divergence structure is more complicated when $H \not\equiv 0$. In fact, the main reason for imposing that $H$ is a multiple of the identity matrix in Theorem F is to ensure a more favourable divergence structure, which turns out to be useful in exploiting a cancellation phenomenon between higher order terms in our estimates – this will be explained in more detail in §2.2. That said, it turns out that for $k = 2$, one retains some divergence structure even when $H$ is not necessarily a multiple of the identity matrix. We prove (concurrently with Theorem F) the following:

**Theorem G** ([**DN20**]). *Let $\Omega$ be a domain in $\mathbb{R}^n$ ($n \geq 3$), $f \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ a positive function and $H \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,; \text{Sym}_n(\mathbb{R}))$. Suppose $p > \frac{3n}{2}$ and $u \in W^{2,p}_{\text{loc}}(\Omega)$ is a solution to (2.1.2) with $k = 2$. Then $u \in C^{1,1}_{\text{loc}}(\Omega)$, and for any concentric balls $B_R \subset B_{2R} \Subset \Omega$ we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where $C$ is a constant depending only on $n, p, R, f, H$ and an upper bound for $\|u\|_{W^{2,p}(B_{2R})}$.*

When $k \geq 3$, the divergence structure is less tractable, although we are still able to obtain a generalisation of Theorem G assuming $p > kn$ - see Theorem H in §2.6 for the details. Once again, we stress that in Theorems G and H, there are no convexity assumptions on $H$. We also refer the reader to Theorem I in §2.7 for a partial improvement on Theorem F; our proof of Theorem I uses the Alexandrov-Bakelman-Pucci estimate in a way that bypasses the divergence structure altogether.

### 2.1.2 Background and new results for the $\sigma_k$-Yamabe equation

We briefly introduced the $\sigma_k$-Yamabe problem in §1.1.2 and stated our new results on the $\sigma_k$-Yamabe equation (Theorems A and B) in §1.2.1. In what follows, we give a more extensive review of the current status of the $\sigma_k$-Yamabe problem, and explain how Theorems A and B fit directly into the framework of Theorem F.

It will be useful to first recall the classical Yamabe problem, which asks whether every conformal class of metrics on a closed Riemannian $n$-manifold $(\mathcal{M}^n, g_0)$ $(n \geq 3)$ admits a metric of constant scalar curvature. This question was answered in the affirmative through the combined works of Yamabe [**Yam60**], Trudinger [**Tru68**], Aubin [**Aub76**] and Schoen [**Sch84**] (see also [**LP87**]); moreover, the sign of any constant scalar curvature metric in $[g_0]$ coincides with the sign of the so-called *Yamabe invariant*, the conformal invariant defined by

$$Y(\mathcal{M}^n, [g_0]) = \inf_{g\in[g_0]} \frac{\int_{\mathcal{M}^n} R_g \, dv_g}{(\mathrm{Vol}(\mathcal{M}^n, g))^{\frac{n-2}{n}}}. \tag{2.1.4}$$

Following the work of Viaclovsky [**Via00a**] and Chang, Gursky & Yang [**CGY02b**] at the turn of the century, there developed a significant interest in the $\sigma_k$-Yamabe problem which, as we have seen in §1.1.2, is a fully nonlinear generalisation of the Yamabe problem. We recall that in the positive case, the $\sigma_k$-Yamabe problem asks whether, given a closed Riemannian manifold $(\mathcal{M}^n, g_0)$ of dimension $n \geq 3$, there exists an elliptic solution to $\sigma_k^{1/k}(g^{-1}A_g) = \text{constant} > 0$ in the conformal class $[g_0]$, that is, a metric $g \in [g_0]$ satisfying

$$\sigma_k^{1/k}\big(g^{-1}A_g(x)\big) = \text{constant} > 0, \quad g^{-1}A_g(x) \in \Gamma_k^+ \quad \text{for } x \in \mathcal{M}^n. \tag{2.1.5}$$

We recall here that

$$A_g = \frac{1}{n-2}\bigg(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\bigg) \tag{2.1.6}$$

is the Schouten tensor of $g$.

As we explained in the introduction, if we write our conformal metrics in the form $g = e^{-2u}g_0$, then the equation on the left of (2.1.5) can be written in terms of $u$ and the background metric $g_0$ using the conformal transformation law

$$A_g = \nabla_0^2 u + du \otimes du - \frac{|\nabla_0 u|_0^2}{2} g_0 + A_0 \tag{2.1.7}$$

(see [**Bes87**]), giving

$$\sigma_k^{1/k}\bigg(g_0^{-1}\bigg(\nabla_0^2 u + du \otimes du - \frac{|\nabla_0 u|_0^2}{2} g_0 + A_0\bigg)\bigg) = ce^{-2u}. \tag{2.1.8}$$

We also recall that the $\sigma_k$-Yamabe problem is usually posed under the admissibility condition $g_0^{-1}A_0 \in \Gamma_k^+$, in which case the ellipticity condition on the right of (2.1.5) is superfluous (see [**Via02, Prop. 2**]).

Similarly, for the negative case of the $\sigma_k$-Yamabe problem we seek a solution to

$$\sigma_k^{1/k}\big(-g^{-1}A_g(x)\big) = \text{constant} > 0, \quad -g^{-1}A_g(x) \in \Gamma_k^+ \quad \text{for } x \in \mathcal{M}^n. \tag{2.1.9}$$

As with the positive case, this problem is usually considered with an admissibility assumption on the background metric, namely $-g_0^{-1}A_0 \in \Gamma_k^+$.

We remark that the choice to consider the Schouten tensor as defined in (2.1.6) is not arbitrary. Rather, the Schouten tensor arises naturally in the *Ricci decomposition* of the full curvature tensor of a metric $g$,

$$\mathrm{Riem}_g = W_g + A_g \owedge g, \tag{2.1.10}$$

where $W_g$ is the Weyl tensor and $\owedge$ is the Kulkarni-Nomizu product on symmetric $(0,2)$-tensors defined by

$$\begin{aligned}(h \owedge k)(X_1, X_2, X_3, X_4) = h(X_1, X_3)k(X_2, X_4) + h(X_2, X_4)k(X_1, X_3) \\ - h(X_1, X_4)k(X_2, X_3) - h(X_2, X_3)k(X_1, X_4)\end{aligned} \tag{2.1.11}$$

(see [**Bes87**]). It is a well-known fact that the $(1,3)$-Weyl tensor is conformally invariant, so it follows from (2.1.10) that the conformal transformation properties of the

Riemann curvature tensor (involving derivatives of the conformal factor) are completely determined by those of the Schouten tensor. Thus, the Schouten tensor is a natural quantity to consider in the context of conformal geometry.

#### 2.1.2.1 The positive case of the $\sigma_k$-Yamabe problem

There has been significant progress on the $\sigma_k$-Yamabe problem in the positive case since its inception, largely under the admissibility assumption $g_0^{-1}A_0 \in \Gamma_k^+$. We first summarise the developments in the existence theory, in roughly chronological order. In the special case $k = 2$, $n = 4$, existence has been known since the work of Chang, Gursky & Yang [**CGY02a**]. For the case that $(\mathcal{M}^n, g_0)$ is locally conformally flat (LCF), existence was proved by Li & Li in [**LL03**], and independently by Guan & Wang [**GW03a**] ($k \neq \frac{n}{2}$) and Brendle & Viaclovsky [**BV04**] ($k = \frac{n}{2}$). The method of [**GW03a**] (and extended in [**BV04**]) is to obtain the solution as the limiting metric of a specific conformal flow, which in turn uses the fact that the problem is variational in the LCF setting. A flow method was also utilised by Sheng, Trudinger & Wang [**STW07**] to prove existence in *all* cases that the problem is variational, which was known at the time to include the case $k = 2$ and the LCF case (see also [**GW06**] for the case $k = 2$, $n > 8$). Branson & Gover [**BG08**] later showed that the problem is variational if and only if either $k = 2$ or the manifold is LCF. Finally, existence is known when $k \geq \frac{n}{2}$: for the case $k > \frac{n}{2}$, see Gursky & Viaclovsky [**GV07**], and also Trudinger & Wang [**TW09**] for related work. For the case $k \geq \frac{n}{2}$, see Li & Nguyen [**LN14**]. The existence of solutions for $2 < k < \frac{n}{2}$ with $(\mathcal{M}^n, g_0)$ not LCF remains a major open problem.

As with the Yamabe problem, the main difficulty in the existence theory for (2.1.5) is obtaining a uniform *a priori* $C^0$ estimate on solutions. By the work of Viaclovsky [**Via02**], a $C^0$ estimate on $C^4$ solutions implies a $C^2$ estimate, from which Hölder estimates of any order can be obtained (see the discussion in §1.1.1). A degree or continuity argument then typically yields existence.

The desired $C^0$ estimates were obtained in the existence proofs of the cases considered by [**CGY02a, LL03, GV07, LN14**], assuming that $(\mathcal{M}^n, g_0)$ is not conformally equivalent to the round $n$-sphere (in this case, $C^0$ compactness fails due to the non-compactness of the conformal group, but existence is trivial). In each of these papers, $C^0$ compactness is derived from a detailed blow-up analysis for blow-up sequences of solutions to (2.1.5). This analysis in turn relies on a combination of Liouville-type theorems (see e.g. [**Via00a, Via00b, LL03, LL05, Li07, Li09**]) and local *a priori* first and second derivative estimates depending on one-sided $C^0$ bounds (i.e. either an upper bound or a lower bound). These local estimates have been established in varying levels of generality in the works of Li & Li [**LL03**], Guan & Wang [**GW03b**], Chen [**Che05**], Wang [**Wan06**], Jin, Li & Li [**JLL07**] and Li [**Li09**], for example.

**Remark 2.1.4.** Many of the results mentioned above apply more generally if the RHS of (2.1.5) is variable, say $f = f(x)$ or $f = f(x, u)$. For variable $f$, there are known obstructions to existence on the round sphere when $k = 1$ (see e.g. [**Li95**], [**CL01**]), so one still excludes conformal equivalence to the round sphere when considering $C^0$ compactness in this setting. See the work of Li, Nguyen & Wang [**LNW20b**] for recent existence results on the sphere when $k \geq 2$.

**Remark 2.1.5.** The existence result of [**STW07**] exploits the variational structure and bypasses the need for a $C^0$ compactness result for (2.1.5). Therefore, the compactness of the solution set remains a major open problem when $2 \leq k < \frac{n}{2}$ with $(\mathcal{M}^n, g_0)$ not LCF. Now, compactness of the solution set should lead one to an existence theory by a degree argument, but it may be the case that there is a non-compact set of solutions. Indeed, when $k = 1$, existence is known by the work of [**Yam60, Tru68, Aub76, Sch84**], but the compactness of the solution set can fail when $n \geq 25$, by the work of Brendle [**Bre08**] and Brendle & Marques [**BM09**]. In dimensions $n \leq 24$, it is known due to Khuri, Marques & Schoen [**KMS09**] that $C^2$ compactness holds assuming the positive mass theorem; for earlier results in lower dimensions, see e.g.

[**LZ99, Dru04, Mar05, LZ05, LZ07**], and see [**Sch91**] for compactness in the LCF case. It is an intriguing question as to whether a similar phenomenon is present for $2 \leq k < \frac{n}{2}$, but this falls beyond the scope of this thesis.

Continuing the theme of §2.1.1, we ask whether second derivative estimates for the $\sigma_k$-Yamabe equation can be obtained under lower regularity assumptions on the conformal factor $u$. To fit into the framework of §2.1.1, we restrict to the case that $(\mathcal{M}^n, g_0)$ is LCF. Then, restricting to a suitable coordinate chart, we may locally view (2.1.8) as an equation on a domain $\Omega \subset \mathbb{R}^n$ with $g_0$ equal to the flat metric, i.e.

$$\sigma_k^{1/k}\left(\nabla_0^2 u + du \otimes du - \frac{|\nabla_0 u|_0^2}{2} I\right) = ce^{-2u} \quad \text{in } \Omega. \tag{2.1.12}$$

Making the substitution $e^{-u} = w^{-1}$ and multiplying through by a suitable factor of $w$, we can rewrite (2.1.12) in the form

$$\sigma_k^{1/k}\left(\nabla_0^2 w - \frac{|\nabla_0 w|_0^2}{2w} I\right) = f(x, w) > 0 \quad \text{in } \Omega, \tag{2.1.13}$$

where $w > 0$. This equation is of the form seen in Case 1 of Theorem F, hence our local $C^{1,1}$ estimate applies if we assume $u \in W^{2,p}(\mathcal{M}^n, g_0)$ with $p > \frac{kn}{2}$. In particular, we obtain Theorem A as stated in the introduction.

We remark that although Theorem A applies only in the LCF case, we hope that a more developed regularity theory for the $\sigma_k$-Yamabe equation could be of use in relation to the open existence problem for $2 < k < \frac{n}{2}$ with $(\mathcal{M}^n, g_0)$ not LCF. We have not explored this possibility yet, but of course it would also be of independent interest to generalise the estimate of Theorem A to arbitrary closed manifolds (see also §5.6).

#### 2.1.2.2 The negative case of the $\sigma_k$-Yamabe problem

There are fewer results on the $\sigma_k$-Yamabe problem in the negative case. In fact, there are no general existence results for smooth solutions to (2.1.9) when $k \geq 2$. This is in stark contrast to the case $k = 1$, where the case of negative Yamabe invariant is easier than the positive case. Whilst $C^1$ estimates are known in the negative case due to Gursky & Viaclovsky [**GV03b**], the main difficulty is in obtaining *a priori* second derivative estimates – one should compare this to the positive case, where $C^2$ estimates are known to follow from $C^0$ estimates [**Via02**], but $C^0$ estimates are not known in general for $2 \leq k < \frac{n}{2}$.

However, there has been some interesting progress in the negative case for non-smooth solutions. As pointed out by Li & Nguyen in [**LN21a**], the $C^1$ estimates of [**GV03b**] imply the existence of a Lipschitz viscosity solution to (2.1.9), assuming $-g_0^{-1}A_0 \in \Gamma_k^+$. On the other hand, the maximum principle implies that if a smooth solution to (2.1.9) does exist, then it is unique, and a strong maximum principle of Caffarelli, Li & Nirenberg [**CLN13**] implies that this smooth solution is also the unique viscosity solution to (2.1.9). In particular, if there are two Lipschitz viscosity solutions to (2.1.9), then there does not exist a smooth solution.

For the closely related $\sigma_k$-Loewner-Nirenberg problem, it is shown by González, Li & Nguyen in [**GLN18**] that for a smooth bounded domain in $\mathbb{R}^n$, there exists a unique continuous viscosity solution which, moreover, is locally Lipschitz. In [**LN21a**], Li & Nguyen show that on an annular domain, this unique viscosity solution is not differentiable; the authors also announce that in forthcoming work with Xiong, this result is generalised to domains with boundaries containing two or more connected components [**LNX**]. Thus, the question of regularity (and uniqueness) in relation to the $\sigma_k$-Yamabe equation in the negative case appears to be a delicate matter. For work on equations similar to (2.1.9), e.g. with the Ricci tensor replacing the Schouten tensor, see for instance [**GV03b, LS05, Gua08, GSW11, Sui17**].

In light of the lack of existence results for smooth solutions to (2.1.9), and the results discussed above, the question of regularity for the $\sigma_k$-Yamabe equation in the negative case is clearly important. Now, by the same reasoning as in the positive case, if $(\mathcal{M}^n, g_0)$ is LCF then we may locally view the equation in (2.1.9) as an equation on a domain $\Omega \subset \mathbb{R}^n$, with $g_0$ equal to the flat metric:

$$\sigma_k^{1/k}\bigg( -\nabla_0^2 u - du \otimes du + \frac{|\nabla_0 u|_0^2}{2} I \bigg) = ce^{-2u} \quad \text{in } \Omega. \tag{2.1.14}$$

Making the substitution $e^{-u} = -w^{-1}$, we can rewrite (2.1.14) in the same form (2.1.13), but now with $w < 0$. Therefore Case 2 of Theorem F applies, and in particular we obtain Theorem B as stated in the introduction.

As far as the author is aware, Theorem B currently provides the only available pointwise second derivative estimate for solutions to the $\sigma_k$-Yamabe problem in the negative case. It also has the added strength of being a regularity result, rather than just an *a priori* estimate.

## 2.2 An outline of the proofs of Theorems F and G

In this section, we provide the reader with an outline of the proofs of Theorems F and G, meanwhile establishing notation that will be used throughout the chapter. We start in §2.2.1 with a comparison to the $k$-Hessian case considered by Urbas [**Urb00**], which will highlight the new steps required to prove Theorems F and G. In Section §2.2.2, we provide a more detailed outline of the proof of Theorems F and G.

### 2.2.1 Difficulties compared with the $k$-Hessian case

In adapting the methods of [**Urb00**] to prove Theorems F and G, we will need to deal with the term $H[u]$ which, whilst being of lower order in the definition of $A_H[u]$, creates new higher order terms in our estimates. Roughly speaking, the two terms which are formally problematic consist of:

**(i)** a contraction of the linearised operator

$$F[u]^{ij} = \frac{\partial \sigma_k(A_H[u])}{\partial (A_H[u])_{ij}} \tag{2.2.1}$$

with double difference quotients of $H[u]_{ij}$ (this arises as a result of taking difference quotients of the equation (2.1.2) twice), and

**(ii)** the divergence of $F[u]^{ij}$ multiplied by a term formally of third order in $u$ (this arises after integrating by parts).

In [**Urb00**], neither of these terms exist since $F[u]^{ij}$ is divergence-free when $H \equiv 0$. In the more general case that we are considering, it is unclear whether these third order terms have a favourable sign individually. However, we will estimate them so as to show that, when combined, they yield a cancellation phenomenon that ensures the overall higher order contribution is positive. For the estimates of the higher order terms arising from the divergence of $F[u]^{ij}$, see Lemmas 2.4.4 and 2.4.5, and for those arising from the double difference quotients of $H[u]$, see Lemma 2.4.10. For the resulting cancellation phenomena, see Corollaries 2.4.12, 2.4.13 and 2.4.14.

### 2.2.2 An outline of the proofs of Theorems F and G

Our proofs of Theorems F and G use an integrability improvement argument, from which the $C^{1,1}_{\rm loc}$ estimates are obtained by Moser iteration. In Case 1 of Theorem F, we will obtain, for a solution $u \in W^{2,q+k-1}_{\rm loc}(\Omega)$ to (2.1.2) with $q > \frac{kn}{2} - k + 1$, the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q}\right)^{1/\beta} \leq \frac{Cq}{\rho^2}\int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1}, \tag{2.2.2}$$

where $\rho \in (0, \frac{R}{3}]$, $\beta = \frac{kn}{kn-2k+2}$ and $C_1$ is a positive constant ensuring $\Delta u + C_1 \geq 1$ a.e. (see the paragraph after Remark 2.2.3 for the justification of the existence of $C_1$). Similarly, in Case 2 of Theorem F and in Theorem G, we will obtain, for a solution $u \in W^{2,q+k}_{\rm loc}(\Omega)$ to (2.1.2) with $q > \frac{(k+1)n}{2} - k$, the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q}\right)^{1/\beta} \leq \frac{Cq}{\rho^2}\int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k}, \tag{2.2.3}$$

now with $\beta = \frac{(k+1)n}{(k+1)n-2(k+1)+2}$. The estimates (2.2.2) and (2.2.3) then yield an improvement in integrability under the respective lower bounds on $q$, which can then be iterated to yield the desired $C^{1,1}_{\rm loc}$ estimates.[1]

[1] One might ask whether a reverse Hölder-type inequality for a single second derivative $\nabla_l\nabla_l u$, similar to (2.2.2) and (2.2.3), can be established. We have been unable to show this.

In the rest of this subsection we explain how the estimates (2.2.2) and (2.2.3) are obtained. Due to the lack of regularity, we derive our estimates through taking difference quotients of the equation (2.1.2). For an index $l \in \{1, \ldots, n\}$ and increment $h \in \mathbb{R}\backslash\{0\}$, we recall the first order difference quotient $\nabla_l^h u(x) = h^{-1}(u(x+he_l) - u(x))$ and the second order difference quotient

$$\Delta_{ll}^h u(x) := \nabla_l^h(\nabla_l^{-h} u(x)) = \frac{u(x+he_l) - 2u(x) + u(x-he_l)}{h^2}. \tag{2.2.4}$$

We also denote

$$v_h(x) = \sum_{l=1}^n \Delta_{ll}^h u(x).$$

The above expressions are well-defined for $x \in \Omega_h = \{y \in \Omega : \mathrm{dist}(y, \partial\Omega) > |h|\}$.

It is well-known (see, for instance, [**GT01, Lemma 7.23**]) that

$$\|\nabla_l^h u\|_{L^s(\Omega')} \leq \|\nabla_l u\|_{L^s(\Omega)} \quad \text{for all } s \geq 1 \text{ and } \Omega' \Subset \Omega \text{ s.t. } \mathrm{dist}(\Omega', \partial\Omega) > |h|. \tag{2.2.5}$$

It follows from (2.2.4) and (2.2.5) that there exists a constant $C = C(n)$ such that

$$\|v_h\|_{L^s(\Omega')} \leq C\|\nabla^2 u\|_{L^s(\Omega)} \quad \text{for all } s \geq 1. \tag{2.2.6}$$

We will also use the following well-known fact – see Appendix 2.A for a proof:

**Lemma 2.2.1.** *Suppose $u \in W^{2,s}(\Omega)$ for some $s \geq 1$. Then $v_h \to \Delta u$ in $L^s_{\mathrm{loc}}(\Omega)$ as $h \to 0$.*

We assume for now that both the increment $h$ and our solution $u$ are fixed, and write $v$ as shorthand for $v_h$. Taking difference quotients of the equation $\sigma_k^{1/k}(A_H[u](x)) = f[u](x) := f(x, u(x), \nabla u(x))$ and appealing to the concavity of $\sigma_k^{1/k}$ in $\Gamma_k^+$, we will derive (at the start of §2.4) the pointwise estimate

$$\sum_l k(f[u])^{k-1}\Delta_{ll}^h f[u] \leq F[u]^{ij}\nabla_i\nabla_j v - \sum_l F[u]^{ij}\Delta_{ll}^h(H[u])_{ij} \quad \text{a.e. in } \Omega_h, \tag{2.2.7}$$

where we recall $F[u]^{ij} = \partial\sigma_k(A_H[u])/\partial(A_H[u])_{ij}$ is the linearised operator.

**Remark 2.2.2.** In (2.2.7), and for the remainder of this chapter, summation notation is employed *only over repeated indices which appear in both upper and lower positions.* Positioning of indices is purely to indicate whether summation convention is being utilised; since we are working with the Euclidean metric, we are free to raise and lower indices at will. For instance, $A_{ij}$, $A^i_j$, $A^j_i$ and $A^{ij}$ all denote the $(i,j)$-entry of a symmetric matrix $A$. Similarly, we do not distinguish between the derivatives $\nabla^i$ and $\nabla_i$ when using index notation.

**Remark 2.2.3.** Since $u$ is fixed, we write $f[u], H[u], A_H[u], F[u]^{ij}$ etc. to emphasise that these are to be considered as functions of $x$. If it is clear from the context (e.g. if there are no derivatives involved), we will simply write $f, H, A_H, F^{ij}$ etc.

The estimates (2.2.2) and (2.2.3) are derived by testing (2.2.7) against suitable test functions. First fix a ball $B_{2R} \Subset \Omega_h$. Since $A_H \in \Gamma_2^+$ is equivalent to positivity of both $\mathrm{tr}(A_H) = \Delta u - \mathrm{tr}(H)$ and $\sigma_2(A_H) = \frac{1}{2}(\mathrm{tr}(A_H)^2 - |A_H|^2)$, there exists a constant $C_1 \geq 0$ (depending on an upper bound for $\|H\|_{C^0(\Sigma)}$ - see Remark 2.1.3) for which $\Delta u + C_1 \geq 1$ and $|\nabla^2 u| \leq \Delta u + C_1$ a.e. in $B_{2R}$. We define $\tilde{v} = v + C_1$, and for a small parameter $\delta > 0$ (that we eventually take to zero) we denote

$$Q_\delta = \big((\tilde{v}^+)^2 + \delta^2\big)^{1/2}.$$

For $\rho \in (0, \frac{R}{3}]$ we also let $\eta \in C_c^\infty(B_{R+2\rho})$ be a standard non-negative cutoff function. Testing (2.2.7) against $\eta Q_\delta^{q-1}$ (for $q > 1$ to be chosen later) then yields

$$\sum_l \int_{B_{R+2\rho}} k\eta Q_\delta^{q-1} f^{k-1} \Delta_{ll}^h f[u] \leq \int_{B_{R+2\rho}} \eta Q_\delta^{q-1} F^{ij} \nabla_i \nabla_j \tilde{v} - \sum_l \int_{B_{R+2\rho}} \eta Q_\delta^{q-1} F^{ij} \Delta_{ll}^h (H[u])_{ij} \qquad (2.2.8)$$

for all $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ solving (2.1.2).

For ease of outlining our argument, let us suppose that $f = f(x,z)$ (the general case $f = f(x,z,\xi)$ will only require minor changes - see §2.5.3). Then the integrand on the left hand side (LHS) of (2.2.8) is a lower order term, whereas the integrands on the RHS of (2.2.8) involve higher order terms, formally of fourth and third order in the limit $h \to 0$, and thus need to be treated.

In §2.4, we integrate by parts in the first integral on the RHS of (2.2.8), using a result of §2.3 that tells us $\nabla_i F[u]^{ij}$ is a regular distribution belonging to $L^{(q+k-1)/(k-1)}_{\mathrm{loc}}(\Omega)$ if $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$. After taking $\delta \to 0$ and carrying out some further calculations (see Lemmas 2.4.2 and 2.4.3), we will obtain the estimate

$$\begin{aligned} \frac{q-1}{Cq^2}\int_{B_{R+\rho}} f^k \frac{\left|\nabla\big((\tilde{v}^+)^{q/2}\big)\right|^2}{\mathrm{tr}(A_H)} &+ \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j \tilde{v} \\ &+ \sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\Delta^h_{ll}(H[u])_{ij} \\ &\leq \frac{C}{\rho^2}\bigg(\int_{B_{R+2\rho}} (\tilde{v}^+)^{q+k-1} + \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1}\bigg), \end{aligned} \tag{2.2.9}$$

where $C$ is a constant independent of $h, q$ and $\rho$.

Whilst the first integral on the LHS of (2.2.9) is a favourable positive higher order term, the other two integrals on the LHS (which we denote by $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$, respectively) involve higher order terms which are, *a priori*, of unknown sign. Treating $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ is the most technical part of the proof.

Now, if we momentarily assume sufficiently high regularity on $u$, say $u \in W^{2,q+2k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$, the issue of dealing with $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ is largely simplified. As will be detailed in the proof of Theorem H in §2.6, one may apply the Cauchy inequality to each of the integrands and absorb the resulting third order terms into the positive term on the LHS of (2.2.9). Under the stated integrability assumption, this crude estimation is sufficient to show

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^k \frac{\left|\nabla\big((\tilde{v}^+)^{q/2}\big)\right|^2}{\mathrm{tr}(A_H)} \leq \frac{C}{\rho^2}\bigg(\int_{B_{R+2\rho}} (\tilde{v}^+)^{q+2k-1} + \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+2k-1}\bigg).$$

An estimate analogous to (2.2.2) and (2.2.3) can then be obtained, assuming $q > kn - 2k + 1$.

The difficulty is to therefore deal with $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ under the *weaker* integrability assumptions of Theorems F and G. At this point, we make the distinction between the various cases. In each case, we estimate $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ so as to produce a cancellation phenomenon when combined, leaving only lower order terms; see Lemmas 2.4.4 and 2.4.5 for the estimates on $(\mathrm{I}_2)_h$, Lemma 2.4.10 for the estimates on $(\mathrm{I}_3)_h$, and Corollaries 2.4.12, 2.4.13 and 2.4.14 for the resulting cancellations. It will then follow from (2.2.9) that, in Case 1 of Theorem F with the relaxed assumption $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$, we have the estimate

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\mathrm{tr}(A_H)} \leq \frac{C}{\rho^2}\left(\int_{B_{R+2\rho}} (\tilde{v}^+)^{q+k-1} + \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1}\right). \tag{2.2.10}$$

Similarly, in the remaining cases with $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$, we will obtain

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\mathrm{tr}(A_H)} \leq \frac{C}{\rho^2}\left(\int_{B_{R+2\rho}} (\tilde{v}^+)^{q+k} + \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k}\right). \tag{2.2.11}$$

To obtain (2.2.2) from (2.2.10) (resp. (2.2.3) from (2.2.11)), we proceed as follows (the details can be found in §2.5). We first obtain an integral estimate for $\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2$, to which we can apply the Sobolev inequality. We then justify taking the limit $h \to 0$ and impose the lower bound $q + k - 1 > \frac{kn}{2}$ (resp. $q + k > \frac{(k+1)n}{2}$), from which we obtain (2.2.2) (resp. (2.2.3)).

## 2.3 Divergence structure of the linearised operator $F[u]^{ij}$

In this section we derive a divergence formula for the linearised operator $F[u]^{ij}$ (defined in (2.2.1)), which we will use at various stages of our proof.

We note that in the case that $A_H[u] = \nabla^2 u$ or $A_H[u] = A_u = \nabla^2 u - \frac{|\nabla u|^2}{2u} I$, the divergence properties of $F[u]^{ij}$ are well-documented (for smooth $u$). In the former case, $F[u]^{ij}$ is divergence-free with respect to the flat metric (see [**Rei73**]), and in the latter case, $u^{1-k}F[u]^{ij}$ is divergence-free with respect to the conformal metric $g_{ij} = u^{-2}\delta_{ij}$

(see [**Via00a**]). For related discussions, see also [**GW03a, BV04, Gon05, Han06, STW07, BG08**].

For $A \in \mathrm{Sym}_n(\mathbb{R})$ and $1 \le k \le n$, define the *k'th Newton tensor of A* inductively by

$$T_k(A) = \sigma_k(A)I - T_{k-1}(A)A, \quad T_0(A)^{ij} = \delta^{ij}. \tag{2.3.1}$$

It is well-known (see [**Rei73**]) that

$$\frac{\partial \sigma_k(A)}{\partial A_{ij}} = T_{k-1}(A)^{ij} \tag{2.3.2}$$

and

$$\mathrm{tr}(T_k(A)) = (n-k)\sigma_k(A), \tag{2.3.3}$$

and moreover $T_{k-1}(A)^{ij}$ is positive definite when $A \in \Gamma_k^+$ (see [**CNS85**]). In particular, by (2.2.1) and (2.3.2), $F[u]^{ij} = T_{k-1}(A_H[u])^{ij}$.

We start by deriving a divergence formula in the case that $u \in C^3(\Omega)$:

**Lemma 2.3.1.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in C^3(\Omega)$. Then for $H \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\, \mathrm{Sym}_n(\mathbb{R}))$ and $2 \le k \le n$,*

$$\nabla_i F[u]^{ij} = \sum_{p=1}^{k-1} (-1)^{p+1} T_{k-p-1}(A_H)^{ab}\Big(\nabla_a (H[u])_b^c - \nabla^c (H[u])_{ab}\Big)(A_H^{p-1})_c^j =: V[u]^j. \tag{2.3.4}$$

*Moreover, if $H(x,z,\xi) = H_2(x,z,\xi)I$, then*

$$\nabla_i F[u]^{ij} = -(n-k+1)\nabla_i (H_2[u])\, T_{k-2}(A_H)^{ij}. \tag{2.3.5}$$

*Proof.* The identity (2.3.4) will follow once we show that for $1 \le k \le n-1$,

$$\nabla_i T_k(A_H[u])^{ij} = \sum_{p=1}^{k} (-1)^{p+1} T_{k-p}(A_H)^{ab}\Big(\nabla_a (H[u])_b^c - \nabla^c (H[u])_{ab}\Big)(A_H^{p-1})_c^j. \tag{2.3.6}$$

Similarly, (2.3.5) will follow once we show that for $1 \le k \le n-1$ and $H(x,z,\xi) = H_2(x,z,\xi)I$,

$$\nabla_i T_k(A_H[u])^{ij} = -(n-k)\nabla_i (H_2[u])\, T_{k-1}(A_H)^{ij}. \tag{2.3.7}$$

To this end, we take the divergence of both sides in (2.3.1), which yields

$$
\begin{aligned}
\nabla_i T_k(A_H[u])^{ij} &= \nabla^j \sigma_k(A_H[u]) - \nabla_i\big(T_{k-1}(A_H[u])^{il}(A_H[u])_l^j\big) \\
&= \frac{\partial \sigma_k(A_H)}{\partial (A_H)_{il}} \nabla^j (A_H[u])_{il} - \nabla_i (T_{k-1}(A_H[u]))^{il} (A_H)_l^j - T_{k-1}(A_H)^{il} \nabla_i (A_H[u])_l^j \\
&\overset{(2.3.2)}{=} T_{k-1}(A_H)^{il}\big(\nabla^j (A_H[u])_{il} - \nabla_i (A_H[u])_l^j\big) - \nabla_i (T_{k-1}(A_H[u]))^{il}(A_H)_l^j \\
&= T_{k-1}(A_H)^{il}\big(\nabla_i (H[u])_l^j - \nabla^j (H[u])_{il}\big) - \nabla_i\big(T_{k-1}(A_H[u])\big)^{il}(A_H)_l^j. \quad (2.3.8)
\end{aligned}
$$

Then (2.3.6) is readily seen by applying (2.3.8) iteratively.

We now turn to (2.3.7), for which we apply an induction argument on $k$ using (2.3.8). The base case $k = 1$ is clear. We suppose that for some $k \geq 2$ we have the identity

$$\nabla_i T_{k-1}(A_H[u])^{ij} = -(n-k+1)\nabla_i (H_2[u])\, T_{k-2}(A_H)^{ij}, \quad (2.3.9)$$

and we show that (2.3.7) then follows. First observe that, by (2.3.9) and the fact $H_{ij} = H_2 \delta_{ij}$, (2.3.8) simplifies to

$$
\begin{aligned}
\nabla_i T_k(A_H[u])^{ij} = \nabla_i (H_2[u]) T_{k-1}(A_H)^{ij} - \nabla^j (H_2[u])\, \mathrm{tr}(T_{k-1}(A_H)) \\
+ (n-k+1)\nabla_i (H_2[u])(T_{k-2}(A_H)A_H)^{ij}. \quad (2.3.10)
\end{aligned}
$$

After substituting (2.3.1) and (2.3.3) into the last term and the penultimate term in (2.3.10), respectively, we arrive at (2.3.7). □

Note that $V[u]^j$ (defined in (2.3.4)) contains at most second order derivatives of $u$. As a consequence, $\nabla_i F[u]^{ij}$ is a regular distribution for $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$. More precisely, we have:

**Lemma 2.3.2.** *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ with $q > 1$ and $2 \leq k \leq n$. Then for $H \in C^{0,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n; \mathrm{Sym}_n(\mathbb{R}))$ and $\varphi \in W^{1,s}_0(\Omega; \mathbb{R}^n)$, $s = \frac{q+k-1}{q}$, we have*

$$\int_\Omega F[u]^{ij} \nabla_i \varphi_j = -\int_\Omega V[u]^j \varphi_j, \quad (2.3.11)$$

*where* $V[u]^j$ *is defined in* (2.3.4). *In particular,* $\nabla_i F[u]^{ij} = V[u]^j \in L^{(q+k-1)/(k-1)}_{\mathrm{loc}}(\Omega)$ *and*

$$\big|\nabla_i F[u]^{ij}\big| \leq C\big(1 + |\nabla^2 u|^{k-1}\big) \quad \text{a.e. in } B_{2R}, \tag{2.3.12}$$

*where* $C$ *is a constant depending on an upper bound for* $\|H\|_{C^{0,1}(\Sigma)}$.

*Proof.* It is clear that $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ implies $V[u]^j \in L^{(q+k-1)/(k-1)}_{\mathrm{loc}}(\Omega)$. Since $\frac{1}{s} + \frac{k-1}{q+k-1} = 1$, it suffices to prove (2.3.11) for $\varphi \in C^\infty_0(\Omega;\mathbb{R}^n)$. Let $u_{(m)} \in C^3(\Omega)$ be such that $u_{(m)} \to u$ in $W^{2,q+k-1}_{\mathrm{loc}}(\Omega)$. Then by (2.3.4), we have for each $m \in \mathbb{N}$ the identity $\nabla_i F[u_{(m)}]^{ij} = V[u_{(m)}]^j$, and it follows that

$$\int_\Omega F[u_{(m)}]^{ij}\nabla_i\varphi_j = -\int_\Omega V[u_{(m)}]^j\varphi_j. \tag{2.3.13}$$

Now, since $u_{(m)} \to u$ in $W^{2,q+k-1}_{\mathrm{loc}}(\Omega)$, we have both $F[u_{(m)}] \to F[u]$ and $V[u_{(m)}] \to V[u]$ in $L^{(q+k-1)/(k-1)}_{\mathrm{loc}}(\Omega)$. In particular, we can take $m \to \infty$ in (2.3.13) to get (2.3.11).

The estimate (2.3.12) follows from the definition of $V[u]^j$. □

## 2.4 Main estimates for the proofs of Theorems F and G

### 2.4.1 Initial integral estimates: isolating higher order terms

The following lemma provides the starting point for our integral estimates:

**Lemma 2.4.1.** *Suppose* $f \in C^0(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ *is positive,* $H \in C^0(\Omega\times\mathbb{R}\times\mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ *and* $u$ *is a solution to* (2.1.2). *Then for fixed* $h$,

$$\sum_l kf^{k-1}\Delta^h_{ll}f[u] \leq F^{ij}\nabla_i\nabla_j v - \sum_l F^{ij}\Delta^h_{ll}(H[u])_{ij} \quad \text{a.e. in } \Omega_h. \tag{2.4.1}$$

*Proof.* The proof follows [**Urb00**], with some adjustments. For $A \in \mathrm{Sym}_n(\mathbb{R})$, let $G^{ij}(A) = \partial\sigma_k^{1/k}(A)/\partial A_{ij} = k^{-1}\sigma_k(A)^{(1-k)/k}F^{ij}(A)$, and denote $G^{ij} := G^{ij}(A_H[u])$. Fix $l \in \{1,\ldots,n\}$ and $h \in \mathbb{R}\backslash\{0\}$. Then there exists a set $S_{h,l} \subset \Omega_h$ with $\mathcal{L}(\Omega_h\backslash S_{h,l}) = 0$

(where $\mathcal{L}$ is the Lebesgue measure) such that $A_H[u](x), A_H[u](x \pm he_l) \in \Gamma_k^+$ for all $x \in S_{h,l}$. By concavity of $\sigma_k^{1/k}$ in $\Gamma_k^+$, it follows that for $x \in S_{h,l}$ we have

$$\sigma_k^{1/k}\big(A_H[u](x \pm he_l)\big) - \sigma_k^{1/k}(A_H[u](x)) \leq G^{ij}(x)\big(A_H[u](x \pm he_l) - A_H[u](x)\big)_{ij}. \quad (2.4.2)$$

Adding the two equations in (2.4.2), dividing through by $h^2$ and summing over $l$, we have

$$\sum_l \Delta_{ll}^h \sigma_k^{1/k}\big(A_H[u](x)\big) \leq \sum_l G^{ij}(x)\Delta_{ll}^h\big(A_H[u](x)\big)_{ij} \quad \text{for } x \in S_h = \bigcap_{l=1}^n S_{h,l}, \quad (2.4.3)$$

with $S_h$ clearly satisfying $\mathcal{L}(\Omega_h \backslash S_h) = 0$. Substituting $G^{ij} = k^{-1}\sigma_k^{(1-k)/k}F^{ij}$ into (2.4.3) and recalling that $A_H[u] = \nabla^2 u - H[u]$, we obtain

$$\sum_l k\sigma_k^{\frac{k-1}{k}}(A_H)\Delta_{ll}^h \sigma_k^{1/k}(A_H[u]) \leq \sum_l F^{ij}\Delta_{ll}^h\big(\nabla^2 u - H[u]\big)_{ij} \quad \text{in } S_h. \quad (2.4.4)$$

Substituting the equation $\sigma_k^{1/k}(A_H) = f$ into the LHS of (2.4.4), and commuting difference quotients with derivatives on the RHS of (2.4.4), we arrive at (2.4.1). □

As outlined in §2.2, we proceed to derive a series of integral estimates by multiplying (2.4.1) by suitable test functions and integrating by parts using the divergence structure proved in Lemma 2.3.2. Recall that for a fixed increment $h > 0$, we defined $v(x) = \sum_l \Delta_{ll}^h u(x)$, and that we fixed a ball $B_{2R} \Subset \Omega_h$ and a constant $C_1$ (depending on an upper bound for $\|H\|_{C^0(\Sigma)}$) such that $\Delta u + C_1 \geq 1$ and $|\nabla^2 u| \leq \Delta u + C_1$ a.e. in $B_{2R}$. The existence of such a constant is guaranteed by the assumption $A_H \in \Gamma_2^+$. We then defined $\tilde{v} = v + C_1$, and for a small parameter $\delta > 0$ (that we eventually take to zero) we defined $Q_\delta = \big((\tilde{v}^+)^2 + \delta^2\big)^{1/2}$. For $\rho \in (0, \frac{R}{3}]$, we also fix a cutoff function $\eta \in C_c^\infty(B_{R+2\rho})$ satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{R+\rho}$ and $|\nabla^l \eta| \leq C(n)\rho^{-l}$ for $l = 1, 2$.

Suppose $u \in W_{\text{loc}}^{2,q+k-1}(\Omega) \cap W_{\text{loc}}^{1,\infty}(\Omega)$ $(q > 1)$ is a solution to (2.1.2). Multiplying (2.4.1) by $\eta Q_\delta^{q-1}$ and integrating over the domain $B_{R+2\rho}$, we see

$$\sum_l \int_{B_{R+2\rho}} k\eta Q_\delta^{q-1} f^{k-1}\Delta_{ll}^h f[u]$$
$$\leq \int_{B_{R+2\rho}} \eta Q_\delta^{q-1}F^{ij}\nabla_i\nabla_j\tilde{v} - \sum_l \int_{B_{R+2\rho}} \eta Q_\delta^{q-1}F^{ij}\Delta_{ll}^h(H[u])_{ij}, \quad (2.4.5)$$

which is just the estimate (2.2.8) in §2.2, repeated here for convenience.

We are now in a position to prove our first integral estimate. In what follows, let

$$J_h^{(s)} = \int_{B_{R+2\rho}} (\tilde{v}^+)^s + \int_{B_{R+3\rho}} (\Delta u + C_1)^s.$$

Roughly speaking, if $u \in W^{2,s}_{\mathrm{loc}}(\Omega)$ then $J_h^{(s)}$ should be interpreted as a lower order term, and terms bounded by $J_h^{(s)}$ are consequently considered 'good terms'.

We will first address the case $f = f(x,z)$ in Theorems F and G for simplicity and postpone the more general case until §2.5.3. The relevant equation is therefore

$$\sigma_k^{1/k}\big(A_H[u](x)\big) = f(x,u(x)) > 0, \quad A_H[u](x) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega. \tag{2.4.6}$$

Throughout §2.4, unless otherwise stated, $C$ will denote a generic positive constant which may vary from line to line, depending only on $n, R, f, H$ and an upper bound for $\|u\|_{W^{1,\infty}(B_{2R})}$. In particular, $C$ is independent of $h$, $q$ and $\rho$, and any norm of $\nabla^2 u$. In addition, we will often use the inequalities $\Delta u + C_1 \geq 1$ and $|\nabla^2 u| \leq \Delta u + C_1$ without explicit reference.

**Lemma 2.4.2.** *Suppose $f \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R})$ is positive, $H \in C^{0,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ and $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$) is a solution to (2.4.6). Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(q-1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-2}F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v} + \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v} + \sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\Delta^h_{ll}(H[u])_{ij} \leq C\rho^{-2}J_h^{(q+k-1)}. \tag{2.4.7}$$

*Proof of Lemma 2.4.2.* Appealing to Lemma 2.3.2 with $\varphi_j = \eta Q_\delta^{q-1}\nabla_j\tilde{v}$, and noting that

$$\nabla_i\varphi_j = Q_\delta^{q-1}\nabla_i\eta\nabla_j\tilde{v} + (q-1)\tilde{v}^+Q_\delta^{q-3}\nabla_i\tilde{v}\nabla_j\tilde{v} + \eta Q_\delta^{q-1}\nabla_i\nabla_j\tilde{v},$$

we have

$$\int_{B_{R+2\rho}} F^{ij}\Big(Q_\delta^{q-1}\nabla_i\eta\nabla_j\tilde{v} + (q-1)\tilde{v}^+Q_\delta^{q-3}\nabla_i\tilde{v}\nabla_j\tilde{v} + \eta Q_\delta^{q-1}\nabla_i\nabla_j\tilde{v}\Big)$$
$$= -\int_{B_{R+2\rho}} \eta Q_\delta^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v}. \qquad (2.4.8)$$

Rearranging (2.4.8) to get the desired integration by parts formula for $\int_{B_{R+2\rho}}\eta Q_\delta^{q-1}F^{ij}\nabla_i\nabla_j\tilde{v}$, and substituting this back into (2.4.5), we obtain

$$(q-1)\int_{B_{R+2\rho}}\eta\tilde{v}^+Q_\delta^{q-3}F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v} + \int_{B_{R+2\rho}}\eta Q_\delta^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v}$$
$$+\sum_l\int_{B_{R+2\rho}}\eta Q_\delta^{q-1}F^{ij}\Delta_{ll}^h(H[u])_{ij} \leq -\int_{B_{R+2\rho}} Q_\delta^{q-1}F^{ij}\nabla_i\eta\nabla_j\tilde{v}$$
$$-\sum_l\int_{B_{R+2\rho}} k\eta Q_\delta^{q-1}f^{k-1}\Delta_{ll}^h f[u]. \qquad (2.4.9)$$

We now take $\delta\to 0$ in (2.4.9), using Fatou's lemma for the first integral (which is positive) and the dominated convergence theorem elsewhere (which is justified since $q>1$). This yields

$$(q-1)\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-2}F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v} + \int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v}$$
$$+\sum_l\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-1}F^{ij}\Delta_{ll}^h(H[u])_{ij} \leq -\int_{B_{R+2\rho}}(\tilde{v}^+)^{q-1}F^{ij}\nabla_i\eta\nabla_j\tilde{v}$$
$$-\sum_l\int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1}f^{k-1}\Delta_{ll}^h f[u]. \qquad (2.4.10)$$

To conclude the proof of Lemma 2.4.2, we must bound the RHS of (2.4.10) from above by $C\rho^{-2}J_h^{(q+k-1)}$. We begin with the first integral on the RHS of (2.4.10). Appealing again to Lemma 2.3.2, now with $\varphi_j = \frac{1}{q}(\tilde{v}^+)^q\nabla_j\eta$ and $\nabla_i\varphi_j = (\tilde{v}^+)^{q-1}\nabla_i\tilde{v}\nabla_j\eta + \frac{1}{q}(\tilde{v}^+)^q\nabla_i\nabla_j\eta$, we have

$$\int_{B_{R+2\rho}} F^{ij}\Big((\tilde{v}^+)^{q-1}\nabla_i\eta\nabla_j\tilde{v} + \frac{1}{q}(\tilde{v}^+)^q\nabla_i\nabla_j\eta\Big) = -\frac{1}{q}\int_{B_{R+2\rho}}(\tilde{v}^+)^q\nabla_i F[u]^{ij}\nabla_j\eta.$$

Therefore,

$$\bigg|\int_{B_{R+2\rho}}(\tilde{v}^+)^{q-1}F^{ij}\nabla_i\eta\nabla_j\tilde{v}\bigg| \leq \bigg|\frac{1}{q}\int_{B_{R+2\rho}}(\tilde{v}^+)^q F^{ij}\nabla_i\nabla_j\eta\bigg| + \bigg|\frac{1}{q}\int_{B_{R+2\rho}}(\tilde{v}^+)^q\nabla_i F[u]^{ij}\nabla_j\eta\bigg|$$
$$\leq \frac{C}{\rho^2}\int_{B_{R+2\rho}}(\tilde{v}^+)^q|F| + \frac{C}{\rho}\int_{B_{R+2\rho}}(\tilde{v}^+)^q\big|\mathrm{div}\,F[u]\big|. \qquad (2.4.11)$$

Recalling $|F| \leq C(\Delta u + C_1)^{k-1}$ and applying Hölder's inequality to the penultimate integral in (2.4.11), we see that $\int_{B_{R+2\rho}} (\tilde{v}^+)^q |F| \leq C J_h^{(q+k-1)}$. The final integral in (2.4.11) satisfies the same estimate, since $|\operatorname{div} F[u]| \leq C(\Delta u + C_1)^{k-1}$ by (2.3.12).

It remains to estimate the second term on the RHS of (2.4.10). Keeping in mind that $f = f(x,z) \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R})$, we apply Hölder's inequality followed by (2.2.6) to obtain

$$\begin{aligned}\left|\sum_l \int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1} \Delta^h_{ll} f[u]\right| &\leq C\Big(\int_{B_{R+2\rho}} (\tilde{v}^+)^q\Big)^{\frac{q-1}{q}} \Big(\int_{B_{R+2\rho}} \Big|\sum_l \Delta^h_{ll} f[u]\Big|^q\Big)^{\frac{1}{q}} \\ &\overset{(2.2.6)}{\leq} C\Big(\int_{B_{R+2\rho}} (\tilde{v}^+)^q\Big)^{\frac{q-1}{q}} \Big(\int_{B_{R+3\rho}} \big|\Delta f[u]\big|^q\Big)^{\frac{1}{q}} \leq C J_h^{(q)}. \end{aligned} \tag{2.4.12}$$

This concludes the proof. □

To clear up notation, we denote the three integrals on the LHS of (2.4.7) involving higher order terms by

$$\begin{aligned}(\mathrm{I}_1)_h &= (q-1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-2} F^{ij} \nabla_i \tilde{v} \nabla_j \tilde{v}, \\ (\mathrm{I}_2)_h &= \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} \nabla_i F[u]^{ij} \nabla_j \tilde{v} \quad \text{and} \\ (\mathrm{I}_3)_h &= \sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} F^{ij} \Delta^h_{ll} (H[u])_{ij}.\end{aligned}$$

The terms $(\mathrm{I}_1)_h$, $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ will be considered in turn. In §2.4.2, we prove an estimate for $(\mathrm{I}_1)_h$. In §2.4.3, we consider the term $(\mathrm{I}_2)_h$: in §2.4.3.1, we estimate $(\mathrm{I}_2)_h$ in the case that $H$ is a multiple of the identity, and in §2.4.3.2 we estimate $(\mathrm{I}_2)_h$ for general $H$ when $k = 2$. In §2.4.4, we consider $(\mathrm{I}_3)_h$. Since the estimate for $(\mathrm{I}_3)_h$ in the general case is slightly involved, for illustrative purposes we first address in §2.4.4.1 the simpler case when $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$, which includes the $\sigma_k$-Yamabe equation in the positive case. The estimate for $(\mathrm{I}_3)_h$ in the general case is proved in §2.4.4.2. In the process, we will prove the cancellation phenomenon between $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ alluded to earlier – see Corollaries 2.4.12, 2.4.13 and 2.4.14.

### 2.4.2 A pointwise lower bound for $F[u]^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v}$

The term $F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v}$ in $(\mathrm{I}_1)_h$ can be bounded in the same way as in [**Urb00**] (see equation (3.6) therein). We reproduce the argument here for the reader's convenience.

**Lemma 2.4.3.** *Suppose $f \in C^0(\Omega \times \mathbb{R})$ is positive, $H \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\,\mathrm{Sym}_n(\mathbb{R}))$ and $u$ is a solution to* (2.4.6)*. Then for $q > 0$,*

$$(v^+)^{q-2}F^{ij}\nabla_i\tilde{v}\nabla_j\tilde{v} \geq \frac{4f^k}{q^2}\frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \quad \text{a.e. in } \Omega_h. \tag{2.4.13}$$

*In particular, for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$, $q > 1$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_1)_h \geq \frac{q-1}{Cq^2}\int_{B_{R+\rho}} f^k\frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)}. \tag{2.4.14}$$

*Proof.* For $1 \leq l \leq n$, denote by $F^{ij}_{(l)}(A)$ the matrix with entries $\partial\sigma_l(A)/\partial A_{ij}$. Using the concavity of $\sigma_k(A)/\sigma_{k-1}(A)$ on $\Gamma_k^+$, we have

$$\frac{F^{ij}_{(k)}(A)}{\sigma_k(A)} \geq \frac{F^{ij}_{(k-1)}(A)}{\sigma_{k-1}(A)} \quad \text{for all } A \in \Gamma_k^+ \tag{2.4.15}$$

(see e.g. [**Urb00, LT94**]). Applying (2.4.15) inductively, it follows that

$$\frac{F^{ij}_{(k)}(A)}{\sigma_k(A)} \geq \cdots \geq \frac{F^{ij}_{(1)}(A)}{\sigma_1(A)} = \frac{\delta^{ij}}{\mathrm{tr}(A)} \quad \text{for all } A \in \Gamma_k^+. \tag{2.4.16}$$

Taking $A = A_H[u]$ in (2.4.16), where $u$ is a solution to (2.4.6), we obtain

$$\frac{F[u]^{ij}(x)}{f^k[u](x)} \geq \frac{\delta^{ij}}{\Delta u(x) - \mathrm{tr}(H[u](x))} \quad \text{for a.e. } x \in \Omega,$$

from which (2.4.13) is readily seen. The estimate (2.4.14) then follows from properties of $\eta$. □

### 2.4.3 Integral estimates for $\nabla_i F[u]^{ij}\nabla_j\tilde{v}$

In this section we obtain estimates for the quantity $(\mathrm{I}_2)_h = \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_i F[u]^{ij}\nabla_j\tilde{v}$. The case in which $H$ is a multiple of the identity matrix will be dealt with first, in §2.4.3.1. The case for general $H$ when $k = 2$ will then be addressed in §2.4.3.2.

#### 2.4.3.1 The case $H = H_2(x,z,\xi)I$

In this section we prove the following two lemmas:

**Lemma 2.4.4.** *Suppose $f \in C^0(\Omega \times \mathbb{R})$ is positive, $H \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ with $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$, and that $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$) is a solution to* (2.4.6)*. Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h \geq -\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial(H_1|\xi|^2)}{\partial \xi_a}[u]\nabla_a\tilde{v} - C\rho^{-1}J_h^{(q+k-1)}. \tag{2.4.17}$$

**Lemma 2.4.5.** *Suppose $H \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ with $H(x,z,\xi) = H_2(x,z,\xi)I$, and that $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$). Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h \geq -\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial H_2}{\partial \xi_a}[u]\nabla_a\tilde{v} - C\rho^{-1}J_h^{(q+k)}. \tag{2.4.18}$$

**Remark 2.4.6.** Note that in Lemma 2.4.5, we do not assume that $u$ solves (2.4.6). In contrast, the weaker integrability assumption in Lemma 2.4.4 relies on both the fact that $u$ solves (2.4.6) *and* that $H_2$ depends quadratically on $\nabla u$.

**Remark 2.4.7.** The first term on the RHS of (2.4.17) and (2.4.18) will later be shown to cancel with a term arising from our estimate for $(\mathrm{I}_3)_h$.

*Proof of Lemmas 2.4.4 and 2.4.5.* The proof consists of three steps. In Step 1, we prove a preliminary estimate assuming only $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and $H = H_2(x,z,\xi)I$, but we do not assume at this point that $u$ necessarily solves (2.4.6). Only in Steps 2 and 3 will we appeal to the specific hypotheses of Lemmas 2.4.4 and 2.4.5.

Our starting point is the following expression for $(\mathrm{I}_2)_h$, which follows from (2.3.5):

$$(\mathrm{I}_2)_h = -(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla_j(H_2[u])\,T_{k-2}(A_H)^{ij}\nabla_i\tilde{v}.$$

***Step 1:*** In this step, we show that for every $u \in W^{2,q+k-1}_{\rm loc}(\Omega) \cap W^{1,\infty}_{\rm loc}(\Omega)$,

$$(\mathrm{I}_2)_h \geq -\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial H_2}{\partial \xi_a}[u]\nabla_a \tilde{v} - \frac{n-k+1}{q}\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^q F^i_a \frac{\partial^2 H_2}{\partial \xi_a \partial \xi_b}[u](A_H)_{ib} \\ - C\rho^{-1} J_h^{(q+k-1)}. \tag{2.4.19}$$

Note that the first integral on the RHS of (2.4.19) is the desired integral seen on the RHS of (2.4.17) and (2.4.18).

To prove (2.4.19), first observe that by the chain rule,

$$\begin{aligned}(\mathrm{I}_2)_h = &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial x^j}[u]\,T_{k-2}(A_H)^{ij}\nabla_i \tilde{v} \\ &- (n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial z}[u]\,T_{k-2}(A_H)^{ij}\nabla_j u \nabla_i \tilde{v} \\ &- (n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\,T_{k-2}(A_H)^{ij}\nabla_j \nabla_a u\,\nabla_i \tilde{v}.\end{aligned} \tag{2.4.20}$$

Denote the top two lines of the RHS of (2.4.20) collectively by $L_1$, and the bottom line by $L_2$. Recalling that $\nabla_j\nabla_a u = H_2\delta_{ja} + (A_H)_{ja}$ and, in view of (2.3.1) and (2.3.3), that

$$\big(T_{k-2}(A_H)A_H\big)_{ia} = -F_{ia} + \frac{1}{n-k+1}\,\mathrm{tr}(F)\delta_{ia}, \tag{2.4.21}$$

we have

$$\begin{aligned}L_2 = &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\,T_{k-2}(A_H)_{ia}H_2\nabla^i\tilde{v} \\ &- (n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\,\big(T_{k-2}(A_H)A_H\big)_{ia}\nabla^i\tilde{v} \\ \overset{(2.4.21)}{=} &-(n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\,T_{k-2}(A_H)_{ia}H_2\nabla^i\tilde{v} \\ &+ (n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\,F_{ia}\nabla^i\tilde{v} - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial H_2}{\partial \xi_a}[u]\,\nabla_a\tilde{v}.\end{aligned}$$

Substituting this identity for $L_2$ into (2.4.20) yields

$$\begin{aligned}(\mathrm{I}_2)_h = &\, L_1 - (n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\,T_{k-2}(A_H)_{ia}H_2\nabla^i\tilde{v} \\ &+ (n-k+1)\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial \xi_a}[u]\,F_{ia}\nabla^i\tilde{v} - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\frac{\partial H_2}{\partial \xi_a}[u]\,\nabla_a\tilde{v}.\end{aligned} \tag{2.4.22}$$

We claim that the terms on the top line of the RHS of (2.4.22) are bounded from below by $-C\rho^{-1}J_h^{(q+k-1)}$. Indeed, as $T_{k-2}(A_H)^{ij} = \partial\sigma_{k-1}(A_H)/\partial(A_H)_{ij}$, by Lemma 2.3.2 we have $|\nabla_i T_{k-2}(A_H[u])^{ij}| \leq C(\Delta u + C_1)^{k-2}$. It is also clear that $|T_{k-2}(A_H)^{ij}| \leq C(\Delta u + C_1)^{k-2}$. Thus, after integrating by parts using Lemma 2.3.2 and applying Hölder's inequality, the lower bound for these terms follows.

To estimate the penultimate integral in (2.4.22), we integrate by parts using Lemma 2.3.2 and apply the following identity resulting from the chain rule:

$$\nabla_i\bigg(\frac{\partial H_2}{\partial\xi_a}[u](x)\bigg) = \bigg(\frac{\partial^2 H_2}{\partial\xi_a\partial\xi_b}[u](x)\bigg)\big((A_H)_{ib}+H_{ib}\big)+\bigg(\frac{\partial^2 H_2}{\partial z\partial\xi_a}[u](x)\bigg)\nabla_i u(x)+\frac{\partial^2 H_2}{\partial x^i\partial\xi_a}[u](x).$$

After an application of Hölder's inequality, this gives

$$\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-1}\frac{\partial H_2}{\partial\xi_a}[u]\,F_{ia}\nabla^i\tilde{v} \geq -\frac{1}{q}\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^q F_a^i\frac{\partial^2 H_2}{\partial\xi_a\partial\xi_b}[u]\,(A_H)_{ib} - C\rho^{-1}J_h^{(q+k-1)},$$

from which (2.4.19) follows.

***Step 2:*** In this step we prove Lemma 2.4.5. Indeed, for $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ (not necessarily solving (2.4.6)) we have the estimate

$$-\frac{n-k+1}{q}\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^q F_a^i\frac{\partial^2 H_2}{\partial\xi_a\partial\xi_b}[u]\,(A_H)_{ib} \geq -C\int_{B_{R+2\rho}}(\tilde{v}^+)^q|F||A_H| \geq -CJ_h^{(q+k)},$$

with the last inequality following once again from the estimate $|F| \leq C(\Delta u + C_1)^{k-1}$ and Hölder's inequality. Substituting this into (2.4.19) then yields the desired estimate (2.4.18).

***Step 3:*** In this step we prove Lemma 2.4.4. Since we assume in this case that $H_2(x,z,\xi) = H_1(x,z)|\xi|^2$ and that $u$ solves (2.4.6), rather than estimating as in Step 2 we observe

$$F_a^i\frac{\partial^2 H_2}{\partial\xi_a\partial\xi_b}[u](A_H)_{ib} = 2H_1F_a^i\delta^{ab}(A_H)_{ib} = 2H_1F_a^i(A_H)_i^a = 2H_1k\sigma_k(A_H) = 2H_1kf^k. \tag{2.4.23}$$

Substituting (2.4.23) into the second integral in (2.4.19), we arrive at (2.4.17). □

#### 2.4.3.2 The case $k = 2$ for general $H$

In this section we obtain an estimate in the case $k = 2$ analogous to (2.4.17) and (2.4.18). We do not assume that $H$ is a multiple of the identity and, as in Lemma 2.4.5, we do not assume that $u$ solves (2.4.6):

**Lemma 2.4.8.** *Suppose $H \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,; \mathrm{Sym}_n(\mathbb{R}))$, $k = 2$ and $u \in W^{2,q+2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$. Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h \geq \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} \frac{\partial H^{ij}}{\partial \xi_a}[u]\, \nabla_i \nabla_a u \, \nabla_j \tilde{v} - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} \frac{\partial \operatorname{tr}(H)}{\partial \xi_a}[u] \, \operatorname{tr}(A_H) \nabla_a \tilde{v} \\ - C\rho^{-1} J_h^{(q+2)}. \tag{2.4.24}$$

**Remark 2.4.9.** The first two terms on the RHS of (2.4.24) will later be shown to cancel with a term arising from our estimate for $(\mathrm{I}_3)_h$ (cf. Remark 2.4.7).

*Proof of Lemma 2.4.8.* As $k = 2$, we have $\nabla_i F[u]^{ij} = \nabla_i H[u]^{ij} - \nabla^j \operatorname{tr}(H[u])$ (by (2.3.4)) and $\nabla^j \nabla^a u = \operatorname{tr}(A_H)\delta^{ja} - F^{ja} - H^{ja}$. Using the chain rule to calculate $\nabla_i H[u]^{ij} - \nabla^j \operatorname{tr}(H[u])$ and then substituting in the identity for $\nabla^j \nabla^a u$, it follows that

$$\begin{aligned}
(\mathrm{I}_2)_h &= \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} \big(\nabla_i H[u]^{ij} - \nabla^j \operatorname{tr}(H[u])\big) \nabla_j \tilde{v} \\
&= \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} \bigg( \frac{\partial H^{ij}}{\partial \xi_a}[u]\, \nabla_i \nabla_a u - \frac{\partial \operatorname{tr}(H)}{\partial \xi^a}[u]\, \nabla^j \nabla^a u \bigg) \nabla_j \tilde{v} \\
&\quad + \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} \bigg( \frac{\partial H^{ij}}{\partial x^i}[u] + \frac{\partial H^{ij}}{\partial z}[u]\, \nabla_i u - \frac{\partial \operatorname{tr}(H)}{\partial x_j}[u] - \frac{\partial \operatorname{tr}(H)}{\partial z}[u]\, \nabla^j u \bigg) \nabla_j \tilde{v} \\
&= \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} \bigg( \frac{\partial H^{ij}}{\partial \xi_a}[u]\, \nabla_i \nabla_a u - \frac{\partial \operatorname{tr}(H)}{\partial \xi^a}[u]\, \operatorname{tr}(A_H)\delta^{ja} \bigg) \nabla_j \tilde{v} \\
&\quad + \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1} \bigg( \frac{\partial H^{ij}}{\partial x^i}[u] + \frac{\partial H^{ij}}{\partial z}[u]\, \nabla_i u - \frac{\partial \operatorname{tr}(H)}{\partial x_j}[u] - \frac{\partial \operatorname{tr}(H)}{\partial z}[u]\, \nabla^j u \\
&\qquad\qquad + \frac{\partial \operatorname{tr}(H)}{\partial \xi^a}[u](F^{ja} + H^{ja}) \bigg) \nabla_j \tilde{v}. 
\end{aligned} \tag{2.4.25}$$

The integral on the last two lines of (2.4.25) can be bounded from below by $-C\rho^{-1}J_h^{(q+2)}$ in exactly the same way as in the proof of Lemmas 2.4.4 and 2.4.5:

we integrate by parts using Lemma 2.3.2, estimate the relevant quantities in terms of $\Delta u + C_1$ and apply Hölder's inequality. The estimate (2.4.24) then follows. $\square$

### 2.4.4 Integral estimates for $F[u]^{ij}\Delta_{ll}^h H[u]_{ij}$

In this section we obtain estimates for $(\mathrm{I}_3)_h = \sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\Delta_{ll}^h(H[u])_{ij}$. More precisely, we will prove the following lemma:

**Lemma 2.4.10.** *Suppose $H \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$, $R > 0$ is such that $B_{2R} \Subset \Omega$ and $\rho \in (0, \frac{R}{3}]$.*

*a) If $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$), then for $|h|$ sufficiently small, we have*

$$(\mathrm{I}_3)_h \geq \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\frac{\partial H_{ij}}{\partial \xi_a}[u]\,\nabla_a\tilde{v} - CJ_h^{(q+k)}. \tag{2.4.26}$$

*b) If $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$) and $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$, then for $|h|$ sufficiently small, we have*

$$(\mathrm{I}_3)_h \geq \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}F^{ij}\frac{\partial H_{ij}}{\partial \xi_a}[u]\,\nabla_a\tilde{v} - CJ_h^{(q+k-1)}. \tag{2.4.27}$$

**Remark 2.4.11.** Neither estimate in Lemma 2.4.10 requires $u$ to be a solution to (2.4.6).

Before proving Lemma 2.4.10 we first discuss its consequences, namely the resulting cancellations between $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$. We first consider the case $H = H_1(x,z)|\xi|^2 I$ with $H_1 \geq 0$:

**Corollary 2.4.12.** *Suppose $f \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R})$ is positive, $H \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$ with $H = H_1(x,z)|\xi|^2 I$ and $H_1 \geq 0$, and that $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ ($q > 1$) is a solution to (2.4.6). Then for $R > 0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h + (\mathrm{I}_3)_h \geq -C\rho^{-1}J_h^{(q+k-1)}. \tag{2.4.28}$$

*In particular,*

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \operatorname{tr}(H)} \leq C\rho^{-2} J_h^{(q+k-1)}. \tag{2.4.29}$$

*Proof.* The estimate (2.4.28) follows from combining the estimates (2.4.17) and (2.4.27). The estimate (2.4.29) is then obtained by substituting (2.4.14) and (2.4.28) into (2.4.7).

□

Similarly, we obtain the following in the case that $H = H_2(x,z,\xi)I$:

**Corollary 2.4.13.** *Suppose $H \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\operatorname{Sym}_n(\mathbb{R}))$ with $H = H_2(x,z,\xi)I$, and $u \in W^{2,q+k}_{\text{loc}}(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega)$ $(q>1)$. Then for $R>0$ with $B_{2R} \Subset \Omega$, $\rho \in (0, \frac{R}{3}]$ and $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h + (\mathrm{I}_3)_h \geq -C\rho^{-1} J_h^{(q+k)}. \tag{2.4.30}$$

*If, in addition, $u$ solves* (2.4.6) *for some positive $f \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R})$, then*

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \operatorname{tr}(H)} \leq C\rho^{-2} J_h^{(q+k)}. \tag{2.4.31}$$

*Proof.* The estimate (2.4.30) follows from combining the estimates (2.4.18) and (2.4.26). The estimate (2.4.31) is then obtained by substituting (2.4.14) and (2.4.30) into (2.4.7).

□

A similar cancellation also holds in the setting of Theorem G, although this requires a little more work:

**Corollary 2.4.14.** *Suppose $H \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\operatorname{Sym}_n(\mathbb{R}))$, $k=2$ and $u \in W^{2,q+2}_{\text{loc}}(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega)$ $(q>1)$. Then for $|h|$ sufficiently small, we have*

$$(\mathrm{I}_2)_h + (\mathrm{I}_3)_h \geq -C\rho^{-2} J_h^{(q+2)}. \tag{2.4.32}$$

*If, in addition, $u$ solves* (2.4.6) *for some positive $f \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R})$, then*

$$\frac{q-1}{q^2}\int_{B_{R+\rho}} f^2 \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \operatorname{tr}(H)} \leq C\rho^{-1} J_h^{(q+2)}. \tag{2.4.33}$$

*Proof.* The estimate (2.4.33) will immediately follow once (2.4.32) is established, by substituting (2.4.14) and (2.4.32) into (2.4.7).

Taking $k = 2$ in Lemma 2.4.10 a) and using $F^{ij} = \mathrm{tr}(A_H)\delta^{ij} - \nabla^i\nabla^j u - H^{ij}$, we see

$$(\mathrm{I}_3)_h \geq \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(A_H)\frac{\partial\,\mathrm{tr}(H)}{\partial\xi_a}[u]\,\nabla_a\tilde{v} - \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\nabla^i\nabla^j u\frac{\partial H_{ij}}{\partial\xi_a}[u]\,\nabla_a\tilde{v}$$
$$- \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}H^{ij}\frac{\partial H_{ij}}{\partial\xi_a}[u]\,\nabla_a\tilde{v} - CJ_h^{(q+2)}. \tag{2.4.34}$$

Now, the first term on the RHS of (2.4.34) cancels with the second term on the RHS of (2.4.24), and the first term on the last line of (2.4.34) can be estimated from below by $-C\rho^{-1}J_h^{(q+2)}$, after integrating by parts and applying Hölder's inequality. Therefore, combining (2.4.24) and (2.4.34), we obtain

$$(\mathrm{I}_2)_h + (\mathrm{I}_3)_h \geq \frac{1}{q}\int_{B_{R+2\rho}} \eta\frac{\partial H_{ij}}{\partial\xi_a}[u]\left(\nabla^i\nabla_a u\,\nabla^j(\tilde{v}^+)^q - \nabla^i\nabla^j u\,\nabla_a(\tilde{v}^+)^q\right) - C\rho^{-1}J_h^{(q+2)}.$$

Now, if $u$ were to have enough regularity, we could integrate by parts here, observe that the third derivatives of $u$ cancel, and obtain (2.4.32) by estimating the remaining terms in the usual way. To circumvent the lack of regularity, we instead apply the following lemma:

**Lemma 2.4.15.** *Let $U \subset \mathbb{R}^n$ be a smooth bounded domain and let $B \in L^\infty(U;\mathbb{R}^{n\times n})$ be an antisymmetric matrix with $\mathrm{supp}(B) \Subset U$. For $1 \leq p < \infty$ and $p' = \frac{p}{p-1}$, consider the bilinear form $\mathcal{B} : W^{1,p}(U) \times W^{1,p'}(U) \to \mathbb{R}$ given by*

$$\mathcal{B}(g,h) = \int_U B^a_j\,\nabla_a g\,\nabla^j h. \tag{2.4.35}$$

*If $\mathrm{div}\,B \in L^q(U;\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1 - \frac{1}{r}$ for some $1 \leq q, r \leq \infty$, then we have the estimate*

$$|\mathcal{B}(g,h)| \leq \int_U |\,\mathrm{div}\,B||\nabla g||h| \tag{2.4.36}$$

*for all $g \in W^{1,p}(U)$ and $h \in W^{1,p'}(U) \cap L^r(U)$.*

Before proving Lemma 2.4.15, we use it to complete the proof of (2.4.32): for each $i \in \{1, \ldots, n\}$, taking $B_j^a = \eta \frac{\partial H_{ij}}{\partial \xi_a}[u] - \eta \frac{\partial H_i^a}{\partial \xi^j}[u]$, $g = \nabla_i u$ and $h = (\tilde{v}^+)^q$ in Lemma 2.4.15 we obtain

$$\int_{B_{R+2\rho}} \eta \frac{\partial H_{ij}}{\partial \xi_a}[u] \left( \nabla^i \nabla_a u \nabla^j (\tilde{v}^+)^q - \nabla^i \nabla^j u \nabla_a (\tilde{v}^+)^q \right) \overset{(2.4.36)}{\leq} C\rho^{-1} \int_{B_{R+2\rho}} (\Delta u + C_1)^2 (\tilde{v}^+)^q$$
$$\leq C\rho^{-1} J_h^{(q+2)}.$$

It remains to prove Lemma 2.4.15. By a standard approximation argument, it suffices to prove (2.4.36) for $g, h \in C^\infty(U)$. We are then justified in integrating by parts in (2.4.35), giving

$$|\mathcal{B}(g,h)| = \left| \int_U \Big( \nabla^j B_j^a \nabla_a g + \underbrace{B_j^a \nabla_a \nabla^j g}_{=0} \Big) h \right| \leq \int_U |\operatorname{div} B| |\nabla g| |h|,$$

where we have used antisymmetry of $B$ to assert that $B_j^a \nabla_a \nabla^j g = 0$. $\square$

#### 2.4.4.1 Proof of Lemma 2.4.10 b)

We now turn our attention back to the proof of Lemma 2.4.10. Whilst the two estimates (2.4.26) and (2.4.27) can be dealt with simultaneously (see the proof of Lemma 2.4.10 in §2.4.4.2), for illustrative purposes we first provide a more direct proof of (2.4.27), which includes the $\sigma_k$-Yamabe equation in the positive case. Indeed, when $H = H_1(x,z)|\xi|^2 I$ we are able to calculate the third order contribution of $\Delta_{ll}^h (H[u])_{ij}$ explicitly by deriving the following discrete version of the Bochner identity, avoiding the more involved estimates required for the general case. In what follows, we denote

$$u_l^h(x) = u(x + he_l).$$

**Lemma 2.4.16** (Discrete Bochner identity)**.** *Suppose* $H_1 \in C^0(\Omega \times \mathbb{R})$ *and* $l \in \{1, \ldots, n\}$*. Then*

$$\begin{aligned} \Delta_{ll}^h \big( H_1[u] |\nabla u|^2 \big) &= 2 H_1 \nabla^i u \nabla_i \Delta_{ll}^h u + (H_1[u])_l^{-h} \big| \nabla \nabla_l^{-h} u \big|^2 + (H_1[u])_l^h \big| \nabla \nabla_l^h u \big|^2 \\ &\quad + \nabla_l^{-h} \nabla_i u \nabla^i u \nabla_l^{-h} H_1[u] + \nabla_l^h \nabla^i u \nabla_i u \nabla_l^h H_1[u] \\ &\quad + \nabla_l^h \Big( \nabla_i u (\nabla^i u)_l^{-h} \nabla_l^{-h} H_1[u] \Big). \end{aligned} \tag{2.4.37}$$

Assuming the validity of Lemma 2.4.16, the proof of (2.4.27) in Lemma 2.4.10 b) is then straightforward:

*Proof of Lemma 2.4.10 b).* Substituting the discrete Bochner identity (2.4.37) into the definition of $(\mathrm{I}_3)_h$ and dropping the two positive terms, we obtain

$$(\mathrm{I}_3)_h \geq 2\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)H_1\nabla^i u\nabla_i\tilde{v}$$
$$+\sum_l \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}\,\mathrm{tr}(F)\Big(\nabla_l^{-h}\nabla_i u\nabla^i u\nabla_l^{-h}H_1[u] + \nabla_l^h\nabla^i u\nabla_i u\nabla_l^h H_1[u]$$
$$+\nabla_l^h\Big(\nabla_i u(\nabla^i u)_l^{-h}\nabla_l^{-h}H_1[u]\Big)\Big). \tag{2.4.38}$$

After applying the product rule for difference quotients,

$$\nabla_l^h(uv)(x) = u_l^h(x)\nabla_l^h v(x) + v(x)\nabla_l^h u(x), \tag{2.4.39}$$

to the integrand in the last line of (2.4.38), we may then estimate the last two lines of (2.4.38) in the usual way. Namely, after applying the bound $\mathrm{tr}(F)\leq C(\Delta u + C_1)^{k-1}$, using Hölder's inequality and appealing to (2.2.5), we see that the last two lines of (2.4.38) are collectively bounded from below by $-CJ_h^{(q+k-1)}$. The estimate (2.4.27) then follows. □

*Proof of the discrete Bochner identity (Lemma 2.4.16).* Using the product rule (2.4.39) to first calculate $\nabla_l^{-h}(H_1[u]|\nabla u|^2)$, we see

$$\Delta_{ll}^h\big(H_1[u]|\nabla u|^2\big) = \nabla_l^h\Big(\nabla_l^{-h}\big(H_1[u]\nabla^i u\nabla_i u\big)\Big)$$
$$= \nabla_l^h\Big((H_1[u]\nabla^i u)_l^{-h}\nabla_l^{-h}\nabla_i u\Big) + \nabla_l^h\Big(H_1[u]\nabla_i u\nabla_l^{-h}\nabla^i u\Big) + \nabla_l^h\Big(\nabla_i u(\nabla^i u)_l^{-h}\nabla_l^{-h}H_1[u]\Big).$$

On the other hand, noting that $\nabla_l^h u_l^{-h}u(x) = \nabla_l^{-h}u(x)$ and $(\nabla_l^{-h}u)_l^h(x) = \nabla_l^h u(x)$, we also have by (2.4.39) the identities

$$\nabla_l^h\Big((H_1[u]\nabla^i u)_l^{-h}\nabla_l^{-h}\nabla_i u\Big) = H_1\nabla^i u\nabla_l^h\nabla_l^{-h}\nabla_i u + \nabla_l^{-h}\nabla_i u\nabla_l^{-h}(H_1[u]\nabla^i u)$$
$$= H_1\nabla^i u\nabla_i\Delta_{ll}^h u + (H_1[u])_l^{-h}\big|\nabla\nabla_l^{-h}u\big|^2 + \nabla_l^{-h}\nabla_i u\nabla^i u\nabla_l^{-h}H_1[u]$$

and

$$\nabla_l^h\Big(H_1[u]\nabla_i u\nabla_l^{-h}\nabla^i u\Big) = \nabla_l^h\nabla^i u\nabla_l^h\Big(H_1[u]\nabla_i u\Big) + H_1\nabla_i u\nabla_l^h\nabla_l^{-h}\nabla^i u$$
$$= (H_1[u])_l^h\big|\nabla\nabla_l^h u\big|^2 + \nabla_l^h\nabla^i u\nabla_i u\nabla_l^h H_1[u] + H_1\nabla_i u\nabla^i\Delta_{ll}^h u.$$

Putting these three identities together, we arrive at (2.4.37). □

#### 2.4.4.2 Proof of Lemma 2.4.10 in the general case

We now prove Lemma 2.4.10 in the general case. To simplify our analysis, we will make use of the following semi-convexity property of $H \in C^{1,1}_{\text{loc}}(\Omega\times\mathbb{R}\times\mathbb{R}^n;\text{Sym}_n(\mathbb{R}))$: there exists a constant $C_\Sigma > 0$ such that the mapping $\xi \longmapsto H(x,z,\xi) + C_\Sigma|\xi|^2 I$ is convex for all $(x,z,\xi)\in\Sigma$ (this is an immediate consequence of the $C^{1,1}_{\text{loc}}$ regularity of $H$). We refer the reader to Remark 2.1.3 for the definition of $\Sigma$. We will make use of this property in the form

$$H_{ij}(x,z,\xi) \geq H_{ij}(x,z,\zeta) + \frac{\partial H_{ij}}{\partial\xi_a}(x,z,\zeta)(\xi-\zeta)_a - C_\Sigma\delta_{ij}|\xi-\zeta|^2 \tag{2.4.40}$$

for all $(x,z,\xi),(x,z,\zeta)\in\Sigma$. Note that in Case 1 of Theorem F, we may take $C_\Sigma = 0$ in (2.4.40), as $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ is convex with respect to $\xi$ when $H_1 \geq 0$. The inequality (2.4.40) will play a role similar to that of the discrete Bochner identity used in the previous subsection (see Lemma 2.4.16).

*Proof of Lemma 2.4.10.* We first prove Lemma 2.4.10 a). It suffices to show that

$$F^{ij}\Delta_{ll}^h(H[u])_{ij} \geq F^{ij}\frac{\partial H_{ij}}{\partial\xi_a}[u]\,\nabla_a\Delta_{ll}^h u + \text{error terms}\quad \forall l\in\{1,\ldots,n\}, \tag{2.4.41}$$

where the error terms satisfy

$$\int_{B_{R+2\rho}}\eta(\tilde v^+)^{q-1}|\text{error terms}| \leq CJ_h^{(q+k)}. \tag{2.4.42}$$

To keep notation succinct, we denote $x^\pm = x\pm he_l$ in what follows.

***Step 1:*** We first prove a lower bound for $F^{ij}(x)\Delta_{ll}^h(H[u](x))_{ij}$, identifying the error terms in (2.4.41). Observe that by (2.4.40) and the fact that $F^{ij}$ is positive definite in

$\Gamma_k^+$, we have

$$\begin{aligned}
&\frac{F^{ij}(x)}{h^2}\bigg[(H[u](x^\pm))_{ij} - H(x^\pm, u(x^\pm), \nabla u(x))_{ij}\bigg]\\
&\geq \frac{F^{ij}(x)}{h^2}\frac{\partial H_{ij}}{\partial \xi_a}\big(x^\pm, u(x^\pm), \nabla u(x)\big)\big(\nabla_a u(x^\pm) - \nabla_a u(x)\big) - \frac{C_\Sigma |F|}{h^2}\big|\nabla u(x^\pm) - \nabla u(x)\big|^2\\
&\geq \frac{F^{ij}(x)}{h^2}\frac{\partial H_{ij}}{\partial \xi_a}[u](x)\big(\nabla_a u(x^\pm) - \nabla_a u(x)\big) - \frac{C_\Sigma |F|}{h^2}\big|\nabla u(x^\pm) - \nabla u(x)\big|^2\\
&\quad - \frac{C|F|}{|h|}\big|\nabla u(x^\pm) - \nabla u(x)\big| \quad \text{for a.e. } x \in B_{R+2\rho},
\end{aligned}$$

where to obtain the second inequality we have estimated

$$\left|\frac{\partial H_{ij}}{\partial \xi_a}\big(x^\pm, u(x^\pm), \nabla u(x)\big) - \frac{\partial H_{ij}}{\partial \xi_a}[u](x)\right| \leq \|H\|_{C^{1,1}(\Sigma)}\big(|x^\pm - x| + |u(x^\pm) - u(x)|\big) \leq C|h|.$$

Recalling (2.2.4), we therefore see that for a.e. $x \in B_{R+2\rho}$,

$$\begin{aligned}
&F^{ij}(x)\Delta_{ll}^h(H[u](x))_{ij}\\
&\qquad \geq F^{ij}(x)\frac{\partial H_{ij}}{\partial \xi_a}[u](x)\nabla_a \Delta_{ll}^h u(x)\\
&\qquad\quad + \frac{F^{ij}(x)}{h^2}\bigg(H(x^+, u(x^+), \nabla u(x))_{ij} - 2(H[u](x))_{ij} + H(x^-, u(x^-), \nabla u(x))_{ij}\bigg)\\
&\qquad\quad - C_\Sigma|F||\nabla_l^h \nabla u|^2 - C_\Sigma|F||\nabla_l^{-h}\nabla u|^2 - C|F||\nabla_l^h \nabla u| - C|F||\nabla_l^{-h}\nabla u|.
\end{aligned} \tag{2.4.43}$$

***Step 2:*** To prove (2.4.26), we need to show that the error terms in last two lines of (2.4.43) satisfy (2.4.42). Formally, these terms behave like $|F|(|\nabla^2 u|^2 + |\nabla^2 u|)$, and so by the estimate $|F| \leq C(\Delta u + C_1)^{k-1}$, the bound (2.4.42) is then conceivable. We now give the details.

Denote the terms on the penultimate line of (2.4.43) collectively by $E_1$, and the terms on the last line of (2.4.43) collectively by $E_2$. The error terms in $E_2$ are easier to deal with. Indeed, by the bound $|F| \leq C(\Delta u + C_1)^{k-1}$, Hölder's inequality and (2.2.5), we have

$$\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|F||\nabla_l^{\pm h}\nabla u|^2 \leq C(J_h^{(q+k)})^{\frac{q+k-2}{q+k}}\bigg(\int_{B_{R+2\rho}} |\nabla_l^{\pm h}\nabla u|^{q+k}\bigg)^{\frac{2}{q+k}} \leq CJ_h^{(q+k)}. \tag{2.4.44}$$

In exactly the same way, one can show $\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|F||\nabla_l^{\pm h}\nabla u| \leq CJ_h^{(q+k-1)}$, and combining these estimates we obtain $\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|E_2| \leq CJ_h^{(q+k)}$.

We now treat the error terms in $E_1$. We first observe that by the fundamental theorem of calculus followed by the chain rule, we have the identities

$$\begin{aligned}
&H(x^\pm, u(x^\pm), \xi)_{ij} - H(x, u(x), \xi)_{ij} \\
&= \int_0^1 \frac{d}{dt} H(x \pm the_l, u(x^\pm), \xi)_{ij}\, dt + \int_0^1 \frac{d}{dt} H(x, u(x \pm the_l), \xi)_{ij}\, dt \\
&= \pm h \int_0^1 \frac{\partial H_{ij}}{\partial x^l}(x \pm the_l, u(x^\pm), \xi)\, dt \pm h \int_0^1 \frac{\partial H_{ij}}{\partial z}(x, u(x \pm the_l), \xi) \nabla_l u(x \pm the_l)\, dt,
\end{aligned}$$

and therefore

$$\begin{aligned}
&H(x^+, u(x^+), \xi)_{ij} - 2H(x, u(x), \xi)_{ij} + H(x^-, u(x^-), \xi)_{ij} \\
&= h \int_0^1 \left( \frac{\partial H_{ij}}{\partial z}(x, u(x + the_l), \xi) \nabla_l u(x + the_l) - \frac{\partial H_{ij}}{\partial z}(x, u(x - the_l), \xi) \nabla_l u(x - the_l) \right) dt \\
&\quad + h \int_0^1 \left( \frac{\partial H_{ij}}{\partial x^l}(x + the_l, u(x^+), \xi) - \frac{\partial H_{ij}}{\partial x^l}(x - the_l, u(x^-), \xi) \right) dt. \qquad (2.4.45)
\end{aligned}$$

Now, by the $C^{1,1}_{\mathrm{loc}}$ regularity of $H$ and the Lipschitz regularity of the mapping $(x, z, p) \mapsto \frac{\partial H_{ij}}{\partial z}(x, z, \xi) p_l$ for fixed $\xi$ and each $l \in \{1, \ldots, n\}$, we can estimate the last line of (2.4.45) from above by $Ch^2$ and the middle line of (2.4.45) from above by

$$Ch^2 + Ch^2 \int_0^1 \frac{1}{t|h|} \Big| \nabla_l u(x + the_l) - \nabla_l u(x - the_l) \Big|\, dt.$$

Applying these estimates in (2.4.45) and taking $\xi = \nabla u(x)$, we therefore see that

$$|E_1| \leq C|F| + C|F| \int_0^1 |\nabla_l^{th} \nabla_l u(x)|\, dt + C|F| \int_0^1 |\nabla_l^{-th} \nabla_l u(x)|\, dt. \qquad (2.4.46)$$

Using (2.4.46), one readily obtains the estimate $\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|E_1| \leq CJ_h^{(q+k-1)}$, applying the same line of argument as seen above for $E_2$. For example, by Fubini's theorem and Young's inequality, we have

$$\begin{aligned}
\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|F| \left( \int_0^1 |\nabla_l^{\pm th} \nabla_l u(x)|\, dt \right) dx &= \int_0^1 \int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|F||\nabla_l^{\pm th} \nabla_l u(x)| dx\, dt \\
&\leq CJ_h^{(q+k-1)} + C \int_0^1 \int_{B_{R+2\rho}} |\nabla_l^{\pm th} \nabla_l u|^{q+k-1}\, dx\, dt \overset{(2.2.5)}{\leq} CJ_h^{(q+k-1)}.
\end{aligned}$$

This completes the proof of Lemma 2.4.10 a).

***Step 3:*** It remains to prove Lemma 2.4.10 b) (see §2.4.4.1 for an alternative proof which is independent of calculations in Steps 1 and 2 above). Note that in this case, we may take $C_\Sigma = 0$ in (2.4.43) and so the error terms on the last two lines of (2.4.43) formally behave like $|F||\nabla^2 u|$. By the same argument as in Step 2, the error terms $E_1$ and $E_2$ considered in Step 2 therefore satisfy $\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q-1}|E_i| \le C J_h^{(q+k-1)}$, and the conclusion follows. □

## 2.5 Completing the proofs of Theorems F and G

In this section we use Corollaries 2.4.12, 2.4.13 and 2.4.14 to prove Theorems F and G, as outlined at the end of §2.2. We will start in §2.5.1 with a detailed proof of Case 1 of Theorem F when $f = f(x,z)$, and then indicate the necessary adjustments for remaining cases, still when $f = f(x,z)$, in §2.5.2. In §2.5.3, we extend these results to the case $f = f(x,z,\xi)$, completing the proofs of Theorems F and G.

### 2.5.1 Proof of Case 1 of Theorem F when $f = f(x,z)$

In this case, we recall that by Corollary 2.4.12 we have the estimate

$$\int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \le \frac{Cq}{\rho^2} J_h^{(q+k-1)}, \tag{2.5.1}$$

where $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$. Let $\theta \in (0,1)$ be such that $\frac{2-\theta}{\theta} \le q+k-1$ (we will eventually take $\theta = \frac{4}{kn+2}$). Also denote by $(2-\theta)^* := n(2-\theta)/(n-2+\theta)$ the Sobolev conjugate of $2-\theta$. We first obtain from (2.5.1) the following:

**Lemma 2.5.1.** *Suppose $f \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R})$ is positive, $H = H_1(x,z)|\xi|^2 I$ with $H_1 \in C^{1,1}_{\mathrm{loc}}(\Omega\times\mathbb{R})$ and $H_1 \ge 0$, and that $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q>1)$ is a solution to* (2.4.6)*. Then*

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\frac{q(2-\theta)^*}{2}}\right)^{\frac{2}{(2-\theta)^*}} \le \frac{Cq}{\rho^2}\left(\int_{B_{R+3\rho}} (\Delta u + C_1)^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1}. \tag{2.5.2}$$

*Proof.* The estimate (2.5.2) will follow immediately once we establish the estimate

$$\left(\int_{B_{R+\rho}} (\tilde{v}^+)^{\frac{q(2-\theta)^*}{2}}\right)^{\frac{2}{(2-\theta)^*}} \leq \frac{Cq}{\rho^2}\left|J_h^{(\frac{2-\theta}{\theta})}\right|^{\frac{\theta}{2-\theta}} J_h^{(q+k-1)}, \tag{2.5.3}$$

since we can then apply Fatou's lemma and the fact that $\tilde{v}^+ \to \Delta u + C_1$ a.e. as $h \to 0$ to the term on the LHS of (2.5.3), and Lemma 2.2.1 to the terms on the RHS of (2.5.3).

Keeping in mind the lower bound $\inf_{B_{2R}} f > \frac{1}{C} > 0$, we first observe that by Hölder's inequality and (2.5.1), we have

$$\begin{aligned}\left(\int_{B_{R+\rho}} \left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^{2-\theta}\right)^{\frac{2}{2-\theta}} &\leq C\left(\int_{B_{R+\rho}} (\Delta u - \mathrm{tr}(H))^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \\ &\overset{(2.5.1)}{\leq} \frac{Cq}{\rho^2}\left|J_h^{(\frac{2-\theta}{\theta})}\right|^{\frac{\theta}{2-\theta}} J_h^{(q+k-1)}. \end{aligned}\tag{2.5.4}$$

On the other hand, since $\frac{q(2-\theta)}{2} \leq q+k-1$, Hölder's inequality gives

$$\left(\int_{B_{R+\rho}} (\tilde{v}^+)^{\frac{q(2-\theta)}{2}}\right)^{\frac{2}{2-\theta}} \leq \left(\int_{B_{R+\rho}} (\tilde{v}^+)^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+\rho}} (\tilde{v}^+)^{q-1} \leq \left|J_h^{(\frac{2-\theta}{\theta})}\right|^{\frac{\theta}{2-\theta}} J_h^{(q+k-1)}. \tag{2.5.5}$$

Applying the Sobolev inequality to $(\tilde{v}^+)^{q/2} \in W^{1,2-\theta}$, and appealing to (2.5.4) and (2.5.5), we arrive at (2.5.3). □

The inequality (2.5.2) is of reverse Hölder-type if $\theta$ satisfies

$$\frac{2-\theta}{\theta} < q+k-1 < \frac{q(2-\theta)^*}{2}.$$

For example, if we fix $\theta = \frac{4}{kn+2}$ and finally impose the assumption $q+k-1 > \frac{kn}{2}$, we see that $\frac{2-\theta}{\theta} = \frac{kn}{2} < q+k-1$ and

$$\frac{q(2-\theta)^*}{2} - (q+k-1) > \left(\frac{kn}{2} - k + 1\right)\left(\frac{kn}{2+kn-2k} - 1\right) - k + 1 = 0.$$

In what follows, we denote

$$\beta := \frac{(2-\theta)^*}{2} = \frac{kn}{kn+2-2k} > 1.$$

*Proof of Case 1 of Theorem F when $f = f(x, z)$.* With $\theta = \frac{4}{kn+2}$, we obtain from (2.5.2) the estimate

$$\left( \int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q} \right)^{1/\beta} \leq \frac{Cq}{\rho^2} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k-1} \tag{2.5.6}$$

for all $q > \frac{kn}{2} - k + 1$ and $\rho \in (0, \frac{R}{3}]$. The constant $C$ in (2.5.6) and below now depends on $\int_{B_{R+3\rho}} (\Delta u + C_1)^{kn/2}$, which is finite due to our hypotheses.

We now carry out the Moser iteration argument. Let $p > \frac{kn}{2}$ be as in the statement of Theorem F, and define a sequence $q_j$ inductively by

$$q_0 = p - k + 1, \quad q_j = \beta q_{j-1} - k + 1 \text{ for } j \geq 1.$$

Then $q_j = \beta q_{j-1} - (k-1) = \beta^j q_0 - (k-1)(\beta^{j-1} + \cdots + \beta + 1)$, which implies

$$\frac{q_j}{\beta^j} = q_0 - (k-1)\left( \frac{1 - \beta^{-j}}{\beta - 1} \right) \overset{j \to \infty}{\longrightarrow} q_0 - \frac{k-1}{\beta - 1} > 0. \tag{2.5.7}$$

Note that the limit in (2.5.7) is positive by definition of $\beta$ and the fact that $q_0 > \frac{kn}{2} - k + 1$. In particular, $q_j \to \infty$ as $j \to \infty$.

Applying (2.5.6) iteratively with $q = q_j$ and $\rho = 3^{-j-1}R$, we have for each $j \geq 0$

$$\begin{aligned} \left( \int_{B_{(1+3^{-j-1})R}} (\Delta u + C_1)^{\beta q_j} \right)^{\beta^{-j-1}} &\leq \left( 9^j C q_j \int_{B_{(1+3^{-j})R}} (\Delta u + C_1)^{\beta q_{j-1}} \right)^{\beta^{-j}} \\ &\overset{(2.5.7)}{\leq} \prod_{i=0}^{j} \left( (9\beta)^i C \right)^{\beta^{-i}} \int_{B_{2R}} (\Delta u + C_1)^p \\ &\leq (9\beta)^{\sum_{i=0}^{\infty} i\beta^{-i}} C^{\sum_{i=0}^{\infty} \beta^{-i}} \int_{B_{2R}} (\Delta u + C_1)^p. \end{aligned}$$

Letting $j \to \infty$ and appealing once again to (2.5.7), we arrive at

$$\|\Delta u + C_1\|_{L^\infty(B_R)} \leq C \left( \int_{B_{2R}} (\Delta u + C_1)^p \right)^{\left( q_0 - \frac{k-1}{\beta - 1} \right)^{-1}},$$

which implies the desired bound on $\|\nabla^2 u\|_{L^\infty(B_R)}$ by the choice of $C_1$. □

### 2.5.2 Proof of Case 2 of Theorem F and Theorem G when $f = f(x, z)$

In the remaining cases, we recall that by Corollaries 2.4.13 and 2.4.14 we have the estimate

$$\int_{B_{R+\rho}} f^k \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\Delta u - \mathrm{tr}(H)} \leq \frac{Cq}{\rho^2} J_h^{(q+k)}, \tag{2.5.8}$$

where $u \in W^{2,q+k}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$.

*Proof of Case 2 of Theorem F and Theorem G when $f = f(x, z)$.* We let $\theta \in (0, 1)$ be such that $\frac{2-\theta}{\theta} \leq q + k$. Following the same arguments as in §2.5.1, one readily obtains the following counterpart to the estimate (2.5.2):

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\frac{q(2-\theta)^*}{2}}\right)^{\frac{2}{(2-\theta)^*}} \leq \frac{Cq}{\rho^2}\left(\int_{B_{R+3\rho}} (\Delta u + C_1)^{\frac{2-\theta}{\theta}}\right)^{\frac{\theta}{2-\theta}} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k}. \tag{2.5.9}$$

Taking $\theta = \frac{4}{(k+1)n+2}$ and imposing $q + k > \frac{(k+1)n}{2}$, we see

$$\frac{2-\theta}{\theta} = \frac{(k+1)n}{2} < q + k < \frac{q(2-\theta)^*}{2}.$$

We thus obtain from (2.5.9) the estimate

$$\left(\int_{B_{R+\rho}} (\Delta u + C_1)^{\beta q}\right)^{1/\beta} \leq \frac{Cq}{\rho^2} \int_{B_{R+3\rho}} (\Delta u + C_1)^{q+k},$$

where

$$\beta := \frac{(k+1)n}{(k+1)n + 2 - 2(k+1)} > 1$$

and $C$ now depends on $\int_{B_{R+3\rho}} (\Delta u + C_1)^{(k+1)n/2}$. The Moser iteration argument then follows through as before, using $p > \frac{(k+1)n}{2}$ and defining $q_j$ inductively by $q_0 = p - k$ and $q_j = \beta q_{j-1} - k$ for $j \geq 1$. □

### 2.5.3 Proof of Theorems F and G for $f = f(x, z, \xi)$

In this section we explain how the preceding arguments may be adjusted to treat the general case $f = f(x, z, \xi)$, thus completing the proofs of Theorems F and G:

*Proof of Theorems F and G.* The arguments up until (2.4.12) remain valid for $f = f(x, z, \xi)$, but the last term in (2.4.10) can no longer be estimated as in (2.4.12). Consequently, under otherwise the same hypotheses, the conclusion of Lemma 2.4.2 now reads

$$(\mathrm{I}_1)_h + (\mathrm{I}_2)_h + (\mathrm{I}_3)_h + (\mathrm{I}_4)_h \leq C\rho^{-2} J_h^{(q+k-1)},$$

where $(\mathrm{I}_1)_h$, $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ are as before and $(\mathrm{I}_4)_h$ is defined by

$$(\mathrm{I}_4)_h = \sum_l \int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1} \Delta_{ll}^h f[u].$$

The estimates for $(\mathrm{I}_1)_h$, $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ are unchanged (see Lemmas 2.4.3, 2.4.4, 2.4.5, 2.4.8 and 2.4.10), since they do not involve differentiating $f$. The integrand of $(\mathrm{I}_4)_h$ was previously a lower order term, but is now formally of third order in $u$. However, this can be treated using some of the ideas already seen in the proof of Lemma 2.4.10. Indeed, let $C_\Sigma > 0$ be a constant such that the mapping $\xi \longmapsto f(x, z, \xi) + C_\Sigma |\xi|^2$ is convex for all $(x, z, \xi) \in \Sigma$. Then by the same argument leading to (2.4.43), we have for each $l \in \{1, \ldots, n\}$ and a.e. $x \in B_{R+2\rho}$ the estimate

$$\begin{aligned}\Delta_{ll}^h f[u](x) \geq \frac{\partial f}{\partial \xi_a}[u](x) \nabla_a \Delta_{ll}^h u(x) - C_\Sigma |\nabla_l^h \nabla u|^2 - C_\Sigma |\nabla_l^{-h} \nabla u|^2 - C|\nabla_l^h \nabla u| - C|\nabla_l^{-h} \nabla u| \\ + \frac{1}{h^2}\Big(f(x^+, u(x^+), \nabla u(x)) - 2f[u](x) + f(x^-, u(x^-), \nabla u(x))\Big). \quad (2.5.10)\end{aligned}$$

Denoting all but the first term on the RHS of (2.5.10) as error terms, it follows from (2.5.10) that

$$(\mathrm{I}_4)_h \geq \int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1} \frac{\partial f}{\partial \xi_a}[u] \nabla_a \tilde{v} - \int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1} |\text{error terms}|. \quad (2.5.11)$$

Now, in the same way that we dealt with the error terms in Step 2 of the proof of Lemma 2.4.10, one readily obtains $\int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1} |\text{error terms}| \leq C J_h^{(q+1)}$. For the first integral on the RHS of (2.5.11), we integrate by parts and apply Hölder's inequality to obtain

$$\left| \int_{B_{R+2\rho}} k\eta(\tilde{v}^+)^{q-1} f^{k-1} \frac{\partial f}{\partial \xi_a}[u] \nabla_a \tilde{v} \right| \leq C\rho^{-1} J_h^{(q+1)}.$$

Returning to (2.5.11), we therefore obtain $(\mathrm{I}_4)_h \geq -C\rho^{-1}J_h^{(q+1)}$. As a consequence, the estimates (2.5.1) and (2.5.8) hold, and the arguments of §2.5 therefore apply without any changes. □

## 2.6 Extending Theorems F and G: the case $k \geq 3$ for general $H$

In this section we consider a minor extension of Theorems F and G. Recall that our proof of Theorems F and G exploited a cancellation phenomenon between higher order terms arising from $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$, where the divergence structure of $F^{ij}$ played a role in estimating $(\mathrm{I}_2)_h$. When $3 \leq k \leq n$ and $H$ is not necessarily a multiple of the identity, the divergence structure given in (2.3.4) is more involved. That said, if one assumes higher integrability on $\nabla^2 u$ from the outset, the terms $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ may be estimated by using Cauchy's inequality and absorbing the resulting negative higher order terms into the positive term $(\mathrm{I}_1)_h$. This avoids the need to prove any cancellation between $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$. We establish:

**Theorem H.** [**DN20**] *Let $\Omega$ be a domain in $\mathbb{R}^n$ $(n \geq 3)$, $f = f(x,z,\xi) \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ a positive function and $H \in C^{1,1}_{\mathrm{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n\,;\mathrm{Sym}_n(\mathbb{R}))$. Suppose $3 \leq k \leq n$, $p > kn$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ is a solution to (2.1.2). Then $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$, and for any concentric balls $B_R \subset B_{2R} \Subset \Omega$ we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where $C$ is a constant depending only on $n, p, R, f, H$ and an upper bound for $\|u\|_{W^{2,p}(B_{2R})}$.*

*Proof.* Following the proof of Theorem G in §2.5.3 but leaving the terms $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ untreated, we have for $u \in W^{2,q+k-1}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ $(q > 1)$

$$\frac{q-1}{Cq^2}\int_{B_{R+2\rho}} \eta \frac{\left|\nabla\left((\tilde{v}^+)^{q/2}\right)\right|^2}{\mathrm{tr}(A_H)} + (\mathrm{I}_2)_h + (\mathrm{I}_3)_h \leq C\rho^{-2}J_h^{(q+k-1)}. \tag{2.6.1}$$

We now suppose further that $\nabla^2 u \in L^{q+2k-1}_{\rm loc}(\Omega)$ $(q > 1)$. By Cauchy's inequality and the bound $|\operatorname{div} F[u]| \leq C(\Delta u + C_1)^{k-1}$, we see that for all $\delta > 0$

$$
\begin{aligned}
(\mathrm{I}_2)_h &= \frac{2}{q}\int_{B_{R+2\rho}} \eta(\tilde{v}^+)^{q/2}\nabla_i F[u]^{ij}\nabla_j(\tilde{v}^+)^{q/2} \\
&\geq -\frac{\delta(q-1)}{q^2}\int_{B_{R+2\rho}}\eta\frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\operatorname{tr}(A_H)} - \frac{1}{\delta(q-1)}\int_{B_{R+2\rho}}\eta(\tilde{v}^+)^q\operatorname{tr}(A_H)\big|\operatorname{div} F[u]\big|^2 \\
&\geq -\frac{\delta(q-1)}{q^2}\int_{B_{R+2\rho}}\eta\frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\operatorname{tr}(A_H)} - \frac{C}{\delta(q-1)}J_h^{(q+2k-1)}. \qquad (2.6.2)
\end{aligned}
$$

By similar reasoning, it also holds that

$$
\begin{aligned}
(\mathrm{I}_3)_h &\overset{(2.4.26)}{\geq} \int_{B_{R+2\rho}}\eta(\tilde{v}^+)^{q-1}F^{ij}\frac{\partial H_{ij}}{\partial\xi_a}[u]\,\nabla_a\tilde{v} - CJ_h^{(q+k)} \\
&\geq -\frac{\delta(q-1)}{q^2}\int_{B_{R+2\rho}}\eta\frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\operatorname{tr}(A_H)} - \frac{C}{\delta(q-1)}J_h^{(q+2k-1)}. \qquad (2.6.3)
\end{aligned}
$$

Taking $\delta$ sufficiently small in (2.6.2) and (2.6.3), and then substituting these estimates into (2.6.1), we obtain

$$
\frac{q-1}{q^2}\int_{B_{R+2\rho}}\eta\frac{\big|\nabla\big((\tilde{v}^+)^{q/2}\big)\big|^2}{\operatorname{tr}(A_H)} \leq C\rho^{-2}J_h^{(q+2k-1)}. \qquad (2.6.4)
$$

The argument then proceeds as in §2.5.1: we let $\theta \in (0,1)$ be such that $\frac{2-\theta}{\theta} \leq q+2k-1$ and obtain from (2.6.4) the estimate

$$
\bigg(\int_{B_{R+\rho}}(\Delta u + C_1)^{\frac{q(2-\theta)^*}{2}}\bigg)^{\frac{2}{(2-\theta)^*}} \leq \frac{Cq}{\rho^2}\bigg(\int_{B_{R+3\rho}}(\Delta u + C_1)^{\frac{2-\theta}{\theta}}\bigg)^{\frac{\theta}{2-\theta}}\int_{B_{R+3\rho}}(\Delta u + C_1)^{q+2k-1}. \qquad (2.6.5)
$$

Taking $\theta = \frac{2}{kn+1}$ and imposing $q + 2k - 1 > kn$, we see that $\frac{2-\theta}{\theta} = kn < q + 2k - 1 < \frac{q(2-\theta)^*}{2}$, and we therefore obtain from (2.6.5) the estimate

$$
\bigg(\int_{B_{R+\rho}}(\Delta u + C_1)^{\beta q}\bigg)^{1/\beta} \leq \frac{Cq}{\rho^2}\int_{B_{R+3\rho}}(\Delta u + C_1)^{q+2k-1},
$$

where $\beta := kn/(kn+1-2k) > 1$ and $C$ now depends on $\int_{B_{R+3\rho}}(\Delta u + C_1)^{kn}$. The Moser iteration argument then goes through as before, giving the desired conclusion. □

## 2.7 An improvement on Theorem F: regularity via the ABP estimate

### 2.7.1 Introduction and statement of new result: Theorem I

As we have seen, the proof of Theorems F and G relies on a certain divergence structure. It is interesting to ask whether such estimates can be obtained without any knowledge of the divergence structure; in fact, we have already seen one instance of this in the proof of Theorem H, although the integrability requirements there are quite stringent.

In this final section of Chapter 2, we show using an alternative method that the integrability assumptions on $\nabla^2 u$ can be weakened in:

1. Case 1 of Theorem F when $k < \frac{n}{2} + 1$, assuming either $H_1 \equiv 0$ or $H_1 > 0$ on $\Omega$,

2. Case 2 of Theorem F when $k < \frac{n}{2}$, assuming $H_2(x, z, \xi) = H_1(x, z)|\xi|^2$.

We note that in each case, the assumed quadratic dependence on $\xi$ will allow us to apply the discrete Bochner formula derived in §2.4.4.1. This quadratic dependence still encompasses the $\sigma_k$-Yamabe equation in both the positive and negative cases, and in particular we obtain improvements on both Theorems A and B for certain values of $k$.

Our method to obtain these improvements is inspired by the work of Bao et. al. [**BCGJ03,LB05**] on the quotient Hessian equations, and makes use of the Alexandrov-Bakelman-Pucci estimate. No integration by parts is involved, and consequently we do not require any divergence structure. The result is as follows:

**Theorem I.** *Let $\Omega$ be a domain in $\mathbb{R}^n$ ($n \geq 3$), $f = f(x, z) \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R})$ a positive function and $H \in C^{1,1}_{\text{loc}}(\Omega \times \mathbb{R} \times \mathbb{R}^n; \text{Sym}_n(\mathbb{R}))$. Suppose $2 \leq k \leq n$, $p \geq 1$ and $u \in W^{2,p}_{\text{loc}}(\Omega) \cap C^{0,1}_{\text{loc}}(\Omega)$ is a solution to*

$$\sigma_k^{1/k}(A_H[u](x)) = f(x, u(x)) > 0, \quad A_H[u](x) \in \Gamma_k^+ \quad \text{for a.e. } x \in \Omega,$$

*and that one of the following conditions holds:*

*1.* $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ *with* $H_1 \equiv 0$ *or* $H_1 > 0$*, and*

$$\begin{cases} p \geq n & \text{if } k(k-1) < n \\ p > k(k-1) & \text{otherwise.} \end{cases}$$

*2.* $H(x,z,\xi) = H_1(x,z)|\xi|^2 I$ *and* $p > n + k(k-1)$*.*

*Then* $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$*, and for any concentric balls* $B_R \subset B_{3R} \Subset \Omega$ *we have*

$$\|\nabla^2 u\|_{L^\infty(B_R)} \leq C,$$

*where* $C$ *is a constant depending only on* $n, p, R, f, H$ *and an upper bound for* $\|u\|_{W^{2,p}(B_{3R})}$.

**Remark 2.7.1.** When $H \equiv 0$ and $f = f(x)$, Theorem I was proved in [**LB05**] (see Theorem 1.1 and Proposition 4.3 therein).

**Remark 2.7.2.** Recall that in Theorem A we assumed $p > \frac{kn}{2}$. Clearly, $\frac{kn}{2} > k(k-1)$ if and only $k < \frac{n}{2} + 1$. Therefore, Case 1 of Theorem I is stronger than Theorem A when $k < \frac{n}{2} + 1$ and $f = f(x,z)$. Similarly, in Theorem B we assumed $p > \frac{(k+1)n}{2}$. It is easy to see that $\frac{(k+1)n}{2} > n + k(k-1)$ if and only if $k < \frac{n}{2}$. Therefore, Case 2 of Theorem I is stronger than Theorem B when $k < \frac{n}{2}$ and $f = f(x,z)$.

The proof of Case 2 in Theorem I requires only minor adjustments to the proof of Case 1, so for clarity of exposition we consider only Case 1 in the following section. We will explain the changes required for Case 2 in §2.7.3.

### 2.7.2 Proof of Case 1 of Theorem I

We begin with an outline of the proof of Case 1 of Theorem I, which will establish notation and highlight the main steps.

To keep notation concise, we denote $\Delta^h g = \sum_l \Delta^h_{ll} g$ for any function $g$. For $p$ as in the statement of Theorem I and $\beta = \beta(n,k,p) > 2$ a constant to be specified later, we will derive the following pointwise upper bound for $\Delta^h u$ on a ball of radius $R$ (assumed to be centred at the origin):

$$\sup_{B_R} \Delta^h u \leq CR^{-\beta/2}\big(1 + \|\Delta u\|^{p/n}_{L^p(B_{3R})}\big)^{\beta/2}. \tag{2.7.1}$$

Since the RHS of (2.7.1) is finite and independent of $h$, this yields an upper bound for $\sup_{B_R} \Delta u$, and consequently (see the discussion after Remark 2.2.3) a bound for $\sup_{B_R} |\nabla^2 u|$.

To this end, we define on $B_{2R}$ the function $v = \eta\Delta^h u$, where

$$\eta(x) = \left(1 - \frac{|x|^2}{4R^2}\right)^{\beta} \tag{2.7.2}$$

is a cutoff and $\beta > 2$ is a constant to be determined later. We continue to denote by $F^{ij}$ the linearised operator, as defined in (2.2.1). We will first derive an upper bound for $-F^{ij}D_{ij}v$ (which is formally of fourth order in the derivatives of $u$) in terms of $|F||D\Delta^h u|$ (formally of third order), $|F||\Delta^h u|$ (formally second order) and lower order terms. More precisely, we will show:

**Lemma 2.7.3.** *Let $\Lambda_F$ denote the largest eigenvalue of $F$. In Case 1 of Theorem I, we have*

$$-F^{ij}D_{ij}v \leq C\Lambda_F\Big(\eta|D(\Delta^h u)| + \eta|\Delta^h u| + \eta + |D\eta||D(\Delta^h u)| + |\Delta^h u||D^2\eta|\Big) \tag{2.7.3}$$

In fact, Lemma 2.7.3 is the main novel contribution in this section; once (2.7.3) is established, the proof of Case 1 of Theorem I proceeds similarly to [**LB05**], but with extra terms. We explain this process now. The key point is that on the upper contact set of $v$ in $B_{2R}$, defined by

$$\Gamma^+_v(B_{2R}) = \{x \in B_{2R} : v(z) \leq v(x) + \nu\cdot(z - x) \text{ for all } z \in B_{2R}, \text{ for some } \nu \in \mathbb{R}^n\}, \tag{2.7.4}$$

we can actually bound $|D\Delta^h u|$ from above in terms of $|\Delta^h u|$. Combined with (2.7.3) and an algebraic result of [**LB05**], this will give an upper bound for

$-F^{ij}D_{ij}v/(\det F^{ij})^{1/n}$ purely in terms of second order terms on $\Gamma_v^+(B_{2R})$. At this point, we will apply the classical Alexandrov-Bakelman-Pucci (ABP) estimate to get an upper bound for $v$ on $B_{2R}$ (and hence $\Delta^h u$ on $B_R$), from which the desired estimate will follow after a little more work. We recall the ABP estimate as follows:

**Theorem 2.7.4** (see e.g. [**GT01, Chapter 9**]). *Suppose $a^{ij}$ is measurable and positive definite a.e. on a smooth bounded domain $\Sigma \subset \mathbb{R}^n$. Then there exists a constant $C = C(n)$ such that for any $v \in W^{2,n}_{\text{loc}}(\Sigma) \cap C^0(\overline{\Sigma})$ with $v \equiv 0$ on $\partial\Sigma$, one has*

$$\sup_{\Sigma} v \leq Cd\bigg(\int_{\Gamma_v^+(\Sigma)} \frac{(-a^{ij}D_{ij}v)^n}{\det(a^{ij})}\,dx\bigg)^{1/n},$$

*where $d$ is the diameter of $\Sigma$.*

We now provide the details of the proof of Case 1 of Theorem I, starting with the proof of Lemma 2.7.3.

*Proof of Lemma 2.7.3.* Our starting point is again the estimate (2.4.1), which now reads:

$$kf^{k-1}\Delta^h f[u] \leq F^{ij}D_{ij}\Delta^h u - F^{ij}\Delta^h(H_1|\nabla u|^2)\delta_{ij}. \tag{2.7.5}$$

Applying the product rule to calculate $D_{ij}v$, we therefore have

$$\begin{aligned} F^{ij}D_{ij}v &= F^{ij}\Big(\eta D_{ij}\Delta^h u + 2D_i\eta D_j\Delta^h u + (\Delta^h u)D_{ij}\eta\Big) \\ &\overset{(2.7.5)}{\geq} \eta F^{ij}\Delta^h(H_1|\nabla u|^2)\delta_{ij} + k\eta f^{k-1}\Delta^h f + 2F^{ij}D_i\eta D_j\Delta^h u + \Delta^h u F^{ij}D_{ij}\eta. \end{aligned} \tag{2.7.6}$$

To obtain (2.7.3) from (2.7.6), it suffices to show

$$\Delta^h(H_1[u]|\nabla u|^2) \geq -C|D(\Delta^h u)| - C|\Delta^h u| - C \tag{2.7.7}$$

and $\Delta^h f \geq -C|\Delta^h u| - C$. We consider only (2.7.7), since the estimate for $\Delta^h f$ follows the same reasoning and is easier.

If $H \equiv 0$, then (2.7.7) holds trivially, so we assume $H > 0$. Recall that in §2.4.4.1 we derived the following discrete Bochner identity: for each $l \in \{1, \ldots, n\}$,

$$\begin{aligned}\Delta^h_{ll}\big(H_1[u]|\nabla u|^2\big) &= 2H_1\nabla^i u\nabla_i\Delta^h_{ll}u + (H_1[u])^{-h}_l\big|\nabla\nabla^{-h}_l u\big|^2 + (H_1[u])^h_l\big|\nabla\nabla^h_l u\big|^2 \\ &\quad + \nabla^{-h}_l\nabla_i u\nabla^i u\nabla^{-h}_l H_1[u] + \nabla^h_l\nabla^i u\nabla_i u\nabla^h_l H_1[u] \\ &\quad + \nabla^h_l\Big(\nabla_i u(\nabla^i u)^{-h}_l\nabla^{-h}_l H_1[u]\Big). \end{aligned} \tag{2.7.8}$$

Computing the bottom line of (2.7.8) further, by applying the product rule for difference quotients twice, we see

$$\begin{aligned}\nabla^h_l\Big(\nabla_i u(\nabla^i u)^{-h}_l\nabla^{-h}_l H_1[u]\Big) &= \nabla^i u\nabla^h_l H_1[u]\nabla^h_l\nabla_i u + \nabla_i u\nabla^h_l\Big((\nabla^i u)^{-h}_l\nabla^{-h}_l H_1[u]\Big) \\ &= \nabla^i u\nabla^h_l H_1[u]\nabla^h_l\nabla_i u + |\nabla u|^2\Delta^h_{ll}H_1[u] + \nabla_i u\nabla^{-h}_l H_1[u]\nabla^{-h}_l\nabla^i u,\end{aligned}$$

and it follows that

$$\begin{aligned}\Delta^h_{ll}(H_1[u]|\nabla u|^2) &\geq 2H_1\nabla^i u\nabla_i\Delta^h_{ll}u + (H_1[u])^{-h}_l\big|\nabla\nabla^{-h}_l u\big|^2 + (H_1[u])^h_l\big|\nabla\nabla^h_l u\big|^2 \\ &\quad - 2|\nabla\nabla^{-h}_l u||\nabla u||\nabla^{-h}_l H_1[u]| - 2|\nabla\nabla^h_l u||\nabla u||\nabla^h_l H_1[u]| \\ &\quad + |\nabla u|^2\Delta^h_{ll}H_1[u]. \end{aligned} \tag{2.7.9}$$

Now, since $H_1 = H_1(x,z) > 0$ and $u > 0$ is continuous, $H_1[u]$ is bounded uniformly away from zero in $B_{2R}$, and in particular $(H_1[u])^{\pm h}_l \geq C_0 > 0$ in $B_{2R}$. Since $\nabla u$ and $\nabla^{\pm h}_l H_1[u]$ are also bounded, there exists a constant $C_1$ such that

$$\begin{aligned}-2|\nabla\nabla^{-h}_l u||\nabla u||\nabla^{-h}_l H_1[u]| &- 2|\nabla\nabla^h_l u||\nabla u||\nabla^h_l H_1[u]| \\ &\geq -\frac{C_0}{2}|\nabla\nabla^{-h}_l u|^2 - \frac{C_0}{2}|\nabla\nabla^h_l u|^2 - C_1,\end{aligned}$$

and substituting this into (2.7.9) and summing over $l$, we see

$$\begin{aligned}\Delta^h(H_1[u]|\nabla u|^2) &\geq 2H_1\nabla^i u\nabla_i\Delta^h u + |\nabla u|^2\Delta^h H_1[u] - C \\ &\geq -C|D(\Delta^h u)| + |\nabla u|^2\Delta^h H_1[u] - C. \end{aligned} \tag{2.7.10}$$

Finally, the fact that $\Delta^h H_1[u] \geq -C|\Delta^h u| - C$ follows from exactly the same arguments as in §2.4.4.2, so we do not repeat the calculations here. □

**Remark 2.7.5.** We reiterate that, in comparison with the proof of [**LB05**], the main new difficulty when $H_1 \not\equiv 0$ is to obtain the upper bound for $-F^{ij}D_{ij}v$ as in Lemma 2.7.3. As we will see below, it is important that the third order expression in our estimate (2.7.3) of $-F^{ij}D_{ij}v$ is of the form $|D\Delta^h u|$, since we will only be able to control third order terms of this precise form on $\Gamma_v^+(B_{2R})$. It will also be important that the second order expression in our estimate of $-F^{ij}D_{ij}v$ is of the form $|\Delta^h u|$.

We now complete the proof of Theorem I, following the arguments of [**LB05**] whilst accounting for extra terms arising when $H_1 \not\equiv 0$.

*Proof of Case 1 of Theorem I.* We start by observing from the definition of $\eta$ that

$$|D\eta| \leq \frac{C}{R}\eta^{1-\frac{1}{\beta}} \quad \text{and} \quad |D^2\eta| \leq \frac{C}{R^2}\eta^{1-\frac{2}{\beta}}, \tag{2.7.11}$$

where $C = C(n,\beta)$. Using (2.7.11) in (2.7.3), we see that a.e. in $B_{2R}$ we have

$$-F^{ij}D_{ij}v \leq \frac{C\Lambda_F}{R^2\eta^{\frac{2}{\beta}}}\left(R^2\eta^{\frac{2}{\beta}}\Big(\eta|D(\Delta^h u)| + \eta|\Delta^h u| + \eta\Big) + R\eta^{1+\frac{1}{\beta}}|D(\Delta^h u)| + \eta|\Delta^h u|\right). \tag{2.7.12}$$

Now let $\Gamma_v^+(B_{2R})$ be the upper contact set of $v$ in $B_{2R}$, as defined in (2.7.4). As discussed in the outline of the proof, we wish to bound the RHS of (2.7.12) in terms of $v$ on $\Gamma_v^+(B_{2R})$. To this end, first observe that since $u \in C^{0,1}(\Omega)$, $Dv(x)$ exists for a.e. $x \in B_{2R}$. For such $x \in \Gamma_v^+(B_{2R})$, let $z \in \partial B_{2R}$ be such that

$$\frac{z-x}{|z-x|} = -\frac{Dv(x)}{|Dv(x)|}.$$

Since $v = 0$ on $\partial B_{2R}$ and $|z-x| \geq |z| - |x| = 2R - |x| \geq R\eta^{\frac{1}{\beta}}$ (the last inequality following from the definition of $\eta$), we thus have for a.e. $x \in \Gamma_v^+(B_{2R})$ that

$$v(x) \geq v(z) - Dv(x)\cdot(z-x) = -Dv(x)\cdot(z-x) = |z-x||Dv(x)| \geq R\eta^{\frac{1}{\beta}}|Dv(x)|. \tag{2.7.13}$$

In particular, $v = \eta \Delta^h u$ is non-negative a.e. in $\Gamma_v^+(B_{2R})$, and a.e. in $\Gamma_v^+(B_{2R})$ we have

$$\begin{aligned}\eta |D(\Delta^h u)| &= |Dv - (\Delta^h u) D\eta| \\ &\leq |Dv| + \Delta^h u |D\eta| \\ &\overset{(2.7.13)}{\leq} \frac{v}{R\eta^{\frac{1}{\beta}}} + \frac{v}{\eta}\frac{\beta}{R}\eta^{1-\frac{1}{\beta}} = \frac{(1+\beta)v}{R\eta^{\frac{1}{\beta}}}. \end{aligned} \tag{2.7.14}$$

Next, we substitute (2.7.14) back into (2.7.12). Using the fact that $D_{ij}v$ is negative semi-definite a.e. in $\Gamma_v^+(B_{2R})$, and that $F^{ij}$ is positive definite, we obtain a.e. in $\Gamma_v^+(B_{2R})$ the estimate

$$\begin{aligned} 0 \leq -F^{ij} D_{ij} v &\overset{(2.7.12)}{\leq} \frac{C\Lambda_F}{R^2\eta^{\frac{2}{\beta}}}\left(R^2\eta^{\frac{2}{\beta}}\Big(\eta|D(\Delta^h u)| + v + \eta\Big) + R\eta^{1+\frac{1}{\beta}}|D(\Delta^h u)| + v\right) \\ &\overset{(2.7.14)}{\leq} \frac{C\Lambda_F}{R^2\eta^{\frac{2}{\beta}}}\left((1+\beta)R\eta^{\frac{1}{\beta}}v + R^2\eta^{\frac{2}{\beta}}v + R^2\eta^{1+\frac{2}{\beta}} + (1+\beta)v + v\right) \\ &= C\Lambda_F\left(\frac{v}{R\eta^{\frac{1}{\beta}}} + v + \eta + \frac{v}{R^2\eta^{\frac{2}{\beta}}}\right). \end{aligned} \tag{2.7.15}$$

At this point, we apply an algebraic result of [**LB05, Prop. 4.2**], which states that a.e. in $B_{2R}$ we have the inequality[2]

$$\frac{\Lambda_F^n}{\det F^{ij}} \leq C\left(\frac{1}{\sigma_k(A_H)}\right)^{k-1}(\mathrm{tr}(A_H))^{k(k-1)} \leq C(\Delta u - \mathrm{tr}(H))^{k(k-1)} \leq C(\Delta u)^{k(k-1)}, \tag{2.7.16}$$

where $C$ depends on $\inf_{B_{2R}} f$. It follows that a.e. in $\Gamma_v^+(B_{2R})$,

$$\begin{aligned} 0 \leq \frac{-F^{ij}D_{ij}v}{(\det F^{ij})^{1/n}} &\overset{(2.7.15)}{\leq} C\left(\frac{v}{R\eta^{\frac{1}{\beta}}} + v + \eta + \frac{v}{R^2\eta^{\frac{2}{\beta}}}\right)\frac{\Lambda_F}{(\det F^{ij})^{1/n}} \\ &\overset{(2.7.16)}{\leq} C\left(\frac{v}{R\eta^{\frac{1}{\beta}}} + v + \eta + \frac{v}{R^2\eta^{\frac{2}{\beta}}}\right)(\Delta u)^{\frac{k(k-1)}{n}}. \end{aligned} \tag{2.7.17}$$

With (2.7.17) established, we are now in a position to apply the ABP estimate stated in Theorem 2.7.4. This yields

[2] Note that the $F^{ij}$ defined in [**LB05**] corresponds to our $G^{ij}$ defined in the proof of Lemma 2.4.1. However, the inequality (2.7.16) still holds for our definition of $F^{ij}$, since $G^{ij} = k^{-1}f^{1-k}F^{ij}$ and both the numerator and denominator on the LHS of (2.7.16) are homogeneous of degree $n$.

$$\begin{aligned}\sup_{B_{2R}} v &\le CR\Big(\int_{\Gamma_v^+(B_{2R})} \frac{(-F^{ij}D_{ij}v)^n}{\det(F^{ij})}\,dx\Big)^{1/n}\\ &\le C\Big(\int_{\Gamma_v^+(B_{2R})} (\eta^{-\frac{1}{\beta}}v)^n(\Delta u)^{k(k-1)}\Big)^{1/n} + CR\Big(\int_{\Gamma_v^+(B_{2R})} v^n(\Delta u)^{k(k-1)}\Big)^{1/n}\\ &\quad + CR\Big(\int_{\Gamma_v^+(B_{2R})} \eta^n(\Delta u)^{k(k-1)}\Big)^{1/n} + CR^{-1}\Big(\int_{\Gamma_v^+(B_{2R})} (\eta^{-\frac{2}{\beta}}v)^n(\Delta u)^{k(k-1)}\Big)^{1/n}.\end{aligned} \tag{2.7.18}$$

Now, the last integral on the RHS of (2.7.18) is where the largest negative exponent of $\eta$ appears alongside $v$. Writing $\eta^{-\frac{2}{\beta}}v = v^{1-\frac{2}{\beta}}(\Delta^h u)^{\frac{2}{\beta}}$, we see that the overall exponent of $\eta$ in this term is $1-\frac{2}{\beta}$, which is still positive since we assumed $\beta > 2$. With this in mind, we see

$$CR^{-1}\Big(\int_{\Gamma_v^+(B_{2R})} (\eta^{-\frac{2}{\beta}}v)^n(\Delta u)^{k(k-1)}\Big)^{1/n} \le CR^{-1}(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}}\Big(\int_{B_{2R}} |\Delta^h u|^{\frac{2n}{\beta}}(\Delta u)^{k(k-1)}\Big)^{1/n}, \tag{2.7.19}$$

where we have assumed $\sup_{B_{2R}} v \ge 0$ (otherwise we are done). We estimate the remaining three terms on the RHS of (2.7.18) so as to put them on equal footing with (2.7.19). For the first term, note $\eta^{-\frac{1}{\beta}}v = v^{1-\frac{2}{\beta}}(\sqrt{\eta}|\Delta^h u|)^{\frac{2}{\beta}} \le v^{1-\frac{2}{\beta}}|\Delta^h u|^{\frac{2}{\beta}}$, so

$$C\Big(\int_{\Gamma_v^+(B_{2R})} (\eta^{-\frac{1}{\beta}}v)^n(\Delta u)^{k(k-1)}\Big)^{1/n} \le C(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}}\Big(\int_{B_{2R}} |\Delta^h u|^{\frac{2n}{\beta}}(\Delta u)^{k(k-1)}\Big)^{1/n}.$$

Similarly, for the second term, $v = v^{1-\frac{2}{\beta}}(\eta|\Delta^h u|)^{\frac{2}{\beta}} \le v^{1-\frac{2}{\beta}}|\Delta^h u|^{\frac{2}{\beta}}$, so

$$CR\Big(\int_{\Gamma_v^+(B_{2R})} v^n(\Delta u)^{k(k-1)}\Big)^{1/n} \le CR(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}}\Big(\int_{B_{2R}} |\Delta^h u|^{\frac{2n}{\beta}}(\Delta u)^{k(k-1)}\Big)^{1/n}.$$

Finally, observe

$$CR\Big(\int_{\Gamma_v^+(B_{2R})} \eta^n(\Delta u)^{k(k-1)}\Big)^{1/n} \le CR\Big(\int_{B_{2R}} (\Delta u)^{k(k-1)}\Big)^{1/n} \le CR.$$

Also note that since $R$ is bounded we may replace all coefficients in the three estimates above with $CR^{-1}$, and it follows that

$$\sup_{B_{2R}} v \le CR^{-1}(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}}\Big(\int_{B_{2R}} |\Delta^h u|^{\frac{2n}{\beta}}(\Delta u)^{k(k-1)}\Big)^{1/n} + CR^{-1}.$$

We now choose $\beta = \frac{2n}{p-k(k-1)}$, so that $\frac{2n}{\beta} = p - k(k-1)$. After applying Hölder's inequality and (2.2.6), we obtain

$$(\sup_{B_{2R}} v)^{2/\beta} \le CR^{-1}\|\Delta u\|^{p/n}_{L^p(B_{3R})} + \frac{CR^{-1}}{(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}}}.$$

If $\sup_{B_{2R}} v \le 1$, again we are done. So suppose otherwise, then the above implies

$$(\sup_{B_{2R}} v)^{2/\beta} \le CR^{-1}\big(1 + \|\Delta u\|^{p/n}_{L^p(B_{3R})}\big).$$

We therefore arrive at the estimate

$$\sup_{B_R} \Delta^h u \le CR^{-\beta/2}\big(1 + \|\Delta u\|^{p/n}_{L^p(B_{3R})}\big)^{\beta/2}, \tag{2.7.20}$$

from which the result follows as explained in the outline of the proof. □

### 2.7.3 Proof of Case 2 of Theorem I

*Proof of Case 2 of Theorem I.* We explain how to adjust the proof of Case 1. We first note that, without any sign assumption on $H_1$, (2.7.7) is replaced with the estimate

$$\Delta^h(H_1[u]|\nabla u|^2) \ge -C|D(\Delta^h u)| - C|\Delta^h u|^2 - C, \tag{2.7.21}$$

with the additional power of $|\Delta^h u|$ coming from the fact that the second and third terms on the RHS of (2.7.9) are no longer necessarily positive. Keeping track of this $|\Delta^h u|^2$ term in the subsequent estimates, then rather than (2.7.15) we obtain

$$0 \le -F^{ij}D_{ij}v \le C\Lambda_F\bigg(\frac{v}{R\eta^{\frac{1}{\beta}}} + v\Delta^h u + \eta + \frac{v}{R^2\eta^{\frac{2}{\beta}}}\bigg). \tag{2.7.22}$$

a.e. in $\Gamma_v^+(B_{2R})$. The second integral on the second line of (2.7.18) is therefore replaced with

$$CR\bigg(\int_{\Gamma_v^+(B_{2R})} (v\Delta^h u)^n(\Delta u - \operatorname{tr}(H))^{k(k-1)}\bigg)^{1/n}, \tag{2.7.23}$$

and writing $v\Delta^h u = v^{1-\frac{2}{\beta}}\eta^{\frac{2}{\beta}}(\Delta^h u)^{1+\frac{2}{\beta}} \le v^{1-\frac{2}{\beta}}|\Delta^h u|^{1+\frac{2}{\beta}}$, we see that (2.7.23) is bounded from above by

$$CR(\sup_{B_{2R}} v)^{1-\frac{2}{\beta}}\bigg(\int_{B_{2R}} (\Delta^h u)^{n+\frac{2n}{\beta}}(\Delta u - \operatorname{tr}(H))^{k(k-1)}\bigg).$$

Choosing $\beta = \frac{2n}{p-k(k-1)}$ and proceeding as before yields the result. □

# Appendix

## 2.A A convergence property of second order difference quotients

In this appendix we prove Lemma 2.2.1, which we recall here:

**Lemma 2.A.1.** *Suppose $u \in W^{2,s}(\Omega)$ for some $s \geq 1$. Then $v_h \to \Delta u$ in $L^s_{\mathrm{loc}}(\Omega)$ as $h \to 0$.*

*Proof.* The proof is a standard argument using Taylor's theorem. We claim that

$$\begin{aligned}\|v_h - \Delta u\|_{L^s(\Omega')} \leq \sum_{l=1}^{n} \int_0^1 \Big\| \nabla_l \nabla_l u(x + the_l) - \nabla_l \nabla_l u(x) \Big\|_{L^s(\Omega')} dt \\ + \sum_{l=1}^{n} \int_0^1 \Big\| \nabla_l \nabla_l u(x - the_l) - \nabla_l \nabla_l u(x) \Big\|_{L^s(\Omega')} dt \qquad (2.A.1)\end{aligned}$$

for all $u \in W^{2,s}(\Omega)$ and $\Omega' \Subset \Omega$ satisfying $|h| < \mathrm{dist}(\Omega', \partial\Omega)$, from which the conclusion follows by the continuity of the translation operator in $L^s(\Omega)$. By density it suffices to prove (2.A.1) for $u \in C^2(\Omega)$. Let $\Omega'$ be as above. Then for each $x \in \Omega'$ and $l \in \{1, \ldots, n\}$, we have by Taylor's theorem

$$u(x \pm he_l) = u(x) \pm h\nabla_l u(x) + h^2 \int_0^1 (1-t) \nabla_l \nabla_l u(x \pm the_l)\, dt,$$

and thus

$$\begin{aligned}v_h(x) - \Delta u(x) = \sum_{l=1}^{n} \int_0^1 (1-t) \Big( \nabla_l \nabla_l u(x + the_l) - \nabla_l \nabla_l u(x) \Big) dt \\ + \sum_{l=1}^{n} \int_0^1 (1-t) \Big( \nabla_l \nabla_l u(x - the_l) - \nabla_l \nabla_l u(x) \Big) dt. \qquad (2.A.2)\end{aligned}$$

Let $s'$ be such that $\frac{1}{s} + \frac{1}{s'} = 1$. It follows from (2.A.2) and Hölder's inequality that for all $g \in L^{s'}(\Omega')$ satisfying $\|g\|_{L^{s'}(\Omega')} \le 1$, we have

$$\begin{aligned}
&\int_{\Omega'} \big(v_h(x) - \Delta u(x)\big) g(x)\, dx \\
&\qquad = \sum_{l=1}^{n} \int_0^1 (1-t) \int_{\Omega'} \Big(\nabla_l \nabla_l u(x + the_l) - \nabla_l \nabla_l u(x)\Big) g(x)\, dx\, dt \\
&\qquad\quad + \sum_{l=1}^{n} \int_0^1 (1-t) \int_{\Omega'} \Big(\nabla_l \nabla_l u(x - the_l) - \nabla_l \nabla_l u(x)\Big) g(x)\, dx\, dt \\
&\qquad \le \sum_{l=1}^{n} \int_0^1 \Big\| \nabla_l \nabla_l u(x + the_l) - \nabla_l \nabla_l u(x) \Big\|_{L^s(\Omega')} dt \\
&\qquad\quad + \sum_{l=1}^{n} \int_0^1 \Big\| \nabla_l \nabla_l u(x - the_l) - \nabla_l \nabla_l u(x) \Big\|_{L^s(\Omega')} dt.
\end{aligned} \tag{2.A.3}$$

Taking the supremum over such $g$ in (2.A.3), we obtain (2.A.1). □

# Chapter 3

# Local pointwise second derivative estimates for smooth solutions to the $\sigma_2$-Yamabe equation on manifolds

In this chapter we consider the problem of obtaining local pointwise second derivative estimates for smooth solutions to the $\sigma_2$-Yamabe equation on Riemannian manifolds, addressing both the positive and negative cases. Although we could consider more general augmented Hessian equations here (as in Chapter 2), to keep the exposition concise we restrict our attention to the $\sigma_k$-Yamabe equation. On the other hand, the restriction to the case $k = 2$ is due to the resulting divergence structure.

In light of the discussion in the introduction, it is of particular interest to prove such estimates for solutions $g = u^{-2}g_0$ $(u > 0)$ to the $\sigma_2$-Yamabe equation in the negative case

$$\sigma_2^{1/2}(-g^{-1}A_g(x)) = f(x, u(x), \nabla u(x)) > 0, \quad -g^{-1}A_g(x) \in \Gamma_2^+ \quad \text{on } B_{2R} \subset \mathcal{M}^n, \tag{3.0.1}$$

as stated in Theorem C. Indeed, *a priori* second derivative estimates (independent of any $L^p$ norm of $\nabla^2 u$) are already known in the positive case

$$\sigma_2^{1/2}(g^{-1}A_g(x)) = f(x, u(x), \nabla u(x)) > 0, \quad g^{-1}A_g(x) \in \Gamma_2^+ \quad \text{on } B_{2R} \subset \mathcal{M}^n \tag{3.0.2}$$

due to [**Via02, GW03b, LL03, Che05, Wan06, JLL07, Li09**] etc. However, for the sake of completeness, we prove a more general version of Theorem C which takes into account both the positive and negative cases. This will not overly complicate the argument, since most of our estimates will deal with both cases simultaneously.

The plan of the chapter is as follows. We begin in §3.1 by setting up our more general result, Theorem J, which supersedes Theorem C. We will indicate why this new result is not simply a consequence of our previous work in Chapter 2, and we will also give an outline of the proof. In §3.3–3.6, we carry out the proof of Theorem J.

## 3.1 Introduction and statement of new result: Theorem J

We explained in §2.1.2 how the positive and negative cases of the $\sigma_k$-Yamabe equation on Euclidean domains could be addressed simultaneously, by instead considering a class of augmented Hessian equations without any sign assumptions on the solution. By exactly the same reasoning, the equations (3.0.1) and (3.0.2) may be addressed simultaneously. Indeed, recalling the transformation law

$$A_g = \nabla_0^2 v + dv \otimes dv - \frac{|\nabla_0 v|_0^2}{2} g_0 + A_0 \tag{3.1.1}$$

for $g = e^{-2v} g_0$ and making the substitution $e^{-v} = -u^{-1}$ (resp. $e^{-v} = u^{-1}$), we see that to obtain our desired second derivative estimates for $C^4$ solutions $g = u^{-2} g_0$ to (3.0.1) (resp. (3.0.2)), it is equivalent to obtain second derivative estimates for negative (resp. positive) $C^4$ solutions to

$$\sigma_2^{1/2}(A_u(x)) = f(x, u(x), \nabla u(x)) > 0, \quad A_u(x) \in \Gamma_2^+ \quad \text{on } B_{2R} \subset \mathcal{M}^n, \tag{3.1.2}$$

where $A_u$ is the $(1,1)$-tensor defined by

$$A_u = g_0^{-1}\bigg(\nabla_0^2 u - \frac{|\nabla_0 u|_0^2}{2u} g_0 + u A_0\bigg).$$

We note that, unless $(\mathcal{M}^n, g_0)$ is locally conformally flat, (3.1.2) does *not* fall into the framework of Theorem G. Although one could locally write $\nabla_0^2 u$ as the flat Hessian plus lower order terms (which in turn could be incorporated into the matrix $H$ in the statement of Theorem G), there is still a factor of $g_0^{-1}$ to contend with. Since our work in Chapter 2 does not address augmented Hessians of the form $A(x)\nabla^2 u(x) - H[u](x)$ when $A$ is not the identity matrix, it is not immediately clear whether our methods will generalise to give estimates for (3.1.2). That said, our approach to the following result will be to follow the proof of Theorems F and G from Chapter 2, and make the necessary adjustments as we go along:

**Theorem J.** *Let $(\mathcal{M}^n, g_0)$ be a manifold of dimension $n \geq 3$, suppose $f \in C^2_{\mathrm{loc}}(\mathcal{M}^n \times \mathbb{R} \times T\mathcal{M}^n)$ is positive and suppose $p > \frac{3n}{2}$ (resp. $p > n$). Let $B_{2R}$ be a geodesic ball in $(\mathcal{M}^n, g_0)$ of radius $2R < i_0$, where $i_0$ is the injectivity radius of $(\mathcal{M}^n, g_0)$ at the centre of the ball. If $u \in C^4(B_{2R})$ is a negative (resp. positive) solution to* (3.1.2)*, then there exists a constant $C$ depending only on $n, R, g_0, \|\ln|u|\|_{W^{2,p}(B_{2R})}$ and $f$ such that*

$$\sup_{B_R} |\nabla_0^2 u|_0 \leq C.$$

**Notation:** For the remainder of this chapter, all computations will be done with respect to the background metric $g_0$, and all integrals will be with respect to $dv_0$. So as to make the computations involving index notation more readable, we will drop all sub/superscript 0s until otherwise specified.

## 3.2 An outline of the proof of Theorem J

The structure of the proof of Theorem J is as follows. We begin in §3.3 by deriving a pointwise estimate on solutions to (3.1.2), appealing to the concavity of $\sigma_2^{1/2}$ in $\Gamma_2^+$; this essentially amounts to differentiating the equation (3.1.2) twice and commuting derivatives, and is analogous to Lemma 2.4.1 in Chapter 2.

In §3.4, we multiply our pointwise estimate by a suitable test function, integrate over a suitable domain, and estimate the integrals involving lower order terms – it is in this step that we first use the divergence structure of the linearised operator $F_i^{\ j}$.

For a cutoff function $\eta$ and a constant $\alpha$ to be specified below, this will give us for all $q > 1$ the following counterpart to Lemma 2.4.2:

$$
\begin{aligned}
(q-1)\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-2} F_i^{\ j}\nabla^i\Delta u\nabla_j\Delta u - \frac{1}{q}\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q}\nabla_j\nabla^i F_i^{\ j} \\
+ \int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)\Delta\Big(\frac{|\nabla u|^2}{2u}\Big) \leq C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u+\alpha)^{q+1}. \qquad (3.2.1)
\end{aligned}
$$

**Remark 3.2.1.** The second integral on the LHS of (3.2.1) is somewhat analogous to the term $(\mathrm{I}_2)_h$ in Chapter 2. Note, however, that in proving (3.2.1), our smoothness assumption will allow for an extra integration by parts compared to that seen in the proof of Lemma 2.4.2. This is the reason for the term $(\Delta u+\alpha)^q\nabla_j\nabla^i F_i^{\ j}$ in (3.2.1), rather than $(\Delta u+\alpha)^{q-1}\nabla_i F^{ij}\nabla_j\Delta u$, which may have been expected given the definition of $(\mathrm{I}_2)_h$. On the other hand, the third integral on the LHS of (3.2.1) is the direct counterpart to $(\mathrm{I}_3)_h$ in the smooth setting.

Up to this point, we do not need to distinguish between the two cases in Theorem J. Our sign assumptions on $u$ will now come into play when estimating the second and third integrals on the LHS of (3.2.1). After addressing these estimates in §3.5, we will obtain

$$
\int_{B_{R+\rho}} \big|\nabla(\Delta u+\alpha)^{(q-1)/2}\big|^2 \leq \frac{C(q-1)}{\rho^2}\int_{B_{R+2\rho}} (\Delta u+\alpha)^{q+2}
$$

in the negative case and

$$
\int_{B_{R+\rho}} \big|\nabla(\Delta u+\alpha)^{(q-1)/2}\big|^2 \leq \frac{C(q-1)}{\rho^2}\int_{B_{R+2\rho}} (\Delta u+\alpha)^{q+1},
$$

in the positive case. Applying the Sobolev inequality and imposing the respective integrability assumptions in each case, we will obtain a reverse Hölder inequality which gives us an improvement in integrability in each case. At this point, we apply the Moser iteration procedure as in Chapter 2 to complete the proof.

Throughout our proof of Theorem J, we will often use the following basic estimates without explicit reference. First, we recall (see Chapter 2 for an explanation) that the assumption $A_u \in \Gamma_2^+$ implies the existence of a constant $\alpha > 0$ (depending on an upper bound $\|\ln|u|\|_{C^1(B_{2R})}$) for which

$$0 \le |\nabla^2 u| < \Delta u + \alpha \tag{3.2.2}$$

and

$$0 < \operatorname{tr} A_u \le \Delta u + \alpha \tag{3.2.3}$$

in $B_{2R}$. In addition, we also assume that $\alpha$ is chosen such that $\Delta u + \alpha \ge 1$. As in Chapter 2, we denote by $F = \partial\sigma_2(A_u)/\partial A_u$ the linearised operator, given by the first Newton tensor of $A_u$:

$$F_i^{\ j} = \frac{\partial \sigma_2(A_u)}{\partial (A_u)^i_{\ j}} = \sigma_1(A_u)\delta_i^j - (A_u)_i^{\ j} = (\operatorname{tr} A_u)\delta_i^j - (A_u)_i^{\ j}. \tag{3.2.4}$$

It is easy to see from (3.2.2) and (3.2.4) that $|F| \le C(\Delta u + \alpha)$ in $B_{2R}$; here and henceforth, $C$ is a constant which we allow to depend on $n, R, g_0, f$ and an upper bound for $\|\ln|u|\|_{C^1(B_{2R})}$, but we do not allow $C$ to depend on any second derivatives of $u$ until otherwise mentioned. The value of $C$ may change from line to line.

## 3.3 Initial pointwise estimate

Taking $R$ smaller if necessary, we may assume that $B_{2R}$ is contained in a coordinate chart with coordinate functions $x^1, \dots, x^n$. We will utilise summation convention whenever an index appears in both a lower and upper position.

Our first step is to prove a pointwise estimate analogous to Lemma 2.4.1:

**Proposition 3.3.1.** *Suppose $f \in C^2_{\mathrm{loc}}(\mathcal{M}^n \times \mathbb{R} \times T\mathcal{M}^n)$ is positive and let $u \in C^4(B_{2R})$ be either a positive or negative solution to* (3.1.2). *Then*

$$2f\Delta f \le F_i^{\ j}\nabla^i\nabla_j \Delta u - \operatorname{tr}(F)\Delta\left(\frac{|\nabla u|^2}{2u}\right) + C(\Delta u + \alpha)^2 \quad \text{on } B_{2R}. \tag{3.3.1}$$

Proposition 3.3.1 will follow from the next lemma (which essentially amounts to differentiating the equation (3.1.2) twice) and standard formulae for commuting covariant derivatives:

**Lemma 3.3.2.** *Suppose $f \in C^2_{\mathrm{loc}}(\mathcal{M}^n \times \mathbb{R} \times T\mathcal{M}^n)$ is positive and let $u \in C^4(B_{2R})$ be a solution to* (3.1.2). *Then*

$$2f\Delta f \leq F_i^{\ j}\Delta(A_u)^i_{\ j} \quad \text{on } B_{2R}. \tag{3.3.2}$$

*Proof.* We start by observing

$$d\sigma_2^{1/2}(A_u) = \frac{\partial \sigma_2^{1/2}(A_u)}{\partial x^k}dx^k = \frac{\partial \sigma_2^{1/2}(A_u)}{\partial (A_u)^i_{\ j}}\frac{\partial (A_u)^i_{\ j}}{\partial x^k}dx^k.$$

Now, if $\Gamma^m_{lk}$ denote the Christoffel symbols of the Levi-Civita connection of $g_0$, then the covariant derivative of a 1-form $\theta = \theta_k dx^k$ is given by

$$(\nabla\theta)_{lk} = \frac{\partial \theta_k}{\partial x^l} - \Gamma^m_{lk}\theta_m. \tag{3.3.3}$$

It follows that the Hessian of $\sigma_2(A_u)^{1/2}$ is given by

$$\begin{aligned}\nabla^2_{lk}\big(\sigma_2^{1/2}(A_u)\big) &= \frac{\partial}{\partial x^l}\left(\frac{\partial \sigma_2^{1/2}(A_u)}{\partial (A_u)^i_{\ j}}\frac{\partial (A_u)^i_{\ j}}{\partial x^k}\right) - \Gamma^m_{lk}\left(\frac{\partial \sigma_2^{1/2}(A_u)}{\partial (A_u)^i_{\ j}}\frac{\partial (A_u)^i_{\ j}}{\partial x^m}\right)\\ &= \frac{\partial^2 \sigma_2^{1/2}(A_u)}{\partial (A_u)^i_{\ j}\partial (A_u)^p_{\ q}}\frac{\partial (A_u)^i_{\ j}}{\partial x^k}\frac{\partial (A_u)^p_{\ q}}{\partial x^l} + \frac{\partial \sigma_2^{1/2}(A_u)}{\partial (A_u)^i_{\ j}}\left(\frac{\partial^2 (A_u)^i_{\ j}}{\partial x^k \partial x^l} - \Gamma^m_{lk}\frac{\partial (A_u)^i_{\ j}}{\partial x^m}\right).\end{aligned} \tag{3.3.4}$$

By concavity of $\sigma_2^{1/2}$ in $\Gamma_2^+$, the first term after the second equality in (3.3.4) is non-positive. Also observe that the quantity in the final parentheses in (3.3.4) is just $\nabla^2_{lk}(A_u)^i_{\ j}$, and so after contracting the $l, k$ indices and substituting in the equation (3.1.2), we obtain

$$\Delta f \leq \frac{\partial \sigma_2^{1/2}(A_u)}{\partial (A_u)^i_{\ j}}\Delta(A_u)^i_{\ j}. \tag{3.3.5}$$

The estimate (3.3.2) then follows from (3.3.5), since $F_i^{\ j} = \partial\sigma_2(A_u)/\partial(A_u)^i_{\ j}$. □

Expanding out the definition of $A_u$, (3.3.2) reads

$$2f\Delta f \leq F_i^{\ j}\nabla^k\nabla_k\left(\nabla^i\nabla_j u - \frac{|\nabla u|^2}{2u}\delta^i_{\ j} + uA^i_{\ j}\right). \tag{3.3.6}$$

Proposition 3.3.1 will follow from (3.3.6) after commuting derivatives in the term $F_i^{\ j}\nabla^k\nabla_k\nabla^i\nabla_j u$. The reason for doing this is the same as in the proof of Theorems F and G: a term of the form ${F_i}^j\nabla^i\nabla_j\Delta u$ will allow us to integrate by parts and appeal to the divergence structure of $F$.

In the following proof, we demonstrate explicitly the process of commuting derivatives in order to illustrate the error terms that arise in such calculations. However, the precise curvature terms that arise are not important for our purposes, and we will not be so explicit in future proofs.

*Proof of Proposition 3.3.1.* Commuting derivatives, we have

$$\begin{aligned}\nabla^k\nabla_k\nabla^i\nabla_j u &= \nabla^k\nabla^i\nabla_k\nabla_j u - \nabla^k\big({R_k}^i{}_{jp}\nabla^p u\big)\\ &= \nabla^k\nabla^i\nabla_j\nabla_k u - \nabla^k\big({R_k}^i{}_{jp}\nabla^p u\big)\\ &= \nabla^i\nabla^k\nabla_j\nabla_k u - {R^{ik}_{\ \ j}}^p\nabla_p\nabla_k u + R^{ip}\nabla_j\nabla_p u - \nabla^k\big({R_k}^i{}_{jp}\nabla^p u\big)\\ &= \nabla^i\nabla_j\nabla^k\nabla_k u - \nabla^i(R_{jp}\nabla^p u) - {R^{ik}_{\ \ j}}^p\nabla_p\nabla_k u + R^{ip}\nabla_j\nabla_p u\\ &\quad - \nabla^k\big({R_k}^i{}_{jp}\nabla^p u\big), \end{aligned} \tag{3.3.7}$$

the second equality due to the symmetry of the Hessian. Since all the curvature terms in (3.3.7) are with respect to the background metric $g_0$, they are all bounded by constants depending on $g_0$, and so after contracting both sides of (3.3.7) with ${F_i}^j$ we obtain

$${F_i}^j\nabla^k\nabla_k\nabla^i\nabla_j u \le {F_i}^j\nabla^i\nabla_j\Delta u + C(\Delta u+\alpha)^2. \tag{3.3.8}$$

Similarly,

$${F_i}^j\Delta(u{A^i}_j) \le C(\Delta u+\alpha)^2, \tag{3.3.9}$$

and after substituting (3.3.8) and (3.3.9) into (3.3.6), we arrive at (3.3.1). □

## 3.4 Integral estimates for lower order terms

We now proceed as in the proof of Theorems F and G by multiplying (3.3.1) by a suitable test function and integrating by parts. To describe this test function, for $\rho\in(0,\frac{R}{2}]$ we let $\eta\in C_0^\infty(B_{R+2\rho})$ be a non-negative cutoff function satisfying $0\le\eta\le 1$,

$\eta = 1$ on $B_{R+\rho}$ and $|\nabla^l \eta| \leq C(n)\rho^{-l}$ for $l = 1, 2$. Multiplying both sides of (3.3.1) by $\eta(\Delta u + \alpha)^{q-1}$ (where $q > 1$ is to be determined) and integrating over $B_{R+2\rho}$, we obtain

$$\begin{aligned}
2\int_{B_{R+2\rho}} &\eta(\Delta u+\alpha)^{q-1} f\Delta f \\
&\leq \int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1} F_i^{\ j}\nabla^i\nabla_j \Delta u - \int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)\Delta\left(\frac{|\nabla u|^2}{2u}\right) \\
&\quad + C\int_{B_{R+2\rho}} (\Delta u+\alpha)^{q+1}. \qquad (3.4.1)
\end{aligned}$$

Following the outline of the proof given in §3.1, we now prove the integral estimate (3.2.1), which is a direct counterpart to Lemma 2.4.2 in Chapter 2:

**Lemma 3.4.1.** *Suppose $q > 1$, $f \in C^2_{\mathrm{loc}}(\mathcal{M}^n \times \mathbb{R} \times T\mathcal{M}^n)$ is positive and let $u \in C^4(B_{2R})$ be either a positive or negative solution to* (3.1.2)*. Then*

$$\begin{aligned}
(q-1)\int_{B_{R+2\rho}} &\eta(\Delta u+\alpha)^{q-2} F_i^{\ j}\nabla^i\Delta u\nabla_j\Delta u - \frac{1}{q}\int_{B_{R+2\rho}} (\Delta u+\alpha)^q \eta \nabla_j\nabla^i F_i^{\ j} \\
&+ \int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)\Delta\left(\frac{|\nabla u|^2}{2u}\right) \leq C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u+\alpha)^{q+1}. \qquad (3.4.2)
\end{aligned}$$

*Proof.* Integrating by parts in the first integral on the RHS of (3.4.1), we see

$$\begin{aligned}
\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1} F_i^{\ j}\nabla^i\nabla_j\Delta u = &-\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1}\nabla^i F_i^{\ j}\nabla_j \Delta u \\
&- \int_{B_{R+2\rho}} (\Delta u+\alpha)^{q-1} F_i^{\ j}\nabla^i\eta\nabla_j\Delta u \\
&- (q-1)\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-2} F_i^{\ j}\nabla^i\Delta u\nabla_j\Delta u. \qquad (3.4.3)
\end{aligned}$$

Integrating by parts again, we see that the first integral on the RHS of (3.4.3) is

$$\begin{aligned}
-\int_{B_{R+2\rho}} &\eta(\Delta u+\alpha)^{q-1}\nabla^i F_i^{\ j}\nabla_j\Delta u \\
&= -\frac{1}{q}\int_{B_{R+2\rho}} \eta\nabla^i F_i^{\ j}\nabla_j(\Delta u+\alpha)^q \\
&= \frac{1}{q}\int_{B_{R+2\rho}} (\Delta u+\alpha)^q\nabla^i F_i^{\ j}\nabla_j\eta + \frac{1}{q}\int_{B_{R+2\rho}} (\Delta u+\alpha)^q\eta\nabla_j\nabla^i F_i^{\ j}, \qquad (3.4.4)
\end{aligned}$$

and similarly the second integral on the RHS of (3.4.3) satisfies

$$
\begin{aligned}
-\int_{B_{R+2\rho}} & (\Delta u+\alpha)^{q-1} F_i{}^j \nabla^i \eta \nabla_j \Delta u \\
&= -\frac{1}{q}\int_{B_{R+2\rho}} F_i{}^j \nabla^i \eta \nabla_j (\Delta u+\alpha)^q \\
&= \frac{1}{q}\int_{B_{R+2\rho}} (\Delta u+\alpha)^q \nabla^i \eta \nabla_j F_i{}^j + \frac{1}{q}\int_{B_{R+2\rho}} (\Delta u+\alpha)^q F_i{}^j \nabla_j \nabla^i \eta \\
&\geq \frac{1}{q}\int_{B_{R+2\rho}} (\Delta u+\alpha)^q \nabla^i \eta \nabla_j F_i{}^j - C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u+\alpha)^{q+1}. \qquad (3.4.5)
\end{aligned}
$$

Substituting (3.4.4) and (3.4.5) into (3.4.3), then (3.4.3) back into (3.4.1) and rearranging, we obtain

$$
\begin{aligned}
&(q-1)\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-2} F_i{}^j \nabla^i \Delta u \nabla_j \Delta u - \frac{1}{q}\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^q \nabla_j \nabla^i F_i{}^j \\
&+ \int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1} \operatorname{tr}(F) \Delta\bigg(\frac{|\nabla u|^2}{2u}\bigg) \leq \frac{2}{q}\int_{B_{R+2\rho}} (\Delta u+\alpha)^q \nabla^i F_i{}^j \nabla_j \eta \\
&- 2\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1} f\Delta f + C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u+\alpha)^{q+1}. \qquad (3.4.6)
\end{aligned}
$$

To obtain (3.4.2), it remains to estimate the first two integrals on the RHS of (3.4.6) from above by $C\rho^{-2}\int_{B_{R+2\rho}}(\Delta u+\alpha)^{q+1}$. For the first of these, we need to calculate the divergence of $F$:

$$
\begin{aligned}
\nabla^i F_i{}^j &= \nabla^i\big(\operatorname{tr}(A_u)\delta_i^{\ j} - (A_u)_i{}^j\big) \\
&= \nabla^j\bigg(\Delta u - \frac{n|\nabla u|^2}{2u} + u\operatorname{tr}(A)\bigg) - \nabla^i\bigg(\nabla_i\nabla^j u - \frac{|\nabla u|^2}{2u}\delta_i^{\ j} + uA_i{}^j\bigg) \\
&= \nabla^j \Delta u - \nabla^i\nabla_i\nabla^j u - (n-1)\nabla^j\bigg(\frac{|\nabla u|^2}{2u}\bigg) + \nabla^j(u\operatorname{tr}(A)) - \nabla^i(uA_i{}^j) \\
&\leq C(\Delta u+\alpha), \qquad (3.4.7)
\end{aligned}
$$

where to reach the last line, we have commuted derivatives to assert $|\nabla^j\Delta u - \nabla^i\nabla_i\nabla^j u| \leq C|\nabla u| \leq C$. This gives the desired estimate for the first integral on the RHS of (3.4.6).

For the second integral on the RHS of (3.4.6), we observe by the chain rule that

$$
\Delta f = \frac{\partial f}{\partial u_l}\nabla_l \Delta u + \frac{\partial f}{\partial u}\Delta u + \text{lower order terms}, \qquad (3.4.8)
$$

where the lower order terms are bounded by a constant $C$ (depending, as usual, on $f$ and $\|\ln|u|\|_{C^1}$). Therefore,

$$\begin{aligned}
-2\int_{B_{R+2\rho}} &\eta(\Delta u+\alpha)^{q-1} f\Delta f \\
&\leq -2\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1} f\frac{\partial f}{\partial u_l}\nabla_l \Delta u + C\int_{B_{R+2\rho}} (\Delta u+\alpha)^q \\
&= -\frac{2}{q}\int_{B_{R+2\rho}} \eta f\frac{\partial f}{\partial u_l}\nabla_l (\Delta u+\alpha)^q + C\int_{B_{R+2\rho}} (\Delta u+\alpha)^q \\
&\leq C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u+\alpha)^q,
\end{aligned}$$

with the final line again following from integration by parts. □

## 3.5 Integral estimates for higher order terms

We now turn to the integrals on the LHS of (3.4.2), all of which currently contain third derivatives of $u$. The first of these integrals is a positive term that will ultimately give us improved integrability, whereas the remaining two integrals are of unknown sign. We note that, unlike in Chapter 2 where we had to *combine* our estimates for $(\mathrm{I}_2)_h$ and $(\mathrm{I}_3)_h$ in order to obtain a cancellation of higher order terms, our regularity assumptions here will allow us to obtain cancellations for each of the two remaining integrals *individually.*

We start with a pointwise estimate for the integrand in the first integral of (3.4.2) – the proof is almost identical to the proof of Lemma 2.4.3, so we do not repeat it here:

**Lemma 3.5.1.** *Suppose* $q>1$, $f\in C^0_{\mathrm{loc}}(\mathcal{M}^n\times\mathbb{R}\times T\mathcal{M}^n)$ *is positive and let* $u\in C^4(B_{2R})$ *be a solution* (3.1.2). *Then*

$$(\Delta u+\alpha)^{q-2}F_i^{\ j}\nabla^i\Delta u\nabla_j\Delta u \geq \frac{Cf^2}{(q-1)^2}\left|\nabla(\Delta u+\alpha)^{(q-1)/2}\right|^2. \tag{3.5.1}$$

We now turn to the remaining two integrals on the LHS of (3.4.2). Let us start with the second of these, which is analogous to the term denoted by '$(\mathrm{I}_3)_h$' in our proof of Theorems F and G. Just as we derived a discrete Bochner formula to estimate $(\mathrm{I}_3)_h$

(see Section 2.4.4.1), we start by deriving a pointwise estimate for $\Delta(|\nabla u|^2/2u)$, this time appealing to the classical Bochner formula:

**Lemma 3.5.2.** *Suppose $u \in C^3(B_{2R})$ is either positive or negative. Then there exists a constant $C$, depending on $g_0$ and an upper bound for $\| \ln |u| \|_{C^1(B_{2R})}$, such that*

$$\Delta\bigg(\frac{|\nabla u|^2}{2u}\bigg) \geq \frac{|\nabla^2 u|^2}{u} + \frac{\langle \nabla u, \nabla \Delta u\rangle}{u} - C(\Delta u + \alpha). \tag{3.5.2}$$

*Proof.* A direct computation gives

$$\begin{aligned}\Delta\bigg(\frac{|\nabla u|^2}{2u}\bigg) &= -\frac{\Delta u|\nabla u|^2}{2u^2} + \frac{|\nabla u|^4}{u^3} - \frac{\langle \nabla u, \nabla |\nabla u|^2\rangle}{u^2} + \frac{\Delta |\nabla u|^2}{2u} \\ &\geq -C(\Delta u + \alpha) + \frac{\Delta |\nabla u|^2}{2u}.\end{aligned} \tag{3.5.3}$$

The estimate (3.5.2) then follows after substituting the Bochner formula

$$\Delta\bigg(\frac{|\nabla u|^2}{2}\bigg) = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u\rangle + \mathrm{Ric}(\nabla u, \nabla u) \tag{3.5.4}$$

into the last term in (3.5.3) and estimating $|\mathrm{Ric}(\nabla u, \nabla u)| \leq C|\nabla u|^2 \leq C$. ∎

Using (3.5.2), we now derive estimates for the last integral on the LHS of (3.4.2), distinguishing between the positive and negative cases:

**Lemma 3.5.3.** *Suppose $q > 1$, $f \in C^2_{\mathrm{loc}}(\mathcal{M}^n \times \mathbb{R} \times T\mathcal{M}^n)$ is positive and let $u \in C^4(B_{2R})$ be either a positive or negative solution to* (3.1.2). *Then*

$$\begin{aligned}\int_{B_{R+2\rho}} \eta(\Delta u + \alpha)^{q-1}\,\mathrm{tr}(F)\Delta\bigg(\frac{|\nabla u|^2}{2u}\bigg) \geq (n-1)&\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u + \alpha)^q |\nabla^2 u|^2 \\ - \frac{n-1}{q+1}&\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u + \alpha)^{q+2} \\ - C\rho^{-2}&\int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+1}.\end{aligned} \tag{3.5.5}$$

*In particular, if $u$ is negative then*

$$\int_{B_{R+2\rho}} \eta(\Delta u + \alpha)^{q-1}\,\mathrm{tr}(F)\Delta\bigg(\frac{|\nabla u|^2}{2u}\bigg) \geq -C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+2}, \tag{3.5.6}$$

*and if $u$ is positive and $q \geq n-1$, then*

$$\int_{B_{R+2\rho}} \eta(\Delta u + \alpha)^{q-1}\,\mathrm{tr}(F)\Delta\bigg(\frac{|\nabla u|^2}{2u}\bigg) \geq -C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+1}. \tag{3.5.7}$$

*Proof.* By (3.5.2) we see

$$\int_{B_{R+2\rho}} \eta(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)\Delta\left(\frac{|\nabla u|^2}{2u}\right) \geq \int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)|\nabla^2 u|^2 + \int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)\langle\nabla u,\nabla\Delta u\rangle - C\int_{B_{R+2\rho}}(\Delta u+\alpha)^{q+1}. \tag{3.5.8}$$

Now,

$$\operatorname{tr}(F) = (n-1)\operatorname{tr}(A_u) = (n-1)\left((\Delta u+\alpha) - \frac{n}{2u}|\nabla u|^2 + u\operatorname{tr}(A) - \alpha\right),$$

and therefore

$$\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)|\nabla^2 u|^2 \geq (n-1)\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q}|\nabla^2 u|^2 - C\rho^{-2}\int_{B_{R+2\rho}}(\Delta u+\alpha)^{q+1} \tag{3.5.9}$$

and

$$\begin{aligned}
&\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)\langle\nabla u,\nabla\Delta u\rangle \\
&\quad= (n-1)\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q}\langle\nabla u,\nabla\Delta u\rangle \\
&\qquad+ (n-1)\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q-1}\left(-\frac{n}{2u}|\nabla u|^2 + u\operatorname{tr}(A)-\alpha\right)\langle\nabla u,\nabla\Delta u\rangle \\
&\quad= \frac{n-1}{q+1}\int_{B_{R+2\rho}} \eta u^{-1}\langle\nabla u,\nabla(\Delta u+\alpha)^{q+1}\rangle \\
&\qquad+ \frac{n-1}{q}\int_{B_{R+2\rho}} \eta u^{-1}\left(-\frac{n}{2u}|\nabla u|^2 + u\operatorname{tr}(A)-\alpha\right)\langle\nabla u,\nabla(\Delta u+\alpha)^{q}\rangle.
\end{aligned} \tag{3.5.10}$$

The last line of (3.5.10) can be estimated from below by $-C\rho^{-2}\int_{B_{R+2\rho}}(\Delta u+\alpha)^{q+1}$ after integrating by parts. Also integrating by parts in the penultimate line of (3.5.10), we therefore see that

$$\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q-1}\operatorname{tr}(F)\langle\nabla u,\nabla\Delta u\rangle \geq -\frac{n-1}{q+1}\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u+\alpha)^{q+2} - C\rho^{-2}\int_{B_{R+2\rho}}(\Delta u+\alpha)^{q+1}. \tag{3.5.11}$$

Substituting (3.5.9) and (3.5.11) back into (3.5.8), we arrive at (3.5.5).

The estimate (3.5.6) clearly follows from (3.5.5). On the other hand, if $u$ is positive then we use the fact $n|\nabla^2 u|^2 \geq (\Delta u)^2$ to estimate the first integral on the RHS of (3.5.5), giving

$$\int_{B_{R+2\rho}} \eta(\Delta u + \alpha)^{q-1} \operatorname{tr}(F)\Delta\left(\frac{|\nabla u|^2}{2u}\right) \geq (n-1)\left(\frac{1}{n} - \frac{1}{q+1}\right)\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u + \alpha)^{q+2} - C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+1}, \tag{3.5.12}$$

which implies (3.5.7) if $q \geq n-1$. □

Finally, we turn to the second integral on the LHS of (3.4.2):

**Lemma 3.5.4.** *Suppose $q > 1$, $f \in C^2_{\mathrm{loc}}(\mathcal{M}^n \times \mathbb{R} \times T\mathcal{M}^n)$ is positive and let $u \in C^4(B_{2R})$ be either a positive or negative solution to* (3.1.2). *Then*

$$\begin{aligned} -\int_{B_{R+2\rho}} (\Delta u + \alpha)^q \eta \nabla_j \nabla^i F_i^{\ j} \geq (n-1)&\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u + \alpha)^q |\nabla^2 u|^2 \\ &- \frac{n-1}{q+1}\int_{B_{R+2\rho}} \eta u^{-1}(\Delta u + \alpha)^{q+2} \\ &- C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+1}. \end{aligned} \tag{3.5.13}$$

*In particular, if $u$ is negative then*

$$-\int_{B_{R+2\rho}} (\Delta u + \alpha)^q \eta \nabla_j \nabla^i F_i^{\ j} \geq -C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+2}, \tag{3.5.14}$$

*and if $u$ is positive with $q \geq n-1$, then*

$$-\int_{B_{R+2\rho}} (\Delta u + \alpha)^q \eta \nabla_j \nabla^i F_i^{\ j} \geq -C\rho^{-2}\int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+1}. \tag{3.5.15}$$

*Proof.* By (3.4.7), we have

$$\nabla_j \nabla^i F_i^{\ j} = \nabla_j \nabla^j \Delta u - \nabla_j \nabla^i \nabla_i \nabla^j u - (n-1)\Delta\left(\frac{|\nabla u|^2}{2u}\right) + \Delta(u \operatorname{tr}(A)) - \nabla_j \nabla^i (u A_i^{\ j}). \tag{3.5.16}$$

Commuting derivatives to assert $|\nabla_j \nabla^j \Delta u - \nabla_j \nabla^i \nabla_i \nabla^j u| \leq C|\nabla^2 u| \leq C(\Delta u + \alpha)$ and recalling (3.5.2), it follows that

$$-\nabla_j \nabla^i F_i{}^j \geq -C(\Delta u + \alpha) + (n-1)\frac{|\nabla^2 u|^2}{u} + (n-1)\frac{\langle \nabla u, \nabla \Delta u \rangle}{u}. \tag{3.5.17}$$

Therefore,

$$\begin{aligned} -\int_{B_{R+2\rho}} (\Delta u + \alpha)^q \eta \nabla_j \nabla^i F_i{}^j \geq (n-1) & \int_{B_{R+2\rho}} \eta u^{-1} (\Delta u + \alpha)^q |\nabla^2 u|^2 \\ & + (n-1) \int_{B_{R+2\rho}} \eta u^{-1} (\Delta u + \alpha)^q \langle \nabla u, \nabla \Delta u \rangle \\ & - C\rho^{-2} \int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+1}, \end{aligned} \tag{3.5.18}$$

and (3.5.13) then follows after writing the penultimate integrand in (3.5.18) as $\frac{1}{q+1} \eta u^{-1} \langle \nabla u, \nabla (\Delta u + \alpha)^{q+1} \rangle$ and integrating by parts. The estimates (3.5.14) and (3.5.15) follow from (3.5.13) in the same way as in the proof of Lemma 3.5.3. □

## 3.6 Completing the proof of Theorem J

*Proof of Theorem J.* Combining Lemmas 3.4.1, 3.5.1, 3.5.3 and 3.5.4, we obtain the estimate

$$\int_{B_{R+\rho}} \left|\nabla (\Delta u + \alpha)^{(q-1)/2}\right|^2 \leq \frac{C(q-1)}{\rho^2} \int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+2}$$

in the case that $u$ is negative, and

$$\int_{B_{R+\rho}} \left|\nabla (\Delta u + \alpha)^{(q-1)/2}\right|^2 \leq \frac{C(q-1)}{\rho^2} \int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+1}$$

if $u$ is positive and $q \geq n-1$.

It follows from the Sobolev inequality in the form

$$\left( \int_{B_{R+\rho}} (\Delta u + \alpha)^{\frac{n(q-1)}{n-2}} \right)^{\frac{n-2}{n}} \leq C \int_{B_{R+\rho}} \left|\nabla (\Delta u + \alpha)^{(q-1)/2}\right|^2 + C \int_{B_{R+\rho}} (\Delta u + \alpha)^{q-1}$$

that

$$\left( \int_{B_{R+\rho}} (\Delta u + \alpha)^{\frac{n(q-1)}{n-2}} \right)^{\frac{n-2}{n}} \leq \frac{C(q-1)}{\rho^2} \int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+2} \tag{3.6.1}$$

in the negative case, and

$$\left( \int_{B_{R+\rho}} (\Delta u + \alpha)^{\frac{n(q-1)}{n-2}} \right)^{\frac{n-2}{n}} \leq \frac{C(q-1)}{\rho^2} \int_{B_{R+2\rho}} (\Delta u + \alpha)^{q+1} \tag{3.6.2}$$

in the positive case when $q \geq n-1$. By (3.6.1), we therefore obtain an improvement in integrability in the negative case whenever $\frac{n(q-1)}{n-2} > q+2$, i.e. for $q+2 > \frac{3n}{2}$, and by (3.6.2), we obtain an improvement in integrability in the positive case whenever $\frac{n(q-1)}{n-2} > q+1$, i.e. for $q+1 > n$. These improvements may be iterated exactly as in the proof of Theorems F and G to obtain the desired pointwise estimate for $\Delta u + \alpha$, and consequently for $|\nabla^2 u|$. □

# Chapter 4

# The existence of conformal metrics with $g^{-1}A_g^\tau \in \Gamma_2^+$

In this chapter we consider the problem of obtaining conformal metrics $g \in [g_0]$ with trace-modified Schouten tensor

$$A_g^\tau := A_g + (1-\tau)\sigma_1(g^{-1}A_g)g = \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{\tau R_g}{2(n-1)}g\right)$$

satisfying $g^{-1}A_g^\tau \in \Gamma_2^+$. In the context of the $\sigma_2$-Yamabe problem, this is a particularly important question when $\tau = 1$, as we have seen that the $\sigma_k$-Yamabe problem (in the positive case) on $[g_0]$ is usually posed under the admissibility assumption $g_0^{-1}A_0 \in \Gamma_k^+$. When $\tau < 1$, this question retains geometric and topological significance, particularly in three dimensions where $g^{-1}A_g^\tau \in \Gamma_2^+$ implies $\mathrm{Ric}_g > 0$ if $\tau \geq \frac{2}{3}$ (see Appendix 5.A for a proof of this well-known fact). The results in this chapter are the product of joint work with Luc Nguyen.

The plan of the chapter is as follows. We begin in §4.1 with a more detailed literature review on the existence of metrics with $g^{-1}A_g^\tau \in \Gamma_k^+$, with an emphasis on the case $k = 2$ in three dimensions. This discussion will motivate Theorem D, which unifies and extends the work of Ge, Lin & Wang [**GLW10**] (see Theorem 4.1.5 below) and Catino & Djadli [**CD10**] (see Theorem 4.1.7 below). The proof of Theorem D will be carried out in §4.2. Our discussion will also motivate a new approach to Theorem 4.1.5, previously obtained in [**GLW10**] using a flow argument. This new proof will be given in §4.3.

Throughout Chapters 4 and 5, we will only be dealing with the $\sigma_k$-Yamabe equation in the positive case, so to keep the discussion concise we will drop any mention of 'the positive case'. We will write $A_g^\tau \in \Gamma_k^+$ as shorthand for $g^{-1}A_g^\tau \in \Gamma_k^+$, and $\sigma_k(A_g)$ as shorthand for $\sigma_k(g^{-1}A_g)$ etc. If at any point we wish to raise an index using the background metric $g_0$, then we will write this explicitly as $g_0^{-1}A_g$. We will use sub/superscript 0s to denote quantities that are defined with respect to the background metric $g_0$.

## 4.1 Background and new results

In this section we expand on the literature review given in §1.2.2 on the existence of conformal metrics with $A_g^\tau \in \Gamma_2^+$. We start in §4.1.1 by considering the case $\tau = 1$, and in §4.1.2 we consider the case $\tau < 1$.

### 4.1.1 Background on the existence of conformal metrics with $g^{-1}A_g \in \Gamma_2^+$

As explained in §2.1.2, it is natural to assume the admissibility condition $A_0 \in \Gamma_k^+$ when considering the $\sigma_k$-Yamabe problem on $[g_0]$. When $k = 1$, this is equivalent to assuming that the Yamabe invariant

$$Y(\mathcal{M}^n, [g_0]) = \inf_{g\in[g_0]} \frac{\int_{\mathcal{M}^n} R_g \, dv_g}{(\mathrm{Vol}(\mathcal{M}^n, g))^{\frac{n-2}{n}}} \tag{4.1.1}$$

of $[g_0]$ is positive. When $k \geq 2$, this condition is less well-understood, and it is interesting to ask – from both the perspective of the $\sigma_k$-Yamabe problem, and more generally – whether certain conformally invariant conditions on a manifold $(\mathcal{M}^n, g_0)$ imply the existence of a conformal metric $g \in [g_0]$ with $A_g \in \Gamma_k^+$.

One of the first significant results in this direction is due to Chang, Gursky & Yang [**CGY02b**], who addressed the case $k = 2$, $n = 4$:

**Theorem 4.1.1** ([**CGY02b**]). *Let* $(\mathcal{M}^4, g_0)$ *be a closed 4-manifold satisfying* $Y(\mathcal{M}^4, [g_0]) > 0$ *and* $\int_{M^4} \sigma_2(A_0)\, dv_0 > 0$. *Then there exists a metric* $g \in [g_0]$ *satisfying* $A_g \in \Gamma_2^+$ *(necessarily with positive Ricci curvature).*

In other words, Theorem 4.1.1 says that if the Yamabe invariant of $[g_0]$ is positive, and $\sigma_2(A_0)$ is positive in the integral sense, then there exists a conformal metric $g$ with $R_g > 0$ and $\sigma_2(A_g) > 0$ pointwise. The authors also established numerous examples of simply-connected 4-manifolds satisfying the hypotheses of Theorem 4.1.1.

A crucial point is that in four dimensions, the condition $\int_{\mathcal{M}^4} \sigma_2(A_0)\, dv_0 > 0$ in Theorem 4.1.1 is conformally invariant. To see this, observe that the four-dimensional Chern–Gauss–Bonnet formula [**Che45**] may be written as

$$8\pi^2\chi(\mathcal{M}^4) = \int_{\mathcal{M}^4} |W_g|_g^2\, dv_g + 4\int_{\mathcal{M}^4} \sigma_2(A_g)\, dv_g, \tag{4.1.2}$$

where $\chi(\mathcal{M}^4)$ denotes the Euler characteristic of $\mathcal{M}^4$ and $W_g$ is the Weyl tensor of the metric $g$. It is well-known that $\int_{\mathcal{M}^4} |W_g|_g^2\, dv_g$ is a conformal invariant, and since $\chi(\mathcal{M}^4)$ is a conformal invariant (it is a topological invariant), (4.1.2) implies that the same is true of $\int_{\mathcal{M}^4} \sigma_2(A_g)\, dv_g$. In particular, $\int_{\mathcal{M}^4} \sigma_2(A_0)\, dv_0 > 0$ if and only if $\int_{\mathcal{M}^4} \sigma_2(A_g)\, dv_g > 0$ for all $g \in [g_0]$, and the hypotheses of Theorem 4.1.1 are equivalent to imposing the existence of a metric $g \in [g_0]$ satisfying $R_g > 0$ and $\int_{\mathcal{M}^4} \sigma_2(A_g)\, dv_g > 0$.

We note that the conclusion $\mathrm{Ric}_g > 0$ in Theorem 4.1.1 is a special case of a more general result due to Guan, Viaclovsky & Wang [**GVW03**], who showed that if $A_g \in \Gamma_k^+$ for some $k \geq \frac{n}{2}$, then $\mathrm{Ric}_g > 0$. If this is the case, it follows as a well-known consequence of Myers' theorem that the fundamental group of $\mathcal{M}^n$ is finite, thus providing a topological obstruction to the existence of metrics with $A_g \in \Gamma_k^+$ for $k \geq \frac{n}{2}$.

**Remark 4.1.2.** It is well-known that there are topological restrictions to the existence of metrics with positive scalar curvature (equivalently, the existence of metrics with $A_g \in \Gamma_1^+$) – see e.g. [**Lic63**]. In [**GLW05**], the authors established further topological restrictions on the existence of metrics with $A_g \in \Gamma_k^+$ for $2 \leq k < \frac{n}{2}$, namely the vanishing of certain Betti numbers.

It is of particular interest to generalise Theorem 4.1.1 to three dimensions, where the implication $A_g \in \Gamma_2^+ \implies \mathrm{Ric}_g > 0$ still holds by [**GVW03**] (see also Appendix 5.A). There has already been some interesting progress on such generalisations in the literature, and for the remainder of this section (and in §4.1.2) we will discuss some of these results.

We begin with a result of Sheng [**She08**] that provides one possible extension of Theorem 4.1.1 to any dimension – we will then focus more specifically on what this result tells for $n = 3$, and why it is not the end of the story. To describe Sheng's result, we first define for $1 \leq k \leq n$ the functionals $Q_k$ by

$$Q_k(g) = \mathrm{Vol}(\mathcal{M}^n, g)^{-\frac{n-2k}{n}} \int_{\mathcal{M}^n} \sigma_k(A_g)\, dv_g.$$

We will also denote

$$[g_0]_k = \{g \in [g_0] : A_g \in \Gamma_k^+\} \tag{4.1.3}$$

and adopt the convention that $[g_0]_0 = [g_0]$. We then define the conformal invariants

$$\hat{Y}_k([g_0]) = \begin{cases} \inf_{g\in[g_0]_{k-1}} Q_k(g) & \text{if } [g_0]_{k-1} \neq \emptyset \\ -\infty & \text{if } [g_0]_{k-1} = \emptyset. \end{cases}$$

Note that when $k = 1$, $\hat{Y}_1([g_0])$ is just a positive multiple of the Yamabe invariant (as defined in (4.1.1)), and thus the quantities $\hat{Y}_k([g_0])$ may be viewed as generalisations of the Yamabe invariant. The result of Sheng is as follows:

**Theorem 4.1.3** ([**She08**]). *Let $(\mathcal{M}^n, g_0)$ be a closed manifold of dimension $n \geq 3$, and suppose $\hat{Y}_k([g_0]) > 0$ for some $2 \leq k \leq n$. Then $[g_0]_k \neq \emptyset$, i.e. there exists a metric $g \in [g_0]$ satisfying $A_g \in \Gamma_k^+$.*

**Remark 4.1.4.** The proof of Theorem 4.1.3 in [**She08**] is based on a continuity method for a family of equations considered by Gursky & Viaclovsky in [**GV03a**], wherein an alternative proof of Theorem 4.1.1 is given. It is also clear that Theorem 4.1.3 reproduces the result of Theorem 4.1.1 when $k = 2$, $n = 4$, by the conformal invariance of $Q_2(g)$ in four dimensions.

For $k = 2$, $n = 3$, Sheng's theorem asserts the following: if $Y(\mathcal{M}^3, [g_0]) > 0$ (that is, $[g_0]_1 \neq \emptyset$) and

$$\hat{Y}_2([g_0]) = \inf_{g \in [g_0]_1} \mathrm{Vol}(\mathcal{M}^3, g)^{1/3} \int_{\mathcal{M}^3} \sigma_2(A_g)\, dv_g > 0,$$

then there exists a metric $g \in [g_0]$ satisfying $A_g \in \Gamma_k^+$ (necessarily with positive Ricci curvature). This requires that $\int_{\mathcal{M}^3} \sigma_2(A_g)\, dv_g$ is positive for all positive scalar curvature metrics in $[g_0]$, and it is natural to ask whether the same conclusion holds if this integral positivity is only satisfied by some positive scalar metric. This question was answered in the affirmative by Ge, Lin & Wang in [**GLW10**]:

**Theorem 4.1.5** ([**GLW10**])**.** *Let $(\mathcal{M}^3, g_0)$ be a closed 3-manifold satisfying $Y(\mathcal{M}^3, [g_0]) > 0$ and $\sup_{g \in [g_0]_1} \int_{\mathcal{M}^3} \sigma_2(A_g)\, dv_g > 0$. Then there exists a metric $g \in [g_0]$ satisfying $A_g \in \Gamma_2^+$ (necessarily with positive Ricci curvature).*

We wish to indicate two differences between Theorems 4.1.1 and 4.1.5. First, as pointed out in [**GLW10**], positivity of $\int_{\mathcal{M}^3} \sigma_2(A_0)\, dv_0$ is no longer a conformally invariant condition in three dimensions. Thus, the requirement in Theorem 4.1.5 that there exists a metric $g \in [g_0]$ satisfying both $R_g > 0$ and $\int_{\mathcal{M}^3} \sigma_2(A_g)\, dv_g > 0$ *simultaneously* is, *a priori*, more restrictive than in Theorem 4.1.1. It would be interesting to determine, for instance, whether the supremum over $[g_0]_1$ in Theorem 4.1.5 can be replaced with a supremum over $[g_0]$, although this possibility will not be addressed directly in this thesis. Instead, we would like to address another distinction between Theorems 4.1.1 and 4.1.5, namely in relation to their conclusions on positive Ricci curvature. This leads us to consider the *trace-modified Schouten tensor*.

### 4.1.2 Background on the trace-modified Schouten tensor $A_g^\tau$

As we have stated, metrics satisfying $A_g \in \Gamma_k^+$ for $k \geq \frac{n}{2}$ possess an important geometric property, namely they have positive Ricci curvature [**GVW03**]. A natural question to ask is whether this condition on the Schouten tensor can be weakened, whilst retaining the implication of positive Ricci curvature.

To this end, we denote by $A_g^\tau$ ($\tau \in \mathbb{R}$) the trace-modified Schouten tensor

$$A_g^\tau := A_g + (1-\tau)\sigma_1(A_g)g = \frac{1}{n-2}\bigg(\operatorname{Ric}_g - \frac{\tau R_g}{2(n-1)}g\bigg),$$

as introduced in [**GV03a, LL03**] (see also the discussion below Question 1 in the introduction). When $\tau = 1$, $A_g^\tau$ coincides with the Schouten tensor $A_g$, and clearly $A_g^\tau > A_g$ (in the sense that $g^{-1}A_g^\tau - g^{-1}A_g$ is positive definite on each tangent space) whenever $R_g > 0$ and $\tau < 1$. Now, a simple algebraic argument (see Proposition 5.A.1 in Appendix 5.A) shows that if $R_g > 0$ and $\sigma_2(A_g^\tau) > 0$ for some $\tau \in [2 - \frac{4}{n}, 2]$, then $\operatorname{Ric}_g > 0$. In particular, when $\tau = 1$ and $n = 3$ or 4, we recover two special cases of the result of [**GVW03**]. But when $n = 3$, we in fact obtain the (well-known) stronger result that $A_g^\tau \in \Gamma_2^+$ implies $\operatorname{Ric}_g > 0$ whenever $\tau \geq \frac{2}{3}$. In light of Theorem 4.1.5, we therefore ask:

**Question 1.** *Fix $\tau \in [\frac{2}{3}, 1)$ and suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold satisfying $Y(\mathcal{M}^3, [g_0]) > 0$ and $\sup_{g\in[g_0]_1}\int_{\mathcal{M}^3}\sigma_2(A_g^\tau)\,dv_g > 0$. Does there exist a metric $g \in [g_0]$ with $A_g^\tau \in \Gamma_2^+$ (necessarily with positive Ricci curvature)?*

**Remark 4.1.6.** We note that the case $\tau = 1$, where $A_g^\tau$ coincides with the Schouten tensor $A_g$, is critical for a number of reasons. As we will expand on later (see e.g. Remark 4.A.3 in Appendix 4.A), $\sigma_2(A_g^\tau)$ is not variational when $\tau < 1$, but in some sense one has additional control on ellipticity when $\tau < 1$, which often aids in obtaining estimates. This phenomenon will be reflected in our proofs of Theorems D and E.

A first step towards answering Question 1 is to see whether the conclusion $\mathrm{Ric}_g > 0$ in Theorem 4.1.5 can be obtained under *any* weakening of the hypotheses. The following result of Catino & Djadli [**CD10**] gives an affirmative answer in this direction:

**Theorem 4.1.7** ([**CD10**]). *Let $(\mathcal{M}^3, g_0)$ be a closed 3-manifold satisfying $R_0 > 0$ and $\int_{\mathcal{M}^3} \sigma_2(A_0)\, dv_0 \geq 0$. Then for all $\tau \leq \tau_0 \approx \frac{7}{10}$, there exists a metric $g \in [g_0]$ satisfying $A_g^\tau \in \Gamma_2^+$ (necessarily with positive Ricci curvature if $\tau \geq \frac{2}{3}$).*

**Remark 4.1.8.** The proof of Theorem 4.1.7 in [**CD10**] is based on a continuity method for the family of equations considered in [**GV03a**] and [**She08**] (see also Remark 4.1.4 above). This path of equations will also be considered in our proof of Theorem D in §4.2.

We now address our new results stemming from the discussion above.

### 4.1.3 New results on the existence of conformal metrics with $g^{-1}A_g^\tau \in \Gamma_2^+$

Two natural questions arise from Theorem 4.1.7. First of all, we ask whether the conclusion of Theorem 4.1.7 can be strengthened under the same hypotheses; in light of Theorem 4.1.5, it is reasonable to expect that for any $\tau < 1$, one should be able to obtain a metric $g \in [g_0]$ satisfying $A_g^\tau \in \Gamma_2^+$. In fact, in joint work with Luc Nguyen, we have been able to provide an affirmative answer to this question under an even weaker assumption than in Theorem 4.1.7. This new result – as given in Theorem D in the introduction – puts Theorems 4.1.5 and 4.1.7 on an equal footing, and we restate it here:

**Theorem D.** *Let $(\mathcal{M}^3, g_0)$ be a closed 3-manifold satisfying $Y(\mathcal{M}^3, [g_0]) > 0$ and suppose*

$$\lambda := \sup_{g = e^{-2u} g_0 \in [g_0]_1} \frac{\int_{\mathcal{M}^3} \sigma_2(A_g)\, dv_g}{\int_{\mathcal{M}^3} e^{4u}\, dv_g} \geq 0.$$

*Then for all $\tau < 1$ there exists a metric $g \in [g_0]$ satisfying $A_g^\tau \in \Gamma_2^+$ (necessarily with positive Ricci curvature if $\tau \geq \frac{2}{3}$).*

The proof of Theorem D will be given in §4.2.

The second question arising from Theorem 4.1.7 is whether the continuity method of [**CD10**] can be adapted to answer Question 1 for any $\tau < 1$. At the time of writing, we have been unable to achieve this. We therefore ask whether the method of [**GLW10**] to prove Theorem 4.1.5, or the method of [**CGY02b**] to prove Theorem 4.1.1, may be more amenable for this purpose. We make partial progress on both of these possibilities in this thesis, although in this chapter we only consider the approach of [**GLW10**] (we refer the reader to Chapter 5 for more detail on the approach of [**CGY02b**]). We summarise our new contribution here as follows.

The proof of Theorem 4.1.5 in [**GLW10**] uses a conformal flow to obtain solutions $g_{u_\nu} = e^{-2u_\nu} g_0 \in [g_0]_1$ to the family of quotient equations

$$\frac{\sigma_2(A_{g_{u_\nu}}) - \nu e^{4u_\nu}}{\sigma_1(A_{g_{u_\nu}})} = c_\nu < 0 \tag{4.1.4}$$

for all $\nu > \lambda$, where $c_\nu$ are negative constants depending on $\nu$ and

$$\lambda = \sup_{g = e^{-2u} g_0 \in [g_0]_1} \frac{\int_{\mathcal{M}^3} \sigma_2(A_g)\, dv_g}{\int_{\mathcal{M}^3} e^{4u}\, dv_g}. \tag{4.1.5}$$

Note that $\lambda$ is positive by the assumptions of Theorem 4.1.5. By establishing *a priori* estimates and a suitable rescaling argument, the authors then show that in the limit $\nu \to \lambda^+$, the solutions obtained converge to a smooth metric $g = e^{-2u} g_0 \in [g_0]_1$ satisfying $\sigma_2(A_g) = \lambda e^{4u}$. In particular, $A_g \in \Gamma_2^+$.

Now, the flow method constructed in [**GLW10**] uses the variational properties of $\sigma_2(A_g^\tau)$ specific to the case $\tau = 1$ (see Remark 4.A.3 in Appendix 4.A). With Question 1 in mind, we ask whether we can establish existence of positive scalar curvature solutions to a family of quotient equations similar to (4.1.4) using elliptic methods, and without appealing to any variational structure. The family of quotient equations should be chosen so that, with suitable estimates in hand (and, if necessary, a scaling argument similar to that of [**GLW10**]), one can extract a limiting metric satisfying $A_g \in \Gamma_2^+$. We have made partial progress on this problem, in that we have been able to obtain the following result using elliptic methods, although not yet independently of the variational structure:

**Proposition 4.1.9.** *Let $(\mathcal{M}^3, g_0)$ be a closed 3-manifold with $R_0 > 0$ and suppose $\lambda > 0$ (where $\lambda$ is defined in (4.1.5)). Fix $\alpha \ll 0$ such that $\nu_0(x) := \sigma_2(A_0) + e^{-2\alpha}\sigma_1(A_0) > \lambda$, and define $\nu_t(x) = (1-t)\nu_0(x) + t\lambda$ for $t \in [0,1]$. Then for all $t < 1$, there exists a smooth solution $g_t = e^{-2u_t}g_0$ to*

$$\frac{\sigma_2(A_{g_t}) - \nu_t(x)e^{4u_t}}{\sigma_1(A_{g_t})} = -1. \tag{$P_t$}$$

We will explain in §4.3 why Proposition 4.1.9, in combination with *a priori* estimates and a rescaling argument, yields the desired metric with $A_g \in \Gamma_2^+$ in the limit $t \to 1$, thus giving an alternative proof of Theorem 4.1.5.

## 4.2 Proof of Theorem D

In this section we prove Theorem D. Our proof combines the continuity method of [**CD10**] with an existence result of Ge, Lin & Wang [**GLW10**] and a strong comparison principle based on the methods of Li, Nguyen & Wang [**LNW20a**]. The existence result of Ge, Lin & Wang we will use is as follows:

**Theorem 4.2.1** ([**GLW10**])**.** *Let $(\mathcal{M}^3, g_0)$ be a closed 3-manifold satisfying $Y(M^3, [g_0]) > 0$ and suppose*

$$\lambda := \sup_{g = e^{-2u}g_0 \in [g_0]_1} \frac{\int_{\mathcal{M}^3} \sigma_2(A_g)\, dv_g}{\int_{\mathcal{M}^3} e^{4u}\, dv_g} = 0.$$

*Then $\lambda$ is attained by a $C^{1,1}$ metric $g = e^{-2u}g_0$ satisfying $R_g \geq 0$ and $\sigma_2(A_g) \equiv 0$ a.e.*

We now state the strong comparison principle that we will use. In order to obtain the following, only minor modifications to the arguments of [**LNW20a**] (see Theorem 2.3 therein) are required, and we refer the reader to Appendix 4.B for the details.

**Proposition 4.2.2.** *Suppose $\tau < 1$ and let $\Omega \subset \mathbb{R}^n$ be a domain equipped with a Riemannian metric $g_0$. For a function $u$ on $\Omega$, denote $g_u = e^{-2u}g_0$. Suppose $u_1 \in C^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies*

$$\sigma_k^{1/k}(A^\tau_{g_{u_1}}) \geq 0, \quad A^\tau_{g_{u_1}} \in \overline{\Gamma_k^+} \quad \text{a.e. in } \Omega,$$

*and that $u_2 \in C^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies*

$$\sigma_k^{1/k}(A^\tau_{g_{u_2}}) = 0, \quad A^\tau_{g_{u_2}} \in \overline{\Gamma_k^+} \quad \text{a.e. in } \Omega.$$

*If $u_1 \leq u_2$ in $\Omega$, then either $u_1 = u_2$ in $\Omega$ or $u_1 < u_2$ in $\Omega$.*

*Proof.* See Appendix 4.B. □

We now proceed to give the proof of Theorem D.

*Proof of Theorem D.* We consider the same path of equations as in [**CD10**]: using the assumption of positive scalar curvature, we fix some $\delta \ll 0$ for which $A_0^\delta = \mathrm{Ric}_0 - \frac{\delta}{4}R_0 g_0$ is positive definite, set $f = \sigma_2^{1/2}(A_0^\delta) > 0$, and consider for $t \geq \delta$ and $g_t = e^{-2u_t}g_0$ the equations

$$\sigma_2^{1/2}(A^t_{g_t}) = fe^{4u_t}. \tag{4.2.1}$$

When $t = \delta$, $u_\delta \equiv 0$ is a solution to (4.2.1) with $A^\delta_{g_\delta} \in \Gamma_2^+$, and therefore for any $T \in [\delta, 1)$, the set

$$\mathcal{U}_T = \big\{t \in [\delta, T] : \text{(4.2.1) has a solution } u_t \in C^{2,\alpha}(\mathcal{M}^3) \text{ with } A^t_{g_t} \in \Gamma_2^+\big\}$$

is non-empty. By the proof of Proposition 2.2 in [**GV03a**], $\mathcal{U}_T$ is also open. It remains to show that $\mathcal{U}_T$ is closed, for which we require *a priori* $C^{2,\alpha}$ estimates on solutions to (4.2.1) for $t \in [\delta, T]$ that are independent of $t$.

Now, the proof of Proposition 2.4 in [**CD10**] yields a uniform upper bound $u_t \leq C$ on solutions to (4.2.1) for all $t \leq 1$. A uniform gradient estimate $|\nabla u_t| \leq C$ for all $t \leq 1$ then follows, as in Proposition 4.1 of [**GV03a**]. We now suppose for a contradiction that the uniform lower bound on solutions to (4.2.1) fails for some $t = \tau < 1$. Fixing $t = \tau$ in (4.2.1), we therefore have a sequence of solutions $u_i$ satisfying $\min_{\mathcal{M}^3} u_i \to -\infty$. By the uniform gradient estimate, it follows that $\max_{\mathcal{M}^3} u_i \to -\infty$ as well.

We denote $g_i = e^{-2u_i} g_0$ and define the rescaled sequence

$$\hat{u}_i = u_i - \frac{1}{\mathrm{Vol}(\mathcal{M}^3, g_0)} \int_{\mathcal{M}^3} u_i \, dv_0,$$

which is now bounded in $C^1$. Denoting $\hat{g}_i = e^{-2\hat{u}_i} g_0 = e^{2\,\mathrm{Vol}(\mathcal{M}^3, g_0)^{-1} \int u_i \, dv_0} g_i$, we see that by the scaling properties of (4.2.1), the $\hat{u}_i$ satisfy

$$\sigma_2^{1/2}(A^\tau_{\hat{g}_i}) = f \exp\left(4u_i - \frac{2}{\mathrm{Vol}(\mathcal{M}^3, g_0)} \int_{\mathcal{M}^3} u_i \, dv_0\right) > 0. \tag{4.2.2}$$

It follows from the local estimates of [**LL03, GW03b, Che05, Wan06, JLL07, Li09**] (discussed in §1.2.1) that the $\hat{u}_i$ are therefore bounded in $C^2$. (We note that the second derivative estimates in these works are independent of any lower bound on the RHS of (4.2.2), and so the fact that $\exp(4u_i - 2\,\mathrm{Vol}(\mathcal{M}^3, g_0)^{-1} \int u_i \, dv_0) \to 0$ as $i \to \infty$ does not preclude uniform $C^2$ estimates here).

As a consequence of these $C^2$ estimates, we may take $i \to \infty$ along a suitable sequence in (4.2.2) to obtain $\hat{u} \in C^{1,1}(\mathcal{M}^3)$, $\hat{g} = e^{-2\hat{u}} g_0$, satisfying

$$A^\tau_{\hat{g}} \in \overline{\Gamma_2^+}, \quad \sigma_2(A^\tau_{\hat{g}}) = 0 \quad \text{a.e. in } \mathcal{M}^3. \tag{4.2.3}$$

On the other hand, by Theorem 4.2.1 we also have $u \in C^{1,1}(\mathcal{M}^3)$, $g_u = e^{-2u} g_0$, satisfying

$$A_{g_u} \in \overline{\Gamma_2^+}, \quad \sigma_2(A_{g_u}) = 0 \quad \text{a.e. in } \mathcal{M}^3. \tag{4.2.4}$$

Since $\sigma_2(A^\tau_{g_u}) = \sigma_2(A_{g_u}) + \frac{1}{16}(1-\tau)(5-3\tau) R^2_{g_u} \geq 0$, it follows that $u \in C^{1,1}(\mathcal{M}^3)$ satisfies

$$A^\tau_{g_u} \in \overline{\Gamma_2^+}, \quad \sigma_2(A^\tau_{g_u}) \geq 0 \quad \text{a.e. in } \mathcal{M}^3.$$

We wish to show that $u$ and $\hat{u}$ differ by a translation. We start by translating $\hat{u}$ by a constant $c \in \mathbb{R}$ so that $u \leq \hat{u} + c$ on $\mathcal{M}^3$ and $u(x) = \hat{u}(x) + c$ for some $x \in \mathcal{M}^3$. For this constant $c$, the set

$$\mathcal{C} = \{x \in \mathcal{M}^3 : u(x) = \hat{u}(x) + c\}$$

is therefore non-empty and, by continuity of $u$ and $\hat{u}$, $\mathcal{C}$ is also closed. Moreover, for any $x \in \mathcal{C}$ we may apply the strong comparison principle of Proposition 4.2.2 to a sufficiently small ball centred at $x$ and conclude that $\mathcal{C}$ is open. Therefore $\mathcal{C} = \mathcal{M}^3$, i.e. $u = \hat{u} + c$ on $\mathcal{M}^3$.

Substituting $u = \hat{u} + c$ into (4.2.4), we therefore see that $\sigma_2(A_{\hat{g}}) = 0$, which is compatible with (4.2.3) if and only if $R_{\hat{g}} \equiv 0$ a.e. on $\mathcal{M}^3$. Since the scalar curvature of a conformal metric $g_w = e^{-2w} g_0$ is related to the scalar curvature of $g_0$ by the transformation law

$$R_{g_w} e^{-2w} = R_0 + 4\Delta_0 w - 2|\nabla_0 w|_0^2 \tag{4.2.5}$$

(see [**Bes87**]), the condition $R_{\hat{g}} \equiv 0$ corresponds to $\hat{u} \in C^{1,1}(\mathcal{M}^3)$ satisfying a uniformly elliptic equation on $\mathcal{M}^3$. Standard elliptic regularity (see e.g. [**GT01, Theorem 9.19**]) therefore implies $\hat{u} \in C^\infty(\mathcal{M}^3)$. But positivity of the Yamabe invariant $Y(\mathcal{M}^3, [g_0])$ implies there are no smooth scalar-flat metrics in $[g_0]$, thus we arrive at a contradiction.

With the uniform lower bound on $u_t$ established, we may then proceed exactly as in [**CD10**] to obtain the desired $C^{2,\alpha}$ estimate. It follows that $\mathcal{U}_T = [\delta, T]$, and since $T < 1$ was arbitrary, this completes the proof. □

## 4.3 The continuity method for a perturbed $\sigma_2$ quotient equation

In this section, we prove Proposition 4.1.9 and use this to prove Theorem 4.1.5 (which was previously obtained in [**GLW10**]). We remind the reader that ultimately, our goal is to prove Theorem 4.1.5 without using the variational structure of $\sigma_2(A_g)$ – the hope is that such a method should be more amenable to the case $\tau < 1$ (see Question 1), where we do not have nice variational properties (see Remark 4.A.3 in Appendix 4.A). However, our progress in this direction is partial, in that our proof of Proposition 4.1.9, whilst avoiding a flow argument, still uses some variational structure.

### 4.3.1 Setting up the problem

Let us begin by recalling the equation under consideration. Using the assumption of positive scalar curvature, we fix $\alpha \ll 0$ such that $\nu_0(x) := \sigma_2(A_0) + e^{-2\alpha}\sigma_1(A_0) > \lambda$, and define $\nu_t(x) = (1-t)\nu_0(x) + t\lambda$ for $t \in [0,1]$. Then for $t < 1$, we consider the path of equations

$$\frac{\sigma_2(A_{g_t}) - \nu_t(x)e^{4u_t}}{\sigma_1(A_{g_t})} = -1, \qquad (P_t)$$

where $g_t = e^{-2u_t}g_0$.

Some remarks are in order, which we will refer back to at various stages of our argument.

**Remark 4.3.1.** It is shown in [**GLW10, Lemma 2**] that $(P_t)$ is an elliptic equation in $\Gamma_1^+$, with ellipticity constants depending on both an upper and lower bound on $\sigma_1(A_{g_t})$. It is also shown that the operator on the LHS of $(P_t)$ is strictly concave in $\Gamma_1^+$.

**Remark 4.3.2.** We are free to choose the constant on the RHS of $(P_t)$, although it must be negative (by definition of $\lambda$ and the fact that $\nu_t > \lambda$ for $t < 1$). Indeed, since the LHS of $(P_t)$ scales by $e^{2\alpha}$ under a translation $u \mapsto u + \alpha$, it is clear that any negative constant on the RHS can be achieved once Proposition 4.1.9 is established. We will make use of this scaling property later.

**Remark 4.3.3.** It is also shown in [**GLW10, Prop. 1**] – independently of any flow argument – that if a solution to $(P_t)$ exists, then it is unique.

We will prove Proposition 4.1.9 using a continuity method. Note that when $t = 0$, $u_0 \equiv \alpha$ is a solution to $(P_t)$ with $A_{g_0} \in \Gamma_1^+$. To obtain solutions for $t \in (0,1)$, we define for $T \in [0,1)$ the set

$$\mathcal{S}_T = \{t \in [0,T] : (P_t) \text{ has a solution } u_t \in C^{2,\alpha}(\mathcal{M}^3) \text{ with } A_{g_t} \in \Gamma_1^+\},$$

and show that $\mathcal{S}_T$ is both open and closed. For this purpose, we will need two results from [**GLW10**]. The first of these (see Corollary 4 therein) provides first and second derivative estimates for solutions to $(P_t)$, and is obtained independently of any flow argument or variational structure:

**Theorem 4.3.4** ([**GLW10**]). *Suppose $\nu$ is a positive function, $\kappa \leq 0$ and $A_0 \in \Gamma_1^+$. Let $B_R$ be a geodesic ball of radius $R < \tau_0$, where $\tau_0$ is the injectivity radius of $(\mathcal{M}^3, g_0)$. Then there exists a constant $C$ depending only on $(B_R, g_0)$ (independent of $\nu$) such that for any solution $g_u = e^{-2u}g_0$, $A_{g_u} \in \Gamma_1^+$, to*

$$\frac{\sigma_2(A_{g_u}) - \nu e^{4u}}{\sigma_1(A_{g_u})} = \kappa$$

*one has*

$$\sup_{B_{R/2}} \big(|\nabla u|^2 + |\nabla^2 u|\big) \leq C.$$

**Remark 4.3.5.** We note in [**GLW10, Cor. 4**], which corresponds to Theorem 4.3.4 above, it assumed that $\nu$ is a positive constant. However, their argument goes through without change if $\nu$ is a positive function.

To describe the second result of [**GLW10**] that we will use, we define

$$\mathcal{H}_{\nu_t}[u] = -2\bigg(\int_{\mathcal{M}^3} \big(\sigma_2(A_{g_u}) - \nu_t e^{4u}\big)\, dv_{g_u}\bigg)\bigg(\int_{\mathcal{M}^3} \sigma_1(A_{g_u})\, dv_{g_u}\bigg). \tag{4.3.1}$$

The relevance of the functional $\mathcal{H}_{\nu_t}$ is that the Euler-Lagrange equation satisfied by its critical points is precisely $(P_t)$ – we refer the reader to the calculations in Appendix 4.A for a justification of this claim. The result of [**GLW10**] is then as follows:

**Proposition 4.3.6** ([**GLW10**]). *Let $t < 1$. Then every critical point of $\mathcal{H}_{\nu_t}$ is a global minimiser.*

**Remark 4.3.7.** We note that in [**GLW10**], it is shown (using their flow) that for all constants $\nu > \lambda$, every critical point of $\mathcal{H}_\nu$ is a global minimiser, but their argument goes through without change if $\nu > \lambda$ is a positive function. We reiterate that we would like to prove Proposition 4.1.9 independently of Proposition 4.3.6, but this will be the subject of future study.

The plan for the remainder of the chapter is as follows. In §4.3.2 we show that $\mathcal{S}_T$ is closed, and in §4.3.3 we show that $\mathcal{S}_T$ is open. In §4.3.4 we complete the proof of Proposition 4.1.9 and use it to prove Theorem 4.1.5.

### 4.3.2 $\mathcal{S}_T$ is closed

**Proposition 4.3.8.** *$\mathcal{S}_T$ is closed for all $T \in [0, 1)$.*

*Proof.* For a sequence $\{t_k\} \subset \mathcal{S}_T$ with $t_k \to t_*$, we wish to show $t_* \in \mathcal{S}_T$, i.e $(P_{t_*})$ has a $C^2$ solution $g_{t_*}$ with $A_{g_{t_*}} \in \Gamma_1^+$. Now, whilst Theorem 4.3.4 implies that solutions to $(P_t)$ satisfy an *a priori* estimate

$$|\nabla_0 u_t|_0 + |\nabla_0^2 u_t|_0 \leq C \quad \text{on } \mathcal{M}^3, \tag{4.3.2}$$

it is not clear how to obtain the $C^0$ estimate and a positive lower bound on $\sigma_1(A_{g_t})$ (both are needed to establish uniform ellipticity – see Remark 4.3.1). We will instead obtain the desired *a priori* estimates for a rescaled equation, take the limit $t_k \to t_*$ on these rescaled solutions, and then rescale back to obtain a solution to $(P_{t_*})$.

To this end, for each solution $u_{t_k}$ in our sequence, define

$$h(\nu_{t_k}) = 2\bigg(\int_{\mathcal{M}^3} \sigma_1(A_{g_{t_k}})\, dv_{g_{t_k}}\bigg)^2 > 0.$$

By the scaling property of the LHS of $(P_t)$, $u_{t_k}$ solves $(P_{t_k})$ if and only if the function

$$\tilde{u}_{t_k} = u_{t_k} + \ln\bigg(\sqrt{2}\int_{\mathcal{M}^3} \sigma_1(A_{g_{t_k}})\, dv_{g_{t_k}}\bigg)$$

satisfies

$$\frac{\sigma_2(A_{\tilde{g}_{t_k}}) - \nu_{t_k}(x)e^{4\tilde{u}_{t_k}}}{\sigma_1(A_{\tilde{g}_{t_k}})} = -h(\nu_{t_k}) < 0, \tag{4.3.3}$$

where $\tilde{g}_{t_k} = e^{-2\tilde{u}_{t_k}} g_0$.

In light of Remarks 4.3.2 and 4.3.3, the estimates in Theorem 4.3.4 are independent of the constant $\kappa$ when $\kappa < 0$, so applying Theorem 4.3.4 to (4.3.3) we have the estimate

$$|\nabla_0 \tilde{u}_{t_k}|_0 + |\nabla_0^2 \tilde{u}_{t_k}|_0 \leq C \quad \text{on } \mathcal{M}^3. \tag{4.3.4}$$

On the other hand, $|\tilde{u}_{t_k}| \leq C$ on $\mathcal{M}^3$ if and only if $0 < C^{-1} \leq e^{\tilde{u}_{t_k}} \leq C$, or equivalently

$$0 < C^{-1} \leq e^{u_{t_k}} \int_{\mathcal{M}^3} \sigma_1(A_{g_{t_k}})\, dv_{g_{t_k}} \leq C. \tag{4.3.5}$$

We derive (4.3.5) as follows. First, we note that the first derivative estimate in (4.3.2) implies there exists a uniform constant $C$ such that $|u_{t_k}(x) - u_{t_k}(y)| \leq C$ for all $x, y \in \mathcal{M}^3$, or equivalently $-C - u_{t_k}(y) \leq -u_{t_k}(x) \leq C - u_{t_k}(y)$ for all $x, y \in \mathcal{M}^3$. Exponentiating and integrating with respect to $dv_0(y)$, we have

$$0 < C^{-1} \int_{\mathcal{M}^3} e^{-u_{t_k}}\, dv_0 \leq e^{-u_{t_k}} \leq C \int_{\mathcal{M}^3} e^{-u_{t_k}}\, dv_0,$$

and therefore

$$0 < C^{-1} \leq e^{u_{t_k}} \int_{\mathcal{M}^3} e^{-u_{t_k}}\, dv_0 \leq C. \tag{4.3.6}$$

The upper bound in (4.3.5) then follows from (4.3.6) and the estimate

$$\begin{aligned}\int_{\mathcal{M}^3} \sigma_1(A_{g_{t_k}})\, dv_{g_{t_k}} &= \int_{\mathcal{M}^3} \sigma_1(A_{g_{t_k}}) e^{-3u_{t_k}}\, dv_0 \\ &\overset{(4.2.5)}{=} \int_{\mathcal{M}^3} e^{-u_{t_k}} \underbrace{\left(\sigma_1(A_0) + \Delta_0 u_{t_k} - \frac{1}{2}|\nabla_0 u_{t_k}|_0^2\right)}_{\leq C \text{ by } (4.3.2)} dv_0,\end{aligned}$$

and the lower bound in (4.3.5) follows from (4.3.6) and positivity of the Yamabe invariant:

$$\int_{\mathcal{M}^3} \sigma_1(A_{g_{t_k}})\, dv_{g_{t_k}} \geq C^{-1} \left(\int_{\mathcal{M}^3} e^{-3u_{t_k}}\, dv_0\right)^{1/3} \geq C^{-1} \int_{\mathcal{M}^3} e^{-u_{t_k}}\, dv_0.$$

At this point, we have shown $\|\tilde{u}_{t_k}\|_{C^2(\mathcal{M}^3,g_0)} \leq C$. To obtain uniform ellipticity, it remains to show uniform positivity of $\sigma_1(A_{\tilde{g}_{t_k}})$. For this, we use the fact that each $u_t$, being a solution to $(P_t)$, is a critical point for $\mathcal{H}_{\nu_t}$ with critical value $h(\nu_t)$. Therefore, by Proposition 4.3.6 and the definition of $\mathcal{H}_{\nu_t}$, the mapping $t \mapsto h(\nu_t)$ is non-increasing, so $h(\nu_{t_k}) \leq h(\nu_0) \leq C$. Using the equation (4.3.3) and our $C^0$ estimate, it follows that

$$C\sigma_1(A_{\tilde{g}_{t_k}}) \geq h(\nu_{t_k})\sigma_1(A_{\tilde{g}_{t_k}}) = -\sigma_2(A_{\tilde{g}_{t_k}}) + \nu_{t_k}(x)e^{4\tilde{u}_{t_k}} \geq -\sigma_2(A_{\tilde{g}_{t_k}}) + C^{-1}\lambda,$$

from which we obtain the desired positive lower bound using $\sigma_2(A) = \frac{1}{2}(\sigma_1(A)^2 - |A|^2) \leq \frac{1}{2}\sigma_1(A)^2$.

With uniform ellipticity established, concavity of the operator on the LHS of $(P_t)$ in $\Gamma_1^+$ (see Remark 4.3.1) means that the regularity theory of Evans-Krylov [**Eva82, Kry82**] applies, and we have a uniform estimate $\|\tilde{u}_{t_k}\|_{C^{2,\alpha}(\mathcal{M}^3)} \leq C$. We can then extract a subsequence (still denoted $\{\tilde{u}_{t_k}\}$) which converges in $C^2$ to some $\tilde{u}_* \in C^2(\mathcal{M}^3)$ satisfying

$$\frac{\sigma_2(A_{\tilde{g}_*}) - \nu_{t_*}(x)e^{4\tilde{u}_*}}{\sigma_1(A_{\tilde{g}_*})} = -\alpha_* := -\lim_{k\to\infty} h(\nu_{t_k}) \leq 0 \quad \text{and} \quad A_{\tilde{g}_*} \in \Gamma_1^+.$$

Recall that we wish to obtain a solution to $(P_{t_*})$ in $\Gamma_1^+$. It suffices to show that $\alpha_* < 0$, for then we can repeat the scaling argument used at the beginning of the proof to obtain our desired solution from $\tilde{u}_*$. To this end, suppose for a contradiction that $\alpha_* = 0$. Then $u_*$ satisfies $\sigma_2(A_{\tilde{g}_*}) = \nu_{t_*}e^{4\tilde{u}_*}$ with $A_{\tilde{g}_*} \in \Gamma_1^+$. But since $t_* \in [0,T]$ and $T < 1$, we have $\nu_{t_*}(x) = (1-t)\nu_0(x) + t\lambda > \lambda$ for $x \in \mathcal{M}^3$. Thus we obtain a contradiction to the definition of $\lambda$, and it must be that $\alpha_* < 0$. □

### 4.3.3 $\mathcal{S}_T$ is open

**Proposition 4.3.9.** *$\mathcal{S}_T$ is open for all $T \in [0,1)$.*

*Proof.* To show that $\mathcal{S}_T$ is open, we first note that $u_t$ is a solution to $(P_t)$ if and only if $u_t$ satisfies

$$F[u_t] := \sigma_2(A_{g_t}) + \sigma_1(A_{g_t}) - \nu_t e^{4u_t} = 0.$$

It therefore suffices to show that the linearisation of $F$ (at $u_t$) is invertible. For the following calculation, we recall from Chapter 2 that if $A$ is a $(1,1)$-tensor, then $\partial\sigma_2(A)/\partial A_i^j = T_1(A)_j^i$, where $T_1(A)_j^i = \sigma_1(A)\delta_j^i - A_j^i$ is the first Newton tensor of $A$. We have

$$\begin{aligned}
\mathcal{L}(\varphi) &:= \frac{d}{ds}\bigg|_{s=0} F[u_t + s\varphi] \\
&= \big(T_1(g_t^{-1}A_{g_t})_j^i + \delta_j^i\big)\frac{d}{ds}\bigg|_{s=0}\big((e^{-2s\varphi}g_t)^{-1}A_{e^{-2s\varphi}g_t}\big)_i^j - 4\nu_t\varphi e^{4u_t} \\
&= \big(T_1(g_t^{-1}A_{g_t})_j^i + \delta_j^i\big)\frac{d}{ds}\bigg|_{s=0}\left[e^{2s\varphi}\Big(g_t^{-1}\big(A_{g_t} + s\nabla_{g_t}^2\varphi + s^2 d\varphi\otimes d\varphi - \frac{1}{2}s^2|\nabla_{g_t}\varphi|_{g_t}^2\, g_t\big)\Big)_i^j\right] \\
&\quad - 4\nu_t\varphi e^{4u_t} \\
&= \big(T_1(g_t^{-1}A_{g_t})_j^i + \delta_j^i\big)\big(2\varphi g_t^{-1}A_{g_t} + g_t^{-1}\nabla_{g_t}^2\varphi\big)_i^j - 4\nu_t\varphi e^{4u_t}.
\end{aligned}$$

Now, $T_1(g_t^{-1}A_{g_t})_j^i(g_t^{-1}A_{g_t})_i^j = 2\sigma_2(g_t^{-1}A_{g_t})$ by (2.3.1)–(2.3.3), and $\delta_j^i(g_t^{-1}A_{g_t})_i^j = \sigma_1(g_t^{-1}A_{g_t})$, so

$$\begin{aligned}
\mathcal{L}(\varphi) &= \big(T_1(A_{g_t}) + I\big) : \nabla_{g_t}^2\varphi + \varphi\big(4\sigma_2(A_{g_t}) + 2\sigma_1(A_{g_t}) - 4\nu_t e^{4u_t}\big) \\
&= \big(T_1(A_{g_t}) + I\big) : \nabla_{g_t}^2\varphi - 2\varphi\sigma_1(A_{g_t}).
\end{aligned}$$

Using the fact that $T_1(A_{g_t})$ is divergence free with respect to $g_t$ (this follows from a direct computation and the twice-contracted Bianchi identity $\nabla^i \operatorname{Ric}_{ij} = \frac{1}{2}\nabla_j R$), we may also write the above as

$$\mathcal{L}(\varphi) = \operatorname{div}\Big(\big(T_1(A_{g_t}) + I\big)\nabla\varphi\Big) - 2\varphi\sigma_1(A_{g_t}),$$

from which it follows that

$$\begin{aligned}
\langle\varphi, -\mathcal{L}\varphi\rangle_{L^2(\mathcal{M}^3,\,g_t)} &= \int_{\mathcal{M}^3}\Big(a_j^i\nabla_i\varphi\nabla^j\varphi + 2\varphi^2\sigma_1(A_{g_t})\Big)\,dv_{g_t} \\
&\geq C_1\|\nabla\varphi\|_{L^2(\mathcal{M}^3,\,g_t)} + C_2\|\varphi\|_{L^2(\mathcal{M}^3,\,g_t)}. \qquad (4.3.7)
\end{aligned}$$

Here, $C_1 > 0$ depends on the lower ellipticity constant for the positive definite matrix $a_j^i = T_1(A_{g_t})_j^i + \delta_j^i$, and $C_2 > 0$ depends on a positive lower bound for $\sigma_1(A_{g_t})$. Note that we have implicitly used [**GLW10, Lemma 2**] to assert that $a_j^i$ is positive definite.

It follows from (4.3.7) that $\ker \mathcal{L}$ is trivial. Since $\mathcal{L}$ is of divergence form, the Fredholm alternative then implies that $\mathcal{L}$ is invertible, as required. □

### 4.3.4 Completing the proofs of Proposition 4.1.9 and Theorem 4.1.5

*Proof of Proposition 4.3.4.* Since $0 \in \mathcal{S}_T$, it follows from Propositions 4.3.8 and 4.3.9 that $\mathcal{S}_T = [0,T]$ for any $T < 1$. Thus $(P_t)$ admits a solution $u_t \in C^{2,\alpha}(\mathcal{M}^3)$ with $A_{g_t} \in \Gamma_1^+$ for any $t < 1$. By standard elliptic theory, these solutions are smooth. □

*Proof of Theorem 4.1.5.* As a consequence of Proposition 4.3.4, for $t \in [0,1)$ we have existence of solutions to the rescaled equation (4.3.3), and by the proof of Proposition 4.3.8, we have $C^{2,\alpha}$ estimates and a positive lower bound on $\sigma_1(A_{g_t})$ that are independent of $t \in [0,1)$. Taking $t \to 1$ along a suitable sequence $t_k$, we therefore obtain $u_\lambda \in C^2(\mathcal{M}^3)$ satisfying

$$\frac{\sigma_2(A_{g_{u_\lambda}}) - \lambda e^{4u_\lambda}}{\sigma_1(A_{g_{u_\lambda}})} = -\kappa_0 := -\lim_{k\to\infty} h(\nu_{t_k}) \leq 0 \quad \text{and} \quad A_{g_{u_\lambda}} \in \Gamma_1^+. \tag{4.3.8}$$

Now, being the $C^2$ limit of a sequence of absolute minimisers for $\mathcal{H}_{\nu_{t_k}}$ (by Proposition 4.3.6), $u_\lambda$ is an absolute minimiser for $\mathcal{H}_\lambda$. On the other hand, by definition of $\lambda$, there is a sequence $\{u_i\}$ with $e^{-2u_i}g_0 \in \Gamma_1^+$, normalised so that $\int \sigma_1(A_{g_{u_i}})\, dv_{g_{u_i}} = 1$ for all $i$, such that $\int (\sigma_2(A_{g_{u_i}}) - \lambda e^{4u_i})\, dv_{g_{u_i}} \to 0$. Therefore $\inf_{e^{-2u}g_0 \in [g_0]_1} \mathcal{H}_\lambda[u] = 0$. Combining these facts, it follows that

$$\mathcal{H}_\lambda[u_\lambda] = \inf_{e^{-2u}g_0 \in [g_0]_1} \mathcal{H}_\lambda[u] = 0, \tag{4.3.9}$$

which is compatible with (4.3.8) if and only if $\kappa_0 = 0$. Therefore, $A_{g_{u_\lambda}} \in \Gamma_1^+$ with $\sigma_2(A_{g_{u_\lambda}}) = \lambda e^{4u_\lambda} > 0$, and in particular $A_{g_{u_\lambda}} \in \Gamma_2^+$. □

# Appendix

## 4.A Some first variation formulas

In this appendix, we derive first variation formulas for two functionals defined on the conformal class of a fixed metric $g_0$ on a closed 3-manifold, namely the total $\sigma_1$ curvature and total $\sigma_2$ curvature of a conformal metric $g_u = e^{-2u}g_0$. The formulas below can be used to verify that $(P_t)$ is the Euler Lagrange equation of the functional $\mathcal{H}_{\nu_t}$ defined in (4.3.1).

**Lemma 4.A.1.**

$$\frac{d}{ds}\bigg|_{s=0}\int_{\mathcal{M}^3}\sigma_1(A_{g_{u+s\varphi}})\,dv_{g_{u+s\varphi}} = -\int_{\mathcal{M}^3}\varphi\sigma_1(A_{g_u})\,dv_{g_u}.$$

*Proof.* Recalling (4.2.5), which tells us $4\sigma_1(A_{g_u}) = R_{g_u} = e^{2u}(R_0 + 4\Delta_0 u - 2|\nabla_0 u|_0^2)$, and noting that $dv_{g_u} = e^{-3u}\,dv_0$, we have

$$\begin{aligned}\int_{\mathcal{M}^3}\sigma_1(A_{g_{u+s\varphi}})\,dv_{g_{u+s\varphi}} &= \int_{\mathcal{M}^3}e^{2s\varphi}\big(\sigma_1(A_{g_u}) + s\Delta_{g_u}\varphi + o(s^2)\big)\,dv_{g_{u+s\varphi}}\\ &= \int_{\mathcal{M}^3}e^{-s\varphi}\big(\sigma_1(A_{g_u}) + s\Delta_{g_u}\varphi + o(s^2)\big)\,dv_{g_u} \text{ as } s\to 0.\end{aligned}$$

Therefore,

$$\frac{d}{ds}\bigg|_{s=0}\int_{\mathcal{M}^3}\sigma_1(A_{g_{u+s\varphi}})\,dv_{g_{u+s\varphi}} = \int_{\mathcal{M}^3}\big(-\varphi\sigma_1(A_{g_u}) + \Delta_{g_u}\varphi\big)\,dv_{g_u} = -\int_{\mathcal{M}^3}\varphi\sigma_1(A_{g_u})\,dv_{g_u}.$$

□

**Lemma 4.A.2.**

$$\frac{d}{ds}\bigg|_{s=0}\int_{\mathcal{M}^3}\sigma_2(A_{g_{u+s\varphi}})\,dv_{g_{u+s\varphi}} = \int_{\mathcal{M}^3}\varphi\sigma_2(A_{g_u})\,dv_{g_u}.$$

*Proof.* Using $\sigma_2(A) = \frac{1}{2}(\sigma_1(A)^2 - |A|^2)$, we first observe that $\sigma_2(A_{g_u}) = -\frac{1}{2}|\operatorname{Ric}_{g_u}|^2_{g_u} + \frac{3}{16}R^2_{g_u}$. Again by (4.2.5), we have

$$R^2_{g_u} = e^{4u}\big(R_0^2 + 8R_0\Delta_0 u + \dots\big),$$

and since $\operatorname{Ric}_{g_u} = \operatorname{Ric}_0 + \nabla_0^2 u + du \otimes du + \Delta_0 u\, g_0 - |\nabla_0 u|_0^2 g_0$ (see [**Bes87**]), we also have

$$|\operatorname{Ric}_{g_u}|^2_{g_u} = e^{4u}\big(|\operatorname{Ric}_0|_0^2 + 2\operatorname{Ric}_0 : \nabla_0^2 u + 2R_0\Delta_0 u + \dots\big).$$

It follows that

$$\begin{aligned}
&\int_{\mathcal{M}^3} \sigma_2(A_{g_{u+s\varphi}})\, dv_{g_{u+s\varphi}} \\
&\qquad = -\frac{1}{2}\int_{\mathcal{M}^3} e^{s\varphi}\big(|\operatorname{Ric}_{g_u}|^2_{g_u} + 2s\operatorname{Ric}_{g_u} : \nabla^2_{g_u}\varphi + 2sR_{g_u}\Delta_{g_u}\varphi + o(s^2)\big)\, dv_{g_u} \\
&\qquad\quad + \frac{3}{16}\int_{\mathcal{M}^3} e^{s\varphi}\big(R^2_{g_u} + 8sR_{g_u}\Delta_{g_u}\varphi + o(s^2)\big)\, dv_{g_u} \text{ as } s \to 0
\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{d}{ds}\bigg|_{s=0}\int_{\mathcal{M}^3} \sigma_2(A_{g_{u+s\varphi}})\, dv_{g_{u+s\varphi}} \\
&\qquad = -\frac{1}{2}\int_{\mathcal{M}^3} \big(\varphi|\operatorname{Ric}_{g_u}|^2_{g_u} + 2\operatorname{Ric}_{g_u} : \nabla^2_{g_u}\varphi + 2R_{g_u}\Delta_{g_u}\varphi\big)\, dv_{g_u} \\
&\qquad\quad + \frac{3}{16}\int_{\mathcal{M}^3} \big(\varphi R^2_{g_u} + 8R_{g_u}\Delta_{g_u}\varphi\big)\, dv_{g_u}.
\end{aligned} \tag{4.A.1}$$

But integrating by parts and using the twice-contracted Bianchi identity, we know

$$\begin{aligned}
-\int_{\mathcal{M}^3} \operatorname{Ric}_{g_u} : \nabla^2_{g_u}\varphi\, dv_{g_u} &= \int_{\mathcal{M}^3} \nabla_i \operatorname{Ric}^{ij}_{g_u} \nabla_j\varphi\, dv_{g_u} \\
&= \frac{1}{2}\int_{\mathcal{M}^3} \nabla^j R_{g_u}\nabla_j\varphi\, dv_{g_u} = -\frac{1}{2}\int_{\mathcal{M}^3} R_{g_u}\Delta_{g_u}\varphi\, dv_{g_u},
\end{aligned}$$

and thus (4.A.1) simplifies to

$$\frac{d}{ds}\bigg|_{s=0}\int_{\mathcal{M}^3} \sigma_2(A_{g_{u+s\varphi}})\, dv_{g_{u+s\varphi}} = \int_{\mathcal{M}^3}\left(-\frac{1}{2}|\operatorname{Ric}_{g_u}|^2_{g_u} + \frac{3}{16}R^2_{g_u}\right)\varphi\, dv_{g_u} = \int_{\mathcal{M}^3} \varphi\sigma_2(A_{g_u})\, dv_{g_u}.$$

□

**Remark 4.A.3.** Carrying out the above calculation with $\sigma_2(A^\tau_{g_u})$ in place of $\sigma_2(A_{g_u})$ for $\tau < 1$, one loses the cancellation of the terms involving $R_{g_u}\Delta_{g_u}\varphi$. The resulting Euler-Lagrange equation is then of fourth order, involving a term of the form $\Delta_{g_u}R_{g_u}$.

## 4.B A strong comparison principle when $\tau < 1$

In this appendix we prove the strong comparison principle stated in Proposition 4.2.2, which we recall here:

**Proposition 4.B.1.** *Suppose $\tau < 1$ and let $\Omega \subset \mathbb{R}^n$ be a domain equipped with a Riemannian metric $g_0$. For a function $u$ on $\Omega$, denote $g_u = e^{-2u} g_0$. Suppose $u_1 \in C^{1,1}_{\text{loc}}(\Omega)$ satisfies*

$$\sigma_k^{1/k}(A^\tau_{g_{u_1}}) \geq 0, \quad A^\tau_{g_{u_1}} \in \overline{\Gamma_k^+} \quad \text{a.e. in } \Omega,$$

*and that $u_2 \in C^{1,1}_{\text{loc}}(\Omega)$ satisfies*

$$\sigma_k^{1/k}(A^\tau_{g_{u_2}}) = 0, \quad A^\tau_{g_{u_2}} \in \overline{\Gamma_k^+} \quad \text{a.e. in } \Omega.$$

*If $u_1 \leq u_2$ in $\Omega$, then either $u_1 = u_2$ in $\Omega$ or $u_1 < u_2$ in $\Omega$.*

The proof of Proposition 4.B.1 follows the proof of [**LNW20a, Theorem 2.3**], with some small modifications; the reason we cannot apply their result directly is that condition (2.3) therein is not satisfied in our case, only the weaker condition (2.6).

An important point for the proof of Proposition 4.B.1 is that the equation

$$\sigma_k^{1/k}(A^\tau_{g_u}) = 0, \quad A^\tau_{g_u} \in \overline{\Gamma_k^+}$$

is locally strictly elliptic when $\tau < 1$. That is, for any compact subset $K$ of $\overline{\Gamma_k^+}$, there exists a positive constant $\lambda = \lambda(K) > 0$ such that for any $(1,1)$-tensor $A$ satisfying

$$A^\tau := A + (1-\tau)\sigma_1(A) I \in K$$

and any nonnegative definite $N$, it holds that

$$\sigma_k^{1/k}(A^\tau + N^\tau) - \sigma_k^{1/k}(A^\tau) \geq \lambda \|N\|. \tag{4.B.1}$$

Indeed, (4.B.1) is easy to check using the conformal transformation law for $A^\tau_{g_u}$

$$A^\tau_{g_u} = \nabla_0^2 u + (1-\tau)\Delta_0 u\, g_0 - \frac{2-\tau}{2}|du|_0^2\, g_0 + du \otimes du + A^\tau_0 \tag{4.B.2}$$

(which follows from (4.2.5) and the formula for $\mathrm{Ric}_{g_u}$ given in Appendix 4.A) and concavity of $\sigma_k^{1/k}$ on $\overline{\Gamma_k^+}$. Crucial here is the fact that $\tau < 1$.

We now turn to the proof of Proposition 4.B.1. Alongside the strict ellipticity discussed above, our main tool will be the following 'preliminary' strong comparison principle of [**LNW20a**], which is an immediate consequence of Lemma 2.5 and Proposition 2.7 therein:

**Proposition 4.B.2** ([**LNW20a**]). *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\tau < 1$. Suppose $u_1 \in C^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies for some constant $a > 0$*

$$\sigma_k^{1/k}(A^\tau_{g_{u_1}}) \geq a > 0, \quad A^\tau_{g_{u_1}} \in \Gamma_k^+ \quad \text{a.e. in } \Omega,$$

*and that $u_2 \in C^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies*

$$\sigma_k^{1/k}(A^\tau_{g_{u_2}}) = 0, \quad A^\tau_{g_{u_2}} \in \overline{\Gamma_k^+} \quad \text{a.e. in } \Omega.$$

*If $u_1 \leq u_2$ in $\Omega$ and $u_1 < u_2$ near $\partial\Omega$, then $u_1 < u_2$ in $\Omega$.*

*Proof of Proposition 4.B.1.* We follow [**LNW20a**], arguing by contradiction. If the conclusion is false, then we can find a closed ball $\overline{B} \subset \Omega$ of some radius $R > 0$ for which there exists $\hat{x} \in \partial B$ with

$$u_1 < u_2 \text{ in } \overline{B}\backslash\{\hat{x}\} \quad \text{and} \quad u_1(\hat{x}) = u_2(\hat{x}).$$

Without loss of generality, we may assume that $B$ is centred at the origin.

We will deform $u_1$ into a strict subsolution $\tilde{u}_1$ in some open ball $A$ around $\hat{x}$ such that $\tilde{u}_1 < u_2$ on $\partial A$ and $\inf_A(u_2 - \tilde{u}_1) = 0$. These conditions imply that $u_2 - \tilde{u}_1$ must attain its infimum (of zero) in $A$, which contradicts the conclusion of Proposition 4.B.2 that $\tilde{u}_1 < u_2$ in $A$.

We construct $\tilde{u}_1$ as follows. For a constant $\alpha > 1$ to be determined later, we define

$$E(x) = e^{-\alpha|x|^2},$$

$$h(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2},$$

$$\xi(x) = \cos(\alpha^{1/2}(x_1 - \hat{x}_1)).$$

For constants $\mu > 0$ and $\nu > 0$ to be determined later, we also define

$$\tilde{u}(x) = \tilde{u}_{\mu,\nu}(x) := u_1(x) + \mu\big(h(x) - \nu\big)\xi(x).$$

For suitable $\alpha, \mu$ and $\nu$, the function $\tilde{u} = \tilde{u}_{\mu,\nu}$ will determine our strict subsolution $\tilde{u}_1$.

Now, for given $\alpha$, we may take a ball $A$ centred at $\hat{x}$ of sufficiently small radius such that $\xi > \frac{1}{2}$ on $A$. We let $\nu_0 = \sup_A h > 0$, which is positive since $h(x) > 0$ whenever $|x| < R$ (e.g. in $A \cap B$). Moreover, for $0 \leq \nu \leq \nu_0$ and sufficiently small $\mu > 0$, it is easy to see that

$$\tilde{u} < u_2 \text{ on } \partial A.$$

We claim that for suitable $\alpha$ and $\mu$, the function $\tilde{u}$ satisfies for some constant $a > 0$

$$\sigma_k^{1/k}(A_{g_{\tilde{u}}}^{\tau}) \geq a > 0, \quad A_{g_{\tilde{u}}}^{\tau} \in \Gamma_k^+ \quad \text{a.e. in } A.$$

To this end, we observe as in [**LNW20a**] that

$$\partial_i \tilde{u}(x) = \partial_i u_1(x) - 2\mu\alpha E(x)\xi(x)x_i - \mu\alpha^{1/2}(h(x) - \nu)\sin(\alpha^{1/2}(x_1 - \hat{x}_1))\delta_{i1} \quad (4.B.3)$$

and

$$\begin{aligned}\partial_i\partial_j\tilde{u}(x) &= \partial_i\partial_j u_1(x) + 2\mu\alpha E(x)\xi(x)\big(2\alpha x_i x_j - \delta_{ij}\big) \\ &\quad + 4\mu\alpha^{3/2}E(x)\sin(\alpha^{1/2}(x_1 - \hat{x}_1))\delta_{i1}x_j \\ &\quad - \mu\alpha(h(x) - \nu)\xi(x)\delta_{i1}\delta_{j1} \qquad (4.B.4)\end{aligned}$$

We wish to calculate $A_{g_{\tilde{u}}}^{\tau}$ using the formula (4.B.2). We start by using (4.B.4) to calculate

$$\begin{aligned}&\partial_i\partial_j\tilde{u}(x) + (1-\tau)\textstyle\sum_p \partial_p\partial_p\tilde{u}(x)\,(g_0)_{ij}(x) \\ &= \partial_i\partial_j u_1(x) + \boxed{4\mu\alpha^2 E(x)\xi(x)x_i x_j} - 2\mu\alpha E(x)\xi(x)\delta_{ij} \\ &\quad + 4\mu\alpha^{3/2}E(x)\sin(\alpha^{1/2}(x_1 - \hat{x}_1))\delta_{i1}x_j - \mu\alpha h(x)\xi(x)\delta_{i1}\delta_{j1} + \boxed{\mu\alpha\nu\xi(x)\delta_{i1}\delta_{j1}} \\ &\quad + (1-\tau)\textstyle\sum_p \partial_p\partial_p u_1(x)(g_0)_{ij}(x) + \boxed{4(1-\tau)\mu\alpha^2 E(x)\xi(x)|x|^2(g_0)_{ij}(x)} \\ &\quad - 2n(1-\tau)\mu\alpha E(x)\xi(x)(g_0)_{ij}(x) + 4(1-\tau)\mu\alpha^{3/2}E(x)\sin(\alpha^{1/2}(x_1 - \hat{x}_1))x_1\,(g_0)_{ij}(x) \\ &\quad - (1-\tau)\mu\alpha h(x)\xi(x)\,(g_0)_{ij}(x) + \boxed{(1-\tau)\mu\alpha\nu\xi(x)(g_0)_{ij}(x)}\,. \qquad (4.B.5)\end{aligned}$$

Recalling the definitions of $E(x)$ and $h(x)$ (in particular, observing their exponential decay as $\alpha \to \infty$), inspecting the unboxed terms in (4.B.5), and noting that

$$\partial_i\partial_j\tilde{u} = (\nabla_0^2)_{ij}\tilde{u} + \Gamma_{ij}^k\partial_k\tilde{u}$$
$$\partial_i\partial_j u_1 = (\nabla_0^2)_{ij}u_1 + \Gamma_{ij}^k\partial_k u_1$$

it follows from (4.B.3) and (4.B.5) that

$$\begin{aligned}
&\nabla_0^2\tilde{u}(x) + (1-\tau)\Delta_0\tilde{u}\, g_0(x) \\
&\quad = \nabla_0^2 u_1(x) + (1-\tau)\Delta_0 u_1(x)\, g_0(x) + 4\mu\alpha^2 E(x)\xi(x)\, x\otimes x + \mu\alpha\nu\xi(x)e_1\otimes e_1 \\
&\qquad + 4(1-\tau)\mu\alpha^2 E(x)\xi(x)|x|^2 g_0(x) + (1-\tau)\mu\alpha\nu\xi(x)g_0(x) \\
&\qquad + O\big(\mu(\alpha^{3/2}E(x) + \alpha^{1/2}\nu)\big) \quad \text{as } \alpha\to\infty. \qquad (4.B.6)
\end{aligned}$$

Also by (4.B.3) we have

$$-\frac{2-\tau}{2}|d\tilde{u}|_0^2\, g_0 + d\tilde{u}\otimes d\tilde{u} = O\big(\mu(\alpha^{3/2}E(x) + \alpha^{1/2}\nu)\big) \quad \text{as } \alpha\to\infty, \qquad (4.B.7)$$

and so combining (4.B.6) and (4.B.7) we obtain

$$\begin{aligned}
A_{g_{\tilde{u}}}^\tau(x) &= A_{g_{u_1}}^\tau(x) + 4\mu\alpha^2 E(x)\xi(x)\, x\otimes x + \mu\alpha\nu\xi(x)e_1\otimes e_1 \\
&\quad + 4(1-\tau)\mu\alpha^2 E(x)\xi(x)|x|^2 g_0(x) + (1-\tau)\mu\alpha\nu\xi(x)g_0(x) \\
&\quad + O\big(\mu(\alpha^{3/2}E(x) + \alpha^{1/2}\nu)\big) \quad \text{as } \alpha\to\infty. \qquad (4.B.8)
\end{aligned}$$

Taking $\alpha$ sufficiently large, so that the error terms in (4.B.8) are absorbed by the strictly positive terms of order $\mu\alpha^2 E$ and $\mu\alpha\nu$, and raising an index using $g_{\tilde{u}}$, we obtain for some positive function $p(x) = p_{\alpha,\mu,\nu}(x) \geq C^{-1} > 0$ that

$$g_{\tilde{u}}^{-1}A_{g_{\tilde{u}}}^\tau(x) \geq g_{\tilde{u}}^{-1}A_{g_{u_1}}^\tau(x) + p(x)g_{\tilde{u}}^{-1}g_0(x) = g_{\tilde{u}}^{-1}A_{g_{u_1}}^\tau(x) + p(x)e^{2\tilde{u}}I$$

It then follows from the strict ellipticity (4.B.1) that for all $1 \leq l \leq k$,

$$\sigma_l^{1/l}\big(g_{\tilde{u}}^{-1}A_{g_{\tilde{u}}}^\tau(x)\big) \geq \sigma_l^{1/l}\big(g_{\tilde{u}}^{-1}A_{g_{u_1}}^\tau(x)\big) + \lambda\sigma_l^{1/l}\big(p(x)e^{2\tilde{u}}I\big) \geq \lambda p(x)e^{2\tilde{u}}\sigma_l^{1/l}(I). \qquad (4.B.9)$$

To complete the proof, we now choose the parameter $\nu \in [0, \nu_0]$ so that $\inf_A(u_2 - \tilde{u}) = 0$. Indeed, $\inf_A(u_2 - \tilde{u}_{\mu,0}) \leq 0 \leq \inf_A(u_2 - \tilde{u}_{\mu,\nu_0})$, so such a value of $\nu \in [0, \nu_0]$ exists. Once all the parameters in the definition of $\tilde{u}$ have been fixed, it follows from (4.B.9) that $\sigma_l^{1/l}\big(g_{\tilde{u}}^{-1} A_{g_{\tilde{u}}}^{\tau}(x)\big) \geq C^{-1} > 0$ for each $1 \leq l \leq k$. $\square$

# Chapter 5

# Integral estimates for a fourth order perturbation of the $\sigma_2$-Yamabe equation in three dimensions

In this chapter we obtain *a priori* integral estimates for solutions $g = e^{2w}g_0$ to equations of the form

$$\sigma_2(A_g^\tau) = \frac{\delta}{4}\Delta_g R_g + f(x,w), \tag{5.0.1}$$

where $\tau \in [\frac{2}{3}, 1]$, $f$ is a positive function and $\delta > 0$ is a real parameter. We restrict our attention to solutions to (5.0.1) of positive scalar curvature on closed 3-manifolds.

Equations of the form (5.0.1) with $\tau = 1$ were previously studied in four dimensions by Chang, Gursky & Yang [**CGY02b**], where they were used to establish the existence of conformal metrics satisfying $A_g \in \Gamma_2^+$ under certain conformally invariant conditions on the background metric. We refer the reader to the discussion surrounding Theorem 4.1.1 in Chapter 4 for more details. Our motivation for studying (5.0.1) in part comes from a desire to generalise the existence result of [**CGY02b**] to three dimensions and address the case $\tau < 1$ – again, we refer the reader to Chapter 4 for a more detailed discussion. On the other hand, it is also natural and of inherent interest to consider fourth order regularisations of non-uniformly elliptic equations such as $\sigma_2(A_g^\tau) = f > 0$, and it is also from this perspective that we will study (5.0.1).

The plan of this chapter is as follows. We begin in §5.1 by reminding the reader of Theorem E. We will also provide a brief description of the work of [**CGY02b**] to put

Theorem E into more context. In §5.2 we introduce a preliminary result, Theorem E′, the proof of which will take up most of the chapter. In §5.3 we establish our notation and conventions for the rest of the chapter. In §5.4 we carry out the proof of Theorem E′, and then in §5.5 we complete the proof of Theorem E. Finally, in §5.6 we give an application of Theorem A.

**Notation:** In keeping with the conventions of [**CGY02b**], we will denote conformal metrics in the form $g_w = e^{2w} g_0$ throughout this chapter.

## 5.1 Statement of Theorem E

We recall from the introduction that we will prove the following result:

**Theorem E.** *Suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold satisfying $Y(\mathcal{M}^3, [g_0]) > 0$. Then for given $s < 6$, $\tau \in [\frac{2}{3}, 1]$, $C_1 > 0$ and positive $f \in C^1(\mathcal{M}^3 \times \mathbb{R})$, there exist constants $\delta_0 > 0$ and $C > 0$ (depending only on $g_0$, $s$, $\tau$, $C_1$, $\sup_{\mathcal{M}^3 \times [-C_1, C_1]}(f + |\nabla_x f|) < \infty$ and $\inf_{\mathcal{M}^3 \times [-C_1, C_1]} f > 0$) such that for every $C^4$ solution $g = e^{2w} g_0$ to*

$$\sigma_2(A_g^\tau) = \frac{\delta}{4} \Delta_g R_g + f(x, w) \tag{5.1.1}$$

*with $0 \leq \delta < \delta_0$ satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,6}(\mathcal{M}^3, g_0)} \leq C_1 \quad \text{and} \quad R_g \geq 0, \tag{5.1.2}$$

*one has*

$$\|w\|_{W^{2,s}(\mathcal{M}^3, g_0)} \leq C. \tag{5.1.3}$$

*If $\tau \in [\frac{2}{3}, 1)$, we may replace the $W^{1,6}$ norm in (5.1.2) with the $W^{1,4}$ norm and the same conclusion holds for all $s < 12$.*

**Remark 5.1.1.** It is important to bear in mind that the constant $C$ in (5.1.3) is independent of $\delta$. Also note that the addition of $\|w\|_{L^\infty(\mathcal{M}^3)}$ in (5.1.2) is superfluous in view of the Morrey embedding $W^{1,4} \hookrightarrow C^{0,\alpha}$ in three dimensions, but we include it in (5.1.2) to make clear the dependence of $C$ and $\delta_0$ on $f$.

Let us briefly describe how our result fits in with the work of Chang, Gursky & Yang [**CGY02b**] in four dimensions. Therein, on a closed 4-manifold $(\mathcal{M}^4, g_0)$ of positive Yamabe invariant with $\int_{\mathcal{M}^4} \sigma_2(A_0)\, dv_0 > 0$, the authors consider the equation (5.1.1) for $\tau = 1$. For a particular choice of $f$, they establish the existence of a smooth solution of positive scalar curvature when $\delta = 1$, calling upon earlier existence results in spectral geometry – see [**CY95, CGY99**]. A series of *a priori* integral estimates, a regularity result and a proof that the linearised operator is invertible then yields existence of smooth solutions of positive scalar curvature for all $\delta \in (0,1]$, via the continuity method. Now, the *a priori* integral estimates obtained in carrying out the continuity method are not sufficient to take $\delta \to 0$ directly in (5.1.1), so the authors carry out an integrability improvement argument to obtain $W^{2,s}$ estimates for all $s < 5$ when $\delta > 0$ is sufficiently small. Crucially, these estimates are independent of $\delta$, and they imply a $C^{1,\alpha}$ estimate by the Morrey embedding theorem. Finally, using these estimates in combination with a Yamabe flow argument, the authors perturb their solutions obtained for small $\delta$ to obtain a conformal metric satisfying $A_g \in \Gamma_2^+$.

Given the above outline, we see that the argument of Chang, Gursky & Yang can be roughly split into three components: an existence result for $\delta \in (0,1]$, an integrability improvement argument for small $\delta > 0$, and a Yamabe flow argument. Our result, Theorem E, can be interpreted as generalising the second of these steps – that is, the integrability improvement argument – to three dimensions, allowing also for $\tau < 1$. Furthermore, an application of Theorem A that we will give in §5.6 provides an alternative to the third step in [**CGY02b**] – that is, the Yamabe flow argument – when $(\mathcal{M}^4, g_0)$ is locally conformally flat. Roughly speaking, we will obtain a $W^{2,s}$-strong solution to $\sigma_2(A_g) = f > 0$ in the limit $\delta \to 0$, and Theorem A will then imply that this solution is smooth.

It remains an interesting open problem as to how to generalise the first step of [**CGY02b**] – that is, the existence problem for $\delta \in (0,1]$ – to three dimensions. Indeed, since $\int_{\mathcal{M}^3} \sigma_2(A_g)\, dv_g$ is not a conformal invariant in three dimensions, it is not quite clear as to what conformally invariant conditions are the most natural replacements for those seen in Theorem 4.1.1. These questions remain the subject of future study.

## 5.2 Theorem E′: a provisional result

The proof of Theorem E is essentially carried out in two stages. First, under the hypotheses of Theorem E, we obtain a uniform $W^{2,3}$ estimate when $\tau = 1$ and a uniform $W^{2,6}$ estimate when $\tau < 1$. The second step is to bootstrap these estimates to obtain the $W^{2,s}$ estimates claimed in Theorem E. Since the majority of the proof of Theorem E is contained in the first step, we present this step as a separate theorem, which will be our initial focus:

**Theorem E′.** *Suppose* $(\mathcal{M}^3, g_0)$ *is a closed 3-manifold satisfying* $Y(\mathcal{M}^3, [g_0]) > 0$. *Then for given* $\tau \in [\frac{2}{3}, 1]$, $C_1 > 0$ *and positive* $f \in C^1(\mathcal{M}^3 \times \mathbb{R})$, *there exist constants* $\delta_0 > 0$ *and* $C > 0$ *(depending only on* $g_0$, $s$, $\tau$, $C_1$, $\sup_{\mathcal{M}^3 \times [-C_1, C_1]}(f + |\nabla_x f|) < \infty$ *and* $\inf_{\mathcal{M}^3 \times [-C_1, C_1]} f > 0$*) such that for every* $C^4$ *solution* $g = e^{2w} g_0$ *to*

$$\sigma_2(A_g^\tau) = \frac{\delta}{4}\Delta_g R_g + f(x, w) \tag{5.2.1}$$

*with* $0 \leq \delta < \delta_0$ *satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,6}(\mathcal{M}^3, g_0)} \leq C_1 \quad \text{and} \quad R_g \geq 0, \tag{5.2.2}$$

*one has*

$$\|w\|_{W^{2,3}(\mathcal{M}^3, g_0)} \leq C. \tag{5.2.3}$$

*If* $\tau \in [\frac{2}{3}, 1)$, *we may replace the* $W^{1,6}$ *norm in* (5.2.2) *with the* $W^{1,4}$ *norm, and the* $W^{2,3}$ *norm in* (5.2.3) *with the* $W^{2,6}$ *norm.*

We note that $W^{2,3}$ estimates for equations similar to (5.2.1) were also considered in three dimensions in the thesis of Reichert [**Rei14**], although the goals of Reichert are different to ours and there is not a significant overlap in the estimates obtained.

## 5.3 Notation and conventions for the chapter

For the remainder of this chapter, metric quantities that are defined with respect to the background metric $g_0$ will be denoted with sub/superscript 0s. Quantities that appear without any sub/superscripts are assumed to be defined with respect to the conformal metric $g = e^{2w}g_0$ (where for each $\delta > 0$, $w$ is a fixed solution to (5.2.1)). For instance, we will now write $\sigma_2(A_\tau)$ rather than $\sigma_2(A_g^\tau)$, and $dv$ rather than $dv_g$. All estimates in this chapter will be global, and we therefore write $\int$ as shorthand for $\int_{\mathcal{M}^3}$. All $\delta$-uniform constants will be denoted by $C$, and we allow these constants to vary from line to line. Whether these constants appear within or outside any integrals is irrelevant, since $dv = e^{3w}\,dv_0$ and by assumption we have a $\delta$-uniform $C^0$ estimate on solutions $w$.

We will often refer to the conformal transformation laws for the scalar curvature and Ricci curvature in three dimensions. For $g = e^{2w}g_0$, these are given (see e.g. [**Bes87**]) with respect to $g_0$ by

$$R = e^{-2w}\big(R_0 - 4\Delta_0 w - 2|\nabla_0 w|_0^2\big), \tag{5.3.1}$$

$$\mathrm{Ric} = \mathrm{Ric}_0 - \nabla_0^2 w + dw \otimes dw - \Delta_0 w g_0 - |\nabla_0 w|_0^2 g_0, \tag{5.3.2}$$

or with respect to $g$ as

$$R = e^{-2w}R_0 - 4\Delta w + 2|\nabla w|^2, \tag{5.3.3}$$

$$\mathrm{Ric} = \mathrm{Ric}_0 - \nabla^2 w - dw \otimes dw - \Delta w\, g + |\nabla w|^2 g. \tag{5.3.4}$$

Beyond the notation already introduced in this thesis, various new quantities will be introduced in this chapter. We collect a list of these here for later reference, along with the first page on which the notation appears:

| | | |
|---|---|---|
| $\mathring{\mathrm{Ric}}$ | $(0,2)$-traceless Ricci tensor, $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{R}{n}g$ | p.122 |
| $G$ | $(0,2)$-Einstein tensor, $G = -\mathrm{Ric} + \frac{R}{2}g$ | p.119 |
| $C$ | $(0,3)$-Cotton tensor, $C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik}$ | p.125 |
| $B$ | $(0,2)$-Bach tensor, $B_{ik} = \frac{1}{n-2}(\nabla^j C_{ijk} - \mathrm{Ric}^{jl}\, W_{ijkl})$ | p.125 |

| | | |
|---|---|---|
| $V$ | $V = \frac{1}{2}\|\nabla w\|^2$, where $e^{2w}$ is the conformal factor as above | p.136 |
| $\mathrm{I}_1^p, \mathrm{I}_2^p$ | see equation (5.4.13) | p.123 |
| $\mathrm{I}_{1,\delta}^p$ | see equation (5.4.14) | p.123 |
| $\mathrm{II}_1^p, \mathrm{II}_2^p$ | see equation (5.4.55) | p.136 |
| $q(\tau)$ | the polynomial $q(\tau) = \frac{3}{8}(\tau - \frac{4}{3})^2 \geq 0$ | p.119 |

## 5.4 Proof of Theorem E′

The plan for §5.4 is as follows. In §5.4.1 we state four propositions and prove Theorem E′ assuming these propositions. §5.4.2–5.4.6 are then dedicated to proving the four propositions assumed in §5.4.1.

### 5.4.1 Proof of Theorem E′ assuming Propositions I–IV

In this section, we state four results (whose proofs will be delayed until later) and use them to prove Theorem E′.

First, we recall that in Theorem E′, we have assumed that our solution $g = e^{2w}g_0$ has non-negative scalar curvature $R$. However, it will be necessary in many of our calculations to divide by $R$, and this will be justified by the first of our propositions:

**Proposition I.** *Suppose $Y(\mathcal{M}^n, [g_0]) > 0$, $\tau \in \mathbb{R}$ and $f \in C(\mathcal{M}^n \times \mathbb{R})$. Then any $C^4$ solution $g = e^{2w}g_0$ to* (5.2.1) *with either*

i) *$\delta > 0$, $f \geq 0$ and $R \geq 0$, or*

ii) *$\delta = 0$ and $f > 0$*

*satisfies $R > 0$. In particular, if $\delta \geq 0$, $f > 0$ and $R \geq 0$, then $R > 0$.*

Next, define the Einstein tensor by $G = -\operatorname{Ric} + \frac{R}{2}g$. Recalling (2.3.1) and (2.3.2), we see that $G$ is precisely the first Newton tensor of $A$, i.e. the linearisation of $\sigma_2(A)$. Moreover, $G$ is divergence-free by the twice-contracted Bianchi identity. Thus, $G$ is a natural tensor to consider in the context of obtaining estimates for (5.2.1), and we will exploit its divergence structure in obtaining the next two results.

In what follows, we define the polynomial $q(\tau) = \frac{3}{8}(\tau - \frac{4}{3})^2 \geq 0$. The precise relevance of $q(\tau)$ will become clear in §5.4.2, but for now the reader should bear in mind that $q(1) = \frac{1}{24}$ and $q(\tau) > \frac{1}{24}$ when $\tau < 1$ – this should give some indication as to why the following estimates are stronger when $\tau < 1$:

**Proposition II.** *Suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold satisfying $Y(\mathcal{M}^3, [g_0]) > 0$. Then for given $\tau \in [\frac{2}{3}, 1]$, $C_1 > 0$ and positive $f \in C^1(\mathcal{M}^3 \times \mathbb{R})$ there exist constants $\delta_0 > 0$ and $C > 0$ (depending only on $g_0$, $\tau$, $C_1$, $\sup_{\mathcal{M}^3\times[-C_1,C_1]}(f + |\nabla_x f|) < \infty$ and $\inf_{\mathcal{M}^3\times[-C_1,C_1]} f > 0$) such that for every $C^4$ solution $g = e^{2w}g_0$ to* (5.2.1) *with $0 \leq \delta < \delta_0$ satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,4}(\mathcal{M}^3,g_0)} \leq C_1 \quad \textit{and} \quad R > 0,$$

*one has*

$$0 \geq \int \left(4\left(q(\tau) - \frac{1}{24}\right)|\nabla R|^2 + \frac{R^3}{C} + 12\,\mathrm{tr}\,\mathring{\mathrm{Ric}}^3 - CR^2 - C\right) dv. \tag{5.4.1}$$

Now, the first two terms on the RHS of (5.4.1) are of the right sign and will ultimately give us improved integrability. But $\mathrm{tr}\,\mathring{\mathrm{Ric}}^3$ is of unknown sign, and so must be dealt with. This is the content of the next proposition:

**Proposition III.** *Suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold satisfying $Y(\mathcal{M}^3, [g_0]) > 0$. Then for given $\tau \in [\frac{2}{3}, 1]$, $C_1 > 0$ and positive $f \in C^1(\mathcal{M}^3 \times \mathbb{R})$, there exist constants $\delta_0 > 0$ and $C > 0$ (depending only on $g_0$, $\tau$, $C_1$, $\sup_{\mathcal{M}^3\times[-C_1,C_1]}(f + |\nabla_x f|) < \infty$ and $\inf_{\mathcal{M}^3\times[-C_1,C_1]} f > 0$) such that for every $C^4$ solution $g = e^{2w}g_0$ to* (5.2.1) *with $0 \leq \delta < \delta_0$ satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,6}(\mathcal{M}^3,g_0)} \leq C_1 \quad \textit{and} \quad R > 0,$$

*one has*

$$0 \geq \int \left(-\,\mathrm{tr}\,\mathring{\mathrm{Ric}}^3 + \frac{R^3}{C} - CR|\nabla w|^4 - C\delta R^3 - CR^2 - C\right) dv. \tag{5.4.2}$$

Propositions I–III will be enough to prove Theorem E′ when $\tau = 1$. For the stronger result when $\tau < 1$, we will need one last result:

**Proposition IV.** *Suppose* $(\mathcal{M}^3, g_0)$ *is a closed 3-manifold. Then for given* $\tau < 1$, $\delta \geq 0$, $C_1 > 0$ *and* $f \in C(\mathcal{M}^3 \times \mathbb{R})$, *there exists a constant* $C > 0$ *(depending only on* $g_0$, $\tau$, $C_1$ *and* $\sup_{\mathcal{M}^3 \times [-C_1, C_1]} f < \infty$*) such that for every* $C^4$ *solution* $g = e^{2w} g_0$ *to* (5.2.1) *satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,4}(\mathcal{M}^3, g_0)} \leq C_1 \quad \text{and } R > 0,$$

*one has*

$$\|w\|_{W^{2,2}(\mathcal{M}^3, g_0)} \leq C.$$

In particular, we get from Proposition IV and the Sobolev embedding theorem a $\delta$-uniform $W^{1,6}$ estimate when $\tau < 1$, assuming only a $\delta$-uniform $W^{1,4}$ estimate.

Assuming Propositions I–IV for now, we proceed to give the proof of Theorem E′:

*Proof of Theorem E′.* Adding the RHS of (5.4.1) to twelve times the RHS of (5.4.2), we get a cancellation of the $\operatorname{tr} \overset{\circ}{\operatorname{Ric}}^3$ terms which yields the estimate

$$0 \geq \int \left( 4\left( q(\tau) - \frac{1}{24} \right) |\nabla R|^2 + \frac{R^3}{C} - CR|\nabla w|^4 - C\delta R^3 - CR^2 - C \right) dv. \tag{5.4.3}$$

Now, Young's inequality implies that

$$R|\nabla w|^4 \leq \varepsilon R^3 + C\varepsilon^{-1} |\nabla w|^6$$

for any $\varepsilon > 0$. By our assumed $\delta$-uniform $W^{1,6}$ bound when $\tau = 1$, and the assumed $\delta$-uniform $W^{1,4}$ bound when $\tau < 1$ combined with Proposition IV, we therefore obtain (after fixing $\varepsilon > 0$ small enough) from (5.4.3) the estimate

$$0 \geq \int \left( 4\left( q(\tau) - \frac{1}{24} \right) |\nabla R|^2 + \frac{R^3}{C} - C\delta R^3 - CR^2 - C \right) dv. \tag{5.4.4}$$

Thus for sufficiently small $\delta$, it follows that

$$\int \left( R^3 + 4\left( q(\tau) - \frac{1}{24} \right) |\nabla R|^2 \right) dv \leq C, \tag{5.4.5}$$

and therefore

$$\int \left( R^3 + 4\left( q(\tau) - \frac{1}{24} \right) |\nabla_0 R|_0^2 \right) dv_0 \le C. \tag{5.4.6}$$

Note that (5.4.6) follows from (5.4.5) by our assumed $\delta$-uniform $C^0$ estimate, for then

$$dv = e^{3w}\, dv_0 \ge \frac{1}{C}\, dv_0$$

and

$$|\nabla R|^2 = g(\nabla R, \nabla R) = e^{-2w} g_0(\nabla_0 R, \nabla_0 R) \ge \frac{1}{C} |\nabla_0 R|_0^2.$$

Now, when $\tau = 1$ the coefficient of $|\nabla_0 R|_0^2$ in (5.4.6) vanishes and we arrive at $\int R^3\, dv_0 \le C$. By the transformation law (5.3.1) for the scalar curvature, this implies

$$\int \left( |\Delta_0 w|^3 - C|\nabla_0 w|_0^6 \right) dv_0 \le C,$$

and therefore $\int |\Delta_0 w|^3\, dv_0 \le C$ by the assumed $\delta$-uniform $W^{1,6}$ bound. The full $W^{2,3}$ estimate then follows by standard elliptic theory, completing the proof of Theorem E$'$ when $\tau = 1$.

If $\tau < 1$, then the coefficient of $|\nabla_0 R|_0^2$ in (5.4.6) is positive and we can rewrite (5.4.6) as

$$\int (R^2 + |\nabla_0 R|_0^2)\, dv_0 \le C.$$

By the Sobolev embedding $W^{1,2} \hookrightarrow L^6$ in three dimensions, this implies $\int R^6\, dv_0 \le C$ and thus

$$\int \left( |\Delta_0 w|^6 - C|\nabla_0 w|_0^{12} \right) dv_0 \le C, \tag{5.4.7}$$

again by (5.3.1). But for the same reason as in the case $\tau = 1$, we have a $W^{2,3}$ estimate from (5.4.6) (we simply drop the $|\nabla_0 R|_0^2$ term), and hence a $W^{1,r}$ estimate for all $r < \infty$ by the Sobolev embedding theorem. In particular we have a $W^{1,12}$ estimate, and the $W^{2,6}$ estimate then follows from (5.4.7) as before, completing the proof of Theorem E$'$. □

### 5.4.2 Proof of Proposition I: positivity of scalar curvature

Proposition I will follow from an application of the maximum principle when $\delta > 0$ and a simple algebraic argument when $\delta = 0$:

*Proof of Proposition I.* Letting $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $A_\tau$, we first compute

$$\sigma_2(A_\tau) = \frac{1}{2}\Big(\operatorname{tr}(A_\tau)^2 - |A_\tau|^2\Big) = -\frac{|\operatorname{Ric}|^2}{2} + \frac{1}{2}\left(1 - \tau + \frac{n\tau^2}{4(n-1)}\right)R^2. \tag{5.4.8}$$

Now, if $\overset{\circ}{\operatorname{Ric}} = \operatorname{Ric} - \frac{R}{n}g$ is the traceless Ricci tensor, then $|\overset{\circ}{\operatorname{Ric}}|^2 = |\operatorname{Ric}|^2 - \frac{R^2}{n}$ and it follows from (5.4.8) that

$$\sigma_2(A_\tau) = -\frac{|\overset{\circ}{\operatorname{Ric}}|^2}{2} + \frac{1}{2}q_n(\tau)R^2, \tag{5.4.9}$$

where

$$q_n(\tau) = \frac{n}{4(n-1)}\left(\tau - \frac{2(n-1)}{n}\right)^2 \geq 0. \tag{5.4.10}$$

First we prove case i). Substituting (5.4.9) into (5.2.1) we obtain, under the assumption that $f \geq 0$,

$$-\frac{\delta}{4}\Delta R + \frac{1}{2}q_n(\tau)R^2 = f + \frac{|\overset{\circ}{\operatorname{Ric}}|^2}{2} \geq 0. \tag{5.4.11}$$

If we were to have $R = 0$ at some point, then the strong maximum principle applied to (5.4.11) would imply that $R \equiv 0$, contradicting $Y(\mathcal{M}^n, [g_0]) > 0$. Therefore $R > 0$.

Next we prove case ii). First note that $\tau \neq \frac{2(n-1)}{n}$, for then (5.4.9) would imply

$$0 < f = \sigma_2(A_\tau) = -\frac{|\overset{\circ}{\operatorname{Ric}}|^2}{2} \leq 0,$$

which is clearly a contradiction. Hence $q_n(\tau) > 0$, and so from (5.4.11) we obtain $R^2 > 0$. Therefore either $R > 0$ or $R < 0$, and positivity of $Y(\mathcal{M}^n, [g_0])$ rules out the latter case. $\square$

From here on, we write $q(\tau)$ as shorthand for $q_3(\tau)$. The relevance of Proposition I is that under the assumptions of Theorem E′, we can assume our solutions to (5.2.1) have positive scalar curvature. Finally, we remark that case ii) in Proposition 1 can be rephrased as follows: if $Y(\mathcal{M}^n, [g_0]) > 0$ and $\sigma_2(A_\tau) > 0$ (necessarily with $\tau \neq \frac{2(n-1)}{n}$), then it holds that $R > 0$.

### 5.4.3 Proof of Proposition II: an $L^3$ bound for $R$ in terms of $\operatorname{tr} \overset{\circ}{\mathrm{Ric}}^3$

Before stating Proposition II, we remarked that the proof would stem from the fact that the Einstein tensor is divergence-free. More precisely, consider the quantity $G^{ij}\nabla_i\nabla_j R$. Integrating by parts, it follows that

$$\int G^{ij}\nabla_i\nabla_j R\, dv = 0, \tag{5.4.12}$$

and it is this identity that the estimate (5.4.1) will be based off. In fact, in this section we will prove a more general version of Proposition II that allows for later application in our bootstrapping argument for Theorem E. Indeed, for the same reason as above we have for any $p \geq 0$

$$\begin{aligned} 0 &= \int G^{ij}\nabla_i\big((p+1)R^p\nabla_j R\big)\, dv \\ &= (p+1)\int R^p G^{ij}\nabla_i\nabla_j R\, dv + p(p+1)\int R^{p-1}G^{ij}\nabla_i R\,\nabla_j R\, dv \\ &=: \mathrm{I}_1^p + \mathrm{I}_2^p, \end{aligned} \tag{5.4.13}$$

and since $\mathrm{I}_1^0 = \int G^{ij}\nabla_i\nabla_j R\, dv$, any estimate for $\mathrm{I}_1^p$ gives an estimate for the integral in (5.4.12) by taking $p = 0$. This is exactly the approach we take. Denoting

$$\mathrm{I}_{1,\delta}^p = \delta(p+1)\int \Big(\Delta R\Delta R^p + 2R^{p-1}(\Delta R)^2 + 2pR^{p-2}|\nabla R|^2\Delta R\Big)\, dv, \tag{5.4.14}$$

we will prove:

**Proposition II$^+$.** *Suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold satisfying $Y(\mathcal{M}^3, [g_0]) > 0$. Then for given $\tau \in [\frac{2}{3}, 1]$, $r \geq 4$, $C_1 > 0$, $0 \leq p \leq \frac{1}{2}(r-4)$ and positive $f \in$*

$C^1(\mathcal{M}^3 \times \mathbb{R})$, *there exist constants* $\delta_0 > 0$ *and* $C > 0$ *(depending only on* $g_0$, $\tau$, $C_1$, $p$, $\sup_{\mathcal{M}^3\times[-C_1,C_1]}(f + |\nabla_x f|) < \infty$ *and* $\inf_{\mathcal{M}^3\times[-C_1,C_1]} f > 0$*) such that for every* $C^4$ *solution* $g = e^{2w}g_0$ *to* (5.2.1) *with* $0 \le \delta < \delta_0$ *satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,r}(\mathcal{M}^3,g_0)} \le C_1 \quad \textit{and} \quad R > 0, \tag{5.4.15}$$

*one has*

$$\begin{aligned} \mathrm{I}_1^p \ge \mathrm{I}_{1,\delta}^p &+ (p+1)\int \left(12R^p \operatorname{tr} \mathring{\mathrm{Ric}}^3 + 2q(\tau)R^{p+3}\right) dv \\ &+ 4(p+1)^2 \int \left(q(\tau) - \frac{1}{24}\right) R^p|\nabla R|^2\, dv \\ &+ \frac{1}{C}\int R^{p-2}|\nabla R|^2\, dv - C\int R^{p+2}\, dv - C. \end{aligned} \tag{5.4.16}$$

**Remark 5.4.1.** It is easy to check that taking $p = 0$ in (5.4.16) implies (5.4.1). Note, however, that we have retained two positive terms in (5.4.16) that have been implicitly dropped in (5.4.1): these are the middle term in the definition of $\mathrm{I}_{1,\delta}^p$, and the term $\frac{1}{C}\int R^{p-2}|\nabla R|^2\, dv$ on the last line of (5.4.16). The former term cannot give us improved $\delta$-uniform integral bounds due to the factor of $\delta$, but is retained for later use (see Corollary 5.4.4 and Lemma 5.4.5). However, the latter gives rise to a $\frac{1}{C}\int R^{3p}\, dv_0$ term by the Sobolev embedding theorem, and is therefore the highest order positive term in (5.4.16) when $\tau = 1$ and $p > \frac{3}{2}$. Since $p = 0$ in Proposition II, this term can be dropped there without weakening the result.

At the heart of Proposition II$^+$ are two lemmas:

- Lemma 5.4.2 will be an identity for the quantity $G^{ij}\nabla_i\nabla_j R$ that results from differentiating a formula for $\sigma_2(A_\tau)$ twice. This identity holds on any 3-manifold (i.e. we don't need to use the equation (5.2.1)) and is a direct counterpart to Lemma 5.4 in [**CGY02b**].
- Lemma 5.4.3 will be an inequality obtained by differentiating our equation (5.2.1) and integrating by parts, and is a variation on Lemma 5.6 in [**CGY02b**].

Morally speaking, the identity in Lemma 5.4.2 will be a result of commuting covariant derivatives, so it will be helpful to recall the definitions of some terms which naturally arise in such calculations. The first such quantity is the *Cotton tensor*, defined

$$C_{ijk} = \nabla_k A_{ij} - \nabla_j A_{ik}.$$

The Cotton tensor plays a special role in three dimensions, where it is conformally invariant and vanishes if and only if the manifold is locally conformally flat. A related quantity is the Bach tensor, defined in $n \geq 3$ dimensions by

$$B_{ik} = \frac{1}{n-2}(\nabla^j C_{ijk} - \mathrm{Ric}^{jl} W_{ijkl}), \tag{5.4.17}$$

where $W$ is the Weyl tensor. When $n \geq 4$, the Weyl tensor vanishes if and only if the manifold is locally conformally flat, and is identically zero when $n = 3$. We refer the reader to [**Bes87**] for proofs of these well-known facts.

In [**CGY02b**], the conformal invariance of the Bach tensor is exploited, although this invariance is unique to four dimensions. That said, in three dimensions the vanishing of the Weyl tensor implies $B_{ik} = \nabla^j C_{ijk}$, from which a relatively simple conformal transformation law follows:

$$B_{ik} = e^{-2w}\Big((B_0)_{ik} + O\big(|\nabla_0 w|_0\big)\Big), \tag{5.4.18}$$

where $O(|\nabla_0 w|_0)$ denotes terms bounded above by $C(g_0)|\nabla_0 w|_0$. A short proof of (5.4.18) is provided in Appendix 5.B.

We now state and prove Lemma 5.4.2:

**Lemma 5.4.2.** *On any 3-manifold* $(\mathcal{M}^3, g)$,

$$\begin{aligned} G^{ij}\nabla_i\nabla_j R = 4\Delta\sigma_2(A_\tau) + 4\Big(|\nabla\mathring{\mathrm{Ric}}|^2 - q(\tau)|\nabla R|^2\Big) + 12\,\mathrm{tr}\,\mathring{\mathrm{Ric}}^3 + 2R|\mathring{\mathrm{Ric}}|^2 \\ - 4\mathring{\mathrm{Ric}}^{ij} B_{ij} - 4\Big(q(\tau) - \frac{1}{24}\Big)R\Delta R. \end{aligned} \tag{5.4.19}$$

*Proof.* Taking the viewpoint that (5.4.19) is an identity for $\Delta\sigma_2(A_\tau)$, we start by recalling (5.4.9) and computing

$$\begin{aligned}\Delta\sigma_2(A_\tau) &= \Delta\bigg(-\frac{|\mathring{\mathrm{Ric}}|^2}{2}+\frac{1}{2}q(\tau)R^2\bigg)\\ &= -\frac{1}{2}\Delta|\mathring{\mathrm{Ric}}|^2+q(\tau)R\Delta R+q(\tau)|\nabla R|^2\\ &= -\mathring{\mathrm{Ric}}^{ij}\Delta\mathring{\mathrm{Ric}}_{ij}-|\nabla\mathring{\mathrm{Ric}}|^2+q(\tau)R\Delta R+q(\tau)|\nabla R|^2. \qquad (5.4.20)\end{aligned}$$

The terms $|\nabla\mathring{\mathrm{Ric}}|^2$, $R\Delta R$ and $|\nabla R|^2$ are of a form already appearing in (5.4.19), so we focus on $-\mathring{\mathrm{Ric}}^{ij}\Delta\mathring{\mathrm{Ric}}_{ij}$. On the other hand, a routine calculation (see Appendix 5.B) yields

$$B_{ij}=3\,\mathrm{Ric}_i^{\ k}\,\mathrm{Ric}_{jk}-\frac{3R}{2}\,\mathrm{Ric}_{ij}-|\,\mathrm{Ric}\,|^2g_{ij}+\frac{1}{2}R^2g_{ij}+\frac{1}{4}\nabla_i\nabla_jR-\Delta A_{ij} \qquad (5.4.21)$$

which implies, after substituting in $\mathrm{Ric}=\mathring{\mathrm{Ric}}+\frac{Rg}{3}$ and $A=\mathring{\mathrm{Ric}}+\frac{Rg}{12}$, that

$$\begin{aligned}B_{ij} &= 3\bigg(\mathring{\mathrm{Ric}}_i^{\ k}+\frac{R}{3}\delta_i^{\ k}\bigg)\bigg(\mathring{\mathrm{Ric}}_{jk}+\frac{R}{3}g_{jk}\bigg)-\frac{3R}{2}\bigg(\mathring{\mathrm{Ric}}_{ij}+\frac{R}{3}g_{ij}\bigg)-\bigg(|\mathring{\mathrm{Ric}}|^2+\frac{R^2}{3}\bigg)g_{ij}\\ &\quad+\frac{1}{2}R^2g_{ij}+\frac{1}{4}\nabla_i\nabla_jR-\Delta\bigg(\mathring{\mathrm{Ric}}_{ij}+\frac{R}{12}g_{ij}\bigg)\\ &= -\Delta\mathring{\mathrm{Ric}}_{ij}+\frac{1}{4}\nabla_i\nabla_jR-\frac{1}{12}\Delta Rg_{ij}+3\mathring{\mathrm{Ric}}_i^{\ k}\mathring{\mathrm{Ric}}_{jk}-|\mathring{\mathrm{Ric}}|^2g_{ij}+\frac{R}{2}\mathring{\mathrm{Ric}}_{ij}. \qquad (5.4.22)\end{aligned}$$

Thus after contracting (5.4.22) with $\mathring{\mathrm{Ric}}^{ij}$ and using the fact that $\mathring{\mathrm{Ric}}$ is trace-free we get

$$-\mathring{\mathrm{Ric}}^{ij}\Delta\mathring{\mathrm{Ric}}_{ij}=-\frac{1}{4}\mathring{\mathrm{Ric}}^{ij}\nabla_i\nabla_jR-3\,\mathrm{tr}\,\mathring{\mathrm{Ric}}^3-\frac{1}{2}R|\mathring{\mathrm{Ric}}|^2+\mathring{\mathrm{Ric}}^{ij}B_{ij}. \qquad (5.4.23)$$

Substituting (5.4.23) back into (5.4.20) and using $\mathring{\mathrm{Ric}}=-G+\frac{Rg}{6}$ then gives

$$\begin{aligned}\Delta\sigma_2(A_\tau) &= -\frac{1}{4}\mathring{\mathrm{Ric}}^{ij}\nabla_i\nabla_jR-3\,\mathrm{tr}\,\mathring{\mathrm{Ric}}^3-\frac{1}{2}R|\mathring{\mathrm{Ric}}|^2+\mathring{\mathrm{Ric}}^{ij}B_{ij}-|\nabla\mathring{\mathrm{Ric}}|^2\\ &\quad+q(\tau)R\Delta R+q(\tau)|\nabla R|^2\\ &= \frac{1}{4}G^{ij}\nabla_i\nabla_jR-3\,\mathrm{tr}\,\mathring{\mathrm{Ric}}^3-\frac{1}{2}R|\mathring{\mathrm{Ric}}|^2+\mathring{\mathrm{Ric}}^{ij}B_{ij}-|\nabla\mathring{\mathrm{Ric}}|^2\\ &\quad+\bigg(q(\tau)-\frac{1}{24}\bigg)R\Delta R+q(\tau)|\nabla R|^2,\end{aligned}$$

which is precisely (5.4.19) after rearranging. ☐

We reiterate that at no point in the proof of Lemma 5.4.2 did we refer to the equation (5.2.1). In contrast, the inequality in the next lemma will be derived directly from (5.2.1) by differentiating (5.2.1), integrating by parts and carrying out some simple estimates:

**Lemma 5.4.3.** *Suppose* $(\mathcal{M}^3, g_0)$ *is a closed 3-manifold satisfying* $Y(\mathcal{M}^3,[g_0]) > 0$. *Then for given* $\tau \in [\frac{2}{3}, 1]$, $\delta \geq 0$ *and* $p \geq 0$, *and any* $C^4$ *solution* $g = e^{2w}g_0$ *to* (5.2.1) *with* $R > 0$, *one has*

$$\int 4R^p\Big(|\nabla\mathring{\text{Ric}}|^2 - q(\tau)|\nabla R|^2\Big)\,dv \geq \int \Big(2\delta R^{p-1}(\Delta R)^2 + 2\delta p R^{p-2}|\nabla R|^2\Delta R - 8R^{p-1}\langle\nabla R,\nabla f\rangle + 8fR^{p-2}|\nabla R|^2\Big)\,dv. \tag{5.4.24}$$

*Proof.* Again using (5.4.9), we can write (5.2.1) in the form

$$-\frac{|\mathring{\text{Ric}}|^2}{2} + \frac{1}{2}q(\tau)R^2 = \sigma_2(A_\tau) = \frac{\delta}{4}\Delta R + f, \tag{5.4.25}$$

which we differentiate to obtain

$$0 = \delta\nabla(\Delta R) + 4\nabla f + 4|\mathring{\text{Ric}}|\nabla|\mathring{\text{Ric}}| - 4q(\tau)R\nabla R.$$

Contracting with $R^{p-1}\nabla R$ and integrating, we get

$$0 = \int \Big(\delta R^{p-1}\langle\nabla R,\nabla(\Delta R)\rangle + 4R^{p-1}\langle\nabla R,\nabla f\rangle + 4R^{p-1}|\mathring{\text{Ric}}|\langle\nabla R,\nabla|\mathring{\text{Ric}}|\rangle - 4q(\tau)R^p|\nabla R|^2\Big)\,dv. \tag{5.4.26}$$

Integrating by parts on the first term in (5.4.26) gives

$$\int \delta R^{p-1}\langle\nabla R,\nabla(\Delta R)\rangle\,dv = \int\Big(-\delta R^{p-1}(\Delta R)^2 - \delta(p-1)R^{p-2}|\nabla R|^2\Delta R\Big)\,dv \tag{5.4.27}$$

and using Cauchy's inequality on the third term gives

$$\begin{aligned}\int 4R^{p-1}|\mathring{\text{Ric}}|\langle\nabla R,\nabla|\mathring{\text{Ric}}|\rangle\,dv &\leq \int\Big(2R^p|\nabla|\mathring{\text{Ric}}||^2 + 2R^{p-2}|\mathring{\text{Ric}}|^2|\nabla R|^2\Big)\,dv\\ &\leq \int\Big(2R^p|\nabla\mathring{\text{Ric}}|^2 + 2R^{p-2}|\mathring{\text{Ric}}|^2|\nabla R|^2\Big)\,dv,\end{aligned} \tag{5.4.28}$$

where to reach the second line we have applied Kato's inequality $|\nabla|\mathring{\mathrm{Ric}}||^2 \leq |\nabla\mathring{\mathrm{Ric}}|^2$. Rearranging (5.4.25) to get an equation for $|\mathring{\mathrm{Ric}}|^2$, we see that the last integral in (5.4.28) can be written as

$$\int 2R^{p-2}|\mathring{\mathrm{Ric}}|^2|\nabla R|^2\,dv = \int \Big( -\delta R^{p-2}|\nabla R|^2\Delta R - 4fR^{p-2}|\nabla R|^2 + 2q(\tau)R^p|\nabla R|^2 \Big)\,dv. \tag{5.4.29}$$

Substituting (5.4.29) into (5.4.28), and then both (5.4.28) and (5.4.27) into (5.4.26), we obtain

$$\begin{aligned}\int \Big( 2R^p|\nabla\mathring{\mathrm{Ric}}|^2 - 2q(\tau)R^p|\nabla R|^2 \Big)\,dv \geq \int \Big( &\delta R^{p-1}(\Delta R)^2 + \delta p R^{p-2}|\nabla R|^2\Delta R \\ &- 4R^{p-1}\langle\nabla R, \nabla f\rangle + 4fR^{p-2}|\nabla R|^2 \Big)\,dv,\end{aligned}$$

which is precisely (5.4.24) after multiplying both sides by two. □

We are now in a position to prove Proposition II$^+$, following [**CGY02b**] (see equation (6.10) therein):

*Proof of Proposition II$^+$.* Multiplying the identity (5.4.19) of Lemma 5.4.2 by $R^p$, integrating over $\mathcal{M}^3$ and integrating by parts on the last resulting term on the RHS, we see

$$\begin{aligned}\mathrm{I}_1^p &= (p+1)\int R^p G^{ij}\nabla_i\nabla_j R\,dv \\ &= (p+1)\int R^p\Big[4\Delta\sigma_2(A_\tau) + 4\Big(|\nabla\mathring{\mathrm{Ric}}|^2 - q(\tau)|\nabla R|^2\Big) + 12\,\mathrm{tr}\,\mathring{\mathrm{Ric}}^3 + 2R|\mathring{\mathrm{Ric}}|^2 \\ &\qquad - 4\mathring{\mathrm{Ric}}^{ij}B_{ij} + 4(p+1)\Big(q(\tau) - \frac{1}{24}\Big)|\nabla R|^2\Big]\,dv.\end{aligned} \tag{5.4.30}$$

We then integrate by parts twice on the first term in the square brackets in (5.4.30) and apply Lemma 5.4.3 to the second term in the square brackets, to obtain the lower bound

$$
\begin{aligned}
\mathrm{I}_1^p \geq (p+1) \int \Big[ & 4\Delta R^p \Big(\frac{\delta}{4}\Delta R + f\Big) + 2\delta R^{p-1}(\Delta R)^2 + 2\delta p R^{p-2}|\nabla R|^2 \Delta R \\
& - 8R^{p-1}\langle \nabla R, \nabla f\rangle + 8fR^{p-2}|\nabla R|^2 + 12R^p \operatorname{tr} \mathring{\mathrm{Ric}}^3 + 2R^{p+1}|\mathring{\mathrm{Ric}}|^2 \\
& - 4R^p \mathring{\mathrm{Ric}}^{ij} B_{ij} + 4(p+1)\Big(q(\tau) - \frac{1}{24}\Big) R^p |\nabla R|^2 \Big] \, dv \\
= (p+1) \int & \Big( \delta \Delta R \Delta R^p + 2\delta R^{p-1}(\Delta R)^2 + 2\delta p R^{p-2}|\nabla R|^2 \Delta R \Big) \, dv \\
& + (p+1) \int \Big( 4f\Delta R^p - 8R^{p-1}\langle \nabla R, \nabla f\rangle + 8fR^{p-2}|\nabla R|^2 + 12R^p \operatorname{tr} \mathring{\mathrm{Ric}}^3 \\
& + 2R^{p+1}|\mathring{\mathrm{Ric}}|^2 - 4R^p \mathring{\mathrm{Ric}}^{ij} B_{ij} \Big) \, dv + 4(p+1)^2 \int \Big( q(\tau) - \frac{1}{24}\Big) R^p |\nabla R|^2 \, dv \\
= \underbrace{\mathrm{I}_{1,\delta}^p + (p+1) \int 12R^p \operatorname{tr} \mathring{\mathrm{Ric}}^3 \, dv + 4(p+1)^2 \int \Big( q(\tau) - \frac{1}{24}\Big) R^p |\nabla R|^2 \, dv}_{(*)} & \\
+ (p+1) \int \Big( & \underbrace{-4R^p \mathring{\mathrm{Ric}}^{ij} B_{ij}}_{(1)} + \underbrace{4f\Delta R^p - 8R^{p-1}\langle \nabla R, \nabla f\rangle + 8fR^{p-2}|\nabla R|^2}_{(2)} \\
& + \underbrace{2R^{p+1}|\mathring{\mathrm{Ric}}|^2}_{(3)} \Big) \, dv, \qquad (5.4.31)
\end{aligned}
$$

where $\mathrm{I}_{1,\delta}^p$ is given in (5.4.14). Note that both equalities in (5.4.31) are just a result of rearranging and labelling terms.

The terms denoted by $(*)$ in the second equality appear in our desired estimate (5.4.16), so require no further work. Of the remaining terms, (3) is positive and of the highest order (formally it behaves like $|\nabla^2 w|^{p+3}$) whereas the components of (1) and (2) that are of unknown sign will be of lower order. For instance, we can see immediately from (5.4.18) that $B_{ij}$ does not contribute any second derivatives of $w$, and so any negative contribution from (1) formally behaves (at worst) like $|\nabla^2 w|^{p+1}|\nabla w|$. After integrating by parts and applying Young's inequality, we will see that any negative contribution of (2) formally behaves (at worst) like $|\nabla^2 w|^p |\nabla w|^2$. We carry out these estimates rigorously now.

**(1)** We start with an application of Young's inequality, which yields

$$R^p \mathring{\mathrm{Ric}}^{ij} B_{ij} \leq C\Big(R^{p+2} + |\mathring{\mathrm{Ric}}|^{p+2} + |B|^{p+2}\Big). \tag{5.4.32}$$

First consider $|\mathring{\mathrm{Ric}}|^{p+2}$. By our assumed $C^0$ estimate we have $|\mathring{\mathrm{Ric}}| \leq C|\mathring{\mathrm{Ric}}|_0$, and combining (5.3.1) with (5.3.2) also we see

$$|\mathring{\mathrm{Ric}}|_0 \leq C\big(|\nabla_0^2 w|_0 + |\Delta_0 w| + |\nabla_0 w|_0^2 + 1\big). \tag{5.4.33}$$

Therefore

$$\begin{aligned}\int |\mathring{\mathrm{Ric}}|^{p+2}\, dv &\leq C\int |\mathring{\mathrm{Ric}}|_0^{p+2}\, dv_0 \\ &\leq C\int \big(|\nabla_0^2 w|_0^{p+2} + |\Delta_0 w|^{p+2} + |\nabla_0 w|_0^{2(p+2)} + 1\big)\, dv_0 \\ &\leq C\int |\Delta_0 w|^{p+2}\, dv_0 + C, \end{aligned}\tag{5.4.34}$$

the last line following from standard elliptic theory applied to the Hessian term, and the assumption $p \leq \frac{1}{2}(r-4)$ applied to the gradient term. Now, by the transformation law (5.3.1), we also have

$$|\Delta_0 w| \leq C\big(R + |\nabla_0 w|_0^2 + 1\big),$$

which when combined with (5.4.34) implies

$$\begin{aligned}\int |\mathring{\mathrm{Ric}}|^{p+2}\, dv &\leq C\int \big(R^{p+2} + |\nabla_0 w|_0^{2(p+2)}\big)\, dv_0 + C \\ &\leq C\int R^{p+2}\, dv + C. \end{aligned}\tag{5.4.35}$$

Next consider $|B|^{p+2}$. This is dealt with in the same way as $|\mathring{\mathrm{Ric}}|^{p+2}$, although the estimate is simpler: we have $|B| \leq C|B|_0$, and the transformation law (5.4.18) implies

$$|B|_0 \leq C|\nabla_0 w|_0 + C,$$

so it follows that

$$\int |B|^{p+2}\, dv \leq C\int |B|_0^{p+2}\, dv_0 \leq C\int |\nabla_0 w|_0^{p+2}\, dv_0 + C \leq C. \tag{5.4.36}$$

Combining (5.4.32), (5.4.35) and (5.4.36), it follows that

$$\int -4R^p \mathring{\mathrm{Ric}}^{ij} B_{ij}\, dv \geq -C \int R^{p+2}\, dv - C. \tag{5.4.37}$$

**(2)** To deal with (2), we integrate by parts (IBP), collect terms and apply Cauchy's inequality (CI) followed by Young's inequality (YI) to get

$$\begin{aligned}
&(p+1)\int \left(4f\Delta R^p - 8R^{p-1}\langle \nabla R, \nabla f\rangle + 8fR^{p-2}|\nabla R|^2\right) dv \\
&\overset{\text{IBP}}{=} (p+1)\int \Big( - 4pR^{p-1}\langle \nabla R, \nabla f\rangle - 8R^{p-1}\langle \nabla R, \nabla f\rangle + 8fR^{p-2}|\nabla R|^2\Big)\, dv \\
&= (p+1)\int \Big( - 8(p+2)f^{1/2}R^{p-1}\langle \nabla R, \nabla f^{1/2}\rangle + 8fR^{p-2}|\nabla R|^2\Big)\, dv \\
&\overset{\text{CI}}{\geq} (p+1)\int \left[-8(p+2)f^{1/2}R^{p-1}\left(\frac{f^{1/2}|\nabla R|^2}{2(p+2)R} + \frac{R(p+2)\left|\nabla f^{1/2}\right|^2}{2f^{1/2}}\right) + 8fR^{p-2}|\nabla R|^2\right] \\
&= (p+1)\int -\left(4(p+2)^2R^p\left|\nabla f^{1/2}\right|^2 + 4fR^{p-2}|\nabla R|^2\right) dv \\
&\geq -C\int R^p\left|\nabla f^{1/2}\right|^2 dv + \frac{1}{C}\int R^{p-2}|\nabla R|^2\, dv \\
&\overset{\text{YI}}{\geq} -C\int R^{p+2}\, dv - C\int \left|\nabla f^{1/2}\right|^{p+2} dv + \frac{1}{C}\int R^{p-2}|\nabla R|^2\, dv \\
&\geq -C\int R^{p+2}\, dv + \frac{1}{C}\int R^{p-2}|\nabla R|^2\, dv - C.
\end{aligned} \tag{5.4.38}$$

Note that we have used to uniform positivity of $f$ to retain the integral $\int R^{p-2}|\nabla R|^2\, dv$ in (5.4.38).

**(3)** Finally, to deal with (3) observe that in view of (5.4.25),

$$\begin{aligned}
\int 2R^{p+1}|\mathring{\mathrm{Ric}}|^2\, dv &= \int 2R^{p+1}\left(q(\tau)R^2 - \frac{\delta}{2}\Delta R - 2f\right) dv \\
&= \int 2q(\tau)R^{p+3}\, dv + \delta(p+1)\int R^p|\nabla R|^2\, dv - 4\int fR^{p+1}\, dv \\
&\geq 2q(\tau)\int R^{p+3}\, dv - C\int R^{p+1}\, dv \\
&\geq 2q(\tau)\int R^{p+3}\, dv - C\int R^{p+2}\, dv - C.
\end{aligned} \tag{5.4.39}$$

Substituting (5.4.37), (5.4.38) and (5.4.39) into (5.4.31) then gives the desired estimate (5.4.16). □

We conclude this section with the following corollary, which is essentially a result of taking $p = 0$ in the above calculations, and will be the basis for our estimate in the next section:

**Corollary 5.4.4.** *Under the same hypotheses as Proposition II,*

$$0 \geq \int \left( 2\delta \frac{(\Delta R)^2}{R} + 12 \operatorname{tr} \mathring{\mathrm{Ric}}^3 + 2R|\mathring{\mathrm{Ric}}|^2 - CR^2 - C \right) dv. \tag{5.4.40}$$

*Proof.* Integrating the identity (5.4.19) of Lemma 5.4.2 and using the fact that $G$ is divergence free yields

$$\begin{aligned} 0 = \int \Big( & 4\Big(|\nabla \mathring{\mathrm{Ric}}|^2 - q(\tau)|\nabla R|^2\Big) + 12 \operatorname{tr} \mathring{\mathrm{Ric}}^3 + 2R|\mathring{\mathrm{Ric}}|^2 - 4\mathring{\mathrm{Ric}}^{ij} B_{ij} \\ & - 4\Big(q(\tau) - \frac{1}{24}\Big) R\Delta R \Big) \, dv. \end{aligned} \tag{5.4.41}$$

The terms $12 \operatorname{tr} \mathring{\mathrm{Ric}}^3$ and $2R|\mathring{\mathrm{Ric}}|^2$ are already of a form appearing in (5.4.40), so we do not address these further. Moreover, the final term in (5.4.41) can be integrated by parts to give a positive term, so can be neglected. This leaves $-4\mathring{\mathrm{Ric}}^{ij} B_{ij}$ and $4(|\nabla \mathring{\mathrm{Ric}}|^2 - q(\tau)|\nabla R|^2)$, which are also easy to deal with: taking $p = 0$ in (5.4.37) gives

$$\int -4\mathring{\mathrm{Ric}}^{ij} B_{ij} \, dv \geq -C \int R^2 \, dv - C, \tag{5.4.42}$$

and similarly taking $p = 0$ in Lemma 5.4.3 yields

$$\begin{aligned} \int 4\Big(|\nabla \mathring{\mathrm{Ric}}|^2 - q(\tau)|\nabla R|^2\Big) \, dv &\geq \int \left( 2\delta \frac{(\Delta R)^2}{R} - 8R^{-1}\langle \nabla R, \nabla f \rangle + 8fR^{-2}|\nabla R|^2 \right) dv \\ &\geq \int 2\delta \frac{(\Delta R)^2}{R} \, dv - C, \end{aligned} \tag{5.4.43}$$

where we have estimated the tail terms in (5.4.43) as in (5.4.38). Substituting (5.4.42) and (5.4.43) back into (5.4.41), we arrive at (5.4.40). □

### 5.4.4 An integral bound for $\delta|\nabla R|^2$

In this transitional section, we prove an integral bound for $\delta|\nabla R|^2$ which will come into use when we prove Proposition III in the next section. We present this estimate separately, as the proof is again based on manipulating $G^{ij}\nabla_i\nabla_j R$ (as in the proof of Proposition II), although the result is only needed for proving Proposition III. The reader may wish to skim over the details for now and refer back to the result when it is utilised later.

**Lemma 5.4.5.** *Under the same hypotheses as Proposition II,*

$$\delta\int|\nabla R|^2\,dv\leq C\int\big(\delta R^3+R^2+1\big)\,dv. \tag{5.4.44}$$

The proof of Lemma 5.4.5 stems from the estimate (5.4.40) of Corollary 5.4.4, and the idea is as follows. First note that by integrating by parts and applying Cauchy's inequality, we have

$$\int\delta|\nabla R|^2\,dv=-\int\delta R\Delta R\,dv\leq\int\left(\delta\frac{(\Delta R)^2}{R}+\frac{1}{4}\delta R^3\right)dv,$$

from which we see that

$$\int\delta\frac{(\Delta R)^2}{R}\,dv\geq\int\left(\delta|\nabla R|^2-\frac{1}{4}\delta R^3\right)dv. \tag{5.4.45}$$

Substituting (5.4.45) into (5.4.40) then yields

$$\delta\int|\nabla R|^2\,dv+\int(12\operatorname{tr}\mathring{\mathrm{Ric}}^3+2R|\mathring{\mathrm{Ric}}|^2)\,dv\geq C\int(\delta R^3+R^2+1)\,dv,$$

which is the desired estimate (5.4.44), except for the integral $\int(12\operatorname{tr}\mathring{\mathrm{Ric}}^3+2R|\mathring{\mathrm{Ric}}|^2)\,dv$ on the LHS. The bulk of the proof of Lemma 5.4.5 is then dedicated to estimating this integral, so as to create terms only of a form appearing in (5.4.44).

*Proof of Lemma 5.4.5.* As indicated above, we focus first on estimating $\int(12\operatorname{tr}\mathring{\mathrm{Ric}}^3+2R|\mathring{\mathrm{Ric}}|^2)\,dv$. Since $\mathring{\mathrm{Ric}}$ is trace-free we have the inequality

$$\operatorname{tr}\mathring{\mathrm{Ric}}^3\geq-\frac{1}{\sqrt{6}}|\mathring{\mathrm{Ric}}|^3,$$

(see [**GV01, Lemma 4.2**] for a proof), and thus

$$12\operatorname{tr}\mathring{\mathrm{Ric}}^3+2R|\mathring{\mathrm{Ric}}|^2\geq|\mathring{\mathrm{Ric}}|^2\big(-2\sqrt{6}|\mathring{\mathrm{Ric}}|+2R\big). \tag{5.4.46}$$

But by Cauchy's inequality,

$$-2\sqrt{6}|\overset{\circ}{\mathrm{Ric}}| = -2\Big(\sqrt{6}|\overset{\circ}{\mathrm{Ric}}|R^{-\frac{1}{2}}\Big)R^{\frac{1}{2}} \geq -6|\overset{\circ}{\mathrm{Ric}}|^2R^{-1} - R$$

and substituting this into (5.4.46) we see

$$\begin{aligned}\int \big(12\,\mathrm{tr}\,\overset{\circ}{\mathrm{Ric}}^3 + 2R|\overset{\circ}{\mathrm{Ric}}|^2\big)\,dv &\geq \int \frac{|\overset{\circ}{\mathrm{Ric}}|^2}{R}\Big( - 6|\overset{\circ}{\mathrm{Ric}}|^2 + R^2\Big)\,dv \\ &\geq \int \frac{|\overset{\circ}{\mathrm{Ric}}|^2}{R}\Big( - 6|\overset{\circ}{\mathrm{Ric}}|^2 + 6q(\tau)R^2\Big)\,dv \\ &\overset{(5.4.25)}{=} \int \frac{|\overset{\circ}{\mathrm{Ric}}|^2}{R}\big(3\delta\Delta R + 12f\big)\,dv \geq \int 3\delta\frac{\Delta R}{R}|\overset{\circ}{\mathrm{Ric}}|^2\,dv.\end{aligned} \tag{5.4.47}$$

To reach the middle line in (5.4.47), we have used the fact that $q(\tau) = \frac{3}{8}(\tau - \frac{4}{3})^2$ and the assumption $\tau \in [\frac{2}{3}, 1]$, which implies $q(\tau) \leq \frac{1}{6}$. Next, Cauchy's inequality again gives

$$\begin{aligned}\int 3\delta\frac{\Delta R}{R}|\overset{\circ}{\mathrm{Ric}}|^2\,dv &\geq \int \Big( - \delta\frac{(\Delta R)^2}{4R} - 9\delta\frac{|\overset{\circ}{\mathrm{Ric}}|^4}{R}\Big)\,dv \\ &\overset{(5.4.25)}{=} \int \Big[ - \delta\frac{(\Delta R)^2}{4R} - 9\delta\frac{|\overset{\circ}{\mathrm{Ric}}|^2}{R}\Big(q(\tau)R^2 - \frac{\delta}{2}\Delta R - 2f\Big)\Big]\,dv \\ &\geq \int \Big( - \delta\frac{(\Delta R)^2}{4R} + \frac{9\delta^2}{2}\frac{\Delta R}{R}|\overset{\circ}{\mathrm{Ric}}|^2 - 9q(\tau)\delta R|\overset{\circ}{\mathrm{Ric}}|^2\Big)\,dv,\end{aligned}$$

which after collecting terms becomes

$$\int 3\delta\Big(1 - \frac{3\delta}{2}\Big)\frac{\Delta R}{R}|\overset{\circ}{\mathrm{Ric}}|^2\,dv \geq \int \Big( - \delta\frac{(\Delta R)^2}{4R} - 9q(\tau)\delta R|\overset{\circ}{\mathrm{Ric}}|^2\Big)\,dv.$$

Therefore for $\delta \in [0, \frac{2}{3})$,

$$\int 3\delta\frac{\Delta R}{R}|\overset{\circ}{\mathrm{Ric}}|^2\,dv \geq \int \Big[ - \delta\Big(1 - \frac{3\delta}{2}\Big)^{-1}\frac{(\Delta R)^2}{4R} - 9\delta q(\tau)\Big(1 - \frac{3\delta}{2}\Big)^{-1}R|\overset{\circ}{\mathrm{Ric}}|^2\Big]\,dv,$$

and substituting this into (5.4.47) then yields

$$\begin{aligned}&\int \big(12\,\mathrm{tr}\overset{\circ}{\mathrm{Ric}}^3 + 2R|\overset{\circ}{\mathrm{Ric}}|^2\big)\,dv \\ &\qquad\geq \int \Big[ - \delta\Big(1 - \frac{3\delta}{2}\Big)^{-1}\frac{(\Delta R)^2}{4R} - 9\delta q(\tau)\Big(1 - \frac{3\delta}{2}\Big)^{-1}R|\overset{\circ}{\mathrm{Ric}}|^2\Big]\,dv.\end{aligned} \tag{5.4.48}$$

The next step is to substitute (5.4.48) into the estimate (5.4.40) of Corollary 5.4.4. After collecting terms, this reads

$$0 \geq \int \left[ \delta \frac{7-12\delta}{2-3\delta} \frac{(\Delta R)^2}{2R} - 9\delta q(\tau) \left( 1 - \frac{3\delta}{2} \right)^{-1} R |\mathring{\mathrm{Ric}}|^2 - CR^2 - C \right] dv, \tag{5.4.49}$$

which can be simplified by observing that for sufficiently small $\delta$, one has $\delta \frac{7-12\delta}{2-3\delta} \geq 2\delta$ and $9\delta q(\tau)(1 - \frac{3\delta}{2})^{-1} \leq 2\delta$. It follows that for sufficiently small $\delta$, (5.4.49) implies

$$0 \geq \int \left( \delta \frac{(\Delta R)^2}{R} - 2\delta R |\mathring{\mathrm{Ric}}|^2 - CR^2 - C \right) dv. \tag{5.4.50}$$

It remains to estimate $\int R|\mathring{\mathrm{Ric}}|^2\, dv$ and $\int \delta \frac{(\Delta R)^2}{R}\, dv$ in (5.4.50), in such a way so as to produce terms of the form appearing in (5.4.44). First observe that

$$\int \delta |\nabla R|^2 \, dv = - \int \delta R \Delta R \, dv \overset{(5.4.25)}{=} \int R \left( 4f + 2|\mathring{\mathrm{Ric}}|^2 - 2q(\tau) R^2 \right) dv \tag{5.4.51}$$

and thus

$$\int R |\mathring{\mathrm{Ric}}|^2 \, dv \leq \int \left( \frac{1}{2} \delta |\nabla R|^2 + q(\tau) R^3 \right) dv. \tag{5.4.52}$$

Similarly, applying Cauchy's inequality instead of (5.4.25) in (5.4.51) yields

$$\int \delta |\nabla R|^2 \, dv = - \int \delta R \Delta R \, dv \leq \int \left( \delta \frac{(\Delta R)^2}{R} + \frac{1}{4} \delta R^3 \right) dv$$

and thus

$$\int \delta \frac{(\Delta R)^2}{R} \, dv \geq \int \left( \delta |\nabla R|^2 - \frac{1}{4} \delta R^3 \right) dv. \tag{5.4.53}$$

Substituting (5.4.52) and (5.4.53) into (5.4.50) then gives

$$0 \geq \int \left( \delta(1-\delta) |\nabla R|^2 - C\delta R^3 - CR^2 - C \right) dv,$$

which implies (5.4.44) for small $\delta$. □

### 5.4.5 Proof of Proposition III: an integral bound for $\mathrm{tr}\, \mathring{\mathrm{Ric}}^3$

The estimate (5.4.1) in Proposition II gives us an $L^2$ bound for $\nabla R$ when $\tau < 1$ and an $L^3$ bound for $R$ when $\tau = 1$. However, in their current state these bounds are not useful, due to the presence of the term $\operatorname{tr}\overset{\circ}{\mathrm{Ric}}^3$ which is of unknown sign. So one needs a bound for $\int \operatorname{tr}\overset{\circ}{\mathrm{Ric}}^3\, dv$, which is achieved in the estimate (5.4.2) of Proposition III (we refer the reader back to the proof of Theorem E′ to see how the estimates (5.4.1) and (5.4.2) then fit together).

In this section we prove the estimate (5.4.2), which is the most technical step in the proof of Theorem E′. We would still like to exploit the divergence structure of the Einstein tensor $G$, say be replacing the scalar curvature in $G^{ij}\nabla_i\nabla_j R$ by some other smooth function $V$. It turns out that $V = \frac{1}{2}|\nabla w|^2$ is a good choice, in the sense that this will produce a *negative* $\operatorname{tr}\overset{\circ}{\mathrm{Ric}}^3$ term which we can use to cancel out the positive one appearing in (5.4.2). Integrating by parts, we now have

$$\int G^{ij}\nabla_i\nabla_j V\, dv = 0, \tag{5.4.54}$$

and it is this identity that the estimate (5.4.2) will be based off.

Now, in a similar vein to (5.4.13), one could be tempted to consider more generally the quantity $G^{ij}\nabla_i(R^p\nabla_j V)$ for $p \geq 0$, so that

$$\begin{aligned} 0 &= \int G^{ij}\nabla_i(R^p\nabla_j V)\, dv \\ &= p\int R^{p-1}G^{ij}\nabla_i R\nabla_j V\, dv + \int R^p G^{ij}\nabla_i\nabla_j V\, dv \\ &=: \mathrm{II}_1^p + \mathrm{II}_2^p\,. \end{aligned} \tag{5.4.55}$$

Then since $\mathrm{II}_2^0 = \int G^{ij}\nabla_i\nabla_j V\, dv$, any estimate for $\mathrm{II}_2^p$ would give an estimate for the integral in (5.4.54) by taking $p = 0$. However, we cannot seem to obtain a counterpart to Proposition II$^+$ for $\mathrm{II}_2^p$ which is useful given anything less than or equal to a $W^{1,6}$ starting estimate. This is in contrast to Proposition II$^+$ which is clearly applicable from the outset (indeed we only required $r \geq 4$ in the hypotheses). Therefore, in this section we settle for proving Proposition III as it is, and refer the reader to §5.5.1 for its higher order version (Proposition III$^+$).

The first step is a purely algebraic result, giving a lower bound for $G^{ij}\nabla_i\nabla_j V$:

**Lemma 5.4.6.** *On any 3-manifold* $(\mathcal{M}^3, g)$ *where* $g = e^{2w}g_0$,

$$G^{ij}\nabla_i\nabla_j V \geq -\operatorname{tr}\mathring{\operatorname{Ric}}^3 + \frac{1}{288}R^3 - \frac{1}{8}R|\nabla w|^4 - \langle \nabla w, \nabla\sigma_2(A)\rangle - G^{ij}\nabla_i|\nabla w|^2\nabla_j w$$
$$- C|\operatorname{Ric}|^2 - C|\operatorname{Ric}||\nabla w|^2 - C. \tag{5.4.56}$$

The first three terms on the RHS of (5.4.56) are already of a form appearing in the desired estimate (5.4.2), and the final three terms (i.e. those on the bottom line of (5.4.56)) are clearly of lower order. So it remains to estimate the two middle terms, and for this purpose we will use the equation (5.2.1). The estimates are as follows.

**Lemma 5.4.7.** *Under the same hypotheses as Proposition II,*

$$\int \big( - \langle \nabla w, \nabla\sigma_2(A)\rangle - G^{ij}\nabla_i|\nabla w|^2\nabla_j w\big)\, dv$$
$$\geq \int \bigg( - C\delta|\nabla^2 w|^2|\nabla w|^2 - \frac{1}{4}R|\nabla w|^4 - C|\operatorname{Ric}||\nabla w|^2 - C|\operatorname{Ric}|^2 - C\bigg)\, dv. \tag{5.4.57}$$

Now all but the first term on the RHS of (5.4.57) are either of a form appearing in (5.4.2) or of lower order, so it remains to estimate this term. This is achieved in the final lemma of the section:

**Lemma 5.4.8.** *Under the same hypotheses as Proposition III,*

$$\int -\delta|\nabla^2 w|^2|\nabla w|^2\, dv \geq C\int \big( - \delta R^3 - R^2 - 1\big)\, dv. \tag{5.4.58}$$

Assuming Lemmas 5.4.6–5.4.8 for now, we see that Proposition III follows relatively quickly:

*Proof of Proposition III.* Substituting (5.4.58) into (5.4.57), then (5.4.57) into (5.4.56) (after integrating both sides) we obtain

$$\int G^{ij}\nabla_i\nabla_j V\, dv \geq \int \bigg( -\operatorname{tr}\mathring{\operatorname{Ric}}^3 + \frac{1}{288}R^3 - \frac{3}{8}R|\nabla w|^4 - C|\operatorname{Ric}||\nabla w|^2$$
$$- C|\operatorname{Ric}|^2 - C\delta R^3 - CR^2 - C\bigg)\, dv. \tag{5.4.59}$$

But in a similar vein to the calculations in (5.4.33)–(5.4.35), we have

$$\int -C|\operatorname{Ric}|^2\,dv \geq \int \big(-CR^2 - C\big)\,dv$$

and

$$\begin{aligned}\int -|\operatorname{Ric}||\nabla w|^2\,dv &\geq \int \big(-|\operatorname{Ric}|^2 - |\nabla w|^4\big)\,dv\\ &\geq \int \big(-CR^2 - C\big)\,dv.\end{aligned}$$

Combining these estimates with (5.4.59) then implies

$$\int G^{ij}\nabla_i\nabla_j V\,dv \geq \int \left(-\operatorname{tr}\mathring{\operatorname{Ric}}^3 + \frac{R^3}{C} - CR|\nabla w|^4 - C\delta R^3 - CR^2 - C\right) dv,$$

as required. □

For the remainder of §5.4.5, we focus on proving Lemmas 5.4.6–5.4.8.

#### 5.4.5.1 Proof of Lemma 5.4.6

The proof of Lemma 5.4.6 consists of a series of computations. We will begin (under the heading **(0)** below) by showing that

$$\begin{aligned}G^{ij}\nabla_i\nabla_j V = &\underbrace{G^{ij}\nabla_i\nabla^k w\nabla_j\nabla_k w}_{\mathrm{II}_1}\underbrace{-G^{ij}\nabla^k w\nabla_k A_{ij}}_{\mathrm{II}_2} + \underbrace{\frac{R}{4}\langle\nabla w, \nabla|\nabla w|^2\rangle}_{\mathrm{II}_3}\\ &+\underbrace{R_{ikjm}G^{ij}\nabla^m w\nabla^k w}_{\mathrm{II}_4}\underbrace{-G^{ij}\nabla_i|\nabla w|^2\nabla_j w}_{\mathrm{II}_5}+\underbrace{G^{ij}\nabla^k w\nabla_k A^0_{ij}}_{\mathrm{II}_6},\end{aligned} \tag{5.4.60}$$

which will follow from directly computing $\nabla_i\nabla_j V$ in terms of $w$.

Inspecting the terms in (5.4.60) we see that $\mathrm{II}_5$ is already of a form appearing in (5.4.56), so this term will need no further work. Furthermore, $\mathrm{II}_6$ is of lower order compared to the remaining terms, so can also be left alone. But by continuing to substitute various transformation laws and identities into $\mathrm{II}_1$–$\mathrm{II}_4$ (dealt with under the headings **(1)**-**(4)** in the proof below), we will start to see some of the other desired terms in (5.4.56) appearing. In particular, we will have

$$\begin{aligned}
\mathrm{II}_1 &= -\operatorname{tr}\mathring{\mathrm{Ric}}^3 + \frac{1}{288}R^3 + \frac{R}{8}|\nabla w|^4 + \cdots,\\
\mathrm{II}_2 &= -\langle \nabla w, \nabla\sigma_2(A)\rangle,\\
\mathrm{II}_3 &= -\frac{R}{4}|\nabla w|^4 + \cdots,
\end{aligned}$$

whereas $\mathrm{II}_4$ will completely cancel with terms appearing elsewhere, which we indicate using boxes around such terms. It will then remain to estimate the lower order terms (carried out under the heading **(5)**), which will complete the proof.

*Proof of Lemma 5.4.6.* **(0)** We start by deriving (5.4.60). Differentiating $V = \frac{1}{2}|\nabla w|^2$ once gives

$$\nabla_j V = \frac{1}{2}\nabla_j(\nabla^k w \nabla_k w) = \nabla_j\nabla_k w \nabla^k w,$$

and differentiating again we obtain

$$\nabla_i\nabla_j V = \nabla_i\nabla^k w \nabla_j\nabla_k w + \nabla_i\nabla_j\nabla_k w\nabla^k w.$$

Now, by symmetry of the Hessian and standard formulas for commuting covariant derivatives,

$$\nabla_i\nabla_j\nabla_k w = \nabla_i\nabla_k\nabla_j w = \nabla_k\nabla_i\nabla_j w + R_{ikjm}\nabla^m w$$

and thus

$$\nabla_i\nabla_j V = \nabla_i\nabla^k w\nabla_j\nabla_k w + \nabla_k\nabla_i\nabla_j w\nabla^k w + R_{ikjm}\nabla^m w\nabla^k w. \tag{5.4.61}$$

Roughly speaking, our approach throughout this proof is to replace second and third order derivatives of $w$ appearing in terms such as (5.4.61) by more relevant tensors such as the Schouten tensor. We start by considering $\nabla_k\nabla_i\nabla_j w$ in (5.4.61). By combining (5.3.3) and (5.3.4), the Schouten tensor $A = \mathrm{Ric} - \frac{R}{4}g$ transforms as

$$A = A_0 - \nabla^2 w - dw\otimes dw + \frac{1}{2}|\nabla w|^2 g,$$

and rearranging this we see

$$\nabla_i \nabla_j w = A^0_{ij} - A_{ij} - \nabla_i w \nabla_j w + \frac{1}{2}|\nabla w|^2 g_{ij}. \tag{5.4.62}$$

Therefore

$$\nabla_k \nabla_i \nabla_j w = \nabla_k A^0_{ij} - \nabla_k A_{ij} - \nabla_k \nabla_i w \nabla_j w - \nabla_i w \nabla_k \nabla_j w + \frac{1}{2}\nabla_k |\nabla w|^2 g_{ij},$$

which when substituted into (5.4.61) yields

$$\begin{aligned} G^{ij}\nabla_i\nabla_j V = G^{ij}\Big[&\nabla_i \nabla^k w \nabla_j \nabla_k w + \nabla_k A^0_{ij}\nabla^k w - \nabla_k A_{ij}\nabla^k w - \nabla_k \nabla_i w \nabla_j w \nabla^k w \\ &- \nabla_i w \nabla^k w \nabla_k \nabla_j w + \frac{1}{2}\nabla_k |\nabla w|^2 \nabla^k w\, g_{ij} + R_{ikjm}\nabla^m w \nabla^k w\Big] \\ = G^{ij}\Big[&\nabla_i \nabla^k w \nabla_j \nabla_k w + \nabla_k A^0_{ij}\nabla^k w - \nabla_k A_{ij}\nabla^k w - \nabla_i |\nabla w|^2 \nabla_j w \\ &+ \frac{1}{2}\langle \nabla w, \nabla |\nabla w|^2\rangle\, g_{ij} + R_{ikjm}\nabla^m w \nabla^k w\Big], \end{aligned}$$

which is precisely (5.4.60) after rearranging.

Now we address the terms $\mathrm{II}_1$–$\mathrm{II}_4$:

**(1)** Again using the transformation law (5.4.62) for the Schouten tensor, we have

$$\begin{aligned} \mathrm{II}_1 &= G^{ij}\nabla_i \nabla^k w \nabla_j \nabla_k w \\ &= G^{ij}\Big(-A_i{}^k + (A^0)_i{}^k - \nabla_i w \nabla^k w + \frac{1}{2}|\nabla w|^2 \delta_i{}^k\Big)\Big(-A_{jk} + A^0_{jk} - \nabla_j w \nabla_k w + \frac{1}{2}|\nabla w|^2 g_{jk}\Big) \\ &= G^{ij}A_i{}^k A_{jk} - 2G^{ij}A_i{}^k A^0_{jk} + G^{ij}(A^0)_i{}^k A^0_{jk} + 2G^{ij}A_{jk}\nabla_i w \nabla^k w - 2G^{ij}(A^0)_i{}^k \nabla_j w \nabla_k w \\ &\quad - G^{ij}A_{ij}|\nabla w|^2 + G^{ij}A^0_{ij}|\nabla w|^2 + \frac{R}{8}|\nabla w|^4. \end{aligned} \tag{5.4.63}$$

Now,

$$\begin{aligned} G^{ij}A_i{}^k A_{jk} &= \Big(-\mathring{\mathrm{Ric}}^{ij} + \frac{R}{6}g^{ij}\Big)\Big(\mathring{\mathrm{Ric}}_i{}^k + \frac{R}{12}\delta_i{}^k\Big)\Big(\mathring{\mathrm{Ric}}_{jk} + \frac{R}{12}g_{jk}\Big) \\ &= -\mathring{\mathrm{Ric}}^{ij}\mathring{\mathrm{Ric}}_i{}^k\mathring{\mathrm{Ric}}_{jk} - \frac{R}{12}\mathring{\mathrm{Ric}}^{ij}\mathring{\mathrm{Ric}}_i{}^k g_{jk} - \frac{R}{12}\mathring{\mathrm{Ric}}^{ij}\mathring{\mathrm{Ric}}_{jk}\delta_i{}^k - \frac{R^2}{144}\mathring{\mathrm{Ric}}^{ij}\delta_i{}^k g_{jk} \\ &\quad + \frac{R}{6}\mathring{\mathrm{Ric}}_i{}^k\mathring{\mathrm{Ric}}_{jk}g^{ij} + \frac{R^2}{72}\mathring{\mathrm{Ric}}_i{}^k g^{ij}g_{jk} + \frac{R^2}{72}\mathring{\mathrm{Ric}}_{jk}g^{ij}\delta_i{}^k + \frac{R^3}{864}g^{ij}\delta_i{}^k g_{jk} \\ &= -\operatorname{tr}\mathring{\mathrm{Ric}}^3 + \frac{1}{288}R^3, \end{aligned} \tag{5.4.64}$$

where to reach the last line we have used the fact that $\mathring{\mathrm{Ric}}$ is trace-free to assert

$$\mathring{\mathrm{Ric}}^{ij}\delta_i{}^k g_{jk} = \mathring{\mathrm{Ric}}_i{}^k g^{ij} g_{jk} = \mathring{\mathrm{Ric}}_{jk} g^{ij}\delta_i{}^k = 0,$$

and also observed that

$$-\frac{R}{12}\mathring{\mathrm{Ric}}^{ij}\mathring{\mathrm{Ric}}_i{}^k g_{jk} - \frac{R}{12}\mathring{\mathrm{Ric}}^{ij}\mathring{\mathrm{Ric}}_{jk}\delta_i{}^k + \frac{R}{6}\mathring{\mathrm{Ric}}_i{}^k\mathring{\mathrm{Ric}}_{jk} g^{ij} = 0.$$

We also have

$$G^{ij}A_{ij} = \Big(-\mathrm{Ric}^{ij} + \frac{R}{2}g^{ij}\Big)\Big(\mathrm{Ric}_{ij} - \frac{R}{4}g_{ij}\Big) = 2\Big(-\frac{|\mathrm{Ric}|^2}{2} + \frac{3}{16}R^2\Big) \overset{(5.4.8)}{=} 2\sigma_2(A), \tag{5.4.65}$$

and substituting (5.4.64) and (5.4.65) back into (5.4.63) we arrive at

$$\begin{aligned}\mathrm{II}_1 = &-\mathrm{tr}\,\mathring{\mathrm{Ric}}^3 + \frac{1}{288}R^3 + \frac{R}{8}|\nabla w|^4 \boxed{+2G^{ij}A_{jk}\nabla_i w\nabla^k w - 2|\nabla w|^2\sigma_2(A)} \\ &- 2G^{ij}A_i{}^k A^0_{jk} + G^{ij}(A^0)_i{}^k A^0_{jk} - 2G^{ij}A^0_{ik}\nabla_j w\nabla^k w + G^{ij}A^0_{ij}|\nabla w|^2.\end{aligned}$$

**(2)** For $\mathrm{II}_2$, we simply observe that

$$G^{ij}\nabla_k A_{ij} = \frac{\partial\sigma_2(A)}{\partial A_{ij}}\nabla_k A_{ij} = \nabla_k\sigma_2(A).$$

It follows immediately that $\mathrm{II}_2 = -\nabla^k w\nabla_k\sigma_2(A) = -\langle\nabla w, \nabla\sigma_2(A)\rangle$.

**(3)** For $\mathrm{II}_3$, we again have by (5.4.62) that

$$\begin{aligned}\mathrm{II}_3 &= \frac{R}{4}\langle\nabla w, \nabla|\nabla w|^2\rangle = \frac{R}{2}\nabla^i\nabla^j w\nabla_i w\nabla_j w \\ &= \frac{R}{2}\Big((A^0)^{ij} - A^{ij} - \nabla^i w\nabla^j w + \frac{1}{2}|\nabla w|^2 g^{ij}\Big)\nabla_i w\nabla_j w \\ &= -\frac{R}{4}|\nabla w|^4 \boxed{-\frac{R}{2}A^{ij}\nabla_i w\nabla_j w} + \frac{R}{2}(A^0)^{ij}\nabla_i w\nabla_j w.\end{aligned}$$

**(4)** For $\mathrm{II}_4$, recall that the full curvature tensor decomposes as $\mathrm{Riem} = A \owedge g$ in three dimensions since the Weyl tensor is identically zero. Thus, using the definition of $\owedge$ in (2.1.11),

$$\begin{aligned}\mathrm{II}_4 &= \Big(g_{ij}A_{km} - g_{im}A_{jk} - g_{jk}A_{im} + g_{km}A_{ij}\Big)G^{ij}\nabla^m w\nabla^k w \\ &= \frac{R}{2}A^{ij}\nabla_i w\nabla_j w - 2G_i{}^k A_{jk}\nabla^i w\nabla^j w + A_{ij}G^{ij}|\nabla w|^2 \\ &\overset{(5.4.65)}{=} -2G_i{}^k A_{jk}\nabla^i w\nabla^j w + \frac{R}{2}A^{ij}\nabla_i w\nabla_j w + 2\sigma_2(A)|\nabla w|^2.\end{aligned}$$

**(5)** We now wrap up the argument. First observe that the boxed terms above cancel exactly with $\mathrm{II}_4$, so after substituting the above expressions for $\mathrm{II}_1$–$\mathrm{II}_4$ back into (5.4.60), it's easy to verify that cancellation of terms yields

$$\begin{aligned} G^{ij}\nabla_i\nabla_j V = &-\operatorname{tr}\mathring{\mathrm{Ric}}^3 + \frac{1}{288}R^3 - \frac{1}{8}R|\nabla w|^4 - \langle \nabla w, \nabla\sigma_2(A)\rangle - G^{ij}\nabla_i|\nabla w|^2\nabla_j w \\ &- 2G^{ij}A_i{}^k A^0_{jk} + G^{ij}(A^0)_i{}^k A^0_{jk} - 2G^{ij}(A^0)_i{}^k\nabla_j w\nabla^k w + G^{ij}A^0_{ij}|\nabla w|^2 \\ &+ G^{ij}\nabla^k w\nabla_k A^0_{ij} + \frac{R}{2}(A^0)^{ij}\nabla_i w\nabla_j w. \end{aligned} \tag{5.4.66}$$

Comparing (5.4.66) with (5.4.56), it remains to estimate the terms on the bottom two lines of (5.4.66). In three dimensions, $|A|^2 = |\operatorname{Ric}|^2 - \frac{5}{8}R^2$ and $|G|^2 = |\operatorname{Ric}|^2 - \frac{1}{4}R^2$, therefore $|A| \le |\operatorname{Ric}|$ and $|G| \le |\operatorname{Ric}|$. Furthermore, $|\nabla A^0| \le C|\nabla w| + C$ since $\nabla_k A^0_{ij} = \partial_k A^0_{ij} - \Gamma^m_{ik}A^0_{mj} - \Gamma^m_{jk}A^0_{im}$ and the Christoffel symbols $\Gamma^m_{jk}$ with respect to $g$ satisfy the transformation law

$$\Gamma^m_{jk} = (\Gamma^m_{jk})_0 + \delta^m_k\partial_j w + \delta^m_j\partial_k w - (g_0)^{ms}(g_0)_{jk}\partial_s w$$

(see e.g. [**Bes87**]). The estimate (5.4.56) then follows from (5.4.66) and the above estimates for $|A|, |G|$ and $|\nabla A^0|$ as in the proof of Proposition 5.16 in [**CGY02b**], namely because the last six terms of (5.4.66) then satisfy:

$$\begin{aligned} -2G^{ij}A_i{}^k A^0_{jk} &\ge -C|\mathrm{Ric}|^2, \\ G^{ij}(A^0)_i{}^k A^0_{jk} &\ge -C|\mathrm{Ric}|, \\ -2G^{ij}(A^0)_i{}^k\nabla_j w\nabla^k w &\ge -C|\mathrm{Ric}||\nabla w|^2, \\ G^{ij}A^0_{ij}|\nabla w|^2 &\ge -C|\mathrm{Ric}||\nabla w|^2, \\ G^{ij}\nabla^k w\nabla_k A^0_{ij} &\ge -C|\mathrm{Ric}||\nabla w| - C|\operatorname{Ric}||\nabla w|^2, \\ \frac{1}{2}R(A^0)^{ij}\nabla_i w\nabla_j w &\ge -C|\mathrm{Ric}||\nabla w|^2. \end{aligned}$$

□

#### 5.4.5.2 Proof of Lemma 5.4.7

*Proof of Lemma 5.4.7.* Recall that we want to prove the estimate (5.4.57):

$$\begin{aligned}\int \big( - \langle \nabla w, \nabla \sigma_2(A) \rangle - G^{ij} \nabla_i |\nabla w|^2 \nabla_j w \big)\, dv \\ \geq \int \Big( - C\delta |\nabla^2 w|^2 |\nabla w|^2 - \frac{1}{4} R |\nabla w|^4 - C |\operatorname{Ric}| |\nabla w|^2 - C |\operatorname{Ric}|^2 - C \Big)\, dv,\end{aligned}$$

which we carry out in three steps. The first step **(1)** will be an estimate for the integral $\int -\langle \nabla w, \nabla \sigma_2(A) \rangle \, dv$, the second **(2)** will be an estimate for $\int -G^{ij} \nabla_i |\nabla w|^2 \nabla_j w \, dv$, and in the final step **(3)** we combine these two estimates to obtain (5.4.57).

**(1)** Integrating by parts and using the transformation law (5.3.1) for scalar curvature, we compute

$$\begin{aligned}\int -\langle \nabla w, \nabla \sigma_2(A) \rangle \, dv &= \int \sigma_2(A) \Delta w \, dv \\ &= \int \sigma_2(A) \Big( - \frac{R}{4} + \frac{R_0}{4} e^{-2w} + \frac{1}{2} |\nabla w|^2 \Big)\, dv \\ &= \int \Big( - \frac{1}{4} R \sigma_2(A) + \frac{1}{4} R_0 e^{-2w} \sigma_2(A) + \frac{1}{2} |\nabla w|^2 \sigma_2(A) \Big)\, dv.\end{aligned} \tag{5.4.67}$$

Consider the first term on the last line of (5.4.67). Writing the equation (5.2.1) in the form

$$\sigma_2(A_g) = \frac{\delta}{4} \Delta_g R_g + f(x, w) - \frac{1}{2} \Big( q(\tau) - \frac{1}{24} \Big) R^2 \tag{5.4.68}$$

and using the fact that $q(\tau) \geq \frac{1}{24}$ for $\tau \leq 1$, we have

$$\begin{aligned}\int -\frac{1}{4} R \sigma_2(A) \, dv &= \int -\frac{1}{4} R \Big( \frac{\delta}{4} \Delta R + f - \frac{1}{2} \Big( q(\tau) - \frac{1}{24} \Big) R^2 \Big)\, dv \\ &= \int \Big( - \frac{\delta}{16} R \Delta R - \frac{1}{4} f R + \frac{1}{8} \Big( q(\tau) - \frac{1}{24} \Big) R^3 \Big)\, dv \\ &\geq \int \Big( \frac{\delta}{16} |\nabla R|^2 - \frac{1}{4} f R \Big)\, dv \\ &\geq \int \Big( \frac{\delta}{16} |\nabla R|^2 - C |\operatorname{Ric}|^2 - C \Big)\, dv.\end{aligned} \tag{5.4.69}$$

Similarly,

$$\begin{aligned}
\int \frac{1}{4} R_0 e^{-2w} \sigma_2(A)\, dv &\overset{(5.4.68)}{=} \int \frac{1}{4} R_0 e^{-2w} \Big( \frac{\delta}{4} \Delta R + f - \frac{1}{2} \Big( q(\tau) - \frac{1}{24} \Big) R^2 \Big)\, dv \\
&= \int \Big( \frac{\delta}{16} R_0 e^{-2w} \Delta R + \frac{1}{4} R_0 e^{-2w} f - \frac{1}{8} R_0 e^{-2w} \Big( q(\tau) - \frac{1}{24} \Big) R^2 \Big)\, dv \\
&= \int \Big[ - \frac{\delta}{16} \Big( e^{-2w} \langle \nabla R_0, \nabla R \rangle + R_0 \langle \nabla e^{-2w}, \nabla R \rangle \Big) \\
&\qquad + \frac{1}{4} R_0 e^{-2w} f - \frac{1}{8} R_0 e^{-2w} \Big( q(\tau) - \frac{1}{24} \Big) R^2 \Big]\, dv \\
&\geq \int \big( - C\delta |\nabla R| - C\delta |\nabla w| |\nabla R| - C |\operatorname{Ric}|^2 - C \big)\, dv \\
&\geq \int \Big( - \frac{\delta}{32} |\nabla R|^2 - C |\nabla w|^2 - C |\operatorname{Ric}|^2 - C \Big)\, dv \\
&\geq \int \Big( - \frac{\delta}{32} |\nabla R|^2 - C |\operatorname{Ric}|^2 - C \Big)\, dv. \qquad (5.4.70)
\end{aligned}$$

Substituting (5.4.69) and (5.4.70) into (5.4.67), we therefore get

$$\int -\langle \nabla w, \nabla \sigma_2(A) \rangle\, dv \geq \int \Big( \frac{\delta}{32} |\nabla R|^2 + \frac{1}{2} |\nabla w|^2 \sigma_2(A) - C |\operatorname{Ric}|^2 - C \Big)\, dv. \quad (5.4.71)$$

**(2)** Integrating by parts and using the fact that $G$ is divergence-free, we get

$$\begin{aligned}
\int -G^{ij} \nabla_i |\nabla w|^2 \nabla_j w\, dv &= \int |\nabla w|^2 G^{ij} \nabla_i \nabla_j w\, dv \\
&\overset{(5.4.62)}{=} \int |\nabla w|^2 G^{ij} \Big( A^0_{ij} - A_{ij} - \nabla_i w \nabla_j w + \frac{1}{2} |\nabla w|^2 g_{ij} \Big)\, dv \\
&\overset{(5.4.65)}{=} \int \Big( |\nabla w|^2 G^{ij} A^0_{ij} - 2\sigma_2(A) |\nabla w|^2 - |\nabla w|^2 G^{ij} \nabla_i w \nabla_j w + \frac{R}{4} |\nabla w|^4 \Big)\, dv \\
&= \int \big( |\nabla w|^2 G^{ij} A^0_{ij} - 2\sigma_2(A) |\nabla w|^2 + |\nabla w|^2 A^{ij} \nabla_i w \nabla_j w \big)\, dv, \qquad (5.4.72)
\end{aligned}$$

where to reach the final line we have used $-G = A - \frac{R}{4} g$. But since $|G| \leq |\operatorname{Ric}|$,

$$\int |\nabla w|^2 G^{ij} A^0_{ij}\, dv \geq \int -C |\operatorname{Ric}| |\nabla w|^2\, dv,$$

and substituting this into (5.4.72) therefore gives

$$\begin{aligned}
&\int -G^{ij} \nabla_i |\nabla w|^2 \nabla_j w\, dv \\
&\qquad \geq \int \big( - 2\sigma_2(A) |\nabla w|^2 + |\nabla w|^2 A^{ij} \nabla_i w \nabla_j w - C |\operatorname{Ric}| |\nabla w|^2 \big). \qquad (5.4.73)
\end{aligned}$$

**(3)** Combining (5.4.71) and (5.4.73) then yields

$$\int \big( -\langle \nabla w, \nabla \sigma_2(A) \rangle - G^{ij} \nabla_i |\nabla w|^2 \nabla_j w \big)\, dv$$

$$\geq \int \bigg( \frac{\delta}{32} |\nabla R|^2 - \frac{3}{2} |\nabla w|^2 \sigma_2(A) + |\nabla w|^2 A^{ij} \nabla_i w \nabla_j w$$

$$- C|\operatorname{Ric}||\nabla w|^2 - C|\operatorname{Ric}|^2 - C \bigg)\, dv. \tag{5.4.74}$$

The first three terms on the RHS of (5.4.74) are still not of a form seen in (5.4.57), but we cancel out the first term by suitably estimating the second:

$$\int -\frac{3}{2} |\nabla w|^2 \sigma_2(A)\, dv \overset{(5.4.68)}{=} \int -\frac{3}{2} |\nabla w|^2 \bigg( \frac{\delta}{4} \Delta R + f - \frac{1}{2} \Big( q(\tau) - \frac{1}{24} \Big) R^2 \bigg)\, dv$$

$$\geq \int \bigg( -\frac{3\delta}{8} |\nabla w|^2 \Delta R - \frac{3}{2} |\nabla w|^2 f \bigg)\, dv$$

$$\geq \int \bigg( \frac{3\delta}{8} \langle \nabla |\nabla w|^2, \nabla R \rangle - C |\nabla w|^2 \bigg)\, dv$$

$$\geq \int \bigg( -\frac{3\delta}{4} |\nabla^2 w| |\nabla w| |\nabla R| - C \bigg)\, dv$$

$$\geq \int \bigg( -\frac{\delta}{32} |\nabla R|^2 - \frac{9}{2} \delta |\nabla^2 w|^2 |\nabla w|^2 - C \bigg)\, dv, \tag{5.4.75}$$

where to reach the last line we have used Cauchy's inequality.

Substituting (5.4.75) into (5.4.74) then gives

$$\int \big( -\langle \nabla w, \nabla \sigma_2(A) \rangle - G^{ij} \nabla_i |\nabla w|^2 \nabla_j w \big)\, dv$$

$$\geq \int \bigg( -\frac{9}{2} \delta |\nabla^2 w|^2 |\nabla w|^2 + |\nabla w|^2 A^{ij} \nabla_i w \nabla_j w - C|\operatorname{Ric}||\nabla w|^2 - C|\operatorname{Ric}|^2 - C \bigg)\, dv, \tag{5.4.76}$$

and it remains to estimate $\int |\nabla w|^2 A^{ij} \nabla_i w \nabla_j w\, dv$. Using $A = \operatorname{Ric} - \frac{R}{4} g$ followed by (5.A.1) in Appendix 5.A, we have

$$\int |\nabla w|^2 A^{ij} \nabla_i w \nabla_j w\, dv = \int |\nabla w|^2 \operatorname{Ric}^{ij} \nabla_i w \nabla_j w\, dv - \int \frac{1}{4} R |\nabla w|^4\, dv$$

$$\overset{(5.A.1)}{\geq} \int \frac{2}{KR} \sigma_2(A_\tau) |\nabla w|^4\, dv - \int \frac{1}{4} R |\nabla w|^4\, dv$$

$$\overset{(5.2.1)}{=} \int \frac{2}{KR} \bigg( \frac{\delta}{4} \Delta R + f \bigg) |\nabla w|^4\, dv - \int \frac{1}{4} R |\nabla w|^4\, dv$$

$$\geq \int \frac{\delta}{2K} \frac{\Delta R}{R} |\nabla w|^4\, dv - \int \frac{1}{4} R |\nabla w|^4\, dv. \tag{5.4.77}$$

But integrating by parts and applying Cauchy's inequality, we see

$$\begin{aligned}
\int \frac{\delta}{2K}\frac{\Delta R}{R}|\nabla w|^4\,dv &= \int \Big( -\frac{\delta}{2K}\langle \nabla R, \nabla R^{-1}\rangle |\nabla w|^4 - \frac{\delta}{2KR}\langle \nabla R, \nabla |\nabla w|^4\rangle \Big)\,dv \\
&= \int \Big[ \frac{\delta}{2K}\frac{|\nabla R|^2}{R^2}|\nabla w|^4 - \frac{2\delta}{K}|\nabla w|^2 \nabla^2 w\Big(\frac{\nabla R}{R}, \nabla w\Big)\Big]\,dv \\
&\geq \int \Big( \frac{\delta}{2K}\frac{|\nabla R|^2}{R^2}|\nabla w|^4 - \frac{\delta}{2K}\frac{|\nabla R|^2}{R^2}|\nabla w|^4 - \frac{2\delta}{K}|\nabla^2 w|^2 |\nabla w|^2\Big)\,dv \\
&= \int -\frac{2\delta}{K}|\nabla^2 w|^2|\nabla w|^2\,dv \qquad (5.4.78)
\end{aligned}$$

and substituting (5.4.78) into (5.4.77), and (5.4.77) into (5.4.76), we get (5.4.57). □

#### 5.4.5.3 Proof of Lemma 5.4.8

*Proof of Lemma 5.4.8.* Again by (5.4.62),

$$|\nabla^2 w|^2 \leq C\Big(|A|^2 + |\nabla w|^4 + 1\Big)$$

and hence by our assumed $W^{1,6}$ estimate,

$$\begin{aligned}
\int -\delta|\nabla^2 w|^2|\nabla w|^2\,dv &\geq \int \Big( -C\delta|A|^2|\nabla w|^2 - C\delta|\nabla w|^6 - C\delta|\nabla w|^2\Big)\,dv \\
&\geq \int \big( -C\delta|A|^2|\nabla w|^2 - C\delta\big)\,dv. \qquad (5.4.79)
\end{aligned}$$

But $\sigma_2(A) = -\frac{|A|^2}{2} + \frac{1}{32}R^2$, so (5.4.79) becomes

$$\begin{aligned}
\int \big( -\delta|\nabla^2 w|^2|\nabla w|^2\big) &\geq \int \Big[ -C\delta|\nabla w|^2\Big( -2\sigma_2(A) + \frac{1}{16}R^2\Big) - C\delta\Big]\,dv \\
&= \int \big( C\delta|\nabla w|^2\sigma_2(A) - C\delta|\nabla w|^2R^2 - C\delta\big)\,dv \\
&\overset{(5.4.68)}{=} \int \Big[ C\delta|\nabla w|^2\Big(\frac{\delta}{4}\Delta R + f - \frac{1}{2}\Big(q(\tau) - \frac{1}{24}\Big)R^2\Big) - C\delta|\nabla w|^2R^2 - C\delta\Big]\,dv \\
&\geq \int \big( C\delta^2|\nabla w|^2\Delta R + C\delta|\nabla w|^2 f - C\delta|\nabla w|^2R^2 - C\delta\big)\,dv \\
&\geq \int \big( -C\delta^2\langle \nabla|\nabla w|^2, \nabla R\rangle - C\delta|\nabla w|^2R^2 - C\delta\big)\,dv \\
&\geq \int \big( -C\delta^2|\nabla^2 w||\nabla w||\nabla R| - C\delta|\nabla w|^2R^2 - C\delta\big)\,dv \\
&\geq \int \big( -C\delta^2|\nabla R|^2 - C\delta^2|\nabla^2 w|^2|\nabla w|^2 - C\delta|\nabla w|^2R^2 - C\delta\big)\,dv.
\end{aligned}$$

Therefore

$$\int \delta(C\delta - 1)|\nabla^2 w|^2|\nabla w|^2\, dv \geq \int \big( - C\delta^2|\nabla R|^2 - C\delta|\nabla w|^2 R^2 - C\delta\big)\, dv$$

which implies that for $\delta$ small enough

$$\int -\delta|\nabla^2 w|^2|\nabla w|^2\, dv \geq \int \big( - C\delta^2|\nabla R|^2 - C\delta|\nabla w|^2 R^2 - C\delta\big)\, dv. \tag{5.4.80}$$

But we know from Lemma 5.4.5 that

$$\int -C\delta^2|\nabla R|^2\, dv \geq \int \big( - C\delta^2 R^3 - C\delta R^2 - C\delta\big)\, dv,$$

and finally since Young's inequality implies $R^2|\nabla w|^2 \leq CR^3 + C|\nabla w|^6$, (5.4.80) reduces to (5.4.58) and the lemma is proved. □

### 5.4.6 Proof of Proposition IV: a $W^{2,2}$ estimate when $\tau < 1$

To complete the proof of Theorem E′, it remains to show that under the assumption of a $\delta$-uniform *a priori* $W^{1,4}$ estimate on solutions to (5.2.1), one can obtain a $\delta$-uniform *a priori* $W^{2,2}$ estimate (and hence the $\delta$-uniform *a priori* $W^{1,6}$ estimate) when $\tau < 1$. The proof is similar in style to our previous estimates, but much shorter:

*Proof of Proposition IV.* As $G$ is divergence-free, we compute

$$\begin{aligned} 0 = \int G^{ij}\nabla_i\nabla_j w\, dv &\overset{(5.4.62)}{=} \int G^{ij}\Big(A^0_{ij} - A_{ij} - \nabla_i w\nabla_j w + \frac{1}{2}|\nabla w|^2 g_{ij}\Big)\, dv \\ &\overset{(5.4.65)}{=} \int \Big(G^{ij}A^0_{ij} - 2\sigma_2(A) - G^{ij}\nabla_i w\nabla_j w + \frac{R}{4}|\nabla w|^2\Big)\, dv. \end{aligned} \tag{5.4.81}$$

Now, by (5.4.68) we have

$$-2\int \sigma_2(A)\, dv = \Big(q(\tau) - \frac{1}{24}\Big)\int R^2\, dv - 2\int f\, dv, \tag{5.4.82}$$

and it follows from substituting (5.4.82) into (5.4.81) that

$$\begin{aligned} \Big(q(\tau) - \frac{1}{24}\Big)\int R^2\, dv &= \int \Big(2f - G^{ij}A^0_{ij} + G^{ij}\nabla_i w\nabla_j w - \frac{R}{4}|\nabla w|^2\Big)\, dv \\ &\leq C\int \big(|\operatorname{Ric}||\nabla w|^2 + |\operatorname{Ric}| + 1\big)\, dv. \end{aligned} \tag{5.4.83}$$

Now, for any $\varepsilon > 0$ we have $|\operatorname{Ric}| \leq C\varepsilon|\operatorname{Ric}|^2 + C\varepsilon^{-1}$ and $|\operatorname{Ric}||\nabla w|^2 \leq C\varepsilon|\operatorname{Ric}|^2 + C\varepsilon^{-1}|\nabla w|^4$, which when combined with (5.4.83) implies

$$\begin{aligned}\Big(q(\tau) - \frac{1}{24}\Big)\int R^2\,dv &\leq C\varepsilon\int|\operatorname{Ric}|^2\,dv + C\varepsilon^{-1}\int|\nabla w|^4\,dv + C\varepsilon^{-1}\\ &\leq C\varepsilon\int|\operatorname{Ric}|^2\,dv + C\varepsilon^{-1}\\ &\leq C\varepsilon\int R^2\,dv + C\varepsilon + C\varepsilon^{-1}, \end{aligned} \tag{5.4.84}$$

the last line following from arguments as in (5.4.33)–(5.4.35) with $p = 0$. Taking $\varepsilon > 0$ sufficiently small in (5.4.84), we conclude that

$$\int R^2\,dv \leq C,$$

and the $W^{2,2}$ estimate then follows by the transformation law (5.3.1). □

## 5.5 Proof of Theorem E

Let us summarise where we stand. In Theorem E′ we showed that we could obtain a $\delta$-uniform $W^{2,3}$ *a priori* estimate assuming a $\delta$-uniform $W^{1,6}$ *a priori* estimate when $\tau = 1$, and a $W^{2,6}$ estimate from a $W^{1,4}$ estimate when $\tau < 1$. We note that in the latter setting, this implies a $C^{1,\alpha}$ estimate by the Sobolev embedding theorem, although the former result does not quite get us beyond the critical Sobolev exponent. Next, we bootstrap the estimates of Theorem E′ to obtain up to (but not including) a $W^{2,6}$ estimate when $\tau = 1$ (which gives a $C^{1,\alpha}$ estimate) and up to a $W^{2,12}$ estimate when $\tau < 1$. This will complete the proof of Theorem E.

### 5.5.1 Proof of Theorem E assuming two further propositions (V, III$^+$)

Following a similar format to the proof of Theorem E′, we will first state two propositions without proof, and use them to complete the proof of Theorem E. The rest of the chapter will then be dedicated to proving these two propositions.

Before stating these propositions, recall from (5.4.13) that we have

$$0 = (p+1)\int R^p G^{ij}\nabla_i\nabla_j R\, dv + p(p+1)\int R^{p-1}G^{ij}\nabla_i R\,\nabla_j R\, dv =: \mathrm{I}_1^p + \mathrm{I}_2^p$$

and from (5.4.55)

$$0 = p\int R^{p-1}G^{ij}\nabla_i R\nabla_j V\, dv + \int R^p G^{ij}\nabla_i\nabla_j V\, dv =: \mathrm{II}_1^p + \mathrm{II}_2^p\,.$$

Bootstrapping our previous estimates will require estimates for each of $\mathrm{I}_1^p$, $\mathrm{I}_2^p$, $\mathrm{II}_1^p$ and $\mathrm{II}_2^p$. An estimate for $\mathrm{I}_1^p$ was already obtained in Proposition II$^+$, although this still involves a potentially troublesome term $\mathrm{I}_{1,\delta}^p$, defined in (5.4.14). The next proposition gives an estimate for $\mathrm{I}_{1,\delta}^p$, $\mathrm{I}_1^p$ and $\mathrm{II}_1^p$ collectively. For this, it will be useful to denote the integral quantities

$$\begin{aligned} A_p &= \int R^{p-1}(\Delta R)^2\, dv \geq 0,\\ B_p &= \int R^{p-2}|\nabla R|^2\Delta R\, dv,\\ C_p &= \int R^{p-3}|\nabla R|^4\, dv \geq 0. \end{aligned} \tag{5.5.1}$$

**Proposition V.** *Suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold satisfying $Y(\mathcal{M}^3,[g_0]) > 0$. Then given $\tau \in [\frac{2}{3}, 1]$, $r > 12$, $C_1 > 0$, $1 < p < 2$ and positive $f \in C^1(\mathcal{M}^3 \times \mathbb{R})$, there exist constants $\delta_0 > 0$ and $C > 0$ (depending only on $g_0$, $\tau$, $C_1$, $p$, $\sup_{\mathcal{M}^3\times[-C_1,C_1]}(f + |\nabla_x f|) < \infty$ and $\inf_{\mathcal{M}^3\times[-C_1,C_1]} f > 0$) such that for every $C^4$ solution $g = e^{2w}g_0$ to (5.2.1) with $0 \leq \delta < \delta_0$ satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,r}(\mathcal{M}^3,g_0)} \leq C_1 \quad \textit{and} \quad R > 0,$$

*one has*

$$\begin{aligned} \mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 12(p+1)\,\mathrm{II}_1^p \geq \frac{1}{C}\delta(A_p + C_p) - C\int R^{\frac{4r}{r-4}}\, dv - C\Big(\int R^{p+3}\, dv\Big)^{\frac{p+2}{p+3}} \\ - 3p(p+1)\Big(q(\tau) - \frac{1}{24}\Big)\int R^p|\nabla R|^2\, dv - C. \end{aligned} \tag{5.5.2}$$

The important point is that even though $A_p$ and $C_p$ are of higher order, their coefficients in (5.5.2) are positive and so these terms can eventually be neglected (the factor of $\delta$ prevents either from being useful in obtaining better uniform estimates). Using our current methods, it seems that we cannot ensure the coefficient of $C_p$ is positive once $p \geq 2$, hence the restriction to $p < 2$; this phenomenon also appears in the work of [**CGY02b**]. The restriction $p > 1$ is also a technical one, but does not affect the proof of Theorem E. The remaining terms in (5.5.2) can then be dominated by corresponding terms with positive sign in the estimate (5.4.16) of Proposition II$^+$. This will become clear when we prove Theorem E below.

Now, since $\mathrm{I}^p_{1,\delta}$ is positive when $p = 0$, and $\mathrm{I}^p_2$ and $\mathrm{II}^p_1$ are both zero when $p = 0$, Proposition V cannot be viewed as a higher order extension of any of Propositions II–IV that we studied before. However, the final proposition we will need in order to prove Theorem E is a direct generalisation of Proposition III:

**Proposition III$^+$.** *Suppose $(\mathcal{M}^3, g_0)$ is a closed 3-manifold satisfying $Y(\mathcal{M}^3, [g_0]) > 0$. Then given $\tau \in [\frac{2}{3}, 1]$, $r > 6$, $C_1 > 0$, $0 \leq p \leq \frac{1}{2}(r-6)$, $\beta, \gamma, \varepsilon > 0$ and positive $f \in C^1(\mathcal{M}^3 \times \mathbb{R})$, there exist constants $\delta_0 > 0$, $C_1 > 0$ (depending only on $g_0$, $\tau$, $C_1$, $p$, $\sup_{\mathcal{M}^3\times[-C_1,C_1]}(f + |\nabla_x f|) < \infty$ and $\inf_{\mathcal{M}^3\times[-C_1,C_1]} f > 0$) and $C'$ (depending only on $g_0$, $\tau$, $C_1$, $p$, $\sup_{\mathcal{M}^3\times[-C_1,C_1]}(f + |\nabla_x f|) < \infty, \inf_{\mathcal{M}^3\times[-C_1,C_1]} f > 0$, $\beta$, $\gamma$ and $\varepsilon$) such that for every $C^4$ solution $g = e^{2w}g_0$ to* (5.2.1) *with $0 \leq \delta < \delta_0$ satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,r}(\mathcal{M}^3,g_0)} \leq C_1 \quad \textit{and} \quad R > 0,$$

*one has*

$$\begin{aligned}\mathrm{II}^p_2 &= \int R^p G^{ij}\nabla_i\nabla_j V\,dv \\ &\geq \int R^p\Big(-\operatorname{tr}\mathring{\mathrm{Ric}}^3 + \frac{1}{288}R^3\Big)\,dv - C\gamma\delta A_p - C\gamma\delta C_p - C\delta\gamma^{-3}\int R^{p+3}\,dv \\ &\quad - C\beta\int R^{p+3}\,dv - C\Big(q(\tau) - \frac{1}{24}\Big)\Big(\varepsilon\int R^p|\nabla R|^2\,dv + \beta\varepsilon^{-1}\int R^{p+3}\,dv\Big) - C'.\end{aligned} \tag{5.5.3}$$

Assuming Propositions V and III$^+$, we proceed to give the proof of Theorem E:

*Proof of Theorem E.* The approach is the same as in the proof of Theorem E$'$ on p.120: we start with the identity

$$0 = \mathrm{I}_1^p + \mathrm{I}_2^p + 12(p+1)\,\mathrm{II}_1^p + 12(p+1)\,\mathrm{II}_2^p \tag{5.5.4}$$

(with the coefficients chosen in (5.5.4) so as to cancel out the $\operatorname{tr}\overset{\circ}{\mathrm{Ric}}^3$ terms appearing in Propositions II$^+$ and III$^+$), and then apply the estimates in Propositions II$^+$, V and III$^+$. We can then choose the parameters $\beta, \gamma, \varepsilon$ carefully so as to force coefficients to have the desired sign, and finally take $\delta$ small enough to get the desired estimate.

For simplicity we first assume that $\tau = 1$. First recall that by the $W^{2,3}$ estimate of Theorem E$'$ we have a $\delta$-uniform $W^{1,r}$ estimate for all $r < \infty$. Therefore by Propositions II$^+$ and III$^+$, we have for all $p$ the estimate

$$\begin{aligned}
0 &= \mathrm{I}_1^p + \mathrm{I}_2^p + 12(p+1)\,\mathrm{II}_1^p + 12(p+1)\,\mathrm{II}_2^p \\
&\geq \underbrace{\Big[\mathrm{I}_{1,\delta}^p + (p+1)\int \Big(12R^p \operatorname{tr}\overset{\circ}{\mathrm{Ric}}^3 + \frac{1}{12}R^{p+3}\Big)dv + \frac{1}{C}\int R^{p-2}|\nabla R|^2 dv - C\int R^{p+2}\,dv - C\Big]}_{\text{The estimate for } \mathrm{I}_1^p \text{ in Proposition II}^+} \\
&\quad + \mathrm{I}_2^p + 12(p+1)\,\mathrm{II}_1^p + 12(p+1)\underbrace{\Big[\int R^p\Big(-\operatorname{tr}\overset{\circ}{\mathrm{Ric}}^3 + \frac{1}{288}R^3\Big)dv - C\gamma\delta A_p - C\gamma\delta C_p \cdots}_{} \\
&\quad \underbrace{\cdots - C\delta\gamma^{-3}\int R^{p+3}\,dv - C\beta\int R^{p+3}\,dv - C'}_{\text{The estimate for } \mathrm{II}_2^p \text{ in Proposition III}^+}\Big] \\
&\geq \underbrace{\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 12(p+1)\,\mathrm{II}_1^p}_{\text{To be estimated using Proposition V}} + \frac{1}{C}\int R^{p-2}|\nabla R|^2\,dv + \frac{1}{C}\int R^{p+3}\,dv - C\gamma\delta A_p - C\gamma\delta C_p \\
&\quad - C\delta\gamma^{-3}\int R^{p+3}\,dv - C\beta\int R^{p+3}\,dv - C'.
\end{aligned} \tag{5.5.5}$$

Applying Proposition V to the collective term $\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 12(p+1)\,\mathrm{II}_1^p$ in (5.5.5) and estimating $(\int R^{p+3}\,dv)^{\frac{p+2}{p+3}} \leq \beta\int R^{p+3}\,dv + C'$, we get for $1 < p < 2$ the estimate

$$\begin{aligned}
0 \geq \frac{1}{C}\delta(A_p + C_p) - C\int R^{\frac{4r}{r-4}}\,dv + \frac{1}{C}\int R^{p-2}|\nabla R|^2\,dv + \frac{1}{C}\int R^{p+3}\,dv - C\gamma\delta A_p \\
- C\gamma\delta C_p - C\delta\gamma^{-3}\int R^{p+3}\,dv - C\beta\int R^{p+3}\,dv - C'.
\end{aligned} \tag{5.5.6}$$

Since $p > 1$ and we are free to choose $r$ as large as we like, we can immediately absorb the term $-C \int R^{\frac{4r}{r-4}}\, dv$ into the positive term $\int R^{p+3}\, dv$ in (5.5.6). We also fix $\gamma > 0$ small enough so that the overall coefficients of $A_p$ and $C_p$ in (5.5.6) are non-negative and so can be neglected. Then it follows that

$$0 \geq \frac{1}{C} \int R^{p-2} |\nabla R|^2\, dv + \frac{1}{C} \int R^{p+3}\, dv - C\delta \int R^{p+3}\, dv - C\beta \int R^{p+3}\, dv - C'.$$

Finally we can take the remaining parameters $\beta$ and $\delta$ small enough to obtain

$$\int R^{p-2} |\nabla R|^2\, dv + \int R^{p+3}\, dv \leq C \tag{5.5.7}$$

for $p < 2$ and sufficiently small $\delta$, which implies

$$\int R^{3p}\, dv_0 \leq C$$

for $p < 2$ and sufficiently small $\delta$ by the Sobolev embedding theorem. The desired estimate then follows from (5.3.1) as in the proof of Theorem E′.

If $\tau < 1$, then by taking $\varepsilon$ small enough in Proposition III$^+$ we can absorb the additional terms in (5.5.2) and (5.5.3) into the additional *positive* term in (5.4.16), which is possible since $3p(p+1) < 4(p+1)^2$. Carrying out a similar argument to the above, we obtain in place of (5.5.7) the estimate

$$\int R^{p} |\nabla R|^2\, dv + \int R^{p+3}\, dv \leq C \tag{5.5.8}$$

for $p < 2$ and sufficiently small $\delta$, and the estimate

$$\int R^{3p+6}\, dv_0 \leq C$$

then follows from the Sobolev embedding theorem. □

Having now established Theorem E using Propositions V and III$^+$, the rest of §5.5 will be dedicated to proving these two propositions.

### 5.5.2 Proof of Proposition V

The structure of this section follows a similar format to before: we first prove an identity that holds on any three-manifold, and then we obtain an estimate using this identity and the equation (5.2.1). The identity and estimate will be for the quantity $B_p$. In what follows, we denote

$$D_p = \int R^{p-2}\nabla^2 R(\nabla R, \nabla R) = \int R^{p-2}\nabla^i\nabla^j R\,\nabla_i R\,\nabla_j R,$$

and we also denote the traceless Hessian of $R$ by

$$\mathring{\nabla}^2 R = \nabla^2 R - \frac{1}{3}(\Delta R)g.$$

We then define two further integral quantities by

$$\mathring{A}_p = \int R^{p-1}|\mathring{\nabla}^2 R|^2 \quad \text{and} \quad \mathring{D}_p = \int R^{p-2}\mathring{\nabla}^2 R(\nabla R, \nabla R).$$

The following lemma is a three-dimensional counterpart to Lemma 6.3 in [**CGY02b**], and gives an identity for $B_p$ in terms of some of the integral quantities defined above.

**Lemma 5.5.1.** *On any 3-manifold $(\mathcal{M}^3, g_0)$ and for any $p > 0$,*

$$\frac{8p+2}{3}B_p = 4\mathring{A}_p - \frac{8}{3}A_p - 2(p-2)C_p + 4(p-2)\mathring{D}_p + 4\int R^{p-1}\,\mathrm{Ric}(\nabla R, \nabla R). \tag{5.5.9}$$

*Proof.* Integrating by parts in the definition of $B_p$ and applying the Bochner formula

$$\frac{1}{2}\Delta|\nabla R|^2 = |\nabla^2 R|^2 + \langle\nabla R, \nabla\Delta R\rangle + \mathrm{Ric}(\nabla R, \nabla R),$$

we see

$$\begin{aligned}
B_p &= \int R^{p-2}|\nabla R|^2\Delta R = \int \Delta(R^{p-2}|\nabla R|^2)R \\
&= \int \Delta R^{p-2}|\nabla R|^2 R + 2\int\langle\nabla R^{p-2}, \nabla|\nabla R|^2\rangle R + \int \Delta|\nabla R|^2 R^{p-1} \\
&= \underbrace{\int \Delta R^{p-2}|\nabla R|^2 R}_{\text{(I)}} + \underbrace{2\int\langle\nabla R^{p-2}, \nabla|\nabla R|^2\rangle R}_{\text{(II)}} + \underbrace{2\int |\nabla^2 R|R^{p-1}}_{\text{(III)}} \\
&\quad \underbrace{+2\int R^{p-1}\langle\nabla R, \nabla\Delta R\rangle}_{\text{(IV)}} + 2\int R^{p-1}\,\mathrm{Ric}(\nabla R, \nabla R).
\end{aligned} \tag{5.5.10}$$

We now estimate each of the braced terms:

**(I)**: Calculating $\Delta R^{p-2}$ using the identity

$$\Delta R^q = qR^{q-1}\Delta R + q(q-1)R^{q-2}|\nabla R|^2, \tag{5.5.11}$$

we see that

$$\begin{aligned}(\mathrm{I}) = \int \Delta R^{p-2}|\nabla R|^2 R &= (p-2)\int R^{p-2}|\nabla R|^2\Delta R + (p-2)(p-3)\int R^{p-3}|\nabla R|^4 \\ &= (p-2)B_p + (p-2)(p-3)C_p.\end{aligned}$$

**(II)**: Observing that

$$\langle \nabla R^{p-2}, \nabla|\nabla R|^2\rangle R = (p-2)R^{p-2}\langle \nabla R, \nabla|\nabla R|^2\rangle = 2(p-2)R^{p-2}\nabla^i\nabla^j R\,\nabla_i R\,\nabla_j R,$$

we have

$$(\mathrm{II}) = 2\int \langle \nabla R^{p-2}, \nabla|\nabla R|^2\rangle R = 4(p-2)\int R^{p-2}\nabla^i\nabla^j R\,\nabla_i R\,\nabla_j R = 4(p-2)D_p.$$

**(III)**: We calculate

$$\begin{aligned}2\mathring{A}_p = 2\int R^{p-1}|\mathring{\nabla}^2 R|^2 &= 2\int R^{p-1}\Big|\nabla^2 R - \frac{1}{3}\Delta R\, g\Big|^2 \\ &= 2\int R^{p-1}\Big(|\nabla^2 R|^2 - \frac{1}{3}(\Delta R)^2\Big) \\ &= 2\int R^{p-1}|\nabla^2 R|^2 - \frac{2}{3}A_p,\end{aligned}$$

from which it follows that

$$(\mathrm{III}) = 2\int R^{p-1}|\nabla^2 R|^2 = 2\mathring{A}_p + \frac{2}{3}A_p.$$

**(IV)**: Integrating by parts and applying (5.5.11), we see

$$\begin{aligned}(\mathrm{IV}) = 2\int R^{p-1}\langle \nabla R, \nabla\Delta R\rangle = \frac{2}{p}\int \langle \nabla R^p, \nabla \Delta R\rangle &= -\frac{2}{p}\int \Delta R^p\,\Delta R \\ &\overset{(5.5.11)}{=} -\frac{2}{p}\int \big(pR^{p-1}\Delta R + p(p-1)R^{p-2}|\nabla R|^2\big)\Delta R \\ &= -2(A_p + (p-1)B_p).\end{aligned}$$

Substituting the above identities for (I)–(IV) back into (5.5.10) then gives

$$B_p = -\frac{4}{3}A_p + 2\mathring{A}_p - pB_p + (p-2)(p-3)C_p + 4(p-2)D_p + 2\int R^{p-1}\operatorname{Ric}(\nabla R, \nabla R). \tag{5.5.12}$$

Now, if we substitute $D_p = \mathring{D}_p + \frac{1}{3}B_p$ into (5.5.12) and collect terms, we obtain

$$\left(\frac{11-p}{3}\right)B_p = -\frac{4}{3}A_p + 2\mathring{A}_p + 4(p-2)\mathring{D}_p + (p-2)(p-3)C_p + 2\int R^{p-1}\operatorname{Ric}(\nabla R, \nabla R). \tag{5.5.13}$$

On the other hand, if we substitute

$$D_p = \frac{1}{2}\int R^{p-2}\nabla^i R\,\nabla_i|\nabla R|^2 = -\frac{1}{2}(B_p + (p-2)C_p)$$

into (5.5.12) and collect terms, we also see that

$$3(p-1)B_p = -\frac{4}{3}A_p + 2\mathring{A}_p - (p-1)(p-2)C_p + 2\int R^{p-1}\operatorname{Ric}(\nabla R, \nabla R). \tag{5.5.14}$$

Adding (5.5.13) and (5.5.14) gives precisely (5.5.9). □

Using the equation (5.2.1) and the above identity for $B_p$, we now derive the following estimate for $B_p$, which corresponds to Lemma 6.4 in [**CGY02b**].

**Corollary 5.5.2.** *Let $w \in C^4$ be a solution of positive scalar curvature to* (5.2.1) *and define $K$ as in Proposition 5.A.1. Then for all $p > 0$,*

$$\left(1 + 4p - \frac{3\delta}{K}\right)B_p \geq -4A_p + (2-p)(1+p)C_p. \tag{5.5.15}$$

*Proof.* We claim that on any 3-manifold, we have the estimate $4\mathring{A}_p \geq 4(2-p)\mathring{D}_p - \frac{2}{3}(2-p)^2C_p$. Assuming this estimate for now, and substituting it into (5.5.9) and collecting terms, we see that

$$\frac{8p+2}{3}B_p \geq -\frac{8}{3}A_p + \frac{2}{3}(2-p)(1+p)C_p + 4\int R^{p-1}\operatorname{Ric}(\nabla R, \nabla R). \tag{5.5.16}$$

Applying Proposition 5.A.1 in Appendix 5.A with $X = \nabla R$ to the last integral in (5.5.16), we therefore have

$$\begin{aligned}
\frac{8p+2}{3}B_p &\geq -\frac{8}{3}A_p + \frac{2}{3}(2-p)(1+p)C_p + \frac{8}{K}\int R^{p-2}\sigma_2(A_\tau)|\nabla R|^2 \\
&\overset{(5.2.1)}{=} -\frac{8}{3}A_p + \frac{2}{3}(2-p)(1+p)C_p + \frac{8}{K}\int\left(\frac{\delta}{4}\Delta R + f\right)R^{p-2}|\nabla R|^2 \\
&\geq -\frac{8}{3}A_p + \frac{2}{3}(2-p)(1+p)C_p + \frac{2\delta}{K}\int \Delta R|\nabla R|^2 R^{p-2} \\
&= -\frac{8}{3}A_p + \frac{2}{3}(2-p)(1+p)C_p + \frac{2\delta}{K}B_p.
\end{aligned}$$

Multiplying through by $\frac{3}{2}$ and collecting terms, we arrive at (5.5.15).

It remains to prove the claim. As explained in the proof of Proposition 5.A.1 in Appendix 5.A, the fact that $\mathring{\nabla}^2 R$ is trace-free implies

$$|\mathring{\nabla}^2 R(\nabla R, \nabla R)| \leq \sqrt{\frac{2}{3}}\,|\mathring{\nabla}^2 R||\nabla R|^2.$$

It follows from Hölder's inequality that

$$\begin{aligned}
4(2-p)\mathring{D}_p &\leq 4\left|(2-p)\int R^{p-2}\mathring{\nabla}^2 R(\nabla R, \nabla R)\right| \\
&\leq 4|2-p|\sqrt{\frac{2}{3}}\int R^{p-2}|\mathring{\nabla}^2 R||\nabla R|^2 \leq 4|2-p|\sqrt{\frac{2}{3}}\mathring{A}_p^{1/2}C_p^{1/2}
\end{aligned}$$

and finally applying Cauchy's inequality we obtain

$$4(2-p)\mathring{D}_p \leq 4\left(\mathring{A}_p + \frac{1}{6}(2-p)^2 C_p\right) = 4\mathring{A}_p + \frac{2}{3}(2-p)^2 C_p. \qquad \square$$

We next present a technical lemma, adapted from Lemma 6.6 in [**CGY02b**], that we will refer back to at numerous points in our later arguments.

**Lemma 5.5.3.** *Fix a background metric $g_0$ and suppose $\mathcal{A} \subseteq C^\infty(\mathcal{M}^3)$ is such that $\|w\|_{W^{1,r}(\mathcal{M}^3, g_0)} \leq C_1$ and $R = R_{e^{2w}g_0} \geq C_2^{-1} > 0$ for all $w \in \mathcal{A}$, for some uniform constants $0 < C_1, C_2 < \infty$. If $r > 6$, then for $1 < p \leq \frac{1}{2}(r-4)$,*

$$\int R^{p-1}|\nabla^2 w|^2|\nabla w|^2 \leq C\left(\int R^{p+3}\right)^{\frac{p+2}{p+3}} + C, \tag{5.5.17}$$

*and for* $0 < p \le \frac{1}{2}(r-6)$,

$$\int R^p |\nabla^2 w|^2 |\nabla w|^2 \le C\Big(\int R^{p+3}\Big)^{\frac{p+2}{p+3}} + C. \qquad (5.5.18)$$

*If* $r > 8$*, then for* $1 < p \le \frac{1}{2}(r-6)$,

$$\int R^{p-1} |\nabla^2 w|^3 |\nabla w|^2 \le C\Big(\int R^{p+3}\Big)^{\frac{p+2}{p+3}} + C \qquad (5.5.19)$$

*and*

$$\int R^{p-1} |\nabla^2 w|^2 |\nabla w|^4 \le C\Big(\int R^{p+3}\Big)^{\frac{p+2}{p+3}} + C. \qquad (5.5.20)$$

*Finally, if* $r > 12$ *and* $p \le 3$*, then there exists* $4 \le \alpha < 6$ *depending on* $r$ *and* $g_0$ *but not* $p$ *for which*

$$\int R^{p-3} |\nabla^2 w|^4 |\nabla w|^4 \le C \int R^\alpha + C. \qquad (5.5.21)$$

*Proof.* The transformation law (5.3.3) implies

$$|R| \le C\big(|R_0| + |\Delta w| + |\nabla w|^2\big) \le C\big(1 + |\nabla^2 w| + |\nabla w|^2\big).$$

Therefore if $a, b, c > 0$ and $s = a + b + \frac{c}{2}$, it follows from Hölder's inequality that

$$\begin{aligned}\int R^a |\nabla^2 w|^b |\nabla w|^c &\le C \int |\nabla^2 w|^{a+b} |\nabla w|^c + |\nabla^2 w|^b |\nabla w|^{2a+c} + |\nabla^2 w|^b |\nabla w|^c \\ &\le C\Big(\int |\nabla^2 w|^s\Big)^{\frac{a+b}{s}} \Big(\int |\nabla w|^{2s}\Big)^{\frac{c}{2s}} + C\Big(\int |\nabla^2 w|^s\Big)^{\frac{b}{s}} \Big(\int |\nabla w|^{2s}\Big)^{\frac{2a+c}{2s}} \\ &\quad + C\Big(\int |\nabla^2 w|^s\Big)^{\frac{b}{s}} \Big(\int |\nabla w|^{\frac{2cs}{2a+c}}\Big)^{\frac{2a+c}{2s}}.\end{aligned}$$

If $2s \le r$, then it follows from Young's inequality and our assumed $W^{1,r}$ bound that

$$\int R^a |\nabla^2 w|^b |\nabla w|^c \le C\Big(\int |\nabla^2 w|^s\Big)^{\frac{a+b}{s}} + C.$$

But by (5.3.4),

$$|\nabla^2 w| \le C\big(1 + |\operatorname{Ric}| + |\nabla w|^2\big), \qquad (5.5.22)$$

so for such values of $s$ we have

$$\int R^a|\nabla^2 w|^b|\nabla w|^c \leq C\bigg(\int |\operatorname{Ric}|^s\bigg)^{\frac{a+b}{s}} + C \leq C\bigg(\int R^s\bigg)^{\frac{a+b}{s}} + C.$$

The first four estimates claimed in Lemma 5.5.3 then follow.

Finally, if $p \leq 3$, then using the assumption $0 < R^{-1} \leq C_2$ and (5.5.22), we obtain

$$\begin{aligned}\int R^{p-3}|\nabla^2 w|^4|\nabla w|^4 \leq C\int |\nabla^2 w|^4|\nabla w|^4 &\leq C\int |\operatorname{Ric}|^4|\nabla w|^4 + |\nabla w|^{12} + |\nabla w|^4\\ &\leq C\int |\operatorname{Ric}|^\alpha + |\nabla w|^r + C\\ &\leq C\int R^\alpha + C.\end{aligned}$$

Note that we have used Young's inequality on the term $|\operatorname{Ric}|^4|\nabla w|^4$ and the fact that $r > 12$ to ensure that $4 \leq \alpha < 6$. □

Recall that we are looking to obtain an estimate for $\mathrm{I}^p_{1,\delta} + \mathrm{I}^p_2 + 12(p+1)\,\mathrm{II}^p_1$ in Proposition V, but we are yet to consider the term $\mathrm{II}^p_1$. The next lemma is an estimate for $\mathrm{II}^p_1$, and corresponds to Lemma 6.5 in [**CGY02b**]:

**Lemma 5.5.4.** *Fix $\tau \in [\frac{2}{3}, 1]$, $r > 12$ and $\varepsilon, \eta > 0$, and suppose $Y(\mathcal{M}^3, [g_0]) > 0$, $C_1 > 0$ and $1 < p < 3$. Then there exists $\delta_0 = \delta_0(g_0, C_1, p, \tau) > 0$, $\alpha = \alpha(g_0, C_1, \tau) < 6$ and constants $C = C(g_0, C_1, p, \tau)$, $C' = C'(g_0, C_1, p, \tau, \varepsilon, \eta)$ such that for every $C^4$ solution $g = e^{2w}g_0$ to (5.2.1) with $0 \leq \delta < \delta_0$ satisfying*

$$\|w\|_{L^\infty(\mathcal{M}^3)} + \|w\|_{W^{1,r}(\mathcal{M}^3,g_0)} \leq C_1 \quad \textit{and} \quad R > 0,$$

*one has*

$$\begin{aligned}\mathrm{II}^p_1 \geq &-\frac{1}{2}p\varepsilon^2\int R^{p-1}G^{ij}\nabla_i R\,\nabla_j R - C\delta\varepsilon^2\eta A_p - \frac{C\delta\varepsilon^2}{\eta}C_p - C\varepsilon^2\bigg(q(\tau) - \frac{1}{24}\bigg)\int R^p|\nabla R|^2\\ &- \frac{C}{\varepsilon^6\eta}\int R^\alpha - \frac{C}{\varepsilon^2}\bigg(\int R^{p+3}\bigg)^{\frac{p+2}{p+3}} - C'.\end{aligned} \tag{5.5.23}$$

*Proof.* For any $\varepsilon > 0$, we may write

$$
\begin{aligned}
\mathrm{II}_1^p &= p\int R^{p-1}G^{ij}\nabla_i R\nabla_j V \\
&= \frac{1}{2}p\int R^{p-1}G^{ij}\nabla_i\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\nabla_j\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big) \\
&\quad - \frac{1}{2}p\varepsilon^2\int R^{p-1}G^{ij}\nabla_i R\,\nabla_j R - \frac{1}{2}p\varepsilon^{-2}\int R^{p-1}G^{ij}\nabla_i V\nabla_j V. \qquad (5.5.24)
\end{aligned}
$$

We immediately see that the penultimate term in (5.5.24) is precisely the first term on the RHS of (5.5.23), so it does not need to be addressed. We start by considering the middle line of (5.5.24), and towards the end of the proof we will estimate the last term in (5.5.24).

Taking $\tau = 1$ in (5.A.4) so that $K = \frac{1}{2}$, we see that the integral on the middle line of (5.5.24) can be estimated as

$$
\begin{aligned}
&\int R^{p-1}G^{ij}\nabla_i\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\nabla_j\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big) \geq 4\int R^{p-2}\sigma_2(A)\Big|\nabla\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\Big|^2 \\
&\overset{(5.4.68)}{=} 4\int R^{p-2}\Big(\frac{\delta}{4}\Delta R + f - \frac{1}{2}\Big(q(\tau) - \frac{1}{24}\Big)R^2\Big)\Big|\nabla\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\Big|^2 \\
&\geq \delta\int R^{p-2}\Delta R\Big|\nabla\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\Big|^2 - 2\Big(q(\tau) - \frac{1}{24}\Big)\int R^p\Big|\nabla\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\Big|^2. \qquad (5.5.25)
\end{aligned}
$$

Now,

$$
\int R^p\Big|\nabla\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\Big|^2 \leq C\int\Big(\varepsilon^2 R^p|\nabla R|^2 + \frac{1}{\varepsilon^2}R^p|\nabla V|^2\Big) \qquad (5.5.26)
$$

and for any $\eta > 0$

$$
\begin{aligned}
\delta\int R^{p-2}\Delta R\Big|\nabla\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\Big|^2 &\geq -\delta A_p^{1/2}\Big(\int R^{p-3}\Big|\nabla\Big(\varepsilon R + \frac{1}{\varepsilon}V\Big)\Big|^4\Big)^{\frac{1}{2}} \\
&\geq -C\delta A_p^{1/2}\Big(\varepsilon^2 C_p^{1/2} + \frac{1}{\varepsilon^2}\Big(\int R^{p-3}|\nabla V|^4\Big)^{\frac{1}{2}}\Big) \\
&= -C\delta\varepsilon^2 A_p^{1/2}C_p^{1/2} - \frac{C\delta A_p^{1/2}}{\varepsilon^2}\Big(\int R^{p-3}|\nabla V|^4\Big)^{\frac{1}{2}} \\
&\geq -C\delta\varepsilon^2\eta A_p - \frac{C\delta\varepsilon^2}{\eta}C_p - \frac{C}{\varepsilon^6\eta}\int R^{p-3}|\nabla V|^4. \qquad (5.5.27)
\end{aligned}
$$

Substituting (5.5.26) and (5.5.27) back into (5.5.25), we obtain for all $\varepsilon, \eta > 0$

$$\begin{aligned}
&\int R^{p-1} G^{ij} \nabla_i \Big( \varepsilon R + \frac{1}{\varepsilon} V \Big) \nabla_j \Big( \varepsilon R + \frac{1}{\varepsilon} V \Big) \\
&\geq -C\delta\varepsilon^2 \eta A_p - \frac{C\delta\varepsilon^2}{\eta} C_p - \frac{C}{\varepsilon^6 \eta} \int R^{p-3} |\nabla V|^4 \\
&\quad - C\Big( q(\tau) - \frac{1}{24} \Big) \int \Big( \varepsilon^2 R^p |\nabla R|^2 + \frac{1}{\varepsilon^2} R^p |\nabla V|^2 \Big). \qquad (5.5.28)
\end{aligned}$$

Now, since we assume uniform $W^{1,r}$ estimates for some $r > 12$, and $\nabla_i V = \nabla_i \nabla^j w \nabla_j w$ implies $|\nabla V| \leq C |\nabla^2 w| |\nabla w|$, we can apply (5.5.21) using the fact that $p < 3$ to get

$$\int R^{p-3} |\nabla V|^4 \leq C \int R^{p-3} |\nabla^2 w|^4 |\nabla w|^4 \leq C \int R^\alpha + C,$$

where $\alpha$ is as in the statement of Lemma 5.5.3. Similarly, we have by (5.5.18)

$$\int R^p |\nabla V|^2 \leq C \int R^p |\nabla^2 w|^2 |\nabla w|^2 \leq C \Big( \int R^{p+3} \Big)^{\frac{p+2}{p+3}} + C.$$

Substituting these two estimates back into (5.5.28), we therefore have

$$\begin{aligned}
&\int R^{p-1} G^{ij} \nabla_i \Big( \varepsilon R + \frac{1}{\varepsilon} V \Big) \nabla_j \Big( \varepsilon R + \frac{1}{\varepsilon} V \Big) \\
&\quad \geq -C\delta\varepsilon^2 \eta A_p - \frac{C\delta\varepsilon^2}{\eta} C_p - \frac{C}{\varepsilon^6 \eta} \int R^\alpha - C\varepsilon^2 \Big( q(\tau) - \frac{1}{24} \Big) \int R^p |\nabla R|^2 \\
&\quad\quad - \frac{C}{\varepsilon^2} \Big( \int R^{p+3} \Big)^{\frac{p+2}{p+3}} - C'. \qquad (5.5.29)
\end{aligned}$$

Having estimated the middle line of (5.5.24), it now remains to estimate the last integral in (5.5.24). Given the transformation law for $G$ and the estimates (5.5.17), (5.5.19) and (5.5.20) of Lemma 5.5.3, we see that

$$\begin{aligned}
\int R^{p-1} G^{ij} \nabla_i V \nabla_j V &\leq C \int R^{p-1} \big( |G_0| + |\nabla^2 w| + |\nabla w|^2 \big) |\nabla^2 w|^2 |\nabla w|^2 \\
&\leq C \int R^{p-1} |\nabla^2 w|^2 |\nabla w|^2 + C \int R^{p-1} |\nabla^2 w|^3 |\nabla w|^2 \\
&\quad + C \int R^{p-1} |\nabla^2 w|^2 |\nabla w|^4 \\
&\leq C \Big( \int R^{p+3} \Big)^{\frac{p+2}{p+3}} + C. \qquad (5.5.30)
\end{aligned}$$

Substituting (5.5.29) and (5.5.30) back into (5.5.24), we arrive at (5.5.23). □

We now combine the results above to prove Proposition V.

*Proof of Proposition V.* Using the definition of $\mathrm{I}^p_{1,\delta}$ in (5.4.14) and the identity (5.5.11),

$$\begin{aligned}\mathrm{I}^p_{1,\delta} \overset{(5.5.11)}{=} \delta(p+1) \int \Delta R\big(pR^{p-1}\Delta R + p(p-1)R^{p-2}|\nabla R|^2\big) \\ + 2R^{p-1}(\Delta R)^2 + 2pR^{p-2}|\nabla R|^2\Delta R \\ = \delta(p+1)\Big((p+2)A_p + p(p+1)B_p\Big).\end{aligned}$$

Also recall that $\mathrm{I}^p_2 = p(p+1)\int R^{p-1}G^{ij}\nabla_i R\,\nabla_j R$. Using these identities and the estimate for $\mathrm{II}^p_1$ in Lemma 5.5.4, we therefore have

$$\begin{aligned}&\mathrm{I}^p_{1,\delta} + \mathrm{I}^p_2 + 12(p+1)\,\mathrm{II}^p_1 \\ &\geq \delta(p+1)\Big((p+2)A_p + p(p+1)B_p\Big) + p(p+1)(1-6\varepsilon^2)\int R^{p-1}G^{ij}\nabla_i R\,\nabla_j R \\ &\quad - C\delta\varepsilon^2\eta A_p - \frac{C\delta\varepsilon^2}{\eta}C_p - C\varepsilon^2\Big(q(\tau) - \frac{1}{24}\Big)\int R^p|\nabla R|^2 - \frac{C}{\varepsilon^6\eta}\int R^\alpha \\ &\quad - \frac{C}{\varepsilon^2}\Big(\int R^{p+3}\Big)^{\frac{p+2}{p+3}} - C'. \end{aligned} \tag{5.5.31}$$

Next, we use (5.A.4) to assert

$$\begin{aligned}\int R^{p-1}G^{ij}\nabla_i R\,\nabla_j R &\geq \int 4R^{p-2}\sigma_2(A)|\nabla R|^2 \\ &\overset{(5.4.68)}{=} \int 4R^{p-2}\Big(\frac{\delta}{4}\Delta R + f - \frac{1}{2}\Big(q(\tau) - \frac{1}{24}\Big)R^2\Big)|\nabla R|^2 \\ &\geq \delta\int R^{p-2}|\nabla R|^2\Delta R - 2\Big(q(\tau) - \frac{1}{24}\Big)\int R^p|\nabla R|^2 \\ &= \delta B_p - 2\Big(q(\tau) - \frac{1}{24}\Big)\int R^p|\nabla R|^2, \end{aligned} \tag{5.5.32}$$

and it follows that if $6\varepsilon^2 < 1$, we can substitute (5.5.32) into (5.5.31) to get

$$\begin{aligned}&\mathrm{I}^p_{1,\delta} + \mathrm{I}^p_2 + 12(p+1)\,\mathrm{II}^p_1 \\ &\geq \delta(p+1)\Big((p+2)A_p + p(p+1)B_p + p(1-6\varepsilon^2)B_p\Big) \\ &\quad - C\delta\varepsilon^2\eta A_p - \frac{C\delta\varepsilon^2}{\eta}C_p - \Big[2p(p+1)(1-6\varepsilon^2) + C\varepsilon^2\Big]\Big(q(\tau) - \frac{1}{24}\Big)\int R^p|\nabla R|^2 \\ &\quad - \frac{C}{\varepsilon^6\eta}\int R^\alpha - \frac{C}{\varepsilon^2}\Big(\int R^{p+3}\Big)^{\frac{p+2}{p+3}} - C'. \end{aligned} \tag{5.5.33}$$

Let us first consider the second line of (5.5.33). Corollary 5.5.2 tells us

$$\left(1 + 4p - \frac{3\delta}{K}\right) B_p \geq -4A_p + C_p(2-p)(1+p), \tag{5.5.34}$$

but since $1 - \frac{\sqrt{3}}{2} \leq K \leq 1$ for $\tau \in [\frac{2}{3}, 1]$, we can take $K = 1$ and the inequality will hold independently of $\tau$ in this range. We do this in order to simplify the following calculation. With this in mind, (5.5.34) implies

$$B_p \geq -\frac{4A_p}{1 + 4p - 3\delta} + \frac{C_p(2-p)(1+p)}{1 + 4p - 3\delta},$$

and therefore the second line of (5.5.33) can be estimated as

$$\begin{aligned}
&\delta(p+1)\Big((p+2)A_p + p(p+1)B_p + p(1-6\varepsilon^2)B_p\Big) \\
&\geq \delta(p+1)\Bigg[A_p\left(p + 2 - \frac{4p(p+1)}{1+4p-3\delta} - \frac{4p(1-6\varepsilon^2)}{1+4p-3\delta}\right) \\
&\qquad + C_p\left(\frac{p(p+1)^2(2-p)}{1+4p-3\delta} + \frac{p(1-6\varepsilon^2)(2-p)(1+p)}{1+4p-3\delta}\right)\Bigg] \\
&= A_p\left(\frac{24p(p+1)\delta\varepsilon^2}{1+4p-3\delta} - \frac{3\delta^2(p+1)(p+2)}{1+4p-3\delta} + \frac{\delta(p+1)(p+2)}{1+4p-3\delta}\right) \\
&\qquad + C_p\left(\frac{\delta p(p+1)^2(2-p)(p+2-6\varepsilon^2)}{1+4p-3\delta}\right) \\
&\geq A_p\left(\frac{24p(p+1)\delta\varepsilon^2}{1+4p-3\delta} - \frac{3\delta^2(p+1)(p+2)}{1+4p-3\delta}\right) + C_p\left(\frac{\delta p(p+1)^2(2-p)(p+2-6\varepsilon^2)}{1+4p-3\delta}\right).
\end{aligned}$$

Since we have assumed $6\varepsilon^2 < 1$, clearly $p + 2 - 6\varepsilon^2 > 1$. Moreover, since we assume $p > 1$ and $\delta < 1$, we can bound

$$\frac{p}{1+4p-3\delta} \geq \frac{p}{1+4p} \geq \frac{1}{5} \quad \text{and} \quad \frac{-1}{1+4p-3\delta} \geq -1.$$

It follows that if we additionally assume $p < 2$ (and this is crucial, otherwise the coefficient of $C_p$ in the above calculation is negative), then

$$\begin{aligned}
&\delta(p+1)\Big((p+2)A_p + p(p+1)B_p + p(1-6\varepsilon^2)B_p\Big) \\
&\qquad \geq \frac{24}{5}\delta\varepsilon^2(p+1)A_p - 3\delta^2(p+1)(p+2)A_p + \frac{1}{5}\delta(p+1)^2(2-p)C_p \\
&\qquad = \frac{24}{5}\delta\varepsilon^2(p+1)A_p - \alpha_p\delta^2 A_p + \beta_p\delta C_p
\end{aligned} \tag{5.5.35}$$

where $\alpha_p$ and $\beta_p$ are positive. Then, substituting (5.5.35) into (5.5.33) we obtain

$$\begin{aligned}\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 12(p+1)\,\mathrm{II}_1^p \geq& \left[\frac{24}{5}\delta\varepsilon^2(p+1) - C\delta\varepsilon^2\eta\right]A_p - \alpha_p\delta^2 A_p + \left[\beta_p\delta - \frac{C\delta\varepsilon^2}{\eta}\right]C_p \\ &- \left[2p(p+1)(1-6\varepsilon^2) + C\varepsilon^2\right]\left(q(\tau) - \frac{1}{24}\right)\int R^p|\nabla R|^2 \\ &- \frac{C}{\varepsilon^6\eta}\int R^\alpha - \frac{C}{\varepsilon^2}\left(\int R^{p+3}\right)^{\frac{p+2}{p+3}} - C'. \end{aligned} \tag{5.5.36}$$

Taking $\eta$ sufficiently small so that $C\delta\varepsilon^2\eta < \frac{24}{5}\delta\varepsilon^2(p+1)$, and then $\varepsilon$ sufficiently small so that $\frac{C\delta\varepsilon^2}{\eta} < \beta_p\delta$ and $C\varepsilon^2 < p(p+1)$, we get

$$\begin{aligned}\mathrm{I}_{1,\delta}^p + \mathrm{I}_2^p + 12(p+1)\,\mathrm{II}_1^p \geq& \frac{1}{C}\delta(A_p + C_p) - \alpha_p\delta^2 A_p - C\int R^\alpha - C\left(\int R^{p+3}\right)^{\frac{p+2}{p+3}} - C \\ &- 3p(p+1)\left(q(\tau) - \frac{1}{24}\right)\int R^p|\nabla R|^2.\end{aligned}$$

Finally, we observe that for $\delta > 0$ small enough, the overall coefficient of $A_p$ is positive, so we obtain (5.5.2) for sufficiently small $\delta$. □

**Remark 5.5.5.** The importance of taking $p < 2$ here is to ensure that $\beta_p$ is positive, which in turn allows us to pick $\varepsilon$ sufficiently small so that the overall coefficient of $C_p$ in (5.5.36) is positive. As seen in the proof of Theorem E on p.151, the positivity of the coefficients in front of $A_p$ and $C_p$ allows us to drop these terms, which is crucial since they are of higher order.

### 5.5.3 Proof of Proposition III$^+$

To complete the proof of Theorem E, it remains to prove Proposition III$^+$.

*Proof of Proposition III*$^+$. Multiplying both sides of (5.4.56) by $R^p$ and integrating, we obtain

$$\begin{aligned}\mathrm{II}_2^p =& \int R^p G^{ij}\nabla_i\nabla_j V \\ \geq& \int R^p\left(-\operatorname{tr}\mathring{\mathrm{Ric}}^3 + \frac{1}{288}R^3\right) - \frac{1}{8}R^{p+1}|\nabla w|^4 - R^p\langle\nabla w, \nabla\sigma_2(A)\rangle \\ &- R^p G^{ij}\nabla_i|\nabla w|^2\nabla_j w - CR^p|\operatorname{Ric}|^2 - CR^p|\operatorname{Ric}||\nabla w|^2 - CR^p.\end{aligned} \tag{5.5.37}$$

Comparing (5.5.37) with the desired estimate (5.5.3), we see that we do not need to address the term $\int R^p(-\operatorname{tr}\overset{\circ}{\operatorname{Ric}}^3 + \frac{1}{288}R^3)$ in (5.5.37). In fact, most of the proof will focus on estimating the term $\int -R^p\langle\nabla w, \nabla\sigma_2(A)\rangle$ in (5.5.37); our estimates for the remaining terms will be easier. We claim that for all $\beta, \gamma, \varepsilon > 0$, we have the estimate

$$-\int R^p\langle\nabla w, \nabla\sigma_2(A)\rangle \geq -C\delta\gamma A_p - C\delta\gamma C_p - C\delta\gamma^{-3}\int R^{p+3} - C\beta\int R^{p+3} \\ - C\Big(q(\tau) - \frac{1}{24}\Big)\Big(\varepsilon\int R^p|\nabla R|^2 + \beta\varepsilon^{-1}\int R^{p+3}\Big) - C'. \tag{5.5.38}$$

We reiterate that, as in the statement of Proposition III$^+$, $C$ is a constant independent of $\beta, \gamma$ and $\varepsilon$, but $C'$ is allowed to depend on these quantities.

To obtain (5.5.38), we start by integrating by parts and appealing to the equation (5.4.68) to get

$$\begin{aligned}
&-\int R^p\langle\nabla w, \nabla\sigma_2(A)\rangle \\
&\quad= \int \sigma_2(A)R^p\Delta w + \int \sigma_2(A)\langle\nabla w, \nabla R^p\rangle \\
&\quad\overset{(5.4.68)}{=} \int \frac{\delta}{4}R^p\Delta R\Delta w + \int fR^p\Delta w - \frac{1}{2}\Big(q(\tau) - \frac{1}{24}\Big)\int R^{p+2}\Delta w \\
&\qquad + \int \frac{\delta}{4}\Delta R\langle\nabla w, \nabla R^p\rangle + \int f\langle\nabla w, \nabla R^p\rangle - \frac{1}{2}\Big(q(\tau) - \frac{1}{24}\Big)\int R^2\langle\nabla w, \nabla R^p\rangle \\
&\quad= \underbrace{\int \frac{\delta}{4}R^p\Delta R\Delta w}_{\text{(I)}} + \underbrace{\int \frac{\delta}{4}pR^{p-1}\Delta R\langle\nabla w, \nabla R\rangle}_{\text{(II)}} \underbrace{- \int R^p\langle\nabla w, \nabla f\rangle}_{\text{(III)}} \\
&\qquad \underbrace{-\frac{1}{2}\Big(q(\tau) - \frac{1}{24}\Big)\int R^{p+2}\Delta w}_{\text{(IV)}} \underbrace{- \frac{p}{2}\Big(q(\tau) - \frac{1}{24}\Big)\int R^{p+1}\langle\nabla w, \nabla R\rangle}_{\text{(V)}},
\end{aligned} \tag{5.5.39}$$

We now estimate each of the braced terms.

**(I)**: By Hölder's inequality and the definition of $A_p$,

$$\begin{aligned}
\text{(I)} \geq -\delta\int R^p|\Delta R||\Delta w| &\geq -\delta A_p^{1/2}\Big(\int R^{p+1}(\Delta w)^2\Big)^{\frac{1}{2}} \\
&\overset{(5.3.3)}{\geq} -C\delta A_p^{1/2}\Big(\int R^{p+3} + \int R^{p+1}|\nabla w|^4 + \int R^{p+1}\Big)^{\frac{1}{2}}.
\end{aligned}$$

But by Young's inequality,

$$\int R^{p+1}|\nabla w|^4 \leq C\int R^{p+3} + C\int |\nabla w|^{2(p+3)}, \tag{5.5.40}$$

and so it follows that for any $\gamma > 0$,

$$\begin{aligned}
(\mathrm{I}) &\geq -C\delta A_p^{1/2}\Big(\int R^{p+3} + \int |\nabla w|^{2(p+3)} + C\Big)^{\frac{1}{2}} \\
&\geq -C\delta A_p^{1/2}\Big(\int R^{p+3} + C\Big)^{\frac{1}{2}} \\
&\geq -C\delta\gamma A_p - C\delta\gamma^{-1}\int R^{p+3} - C'.
\end{aligned}$$

**(II)**: Estimating in a similar manner as above, we have for any $\gamma > 0$

$$\begin{aligned}
(\mathrm{II}) \geq -C\delta\int R^{p-1}|\Delta R||\nabla w||\nabla R| &\geq -C\delta A_p^{1/2}\Big(\int R^{p-1}|\nabla R|^2|\nabla w|^2\Big)^{\frac{1}{2}} \\
&\geq -C\delta A_p^{1/2}C_p^{1/4}\Big(\int R^{p+1}|\nabla w|^4\Big)^{\frac{1}{4}} \\
&\geq -C\delta A_p^{1/2}C_p^{1/4}\Big(\int R^{p+3} + C\Big)^{\frac{1}{4}} \\
&\geq -C\delta\gamma A_p - C\delta\gamma^{-1}C_p^{1/2}\Big(\int R^{p+3} - C\Big)^{\frac{1}{2}} + C' \\
&\geq -C\delta\gamma A_p - C\delta\gamma C_p - C\delta(\gamma^{-1} + \gamma^{-3})\int R^{p+3} - C'.
\end{aligned}$$

**(III)**: Here we only require an application of Young's and Cauchy's inequalities:

$$\begin{aligned}
(\mathrm{III}) \geq -\int R^p|\nabla w||\nabla f| &\geq -C\int R^{p+2} - C\int |\nabla w|^{\frac{p+2}{2}}|\nabla f|^{\frac{p+2}{2}} \\
&\geq -C\int R^{p+2} - C\int |\nabla w|^{p+2} - C\int |\nabla f|^{p+2} \\
&\geq -C\int R^{p+2} - C.
\end{aligned}$$

**(IV)**: Again referring to the transformation law (5.3.3) for $R$, and then applying Young's inequality, we see that for any $\beta > 0$

$$\begin{aligned}
\int R^{p+2}\Delta w &= \frac{1}{4}\int R^{p+2}\Big(-R + e^{-2w}R_0 + 2|\nabla w|^2\Big) \\
&\leq C\int R^{p+2}(1 + |\nabla w|^2) \\
&\leq C\beta\int R^{p+3} + C'.
\end{aligned} \tag{5.5.41}$$

Therefore,

$$(\mathrm{IV}) = -\frac{1}{2}\left(q(\tau) - \frac{1}{24}\right)\int R^{p+2}\Delta w \geq -C\beta\int R^{p+3} - C'$$

**(V)**: By Young's inequality, we have for all $\varepsilon, \beta > 0$

$$\begin{aligned}(\mathrm{V}) &\geq -C\left(q(\tau) - \frac{1}{24}\right)\int R^{p+1}|\nabla w||\nabla R| \\ &\geq -C\left(q(\tau) - \frac{1}{24}\right)\left(\varepsilon\int R^{p}|\nabla R|^2 + \varepsilon^{-1}\int R^{p+2}|\nabla w|^2\right) \\ &\geq -C\left(q(\tau) - \frac{1}{24}\right)\left(\varepsilon\int R^{p}|\nabla R|^2 + \beta\varepsilon^{-1}\int R^{p+3}\right) - C'.\end{aligned}$$

Substituting the above estimates for (I)–(V) back into (5.5.39), we therefore obtain (5.5.38).

With (5.5.38) established, it remains to estimate the terms on the bottom line of (5.5.37) (note that the term $\int R^{p+1}|\nabla w|^4$ on the top line of (5.5.37) is already addressed using (5.5.40)). Indeed, to obtain the desired estimate (5.5.3) of Proposition III$^+$, it suffices to show that the bottom line of (5.5.37) satisfies the estimate

$$\left|\int -R^p G^{ij}\nabla_i|\nabla w|^2\nabla_j w - CR^p|\operatorname{Ric}|^2 - CR^p|\operatorname{Ric}||\nabla w|^2 - CR^p\right| \leq C\beta\int R^{p+3} + C' \tag{5.5.42}$$

for arbitrary $\beta > 0$ and $C'$ depending on $\beta$.

To this end, using $|G| \leq |\operatorname{Ric}|$ and (5.5.22), we first observe that

$$\begin{aligned}\left|\int R^p G^{ij}\nabla_i|\nabla w|^2\nabla_j w\right| &\leq C\int R^p|\operatorname{Ric}||\nabla^2 w||\nabla w|^2 \\ &\leq C\int R^p|\operatorname{Ric}|^2|\nabla w|^2 + C\int R^p|\operatorname{Ric}||\nabla w|^4 + C\int R^p|\operatorname{Ric}||\nabla w|^2. \end{aligned}\tag{5.5.43}$$

It is then easy to see that the integrals on the bottom line of (5.5.43) can be estimated in exactly the same way as we have done before. Explicitly, for any $\beta > 0$ we have

$$\begin{aligned}\int R^p|\operatorname{Ric}|^2|\nabla w|^2 &\leq C\int|\operatorname{Ric}|^{p+2}|\nabla w|^2 \leq C\beta\int R^{p+3} + C\beta^{-1}\int|\nabla w|^{2(p+3)}, \\ \int R^p|\operatorname{Ric}||\nabla w|^4 &\leq C\int|\operatorname{Ric}|^{p+1}|\nabla w|^4 \leq C\beta\int R^{p+3} + C\beta^{-1}\int|\nabla w|^{2(p+3)}, \\ \int R^p|\operatorname{Ric}||\nabla w|^2 &\leq C\int|\operatorname{Ric}|^{p+1}|\nabla w|^2 \leq C\beta\int R^{p+3} + C\beta^{-1}\int|\nabla w|^{p+3}.\end{aligned}$$

Combining the above we see that for any $\beta > 0$,

$$\left| \int R^p G^{ij} \nabla_i |\nabla w|^2 \nabla_j w \right| \leq C\beta \int R^{p+3} + C'. \tag{5.5.44}$$

Inspecting the remaining three terms on the LHS of (5.5.42), we see that these have already been dealt with by our estimates preceding (5.5.44), and (5.5.42) follows.

Substituting (5.5.38) and (5.5.42) back into (5.5.37), we obtain (5.5.3), completing the proof of Proposition III$^+$ and therefore Theorem E. □

## 5.6 An application of Theorem A

In this section we provide an application of Theorem A, relevant to the work of [**CGY02b**] discussed in §5.1. We recall that in [**CGY02b**], the authors establish the existence of smooth solutions $g_{w_\delta} = e^{2w_\delta} g_0$ of positive scalar curvature to the equations

$$\sigma_2(A_{g_{w_\delta}}) = \frac{\delta}{4} \Delta_{g_{w_\delta}} R_{g_{w_\delta}} + f(x, w_\delta) \tag{5.6.1}$$

for each $\delta \in (0, 1]$, where $f \in C^\infty(\mathcal{M}^4 \times \mathbb{R})$ is a carefully chosen positive function that we do not specify here. Moreover, solutions are shown to satisfy the uniform estimates

$$\|w_\delta\|_{W^{2,s}(\mathcal{M}^4, g_0)} \leq C \quad \text{for all } \delta \in (0, 1], \quad 1 \leq s < 5, \tag{5.6.2}$$

where the constant $C = C(s)$ is independent of $\delta$. A Yamabe flow argument is then applied to obtain a conformal metric $g$ with $A_g \in \Gamma_2^+$.

Using Theorem A, we now provide an alternative to the flow argument of [**CGY02b**] in the case that $(\mathcal{M}^4, g_0)$ is locally conformally flat. Roughly speaking, we will use a result of Li & Nguyen [**LN21b**] to assert that, along a subsequence, the solutions $w_\delta$ converge weakly to a $W^{2,s}$-strong solution of $\sigma_2(A_{g_w}) = f(x, w) > 0$. Theorem A will then imply that this solution is smooth.

To this end, fix $4 < s < 5$. By (5.6.2), there is a sequence $\delta_i \to 0$ for which $w_i := w_{\delta_i}$ converges weakly in $W^{2,s}(\mathcal{M}^4, g_0)$ to some $w \in W^{2,s}(\mathcal{M}^4, g_0)$. By the Morrey embedding $W^{2,s}(\mathcal{M}^4, g_0) \hookrightarrow C^{1,1-\frac{4}{s}}(\mathcal{M}^4, g_0)$, we may assume $w_i \to w$ in $C^{1,\alpha}(\mathcal{M}^4, g_0)$ for some $\alpha > 0$. It then follows from [**LN21b, Prop. 5.3**] that for all $\varphi \in C^0(\mathcal{M}^4)$,

$$\lim_{i\to\infty} \int_{\mathcal{M}^4} \sigma_2(A_{g_{w_i}})\varphi \, dv_0 = \int_{\mathcal{M}^4} \sigma_2(A_{g_w})\varphi \, dv_0. \tag{5.6.3}$$

Substituting the equation (5.6.1) into (5.6.3) and integrating by parts, we see that

$$\int_{\mathcal{M}^4} \sigma_2(A_{g_w})\varphi \, dv_0 = \lim_{i\to\infty} \int_{\mathcal{M}^4} \left(\frac{\delta_i}{4} R_{g_{w_i}} \Delta_{g_{w_i}} \varphi + f(x, w_i)\varphi\right) dv_0 = \int_{\mathcal{M}^4} f(x, w)\varphi \, dv_0$$

for all $\varphi \in C^2(\mathcal{M}^4, g_0)$, and it follows that $w \in W^{2,s}(\mathcal{M}^4, g_0)$ satisfies

$$\sigma_2(A_{g_w}) = f(x, w) > 0 \quad \text{a.e. in } \mathcal{M}^4. \tag{5.6.4}$$

In order to apply Theorem A, it remains to show that $R_{g_w} > 0$ a.e. in $\mathcal{M}^4$. Clearly, $R_{g_w} \geq 0$ since $R_{g_{w_i}} > 0$ for all $i$. On the other hand, by (5.6.4)

$$0 < \sigma_2(A_{g_w}) = \frac{1}{2}\big(\sigma_1(A_{g_w})^2 - |A_{g_w}|^2\big) \leq \frac{1}{2}\sigma_1(A_{g_w})^2, \tag{5.6.5}$$

which implies either $R_{g_w} > 0$ or $R_{g_w} < 0$. Only the former possibility is compatible with $R_{g_w} \geq 0$.

Therefore, if $(\mathcal{M}^4, g_0)$ is locally conformally flat, we obtain from Theorem A that $u := e^{-w} \in C^{1,1}(\mathcal{M}^4, g_0)$, and consequently (5.6.4) is uniformly elliptic at $w$. We wish to apply the Evans-Krylov theorem to assert that $u \in C^{2,\alpha}(\mathcal{M}^4, g_0)$. Indeed, by the proof of [**CC95, Theorem 6.6**], it suffices to observe that, by Lemmas 2.4.1 and 2.4.16, $v = \sum_l \Delta_{ll}^h u$ is a subsolution to a uniformly elliptic linear equation, namely

$$F^{ij}\nabla_i\nabla_j v + B^i D_i v \geq C,$$

where $F^{ij}$ is uniformly elliptic and $F^{ij}$, $B^i$ and $C$ are essentially bounded. Thus $u \in C^{2,\alpha}(\mathcal{M}^4, g_0)$, and since $f^{1/2} \in C^\infty(\mathcal{M}^4, g_0)$, standard elliptic regularity ensures that $u$ (and hence $w$) belongs to $C^\infty(\mathcal{M}^4, g_0)$.

# Appendix

## 5.A Inequalities involving the trace-modified Schouten tensor

In this appendix, we first show that that in three dimensions, $g^{-1}A_g^\tau \in \Gamma_2^+$ implies $\mathrm{Ric}_g > 0$ if $\tau \in [\frac{2}{3}, 1]$. We in fact obtain an explicit lower bound that will be used throughout our analysis in Chapter 5:

**Proposition 5.A.1.** *Suppose $(\mathcal{M}^n, g)$ satisfies $R_g > 0$ and fix $\tau \in [2 - \frac{4}{n}, 2]$. Then there exists a constant $K = K(n, \tau) > 0$ such that*

$$\mathrm{Ric}_g(X, X) \geq \frac{n-1}{KR_g}\sigma_2(A_g^\tau)|X|^2. \tag{5.A.1}$$

*Therefore, if $R_g > 0$ and $\sigma_2(A_g^\tau) > 0$ for some $\tau \in [2 - \frac{4}{n}, 2]$ (necessarily with $\tau \neq \frac{2(n-1)}{n}$ by the proof of Proposition I), it holds that* $\mathrm{Ric}_g > 0$.

*Proof.* We will use the basic result (see e.g. [**SW71, p.234**]) that for a traceless $(n \times n)$-matrix $M = (m_{ij})$ with Hilbert-Schmidt norm

$$|M| = \left(\sum_{i,j} m_{ij}^2\right)^{\frac{1}{2}},$$

it holds that

$$\max_{v \in \mathbb{S}^{n-1}} |Mv|^2 \leq \frac{n-1}{n}|M|^2.$$

Applying this to the traceless Ricci tensor $\mathring{\mathrm{Ric}}_g = \mathrm{Ric}_g - \frac{1}{n}R_g g$ gives

$$|\mathring{\mathrm{Ric}}_g(X, X)| \leq \sqrt{\frac{n-1}{n}}|\mathring{\mathrm{Ric}}_g||X|^2.$$

Using this inequality and subsequently applying Cauchy's inequality, we see that for any $K > 0$ it holds that

$$
\begin{aligned}
\operatorname{Ric}_g(X,X) &\geq -\sqrt{\frac{n-1}{n}}|\mathring{\operatorname{Ric}}_g||X|^2 + \frac{R_g}{n}|X|^2 \\
&\geq -\left(|\mathring{\operatorname{Ric}}_g|\sqrt{\frac{n-1}{2KR_g}}\right)^2|X|^2 - \left(\sqrt{\frac{R_gK}{2n}}\right)^2|X|^2 + \frac{R_g}{n}|X|^2 \\
&= \frac{n-1}{KR_g}\left( - \frac{|\mathring{\operatorname{Ric}}_g|^2}{2} - \frac{R_g^2K^2}{2n(n-1)} + \frac{R_g^2K}{n(n-1)}\right)|X|^2. \qquad (5.A.2)
\end{aligned}
$$

Note that we have used the fact $R_g > 0$ in applying Cauchy's inequality. Since $\sigma_2(A_g^\tau) = -\frac{|\mathring{\operatorname{Ric}}_g|^2}{2} + \frac{1}{2}q_n(\tau)$ (see (5.4.9) and (5.4.10)), the estimate (5.A.1) will follow from (5.A.2) if we can choose $K$ real and positive such that

$$
\frac{1}{2}q_n(\tau) = \frac{-K^2}{2n(n-1)} + \frac{K}{n(n-1)}. \qquad (5.A.3)
$$

It is routine to check that (5.A.3) has solutions

$$
K = \frac{2 \pm \sqrt{4 - 4(n-1)^2 + 4n(n-1)\tau - n^2\tau^2}}{2},
$$

so that $K$ can be chosen real and positive if and only if $\tau \in [2 - \frac{4}{n}, 2]$. □

**Remark 5.A.2.** Taking $\tau = 1$ in Proposition 5.A.1, we see that $A_g \in \Gamma_2^+$ implies $\operatorname{Ric}_g > 0$ when $n = 3, 4$, in agreement with [**GVW03**]

A similar lower bound can be obtained for the Einstein tensor $G_g = -\operatorname{Ric}_g + \frac{R_g}{2}g$. Observing that $G_g = -\mathring{\operatorname{Ric}}_g + \frac{n-2}{2n}R_g g$, a calculation analogous to the one above yields

$$
\begin{aligned}
G_g(X,X) &\geq -\sqrt{\frac{n-1}{n}}|\mathring{\operatorname{Ric}}_g||X|^2 + \frac{n-2}{2n}R_g|X|^2 \\
&\geq \frac{n-1}{KR_g}\left( - \frac{|\mathring{\operatorname{Ric}}_g|^2}{2} - \frac{R_g^2K^2}{2n(n-1)} + \frac{(n-2)R_g^2K}{2n(n-1)}\right)
\end{aligned}
$$

for all $K > 0$. Then $\frac{1}{2}q_n(\tau) = -\frac{K^2}{2n(n-1)} + \frac{(n-2)K}{2n(n-1)}$ has a positive real solution for $K$ if and only if $\tau \in [1, 3 - \frac{4}{n}]$, which gives us a counterpart to Proposition 5.A.1:

**Proposition 5.A.3.** *Suppose $(\mathcal{M}^n, g)$ satisfies $R > 0$ and fix $\tau \in [1, 3 - \frac{4}{n}]$. Then there exists a constant $K = K(n,\tau) > 0$ such that*

$$G_g(X,X) \geq \frac{2}{KR_g}\sigma_2(A_g^\tau)|X|^2. \tag{5.A.4}$$

We see that in contrast to Proposition 5.A.1, in no dimension does Proposition 5.A.3 allow us to take $\tau < 1$ and obtain positivity of the Einstein tensor from positive $\sigma_2(A_g^\tau)$. This is not surprising in light of the ellipticity properties discussed in §2.3 and the fact that $G_g$ is precisely the first Newton tensor of $A_g$. However, by taking $\tau = 1$ and appealing to (5.4.68), when $n = 3$ we have

$$G_g(X,X) \geq \frac{4}{R_g}\sigma_2(A_g)|X|^2 = \frac{4}{R}\left[\sigma_2(A_g^\tau) - \frac{1}{2}\left(q(\tau) - \frac{1}{24}\right)R_g^2\right]|X|^2. \tag{5.A.5}$$

The negative $R_g^2$ term on the RHS of (5.A.5) will introduce some unfavourable terms in the proof of Theorem E when $\tau < 1$. However, by keeping a careful track of constants (see the paragraph above (5.5.8)), these negative terms will be dominated by positive terms also appearing when $\tau < 1$.

# 5.B Some properties of the Bach tensor

## 5.B.1 The conformal transformation law for the Bach tensor

Here we prove the transformation law (5.4.18) for the Bach tensor in three dimensions,

$$B_{ik} = e^{-2w}\Big((B_0)_{ik} + O(|\nabla_0 w|_0)\Big), \tag{5.B.1}$$

whee $O(|\nabla_0 w|_0)$ denotes terms bounded above by $C(g_0)|\nabla_0 w|_0$.

*Proof.* Using the standard formula for the covariant derivative of a $(0,3)$-tensor in terms of partial derivatives and Christoffel symbols (see e.g. [**Bes87**]), we observe

$$\begin{aligned} B_{ik} = \nabla^j C_{ijk} &= e^{-2w}g_0^{pj}\nabla_p C_{ijk} \\ &= e^{-2w}g_0^{pj}\Big[\partial_p C_{ijk} - C_{djk}\Gamma_{pi}^d - C_{idk}\Gamma_{pj}^d - C_{ijd}\Gamma_{pk}^d\Big] \\ &= e^{-2w}g_0^{pj}\Big[\partial_p C_{ijk}^0 - C_{djk}^0\Gamma_{pi}^d - C_{idk}^0\Gamma_{pj}^d - C_{ijd}^0\Gamma_{pk}^d\Big], \end{aligned} \tag{5.B.2}$$

where to reach the last line we have used the conformal invariance of the Cotton tensor.

On the other hand, we have the following standard formula for the Christoffel symbols under a conformal change of metric $g = e^{2w} g_0$ (see e.g. [**Bes87**]),

$$\Gamma^m_{jk} = (\Gamma^m_{jk})_0 + \delta^m_k \partial_j w + \delta^m_j \partial_k w - (g_0)^{ms}(g_0)_{jk}\partial_s w = (\Gamma^m_{jk})_0 + O(|\nabla_0 w|_0), \qquad (5.B.3)$$

and substituting (5.B.3) into each Christoffel symbol on the last line of (5.B.2), we obtain (5.B.1). □

### 5.B.2 An identity for the Bach tensor

In this section we prove the identity (5.4.21) for the Bach tensor in three dimensions, which we recall here:

$$B_{ij} = 3 \operatorname{Ric}_i{}^k \operatorname{Ric}_{jk} - \frac{3R}{2} \operatorname{Ric}_{ij} - |\operatorname{Ric}|^2 g_{ij} + \frac{1}{2} R^2 g_{ij} + \frac{1}{4} \nabla_i \nabla_j R - \Delta A_{ij} \qquad (5.B.4)$$

*Proof.* Directly from the definitions of $A$, $B$ and $C$, we see

$$B_{ij} = \nabla^p C_{ipj} = \nabla^p \nabla_j A_{ip} - \Delta A_{ij} = \nabla^p \nabla_j \operatorname{Ric}_{ip} - \frac{1}{4} \nabla_i \nabla_j R - \Delta A_{ij}. \qquad (5.B.5)$$

The remainder of the calculation amounts to computing $\nabla^p \nabla_j \operatorname{Ric}_{ip}$. Commuting derivatives and applying the twice contracted Bianchi identity, we first have

$$\begin{aligned} \nabla^p \nabla_j \operatorname{Ric}_{ip} &= \nabla_j \nabla^p \operatorname{Ric}_{ip} + R_{pjik} \operatorname{Ric}^{kp} + R^p{}_{jpk} \operatorname{Ric}_i{}^k \\ &= \frac{1}{2} \nabla_j \nabla_i R + R_{pjik} \operatorname{Ric}^{kp} + \operatorname{Ric}_{jk} \operatorname{Ric}_i{}^k . \end{aligned} \qquad (5.B.6)$$

Next, we recall from (2.1.10) (keeping in mind that $W \equiv 0$) that $R_{pjik} = g_{pi} A_{jk} - g_{pk} A_{ij} - g_{ij} A_{pk} + g_{jk} A_{pi}$, from which we obtain

$$\begin{aligned} R_{pjik} \operatorname{Ric}^{kp} &= \operatorname{Ric}^{kp} \Big[ g_{pi} \Big( \operatorname{Ric}_{jk} - \frac{1}{4} R g_{jk} \Big) - g_{pk} \Big( \operatorname{Ric}_{ij} - \frac{1}{4} R g_{ij} \Big) \\ &\qquad - g_{ij} \Big( \operatorname{Ric}_{pk} - \frac{1}{4} R g_{pk} \Big) + g_{jk} \Big( \operatorname{Ric}_{pi} - \frac{1}{4} R g_{pi} \Big) \Big] \\ &= 2 \operatorname{Ric}_{jk} \operatorname{Ric}_i{}^k - \frac{3R}{2} \operatorname{Ric}_{ij} + \frac{1}{2} R^2 g_{ij} - |\operatorname{Ric}|^2 g_{ij}. \end{aligned} \qquad (5.B.7)$$

Substituting (5.B.7) into (5.B.6), then (5.B.6) back into (5.B.5), we arrive at (5.B.4).

□

# Chapter 6

# Closing remarks

In this thesis we have obtained new estimates and regularity results for some fully nonlinear equations arising in conformal geometry. In Chapter 2 we obtained local pointwise second derivative estimates for $W^{2,p}$-strong solutions to a class of augmented Hessian equations. This class included the $\sigma_k$-Yamabe equation on Euclidean domains in both the positive and negative cases. In Chapter 3 we obtained similar estimates for smooth solutions on manifolds when $k = 2$. Our work contributes to a growing literature on the regularity theory for the $\sigma_k$-Yamabe equation and, from a broader perspective, the regularity theory for fully nonlinear, non-uniformly elliptic equations.

In Chapter 4 we established the existence of conformal metrics satisfying $A_g^\tau \in \Gamma_2^+$ in three dimensions, under natural assumptions on the background metric. We obtained a new existence result when $\tau < 1$, and developed a new proof of an existing result when $\tau = 1$, which we hope will be useful in tackling some related problems. The existence problems considered here are of relevance in the context of the $\sigma_k$-Yamabe problem, and are also of independent geometric and topological interest, with direct implications on the existence of conformal metrics with positive Ricci curvature on closed 3-manifolds.

Finally, in Chapter 5 we obtained *a priori* integral estimates for fourth order perturbations of the (trace-modified) $\sigma_2$-Yamabe equation. Our study of these equations was partly motivated by the existence problems considered in Chapter 4, but was also motivated from the analytical viewpoint of using fourth order regularisations to study non-uniformly elliptic equations of second order. In this context we also gave an application of our work in Chapter 2.

To conclude, we touch upon what we believe to be some pertinent questions leading on from the results of this thesis.

- *Can we extend the local pointwise second derivative estimates of Theorems A, B etc. to Riemannian manifolds that are not locally conformally flat?*

It is not immediately clear how to modify our method involving difference quotients to obtain local estimates on arbitrary manifolds. It may be that a different way of approximating the Laplace-Beltrami operator is better suited, e.g. as the average over balls. A restriction to the case $k = 2$ is reasonable if one hopes to use the divergence structure.

- *Can we lower the starting exponent $p$ in Theorems A, B etc.?*

We have seen an improvement on the starting exponent $p$ for certain values of $k$ in §2.7, but it would be particularly interesting to determine the sharp lower bounds for $p$ in Theorems A and B. We note that once $p \leq n$, the Morrey embedding theorem no longer implies our solution is in $C^{0,1}_{\mathrm{loc}}(\Omega)$, and in this case it would also be interesting to explore local pointwise first derivative estimates.

- *Can we obtain a $W^{2,p}$ estimate (depending on a $C^1$ bound) on smooth solutions to the $\sigma_k$-Yamabe equation in the negative case, for any $p \geq 1$?*

As we have discussed, it is a major open problem as to whether one can establish an *a priori* second derivative estimate (depending on a $C^1$ bound) on solutions to the $\sigma_k$-Yamabe equation in the negative case. An intermediary problem is to establish a $W^{2,p}$ estimate for some $p \geq 1$, or to establish conditions under which such an estimate holds. If one can obtain sufficiently strong $W^{2,p}$ estimates, then a result such as Theorem C may yield the full $C^2$ estimate.

• *Can we obtain an affirmative answer to Question 1 in Chapter 4?*

We would like to remove our reliance on the variational structure in the proof of Proposition 4.1.9, in order to extend our method to the case $\tau < 1$. There is also the possibility of addressing Question 1 by extending our work in Chapter 5, which brings us on to our final problem.

• *In three dimensions: can we establish suitable conformally invariant conditions on* $g_0$ *which ensure that* (5.0.1) *admits positive scalar curvature solutions for* $\delta$ *in some positive range* $\delta \in (0, \delta_0]$ *and some positive function* $f$*?*

More generally, it remains an interesting open problem to determine natural conformally invariant conditions on a closed 3-manifold $(\mathcal{M}^3, g_0)$ which ensure the existence of a conformal metric satisfying $A_g^\tau \in \Gamma_2^+$. An existence result for (5.0.1) combined with Theorem E is one possible approach.

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