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Abstract

This paper provides a revealed preference characterisation of quasi-hyperbolic discounting which is designed to be applied to readily-available expenditure surveys. We describe necessary and sufficient conditions for the leading forms of the model and also explore the consequences the restrictions on preferences popularly used in empirical lifecycle consumption models. Using data from a household consumption panel dataset we explore the prevalence of time-inconsistent behaviour. The sophisticated quasi-hyperbolic model provides a significantly more successful account of behaviour than the alternatives considered. We estimate the joint distribution of time preferences and the distribution of discount functions at various time horizons.

Key Words: Quasi-hyperbolic discounting, revealed preference.

JEL Classification: D11, D12, D90.

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1 Introduction

This paper derives a revealed preference characterisation of quasi-hyperbolic discounting which is designed to be applied to expenditure survey data. We show how to use it to evaluate alternative dynamic consumption models and to recover the joint distribution of time preferences. We carry out a substantive application using a large, nationally representative expenditure survey.

For behavioural economists, the fact that a revealed preference condition exists, showing that the hyperbolic model has inherent empirical content which is not driven by auxiliary parametric assumptions, is a significant result. It means that despite its great flexibility the model is falsifiable, and hence meaningful in the Samuelsonian sense, in a manner comparable to the classical utility maximisation model.[1] Furthermore, the fact that our results apply to observational data on realised non-durable expenditures shows that the ability to detect, measure and recover hyperbolic preferences is not just confined to the lab or to artefactual field experiments or to other similarly rich, though arguably artificial, decision-making environments.

For applied theorists interested in the empirical implications of models, the results here help to further extend revealed preference methods beyond simple neo-classical models. This is of interest because revealed preference theory usually exploits some self-consistency property in the individual's behaviour. The hyperbolic model, however, explicitly implies realised choice behaviour that is *inconsistent* in some respects. Nonetheless we show that it is possible to characterise the model using realised choices without requiring knowledge of the agent's plans.

Applied empirical economists interested in modelling expenditure survey data will find in this paper a set of simple empirical procedures, based on the behaviour of certain index numbers, which will allow them to check whether behaviour is consistent with hyperbolic discounting. We also provide easy-to-apply metrics which can be used to compare the empirical performance of alternative consumption models, and we show how our theory-driven restrictions can be applied to recover the joint distribution of time preferences and the distribution of discount functions at different time horizons.

The assumption of exponential discounting with its constant discount rate is parsimonious since it allows a person's time preference to be summarised as a single parameter. It is also relatively easy to work with since it makes the strong prediction that the consumer's intertemporal preferences are time-consistent. However a certain amount of evidence has accrued which indicates that people often do not behave in a time-consistent manner and in fact have a tendency towards present bias.[2] As a result the quasi-hyperbolic discounting model has been

[1]Price-quantity data are consistent with the hypothesis that they were generated by a rational consumer with well-behaved preferences if, and only if, they satisfy the Generalised Axiom of Revealed Preference (GARP). Afriat (1967), Diewert (1973) and Varian (1982). See below.

[2]See Frederick *et al* (2002) for a survey of the empirical literature. Samuelson foreshadowed precisely this in his original article, writing: "Actually, however, as the individual moves along in time there is a sort of perspective phenomenon in that his view of the future in relation to his instantaneous time position remains invariant, rather than his evaluation of any particular year (e.g. 1940). This relativity effect is expressed in the behaviour of men who make irrevocable trusts, in the taking out of life insurance as a compulsory savings

put forward as an alternative form which can incorporate present bias into preferences.[3,4]

In this paper we study quasi-hyperbolic consumption behaviour from a revealed preference perspective in the manner of Samuelson (1948), Houthakker (1950) and Afriat (1967). Rather than describing the implications of the theory in terms of unobserved, and therefore estimated, structural equations (Euler equations for example), revealed preference uses systems of inequalities which depend neither on strong functional form assumptions nor on the behaviour of unobservables. Statistical error terms and special assumptions about the functional form of the economic model may be added but it is not an essential requirement of the approach. The classic example is the Generalised Axiom of Revealed Preference (GARP), which is a necessary and sufficient condition for the standard utility maximisation model with competitive linear pricing. GARP is a simple system of inequalities (or an equivalent linear program) which assumes little about the functional form of preferences but which exhausts the empirical implications of the model. In this paper we are, in essence, asking whether there is a GARP-like condition for the quasi-hyperbolic consumption model which only requires data on realised expenditures, spot prices and interest rates.

We focus on the consumption model for a number of reasons. The first is that consumption behaviour matters in both macro and microeconomics. Consumption by households accounts for around 60% of GDP among OECD countries and it is therefore important that we understand consumption decisions.

The second is that we would like to develop simple nonparametric methods which can be applied to readily-available expenditure survey data. The strongest evidence in favour of hyperbolic discounting behaviour often comes from the laboratory or artefactual field experiments[5] that allow researchers to offer participants the ability to make identifable committed payments and to observe both the subjects' hypothetical plans as well as their realised choices. The data recorded in expenditure surveys, by contrast, are often far less rich – we only ever

measure, etc." (Samuelson, (1937, p. 160)).

[3]Strotz (1955-1956) considered non-exponential discounting and Phelps and Pollak (1968), and then Elster (1979) studied the, now firmly established, $\beta\delta$ form. Laibson (1997,1998) and Harris and Laibson (2001) in particular have analysed the implications of this form extensively.

[4]The model has now been widely adopted and applied to describe a range of phenomena from the role of illiquid assets as commitments (Laibson (1997)), the excess sensitivity of consumption to income and the retirement savings puzzle (Laibson (1998)), the simultaneous holding by households of high pre-retirement wealth, low liquid assets and high credit-card debt (Angeletos et al. (2001)), labour supply and welfare programme participation (Fang and Silverman (2009)), procrastination in a number of contexts (Fischer (1999) and O'Donoghue and Rabin (1999, 2001), addiction/habit formation(O'Donoghue and Rabin (2000), Gruber and Koszegi (2001), and Carrillo (1998)), information acquisition (Carrillo and Mariotti (2000) and Benabou and Tirole (2002)) and the Phillips curve (Graham and Snower (2008)). This is a far from exhaustive list of applications but it serves to give a sense of the breadth of topics to which the model has been fruitfully applied. The emerging consensus seems to be that, compared to the exponential model, quasi-hyperbolic discounting better fits the stylised facts regarding individuals' inter-temporal behaviour and the experimental evidence. Indeed, Frederick *et al* (2002, p361) conclude that "the collective evidence ... seems overwhelmingly to support hyperbolic discounting". Nonetheless, we note that there are some more recent experimental designs and studies whose findings have not unambiguously supported hyperbolic discounting (for example Andersen *et al* (2014), Andreoni and Sprenger (2012), Benhabib, Bisin and Schotter (2010)).

[5]The terminology is from Harrison and List (2004). See for example Thaler (1981), Benzion, Rapoport and Yagil (1989), Kirby and Herrnstein (1996). See also the discussions on experimental methods in Benhabib, Bisin and Schotter (2010) as well as Rubinstein (2001, 2003) and Levitt and List (2007).

see what the surveyed households actually do, never what they plan to do. Our approach is therefore different from, but hopefully complementary to, that of Echenique, Imai and Saito (2013/15) whose characterisations of time-inconsistent behaviour are based on the idea that, for each consumer, researchers can observe and compare several different *planned* and *pre-committed* consumption profiles for a single consumption good resulting from different temporal price paths and budgets, i.e. low dimensional *stated* or experimental preference data. We hope therefore that the conditions described here are likely to be useful for empirical researchers considering how best to capture observed behaviour in consumption survey data.

Thirdly, consumption behaviour is a good focus of study for our purposes because the survey data are often particularly abundant. Time-inconsistency may be an important feature of households' decisions to invest in a house or an individual's decision to invest in education, but such decisions are made rather infrequently. Consumption decisions, on the other hand, are made all the time by households and so, as argued by Angeletos *et al* (p.65, 2001), provide an excellent context in which to study inter-temporal models. In particular the frequency of these types of choices by households mean that we can be vastly more flexible about allowing for preference heterogeneity between households as we can effectively model households using their own idiosyncratic time-series rather than having to pool across different households.

The plan of the paper is as follows. In section 2 we derive necessary and sufficient revealed preference conditions for a number of related models. These are based on a dataset consisting of expenditures on a number of goods and services, their corresponding nominal prices and an interest rate. We first derive the conditions for a "sophisticated" quasi-hyperbolic consumer who is aware that their future self may have different preferences over consumption profiles, but who is not assumed to be able to pre-commit to a consumption plan. We show that the conditions consist of two elements: within-period preferences over goods must satisfy GARP whilst inter-temporal behaviour is characterised by the evolution of certain quantity index numbers. This provides a useful diagnostic: researchers can disentangle violations caused by unstable preference over goods from violations caused by non-hyperbolic inter-temporal choices. We also provide several related results: we explore the number of observations required to reject the model; the identification of time-preferences; the empirical special case where a single composite consumption good is assumed to be observed and the theoretical case in which preferences are assumed to be in the hyperbolic absolute risk aversion family which comprises (amongst others) exponential utility, power utility, and therefore isoelastic utility as special cases. We then provide a set of parallel conditions, including an observational equivalence result, for a model of a "naive" individual who is a hyperbolic discounter but wrongly assumes that his future selves will simply fall into line with the consumption plan which he maps out.

In section 3 we carry out a substantive empirical application using a large, nationally representative expenditure panel survey: the Spanish Continuous Family Expenditure Survey (the *Encuesta Continua de Presupuestos Familiares* - ECPF). The ECPF is a quarterly budget survey of Spanish households which interviews about 3,200 households every quarter and in which it is possible to follow a participating household for up to eight consecutive quarters. We examine the non-durable expenditure behaviour of the survey households for consistency with various forms of quasi-hyperbolic discounting as well as exponential discounting. By treating the data for each household as a separate short time-series, we are able to do so whilst allowing for the maximal degree of preference heterogeneity. We note that, since it contains an extra free parameter compared to the standard exponential model, the hyperbolic model *necessarily* can fit the data no worse than the exponential model. We consider this issue in some detail and provide methods for comparing the empirical performance of these models on the basis of their predictive and informational content.

We show that, even making careful allowance for the extra degree of freedom, quasi-hyperbolic discounting provides a significantly more successful account of behaviour than the standard exponential model. We show that the prevalence of hyperbolic behaviour is sensibly correlated with a number of household attributes and choices related to long-term behaviour such as owner occupation, smoking and health expenditures. We also note a strong, non-linear association with total household expenditure which we use as a rough approximation to the overall resources in the household. As these grow, the prevalence of hyperbolic behaviour declines albeit at a declining rate. We then use the conditions for the model to recover an estimate of the joint distribution of time-preferences. The average value of the exponential factor in the sample is close to 0.95, whilst the hyperbolic discount factor is lower: around 0.84. We find considerable evidence of heterogeneity in discount rates between households. We conclude by providing an estimate of the distribution of quasi-hyperbolic discount functions at various time horizons.

In section 4 we offer some conclusions and discuss avenues for further work.

2 Conditions for the Quasi-hyperbolic Consumption Model

We are interested in the empirical implications of the quasi-hyperbolic discounting model for a finite dataset of interest rates, spot prices and purchases of goods for a mortal, self-aware individual who knows the future course of prices and interest rates but who has no ability to commit to their consumption plans.

Our data consists of a vector of transactions for K market goods in each period, their corresponding prices and the interest rate (denoted $\mathbf{c}_t$, $\mathbf{p}_t$ and r_t respectively, where t indexes the observation) for an individual household over time.

$$\{r_t, \mathbf{p}_t, \mathbf{c}_t\}$$

We make the natural assumption that we will only have data on an individual for part of their life. That is, we assume that the agent lives for $T+1$ periods $\{0, ..., T\}$ but that we only observe a contiguous subset of periods denoted by $\tau \subset \{1, ..., T\}$, rather than their entire lives. We will denote the number of observations by $|\tau|$ and members of the set of observations by t. Where some arguments require discussion of the terminal period T we make it clear that this period is not necessarily observed and that $\max\{\tau\} \leq T$.

2.1 The sophisticated individual

We take what we consider to be a standard version of the quasi-hyperbolic consumption model in which the individual is cast as a composite of temporal selves indexed by their respective periods of control over the consumption decision. During their period of control, self t inherits the current level of total wealth[6] A_t and chooses a consumption bundle for period t, such that

$$\boldsymbol{\rho}_t' \mathbf{c}_t + \Sigma_t = \Delta_t$$

where we denote everything in discounted terms, so $\Delta_t = A_t / \prod_{i=1}^{t} (1 + r_i)$, denotes discounted wealth and $\rho_t^k = p_t^k / \prod_{i=1}^{t} (1 + r_i)$, denotes discounted prices. Discounted savings are denoted Σ_t. Self $t+1$ then inherits wealth equal to $\Delta_{t+1} = \Sigma_t$. The game continues, with self $t+1$ in control. We assume that in the final period of life $\Sigma_T = 0$.[7] The payoff for the t'th player of this game is

$$U(\mathbf{c}_t, \mathbf{c}_{t+1}, ..., \mathbf{c}_T) = u(\mathbf{c}_t) + \beta \sum_{i=1}^{T-t} \delta^i u(\mathbf{c}_{t+i})$$

where u is a concave, continuous and differentiable instantaneous utility function, δ is the standard exponential discount factor and β is the additional discount term that the quasi-hyperbolic model introduces.

The sophisticated hyperbolic model is a non-cooperative game between the temporal selves, and a rigorous derivation of necessary conditions is provided in Harris and Laibson (2001). For our purposes it suffices to use the following Lemma to motivate what follows and to characterise the implications of optimising behaviour in this model using the first order condition on the equilibrium path (see Harris and Laibson (2001)).

Lemma 1. *(Harris and Laibson (2001)). On the equilibrium path:*

$$\frac{\partial u}{\partial c_t^k} = \delta^{-t} \lambda \rho_t^k \prod_{i=1}^{t} \left[1 - (1-\beta) \sum_{k=1}^{K} \left(\rho_i^k \frac{\partial c_i^k}{\partial \Delta_i} \right) \right]^{-1} \qquad \forall k, t$$

and the corresponding Euler equation is

$$\frac{\partial u}{\partial c_t^k} = \delta \frac{\rho_t^k}{\rho_{t+1}^k} \left[1 - (1-\beta) \sum_{k=1}^{K} \left(\rho_{t+1}^k \frac{\partial c_{t+1}^k}{\partial \Delta_{t+1}} \right) \right] \frac{\partial u}{\partial c_{t+1}^k} \qquad \forall k, t$$

where λ is a strictly positive constant (the marginal utility of wealth in period 0), $\beta \in (0,1)$, $\delta \in (0,1]$ and $\partial c_t^k / \partial \Delta_t$ is the marginal propensity to consume the kth good out of current discounted wealth.

Proof. See the Appendix. □

[6] That is, current financial wealth plus discounted future earnings.

[7] This is without loss of generality over $\Sigma_T \geq 0$ since the lifetime budget can always be defined accordingly.

Both expressions are slight multi-good generalisations of those in Harris and Laibson (2001)[8] and readily reduce to those in Harris and Laibson (2001) if we consider a single consumption good, no inflation and a fixed interest rate. These expressions would simplify to the standard exponential discounting case if we were to allow $\beta = 1$, but our characterisation focuses on the strict hyperbolic case where $0 < \beta < 1$. The object

$$\sum_{k=1}^{K}\left(\rho_{t+1}^{k}\frac{\partial c_{t+1}^{k}}{\partial \Delta_{t+1}}\right)$$

is the period $t+1$ marginal propensity to spend out of wealth. We denote this by μ_t. We assume that demands are normal and therefore $\mu_t \in (0,1)$ (except in the last period of life where it equals one and all remaining wealth is consumed).

Following Browning (1989) we define what it means for the model to rationalise the data as follows.

Definition 1. *The sophisticated quasi-hyperbolic discounting model rationalises the data* $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\tau}$ *if there exists a locally non-satiated, differentiable and concave instantaneous utility function* $u(.)$ *and constants* $\lambda > 0$, $\beta \in (0,1)$, $\delta \in (0,1]$, $\mu_T = 1$ *and* $\{\mu_t \in (0,1)\}_{t\in\tau\backslash T}$ *such that*

$$\frac{\partial u}{\partial c_t^k} = \lambda\frac{\rho_t^k}{\delta^t}\prod_{i=1}^{t}\frac{1}{[1-(1-\beta)\,\mu_i]} \qquad \forall k \tag{1}$$

This says that the data are consistent with the theory if there exists a well-behaved instantaneous utility function, the derivatives of which satisfy the sophisticated hyperbolic first order conditions (or equivalently the Euler equation) on the equilibrium path. If such a utility function exists, and we know what it is, then "rationalisability" means that we could precisely replicate the observed choices of the consumer. If we were to set $\beta = 1$ then this definition would simplify to the rationalisability definition given by Browning (1989) for the exponential discounting model. Our main result in this paper is the following:

Proposition 1. *The following statements are equivalent.*
(1) The sophisticated hyperbolic discounting model rationalises the data $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\tau}$.
(2) The data $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\tau}$ *satisfy GARP and the following condition:*

$$\boldsymbol{\rho}_s'\left(\mathbf{c}_s - \mathbf{c}_{s+h}\right) < 0 \Rightarrow \frac{\boldsymbol{\rho}_{s+h}'\left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)}{\boldsymbol{\rho}_s'\left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)} < 1 \tag{RD}$$

for all $s, s+h \in \tau$, $h \geq 1$.

[8]Note that in deriving the first order condition and the Euler equation we follow the heuristic method of Harris and Laibson (2001) - that is we simply make the necessary assumptions regarding the smoothness/differentiability of demands wherever possible. As Harris and Laibson (2001) show, the same results can be derived under less benign conditions. Readers are referred to their paper for a rigorous derivation to which we have nothing to add.

Proof. (1) ⇒ (2)**:** Using Definition 1 and setting

$$\lambda_t = \lambda \frac{1}{\delta^t} \prod_{i=1}^{t} \frac{1}{[1-(1-\beta)\mu_i]} \qquad \forall t \in \tau$$

gives

$$\frac{\partial u(\mathbf{c}_t)}{\partial c_t^k} = \lambda_t \rho_t^k. \tag{2}$$

Note that $\lambda > 0, \beta \in (0,1), \delta > 0, \mu_T = 1$ and $\{\mu_t \in (0,1)\}_{t\in\tau/T}$ together imply that $\{\lambda_t > 0\}_{t\in\tau}$. Equation (2) is thus the first order condition for standard atemporal utility maximisation and hence, by Afriat's Theorem, GARP is satisfied for the data $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\tau}$ which is the first part of condition (2).

To establish (RD) let

$$\Psi_t = \prod_{i=1}^{t} \left(1/\left[1-(1-\beta)\mu_i\right]\right) \tag{3}$$

(with $\Psi_0 = 1$) and rewrite the first order condition for sophisticated hyperbolic discounting in vector notation giving

$$\nabla u(\mathbf{c}_t) = \lambda \frac{\Psi_t}{\delta^t} \boldsymbol{\rho}_t$$

Note that $\beta \in (0,1)$ and $\mu_t \in (0,1] \Rightarrow 1 < \Psi_{t-1} < \Psi_t \;\; \forall t \neq 0$. Concavity of the instantaneous utility function gives

$$u(\mathbf{c}_s) \leq u(\mathbf{c}_t) + \nabla u(\mathbf{c}_t)'(\mathbf{c}_s - \mathbf{c}_t) \qquad \forall s, t \in \tau$$

Substituting in the first order conditions gives

$$u(\mathbf{c}_s) \leq u(\mathbf{c}_t) + \lambda \frac{\Psi_t}{\delta^t} \boldsymbol{\rho}_t'(\mathbf{c}_s - \mathbf{c}_t)$$

and thus the data satisfying sophisticated hyperbolic discounting implies being able to find real numbers $\{u_t, \lambda > 0, \delta \in (0,1], \Psi_t\}_{t\in\tau}$ such that

$$u_s \leq u_t + \lambda \frac{\Psi_t}{\delta^t} \boldsymbol{\rho}_t'(\mathbf{c}_s - \mathbf{c}_t) \qquad \forall s, t \in \tau \tag{4}$$

$$\Psi_0 = 1,\; 1 < \Psi_{t-1} < \Psi_t \quad \forall t \in \tau \tag{5}$$

Writing equation (4) for any pair of chronologically ordered periods $s, s+h \in \tau$ gives

$$u_s \leq u_{s+h} + \lambda \frac{\Psi_{s+h}}{\delta^{s+h}} \boldsymbol{\rho}_{s+h}'(\mathbf{c}_s - \mathbf{c}_{s+h}) \tag{6}$$

$$u_{s+h} \leq u_s + \lambda \frac{\Psi_t}{\delta^t} \boldsymbol{\rho}_s'(\mathbf{c}_{s+h} - \mathbf{c}_s) \tag{7}$$

$$\Rightarrow \delta^h \Psi_s \boldsymbol{\rho}_s'(\mathbf{c}_s - \mathbf{c}_{s+h}) \leq \Psi_{s+h} \boldsymbol{\rho}_{s+h}'(\mathbf{c}_s - \mathbf{c}_{s+h}) \tag{8}$$

Equation (8) gives us the following restrictions on δ.

$$\text{If } \boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h}) > 0 \text{ then } \delta \in \left(0, \left(\frac{\Psi_{s+h}}{\Psi_s} \frac{\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h})}\right)^{1/h}\right] \tag{9}$$

$$\text{If } \boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h}) < 0 \text{ then } \delta \in \left[\left(\frac{\Psi_{s+h}}{\Psi_s} \frac{\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h})}\right)^{1/h}, 1\right] \tag{10}$$

With $|\tau|$ periods, this gives $(|\tau|^2 - |\tau|)/2$ interval restrictions on the exponential discount rate δ which must be satisfied simultaneously. Note that $\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h}) > 0 \Rightarrow u_{s+h} < u_s \Rightarrow \boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h}) > 0$ so the right hand side of equation (9) is always positive. If, in equation (10) $\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h}) > 0$, then the right hand side is negative, but even though we cannot take a real root of these values, they merely imply that $\delta > 0$ (as is obvious from equation (8)) and so we can simply eliminate them from our system of inequalities. Thus equation (10) only gives us new information when $\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h}) < 0$, i.e. when $u_{s+h} > u_s$.

In order for the interval restriction in (10) to be satisfied, it must be the case (since $\Psi_{s+h} > \Psi_s$ when $h \geq 1$) that for all $s, s+h \in \tau$, $h \geq 1$:

$$\begin{aligned} \boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h}) < 0 &\Rightarrow \left(\frac{\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h})}\right)^{1/h} < 1 \\ &\Rightarrow \frac{\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h})} < 1 \end{aligned}$$

which is condition (RD).

(2) $\Rightarrow$ (1): We first show that the data passing GARP implies that the sophisticated hyperbolic discounting model rationalises the data for some $\delta > 0$ (where δ is not necessarily ≤ 1). By Afriat's Theorem, GARP is equivalent to the existence of a well-behaved utility function $u(\mathbf{c})$ and constants $\{\lambda_t > 0\}_{t \in \tau}$ such that

$$\frac{\partial u(\mathbf{c}_t)}{\partial c_t^k} = \lambda_t \rho_t^k \tag{11}$$

Now set a δ defined by

$$\delta > \max \left\{ \frac{\lambda_t}{\lambda_{t+1}} \right\} \quad \forall t, t+1 \in \tau$$

and for all $t \in \tau$ define

$$1 - (1 - \beta) \mu_{t+1} = \frac{1}{\delta} \frac{\lambda_t}{\lambda_{t+1}} \tag{12}$$

Note that, since $\delta > \max \left\{ \frac{\lambda_t}{\lambda_{t+1}} \right\}$, then

$$1 - (1 - \beta) \mu_{t+1} = \frac{1}{\delta} \frac{\lambda_t}{\lambda_{t+1}} < 1$$

Rearranging equation (12) and solving recursively gives:

$$\lambda_t = \frac{\lambda_0}{\delta^t} \prod_{i=1}^{t} \frac{1}{[1 - (1-\beta)\,\mu_i]} \qquad \forall t \in \tau$$

Now we can set $\lambda = \lambda_0$ and substitute into equation (11) to give

$$\frac{\partial u(\mathbf{c}_t)}{\partial c_t^k} = \frac{\lambda}{\delta^t} \prod_{i=1}^{t} \frac{1}{[1 - (1-\beta)\,\mu_i]} \qquad \forall t \in \tau$$

Apart from the restriction that $\delta \leq 1$, this gives us the definition of *rationalise* in Definition 1, since our choice of δ gives $(1 - (1-\beta)\,\mu_{t+1}) < 1$ which means that we can always find a $\beta \in (0,1)$ and $\mu_{t+1} \in (0,1]$ to satisfy the definition.

We have shown that if the data pass GARP then sophisticated hyperbolic discounting rationalises the data for some $\delta > 0$. But if the sophisticated quasi-hyperbolic discounting model rationalises the data then we know that the lower bound on δ is given by equation (10). Equation (10) and the fact we know we must have $\Psi_{s+h} > \Psi_s$ when $h \geq 1$ implies

$$\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h}) < 0 \Rightarrow \delta > \left(\frac{\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h})} \right)^{1/h}$$

If the data satisfy

$$\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h}) < 0 \Rightarrow \left(\frac{\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h})} \right) < 1$$

for all $s, s+h \in \tau$, $h \geq 1$, then this clearly implies

$$\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h}) < 0 \Rightarrow \left(\frac{\boldsymbol{\rho}_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s' (\mathbf{c}_s - \mathbf{c}_{s+h})} \right)^{1/h} < 1 \tag{13}$$

and so if the data satisfy GARP and equation (13) then they can be rationalised by the sophisticated hyperbolic discounting model with a lower bound on δ that is ≤ 1. □

Proposition 1 shows that the hyperbolic model implies GARP plus a further condition (RD). The requirement that the dataset of demands and discounted prices $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t \in \tau}$ satisfies GARP is, in fact, equivalent to the data on demands and spot prices $\{\mathbf{p}_t, \mathbf{c}_t\}_{t \in \tau}$ passing GARP. This is because the discounting of the spot prices does not affect relative prices within periods and the demands themselves are fixed - thus the budget constraints are effectively identical whether prices are discounted or not and are simply expressed in different units. The fact that the within-period data satisfy GARP is essentially a consequence of the inter-temporal separability in the model. Since weak separability is necessary and sufficient for the second stage of two-stage budgeting (Gorman (1959)) this means that however the consumer decides to allocate expenditure across time, expenditure *within* each period is allocated across goods according to the maximisation of stable within-period (instantaneous) preferences over goods. Thus we should expect the GARP condition to arise simply from inter-temporal weak separability and stable within-period preferences, and it is the (RD) condition that is the additional implication arising from hyperbolic discounting.

To gain some insight into (RD), consider equations (9) and (10). Equation (9) applies when an earlier bundle is revealed preferred to a later one (that is, $\boldsymbol{\rho}_s'\mathbf{c}_s > \boldsymbol{\rho}_s'\mathbf{c}_{s+h}$ or, equivalently, when the Laspeyres consumption index falls between these periods), and so the agent has revealed some level of impatience for consumption today over tomorrow. But because impatience due to the exponential discount factor δ and the extra hyperbolic factor β are confounded in observed behaviour, it transpires that seeing $\boldsymbol{\rho}_s'\mathbf{c}_s > \boldsymbol{\rho}_s'\mathbf{c}_{s+h}$ does not allow us to infer anything about discounting. However, equation (10) *does* give us information (beyond $\delta > 0$) for the case when a later bundle is revealed preferred to an earlier one ($\boldsymbol{\rho}_{s+h}'\mathbf{c}_{s+h} > \boldsymbol{\rho}_{s+h}'\mathbf{c}_s$, or equivalently, when the Paasche consumption index rises), since this tells us the agent was willing to postpone consumption given the prices they faced. This tells us something about how patient they are regarding the exponential part of discounting despite the fact that this has been contaminated by extra discounting due to β – i.e. we can still use observed behaviour to put a lower bound on δ because hyperbolic discounting makes overall discounting appear greater than the exponential part. Thus (RD) can be interpreted as saying that if the agent increases consumption between periods s and $s+h$ then the price index between periods s and $s+h$ (defined with respect to changes in consumption) must be falling. Specifically, the parameters that are added to the exponential model by introducing quasi-hyperbolic discounting are $\{\Psi_t\}$, as defined in equation (3), and all that we really know about them is that $\Psi_{s+h}/\Psi_s > 1$, and so equation (10):

$$\delta \geq \left(\frac{\Psi_{s+h}}{\Psi_s} \frac{\boldsymbol{\rho}_{s+h}' \left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)}{\boldsymbol{\rho}_s' \left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)} \right)^{1/h}$$

also tells us that

$$\delta \geq \left(\frac{\boldsymbol{\rho}_{s+h}' \left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)}{\boldsymbol{\rho}_s' \left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)} \right)^{1/h}$$

But equation (9):

$$\delta \leq \left(\frac{\Psi_{s+h}}{\Psi_s} \frac{\boldsymbol{\rho}_{s+h}' \left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)}{\boldsymbol{\rho}_s' \left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)} \right)^{1/h}$$

does not imply that

$$\delta \leq \left(\frac{\boldsymbol{\rho}_{s+h}' \left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)}{\boldsymbol{\rho}_s' \left(\mathbf{c}_s - \mathbf{c}_{s+h}\right)} \right)^{1/h}$$

and so has no bite.

A few further remarks on Proposition 1 are in order. The first is that, for data of the kind we consider, these conditions, being both necessary and sufficient, are *sharp*; they exhaust all of the empirical implications of the model of interest and they apply to all specific instances of sophisticated hyperbolic models that satisfy the general properties we have stated. Second, like all revealed preference type characterisations, it is exact: because the model is deterministic there should be no random variation in the choices the consumer makes and therefore the individual either passes the condition or does not.[9] We return to this point in our conclusions. Third, the conditions are computationally very straightforward to check empirically: they simply require the calculation of various quantity and price indices. In particular there is no need for linear or integer programming, nor to conduct a grid search for rationalising parameter values for β or the exponential discount rate δ.See https://sites.google.com/site/laurablow/working-papers/exponential.pdf for details. Fourthly the restriction that $\delta \in (0, 1]$ is material; if we were to allow consumers to prefer future to present consumption, then the inter-temporal condition (RD) would be lost and the model would only require within-period GARP.[10] In principle there is no reason why individuals might not value the future more than the here-and-now, however there is little empirical support for it: Frederick, *et al*'s (2002) review of the empirical evidence finds that $\delta \in (0, 1]$ in all studies irrespective of the time horizon considered.

We now offer three further simple results concerning (i) the rejectability of the model, (ii) the special case where there is one aggregate consumption good and (iii) the identification of aspects of the model. We begin with the number of observations needed to detect a violation.

Corollary 1. *Assuming that the demand and prices data satisfy GARP, the sophisticated hyperbolic discounting model could in principle be rejected with only two periods of data.*

Proof. See the Appendix. □

Plainly the model could be rejected with just two observations because that is all that is required to reject the GARP-clause of Proposition 1. Nonetheless the result shows that, even assuming the data satisfy GARP, violations of the conditions can still be detected with just two observations. Of course the ability to detect violations of the conditions in Proposition 1 is weakly increasing in the number of observations. The fact that the conditions in Proposition 1 break down nicely into within- and between-period elements like this gives a useful diagnostic: it is possible for the researcher to differentiate between violations which stem from the assumption of inter-temporal separability of preferences, which is common to many inter-temporal models, from violations which stem from condition (RD) which is specific to this particular model.

We now consider a one-good world. Many applications of inter-temporal consumption models use data on a single aggregate consumption good. For such data the GARP requirement for efficient within-period allocation across goods is not directly relevant (although we do note that time-separable rational preferences over goods within each period *are* required for the construction of an economically meaningful consumption aggregate/index number in the first place (Gorman (1959)) and the conditions in Proposition 1 simplify to the following where $K = 1$.

[9] Measurement error is, of course, a potential source of apparently random variation in behaviour. It can be incorporated into a revealed preference approach using the ideas in Varian (1985).

[10] See the discussion in the proof of Proposition 1.

Corollary 2. *The following statements are equivalent.*
(1) The sophisticated hyperbolic discounting model rationalises the data $\{\rho_t, c_t\}_{t\in\tau}$ *.*
(2) The data satisfy the condition: $(c_s - c_{s+h}) < 0 \Rightarrow (\rho_s - \rho_{s+h}) > 0$ *for all* $s, s+h \in \tau$, $h \geq 1$.

Proof. See the Appendix. □

This result is useful in macroeconomic applications of the model and it also serves to shed some light on condition (RD) in Proposition 1: it shows that when consumption is rising it must be the case that discounted prices are falling. This is essentially a Law of Demand type result[11] in an inter-temporal context. In Proposition 1 the intuition is much the same but the presence of vectors of demands and prices means that it is couched in terms of price and quantity index numbers.

Our third result concerns the identification of the discount terms in the model.

Corollary 3. *If the data satisfies Proposition 1 then for the exponential discount factor*

$$\delta \in \left(\max \left\{ \left(\frac{\boldsymbol{\rho}'_{t+j}\left(\mathbf{c}_t - \mathbf{c}_{t+j}\right)}{\boldsymbol{\rho}'_t\left(\mathbf{c}_t - \mathbf{c}_{t+j}\right)} \right)^{1/j} \right\}, 1 \right]$$

for all t, $t{+}j \in \tau$, $\boldsymbol{\rho}'_t\left(\mathbf{c}_t - \mathbf{c}_{t+j}\right) < 0$. *And for the hyperbolic discount factor, given the choice for* δ, *we have bounds which depend on the sign of* $\boldsymbol{\rho}'_{T-1}\left(\mathbf{c}_{T-1} - \mathbf{c}_T\right)$:

$$\beta \in \left(0, \frac{1}{\delta} \min \left\{ \left(\frac{\boldsymbol{\rho}'_{t+j}\left(\mathbf{c}_t - \mathbf{c}_{t+j}\right)}{\boldsymbol{\rho}'_t\left(\mathbf{c}_t - \mathbf{c}_{t+j}\right)} \right)^{1/j} \right\} \right]$$

for all $t, t+j \in \tau$, *if* $\boldsymbol{\rho}'_{T-1}\left(\mathbf{c}_{T-1} - \mathbf{c}_T\right) > 0$.

$$\beta \in \left(\max \left\{ 0, \frac{\boldsymbol{\rho}'_T\left(\mathbf{c}_{T-1} - \mathbf{c}_T\right)}{\boldsymbol{\rho}'_{T-1}\left(\mathbf{c}_{T-1} - \mathbf{c}_T\right)} \right\}, \frac{1}{\delta} \min \left\{ \left(\frac{\boldsymbol{\rho}'_{t+j}\left(\mathbf{c}_t - \mathbf{c}_{t+j}\right)}{\boldsymbol{\rho}'_t\left(\mathbf{c}_t - \mathbf{c}_{t+j}\right)} \right)^{1/j} \right\} \right]$$

for all $t, t+j \in \tau$, *if* $\boldsymbol{\rho}'_{T-1}\left(\mathbf{c}_{T-1} - \mathbf{c}_T\right) < 0$

Proof. See the Appendix. □

We also note that in the case where we have a single consumption good we have the consequence that rising consumption ($c_s - c_{s+h} < 0$) implies that $\delta \in \left(\left(\rho_{s+h}/\rho_s\right)^{1/h}, 1 \right]$ for all $s, s+h \in \tau$, $h \geq 1$ – thus in practice we simply take the maximum over the relevant ratios of discounted prices. The bounds on β follow in a natural way.

As noted above, the conditions in Proposition 1 exhaust all of the empirical implications of the sophisticated hyperbolic model that satisfies the general properties we have stated. It may be of some interest however to consider the kinds of additional requirements which particular special instances of the model imply. We have, thus far, only assumed concavity

[11] We assume that aggregate consumption is a normal good.

of the within-period utility function, but, for example, applied work on lifecycle consumption and saving very commonly use an iso-elastic (constant relative risk aversion (CRRA)) form for the instantaneous utility function, and so it may be useful to ask if restrictions like this have any additional implications for our results.

Our strategy for identifying such implications is based on equations (9) and (10) which tell us that, if we have observations such that $\boldsymbol{\rho}_t'\mathbf{c}_t > \boldsymbol{\rho}_t'\mathbf{c}_{t+j}$ and $\boldsymbol{\rho}_s'\mathbf{c}_s < \boldsymbol{\rho}_s'\mathbf{c}_{s+h}$ (where $t+j$ and $s+h$ denote chronologically later observations than t and s respectively), then this restricts the discount factor in the following manner:

$$0 \leq \left(\frac{\Psi_{s+h}}{\Psi_s}\frac{\boldsymbol{\rho}_{s+h}'(\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s'(\mathbf{c}_s - \mathbf{c}_{s+h})}\right)^{1/h} \leq \delta \leq \left(\frac{\Psi_{t+j}}{\Psi_t}\frac{\boldsymbol{\rho}_{t+j}'(\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t'(\mathbf{c}_t - \mathbf{c}_{t+j})}\right)^{1/j} \leq 1 \tag{14}$$

remembering that the definition of Ψ_t implies $1 < \Psi_{t-1} < \Psi_t \quad \forall t \neq 0$ with $\Psi_0 = 1$. Equation (14) allows us to exploit restrictions on the consumption function, and in particular the marginal propensity to consume out of wealth, which will introduce empirical implications for the sophisticated model beyond GARP. For example, if we observe $\boldsymbol{\rho}_t'\mathbf{c}_t > \boldsymbol{\rho}_t'\mathbf{c}_{t+j}$ and $\boldsymbol{\rho}_s'\mathbf{c}_s < \boldsymbol{\rho}_s'\mathbf{c}_{s+h}$ along with the following:

$$\left(\frac{\boldsymbol{\rho}_{t+j}'(\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t'(\mathbf{c}_t - \mathbf{c}_{t+j})}\right)^{1/j} < \left(\frac{\boldsymbol{\rho}_{s+h}'(\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s'(\mathbf{c}_s - \mathbf{c}_{s+h})}\right)^{1/h} \tag{15}$$

then in order for there to be a δ that satisfies equation (14) it must be that

$$\left(\frac{\Psi_{t+j}}{\Psi_t}\right)^{1/j} > \left(\frac{\Psi_{s+h}}{\Psi_s}\right)^{1/h}$$

which, in turn, implies some restriction on the marginal propensity to spend out of wealth terms, namely:

$$\max\{\mu_{t+i}\}_{i=1,\ldots,j} > \min\{\mu_{s+g}\}_{g=1,\ldots,h} \tag{16}$$

Thus to find further restrictions for the model we need to characterise the circumstances under which requiring $\max\{\mu_{t+i}\}_{i=1,\ldots,j} > \min\{\mu_{s+i}\}_{i=1,\ldots,h}$ also leads to a restriction on observable behaviour.

In the general case, a restriction on the derivative of a within-period expenditure function (i.e. on μ_t) at one point does not tell us anything about the shape of the rest of this function and therefore does not have any implications for observed expenditure. But, if we restrict ourselves to cases where the expenditure function is linear in all periods then requiring $\mu_t > \mu_s$ *would* have some implications for observable behaviour. Because discounted assets decline over time, then when $t < s$ it must be the case that $\Delta_t > \Delta_s$. In this case, if the expenditure function is linear in all periods then requiring $\mu_t > \mu_s$ would imply $\boldsymbol{\rho}_t'\mathbf{c}_t > \boldsymbol{\rho}_s'\mathbf{c}_s$, i.e. that within-period expenditure declines between periods t and s.

Denote the instantaneous indirect utility function by $v(\boldsymbol{\rho}, y)$ where y is total within-period expenditure. Let v' denote $\partial v/\partial y$ and so on for higher derivatives. We can show (see the proof of Proposition 2 in the Appendix) that the expenditure function $y_t(\Delta_t)$ for the sophisticated hyperbolic model is linear in all periods if the ratio of the coefficient of prudence to the coefficient of risk aversion[12] (absolute or relative) is constant. This class includes, for example, the whole of the hyperbolic absolute risk aversion (HARA) family of utility functions which comprises (amongst others) exponential utility, power utility, and therefore iso-elastic utility as special cases. This leads us to the following proposition:

Proposition 2. *If the sophisticated hyperbolic discounting model rationalises the data $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\tau}$ for an agent with an instantaneous utility function such that $v'''v'/(v'')^2$ is constant, then the data $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\tau}$ satisfy the conditions in Proposition 1 and*

$$\forall \left\{\{t+i\}_{i=0,\ldots,j}\ ,\ \{s+g\}_{g=0,\ldots,h}\right\} \in \tau\ ,\ t<s,\ t+j\leq s \text{ and } h\geq 1:$$

$$\boldsymbol{\rho}_t'\mathbf{c}_t > \boldsymbol{\rho}_t'\mathbf{c}_{t+j},\ \boldsymbol{\rho}_s'\mathbf{c}_s < \boldsymbol{\rho}_s'\mathbf{c}_{s+h} \text{ and } \left(\frac{\boldsymbol{\rho}_{t+j}'(\mathbf{c}_t-\mathbf{c}_{t+j})}{\boldsymbol{\rho}_t'(\mathbf{c}_t-\mathbf{c}_{t+j})}\right)^{1/j} < \left(\frac{\boldsymbol{\rho}_{s+h}'(\mathbf{c}_s-\mathbf{c}_{s+h})}{\boldsymbol{\rho}_s'(\mathbf{c}_s-\mathbf{c}_{s+h})}\right)^{1/h} \tag{RS}$$

$$\Rightarrow \max\left\{\boldsymbol{\rho}_{t+i}'\mathbf{c}_{t+i}\right\}_{i=1,\ldots,j} > \min\left\{\boldsymbol{\rho}_{s+g}'\mathbf{c}_{s+g}\right\}_{g=1,\ldots,h}$$

Proof. See the Appendix. □

The intuition for (RS) is best understood in a one-good world setting with a one-period difference in the dates at which consumption is observed ($j=h=1$) where it implies that if consumption decreases between periods t and $t+1$ but increases between periods s and $s+1$, and prices fall more (or increase less) between periods t and $t+1$ than between s and $s+1$, then it must be the case that spending in period $t+1$ is bigger than in period $s+1$. Such behaviour is, of course, ruled out entirely in the exponential model – if consumption increases between any two periods then it must also increase between any other two periods with a larger (smaller) price fall (increase) otherwise no δ exists that can satisfy the model. In the unrestricted sophisticated hyperbolic model this behaviour is allowed (conditional on (RD) being satisfied) since it simply requires choosing $\Psi_{t+1}/\Psi_t > \Psi_{s+1}/\Psi_s$ and therefore $\mu_{t+1} > \mu_{s+1}$ to satisfy equation (14). But if we require that $\mu_{t+1} > \mu_{s+1}$ implies $\boldsymbol{\rho}_{t+1}'\mathbf{c}_{t+1} > \boldsymbol{\rho}_{s+1}'\mathbf{c}_{s+1}$ then we can count $\boldsymbol{\rho}_{t+1}'\mathbf{c}_{t+1} \leq \boldsymbol{\rho}_{s+1}'\mathbf{c}_{s+1}$ as a violation of sophisticated hyperbolic discounting.

2.2 The naive individual

So far we have only considered sophisticated consumers who understand that their future selves will have different preferences to their current self. We now look at the case of an individual who is a hyperbolic discounter but wrongly assumes that his future selves will simply fall into line with the consumption plan which he maps out. The literature (e.g. O'Donoghue and Rabin (1999)) describes this as naivety and means by this that the current self knows themselves to be a hyperbolic discounter with an inclination for immediate gratification, but believes that future selves do not have present-biased preferences and will behave as exponential discounters

[12] That is $v'''v'/(v'')^2$

with $\beta = 1$. That is, in period t they maximise

$$u(\mathbf{c}_t) + \beta \sum_{i=1}^{T-t} \delta^i u(\mathbf{c}_{t+i})$$

but believe that in periods $\varsigma = t+1...T-1$ their future selves will maximise

$$u(\mathbf{c}_\varsigma) + \sum_{i=1}^{T-\varsigma} \delta^i u(\mathbf{c}_{\varsigma+i})$$

The implications for the first order conditions and Euler equation of individuals who behave in this way are given in the next Lemma.

Lemma 2. *On the equilibrium path:*

$$\frac{\partial u}{\partial c_t^k} = \lambda \frac{\rho_t^k}{\delta^t} \Omega_t \quad \forall k, t$$

where $\Omega_0 = 1$, $\Omega_{t-1} < \Omega_t$. *The corresponding Euler equation is*

$$\frac{\partial u}{\partial c_t^k} = \delta \frac{\rho_t^k}{\rho_{t+1}^k} \frac{\Omega_t}{\Omega_{t+1}} \frac{\partial u}{\partial c_{t+1}^k} \qquad \forall k, t$$

where λ *is a strictly positive constant and* $\delta \in (0, 1]$.

Proof. See the Appendix. □

Lemma 2 shows that the equilibrium conditions for the naive individual are structurally very similar (and mathematically identical) to that of the self-aware hyperbolic discounter, where $\Psi_t = \prod_{i=1}^{t} (1/[1-(1-\beta)\mu_i])$ is now replaced by the sequence of constants Ω_t. We have denoted this parameter with a different symbol as Ω_t does not have the same interpretation as Ψ_t. As the naive discounter believes he will follow today's plan tomorrow, he always underestimates how much tomorrow's self will consume. Thus Ω_t is merely a balancing term, devoid of meaningful economic content, which makes the naive first order condition and Euler equation equalities rather than inequalities, whereas Ψ_t relates closely to the marginal propensity to consume out of remaining assets, of which the sophisticated discounter is fully aware. Paralleling Definition 1 we describe the requirements for rationalisability below.

Definition 2. *The naive quasi-hyperbolic discounting model rationalises the data* $\{r_t, \mathbf{p}_t, \mathbf{c}_t\}_{t\in\tau}$ *if there exists a locally non-satiated, differentiable and concave instantaneous utility function* $u(.)$ *and constants* $\lambda > 0$, $\delta \in (0, 1]$, *and* $\{\Omega_t\}_{t\in\tau}$ *such that*

$$\frac{\partial u}{\partial c_t^k} = \lambda \frac{\rho_t^k}{\delta^t} \Omega_t \qquad \forall k \tag{17}$$

$$\Omega_0 = 1,\ 1 < \Omega_{t-1} < \Omega_t \qquad \forall t \tag{18}$$

We then have our main result concerning the naive discounter which is that, perhaps surprisingly[13], the sophisticated and naive hyperbolic discounting models are nonparametrically indistinguishable:

Proposition 3. *The following statements are equivalent.*
(1) The naive quasi-hyperbolic discounting model rationalises the data $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\mathcal{T}}$
(2) The sophisticated quasi-hyperbolic discounting model rationalises the data $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\mathcal{T}}$.

Proof. See the Appendix. □

Thus, without further restrictions (which we will consider below), sophisticated and naive hyperbolic discounting models are nonparametrically indistinguishable given our observables. It is important to note that this does not mean that two individuals with identical instantaneous utility functions, discount factors and budgets and facing identical prices one of whom is a naive hyperbolic discounter and the other a sophisticated hyperbolic discounter would have identical lifetime consumption paths, but it does imply that the difference between them are of degree rather than of kind. What it says is that if, as is usually the case, all we observe is standard consumption and price data $\{r_t, \mathbf{p}_t, \mathbf{c}_t\}_{t\in\mathcal{T}}$ over a period of time, then the nonparametric empirical implications of sophisticated hyperbolic discounting and naive hyperbolic discounting for these data are identical. Since the first order conditions for the naive hyperbolic discounter are observationally identical to those of the self-aware discounter, the parallels of Corollaries 1 to 3 also apply to the naive hyperbolic discounter.

Note that the Euler equation for the sophisticated hyperbolic discounting can be written as

$$\lambda_t = \delta \frac{\Psi_t}{\Psi_{t+1}} \lambda_{t+1}$$

and since $1 < \Psi_{t-1} < \Psi_t \;\; \forall t$, this implies

$$\lambda_t < \delta\lambda_{t+1} \tag{19}$$

Similarly the Euler equation for the naive hyperbolic discounter is

$$\lambda_t = \delta \frac{\Omega_t}{\Omega_{t+1}} \lambda_{t+1}$$

and again the condition $1 < \Omega_{t-1} < \Omega_t \;\; \forall t$ implies

$$\lambda_t < \delta\lambda_{t+1} \tag{20}$$

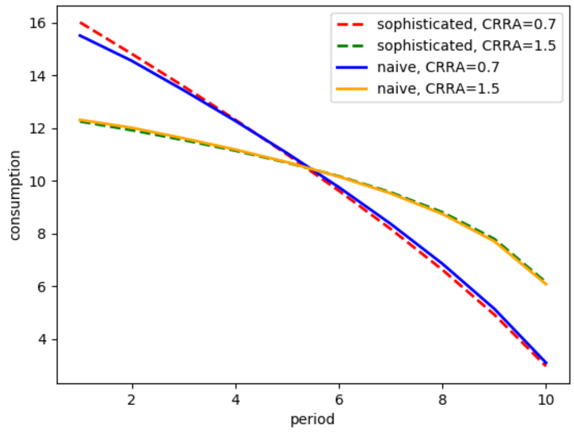
Noting that an exponential discounter would set the terms on the left and right hand sides of (19) (equivalently (20)) equal leads to the following, perhaps obvious, remark:

[13]Papers that hypothesise that observed behaviour distinguishes sophistication from naivety generally need some kind of committed upfront costs. For example, Della Vigna and Malmendier (2006) in an analysis of gym membership, find there are people who choose a flat monthly fee and then use the gym so little that their per-gym visits cost more than would a ten-visit pass, something which a sophisticate (who recognises they will not use the gym enough in the future) would not do.

Remark 1. *With diminishing marginal utility, the lifetime consumption path of the self-aware hyperbolic discounting and the naive model will always decline more (increase less) than the otherwise equivalent exponential discounter.*

Some examples of this are illustrated in Figure 1, which shows simulated consumption paths for a simple ten period model with CRRA preferences. We have kept the exponential discount rate at unity and discounted prices constant (i.e. zero real interest rate) so that an exponential discounter would simply equalise consumption across periods. For the lower relative risk aversion parameter of 0.7, consumption is brought forward more than for the higher level of 1.5 for both the sophisticated and naive types. For lower relative risk aversion the sophisticated discounter front-loads consumption more than the naive discounter, and this reverses as relative risk aversion decreases (although the paths are very similar when the coefficient of relative risk aversion is 1.5, but the naive discounter does start at a slightly higher level than the sophisticated discounter).

Figure 1: Simulated consumption paths for naive and sophisticated discounters



As with sophisticated behaviour, we can ask what further functional restrictions on the naive discounter would give rise to additional empirical restrictions. As the unrestricted sophisticated and naive models are identical, the strategy here will be the same as in the sophisticated case, since Ψ_{s+h}/Ψ_s and Ψ_{t+j}/Ψ_t in equation (14) are simply replaced with Ω_{s+h}/Ω_s and Ω_{t+j}/Ω_t. There is a difference though: since the term Ω_t in the naive model does not have an economic interpretation, thinking of types of instantaneous utility function for the naive model where requiring $(\Omega_{t+j}/\Omega_t)^{1/j} > (\Omega_{s+h}/\Omega_s)^{1/h}$ leads to a restriction on observed behaviour is somewhat harder than for sophisticated hyperbolic discounting. However, a necessary condition is given in Proposition 4.

Proposition 4. *If the naive model rationalises the data* $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\mathcal{T}}$ *for an agent with an instantaneous utility function exhibiting decreasing absolute risk aversion,* $v'''v'/(v'')^2 < 1$, *where* $v'''v'/(v'')^2$ *is independent of prices* $\boldsymbol{\rho}$ *then the data* $\{\boldsymbol{\rho}_t, \mathbf{c}_t\}_{t\in\mathcal{T}}$ *satisfy satisfy the conditions* *in Proposition 1 and the following conditions:*

$$\forall \ \left\{ \{t+i\}_{i=0,...,j} \ , \ \{s+g\}_{g=0,...,h} \right\} \in \tau \ , \ t < s, \ t+j \leq s \ \textit{and} \ h \geq 1 \ :$$

$$\boldsymbol{\rho}_t'\mathbf{c}_t > \boldsymbol{\rho}_t'\mathbf{c}_{t+j}, \ \boldsymbol{\rho}_s'\mathbf{c}_s < \boldsymbol{\rho}_s'\mathbf{c}_{s+h} \ \textit{and} \ \left(\frac{\boldsymbol{\rho}_{t+j}'(\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t'(\mathbf{c}_t - \mathbf{c}_{t+j})} \right)^{1/j} < \left(\frac{\boldsymbol{\rho}_{s+h}'(\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s'(\mathbf{c}_s - \mathbf{c}_{s+h})} \right)^{1/h} \Rightarrow \tag{RN}$$

$$\left(\max \left\{ \boldsymbol{\rho}_{t+i}'\mathbf{c}_{t+i} \right\}_{i=1,...,j} > \min \left\{ \boldsymbol{\rho}_{s+i}'\mathbf{c}_{s+i} \right\}_{i=1,...,h} \right) \quad \textit{and}$$

$$\textit{not} \left(\begin{array}{c} \boldsymbol{\rho}_{s+g}'\mathbf{c}_{s-1+g} > \boldsymbol{\rho}_{s+g}'\mathbf{c}_{s+g} \ \textit{and} \ \boldsymbol{\rho}_{t+i}'\mathbf{c}_{t+i} < \boldsymbol{\rho}_{s+g}'(\mathbf{c}_{s+g} - \mathbf{c}_{s-1+g}) \\ \textit{for any} \ i = 1,...,j \ \textit{and} \ g = 1,...,h \end{array} \right)$$

Proof. See the Appendix. □

The first part of (RN) is identical to (RS) and the second (additional) restriction seems *a priori* very weak, in that violating it and thus rejecting the model would appear to be unlikely. For example $\boldsymbol{\rho}_{t+i}'\mathbf{c}_{t+i} < \boldsymbol{\rho}_{s+g}'(\mathbf{c}_{s+g} - \mathbf{c}_{s-1+g})$ means spending has increased so much between $t+i$ and $s+g$ that $\boldsymbol{\rho}_{t+i}'\mathbf{c}_{t+i}$ is even smaller than $\boldsymbol{\rho}_{s+g}'(\mathbf{c}_{s+g} - \mathbf{c}_{s-1+g})$.

For data that pass Proposition 1, the further restrictions needed to reject the naive model are slightly different from those for our restricted version of the sophisticated hyperbolic discounting. The empirical restrictions for both models are almost identical: the observed behaviour needed to reject involves the agent increasing spending in response to a (discounted) price decrease but then decreasing spending in response to a bigger (relative) price decrease - something that is completely ruled out for the exponential discounter. For both models the spending decrease must come earlier in time than the increase. The theoretical, functional restrictions, though, differ across the two models. For the naive case, the extra restriction on the (indirect) utility function is that the ratio of the coefficient of prudence to the coefficient of risk aversion is less than one, which is equivalent to

$$\frac{v'''v' - v''}{(v'')^2} < 0$$

i.e. decreasing absolute risk aversion. For the sophisticated hyperbolic discounting the restriction is that this quantity is constant, so, although there are overlaps, neither is a subset of the other. Proposition 4 also adds the extra restriction that the ratio of the coefficient of prudence to the coefficient of risk aversion is independent of prices. This allows a weak kind of distinction between naive and sophisticated behaviour. For example, suppose we insist that the instantaneous utility function be isoelastic, as in the case in most macro or lifecycle models of consumption. Then if the data do not pass Proposition 2, they could not have been generated by a sophisticated discounter, but they *could* have been generated by a naive discounter.

3 Empirical Application: Hyperbolic Behaviour in Household Panel Data

Much of the empirical evidence for hyperbolic behaviour comes from laboratory studies. It is sometimes argued (and indeed has been argued to us) that the key insight that has emerged over the past decade of research into time-inconsistent behaviour is that to identify β, δ discounting one must have access to observations made in the kind of decision-making environments which only labs can produce. Such environments allow us to observe subjects making a mixture of decisions, some governed by short-term consideration influenced by β, others governed by longer-term impatience influenced by δ, and in which they may be able to make partial commitments and so on. Echenique, Imai and Saito (2013/15), for example, use Convex Time Budget experimental design, in which subjects are asked to allocate monetary amounts between a "sooner" and a "later" time whilst facing an interest rate with each subject asked to make several choices as the timings and rate of return are varied. Even better would be data in which we could observe consumption plans made by subjects at different dates and then, subsequently, observe whether and how those plans are revised.

Some experimental data, though, is not without problems. Excellent discussions of various elicitation techniques, and methods used to analyse data, are presented in, amongst others, Andreoni, Kuhn and Sprenger (2015), Andersen *et al* (2013, 2014), Andreoni and Sprenger (2012) and Benhabib, Bisin and Schotter (2010). For example, many of the experimental studies referenced in Federick et al (2002) (see, for example, their Table 1) are hypothetical, i.e. questions of the type "What amount of money, $\$x$, if paid to you today would make you indifferent to $\$y$ paid to you in t days" are asked of the subjects purely hypothetically. Many studies also do not consider curvature of the utility function. The Convex Time Budget data mentioned above comes from Andreoni and Sprenger (2012), and, though meticulously designed, with real payoffs made to the participating students with the possibility of real time delays (by selecting one of their experimental choices at random), necessarily involved small monetary amounts – each participant was asked to allocate 100 tokens with values varying from \$0.10 to \$0.20.

In addition, even if experimental data of this kind could provide the ideal conditions for studying time inconsistency, if that is the *only* data in which time inconsistency might be empirically relevant then this seems tantamount to saying that β, δ discounting as an empirical phenomenon is confined to the lab and has little to say about the kinds of real world data with which economists interested in empirically modelling consumption dynamics normally deal. Therefore, we are interested in the empirical consequences of hyperbolic behaviour in standard, widely-available household survey data on consumption behaviour across multiple goods which contain none of these "rich" features. Our theoretical results above indicate that hyperbolic behaviours *do* have falisfiable consequences even in such environments.

Our empirical application uses the theoretical results to consider a number of substantive issues. Firstly, how well does the hyperbolic model represent behaviour in these sorts of data?

Secondly how does hyperbolic behaviour relate to the observable characteristics of different households? Thirdly how can we properly evaluate the hyperbolic model against the standard exponential model when the hyperbolic model *necessarily* fits the data better due to the presence of a free parameter? Finally what are the joint distributions of time preferences?

3.1 Data

The data used here to investigate the empirical implementation of the ideas outlined above is a large, nationally representative consumption panel: the Spanish Continuous Family Expenditure Survey (the *Encuesta Continua de Presupuestos Familiares* - ECPF). The ECPF is a quarterly budget survey of Spanish households which interviews about 3,200 households every quarter. These households are randomly rotated at a rate of 12.5% each quarter. Thus it is possible to follow a participating household for up to eight consecutive quarters. This dataset is a much studied survey which has often been used for the analysis of inter-temporal models and particularly, latterly, the analysis of habits models (for example, Browning and Collado (2001, 2007)). The data used here are drawn from the years 1985 to 1997 and are the selected sub-sample of couples with and without children, in which the husband is in full-time employment in a non-agricultural activity and the wife is out of the labour force (this is to minimise the effects of non-separabilities between consumption demands and leisure which the empirical application does not otherwise allow for). The dataset consists of 21866 observations on 3134 households. The data record household non-durable expenditures and these are disaggregated into six commodity groups: food consumed at home, alcohol, tobacco, clothing and footwear, restaurants and bars and domestic energy. The discounted price data are calculated from published prices aggregated to correspond to the expenditure categories and the average interest rate on consumer loans.[14]

3.2 Hyperbolic behaviour

3.2.1 Consistency Results

We examined the consistency of the data with each of the following models: atemporal (i.e. within period) utility maximisation (this is a useful benchmark but also a necessary condition for any inter-temporal model in which preferences are weakly separable over time); hyperbolic discounting as characterised in Propositions 1 and 3; the restricted versions of the sophisticated and naive hyperbolic discounting models as characterised in Propositions 2 and 4 respectively; and finally the exponential discounting consumption model[15].

[14]It should also be noted that these results treat the household as a unitary entity and abstracts from issues to do with collective household behaviour. For a discussion of the issues raised by collective models of households in a revealed-preference framework see Cherchye, De Rock and Vermeulen (2007) and especially Adams *et al* (2014) who suggest that the coexistence of exponential discounters with differing discount rates within a household as another potential source of time-inconsistency in aggregate (household) behaviour.

[15]We use a previously unexploited (we believe) method of testing for exponential discounting which uses the conditions for δ implied from equation (14) with $\Psi_t = 1\ \forall t$. This enjoys a great computational advantage over linear programming combined with a grid search over δ. See https://sites.google.com/site/laurablow/working-papers/exponential.pdf for details.

We examine the behaviour of each individual household for consistency with each of these models separately. This is a one-at-a-time approach to the data; the data are never pooled across households so we therefore allow for unrestricted heterogeneity within the classes of models studied: households may differ arbitrarily with respect to whether they are rationalisable by a given model, their time preferences and their preferences for different goods and services.

Table 1: Rationalisability results

Model	GARP	Hyperbolic	Restricted Sophisticated	DARA Naive	Exponential
Pass Rate	0.9371	0.8149	0.2964	0.2964	0.0230
(Std. Error)	(0.0044)	(0.0071)	(0.0081)	(0.0081)	(0.0027)

Table 1 reports the pass rates for each model considered. We see that nearly 94% of the households in these data pass GARP and therefore their behaviour is consistent with within-period utility maximisation. When we add condition (RD) and thus test hyperbolic discounting (sophisticated or naive), as specified in Propositions 1 and 3, we find a pass rate of 81% - four out of five of the households in our data behave precisely consistently with the predictions of the β, δ model. The models of Proposition 2 and 4, which impose restrictions on preferences as described in these Propositions, reduce the consistency to about one third of households. The sophisticated and naive versions of this model perform identically because the restrictions the models impose on observable behaviour are almost the same: the first restriction in (RN) is identical to that of (RS), and the second restriction of (RN) is such that, in this data, nobody fails it.[16] Thus it does appear that these functional form assumptions, so common in applied work, have a material impact on the fit of the model.

The standard exponential model reported in the fifth column rationalises very few households (less than 2.5 percent). Given that within-period consistency with utility maximisation is a necessary but not sufficient condition for the exponential model, and that most households do pass GARP, this poor performance must therefore be due, in the main, to the inter-temporal behaviour displayed by the households in the sample. The standard errors for each of these pass rates is given in brackets in Table 1. In each case the effects of sampling variation appear to be quite modest and we would therefore expect the proportions of households consistent with each model in another random sample of a similar size from this population to be close to those in Table 1.

Our revealed-preference results reported in Table 1 (like all revealed preference exercises we are aware of) only utilises the choice behaviour of agents. It does not resort to other variables to explain those choices. By contrast, regression-based approaches typically condition on

[16]Note that for the restricted models (Propositions 2 and 4) these are upper bounds on the pass rates for the proposed restrictions on utility functions since the restrictions on observed behaviour are necessary but not sufficient.

Table 2: The probability of Hyperbolic behaviour - probit

Hyperbolic=1	Coefficient	Std. Error
Age (Head of Household)	-0.016^{*}	(0.008)
Age (Spouse)	0.017^{**}	(0.008)
Children aged 0-2	0.009	(0.070)
Children aged 3-6	0.075	(0.058)
Children aged 7-13	-0.065^{*}	(0.034)
Children aged 14-15	−0.017	(0.061)
Children aged 16-17	0.057	(0.066)
Owner Occupier	0.112^{*}	(0.064)
Car Owner	-0.169^{*}	(0.091)
University level Education	−0.061	(0.104)
High School Education	0.059	(0.065)
Low spending on health	0.219^{**}	(0.099)
Heavy Smoker	0.198^{*}	(0.109)
Heavy Drinker	−0.011	(0.094)
log Total Expenditure	-3.696^{*}	(2.205)
log Total Expenditure-squared	0.127^{*}	(0.073)
Number of Obs	-0.157^{***}	(0.028)
Constant	28.893^{*}	(16.709)
Observations	3,134	
Log Likelihood	−1,464.158	
Akaike Inf. Crit.	2,964.316	
Note:	*p<0.1; **p<0.05; ***p<0.01	

"taste-shifters" like demographic variables to help to explain departures from the baseline model. It is thus interesting to then see how closely (or otherwise) the revealed preference results are associated with other standard observables. To this end, Table 2 reports the results from a probit model of the conditional probability of a household displaying hyperbolic behaviour. The most significant variables seem to relate the hyperbolic behaviour we detect to variables which reflect long-term behaviour. Being an owner-occupier, for example, (compared to the omitted category which comprises mainly renters) seems to be positively associated with the rate of hyperbolic behaviour with respect to non-durable consumption. This may be evidence that self-aware sophisticated hyperbolic discounters might use durable investment as a commitment device, however, we note that car ownership does not seem to have an independent association with hyperbolic behaviour. Another evidently important correlate is low out-of-pocket expenditure on health (durable and non-durable expenditure on medical products and spending on medical services). The explanatory variable here is a dummy variable indicating that the household is in the bottom 10% of the sample distribution of these expenditures: these households are relatively low health-spenders and are either presumably relatively healthy or treat medical investments in their health differently from the way in which they treat housing and so under-invest. Along the same lines, being a heavy smoker (expenditure on tobacco at or above the 90'th percentile in the sample) is positively associated with time inconsistency. Drinking (defined in the same way) seems to have little explanatory power. We also note a strong, non-linear association with total household expenditure which we use as a rough approximation to the overall resources in the household. As these grow, the rate of hyperbolic behaviour declines albeit at a declining rate. The variable recording the number of observations we have on the household is included as the ability of revealed preference tests to detect non-model behaviour is necessarily increasing in the number of observations and it is therefore important to control for this effect.

3.3 Model comparison

As Angeletos *et al* (2001) conclude, on the basis of their own empirical work and a sense of the literature, "All in all, a model of consumption based on a hyperbolic discount function consistently better approximates the data than does a model based on an exponential discount function." On the basis of our results in Table 1 it would be hard to disagree. Nonetheless, is important to remember when looking at measures of fit that these models are not equally flexible. The atemporal model of utility maximisation does not consider inter-temporal planning at all and takes the budget allocated to each period as exogenously given. It thus places no restrictions whatsoever concerning how spending is allocated across time and therefore, as long as within-period preferences are time-separable and satisfy GARP, *any* inter-temporal allocation is, in that sense, rationalisable with the atemporal model. The exponential model, in contrast, is much more demanding. It constrains both within-period and inter-temporal choices: within-period choices must be rational and stable, and inter-temporal choices must be time-consistent. The hyperbolic model is, in a sense, intermediate. It too requires within-period rationality and stability, but whilst it does not require time-consistency, inter-temporal behaviour is constrained by the form of the hyperbolic Euler equation. Thus, whilst much is sometimes made of the ability of the hyperbolic model to explain observed behaviour which the exponential model cannot[17], the fact that our (or anyone else's) results show that the hyperbolic model fits better should come as little surprise. How, then, should we make sense of the results in Table 1? We consider two approaches. The first is based on Selten's Index of predictive success, the second on the Kullback-Leibler information criterion.

3.3.1 Predictive Success

In their revealed preference guise, stripped of special functional form assumptions, all three models generate restrictions in the form of *sets* of choices which are consistent with the model of interest (for example, given any collection of budget constraints there will be a *set* of demands which satisfy GARP). To investigate the performance of models which predict sets, it is useful to think about two objects. The first is the feasible outcome space (for example, the set of choices which satisfy the inter-temporal budget constraint) which we will denote F. The

[17] See Frederick *et al* (2001) for example.

second is the subset of model-consistent choices (the subset of feasible choices which satisfy the restrictions of the model), denoted P, with $P \subseteq F$. When one conducts a particular empirical revealed preference test one is, in essence, checking to see whether the observed choices lie within P.

With this in mind it becomes clear that it is necessary to allow for the size theoretically consistent subset relative to the set of possible outcomes. The essential idea - which is due to Selten and Krischker (1983) and Selten (1991) - is that if the subset of observations consistent with the model (P) is a large proportion of the set of behaviours which the consumer could possibly display (F) then we should be little surprised if we find that many of the observed choices lie in P - they could hardly have done otherwise. For example, if we are testing the atemporal model and the collection of budget constraints never cross then all feasible choices, necessarily, satisfy GARP: it would be impossible to make a choice that was not in P because $P = F$. This means that empirical fit alone (the proportion of the sample which passes the relevant test) is not a sufficient basis for ranking the performance of alternative theories: if it were, then no theory could out-perform a meaningless theory like "anything goes". A better approach would be to consider the trade-off between the pass rate and some sort of measure of how demanding the theory is. Following Selten (1991) let a denote the size of the theory-consistent subset P *relative* to the outcome space F for the model of interest. The relative area of the empty set is zero and the relative area of all outcomes is one so $a \in [0, 1]$. Now suppose that we have some choice/outcome data. Let r denote the pass rate; this is simply the proportion of the data that lies in P and hence satisfies the restrictions of the model of interest (i.e. the numbers in Table 1). Selten (1991) argues that *both* the pass rate and the area should be taken into account when comparing models into an overall measure of predictive success $m(r, a)$. He further suggests that demanding theories are characterised by small values for a; and empirically successful theories combine small values of a with a high degree of agreement between the data and theory (large r). He also argues that the trade-off between the ability to fit the data and the restrictiveness of the theory should be the difference measure[18]:

$$m(r, a) = r - a$$

The Selten index for the models is shown in Table 3.

Table 3: Selten's Index: Predictive Success

Model	GARP	Hyperbolic	Restricted Sophisticated	DARA Naive	Exponential
Selten Index	0.0092	0.0550	0.0494	0.0494	0.0012
(Std. Error)	(0.0044)	(0.0070)	(0.0079)	(0.0079)	(0.0026)

[18] In brief, Selten's main requirements are monotonicity $m(1, 0) > m(0, 1)$, equivalence of meaningless theories $m(0, 0) = m(1, 1)$, and the requirement that the performance of the mean (across subjects) is equal to the mean performance (across subjects) $m(\bar{\mathbf{r}}, \bar{\mathbf{a}}) = \overline{\mathbf{m}}$. This last assumption is strong and responsible for the linearity of the resulting index.

These results indicate that even allowing for the stronger restrictions in the exponential model the hyperbolic models (particularly the restricted versions) out-perform it. In other words, whilst the hyperbolic model must fit better than the exponential alternative it does so without becoming vastly more permissive and the improved fit outweighs the effects of having an extra free parameter. The exponential model on the other hand appears to have a predictive success which is not statistically significantly different from zero. In this case the model is more demanding than the hyperbolic models yet the pass rate is far lower than you would expect even allowing for this.

3.3.2 Kullback-Leibler Information Criterion

An alternative approach to assessing the fit of the model is to use the fact that since both the relative size of the theory-consistent set and the empirical pass rate satisfy all of the necessary properties of probabilities[19] we can be justified in thinking about the problem of comparing them as the problem of comparing probability distributions. Here a is the probability that a random uniform choice over the feasible set will satisfy the restrictions of the model. Similarly r is the probability that a randomly drawn subject will exhibit behaviour consistent with the model. A simple way in which to make a comparison between these two distributions in units which are meaningful is to use the Kullback-Leibler divergence (Kullback and Leibler (1951))

$$KL(r,a) = r\log_2\left(\frac{r}{a}\right) + (1-r)\log_2\left(\frac{1-r}{1-a}\right)$$

One interpretation of the Kullback-Leibler divergence of one distribution from another is the information gained from revising beliefs from a prior to a posterior. Hence in this example, the Kullback-Leibler divergence measures the information (in bits) conveyed by the empirical distribution of outcomes, $\{r, 1-r\}$, relative to a prior model of uniform random choice over the set of feasible outcomes, $\{a, 1-a\}$. For example, suppose that the pass rate was 0.7 and this matched the size of theory-consistent set predicted by the model precisely (i.e. the proportion of all possible observed choices which satisfy the theory was also 0.7) then the observed pass rate would convey no surprise at all and the data would generate zero bits of information about the empirical performance of the model. If, on the other hand, the set of outcomes which are rationalisable represent very little of the outcome space ($a \to 0$) (i.e. the model makes very precise predictions) and most of the data generally satisfy the theoretical restrictions ($r \to 1$) then the outcome of the empirical test is extremely informative and $KL(r,a)$ is large.

Table 4 presents the Kullback-Leibler information criteria[20] for the five models. It concurs with the Selten index results.

[19]They are between zero and one and the area of two non-overlapping subsets is the sum of their individual areas.

[20]Measured in bits $\times 10^3$.

Table 4: Kullback-Leibler divergence

Model	GARP	Hyperbolic	Restricted Sophisticated	DARA Naive	Exponential
KLIC	0.9437	12.6878	9.0939	9.0939	0.0462
(Std. Error)	(0.9398)	(3.2722)	(2.8243)	(2.8243)	(0.1965)

The very high pass rate of the atemporal model is revealed to be quite uninformative about the success or otherwise of that model - and indeed the standard error indicates that the observed non-zero value in the sample is very likely to be due to sampling variation rather than a feature of the population: another sample could easily produce zero information about the model. The KLIC being essentially zero indicates that the predictive success of the model is due, almost entirely, to its permissiveness. Turning to the comparison of interest, that between the exponential and the hyperbolic models, we see that, even allowing for the less demanding nature of the hyperbolic models, their performance is better than the exponential model. The exponential model does badly because, even though it is very restrictive, few people pass the test. The unrestricted hyperbolic model seems to be the most informative about household behaviour.

3.4 Identification: the joint distribution of time preferences

Our conclusion from the analysis of model consistency is that, even making proper allowance for the relative restrictiveness of the alternative models, hyperbolic behaviour provides the best explanation of the consumption behaviour we observe in these data. Given this, for those four out-of five households that behave consistently with the hyperbolic model, we can apply the inequalities in corollary 3 to recover their time preference. For each household we let δ vary across the bounded set given in corollary 3, and, given δ, we assign the upper bound to β given in corollary 3 (as shown in the corollary, the lower bound would usually be zero), so that the household is the "least hyperbolic" it can be given δ. We then estimate a kernel density using a uniform kernel function with an individual-specific bandwidth which matches the identified intervals for each parameter for that individual. Figures 2 and 3 show estimates of the two marginal distributions over the full 0-1 support, and Figure 4 shows the joint distribution over the restricted range $\delta, \beta \in [0.8, 1] \times [0.5, 1]$.

The identified intervals for the exponential parameter are bounded from above by 1 for all households (see corollary 3). This explains the monotonic estimated marginal density of δ since the upper end of all of the kernel functions include this end point. Nonetheless it is clear that the exponential discount rates are above 0.9 for the most part. Table 5 shows the descriptive statistics based on numerically integrating the estimated densities. This shows that the mean value for the exponential parameter δ is around 0.95. Frederick *et al* (2002) review a large number of studies which have attempted to measure δ and whilst they argue that there appears to be remarkably little consensus in the literature, a value of around 0.95 is probably reasonable. The range is fairly narrow: the inter-quartile range is [0.9259, 0.9770].

Figure 3 shows the estimated marginal distribution of the hyperbolic parameter. The

Figure 2: The density of the exponential parameter δ

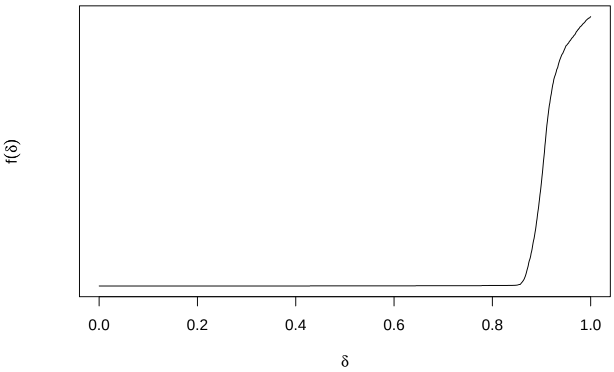
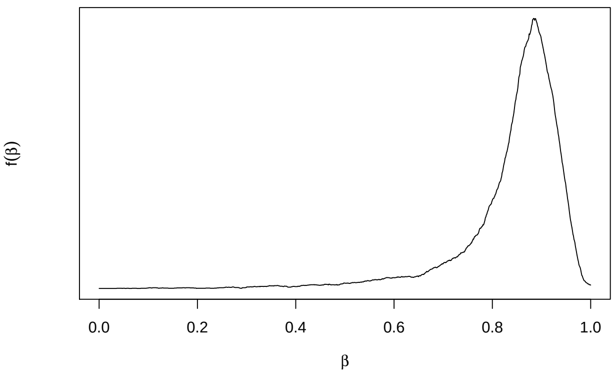


Figure 3: The density of the hyperbolic parameter β



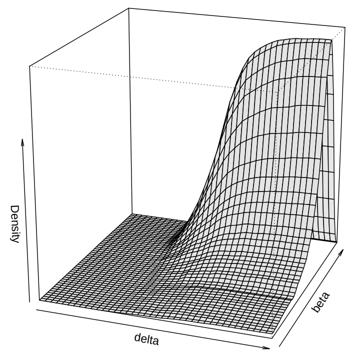
mean value of the hyperbolic parameter calculated from the density is 0.8438. Compared to the exponential rate there is more dispersion: the inter-quartile range is $[0.8168, 0.9059]$. Indeed 10 percent of the distribution shows quite pronounced hyperbolism with $\beta < 0.72$.

Table 5: The distribution of time preferences: Descriptive statistics

Parameter	Mean	10th p'ctile	1st quartile	Median	3rd quartile	90th p'tile
δ	0.9497	0.9049	0.9259	0.9530	0.9770	0.9910
β	0.8438	0.7267	0.8168	0.8689	0.9059	0.9339

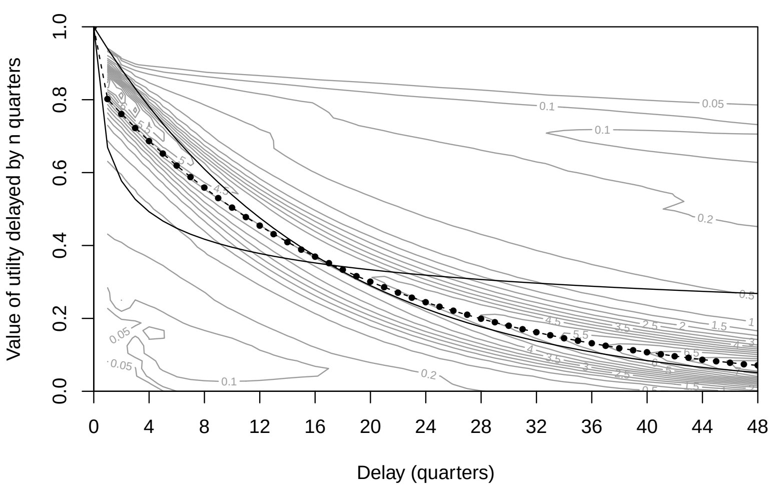
Figure 4 shows the estimated joint distribution - recall that in order to make the shape of the distribution a little easier to see we have plotted it over a restricted range $\delta, \beta \in [0.8, 1] \times [0.5, 1]$. The main point which seems to emerge from the joint distribution is the lack of any particularly strong correlation between β and δ. It does not seem to be the case that,

Figure 4: The joint density of time preferences δ, β



for example, that low δ tends to be associated with high (or low) values for the hyperbolic parameter.

Figure 5: The distribution of discount functions by horizon (contour plot)



The important aspect of the β, δ time preference parameters is how they combine to form the individual's discount function. Figure 5 provides a contour plot of an estimate of the distribution of discount functions at various time horizons among the hyperbolic households. Each household's discount function is calculated using a parameterisation which applies the β, δ parameter values at the centre of that household's identified set. We then estimated cross-sectional density of the quasi-hyperbolic sequence $\left[1, \beta\delta, \beta\delta^2, \beta\delta^3, ...\right]$ at each time horizon. For comparison Figure 5 also plots (dotted line) the quasi-hyperbolic sequence using $\beta = 0.8438$ and $\delta = 0.9497$ which are the mean values from Table 5, the exponential function δ^t for $\delta = 0.94$ and the true hyperbolic function $(1 + \alpha t)^{-\gamma/\alpha}$ with $\alpha = 4$ and $\gamma = 1$. There is evident heterogeneity but on average postponing an immediate reward by a quarter reduces the value of that reward by approximately one-fifth. By contrast, delaying a distant reward by an additional quarter reduces the value of that reward by a much smaller proportion (approximately one-twentieth).

4 Conclusion

We provide a choice-revealed preference characterisation of quasi-hyperbolic consumption behaviour in the kind of decision-making and measurement environment of the kind routinely provided by household expenditure surveys. We describe conditions which breakdown neatly into a within- and between-period component which allows for useful diagnostics regarding the source of any violations. We also explore whether preference restrictions of the type commonly used in lifecycle consumption modelling, such as an iso-elastic instantaneous utility function, have further revealed preference implications. We have also applied this characterisation to a large, nationally representative consumption panel to explore a number of substantive issues including the joint distribution of time preferences, the distribution of discount functions at various horizons and the relationship between the prevalence of hyperbolic preferences in the household population and household characteristics.

In this paper we have focussed on the perfect foresight version of the model. This is a limitation and we leave to future research the extension of the conditions to deal with uncertainty. We hypothesise that this could be done in the exponential model, for example, by requiring, not that the path of the discounted marginal utility of lifetime wealth is flat (as in Browning (1989)), but that it follows a martingale. Similarly, as the test for the sophisticated model can be written in terms of the path of the discounted marginal utility of lifetime wealth increasing over time, this could be reformulated as a sub-martingale under a situation of uncertainty. In other words part of the combinatorial GARP-like components would be replaced by a probabilistic counterpart. However we leave this for future work. In the present context we only have a short duration of observations on each household so a nonparametric statistical test of, say, the sub-martingale hypothesis for exponential discounting (for example along the lines of Phillips and Jin (2013)) would have no power to reject. We therefore focus on the perfect certainty version of the models in our empirical work and also our theoretical work.

Another limitation is that we treat the household as a unitary decision-making entity and ignore the issues raised by collective household behaviour. The revealed preference implications of atemporal collective models are discussed in Cherchye, De Rock and Vermeulen (2007), and inter-temporal models in Adams *et al* (2014). Adams *et al* (2014) speculate that households composed of two exponential individuals with heterogeneous discount rates may give rise to aggregate (household) behaviour which is time-inconsistent. Since our empirical work looks at couples this may, if true, account for the apparent prevalence of hyperbolic household behaviours. It may be important therefore to extend the ideas presented here to collective models of households, but at present we have not done so.

We have focussed on quasi-hyperbolic discounting, but hypothesise that a similar approach could be used to see if nonparametric tests of other models of temporal discounting can be formulated. Of course, phenomena such as the magnitude effect might be difficult to test on survey data as we do not see the same agent making repeated choices over different principals. There is no reason, though, that the methods we use could not be used on experimental data. For example, in experimental data eliciting choices over time-dated monetary flows we see the agent make committed choices; the test for hyperbolic discounting would therefore be exponential discounting between future payments, with an extra discount factor between immediate and future payments, and where we observe an agent making multiple choices they must have consistent discount factors across those choices[21].

The empirical conditions we find are quite easy to apply since they require nothing more complicated than checking inequalities. In an empirical application we consider in some detail how to interpret the revealed preference performance of alternative models which differ in their restrictiveness. We suggest Selten's index of predictive success and the Kullback-Leibler divergence as sensible means of doing this. This seems to be quite a fruitful way of describing the performance of the models in question - and possibly economic models in general. We compare tests of atemporal behaviour (a GARP test), hyperbolic discounting and exponential discounting. We find that, although most households pass GARP, when we consider the trade-off between the pass rate and measures of how demanding the theory is, GARP does not do very well because of its permissiveness. Exponential discounting also does not do well predictively because, although it is very restrictive, few people pass the test. However, we find that the hyperbolic discounting models perform well.

We show that the hyperbolic behaviour detected by our restrictions is sensibly correlated with household characteristics related to long-term decision making and other behaviours in which inter-temporal considerations are important like smoking and health investments. Using our characteristation of the model we are able to provide estimates of the joint distribution of time preference and the distribution of discount functions. We find average exponential discount factors around 0.95 and hyperbolic factors around 0.84.

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Appendix - Proofs

Proof of Lemma 1

Denote savings as S_t (with the assumption that $S_T = 0$). In period t, person t's program is:

$$V(A_t) = \max_{c_t} u(c_t) + \beta \sum_{i=1}^{T-t} \delta^i u(c_{t+i}) \tag{21}$$

$$\begin{aligned} s.t. \quad & \mathbf{p}_t' \mathbf{c}_t + S_t = A_t \\ & A_{t+1} = (1 + r_{t+1}) S_t \end{aligned}$$

Expressing everything in discounted terms gives

$$V(\Delta_t) = \max_{c_t} u(c_t) + \beta \sum_{i=1}^{T-t} \delta^i u(c_{t+i}) \tag{22}$$

$$s.t. \quad \Delta_{t+1} = \Delta_t - \boldsymbol{\rho}_t' \mathbf{c}_t$$

where $\Delta_t = A_t / \prod_{i=1}^{t} (1 + r_i)$ and $\rho_t^k = p_t^k / \prod_{i=1}^{t} (1 + r_i)$.

Person t knows that each person $\varsigma = t + 1 ... T$ (his future selves) will have program:

$$V(\Delta_\varsigma) = \max_{c_\varsigma} u(\mathbf{c}_\varsigma) + \beta \sum_{i=1}^{T-\varsigma} \delta^i u(\mathbf{c}_{\varsigma+i}) \tag{23}$$

$$s.t. \quad \Delta_{\varsigma+1} = \Delta_\varsigma - \boldsymbol{\rho}_\varsigma' \mathbf{c}_\varsigma$$

Thus we can re-write the value function in equation (22) as

$$\begin{aligned} V(\Delta_t) &= \max_{\mathbf{c}_t ... \mathbf{c}_T} u(\mathbf{c}_t) + \delta u(\mathbf{c}_{t+1}) + \beta \sum_{i=2}^{T-t} \delta^i u(\mathbf{c}_{t+i}) + \beta\delta u(\mathbf{c}_{t+1}) - \delta u(\mathbf{c}_{t+1}) \\ &= \max_{\mathbf{c}_t} u(\mathbf{c}_t) + \delta [V(\Delta_{t+1}) - (1 - \beta) u(\mathbf{c}_{t+1})] \\ s.t.\ \ \Delta_{t+1} &= \Delta_t - \boldsymbol{\rho}_t' \mathbf{c}_t \end{aligned} \tag{24}$$

The first order condition from equation (24) is

$$\frac{\partial u_t}{\partial c_t^k} - \delta \rho_t^k \left[V_{\Delta_{t+1}} - (1 - \beta) \sum_{k=1}^{K} \frac{\partial u}{\partial c_{t+1}^k} \frac{\partial c_{t+1}^k}{\partial \Delta_{t+1}} \right] = 0 \qquad \forall k \tag{25}$$

where $V_{\Delta_{t+1}}$ denotes $\frac{\partial V(\Delta_{t+1})}{\partial \Delta_{t+1}}$, and the envelope theorem gives

$$V_{\Delta_t} = \delta \left[V_{\Delta_{t+1}} - (1 - \beta) \sum_{k=1}^{K} \frac{\partial u}{\partial c_{t+1}^k} \frac{\partial c_{t+1}^k}{\partial \Delta_{t+1}} \right] \qquad \forall k \tag{26}$$

Equations (25) and (26) give

$$V_{\Delta_t} = \frac{\partial u}{\partial c_t^k}\frac{1}{\rho_t^k} = \forall k \tag{27}$$

Updating (27) and substituting into (25) gives

$$\begin{aligned}
\frac{\partial u}{\partial c_t^k} &= \delta \rho_t^k \left[V_{\Delta_{t+1}} - (1-\beta) \sum_{k=1}^{K} V_{\Delta_{t+1}} \rho_{t+1}^k \frac{\partial c_{t+1}^k}{\partial \Delta_{t+1}} \right] \qquad \forall k \\
&= \delta V_{\Delta_{t+1}} \rho_t^k \left[1 - (1-\beta) \sum_{k=1}^{K} \rho_{t+1}^k \frac{\partial c_{t+1}^k}{\partial \Delta_{t+1}} \right] \qquad \forall k
\end{aligned}$$

and using (27) once more gives

$$\frac{\partial u}{\partial c_t^k} = \delta \frac{\partial u}{\partial c_{t+1}^k} \frac{\rho_t^k}{\rho_{t+1}^k} \left[1 - (1-\beta) \sum_{k=1}^{K} \rho_{t+1}^k \frac{\partial c_{t+1}^k}{\partial \Delta_{t+1}} \right] \qquad \forall k$$

which is the Euler equation in Lemma 1.

Define

$$\mu_{t+1} = \sum_{k=1}^{K} \rho_{t+1}^k \frac{\partial c_{t+1}^k}{\partial \Delta_{t+1}} \tag{28}$$

Note that this is the period $t+1$ marginal propensity to spend out of wealth. Given consumption in each period is normal then it follows that

$$\mu_t \in (0,1) \qquad \forall t \neq T$$

with $\mu_T = 1$ by exhaustion of the lifetime budget. Solving for $\partial u / \partial c_{t+1}^k$ gives

$$\frac{\partial u}{\partial c_{t+1}^k} = \frac{\partial u}{\partial c_t^k} \frac{1}{\delta} \frac{\rho_{t+1}^k}{\rho_t^k} \left[1 - (1-\beta)\, \mu_{t+1} \right]^{-1} \qquad \forall k$$

and solving recursively gives

$$\frac{\partial u}{\partial c_t^k} = \frac{\partial u}{\partial c_0^k} \frac{\rho_t^k}{\rho_0^k} \frac{1}{\delta^t} \prod_{i=1}^{t} \frac{1}{\left[1 - (1-\beta)\, \mu_i \right]} \qquad \forall k \tag{29}$$

Using (27) dated in $t = 0$ and denoting $V_{\Delta_0} = \lambda$ gives

$$\lambda = \frac{\partial u}{\partial c_0^k} \frac{1}{\rho_0^k} \qquad \forall k$$

On substitution into (29) this gives the condition

$$\frac{\partial u}{\partial c_t^k} = \lambda \frac{1}{\delta^t} \rho_t^k \prod_{i=1}^{t} \frac{1}{\left[1 - (1-\beta)\, \mu_i \right]} \qquad \forall k \tag{30}$$

which is Definition 1. Reinserting the definition of μ_i (as given in equation (28)) into equation (30) gives the first equation of Lemma 1

$$\frac{\partial u}{\partial c_t^k} = \lambda \frac{1}{\delta^t} \rho_t^k \prod_{i=1}^{t} \left[1 - (1-\beta) \sum_{k=1}^{K} \rho_t^k \frac{\partial c_t^k}{\partial \Delta_t} \right]^{-1} \qquad \forall k$$

■

Proof of Proposition 2

The proof of Proposition 1 shows that the sophisticated hyperbolic discounting model rationalising the data ⇒ equations (9) and (10). Therefore, if in the data we observe

$$\boldsymbol{\rho}_t'\mathbf{c}_t > \boldsymbol{\rho}_t'\mathbf{c}_{t+j},\ \boldsymbol{\rho}_s'\mathbf{c}_s < \boldsymbol{\rho}_s'\mathbf{c}_{s+h} \text{ and } \left(\frac{\boldsymbol{\rho}_{t+j}'(\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t'(\mathbf{c}_t - \mathbf{c}_{t+j})} \right)^{1/j} < \left(\frac{\boldsymbol{\rho}_{s+h}'(\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s'(\mathbf{c}_s - \mathbf{c}_{s+h})} \right)^{1/h}$$

then using equations (9) and (10) gives

$$\begin{aligned}
\delta \left(\frac{\Psi_t}{\Psi_{t+j}} \right)^{1/j} \leq \left(\frac{\boldsymbol{\rho}_{t+j}'(\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t'(\mathbf{c}_t - \mathbf{c}_{t+j})} \right)^{1/j} &< \left(\frac{\boldsymbol{\rho}_{s+h}'(\mathbf{c}_s - \mathbf{c}_{s+h})}{\boldsymbol{\rho}_s'(\mathbf{c}_s - \mathbf{c}_{s+h})} \right)^{1/h} \leq \delta \left(\frac{\Psi_s}{\Psi_{s+h}} \right)^{1/h} \\
\Rightarrow \delta \left(\prod_{i=1}^{j} \left[1 - (1-\beta)\mu_{t+i} \right] \right)^{1/j} &< \delta \left(\prod_{i=1}^{h} \left[1 - (1-\beta)\mu_{s+i} \right] \right)^{1/h} \\
\Rightarrow \max \left\{ \mu_{t+i} \right\}_{i=1,\ldots,j} &> \min \left\{ \mu_{s+i} \right\}_{i=1,\ldots,h} \qquad (31)
\end{aligned}$$

Denote the instantaneous indirect utility function by $v(\boldsymbol{\rho}, y) = \max u(\mathbf{c})\ \ s.t. \boldsymbol{\rho}'\mathbf{c} = y$. Here, and in subsequent proofs in the Appendix, we will make use of the fact that

$$\lambda_t = \frac{\partial v(\boldsymbol{\rho}_t, y_t)}{\partial y_t} = \frac{1}{\rho_t^k} \frac{\partial u}{\partial c_t^k}$$

which we denote by v_t', and thus the first order conditions for the sophisticated hyperbolic discounting can be written as

$$v_t' = \delta v_{t+1}' \left(1 - (1-\beta)\mu_{t+1} \right)$$

For the final decision making period, totally differentiating the first order conditions $v_{T-1}' = \beta\delta v_T'$ with respect to Δ_{T-1} (remembering that $\Delta_T = \Delta_{T-1} - y_{T-1}$) gives

$$\frac{\partial y_{T-1}}{\partial \Delta_{T-1}} = \frac{\sigma_T}{(\sigma_{T-1} + \sigma_T)}$$

where σ_t denotes v_t''/v_t'. Totally differentiating again gives

$$\frac{\partial^2 y_{T-1}}{\partial \Delta_{T-1}^2} = \left(\frac{1}{(\sigma_{T-1} + \sigma_T)} \right)^3 \left(\frac{v_T'''}{v_T'} (\sigma_{T-1})^2 - \frac{v_{T-1}'''}{v_{T-1}'} (\sigma_T)^2 \right)$$

and thus

$$\frac{\partial^2 y_{T-1}}{\partial \Delta_{T-1}^2} = 0 \Rightarrow$$
$$\frac{v_T'''}{v_T'} \left(\sigma_{T-1}\right)^2 = \frac{v_{T-1}'''}{v_{T-1}'} \left(\sigma_T\right)^2 \Rightarrow$$
$$\frac{v_T''' v_T'}{\left(v_T''\right)^2} = \frac{v_{T-1}''' v_{T-1}'}{\left(v_{T-1}''\right)^2}$$

Therefore if $v'''v'/\left(v''\right)^2$ is constant, the period $T-1$ expenditure function, $y_{T-1}\left(\Delta_{T-1}\right)$, is linear. It is intuitively obvious that if in the final period

$$\begin{aligned} v_{T-1}' &= \beta \delta v_T' \\ &= \varkappa_T v_T' \end{aligned}$$

where $\varkappa_T$ is (obviously) a constant, gives a linear consumption rule (i.e. a constant μ_{T-1}) then

$$\begin{aligned} v_{T-2}' &= \delta v_{T-1}' \left(1 - (1-\beta)\mu_{T-1}\right) \\ &= \varkappa_{T-1} v_{T-1}' \end{aligned}$$

where $\varkappa_{T-1}$ is a constant (since μ_{T-1} is constant) will also give a linear expenditure rule, and so on backwards through time. We prove this by induction. Differentiating

$$v_t' = \delta v_{t+1}' \left(1 - (1-\beta) \frac{\partial y_{t+1}}{\partial \Delta_{t+1}}\right)$$

(remembering that $\mu_t \equiv \sum_{k=1}^{K} \rho_t^k \frac{\partial c_t^k}{\partial \Delta_t} = \frac{\partial y_t}{\partial \Delta_t}$) twice with respect to Δ_t gives

$$\frac{\partial^2 y_t}{\partial \Delta_t^2} = \frac{\left(\frac{v_{t+1}'''}{v_{t+1}'}\left(\frac{\partial y_{t+1}}{\partial \Delta_{t+1}}\right)^2 + \sigma_{t+1}\frac{\partial^2 y_{t+1}}{\partial \Delta_{t+1}^2} - \frac{(1-\beta)}{\psi_{t+1}}\frac{\partial^3 y_{t+1}}{\partial \Delta_{t+1}^3}\right)\left(\sigma_t\right)^2 - \frac{v_t'''}{v_t'}\left(\sigma_{t+1}\frac{\partial y_{t+1}}{\partial \Delta_{t+1}} - \frac{(1-\beta)}{\psi_{t+1}}\frac{\partial^2 y_{t+1}}{\partial \Delta_{t+1}^2}\right)^2}{\left(\sigma_t + \sigma_{t+1}\frac{\partial y_{t+1}}{\partial \Delta_{t+1}} - \frac{(1-\beta)}{\psi_{t+1}}\frac{\partial^2 y_{t+1}}{\partial \Delta_{t+1}^2}\right)^3} \tag{32}$$

where $\psi_{t+1} = \left(1 - (1-\beta)\frac{\partial y_{t+1}}{\partial \Delta_{t+1}}\right)$, and so if $y_{t+1}\left(\Delta_{t+1}\right)$ is linear so that $\frac{\partial y_{t+1}^2}{\partial \Delta_{t+1}^2} = \frac{\partial y_{t+1}^3}{\partial \Delta_{t+1}^3} = 0$ then the top of the right hand side becomes

$$\left(\frac{\partial y_{t+1}}{\partial \Delta_{t+1}}\right)^2 \left(\frac{v_{t+1}'''}{v_{t+1}'}\left(\sigma_t\right)^2 - \frac{v_t'''}{v_t'}\left(\sigma_{t+1}\right)^2\right)$$

which $= 0$ if $v'''v'/\left(v''\right)^2$ is constant. Thus if $y_{t+1}\left(\Delta_{t+1}\right)$ is linear and $v'''v'/\left(v''\right)^2$ is constant then $y_t\left(\Delta_t\right)$ is linear. But we have shown that when $v'''v'/\left(v''\right)^2$ is constant then $y_{T-1}\left(\Delta_{T-1}\right)$ is linear. Therefore, by induction, $y_t\left(\Delta_t\right)$ is linear $\forall t$.

With a linear expenditure rule (so μ_t is also the average propensity to spend) then for any

$t_1 < t_2$ we can write

$$\boldsymbol{\rho}_{t_1}'\mathbf{c}_{t_1} = \mu_{t_1}\Delta_{t_1}$$
$$\boldsymbol{\rho}_{t_2}'\mathbf{c}_{t_2} = \mu_{t_2}\Delta_{t_2} = \mu_{t_2}\Delta_{t_1}\prod_{i=t_1}^{t_2-1}(1-\mu_i)$$

so that

$$\boldsymbol{\rho}_{t_1}'\mathbf{c}_{t_1} > \boldsymbol{\rho}_{t_2}'\mathbf{c}_{t_2} \Leftrightarrow \mu_{t_1} > \mu_{t_2}\prod_{i=t_1}^{t_2-1}(1-\mu_i)$$
$$\boldsymbol{\rho}_{t_1}'\mathbf{c}_{t_1} < \boldsymbol{\rho}_{t_2}'\mathbf{c}_{t_2} \Leftrightarrow \mu_{t_1} < \mu_{t_2}\prod_{i=t_1}^{t_2-1}(1-\mu_i)$$

and, since $\mu_t \in (0,1]\ \forall t$:

$$\mu_{t_1} > \mu_{t_2} \Rightarrow \mu_{t_1} > \mu_{t_2}\prod_{i=t_1}^{t_2-1}(1-\mu_i)$$
$$\Rightarrow \boldsymbol{\rho}_{t_1}'\mathbf{c}_{t_1} > \boldsymbol{\rho}_{t_2}'\mathbf{c}_{t_2}$$

Therefore for a linear consumption function, if $t < s$ and $t + j \leq s$ then

$$\max\{\mu_{t+i}\}_{i=1,\ldots,j} > \min\{\mu_{s+i}\}_{i=1,\ldots,h} \Rightarrow \max\left\{\boldsymbol{\rho}_{t+i}'\mathbf{c}_{t+i}\right\}_{i=1,\ldots,j} > \min\left\{\boldsymbol{\rho}_{s+i}'\mathbf{c}_{s+i}\right\}_{i=1,\ldots,h}.$$

■

Proof of Corollary 1

Equation (10) in the proof of Proposition 1 shows that (conditional on the data satisfying GARP) if $\boldsymbol{\rho}_1'(\mathbf{c}_1 - \mathbf{c}_2) < 0$ then we must have:

$$\delta \geq \frac{\Psi_2}{\Psi_1}\frac{\boldsymbol{\rho}_2'(\mathbf{c}_1 - \mathbf{c}_2)}{\boldsymbol{\rho}_1'(\mathbf{c}_1 - \mathbf{c}_2)}$$

Since we know $\Psi_2/\Psi_1 > 1$, then if

$$\frac{\boldsymbol{\rho}_2'(\mathbf{c}_1 - \mathbf{c}_2)}{\boldsymbol{\rho}_1'(\mathbf{c}_1 - \mathbf{c}_2)} \geq 1$$

this implies $\delta > 1$ thus rejecting the model.■

Proof of Corollary 2

In a one-good world all data pass GARP therefore part (2) of Proposition 1 reduces to equation (RD)

$$\forall\ s, s+h \in \tau,\ h \geq 1:\ \rho_s' \mathbf{c}_s < \rho_s' \mathbf{c}_{s+h} \Rightarrow \frac{\rho_{s+h}' (\mathbf{c}_s - \mathbf{c}_{s+h})}{\rho_s' (\mathbf{c}_s - \mathbf{c}_{s+h})} < 1$$

In a one-good case this reduces to

$$\forall\ s, s+h \in \tau,\ h \geq 1:\ c_s < c_{s+h} \Rightarrow \frac{\rho_{s+h}}{\rho_s} < 1$$

or, equivalently $(c_s - c_{s+h}) < 0 \Rightarrow (\rho_s - \rho_{s+h}) > 0$. In addition, the one-good case of equation (10) is

$$(c_s - c_{s+h}) < 0 \Rightarrow \delta \geq \left(\frac{\Psi_{s+h}}{\Psi_s} \frac{\rho_{s+h}}{\rho_s} \right)^{1/h}$$

Since $h \geq 1$ we know $\Psi_{s+h} > \Psi_s$ and therefore we have

$$(c_s - c_{s+h}) < 0 \Rightarrow \delta > \left(\frac{\rho_{s+h}}{\rho_s} \right)^{1/h}$$

■

Proof of Corollary 3

In the proof of Proposition 1 we showed that if the data satisfy the Afriat inequalities then we can satisfy the sophisticated hyperbolic discounting model by setting

$$\delta > \max \left\{ \frac{\lambda_t}{\lambda_{t+1}} \right\} \quad \forall t, t+1 \in \tau \tag{33}$$

and then setting

$$1 - (1-\beta)\, \mu_{t+1} = \frac{\lambda_t}{\lambda_{t+1}} \frac{1}{\delta} < 1$$

Conditional on our choice of δ, this gives us $\tau_2 - \tau_1$ equations:

$$\beta = 1 - \frac{\left(1 - \frac{\lambda_t}{\lambda_{t+1}} \frac{1}{\delta}\right)}{\mu_{t+1}} \tag{34}$$

Since $\mu_{t+1} \in (0,1)$ for $t+1 \neq T$ and $\frac{\lambda_t}{\lambda_{t+1}} \frac{1}{\delta} < 1$, then $\left(1 - \frac{\lambda_t}{\lambda_{t+1}} \frac{1}{\delta}\right) / \mu_{t+1} > \left(1 - \frac{\lambda_t}{\lambda_{t+1}} \frac{1}{\delta}\right)$, so equation (34) (plus $\beta > 0$ by assumption) gives us:

$$\begin{aligned} & 0 < \beta < \frac{\lambda_t}{\lambda_{t+1}} \frac{1}{\delta} \Rightarrow \\ & 0 < \beta < \frac{1}{\delta} \min \left\{ \frac{\lambda_t}{\lambda_{t+1}} \right\} \quad \forall t, t+1 \in \tau, t+1 \neq T \end{aligned} \tag{35}$$

By Afriat's Theorem, the data satisfying GARP is equivalent to being able to find real numbers $\{u_t, \lambda_t > 0\}_{t \in \tau}$ such that:

$$u_s \leq u_t + \lambda_t \rho_t' (\mathbf{c}_s - \mathbf{c}_t) \qquad \forall t, s \in \tau \tag{36}$$

Writing equation (36) for any pair of periods $t, t+j \in \tau$, $j > 0$ gives:

$$\begin{aligned} u_t &\leq u_{t+j} + \lambda_{t+j} \boldsymbol{\rho}_{t+j}' (\mathbf{c}_t - \mathbf{c}_{t+j}) \\ u_{t+j} &\leq u_t + \lambda_t \boldsymbol{\rho}_t' (\mathbf{c}_{t+j} - \mathbf{c}_t) \\ &\Rightarrow \frac{\lambda_t}{\lambda_{t+j}} \boldsymbol{\rho}_t' (\mathbf{c}_t - \mathbf{c}_{t+j}) \leq \boldsymbol{\rho}_{t+j}' (\mathbf{c}_t - \mathbf{c}_{t+j}) \end{aligned} \tag{37}$$

Equation (37) gives:

$$\boldsymbol{\rho}_t' (\mathbf{c}_t - \mathbf{c}_{t+j}) < 0 \Rightarrow \frac{\lambda_t}{\lambda_{t+j}} \geq \frac{\boldsymbol{\rho}_{t+j}' (\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t' (\mathbf{c}_t - \mathbf{c}_{t+j})} \tag{38}$$

$$\boldsymbol{\rho}_t' (\mathbf{c}_t - \mathbf{c}_{t+j}) > 0 \Rightarrow \frac{\lambda_t}{\lambda_{t+j}} \leq \frac{\boldsymbol{\rho}_{t+j}' (\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t' (\mathbf{c}_t - \mathbf{c}_{t+j})} \tag{39}$$

Since

$$\frac{\lambda_t}{\lambda_{t+j}} = \frac{\lambda_t}{\lambda_{t+1}} \frac{\lambda_{t+1}}{\lambda_{t+2}} ... \frac{\lambda_{t+j-1}}{\lambda_{t+j}} \tag{40}$$

then

$$\max \left\{ \frac{\lambda_{t+i}}{\lambda_{t+i+1}} \right\}_{i=0,...,j-1} \geq \left(\frac{\lambda_t}{\lambda_{t+j}} \right)^{1/j} \tag{41}$$

Taking equation (41) over all observations implies

$$\left(\max \left\{ \frac{\lambda_t}{\lambda_{t+1}} \right\} \quad \forall t, t+1 \in \tau \right) \geq \left(\max \left\{ \left(\frac{\lambda_t}{\lambda_{t+j}} \right)^{1/j} \right\} \quad \forall t, t+j \in \tau \right) \tag{42}$$

Substituting (42) into (33) and using (38) gives

$$\delta > \max \left\{ \left(\frac{\boldsymbol{\rho}_{t+j}' (\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t' (\mathbf{c}_t - \mathbf{c}_{t+j})} \right)^{1/j} \right\} \quad \forall t, t+j \in \tau \ , \ \boldsymbol{\rho}_t' (\mathbf{c}_t - \mathbf{c}_{t+j}) < 0$$

which is the first part of Corollary 3.

Equation (40) also implies

$$\min \left\{ \frac{\lambda_{t+i}}{\lambda_{t+i+1}} \right\}_{i=0,...,j-1} \leq \left(\frac{\lambda_t}{\lambda_{t+j}} \right)^{1/j} \tag{43}$$

Therefore equations (39), (38) and (43) imply that

$$\left(\min \left\{ \frac{\lambda_t}{\lambda_{t+1}} \right\} \quad \forall t, t+1 \in \tau \right) \leq \left(\min \left\{ \left(\frac{\boldsymbol{\rho}_{t+j}' (\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}_t' (\mathbf{c}_t - \mathbf{c}_{t+j})} \right)^{1/j} \right\} \quad \forall t, t+j \in \tau \right)$$

and substituting this into equation (35) gives

$$0 < \beta < \frac{1}{\delta} \min \left\{ \left(\frac{\boldsymbol{\rho}'_{t+j} \left(\mathbf{c}_t - \mathbf{c}_{t+j} \right)}{\boldsymbol{\rho}'_t \left(\mathbf{c}_t - \mathbf{c}_{t+j} \right)} \right)^{1/j} \right\} \quad \forall t, t+j \in \tau$$

If we observe the final period T, then $\beta = \frac{\lambda_{T-1}}{\lambda_T} \frac{1}{\delta}$, which by equation (37) implies

$$\boldsymbol{\rho}'_{T-1} \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right) > 0 \Rightarrow \beta \leq \frac{\boldsymbol{\rho}'_T \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right)}{\boldsymbol{\rho}'_{T-1} \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right)}$$
$$\boldsymbol{\rho}'_{T-1} \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right) < 0 \Rightarrow \beta \geq \frac{\boldsymbol{\rho}'_T \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right)}{\boldsymbol{\rho}'_{T-1} \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right)}$$

Therefore if $\boldsymbol{\rho}'_{T-1} \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right) > 0$ we have

$$0 < \beta \leq \frac{1}{\delta} \min \left\{ \left(\frac{\boldsymbol{\rho}'_{t+j} \left(\mathbf{c}_t - \mathbf{c}_{t+j} \right)}{\boldsymbol{\rho}'_t \left(\mathbf{c}_t - \mathbf{c}_{t+j} \right)} \right)^{1/j} \right\} \quad \forall t, t+j \in \tau$$

with a strict inequality if $\min \left\{ \left(\frac{\boldsymbol{\rho}'_{t+j}(\mathbf{c}_t - \mathbf{c}_{t+j})}{\boldsymbol{\rho}'_t(\mathbf{c}_t - \mathbf{c}_{t+j})} \right)^{1/j} \right\} \neq \frac{\boldsymbol{\rho}'_T(\mathbf{c}_{T-1} - \mathbf{c}_T)}{\boldsymbol{\rho}'_{T-1}(\mathbf{c}_{T-1} - \mathbf{c}_T)}$. And if $\boldsymbol{\rho}'_{T-1} \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right) < 0$ we have

$$\max \left\{ 0, \frac{\boldsymbol{\rho}'_T \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right)}{\boldsymbol{\rho}'_{T-1} \left(\mathbf{c}_{T-1} - \mathbf{c}_T \right)} \right\} \leq \beta < \frac{1}{\delta} \min \left\{ \left(\frac{\boldsymbol{\rho}'_{t+j} \left(\mathbf{c}_t - \mathbf{c}_{t+j} \right)}{\boldsymbol{\rho}'_t \left(\mathbf{c}_t - \mathbf{c}_{t+j} \right)} \right)^{1/j} \right\} \quad \forall t, t+j \in \tau$$

with a strict inequality if $\max \left\{ 0, \frac{\boldsymbol{\rho}'_T(\mathbf{c}_{T-1} - \mathbf{c}_T)}{\boldsymbol{\rho}'_{T-1}(\mathbf{c}_{T-1} - \mathbf{c}_T)} \right\} = 0.$■

Proof of Lemma 2

We proceed by working backwards from the agent's last decision making period. In the exposition we will use a superscript to denote whose beliefs/plans we are using, i.e. $\mathbf{c}_{T-1}^{T-1}$ denotes the consumption in period $T-1$ that person $T-1$ chooses, whereas $\mathbf{c}_{T-1}^{T-2}$ will denote the consumption in period $T-1$ that person $T-2$ believes person $T-1$ will choose. Note that this implies that actual consumption in a period t is the same as $\mathbf{c}_t^t$ but different from all $\mathbf{c}_t^{t-i}$ for $i > 1$ (i.e. different from what previous selves thought the current self would do).

Period $T-1$

In the last decision making period, person $T-1$'s program is:

$$\max_{\mathbf{c}_{T-1}^{T-1}} u\left(\mathbf{c}_{T-1}^{T-1}\right)+\beta\delta u\left(\mathbf{c}_{T}^{T-1}\right)$$
$$s.t.\quad \boldsymbol{\rho}'_{T-1}\mathbf{c}_{T-1}^{T-1}+\boldsymbol{\rho}_T\mathbf{c}_T^{T-1}=\Delta_{T-1}$$
$$\Rightarrow$$
$$\lambda_{T-1}^{T-1}=\frac{1}{\boldsymbol{\rho}_{T-1}^k}\frac{\partial u}{\partial c_{T-1}^{T-1,k}}=\beta\delta\frac{1}{\boldsymbol{\rho}_T^k}\frac{\partial u}{\partial c_T^{T-1,k}}\quad\forall k \tag{44}$$

Period $T-2$

Now moving back, person $T-2$'s program is:

$$\max_{\mathbf{c}_{T-2}^{T-2}} u\left(\mathbf{c}_{T-2}^{T-2}\right)+\beta\delta u\left(\mathbf{c}_{T-1}^{T-2}\right)+\beta\delta^2 u\left(\mathbf{c}_{T}^{T-2}\right)$$
$$s.t.\quad \boldsymbol{\rho}_{T-2}\mathbf{c}_{T-2}^{T-2}+\boldsymbol{\rho}_{T-1}\mathbf{c}_{T-1}^{T-2}+\boldsymbol{\rho}_T\mathbf{c}_T^{T-2}=\Delta_{T-2}$$
$$\text{while believing person } T-1 \text{ will do:}$$
$$\max_{\mathbf{c}_{T-1}^{T-2}} u\left(\mathbf{c}_{T-1}^{T-2}\right)+\delta u\left(\mathbf{c}_{T}^{T-2}\right) \tag{45}$$
$$s.t.\ \boldsymbol{\rho}'_{T-1}\mathbf{c}_{T-1}^{T-2}+\boldsymbol{\rho}_T\mathbf{c}_T^{T-2}=\Delta_{T-1}=\Delta_{T-2}-\boldsymbol{\rho}_{T-2}\mathbf{c}_{T-2}^{T-2}$$

Looking at the last two components of $T-2$'s objective function, $\beta\delta u\left(\mathbf{c}_{T-1}^{T-2}\right)+\beta\delta^2 u\left(\mathbf{c}_{T}^{T-2}\right)$, we see that it is consistent with his beliefs about what person $T-1$ will do (equation (45)), and so the first order conditions are simply the ones they would be if $T-2$ was choosing the whole consumption path:

$$\lambda_{T-2}^{T-2}=\frac{1}{\rho_{T-2}^k}\frac{\partial u}{\partial c_{T-2}^{T-2,k}}=\beta\delta\frac{1}{\rho_{T-1}^k}\frac{\partial u}{\partial c_{T-1}^{T-2,k}}=\beta\delta^2\frac{1}{\rho_T^k}\frac{\partial u}{\partial c_T^{T-2,k}}\qquad\forall k \tag{46}$$

So $T-2$ plans for the path implied by (46), but when it comes to $T-1$'s turn, he does:

$$\frac{1}{\rho_{T-1}^k}\frac{\partial u}{\partial c_{T-1}^{T-1,k}}=\beta\delta\frac{1}{\rho_T^k}\frac{\partial u}{\partial c_T^{T-1,k}}\qquad\forall k$$

as given by (44), instead of what is given by (46), namely:

$$\frac{1}{\rho_{T-1}^k}\frac{\partial u}{\partial c_{T-1}^{T-2,k}}=\delta\frac{1}{\rho_T^k}\frac{\partial u}{\partial c_T^{T-2,k}}\qquad\forall k \tag{47}$$

Since $0 < \beta < 1$, this means that

$$\frac{\partial u}{\partial c_{T-1}^{T-1,k}} < \frac{\partial u}{\partial c_{T-1}^{T-2,k}} \quad \forall k \tag{48}$$

and so

$$\lambda_{T-2}^{T-2} = \frac{1}{\rho_{T-2}^k} \frac{\partial u}{\partial c_{T-2}^{T-2,k}} = \beta\delta \frac{1}{\rho_{T-1}^k} \frac{\partial u}{\partial c_{T-1}^{T-2,k}} > \beta\delta \frac{1}{\rho_{T-1}^k} \frac{\partial u}{\partial c_{T-1}^{T-1,k}} = \beta\delta\lambda_{T-1}^{T-1} \quad \forall k$$

Assuming concave utility (48) also implies

$$c_{T-1}^{T-1,k} > c_{T-1}^{T-2,k} \quad \forall k$$

i.e. person $T-1$ consumes a bigger share of the assets left to him by $T-2$ than $T-2$ planned that he would.

Period $T-3$

We now look at what person $T-3$ does. His program is to

$$\begin{aligned} &\max_{\mathbf{c}_{T-3}^{T-3}} u\left(\mathbf{c}_{T-3}^{T-3}\right) + \beta\delta u\left(\mathbf{c}_{T-2}^{T-3}\right) + \beta\delta^2 u\left(\mathbf{c}_{T-1}^{T-3}\right) + \beta\delta^3 u\left(\mathbf{c}_{T}^{T-3}\right) \\ &s.t.\;\; \boldsymbol{\rho}_T'\mathbf{c}_T^{T-3} + \boldsymbol{\rho}_{T-1}'\mathbf{c}_{T-1}^{T-3} + \boldsymbol{\rho}_{T-2}'\mathbf{c}_{T-2}^{T-3} + \boldsymbol{\rho}_{T-3}'\mathbf{c}_{T-3}^{T-3} = \Delta_{T-3} \end{aligned}$$

believing that $T-2$ and $T-1$ will behave as simple exponential discounters, i.e. that person $T-2$ will do:

$$\begin{aligned} &\max u\left(c_{T-2}^{T-3}\right) + \delta\left(c_{T-1}^{T-3}\right) + \delta^2 u\left(c_{T}^{T-3}\right) \\ s.t.\;\; &\boldsymbol{\rho}_{T-2}\mathbf{c}_{T-2}^{T-3} + \boldsymbol{\rho}_{T-1}\mathbf{c}_{T-1}^{T-3} + \boldsymbol{\rho}_{T}\mathbf{c}_{T}^{T-3} = \Delta_{T-2} = \Delta_{T-3} - \boldsymbol{\rho}_{T-3}'\mathbf{c}_{T-3}^{T-3} \end{aligned}$$

and that person $T-1$ will also follow this plan.

Again, his objective function is consistent with his beliefs about $T-2$ and $T-1$, so $T-3$ plans for

$$\begin{aligned} \lambda_{T-3}^{T-3} &= \frac{1}{\rho_{T-3}^k} \frac{\partial u}{\partial c_{T-3}^{T-3,k}} = \beta\delta \frac{1}{\rho_{T-2}^k} \frac{\partial u}{\partial c_{T-2}^{T-3,k}} \\ &= \beta\delta^2 \frac{1}{\rho_{T-1}^k} \frac{\partial u}{\partial c_{T-1}^{T-3,k}} = \beta\delta^3 \frac{1}{\rho_{T}^k} \frac{\partial u}{\partial c_{T}^{T-3,k}} \quad \forall k \end{aligned}$$

since he does believe that $T-2$ will do

$$\frac{1}{\rho_{T-2}^k} \frac{\partial u}{\partial c_{T-2}^{T-3,k}} = \delta \frac{1}{\rho_{T-1}^k} \frac{\partial u}{\partial c_{T-1}^{T-3,k}} = \delta^2 \frac{1}{\rho_{T}^k} \frac{\partial u}{\partial c_{T}^{T-3,k}} \tag{49}$$

But when it comes to $T-2$'s period of control it turns out instead, as shown by (46), that

$T-2$ plans for

$$\frac{1}{\rho^k_{T-2}}\frac{\partial u}{\partial c^{T-2,k}_{T-2}} = \beta\delta\frac{1}{\rho^k_{T-1}}\frac{\partial u}{\partial c^{T-2,k}_{T-1}} = \beta\delta^2\frac{1}{\rho^k_T}\frac{\partial u}{\partial c^{T-2,k}_T} \tag{50}$$

Again this implies that

$$\frac{\partial u}{\partial c^{T-2,k}_{T-2}} < \frac{\partial u}{\partial c^{T-3,k}_{T-2}}$$
$$\lambda^{T-3}_{T-3} = \frac{1}{\rho^k_{T-3}}\frac{\partial u}{\partial c^{T-3,k}_{T-3}} > \beta\delta\frac{1}{\rho^k_{T-2}}\frac{\partial u}{\partial c^{T-2,k}_{T-2}} = \beta\delta\lambda^{T-2}_{T-2}$$

and so with diminishing marginal utility, $c^{T-2,k}_{T-2} > c^{T-3,k}_{T-2}$.

Periods 0 **to** T

Thus, continuing working backwards through time we get (removing the superscripts for ease, so that c^k_t denotes the consumption at t chosen by person t - i.e. observed consumption):

$$\lambda \equiv \frac{1}{p^k_0}\frac{\partial u}{\partial c^k_0} > \frac{\beta\delta}{\rho^k_1}\frac{\partial u}{\partial c^k_1} > ... > \frac{\beta^t\delta^t}{\rho^k_t}\frac{\partial u}{\partial c^k_t} > ... > \frac{\beta^{T-1}\delta^{T-1}}{\rho^k_{T-1}}\frac{\partial u}{\partial c^k_{T-1}} = \frac{\beta^T\delta^T}{\rho^k_T}\frac{\partial u}{\partial c^k_T} \quad \forall k$$
$$\Rightarrow \exists\ \phi_i > 1\ ,\ i = 1,..,T-1\ \ s.t.:$$
$$\frac{\partial u}{\partial c^k_0} = \beta\delta\phi_1\frac{p^k_0}{\rho^k_1}\frac{\partial u}{\partial c^k_1} = ... = \beta^t\delta^t\left(\prod_{i=1}^{t}\phi_i\right)\frac{p^k_0}{\rho^k_t}\frac{\partial u}{\partial c^k_t} = ...$$
$$... = \beta^{T-1}\delta^{T-1}\left(\prod_{i=1}^{T-1}\phi_i\right)\frac{p^k_0}{\rho^k_{T-1}}\frac{\partial u}{\partial c^k_{T-1}} = \beta^T\delta^T\left(\prod_{i=1}^{T-1}\phi_i\right)\frac{p^k_0}{\rho^k_T}\frac{\partial u}{\partial c^k_T} \tag{51}$$

since

$$\frac{1}{p^k_0}\frac{\partial u}{\partial c^k_0} > \frac{\beta\delta}{\rho^k_1}\frac{\partial u}{\partial c^k_1} \Rightarrow \exists\ \ \phi_1 > 1 \quad s.t.\ \ \frac{1}{p^k_0}\ \frac{\partial u}{\partial c^k_0} = \phi_1\frac{\beta\delta}{\rho^k_1}\frac{\partial u}{\partial c^k_1}$$

and

$$\frac{\beta\delta}{\rho^k_1}\frac{\partial u}{\partial c^k_1} > \frac{\beta^2\delta^2}{\rho^k_2}\frac{\partial u}{\partial c^k_2} \Rightarrow$$
$$\phi_1\frac{\beta\delta}{\rho^k_1}\frac{\partial u}{\partial c^1_1} > \phi_1\frac{\beta^2\delta^2}{\rho^k_2}\frac{\partial u}{\partial c^2_2} \Rightarrow \exists\ \ \phi_2 > 1 \quad s.t. \quad \phi_1\frac{\beta\delta}{\rho^k_1}\frac{\partial u}{\partial c^k_1} = \phi_2\phi_1\frac{\beta^2\delta^2}{\rho^k_2}\frac{\partial u}{\partial c^k_2}$$

and so on

(the equality between the final two periods in (51) holds since person $T-1$ is the final decision maker, although we assume in general that we do not see the final period).

Note that the $\{\phi_t\}$ are the same for all k goods within a period since

$$\frac{1}{\rho^k_t}\frac{\partial u}{\partial c^{t,k}_t} = \lambda^t_t \quad \forall k, t$$

Hence, denoting

$$\Omega_t = \frac{1}{\beta^t} \prod_{i=1}^{t} \frac{1}{\phi_i}$$
$$\lambda \equiv \frac{\partial u}{\partial c_0} \frac{1}{p_0^k}$$

we can write the conditions for the naive hyperbolic discounting model as

$$\frac{\partial u}{\partial c_t^k} = \lambda \frac{\rho_t^k}{\delta^t} \Omega_t \qquad \forall k$$

which is the first line of Lemma 2

The second line of Lemma 2 (the restrictions on Ω_t) is derived as follows. Comparing the plans made by person t and person $t+1$ (now we need to reintroduce the decision maker superscripts), and expressing in terms of the marginal utility of wealth ($\lambda_{t+i}^t = \partial v \left(\boldsymbol{\rho}_t, y_{t+i}^t\right) / \partial y_{t+i}^t = \left(1/\rho_t^k\right) \left(\partial u / \partial c_{t+i}^{t,k}\right)$) we have

$$\begin{aligned}
&t \text{ plans:} \\
&\lambda_t^t = \beta\delta\lambda_{t+1}^t = \beta\delta^2\lambda_{t+2}^t = \beta\delta^3\lambda_{t+3}^t = ... \qquad (52)
\end{aligned}$$

$$\begin{aligned}
&t+1 \text{ plans:} \\
&\lambda_{t+1}^{t+1} = \beta\delta\lambda_{t+2}^{t+1} = \beta\delta^2\lambda_{t+3}^{t+1} = \beta\delta^3\lambda_{t+4}^{t+1} = ... \qquad (53)
\end{aligned}$$

This means that we can **rule out**

$$\lambda_t^t \geq \delta\lambda_{t+1}^{t+1} \qquad (54)$$

since if (54) was to hold, then equations (52) and (53) would imply

$$\begin{aligned}
\beta\delta^2\lambda_{t+2}^t &\geq \beta\delta^2\lambda_{t+2}^{t+1} \\
\beta\delta^3\lambda_{t+3}^t &\geq \beta\delta^3\lambda_{t+3}^{t+1} \\
&... \\
\beta\delta^{T-t}\lambda_T^t &\geq \beta\delta^{T-t}\lambda_T^{t+1}
\end{aligned}$$

which by concavity of $v(\boldsymbol{\rho}, y)$ in y given $\boldsymbol{\rho} \Rightarrow$

$$\begin{aligned}
y_{t+2}^t &\leq y_{t+2}^{t+1} \\
y_{t+3}^t &\leq y_{t+3}^{t+1} \\
&... \\
y_T^t &\leq y_T^{t+1}
\end{aligned}$$

and we already know that $y_{t+1}^t < y_{t+1}^{t+1}$. Hence for periods $t+1$ to T, person $t+1$ would be planning to consume the same or more each period (and strictly more in period $t+1$) of good

k than person t planned and so would violate the lifetime budget constraint.

Therefore we must have

$$\lambda_t^t < \delta \lambda_{t+1}^{t+1} \tag{55}$$

or equivalently

$$\frac{1}{\rho_t^k} \frac{\partial u}{\partial c_t^{t,k}} < \delta \frac{1}{\rho_{t+1}^k} \frac{\partial u}{\partial c_{t+1}^{t+1,k}} \tag{56}$$

The intuition of this is that it tells us that an exponential discounter planning to spend y_{t+1}^{t+1} tomorrow would be spending less today than an naive hyperbolic discounter with otherwise equivalent preference parameters, or, equivalently, a naive hyperbolic discounter spending y_t^t today will spend less tomorrow than an (otherwise equivalent) exponential discounter spending y_t^t today.

Equation (55) gives us the final line of Lemma 2 since

$$\begin{aligned} \lambda_t^t &= \delta \frac{\Omega_t}{\Omega_{t+1}} \lambda_{t+1}^{t+1} < \delta \lambda_{t+1}^{t+1} \\ &\Rightarrow \frac{\Omega_t}{\Omega_{t+1}} < 1 \\ &\Rightarrow \Omega_t > \Omega_{t+1} \end{aligned} \tag{57}$$

■

Proof of Proposition 3

We need to show the empirical conditions for the naive quasi-hyperbolic discounting model rationalising the data are identical to those of the sophisticated quasi-hyperbolic discounting model. This can be done by following the proof of Proposition 1, remembering that by definition $1 - (1 - \beta)\, \mu_{t+1} = \Psi_t / \Psi_{t+1}$, replacing Ψ_t with Ω_t, and noting that, of course (since $\Omega_0 = 1$)

$$\prod_{i=1}^{t} \frac{\Omega_i}{\Omega_{i-1}} = \Omega_t$$

Proof of Proposition 4

Similarly to the proof of Proposition 2, observing

$$\boldsymbol{\rho}_t' \mathbf{c}_t > \boldsymbol{\rho}_t' \mathbf{c}_{t+j},\ \boldsymbol{\rho}_s' \mathbf{c}_s < \boldsymbol{\rho}_s' \mathbf{c}_{s+h} \text{ and } \left(\frac{\boldsymbol{\rho}_{t+j}' \left(\mathbf{c}_t - \mathbf{c}_{t+j} \right)}{\boldsymbol{\rho}_t' \left(\mathbf{c}_t - \mathbf{c}_{t+j} \right)} \right)^{1/j} < \left(\frac{\boldsymbol{\rho}_{s+h}' \left(\mathbf{c}_s - \mathbf{c}_{s+h} \right)}{\boldsymbol{\rho}_s' \left(\mathbf{c}_s - \mathbf{c}_{s+h} \right)} \right)^{1/h}$$

gives

$$\delta\left(\frac{\Omega_t}{\Omega_{t+j}}\right)^{1/j} \leq \left(\frac{\boldsymbol{\rho}'_{t+j}\left(\mathbf{c}_t-\mathbf{c}_{t+j}\right)}{\boldsymbol{\rho}'_t\left(\mathbf{c}_t-\mathbf{c}_{t+j}\right)}\right)^{1/j} < \left(\frac{\boldsymbol{\rho}'_{s+h}\left(\mathbf{c}_s-\mathbf{c}_{s+h}\right)}{\boldsymbol{\rho}'_s\left(\mathbf{c}_s-\mathbf{c}_{s+h}\right)}\right)^{1/h} \leq \delta\left(\frac{\Omega_s}{\Omega_{s+h}}\right)^{1/h} \tag{58}$$

$$\Rightarrow \left(\prod_{i=1}^{j}\phi_{t+i}\right)^{1/j} < \left(\prod_{i=1}^{h}\phi_{s+i}\right)^{1/h} \tag{59}$$

$$\Rightarrow \min\left\{\phi_{t+i}\right\}_{i=1,\ldots,j} < \max\left\{\phi_{s+i}\right\}_{i=1,\ldots,h} \tag{60}$$

Note that ϕ_t does not have an interpretation like μ_t in $\left[1-(1-\beta)\,\mu_t\right]$ does as being the marginal propensity to spend. We simply defined ϕ_t by

$$\lambda_t^t = \beta\delta\lambda_{t+1}^t = \beta\delta\phi_{t+1}\lambda_{t+1}^{t+1} \Rightarrow \lambda_{t+1}^t = \phi_{t+1}\lambda_{t+1}^{t+1}$$

where $\phi_{t+1} > 1$ since person $t+1$ spends more in period $t+1$ than person t planned he would. Thus finding the empirical implications of $\max\left\{\phi_{s+i}\right\}_{i=1,\ldots,h} > \min\left\{\phi_{t+i}\right\}_{i=1,\ldots,j}$ requires a different strategy from the one we used for the sophisticated discounter. We need to look at adjacent period pairs $\left\{(t+i, t+1+i)\right\}_{i=1,\ldots,j}$ and $\left\{(s+i, s+1+i)\right\}_{i=1,\ldots,h}$. In what follows we will use $t, t+1$ and $s, s+1$ as examples or else the notation becomes very long. We have

$$\lambda_{t+1}^t = \phi_{t+1}\lambda_{t+1}^{t+1} \Rightarrow \lambda_{t+1}^t > \lambda_{t+1}^{t+1} \tag{61}$$

and

$$\lambda_{s+1}^s = \phi_{s+1}\lambda_{s+1}^{s+1} \Rightarrow \lambda_{s+1}^s > \lambda_{s+1}^{s+1} \tag{62}$$

and also if $\phi_{s+1} > \phi_{t+1}$ then equations (61) and (62) imply

$$\frac{\lambda_{s+1}^s}{\lambda_{s+1}^{s+1}} > \frac{\lambda_{t+1}^t}{\lambda_{t+1}^{t+1}} \tag{63}$$

we will come back to equation (63) shortly.

Comparing the plans of t and $t+1$ (as in equations (52) and (53) implies

$$\lambda_{t+1}^t = \delta^m\lambda_{t+1+m}^t \quad \forall\ m = 1, .., T-t-1$$
$$\lambda_{t+1}^{t+1} = \beta\delta^m\lambda_{t+1+m}^{t+1} \ \forall\ m = 1, .., T-t-1$$

and these and equation (61) give

$$\lambda_{t+1+m}^t = \beta\phi_{t+1}\lambda_{t+1+m}^{t+1} \quad \forall\ m = 1, .., T-t-1 \tag{64}$$

and similarly for s we have

$$\lambda_{s+1+n}^s = \beta\phi_{s+1}\lambda_{s+1+n}^{s+1} \quad \forall\ n = 1, .., T-s-1 \tag{65}$$

Remembering that by definition $\beta\phi_{t+1} = \Omega_t/\Omega_{t+1}$ and thus equation (57) implies $\beta\phi_{t+1} < 1$, then equation (64) and concavity of $v(\boldsymbol{\rho}_t, y_t)$ with respect to expenditure give

$$y^t_{t+1+m} > y^{t+1}_{t+1+m} \quad \forall\ m = 1, .., T - t - 1 \tag{66}$$

Thus whatever person $t+1$ overspends in period $t+1$ compared to person t's plans, he must save by underspending in periods $t+2, ..., T$ compared to t's plans in order to preserve the budget constraint. And the same argument applies to person $s+1$ compared to person s. Thus we have (remembering that $t < s$):

$$\left(y^{t+1}_{t+1} - y^t_{t+1}\right) = \sum_{m=1}^{T-t-1} \left(y^t_{t+1+m} - y^{t+1}_{t+1+m}\right) > \sum_{n=1}^{T-s-1} \left(y^t_{s+1+n} - y^{t+1}_{s+1+n}\right) \tag{67}$$

$$\left(y^{s+1}_{s+1} - y^s_{s+1}\right) = \sum_{n=1}^{T-s-1} \left(y^s_{s+1+n} - y^{s+1}_{s+1+n}\right) \tag{68}$$

So now we would like to be able to say something about the size of $\left(y^t_{s+1+n} - y^{t+1}_{s+1+n}\right)$ versus $\left(y^s_{s+1+n} - y^{s+1}_{s+1+n}\right)$, $n = 1, .., T - s - 1$.

Repeating equation (66) over time (for $m = s - t + n$) gives

$$\begin{aligned} &y^t_{s+1+n} > y^{t+1}_{s+1+n} > y^{t+2}_{s+1+n} > ... > y^{s+1}_{s+1+n} \Rightarrow \\ &y^{t+1}_{s+1+n} > y^{s+1}_{s+1+n} \quad \forall\ n = 1, .., T - s - 1 \end{aligned} \tag{69}$$

Equations (64) and (65) with $\phi_{s+1} > \phi_{t+1}$ imply

$$\begin{aligned} &\frac{v'^{t+1}_{s+1+n}}{v'^t_{s+1+n}} = \frac{1}{\beta\phi_{t+1}} > \frac{1}{\beta\phi_{s+1}} = \frac{v'^{s+1}_{s+1+n}}{v'^s_{s+1+n}} \Rightarrow \\ &\ln v'^{s+1}_{s+1+n} - \ln v'^s_{s+1+n} < \ln v'^{t+1}_{s+1+n} - \ln v'^t_{s+1+n} \quad \forall\ n = 1, .., T - s \end{aligned} \tag{70}$$

Equation (70) involves differences in $\ln v'$, and we know from the properties of $v(\boldsymbol{\rho}, y)$ that $\partial(\ln v')/\partial y = v''/v' < 0$. We also know from equation (69) that $y^{s+1}_{s+1+n} < y^s_{s+1+n} < y^{t+1}_{s+1+n} < y^t_{s+1+n}$. Therefore if $\ln v'$ is convex, so that

$$\frac{\partial^2(\ln v')}{\partial y^2} = \frac{\partial(v''/v')}{\partial y} \geq 0$$

(i.e. decreasing absolute risk aversion) then equation (70) $\Rightarrow$

$$y^s_{s+1+n} - y^{s+1}_{s+1+n} < y^t_{s+1+n} - y^{t+1}_{s+1+n} \quad \forall n = 1, .., T - s \tag{71}$$

(under increasing relative risk aversion we could have $y^s_{s+1+n} - y^{s+1}_{s+1+n} \gtrless y^t_{s+1+n} - y^{t+1}_{s+1+n}$).

Equation (71) implies

$$\sum_{n=1}^{T-s-1}\left(y^t_{s+1+n}-y^{t+1}_{s+1+n}\right)>\sum_{n=1}^{T-s-1}\left(y^s_{s+1+n}-y^{s+1}_{s+1+n}\right)$$

which implies by equations (67) and (68) that we must have

$$\left(y^{t+1}_{t+1}-y^t_{t+1}\right)>\left(y^{s+1}_{s+1}-y^s_{s+1}\right) \tag{72}$$

Now we want to ask what conditions will generate a violation of (72). Equation (63) gives

$$\ln v'^t_{t+1}-\ln v'^{t+1}_{t+1}<\ln v'^s_{s+1}-\ln v'^{s+1}_{s+1} \tag{73}$$

As with equation (70), equation (73) involves differences in $\ln v'$. Again, we know that $y^t_{t+1}<y^{t+1}_{t+1}$ and $y^s_{s+1}<y^{s+1}_{s+1}$ and so if $\partial\left(v''/v'\right)/\partial y\geq 0$ we would like to be able to say that having $y^{t+1}_{t+1}<y^{s+1}_{s+1}$ implies $\left(y^{t+1}_{t+1}-y^t_{t+1}\right)<\left(y^{s+1}_{s+1}-y^s_{s+1}\right)$, and therefore equation (72) is violated. However, unlike equation (70), the comparison in equation (73) is across different time periods $s+1$ and $t+1$ and thus we are not holding prices constant. Therefore we cannot say that equation (73) with $\partial\left(v''/v'\right)/\partial y\geq 0$ and $y^{s+1}_{s+1}>y^{t+1}_{t+1}$ implies $\left(y^{s+1}_{s+1}-y^s_{s+1}\right)>\left(y^{t+1}_{t+1}-y^t_{t+1}\right)$ without also assuming that $\partial\left(v''/v'\right)/\partial y$ is independent of $\boldsymbol{\rho}$.

The other circumstance under which equation (72) is contradicted is as follows: if in the data we have $\boldsymbol{\rho}'_{s+1}\mathbf{c}_s>\boldsymbol{\rho}'_{s+1}\mathbf{c}_{s+1}\left(=y^{s+1}_{s+1}\right)$ then, since $y^{s+1}_{s+1}>y^s_{s+1}$, it must be the case that $\boldsymbol{\rho}'_{s+1}\mathbf{c}_s>y^s_{s+1}$. Hence equation (72) becomes

$$\left(y^{t+1}_{t+1}-y^t_{t+1}\right)>\left(y^{s+1}_{s+1}-y^s_{s+1}\right)>\left(y^{s+1}_{s+1}-\boldsymbol{\rho}'_{s+1}\mathbf{c}^s_s\right)$$

and, of course $y^{t+1}_{t+1}>\left(y^{t+1}_{t+1}-y^t_{t+1}\right)$, so we can write

$$y^{t+1}_{t+1}>\left(y^{t+1}_{t+1}-y^t_{t+1}\right)>\left(y^{s+1}_{s+1}-y^s_{s+1}\right)>\left(y^{s+1}_{s+1}-\boldsymbol{\rho}'_{s+1}\mathbf{c}^s_s\right)$$

Therefore if we see

$$\boldsymbol{\rho}'_{s+1}\mathbf{c}^s_s>y^{s+1}_{s+1}$$
and
$$y^{t+1}_{t+1}<\left(y^{s+1}_{s+1}-\boldsymbol{\rho}'_{s+1}\mathbf{c}^s_s\right)$$

i.e.

$$\boldsymbol{\rho}'_{s+1}\mathbf{c}_s>\boldsymbol{\rho}'_{s+1}\mathbf{c}_{s+1}$$
and
$$\boldsymbol{\rho}'_{t+1}\mathbf{c}_{t+1}<\left(\boldsymbol{\rho}'_{s+1}\mathbf{c}_{s+1}-\boldsymbol{\rho}'_{s+1}\mathbf{c}_s\right)$$

then equation (72) cannot hold.

Recall that we used $(t, t+1)$ and $(s, s+1)$ as an example, and that to investigate $\max \{\phi_{s+i}\}_{i=1,...,h} > \min \{\phi_{t+i}\}_{i=1,...,j}$ we actually need to look at all adjacent period pairs $\{(t-1+i, t+i)\}_{i=1,...,j}$ and $\{(s-1+i, s+i)\}_{i=1,...,h}$. Thus to apply the above analysis to the generalised version of equation (70) comparing periods $(t-1+i, t+i)$ and $(s-1+g, s+g)$ we need to be able to replace $y^{s+1}_{s+1+n} < y^{t+1}_{s+1+n}$ (which comes from equation (69)) with $y^{s+g}_{s+g+n} < y^{t+i}_{s+g+n}$, and hence (again referring to equation (69)) we need all $t+i$ $(i = 1, ..., j)$ to come before all $s+g$ $(g = 1, ..., h)$ and hence we need $t + j \leq s$.■

Proof of Corollaries 1-3 for the naive discounter

These follow the proofs of Corollaries 1-3 replacing Ψ_t with Ω_t. For Corollary 3 we again remember that by definition $1 - (1-\beta)\mu_{t+1} = \Psi_t/\Psi_{t+1}$ and $\Omega_t/\Omega_{t+1} = \beta\phi_{t+1}$, thus

$$\beta = \frac{\lambda_t}{\lambda_{t+1}} \frac{1}{\delta\phi_{t+1}}$$

and therefore, as before, $\beta < \frac{\lambda_t}{\lambda_{t+1}} \frac{1}{\delta}$ since $\phi_{t+1} > 1$. Again, since $\Omega_{T-1}/\Omega_T = 1$, then if we observe the final period

$$\beta = \frac{\lambda_{T-1}}{\lambda_T} \frac{1}{\delta}$$

■