Stability and perturbation analysis of non-negative super-resolution

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Abstract

The convolution of a discrete measure, $x = \sum_{i=1}^{k} a_i \delta_{t_i}$, with a local window function, $\phi(s-t)$, is a common model for a measurement device whose resolution is substantially lower than that of the objects being observed. Super-resolution concerns localising the point sources $\{a_i, t_i\}_{i=1}^{k}$ with an accuracy beyond the essential support of $\phi(s-t)$, typically from m samples $y(s_j) = \sum_{i=1}^{k} a_i \phi(s_j - t_i) + w_j$, where w_j indicates an inexactness in the sample value. We consider the setting of x being non-negative and study two aspects of this problem: stability of the solutions with respect to measurement noise and perturbation of the solutions with respect to inaccuracies in the dual variable when solving the dual problem.

In Part I, we characterise non-negative solutions $\hat{x}$ consistent with the samples within the bound $\|w\|_2 \leq \delta$. We show that the integrals of $\hat{x}$ and x over $(t_i - \epsilon, t_i + \epsilon)$ are close, converging to one another as ϵ and δ approach zero. We then show how to make this general result, for windows that form a Chebyshev system, precise for the case of $\phi(s-t)$ being a Gaussian window, in which case the average error between $\hat{x}$ and x is $\mathcal{O}(\delta^{1/6})$. The main innovation of this result is that non-negativity alone is sufficient to localise point sources beyond the sensor resolution and that, while regularisers such as total variation might be particularly effective, they are not required in the non-negative setting.

A practical approach for solving the problem is to consider its dual. In Part II, we study the stability of solutions with respect to the solutions to the dual problem. In particular, we establish a relationship between perturbations in the primal variable and perturbations the dual variable around the optimiser. We then establish a similar relationship between perturbations in the dual variable around the optimiser and the magnitude of the additive noise $\|w\|_2$ in the measurements.

Contents

Chapter 1

Introduction

1.1 Background and description of the problem

The problem of super-resolution concerns recovering a resolution beyond the size of the point spread function of a sensor. For instance, a particularly stylised example is that of recovering multiple point sources which, because of the finite *resolution* or *bandwidth* of the sensor, may not be visually distinguishable. Various instances of this problem exist in applications such as astronomy [69], imaging in chemistry, medicine and neuroscience [6, 46, 71, 59, 44, 36, 45, 83], spectral estimation [82, 80], geophysics [51], and system identification [76]. Often in these applications, much is known about the point spread function of the sensor, or can be estimated and, given such model information, it is possible to identify point source locations with accuracy substantially below the essential width of the sensor point spread function. Recently there has been substantial interest from the mathematical community in posing algorithms and proving super-resolution guarantees in this setting, see for instance [12, 81, 42, 23, 31, 24, 2, 4]. Typically, these approaches borrow notions from compressed sensing [26, 13, 14].

This project in collaboration with the National Physical Laboratory (NPL) is motivated by the EU Marine Strategy Framework Directive which requires EU member states to assess the level of marine noise in their seas. To achieve this, NPL are interested in using mathematical modelling and numerical techniques to estimate the positions and sound levels of the ships in a particular shipping lane, given only a small number of observations from hydrophones, or underwater microphones (see an illustration of the problem in Figure 1.1). Solving this problem may have other potential applications, for example monitoring ship activity in a harbour.

The focus of this thesis is on the problem of reconstructing a non-negative sparse signal from blurred and noisy measurements. Mathematically, this is a sparse de-

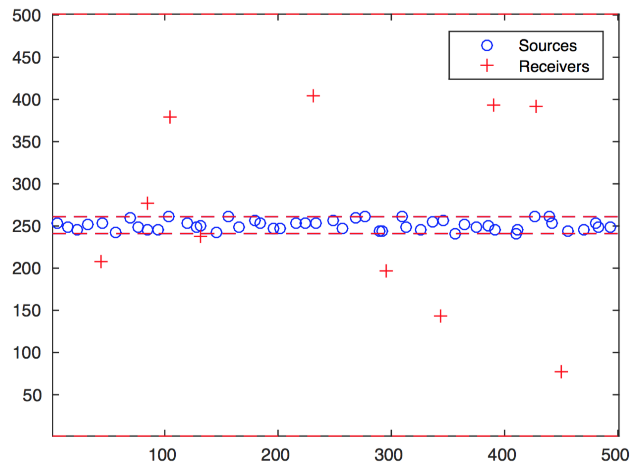


Figure 1.1: Example of a receiver and source configuration (or hydrophones and ships respectively) in the shipping lane. Image taken with permission from [64].

convolution problem, a type of inverse problem where the unknown signal x is a 1D k-discrete non-negative Borel measure defined on the interval $I = [0, 1] \subset \mathbb{R}$, or a weighted sum of Dirac delta functions located at $\{t_i\}_{i=1}^k$ with weights $\{a_i\}_{i=1}^k$:

$$x = \sum_{i=1}^{k} a_i \cdot \delta_{t_i} \quad \text{with} \;\; a_i > 0 \quad \text{and } t_i \in int(I) \quad \text{for all } i, \tag{1.1}$$

and the measurements $\{y_j\}_{j=1}^m$ consist of samples at locations $\{s_j\}_{j=1}^m$ of the convolution of x with a known point spread function[1] ϕ (which can be, for example, a Gaussian):

$$y_j = y(s_j) = \int_I \phi(t - s_j)x(\mathrm{d}t) + w_j = \sum_{i=1}^{k} a_i \phi(t_i - s_j) + w_j, \tag{1.2}$$

where w_j with $\|w\|_2 \leq \delta$ can represent additive noise. The aim is to recover x from the measurements $\{y_j\}_{j=1}^m$.

This model can be interpreted in multiple ways. On one hand, ϕ describes a physical phenomenon such as propagation of sound from a number of sources, which is captured by the sum formulation in (1.2). Alternatively, the underlying signal is only observable through convolution with a kernel ϕ, which is expressed in the integral formulation of (1.2).

We observe the signal $y(s)$ at sample points $s_j \in S$ with $|S| > 2k$ and we want to determine the values of t_i, a_i for all $i = 1, \dots, k$ from these observations. The number

[1] We will use the terms *point spread function*, *window function* and *(convolution) kernel* interchangeably in this thesis. The same applies to the terms *point sources* and *spikes*, which we will use to refer to the discrete signal, modelled by Dirac delta functions.

of sources k is also unknown. The goal of finding $y(s)$ given a finite number of samples can be viewed as an interpolation problem or as a classical Shannon sampling problem [77]. However, in our case we are interested to find not only the value of $y(s)$, but also the parameters a_i and t_i, which form the structure of the underlying measure x. In the field of compressed sensing, this is the difference between basis pursuit and signal approximation.

We seek to determine the parameters of the model consistent with the observations y_j. Writing the samples from (1.2) in vector notation:

$$y := [y_1, \ldots, y_m]^T \in \mathbb{R}^m, \qquad \Phi(t) := [\phi(t - s_1), \ldots, \phi(t - s_m)]^T \in \mathbb{R}^m \tag{1.3}$$

allows us to state the program we investigate:

$$\underset{\{\hat{a}_i\},\{\hat{t}_i\}_{i=1}^k}{\operatorname{argmin}} \left\| y - \sum_{i=1}^{k} \hat{a}_i \Phi(\hat{t}_i) \right\|_2 . \tag{1.4}$$

Provided that y satisfies model (1.2) and ϕ satisfies certain properties, there exists a unique solution to (1.4) with zero objective [74].

If the locations of the point sources, $t_1, \ldots, t_k$, were known, the problem would be equivalent to simply solving a linear system to find the amplitudes $a_1, \ldots, a_k$. If $t_1, \ldots, t_k$ were unknown but known to be sampled from a discrete finite set, then the problem would be a traditional compressed sensing problem. However, because $t_1, \ldots, t_k$ are unknown and take continuous values, the problem we solve is a continuous optimisation problem. Like the traditional compressed sensing problem, it is a non-convex global optimisation problem. However, unlike the typical setting for compressed sensing, the sampling matrix in this case would be highly coherent, due to the continuity of the kernel ϕ. Moreover, higher accuracy would require a finer grid for the potential locations of the points sources and sampling locations, which would lead to higher correlation of the sampling matrix. This is far from the ideal compressed sensing setting, where the sampling matrix is random. Therefore, it is desirable that we work with a model which takes into account the continuity of the source locations at a fundamental level as opossed to a discrete model with a fine discretisation.

As an illustration, Figure 1.2 shows the discrete measure x in blue for $k = 3$, the continuous function $y(s) = \int_I \phi(t - s)x(\mathrm{d}t)$ in red and the noisy samples $y_j = y(s_j)$ at the sample locations S represented as the black circles.

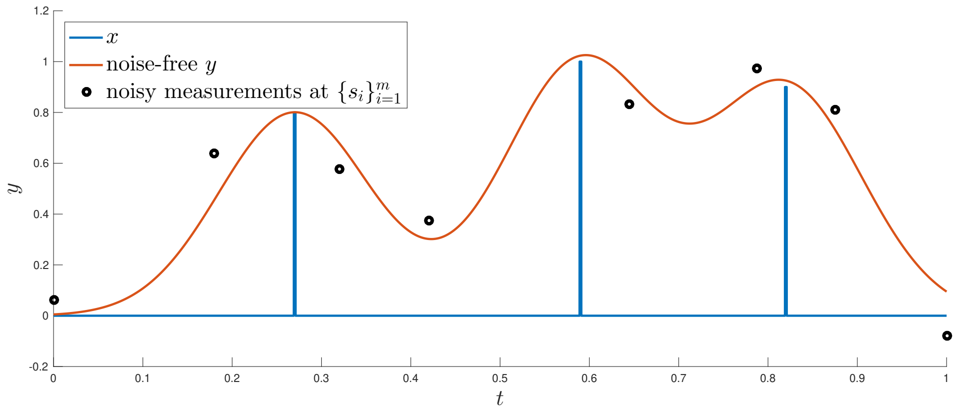


Figure 1.2: Example of discrete measure x and measurements y with the Gaussian kernel $\phi(t) = e^{-\frac{t^2}{\sigma^2}}$.

There is a growing literature (see Section 1.2) on solving this problem using algorithms that are guaranteed to find the optimal solution. The approach shared by all these methods is to impose structure on ϕ in addition to the observations. Algorithms proposed for this problem include a class of methods that consider a convex relaxation of (1.4). Compelling theoretical results have been obtained for this approach and we will extend some of them in the next chapters.

In this thesis, we study the problem of super-resolution in the non-negative setting, namely when the weights a_i of the Dirac delta functions in the measure x are non-negative, and we investigate the implications of the non-negative assumption, both from a purely theoretical point of view, in Part I, and from an algorithmic point of view, in Part II. The non-negativity assumption is motivated in part by a number of applications where the spikes in the signal are positive, like the ship localisation problem described above or fluorescence microscopy [6], and in part by the mathematical properties of such a model. Specifically, this assumption simplifies the conditions that the dual certificate needs to satisfy, which allows us to use pre-existing results for the construction of the certificate.

First we consider the *feasibility problem*:

$$\text{Find } z \geq 0 \text{ subject to } \left\| y - \int_I \Phi(t) z(\mathrm{d}t) \right\|_2 \leq \delta, \tag{1.5}$$

where z is a non-negative Borel measure on $[0, 1]$ and δ is an upper bound on the ℓ_2 norm of the noise w in the measurements (1.2). Herein we characterise non-negative measures consistent with measurements (1.2) in relation to the discrete measure (1.1). That is, we consider any non-negative Borel measure z from the Program (1.5) and show that any such z is close to x given by (1.1) in an appropriate metric, which means that the super-resolution problem is stable, see Theorems 9, 10, 19 and Corollary 20. Program (1.5) is particularly notable in that there is no regulariser of z beyond imposing non-negativity and, rather than specify an algorithm to select a z which satisfies Program (1.5), we consider all admissible solutions. The admissible solutions of Program (1.5) are determined by the source and sample locations, which we denote as

$$T = \{t_i\}_{i=1}^k \subset \text{int}(I) \quad \text{and} \quad S = \{s_j\}_{j=1}^m \subseteq I \tag{1.6}$$

respectively, as well as the particular function $\phi(t)$ used to sample the k-sparse non-negative measure x from (1.1).

In the second part of the thesis, we consider the dual approach to solving the super-resolution problem for Gaussian ϕ:

$$\phi(t) = e^{-t^2/\sigma^2}. \tag{1.7}$$

While non-negativity is a sufficient condition to show stability, in practice it is useful to use the knowledge about the discrete measure and use a regulariser like the total variation (TV) norm to impose sparsity. Therefore, we consider the TV norm minimisation problem and analyse how the locations $\{t_i\}_{i=1}^k$ and weights $\{a_i\}_{i=1}^k$ of the sources in x vary around their true values as λ is perturbed around its optimiser λ^*. This inaccuracy in λ can be due to algorithmic errors or noise in the data, so it is of practical interest to study how such perturbations affect the measure x.

We start with an analysis in the noise-free setting, when the measurements y_j are exact (for $w_j = 0$). The signal x can be recovered by solving:

$$\min_{z \geq 0} \|z\|_{TV} \quad \text{subject to} \quad y = \int_I \Phi(t) z(\mathrm{d}t), \tag{1.8}$$

over all non-negative measures z on $I = [0, 1]$. The TV norm for measures can be seen as the continuous analogue of the ℓ_1 norm for vectors. Moreover, the TV norm of the discrete measure x in (1.1) is equal to the ℓ_1 norm of the vector of weights $\|x\|_{TV} = \sum_{i=1}^k |a_i|$.

As we will see in the literature review in Section 1.2, problem (1.8) recovers exactly the solution (1.1) under various assumptions on the minimum separation of the spikes in x or the kernel ϕ, depending on the model used.

Specifically, in Part II we consider the dual of (1.8):

$$\max_{\lambda \in \mathbb{R}^m} y^T \lambda \quad \text{subject to} \quad \lambda^T \Phi(t) \leq 1 \quad \forall t \in I, \tag{1.9}$$

which is a finite-dimensional problem with infinitely many constraints, known as a semi-infinite program (see Appendix B.1 for the derivation of the dual). Such problems can be solved using a number of algorithms including exchange methods [35] and sequential quadratic programming [58]. The advantage over algorithms that solve the primal problem (for example the ADCG algorithm [9], see Section 1.2.2) is working in a finite dimensional space, which simplifies the analysis of the algorithms used. Note that, since the primal problem (1.8) has only equality constraints, Slater's condition holds and therefore we have strong duality.

For problems (1.8) and (1.9), we give bounds on the perturbations in t_i and a_i as λ is perturbed around its true value, see Theorems 28 and 29.

We then extend the perturbation analysis to the case when the measurements are corrupted by additive noise and consider the slightly modified version of the dual, where we include an additional box constraint on λ:

$$\max_{\lambda \in \mathbb{R}^m} y^T \lambda \quad \text{such that} \quad \lambda^T \Phi(t) \leq 1, \quad \forall t \in I, \\ \text{and} \quad \|\lambda\|_\infty \leq \tau, \tag{1.10}$$

where the bound τ of the box constraint is fixed. In Appendix B.2, we see that the box constraint in the dual leads to an ℓ_1 norm minimisation objective in the primal problem:

$$\min_{z \geq 0} \left\| y - \int_I \Phi(t) z(\mathrm{d}t) \right\|_1 \quad \text{such that} \quad \|z\|_{TV} \leq \Pi, \tag{1.11}$$

for some $\Pi > 0$, which is a reasonable choice of the primal problem in the noisy setting, as this formulation accounts for noise in the measurements.

Here we give a similar bound on the perturbation of λ around λ^* in terms of the noise w, see Theorem 33. While the bounds given in these theorems apply only to the case when the convolution kernel is Gaussian, the same techniques can be applied to obtain perturbation bounds for other kernels, with a few differences in the way specific sums in the proofs are bounded, which would depend on the kernel used.

Before starting to discuss the stability results in more detail in Chapter 2, we give a survey of the literature of super-resolution in Section 1.2 to help the reader gain a better understanding of where our work is situated in the landscape of super-resolution research.

1.2 Literature review

1.2.1 Recovery guarantees for super-resolution

While still a new field, there has been a considerable amount of research on super-resolution during the last couple of years. In this section, we give an overview of the main directions in the literature, grouped on the measurement model used and on the problem solved in order to recover the signal.

Low pass filter

A seminal paper is [12], which considers a superposition of Dirac delta functions situated on the 1-dimensional torus $\mathbb{T}$ obtained by bending the $[0,1]$ segment to form a circle:

$$x = \sum_i a_j \delta_{t_j}, \tag{1.12}$$

where $a_j \in \mathbb{C}$, $\{t_j\}$ are the locations of the spikes on $\mathbb{T}$ and δ_τ is a Dirac measure at τ. The measurement consists of convolving the signal with a low pass filter so that only the lowest $2f_c + 1$ frequencies (for some $f_c \in \mathbb{N}$) are known:

$$y(k) = \int_0^1 e^{-i2\pi kt} x(\mathrm{d}t) = \sum_j a_j e^{-i2\pi k t_j}, \quad k \in \mathbb{Z}, |k| \leq f_c. \tag{1.13}$$

For convenience, the low pass filter chosen in this model (corresponding to the point spread function ϕ in (1.2)) is the periodic Dirichlet kernel, whose Fourier transform is 1 at any $k \in \mathbb{Z}$ such that $|k| \leq f_c$ and zero at $|k| > f_c$, such that the representation of the measured signal in frequency domain is given by (1.13) [11].

An essential condition in most of the work on super-resolution is that there is a minimum distance between each pair of spikes. Let T be a set of points on $\mathbb{T}$, then the minimum separation is defined as the closest distance between any two elements of T:

$$\Delta(T) = \inf_{(t,t') \in T : t \neq t'} |t - t'| \tag{1.14}$$

where $|t - t'|$ is the wrap-around distance (for example, the distance between $t = 0$ and $t' = 3/4$ is equal to $1/4$).

The authors of [12] show that, as long as there is a minimum separation distance of $2\lambda_c = 2/f_c$ between the spikes, the input signal x is the solution to:

$$\min_z \|z\|_{TV} \quad \text{subject to} \quad \mathcal{F}z = y, \tag{1.15}$$

where $\mathcal{F}$ is the linear map collecting the lowest $2f_c + 1$ frequency coefficients and $\|z\|_{TV}$ is the total variation norm (or the TV norm) of the measure z. The TV norm of a complex measure z is defined as

$$\|z\|_{TV} = \sup_{\pi} \sum_{B_i \in \pi} |z(B_i)|, \tag{1.16}$$

where the supremum is taken over all partitions π of $\mathbb{T}$ into a countable number of disjoint measurable subsets.

Note that if z has the form in (1.12), then its TV norm is equal to the ℓ_1 norm of the amplitudes $\|z\|_{TV} = \sum_j |a_j|$. However, the minimisation in (1.15) is carried out over the set of all finite complex measures z supported on $[0,1]$, not only ones consisting of spikes like (1.12). It is then shown that the solution to (1.15) can be found by solving a semidefinite program.

To account for potential noise in the measurements, a modified version of (1.15) is discussed in [11]:

$$\min_z \|z\|_{TV} \quad \text{subject to} \quad \|y - \mathcal{F}z\|_1 \leq \delta, \tag{1.17}$$

where τ bounds the ℓ_2 norm of the noise:

$$y = \mathcal{F}x + w, \quad \|w\|_2 \leq \delta. \tag{1.18}$$

In [12], the authors analyse the discrete version of this problem (so the measures become vectors and the TV norm becomes the vector ℓ_1 norm) and show that the error in the estimated signal x grows linearly with the noise level δ. Then, in [11], the authors show a similar result without discretising on a grid by bounding the convolution of the error with a Fejer kernel with a higher cut-off frequency than the low pass filter used in the measurement step. However, this leads to masking the error in higher frequencies. More standard convergence results are proved in [39], where it is shown that the error between the estimated locations of the spikes and the true locations is proportional to $\sqrt{\delta}$ and the error between the estimated amplitudes and the true amplitudes is proportional to δ. In a subsequent paper by the same author, the minimum separation distance required for perfect recovery is reduced to $\frac{1.26}{f_c}$ [40]. Moreover, in [20] it is conjectured that a phase transition occurs when the minimum separation distance is $\frac{1}{f_c}$ and proved that TV norm minimisation can fail when $\Delta(T)$ is asymptotically equal to $\frac{1}{f_c}$.

Another model for noise is given in [41], where the measurement model is extended to account for the case when a small number of samples are completely corrupted:

$$y = \mathcal{F}x + v + w, \quad \|w\|_2 \leq \delta, \tag{1.19}$$

where $v \in \mathbb{C}^n$ is a sparse vector where each entry is non-zero with probability $\frac{s}{n}$. The input signal x is reconstructed by solving the optimisation problem

$$\min_{z,v} \|z\|_{TV} + \lambda \|v\|_1 \quad \text{subject to} \quad \|y - \mathcal{F}z + v\|_2 \leq \delta, \tag{1.20}$$

For the case when no dense noise is present ($w = 0$), the author shows that the solution to Problem (1.20) with $\lambda = 1/\sqrt{n}$ is equal to x and v with high probability, depending on n, k, and s. In the same paper, two methods for solving (1.20) are presented (that work in the noisy case $w \neq 0$), one based on semidefinite programming, and one based on a greedy algorithm.

The papers [80] and [81] are similar to [12] in terms of the model used, the method of finding the signal (semidefinite programming) and the convergence bounds, except that the norm used is the atomic norm. In the same setting of atomic norm minimisation, the authors of [19] consider low dimensional projections of the samples to improve scalability of the SDP and give conditions and theoretical guarantees for exact recovery.

Lastly, the author of [16] proposes an algorithm called *AtomicLift* based on semidefinite programming to solve a related problem in which the kernel function ϕ in the convolution is also unknown (a problem known as *blind deconvolution*, [1, 54, 55, 56, 79]).

Classical spectral methods

Before we move on to other measurement models, we mention that there are other approaches for analysing and solving the problem of estimating a sparse signal from its Fourier coefficients that predate the optimisation-based methods described so far. Prony's method [22] is one of the first known and it can recover the signal exactly from $2k + 1$ samples of the type (1.13) if there is no noise. However, it is known to perform poorly in the presence of noise [73]. Not unrelated is the theory of signals with finite rate of innovation ([85],[28]), which considers more general signals that have a parametric representation and a finite number of degrees of freedom. In the case of weighted sums of Diracs like in (1.12), it reduces to Prony's method. In these approaches, no minimum separation condition is required for exact recovery in the noiseless case. However, they are built on polynomial rooting techniques, which makes them harder to extend to multiple dimensions and also less stable to noise.

Another similar technique for solving the same problem is the matrix pencil method [49], which reduces to solving a generalised eigenvalue problem. In [60], the author uses bounds on the condition number of the Vandermonde matrix to give noise tolerance bounds. They show that, if the cut-off frequency f_c satisfies

$$f_c > 1/\Delta + 1, \tag{1.21}$$

then there exists an algorithm that can recover the spike locations and magnitudes at an inverse polynomial rate in the magnitude of the noise. On the other hand, if the cut-off frequency satisfies

$$f_c < (1-\epsilon)/\Delta \tag{1.22}$$

then we can find two signals that can only be recognised as different if the noise magnitude is inverse exponential in ϵ, so if we do not have enough samples, we need exponentially small noise for successful recovery.

General convolution kernel

The problem is considered in [4] for general kernels ϕ of the convolution operator in both 1D and 2D cases:

$$\min_z \|z\|_{TV} \quad \text{subject to} \quad y(s) = \int \phi(s-t)z(\mathrm{d}t), \tag{1.23}$$

where the true discrete signal x has real weights and the measurements are taken continuously in the domain. The authors impose specific properties on the kernel, related to the behaviour around the source and away from it (to ensure that the function is symmetric around the source point, decays rapidly as it moves away from the source and then tends to zero without much variation). Examples of such kernels are the Gaussian kernel $e^{-\frac{t^2}{2}}$ and the Cauchy kernel $\frac{1}{1+t^2}$.

Then, the minimum separation condition is discarded in [3] for real non-negative measures, where a similar setup to [4] is considered, but where the signal is assumed to be on a grid. The minimum separation condition is replaced by a "Rayleigh regularity" condition related to the density of the spikes.

More recently, in [5], the authors solve a more general version of problem (1.23) by allowing the weights of the discrete measure x to be both positive and negative, and by taking discrete samples, when the convolution kernel is the Gaussian kernel or the Ricker wavelet. One of the conditions required by the proof of the recovery guarantees is that each spike has a pair of nearby samples, which is similar to our work presented in Chapter 2.

In [24] and [31], a slightly different problem is solved:

$$\min_z \frac{1}{2}\|\Phi z - y\|_2^2 + \lambda \|z\|_{TV}, \tag{1.24}$$

for $\lambda > 0$ and Φ a general convolution operator with a continuous kernel ϕ, where y represents the noisy measurements with noise w. The input signal x is of the form (1.12) with real amplitudes $a_j \in \mathbb{R}$.

In [31], the authors prove a number of results regarding the support of the recovered solution in the presence of noise when λ and $\|w\|_2/\lambda$ are small enough. Firstly, the recovered signal is supported in the neighbourhood of the support of the true signal. Moreover, under an additional condition (the *non-degenerate source condition*), the number of spikes of the recovered signal is equal to the number of spikes of the true signal and the locations and amplitudes of the spikes of the recovered signal converge linearly with respect to the noise level to those of the true signal. Finally, when the problem is discretised, the solution is supported on pairs of grid points adjacent to the location of the true spikes.

In addition, it is shown in [24] that, for non-negative amplitudes $a_j \geq 0$, if the signal-to-noise ratio is of the order $1/\Delta^{2k-1}$ (where Δ is the minimum separation and k is the number of spikes), the exact solution is recovered by solving (1.24).

Lastly, the authors of [67] prove exact recovery and stability from sub-sampled measurements in a similar setting as above, and in [68] an extension to the two dimensional case of the analysis above is presented.

General convolution kernel, non-negative measures

Finally, in the work [74], based on [21], the authors consider the problem:

$$\begin{aligned}
\underset{x}{\text{minimise}} \quad & \int_I x(\mathrm{d}t) && \text{(1.25a)}\\
\text{subject to} \quad & y(s) = \int_I \phi(s-t)x(\mathrm{d}t), \quad s \in \mathcal{S} && \text{(1.25b)}\\
& \operatorname{supp} x \subset I, && \text{(1.25c)}\\
& x \geq 0, && \text{(1.25d)}
\end{aligned}$$

where we note that, for non-negative measures, $\|x\|_{TV} = \int_I x(\mathrm{d}t)$. It is shown that, in the noiseless case, the minimum separation condition is not required, provided that the point spread function (or convolution kernel) satisfies certain properties (which the Gaussian point spread function does satisfy). However, robustness to noise is not analysed in this case. One of the key tools used in [74] to replace the minimum separation is that of a Chebyshev system, or T-system [50], which we will define in

Chapter 2. T-systems have previously been used for the same purpose in [21], which [74] extends. While the problem formulation in [21] is more general and the conditions they require are more simple than in [74], one drawback is that they rely on having a measurement of the form $y(s_0) = \int x(\mathrm{d}t)$, in addition to the ones in (1.25), which is not feasible in practice.

In [74], the point spread function ϕ must be integrable and positive, so that the problem is well defined. Furthermore, the matrix $[v(s_1) \dots v(s_{2k})]$ must be invertible for any $s_1, \dots, s_{2k} \in S$, where

$$v(s) = [\phi(s - t_1) \quad \dots \quad \phi(s - t_k) \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi(s - t_1) \quad \dots \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi(s - t_k)]^T, \tag{1.26}$$

which allows us to calculate the amplitudes $a_1, \dots, a_k$ if the locations of the sources $t_1, \dots, t_k$ were known. Lastly, the matrix defined as

$$\Lambda(p_1, \dots, p_{2k+1}) = \begin{bmatrix} \kappa(p_1) & \dots & \kappa(p_{2k+1}) \\ 1 & \dots & 1 \end{bmatrix} \tag{1.27}$$

where

$$\kappa(t) = \int_I \phi(t - s) v(s) \, \mathrm{d}P(s), \tag{1.28}$$

and $P(s)$ is the uniform measure on the set S of sample points, must be nonsingular for certain values located in the neighbourhoods of the source locations.

In the recent paper [30], it is shown that the conditions required by [74] are equivalent to the *non-degenerate source condition* from [31] and [24]. In addition, the author shows stability of the TV norm minimisation problem with respect to additive noise in the measurements, in the non-negative setting. This is similar to our main result from Part I, with two main differences: it relies on the TV norm regulariser, while we only consider the feasibility problem, and as a consequence the stability bounds from [30] depend linearly on the noise, while for the feasibility problem this is less favourable.

While most of the literature on super-resolution focuses on the TV norm regulariser, the authors of [27] consider the feasibility problem, in a similar setting as the one introduced at the beginning of the current chapter. The results have a similar flavour as our results from Part I, with the difference in the way the error is measured (the Prokhorov metric) and the fact that they work with probability measures, and therefore the measures are constrained to have total mass equal to one. Therefore, the methods and the exact formulation of the noise bound in [27] are technically different from ours.

Dual certificate

In all the optimisation based methods, an important step in guaranteeing that the true signal x is the unique solution of the minimisation problem ((1.25) for example) is the existence and construction of a *dual certificate*. A dual certificate (or dual polynomial, as it is also called) is a function $q(t)$ which is a solution to the dual problem of the minimisation problem that we solve, such that

$$q(t_i) = 1, \quad \forall t_i \in T, \tag{1.29a}$$

$$|q(t)| < 1, \quad \forall t \in \mathbb{T} \setminus T, \tag{1.29b}$$

where $T = \{t_i\}_{i=1}^k$ is the support of the signal x. In general, the dual certificate is constructed by taking a linear combination of shifted versions of a kernel K and its derivative at $\{t_i\}_{i=1}^k$:

$$q(t) = \sum_{t_i \in T} \alpha_i K(t - t_i) + \beta_i K'(t - t_i), \tag{1.30}$$

with the coefficients α_i and β_i chosen such that:

$$q(t_i) = 1, \quad \forall t_i \in T, \tag{1.31a}$$

$$q'(t_i) = 0, \quad \forall t_i \in T. \tag{1.31b}$$

In work based on [12], the kernel K is chosen to be the Fejer kernel, and the construction of q depends on the minimum separation between spikes, Δ. In the more recent work [74], K is the Gaussian kernel, and the minimum separation condition is replaced by the conditions on the point spread function ϕ described above and the construction also uses the idea of a T-system.

This method of proof is a common theme in the literature, as are the minimum separation distance and the behaviour of the support and amplitudes of the solution in the presence of noise.

1.2.2 Algorithms for super-resolution

While the idea of super-resolution and the specific mathematical formulation that we described so far is fairly recent, estimating the frequencies and amplitudes of a signal is not a new problem.

Therefore, one of the first algorithms to solve it has been known since 1795, namely Prony's method [22], which is based on polynomial root finding. More recent algorithms, like the matrix pencil method [49], are based on solving a generalised eigenvalue problem. While in the noise-free case Prony's method can recover the signal exactly from $2k+1$ samples, it has been known to perform poorly in the presence of noise, while the matrix pencil method is more stable to noise [73]. Other well known algorithms in the same category are MUSIC [75] and ESPRIT [70].

Similarly, optimisation based methods for deconvolution problems, which have been used in the geophysics community since the 1980s (see, for example, [53]), are robust when the samples are corrupted by noise. However, these techniques are used in a discrete setting.

In [29], two algorithms are proposed for the case where the measurements are further projected into a lower dimension, for example using a Gaussian matrix. One is a greedy approach using an oversampled grid, while the other performs gradient descent in the discrete projected measurement space combined with one of the classical line spectral algorithms (e.g. MUSIC) to identify the off-the-grid frequencies.

In this section, we will describe two optimisation based methods for solving the super-resolution problem without discretising, namely an approach based on semidefinite programming (SDP), and a greedy algorithm that finds the spikes in an iterative fashion. Both these methods make use of specific tricks to avoid solving an optimisation problem over an infinite dimensional space. Most of the algorithms in the recent literature are variations of these ideas.

Semidefinite programming approach

The SDP approach was first proposed in [12] to solve problem (1.15) with the low pass measurement model without noise (1.13). The same authors extend it to the noisy model (1.18) in [11], and then to the model with corrupted samples (1.19) in [41]. We will describe the main ideas for the noise-free model, as the methods for the noisy problems are extensions of these ideas.

We first start by writing the dual of problem (1.15):

$$\max_{c} \operatorname{Re}\langle y, c\rangle \quad \text{subject to} \quad \|\mathcal{F}^* c\|_\infty \leq 1, \tag{1.32}$$

where $\mathcal{F}^* c$ is the trigonometric polynomial

$$(\mathcal{F}^* c)(t) = \sum_{|k| \leq f_c} c_k e^{i2\pi k t}. \tag{1.33}$$

Then, it is shown that the constraint in (1.32) is equivalent to finding a Hermitian matrix $Q \in \mathbb{C}^{n\times n}$ (where $n = 2f_c + 1$) such that

$$\begin{bmatrix} Q & c \\ c^* & 1 \end{bmatrix} \succeq 0, \quad \sum_{i=1}^{n-j} Q_{i,i+j} = \begin{cases} 1, & j = 0 \\ 0, & j = 1, 2, \ldots, n-1. \end{cases} \tag{1.34}$$

Therefore, we can find the solution to the dual problem (1.32) by solving the following semidefinite program:

$$\max_{c,Q} \operatorname{Re} \langle y, c \rangle \quad \text{subject to} \quad (1.34) \tag{1.35}$$

over Hermitian matrices $Q \in \mathbb{C}^{n\times n}$ and vectors of coefficients $c \in \mathbb{C}^n$.

The spike locations are then given by the points t where $|\left(\mathcal{F}^* c\right)(t)| = 1$, and for known spike locations, we find the amplitudes by solving the linear least squares problem given by the measurements (1.13).

For solving the noisy problems (1.17) and (1.20), the SDP approach is similar, with the dual problem (1.32) being slightly different.

An extension of this approach which works in multiple dimensions is given in [15], where a penalised formulation with low rank solutions is considered. The authors introduce the Fourier-based Frank-Wolfe algorithm, which takes advantage of the Fourier and low rank structure of the problem to ensure scalability to large matrices.

Greedy Approach

In [9], the authors present a version of the conditional gradient method (CGM, also called Frank-Wolfe) called the Alternating Descent Conditional Gradient Method (ADCG) for solving the following reformulation of (1.25):

$$\begin{aligned} &\text{minimise } \ell(\Phi x - y) \\ &\text{subject to } x \geq 0 \\ &\qquad\qquad x(\mathbb{T}) \leq \tau \end{aligned} \tag{1.36}$$

where $\Phi x = \int \phi(S-t)x(\mathrm{d}t)$ ($\phi(S-t)$ as defined in (1.4)), ℓ represents the convex loss function (in our case the squared ℓ_2 norm of the vector) and $\tau > 0$ is a parameter that empirically controls the cardinality of the solution. This is a constrained optimisation problem in the space of measures.

Before introducing the ADCG algorithm, we remind the reader that the conditional gradient method works by linearising the objective function f around the current iterate s^j

$$\bar{f}(s) = f(s^j) + \left\langle \nabla f(s^j), s - s^j \right\rangle \tag{1.37}$$

and finding a minimiser of the linearisation $\bar{f}$ in the feasible set $\mathcal{D}$

$$\operatorname*{argmin}_{s\in\mathcal{D}} \bar{f}(s) = \operatorname*{argmin}_{s\in\mathcal{D}} \left\langle \nabla f(s^j), s \right\rangle. \tag{1.38}$$

Then, we select the new iterate s^{j+1} by taking a step from s^j in the direction of s, where the step size is selected, for example, by doing a line search.

Back to our problem, let $f(x) = \ell(\Phi x - y)$. In the space of measures, we think of $\left\langle \nabla f(x^j), s \right\rangle$ as a directional derivative:

$$\left\langle \nabla f(x^j), s \right\rangle = D_s f(x^j) = \lim_{t \to 0} \frac{f(x^j + ts) - f(x^j)}{t} = \lim_{t \to 0} \frac{\ell(\Phi(x^j + ts) - y) - \ell(\Phi x^j - y)}{t} \tag{1.39}$$

and, by writing the residual as $r^j = \Phi x^j - y$:

$$\left\langle \nabla f(x^j), s \right\rangle = \lim_{t \to 0} \frac{\ell(r^j + t\Phi s) - \ell(r^j)}{t} = D_{\Phi s} \ell(r^j) = \left\langle \nabla \ell(r^j), \Phi s \right\rangle, \tag{1.40}$$

where the inner product in the last term is the vector dot product. By interchanging the integral in Φ with the inner product and by writing $F(\theta) = \left\langle \nabla \ell(r^j), \phi(S - \theta) \right\rangle$, the function to minimise is:

$$\min_{s \geq 0, s(\mathbb{T}) \leq \tau} \int F(\theta) \, \mathrm{d}s(\theta) \tag{1.41}$$

and this is achieved by setting $s = \tau \delta_{\theta^*}$, where $\theta^* = \operatorname{argmin}_\theta F(\theta)$ (except the case when $F(\theta) \geq 0, \forall \theta$, when the optimal solution is the zero measure), so at each step we add a new spike. Once this is done, we calculate the weights given the new support, and then we fix the weights and optimise over the spike locations around the current support. While there is no analysis to show the need for the last step, in practice the algorithm converges very slowly (or fails to converge) if this step is omitted. We now present the full algorithm.

The ADCG algorithm is iterative and it takes the following steps at each iteration:

1. Compute the gradient of the loss: $g^j \leftarrow \nabla \ell(\Phi x^{j-1} - y)$
2. Compute next source: Choose $t^j \in \underset{t}{\operatorname{argmin}} \left\langle \phi(S - t), g^j \right\rangle$
3. Update support: $S^j \leftarrow S_{j-1} \cup \{t^j\}$
4. Coordinate descent on non-convex objective:
   (a) Compute weights: $x^j \leftarrow \underset{x \geq 0, x(S^j) \leq \tau, x({S^j}^C) = 0}{\operatorname{argmin}} \ell(\sum_{t \in S^j} x(\{t\}) \phi(S - t) - y)$
   (b) Prune support: $S^j = \operatorname{support}(x^j)$
   (c) Locally improve support: holding the weights fixed, solve:

$$\underset{t_i \in S^j}{\operatorname{argmin}} \ \ell(\sum_{i=1}^{|S^j|} w_i \phi(S - t_i) - y)$$

Steps 1 and 2 correspond to linearising the objective and minimising the linearisation in the classical CGM algorithm, as described above. In ADCG, these two steps conclude with adding one extra point to the support of the measure x in Step 3. Step 4 corresponds to choosing the step size in CGM. We alternate between adjusting the weights while holding the support fixed (Step 4(a)) and adjusting the support (the locations of t_i) while holding the weights and the number of sources fixed (Step 4(c)). In Step 4(b) we ensure that if any source has weight zero after the minimisation in Step 1(a), we remove it completely.

ADCG is one of a number of greedy algorithms that solve the super-resolution problem in a setting similar to the one in this thesis. Another version of the Frank-Wolfe algorithm for measures is given in [10], which predates ADCG and considers a Tikhonov regularised version of super-resolution. More recently, the algorithms introduced in [25] and [37], the Sliding Frank-Wolfe algorithm and a version of Orthogonal Matching Pursuit, have been shown to converge towards the unique solution in a finite number of iterations.

1.3 A roadmap of the thesis

So far, we introduced the problem of super-resolution, the exact mathematical setting in which we work and we briefly described the main results of this thesis, followed by an overview of the existing literature so that we have a clearer idea of where our results for non-negative measures fit in the general landscape of mathematical super-resolution.

The rest of the thesis is divided into two parts, Part I focuses on stability results for the feasibility problem and Part II focuses on perturbation analysis of the source locations and weights with respect to perturbations in the dual variable when we solve the dual problem. More specifically:

Part I

In the first part of the thesis, we focus on the feasibility problem and show that, under specific conditions, any measure which satisfies the measurements is close in a certain sense to the true discrete measure that generated the measurements. The work in Chapters 2, 3 and 4 has been published as part of a larger body of work in a journal publication [34] and a subset of this work has also been presented at and published in the proceedings of the *IEEE Data Science Workshop (DSW) 2018* [33].

- In Chapter 2, we describe the specific setup for our stability results for the feasibility problem in the non-negative setting. We then discuss the first result of the thesis, namely the stability properties of the feasibility problem for a general convolution kernel ϕ.

- In Chapter 3, we make the stability results specific to the case when ϕ is Gaussian. These are given explicitly in terms of the parameters of the problem, namely the sampling locations, the width of the Gaussian kernel and the minimum separation of sources. We end the chapter with simulations of the sample proximity condition, which is solved numerically to show the relationships between the minimum separation of sources, the width of the convolution kernel and the sample proximity. This is done both for the Gaussian kernel and for an example of a non-differentiable kernel.

- Chapter 4 contains the detailed proofs of the results in Chapter 3.

Part II

In the second part of the thesis, we focus on the dual problem of the TV norm minimisation formulation of super-resolution. The results in Chapter 5 have been presented at and published in the proceedings of the *13th International Conference on Sampling Theory and Applications (SampTA) 2019* [17].

- In Chapter 5, we discuss the perturbation of the source locations and weights around their true values as the dual variable is perturbed around its optimal value, due to noise in the data or inaccuracy in the solution of the dual problem.

- Chapter 6 takes this work one step further by giving bounds on the perturbation of the dual variable around its optimal value as a consequence of the noise in the measurements.

- In Chapter 7, we solve the exact penalty formulation of the dual problem using the level method and show how the bounds from Chapters 5 and 6 compare with what is observed in practice.

We end the thesis with Chapter 8, where we draw conclusions from our work and give a few directions in which it may be extended.

Lastly, the Appendices section contains proofs of the preliminary results from the above publications that were the work of my collaborators, specifically Lemma 5, Lemma 11 and Proposition 12. The other appendices contain derivations of dual problems which involve standard techniques and we considered they would distract from the main contributions of the thesis.

Part I

Stability of non-negative super-resolution

Chapter 2

Stability of non-negative super-resolution – general point spread function

In this chapter, we discuss the first contribution of the thesis: an analysis of the stability of non-negative super-resolution, for a general convolution kernel ϕ. While most of the existing results rely on using the total variation norm as a regulariser, we show that, under the non-negativity constraint, this is not required to prove stability with respect to additive noise in the measurements. For convenience, we rewrite the feasibility problem (1.5), on which we will focus in Part I of the thesis:

$$\text{Find } z \geq 0 \text{ subject to } \left\| y - \int_I \Phi(t) z(\mathrm{d}t) \right\|_2 \leq \delta,$$

over all non-negative measures z on $I = [0, 1]$. We show that any solution $\hat{x}$ to (1.5) is "close" (in a certain sense which we will define) to the true measure x which generated the measurements y, under specific conditions on x, y and ϕ.

We start the chapter by introducing uniqueness results for the feasibility problem in the noise-free setting in Section 2.1. We then introduce preliminary notions in Section 2.2, which then allow us to state the main stability results for general ϕ in Section 2.3, namely Theorems 9 and 10. In Section 2.4, we give a high level proof of Theorem 9 and we finish the chapter with Section 2.5, which includes detailed proofs of all the results used in Section 2.4.

2.1 Motivation: uniqueness of sparse measures

We begin this chapter with a short section on the uniqueness of the solution to the feasibility problem (1.5) with $\delta = 0$ in the case when the measurements are exact, namely y_j in (1.2) with $w_j = 0$ for $j = 1, \ldots, m$. That is, the true measure x in (1.1) is the unique solution to (1.5) as long as $m \geq 2k + 1$.

For $j = 1, \ldots, m$, let

$$\phi_j(t) = \phi(t - s_j). \tag{2.1}$$

We assume that the functions $\{\phi_j\}_{j=1}^m$ form a Chebyshev system, or T-system [50], which we define below:

Definition 1. ***(Chebyshev, T-system [50])*** *Real-valued and continuous functions $\{\phi_j\}_{j=1}^m$ form a T-system on the interval I if the $m \times m$ matrix $[\phi_j(\tau_l)]_{l,j=1}^m$ is nonsingular for any increasing sequence $\{\tau_l\}_{l=1}^m \subset I$.*

Example of T-systems include the monomials $\{1, t, \cdots, t^{m-1}\}$ on any closed interval of the real line. In fact, T-systems generalise monomials and in many ways preserve their properties. For instance, any "polynomial" $\sum_{j=1}^m b_j \phi_j$ of a T-system $\{\phi_j\}_{j=1}^m$ has at most $m - 1$ distinct zeros on I. Or, given m distinct points on I, there exists a unique polynomial in $\{\phi_j\}_{j=1}^m$ that interpolates these points. Note also that linear independence of $\{\phi_j\}$ is a necessary condition for forming a T-system, but not sufficient. Let us emphasise that T-system is a broad and general concept with a range of applications in classical approximation theory and modern signal processing. In the context of super-resolution, translated copies of the Gaussian window form a T-system on any interval. We refer the interested reader to [50, 52] for the role of T-systems in classical approximation theory and to [65] for their relationship to *totally positive kernels*.

Our analysis based on T-systems has been inspired by the work by Schiebinger et al. [74], where the authors use the T-system property in order to construct the dual certificate for the spike deconvolution problem and to show uniqueness of the solution to the TV norm minimisation problem without the need of a minimum separation. The theory of T-systems has also been used in the same context by De Castro and Gamboa in [21]. However, both [74] and [21] focus on the noise-free problem exclusively, while we will extend the T-systems approach to the noisy case as well, as we will see later.

The following result, given in Proposition 2, simplifies the prior analysis by using readily available results on T-systems. Moreover, this shows uniqueness of the solution to the feasibility problem, which removes the need for TV norm regularisation in the results of Schiebinger et al. [74].

Proposition 2. ***(Uniqueness of exactly sampled sparse non-negative measures)*** *Let x be a non-negative k-sparse discrete measure supported on I as given in*

(1.1). *Let* $\{\phi_j\}_{j=1}^m$ *form a T-system on* I*, and given* $m \geq 2k+1$ *measurements as in* (1.2)*, then* x *is the unique solution of Program* (1.5) *with* $\delta = 0$*.*

Proposition 2 states that Program (1.5) successfully localises the k impulses present in x given only $2k+1$ measurements when $\{\phi_j\}_{j=1}^m$ form a T-system on I. Note that $\{\phi_j\}_{j=1}^m$ only need to be continuous and no minimum separation is required between the impulses. Moreover, the noise-free analysis here is simpler than in [74] as it avoids the introduction of the TV norm minimisation and is more insightful in that it shows that it is not the sparsifying property of TV minimisation which implies the result, but rather it follows from the non-negativity constraint and the T-system property.

Proposition 2 states that if x is a non-negative k-sparse discrete measure supported on I, see (1.1), provided $m \geq 2k+1$ and $\{\phi_j\}_{j=1}^m$ are a T-system, then x is the unique non-negative solution to Program (1.5) with $\delta = 0$. This follows from the existence of a dual polynomial, as stated in Lemma 5 below. Let us first define the notion of dual certificate, or dual polynomial, for the feasibility problem (1.5) in the noise-free setting $\delta = 0$.

Definition 3. ***(Dual certificate for the feasibility problem in the noise-free setting)*** *A dual certificate for* (1.5) *with* $\delta = 0$ *is a function of the form*

$$q(t) = \sum_{j=1}^{m} b_j \phi(t - s_j), \tag{2.2}$$

with $b_j \in \mathbb{R}$*, which satisfies the conditions:*

$$q(t_i) = 0, \quad \forall i = 1, \dots, k, \tag{2.3}$$

$$q(t) > 0, \quad \forall t \neq t_i, \forall i = 1, \dots, k. \tag{2.4}$$

Figure 2.1 shows an example of such a dual certificate using a Gaussian ϕ. Under the assumption that the functions $\{\phi(\cdot - s_j)\}_{j=1}^m$ form a T-system, such a dual certificate exists, see Karlin and Studden [50], Theorem 5.1, pp. 28 for a constructive proof. We give below a simplified version of this result.

Lemma 4. (Dual polynomial existence for T-systems) [50, Theorem 5.1, pp. 28] *With* $m \geq 2k+1$*, suppose that* $\{\phi_j\}_{j=1}^m$ *form a T-system on* I *and let* $T \subset I$ *with* $|T| = k$*. Then there exists a polynomial* $q(t) = \sum_{j=1}^m b_j \phi_j(t)$ *that is non-negative on* I *and vanishes only on* T*.*

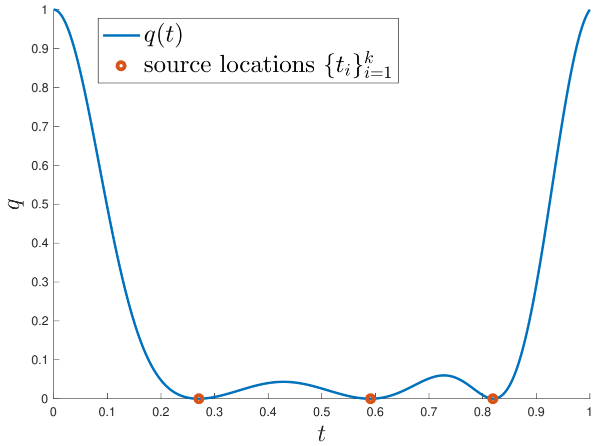


Figure 2.1: Example of dual certificate $q(t)$ required in Lemma 5. Here, we have $k = 3$ and $t_1 = 0.27, t_2 = 0.59$ and $t_3 = 0.82$ and the convolution kernel ϕ is Gaussian $\phi(t) = e^{-t^2/\sigma^2}$.

We can now state the uniqueness result independently of the notion of T-system. Proposition 2 above is a consequence of Lemma 5, proved in Appendix A.2, together with the existence of the dual certificate:

Lemma 5. ***(Uniqueness of non-negative sparse measure)*** *Let x be a non-negative k-sparse discrete measure supported on I, see (1.1). Then, x is the unique solution of problem* (1.5) *with $\delta = 0$ if*

- *the $k \times m$ matrix $[\phi_j(t_i)]_{i=1,j=1}^{i=k,j=m}$ is full rank, and*
- *a dual certificate q as given in Definition 3 exists.*

Note that, if the non-negativity assumption is dropped, the above uniquenes result does not hold. Intuitively, if we have two point sources located close to each other and with weights of equal magnitude and opposite signs, the measurements of their convolution with a wide kernel (wider than the distance between the point sources) does not contain much information. More technically, the conditions that the dual certificate must satisfy would not be the same as the ones in Definition 3 (see, for example [12]) and therefore we would not be able to use the construction of the certificate by Karlin from [50]. This would also invalidate our stability results in the next sections, as we rely on the construction by Karlin.

2.2 Preliminary definitions

In this section we introduce the mathematical setting of our problem in more detail. Specifically, we discuss two of the main concepts that our results rely on, namely the minimum separation of the sources and the proximity of the sampling locations to the source locations. Then we introduce the notion of T*-system, an extension to the idea of a T-system, and the dual polynomial separators, which allow us to state the stability results in the presence of noise in Section 2.3.

Firstly, the minimum separation is a common notion in the literature of super-resolution (introduced in [12]) and, while we have seen in the previous section that it is not needed to show uniqueness of the solution to the feasibility problem in the non-negative case, it does play an important role when the samples are corrupted by noise. Similarly, we need a notion of sample proximity to ensure that we collect enough information through our sampling, otherwise the samples may be too far from the source locations when the convolution kernel decays fast. A similar notion is discussed in [5].

Definition 6. ***(Minimum separation and sample proximity)*** *For finite $\widetilde{T} = T \cup \{0,1\} \subset I$, let $\Delta(T) > 0$ be the minimum separation between the points in T along with the endpoints of I, namely*

$$\Delta(T) = \min_{T_i, T_j \in \widetilde{T}, i \neq j} |T_i - T_j|. \tag{2.5}$$

We define the sample proximity to be the number $\lambda \in (0, \frac{1}{2})$ such that, for each source location t_i, there exists a closest sample location $s_{l(i)} \in S$ to t_i with

$$|t_i - s_{l(i)}| \leq \lambda \Delta(T). \tag{2.6}$$

We describe the nearness of solutions to Program (1.5) in terms of an additional parameter ϵ associated with intervals around the sources T; that is we let $\epsilon \leq \Delta(T)/2$ and define intervals as:

$$T_{i,\epsilon} := \left\{ t : |t - t_i| \leq \epsilon \right\} \cap I \quad (i = 1, \ldots, k), \qquad T_\epsilon := \bigcup_{i=1}^{k} T_{i,\epsilon}, \tag{2.7}$$

and set $T_{i,\epsilon}^C$ and T_ϵ^C to be the complements of these sets with respect to I. In order to make the most general result of Theorem 9 more interpretable, we turn to presenting them in Chapter 3 for the case of $\phi_j(t)$ being shifted Gaussians.

While Proposition 2 implies that T-systems ensure unique non-negative solutions, more is needed to ensure stability of these results with respect to inexact samples, that is $\delta > 0$. This is to be expected, as T-systems imply invertibility of the linear system Φ in (1.3) for any configuration of sources and samples as given in (1.6), but do not limit the condition number of such a system. We control the condition number of Φ by imposing further conditions on the source and sample configuration, which is analogous to imposing conditions that there exists a dual polynomial which is sufficiently bounded away from zero in regions away from sources, see Section 2.3. In particular, we extend the notion of T-system in Definition 1 to a T*-system which includes conditions on samples at the boundary of the interval, additional conditions on the window function, and a condition ensuring that there exist samples sufficiently near sources as given by the notation (2.7) but stated in terms of a new variable ρ so as to highlight its different role here.

Definition 7. ***(T*-system)*** *For an even integer m, real-valued functions $\{\phi_j\}_{j=0}^m$ form a T*-system on $I = [0,1]$ if the following holds for every $T = \{t_1, t_2, \ldots, t_k\} \subset I$ when $\rho > 0$ is sufficiently small. For any increasing sequence $\tau = \{\tau_l\}_{l=0}^m \subset I$ such that*

- $\tau_0 = 0$, $\tau_m = 1$,
- *except exactly three points, namely τ_0, τ_m, and say $\tau_{\underline{l}} \in \mathrm{int}(I)$, the other points belong to T_ρ,*
- *every $T_{i,\rho}$ contains an even number of points,*

we have that

1. *the determinant of the $(m+1) \times (m+1)$ matrix $M_\rho := [\phi_j(\tau_l)]_{l,j=0}^m$ is positive, and*
2. *the magnitudes of all minors of M_ρ along the row containing $\tau_{\underline{l}}$ approach zero at the same rate[1] when $\rho \to 0$.*

Let us briefly discuss T*-systems as an alternative to T-systems in Definition 1. The key property of a T-system to our purpose is that an arbitrary polynomial $\sum_{j=0}^m b_j\phi_j$ of a T-system $\{\phi_j\}_{j=0}^m$ on I has at most m zeros. Polynomials of a T*-system may not have such a property as T-systems allow arbitrary configurations of points τ while T*-systems only ensure the determinant in Condition 1 of Definition 7 be positive for configurations where the majority of points in τ are paired in T_ρ. However, as

[1] A function $u : \mathbb{R} \to \mathbb{R}^+$ approaches zero at the rate ρ^P when $u(\rho) = \Theta(\rho^P)$. See, for example [18], page 44.

the analysis later shows, Condition 1 in Definition 7 is designed for constructing dual certificates for Program (1.5). We will also see later that Condition 2 in Definition 7 is meant to exclude trivial polynomials that do not qualify as dual certificates. Lastly, rather than any increasing sequence $\{\tau_l\}_{l=0}^m \subset I$, Definition 7 only considers subsets τ that mainly cluster around the support T, whereas in our use all but one entry in τ is taken from the set of samples S; this is only intended to simplify the burden of verifying whether a family of functions form a T*-system. While the first and third bullet points in Definition 7 require that there need to be at least two samples per interval $T_{i,\rho}$ as well as samples which define the interval endpoints which gives a sampling complexity $m = 2k + 2$, we typically require S to include additional samples, $m > 2k + 2$, due to the locations of T being unknown. In fact, as T is unknown, the third bullet point imposes a sampling density of m being proportional to the inverse of the minimum separation of the sources $\Delta(T)$. The additional point $\tau_{\underline{l}}$ is not taken from the set S, it instead acts as a free parameter to be used in the dual certificate. In Figure 2.2, we show an example of points $\{\tau_l\}_{l=0}^{10}$ which satisfy the conditions in Definition 7 for $k = 3$ sources.

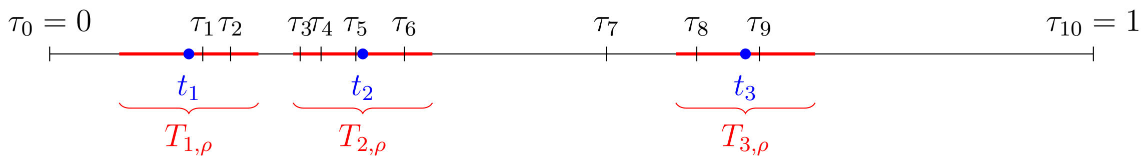


Figure 2.2: Example of $\{\tau_l\}_{l=1}^m$ that satisfy the conditions in Definition 7 for $m = 10$ and $k = 3$.

Note the different roles that the related quantities $\lambda\Delta(T)$, ϵ and ρ play. Firstly, ρ in the defnition of the T*-system above is used in the construction of the dual certificate to determine an interval $T_{i,\rho}$ so that $T_{i,\rho} \to \{t_i\}$ as $\rho \to 0$, which forces the dual certicate to be non-negative (see the construction of the dual certificate in in Appendix A.4). On the other hand, ϵ determines the intervals $T_{i,\epsilon}$ over which we calculate the error between the true signal x and the estimated signal $\hat{x}$, which we bound in Theorems 9 and 19. Lastly, $\lambda\Delta(T)$ is the the maximum distance that is required between each source location t_i and its closest sample $s_{l(i)}$.

A related notion is that of a dual polynomial separator. We control the stability to inexact measurements by introducing a function in Definition 8 which quantifies the dual polynomial $q(t)$ associated with Program (1.5) to be at least $\bar{f}$ away from the necessary constraints for all values of t at least ϵ away from the sources. Specifically, for F defined below, we will require that $q(t) \geq F(t)$ for all $t \in [0, 1]$.

Definition 8. ***(Dual polynomial separators)*** *Let $f : \mathbb{R} \to \mathbb{R}_+$ be a bounded function with $f(0) = 0$, $\bar{f}, f_0, f_1$ be positive constants, and $\{T_{i,\epsilon}\}_{i=1}^k$ the neighbourhoods as defined in* (2.7). *We then define*

$$F(t) := \begin{cases} f_0, & t = 0, \\ f_1, & t = 1, \\ f(t - t_i), & \text{when there exists } i \in \{1, \dots, k\} \text{ such that } t \in T_{i,\epsilon}, \\ \bar{f}, & \text{elsewhere on } int(I). \end{cases} \tag{2.8}$$

We will see later that the values of f_0, f_1 and $\bar{f}$ play a role in the bounds in Theorems 9 and 19. The function f can be chosen freely as long as the conditions in the theorems are satisfied. In Chapter 3, we take $f(t) = 0, \forall t$, so F becomes a piecewise constant function (see an example in Figure 2.3) and we prove that the T*-system property is satisfied for this choice of f.

We are now ready to introduce in the next section the first significant result of this thesis, namely the stability of non-negative super-resolution for a general convolution kernel ϕ.

2.3 Stability of non-negative super-resolution for general point spread function

Equipped with the definitions of T and T*-systems, Definitions 1 and 7 respectively, we are able to characterise any solution to Program (1.5) for $\phi_j(t)$ which form a T-system and suitable source and sample configurations (1.6).

We defer a detailed discussion on the dual polynomial q and the precise role of the above dual polynomial separator to Section 2.4 and state our most general result characterising the solutions to Program (1.5) in terms of this separator. Theorem 19 for Gaussian ϕ in Chapter 3 follows from Theorem 9 by introducing alternative conditions on the source and sample configuration.

Theorem 9. (Average stability for problem (1.5) **for $\phi_j(t)$ a T-system)** *Let $\hat{x}$ be a solution of problem* (1.5) *and consider the function $F(t)$ as given in Definition 8. Suppose that:*

- *$\{\phi_j\}_{j=1}^m$ form a T-system on I,*

- *$\{F\} \cup \{\phi_j\}_{j=1}^m$ form a T*-system on I, and*

- *$\Delta = \Delta(T)$ and $\lambda = \lambda_0 \in (0, 1/2)$ from Definition 6 satisfy the following sample proximity condition:*

$$\phi(\lambda\Delta) = \phi(\Delta - \lambda\Delta) + \phi(\Delta + \lambda\Delta) + \frac{1}{\Delta}\int_{\Delta-\lambda\Delta}^{1/2-\lambda\Delta} \phi(x)\,\mathrm{d}x + \frac{1}{\Delta}\int_{\Delta+\lambda\Delta}^{1/2+\lambda\Delta} \phi(x)\,\mathrm{d}x. \tag{2.9}$$

Then, for any $\epsilon \in (0, \Delta/2)$ and for all $i = 1, \ldots, k$,

$$\int_{T_\epsilon^C} \hat{x}(\mathrm{d}t) \leq \frac{2\|b\|_2\delta}{\bar{f}}, \tag{2.10}$$

$$\left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right| \leq \left(2\left(1 + \frac{\phi^\infty\|b\|_2}{\bar{f}}\right)\cdot\delta + L\|\hat{x}\|_{TV}\cdot\epsilon\right)\sum_{j=1}^{k}(A^{-1})_{ij}, \tag{2.11}$$

where:

- *$b \in \mathbb{R}^m$ is the vector of coefficients of the dual certificate q associated with Program (1.5) and $\bar{f}$ is given in Definition 8, which is used to construct the dual certificate q, as described in Lemma 11 in Section 2.4,*
- *$\phi^\infty = \max_{s,t\in I}|\phi(s-t)|$,*
- *L is the Lipschitz constant of ϕ,*
- *$A \in \mathbb{R}^{k\times k}$ is the matrix*

$$A = \begin{bmatrix} |\phi_1(t_1)| & -|\phi_1(t_2)| & \ldots & -|\phi_1(t_k)| \\ -|\phi_2(t_1)| & |\phi_2(t_2)| & \ldots & -|\phi_2(t_k)| \\ \vdots & \vdots & \ddots & \vdots \\ -|\phi_k(t_1)| & -|\phi_k(t_2)| & \ldots & |\phi_k(t_k)| \end{bmatrix}, \tag{2.12}$$

with $\phi_i(t_i) = \phi(t_i - s_{l(i)})$ evaluated at $s_{l(i)}$ as defined in (2.6).

In (2.11), Theorem 9 bounds the difference between the average over the interval $T_{i,\epsilon}$ of any solution $\hat{x}$ to problem (1.5) and the discrete measure x, whose average is simply a_i. Similarly, (2.10) is a bound on the integral of $\hat{x}$ over T_ϵ^C, away from the sources. The condition that there exists $\lambda_0 \in (0, \frac{1}{2})$ satisfying (2.9) is used to ensure that the matrix (2.12) is strictly diagonally dominant. It relies on the windows $\phi_j(t)$ being sufficiently localised about zero. For a given window ϕ and minimum separation $\Delta(T)$, once we found λ_0 satisfying the equality (2.9), we have that the matrix (2.12) is diagonally dominant for all $\lambda \leq \lambda_0$. If diagonal dominance is lost, we cannot guarantee that the entries of A^{-1} are positive.

Though Theorem 9 explicitly states that the location of the closest samples to each source is less than $\lambda_0\Delta(T)$, this can be achieved without knowing the locations of the sources by placing the samples uniformly at interval $2\lambda_0\Delta(T)$, which gives a sampling complexity of $m = (2\lambda_0\Delta(T))^{-1}$. Lastly, while we require λ_0 to satisfy condition (2.9), the error bounds in (2.10) and (2.11) do not explicitly depend on λ_0. However, the choice of the locations of the closest samples $s_{l(i)}$ to each source t_i (which λ_0 determines) is reflected in the dual certificate, and therefore in the norm of its coefficients $\|b\|_2$.

Clustering of indistinguishable sources

Theorem 9 gives uniform guarantees for all sources in terms of the minimum separation condition $\Delta(T)$, which measures the worst proximity of sources. One might imagine that, for example, if all but two sources are sufficiently well separated, then Theorem 9 might hold for the sources that are well separated; moreover, assuming ϵ is fixed, then if two sources t_i and t_{i+1} with magnitudes a_i and a_{i+1} are closer than 2ϵ, namely $|t_i - t_{i+1}| < 2\epsilon$, we might imagine that a variant of Theorem 9 might hold but with sources t_i and t_{i+1} approximated with source t_ξ near t_i and t_{i+1} and with $a_\xi = a_i + a_{i+1}$.

In this section we extend Theorem 9 to this setting by considering ϵ fixed and alternative intervals $\{\tilde{T}_{i,\epsilon}\}_{i=1}^{\tilde{k}}$ a partition of T_ϵ such that each $\tilde{T}_{i,\epsilon}$ contains a group of consecutive sources $t_{i1}, \dots, t_{ik_i}$ (with weights $a_{i1}, \dots, a_{ik_i}$ respectively) which are within at most 2ϵ of each other. Define

$$\tilde{T}_{i,\epsilon} = \bigcup_{l=1}^{k_i} T_{il,\epsilon}, \qquad \text{where } t_{il} \in T_{il,\epsilon} \quad \text{and} \quad |t_{il+1} - t_{il}| < 2\epsilon, \quad \forall l = 1, \dots, k_i - 1, \tag{2.13}$$

for $\sum_{i=1}^{\tilde{k}} k_i = k$, so that we have

$$T_\epsilon = \bigcup_{i=1}^{\tilde{k}} \tilde{T}_{i,\epsilon} \quad \text{and} \quad \tilde{T}_{i,\epsilon} \bigcap \tilde{T}_{j,\epsilon} \neq \emptyset, \quad \forall i \neq j. \tag{2.14}$$

Theorem 10. ***(Average stability for Program*** (1.5)***: grouped sources)*** *Let $\hat{x}$ be a solution of Program* (1.5) *and $I = [0, 1]$ be partitioned as described by* (2.13)*. If the samples are placed uniformly at interval $2\lambda_0\epsilon$ where $\lambda = \lambda_0$ satisfies* (2.9) *with $\Delta = 2\epsilon$, then there exist $\{\xi_i\}_{i\in[\tilde{k}]}$ with $\xi_i \in \tilde{T}_{i,\epsilon}$ such that*

$$\left| \int_{\tilde{T}_{i,\epsilon}} \hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_i} a_{ir} \right| \leq \left(2\left(1 + \frac{\phi^\infty \|b\|_2}{\bar{f}}\right) \cdot \delta + (2k-1)L\|\hat{x}\|_{TV} \cdot \epsilon \right) \sum_{j=1}^{\tilde{k}} (\tilde{A}^{-1})_{ij}, \tag{2.15}$$

where the constants are the same as in (9) *and the matrix* $\tilde{A} \in \mathbb{R}^{\tilde{k}\times\tilde{k}}$ *is*

$$\tilde{A} = \begin{bmatrix} |\phi_1(\xi_1)| & -|\phi_1(\xi_2)| & \dots & -|\phi_1(\xi_{\tilde{k}})| \\ -|\phi_2(\xi_1)| & |\phi_2(\xi_2)| & \dots & -|\phi_2(\xi_{\tilde{k}})| \\ \vdots & \vdots & \ddots & \vdots \\ -|\phi_{\tilde{k}}(\xi_1)| & -|\phi_{\tilde{k}}(\xi_2)| & \dots & |\phi_{\tilde{k}}(\xi_{\tilde{k}})| \end{bmatrix}.$$

Note that the bound on the error away from the sources from (2.10) in Theorem 9 still holds if we replace any group of sources from an interval $\tilde{T}_{i,\epsilon}$ with some $\xi_i \in \tilde{T}_{i,\epsilon}$, so the bound on T_ϵ^C remains valid without modification.

As an exemplar source location where Theorem 10 might be applied, consider the situation where the k source locations comprising T are drawn uniformly at random in $(0,1)$, where we have that (from [38] page 42, Exercise 22)

$$P(\Delta(T) > \theta) = [1-(k+1)\theta]^k, \quad \theta \in \left[0, \frac{1}{k+1}\right].$$

Then, the cumulative distribution function is

$$F(\theta) = P(\Delta(T) \leq \theta) = 1 - [1-(k+1)\theta]^k,$$

and so the distribution of $\Delta(T)$ is

$$f(\theta) = F'(\theta) = (k+1)k[1-(k+1)\theta]^{k-1},$$

with an expectation of

$$E(\Delta(T)) = \int_0^{\frac{1}{k+1}} P(\Delta(T) > \theta)\,\mathrm{d}\theta = \frac{1}{(k+1)^2}. \tag{2.16}$$

That is, for x from (1.1) with sources T drawn uniformly at random in $(0,1)$, the expected value of $\Delta(T)$ is given by (2.16) and, in Theorem 9, the corresponding number of samples m would scale quadratically with the number of sources k due to the scaling of $m \sim \Delta(T)^{-1}$. Alternatively, Theorem 10 allows meaningful results for m proportional to k by grouping the sources that are within k^{-2} of one another.

Lastly, one does not need to know where the sources are clustered in order to take advantage of this result. For an interval of fixed length, Theorem 10 gives an error bound between all the sources within that interval and one potential source that approximates the group, regardless of where they are located and where the interval is taken.

2.4 Proof of Theorem 9 (Average stability for the feasibility problem)

The results in Section 2.3 are developed by establishing a dual polynomial for problem (1.5) which is non-negative except at the source locations. This implies that the solution to problem (1.5) is unique when $\delta = 0$, see Proposition 2, and we show that the dual polynomial is sufficiently non-negative away from the source locations, a property which is then used to develop Theorems 9 and 10. In this section we state the key ideas and lemmas used to prove the aforementioned results.

We first discuss how to bound the error away from the sources, given in (2.10), followed by the proof of the bound (2.11) on the error around the sources.

2.4.1 Error away from the source locations

To start with, we introduce a dual polynomial $q(t)$ similar to the one in Definition 3 with the added constraint that $q(t)$ is far enough from zero for t at least ϵ away from the source locations $\{t_i\}_{i=1}^k$. This allows us to control the error $h = \hat{x} - x$ away from the source locations, namely over the interval T_ϵ^C, a result given below and proved in Appendix A.3.

Lemma 11. ***(Error away from the support)*** *Let $\hat{x}$ be a solution of Program (1.5) with $\delta \geq 0$ and set $h = \hat{x} - x$ to be the error. Consider $F(t)$ given in Definition 8 and suppose that there exist a positive $\epsilon \leq \Delta(T)/2$, real coefficients $\{b_j\}_{j=1}^m$, and a polynomial $q = \sum_{j=1}^m b_j \phi_j$ such that*

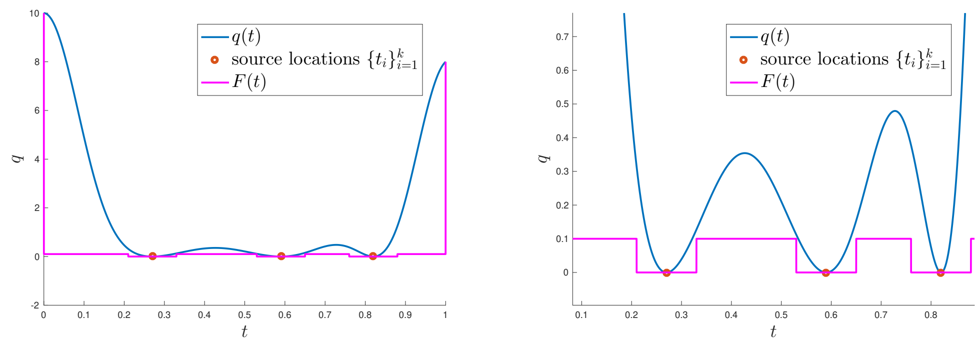
$$q(t) \geq F(t),$$

where the equality holds on T. Then we have that

$$\bar{f} \int_{T_\epsilon^C} h(\mathrm{d}t) + \sum_{i=1}^{k} \int_{T_{i,\epsilon}} f\left(t - t_i\right) h(\mathrm{d}t) \leq 2\|b\|_2 \delta, \tag{2.17}$$

where $b \in \mathbb{R}^m$ is the vector formed by the coefficients $\{b_j\}_{j=1}^m$.

There is a natural analogy here with the case of exact samples. In the setting where $w_j = 0$ in (1.2), the dual certificate q in Lemma 5 was required to be positive off the support T. In the presence of inexact samples however, Lemma 11 loosely-speaking requires the dual certificate to be bounded[2] *away* from zero (see example in Figure 2.3) for $t \in T_\epsilon^C$.



(a) (b)

Figure 2.3: Example of dual certificate $q(t)$ that satisfies the conditions in Lemma 11, where the window function is the Gaussian kernel $\phi(t) = e^{-t^2/\sigma^2}$. We take $k = 3$, $t_i \in \{0.27, 0.59, 0.82\}$ and the function $F(t)$ such that $q(t) \geq F(t)$.

Consequently, Lemma 11 controls the error h away from the support T, as it guarantees that

$$\int_{T_\epsilon^C} h(\mathrm{d}t) \leq \frac{2\|b\|_2 \delta}{\bar{f}}, \tag{2.18}$$

if the dual certificate q exists. Indeed, (2.18) follows directly from (2.17) because the sum in (2.17) is non-negative. This is in turn the case because $f(0) = 0$ and the error h is non-negative off the support T.

Let us now study the existence of this certificate. Proposition 12, proved in Appendix A.4, guarantees the existence of the dual certificate q required in Lemma 11 and heavily relies on the concept of T*-system in Definition 7. We remark that the proof benefits from the ideas in [50].

Proposition 12. ***(Existence of*** q***)*** *For $m \geq 2k+2$, suppose that $\{\phi_j\}_{j=1}^m$ form a T-system on I and that $\{F\} \cup \{\phi_j\}_{j=1}^m$ form a T*-system on I, where $F(t)$ is the function given in Definition 8. Then the dual certificate q in Lemma 11 exists and consequently Program (1.5) is stable in the sense that (2.17) holds.*

Note that to ensure the conclusion of Lemma 11 holds, it suffices that there exists a polynomial $q = \sum_{j=1}^m b_j \phi_j$ such that $q(t) \geq F(t)$ with the equality met on

[2]Note the scale invariance of (2.17) under scaling of f and $\bar{f}$. Indeed, by changing $f, \bar{f}$ to $\alpha f, \alpha \bar{f}$ for positive α, the proof dictates that b changes to αb and consequently α cancels out from both sides of (2.17). Similarly, if we change Φ to $\alpha\Phi$ in (1.3), the proof dictates that b changes to b/α and α again cancels out, leaving (2.17) unchanged.

the support T. Equivalently, it suffices that there exists a non-negative polynomial $\dot{q} = -b_0 F + \sum_{j=1}^{m} b_j \phi_j$ that vanishes on T such that $b_0 > 0$ and at least one other coefficient, say b_{j_0}, is nonzero. This situation is reminiscent of Lemma 4. In contrast to Lemma 4, however, such $\dot{q}$ exists when $\{F\} \cup \{\phi_j\}_{j=1}^{m}$ is a T*-system rather than a T-system. The more subtle T*-system requirement is to avoid trivial or unbounded polynomials.

One key observation is that Lemma 11 is almost silent about the error near the impulses in x. Indeed, because $f(0) = 0$ by assumption, (2.17) completely fails to control the error on the support T. In order to control the error in $T_{i,\epsilon}$ (near each source location t_i), we need to approach the problem differently, as shown in the next subsection.

2.4.2 Error around the source locations

Let $A \in \mathbb{R}^{k \times k}$ be defined as in (2.12):

$$A = \begin{bmatrix} |\phi_1(t_1)| & -|\phi_1(t_2)| & \dots & -|\phi_1(t_k)| \\ -|\phi_2(t_1)| & |\phi_2(t_2)| & \dots & -|\phi_2(t_k)| \\ \vdots & \vdots & \ddots & \vdots \\ -|\phi_k(t_1)| & -|\phi_k(t_2)| & \dots & |\phi_k(t_k)| \end{bmatrix},$$

where $\phi_i(t_i) = \phi(t_i - s_{l(i)})$ is evaluated at the source t_i and the closest sample to it, as defined in (2.6).

The proof of (2.11) in Theorem 9 consists of two steps. We first show that we can bound the error if the matrix A is strictly diagonally dominant. It is easy to see that, if the window function ϕ is localised, then the entries on the main diagonal are larger in absolute value than the off-diagonal entries. If, moreover, we choose the sampling locations $\{s_j\}_{j=1}^{m}$ such that A is strictly diagonally dominant (which means that for each source, there is a sampling location that is "close enough" to it), then the bound (2.11) is guaranteed.

Proposition 13. ***(Bound on the error around the support)*** *For each source t_i, select $s_{l(i)}$ to be the closest sample as defined in* (2.6)*, and define the matrix A in* (2.12) *using the sequences $\{t_i\}_{i=1}^{k}$, $\{s_{l(i)}\}_{i=1}^{k}$. If A is strictly diagonally dominant, then the error around the support is bounded according to* (2.11)*.*

Then, we want to go further and see what it means exactly for A to be strictly diagonally dominant, so the second step in the proof of Theorem 9 is to give an upper bound for the distance between the sources $\{t_i\}_{i=1}^k$ and the closest sampling locations $\{s_{l(i)}\}_{i=1}^k$ such that A is strictly diagonally dominant.

Given an even positive function ϕ that is localised at 0 and with fast decay, let Δ and λ as given in Definition 6, so

$$|t_i - s_{l(i)}| \leq \lambda\Delta \tag{2.19}$$

We want to find λ_0 such that

$$\phi(s_{l(i)} - t_i) \geq \sum_{j\neq i} \phi(s_{l(i)} - t_j), \quad \forall\lambda \in (0, \lambda_0), \quad \forall i = 1, \dots, k, \tag{2.20}$$

namely, we want the matrix A to be strictly diagonally dominant. From the conditions (2.20), we can obtain a more general inequality involving ϕ and Δ that λ_0 must satisfy such that, for any λ with $\lambda < \lambda_0$, A is strictly diagonally dominant. The equality is given by (2.9):

$$\phi(\lambda_0\Delta) = \phi(\Delta - \lambda_0\Delta) + \phi(\Delta + \lambda_0\Delta) + \frac{1}{\Delta}\int_{\Delta-\lambda_0\Delta}^{1/2-\lambda_0\Delta} \phi(x)\,\mathrm{d}x + \frac{1}{\Delta}\int_{\Delta+\lambda_0\Delta}^{1/2+\lambda_0\Delta} \phi(x)\,\mathrm{d}x.$$

Proposition 14. *(A **is strictly diagonally dominant**) Let $\lambda_0 \in (0, \frac{1}{2})$ such that $\left|t_i - s_{l(i)}\right| \leq \lambda_0\Delta$ for all $i = 1, \dots, k$. If λ_0 satisfies* (2.9)*, then the matrix A defined in* (2.12) *is strictly diagonally dominant.*

Finally, we note that the proof of Theorem 10 involves the same ideas as the ones discussed in this section, with a few modifications. The detailed proofs of Proposition 13 and Proposition 14 are given in Section 2.5.1 and Section 2.5.2 respectively. The proof of Theorem 10 is similar to the proof presented in the current section, so we only show the differences in Section 2.5.3.

2.5 Detailed proofs of the results in this chapter

In this section we present the detailed proofs of the intermediate results that were used in Section 2.4.

2.5.1 Proof of Proposition 13 (Bound on the error around the support)

In this proof, we will use the following result for strictly diagonally dominant matrices from [66], based on the theory of M-matrices:

Lemma 15. *If A is a strictly diagonally dominant matrix with positive entries on the main diagonal and negative entries otherwise, then A is invertible and A^{-1} has non-negative entries.*

Proof of Proposition 13

Let $\hat{x}$ be a solution of (1.5) and $h = x - \hat{x}$. Then, with $\phi_j(t) = \phi(t - s_j)$ for some j, by reverse triangle inequality we have

$$\begin{aligned}
\delta \geq \left(\sum_{j=1}^{m} \left(y(s_j) - \int_0^1 \phi_j(t)\hat{x}(\mathrm{d}t) \right)^2 \right)^{1/2} &\geq \left(\sum_{j=1}^{m} \big(\phi_j(t)h(\mathrm{d}t) + w_j \big)^2 \right)^{1/2} \\
&\geq \left(\sum_{j=1}^{m} \big(\phi_j(t)h(\mathrm{d}t) \big)^2 \right)^{1/2} - \|w\|_2 \\
&\geq \left(\sum_{j=1}^{m} \big(\phi_j(t)h(\mathrm{d}t) \big)^2 \right)^{1/2} - \delta,
\end{aligned}$$

and so

$$\sum_{j=1}^{m} \left(\int_0^1 \phi_j(t)h(\mathrm{d}t) \right)^2 \leq 4\delta^2 \implies \left| \int_0^1 \phi_j(t)h(\mathrm{d}t) \right| \leq 2\delta, \quad \forall j = 1, \ldots, m. \tag{2.22}$$

We apply the reverse triangle inequality again to find a lower bound of the left-hand side term in (2.22):

$$\left| \int_0^1 \phi_j(t)h(\mathrm{d}t) \right| \geq \left| \int_{T_{i,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - a_i\phi_j(t_i) \right| \tag{2.23a}$$

$$- \sum_{l \neq i} \left| \int_{T_{l,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - a_l\phi_j(t_l) \right| \tag{2.23b}$$

$$- \left| \int_{T_\epsilon^C} \phi_j(t)\hat{x}(\mathrm{d}t) \right|. \tag{2.23c}$$

We now need to lower bound the term in (2.23a) and upper bound the terms in (2.23b), (2.23c). For the first, we obtain:

$$\begin{aligned}
&\left|\int_{T_{i,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - a_i\phi_j(t_i)\right| \\
&= \left|\int_{T_{i,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - a_i\phi_j(t_i) + \phi_j(t_i)\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - \phi_j(t_i)\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t)\right| \\
&\geq \left|\phi_j(t_i)\right| \left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right| - \int_{T_{i,\epsilon}} \left|\phi_j(t) - \phi_j(t_i)\right| \hat{x}(\mathrm{d}t) \\
&\geq \left|\phi_j(t_i)\right| \left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right| - L\int_{T_{i,\epsilon}} |t - t_i|\, \hat{x}(\mathrm{d}t) \\
&\geq \left|\phi_j(t_i)\right| \left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right| - L\epsilon\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t). \qquad (2.24)
\end{aligned}$$

Therefore, from (2.24), we obtain:

$$\left|\int_{T_{i,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - a_i\phi_j(t_i)\right| \geq \left|\phi_j(t_i)\right| \left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right| - L\epsilon\|\hat{x}\|_{TV}. \qquad (2.25)$$

For the term (2.23b), we have:

$$\begin{aligned}
&\left|\int_{T_{l,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - a_l\phi_j(t_l)\right| \\
&= \left|\int_{T_{l,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - a_l\phi_j(t_l) + \phi_j(t_l)\int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t) - \phi_j(t_l)\int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t)\right| \\
&\leq \left|\phi_j(t_l)\right| \left|\int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t) - a_l\right| + \int_{T_{l,\epsilon}} \left|\phi_j(t) - \phi_j(t_l)\right| \hat{x}(\mathrm{d}t) \\
&\leq \left|\phi_j(t_l)\right| \left|\int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t) - a_l\right| + L\int_{T_{l,\epsilon}} |t - t_l|\, \hat{x}(\mathrm{d}t) \\
&\leq \left|\phi_j(t_l)\right| \left|\int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t) - a_l\right| + L\epsilon\int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t),
\end{aligned}$$

so

$$\sum_{l\neq i}\left|\int_{T_{l,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - a_l\phi_j(t_l)\right| \leq \sum_{l\neq i}\left(\left|\phi_j(t_l)\right| \left|\int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t) - a_l\right|\right) + L\epsilon\sum_{l\neq i}\int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t). \qquad (2.26)$$

Finally, for the term (2.23c), we have:

$$\left|\int_{T_\epsilon^C} \phi_j(t)\hat{x}(\mathrm{d}t)\right| \leq \max_{t\in T_\epsilon^C} \left|\phi_j(t)\right| \int_{T_\epsilon^C} \hat{x}(\mathrm{d}t) \leq \phi^\infty \left(\frac{2\|b\|_2\delta}{\bar{f}}\right). \tag{2.27}$$

Let us denote

$$z_i = \left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right|,$$

for all $i = 1, \ldots, k$. Then, by combining (2.23) with the bounds (2.24),(2.26) and (2.27), we obtain the j-th row of a linear system:

$$2\delta \geq \left|\phi_j(t_i)\right| z_i - L\epsilon \int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - \sum_{l\neq i} \left|\phi_j(t_l)\right| z_l - L\epsilon \sum_{l\neq i} \int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t) - \phi^\infty \frac{2\|b\|_2}{\bar{f}}\delta. \tag{2.28}$$

By using (2.28) along with

$$\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) + \sum_{l\neq i} \int_{T_{l,\epsilon}} \hat{x}(\mathrm{d}t) = \int_{T_\epsilon} \hat{x}(\mathrm{d}t) \leq \int_I \hat{x}(\mathrm{d}t) = \|\hat{x}\|_{TV},$$

we obtain:

$$2\left(1 + \frac{\phi^\infty \|b\|_2}{\bar{f}}\right)\delta + \epsilon L \|\hat{x}\|_{TV} \geq \left|\phi_j(t_i)\right| z_i - \sum_{l\neq i} \left|\phi_j(t_l)\right| z_l. \tag{2.29}$$

Now, for all i, we select the $j = l(i)$, the index corresponding to the closest sample as defined in Definition 6. The inequalities in (2.29) can be written as

$$Az \leq v, \tag{2.30}$$

where A, z and v are defined as

$$A = \begin{bmatrix} |\phi_1(t_1)| & -|\phi_1(t_2)| & \ldots & -|\phi_1(t_k)| \\ -|\phi_2(t_1)| & |\phi_2(t_2)| & \ldots & -|\phi_2(t_k)| \\ \vdots & \vdots & \ddots & \vdots \\ -|\phi_k(t_1)| & -|\phi_k(t_2)| & \ldots & |\phi_k(t_k)| \end{bmatrix},$$

$$z = \begin{bmatrix} z_1 & z_2 & \ldots & z_k \end{bmatrix}^T$$

$$v = \left(2\left(1 + \frac{\phi^\infty \|b\|_2}{\bar{f}}\right)\delta + \epsilon L \|\hat{x}\|_{TV}\right) \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T.$$

Because A is strictly diagonally dominant, Lemma 15 holds and therefore A^{-1} exists and has non-negative entries, so when we multiply (2.30) by A^{-1}, the sign does not change:

$$z \leq A^{-1} v, \tag{2.31}$$

where we can bound the entries of A^{-1} [84]:

$$\|A^{-1}\|_\infty < \frac{1}{\min_j(\phi_j(t_j) - \sum_{i\neq j} \phi_j(t_i))}.$$

The proof of Proposition 13 is now complete, since (2.31) is equivalent to the error bound (2.11).

2.5.2 Proof of Proposition 14 (The sampling matrix is strictly diagonally dominant)

For the sake of simplicity, let s_i be the closest sample $s_{l(i)}$ to the source t_i, as defined in Definition 6. For a fixed i, assume (without loss of generality, as we will see later) that $s_i < t_i$. It follows that

$$\begin{aligned} |s_i - t_l| &= s_i - t_l \geq (i-l)\Delta - \lambda\Delta, \quad \forall l < i, \\ |s_i - t_l| &= t_l - s_i \geq (l-i)\Delta + \lambda\Delta, \quad \forall l > i, \end{aligned}$$

and so

$$\begin{aligned} \phi(|s_i - t_l|) &\leq \phi((i-l)\Delta - \lambda\Delta), \quad \forall l < i, \\ \phi(|s_i - t_l|) &\leq \phi((l-i)\Delta + \lambda\Delta), \quad \forall l > i. \end{aligned}$$

Then we have that

$$\begin{aligned} \sum_{l\neq i} \phi(|s_i - t_l|) &= \sum_{l=1}^{i-1} \phi(|s_i - t_l|) + \sum_{l=i+1}^{k} \phi(|s_i - t_l|) \\ &\leq \sum_{l=1}^{i-1} \phi((i-l)\Delta - \lambda\Delta) + \sum_{l=i+1}^{k} \phi((l-i)\Delta + \lambda\Delta) \\ &= \sum_{l=1}^{i-1} \phi(l\Delta - \lambda\Delta) + \sum_{l=1}^{k-i} \phi(l\Delta + \lambda\Delta). \end{aligned} \tag{2.32}$$

We now want to find upper bounds for each of the two sums in (2.32). We will derive the bound for the first term, as the second one is similar. We have that

$$\sum_{l=1}^{i-1} \phi(l\Delta - \lambda\Delta) = \phi(\Delta - \lambda\Delta) + \frac{1}{\Delta}\sum_{l=2}^{i-1} \phi(l\Delta - \lambda\Delta)\Delta, \tag{2.33}$$

and the sum in the previous equation is a lower Riemann sum (note that ϕ is decreasing in $[0, 1]$)

$$S = \sum_{l=2}^{i-1} \phi(x_l^*)(x_l - x_{l-1})$$

of $\phi(x)$ over $[\Delta - \lambda\Delta, (i-1)\Delta - \lambda\Delta]$, with partition and x_l^* chosen as follows:

$$[x_l, x_{l-1}] = [l\Delta - \lambda\Delta, (l-1)\Delta - \lambda\Delta], \qquad l = 2\ldots, i-1,$$
$$x_l^* = l\Delta - \lambda\Delta, \qquad l = 2, \ldots, i-1.$$

Therefore, the sum S is less than or equal to the integral:

$$\sum_{l=2}^{i-1} \phi(l\Delta - \lambda\Delta)\Delta \leq \int_{\Delta-\lambda\Delta}^{(i-1)\Delta-\lambda\Delta} \phi(x)\,\mathrm{d}x. \tag{2.34}$$

By substituting (2.34) into (2.33), we obtain

$$\sum_{l=1}^{i-1} \phi(l\Delta - \lambda\Delta) \leq \phi(\Delta - \lambda\Delta) + \frac{1}{\Delta}\int_{\Delta-\lambda\Delta}^{(i-1)\Delta-\lambda\Delta} \phi(x)\,\mathrm{d}x.$$

We can obtain a similar upper bound for the second sum in (2.32) and then

$$\begin{aligned}\sum_{l\neq i} \phi(|s_i - t_l|) \leq{}& \phi(\Delta - \lambda\Delta) + \phi(\Delta + \lambda\Delta) \\ &+ \frac{1}{\Delta}\int_{\Delta-\lambda\Delta}^{(i-1)\Delta-\lambda\Delta} \phi(x)\,\mathrm{d}x + \frac{1}{\Delta}\int_{\Delta+\lambda\Delta}^{(k-i)\Delta+\lambda\Delta} \phi(x)\,\mathrm{d}x, \quad \forall i = 1, \ldots, k.\end{aligned} \tag{2.35}$$

We can further upper bound the right hand side over all $i = 1, \ldots, k$ and this bound corresponds to the case when the source t_i is in the middle of the unit interval (at $\frac{1}{2}$) and the sources t_1 and t_k are at 0 and 1 respectively:

$$\int_{\Delta-\lambda\Delta}^{(i-1)\Delta-\lambda\Delta} \phi(x)\,\mathrm{d}x + \int_{\Delta+\lambda\Delta}^{(k-i)\Delta+\lambda\Delta} \phi(x)\,\mathrm{d}x \leq \int_{\Delta-\lambda\Delta}^{1/2-\lambda\Delta} \phi(x)\,\mathrm{d}x + \int_{\Delta+\lambda\Delta}^{1/2+\lambda\Delta} \phi(x)\,\mathrm{d}x,$$

for all $i = 1, \ldots, k$ and therefore we have

$$\sum_{l\neq i} \phi(|s_i - t_l|) \leq \phi(\Delta - \lambda\Delta) + \phi(\Delta + \lambda\Delta) + \frac{1}{\Delta}\int_{\Delta-\lambda\Delta}^{1/2-\lambda\Delta} \phi(x)\,\mathrm{d}x + \frac{1}{\Delta}\int_{\Delta+\lambda\Delta}^{1/2+\lambda\Delta} \phi(x)\,\mathrm{d}x,$$

for all $i = 1, \ldots, k$. In order to find λ_0, we solve (2.9) since $|s_i - t_i| \leq \lambda\Delta$ implies $\phi(|s_i - t_i|) \geq \phi(\lambda\Delta)$.

We note that if we only have three sources, then the integral terms should not be included, and if we have four sources, then the last integral term should not be included.

2.5.3 Proof of Theorem 10 (Average stability for the feasibility problem: grouped sources)

The proof of Theorem 10 involves the same ideas as Theorem 9. The differences are in the analysis of Proposition 13. We continue this analysis from (2.23), where we lower bound the left-hand side term of (2.22):

$$2\delta \geq \left| \int_0^1 \phi_j(t) h(\mathrm{d}t) \right| \geq \left| \int_{\tilde{T}_{i,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_i} a_{ir}\phi_j(t_{ir}) \right| \tag{2.36a}$$

$$- \sum_{l \neq i} \left| \int_{\tilde{T}_{l,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_l} a_{lr}\phi_j(t_{lr}) \right| \tag{2.36b}$$

$$- \left| \int_{T_\epsilon^C} \phi_j(t)\hat{x}(\mathrm{d}t) \right|, \tag{2.36c}$$

where, in each term, the sum from $r = 1$ to k_i is over all the true sources in $\tilde{T}_{i,\epsilon}$ (for all $i = 1, \ldots, \tilde{k}$). In order to obtain bounds for the terms (2.36a) and (2.36b), we need the following fact:

$$\exists \xi_i \in [\operatorname*{argmin}_{r=1,\ldots,k_i} \phi_j(t_{ir}), \operatorname*{argmax}_{r=1,\ldots,k_i} \phi_j(t_{ir})] \quad \text{such that} \quad \phi_j(\xi_i) \sum_{r=1}^{k_i} a_{ir} = \sum_{r=1}^{k_r} a_{rk}\phi_j(t_{ir}), \tag{2.37}$$

for all $r = 1, \ldots, \tilde{k}$. This comes from the continuity of ϕ_j and the intermediate value theorem, since:

$$\min_{r=1,\ldots,k_i} \phi_j(t_{ir}) \leq \frac{\sum_{r=1}^{k_i} a_{ir}\phi_j(t_{ir})}{\sum_{r=1}^{k_i} a_{ir}} \leq \max_{r=1,\ldots,k_i} \phi_j(t_{ir}).$$

We proceed as before to find a lower bound for (2.36a) and an upper bound for (2.36b), while the upper bound for (2.36c) is the same. For (2.36a):

$$
\begin{aligned}
\left|\int_{\tilde{T}_{i,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_i} a_{ir}\phi_j(t_{ir})\right| &= \left|\int_{\tilde{T}_{i,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - \phi_j(\xi_i)\sum_{r=1}^{k_i} a_{ir}\right| \\
&= \Bigg|\int_{\tilde{T}_{i,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - \phi_j(\xi_i)\sum_{r=1}^{k_i} a_{ir} \\
&\qquad + \phi_j(\xi_i)\int_{\tilde{T}_{i,\epsilon}} \hat{x}(\mathrm{d}t) - \phi_j(\xi_i)\int_{\tilde{T}_{i,\epsilon}} \hat{x}(\mathrm{d}t)\Bigg| \\
&\geq |\phi_j(\xi_i)|\left|\int_{\tilde{T}_{i,\epsilon}} \hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_i} a_{ir}\right| - \int_{\tilde{T}_{i,\epsilon}} |\phi_j(t) - \phi_j(\xi_i)|\,\hat{x}(\mathrm{d}t) \\
&\geq |\phi_j(\xi_i)|\left|\int_{\tilde{T}_{i,\epsilon}} \hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_i} a_{ir}\right| - L\int_{\tilde{T}_{i,\epsilon}} |t - \xi_i|\,\hat{x}(\mathrm{d}t) \\
&\geq |\phi_j(\xi_i)|\left|\int_{\tilde{T}_{i,\epsilon}} \hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_i} a_{ir}\right| - L(2k_i - 1)\epsilon\int_{\tilde{T}_{i,\epsilon}} \hat{x}(\mathrm{d}t),
\end{aligned}
$$

where the width of $\tilde{T}_{i,\epsilon}$ is at most $2k_i\epsilon$ and $\xi_i \in \tilde{T}_{i,\epsilon}$ is chosen according to (2.37), so the distance $|t - \xi_i|$ for $t \in \tilde{T}_{i,\epsilon}$ is at most $(2k_i - 1)\epsilon$. For the second term (2.36b):

$$
\begin{aligned}
\left|\int_{\tilde{T}_{l,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_l} a_{lr}\phi_j(t_{lr})\right| &= \left|\int_{\tilde{T}_{l,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - \phi_j(\xi_l)\sum_{r=1}^{k_l} a_{lr}\right| \\
= \left|\int_{\tilde{T}_{l,\epsilon}} \phi_j(t)\hat{x}(\mathrm{d}t) - \phi_j(\xi_l)\sum_{r=1}^{k_l} a_{lr} + \phi_j(\xi_l)\int_{\tilde{T}_{l,\epsilon}} \hat{x}(\mathrm{d}t) - \phi_j(\xi_l)\int_{\tilde{T}_{l,\epsilon}} \hat{x}(\mathrm{d}t)\right| \\
\leq |\phi_j(\xi_l)|\left|\int_{\tilde{T}_{l,\epsilon}} \hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_l} a_{lr}\right| + \int_{\tilde{T}_{l,\epsilon}} |\phi_j(t) - \phi_j(\xi_l)|\,\hat{x}(\mathrm{d}t) \\
\leq |\phi_j(t_l)|\left|\int_{\tilde{T}_{l,\epsilon}} \hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_l} a_{lr}\right| + L\int_{\tilde{T}_{l,\epsilon}} |t - \xi_l|\,\hat{x}(\mathrm{d}t) \\
\leq |\phi_j(t_l)|\left|\int_{\tilde{T}_{l,\epsilon}} \hat{x}(\mathrm{d}t) - \sum_{r=1}^{k_l} a_{lr}\right| + L(2k_l - 1)\epsilon\int_{\tilde{T}_{l,\epsilon}} \hat{x}(\mathrm{d}t)
\end{aligned}
$$

so

$$\sum_{l\neq i}\left|\int_{\tilde{T}_{l,\epsilon}}\phi_j(t)\hat{x}(\mathrm{d}t)-\sum_{r=1}^{k_l}a_{lr}\phi_j(t_{lr})\right|\leq\sum_{l\neq i}\left(\left|\phi_j(\xi_l)\right|\left|\int_{\tilde{T}_{l,\epsilon}}\hat{x}(\mathrm{d}t)-\sum_{r=1}^{k_l}a_{lr}\right|\right)$$
$$+L\epsilon\sum_{l\neq i}(2k_l-1)\int_{\tilde{T}_{l,\epsilon}}\hat{x}(\mathrm{d}t).$$

Let

$$\tilde{z}_i=\left|\int_{\tilde{T}_{i,\epsilon}}\hat{x}(\mathrm{d}t)-\sum_{r=1}^{k_i}a_{ir}\right|$$

and we obtain an inequality as before:

$$2\left(1+\frac{\phi^{\infty}\|b\|_2}{\bar{f}}\right)\delta+(2k-1)\epsilon L\|\hat{x}\|_{TV}\geq\left|\phi_j(\xi_i)\right|\tilde{z}_i-\sum_{l\neq i}\left|\phi_j(\xi_l)\right|\tilde{z}_l,$$

where we obtained the second constant as follows:

$$\begin{aligned}L\epsilon(2k_i-1)\int_{\tilde{T}_{i,\epsilon}}\hat{x}(\mathrm{d}t)+L\epsilon\sum_{l\neq i}(2k_l-1)\int_{\tilde{T}_{l,\epsilon}}\hat{x}(\mathrm{d}t)&=L\epsilon\sum_{l=1}^{\tilde{k}}(2k_l-1)\int_{\tilde{T}_{l,\epsilon}}\hat{x}(\mathrm{d}t)\\&\leq L\epsilon\|\hat{x}\|_{TV}\sum_{l=1}^{\tilde{k}}(2k_l-1)\\&\leq(2k-1)L\epsilon\|\hat{x}\|_{TV}\end{aligned}$$

and, for all $i=1,\dots,\tilde{k}$, we select $j(i)=\operatorname{argmin}_j|s_j-\xi_i|$. The linear system is

$$\tilde{A}\tilde{z}\leq\tilde{v},$$

with:

$$\begin{aligned}\tilde{A}&=\begin{bmatrix}|\phi_1(\xi_1)| & -|\phi_1(\xi_2)| & \dots & -|\phi_1(\xi_{\tilde{k}})|\\-|\phi_2(\xi_1)| & |\phi_2(\xi_2)| & \dots & -|\phi_2(\xi_{\tilde{k}})|\\\vdots & \vdots & \ddots & \vdots\\-|\phi_{\tilde{k}}(\xi_1)| & -|\phi_{\tilde{k}}(\xi_2)| & \dots & |\phi_{\tilde{k}}(\xi_{\tilde{k}})|\end{bmatrix},\\\tilde{z}&=\begin{bmatrix}z_1 & z_2 & \dots & z_{\tilde{k}}\end{bmatrix}^T,\\\tilde{v}&=\left(2\left(1+\frac{\phi^{\infty}\|b\|_2}{\bar{f}}\right)\delta+(2k-1)L\epsilon\|\hat{x}\|_{TV}\right)\begin{bmatrix}1 & 1 & \dots & 1\end{bmatrix}^T.\end{aligned}$$

In Section 2.5.2, we discussed what the choice of λ should be so that, if $|t_i-s_i|\leq\lambda\Delta$, the matrix A is strictly diagonally dominant. Here, the matrix $\tilde{A}$ is similar to A except that we evaluate ϕ at $|\xi_i-s_i|$, where ξ_i corresponds to a group of sources in $\tilde{T}_{i,\epsilon}$ that are located within distances smaller than 2ϵ and s_i is the closest sample to ξ_i. Given that the minimum separation between ξ_i sources is 2ϵ, the analysis in Section 2.5.2 is the same, so $\tilde{A}$ is strictly diagonally dominant if

$$|\xi_i - s_i| \leq 2\lambda\epsilon, \quad \forall i = 1, \ldots, \tilde{k}$$

and λ is chosen to satisfy (2.9) where we take $\Delta = 2\epsilon$. For the value of λ found this way, we select the sampling locations uniformly at intervals of $2\lambda\epsilon$.

Chapter 3

Stability of non-negative super-resolution – Gaussian point spread function

In this chapter we consider $\phi_j(t)$ to be shifted Gaussians with centres at the sampling locations s_j, specifically

$$\phi_j(t) = g(t - s_j) = e^{-\frac{(t-s_j)^2}{\sigma^2}}. \tag{3.1}$$

Similarly to Chapter 2, we discuss the properties of the solution to the feasibility problem (1.5) when the samples are perturbed by additive noise. In the particular case of ϕ being Gaussian, we are able to give more explicit bounds on the error by further bounding $\|b\|_2$ from (2.10) and (2.11) in Theorem 9. This bound is given in Theorem 19.

We first introduce briefly the uniqueness result in the noise-free setting, together with the conditions on the samples, sources and minimum separation necessary for the Gaussian case in Section 3.1. Then we give the main result, Theorem 19 and a corollary which simplifies the bound in a more constrained setting in Section 3.2, and we give a high level proof of Theorem 19 in Section 3.3. We close the chapter with a discussion of some of the technical details of this result in Section 3.4.

3.1 Motivation: the noise-free case and preliminary definitions

Before stating the stability results in the Gaussian case, it is important to note that, in the setting of exact samples, $w_i = 0$, the solution of Program (1.5) is unique when $\delta = 0$.

Proposition 16. (Uniqueness of exactly sampled sparse non-negative measures for $\phi(t)$ Gaussian) *Let x be a non-negative k-sparse discrete measure supported on I, see (1.1). If $\delta = 0$, $m \geq 2k+1$ and $\{\phi_j\}_{j=1}^m$ are shifted Gaussians as in* (3.1), *then x is the unique solution of Program (1.5) with $\delta = 0$.*

Proposition 16 states that Program (1.5) successfully localises the k impulses present in x given only $2k+1$ measurements when $\phi_j(t)$ are shifted Gaussians whose centres are in I. This is a direct consequence of the more general Proposition 2, combined with the fact that shifted Gaussians form a T-system [50]. Motivated by this result, in the rest of this section we will introduce the necessary notions so that we can give an analogous result to Theorem 9 when ϕ is Gaussian.

In order to give a more explicit bound of $\|b\|_2$ from Theorem 9 in the case when ϕ is Gaussian, we need to impose additional conditions on the source and sampling locations relative to the boundary of the domain $I = [0,1]$ and the separation between them.

Conditions 17. ***(Gaussian window conditions)*** *When the window function is a Gaussian $\phi(t) = e^{-\frac{t^2}{\sigma^2}}$, we require its width σ and the source and sampling locations from* (1.6) *to satisfy the following conditions:*

1. *Samples define the interval boundaries: $s_1 = 0$ and $s_m = 1$,*
2. *Samples near sources: for every $i = 1, \ldots, k$, there exists a pair of samples $s, s' \subset S$, one on each side of t_i, such that $|s - t_i| \leq \eta$ and $s' - s \in [C_1\eta, C_2\eta]$ for some $C_1 \in (0,1]$ and $C_2 \in [1,2)$ and $\eta \leq \sigma^2$ small enough; which is quantified in Lemma 22.*
3. *Sources away from the boundary: $\sigma\sqrt{\log(1/\eta^3)} \leq t_i, s_j \leq 1 - \sigma\sqrt{\log(1/\eta^3)}$ for every $i = 1, \ldots, k$ and $j = 2, \ldots, m-1$,*
4. *Minimum separation of sources: $\sigma \leq \sqrt{2}$ and $\Delta(T) > \sigma\sqrt{\log\left(3 + \frac{4}{\sigma^2}\right)}$, where the minimum separation $\Delta(T)$ of the sources is defined in Definition 6.*

The four properties in Conditions 17 can be interpreted as follows: Property 1 imposes that the sources are within the interval defined by the minimum and maximum sample; Property 2 ensures that there is a pair of samples near each source which translates into a sampling density condition in relation to the minimum separation between sources and in particular requires the number of samples $m \geq 2k+2$; Property 3 constrains the width of the Gaussian σ through the sampling density η

(in particular $\sigma \geq \epsilon/\sqrt{\log(1/\eta^3)}$, where ϵ is the minimum distance between a source and the sampling boundary, which implies that having samples far away from the sources requires a wider point spread function. Conversely, a smaller distance of the sources to the boundary ϵ allows a narrower point spread function); Properties 3 and 4 are technical conditions used in the proof to bound the eigenvalues of an associated stability matrix, and we expect may be improved, though a condition constraining the size of σ as compared to η is likely necessary in the noisy setting as otherwise the samples in (1.2) may have little dependence on the source locations t_i.

Lastly, we introduce a more specific form of the dual polynomial separator $F(t)$ from Definition 8.

Definition 18. ***(Dual polynomial separator for ϕ Gaussian)*** *Let $\bar{f}, f_1$ be positive constants with $\bar{f} < 1$ and let f_0 be greater that both $\bar{f}$ and f_1, with the exact relationship between f_0 and $\bar{f}$ given in the proof of Lemma 21 in Section 3.3. For $\epsilon > 0$, we define*

$$F(t) := \begin{cases} f_0, & t = 0, \\ f_1, & t = 1, \\ 0, & \textit{when there exists } i = 1, \dots, k \textit{ such that } t \in T_{i,\epsilon}, \\ \bar{f}, & \textit{elsewhere on } I. \end{cases} \tag{3.2}$$

3.2 Stability results simplified to Gaussian point spread function

We can now present our main result on the robustness of Program (1.5) as it applies to the Gaussian window; this is Theorem 19, which follows from Theorem 9. Theorem 19 extends this uniqueness condition to show that any solution to Program (1.5) with $\delta > 0$ is proportionally close to the unique solution when $\delta = 0$. We prove in Theorem 19 that any solution to Program (1.5) is locally consistent with the discrete measure in terms of local averages over intervals $T_{i,\epsilon}$ as given in (2.7). Moreover, for Theorem 19, we make Property 2 of Conditions 17 more transparent by using the sample proximity $\lambda\Delta(T)$ from Definition 6; that is, η defined in Conditions 17 is related to the sample proximity from Definition 6 by $\lambda\Delta(T) \leq \eta/2$.

Theorem 19. **(Average stability of Program** (1.5) **for** $\phi(t)$ **Gaussian: source proximity dependence)** *Let* $I = [0,1]$ *and consider a* k*-sparse non-negative measure* x *supported on* T *and sample locations* S *as given in* (1.6) *and for positive* σ*, let* $\{\phi_j(t)\}_{j=1}^m$ *as defined in* (3.1)*. If the Conditions 17 hold, then, in the presence of additive noise, Program* (1.5) *is stable in the sense that, for any solution* $\hat{x}$ *of Program* (1.5)*:*

$$\left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right| \leq \left[(c_1 + F_2)\cdot\delta + c_2\frac{\|\hat{x}\|_{TV}}{\sigma^2}\cdot\epsilon\right] F_3, \tag{3.3}$$

$$\int_{T_\epsilon^C} \hat{x}(\mathrm{d}t) \leq F_2\cdot\delta, \tag{3.4}$$

where the exact expressions of $F_2 = F_2(k, \Delta(T), \frac{1}{\sigma}, \frac{1}{\epsilon})$ *and* $F_3 = F_3(\Delta(T), \sigma, \lambda)$ *are given in the proof (see* (3.26) *in Section 3.3.2), provided that* λ*,* $\Delta(T)$ *and* σ *satisfy* (2.9)*. In particular, for* $\sigma < \frac{1}{\sqrt{3}}$*,* $\Delta(T) > \sigma\sqrt{\log\frac{5}{\sigma^2}}$ *and* $\lambda < 0.4$*, we have* $F_3(\Delta(T), \sigma, \lambda) < c_5$ *and:*

$$F_2(k, \Delta(T), \frac{1}{\sigma}, \frac{1}{\epsilon}) < c_3\frac{kC_2(\frac{1}{\epsilon})}{\sigma^2}\left[\frac{c_4}{\sigma^6(1-3\sigma^2)^2}\right]^k. \tag{3.5}$$

Above, c_1, c_2, c_3, c_4, c_5 *are universal constants and* $C_2(\frac{1}{\epsilon})$ *is given by* (3.19) *in Section 3.3.1.*

The presence of $\|\hat{x}\|_{TV}$ in Theorem 19 is a feature of the proof which we expect can be removed and replaced with $\|x\|_{TV}$ by proving any solution of problem (1.5) is necessarily bounded due to the sampling proximity condition of Definition 6. Theorem 19 follows from the more general result of Theorem 9 and its proof is given in Section 3.3.

Lastly, we give a corollary of Theorem 19 where we show that, for $\delta > 0$ but sufficiently small, one can equate the δ and ϵ dependent terms in Theorem 19 to show that the error approaches zero as δ goes to zero. The proof of the corollary is given in Section 3.3.3.

Corollary 20. ***(Dependence of the error bound on*** δ***)*** *Under the conditions in Theorem 19 and for* $\sigma < \frac{1}{\sqrt{3}}$*,* $\Delta(T) > \sigma\sqrt{\log\frac{5}{\sigma^2}}$ *and* $\lambda < 0.4$*, there exists* δ_0 *such that:*

$$\left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right| \leq \bar{C}_2\cdot\delta^{\frac{1}{6}}, \tag{3.6}$$

for all $\delta \in (0, \delta_0)$*, where* $\bar{C}_2$ *is given in the proof in Section 3.3.3.*

As we will see in Lemma 24 in Section 3.3.3, we have that $C_2 \sim \frac{1}{\epsilon^5}$ and this is determined by the our construction of the dual certificate. Then, in order to eliminate the dependence on ϵ of the bounds in Theorem 19, the best choice of ϵ as a function of δ is $\epsilon = \delta^{\frac{1}{6}}$, which leads to the $\delta^{\frac{1}{6}}$ bound in Corollary 20. In this sense, the exponent $\frac{1}{6}$ is tight. Lastly, we remind the reader that δ is the noise level in the samples and, therefore, it cannot be chosen arbitrarily.

3.3 Proof of Theorem 19 (Gaussian with sparse measure)

In this section, we give the main steps taken to obtain the explicit bounds in Theorem 19 for the Gaussian window function. This is a particular case of the more general Theorem 9, where the window function is taken to be the Gaussian $\phi_j(t) = e^{-(t-s_j)^2/\sigma^2}$, given in (3.1).

3.3.1 Bound on the coefficients of the dual certificate for Gaussian window

We will give an explicit bound on the vector of coefficients $\|b\|_2$ of the dual certificate q from Lemma 11 in terms of the parameters of the problem k, T, S and σ (the width of the Gaussian window).

In addition, we need to ensure that the conditions in Theorem 9 are satisfied. With the dual polynomial separator given in Definition 18, Theorem 9 requires that $\{F\} \cup \{\phi_j\}_{j=1}^m$ form a T*-system on I. We show in Lemma 21 that this requirement is satisfied for the choice in (3.1) of $\phi_j(t) = e^{-(t-s_j)^2/\sigma^2}$. The proof is given in Section 4.1.

Lemma 21. ***($\{F\} \cup \{\phi_j\}_{j=1}^m$ form a T*-system)*** *Consider the function $F(t)$ defined in* (3.2) *and suppose that $m \geq 2k+2$. Then $\{F\} \cup \{\phi_j\}_{j=1}^m$ form a T*-system on I, with ϕ extended totally positive, even Gaussian and ϕ_j defined as in* (3.1)*, provided that $f_0 \gg \bar{f}$, $f_0 \gg f_1$ and $\bar{f}, f_0, f_1 \gg 0$. These requirements are made precise in the proof and are dependent on ϵ.*

In this setting, consider a subset of $m = 2k+2$ samples $\{s_j\}_{j=1}^m \subset S$ (since in the proof of Lemma 21 we select the $2k+2$ samples that are the closest to the sources) such that they satisfy Conditions 17. Therefore, we have that $s_1 = 0$, $s_m = s_{2k+2} = 1$, and

$$|s_{2i} - t_i| \leq \eta, \qquad s_{2i+1} - s_{2i} = \eta, \qquad \forall i = 1, \ldots, k, \tag{3.7}$$

for a small $\eta \leq \sigma^2$, see (3.1). That is, we collect two samples close to each impulse t_i in x, one on each side of t_i. Suppose also that

$$\sigma \leq \sqrt{2}, \quad \Delta > \sigma \sqrt{\log\left(3 + \frac{4}{\sigma^2}\right)}, \quad \eta \leq \sigma^2, \tag{3.8}$$

namely the width of the Gaussian is much smaller than the separation of the impulses in x. Lastly, assume that the impulse locations $T = \{t_i\}_{i=1}^{k}$ and sampling points $\{s_j\}_{j=2}^{m-1}$ are away from the boundary of I, namely

$$\sigma\sqrt{\log(1/\eta^3)} \leq t_i \leq 1 - \sigma\sqrt{\log(1/\eta^3)}, \qquad \forall i = 1, \ldots, k,$$

$$\sigma\sqrt{\log(1/\eta^3)} \leq s_j \leq 1 - \sigma\sqrt{\log(1/\eta^3)}, \qquad \forall j = 2, \ldots, m-1. \tag{3.9}$$

Remark 1. The Property *2. Samples near sources* in Conditions 17 states that for each two samples s' and s near each source, we have that:

$$C_1\eta \leq s' - s \leq C_2\eta, \tag{3.10}$$

but in (3.7) above we simplified the condition to $s_{2i+1} - s_{2i} = \eta$. Throughout the proofs in this paper we will use the simplified condition in (3.7) instead of the more general (3.10) so that we do not obscure the central issues of the proof with extra indices and separate treatment of the upper and lower bounds for $s' - s$. This has implications for Lemma 22 below, and we will point out the places in its proof where using the more general condition (3.10) would require separate treatment (see footnote 1 on page 70 and footnote 3 on page 73).

Remark 2. The Property *3. Sources away from the boundary* in Conditions 17 is necessary due to our method of proof, which imposes that the sources and samples are in the interval $I = [0, 1]$ independently from the width σ of the convolution kernel and the minimum separation of sources Δ. While the exact form of the boundary conditions depend on the specific approach that we took in the proof, there is an interplay between σ and Δ in the noisy setting and due to the fact that the interval $I = [0, 1]$ is fixed, some form of scaling is required, which is what this condition achieves. For example, for $\sigma = 1$, the Gaussian kernel defined in (3.1) varies by at most $\frac{1}{e}$ (between two sources located at $t_1 = 0$ and $t_2 = 1$) and therefore our results for the noisy setting are not meaningful for large σ.

We can now give an explicit bound on $\|b\|_2$ for the Gaussian window function, as required by Theorem 9. The following result is proved in Section 4.2.

Lemma 22. ***(Bounds on*** $\|b\|_2$ ***for*** $\phi(t)$ ***Gaussian)*** *Suppose that the window function* ϕ *is Gaussian, as defined in* (3.1)*, the assumptions* (3.7)*,* (3.8) *and* (3.9) *(namely Conditions 17) hold and* η *satisfies:*

$$\eta \leq \min \left\{ \frac{8F_{\min}(\Delta, \frac{1}{\sigma})}{34(2k+2)\left(80k+8+kP\left(\frac{1}{\sigma}\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right)^{\frac{1}{2}}}, \frac{\bar{C}(f_0,f_1)^{\frac{1}{6}}}{\left(4k+4+\frac{4k}{\sigma^2}\right)^{\frac{1}{3}}} \right\}. \tag{3.11}$$

Then we have the following bound:

$$\|b\|_2 \leq \frac{\sqrt{(2k+2)\left(4k+5+\frac{4k}{\sigma^4}\right)}}{1-\frac{\sqrt{e}}{2}} \bar{C}(f_0,f_1)^{\frac{5}{4}} \left[\frac{F_{\max}\left(\Delta,\frac{1}{\sigma}\right)}{F_{\min}\left(\Delta,\frac{1}{\sigma}\right)^2}\right]^k, \tag{3.12}$$

where

$$\bar{C}(f_0,f_1) = f_0^2+f_1^2+2f_0+2f_1+2, \tag{3.13}$$

$$P\left(\frac{1}{\sigma}\right) = \frac{4}{\sigma^4}+\frac{13}{4}\left(\frac{2}{\sigma^2}+\frac{4}{\sigma^4}\right)^2+\frac{9}{4}\left(\frac{12}{\sigma^4}+\frac{8}{\sigma^6}\right)^2. \tag{3.14}$$

$$F_{\max}\left(\Delta,\frac{1}{\sigma}\right) = \left(8+\left(1+\frac{4}{\sigma^4}\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right)^{\frac{1}{2}}\left(32+\left(\frac{1}{\sigma^4}+\frac{2}{\sigma^6}+\frac{2}{\sigma^8}\right)\frac{16}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right)^{\frac{1}{2}}, \tag{3.15}$$

$$F_{\min}\left(\Delta,\frac{1}{\sigma}\right) = 1-\left(1+\frac{2}{\sigma^2}\right)\frac{2e^{-\frac{\Delta^2}{\sigma^2}}}{1-e^{-\frac{\Delta^2}{\sigma^2}}}. \tag{3.16}$$

To obtain the final bounds for the Gaussian window function, we will substitute the above bounds in the right hand side of (2.11) and (2.10) in Theorem 9. We will then obtain F_2 in Theorem 19 (see (3.27)).

For more clarity, in the following lemma we simplify F_2 further, in the case when stronger conditions apply to σ, $\Delta(T)$ and λ.

Lemma 23. ***(Simplified bounds for*** $\phi(t)$ ***Gaussian)*** *If the conditions in Lemma 22 hold and, in addition,* $\sigma < \frac{1}{\sqrt{3}}$*,* $\Delta > \sigma\sqrt{\log\frac{5}{\sigma^2}}$ *and* $\bar{f} < 1$*, then*

$$\frac{F_{\max}\left(\Delta,\frac{1}{\sigma}\right)}{F_{\min}\left(\Delta,\frac{1}{\sigma}\right)^2} < \frac{c_1}{\sigma^6(1-3\sigma^2)^2}, \tag{3.17}$$

$$\frac{\sqrt{(2k+2)\left(4k+5+\frac{4k}{\sigma^4}\right)}}{1-\frac{\sqrt{e}}{2}}\frac{\bar{C}(f_0,f_1)^{\frac{5}{4}}}{\bar{f}} < c_3\cdot\frac{kC_2(\frac{1}{\epsilon})}{\sigma^2}, \tag{3.18}$$

where

$$C_2\left(\frac{1}{\epsilon}\right) = \frac{\bar{C}(f_0, f_1)^{\frac{5}{4}}}{\bar{f}}. \tag{3.19}$$

Moreover, if λ *in Theorem 9 satisfies* $\lambda < \frac{2}{4}$*, then*

$$\frac{1}{e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} - e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} \cdot \frac{e^{-\frac{\Delta^2}{\sigma^2}} + e^{-\frac{2\Delta^2}{\sigma^2}}}{1-e^{-\frac{\Delta^2}{\sigma^2}}} - e^{-\frac{\Delta^2(1-\lambda)^2}{\sigma^2}}} < c_4. \tag{3.20}$$

Above, c_1, c_2, c_3, c_4 *are universal constants.*

More specifically, (3.17) and (3.18) will be used to bound F_2 to obtain (3.5). Lastly, (3.20) will be used to bound F_3, which appears in the bound given by Theorem 19.

The proof of Lemma 23 is given in Section 4.5. Note that we give C_2 as a function of $\frac{1}{\epsilon}$ because, as $\epsilon \to 0$, C_2 grows at a rate dependent on ϵ, as indicated in the Lemma 24 in Section 3.3.3.

3.3.2 Proof of Theorem 19 (Average stability for the feasibility problem with Gaussian point spread function)

We now apply Theorem 9 with $\phi(t) = g(t) = e^{-t^2/\sigma^2}$. We have that $\phi^\infty = 1$, the Lipschitz constant L of g on $[-1, 1]$ is $L = \frac{2}{\sigma^2}$, and

$$\sum_{j=1}^{k} (A^{-1})_{ij} \le \|A^{-1}\|_\infty < \frac{1}{\min_j \left(g(s_j - t_j) - \sum_{i\neq j} g(s_j - t_i)\right)}, \tag{3.21}$$

for all $i = 1, \ldots, k$. The last inequality comes from the definition of A in (2.12) with $\phi(t) = g(t)$ and $s_j := s_{l(j)}$ as given in Definition 6, and [84]. Then, by assumption, $|s_j - t_j| \le \lambda\Delta$ for an arbitrary $j = 1, \ldots, k$ and g is decreasing, so

$$g(s_j - t_j) \ge g(\lambda\Delta) = e^{-\frac{\lambda^2\Delta^2}{\sigma^2}}.$$

We now assume without loss of generality that $s_j < t_j$. Then, it follows that

$$|s_j - t_i| \ge |j - i|\Delta - \lambda\Delta, \text{ if } i < j \qquad \text{and} \qquad |s_j - t_i| \ge |j - i|\Delta + \lambda\Delta, \text{ if } i > j.$$

This, in turn, leads to

$$\sum_{i\neq j} g(s_j - t_i) = \sum_{i=1}^{j-1} g(s_j - t_i) + \sum_{i=j+1}^{k} g(s_j - t_i)$$
$$\leq \sum_{i=1}^{j-1} g((j-i)\Delta - \lambda\Delta) + \sum_{i=j+1}^{k} g((i-j)\Delta + \lambda\Delta)$$
$$\leq \sum_{i=1}^{\infty} g((i-\lambda)\Delta) + \sum_{i=1}^{\infty} g((i+\lambda)\Delta). \tag{3.22}$$

We now bound each sum in (3.22) as follows

$$\sum_{i=1}^{\infty} g((i-\lambda)\Delta) = \sum_{i=1}^{\infty} e^{-\frac{(i-\lambda)^2\Delta^2}{\sigma^2}}$$
$$= e^{-\frac{\Delta^2(1-\lambda)^2}{\sigma^2}} + e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} \sum_{i=2}^{\infty} \left(e^{-\frac{\Delta^2}{\sigma^2}}\right)^{i^2-2i\lambda}$$
$$\leq e^{-\frac{\Delta^2(1-\lambda)^2}{\sigma^2}} + e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} \sum_{i=2}^{\infty} \left(e^{-\frac{\Delta^2}{\sigma^2}}\right)^{i} \qquad (i^2 - 2i\lambda > i \text{ for } i \geq 2)$$
$$= e^{-\frac{\Delta^2(1-\lambda)^2}{\sigma^2}} + e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} \cdot \frac{e^{-\frac{2\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}, \tag{3.23}$$

and similarly, we have that

$$\sum_{i=1}^{\infty} g((i+\lambda)\Delta) = \sum_{i=1}^{\infty} e^{-\frac{(i+\lambda)^2\Delta^2}{\sigma^2}} \leq e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} \cdot \frac{e^{-\frac{\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}. \tag{3.24}$$

By combining (3.22), (3.23) and (3.24), we obtain:

$$g(s_j - t_j) - \sum_{i\neq j} g(s_j - t_i) \geq e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} - e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} \cdot \frac{e^{-\frac{\Delta^2}{\sigma^2}} + e^{-\frac{2\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} - e^{-\frac{\Delta^2(1-\lambda)^2}{\sigma^2}}. \tag{3.25}$$

The above inequality also holds when we take the minimum over $j = 1, \ldots, k$ and, inserting it in (3.21) and using this result and the bound on $\|b\|_2$ from Lemma 22 in (2.11), we obtain (3.3):

$$\left| \int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i \right| \leq \left[(c_1 + F_2) \cdot \delta + c_2 \frac{\|\hat{x}\|_{TV}}{\sigma^2} \cdot \epsilon \right] F_3, \tag{3.26}$$

where

$$F_2(k, \Delta(T), \frac{1}{\sigma}, \frac{1}{\epsilon}) = \frac{\sqrt{(2k+2)\left(4k+5+\frac{4k}{\sigma^4}\right)}}{1-\frac{\sqrt{\epsilon}}{2}} \frac{\bar{C}(f_0, f_1)^{\frac{5}{4}}}{\bar{f}} \left[\frac{F_{\max}\left(\Delta, \frac{1}{\sigma}\right)}{F_{\min}\left(\Delta, \frac{1}{\sigma}\right)^2}\right]^k, \quad (3.27)$$

$$F_3(\Delta(T), \sigma, \lambda) = \frac{1}{e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} - e^{-\frac{\Delta^2\lambda^2}{\sigma^2}} \cdot \frac{e^{-\frac{\Delta^2}{\sigma^2}} + e^{-\frac{2\Delta^2}{\sigma^2}}}{1-e^{-\frac{\Delta^2}{\sigma^2}}} - e^{-\frac{\Delta^2(1-\lambda)^2}{\sigma^2}}}, \quad (3.28)$$

and $\bar{C}, F_{\max}, F_{\min}$ are given in (3.13), (3.15), (3.16) respectively. The error bound away from the sources (3.4) is obtained by applying Lemma 11 with the same bounds on $\|b\|_2$.

Then, by using Lemma 23 with $\bar{f} < 1$, we obtain (3.5). Note that we can apply Lemma 23 because, for $\sigma < \frac{1}{\sqrt{3}}$, we have that $\frac{5}{\sigma^2} > 3 + \frac{4}{\sigma^2}$ and, therefore, $\Delta > \sigma\sqrt{\log\frac{5}{\sigma^2}} > \sigma\sqrt{\log(3+\frac{4}{\sigma^2})}$.

3.3.3 Proof of Corollary 20 (Dependence of the error bound on noise)

First we give an explicit dependence of $C_2(\frac{1}{\epsilon})$ on ϵ for small $\epsilon > 0$ in the following lemma, proved in Section 4.6.

Lemma 24. ***(Dependence of** C_2 **on** ϵ**)*** *If $f_1 < f_0$, $1 < f_0$ and $\bar{f} < 1$, then there exists $\epsilon_0 > 0$ such that:*

$$C_2\left(\frac{1}{\epsilon}\right) < \bar{c}_2 C_\epsilon^{\frac{5}{2}} \cdot \frac{1}{\epsilon^5}, \quad (3.29)$$

for all $\epsilon \in (0, \epsilon_0)$, where $\bar{c}_2$ is a universal constant and C_ϵ is defined in the proof, see (4.107).

Let $\epsilon = \delta^{\frac{1}{6}}$ in the bound on $C_2(\frac{1}{\epsilon})$ in Lemma 24:

$$C_2\left(\frac{1}{\epsilon}\right) < \bar{c}_2 C_\epsilon^{\frac{5}{2}} \cdot \frac{1}{\delta^{\frac{5}{6}}}, \quad \forall \delta < \epsilon_0^6, \quad (3.30)$$

which we substitute in the bound (3.5) in Theorem 19 to obtain:

$$\left|\int_{T_{i,\epsilon}} \hat{x}(\mathrm{d}t) - a_i\right| \leq \bar{C}_2 \cdot \delta^{\frac{1}{6}}, \quad \forall \delta < \epsilon_0^6,$$

where

$$\bar{C}_2 = c_1 + c_3 \frac{k\bar{c}_2 C_\epsilon^{\frac{5}{2}}}{\sigma^2}\left[\frac{c_4}{\sigma^6(1-3\sigma^2)^2}\right]^k + \frac{c_2\|\hat{x}\|_{TV}}{\sigma^2}. \quad (3.31)$$

Note that we apply Lemma 24 with $\bar{f} < 1$ and that the inequality in the corollary holds for $\delta < \delta_0 = \epsilon_0^6$, where ϵ_0 is given by Lemma 24.

3.4 Discussion

In this final section, we discuss a few issues regarding the robustness of our construction of the dual certificate from Appendix A.4. There are two points that need to be raised: the construction itself and the proof that we indeed have a T*-System.

At the moment, we do not use any samples that are away from sources in the the construction of the dual certificate. If the sources are close enough compared to σ, then this is not an issue. However, for σ small relative to the distance between samples, in light of the proof of Lemma 21 (see Section 4.1), if we consider the dual certificate as the expansion of the determinant N of M^ρ in (4.1) along the $\tau_{\underline{l}}$ row:

$$N = -F(\tau_l)\beta_0 + \sum_{j=1}^{m}(-1)^{j+1}\beta_j g(\tau_l - s_j), \tag{3.32}$$

then the terms $g(\tau_{\underline{l}})$ become exponentially small (as $\tau_{\underline{l}}$ is far from all samples s_j) and, therefore, the value of N is close to $-F(\tau_l)$ (which is $-\bar{f}$ if $\tau_l \in T_\epsilon^C$). This is problematic, as we require that $N > 0$. We can overcome this by adding "fake" sources at intervals η^{-1} so that they cover the regions where we have no true sources, together with two close samples for each extra source. The determinant N becomes:

$$N = \begin{vmatrix} f_0 & f(s_1) & \cdots & g(s_m) \\ 0 & g(t_1 - s_1) & \cdots & g(t_1 - s_m) \\ 0 & g'(t_1 - s_1) & \cdots & g'(t_1 - s_m) \\ \vdots & \vdots & & \vdots \\ \bar{f} & g(\tau_j - s_1) & \cdots & g(\tau_j - s_m) \\ \bar{f} & g'(\tau_j - s_1) & \cdots & g'(\tau_j - s_m) \\ \vdots & \vdots & & \vdots \\ F(\tau_{\underline{l}}) & g(\tau_{\underline{l}} - s_1) & \cdots & g(\tau_{\underline{l}} - s_m) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & g(t_k - s_1) & \cdots & g(t_k - s_m) \\ 0 & g'(t_k - s_1) & \cdots & g'(t_k - s_m) \\ f_1 & g(1 - s_1) & \cdots & g(1 - s_m) \end{vmatrix}. \tag{3.33}$$

Here, the rows are ordered according to the ordering of the set containing $t_i, \tau_j, \tau_{\underline{l}}$. The terms in the expansion of (3.33) along the row with $\tau_{\underline{l}}$ do not approach 0 exponentially with this construction, since for any $\tau_{\underline{l}}$ there exists s_i close enough so that $g(\tau_{\underline{l}} - s_i) > f^*$ for some $f^* > 0$.

More specifically, consider also the expansion of N along the first column:

$$N = f_0 N_{1,1} + f_1 N_{m+1,1} - F(\tau_{\underline{l}}) N_{\tau_{\underline{l}},1} - \bar{f} \sum_{j<\tau_{\underline{l}}} (N_{j,1} - N_{j+1,1}) + \bar{f} \sum_{j>\tau_{\underline{l}}} (N_{j,1} - N_{j+1,1}). \tag{3.34}$$

We use this expansion in the proof of Lemma 21 in Section 4.1 to show that the functions $F \cup \{g_j\}_{j=1}^m$ form a T*-System. For $\tau_{\underline{l}} \in T_\epsilon^C$, $F(\tau_{\underline{l}}) = \bar{f}$ and the setup in Lemma 21, we require that (see (4.6)):

$$\frac{f_0}{\bar{f}} \geq \frac{N_{\tau_{\underline{l}},1}}{\min_{\tau_{\underline{l}} \in T_\epsilon^C} N_{1,1}}. \tag{3.35}$$

In the construction (3.33), if we upper bound the pairs $N_{j,1} - N_{j+1,1}$ in the two sums in (3.34) (a separate problem by itself), then we can impose a similar condition to (3.35) for f_0 and $\bar{f}$. From here, we obtain that $f_0 = C\bar{f}$ where finding $C \geq \frac{N_{\tau_{\underline{l}},1}}{\min_{\tau_{\underline{l}} \in T_\epsilon^C} N_{1,1}}$ involves finding a lower bound on $N_{1,1}$:

$$N = \begin{vmatrix} g(t_1 - s_1) & \cdots & g(t_1 - s_m) \\ g'(t_1 - s_1) & \cdots & g'(t_1 - s_m) \\ \vdots & & \vdots \\ g(\tau_j - s_1) & \cdots & g(\tau_j - s_m) \\ g'(\tau_j - s_1) & \cdots & g'(\tau_j - s_m) \\ \vdots & & \vdots \\ g(\tau_{\underline{l}} - s_1) & \cdots & g(\tau_{\underline{l}} - s_m) \\ \vdots & & \vdots \\ g(t_k - s_1) & \cdots & g(t_k - s_m) \\ g'(t_k - s_1) & \cdots & g'(t_k - s_m) \\ g(1 - s_1) & \cdots & g(1 - s_m) \end{vmatrix}. \tag{3.36}$$

The structure of the above determinant is similar to the denominator in Section 4.2 but only up to the row with $\tau_{\underline{l}}$. The rows after it do not preserve the diagonally dominant structure of the matrix, as each source becomes associated with one close sample to it and the first sample corresponding to the next source. This is an issue in both the construction described in the proof of Proposition 12 in Appendix A.4 (and detailed in the proof of Lemma 21) and the construction described in the current section (which would result from considering a determinant with "fake" sources like (3.33)). However, by adding extra "fake" sources, one could argue that the determinant (3.33) is better behaved, as the distance between a source and the first sample corresponding to the next source is smaller, which we leave for further work.

3.5 Beyond the Gaussian case: solving the sample proximity condition numerically

The source and sample conditions of Theorems 9 and 19 are determined by condition (2.9) used to ensure the matrix A in (2.12) is diagonally dominant. In this chapter, we explore further when the sufficient bound (2.9) is satisfied for both the Gaussian window ϕ as defined in (3.1) and the function

$$\hat{\phi}(t) = (1 - |t|)^\gamma, \quad \gamma > 1, \tag{3.37}$$

as an example of a function that is non-differentiable, has more rapid initial decay and slower decay far from the origin, and moreover is not known to form a T-system. Its use here is motivated by the fact that, in practice, the convolution kernel may describe a physical process (for example light or sound propagation) and it may not necessarily be differentiable. It can be shown that $\hat{\phi}$ is Lipschitz continuous.

Figure 3.1 shows the relationships between λ, $\Delta(T)$ and the window localization parameters γ and σ, in the left and right panels respectively, by solving equation (2.9) numerically. We do this by fixing two of the three parameters and solving (2.9) numerically for the third parameter. Recall that (2.9) considers the worst sampling locations consistent with bound (2.6). The first row of Figure 3.1 (panels (a) and (b)) shows the degree to which samples are needed to become closer, that is λ to decrease, as the window function becomes wider (for small values of γ in (a) and large values of σ in (b)). This also depends on the minimum distance between sources $\Delta(T)$ with λ decaying more quickly for small $\Delta(T)$. The second row of Figure 3.1, in panels (c) and (d), shows the dependence between the width and $\Delta(T)$. When sources are closer to each other, the window function must be narrow for the same value of λ. In both plots we also show the case when $\lambda = 0$, namely when we have samples at the locations of the sources. Going beyond this curve (bottom left in (c) and top left in (d)) leads to not being able to reconstruct the signal. Approximation to these curves as λ approaches zero by taking leading Taylor series in (2.9) gives the following relationships between $\Delta(T)$ and the localization parameters of the windows: $\Delta(T) \approx \frac{2-2^{-\gamma}}{1+\gamma}$ for (3.37) and $\Delta(T) \approx \sqrt{\pi}\sigma \operatorname{erf}\left(\frac{1}{2\sigma}\right)$ for the Gaussian window (3.1).

Finally, in the bottom row of Figure 3.1, we fix the parameters γ and σ of the windows and show the dependence between λ and $\Delta(T)$. As expected, when the minimum distance between sources is greater, the distance between sources and samples can also be greater.

We show a few examples of parameters that satisfy (2.9) in Table 3.1 and signals with sources and sampling locations that have these parameters in Figure 3.2. Here we see $k = 5$ sources generated using the window function $\hat{\phi}$ in (a), (c), (e) and using the window function ϕ in (b), (d), (f). We start with the sources placed at $t_1 = 0.1, t_2 = 0.4, t_3 = 0.5, t_4 = 0.66, t_5 = 0.78$ in (a) and (b) so that we have the minimum distance between sources $\Delta(T) = 0.1$, then in (c) and (d) we keep the same source locations and we increase the width of the window functions, and in (e) and (f) we have the same width as in (a) and (b) but we move the sources to $t_1 = 0.1, t_2 = 0.25, t_3 = 0.4, t_4 = 0.63, t_5 = 0.78$ so that we increase the minimum distance to $\Delta(T) = 0.15$. For each of these configurations, we place the samples uniformly at intervals $2\lambda\Delta(T)$, so that the distance between each source and its closest sample is at most $\lambda\Delta(T)$.

γ	$\Delta(T)$	λ	σ	$\Delta(T)$	λ
25	0.1	0.4245	0.07	0.1	0.4292
15	0.1	0.2401	0.085	0.1	0.3386
25	0.15	0.4720	0.07	0.15	0.4833

Table 3.1: Examples of parameter values that satisfy (2.9) for $\hat{\phi}$ (left) and ϕ (right). Rows correspond to rows in Figure 3.2.

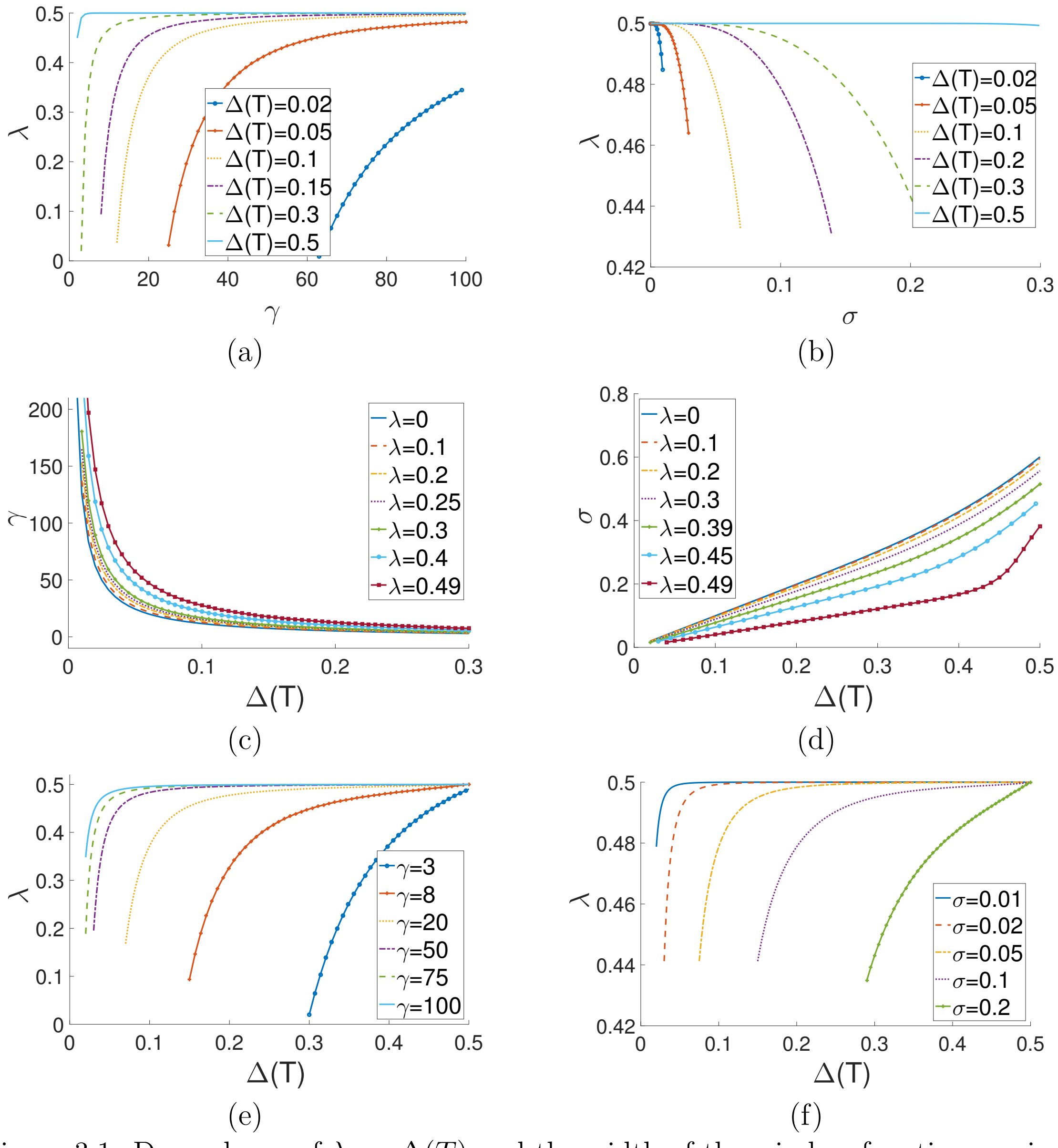
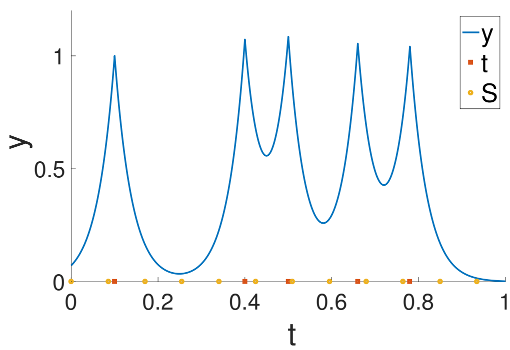
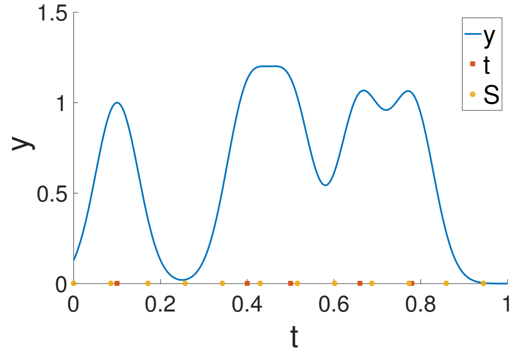


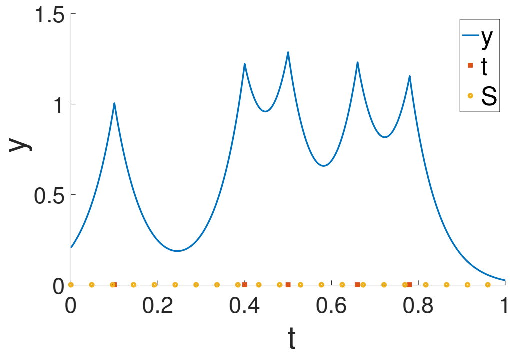
Figure 3.1: Dependence of λ on $\Delta(T)$ and the width of the window function as given by (2.9) for (3.37) (left), and (3.1) (right).



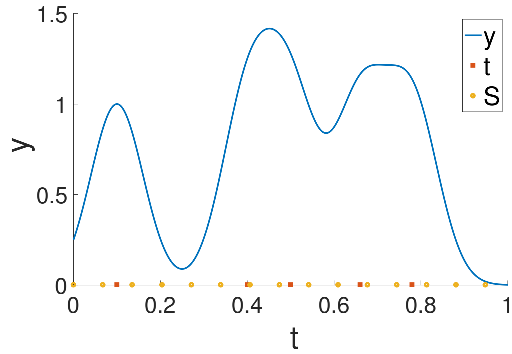
(a)



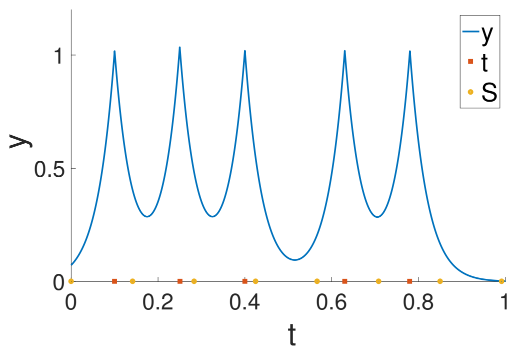
(b)



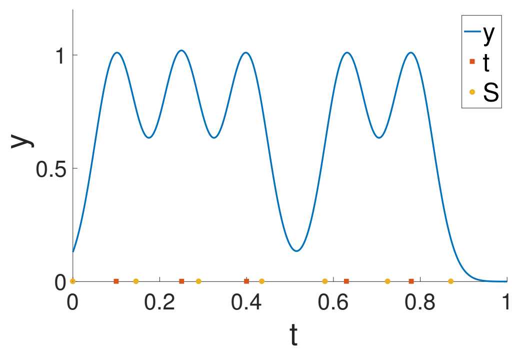
(c)



(d)



(e)



(f)

Figure 3.2: Examples from Table 3.1 for (3.37) (left) and (3.1) (right), where sampling points are located uniformly at interval $2\lambda\Delta(T)$.

Chapter 4

Detailed proofs of the lemmas in Chapter 3

In this chapter we give detailed proofs of the lemmas used in Chapter 3, where they have been omitted for clarity.

In Lemma 21, proved in the first section, we make sure that one of the conditions in Theorem 9, that $\{F\} \cup \{\phi_j\}_{j=1}^m$ form a T*-system, is satisfied so that we can apply Theorem 9 for ϕ Gaussian and the choice of F in Definition 18. Then, in Lemma 22, proved in Section 4.2, we bound $\|b\|_2$ in the case when ϕ is Gaussian so that we then give more explicit bounds on the error in Theorem 9, which allows us to state the bound from Theorem 19. Lemmas 25 and 26, proved in Sections 4.3 and 4.4 respectively, are two results stated and used in the proofs of Lemmas 21 and 22 respectively. Lastly, in Lemma 23, proved in Section 4.5, we simplify the bounds from Theorem 19 in a more constrained setting to help the reader see the main factors in the bound more clearly, obtaining the bound in (3.5), while in Lemma 24, proved in the last section of the chapter, we give a bound on C_2 that is explicit in ϵ, which is then used in the proof of Corollary 20, in Section 3.3.3, where we give a bound on the error from Theorem 19 that does not depend on ϵ.

4.1 Proof of Lemma 21 (The T*-system condition is satisfied)

Following the definition of T*-systems in Definition 7, consider an increasing sequence $\{\tau_l\}_{l=0}^m \subset I$ such that $\tau_0 = 0$, $\tau_m = 1$, and except one more point (say $\tau_{\underline{l}}$), the rest of points belong to T_ρ, the ρ-neighbourhood of the support $T \subset \text{int}(I)$.

We also select the subset of samples S of size $2k+2$ that is closest to the support T (in Hausdorff distance), so without loss of generality, we set $m = 2k+2$. With this assumption, the setup in Definition 7 forces that every neighbourhood $T_{i,\rho}$ contain exactly two points, say t_i and $t_{i,\rho} := t_i + \rho$ to simplify the presentation. Then the determinant in part 1 of Definition 7 can be written as

$$M^\rho = \begin{vmatrix} F(0) & g(s_1) & \cdots & g(s_m) \\ F(t_1) & g(t_1 - s_1) & \cdots & g(t_1 - s_m) \\ F(t_{1,\rho}) & g(t_{1,\rho} - s_1) & \cdots & g(t_{1,\rho} - s_m) \\ \vdots & & & \\ F(\tau_{\underline{l}}) & g(\tau_{\underline{l}} - s_1) & \cdots & g(\tau_{\underline{l}} - s_m) \\ \vdots & & & \\ F(t_k) & g(t_k - s_1) & \cdots & g(\tau - s_m) \\ F(t_{k,\rho}) & g(t_{k,\rho} - s_1) & \cdots & g(t_{k,\rho} - s_m) \\ F(1) & g(1 - s_1) & \cdots & g(1 - s_m) \end{vmatrix}$$

$$= \begin{vmatrix} f_0 & g(s_1) & \cdots & g(s_m) \\ 0 & g(t_1 - s_1) & \cdots & g(t_1 - s_m) \\ 0 & g(t_{1,\rho} - s_1) & \cdots & g(t_{1,\rho} - s_m) \\ \vdots & & & \\ F(\tau_{\underline{l}}) & g(\tau_{\underline{l}} - s_1) & \cdots & g(\tau_{\underline{l}} - s_m) \\ \vdots & & & \\ 0 & g(t_k - s_1) & \cdots & g(\tau - s_m) \\ 0 & g(t_{k,\rho} - s_1) & \cdots & g(t_{k,\rho} - s_m) \\ f_1 & g(1 - s_1) & \cdots & g(1 - s_m) \end{vmatrix}. \qquad \text{(evaluating } F(t) \text{ as in (3.2))} \tag{4.1}$$

We will now need the following lemma in order to simplify the determinant above. The result is proved in Section 4.3.

Lemma 25. ***(Bounds for the determinant of a perturbed matrix)***
Let $A, B \in \mathbb{R}^{m\times m}$ with $m \geq 2$ and $\det(A) > 0$. If $0 \leq \epsilon \leq \frac{8}{34m\rho(A^{-1}B)}$, then

$$\det(A)\left(1 - \frac{17\sqrt{e}}{8}m\epsilon\rho(A^{-1}B)\right) \leq \det(A+\epsilon B) \leq \det(A)\left(1 + \frac{17\sqrt{e}}{8}m\epsilon\rho(A^{-1}B)\right),$$

where $\rho(X)$ is the spectral radius of the matrix X. In particular, for the stated choice of ϵ,

$$\det(A)\left(1 - \frac{\sqrt{e}}{2}\right) \leq \det(A+\epsilon B) \leq \det(A)\left(1 + \frac{\sqrt{e}}{2}\right).$$

As $\rho \to 0$, note that $g(t_{i,\rho} - s_j) = g(t_i - s_j) + \rho \cdot g'(t_i - s_j) + \frac{\rho^2}{2}g''(\xi)$, for some $\xi \in [t_i - s_j, t_i - s_j + \rho]$. After applying this expansion, we subtract the rows with

$g(t_i - s_j)$ from the rows with $g(t_{i,\rho} - s_j)$, take ρ^k outside of the determinant and we can write M^ρ as:

$$M^\rho = \rho^k \det(M_N + \rho M_P),$$

M_N is a matrix with entries independent of ρ and with determinant:

$$N = \det(M_N) = \begin{vmatrix} f_0 & g(s_1) & \cdots & g(s_m) \\ 0 & g(t_1 - s_1) & \cdots & g(t_1 - s_m) \\ 0 & g'(t_1 - s_1) & \cdots & g'(t_1 - s_m) \\ \vdots & & & \\ F(\tau_{\underline{l}}) & g(\tau_{\underline{l}} - s_1) & \cdots & g(\tau_{\underline{l}} - s_m) \\ \vdots & & & \\ 0 & g(t_k - s_1) & \cdots & g(\tau - s_m) \\ 0 & g'(t_k - s_1) & \cdots & g'(t_k - s_m) \\ f_1 & g(1 - s_1) & \cdots & g(1 - s_m) \end{vmatrix}. \tag{4.2}$$

Moreover, while the entries of M_P depend on ρ, the magnitude of each entry can be bounded from above independently of ρ. Consequently, $\|M_P\|_F$ is bounded from above independently of ρ. Let us assume for the moment that $N > 0$ and so M_N is invertible. Since $\rho(M_N^{-1}M_P) \leq \|M_N^{-1}\|_2 \|M_P\|_F$, this implies that $\rho(M_N^{-1}M_P)$ is bounded from above independently of ρ. We can then apply the stronger result of Lemma 25 to M^ρ and obtain:

$$0 < (1 - \rho C_N)\rho^k N \leq M^\rho \leq (1 + \rho C_N)\rho^k N, \tag{4.3}$$

where $C_N > 0$ is a constant that does not depend on ρ. Note that we do not need write the condition on ρ required by Lemma 25 explicitly because $\rho \to 0$ and also that (4.3) applies to the minors of M^ρ and N along the row containing $\tau_{\underline{l}}$. That N is indeed positive (and therefore we can apply Lemma 25) is established below By its definition in (3.2), $F(\tau_{\underline{l}})$ can take two values above. Either

- $F(\tau_{\underline{l}}) = 0$, which happens when there exists $i_0 \in \{1, \ldots, k\}$ such that $\tau_{\underline{l}} \in T_{i_0,\epsilon}$, namely when $\tau_{\underline{l}}$ is close to the support T. In this case, by applying the Laplace expansion to N, we find that

  $$N = f_0 \cdot N_{1,1} + f_1 \cdot N_{m+1,1}, \tag{4.4}$$

  where $N_{1,1}$ are $N_{m+1,1}$ are the corresponding minors in (4.2). Note that both $N_{1,1}$ and $N_{m+1,1}$ are positive because the Gaussian window is *extended totally positive*, see Example 5 in [50]. Recalling that $f_0, f_1 > 0$, we conclude that N is positive. Therefore, when ρ is sufficiently small, (4.2) implies that M^ρ is non-negative when $F(\tau_{\underline{l}}) = 0$. Or

- $F(\tau_{\underline{l}}) = \bar{f}$, which happens when $\tau_{\underline{l}} \in T_\epsilon^C$, namely when $\tau_{\underline{l}}$ is away from the support T. Suppose that $f_0 \gg \bar{f}$ so that N is dominated by its first minor, namely $N_{1,1}$. More precisely, by applying the Laplace expansion to N in (4.2), we find that

$$N = f_0 \cdot N_{1,1} - \bar{f} \cdot N_{\underline{l},1} + f_1 \cdot N_{m+1,1}, \tag{4.5}$$

in which all three minors are positive because the Gaussian window is extended totally positive. Also, note that $N_{\underline{l},1}$ does *not* depend on $\tau_{\underline{l}}$ and recall also that $f_0, \bar{f}, f_1$ are all positive. Therefore N in (4.5) is positive if

$$\frac{f_0}{\bar{f}} > \frac{N_{\underline{l},1}}{\min_{\tau_{\underline{l}}} N_{1,1}}, \tag{4.6}$$

where the minimum is over $\tau_{\underline{l}} \in T_\epsilon^C$. The right-hand side above is well-defined because $N_{1,1} = N_{1,1}(\tau_{\underline{l}})$ is positive for every $\tau_{\underline{l}} \in I$, $N_{1,1}(\tau_{\underline{l}})$ is a continuous function of $\tau_{\underline{l}}$, and I is compact. Indeed, $N_{1,1}(\tau_{\underline{l}})$ is positive because the Gaussian window is extended totally positive. As before, N being positive implies that M^ρ is non-negative when ρ is sufficiently small, see (4.2).

By combining both cases above, we conclude that M^ρ is non-negative for sufficiently small ρ provided that (4.6) holds, thereby verifying part 1 of Definition 7. To verify part 2 of that definition, consider the minors along the row containing $\tau_{\underline{l}}$ in M^ρ, see (4.2). Starting with the first minor along this row and applying the same arguments as before for M^ρ, we observe, after applying Lemma 25, that

$$M^\rho_{\underline{l},1} \geq (1-\rho C_{\underline{l},1})\rho^k \begin{vmatrix} g(s_1) & \cdots & g(s_m) \\ g(t_1 - s_1) & \cdots & g(t_1 - s_m) \\ g'(t_1 - s_1) & \cdots & g'(t_1 - s_m) \\ \vdots & & \vdots \\ g(t_k - s_1) & \cdots & g(t_1 - s_m) \\ g'(t_k - s_1) & \cdots & g'(t_k - s_m) \\ g(1 - s_1) & \cdots & g(1 - s_m) \end{vmatrix} =: (1-\rho C_{\underline{l},1})\rho^k \cdot N_{\underline{l},1}, \tag{4.7}$$

and also $M^\rho_{\underline{l},1} \leq (1+\rho C_{\underline{l},1})\rho^k \cdot N_{\underline{l},1}$ as $\rho \to 0$. Here $C_{\underline{l},1} > 0$ is a constant that does not depend on ρ. Moreover, $N_{\underline{l},1}$ does not depend on ρ and is positive because the Gaussian window is extended totally positive. Therefore $M^\rho_{\underline{l},1}$ in (4.7) approaches zero at the rate ρ^k.

Consider next the $(j+1)$th minor along the row containing $\tau_{\underline{l}}$ of M^ρ in (4.2), namely $M^\rho_{\underline{l},j+1}$ with $j = 1, \ldots, m$. Using the same arguments as before, we obtain after applying Lemma 25 that

$$M^{\rho}_{\underline{l},j+1} \geq$$

$$\geq (1-\rho C_{\underline{l},j+1})\rho^k \begin{vmatrix} f_0 & g(s_1) & \cdots & g(s_{j-1}) & g(s_{j+1}) & \cdots & g(s_m) \\ 0 & g(t_1-s_1) & \cdots & g(t_1-s_{j-1}) & g(t_1-s_{j+1}) & \cdots & g(t_1-s_m) \\ 0 & g'(t_1-s_1) & \cdots & g'(t_1-s_{j-1}) & g'(t_1-s_{j+1}) & \cdots & g'(t_1-s_m) \\ \vdots & & & & & & \\ 0 & g(t_k-s_1) & \cdots & g(t_k-s_{j-1}) & g(t_k-s_{j+1}) & \cdots & g(t_k-s_m) \\ 0 & g'(t_k-s_1) & \cdots & g'(t_k-s_{j-1}) & g'(t_k-s_{j+1}) & \cdots & g'(t_k-s_m) \\ f_1 & g(1-s_1) & \cdots & g(1-s_{j-1}) & g(1-s_{j+1}) & \cdots & g(1-s_m) \end{vmatrix}$$

$$= (1-\rho C_{\underline{l},j+1})\rho^k f_0 \begin{vmatrix} g(t_1-s_1) & \cdots & g\left(t_1-s_{j-1}\right) & g\left(t_1-s_{j+1}\right) & \cdots & g(t_1-s_m) \\ g'(t_1-s_1) & \cdots & g'\left(t_1-s_{j-1}\right) & g'\left(t_1-s_{j+1}\right) & \cdots & g'(t_1-s_m) \\ \vdots & & & & & \\ g(t_k-s_1) & \cdots & g\left(t_k-s_{j-1}\right) & g\left(t_k-s_{j+1}\right) & \cdots & g(t_k-s_m) \\ g'(t_k-s_1) & \cdots & g'\left(t_k-s_{j-1}\right) & g'\left(t_k-s_{j+1}\right) & \cdots & g'(t_k-s_m) \\ g(1-s_1) & \cdots & g\left(1-s_{j-1}\right) & g\left(1-s_{j+1}\right) & \cdots & g(1-s_m) \end{vmatrix}$$

$$- (1-\rho C_{\underline{l},j+1})\rho^k f_1 \begin{vmatrix} g(s_1) & \cdots & g\left(s_{j-1}\right) & g\left(s_{j+1}\right) & \cdots & g(s_m) \\ g(t_1-s_1) & \cdots & g\left(t_1-s_{j-1}\right) & g\left(t_1-s_{j+1}\right) & \cdots & g(t_1-s_m) \\ g'(t_1-s_1) & \cdots & g'\left(t_1-s_{j-1}\right) & g'\left(t_1-s_{j+1}\right) & \cdots & g'(t_1-s_m) \\ \vdots & & & & & \\ g(t_k-s_1) & \cdots & g\left(t_k-s_{j-1}\right) & g\left(t_k-s_{j+1}\right) & \cdots & g(t_k-s_m) \\ g'(t_k-s_1) & \cdots & g'\left(t_k-s_{j-1}\right) & g'\left(t_k-s_{j+1}\right) & \cdots & g'(t_k-s_m) \end{vmatrix}$$

$$=: (1-\rho C_{\underline{l},j+1})\rho^k \left(f_0 \cdot N_{\underline{l},j+1,0} - f_1 \cdot N_{\underline{l},j+1,1}\right) =: (1-\rho C_{\underline{l},j+1})\rho^k \cdot N_{\underline{l},j+1}, \tag{4.8}$$

and also $M^{\rho}_{\underline{l},j+1} \leq (1+\rho C_{\underline{l},j+1})\rho^k \cdot N_{\underline{l},j+1}$ as $\rho \to 0$, provided $N_{\underline{l},j+1} > 0$. Here, $N_{\underline{l},j+1}$ is the determinant on the first line of (4.8) and $C_{\underline{l},j+1} > 0$ is a constant independent of ρ. Note that $N_{\underline{l},j+1,0}$ and $N_{\underline{l},j+1,1}$ are both positive because the Gaussian window is extended totally positive. To ensure $N_{\underline{l},j+1} > 0$, we require $f_0 \gg f_1$, or more precisely, the following to hold.

$$\frac{f_0}{f_1} > \frac{N_{\underline{l},j+1,1}}{N_{\underline{l},j+1,0}}. \tag{4.9}$$

It then follows that $M^{\rho}_{\underline{l},j+1}$ approaches zero at the rate ρ^k for every j, thereby verifying part 2 in Definition 7 for $\tau_{\underline{l}} \in int(I)$. In conclusion, we find that $\{F\} \cup \{\phi_j\}_{j=1}^m$ form a T*-system on I with $\phi_j(t) = g(t-s_j) = e^{-\frac{(t-s_j)^2}{\sigma^2}}$ as in (3.1), provided that (4.6) and (4.9) hold.

Remark 1. While we do not require that $f_1 \gg \bar{f}$, if we impose that both $f_0 \gg \bar{f}$ and $f_1 \gg \bar{f}$, then (4.5) holds for smaller $\frac{f_0}{\bar{f}}$, so it is useful in practice.

Remark 2. From this proof and in light of the first remark, we see that we only need to specify one end point rather than both, so we only need $m = 2k+1$. However, the dual polynomial is better in practice if we have conditions at both end points (if we specify both f_0 and f_1).

4.2 Proof of Lemma 22: (Bounds on the dual certificate coefficients for Gaussian point spread function)

Recall our assumptions that

$$m = 2k+2, \qquad s_1 = 0, \quad s_m = s_{2k+2} = 1, \tag{4.10}$$

and that

$$|s_{2i} - t_i| \le \eta, \qquad s_{2i+1} - s_{2i} = \eta, \qquad \forall i = 1, \dots, k. \tag{4.11}$$

That is, we collect two samples near each impulse in x, supported on T. In addition, we make the following assumptions on η and σ:

$$\sigma \le \sqrt{2}, \quad \Delta > \sigma \sqrt{\log\left(3 + \frac{4}{\sigma^2}\right)}, \quad \eta \le \sigma^2. \tag{4.12}$$

After studying Appendix A.4, it becomes clear that the entries of $b \in \mathbb{R}^m$ are specified as

$$b_j = \lim_{\rho \to 0} (-1)^{j+1} \frac{M^\rho_{\underline{l},j+1}}{M^\rho_{\underline{l},1}} = (-1)^{j+1} \lim_{\rho \to 0} \frac{M^\rho_{\underline{l},j+1}}{M^\rho_{\underline{l},1}}, \qquad j = 1, \dots, m, \tag{4.13}$$

where the numerator and the denominator are the minors $\{M^\rho_{\underline{l},j}\}_{j=1}^{m+1}$ of M^ρ in (4.1) along the row containing $\tau_{\underline{l}}$. Using the upper and lower bounds on these quantities derived earlier, we obtain:

$$\frac{(1 - \rho C_{\underline{l},j+1}) N_{\underline{l},j+1}}{(1 + \rho C_{\underline{l},1}) N_{\underline{l},1}} \le \frac{M^\rho_{\underline{l},j+1}}{M^\rho_{\underline{l},1}} \le \frac{(1 + \rho C_{\underline{l},j+1}) N_{\underline{l},j+1}}{(1 - \rho C_{\underline{l},1}) N_{\underline{l},1}}, \qquad j = 1, \dots, m, \tag{4.14}$$

which in turn implies the following expression for b_j:

$$b_j = (-1)^{j+1} \frac{N_{\underline{l},j+1}}{N_{\underline{l},1}}, \qquad j = 1, \dots, m. \tag{4.15}$$

Recall that $N_{\underline{l},1}, N_{\underline{l},j+1} > 0$ and so, $|b_j| = \frac{N_{\underline{l},j+1}}{N_{\underline{l},1}}$ for $j = 1, \dots, m$. Therefore, in order to upper bound each $|b_j|$, we will respectively lower bound the denominator $N_{\underline{l},1}$ and upper bound the numerators $N_{\underline{l},j+1}$.

4.2.1 Bound on the denominator of the dual certificate coefficients

We now find a lower bound for the first minor, namely $N_{\underline{l},1}$ in (4.15). Let us conveniently assume that the spike locations $T = \{t_i\}_{i=1}^k$ and the sampling points $S = \{s_j\}_{j=2}^{m-1}$ are away from the boundary of interval $I = [0, 1]$, namely

$$\sigma\sqrt{\log(1/\eta^3)} \leq t_i \leq 1 - \sigma\sqrt{\log(1/\eta^3)}, \qquad \forall i = 1, \dots, k,$$

$$\sigma\sqrt{\log(1/\eta^3)} \leq s_j \leq 1 - \sigma\sqrt{\log(1/\eta^3)}, \qquad \forall j = 2, \dots, m-1. \tag{4.16}$$

In particular, (4.16) implies that

$$g(t_i) \leq \eta^3, \quad g(1 - t_i) \leq \eta^3, \qquad i = 1, \dots, k,$$

$$g(s_j) \leq \eta^3, \quad g(1 - s_j) \leq \eta^3, \qquad j = 2, \dots, m-1. \tag{4.17}$$

For the derivatives, we have that:

$$|g'(t_i)| = \frac{2t_i}{\sigma^2} g(t_i) \leq \frac{2\eta^3}{\sigma^2},$$

where we used the fact that $0 \leq t_i \leq 1$ and (4.17). Similarly, for the $1 - t_i$, s_j and $1 - s_j$:

$$|g'(t_i)| \leq \frac{2\eta^3}{\sigma^2}, \quad |g'(1 - t_i)| \leq \frac{2\eta^3}{\sigma^2}, \qquad i = 1, \dots, k,$$

$$|g'(s_j)| \leq \frac{2\eta^3}{\sigma^2}, \quad |g'(1 - s_j)| \leq \frac{2\eta^3}{\sigma^2}, \qquad j = 2, \dots, m-1. \tag{4.18}$$

With the assumptions in (4.10), (4.11), (4.17), (4.18) and the fact that $g(0) = 1$, we have that the determinant $N_{\underline{l},1}$ in (4.7) is equal to

$$N_{\underline{l},1} = \begin{vmatrix} 1 & \cdots & O(\eta^3) & O(\eta^3) & \cdots & O(\eta^3) \\ \vdots & & \vdots & \vdots & & \vdots \\ O(\eta^3) & \cdots & g(t_i - s_{2u}) & g(t_i - s_{2u+1}) & \cdots & O(\eta^3) \\ O(\eta^3/\sigma^2) & \cdots & g'(t_i - s_{2u}) & g'(t_i - s_{2u+1}) & \cdots & O(\eta^3/\sigma^2) \\ \vdots & & \vdots & \vdots & & \vdots \\ O(\eta^3) & \cdots & O(\eta^3) & O(\eta^3) & \cdots & 1 \end{vmatrix}, \tag{4.19}$$

where we wrote

$$g(t_i) = O(\eta^3) \text{ and } g'(t_i) = O(\frac{\eta^3}{\sigma^2}) \iff |g(t_i)| \leq M_1\eta^3 \text{ and } |g'(t_i)| \leq M_2\frac{\eta^3}{\sigma^2}, \tag{4.20}$$

for t_i within the bounds defined in (4.16) and some $M_1, M_2 > 0$. Here, we can take $M_1 = M_2 = 2$ and we use the same notation for $1 - t_i$, s_j and $1 - s_j$ with the same constants $M_1 = M_2 = 2$. We then take the Taylor expansion of $g(t_i - s_{2u+1})$ and $g'(t_i - s_{2u+1})$ around $t_i - s_{2u}$, subtract the columns with $t_i - s_{2u}$ from the columns where we performed the expansion, take η^k outside of the determinant and we obtain[1]:

$$N_{\underline{l},1} = \eta^k |C + \eta C'|, \tag{4.21}$$

where

$$C = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & g(t_i - s_{2u}) & -g'(t_i - s_{2u}) & \cdots & 0 \\ 0 & \cdots & g'(t_i - s_{2u}) & -g''(t_i - s_{2u}) & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{4.22}$$

$$C' = \begin{bmatrix} 0 & \cdots & O(\eta^2) & O(2\eta) & \cdots & O(\eta^2) \\ \vdots & & \vdots & \vdots & & \vdots \\ O(\eta^2) & \cdots & 0 & \frac{1}{2}g''(\xi_{i,u}) & \cdots & O(\eta^2) \\ O(\eta^2/\sigma^2) & \cdots & 0 & \frac{1}{2}g'''(\xi'_{i,u}) & \cdots & O(\eta^2/\sigma^2) \\ \vdots & & \vdots & \vdots & & \vdots \\ O(\eta^2) & \cdots & O(\eta^2) & O(2\eta) & \cdots & 0 \end{bmatrix}, \tag{4.23}$$

for some $\xi_{i,u}, \xi'_{i,u} \in [t_i - s_{2u} - \eta, t_i - s_{2u}]$ for all $i, u = 1, \ldots, k$. Note that, using the notation in (4.20), if we subtract a function that is $O(\eta)$ with constant $M_1 > 0$ from another function that is $O(\eta)$ with constant $M_2 > 0$, we obtain a function that is $O(\eta)$ with constant $M_1 + M_2$, which is why we wrote $O(2\eta)$ on the first and last columns where we subtracted two functions $O(\eta)$ with the same constant M_1 (so $O(2\eta)$ implies $\leq 2M_1\eta$). Next, we apply Taylor expansion around $t_i - t_u$ in the terms with $t_i - s_{2u}$ in C as follows:

$$g(t_i - s_{2u}) = g(t_i - t_u + t_u - s_{2u}) = g(t_i - t_u) + (t_u - s_{2u})g'(\xi^*_{i,u}), \tag{4.24}$$

for some $\xi^*_{i,u} \in [t_i - t_u - |t_u - s_{2u}|, t_i - t_u + |t_u - s_{2u}|]$ for all $i, u = 1, \ldots, k$. Note that $|t_u - s_{2u}| \leq \eta$ according to (4.11). By applying a similar Taylor expansion to g' and g'', we can write

$$C = A + \eta A', \tag{4.25}$$

[1]Note that the equality in (4.21) is due to the fact that $s_{2u+1} - s_{2u} = \eta$. If we relax this condition to (3.10), where $C_1\eta$ and $C_2\eta$ take the roles of lower and maximum upper bounds of η, we obtain:

$$C_1^k \eta^k |C + \bar{C}'| \leq N_{\underline{l},1} \leq C_2^k \eta^k |C + \bar{C}'|,$$

where $\bar{C}'$ is the same as C' in (4.23) but every entry multiplied by η or $s_{2u+1} - s_{2u} \in [C_1\eta, C_2\eta]$, so the order of its entries is the same as in $\eta C'$ in (4.21). The computations in this subsection will then follow in a similar way except that we will work with the lower bound involving $C_1^k \eta^k$ instead of the term involving η^k.

where

$$A = \begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & g(t_i - t_u) & -g'(t_i - t_u) & \cdots & 0 \\ 0 & \cdots & g'(t_i - t_u) & -g''(t_i - t_u) & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{4.26}$$

and

$$A' = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \frac{t_u - s_{2u}}{\eta} g'(\xi^*_{i,u}) & -\frac{t_u - s_{2u}}{\eta} g''(\xi^{*\prime}_{i,u}) & \cdots & 0 \\ 0 & \cdots & \frac{t_u - s_{2u}}{\eta} g''(\xi^{*\prime}_{i,u}) & -\frac{t_u - s_{2u}}{\eta} g'''(\xi^{*\prime\prime}_{i,u}) & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.27}$$

for some $\xi^*_{i,u}, \xi^{*\prime}_{i,u}, \xi^{*\prime\prime}_{i,u} \in [t_i - t_u - |t_u - s_{2u}|, t_i - t_u + |t_u - s_{2u}|]$ for all $i, u = 1, \ldots, k$. We now substitute (4.25) into (4.21) and we obtain:

$$N_{\underline{l},1} = \eta^k |A + \eta(A' + C')|, \tag{4.28}$$

Assuming for the moment that $|A| > 0$ holds, we obtain via Lemma 25 the following bound:

$$\eta^k \left(1 - \frac{\sqrt{e}}{2}\right) \det(A) \leq N_{\underline{l},1} \leq \eta^k \left(1 + \frac{\sqrt{e}}{2}\right) \det(A) \tag{4.29}$$

if

$$\eta \leq \frac{8}{34(2k+2)\rho(A^{-1}(A' + C'))}. \tag{4.30}$$

We look closer at the condition (4.30) on η in Section 4.2.3. That $|A| > 0$ (and therefore our application of Lemma 25 above is valid) is established below. We can in fact write A more compactly as follows. For scalar t, let

$$H(t) := \begin{bmatrix} g(t) & -g'(t) \\ g'(t) & -g''(t) \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{4.31}$$

which allows us to rewrite A as

$$A = \begin{bmatrix} 1 & 0_{1\times 2k} & 0 \\ 0_{2k\times 1} & B & 0_{2k\times 1} \\ 0 & 0_{1\times 2k} & 1 \end{bmatrix} \in \mathbb{R}^{m\times m}, \tag{4.32}$$

$$B := \begin{bmatrix} H(0) & H(t_1 - t_2) & H(t_1 - t_3) & \cdots & H(t_1 - t_k) \\ H(t_2 - t_1) & H(0) & H(t_2 - t_3) & \cdots & H(t_2 - t_k) \\ \vdots & & & & \\ H(t_k - t_1) & H(t_k - t_2) & H(t_k - t_3) & \cdots & H(0) \end{bmatrix} \in \mathbb{R}^{2k\times 2k}, \tag{4.33}$$

where $0_{a\times b}$ is the matrix of zeros of size $a \times b$. It follows from (4.32) that $|A| = |B|$ by Laplace expansion of $|A|$. In particular, the eigenvalues of A are: $1, 1$ and the eigenvalues of B. Let us note that B is a symmetric matrix[2], since $H(-t) = H(t)^T$; hence, A is also symmetric. We now proceed to lower bound $|B|$, the details of which are given in Section 4.4. The main observation is that $H(0)$ is a diagonal matrix while the entries of $H(t_i - t_j)$ for $i \neq j$ decay with the separation of sources Δ.

Lemma 26. ***(Lower bound on the eigenvalues of** B**)*** *Let* $\sigma \leq \sqrt{2}$, $\Delta > \sigma\sqrt{\log\left(3 + \frac{4}{\sigma^2}\right)}$ *and*

$$0 < \quad F_{\min}\left(\Delta, \frac{1}{\sigma}\right) = 1 - \left(1 + \frac{2}{\sigma^2}\right)\frac{2e^{-\frac{\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} \quad < 1.$$

Then, for each $i = 1, \ldots, 2k$, *it holds that*

$$\lambda_i(B) \geq F_{\min}\left(\Delta, \frac{1}{\sigma}\right).$$

Since $|A| = |B|$, we obtain via Lemma 26 that $|A| \geq F_{\min}\left(\Delta, \frac{1}{\sigma}\right)^{2k} > 0$. Using this in (4.29), leads to the following bound:

$$N_{\underline{l},1} \geq \eta^k \left(1 - \frac{\sqrt{e}}{2}\right) F_{\min}\left(\Delta, \frac{1}{\sigma}\right)^{2k}. \tag{4.34}$$

4.2.2 Bound on the numerator of the dual certificate coefficients

Since (4.3) holds for all the minors $M^{\rho}_{\underline{l},j}$ and $N_{\underline{l},j}$, let us now upper bound the $N_{\underline{l},j}$ for $j = 2, \ldots, m+1$ in the numerator of (4.15). Note that we distinguish two cases: $j \in \{3, \ldots, m\}$ and $j \in \{2, m+1\}$. To simplify the presentation for the first case, suppose, for example, that $j = 3$. Using the assumptions in (4.10), (4.11), (4.17), (4.18) and the fact that $g(0) = 1$,

$$N_{\underline{l},3} = \begin{vmatrix} f_0 & 1 & O(\eta^3) & \cdots & O(\eta^3) & O(\eta^3) & \cdots & O(\eta^3) \\ \vdots & & & & & & & \\ 0 & O(\eta^3) & g(t_i - s_3) & \cdots & g(t_i - s_{2u}) & g(t_i - s_{2u+1}) & \cdots & O(\eta^3) \\ 0 & O(\eta^3/\sigma^2) & g'(t_i - s_3) & \cdots & g'(t_i - s_{2u}) & g'(t_i - s_{2u+1}) & \cdots & O(\eta^3/\sigma^2) \\ \vdots & & & & & & & \\ f_1 & O(\eta^3) & O(\eta^3) & \cdots & O(\eta^3) & O(\eta^3) & \cdots & 1 \end{vmatrix}. \tag{4.35}$$

[2]Indeed, g, g'' are even functions and g' is an odd function.

We now expand $g(t_i - s_{2u+1})$ around $g(t_i - s_{2u})$, subtract the columns and take η out of the determinant as before[3]:

$$N_{\underline{l},3} = \eta^{k-1} \begin{vmatrix} f_0 & 1 & O(\eta^3) & \cdots & O(\eta^3) & O(2\eta^2) & \cdots & O(\eta^3) \\ \vdots & & & & & & & \\ 0 & O(\eta^3) & g(t_i - s_3) & \cdots & g(t_i - s_{2u}) & -g'(\xi_{i,u}) & \cdots & O(\eta^3) \\ 0 & O(\eta^3/\sigma^2) & g'(t_i - s_3) & \cdots & g'(t_i - s_{2u}) & -g''(\xi'_{i,u}) & \cdots & O(\eta^3/\sigma^2) \\ \vdots & & & & & & & \\ f_1 & O(\eta^3) & O(\eta^3) & \cdots & O(\eta^3) & O(2\eta^2) & \cdots & 1 \end{vmatrix}$$
$$=: \eta^{k-1} \det(\tilde{N}_3), \tag{4.36}$$

where $\xi_{i,u}, \xi'_{i,u} \in [t_i - s_{2u} - \eta, t_i - s_{2u}]$ and denote the matrix in (4.36) by $\tilde{N}_3$. Note the η^{k-1} since we only perform column operations on $k-1$ columns and $\det(\tilde{N}_3) > 0$ (due to the choice of f_0 and f_1). Next let $C \in \mathbb{R}^{m\times 3}$ consist of the first, second, and last columns of $\tilde{N}_3$ and $D \in \mathbb{R}^{m\times(m-3)}$ consist of the rest of the columns of $\tilde{N}_3$. Then, we may write that

$$\det([CD])^2 = \det\left(\begin{bmatrix} C^* \\ D^* \end{bmatrix} [CD]\right) = \det\left(\begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D \end{bmatrix}\right) \leq \det(C^*C)\det(D^*D), \tag{4.37}$$

where we note that swapping columns only changes the sign in a determinant (here, $\det([CD]) = -\det(\tilde{N}_3)$) and in the last inequality we applied Fischer's inequality (see, for example, Theorem 7.8.3 in [48]), which works because the matrix $[CD]^*[CD]$ is Hermitian positive definite. Therefore, we have that

$$\det(\tilde{N}_3) = |\det([CD])| \leq \det(C^*C)^{\frac{1}{2}} \det(D^*D)^{\frac{1}{2}}, \tag{4.38}$$

and it suffices to bound the determinants on the right-hand side above.

[3]Similarly to the issue addressed in footnote 1, if instead of $s_{2u+1} - s_{2u} = \eta$ we have (3.10), then:

$$C_1^{k-1}\eta^{k-1}\det(\tilde{N}_3) \leq N_{\underline{l},3} \leq C_2^{k-1}\eta^{k-1}\det(\tilde{N}_3),$$

and then the proof will continue in a similar way except that we work with the upper bound of $N_{\underline{l},3}$ above and some of the calculations will involve C_1 and C_2 as well.

Bounding $\det(C^*C)$

We now write C as follows:

$$C = \begin{bmatrix} f_0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ f_1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & O(\eta^3) \\ \vdots & \vdots & \vdots \\ 0 & O(\eta^3) & O(\eta^3) \\ 0 & O(\eta^3/\sigma^2) & O(\eta^3/\sigma^2) \\ \vdots & \vdots & \vdots \\ 0 & O(\eta^3) & 0 \end{bmatrix} =: X + \widetilde{Z}, \tag{4.39}$$

where we denote the first matrix by X and the second matrix by $\widetilde{Z}$. We have that

$$\det(C^*C) = \det((X + \widetilde{Z})^*(X + \widetilde{Z})) = \det(X^*X + \widetilde{Z}'), \tag{4.40}$$

with $\widetilde{Z}' = X^*\widetilde{Z} + \widetilde{Z}^*X + \widetilde{Z}^*\widetilde{Z}$ and we apply Weyl's inequality to C^*C to obtain:

$$\det(C^*C) \leq (\lambda_1(X^*X) + \lambda_{\max}(\widetilde{Z}'))(\lambda_2(X^*X) + \lambda_{\max}(\widetilde{Z}'))(\lambda_3(X^*X) + \lambda_{\max}(\widetilde{Z}')). \tag{4.41}$$

Then

$$X^*X = \begin{bmatrix} f_0^2 + f_1^2 & f_0 & f_1 \\ f_0 & 1 & 0 \\ f_1 & 0 & 1 \end{bmatrix} \quad \text{with} \quad \lambda_3(X^*X) = 0, \tag{4.42}$$

and

$$\|X^*X\|_F \leq \bar{C}(f_0, f_1), \quad \text{where} \quad \bar{C}(f_0, f_1) = f_0^2 + f_1^2 + 2f_0 + 2f_1 + 2, \tag{4.43}$$

so

$$\|X\|_2^2 = \|X^*X\|_2 \leq \|X^*X\|_F \leq \bar{C}(f_0, f_1) \quad \text{so} \quad \|X\|_2 = \sqrt{\bar{C}(f_0, f_1)} \tag{4.44}$$

and

$$\begin{aligned} \|\widetilde{Z}'\|_2 &= \|X^*\widetilde{Z} + \widetilde{Z}^*X + \widetilde{Z}^*\widetilde{Z}\|_2 \\ &\leq 2\|X\|_2\|\widetilde{Z}\|_2 + \|\widetilde{Z}\|_2^2 = \left(2\sqrt{\bar{C}(f_0, f_1)} + \|\widetilde{Z}\|_2\right)\|\widetilde{Z}\|_2 \\ &\leq 3\sqrt{\bar{C}(f_0, f_1)}\|\widetilde{Z}\|_2, \end{aligned} \tag{4.45}$$

where the last inequality holds if

$$\|\widetilde{Z}\|_2 \leq \sqrt{\bar{C}(f_0, f_1)}. \tag{4.46}$$

Noting that $\widetilde{Z}'$ is symmetric and $\lambda_{\max}(\widetilde{Z}') \leq \|\widetilde{Z}'\|_2$, we substitute (4.42) and (4.45) into (4.41) and obtain:

$$\det(C^*C) \leq \left(\bar{C}(f_0, f_1) + 3\sqrt{\bar{C}(f_0, f_1)}\|\widetilde{Z}\|_2\right)^2 3\sqrt{\bar{C}(f_0, f_1)}\|\widetilde{Z}\|_2, \tag{4.47}$$

if (4.46) holds. Further applying (4.46) in the parentheses, we obtain:

$$\det(C^*C) \leq 48\bar{C}(f_0, f_1)^{\frac{5}{2}}\|\widetilde{Z}\|_2. \tag{4.48}$$

Now, using (4.20) with $M_1 = M_2 = 2$, we are able to upper bound $\|\widetilde{Z}\|_F$, which is also an upper bound for $\|\widetilde{Z}\|_2$:

$$\begin{aligned}\|\widetilde{Z}\|_2 \leq \|\widetilde{Z}\|_F &\leq \sqrt{(2k+2)(2\eta^3)^2 + 2k\left(\frac{2\eta^3}{\sigma^2}\right)^2} \\ &\leq (2k+2)2\eta^3 + 2k\frac{2\eta^3}{\sigma^2} = \eta^3\left(4k + 4 + \frac{4k}{\sigma^2}\right).\end{aligned} \tag{4.49}$$

Therefore, to satisfy (4.46), it is sufficient to find η such that:

$$\eta^3\left(4k + 4 + \frac{4k}{\sigma^2}\right) \leq \sqrt{\bar{C}(f_0, f_1)}. \tag{4.50}$$

With this choice of η, by substituting (4.49) into (4.48), we obtain:

$$\det(C^*C) \leq 48\bar{C}(f_0, f_1)^{\frac{5}{2}}\left(4k + 4 + \frac{4k}{\sigma^2}\right)\eta^3. \tag{4.51}$$

Bounding $\det(D^*D)$

Then, since D^*D is Hermitian positive definite, we can apply Hadamard's inequality (Theorem 7.8.1 in [48]) to bound its determinant by the product of its main diagonal entries (i.e. the squared 2-norms of the columns of D), so we obtain, after we use (4.20) with $M_1 = M_2 = 2$:

$$\begin{aligned}\det(D^*D) \leq &\left(8\eta^6 + \sum_{i=1}^k g(t_i - s_3)^2 + \sum_{i=1}^k g'(t_i - s_3)^2\right) \\ &\cdot \prod_{u=2}^k \left(8\eta^6 + \sum_{i=1}^k g(t_i - s_{2u})^2 + \sum_{i=1}^k g'(t_i - s_{2u})^2\right) \\ &\cdot \prod_{u=2}^k \left(32\eta^4 + \sum_{i=1}^k g'(\xi_{i,u})^2 + \sum_{i=1}^k g''(\xi'_{i,u})^2\right),\end{aligned} \tag{4.52}$$

where $\xi_{i,u}, \xi'_{i,u} \in [t_i - s_{2u} - \eta, t_i - s_{tu}]$.

For fixed $u \in \{1, \ldots, k\}$, we may write that

$$\begin{aligned} \sum_{i=1}^{k} g(t_i - s_{2u})^2 \leq 2 \sum_{i=0}^{\infty} g(i\Delta)^2 \leq 2 \sum_{i=0}^{\infty} g(i\Delta) = 2 \sum_{i=0}^{\infty} e^{-\frac{i^2\Delta^2}{\sigma^2}} \\ \leq 2 \sum_{i=0}^{\infty} \left(e^{-\frac{\Delta^2}{\sigma^2}} \right)^i = \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}. \end{aligned} \tag{4.53}$$

To see the inequality above, if we note that $s_{2i} \leq t_i \leq s_{2i+1}$, we have that

$$\begin{aligned} g(t_{u-1} - s_{2u}) \leq g(0), \qquad & g(t_u - s_{2u}) \leq g(0), \\ g(t_{u-2} - s_{2u}) \leq g(\Delta), \qquad & g(t_{u+1} - s_{2u}) \leq g(\Delta), \\ \vdots \qquad & \vdots \end{aligned}$$

and by adding the inequalities we obtain (4.53). Likewise, it holds that

$$\sum_{i=1}^{k} g'(t_i - s_{2u})^2 \leq \frac{4}{\sigma^4} \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}},$$

$$\sum_{i=1}^{k} g''(t_i - s_{2u})^2 \leq \left(\frac{2}{\sigma^2} + \frac{4}{\sigma^4} \right)^2 \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}. \tag{4.54}$$

and, for $\xi_{i,u}, \xi'_{u,u} \in [t_i - s_{2u} - \eta, t_i - s_{2u}]$:

$$\sum_{i=1}^{k} g'(\xi_{i,u})^2 \leq \frac{4}{\sigma^4} \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}},$$

$$\sum_{i=1}^{k} g''(\xi'_{i,u})^2 \leq \left(\frac{2}{\sigma^2} + \frac{4}{\sigma^4} \right)^2 \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}. \tag{4.55}$$

Note that we obtained (4.55) in the same way as (4.53),

$$\begin{aligned} g(\xi_{u,u}) \leq g(0) \qquad & g(\xi_{u-1,u}) \leq g(0) \\ g(\xi_{u+1,u}) \leq g(\Delta) \qquad & g(\xi_{u-2,u}) \leq g(\Delta) \\ g(\xi_{u+2,u}) \leq g(2\Delta) \qquad & g(\xi_{u-3,u}) \leq g(2\Delta) \\ \vdots \qquad & \vdots \end{aligned}$$

Lastly, the above bounds also hold if we have $g(t_i - s_{2u+1})$ instead of $g(t_i - s_{2u})$. Substituting these bounds back into (4.52) and using (4.20) with $M_1 = M_2 = 2$, we obtain:

$$
\begin{aligned}
\det(D^*D) \leq & \left[8\eta^6 + \left(1+\frac{4}{\sigma^4}\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right] \\
& \cdot \prod_{u=2}^{k}\left[8\eta^6 + \left(1+\frac{4}{\sigma^4}\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right] \\
& \cdot \prod_{u=2}^{k}\left[32\eta^4 + \left(\frac{4}{\sigma^4}+\left(\frac{2}{\sigma^2}+\frac{4}{\sigma^4}\right)^2\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right] \\
\leq & \left[8 + \left(1+\frac{4}{\sigma^4}\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right]^k \left[32 + \left(\frac{4}{\sigma^4}+\left(\frac{2}{\sigma^2}+\frac{4}{\sigma^4}\right)^2\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right]^{k-1} \\
= & F_1\left(\Delta,\frac{1}{\sigma}\right)^k \cdot F_2\left(\Delta,\frac{1}{\sigma}\right)^{k-1}, \qquad (4.56)
\end{aligned}
$$

where

$$
\begin{aligned}
F_1\left(\Delta,\frac{1}{\sigma}\right) &= 8 + \left(1+\frac{4}{\sigma^4}\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}, \\
F_2\left(\Delta,\frac{1}{\sigma}\right) &= 32 + \left(\frac{1}{\sigma^4}+\frac{2}{\sigma^6}+\frac{2}{\sigma^8}\right)\frac{16}{1-e^{-\frac{\Delta^2}{\sigma^2}}}. \qquad (4.57)
\end{aligned}
$$

Note that the bound (4.56) on $\det(D^*D)$ is the same for all $j = 3, \ldots, m$. Combining (4.51) with (4.56) in (4.36), we obtain:

$$
N_{\underline{l},j} \leq \eta^{k+\frac{1}{2}} C(f_0, f_1)^{\frac{1}{2}} \left(4k + 4 + \frac{4k}{\sigma^2}\right)^{\frac{1}{2}} F_1\left(\Delta,\frac{1}{\sigma}\right)^{\frac{k}{2}} F_2\left(\Delta,\frac{1}{\sigma}\right)^{\frac{k-1}{2}} \qquad (4.58)
$$

for $j = 3, \ldots, m$ if (4.50) holds.

Finally, we need to upper bound $N_{\underline{l},j}$ for $j = 2$ and $j = m+1$. For simplicity, consider $j = 2$. Applying the same assumptions and operations as in (4.36), we have

$$
N_{\underline{l},2} = \eta^k \begin{vmatrix}
f_0 & O(\eta^3) & \cdots & O(\eta^3) & O(2\eta^2) & \cdots & O(\eta^3) \\
\vdots & & & & & & \\
0 & g(t_i - s_2) & \cdots & g(t_i - s_{2u}) & -g'(\xi_{i,u}) & \cdots & O(\eta^3) \\
0 & g'(t_i - s_2) & \cdots & g'(t_i - s_{2u}) & -g''(\xi'_{i,u}) & \cdots & O(\eta^3/\sigma^2) \\
\vdots & & & & & & \\
f_1 & O(\eta^3) & \cdots & O(\eta^3) & O(2\eta^2) & \cdots & 1
\end{vmatrix}, \qquad (4.59)
$$

where $\xi_{i,u}, \xi'_{i,u} \in [t_i - s_{2u} - \eta, t_i - s_{2u}]$. We bound $N_{\underline{l},2}$ by using Hadamard's inequality (the more general version, see [43]) and, after we use (4.20) with $M_1 = M_2 = 2$, we obtain:

$$N_{\underline{l},2} \leq \eta^k \sqrt{f_0^2 + f_1^2} \left[1 + 4(k+1)\eta^6 + 4k\frac{\eta^6}{\sigma^4}\right]^{\frac{1}{2}}$$

$$\cdot \prod_{u=1}^{k} \left[8\eta^6 + \sum_{i=1}^{k} g(t_i - s_{2u})^2 + \sum_{i=1}^{k} g'(t_i - s_{2u})^2\right]^{\frac{1}{2}}$$

$$\cdot \prod_{u=1}^{k} \left[32\eta^4 + \sum_{i=1}^{k} g'(\xi_{i,u})^2 + \sum_{i=1}^{k} g''(\xi'_{i,u})^2\right]^{\frac{1}{2}}, \tag{4.60}$$

and, by applying the bounds on the sums and $\eta \leq 1$, we obtain:

$$N_{\underline{l},2} \leq \eta^k \sqrt{f_0^2 + f_1^2} \left(4k + 5 + \frac{4k}{\sigma^4}\right)^{\frac{1}{2}} F_1\left(\Delta, \frac{1}{\sigma}\right)^{\frac{k}{2}} F_2\left(\Delta, \frac{1}{\sigma}\right)^{\frac{k}{2}}, \tag{4.61}$$

for F_1 and F_2 defined as in (4.57), and note that the same bound also holds for $N_{\underline{l},m+1}$.

To conclude, from (4.58) and (4.61), we can derive a general bound valid for all j:

$$N_{\underline{l},j} \leq \eta^k C(f_0, f_1)^{\frac{1}{2}} \left(4k + 5 + \frac{4k}{\sigma^4}\right)^{\frac{1}{2}} F_{\max}\left(\Delta, \frac{1}{\sigma}\right)^{k} \tag{4.62}$$

for all $j = 2, \ldots, m+1$ if (4.50) holds, where

$$F_{\max}\left(\Delta, \frac{1}{\sigma}\right) = \left(8 + \left(1 + \frac{4}{\sigma^4}\right)\frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}\right)^{\frac{1}{2}} \left(32 + \left(\frac{1}{\sigma^4} + \frac{2}{\sigma^6} + \frac{2}{\sigma^8}\right)\frac{16}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}\right)^{\frac{1}{2}}. \tag{4.63}$$

4.2.3 Condition on sample proximity

We now return to the condition (4.30) that η must satisfy so that our application of Lemma 25 is valid. Since A is positive definite, we have $\left\|A^{-1}\right\|_2 \leq 1/\lambda_{\min}(A)$. Using this, we obtain:

$$\rho(A^{-1}(A'+C')) \leq \|A^{-1}(A'+C')\|_2 \leq \|A^{-1}\|_2\|A'+C'\|_2 \leq \frac{1}{\lambda_{\min}(A)} \cdot \|A'+C'\|_F, \tag{4.64}$$

and

$$A' + C' = \begin{bmatrix} 0 & \cdots & O(\eta^2) & O(2\eta) & \cdots & O(\eta^2) \\ \vdots & & \vdots & \vdots & & \vdots \\ O(\eta^2) & \cdots & \frac{t_u - s_{2u}}{\eta} g'(\xi^*_{i,u}) & -\frac{t_u - s_{2u}}{\eta} g''(\xi^{*\prime}_{i,u}) + \frac{1}{2} g''(\xi_{i,u}) & \cdots & O(\eta^2) \\ O(\eta^2/\sigma^2) & \cdots & \frac{t_u - s_{2u}}{\eta} g''(\xi^{*\prime}_{i,u}) & -\frac{t_u - s_{2u}}{\eta} g'''(\xi^{*\prime\prime}_{i,u}) + \frac{1}{2} g'''(\xi'_{i,u}) & \cdots & O(\eta^2/\sigma^2) \\ \vdots & & \vdots & \vdots & & \vdots \\ O(\eta^2) & \cdots & O(\eta^2) & O(2\eta) & \cdots & 0 \end{bmatrix}, \tag{4.65}$$

where

$$\xi_{i,u}, \xi'_{i,u} \in [t_i - s_{2u} - \eta, t_i - s_{2u}],$$
$$\xi^*_{i,u}, \xi^{*\prime}_{i,u}, \xi^{*\prime\prime}_{i,u} \in [t_i - t_u - |t_u - s_{2u}|, t_i - t_u + |t_u - s_{2u}|]. \tag{4.66}$$

for all $i, u = 1, \ldots, k$.

Because the eigenvalues of A are $1, 1$ and the eigenvalues of B and $\lambda_{\min}(B) \geq F_{\min}(\Delta, \frac{1}{\sigma})$ with $0 < F_{\min}(\Delta, \frac{1}{\sigma}) < 1$, then we have that that $\lambda_{\min}(A) \geq F_{\min}\left(\Delta, \frac{1}{\sigma}\right)$. Moreover, after applying (4.20) with $M_1 = M_2 = 2$, we have that:

$$\begin{aligned}
\|A' + C'\|_F^2 \leq\ & 8(k+1)\eta^4 + 8k\frac{\eta^4}{\sigma^4} \qquad \text{(from the first and last columns)} \\
& + 8k\eta^4 + 32k\eta^2 + \qquad \text{(from the first and last rows)} \\
& + \sum_{i,u=1}^{k} g'(\xi^*_{i,u})^2 + \sum_{i,u=1}^{k} g''(\xi^{*\prime}_{i,u})^2 \\
& + \sum_{i,u=1}^{k} \left(-\frac{t_u - s_{2u}}{\eta} g''(\xi^{*\prime}_{i,u}) + \frac{1}{2} g''(\xi_{i,u}) \right)^2 \\
& + \sum_{i,u=1}^{k} \left(-\frac{t_u - s_{2u}}{\eta} g'''(\xi^{*\prime\prime}_{i,u}) + \frac{1}{2} g'''(\xi'_{i,u}) \right)^2 .
\end{aligned} \tag{4.67}$$

We upper bound this using the inequalities in (4.55) and, similarly, for g''':

$$\sum_{i=1}^{k} g'''(\xi_{i,u})^2 \leq \left(\frac{12}{\sigma^4} + \frac{8}{\sigma^6} \right)^2 \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}, \tag{4.68}$$

and note that these inequalities hold for all numbers ξ in (4.66). By expanding the parentheses and applying Cauchy-Schwartz to the products, and using that $|t_u - s_{2u}| \leq \eta$, we obtain:

$$\begin{aligned}
\|A' + C'\|_F^2 \leq\ & 8(2k+1)\eta^4 + 32k\eta^2 + 8k\frac{\eta^4}{\sigma^4} \\
& + k \cdot \frac{4}{\sigma^4} \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} + k \cdot \left(\frac{2}{\sigma^2} + \frac{4}{\sigma^4} \right)^2 \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} \\
& + \frac{9k}{4} \cdot \left(\frac{2}{\sigma^2} + \frac{4}{\sigma^4} \right)^2 \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} \\
& + \frac{9k}{4} \cdot \left(\frac{12}{\sigma^4} + \frac{8}{\sigma^6} \right)^2 \cdot \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}
\end{aligned} \tag{4.69}$$

Using the assumption that $\eta \leq \sigma^2$ to write $\frac{\eta^4}{\sigma^4} \leq \sigma^4$, and then by applying $\eta \leq 1$ and $\sigma \leq \sqrt{2}$, we can write

$$\|A' + C'\|_F^2 \leq 80k + 8 + kP\left(\frac{1}{\sigma}\right) \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}, \tag{4.70}$$

where $P\left(\frac{1}{\sigma}\right)$ is a polynomial in $\frac{1}{\sigma}$ defined as follows:

$$P\left(\frac{1}{\sigma}\right) = \frac{4}{\sigma^4} + \frac{13}{4}\left(\frac{2}{\sigma^2} + \frac{4}{\sigma^4}\right)^2 + \frac{9}{4}\left(\frac{12}{\sigma^4} + \frac{8}{\sigma^6}\right)^2. \tag{4.71}$$

Inserting the above observations in the condition (4.30), we finally obtain

$$\eta \leq \frac{8F_{\min}(\Delta, \frac{1}{\sigma})}{34(2k+2)\left(80k + 8 + kP\left(\frac{1}{\sigma}\right)\frac{2}{1-e^{-\frac{\Delta^2}{\sigma^2}}}\right)^{\frac{1}{2}}}. \tag{4.72}$$

With this choice of η, the condition (4.30) is satisfied.

4.2.4 Bound on dual certificate coefficients

We now give a final bound for (4.13). Combining the results from 4.2.1 and 4.2.2, we arrive at

$$\begin{aligned} |b_j| &= \frac{N_{\underline{l},j+1}}{N_{\underline{l},1}} \qquad \text{(see (4.15))} \\ &\leq \frac{\bar{C}(f_0, f_1)^{\frac{5}{4}}\left(4k + 5 + \frac{4k}{\sigma^4}\right)^{\frac{1}{2}}}{1 - \frac{\sqrt{e}}{2}}\left[\frac{F_{\max}\left(\Delta, \frac{1}{\sigma}\right)}{F_{\min}\left(\Delta, \frac{1}{\sigma}\right)^2}\right]^k, \text{(see (4.62),(4.34))} \end{aligned} \tag{4.73}$$

for every $j = 1, \ldots, m$, provided that the conditions (4.50) and (4.72) hold. Consequently,

$$\|b\|_2 = \sqrt{\sum_{j=1}^{m} b_j^2} \leq \frac{\sqrt{(2k+2)\left(4k + 5 + \frac{4k}{\sigma^4}\right)}}{1 - \frac{\sqrt{e}}{2}}\bar{C}(f_0, f_1)^{\frac{5}{4}}F\left(\Delta, \frac{1}{\sigma}\right)^k, \tag{4.74}$$

where we write $F\left(\Delta, \frac{1}{\sigma}\right) = \frac{F_{\max}\left(\Delta, \frac{1}{\sigma}\right)}{F_{\min}\left(\Delta, \frac{1}{\sigma}\right)^2}$ if, $\sigma \leq \sqrt{2}$ and $\Delta > \sigma\sqrt{\log\left(3 + \frac{4}{\sigma^2}\right)}$ and the conditions (4.50) and (4.72) hold. This completes the proof of Lemma 22 since $m = 2k + 2$ by assumption.

4.3 Proof of Lemma 25 (Bounds for the determinant of a perturbed matrix)

To begin with, let us note that

$$\det(A + \epsilon B) = \det(A)\det(I + \epsilon A^{-1}B)$$

We will now upper and lower bound the second term in this equality. Denoting $C = A^{-1}B$, consider

$$\log\det(I+\epsilon C) = \sum_{i\in R}\log(1+\epsilon\lambda_i(C)) + \sum_{i\in I}\log(1+2\epsilon\operatorname{Re}(\lambda_i(C))+\epsilon^2|\lambda_i(C)|^2),$$

where

$R = \{i = 1, \ldots, m : \operatorname{Im}(\lambda_i(C)) = 0\}$ and
$I = \{i = 1, \ldots, m : \lambda_{i_1}(C), \lambda_{i_2}(C)$ are complex conjugate and $\operatorname{Im}(\lambda_{i_1}(C)) \neq 0\}$.

1. For $i \in R$, if $\epsilon < \frac{1}{|\lambda_i(C)|}$, use apply Taylor expansion for $\log(1+x)$ and obtain:

$$\log(1+\epsilon\lambda_i(C)) = \epsilon\lambda_i(C) - \frac{\epsilon^2\lambda_i^2(C)}{2\xi_{i,\epsilon}^2}, \qquad \text{where } \xi_{i,\epsilon} \in \left[1-\epsilon|\lambda_i(C)|, 1+\epsilon|\lambda_i(C)|\right]. \tag{4.75}$$

2. For $i \in C$, we apply the same Taylor expansion and, writing $y = 2\epsilon\operatorname{Re}(\lambda_i(C)) + \epsilon^2|\lambda_i(C)|^2$, for $|y| < 1$ we obtain:

$$\log(1+y) = \frac{y}{\xi_{i,\epsilon}}, \qquad \text{where } \xi_{i,\epsilon} \in \left[1-|y|, 1+|y|\right].$$

   Then, we have that

$$2\epsilon|\lambda_i(C)| + \epsilon^2|\lambda_i(C)|^2 \leq 4\epsilon|\lambda_i(C)| < 1,$$

   where the first inequality is true if $\epsilon \leq \frac{2}{|\lambda_i(C)|}$ and the second inequality is true if $\epsilon < \frac{1}{4|\lambda_i(C)|}$. From the condition on $\xi_{i,\epsilon}$ and noting that $|\operatorname{Re}(\lambda_i(C))| \leq |\lambda_i(C)|$, we have that

$$\xi_{i,\epsilon} \leq 1+|y| \leq 1+4\epsilon|\lambda_i(C)| \qquad \text{and} \qquad \xi_{i,\epsilon} \geq 1-|y| \geq 1-4\epsilon|\lambda_i(C)| \geq \frac{1}{2}, \tag{4.76}$$

   where the last inequality holds if $\epsilon \leq \frac{1}{8|\lambda_i(C)|}$. Therefore, $\frac{1}{\xi_{i,\epsilon}} \leq 2$ if $\epsilon \leq \frac{1}{8|\lambda_i(C)|}$.

We now use (4.75), (4.76) and $\epsilon \leq \frac{1}{8|\lambda_i(C)|}$. Let $\epsilon \leq \frac{1}{8|\lambda_i(C)|}$ for all eigenvalues $\lambda_i(C)$, then:

$$\log\det(I+\epsilon C) = \sum_{i\in R}\epsilon\lambda_i(C) - \frac{\epsilon^2}{2}\sum_{i\in R}\frac{\lambda_i^2(C)}{\xi_{i,\epsilon}^2} + \sum_{i\in I}\frac{1}{\xi_{i,\epsilon}}\left(2\epsilon\operatorname{Re}(\lambda_i(C)) + \epsilon^2|\lambda_i(C)|^2\right), \tag{4.77}$$

and, by using that $\xi_{i,\epsilon} \geq 1 - \epsilon|\lambda_i(C)| \geq \frac{7}{8}$ for $i \in R$ and $\frac{1}{\xi_{i,\epsilon}} \leq 2$ for $i \in C$ and the fact that the index $i \in I$ accounts for two eigenvalues, we obtain:

$$\begin{aligned}
|\log\det(I+\epsilon C)| &\leq \epsilon \sum_{i\in R} |\lambda_i(C)| + \frac{32\epsilon^2}{49} \sum_{i\in R} |\lambda_i(C)|^2 + 4\epsilon \sum_{i\in I} |\lambda_i(C)| + 2\epsilon^2 \sum_{i\in I} |\lambda_i(C)|^2 \\
&\leq 2\epsilon \sum_{i=1}^{m} |\lambda_i(C)| + \epsilon^2 \sum_{i=1}^{m} |\lambda_i(C)|^2 \\
&\leq m\epsilon\rho(C)(2+\epsilon\rho(C)) \\
&\leq m\epsilon\rho(C)(2+\frac{1}{8}) = \frac{17}{8} m\epsilon\rho(C) \\
&\leq \frac{1}{2},
\end{aligned} \tag{4.78}$$

where the second last inequality holds if $\epsilon \leq \frac{1}{8\rho(C)}$ and the last inequality holds if $\epsilon \leq \frac{8}{34m\rho(C)}$, which is also the dominating condition for ϵ in the proof if $m \geq 2$.

Now, note that for $|x| \leq \frac{1}{2}$, we have that $e^x = 1 + xe^{\xi}$ for some $\xi \in [-|x|, |x|] \subseteq [-\frac{1}{2}, \frac{1}{2}]$ by Taylor expansion, and by taking $x = \log\det(I+\epsilon C)$, we obtain:

$$\det(I+\epsilon C) = e^x \leq 1 + |x|e^{\frac{1}{2}} \leq 1 + \frac{17\sqrt{e}}{8} m\epsilon\rho(C) \leq 1 + \frac{\sqrt{e}}{2},$$

and similarly

$$\det(I+\epsilon C) = e^x \geq 1 - |x|e^{\frac{1}{2}} \geq 1 - \frac{17\sqrt{e}}{8} m\epsilon\rho(C) \geq 1 - \frac{\sqrt{e}}{2}.$$

From the last two inequalities, by multiplying by $\det(A)$, we obtain the result of our lemma.

4.4 Proof of Lemma 26 (Eigenvalues lower bound)

Recalling the definition of B in (4.33), we apply Gershgorin disc theorem to find the discs $D(a_{ii}, \sum_{i\neq j} |a_{ij}|)$ which contain the eigenvalues of B. Due to the structure of H, we consider two cases:

1. On odd rows, the centre of the disc is $a_{\text{odd}_{ii}} = g(0) = 1$ and the radius is

$$\begin{aligned}
R_{\text{odd}_i} &= \sum_{\substack{j=1\\j\neq i}}^{k} |g(t_i - t_j)| + |-g'(t_i - t_j)| && \text{(4.79a)} \\
&= \sum_{\substack{j=1\\j\neq i}}^{k} g(t_i - t_j)\left(1 + \frac{2|t_i - t_j|}{\sigma^2}\right), \qquad \text{for} \quad i = 1,\ldots,k. && \text{(4.79b)}
\end{aligned}$$

2. On even rows, the centre of the disc is $a_{\text{even}_{ii}} = -g''(0) = \frac{2}{\sigma^2}$ and the radius is

$$R_{\text{even}_i} = \sum_{\substack{j=1\\ j\neq i}}^{k} |g'(t_i - t_j)| + |-g''(t_i - t_j)| \tag{4.80a}$$

$$= \sum_{\substack{j=1\\ j\neq i}}^{k} g(t_i - t_j)\left(\frac{2|t_i - t_j|}{\sigma^2} + \left|-\frac{2}{\sigma^2} + \frac{4(t_i - t_j)^2}{\sigma^4}\right|\right), \quad \text{for} \quad i = 1, \ldots, k. \tag{4.80b}$$

Since the eigenvalues of B are real, they are lower bounded by

$$\min_{i=1,\ldots,k} 1 - R_{\text{odd}_i} \quad \text{or} \quad \min_{i=1,\ldots,k} \frac{2}{\sigma^2} - R_{\text{even}_i}. \tag{4.81}$$

Because $|t_i - t_j| \leq 1$, we have that

$$1 - R_{\text{odd}_i} \geq 1 - \left(1 + \frac{2}{\sigma^2}\right) \sum_{\substack{j=1\\ j\neq i}}^{k} g(t_i - t_j) \tag{4.82}$$

Since $\frac{4(t_i-t_j)^2}{\sigma^4} - \frac{2}{\sigma^2} > \frac{4\Delta^2}{\sigma^4} - \frac{2}{\sigma^2} > 0$ due to our assumptions on σ (see (4.12)), we obtain

$$\frac{2}{\sigma^2} - R_{\text{even}_i} \geq \frac{2}{\sigma^2} - \frac{4}{\sigma^4} \sum_{\substack{j=1\\ j\neq i}}^{k} g(t_i - t_j), \tag{4.83}$$

for all $i = 1, \ldots, k$. Using the fact that g is decreasing and $|t_i - t_j| \geq |i - j|\Delta$, we obtain

$$\sum_{\substack{j=1\\ j\neq i}}^{k} g(t_i - t_j) \leq \sum_{\substack{j=1\\ j\neq i}}^{k} g(|i - j|\Delta) \tag{4.84a}$$

$$\leq 2\sum_{j=1}^{\infty} g(j\Delta) = 2\sum_{j=1}^{\infty} e^{-\frac{j^2\Delta^2}{\sigma^2}} \leq 2\sum_{j=1}^{\infty}\left(e^{-\frac{\Delta^2}{\sigma^2}}\right)^j \tag{4.84b}$$

$$= \frac{2e^{-\frac{\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}, \qquad \text{for } i = 1, \ldots, k. \tag{4.84c}$$

The sum of of the series is valid because $e^{-\frac{\Delta^2}{\sigma^2}} < 1$. Combining this with (4.82) and (4.83), we obtain:

$$1 - R_{\text{odd}_i} \geq 1 - \left(1 + \frac{2}{\sigma^2}\right) \frac{2e^{-\frac{\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}, \tag{4.85a}$$

$$\frac{2}{\sigma^2} - R_{\text{even}_i} \geq \frac{2}{\sigma^2} - \frac{4}{\sigma^4} \frac{2e^{-\frac{\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}, \tag{4.85b}$$

for all $i = 1, \dots, k$. By using the assumption that $\Delta > \sigma\sqrt{\log\left(3 + \frac{4}{\sigma^2}\right)}$, we can check that the lower bound in (4.85a) is greater than zero, and by also using that $\sigma \leq \sqrt{2}$, we can check that it is smaller than than the lower bound in (4.85b).

To conclude, since all the eigenvalues of the matrix B are real and in the union of the discs $D(1, R_{\text{odd}_i})$ and $D(\frac{2}{\sigma^2}, R_{\text{even}_i})$ for $i = 1, \dots, k$, then, by using the above observation and the lower bound in (4.85a), we obtain a lower bound of all the eigenvalues of B:

$$\lambda_j(B) \geq 1 - \left(1 + \frac{2}{\sigma^2}\right) \frac{2e^{-\frac{\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}}, \qquad \forall j = 1, \dots, 2k. \tag{4.86}$$

Note that we may be able to obtain better bounds given by (4.85b) for k eigenvalues if we scale the Gershgorin discs so that $D(1, R_{\text{odd}_i})$ and $D(1, R_{\text{even}_i})$ become disjoint.

4.5 Proof of Lemma 23 (Simplified bounds for Gaussian point spread function)

Because $\Delta > \sigma\sqrt{\log \frac{5}{\sigma^2}}$, we have that

$$e^{-\frac{\Delta^2}{\sigma^2}} < \frac{\sigma^2}{5}, \tag{4.87}$$

which implies that

$$\frac{1}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} < \frac{5}{5 - \sigma^2}. \tag{4.88}$$

Then, $-\frac{2e^{-\frac{\Delta^2}{\sigma^2}}}{1-e^{-\frac{\Delta^2}{\sigma^2}}} > -\frac{2\sigma^2}{5-\sigma^2}$, so

$$1 - (1 + \frac{2}{\sigma^2})\frac{2e^{-\frac{\Delta^2}{\sigma^2}}}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} > \frac{1 - 3\sigma^2}{5 - \sigma^2}, \quad \text{i.e.} \quad F_{\min}\left(\Delta, \frac{1}{\sigma}\right) > \frac{1 - 3\sigma^2}{5 - \sigma^2}. \tag{4.89}$$

Similarly, we have that:

$$8 + \left(1 + \frac{4}{\sigma^4}\right) \frac{2}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} < \frac{-8\sigma^6 + 50\sigma^4 + 40}{\sigma^4(5 - \sigma^2)} < \frac{98}{\sigma^4(5 - \sigma^2)} \tag{4.90}$$

and

$$\begin{aligned} 32 + \left(\frac{1}{\sigma^4} + \frac{2}{\sigma^6} + \frac{2}{\sigma^8}\right) \frac{16}{1 - e^{-\frac{\Delta^2}{\sigma^2}}} &< \frac{-32\sigma^{10} + 160\sigma^8 + 80\sigma^4 + 160\sigma^2 + 160}{\sigma^8(5 - \sigma^2)} \\ &< \frac{592}{\sigma^8(5 - \sigma^2)}, \end{aligned} \tag{4.91}$$

where in the last two inequalities we used $\sigma < 1$. Combining (4.89),(4.90) and (4.91), we obtain (3.17):

$$\frac{F_{\max}\left(\Delta, \frac{1}{\sigma}\right)}{F_{\min}\left(\Delta, \frac{1}{\sigma}\right)^2} < \frac{c_1}{\sigma^6(1-3\sigma^2)^2}. \tag{4.92}$$

Then, using that $\bar{f} < 1$ and that $\frac{\bar{C}^{\frac{5}{4}}}{\bar{f}} \leq \frac{(\bar{C}+2k)^{\frac{3}{2}}}{\bar{f}}, (\bar{C}+2k)^{\frac{3}{2}} \leq \frac{(\bar{C}+2k)^{\frac{3}{2}}}{\bar{f}}$, we have that:

$$\begin{aligned}
&\left((6+\frac{2}{\bar{f}})\sqrt{4k+5+\frac{4k}{\sigma^4}}\bar{C}^{\frac{5}{4}} + \frac{6}{\eta}(\bar{C}+2k)^{\frac{3}{2}} \right) \frac{\sqrt{2k+2}}{1-\frac{\sqrt{e}}{2}} \\
&\qquad < c_2 \frac{(\bar{C}+2k)^{\frac{3}{2}}}{\bar{f}} \sqrt{2k+2} \frac{\eta\sqrt{4k\sigma^4+5\sigma^4+4k}+\sigma^2}{\eta\sigma^2} \\
&\qquad < c_2 \frac{(\bar{C}+2k)^{\frac{3}{2}}}{\bar{f}} \sqrt{2k+2} \cdot \frac{\sqrt{8k+5}+1}{\eta\sigma^2} \qquad (\sigma < 1) \\
&\qquad < c_2 \frac{(\bar{C}+2k)^{\frac{3}{2}}}{\bar{f}} \frac{k}{\eta\sigma^2},
\end{aligned} \tag{4.93}$$

for a large enough constant c_2. Similarly we show that (3.18) holds:

$$\begin{aligned}
\frac{\sqrt{(2k+2)\left(4k+5+\frac{4k}{\sigma^4}\right)}}{1-\frac{\sqrt{e}}{2}} \frac{\bar{C}(f_0,f_1)^{\frac{5}{4}}}{\bar{f}} &< c_5 \frac{\sqrt{(k+1)(4k\sigma^4+5\sigma^4+4k)}}{\sigma^2} \frac{\bar{C}^{\frac{5}{4}}}{\bar{f}} \\
&< c_5 \cdot \frac{\sqrt{(k+1)(8k+5)}}{\sigma^2} \frac{\bar{C}^{\frac{5}{4}}}{\bar{f}} \qquad (\sigma < 1) \\
&< c_5 \frac{k}{\sigma^2} \frac{\bar{C}^{\frac{5}{4}}}{\bar{f}}.
\end{aligned} \tag{4.94}$$

Finally, from (4.87) and (4.88), we also obtain:

$$\frac{e^{-\frac{\Delta^2}{\sigma^2}} + e^{-\frac{2\Delta^2}{\sigma^2}}}{1-e^{-\frac{\Delta^2}{\sigma^2}}} < \left(\sigma^2 + \frac{\sigma^4}{5}\right) \frac{1}{5-\sigma^2} < \frac{16}{210}, \tag{4.95}$$

where, in the last inequality, we used $\sigma < \frac{1}{\sqrt{3}}$. Furthermore, if $\lambda < \frac{2}{5}$, then $1-2\lambda > \frac{1}{5}$ and

$$e^{-\frac{\Delta^2(1-2\lambda)}{\sigma^2}} < \left(\frac{\sigma^2}{5}\right)^{1-2\lambda} < \frac{1}{15^{1-2\lambda}} < \frac{1}{15^{1/5}}, \tag{4.96}$$

so we can combine the last two inequalities to obtain:

$$1 - \frac{e^{-\frac{\Delta^2}{\sigma^2}} + e^{-\frac{2\Delta^2}{\sigma^2}}}{1-e^{-\frac{\Delta^2}{\sigma^2}}} - e^{-\frac{\Delta^2(1-2\lambda)}{\sigma^2}} > 1 - \frac{16}{210} - \frac{1}{15^{1/5}} > \frac{1}{3}. \tag{4.97}$$

Finally, to bound $e^{\frac{\Delta^2\lambda^2}{\sigma^2}}$, note from the definition of λ and η that $\lambda\Delta < \frac{\eta}{2}$, and from the assumption that $\eta \leq \sigma^2$, we obtain that:

$$e^{\frac{\Delta^2\lambda^2}{\sigma^2}} < e^{\frac{\eta^2}{4\sigma^2}} \leq e^{\sigma^2/4} < c_5, \tag{4.98}$$

where we used that $\sigma < 1$. Combining the last two inequalities, we obtain that (3.20) holds for some constant $c_5 > 0$.

4.6 Proof of Lemma 24 (Dependence on averaging interval length)

In the proof of Lemma 21 in Section 4.1 we require that $f_0 \gg \bar{f}$ and $f_0 \gg f_1$ These conditions come from equations (4.5) and (4.9) respectively. We can, therefore, fix $\bar{f}$ and f_1 such that $\bar{f} < 1$ and $1 < f_1 < f_0$, and give an expression of f_0 as a function of $\bar{f}$ as $\epsilon \to 0$. From equation (4.5), we have that

$$\frac{f_0}{\bar{f}} > \frac{N_{\underline{l},1}}{\min_{\tau_{\underline{l}} \in T_\epsilon^C} N_{1,1}(\tau_{\underline{l}})}. \tag{4.99}$$

which is required by the condition that $\det(M_N)$ in (4.2) is positive when $\tau_{\underline{l}} \in T_\epsilon^C$. While $N_{\underline{l},1}$ does not depend on ϵ, we will argue below that, for $\epsilon \to 0$, the minimum in the denominator is lower bounded by $N_{1,1}(\tau_{\underline{l}})$, where

$$\tau_{\underline{l}} \in \bar{T}_\epsilon = \{t_1 - \epsilon, t_1 + \epsilon, \ldots, t_k - \epsilon, t_k + \epsilon\}. \tag{4.100}$$

Therefore, a sufficient condition to ensure (4.99) is:

$$\frac{f_0}{\bar{f}} > \frac{N_{\underline{l},1}}{\min_{\tau_{\underline{l}} \in \bar{T}_\epsilon} N_{1,1}(\tau_{\underline{l}})}. \tag{4.101}$$

Note that $\min_{\tau_{\underline{l}} \in \bar{T}_\epsilon} N_{1,1}(\tau_{\underline{l}}) \to 0$ as $\epsilon \to 0$, since two rows of the determinant become equal. More explicitly, let us assume, for simplicity, that the minimum over the finite set of points $\bar{T}_\epsilon$ is attained $t_1 + \epsilon$. Then, we subtract the row with $\tau_{\underline{l}}$ from the first row of $N_{1,1}(\tau_{\underline{l}})$, and then expand the determinant along this row. By taking $\tau_{\underline{l}} = t_1 + \epsilon$, we obtain

$$N_{1,1}(t_1 + \epsilon) = \sum_{j=1}^{m} (-1)^{j+1} N_{1,1,j} \left[g(t_1 - s_j) - g(t_1 - s_j + \epsilon) \right],$$

where the minors $N_{1,1,j}$ are fixed (i.e. independent of ϵ). Therefore, as $\epsilon \to 0$, we have that $N_{1,1}(t_1 + \epsilon) \to 0$, with $N_{1,1}(t_1 + \epsilon) > 0, \forall \epsilon$. Then, everything else in $N_{1,1}(t_1 + \epsilon)$

being fixed, there exists $\epsilon_0 > 0$ such that [4]

$$\min_{\tau_{\underline{l}} \in T_\epsilon^C} N_{1,1}(\tau_{\underline{l}}) = N_{1,1}(t_1 + \epsilon), \quad \forall \epsilon < \epsilon_0. \tag{4.102}$$

We will now find the exact rate at which $N_{1,1}(t_1+\epsilon) \to 0$ for $\epsilon < \epsilon_0$. In the row with $\tau_{\underline{l}}$ in $N_{1,1}(t_1+\epsilon)$, we Taylor expand the entries in the columns $j = 1, \ldots, m$ as follows:

$$g(\tau_{\underline{l}} - s_j) = g(t_1 - s_j + \epsilon) = g(t_1 - s_j) + \epsilon g'(t_1 - s_j) + \frac{\epsilon^2}{2} g''(\xi_j), \tag{4.103}$$

for some $\xi_j \in [t_1 - s_j, t_1 - s_j + \epsilon]$, and note that $\xi_j \to t_1 - s_j$ as $\epsilon \to 0$. Then

$$N_{1,1}(t_1+\epsilon) = \epsilon \begin{vmatrix} g\left(t_1 - s_1\right) & \cdots & g\left(t_1 - s_m\right) \\ g'\left(t_1 - s_1\right) & \cdots & g'\left(t_1 - s_m\right) \\ \vdots & & \vdots \\ g'(t_1 - s_1) + \frac{\epsilon}{2} g''(\xi_1) & \cdots & g'(t_1 - s_m) + \frac{\epsilon}{2} g''(\xi_m) \\ \vdots & & \vdots \\ g\left(t_k - s_1\right) & \cdots & g\left(\tau - s_m\right) \\ g'\left(t_k - s_1\right) & \cdots & g'\left(t_k - s_m\right) \\ g\left(1 - s_1\right) & \cdots & g\left(1 - s_m\right) \end{vmatrix}$$

$$= \frac{\epsilon^2}{2} \begin{vmatrix} g\left(t_1 - s_1\right) & \cdots & g\left(t_1 - s_m\right) \\ g'\left(t_1 - s_1\right) & \cdots & g'\left(t_1 - s_m\right) \\ \vdots & & \vdots \\ g''(\xi_1) & \cdots & g''(\xi_m) \\ \vdots & & \vdots \\ g\left(t_k - s_1\right) & \cdots & g\left(\tau - s_m\right) \\ g'\left(t_k - s_1\right) & \cdots & g'\left(t_k - s_m\right) \\ g\left(1 - s_1\right) & \cdots & g\left(1 - s_m\right) \end{vmatrix} =: \frac{\epsilon^2}{2} N_{1,1}^\epsilon, \tag{4.104}$$

for $\epsilon < \epsilon_0$, where in the first equality we subtracted the first row from the row with $\tau_{\underline{l}}$ and in the second equality we subtracted the the second row from the row with $\tau_{\underline{l}}$. Note that swapping the $\tau_{\underline{l}}$ row with the third row involves an even number of adjacent row swaps, so the sign of the determinant remains the same. Also, for $\epsilon < \epsilon_0$, we have that:

$$N_{1,1}^\epsilon \to \begin{vmatrix} g\left(t_1 - s_1\right) & \cdots & g\left(t_1 - s_m\right) \\ g'\left(t_1 - s_1\right) & \cdots & g'\left(t_1 - s_m\right) \\ g''(t_1 - s_1) & \cdots & g''(t_1 - s_m) \\ \vdots & & \vdots \\ g\left(t_k - s_1\right) & \cdots & g\left(\tau - s_m\right) \\ g'\left(t_k - s_1\right) & \cdots & g'\left(t_k - s_m\right) \\ g\left(1 - s_1\right) & \cdots & g\left(1 - s_m\right) \end{vmatrix} =: N'_{1,1} > 0, \tag{4.105}$$

[4] To see this, take $f(\epsilon) = N_{1,1}(t_1+\epsilon)$ and we know that f is continuous, $f(0) = 0$ and $f(\epsilon) > 0, \forall \epsilon$. If $f(\epsilon) = C\epsilon^2$ on $[0, \epsilon']$ for some $\epsilon' > 0$ and $C > 0$, which we show later in the proof (see (4.104)), then take $B = \min_{\epsilon \geq \epsilon'} f(\epsilon)$, and then there exists $\epsilon_0 \leq \epsilon'$ such that $f(\epsilon) \leq B, \forall \epsilon < \epsilon_0$. So we have that $f(\epsilon) \leq \min_{\tau \geq \epsilon} f(\tau), \forall \epsilon < \epsilon_0$, which implies that $\min_{\tau \geq \epsilon} f(\tau) = f(\epsilon), \forall \epsilon < \epsilon_0$.

where the last inequality is true because Gaussians form an extended T-system (see [50]) and the determinant in the limit does not depend on ϵ.[5]

Substituting (4.104) and (4.102) into (4.99), and noting the the minimum in (4.102) can be attained at any $\tau_{\underline{l}} \in \bar{T}_\epsilon$ defined in (4.100) (not necessarily at $t_1 + \epsilon$, which we assumed above for simplicity), we obtain:

$$\frac{f_0}{\bar{f}} > \frac{2N_{\underline{l},1}}{\epsilon^2 \min_{\tau_{\underline{l}} \in \bar{T}_\epsilon} N_{1,1}^\epsilon}, \quad \forall \epsilon < \epsilon_0, \tag{4.106}$$

which is the condition we must impose on $f_0/\bar{f}$ so that $\det(M_N) > 0$ for $\epsilon < \epsilon_0$ instead of (4.6). Therefore, for $\epsilon < \epsilon_0$, we set

$$f_0 = C_\epsilon \cdot \frac{\bar{f}}{\epsilon^2}, \quad \text{where} \quad C_\epsilon = \frac{3N_{\underline{l},1}}{\min_{\tau_{\underline{l}} \in \bar{T}_\epsilon} N_{1,1}^\epsilon} \quad \text{and} \quad \lim_{\epsilon \to 0} C_\epsilon \in (0, +\infty). \tag{4.107}$$

Using that $f_1 < f_0$ and $1 < f_0$, we bound $\bar{C}(f_0, f_1)$ in (3.13):

$$\bar{C}(f_0, f_1) < \bar{c}_1 f_0^2, \tag{4.108}$$

where $\bar{c}_1$ is a universal constant. Finally, we insert (4.107) and (4.108) into (3.19) and use the fact that $\bar{f} < 1$ to obtain

$$\begin{aligned} C_2\left(\frac{1}{\epsilon}\right) &< \frac{\bar{c}_2 f_0^{\frac{5}{2}}}{\bar{f}} = \bar{c}_2 C_\epsilon^{\frac{5}{2}} \bar{f}^{\frac{3}{2}} \cdot \frac{1}{\epsilon^5} \\ &< \bar{c}_2 C_\epsilon^{\frac{5}{2}} \cdot \frac{1}{\epsilon^5}, \quad \forall \epsilon < \epsilon_0, \end{aligned} \tag{4.109}$$

where $\bar{c}_2$ is a universal constant, and this concludes the proof.

[5]Note that in the previous footnote we assumed that $f(\epsilon) = C\epsilon^2$, where C is independent of ϵ, but actually from (4.104) we have that $f(\epsilon) = N_{1,1}(t_1 + \epsilon) = \frac{\epsilon^2}{2} N_{1,1}^\epsilon$, where $N_{1,1}^\epsilon \to N'_{1,1} > 0$ as $\epsilon \to 0$. Our conclusion that the $N_{1,1}$ goes to zero at the rate ϵ^2 does not change because, for small enough $E > 0$, there exists $\epsilon' > 0$ such that:

$$0 < N'_{1,1} - E < N_{1,1}^\epsilon < N'_{1,1} + E, \quad \forall \epsilon < \epsilon',$$

so $f(\epsilon) \geq C_2 \epsilon^2$ for all $\epsilon < \epsilon'$, where $C_2 = (N'_{1,1} - E)/2 > 0$ is independent of ϵ. Then, using the argument in the previous footnote, there exists $\epsilon_0 > 0$ such that $\min_{\tau \geq \epsilon} f(\tau) \geq C_2 \epsilon^2$ for all $\epsilon < \epsilon_0$, and therefore the obtain a version of (4.106) where the factor in front of $1/\epsilon^2$ is independent of ϵ.

Part II

The dual approach to non-negative super-resolution

Chapter 5

Perturbation of the primal variables in the noise-free setting

While in the first part of the thesis we studied the stability property of the solution to the super-resolution problem regardless of how the solution is obtained, in the second part we focus on a practical aspect related to solving the non-negative super-resolution problem.

In this chapter we analyse the primal and dual problem when solving the TV norm minimisation problem over non-negative measures. Specifically, we establish a relationship between perturbations of the primal variable and perturbations of the dual variable around their optimal values. We first describe the setup in more detail in Section 5.1, following which we introduce the main results of the chapter in Section 5.2: a bound on perturbations of the source locations around their true values as the dual variable λ[1] is perturbed around its optimal value in Theorem 28, and a bound on perturbations of the source weights around their true values as the source locations are perturbed around their true values in Theorem 29. In Sections 5.3 and 5.4 we give the proofs of the two theorems respectively.

5.1 Motivation

We now focus on solving the TV norm minimisation problem as given in (1.8) in Chapter 1, which we rewrite here for clarity:

$$\min_{z \geq 0} \|z\|_{TV} \quad \text{subject to} \quad y = \int_I \Phi(t) z(\mathrm{d}t),$$

[1] In Part II of the thesis, we use λ to denote the dual variable. Note that this is completely different from the meaning of λ in Part I, where it denoted the sample proximity.

over non-negative measures z on $[0,1]$, where

$$y = [y_1, \dots, y_m]^T, \tag{5.1}$$

$$\Phi(t) = [\phi(t - s_1), \dots, \phi(t - s_m)]^T, \tag{5.2}$$

when the convolution kernel ϕ is Gaussian:

$$\phi(t) = e^{-t^2/\sigma^2}.$$

In the current chapter we only consider the noise-free setting, namely when $w_j = 0$ in the measurements as given by (1.2):

$$y_j = \int_I \phi(t - s_j) x(\mathrm{d}t) = \sum_{i=1}^{k} a_i \phi(t_i - s_j),$$

for all $j = 1, \dots, m$.

Throughout Part II of the thesis, we approach the super-resolution problem by considering its dual. In the case of (1.8), this is given in (1.9) in Chapter 1 and derived in Appendix B.1:

$$\max_{\lambda \in \mathbb{R}^m} \; y^T \lambda \quad \text{subject to} \quad \lambda^T \Phi(t) \leq 1 \quad \forall t \in I,$$

which is a finite-dimensional problem with infinitely many constraints, known as a semi-infinite program. Such problems can be solved using a number of algorithms including exchange methods [35] and sequential quadratic programming [58]. The inifinte number of constraints are handled, for example, by solving for a finite number of constraints and at each iteration replacing some of these by new constraints, which is the approach taken by the exchange methods. The advantage over algorithms that solve the primal problem (for example the ADCG algorithm [9]) is working in a finite dimensional space, which simplifies the analysis of the algorithms used.

At this point, it is worth mentioning that, while we have previously shown that non-negativity, and not the TV norm, is the main regulariser for ensuring uniqueness and stability, one would be advised to include additional regularisers such as the TV norm or a sparsity constraint in the context of non-convex methods to encourage sparsity, specifically in the noisy setting.

When approaching the super-resolution problem by solving its dual, we are ultimately interested in finding the locations $\{t_i\}_{i=1}^k$ and magnitudes $\{a_i\}_{i=1}^k$ of the sparse measure x once we have the solution λ^* to the dual problem. However, it is possible that there are inaccuracies in the dual solution λ^* caused by algorithmic error or noise in the data, which we see as a perturbed problem in the dual variable λ. Therefore, we require a result which shows that we can control the errors in the primal variable in terms of the errors in the dual variable.

A problem of interest is, therefore, how small perturbations of the dual variable λ around the optimser λ^* affect the solution of the primal problem, specifically the locations t_i and magnitudes a_i of the point sources. The first result in this chapter is a bound on how far the estimated locations t_i are from their true values as λ is perturbed from its optimal value λ^*, given in Theorem 28. Then, we show a similar result on how far the estimated weights a_i are from their true values in Theorem 29. These theorems give us an insight into the size of the error in the locations and weights when we apply an optimisation algorithm to the dual of the super-resolution problem.

While the bounds given in these theorems apply only to the case when the convolution kernel is Gaussian, the same techniques can be applied to obtain perturbation bounds for other kernels, with a few differences in the way sums in the proofs are bounded, which would would be specific to the kernel used.

5.2 Bound on the error in source locations and weights as the dual variable is perturbed

In this section we present the main results of the chapter, namely two theorems that give bounds on the perturbations around the source locations t_k and the weights a_k respectively, as the dual variable is perturbed away from the optimiser λ^*, when the convolution kernel is a Gaussian with known width σ as defined in (1.7).

Before we discuss these results, we need to define the concept of a dual certificate for the TV-norm minimisation problem, which, like in the previous part of the thesis, plays an important role throughout the second part.

Definition 27. ***(Dual certificate for TV-norm minimisation)*** *Consider a solution λ^* of the dual problem* (1.9) *or* (1.10)*. Then a dual certificate is a function of the form*

$$q(t) = \sum_{j=1}^{m} \lambda_j^* \phi(t - s_j) = \lambda^{*T} \Phi(t), \tag{5.3}$$

which satisfies the conditions:

$$q(t_i) = 1, \quad \forall i = 1, \dots, k, \tag{5.4}$$

$$q(t) < 1, \quad \forall t \neq t_i, \forall i = 1, \dots, k. \tag{5.5}$$

As we have seen in Part I, the idea of dual certificate is common in the super-resolution literature and it is known that the global maximisers of $q(t)$ correspond to the source locations $\{t_i\}_{i=1}^k$ (see, for example [12, 74, 34]). Once these are found, amplitudes $\{a_i\}_{i=1}^k$ are obtained by solving a linear system. As in the case of the feasibility problem, the dual certificate is used to show uniqueness of the solution of the TV norm minimisation problem (1.8) in the noise-free setting, see Appendix B.3.

Note the similarity between the dual certificate defined above and the dual certificate for the feasibility problem given in Definition 3 in Chapter 2. The difference between the two is the right hand side of the conditions (5.4) and (5.5) which, in this case, is one instead of zero. Both these definitions are slightly simpler than the definition of the dual certificate when there is no non-negativity constraint. There, it is required that $|q(t)| \leq 1$ for all $t \in [0,1]$, with equality $q(t) = \pm 1$ at the source locations t_i, where the sign of $q(t_i)$ must be the same as the sign of a_i, see for example [12, 31]. Bearing this difference in mind, the results in the current chapter and in Chapter 6 can be extended to the signed case with minor modifications, using the same proof techniques.

We are now ready to discuss the perturbation results in the noise-free setting. In the following theorem we consider the dual (1.9) of (1.8) and quantify how the source locations given by the global maximisers of the dual certificate formed by the dual solution λ^* are affected by perturbations of λ^*. The proof is given in Section 5.3.

Theorem 28. ***(Dependence of*** $|t - t^*|$ ***on*** $\|\lambda - \lambda^*\|_2$***)*** *Let* $\lambda^* \in \mathbb{R}^m$ *be a solution of the dual program* (1.9) *with* ϕ *Gaussian as given in* (1.7) *such that the the dual certificate* $q(s)$ *defined in* (5.3) *satisfies conditions* (5.4) *and* (5.5), λ *a perturbation of* λ^* *in a ball of radius* δ_λ *and* t *an arbitrary local maximiser of* $q_\lambda(s) = \sum_{j=1}^m \lambda_j \phi(s - s_j)$ *so that for* $\lambda = \lambda^*$, *the corresponding local maximiser* t^* *is a true source location in* x. *Let* $R = \frac{\|\lambda^*\|_2}{\sigma}$ *and* $c \approx 3.9036$ *a universal constant. If the radius* δ_λ *is bounded by*

$$\delta_\lambda \leq \frac{|q''(t^*)|^2 \sigma^3 \sqrt{e}}{4\sqrt{2}\,(2 + cR)\, m}, \tag{5.6}$$

then

$$|t - t^*| \leq C_{t^*} \|\lambda - \lambda^*\|_2, \tag{5.7}$$

where

$$C_{t^*} = \frac{1}{4 + cR} \left[1 + \frac{2\sqrt{2m}(2 + cR)}{|q''(t^*)|\sqrt{e}} \right]. \tag{5.8}$$

$$\tag{5.9}$$

One of the main conclusions which can be drawn from this result is that the primal spike location error is controlled in ℓ_∞, but degrades as a function of the number of measurements in the order of $\sqrt{m}$. Alternatively, we can write (5.7) in terms of the ℓ_2 norm of the error between the vector of true source locations $\mathbf{t}^*$ and the perturbed source locations $\tilde{\mathbf{t}}$:

$$\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2 \leq \sqrt{k} C_{t^*} \|\lambda - \lambda^*\|_2.$$

Of crucial importance is the curvature of the dual certificate at the true solution: the flatter the certificate, the worse the estimation error. Specifically, for the same values of λ and λ^*, the constant C_{t^*} is larger and the radius δ_λ where the results hold is smaller for smaller $|q''(t^*)|$. Our theorem also gives important information about the accuracy in the dual variable required to guarantee our upper bound on the error of recovery. This accuracy is of the inverse order of the number of measurements, which is quite a stringent constraint. Both the m and the $\sqrt{m}$ factors are a consequence of the way we bound sums of shifted copies of the kernel, namely $\sum_{j=1}^m \phi(t - s_j) \leq m \max_{t \in \mathbb{R}} \phi(t)$. Given the fast decay of the Gaussian, it is clear that this is not a tight bound. However, any bound would reflect the density of samples close to each source location.

We will now give a result regarding the perturbation of the magnitudes a_i when λ^* is perturbed. Let Φ be the matrix whose entries are defined as

$$\Phi_{ij} = \phi(t_j - s_i), \tag{5.10}$$

and $\mathbf{t}^*$ and $\mathbf{a}^*$ the vectors of source locations and weights:

$$\mathbf{t}^* = [t_1, \dots, t_k]^T, \qquad \mathbf{a}^* = [a_1, \dots, a_k]^T.$$

When we solve (1.9) exactly, we obtain the source locations by finding the global maximisers of $q(s)$. Then the vector of weights $\mathbf{a}^*$ is found by solving the system

$$\Phi \mathbf{a} = \mathbf{y}.$$

When the source locations are perturbed, we denote the resulting perturbed data matrix by:

$$\tilde{\Phi} = \Phi + E, \tag{5.11}$$

and we calculate the vector of perturbed weights $\tilde{\mathbf{a}}$ as the solution of the least squares problem

$$\min_{\mathbf{a}} \|\tilde{\Phi}\mathbf{a} - \mathbf{y}\|_2. \tag{5.12}$$

The following theorem gives a bound on the error $\|\mathbf{a}^* - \tilde{\mathbf{a}}\|_2$ between the vector of true weights $\mathbf{a}^*$ and the vector of weights $\tilde{\mathbf{a}}$ obtained by solving the least squares problem

(5.12) with the perturbed matrix $\tilde{\Phi}$, as a function of the error $\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2$ between the perturbed source locations $\tilde{\mathbf{t}}$ and the true source locations $\mathbf{t}^*$. The proof is given in Section 5.4.

Theorem 29. ***(Dependence of*** $\|\tilde{\mathbf{a}} - \mathbf{a}^*\|_2$ ***on*** $\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2$***)*** *Let* $\mathbf{t}^* \in [0,1]^k$ *be the vector of true source locations and* $\tilde{\mathbf{t}} \in [0,1]^k$ *the perturbed source locations, such that:*

$$\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2 < \frac{\sigma^2 \sigma_{\max}(\Phi)}{4e^{4/\sigma^2}\sqrt{m}} \left(\sqrt{1 + \frac{\sigma^2_{\min}(\Phi)}{\sigma^2_{\max}(\Phi)}} - 1 \right). \tag{5.13}$$

Then the error between the true weights $\mathbf{a}^*$ *and the perturbed weights* $\tilde{\mathbf{a}}$ *obtained by solving problem* (5.12) *is bounded by:*

$$\|\tilde{\mathbf{a}} - \mathbf{a}^*\|_2 \leq C_{a^*} \|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2 + O(\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2^2), \tag{5.14}$$

where

$$C_{a^*} = \frac{4e^{\frac{4}{\sigma^2}}\sqrt{m}\|\mathbf{a}^*\|_2}{\sigma^2 \sigma_{\min}(\Phi)}.$$

Note the connection between Theorem 28 and Theorem 29: Theorem 28 bounds the error in the source locations as a consequence of the perturbations in the dual variable λ, then Theorem 29 bounds the errors in the soure weights as a direct consequence of the perturbations in the source locations, and indirectly as a consequence of the perturbations in λ. One can combine the two theorems to obtain a result of the form:

$$\|\tilde{\mathbf{a}} - \mathbf{a}^*\|_2 \leq \sqrt{k} C_{t^*} C_{a^*} \|\lambda - \lambda^*\|_2, \tag{5.15}$$

which is more similar to Theorem 28. However, we chose to keep the two results separate for more clarity on how the errors in λ propagate to the source locations and weights.

5.3 Proof of Theorem 28 (Dependence of the source location error on dual variable error)

Let t^* be an arbitrary local maximiser of the function $q(t)$ in (5.3), so t^* is also a source location, and λ^* the solution to (1.9). The key step in this proof is applying a quantitative version of the implicit function theorem [57] to the function:

$$F(t, \lambda) = \sum_{j=1}^{m} \lambda_j \phi'(t - s_j), \tag{5.16}$$

where $F(t^*, \lambda^*) = 0$ because t^* is a maximizer of $q(s)$ in (5.3). The theorem allows us to express t as a function $t(\lambda)$ of λ with:

$$\partial_\lambda t(\lambda) = -\left[\partial_t F(t(\lambda), \lambda)\right]^{-1} \partial_\lambda F(t(\lambda), \lambda), \tag{5.17}$$

for t in a ball of radius δ_0 around t^* and for λ in a ball of radius $\delta_1 \leq \delta_0$ around λ^*, where δ_0 is chosen such that

$$\sup_{(t,\lambda)\in V_\delta} \left\| I - \left[\partial_t F(t^*, \lambda^*)\right]^{-1} \partial_t F(t, \lambda) \right\| \leq \frac{1}{2}, \tag{5.18}$$

where $V_\delta = \left\{(t, \lambda) \in \mathbb{R}^{m+1} : |t - t^*| \leq \delta_0, \|\lambda - \lambda^*\| \leq \delta_0\right\}$ and δ_1 is given by

$$\delta_1 = (2M_t B_\lambda)^{-1}\delta_0, \tag{5.19}$$

where

$$B_\lambda = \sup_{(t,\lambda)\in V_\delta} \|\partial_\lambda F(t, \lambda)\|_2,$$
$$M_t = \left\|\partial_t F(t^*, \lambda^*)^{-1}\right\|_2.$$

The following two lemmas, proved in Sections 5.3.1 and 5.3.2 respectively, give us values of δ_0 and δ_1 that define balls around t^* and λ^* respectively which are included in the balls required by the quantitative implicit function theorem with radii defined in (5.18) and (5.19).

Lemma 30. ***(Radius of ball around*** t^****)*** *The condition* (5.18) *is satisfied if*

$$\delta_0 = \frac{\sigma^2|q''(t^*)|}{\sqrt{m}\left(4 + 2c \cdot \frac{\|\lambda^*\|_2}{\sigma}\right)}. \tag{5.20}$$

Lemma 31. ***(Radius of ball around*** λ^****)*** *For* δ_0 *from Lemma 30 and* δ_1 *from condition* (5.19)*, the following choice of* δ_λ*:*

$$\delta_\lambda = \frac{\sigma\sqrt{e}|q''(t^*)|}{2\sqrt{2m}} \cdot \delta_0$$

satisfies $\delta_\lambda < \delta_1$*.*

Given the definition of the function F in (5.16), we have that

$$\partial_t F(t, \lambda) = \sum_{j=1}^{m} \lambda_j \phi''(t - s_j), \tag{5.21}$$

$$\partial_\lambda F(t, \lambda) = [\phi'(t - s_1), \quad \dots \quad, \phi'(t - s_m)]^T. \tag{5.22}$$

By applying Taylor expansion to $t(\lambda)$ around λ^* in the region defined by δ_0 and δ_λ, we have that

$$t(\lambda) = t(\lambda^*) + \left\langle \lambda - \lambda^*, \partial_\lambda t(\lambda_\delta) \right\rangle,$$

for some λ_δ on the line segment determined by λ^* and λ, so

$$\begin{aligned} \left|t(\lambda) - t(\lambda^*)\right| &\leq \|\lambda - \lambda^*\|_2 \cdot \left\|\partial_\lambda t(\lambda_\delta)\right\|_2 \\ &\leq \frac{\delta_0}{\sum_{j=1}^m \lambda_{\delta j} \phi''(t(\lambda_\delta) - s_j)} \cdot \left\| \left[\phi'(t(\lambda_\delta) - s_1), \quad \dots \quad, \phi'(t(\lambda_\delta) - s_m)\right] \right\|_2, \end{aligned} \tag{5.23}$$

where in the last inequality we used that $\|\lambda - \lambda^*\| \leq \delta_0$ and (5.17). We now need to bound the terms in (5.23) for the Gaussian kernel $\phi(t) = e^{-t^2/\sigma^2}$. First, we rewrite the last inequality as

$$\begin{aligned} |t(\lambda) - t(\lambda^*)| &\sum_{j=1}^m (\lambda_{\delta j} + \lambda_j^* - \lambda_j^*) \phi''(t(\lambda_\delta) - s_j) \\ &\leq \delta_0 \cdot \left\| \left[\phi'(t(\lambda_\delta) - s_1), \quad \dots \quad, \phi'(t(\lambda_\delta) - s_m)\right] \right\|_2, \end{aligned} \tag{5.24}$$

we apply the reverse triangle inequality in the sum on the left hand side:

$$\begin{aligned} |t(\lambda) - t(\lambda^*)| &\left[- \left| \sum_{j=1}^m (\lambda_{\delta j} - \lambda_j^*) \phi''(t(\lambda_\delta) - s_j) \right| + \left| \sum_{j=1}^m \lambda_j^* \phi''(t(\lambda_\delta) - s_j) \right| \right] \\ &\leq \delta_0 \cdot \left\| \left[\phi'(t(\lambda_\delta) - s_j)\right]_{j=1}^m \right\|_2 \end{aligned} \tag{5.25}$$

and then we apply the Cauchy-Schwartz inequality to the first sum on the left hand side above to obtain:

$$\begin{aligned} |t(\lambda) - t(\lambda^*)| &\left[- \|\lambda_\delta - \lambda^*\|_2 \cdot \left\| \left[\phi''(t(\lambda_\delta) - s_j)\right]_{j=1}^m \right\|_2 + \left| \sum_{j=1}^m \lambda_j^* \phi''(t(\lambda_\delta) - s_j) \right| \right] \\ &\leq \delta_0 \cdot \left\| \left[\phi'(t(\lambda_\delta) - s_j)\right]_{j=1}^m \right\|_2. \end{aligned} \tag{5.26}$$

To simplify the notation, we write $\delta_t = |t(\lambda) - t(\lambda^*)|$ and

$$A = \left\| \left[\phi''(t(\lambda_\delta) - s_j)\right]_{j=1}^m \right\|_2, \tag{5.27}$$

$$B = \left| \sum_{j=1}^m \lambda_j^* \phi''(t(\lambda_\delta) - s_j) \right|, \tag{5.28}$$

$$C = \left\| \left[\phi'(t(\lambda_\delta) - s_j)\right]_{j=1}^m \right\|_2, \tag{5.29}$$

and by using[2] $\|\lambda_\delta - \lambda^*\|_2 \leq \delta_0$, we have that:

$$\delta_t(-\delta_0 A + B) \leq \delta_0 C, \tag{5.30}$$

which can be further re-written as:

$$\delta_t \leq \frac{C + \delta_t A}{B} \cdot \delta_0. \tag{5.31}$$

The aim now is to obtain a bound on δ_t as a function of δ_0 and the parameters of the problem. Therefore, we need to lower bound B and upper bound $C + \delta_t A$.

Bounding A,B,C

We start with B, for which we want to calculate a lower bound. First, we Taylor expand each term of the sum around $t(\lambda^*) - s_j$ as follows:

$$\begin{aligned} B &= \left| \sum_{j=1}^{m} \lambda_j^* \phi''(t(\lambda^*) - s_j + t(\lambda_\delta) - t(\lambda^*)) \right| \\ &= \left| \sum_{j=1}^{m} \lambda_j^* \phi''(t(\lambda^*) - s_j) + \big(t(\lambda_\delta) - t(\lambda^*)\big) \sum_{j=1}^{m} \lambda_j^* \phi'''(\xi_j) \right| && (5.32) \\ &\geq \left| \sum_{j=1}^{m} \lambda_j^* \phi''(t(\lambda^*) - s_j) \right| - \big|t(\lambda_\delta) - t(\lambda^*)\big| \left| \sum_{j=1}^{m} \lambda_j^* \phi'''(\xi_j) \right|, && (5.33) \end{aligned}$$

where $\xi_j \in \big[t(\lambda^*) - s_j - |t(\lambda_\delta) - t(\lambda^*)|, \quad t(\lambda^*) - s_j + |t(\lambda_\delta) - t(\lambda^*)|\big]$ for $j = 1, \ldots, m$, and on the last line we used the reverse triangle inequality. We calculate an upper bound of the last sum in the previous equation as follows:

$$\begin{aligned} \left| \sum_{j=1}^{m} \lambda_j^* \phi'''(\xi_j) \right| &\leq \|\lambda^*\|_2 \cdot \left\| \big[\phi'''(\xi_j)\big]_{j=1}^{m} \right\|_2, \quad \text{by Cauchy-Schwartz,} && (5.34) \\ &\leq \frac{c\|\lambda^*\|_2 \sqrt{m}}{\sigma^3}, && (5.35) \end{aligned}$$

where in the last line we used the maximum value of $\phi'''(t)$ and c is a constant.[3]

Finally, by using the δ_0 from Lemma 30 as a bound on $|t(\lambda_\delta) - t(\lambda^*)|$ and (5.35), we obtain:

$$B \geq |q''(t^*)| \left[1 - \frac{c\|\lambda^*\|_2}{4\sigma + 2c\|\lambda^*\|_2}\right]. \tag{5.36}$$

[2] Since $\|\lambda - \lambda^*\| \leq \delta_0$ and λ_δ is on the line segment between λ^* and λ, then λ_δ is in the ball centred at λ^* with radius δ_0.

[3] $\max_{t\in\mathbb{R}} \phi'(t) = \frac{\sqrt{2}}{\sigma\sqrt{e}}, \max_{t\in\mathbb{R}} \phi''(t) = \frac{2}{\sigma^2}, \max_{t\in\mathbb{R}} \phi'''(t) = \frac{c}{\sigma^3},$ where $c = \frac{4\sqrt{9-3\sqrt{6}}}{e^{\frac{3-\sqrt{6}}{2}}} \approx 3.9036.$

Note that the last fraction above is subunitary, so the bound is indeed positive.

Lastly, we upper bound $C+\delta_t A$. We bound both A and C using the upper bounds on ϕ' and ϕ'' given in footnote 3 and obtain:

$$A \leq \frac{2\sqrt{m}}{\sigma^2}, \tag{5.37}$$

$$C \leq \frac{\sqrt{2m}}{\sigma\sqrt{e}}, \tag{5.38}$$

and for δ_t we use the bound (5.20). Putting (5.20), (5.36), (5.37) and (5.38) together, we obtain:

$$\left|t(\lambda) - t(\lambda^*)\right| \leq C_{t^*} \cdot \|\lambda - \lambda^*\|_2, \tag{5.39}$$

where

$$C_{t^*} = \frac{2\sqrt{2m}\left(2\sigma + c\|\lambda^*\|_2\right)}{|q''(t^*)|\sigma\sqrt{e}\left(4\sigma + c\|\lambda^*\|_2\right)} + \frac{2\sigma}{4\sigma + F\|\lambda^*\|_2}, \tag{5.40}$$

which can also be written in the form in (5.8) in Theorem 28.

5.3.1 Proof of Lemma 30 (Radius of the ball around source location)

Let us now find the radius δ_0 which satisfies (5.18). Using (5.21), the expression inside the sup in (5.18) is

$$E = \left|1 - \frac{\sum_{j=1}^m \lambda_j \phi''(t-s_j)}{\sum_{j=1}^m \lambda_j^* \phi''(t^*-s_j)}\right| = \frac{\left|\sum_{j=1}^m \lambda_j^* \phi''(t^*-s_j) - \lambda_j \phi''(t-s_j)\right|}{\left|\sum_{j=1}^m \lambda_j^* \phi''(t^*-s_j)\right|}. \tag{5.41}$$

By denoting each term in the sum in the numerator in the last equation above by T_j and then adding and subtracting λ_j^* and t^*, we obtain:

$$\begin{aligned} T_j &= \lambda_j^* \phi''(t^*-s_j) - (\lambda_j - \lambda_j^*)\phi''(t-s_j) - \lambda_j^* \phi''(t^* - s_j + t - t^*) \\ &= -(\lambda_j - \lambda_j^*)\phi''(t-s_j) - \lambda_j^*(t-t^*)\phi'''(\xi_j), \end{aligned} \tag{5.42}$$

for some $\xi_j \in \left[t^* - s_j - |t-t^*|, t^* - s_j + |t-t^*|\right]$. Then:

$$\begin{aligned} E &\leq \frac{\left|\sum_{j=1}^m (\lambda_j - \lambda_j^*)\phi''(t-s_j)\right| + \left|\sum_{j=1}^m \lambda_j^*(t-t^*)\phi'''(\xi_j)\right|}{\left|\sum_{j=1}^m \lambda_j^* \phi''(t^*-s_j)\right|} \\ &\leq \frac{\|\lambda-\lambda^*\|_2 \left\|\left[\phi''(t-s_j)\right]_{j=1}^m\right\|_2 + |t-t^*| \left|\sum_{j=1}^m \lambda_j^* \phi'''(\xi_j)\right|}{\left|\sum_{j=1}^m \lambda_j^* \phi''(t^*-s_j)\right|} = E' \end{aligned} \tag{5.43}$$

We now have that

$$\sup_{(t,\lambda)\in V_{\delta_0}} E \leq \sup_{\substack{|t-t^*|\leq\delta_0,\\ \|\lambda-\lambda^*\|\leq\delta_0}} E' \tag{5.44}$$

$$\leq \delta_0 \cdot \frac{\left\|\left[\phi''(t-s_j)\right]_{j=1}^m\right\|_2 + \left|\sum_{j=1}^m \lambda_j^* \phi'''(\xi_j)\right|}{\left|\sum_{j=1}^m \lambda_j^* \phi''(t^*-s_j)\right|}. \tag{5.45}$$

We now further upper bound the fraction on the last line of the previous equation. The terms in the numerator are bounded by taking the maxima of the functions ϕ'' and ϕ''' from footnote 3 respectively:

$$\left\|\left[\phi''(t-s_j)\right]_{j=1}^m\right\|_2 = \sqrt{\sum_{j=1}^m \phi''(t-s_j)^2} \leq \sqrt{m \cdot \max_j |\phi''(t-s_j)|^2} \leq \frac{2\sqrt{m}}{\sigma^2} \tag{5.46}$$

and

$$\left|\sum_{j=1}^m \lambda_j^* \phi'''(\xi_j)\right| \leq \|\lambda^*\|_2 \left\|\left[\phi'''(\xi_j)\right]_{j=1}^m\right\|_2 \qquad \text{by Cauchy-Schwartz} \tag{5.47}$$

$$= \|\lambda^*\|_2 \sqrt{\sum_{j=1}^m \phi'''(\xi_j)^2} \tag{5.48}$$

$$\leq \|\lambda^*\|_2 \max_j |\phi'''(\xi_j)| \sqrt{m} \tag{5.49}$$

$$= c \cdot \frac{\|\lambda^*\|_2 \sqrt{m}}{\sigma^3}, \tag{5.50}$$

where $c = \frac{4\sqrt{9-3\sqrt{6}}}{e^{\frac{3-\sqrt{6}}{2}}} \approx 3.9036$. By writing

$$q(t) = \sum_{j=1}^m \lambda_j^* \phi(t-s_j) \tag{5.51}$$

and using the above bounds, we have that

$$\sup_{(t,\lambda)\in V_{\delta_0}} E \leq \delta_0 \cdot \frac{\frac{2\sqrt{m}}{\sigma^2} + c \cdot \frac{\|\lambda^*\|_2\sqrt{m}}{\sigma^3}}{\left|q''(t^*)\right|} \tag{5.52}$$

Finally, in order to satisfy condition (5.18), we need to impose the condition that the right hand side of (5.52) is less than or equal to $\frac{1}{2}$. We select δ_0 to be the largest value that satisfies this, so:

$$|t-t^*| \leq \delta_0 = \frac{\left|q''(t^*)\right|}{\frac{4\sqrt{m}}{\sigma^2} + 2c \cdot \frac{\|\lambda^*\|_2\sqrt{m}}{\sigma^3}} = \frac{\sigma^2 |q''(t^*)|}{\sqrt{m}\left(4 + 2c \cdot \frac{\|\lambda^*\|_2}{\sigma}\right)}. \tag{5.53}$$

5.3.2 Proof of Lemma 31 (Radius of the ball around the dual solution)

The radius δ_λ of the perturbation of λ^* is given by:

$$\delta_\lambda = (2M_t B_\lambda)^{-1}\delta_0, \tag{5.54}$$

where

$$B_\lambda = \sup_{(t,\lambda)\in V_\delta} \|\partial_\lambda F(t,\lambda)\|_2, \tag{5.55}$$

$$M_t = \left\|\partial_t F(t^*,\lambda^*)^{-1}\right\|_2. \tag{5.56}$$

For B_λ, we have:

$$\|\partial_\lambda F(t,\lambda)\|_2 = \sqrt{\sum_{j=1}^{m} \phi'(t-s_j)^2} \leq \frac{\sqrt{2m}}{\sigma\sqrt{e}}, \tag{5.57}$$

where we have used the global maximum of the first derivative of the Gaussian from footnote 3, so by taking sup on both sides in the last equation, we obtain:

$$B_\lambda \leq \frac{\sqrt{2m}}{\sigma\sqrt{e}}. \tag{5.58}$$

Note that here we do not use any assumptions on the locations of the sources t_i and the samples s_j. If we did, we would be able to obtain a tighter bound than by only using the absolute maximum of the function.

For M_t, note that we have

$$M_t = |q''(t^*)|^{-1}, \tag{5.59}$$

where $q(t)$ is defined in (5.51), so

$$(2M_t B_\lambda)^{-1}\delta_0 \geq \frac{\sigma\sqrt{e}|q''(t^*)|}{2\sqrt{2m}} \cdot \delta_0, \tag{5.60}$$

We then take δ_λ to be equal to the lower bound in the equation above:

$$\delta_\lambda = \frac{\sigma\sqrt{e}|q''(t^*)|}{2\sqrt{2m}} \cdot \delta_0, \tag{5.61}$$

and, after substituting our choice of δ_0 from (5.20), we obtain the radius (5.6) in Theorem 28.

5.4 Proof of Theorem 29 (Dependence of the source weight error on source location error)

We apply equation (4.2) in [78], with $e = 0$ (the noise in the observations), and obtain

$$\tilde{\mathbf{a}} = \mathbf{a}^* - \Phi^\dagger E \mathbf{a}^* - F^T E \mathbf{a}^*, \tag{5.62}$$

where $\Phi^\dagger = (\Phi^T \Phi)^{-1} \Phi^T$ is the pseudo-inverse of Φ and $F = O(E)$ is the perturbation of the $\Phi^\dagger$ due to the perturbation E of Φ, namely

$$\tilde{\Phi}^\dagger = \Phi^\dagger + F^T.$$

In order to obtain an explicit expression for F, we write $\tilde{\Phi}^\dagger$:

$$\begin{aligned}
\tilde{\Phi}^\dagger &= (\tilde{\Phi}^T \tilde{\Phi})^{-1} \tilde{\Phi}^T \\
&= \left[(\Phi + E)^T (\Phi + E) \right]^{-1} (\Phi + E)^T \qquad \text{by (5.11)} \\
&= (\Phi^T \Phi + \Delta)^{-1} (\Phi^T + E^T),
\end{aligned} \tag{5.63}$$

where

$$\Delta = E^T \Phi + \Phi^T E + E^T E \in \mathbb{R}^{k \times k}. \tag{5.64}$$

In order to compute the first factor in (5.63), consider the QR decomposition of Φ:

$$\Phi = QR, \quad \text{where} \quad Q \in \mathbb{R}^{m \times k} \quad \text{and} \quad R \in \mathbb{R}^{k \times k}, \tag{5.65}$$

with $Q^T Q = I_k$ R upper triangular. We have that:

$$\Phi^\dagger = R^{-1} Q^T, \tag{5.66}$$

$$\Phi^T \Phi = R^T R. \tag{5.67}$$

We then write the first factor in (5.63) as

$$\begin{aligned}
(\Phi^T \Phi + \Delta)^{-1} &= (R^T R + \Delta)^{-1} \\
&= \left[R^T \left(I + R^{-T} \Delta R^{-1} \right) R \right]^{-1} \\
&= R^{-1} \left[I + \sum_{l=1}^{\infty} (-1)^l \left(R^{-T} \Delta R^{-1} \right)^l \right] R^{-T} \\
&= (R^T R)^{-1} + S_\Phi \\
&= (\Phi^T \Phi)^{-1} + S_\Phi,
\end{aligned} \tag{5.68}$$

where

$$S_\Phi = R^{-1}\left[\sum_{l=1}^{\infty}(-1)^l\left(R^{-T}\Delta R^{-1}\right)^l\right]R^{-T} \in \mathbb{R}^{k\times k}, \tag{5.69}$$

and in the second inequality in (5.68) we applied the Neumann series expansion to the matrix $I - R^{-T}\Delta R^{-1}$, which converges if

$$\| - R^{-T}\Delta R^{-1}\|_2 < 1. \tag{5.70}$$

We will return to condition (5.70) at the end of this section. We now substitute (5.68) in (5.63), giving

$$\begin{aligned}\tilde{\Phi}^\dagger &= \left[(\Phi^T\Phi)^{-1} + S_\Phi\right](\Phi^T + E^T)\\ &= \Phi^\dagger + (\Phi^T\Phi)^{-1}E^T + S_\Phi\Phi^T + S_\Phi E^T,\end{aligned}$$

so we have that

$$F^T = (\Phi^T\Phi)^{-1}E^T + S_\Phi\Phi^T + S_\Phi E^T, \tag{5.71}$$

which is indeed $O(E)$, since $S_\Phi = O(\Delta)$ and $\Delta = O(E)$. We next upper bound $\|S_\Phi\|_2$. Firstly, note that, because $\Phi^\dagger = \left(QR^{-1}\right)^T$, we have that:[4]

$$\|\Phi^\dagger\|_2 = \|QR^{-1}\|_2 = \|R^{-1}\|_2. \tag{5.73}$$

Then, by using (5.73), norm submultiplicativity and triangle inequality, from (5.69) we have

$$\|S_\Phi\|_2 \le \|\Phi^\dagger\|_2^2\sum_{l=1}^{\infty}\|\Phi^\dagger\|_2^{2l}\|\Delta\|_2^l. \tag{5.74}$$

Now let D be an upper bound on $\|\Delta\|_2$, obtained by applying the triangle inequality in (5.64), so that

$$\|\Delta\|_2 \le D = 2\|E\|_2\|\Phi\|_2 + \|E\|_2^2. \tag{5.75}$$

Then, from (5.74) we have

$$\begin{aligned}\|S_\Phi\|_2 &\le \|\Phi^\dagger\|_2^2\sum_{l=1}^{\infty}\|\Phi^\dagger\|_2^{2l}D^l\\ &= \|\Phi^\dagger\|_2^2\left(\frac{1}{1 - D\|\Phi^\dagger\|_2^2} - 1\right) = \frac{D\|\Phi^\dagger\|_2^4}{1 - D\|\Phi^\dagger\|_2^2},\end{aligned} \tag{5.76}$$

[4] To see the second equality in (5.73), for a matrix $Q \in \mathbb{R}^{m\times k}$ with $Q^TQ = I$ and any matrix $A \in \mathbb{R}^{k\times k}$ we have that

$$\|QA\|_2 = \sup_{\|v\|_2=1}\|QAv\|_2 = \sup_{\|v\|_2=1}\|Av\|_2 = \|A\|_2,$$

since

$$\|QAv\|_2^2 = v^TA^TQ^TQAv = v^TA^TAv = \|Av\|_2^2. \tag{5.72}$$

where the series converges if $D\|\Phi^\dagger\|_2^2 < 1$, in which case the denominator in the last fraction above is positive. We return to this condition at the end of the section. We also know that[5]

$$\|\Phi^\dagger\|_2 = \frac{1}{\sigma_{\min}(\Phi)}. \tag{5.77}$$

By applying triangle inequality in (5.71) and then using (5.76) and the fact that $\|(\Phi^T\Phi)^{-1}\|_2 = 1/\sigma_{\min}^2(\Phi) = \|\Phi^\dagger\|_2^2$ (from (5.77)), we obtain

$$\|F\|_2 \leq \|E\|_2\|\Phi^\dagger\|_2^2 + \frac{D\|\Phi^\dagger\|_2^4}{1 - D\|\Phi^\dagger\|_2^2}\left(\|\Phi\|_2 + \|E\|_2\right), \tag{5.78}$$

where D is given in (5.75). It remains to establish an upper bound on $\|E\|_F$, and consequently on $\|E\|_2$. The following lemma gives us such a bound.

Lemma 32. ***(Upper bound on** $\|E\|_F$**)*** *Let* $E = \tilde{\Phi} - \Phi$ *for* Φ *and* $\tilde{\Phi}$ *as defined in* (5.10) *and* (5.11) *respectively for* $t_j, \tilde{t}_j \in [0,1]$ *for* $j = 1, \ldots, k$. *Then:*

$$\|E\|_F \leq \frac{4e^{\frac{4}{\sigma^2}}\sqrt{m}}{\sigma^2}\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2. \tag{5.79}$$

By using triangle inequality and norm sub-multiplicativity in (5.62), and then substituting (5.78) and (5.79), we obtain

$$\begin{aligned}
\|\mathbf{a}^* - \tilde{\mathbf{a}}\|_2 &\leq \|E\|_2\|\Phi^\dagger\|_2\|\mathbf{a}^*\|_2 + \|E\|_2^2\|\Phi^\dagger\|_2^2\|\mathbf{a}^*\|_2 \\
&\quad + \frac{\|E\|_2 D\|\Phi^\dagger\|_2^4}{1 - D\|\Phi^\dagger\|_2^2}(\|\Phi\|_2 + \|E\|_2)\|\mathbf{a}^*\|_2 \\
&\leq \frac{4e^{\frac{4}{\sigma^2}}\sqrt{m}\|\mathbf{a}^*\|_2}{\sigma^2\sigma_{\min}(\Phi)}\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2 + O(\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2^2),
\end{aligned}$$

which is the bound given in Theorem 29. Note that because $\|E\|_2 = O(\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2)$ (see (5.79)), the first term is the only term that is $O(\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2)$ in the first inequality above, so the other terms are included in the $O(\|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2^2)$ term at the end.

Lastly, we return to condition (5.70), which must be satisfied in order for the bound above to hold. By using norm sub-multiplicativity and the bound on $\|\Delta\|_2$ from (5.75), we obtain

$$\|{\Phi^\dagger}^T\Delta\Phi^\dagger\|_2 \leq \|\Phi^\dagger\|_2^2\|E\|_2^2 + 2\|\Phi\|_2\|\Phi^\dagger\|_2^2\|E\|_2 \tag{5.80}$$

[5]Using the SVD $\Phi = U\Sigma V^T$, we have $\Phi^\dagger = (\Phi^T\Phi)^{-1}\Phi^T = (V\Sigma^2V^T)^{-1}V\Sigma U^T = V\Sigma^{-1}U^T$, so the conclusion follows.

and by requiring that the right hand side above is less than one, we obtain a quadratic constraint on $\|E\|_2$, satisfied if

$$\|E\|_2 < \sigma_{\max}(\Phi)\left(\sqrt{1+\frac{\sigma^2_{\min}(\Phi)}{\sigma^2_{\max}(\Phi)}}-1\right).$$

By using the bound on $\|E\|_2$ from (5.79), the above holds if (5.13) holds. Note that by imposing this, we also ensure that the condition for the series in (5.76) to converge holds, since $D\|\Phi^\dagger\|_2^2$ is equal to the right hand side of (5.80).

5.4.1 Proof of Lemma 32 (Bound on the sampling matrix perturbation)

Since $E = \tilde{\Phi} - \Phi$, for $\tilde{t}_j$ being a perturbation of t_j, we have that

$$|E_{ij}| = \left| e^{-\frac{(s_i-\tilde{t}_j)^2}{\sigma^2}} - e^{-\frac{(s_i-t_j)^2}{\sigma^2}} \right| = e^{-\frac{(s_i-t_j)^2}{\sigma^2}} \left| e^{\frac{1}{\sigma^2}\left[(s_i-t_j)^2-(s_i-\tilde{t}_j)^2\right]} - 1 \right|.$$

Then the exponent can be written as

$$\left|\frac{1}{\sigma^2}\left[(s_i-t_j)^2-(s_i-\tilde{t}_j)^2\right]\right| = \left|\frac{1}{\sigma^2}\left[2s_i(\tilde{t}_j-t_j)+(t_j+\tilde{t}_j)(t_j-\tilde{t}_j)\right]\right| \leq \frac{4|\tilde{t}_j-t_j|}{\sigma^2},$$

where we used that $s_i, \tilde{t}_j, t_j \in [0,1]$, so

$$e^{-\frac{4}{\sigma^2}|\tilde{t}_j-t_j|} \leq e^{\frac{1}{\sigma^2}\left[(s_i-t_j)^2-(s_i-\tilde{t}_j)^2\right]} \leq e^{\frac{4}{\sigma^2}|\tilde{t}_j-t_j|},$$

which implies that

$$\begin{aligned} |E_{ij}| &\leq \left| e^{\frac{1}{\sigma^2}\left[(s_i-t_j)^2-(s_i-\tilde{t}_j)^2\right]} - 1 \right| \\ &\leq \max\left\{ 1 - e^{-\frac{4}{\sigma^2}|\tilde{t}_j-t_j|}, e^{\frac{4}{\sigma^2}|\tilde{t}_j-t_j|} - 1 \right\} \\ &= e^{\frac{4}{\sigma^2}|\tilde{t}_j-t_j|} - 1 \\ &= \frac{4}{\sigma^2}|\tilde{t}_j-t_j| \cdot e^{\xi}, \end{aligned}$$

for some $\xi \in \left[-\frac{4}{\sigma^2}|\tilde{t}_j-t_j|, \frac{4}{\sigma^2}|\tilde{t}_j-t_j|\right]$ and where in the first inequality we have used that $e^{-\frac{(s_i-t_j)^2}{\sigma^2}} \leq 1$. Then

$$|E_{ij}| \leq \frac{4}{\sigma^2}|\tilde{t}_j-t_j| \cdot e^{\frac{4}{\sigma^2}|\tilde{t}_j-t_j|} \leq \frac{4e^{\frac{4}{\sigma^2}}}{\sigma^2}|\tilde{t}_j-t_j|,$$

where the last inequality holds if $\tilde{t}_j, t_j \in [0,1]$. We can therefore conclude that

$$\|E\|_2 \leq \|E\|_F \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{k} \left(\frac{4e^{\frac{4}{\sigma^2}}}{\sigma^2} \right)^2 |\tilde{t}_j - t_j|^2}$$

$$= \frac{4e^{\frac{4}{\sigma^2}} \sqrt{m}}{\sigma^2} \|\tilde{\mathbf{t}} - \mathbf{t}^*\|_2, \qquad (5.81)$$

provided that $\tilde{t}_j, t_j \in [0,1]$ for all $j = 1, \ldots, k$.

Chapter 6

Perturbation of the dual variable in the noisy setting

In this chapter, we consider a similar setup as in Chapter 5, which we adapt to account for additive noise in the measurements. The main result of the chapter is a bound on the perturbation in the dual variable λ around the minimiser λ^* as a function of the noise w in the measurements.

The aim is to estimate how the source locations $\{t_i\}_{i=1}^k$ and weights $\{a_i\}_{i=1}^k$ are perturbed around their true values by the additive noise w in the measurements. In the previous chapter, we have established how the source locations and weights are perturbed around their true values as the dual variable λ is perturbed around its optimal value λ^*. In the noisy setting, we want to establish a precise quantitative relationship between the perturbations of λ around λ^* and the magnitude of the noise. This, combined with the result in Chapter 5, gives a complete characterisation of the solution of the super-resolution problem when obtained by solving the dual of the TV norm minimisation problem, in the presence of additive noise in the measurements.

We start by discussing in detail the setting under which our results hold in Section 6.1, before giving the main result, Theorem 33, in Section 6.2, and we end the chapter with a detailed proof in Sections 6.3 and 6.4.

6.1 Motivation

As stated in the introduction of the chapter, we now consider the setting where the measurements y_j are corrupted by additive noise, as defined in (1.2):

$$y_j = \int_I \phi_j(t)x(\mathrm{d}t) + w_j = \sum_{i=1}^k a_i\phi_j(t_i) + w_j,$$

for $w_j \neq 0$ and $j = 1, \ldots, m$. Before we state the main result, which gives a relationship between the magnitude of the noise $\|w\|_2$ and the perturbation in λ around its optimal value as a consequence of the noise, we first need to describe the exact mathematical setting under which this result holds. We will then introduce the function $\bar{F}$ in (6.11), whose Jacobian is crucial for this result.

In order to account for noise in the measurements, we consider a slightly modified version of the dual problem (1.9). Specifically, we use an additional box constraint on the dual variable λ and obtain the dual problem given in (1.10) in Chapter 1:

$$\max_{\lambda \in \mathbb{R}^m} y^T \lambda \quad \text{such that} \quad \lambda^T \Phi(t) \leq 1, \quad \forall t \in I,$$
$$\text{and} \quad \|\lambda\|_\infty \leq \tau,$$

where the bound τ in the box constraint is fixed. This is the dual of the problem (see Appendix B.2):

$$\min_{z \geq 0} \left\| y - \int_I \Phi(t) z(\mathrm{d}t) \right\|_1 \quad \text{such that} \quad \|z\|_{TV} \leq \Pi,$$

for some $\Pi > 0$, which is a reasonable choice of the primal problem in the noisy setting, as this formulation accounts for noise in the measurements. This is solved over all non-negative measures z on $I = [0, 1]$.

To motivate the setting in which we apply the implicit function theorem (specifically the form of the function $\bar{F}$ introduced in (6.11) in the next section) to obtain the perturbation result from Theorem 33, consider the exact penalty formulation of (1.10):

$$\min_{\lambda \in \mathbb{R}^m} \Psi_\Pi(\lambda) \quad \text{such that} \quad \|\lambda\|_\infty \leq \tau, \tag{6.1}$$

where

$$\Psi_\Pi(\lambda) = -y^T \lambda + \Pi \cdot \max \left\{ \sup_s \left(\sum_{j=1}^m \lambda_j \phi(s - s_j) - 1 \right), 0 \right\}. \tag{6.2}$$

For a large enough value of Π, a solution to (6.1) which satisfies the constraints in (1.10) is also a solution of (1.10) (see, for example, Section 1.2 in [47]). This is a non-smooth optimisation problem and its solution can be found by using any algorithm based on subgradients, for example the level method [61], which we will discuss briefly in Chapter 7.

A subgradient of $\Psi_\Pi(\lambda)$ has the form:

$$\partial \Psi_\Pi = \begin{cases} -y + \Pi \sum_{i=1}^{k'} \nu_i g(s_i^*), \quad (\nu_1 + \ldots + \nu_{k'} = 1) & \text{if } \sup_s \sum_{j=1}^m \lambda_j \phi(s - s_j) > 1, \\ -y + \Pi \sum_{i=1}^{k'} \nu_i g(s_i^*), \quad (\nu_1 + \ldots + \nu_{k'} \leq 1) & \text{if } \sup_s \sum_{j=1}^m \lambda_j \phi(s - s_j) = 1, \\ -y, & \text{if } \sup_s \sum_{j=1}^m \lambda_j \phi(s - s_j) < 1, \end{cases} \tag{6.3}$$

where $\{s_i^*\}_{i=1}^{k'}$ are the global maximisers of the function $\sum_{j=1}^{m} \lambda_j \phi(s - s_j)$, the vectors $g(s)$ are of the form $g(s) = [\phi(s - s_1), \ldots, \phi(s - s_m)]^T$ and $\nu_i \geq 0$ for all $i = 1, \ldots, k'$. Note that here we apply the formula for the subdifferential of the max function and for the sup function: the subdifferential of max or sup is equal to the convex hull of the subdifferentials of the active functions in the max or the sup respecrively (see for example [47]). The coefficients in the convex combination from the formula for the subgradient of the max function with zero account for the case when $\nu_1 + \ldots + \nu_{k'} < 1$[1].

As in the noise-free setting, we assume here that the dual solution λ^* forms a dual certificate, namely the function $q(s)$ as defined in (5.3) satisfies conditions (5.4) and (5.5). Then, the subdifferential at λ^* has the form:

$$\partial \Psi_\Pi(\lambda^*) = -y + \Pi \sum_{i=1}^{k} \nu_i g(t_i), \tag{6.4}$$

where $\{t_i\}_{i=1}^k$ are the source locations, so the optimality condition for (6.1):

$$0 \in \partial \Psi_\Pi(\lambda^*), \tag{6.5}$$

is equivalent to:

$$y = \Pi \sum_{i=1}^{k} \nu_i g(t_i), \tag{6.6}$$

for some $\nu_1, \ldots, \nu_k \geq 0$ with $\nu_1 + \ldots + \nu_k \leq 1$ and for $w = 0$. Note that, given the definition of y from (1.2), the optimality condition (6.6) is satisfied for

$$\nu_i = \frac{a_i}{\Pi}, \quad \forall i = 1, \ldots, k, \tag{6.7}$$

$$w_j = 0, \quad \forall j = 1, \ldots, m, \tag{6.8}$$

where in order to satisfy $\nu_1 + \ldots + \nu_k \leq 1$, we need Π such that:

$$\Pi \geq a_1 + \ldots + a_k, \tag{6.9}$$

which is the same as the constraint in (1.11).

[1]More specifically, both functions in the max attain their maximum, so we have that $\partial \Psi_\pi = -y + \Pi \left[\alpha_1 \partial \sup_s \left(\sum_{j=1}^{m} \lambda_j \phi(s - s_j) - 1 \right) + \alpha_2 \partial 0 \right]$, with $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$, and therefore $\partial \Psi_\pi = -y + \Pi \sum_{i=1}^{k'} \alpha_1 \nu_i' g(s_i^*)$, with $\nu_1' + \ldots + \nu_{k'}' = 1$ and $0 \leq \alpha_1 \leq 1$.

6.2 Bound on dual variable error as a function of noise

Motivated by the above reasoning, we now apply the quantitative implicit function theorem [57] to a function F of the form:

$$F([\lambda, \nu]^T, w) = \sum_{i=1}^{k} a_i \Phi(t_i^*) - \sum_{i=1}^{k} \nu_i \Phi(t(\lambda)) + w, \tag{6.10}$$

where we know that $F([\lambda^*, a]^T, 0) = 0$ (see equation (6.6)). To simplify the notation, we include the parameter Π in the coefficients ν_i, so in the second sum in F each ν_i actually corresponds to $\Pi\nu_i$, and $\nu_1 + \ldots + \nu_k \leq \Pi$ rather than $\nu_1 + \ldots + \nu_k \leq 1$.

However, note that $F : \mathbb{R}^{m+k} \times \mathbb{R}^m \to \mathbb{R}^m$ and in order to apply the implicit function theorem to obtain the dependence of the first argument of F as a function of the second argument, we need the first argument to be in $\mathbb{R}^m \times \mathbb{R}^m$.

To solve this issue, we first reduce the system by discarding $m-2k$ rows above. The remaining $2k$ rows depend on the conditions that must be met in order for the conditions in the implicit function theorem to be satisfied, and intuitively these correspond to the samples that contain the most information, for example the two samples that are the closest to each source. This leads to the function $F : \mathbb{R}^{m+k} \times \mathbb{R}^{2k} \to \mathbb{R}^{2k}$. The next step is to fix all except k components of λ to the boundary of the box constraint $\|\lambda\|_\infty \leq \tau$. This is justified by the fact that the system $F([\lambda^*, a]^T, 0) = 0$ of $2k$ equations and $m + k$ unknowns $[\lambda^*, a]^T$ is under-determined, and therefore solving it involves setting $m - k$ of the unknowns to arbitrary values. In practice, this is achieved by tuning the box constraint in such a way that the largest components of λ are set to τ or $-\tau$. More specifically, we keep k values of λ which correspond to a subset of the $2k$ samples which have not been discarded. This finally leads to the function which we will denote by $\bar{F}$:

$$\bar{F} : \mathbb{R}^{2k} \times \mathbb{R}^{2k} \to \mathbb{R}^{2k}, \quad \bar{F}([\bar{\lambda}, \nu]^T, \bar{w}) = \sum_{i=1}^{k} a_i \bar{\Phi}(t_i^*) - \sum_{i=1}^{k} \nu_i \bar{\Phi}(t(\lambda)) + \bar{w}, \tag{6.11}$$

where $\bar{\lambda} \in \mathbb{R}^k$ contains only the entries of λ which are not fixed and $\bar{w} \in \mathbb{R}^{2k}$, $\bar{\Phi}(t) \in \mathbb{R}^{2k}$ are the vectors with the entries from w and $\Phi(t)$ respectively corresponding to the equations which have not been dropped. Given the definition of F in (6.11), we can now state the main result of this section, proved in Section 6.3.

Theorem 33. ***(Dependence of*** $\|\lambda - \lambda^*\|_2$ ***on*** w***)*** *Let* λ^* *be the solution to the dual problem* (1.10) *with* $w = 0$*. Given the function* $\bar{F}$ *defined in* (6.11)*, let* J^* *be its Jacobian with respect with the first variable, evaluated at* $([\bar{\lambda}^*, a]^T, 0)$ *and* $\sigma_{\min}(J^*)$ *its smallest singular value. We also assume that the solution* λ^* *forms a dual certificate, namely the function* $q(t)$ *defined in* (5.3) *satisfies the conditions* (5.4) *and* (5.5)*. If* J^* *is invertible and* $\|\bar{w}\|_2 \leq \delta_w$*, then:*

$$\left\|\bar{\lambda} - \bar{\lambda}^*\right\|_2 \leq C_{\lambda^*} \cdot \|\bar{w}\|_2 , \tag{6.12}$$

where

$$C_{\lambda^*} = \frac{2}{\sigma_{\min}(J^*)}, \tag{6.13}$$

$$\delta_w = \frac{\sigma_{\min}(J^*)^2}{4P(m,k,\sigma,\Pi,\tau,C_{t^*})}, \tag{6.14}$$

and

$$P(m,k,\sigma,\Pi,\tau,C_{t^*}) = \sqrt{2}k\left[\frac{1}{\sigma^2}\left(2\sqrt{k}C_{t^*}^2\Pi + 4kC_{t^*}\bar{\Delta}_2\tau\Pi\right)\right.$$
$$\left. + \frac{1}{\sigma}\left(\frac{\sqrt{2k}C_{t^*}}{\sqrt{e}} + 4\sqrt{k}C_{t^*}^2\Pi + \frac{2\sqrt{2}\bar{\Delta}_2\Pi}{\sqrt{e}} + 8kC_{t^*}\bar{\Delta}_2\tau\Pi + \frac{\sqrt{2k}\bar{\Delta}_2\Pi}{\sqrt{e}} + \sqrt{\frac{2}{e}}C_{t^*}\right)\right]$$

where C_{t^*} *is given in* (5.8) *in Theorem 28 and* $\bar{\Delta}_2$ *is given in* (6.72) *in the proof of Lemma 34.*

The theorem above makes explicit the dependence of the perturbation in the dual variable $\bar{\lambda}$ around the solution $\bar{\lambda}^*$ on the additive noise $\bar{w}$ in the measurement vector $\bar{y}$. This is a linear relation where the constant depends on the specific configuration of the problem we are solving, namely the locations and weights of the sources, and width of the Gaussian and the sampling locations. The theorem also gives an upper bound on the magnitude of the noise where this result holds as a function of the same parameters.

Moreover, under the assumption that $\Pi, C_{t^*}, \tau \geq 1$, we can write the order of P as:

$$P \approx \frac{k\sqrt{k}\Pi C_{t^*}}{\sigma^2}\left(C_{t^*} + \sqrt{k}\bar{\Delta}_2\tau\right), \tag{6.15}$$

which gives us a clearer idea about the main parameters that affect the magnitude of the noise for which our perturbation results hold.

One observation we want to make is that, while the above result only applies to a subset of the components of λ and w, which components are selected is not arbitrary.

The choice of the components of λ and w reflects which samples s_j contain the most information, and therefore which components of the noise vector w affect the solution to the optimisation problem the most. More specifically, in order for the Jacobian J^* to be invertible, we are lead to select the samples (and therefore the components of λ and w) that satisfy this condition the best, namely the ones that are the closest to the source locations. We discuss this aspect in more detail in Section 6.2.1.

Lastly, note that the results in Section 5.2 and Section 6.2 refer to different optimisation problems: the duals (1.9) and (1.10) of problems (1.8) and (1.11) respectively. The proofs of the perturbation results rely on the property that the dual solution λ forms a dual certificate, the global maximisers of which give the locations of the point sources in the input signal x, with the additional bound on λ from (1.10) being used in the proof of Theorem 33. Combined with the fact that we do not use the exact form of the primal problems, we conclude that the two results are compatible and the perturbation analysis applies to the dual pair (x, λ^*) as the solution to the noise-free primal and dual problems and their perturbation due to noise in the measurements.

6.2.1 Discussion

One of the conditions in Theorem 33 is that the Jacobian J^* is invertible. While we do not provide a rigorous analysis of the conditions in which this is satisfied, in this section we discuss in more detail what the condition requires and give further motivation for why it is true in a reasonable scenario. Specifically, we assume that the samples that are used for calculating the Jacobian are the closest samples to the sources (for each source location, we select the two samples that are the closest to it), so the rows in the system given by $\bar{F}$ in (6.11) correspond to these samples. Consequently, the entries in λ and the entries in the noise vector w also correspond to the same samples.

Recall that J^* is the Jacobian of the function $\bar{F}$ from (6.11) with respect to the first argument. The entries in J^* are:

$$\partial_{\lambda_l} F([\lambda, \nu]^T, w)\Big|_{\substack{\lambda=\lambda^* \\ \nu=a \\ w=0}} = -\sum_{i=1}^{k} a_i \phi'(t_i^* - s_j) \partial_{\lambda_l} t_i(\lambda^*) \tag{6.16}$$

$$= \sum_{i=1}^{k} \frac{a_i \phi'(t_i^* - s_j) \phi'(t_i^* - s_l)}{q''(t_i^*)}, \tag{6.17}$$

for $l = 1, \dots, k$, $j = 1, \dots, 2k$ and

$$\partial_{\nu_l} F([\lambda, \nu]^T, w)\Big|_{\substack{\lambda=\lambda^* \\ \nu=a \\ w=0}} = -\phi(t_l^* - s_j), \tag{6.18}$$

for $l = 1, \dots, k$, $j = 1, \dots, 2k$, where in the first equality we used (5.17) with (5.21) and (5.22) plugged in, so the result holds under the conditions in Theorem 28, namely for λ with $\|\lambda - \lambda^*\|_2 \leq \delta_\lambda$, where δ_λ is given in (5.6).

Writing J^* as

$$J^* = [J_\lambda J_\nu], \tag{6.19}$$

where the entries in the blocks J_λ and J_ν are given by (6.17) and (6.18) respectively, we have that:

$$J_\lambda = \sum_{i=1}^{k} \frac{a_i}{q''(t_i^*)} \Phi'(t_i^*)\Phi'(t_i^*)^T, \tag{6.20}$$

and

$$J_\nu = -[\Phi(t_1^*) \dots \Phi(t_k^*)], \tag{6.21}$$

where

$$\Phi(t) = [\phi(t - s_1), \dots, \phi(t - s_{2k})]^T, \tag{6.22}$$

$$\Phi'(t) = [\phi'(t - s_1), \dots, \phi'(t - s_{2k})]^T. \tag{6.23}$$

Note that $rank(J_\nu) = k$ by the T-systems property of the Gaussian (assuming that the $t_1 \leq \dots \leq t_k$ and $s_1 \leq \dots s_{2k}$) and in order for the matrix J to be invertible we need $rank(J) = 2k$. By rewriting the columns of J_λ, we have that:

$$J^* = \left[\sum_{i=1}^{k} \frac{a_i \phi'(t_i^* - s_1)}{q''(t_i^*)} \Phi'(t_i^*) \quad \dots \quad \sum_{i=1}^{k} \frac{a_i \phi'(t_i^* - s_k)}{q''(t_i^*)} \Phi'(t_i^*) \quad -\Phi(t_1^*) \quad \dots \quad -\Phi(t_k^*) \right], \tag{6.24}$$

and by taking its determinant and using the multi-linearity property of the determinant with respect to its columns, we have that:

$$\det(J^*) = (-1)^k \frac{a_1 \dots a_k}{q''(t_1^*) \dots q''(t_k^*)} \cdot \sum_{l=1}^{k!} \left(\prod_{i=1}^{k} \phi'(P_l(t_i^*) - s_i) \right) \left| P_l(\Phi'(t_1^*)) \quad \dots \quad P_l(\Phi'(t_k^*)) \quad \Phi(t_1^*) \quad \dots \quad \Phi(t_k^*) \right|, \tag{6.25}$$

where P_l for $l = 1, \dots, k!$ are the permutations of k elements. Note that when we expand the determinant, the terms in the final sum are determinants with all the possible combinations of the vectors in each sum, which results in many determinants having repeated columns, so they are equal to zero. The only non-zero determinants in the resulting sum are the ones where the first k columns are the vectors $\{\Phi'(t_i^*)\}_{i=1}^k$

and their permutations, multiplied by the corresponding constants. We now order the columns of the determinant:

$$\begin{aligned}\det(J^*) &= (-1)^k \frac{a_1 \dots a_k}{q''(t_1^*) \dots q''(t_k^*)} \\ &\quad \sum_{l=1}^{k!} sign(P_i) \left(\prod_{i=1}^{k} \phi'(P_l(t_i^*) - s_i) \right) \left| \Phi(t_1^*) \quad \Phi'(t_1^*) \quad \dots \quad \Phi(t_k^*) \quad \Phi'(t_k^*) \right|, \\ &= (-1)^k \frac{a_1 \dots a_k}{q''(t_1^*) \dots q''(t_k^*)} \left| \Phi(t_1^*) \quad \Phi'(t_1^*) \quad \dots \quad \Phi(t_k^*) \quad \Phi'(t_k^*) \right| \\ &\quad \sum_{l=1}^{k!} sign(P_l) \left(\prod_{i=1}^{k} \phi'(P_l(t_i^*) - s_i) \right), \end{aligned} \tag{6.26}$$

where by $sign(P_i)$ we denote the sign of the determinant corresponding to the permutation P_i after reordering the columns as above. Because of the extended T-system property of the Gaussian function [50], the determinant above is strictly positive. The dominant term in the sum is the one corresponding to the identity permutation, where for each $i = 1, \dots, k$, the sample s_i is the closest sample to the source location t_i^*. As the samples get further, the terms of the sum approach zero. This can be expressed more quantitatively by imposing explicit conditions on the distances between the closest samples and the sources, the separation of sources and the separation of samples, as we have done, for example, in Part I of the thesis.

6.3 Proof of Theorem 33 (Dependence of the dual variable error on noise)

We apply the quantitative implicit function theorem to the function $\bar{F}$ defined in (6.11). However, in order to simplify the notation, we use F, λ, w instead of $\bar{F}, \bar{\lambda}, \bar{w}$ respectively throughout this proof. The partial derivatives of F are:

$$\partial_{\lambda_l} F_j = -\sum_{i=1}^{k} \nu_i \phi'(t_i(\lambda) - s_j) \partial_{\lambda_l} t_i(\lambda) \quad l = 1, \dots, k, \quad j = 1, \dots, 2k, \tag{6.27}$$

$$\partial_{\nu_l} F_j = -\phi(t_l(\lambda) - s_j), \quad l = 1, \dots, k, \quad j = 1, \dots, 2k, \tag{6.28}$$

$$\partial_{w_l} F_j = \begin{cases} 1, & \text{if} \quad l = j, \\ 0, & \text{otherwise}, \end{cases} \qquad l, j = 1, \dots, 2k. \tag{6.29}$$

Let $\gamma = [\lambda, \nu]^T$ and $\gamma^* = [\lambda^*, a]^T$, so that we can write $F([\lambda, \nu]^T, w)$ as $F(\gamma, w)$ and $F(\gamma^*, 0) = 0$. In order to apply the implicit function theorem, the following conditions must be satisfied:

1. $\partial_\gamma F(\gamma^*, 0)$ is invertible,

2. We choose the radius δ_γ of the ball V_{δ_γ} around γ where the result of the quantitative implicit function theorem holds:

$$\sup_{(\gamma,w)\in V_{\delta_\gamma}} \left\| I - \left[\partial_\gamma F(\gamma^*, 0)\right]^{-1} \partial_\gamma F(\gamma, w)\right\|_2 \leq \frac{1}{2}, \tag{6.30}$$

3. The radius δ_w of the ball around $w^* = 0$ that contains w is:

$$\delta_w = (2M_w B_{\delta_\gamma})^{-1}\delta_\gamma, \tag{6.31}$$

where

$$B_{\delta_\gamma} = \sup_{(\gamma,w)\in V_{\delta_\gamma}} \|\partial_w F(\gamma, w)\|_2, \tag{6.32}$$

$$M_w = \|\partial_\gamma F(\gamma^*, 0)^{-1}\|_2. \tag{6.33}$$

The first condition is also one of the conditions in the theorem, and it has been discussed in Section 6.2.1. We now need to establish the two radii for the balls of the perturbations.

Perturbation radii

Before proceeding to calculating the radii of the balls where the implicit function theorem holds, we need to state the following lemma, which allows us to write the Jacobian of F with respect to the first variable as a sum of the Jacobian evaluated at $(\gamma^*, w^*) = ([\lambda^*, a]^T, 0)$ and a perturbation matrix, whose norm is bounded explicitly. The proof of Lemma 34 is given in Section 6.4.

Lemma 34. ***(Bound on the perturbation of the Jacobian of*** F***)*** *Let* $J(\lambda, \nu, w)$ *be the Jacobian of* $F(\gamma, w)$ *with respect to* $\gamma = [\lambda, \nu]^T$ *and* δ_γ *an upper bound on the perturbation of* $\gamma^* = [\lambda^*, a]^T$*, namely:*

$$\left\| \begin{bmatrix} \lambda - \lambda^* \\ \nu - a \end{bmatrix} \right\|_2 \leq \delta_\gamma.$$

Then:

$$J(\lambda, \nu, w) = J(\lambda^*, a, 0) + E, \tag{6.34}$$

with

$$\|E\|_F \leq P(k, \sigma, \Pi, \tau, C_{t^*}) \cdot \delta_\gamma, \tag{6.35}$$

where:

$$P(k,\sigma,\Pi,\tau,C_{t^*}) = \sqrt{2}k\Bigg[\frac{1}{\sigma^2}\left(2\sqrt{k}C_{t^*}^2\Pi + 4kC_{t^*}\bar{\Delta}_2\tau\Pi + 2C_{t^*}\right) + \frac{1}{\sigma}\left(\frac{\sqrt{2k}C_{t^*}}{\sqrt{e}} + 4\sqrt{k}C_{t^*}^2\Pi + \frac{2\sqrt{2}\bar{\Delta}_2\Pi}{\sqrt{e}} + 8kC_{t^*}\bar{\Delta}_2\tau\Pi + \frac{\sqrt{2k}\bar{\Delta}_2\Pi}{\sqrt{e}}\right)\Bigg], \tag{6.36}$$

for $\|\lambda - \lambda^*\| \leq \delta_\lambda$*, where* δ_λ *and* C_{t^*} *are given in* (5.6) *and* (5.8) *respectively in Theorem 28 and* $\bar{\Delta}_2$ *is given in* (6.72) *in the proof.*

We can now use Lemma 34 to write

$$\partial_\gamma F(\gamma, w) = \partial_\gamma F(\gamma^*, 0) + E, \tag{6.37}$$

then

$$\begin{aligned} I - \left[\partial_\gamma F(\gamma^*,0)\right]^{-1}\partial_\gamma F(\gamma,w) &= I - \left[\partial_\gamma F(\gamma^*,0)\right]^{-1}\left[\partial_\gamma F(\gamma^*,0) + E\right] \\ &= -\left[\partial_\gamma F(\gamma^*,0)\right]^{-1}E, \end{aligned} \tag{6.38}$$

so

$$\begin{aligned} \left\|I - \left[\partial_\gamma F(\gamma^*,0)\right]^{-1}\partial_\gamma F(\gamma,w)\right\|_2 &\leq \left\|\left[\partial_\gamma F(\gamma^*,0)\right]^{-1}\right\|_2 \cdot \|E\|_F \\ &\leq \frac{\|E\|_F}{\sigma_{\min}(\partial_\gamma F(\gamma^*,0))} \\ &\leq \frac{P(k,\sigma,\Pi,\tau,C_{t^*})}{\sigma_{\min}(\partial_\gamma F(\gamma^*,0))}\cdot\delta_\gamma, \end{aligned} \tag{6.39}$$

where $P \cdot \delta_\gamma$ is the upper bound on $\|E\|_F$ given in (6.36).

Therefore, from the condition that the right-hand side of the last inequality is less than or equal to $\frac{1}{2}$, we choose the radius δ_γ to be:

$$\delta_\gamma = \frac{\sigma_{\min}(\partial_\gamma F(\gamma^*,0))}{2P(k,\sigma,\Pi,\tau,C_{t^*})}. \tag{6.40}$$

Using (6.29), we have that

$$B_{\delta_\gamma} = 1. \tag{6.41}$$

Then

$$M_w = \frac{1}{\sigma_{\min}(\partial_\gamma F(\gamma^*,0))}, \tag{6.42}$$

so, using (6.40), we obtain

$$\delta_w = \frac{\sigma_{\min}(\partial_\gamma F(\gamma^*,0))^2}{4P(k,\sigma,\Pi,\tau,C_{t^*})}. \tag{6.43}$$

Applying the quantitative implicit function theorem

Having calculated the radii where the quantitative implicit function theorem holds, we apply it to obtain:

$$\partial_w g(w) = -\left[\partial_1 F(g(w), w)\right]^{-1}, \tag{6.44}$$

where ∂_1 is the partial derivative with respect with the first argument and $g(w)$ gives the dependence of $[\lambda, \nu]^T$ on w. Specifically, we write:

$$\lambda_i(w) = g_i(w) \quad \text{for} \quad i = 1, \ldots, k, \tag{6.45}$$

$$\nu_i(w) = g_{k+i}(w) \quad \text{for} \quad i = 1, \ldots, k. \tag{6.46}$$

Let $J(\lambda, \nu, w) = \partial_1 F([\lambda, \nu]^T, w)$, where $\lambda = \lambda(w)$ and $\nu = \nu(w)$ by (6.45) and (6.46). Lemma 34 gives

$$J(\lambda, \nu, w) = J(\lambda^*, a, 0) + E, \tag{6.47}$$

so E is the perturbation of $J(\lambda^*, a, 0)$ due to perturbed λ, ν, w and a bound on $\|E\|_F$ is given in the lemma.

The following result allows us to use the upper bound on the norm of the perturbation given by Lemma 34 in order to lower bound the smallest singular value of J.

Lemma 35. *Let $J \in \mathbb{R}^{m \times n}$. If $J = A + E$, then*

$$\sigma_{\min}(J) \geq \sigma_{\min}(A) - \|E\|_F.$$

Proof. We have that

$$\begin{aligned}
\sigma(J) &= \min_{\|v\|_2=1} \max_{\|u\|_2=1} u^T (A + E) v \\
&\geq \min_{\|v\|_2=1} \max_{\|u\|_2=1} u^T A v - \max_{\|v\|_2=1} \max_{\|u\|_2=1} u^T E v \\
&\geq \sigma_{\min}(A) - \|E\|_F.
\end{aligned}$$

□

We now apply Lemma 35:

$$\begin{aligned}
\frac{1}{\sigma_{\min}(J(\lambda, \nu, w))} &\leq \frac{1}{\sigma_{\min}(J(\lambda^*, a, 0)) - \|E\|_F} = \frac{1}{\sigma_{\min}(J(\lambda^*, a, 0)) \left(1 - \frac{\|E\|_F}{\sigma_{\min}(J(\lambda^*, a, 0))}\right)} \\
&\leq \frac{1}{\sigma_{\min}(J(\lambda^*, a, 0))} \cdot \left(1 + \frac{2\|E\|_F}{\sigma_{\min}(J(\lambda^*, a, 0))}\right),
\end{aligned} \tag{6.48}$$

for

$$\frac{\|E\|_F}{\sigma_{\min}(J(\lambda^*, a, 0))} \leq \frac{1}{2}, \tag{6.49}$$

where we used the fact that $(1-x)^{-1} \leq 1 + 2x$ for $x \in [0, \frac{1}{2}]$. Note that the condition above is the same as the condition that the right hand side of (6.39) is less than or equal to $\frac{1}{2}$, which is satisfied for our choice of δ_γ and δ_w.

From (6.44) and (6.48), we have that:

$$\begin{aligned} \|\partial_w g(w)\|_2 &= \frac{1}{\sigma_{\min}(J(\lambda^*, a, 0) + E)} \\ &\leq \frac{1}{\sigma_{\min}(J(\lambda^*, a, 0))} \left(1 + \frac{2}{\sigma_{\min}(J(\lambda^*, a, 0))} \cdot \|E\|_F\right) \\ &\leq \frac{2}{\sigma_{\min}(J(\lambda^*, a, 0))}, \end{aligned} \tag{6.50}$$

where $\|E\|_F$ is upper bounded in (6.36) and w, λ and ν satisfy

$$\|w\|_2 \leq \delta_w, \quad \|\lambda - \lambda^*\|_2 \leq \delta_\gamma, \quad \|\nu - a\|_2 \leq \delta_\gamma.$$

The first-order Taylor expansion of $g(w)$ around $w = 0$ is:

$$g(w) = g(0) + \partial_w g(w_\delta)^T w, \tag{6.51}$$

for some w_δ on the segment between the zero vector and w. Noting that $g(w)$ is our notation for the vector:

$$g(w) = \begin{bmatrix} \lambda(w) \\ \nu(w) \end{bmatrix}, \tag{6.52}$$

with $\lambda(0) = \lambda^*$ and $\nu(0) = a$, from (6.51) we have that:

$$\begin{aligned} \left\| \begin{bmatrix} \lambda(w) - \lambda^* \\ \nu(w) - a \end{bmatrix} \right\|_2 &= \left\| \partial_w g(w_\delta)^T w \right\|_2 \\ &\leq \|\partial_w g(w_\delta)\|_2 \cdot \|w\|_2, \end{aligned} \tag{6.53}$$

for w, λ and ν such that

$$\|w\|_2 \leq \delta_w, \quad \|\lambda - \lambda^*\|_2 \leq \delta_\gamma, \quad \|\nu - a\|_2 \leq \delta_\gamma,$$

where we use the bound from (6.50).

6.4 Proof of Lemma 34 (Bound on the Jacobian perturbation)

Let $J(\lambda, \nu, w) = \partial_1 F([\lambda, \nu]^T, w)$, where $\lambda = \lambda(w)$ and $\nu = \nu(w)$ by (6.45) and (6.46), and we want to write J in the form

$$J(\lambda, \nu, w) = J(\lambda^*, a, 0) + E \tag{6.54}$$

i.e. E is the perturbation of $J(\lambda^*, a, 0)$ due to perturbed λ, ν, w. In order to apply Lemma 35, we need an upper bound on $\|E\|_F$, so we need to upper bound each entry of E. Let

$$J = [J_1 J_2]\,, \tag{6.55}$$

where J_1 corresponds to the terms (6.27) and J_2 to the terms (6.28) and

$$E = [E_1 E_2] \tag{6.56}$$

the corresponding perturbation terms.

Entries in J_1

For $i = 1, \ldots, k$ and $j = 1, \ldots, 2k$:

$$\begin{aligned}
J_{1_{j,i}} &= -\sum_{p=1}^{k} (\nu_p - a_p + a_p)\phi'(t_p(\lambda) - t_p^* + t_p^* - s_j)\partial_{\lambda_i} t_p(\lambda) \\
&= -\sum_{p=1}^{k} \partial_{\lambda_i} t_p(\lambda) \left[a_p \phi'(t_p^* - s_j + t_p(\lambda) - t_p^*) + (\nu_p - a_p)\phi'(t_p^* - s_j + t_p(\lambda) - t_p^*)\right] \\
&= -\sum_{p=1}^{k} \partial_{\lambda_i} t_p(\lambda) \big[a_p \phi'(t_p^* - s_j) + a_p(t_p(\lambda) - t_p^*)\phi''(\xi_{j,p}) + (\nu_p - a_p)\phi'(t_p^* - s_j) \\
&\quad + (\nu_p - a_p)(t_p(\lambda) - t_p^*)\phi''(\xi_{j,p})\big] \\
&= -\sum_{p=1}^{k} \partial_{\lambda_i} t_p(\lambda) \left(a_p \phi'(t_p^* - s_j) + \Delta_{1_{j,p}}\right),
\end{aligned} \tag{6.57}$$

where

$$\Delta_{1_{j,p}} = a_p(t_p(\lambda) - t_p^*)\phi''(\xi_{j,p}) + (\nu_p - a_p)\phi'(t_p^* - s_j) + (\nu_p - a_p)(t_p(\lambda) - t_p^*)\phi''(\xi_{j,p}), \tag{6.58}$$

for some $\xi_{j,p} \in [t_p^* - s_j - |t_p - t_p^*|, t_p^* - s_j + |t_p - t_p^*|]$. The factor involving the partial derivative in (6.57) has the same form as (5.17) so in order to bound it we write the Taylor expansion of (5.17) around λ^*:

$$\partial_\lambda t_p(\lambda) = \partial_\lambda t_p(\lambda^*) + \partial^2_{\lambda\lambda} t_p(\lambda_\delta)\,(\lambda - \lambda^*)\,, \tag{6.59}$$

for some λ_δ on the segment between λ and λ^*. By using (5.17) with (5.21) and (5.22), the entry i, l in the Hessian matrix $H = \partial^2_{\lambda\lambda} t_p(\lambda_\delta)$ is

$$\left(H^T\right)_{i,l} = \frac{F_{i,l}(\lambda)}{\left[\sum_{j=1}^m \lambda_j \phi''(t_p(\lambda) - s_j)\right]^2}, \tag{6.60}$$

for $i, l = 1, \ldots, k$, where

$$\begin{aligned} F_{i,l}(\lambda) = &- \phi''(t_p(\lambda) - s_i)\partial_{\lambda_l} t_p(\lambda) \sum_{j=1}^m \lambda_j \phi''(t_p(\lambda) - s_j) \\ &+ \phi'(t_p(\lambda) - s_i) \left(\sum_{j=1}^m \lambda_j \phi'''(t_p(\lambda) - s_j)\partial_{\lambda_l} t_p(\lambda) + \phi''(t_p(\lambda) - s_l) \right). \end{aligned} \tag{6.61}$$

Note that in the denominator (6.60) we use all m entries of λ and samples due to how we defined the function from (5.17), and the same is true for the sums in (6.61). From (6.59) and (6.61), we then write:

$$\partial_{\lambda_i} t_p(\lambda) = \partial_{\lambda_i} t_p(\lambda^*) + \Delta_{2_{i,p}}, \tag{6.62}$$

where

$$\Delta_{2_{i,p}} = \sum_{l=1}^k \frac{(\lambda_l - \lambda_l^*) F_{i,l}(\lambda_\delta)}{\left[\sum_{j=1}^m \lambda_j \phi''(t_p(\lambda_\delta) - s_j)\right]^2}. \tag{6.63}$$

Note that l goes up to k because we only work with k entries in λ. Therefore, we have that:

$$J_{1_{j,i}} = -\sum_{p=1}^k \left(\partial_{\lambda_i} t_p(\lambda^*) + \Delta_{2_{i,p}}\right) \left(a_p \phi'(t_p^* - s_j) + \Delta_{1_{j,p}}\right), \tag{6.64}$$

where

$$\Delta_{1_{j,p}} = O(|t_p - t_p^*| + |\nu_p - a_p|), \tag{6.65}$$

$$\Delta_{2_{i,p}} = O(\|\lambda - \lambda^*\|_2), \tag{6.66}$$

for $i = 1, \ldots, k$, $j = 1, \ldots, 2k$ and $p = 1, \ldots, k$. The next step now is to upper bound $|\Delta_{1_{j,p}}|$ and $|\Delta_{2_{i,p}}|$.

Bounding $\Delta_{1_{j,p}}$

By the triangle inequality, we have that:

$$\begin{aligned} |\Delta_{1_{j,p}}| &\leq |a_p||t_p(\lambda) - t_p^*||\phi''(\xi_{j,p})| + |\nu_p - a_p||\phi'(t_p^* - s_j)| + |\nu_p - a_p||t_p(\lambda) - t_p^*||\phi''(\xi_{j,p})| \\ &\leq |a_p||t_p(\lambda) - t_p^*|\frac{2}{\sigma^2} + |\nu_p - a_p|\frac{\sqrt{2}}{\sqrt{e}\sigma} + |\nu_p - a_p||t_p(\lambda) - t_p^*|\frac{2}{\sigma^2} =: \bar{\Delta}_{1_p}, \end{aligned} \tag{6.67}$$

for $j = 1, \ldots, 2k$ and $p = 1, \ldots, k$, where we have used the maxima of the Gaussian and its derivatives given in footnote 3.

Bounding $\Delta_{2_{i,p}}$

By applying the Cauchy-Schwartz inequality, we have that:

$$|\Delta_{2_{i,p}}| \leq \frac{1}{\left|\sum_{j=1}^{m} \lambda_j \phi''(t_p(\lambda_\delta) - s_j)\right|^2} \cdot \left\| \begin{bmatrix} F_{i,1}(\lambda_\delta) \\ \vdots \\ F_{i,k}(\lambda_\delta) \end{bmatrix} \right\|_2 \cdot \|\lambda - \lambda^*\| 2 \tag{6.68}$$

We now bound $|F_{i,l}|$ for $i, l = 1, \ldots, k$:

$$\begin{aligned}
|F_{i,l}(\lambda_\delta)| \leq & \, |\phi''(t_p(\lambda_\delta) - s_i)| \left|\partial_{\lambda_l} t_p(\lambda_\delta)\right| \left|\sum_{j=1}^{m} \lambda_j \phi''(t_p(\lambda_\delta) - s_j)\right| \\
& + |\phi'(t_p(\lambda_\delta) - s_i)| \left(\left|\partial_{\lambda_l} t_p(\lambda_\delta)\right| \left|\sum_{j=1}^{m} \lambda_j \phi'''(t_p(\lambda_\delta) - s_j)\right| + |\phi''(t_p(\lambda_\delta) - s_l)| \right) \\
\leq & \, \frac{2C_{t^*}}{\sigma^2} \|\lambda_\delta\|_2 \cdot \left\| \left[\phi''(t_p(\lambda_\delta) - s_j)\right]_{j=1}^{m} \right\|_2 \\
& + \frac{\sqrt{2}}{\sqrt{e}\sigma} \left(C_{t^*} \|\lambda_\delta\|_2 \cdot \left\| \left[\phi'''(t_p(\lambda_\delta) - s_j)\right]_{j=1}^{m} \right\|_2 + \frac{2}{\sigma^2} \right) \\
\leq & \, \frac{2C_{t^*}}{\sigma^2} \|\lambda_\delta\|_2 \cdot \frac{2\sqrt{m}}{\sigma^2} + \frac{\sqrt{2}}{\sqrt{e}\sigma} \left(C_{t^*} \|\lambda_\delta\|_2 \cdot \frac{c\sqrt{m}}{\sigma^3} + \frac{2}{\sigma^2} \right),
\end{aligned} \tag{6.69}$$

where we used the Cauchy-Schwartz inequality, the bounds in footnote 3 and C_{t^*} from (5.8). Therefore, the above inequality holds for $\lambda_\delta \in \mathcal{B}(\lambda^*, \delta_\lambda)$ with δ_λ from (5.6).

The final bound on $|F_{i,l}|$ is

$$|F_{i,l}| \leq \frac{c_2 C_{t^*} \sqrt{m} \|\lambda_\delta\|_2}{\sigma^4} + \frac{2\sqrt{2}}{\sqrt{e}\sigma^3}, \tag{6.70}$$

where $c_2 = 4 + \frac{c\sqrt{2}}{\sqrt{e}} \approx 7.3484$, for $i, l = 1, \ldots, k$.

The next step is to obtain a lower bound on the denominator in (6.68). By adding and subtracting λ_j^* to λ_j and applying the reverse triangle inequality, we obtain:

$$\begin{aligned}
\left|\sum_{j=1}^{m} \lambda_j \phi''(t_p(\lambda) - s_j)\right| &\geq B' - \left|\sum_{j=1}^{m} (\lambda_j - \lambda_j^*) \phi''(t_p(\lambda) - s_j)\right| \\
&\geq B' - \frac{2\sqrt{m}\|\lambda - \lambda^*\|_2}{\sigma^2},
\end{aligned} \tag{6.71}$$

where B' has the same form as B in (5.28), except that we evaluate it at a different $t(\lambda_\delta)$ and possibly a subset of the entries in λ^*, and then we apply Cauchy-Schwartz inequality and the bound in footnote 3. For B', the bound in (5.36) is valid. By combining (6.68), (6.70) and (6.71), we obtain the final bound on $|\Delta_{2_{i,p}}|$:

$$|\Delta_{2_{i,p}}| \leq \bar{\Delta}_2 \cdot \|\lambda - \lambda^*\|_2 \tag{6.72}$$

for $i = 1, \ldots, k$ and $p = 1, \ldots, k$, where

$$\bar{\Delta}_2 = \frac{\left(c_2 C_t \sqrt{m}\|\lambda_\delta\|_2 + \frac{2\sqrt{2}}{\sqrt{e}}\sigma\right)\sqrt{k}}{\left(\sigma^2 B' - 2\sqrt{m}\|\lambda - \lambda^*\|_2\right)^2}. \tag{6.73}$$

Therefore, from (6.64) and by using the definitions of $\bar{\Delta}_{1_p}$ and $\bar{\Delta}_2$ from (6.67) and (6.72) respectively, we have that:

$$\begin{aligned}
|E_{1_{j,i}}| &= \left|\sum_{p=1}^{k} \partial_{\lambda_i} t_p(\lambda^*)\Delta_{1_{j,p}} + a_p\phi'(t_p^* - s_j)\Delta_{2_{i,p}} + \Delta_{1_{j,p}}\Delta_{2_{i,p}}\right| \\
&\leq C_t \sum_{p=1}^{k} \bar{\Delta}_{1_p} + \|\lambda - \lambda^*\|_2 \bar{\Delta}_2 \|a\|_2 \cdot \left\|\left[\phi'(t_p^* - s_j)\right]_{p=1}^{k}\right\|_2 + \|\lambda - \lambda^*\|_2 \bar{\Delta}_2 \sum_{p=1}^{k} \bar{\Delta}_{1_p} \\
&\leq (C_t + \|\lambda - \lambda^*\|_2 \bar{\Delta}_2)\Big(\frac{2}{\sigma^2}\|a\|_2\|t(\lambda) - t^*\|_2 + \frac{\sqrt{2}}{\sqrt{e}\sigma}\|\nu - a\|_1 \\
&\quad + \frac{2}{\sigma}\|\nu - a\|_2\|t(\lambda) - t^*\|_2\Big) + \frac{\sqrt{2k}}{\sqrt{e}\sigma}\|a\|_2\bar{\Delta}_2 \\
&\leq (C_t + \|\lambda - \lambda^*\|_2 \bar{\Delta}_2)\Big(\frac{2\sqrt{k}C_{t^*}}{\sigma^2}\|a\|_2\|\lambda - \lambda^*\|_2 + \frac{\sqrt{2}}{\sqrt{e}\sigma}\|\nu - a\|_1 \\
&\quad + \frac{2\sqrt{k}C_{t^*}}{\sigma}\|\nu - a\|_2\|\lambda - \lambda^*\|_2\Big) + \frac{\sqrt{2k}}{\sqrt{e}\sigma}\|a\|_2\|\lambda - \lambda^*\|_2\bar{\Delta}_2 =: \bar{E}_1
\end{aligned} \tag{6.74}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, 2k$.

Entries in J_2

By adding and subtracting t_j^* then taking a Taylor expansion like before, we obtain:

$$\begin{aligned}
J_{2_{j,i}} &= -\phi(t_i^* - s_j + t_i(\lambda) - t_i^*) \\
&= -\phi(t_i^* - s_j) - (t_i(\lambda) - t_i^*)\phi'(\xi_j) \\
&= -\phi(t_i^* - s_j) - E_{2_{i,j}},
\end{aligned} \tag{6.75}$$

for some $\xi_j \in [t_i^* - s_j - |t_i(\lambda) - t_i^*|, t_i^* - s_j + |t_i(\lambda) - t_i^*|]$ and $E_{2_{j,i}}$ is the perturbation term. Then:

$$|E_{2_{j,i}}| \leq |t_i(\lambda) - t_i^*| \cdot \frac{\sqrt{2}}{\sigma\sqrt{e}}, \tag{6.76}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, 2k$.

Putting everything together

We have that

$$
\begin{aligned}
\|E\|_F &= \sqrt{\sum_{i=1}^{k}\sum_{j=1}^{2k} E_{1_{j,i}}^2 + \sum_{i=1}^{k}\sum_{j=1}^{2k} E_{2_{j,i}}^2} \\
&\leq \sqrt{2k^2\bar{E}_1^2 + \frac{4k}{\sigma^2 e}\sum_{i=1}^{k}|t_i(\lambda) - t_i^*|^2} \\
&\leq k\bar{E}_1\sqrt{2} + \frac{2\sqrt{k}}{\sigma\sqrt{e}}\|t(\lambda) - t^*\|_2 \\
&\leq k\bar{E}_1\sqrt{2} + \frac{2C_{t^*}k}{\sigma\sqrt{e}}\|\lambda - \lambda^*\|_2, \qquad (6.77)
\end{aligned}
$$

where we have used the bounds on the entries of E_1 and E_2 from (6.74) and (6.76) and Theorem 28, so this result holds for $\lambda \in \mathcal{B}(\lambda^*, \delta_\lambda)$ for δ_λ defined in the theorem. Finally, by substituting the expression of $\bar{E}_1$ from (6.74), we obtain:

$$
\begin{aligned}
\|E\|_F \leq \sqrt{2}k\Bigg[&\left(C_{t^*} + \|\lambda - \lambda^*\|_2\bar{\Delta}_2\right)\left(\frac{2C_{t^*}\sqrt{k}}{\sigma^2}\|a\|_2\|\lambda - \lambda^*\|_2\right. \\
&\left. + \frac{\sqrt{2}}{\sqrt{e}\sigma}\|\nu - a\|_1 + \frac{2C_{t^*}\sqrt{k}}{\sigma}\|\nu - a\|_2\|\lambda - \lambda^*\|_2\right) + \frac{\sqrt{2k}}{\sqrt{e}\sigma}\|a\|_2\|\lambda - \lambda^*\|_2\bar{\Delta}_2\Bigg] \\
&+ \frac{2C_{t^*}k}{\sigma\sqrt{e}}\|\lambda - \lambda^*\|_2. \qquad (6.78)
\end{aligned}
$$

Let δ_γ be a bound on the perturbation:

$$
\left\| \begin{bmatrix} \lambda - \lambda^* \\ \nu - a \end{bmatrix} \right\|_2 \leq \delta_\gamma, \qquad (6.79)
$$

and therefore:

$$
\|\lambda - \lambda^*\|_2 \leq \delta_\gamma \qquad \text{and} \qquad \|\nu - a\|_2 \leq \delta_\gamma. \qquad (6.80)
$$

We also have that:

$$
\|\nu - a\|_2 \leq \|\nu - a\|_1 \leq \|\nu\|_1 + \|a\|_1 \leq 2\Pi, \qquad (6.81)
$$

where we used that $\nu_1 + \ldots + \nu_k \leq \Pi$ and the fact that $x = \sum_{p=1}^{k} a_p\delta_{t_p}$ is the solution to (1.11), so it satisfies $\|x\|_{TV} = \|a\|_1 \leq \Pi$.

Similarly, we have that:

$$
\begin{aligned}
\|\lambda - \lambda^*\|_2 &\leq \|\lambda\|_2 + \|\lambda^*\|_2 \leq \sqrt{k}\|\lambda\|_\infty + \sqrt{k}\|\lambda^*\|_\infty \\
&\leq 2\sqrt{k}\tau, \qquad (6.82)
\end{aligned}
$$

since both λ and λ^* satisfy the constraint in (1.10). In order to write the bound (6.78) as $P \cdot \delta_\gamma$, we expand the parentheses and use the following bounds:

$$\|\lambda - \lambda^*\|_2 \|\nu - a\|_1 \leq 2\Pi \cdot \delta_\gamma \tag{6.83}$$

$$\|\lambda - \lambda^*\|_2^2 \|\nu - a\|_2 \leq 4\sqrt{k}\tau\Pi \cdot \delta_\gamma \tag{6.84}$$

$$\|\lambda - \lambda^*\|_2 \|\nu - a\|_2 \leq 2\Pi \cdot \delta_\gamma \tag{6.85}$$

to obtain:

$$\begin{aligned} \|E\|_F \leq \sqrt{2}k\Bigg(& \frac{2\sqrt{k}C_{t^*}^2\Pi}{\sigma^2} + \frac{\sqrt{2k}C_{t^*}}{\sqrt{e}\sigma} + \frac{4\sqrt{k}C_{t^*}^2\Pi}{\sigma} \\ & + \frac{4kC_{t^*}\bar{\Delta}_2\tau\Pi}{\sigma^2} + \frac{2\sqrt{2}\bar{\Delta}_2\Pi}{\sqrt{e}\sigma} + \frac{8kC_{t^*}\bar{\Delta}_2\tau\Pi}{\sigma} + \frac{\sqrt{2k}\bar{\Delta}_2\Pi}{\sqrt{e}\sigma} \\ & + \frac{\sqrt{2}C_{t^*}}{\sigma\sqrt{e}}\Bigg) \cdot \delta_\gamma, \end{aligned} \tag{6.86}$$

which we rearrange based on σ to obtain $\|E\|_F \leq P(k, \sigma, \Pi, \tau, C_{t^*}) \cdot \delta_\gamma$, where:

$$\begin{aligned} P(k, \sigma, \Pi, \tau, C_{t^*}) = \sqrt{2}k\Bigg[& \frac{1}{\sigma^2}\left(2\sqrt{k}C_{t^*}^2\Pi + 4kC_{t^*}\bar{\Delta}_2\tau\Pi\right) \\ & + \frac{1}{\sigma}\left(\frac{\sqrt{2k}C_{t^*}}{\sqrt{e}} + 4\sqrt{k}C_{t^*}^2\Pi + \frac{2\sqrt{2}\bar{\Delta}_2\Pi}{\sqrt{e}} + 8kC_{t^*}\bar{\Delta}_2\tau\Pi + \frac{\sqrt{2k}\bar{\Delta}_2\Pi}{\sqrt{e}} + \sqrt{\frac{2}{e}}C_{t^*}\right)\Bigg], \end{aligned}$$

which is the final bound in (6.36).

Chapter 7

Perturbation bounds in practice: numerical experiments

In the previous two chapters, we analysed how the estimated source locations and weights are affected by inaccuracies in the solution of the dual variable λ when solving the dual of the TV norm minimisation problem. These inaccuracies can be caused by numerical error due to the algorithm used or noise in the measurement data y. In this final chapter, we illustrate these results through numerical experiments where the exact penalty formulation of the dual problem is solved using a non-smooth optimisation algorithm, the level method.

We first give a brief overview of the level method and how we apply it to our problem in Section 7.1, before showing and discussing the numerical experiments in Section 7.2.

7.1 An overview of the level method

In order to calculate the errors that we bound in Theorem 28, 29 and 33, we need to calculate the solution λ to the dual problem (1.9) or (1.10). Note that (1.10) is the same as (1.9) with an additional box constraint, which does not affect the analysis of (1.9) presented in Chapter 5. Therefore, we focus on solving problem (1.10):

$$\max_{\lambda \in \mathbb{R}^m} y^T \lambda \quad \text{such that} \quad \lambda^T \Phi(t) \leq 1, \quad \forall t \in I,$$
$$\text{and} \quad \|\lambda\|_\infty \leq \tau.$$

More specifically, we solve the exact penalty formulation, given in (6.1):

$$\min_{\lambda \in \mathbb{R}^m} \Psi_\Pi(\lambda) \quad \text{such that} \quad \|\lambda\|_\infty \leq \tau,$$

where

$$\Psi_\Pi(\lambda) = -y^T\lambda + \Pi \cdot \max\left\{\sup_s \left(\sum_{j=1}^m \lambda_j \phi(s - s_j) - 1\right), 0\right\}.$$

As described in Chapter 6, for a large enough value of Π, a solution to the exact penalty problem which satisfies the constraints of the dual problem is also a solution to the dual. This is a non-smooth optimisation problem, which we solve using the level method, as given in [61]. It relies on having a subgradient of Ψ_Π at each iteration, which we calculate as follows:

$$\partial\Psi_\Pi = \begin{cases} -y + \Pi g(s^*), & \text{if } \sup_s \sum_{j=1}^m \lambda_j \phi(s - s_j) \geq 1, \\ -y, & \text{if } \sup_s \sum_{j=1}^m \lambda_j \phi(s - s_j) < 1, \end{cases} \tag{7.1}$$

where s^* is a global maximiser of the function $\sum_{j=1}^m \lambda_j \phi(s - s_j)$ and $g(s) = [\phi(s - s_1), \ldots, \phi(s - s_m)]^T$ and note that this is consistent with the definition of the subdifferential form (6.3).

We now take a step back and describe how the level method works, as described in [61], before showing the results of the numerical experiments in the next section.

The algorithm

We solve the problem:

$$\min_{x \in Q} f(x)$$

for a convex non-smooth function f and a closed convex set Q. The level method relies on computing a piecewise linear model $\hat{f}_k$ of f at every iteration. At the kth iteration, the model is:

$$\hat{f}_k(x) = \max_{0 \leq i \leq k} \left[f(x_i) + \left\langle g(x_i), x - x_i \right\rangle \right], \tag{7.2}$$

where $g(x_i)$ is a subgradient of f at x_i. We then perform two steps at each iteration k:

1. Find the optimal value $\hat{f}_k^*$ of the current model $\hat{f}_k$ and the smallest value f_k^* of the objective f so far. Note that $\hat{f}_k^* \leq f_k^*$. This is a linear program.

2. Calculate the next iterate x_{k+1} as the projection of x_k on the level set given by:

   $$\mathcal{L}_k(\alpha) = \{x \in Q \mid \hat{f}_k(x) \leq l_k(\alpha)\} \quad \text{where} \quad l_k(\alpha) = (1 - \alpha)\hat{f}_k^* + \alpha f_k^*$$

   for some $\alpha \in (0, 1)$. This is a quadratic program.

The level method achieves ϵ accuracy after at most $\frac{4}{\epsilon^2} M_f^2 D^2$ steps, where $D = diam Q$ and

$$M_f = \max\{\|g\| \mid g \in \partial f(x), x \in Q\}. \tag{7.3}$$

We are now ready to discuss the numerical experiments.

7.2 Numerical experiments

Having introduced the level method and the way we apply it to the TV norm minimisation problem in the previous section, we can now discuss the numerical experiments which illustrate the bounds given by Theorems 28, 29 in Chapter 5 and Theorem 33 in Chapter 6.

We take an example of a source and sample configuration and a Gaussian kernel for a given σ and solve the exact penalty formulation of the dual problem (1.10), as discussed in Section 7.1. Then, we introduce inaccuracies in λ by stopping the algorithm early and show how these perturbations affect the source locations and weighs. Next, we add noise to the measurements to show how λ is affected. We are, therefore, able to compare the ratios of the perturbations obtained numerically with the constants in the theorems to show the validity of our results. The specific details are discussed in the next subsections.

Setup

We place three sources at locations $t_i^* \in T = \{0.25, 0.63, 0.889\}$ with weights $a_i^* \in \{0.8, 0.5, 0.9\}$ and $m = 21$ equispaced samples in $[0, 1]$, with a Gaussian kernel $\phi(t) = e^{-t^2/\sigma^2}$ with $\sigma = 0.07$. We show this configuration in Figure 7.1

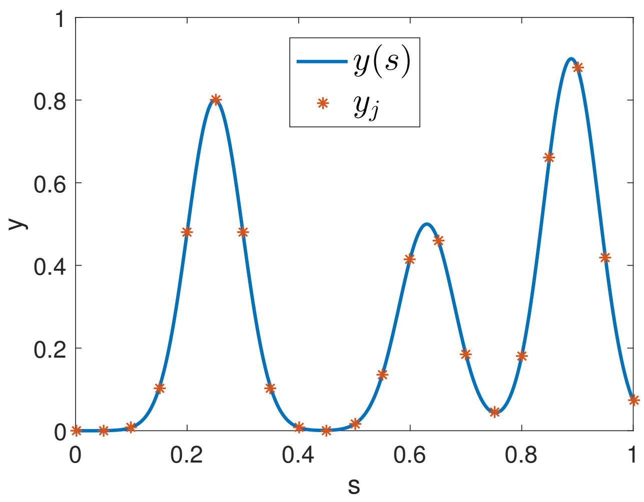


Figure 7.1: Example of source-sample configuration used for numerical experiments in the current chapter.

Effect of λ^* perturbations on t^*

We then solve the dual problem (1.10) in the exact penalty formulation (6.1) with box constraint parameter $\tau = 10^5$ and penalty parameter $\Pi = 100$ and run it for $P = 500$

iterations. This gives an accuracy in the source locations of $|t_i - t_i^*| \leq 10^{-8}$ for $t_i^* \in T$.

While it is possible to optimise the parameters τ, Π and P in order to obtain better accuracy in the source locations t_i and weights a_i, it is not the aim of this chapter. Note that Theorem 28 gives the result (5.7) in the form

$$|t_i - t_i^*| \leq C_{t^*} \|\lambda - \lambda^*\|_2,$$

where $t_i^* \in T$ is an arbitrary true source location, λ^* is the solution to the dual problem (1.10)[1], t is obtained by perturbing t^* as a consequence of the perturbation λ^* in λ.

One way of showing that a relationship of the type of (5.7) holds in practice is to plot the ratio $\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^*\|_2}$, for $p = p_0, \dots, P$ and $i = 1, \dots, k$, where P is the number of iterations the level method is run for, p is the index of each iteration and $t_i^{(p)}$ and $\lambda^{(p)}$ are the values of t_i and λ obtained at iteration p, where $p_0 \geq 1$ is large enough so that $\|\lambda^{(p)} - \lambda^*\|_2$ satisfies the condition in Theorem 28. The level method computes the value $\lambda^{(p)}$ after p iterations and $\{t_i^{(p)}\}_{i=1}^k$ are obtained by calculating the global maxima of the dual certificate $q^{(p)}(s) = \sum_{j=1}^m \lambda_j^{(p)} \phi(s - s_j)$. Since we know the true value of t_i^*, we can find $t_i^{(p)}$ by running a local optimisation algorithm with t_i^* as the initial condition. For a large enough value of p, this will give an accurate value of $t_i^{(p)}$ and we can, therefore, calculate $|t_i^{(p)} - t_i^*|$ for each $p = p_0, \dots, P$ and $t_i^* \in T$. Then we check that:

$$\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^*\|_2} \leq C_{t^*}, \tag{7.4}$$

for $p = p_0, \dots, P$ and $i = 1, \dots, k$. One issue is that the true value of λ^* is not known. The best estimate we have is $\lambda_{best}^* = \lambda^{(P)}$, namely the value of λ^* given by the level method after P iterations. Therefore, the result of Theorem 28 cannot be verified directly in practice, but must be adapted to take into account this inaccuracy. For $i = 1, \dots, k$, we have that:

$$\begin{aligned} |t_i^{(p)} - t_i^*| &\leq C_{t^*} \|\lambda^{(p)} - \lambda^*\|_2 \\ &\leq C_{t^*} \left(\|\lambda^{(p)} - \lambda_{best}^*\|_2 + \|\lambda_{best}^* - \lambda^*\|_2 \right), \end{aligned} \tag{7.5}$$

so

$$\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda_{best}^*\|_2} \leq C_{t^*} \left(1 + \frac{\|\lambda_{best}^* - \lambda^*\|_2}{\|\lambda^{(p)} - \lambda_{best}^*\|_2} \right). \tag{7.6}$$

[1] Note that the analysis of the dual problem (1.9) from Chapter 5 applies to the dual problem (1.10) considered in Chapter 6 as well, as the only difference difference between (1.9) and (1.10) is a box constraint on λ.

For fixed P, which in the experiments in this section is $P = 500$, $\|\lambda^*_{best} - \lambda^*\|_2$ above is fixed and as p approaches P, we have that $\|\lambda^{(p)} - \lambda^*_{best}\|_2 \to 0$, and therefore the right hand side above goes to infinity. An instance of this behaviour is show in Figure 7.2. This is not a problem for our results, as it is not relevant how the ratio $\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^*_{best}\|_2}$ behaves for $\|\lambda^{(p)} - \lambda^*_{best}\|_2 \le \|\lambda^*_{best} - \lambda^*\|_2$.

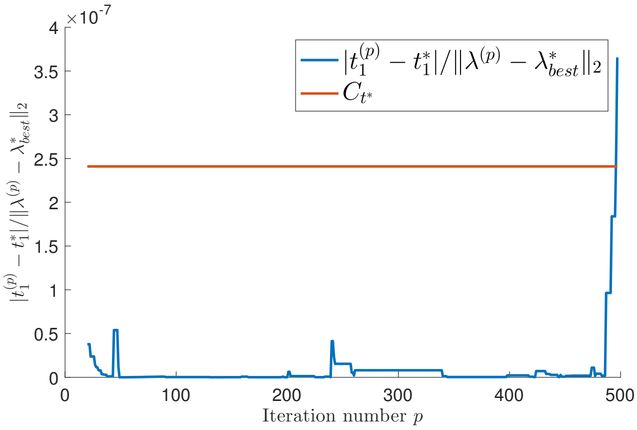


Figure 7.2: The ratio $\frac{|t^{(p)} - t^*|}{\|\lambda^{(p)} - \lambda^*_{best}\|_2}$ starts growing as $\lambda^{(p)}$ approaches λ^*_{best}.

We can then find a range for p where $\frac{\|\lambda^*_{best} - \lambda^*\|_2}{\|\lambda^{(p)} - \lambda^*_{best}\|_2} \le 1$ and where we can see that

$$\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^*_{best}\|_2} \le 2C_{t^*}. \tag{7.7}$$

In Figure 7.3, we plot $\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^*_{best}\|_2}$ for $p = 20, \dots, 270$, where we see that the ratio is less than C_{t^*}. Specifically, in the left hand side panels (a), (c) and (e), we show the errors $\|t_i^{(p)} - t_i^*\|$ and $\|\lambda^{(p)} - \lambda^*_{best}\|_2$ for each $i \in \{1, 2, 3\}$ respectively at each iteration p for $p = 20, \dots, 270$ and $\lambda^*_{best} = \lambda^{(P)}$ at $P = 500$. In the right hand side panels (b), (d) and (f), we show the ratio $\frac{\|t_i^{(p)} - t_i^*\|}{\|\lambda^{(p)} - \lambda^*_{best}\|_2}$ and the constant C_{t^*} from Theorem 28.

Moreover, in Figure 7.4 we show $\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^*_{best}\|_2}$ and the constant C_{t^*} for the same setup as in Figure 7.3, except that the first two sources are closer together, namely $T = \{0.25, 0.49, 0.888\}$ in panels (a), (c) and (e) and $T = \{0.25, 0.35, 0.888\}$ in panels (b), (d) and (f). Similarly, in Figure 7.5 we show $\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^*_{best}\|_2}$ and the constant C_{t^*} for the same setup as in Figure 7.3, with smaller σ: $\sigma = 0.04$ in panels (a), (c) and (e) and $\sigma = 0.01$ in panels (b), (d) and (f). Lastly, in Figure 7.6 we take $m = 11$ measurements in panels (a), (c) and (e) and $m = 31$ measurements in panels (b), (d) and (f).

Effect of t^* perturbations on a^*

In the case of Theorem 29, it is more straightforward to check the ratio of the errors, since we know the true values of the source locations and weights, which we denote by $t^* = [t_1^*, \ldots, t_k^*]^T$ and $a^* = [a_1^*, \ldots, a_k^*]^T$ respectively. The error bound (5.14) given by the theorem is of the form:

$$\|a - a^*\|_2 \leq C_{a^*}\|t - t^*\|_2 + \mathcal{O}(\|t - t^*\|_2^2),$$

where t is the perturbed vector t^* and a is the perturbed vector a^* as a consequence of perturbing t^*. For the values $t_i^{(p)}, i \in \{1, 2, 3\}$, obtained after p iterations of the level method, we now solve the least squares problem $\text{argmin}_{\hat{a}} \|\Phi^{(p)}\hat{a} - y\|_2$ with the entries in the data matrix $\Phi^{(p)}$ given by $\Phi_{j,i}^{(p)} = \phi(t_i^{(p)} - s_j)$ to find the corresponding perturbed weights $a_i^{(p)}$ for $i \in \{1, 2, 3\}$. Then, according to Theorem 29, we have that:

$$\frac{\|a^{(p)} - a^*\|_2}{\|t^{(p)} - t^*\|_2} \leq C_{a^*} + \mathcal{O}(\|t^{(p)} - t^*\|_2). \tag{7.8}$$

In Figure 7.7, panel (a), we show $\|a^{(p)} - a^*\|_2$ and $\|t^{(p)} - t^*\|_2$ in the same setting as in Figure 7.3, for iterations $p = 20, \ldots, 270$, and in panel (b) we see the ratio $\frac{\|a^{(p)} - a^*\|_2}{\|t^{(p)} - t^*\|_2}$.

Then, in Figure 7.8 we show the different scenarios from Figures 7.4, 7.5 and 7.6. Specifically, in panels (a) and (b) we show the ratio when the first two sources are closer together, $T = \{0.25, 0.49, 0.888\}$ and $T = \{0.25, 0.35, 0.888\}$ respectively, in panels (c) and (d) we show the ratio for smaller sigma, $\sigma = 0.04$ and $\sigma = 0.01$ respectively. Lastly, in panel (e) we show the ratio for $m = 11$ and in panel (f) with $m = 31$.

We do not plot the value of C_{a^*}, as it is of the order of 10^5, and therefore (7.8) holds.

Effect of the noise w on λ^* and t^*

As in the case of Theorem 28, where we rely on a best approximation λ_{best}^* of λ^* for the numerical experiments, a similar approach is required to check the validity of the results of Theorem 33 in practice. Theorem 33 gives the bound (6.12) in the form:

$$\|\lambda_w^* - \lambda^*\|_2 \leq C_{\lambda^*} \cdot \|w\|_2,$$

where λ^* is the true solution of the dual problem (1.10) and λ_w^* is the solution to the same problem with y perturbed by the noise w.

As it is not possible to know exactly the values of λ^* and λ^*_w, let $\lambda^*_{best} = \lambda^{(P)}$ be the value of λ given by the level method after P iterations when y is exact and λ_{best} be the value of λ returned by the level method after P iterations when y is corrupted by the additive noise w. Then we can reformulate the bound (6.12) in terms of λ^*_{best} and λ_{best}:

$$\begin{aligned}
\|\lambda_{best} - \lambda^*_{best}\|_2 &= \|\lambda_{best} - \lambda^*_{best} + \lambda^* - \lambda^* + \lambda^*_w - \lambda^*_w\|_2 \\
&\leq \|\lambda_{best} - \lambda^*_w\|_2 + \|\lambda^* - \lambda^*_w\|_2 + \|\lambda^* - \lambda^*_{best}\|_2 \\
&\leq \|\lambda_{best} - \lambda^*_w\|_2 + C_{\lambda^*}\|w\|_2 + \|\lambda^* - \lambda^*_{best}\|_2, \qquad (7.9)
\end{aligned}$$

so

$$\frac{\|\lambda_{best} - \lambda^*_{best}\|_2}{\|w\|_2} \leq C_{\lambda^*} + \frac{\|\lambda_{best} - \lambda^*_w\|_2 + \|\lambda^* - \lambda^*_{best}\|_2}{\|w\|_2}. \qquad (7.10)$$

As before, we plot $\frac{\|\lambda_{best} - \lambda^*_{best}\|_2}{\|w\|_2}$, where λ^*_{best} is the solution we obtain by solving the dual problem (1.10) in its exact penalty formulation using the level method with $P = 100$ iterations and λ_{best} is the 'noisy' solution, which is obtained by solving the problem with $P = 100$ iterations when y is corrupted by additive noise w. We repeat this for different magnitudes of the noise w, which we increase gradually as follows. For each component y_j of y, we add a sample X_j from the standard uniform distribution $U(0, 1)$, multiplied by a coefficient w_c:

$$y_{noisy_j} = y_j + w_c \cdot X_j. \qquad (7.11)$$

We repeat this for different values of the coefficient w_c from the set:

$$\begin{aligned}
w_c \in \{&0.000002, 0.000004, \dots, 0.00001, \\
&0.00002, 0.00004, \dots, 0.0001, \\
&0.0002, 0.0004, \dots, 0.001, \\
&0.002, 0.003, \dots, 0.01, \\
&0.02, 0.03, \dots, 0.1\}. \qquad (7.12)
\end{aligned}$$

Therefore, in Figure 7.9 we show the basic setup described at the beginning of this section. Panel (a) shows $\|\lambda_{best} - \lambda^*_{best}\|_2$ against the norm of the noise $\|w\|_2$, and in order to check that the algorithm actually converges to a useful λ^*_{best}, we also plot plot $\|t_{best} - t^*\|_2$ against $\|w\|_2$ in panel (b), since we know the true value t^*. Then, in panel (c) we plot the ratio $\frac{\|\lambda_{best} - \lambda^*_{best}\|_2}{\|w\|_2}$ and C_{λ^*} as given by Theorem 33, where we see that the ratio is smaller than the constant, as the theorem states. In the same plot, we also show the ratio $\frac{\|t_{best}-t^*\|_2}{\|w\|_2}$ and we see that it does not grow as the magnitude of the noise increases.

In Figures 7.10 and 7.11, we show the same plots as in Figure 7.9, but with the first two sources closer ($T = 0.25, 0.49, 0.8888$) and a more narrow kernel ($\sigma = 0.04$) respectively. Note in these plots, for small noise magnitude $\|w\|_2$, the effect of the ratio $\frac{\|\lambda_{best}-\lambda^*_w\|_2+\|\lambda^*-\lambda^*_{best}\|_2}{\|w\|_2}$ in (7.10) going to infinity.

We end the chapter by mentioning that in these experiments we only take into account $2k$ entries of λ and w, corresponding to the $2k$ samples that are the closest to the k sources, as described in Chapter 6, for which Theorem 33 holds. However, in Figure 7.12 we show the same plot where we calculate the errors using the full vectors $\lambda_{best}, \lambda^*_{best}$ and w and we see that the ratio of the two norms follows a similar pattern.

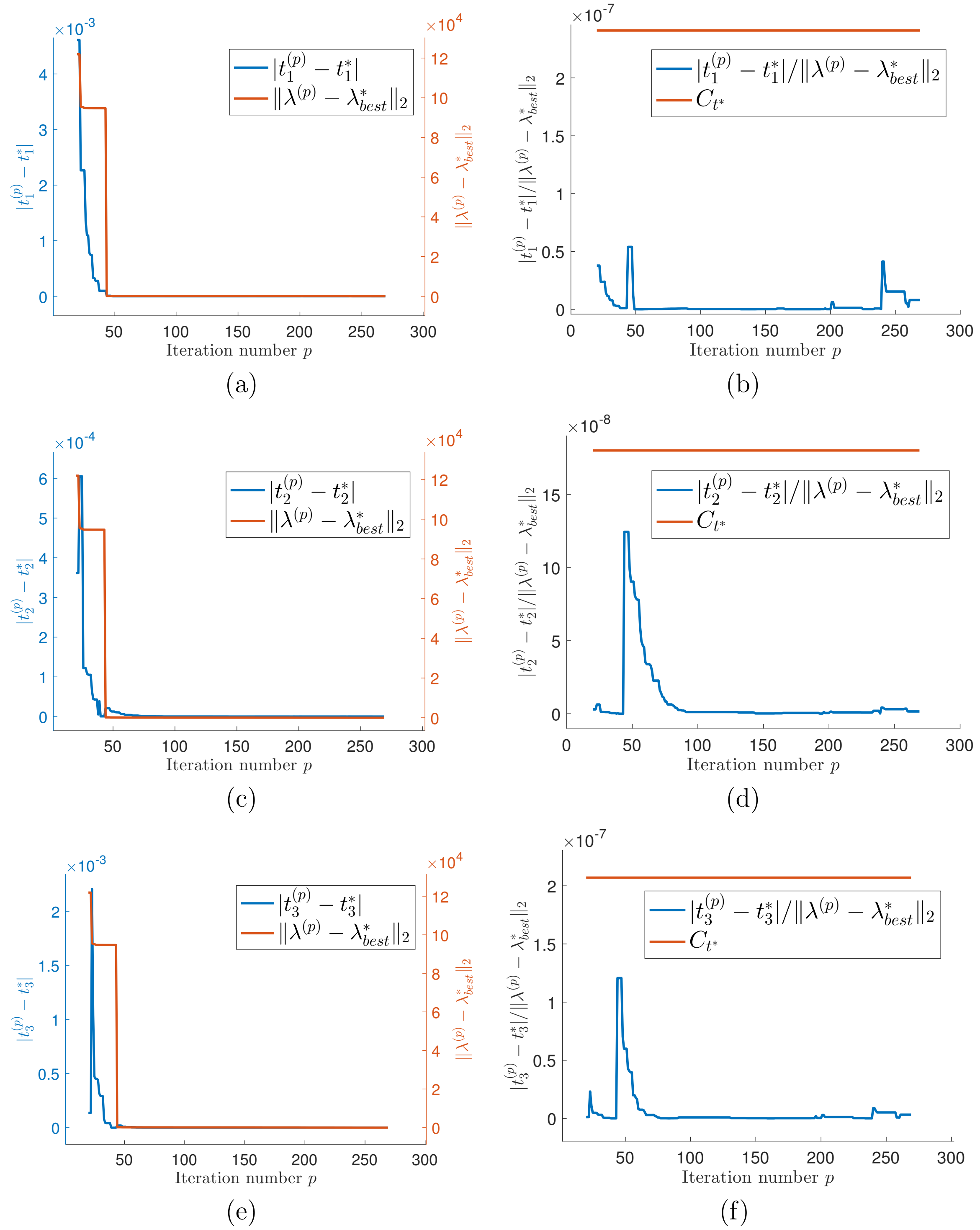


Figure 7.3: The result of Theorem 28 for $T = \{0.25, 0.63, 0.888\}$, $\sigma = 0.07$ and $m = 21$. In panels (a), (c) and (e) we show how $|t_i^{(p)} - t_i^*|$ and $\|\lambda^{(p)} - \lambda^{(p)}_{best}\|_2$ change at each iteration $p = 20, \ldots, 270$ of the level method for t_1, t_2 and t_3 respectively, while in panels (b), (d) and (f) we show their ratio and C_{t^*} given by Theorem 28.

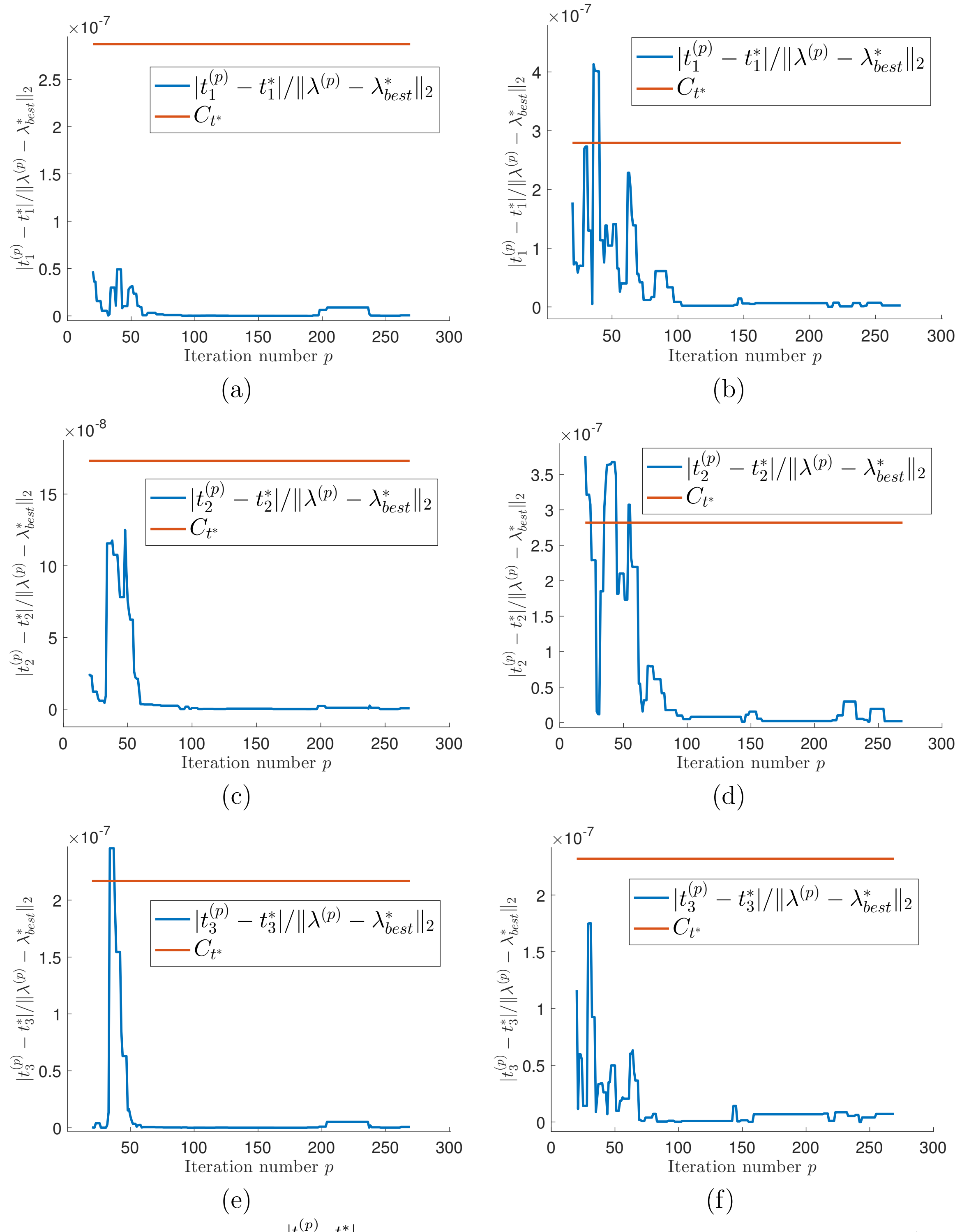


Figure 7.4: The ratio $\frac{|t_i^{(p)}-t_i^*|}{\|\lambda^{(p)}-\lambda^{(p)}_{best}\|_2}$ at each iteration p of the level method for t_1 (top row), t_2 (middle row) and t_3 (bottom row) respectively in the same setting as in Figure 7.3, but with the first two sources closer to each other: $T = \{0.25, 0.49, 0.888\}$ in panels (a), (c), (e) and $T = \{0.25, 0.35, 0.888\}$ in panels (b), (d), (f).

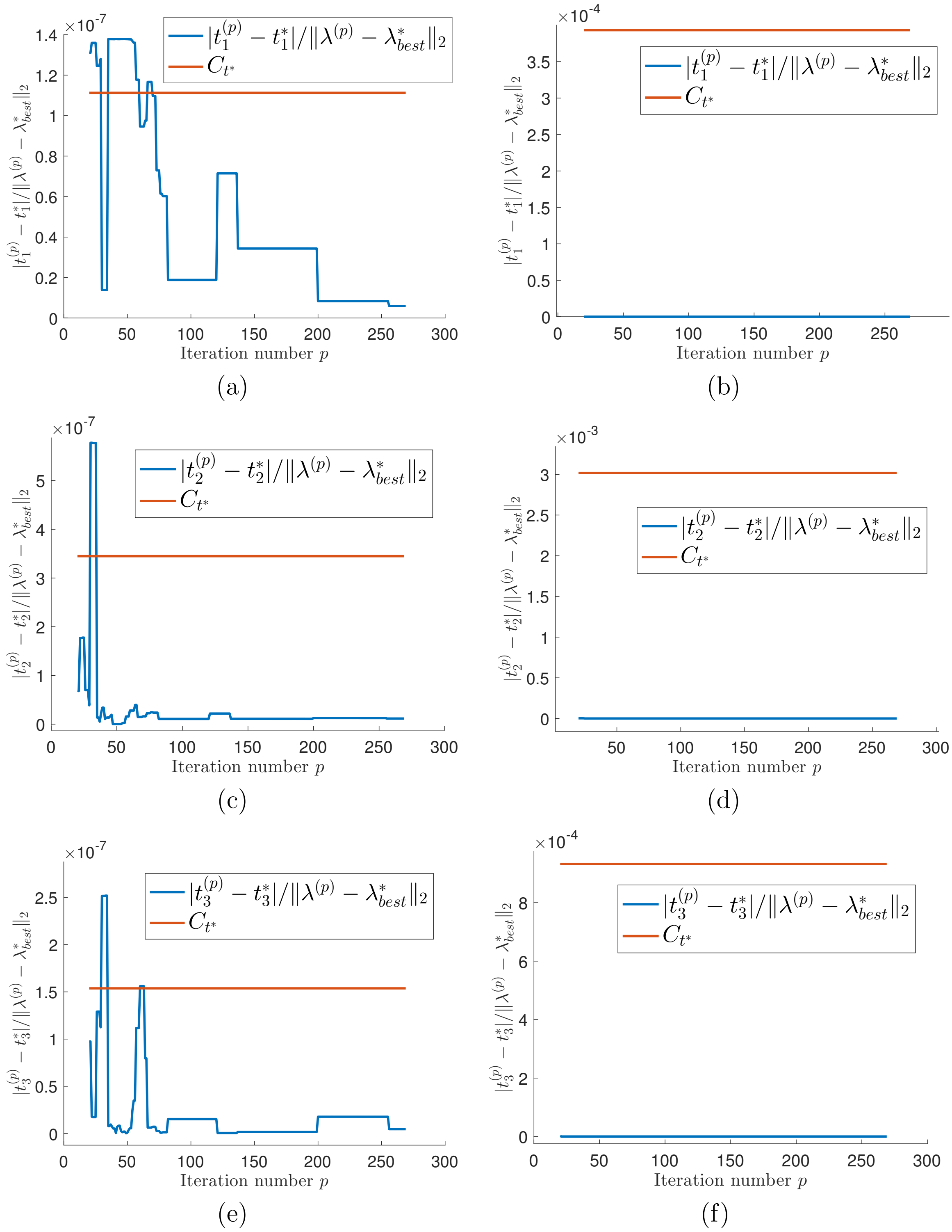


Figure 7.5: The ratio $\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^{(p)}_{best}\|_2}$ at each iteration p of the level method for t_1 (top row), t_2 (middle row) and t_3 (bottom row) respectively in the same setting as in Figure 7.3, but with the the convolution kernel more narrow: $\sigma = 0.04$ in panels (a), (c), (e) and $\sigma = 0.01$ in panels (b), (d), (f).

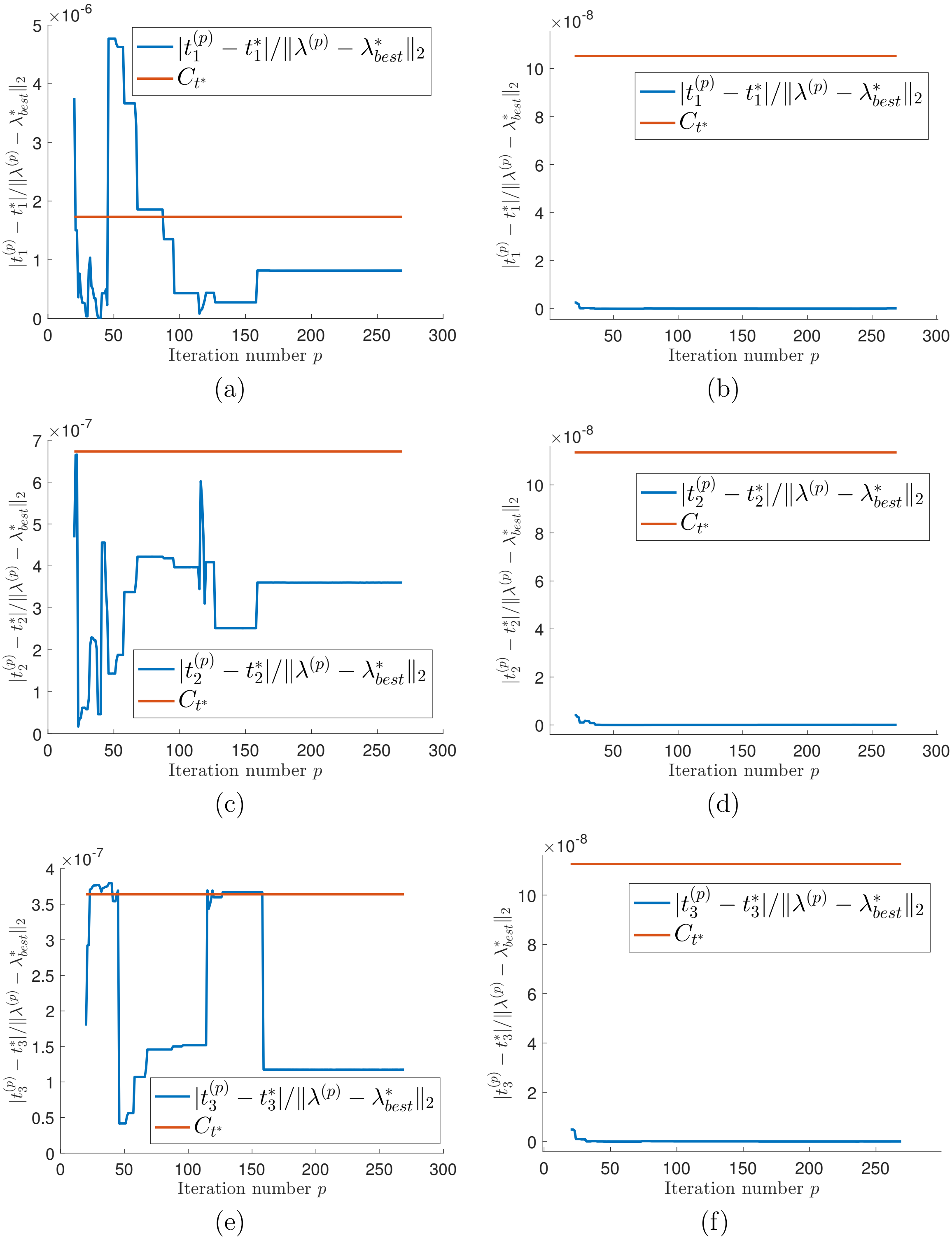


Figure 7.6: The ratio $\frac{|t_i^{(p)} - t_i^*|}{\|\lambda^{(p)} - \lambda^{(p)}_{best}\|_2}$ at each iteration p of the level method for t_1 (top row), t_2 (middle row) and t_3 (bottom row) respectively in the same setting as in Figure 7.3, but with fewer measurements ($m = 11$) in panels (a), (c), (e) and more measurements ($m = 31$) in panels (b), (d), (f).

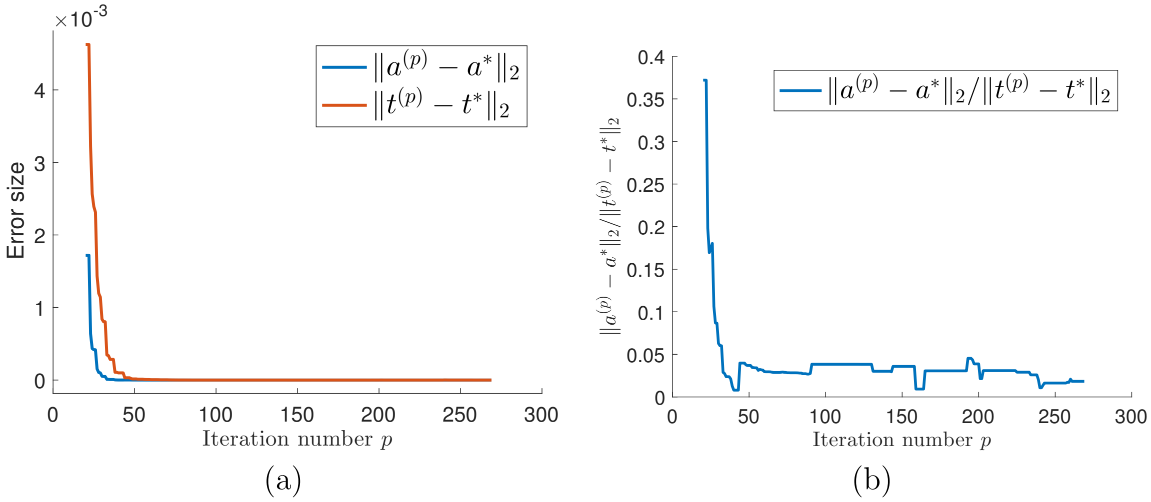


Figure 7.7: Plots of $\|a^{(p)} - a^*\|_2$ and $\|t^{(p)} - t^*\|_2$ (panel (a)) and their ratio (panel (b)) for $p = 20, \ldots, 270$, in the setup described at the beginning of this section.

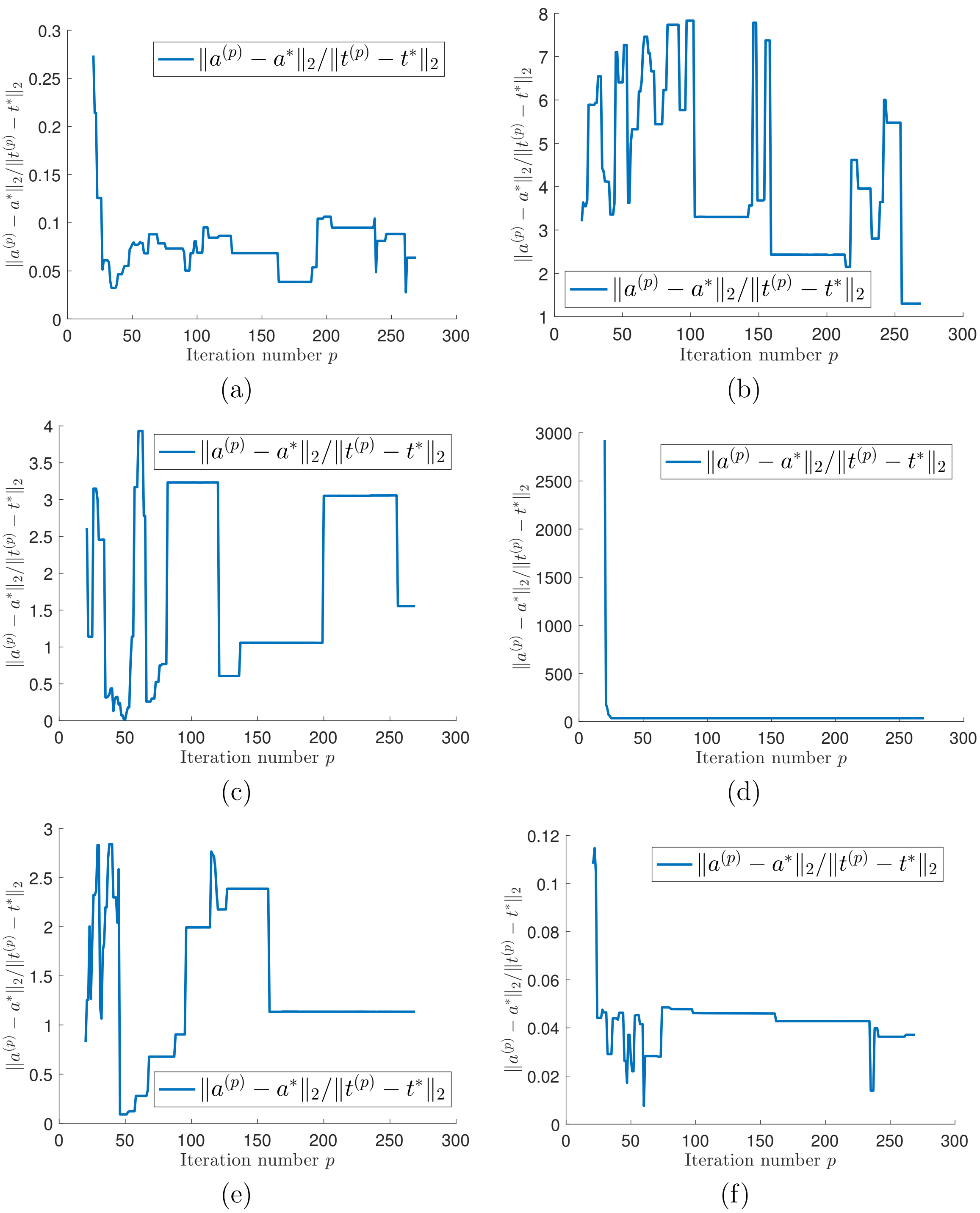
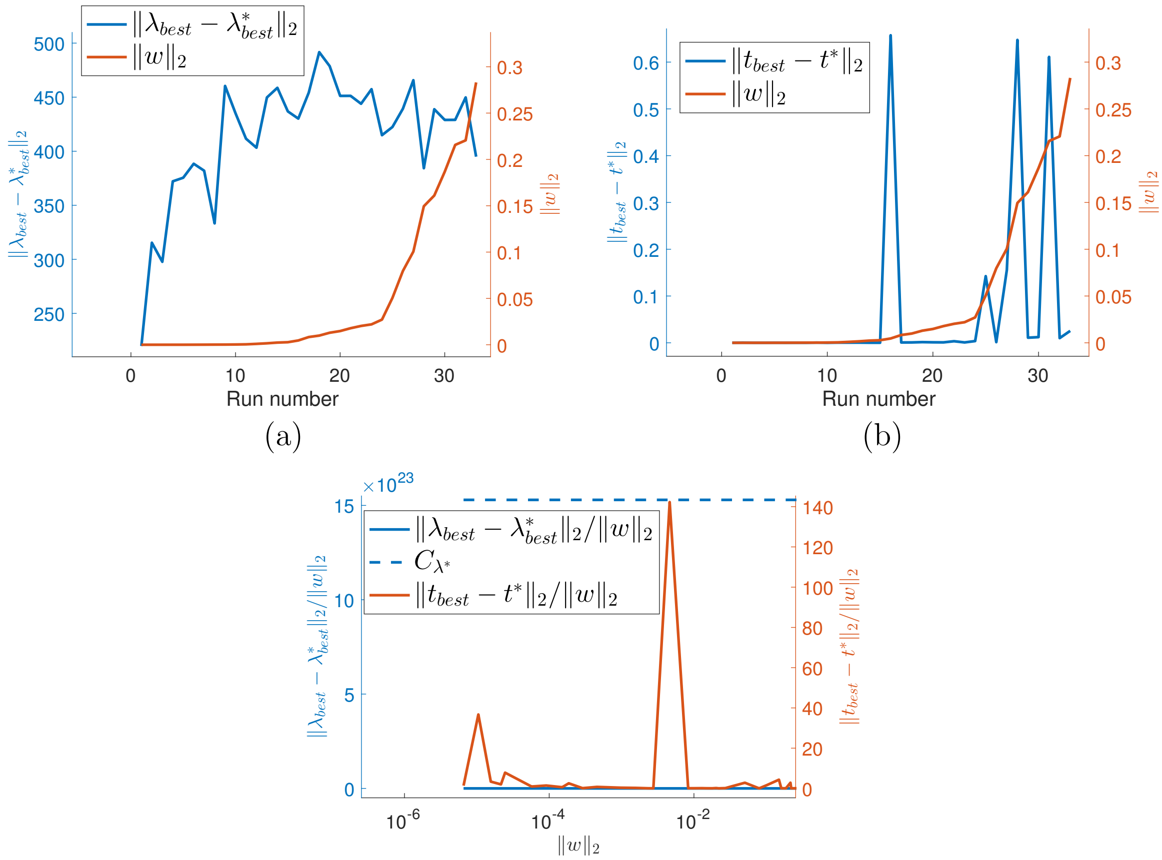


Figure 7.8: The ratio $\frac{\|a^{(p)}-a^*\|_2}{\|t^{(p)}-t^*\|_2}$ for $p = 20, \ldots, 270$ in the setup described at the beginning of this section, except: in panels (a) and (b) we take the first two sources to be closer to each other, $T = \{0.25, 0.49, 0.8888\}$ and $T = \{0.25, 0.35, 0.8888\}$ respectively, in panels (c) and (d) the convolution kernel is more narrow, $\sigma = 0.04$ and $\sigma = 0.01$ respectively, in panel (e) we take fewer measurements, $m = 11$, and in panel (f) we take more measurements, $m = 31$.



(c)

Figure 7.9: Plots of $\|\lambda_{best} - \lambda^*_{best}\|_2$ (panel(a)), $\|t_{best} - t\|_2$ (panel (b)) and their ratio to the noise $\|w\|_2$ (panel(c)) for $\|w\|_2$ in a range as given in (7.11) and (7.12), in the setting described at the beginning of this section.

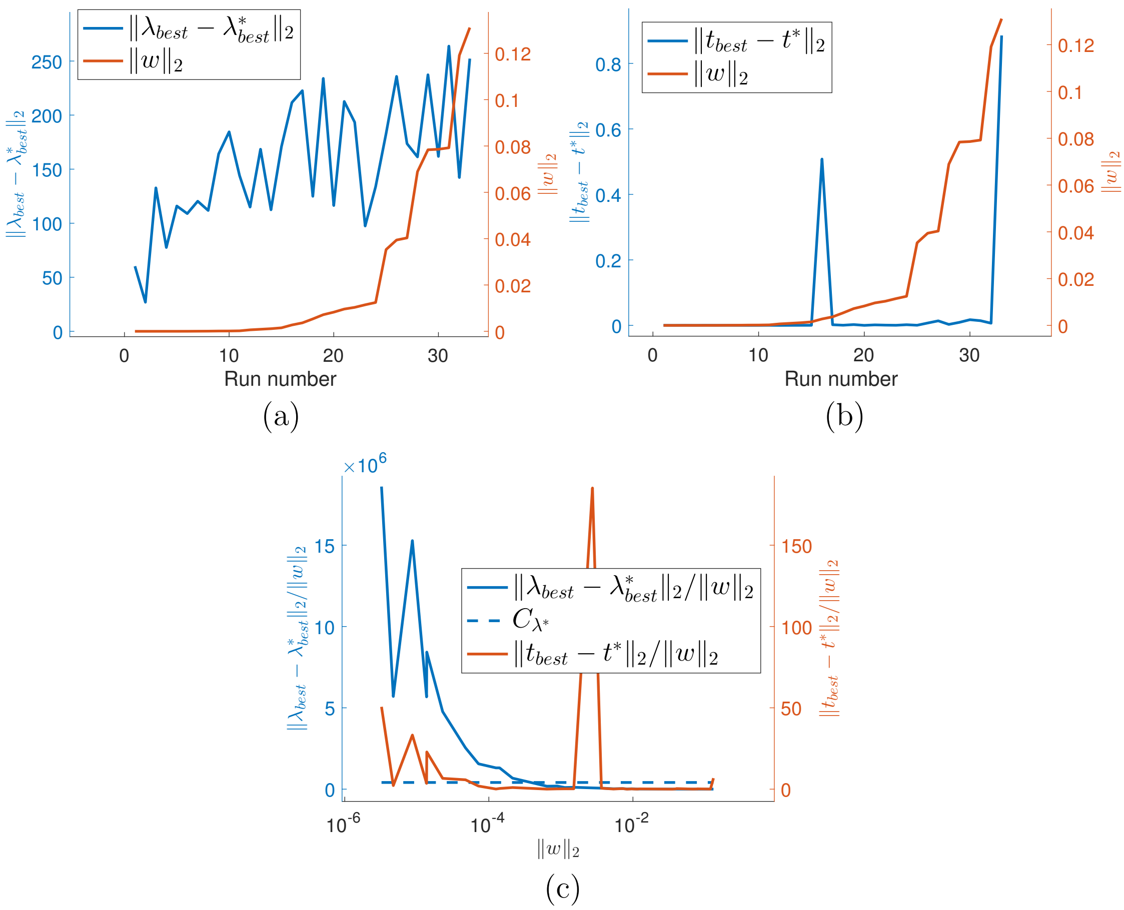
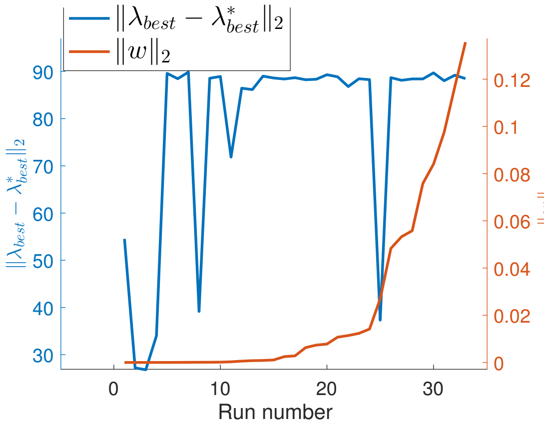
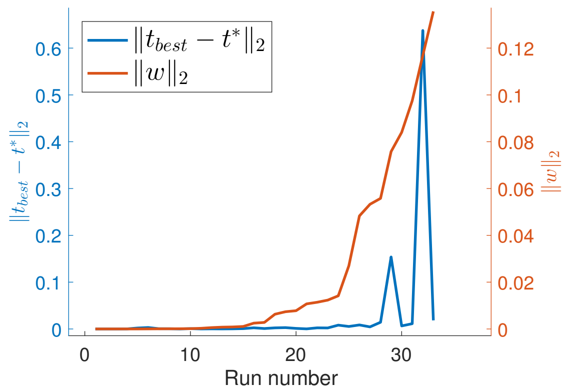


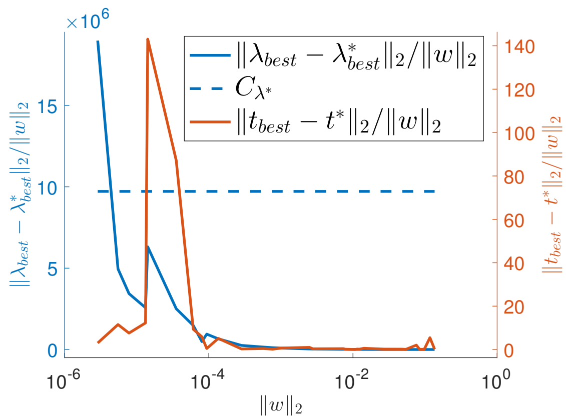
Figure 7.10: Plots of $\|\lambda_{best} - \lambda^*_{best}\|_2$ (panel(a)), $\|t_{best} - t\|_2$ (panel (b)) and their ratio to the noise $\|w\|_2$ (panel(c)) for $\|w\|_2$ in a range as given in (7.11) and (7.12), in the setting described at the beginning of this section, except that the first two point sources are closer to each other $T = \{0.25, 0.49, 0.8888\}$.



(a)



(b)



(c)

Figure 7.11: Plots of $\|\lambda_{best} - \lambda^*_{best}\|_2$ (panel(a)), $\|t_{best} - t\|_2$ (panel (b)) and their ratio to the noise $\|w\|_2$ (panel(c)) for $\|w\|_2$ in a range as given in (7.11) and (7.12), in the setting described at the beginning of this section, except that the kernel is more narrow $\sigma = 0.04$.

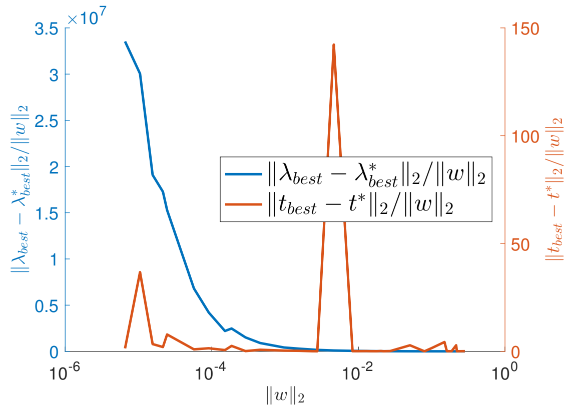


Figure 7.12: $\frac{\|\lambda_{best}-\lambda^*_{best}\|_2}{\|w\|_2}$ and $\frac{\|t_{best}-t\|_2}{\|w\|_2}$ for $\|w\|_2$ in a range as given in (7.11) and (7.12) in the setting described at the beginning of this section. Here, instead of selecting only $2k$ entries of λ and w as described in Chapter 6, we calculate the errors using all the components of the vectors.

Chapter 8

Conclusions and future work

In this thesis, we have considered two aspects of the non-negative super-resolution problem: stability of the feasibility problem with respect to noise in the measurements and perturbations in the source locations and weights when solving the dual problem.

- *Stability analysis:* We have shown that the non-negative super-resolution problem is stable. Specifically, under certain conditions on the source and sampling locations and the window function, any signal consistent with the measurements up to a constant δ is necessarily close to the true signal that generated the measurements, where we define the closeness as the local average error. We have shown that the error depends linearly on the level of noise and the length over which the average is taken, with the constants depending on the parameters of the problem: the minimum separation of sources, the source-sample proximity and the width of the window function. The TV norm is not required as a regulariser and this result holds independently of how the problem is solved.

- *Perturbation analysis:* In the second part of the thesis, we have considered the dual problem of the TV norm minimisation problem for non-negative measures. We have given explicit bounds on the perturbations of the source locations and weights as the optimal value of the dual problem is perturbed due to inaccuracies in the algorithm or noise in the data. We then showed how these bounds hold in practice by applying the level method to the exact penalty formulation of the dual problem.

Future work

To conclude, the work in this thesis has opened up a number of possible directions which may be considered in the future:

- *Extensions to higher dimensions:* The theory in Part I of the thesis has been extended to recovery of point sources in two dimensions in [32]. The most significant difference between our work and [32] is the construction of the dual certificate, which is a sum of products of one dimensional dual certificates.

  We believe it is possible to this even further and develop a similar theory for the recovery of curves in two dimensions. Similar work has been done, see for example [63] and [62], but not in the context of the feasibility problem. Following an initial investigation, there are two main issues we have identified: constructing the dual certificate and showing uniqueness of the solution (showing uniqueness of the support is an almost trivial extension of the one-dimensional case). The techniques used in the perturbation analysis in Part II can be use in the same way in higher dimensions to obtain similar perturbation bounds.

- *Improved level method:* It may be possible to take advantage of the perturbation analysis developed in Chapters 5 and 6 to obtain an improved version of the level method when applied to the exact penalty formulation of the dual problem. Our choice of the subgradient from (7.1) is rather arbitrary. We could instead use the way the source locations and weights are perturbed around their optimal values to choose a better subgradient around the minimiser λ^*, which would lead to an improved constant M_f in (7.3).

- *Iterative hard thresholding algorithm:* In the discrete setting, super-resolution can be formulated as a compressed sensing problem, where one solves an underdetermined linear systems under the assumption that the solution is sparse. This can be solved using an iterative hard thresholding algorithm, see for example [7] and [8], which relies on applying a hard thresholding operator at each iteration to set all except k entries of the vector to zero. In the continuous setting, it is not obvious how the thresholding operator would work. We would want to find the discrete measure with k spikes which is the closest in some metric to the continuous measure calculated at the current iteration. One way of doing this is by using the increasingly popular Wasserstein distance [72], which has not been used in the context of super-resolution, as far as we are aware.

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Appendices

Appendix A

Proofs of the results in Chapter 2

A.1 Derivation of the dual for the feasibility problem

The results presented in Part I of the thesis rely on the existence of a dual certificate. This is a function of the form $q(t) = {\lambda^*}^T \Phi(t)$, where λ is a solution to the dual of the feasibility problem (1.5) in the noise-free setting (i.e. with $\delta = 0$), so that $q(t)$ satisfies the conditions given in Lemma 5. For completeness, we derive here the dual problem:

$$\min_{\lambda \in \mathbb{R}^m} \lambda^T y \quad \text{subject to} \quad \lambda^T \Phi(t) \geq 0, \quad \forall t \in I. \tag{A.1}$$

Note that the dual certificate satisfies the constraint in the problem above.

For x a non-negative measure on $I = [0,1]$, $\lambda \in \mathbb{R}^m$ and $\nu : I \to \mathbb{R}_+$, the Lagrangian of (1.5) is:

$$L(x, \lambda, \nu) = \lambda^T \left(y - \int_I \Phi(t) x(\mathrm{d}t) \right) - \int_I \nu(t) x(\mathrm{d}t) \tag{A.2}$$

and then:

$$\begin{aligned} \max_{\substack{\nu: I \to \mathbb{R}_+ \\ \lambda \in \mathbb{R}^m}} \min_x L(x, \lambda, \nu) &= \max_{\substack{\nu: I \to \mathbb{R}_+ \\ \lambda \in \mathbb{R}^m}} \left[\lambda^T y - \max_x \int_I \left(\lambda^T \Phi(t) + \nu(t) \right) x(\mathrm{d}t) \right] \\ &= \begin{cases} -\infty & \text{if } \lambda^T \Phi(t^*) > -\nu(t^*), \text{ for some } t^* \in I, \\ \max_{\lambda \in \mathbb{R}^m} \lambda^T y & \text{if } \lambda^T \Phi(t) \leq -\nu(t), \forall t \in I, \end{cases} \end{aligned} \tag{A.3}$$

and, since $\nu(t) \geq 0$, the dual problem is:

$$\max_{\lambda \in \mathbb{R}^m} \lambda^T y \quad \text{subject to} \quad \lambda^T \Phi(t) \leq 0, \quad \forall t \in I. \tag{A.4}$$

By doing the substitution $\tilde{\lambda} = -\lambda$, we obtain the dual problem (A.1).

A.2 Proof of Lemma 5 (Uniqueness of the non-negative sparse measure)

Let $\hat{x}$ be a solution of Program (1.5) with $\delta = 0$ and let $h = \hat{x} - x$ be the error. Then, by feasibility of both x and $\hat{x}$ in Program (1.5), we have that

$$\int_I \phi_j(t) h(\mathrm{d}t) = 0, \qquad j \in 1, \ldots, m. \tag{A.5}$$

Let T^C be the complement of $T = \{t_i\}_{i=1}^k$ with respect to I. By assumption, the existence of a dual certificate allows us to write that

$$\begin{aligned}
\int_{T^C} q(t)h(\mathrm{d}t) &= \int_I q(t)h(\mathrm{d}t) - \int_T q(t)h(\mathrm{d}t) \\
&= \int_I q(t)h(\mathrm{d}t) \qquad \big(q\,(t_i) = 0,\ i = 1, \ldots, k\big) \\
&= \sum_{j=1}^m b_j \int_I \phi_j(t) h(\mathrm{d}t) \\
&= 0. \qquad \text{(see (A.5))}
\end{aligned}$$

Since $x = 0$ on T^C, then $h = \hat{x}$ on T^C, so the last equality is equivalent to

$$\int_{T^C} q(t)\hat{x}(\mathrm{d}t) = 0. \tag{A.6}$$

But q is strictly positive on T^C, so it must be that $h = \hat{x} = 0$ on T^C and, therefore, $h = \sum_{i=1}^k c_i \delta_{t_i}$ for some coefficients $\{c_i\}$. Now (A.5) reads $\sum_{i=1}^k c_i \phi_j(t_i) = 0$ for every $j = 1, \ldots, m$. This gives $c_i = 0$ for all $i = 1, \ldots, k$ because $[\phi_j(t_i)]_{i,j}$ is, by assumption, full rank. Therefore $h = 0$ and $\hat{x} = x$ on I, which completes the proof of Lemma 5.

A.3 Proof of Lemma 11 (Error away from the support)

Let $\hat{x}$ be a solution of Program (1.5) and set $h = \hat{x} - x$ to be the error. Then, by feasibility of both x and $\hat{x}$ in Program (1.5) and using the triangle inequality, we have that

$$\left\| \int_I \Phi(t) h(\mathrm{d}t) \right\|_2 \leq 2\delta. \tag{A.7}$$

Next, the existence of the dual certificate q allows us to write that

$$
\begin{aligned}
&\bar{f}\int_{T_\epsilon^C} h(\mathrm{d}t) + \sum_{i=1}^{k}\int_{T_{i,\epsilon}} f\left(t-t_i\right)h(\mathrm{d}t) \\
&\qquad \le \int_{T_\epsilon^C} q(t)h(\mathrm{d}t) + \sum_{i=1}^{k}\int_{T_{i,\epsilon}} q(t)h(\mathrm{d}t) \\
&\qquad = \int_{T_\epsilon^C} q(t)h(\mathrm{d}t) + \int_{T_\epsilon} q(t)h(\mathrm{d}t) \qquad \left(T_\epsilon = \cup_{i=1}^{k} T_{i,\epsilon}\right) \\
&\qquad = \int_I q(t)h(\mathrm{d}t) = \sum_{j=1}^{m} b_j \int_I \phi_j(t)h(\mathrm{d}t) \\
&\qquad \le \|b\|_2 \cdot \left\| \int_I \Phi(t)h(\mathrm{d}t) \right\|_2 \qquad \text{(Cauchy-Schwarz inequality)} \\
&\qquad \le \|b\|_2 \cdot 2\delta, \qquad \text{(see (A.7))}
\end{aligned}
$$

which completes the proof of Lemma 11.

A.4 Proof of Proposition 12 (Existence of the dual certificate)

Without loss of generality and for better clarity, suppose that $T = \{t_i\}_{i=1}^{k}$ is an increasing sequence. Consider a positive scalar ρ such that $\rho \le \epsilon \le \Delta/2$. Consider also an increasing sequence $\{\tau_l\}_{l=1}^{m} \subset I = [0,1]$ such that $\tau_1 = 0$, $\tau_m = 1$, and every $T_{i,\rho}$ contains an even and nonzero number of the remaining points. Let us define the polynomial

$$
q^\rho(t) = \begin{vmatrix}
-F(t) & \phi_1(t) & \cdots & \phi_m(t) \\
-F(\tau_1) & \phi_1(\tau_1) & \cdots & \phi_m(\tau_1) \\
-F(\tau_2) & \phi_1(\tau_2) & \cdots & \phi_m(\tau_2) \\
\vdots & \vdots & \vdots & \vdots \\
-F(\tau_m) & \phi_1(\tau_m) & \cdots & \phi_m(\tau_m)
\end{vmatrix}, \qquad t \in I. \tag{A.8}
$$

Note that $q^\rho(t) = 0$ when $t \in \{\tau_l\}_{l=1}^{m}$. By assumption, $\{F\} \cup \{\phi_j\}_{j=1}^{m}$ form a T*-system on I. Therefore, invoking the first part of Definition 7, we find that q^ρ is non-negative on T_ρ^C. We represent this polynomial with $q^\rho = -\beta_0^\rho F + \sum_{j=1}^{m}(-1)^j \beta_j^\rho \phi_j$ and note that $\beta_0^\rho = |\phi_j(\tau_i)|_{i,j=1}^{m}$. By assumption also $\{\phi_j\}_{j=1}^{m}$ form a T-system on I and therefore $\beta_0^\rho > 0$. This observation allows us to form the normalized polynomial

$$
\dot{q}^\rho := \frac{q^\rho}{\beta_0^\rho} = -F + \sum_{j=1}^{m}(-1)^j \frac{\beta_j^\rho}{\beta_0^\rho}\phi_j =: -F + \sum_{j=1}^{m}(-1)^j b_j^\rho \phi_j.
$$

Note also that the coefficients $\{\beta_j^\rho\}_{j=0}^m$ correspond to the minors in the second part of Definition 7. Therefore, for each $j = 0, \ldots, m$, we have that $|\beta_j^\rho|$ approaches zero at the same rate, as $\rho \to 0$. So for sufficiently small ρ_0, every b_j^ρ is bounded in magnitude when $\rho \leq \rho_0$; in particular, $|b_j^\rho| = \Theta(1)$. This means that for sufficiently small ρ_0, $\{\dot{q}^\rho : \rho \leq \rho_0\}$ is bounded. Therefore, we can find a subsequence $\{\rho_l\}_l \subset [0, \rho_0]$ such that $\rho_l \to 0$ and the subsequence $\{\dot{q}^{\rho_l}\}_l$ converges to the polynomial

$$\dot{q} := -F + \sum_{j=1}^{m} b_j \phi_j.$$

Note that $b_j \neq 0$ for every $j = 1, \ldots, m$; in particular, $|b_j| = \Theta(1)$. Hence the polynomial $\sum_{j=1}^m b_j \phi_j$ is nontrivial, namely does not uniformly vanish on I. (It would have sufficed to have some nonzero coefficient, say b_{j_0}, rather than requiring all $\{b_j\}_{j=1}^m$ to be nonzero. However that would have made the statement of Definition 7 more cumbersome). Lastly observe that $\dot{q}$ is non-negative on I and vanishes on T (as well as on the boundary of I). This completes the proof of Proposition 12.

Appendix B

Duality for TV norm minimisation

B.1 Derivation of the dual problem in the noise-free setting

Here we show the duality of problems (1.8):

$$\min_{x\geq 0} \|x\|_{TV} \quad \text{subject to} \quad y = \int_I \Phi(t)x(\mathrm{d}t),$$

and (1.9):

$$\max_{\lambda\in\mathbb{R}^m} y^T\lambda \quad \text{subject to} \quad \lambda^T\Phi(t) \leq 1 \quad \forall t \in I.$$

We write the Lagrangian of (1.8):

$$L(x,\lambda,\nu) = \|x\|_{TV} + \lambda^T\left(y - \int_I \Phi(t)x(\mathrm{d}t)\right) - \nu\int_I x(\mathrm{d}t), \tag{B.1}$$

and then:

$$\begin{aligned}\max_{\lambda\in\mathbb{R}^m,\nu\geq 0}\min_x L(x,\lambda,\nu) &= \max_{\lambda\in\mathbb{R}^m,\nu\geq 0}\left[\lambda^T y - \max_x \int_I \left(\lambda^T\Phi(t) - 1 + \nu\right)x(\mathrm{d}t)\right] \\ &= \begin{cases} -\infty, & \text{if } \lambda^T\Phi(t^*) - 1 > -\nu, \text{ for some } t^* \in I, \\ \max_{\lambda\in\mathbb{R}^m}\lambda^T y & \text{if } \lambda^T\Phi(t) - 1 \leq -\nu, \forall t \in I. \end{cases}\end{aligned} \tag{B.2}$$

Since $\nu \geq 0$, the constraint in the second case above becomes $\lambda^T\Phi(t) \leq 1, \forall t \in I$, and therefore we obtain the dual (1.9).

B.2 Derivation of the dual problem in the noisy setting

In this section, we show the duality of the following problems:

$$\min_{x\geq 0}\left\|y - \int \Phi(t)x(\mathrm{d}t)\right\|_1 \quad \text{subject to} \quad \|x\|_{TV} \leq \Pi,$$

which is given in (1.11), and

$$\max_{\substack{\beta>0\\ \lambda\in\mathbb{R}^m}} \beta\left(\lambda^T y - \Pi\right) \quad \text{subject to} \quad \lambda^T\Phi(t) \le 1, \quad \forall t\in[0,1] \quad \text{and} \quad \|\lambda\|_\infty \le 1/\beta, \tag{B.3}$$

which is a more general version of the dual problem (1.10). We start from the primal problem (1.11) by introducing a new variable $z = \int \Phi(t)x(\mathrm{d}t)$:

$$\min_{\substack{x\ge 0\\ z\in\mathbb{R}^m}} \|z-y\|_1 \quad \text{subject to} \quad z = \int \Phi(t)x(\mathrm{d}t), \quad \|x\|_{TV} \le \Pi, \tag{B.4}$$

and then we write the Lagrangian:

$$L(x,z,\beta,\lambda) = \|z-y\|_1 + \lambda^T\left(z - \int \Phi(t)x(\mathrm{d}t)\right) + \beta\left(\|x\|_{TV} - \Pi\right), \tag{B.5}$$

so the Lagrangian dual problem is:

$$\begin{aligned}
&\max_{\substack{\beta\ge 0\\ \lambda\in\mathbb{R}^m}} \min_{\substack{x\ge 0\\ z\in\mathbb{R}^m}} L(x,z,\beta,\lambda) = \\
&\quad = \max_{\substack{\beta\ge 0\\ \lambda\in\mathbb{R}^m}} \min_{\substack{x\ge 0\\ z\in\mathbb{R}^m}} \left[\|z-y\|_1 + \lambda^T z + \int \left(\beta - \lambda^T\Phi(t)\right)x(\mathrm{d}t)\right] - \beta\Pi \\
&\quad = \max_{\substack{\beta\ge 0\\ \lambda\in\mathbb{R}^m}} \min_{\substack{x\ge 0\\ w\in\mathbb{R}^m}} \left[\|w\|_1 + \lambda^T w + \int \left(\beta - \lambda^T\Phi(t)\right)x(\mathrm{d}t)\right] + \lambda^T y - \beta\Pi,
\end{aligned} \tag{B.6}$$

where in the last equality we make the substitution $w = z - y$.

The integral on the right hand side is equal to $-\infty$ if there exists $t_0\in[0,1]$ such that $\lambda^T\Phi(t_0) > \beta$, as we can set $x = \infty\cdot\delta_{t_0}$. Therefore, we impose the condition that $\lambda^T\Phi(t)\le\beta$ for all $t\in[0,1]$, in which case the integral is equal to zero by taking x to be zero wherever the integrand is non-zero, and the dual becomes:

$$\max_{\substack{\beta\ge 0\\ \lambda\in\mathbb{R}^m}} \min_{w\in\mathbb{R}^m} \left(\|w\|_1 + \lambda^T w\right) + \lambda^T y - \beta\Pi \quad \text{subject to} \quad \lambda^T\Phi(t)\le\beta, \quad \forall t\in[0,1]. \tag{B.7}$$

which can be rewritten as:

$$\max_{\substack{\beta\ge 0\\ \lambda\in\mathbb{R}^m}} \max_{w\in\mathbb{R}^m} \left(-\lambda^T w - \|w\|_1\right) + \lambda^T y - \beta\Pi \quad \text{subject to} \quad \lambda^T\Phi(t)\le\beta, \quad \forall t\in[0,1]. \tag{B.8}$$

and note that for $f(w) = \|w\|_1$:

$$f^*(\lambda) = \max_{w} \left(\lambda^T w - \|w\|_1\right) = \begin{cases} 0, & \text{if} \quad \|\lambda\|_\infty \leq 1, \\ \infty, & \text{otherwise,} \end{cases} \tag{B.9}$$

is its conjugate [47]. Therefore, we impose the condition that $\|\lambda\|_\infty \leq 1$ and the dual becomes:

$$\max_{\substack{\beta \geq 0 \\ \lambda \in \mathbb{R}^m}} \lambda^T y - \beta \Pi \quad \text{subject to} \quad \lambda^T \Phi(t) \leq \beta, \quad \forall t \in [0,1] \quad \text{and} \quad \|\lambda\|_\infty \leq 1. \tag{B.10}$$

We then make the substitution $\lambda' = \lambda/\beta$ (for $\beta > 0$) to obtain:

$$\max_{\substack{\beta > 0 \\ \lambda' \in \mathbb{R}^m}} \beta \left(\lambda'^T y - \Pi\right) \quad \text{subject to} \quad \lambda'^T \Phi(t) \leq 1, \quad \forall t \in [0,1] \quad \text{and} \quad \|\lambda'\|_\infty \leq 1/\beta, \tag{B.11}$$

which is the problem (B.3).

Note that if we fix β and solve for λ', given that we are interested in the value of λ' rather than the value of the objective function, the problem above becomes:

$$\operatorname*{argmax}_{\lambda' \in \mathbb{R}^m} \lambda'^T y \quad \text{subject to} \quad \lambda'^T \Phi(t) \leq 1, \quad \forall t \in [0,1] \quad \text{and} \quad \|\lambda'\|_\infty \leq 1/\beta, \tag{B.12}$$

which is the problem (1.10) that we consider in Section 6.2.

B.3 Uniqueness of the solution assuming existence of the dual certificate

In Chapter 5, we claim that the existence of the dual certificate given in Definition 27 implies uniqueness of the non-negative TV norm minimisation problem (1.8). We show in this appendix that this is the case. Similarly to the proof in Appendix A.2, let x be the true measure and $\hat{x}$ a solution of (1.8) and let $h = \hat{x} - x$ be the error. Then we have that

$$\int_I \phi(t - s_j) h(\mathrm{d}t) = 0, \quad \forall j = 1, \ldots, m. \tag{B.13}$$

Let T be the support of x and $T^C = I \setminus T$, then:

$$\begin{aligned}
\int_{T^C} (q(t)-1)h(\mathrm{d}t) &= \int_I q(t)h(\mathrm{d}t) - \int_T q(t)h(\mathrm{d}t) - \int_{T^C} h(\mathrm{d}t) \\
&= \sum_{j=1}^m \lambda_j \int_I \phi(t-s_j)h(\mathrm{d}t) - \int_T q(t)h(\mathrm{d}t) - \int_{T^C} h(\mathrm{d}t) \\
&= -\int_T h(\mathrm{d}t) - \int_{T^C} h(\mathrm{d}t) = -\int_I h(\mathrm{d}t) \\
&= -\|\hat{x}\|_{TV} + \|x\|_{TV} \\
&= 0, \qquad \text{(B.14)}
\end{aligned}$$

where in the second line above we used the explicit form of the dual certificate $q(t) = \sum_{j=1}^m \lambda_j \phi(t-s_j)$.

Since $q(t) < 1$ on T^C, this implies that the error h is only supported on T (which also means that $\hat{x}$ is also only supported on T). Then, by the same argument as in A.2 based on T-systems (which the Gaussian kernel satisfies), we have that the weights of the point sources is $\hat{x}$ are the same as the ones in x, and therefore $\hat{x} = x$, so the solution is unique.