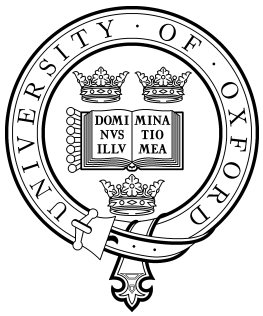
Laplace Transforms, Non-Analytic Growth Bounds and C_0-Semigroups



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St. John's College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

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In this thesis, we study a non-analytic growth bound $\zeta(f)$ associated with an exponentially bounded measurable function $f : \mathbb{R}_+ \to \mathbf{X}$, which measures the extent to which f can be approximated by holomorphic functions. This growth bound is related to the location of the domain of holomorphy of the Laplace transform of f far from the real axis. We study the properties of $\zeta(f)$ as well as two associated abscissas, namely the non-analytic abscissa of convergence, $\zeta_1(f)$ and the non-analytic abscissa of absolute convergence $\kappa(f)$. These new bounds may be considered as non-analytic analogues of the exponential growth bound $\omega_0(f)$ and the abscissas of convergence and absolute convergence of the Laplace transform of f, abs(f) and abs$(\|f\|)$. Analogues of several well known relations involving the growth bound and abscissas of convergence associated with f and abscissas of holomorphy of the Laplace transform of f are established. We examine the behaviour of ζ under regularisation of f by convolution and obtain, in particular, estimates for the non-analytic growth bound of the classical fractional integrals of f. The definitions of ζ, ζ_1 and κ extend to the operator-valued case also. For a C_0-semigroup $\mathbf{T}$ of operators, $\zeta(\mathbf{T})$ is closely related to the critical growth bound of $\mathbf{T}$. We obtain a characterisation of the non-analytic growth bound of $\mathbf{T}$ in terms of Fourier multiplier properties of the resolvent of the generator. Yet another characterisation of $\zeta(\mathbf{T})$ is obtained in terms of the existence of unique mild solutions of inhomogeneous Cauchy problems for which a non-resonance condition holds. We apply our theory of non-analytic growth bounds to prove some results in which $\zeta(\mathbf{T})$ does not appear explicitly; for example, we show that all the growth bounds $\omega_\alpha(\mathbf{T}), \alpha > 0$, of a C_0-semigroup $\mathbf{T}$ coincide with the spectral bound $s(\mathbf{A})$, provided the pseudo-spectrum is of a particular shape. Lastly, we shift our focus from non-analytic bounds to sun-reflexivity of a Banach space with respect to C_0-semigroups. In particular, we study the relations between the existence of certain approximations of the identity on the Banach space $\mathbf{X}$ and that of C_0-semigroups on $\mathbf{X}$ which make $\mathbf{X}$ sun-reflexive.

To my parents and my brother

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Chapter 1

Introduction

1.1 Background

Linear differential equations in Banach spaces are intimately connected with the theory of one-parameter semigroups and vector-valued Laplace transforms. In fact, given a closed linear operator $\mathbf{A}$ with dense domain $D(\mathbf{A}) \subset \mathbf{X}$, where $\mathbf{X}$ is a Banach space, the associated abstract Cauchy problem

$$\begin{cases} u'(t) &= \mathbf{A}u(t) \quad (t \geq 0), \\ u(0) &= x, \end{cases} \tag{ACP}$$

is mildly well posed (that is, for each $x \in \mathbf{X}$ there exists a unique mild solution of (ACP)) if and only if the resolvent of $\mathbf{A}$ is a Laplace transform. This is equivalent to saying that $\mathbf{A}$ generates a strongly continuous semigroup $\mathbf{T}$ on $\mathbf{X}$, and then the mild solution of (ACP) is given by $u(t) = \mathbf{T}(t)x$.

Here, by a *mild solution* of (ACP) we mean a continuous function u defined on the non-negative reals and taking values in $\mathbf{X}$ such that

$$\int_0^t u(s)\,ds \in D(\mathbf{A}) \text{ and } \mathbf{A}\int_0^t u(s)\,ds = u(t) - x \quad (t \geq 0).$$

By a *classical solution* of (ACP) we mean a continuously differentiable, $\mathbf{X}$-valued function u defined on the non-negative reals such that $u(t) \in D(\mathbf{A})$ for all $t \geq 0$ and (ACP) holds. The abstract Cauchy problem is classically well-posed if for each $x \in D(\mathbf{A})$, there exists a unique classical solution of (ACP). A mild solution u is a classical solution if and only if u is continuously differentiable. If u is a continuous, Laplace transformable function, then u is a mild solution of (ACP) if and only if $\hat{u}(\lambda) \in D(\mathbf{A})$ and $\lambda\hat{u}(\lambda) - \mathbf{A}\hat{u}(\lambda) = x$, for $\operatorname{Re}\lambda$ sufficiently large. (ACP) is mildly well posed if and only if $\rho(\mathbf{A}) \neq \emptyset$ and (ACP) is classically well-posed, if and only if $\mathbf{A}$ generates a C_0-semigroup.

These relations between solutions of differentiable equations and semigroups are the primary reasons why semigroups of operators have been studied intensively. We refer the

reader to the books of Hille and Phillips [28], Engel and Nagel [20], Davies [17] and Pazy [42] for the basic theory. The recent monograph by Arendt, Batty, Hieber and Neubrander [2] is particularly useful for our purposes as it presents the theory of linear evolution equations and semigroups via Laplace transforms methods.

For applications, it is useful to describe the properties of a semigroup in terms of its generator, as this gives valuable information about the solutions of the well posed or mildly well posed Cauchy problem even though the solutions may not be known explicitly, which is usually the case. Of particular interest is the asymptotic behaviour of these solutions; this has led to investigations into the behaviour of $\mathbf{T}(t)$ as $t \to \infty$ and more generally to the theory of asymptotics of strongly continuous semigroups. The starting point of this theory is Liapunov's stability theorem for matrices which characterises the 'stability' of the semigroup generated by an $n \times n$ matrix $\mathbf{A}$ in terms of the location of its eigenvalues. A C_0-semigroup $\mathbf{T}$ is called *uniformly exponentially stable* if $\omega_0(\mathbf{T}) < 0$, where $\omega_0(\mathbf{T})$ is the exponential growth bound of $\mathbf{T}$ given by

$$\omega_0(\mathbf{T}) = \inf\{\omega \in \mathbb{R} : \sup_{t \geq 0} e^{-\omega t}\|\mathbf{T}(t)\| < \infty\}.$$

$\mathbf{T}$ being uniformly exponentially stable is equivalent to

$$\lim_{t \to \infty} \|\mathbf{T}(t)\| = 0.$$

For a closed operator $\mathbf{A}$, the spectral bound $s(\mathbf{A})$ is given by

$$s(\mathbf{A}) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathbf{A})\}.$$

In terms of the exponential growth bound and the spectral bound Liapunov's theorem may be stated as follows.

Theorem 1.1.1. *Let $\mathbf{T}$ be the semigroup on $\mathbb{C}^n$ generated by $\mathbf{A} \in M_n(\mathbb{C})$. Then*

$$\omega_0(\mathbf{T}) = s(\mathbf{A}).$$

The above theorem extends to semigroups generated by bounded operators $\mathbf{A}$ defined on any Banach space. This is a direct consequence of the validity of the spectral mapping Theorem $\sigma(e^{t\mathbf{A}}) = e^{t\sigma(\mathbf{A})}, \ \ t \geq 0$, for such semigroups. However, for general C_0-semigroups the growth bound and the spectral bound do not necessarily coincide; in most cases this failure of Liapunov's stability theorem is due to the absence of any kind of spectral mapping theorem.

The exponential growth of the mild solutions of a well posed Cauchy problem is determined by the uniform growth bound $\omega_0(\mathbf{T}), \mathbf{T}$ being the associated semigroup. Thus Liapunov's theorem implies that if $\mathbf{A}$ is bounded, then the exponential growth of the mild

solutions of (ACP) is determined by the location of the spectrum of $\mathbf{A}$. In the case when $\mathbf{A}$ is an unbounded operator, information about the location of the spectrum of $\mathbf{A}$ is no longer enough, and additional assumptions are needed, either on the smoothness of $\mathbf{T}$ or on the geometry of the underlying space $\mathbf{X}$.

For eventually norm-continuous semigroups the spectral mapping theorem $\sigma(\mathbf{T}(t)) \setminus \{0\} = e^{t\sigma(\mathbf{A})}$ holds [28], and therefore, so does Liapunov's stability theorem. The category of eventually norm-continuous semigroups includes all semigroups which are eventually compact, eventually differentiable or holomorphic. Building on preliminary work of Martinez and Mazon [37], Blake [11] introduced the concept of asymptotically norm-continuous semigroups or semigroups which are norm-continuous at infinity. A spectral mapping theorem for the peripheral spectrum holds for such semigroups and this is sufficient for deducing that $\omega_0(\mathbf{T}) = s(\mathbf{A})$. All eventually norm-continuous semigroups with finite growth bounds are asymptotically norm-continuous.

Several other growth bounds and spectral bounds have been introduced in order to further describe the asymptotic behaviour of strongly continuous semigroups. Among these are the growth bound $\omega_1(\mathbf{T})$, which determines the exponential growth of classical solutions of (ACP), higher order analogues $\omega_n(\mathbf{T}), n \in \mathbb{N}$, which estimate the exponential growth of solutions of (ACP) with initial values in $D(\mathbf{A}^n)$, and the more general fractional growth bounds $\omega_\alpha(\mathbf{T}), \alpha \geq 0$. The pseudo-spectral bound $s_0(\mathbf{A})$ is the abscissa of uniform boundedness of the resolvent while the n-th spectral bound $s_n(\mathbf{A})$ is the abscissa of polynomial boundedness of degree n of the resolvent. There is a large literature on the relations between these growth bounds associated with the semigroup $\mathbf{T}$ and the spectral bounds of the generator $\mathbf{A}$. We refer to [40] and [2, Chapter 5] for surveys. Inequalities showing that the growth bounds are not less than spectral bounds are relatively easy to obtain compared with opposite inequalities. The first relation showing a spectral bound dominating a growth bound for arbitrary C_0-semigroups was $\omega_2(\mathbf{T}) \leq s_0(\mathbf{A})$, obtained in [47]. Amongst the most striking results in this direction are the Gearhart-Prüss theorem establishing the equality $\omega_0(\mathbf{T}) = s_0(\mathbf{A})$ for strongly continuous semigroups defined on Hilbert spaces [22], [44] and the theorem of Weis and Wrobel showing $\omega_1(\mathbf{T}) \leq s_0(\mathbf{A})$ for semigroups on general Banach spaces [50]. The analogue of the Gearhart-Prüss Theorem for higher order bounds, involving the equality of $\omega_n(\mathbf{T})$ and $s_n(\mathbf{A}), n \in \mathbb{N}$, for semigroups defined on Hilbert spaces has been obtained in [51].

A new growth bound, the *growth bound of local variation* $\delta(\mathbf{T})$ or the *critical growth bound* $\omega_{\text{crit}}(\mathbf{T})$ has recently been introduced in [11] and [38], building on ideas from [37]. It measures the growth of the uniform local variation of mild solutions of the Cauchy problem, and it is related to $s^\infty(\mathbf{A})$ and $s_0^\infty(\mathbf{A})$, the bounds of the spectrum and the pseudo-spectrum of $\mathbf{A}$ away from the real axis. The spectral bounds $s^\infty(\mathbf{A})$ and $s_0^\infty(\mathbf{A})$ may be considered as

analogues of $s(\mathbf{A})$ and $s_0(\mathbf{A})$ determining the existence and boundedness of the resolvents in those parts of the right half-planes which are away from the real axis. There is an analogue of the Gearhart-Prüss Theorem for these bounds ($\delta(\mathbf{T}) = s_0^\infty(\mathbf{A})$) for semigroups on Hilbert spaces [11]. Applications of the critical growth bound to perturbation theory and to various evolution equations may be found in [9], [13], [14] and [15].

The standard growth and spectral bounds for semigroups are all special cases of bounds and abscissas associated with vector or operator-valued functions on $\mathbb{R}_+$ and their Laplace transforms. For example, for a strongly continuous semigroup $\mathbf{T}$ with generator $\mathbf{A}$, the spectral bounds $s(\mathbf{A})$ and $s_0(\mathbf{A})$ are just the abscissa of holomorphy and boundedness ([2, Section 1.4, Section 1.5]) of the operator-valued function $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ while $\omega_1(\mathbf{T})$ is the abscissa of convergence of the Laplace transform $\hat{\mathbf{T}}$ of $\mathbf{T}$. Most of the general results also extend naturally to exponentially bounded functions, but some, like the Gearhart-Prüss Theorem are confined to semigroups and/or depend on the geometry of the Banach space in question. The Weis-Wrobel Theorem is an example of a semigroup result extending to the case of exponentially bounded functions as shown by Blake [11, Theorem 6.5.9],[5]. However, none of the characterisations of the critical growth bound known so far extends in a useful way to functions.

1.2 A new growth bound

In this thesis, we study a growth bound $\zeta(f)$ associated with an exponentially bounded function defined on $\mathbb{R}_+$, which may be described in a sense, as the growth bound of f modulo functions which are holomorphic and exponentially bounded in a sector about the positive real axis. Therefore, we call this growth bound the *non-analytic growth bound* of f. We work as far as possible in the general setting of vector-valued, exponentially bounded functions defined on $\mathbb{R}_+$ and deduce results for semigroups as special cases.

$\zeta(f)$ may be thought of as the non-analytic analogue of the exponential growth bound $\omega_0(f)$ of f. In fact, it is related to the analytic behaviour of $\hat{f}$ away from the real axis in much the same way as $\omega_0(f)$ is related to $\hat{f}$ in the right half-planes of $\mathbb{C}$. In particular, $\mathrm{hol}_0^\infty(\hat{f}) \leq \zeta(f)$, where $\mathrm{hol}_0^\infty(\hat{f})$ is the analogue of the spectral bound $s_0^\infty(\mathbf{A})$ for functions. We also introduce the *non-analytic abscissas of convergence* and *absolute convergence,* $\zeta_1(f)$ and $\kappa(f)$ associated with f, which are again analogues of the abscissas of convergence and absolute convergence, $\mathrm{abs}(f)$ and $\mathrm{abs}(\|f\|)$ of the Laplace transform of f. We obtain non-analytic analogues of many of the relations between growth bounds and spectral bounds for semigroups and their extensions to exponentially bounded, vector-valued functions. In particular, such an analogue of Blake's extension of the Weis-Wrobel Theorem to functions is obtained.

The non-analytic growth bound coincides with the growth bound of non-integrability of operator-valued functions on $\mathbb{R}_+$ defined in [10]. It has been established in [11], [10] that the critical growth bound and the growth bound of non-integrability associated with a strongly continuous semigroup $\mathbf{T}$ with generator $\mathbf{A}$ defined on a Hilbert space coincide with the spectral bound $s_0^\infty(\mathbf{A})$. Thus, we have an analogue of the Gearhart-Prüss theorem for the non-analytic growth bound also. We derive higher order analogues of this result. For many semigroups $\delta(\mathbf{T}) = \zeta(\mathbf{T})$ (we do not know of any semigroup for which they differ). A comparison of the critical and non-analytic growth bounds shows that unlike the critical growth bound, the concept of the non-analytic growth bound of exponentially bounded vector-valued functions is a useful and interesting study in itself.

An examination of the behaviour of the non-analytic growth bound of an exponentially bounded, measurable function f defined on $\mathbb{R}_+$, under regularisation by convolution yields some interesting results. Besides establishing analogues of known results for growth bounds of convolutions of f with ϕ, we obtain estimates for $\zeta(\phi * f)$ in terms of the growth bound and certain abscissas of holomorphy of the Laplace transform of f, where ϕ is a locally integrable complex-valued function defined on $\mathbb{R}$ satisfying certain additional conditions. A particular case of these results is an estimate for the classical fractional integral of f. This line of study builds up to the definition of the *fractional non-analytic growth bounds* for a function f. In the case of a semigroup $\mathbf{T}$, these bounds are the natural analogues of the fractional growth bounds $\omega_\alpha(\mathbf{T})$. The relation between the fractional non-analytic and uniform growth bounds of $\mathbf{T}$ yields a rather striking result: The uniform growth bounds $\omega_\alpha(\mathbf{T})$ equal the spectral bound $s(\mathbf{A})$ for all $\alpha > 0$ provided the pseudo-spectrum of $\mathbf{A}$ is of a particular shape.

A characterisation of the uniform growth bound in terms of the Fourier multiplier properties of the resolvent has been obtained in [27] and for higher order growth bounds in [32]. In fact, Fourier multipliers have often been used to study stability and hyperbolicity of strongly continuous semigroups (see [30], [34] and [49]). Using ideas from [32], [30] and [34] we obtain a characterisation for $\zeta(\mathbf{T})$ for a strongly continuous semigroup $\mathbf{T}$ in terms of the shape of the pseudo-spectrum of the generator $\mathbf{A}$ and Fourier multiplier properties of functions of the form $s \mapsto \phi(s)\mathbf{R}(w + is, \mathbf{A})$ where $w > s_0^\infty(\mathbf{A})$, $\mathbf{R}(\lambda, \mathbf{A})$ is the resolvent of $\mathbf{A}$ and ϕ is a suitable smooth function. Thus, one is able to obtain information about the non-analytic behaviour of solutions of (ACP) from the pseduo-spectrum and resolvents of the operator $\mathbf{A}$.

In [45] and [43] results have been obtained concerning existence of bounded solutions of the inhomogeneous Cauchy problem

$$u'(t) = \mathbf{A}u(t) + f(t) \quad (t \in \mathbb{R}), \tag{1.1}$$

when f is a bounded function on $\mathbb{R}$ taking values in $\mathbf{X}$, $\mathbf{A}$ is the generator of a C_0-semigroup, and a non-resonance condition between $\mathbf{A}$ and f is satisfied together with some assumptions on the spectrum of $\mathbf{A}$. We study (1.1) when $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X}), 1 \leq p < \infty$ and $\mathbf{A}$ is any closed operator. We obtain a necessary and sufficient condition for a function in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ to be a mild solution of (1.1) for this case. If, in addition, $\mathbf{A}$ generates a C_0-semigroup $\mathbf{T}$ and f satisfies a non-resonance condition with respect to $\mathbf{A}$ then the existence of a unique mild solution of (1.1) is closely related to $\zeta(\mathbf{T})$. In fact, we are able to obtain a necessary and sufficient condition for $\zeta(\mathbf{T}) < 0$ in terms of the shape of the pseudo-spectrum of $\mathbf{A}$ and the existence of such solutions. This characterisation is comparable to characterisations of hyperbolicity of the C_0-semigroup generated by $\mathbf{A}$ in terms of the existence of unique mild solutions of (1.1), for every $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ ([33], [16, Section 4.3]) and for every bounded and continuous f ([44, Theorem 4]). The result in [33] is proven in the context of non-autonomous Cauchy problems. We refer the reader to [16] for the definitions and theory concerning non-autonomous Cauchy problems.

Towards the end of the thesis, we digress from the subject of non-analytic growth bounds and study the relations between sun-reflexivity of a Banach space with respect to a strongly continuous semigroup and the existence of approximations of the identity on the space with some special properties. This study is inspired by [46] where strong Feller semigroups and approximations of the identity on C^*-algebras are studied.

1.3 Overview of thesis

In Chapter 2 we introduce some notations and collect well known results from the vast literature on strongly continuous semigroups and vector-valued Laplace and Fourier transforms. Definitions of some well known abscissas of holomorphy and boundedness of Laplace transforms are recalled, new definitions added and properties of these abscissas deduced.

We begin the first section of Chapter 3 by introducing the non-analytic growth bounds and abscissas of convergence of the Laplace transform of an exponentially bounded vector-valued function defined on $\mathbb{R}_+$. Subsequently, equivalent descriptions of these bounds are derived (Proposition 3.1.4) and their basic properties studied. Section 3.2 is devoted to the particular case of operator-valued functions. In Subsection 3.2.1, we obtain descriptions of the non-analytic bounds of operator valued functions $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ in terms of similar bounds for the orbit maps $t \mapsto \mathbf{T}(t)x,\ x \in \mathbf{X}$. Subsection 3.2.2 deals specifically with the non-analytic bounds for strongly continuous semigroups. A non-analytic version of the Gearhart-Prüss theorem for higher orders on Hilbert spaces is obtained in Theorem 3.2.9. Analogous to the concepts of essential norm continuity and essential measurability of C_0-semigroups [48], we introduce essentially holomorphic C_0-semigroups in Section 3.3.

In Section 3.4 we undertake a comparison of the critical and non-analytic growth bounds. We look at several classes of semigroups for which a non-analytic analogue of the Gearhart-Prüss theorem holds for arbitrary Banach spaces, so that the critical and non-analytic growth bounds coincide.

In Chapter 4 we study the behaviour of $\zeta(f)$ under convolutions. Theorem 4.1.6 is an analogue of Blake's extension [10, Theorem 6.5.7] of the Weis-Wrobel result [50] to functions. In Section 4.3, the fractional non-analytic growth bounds $\zeta_\alpha(f), \alpha > 0$ of f are introduced and estimates for these are obtained in terms of the uniform growth bound and certain abscissas of holomorphy. In particular, we obtain estimates for the non-analytic growth bounds of the Weyl and the Riemann-Liouville fractional integrals of exponentially bounded measurable functions (Corollary 4.3.2 and Theorem 4.3.3). Section 4.4 is devoted to the study of the fractional growth bounds of C_0-semigroups. In Theorem 4.4.2 we show that for a strongly continuous semigroup $\mathbf{T}$ all the growth bounds $w_\alpha(\mathbf{T}), \alpha > 0$ coincide with the spectral bound $s(\mathbf{A})$ of the generator $\mathbf{A}$ provided $s_0^\infty(\mathbf{A}) = -\infty$. Convexity of the function $\alpha \mapsto \zeta_\alpha(f)$ is studied in Section 4.5

We obtain characterisations of the non-analytic growth bound of a strongly continuous semigroup in terms of some properties of the resolvent of the generator, in particular Fourier multiplier properties, in Section 5.1. In Section 5.2 the effect on $\zeta(\mathbf{T})$ due to perturbations of the generator of a semigroup $\mathbf{T}$ is studied. Section 5.3 brings out the connection between $\zeta(\mathbf{T})$ and the existence of unique solutions of some inhomogeneous Cauchy problems.

In Chapter 6 we study relationships between sun-reflexivity and the existence of approximations of the identity on a Banach space. We prove, in particular, the existence of Banach spaces admitting no strongly continuous semigroups with respect to which they are sun-reflexive.

Chapter 2

Preliminaries

2.1 Notation

2.1.1 Sets

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ shall denote the natural numbers, integers, the real numbers and the complex numbers respectively. The half-line $[0, \infty)$ will be denoted by $\mathbb{R}_+$ and the open half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$ by $\mathbb{C}_+$. In general, for $w \in \mathbb{R}$, we define the open half plane H_w by

$$H_w = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > w\}.$$

Further, for $b \geq 0$, we define

$$\begin{aligned} Q_{w,b} &= \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq w, |\operatorname{Im} \lambda| \geq b\}, \\ Q^o_{w,b} &= \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > w, |\operatorname{Im} \lambda| > b\}. \end{aligned}$$

For $b > 0, Q_{w,b}$ is a pair of closed quadrants and $Q^o_{w,b}$ is its interior.

For $\theta > 0$, we shall denote the sector of the complex plane of angle θ, containing the positive reals by

$$\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}.$$

2.1.2 Banach spaces and operators

Throughout this thesis, $\mathbf{X}$ shall be a complex Banach space and $\mathbf{X}^*$ its dual. The Banach algebra of bounded linear operators on $\mathbf{X}$ shall be denoted by $\mathcal{L}(\mathbf{X})$. For an unbounded linear operator $\mathbf{A}$ on $\mathbf{X}$, $D(\mathbf{A}), \operatorname{Ran}(\mathbf{A})$, and $\operatorname{Ker}(\mathbf{A})$ shall denote the domain, range and kernel of $\mathbf{A}$ respectively.

If $\mathbf{B}$ is a linear operator on $\mathbf{X}$ with domain $D(\mathbf{B})$ and $\mathbf{Y}$ is a subspace of $\mathbf{X}$ containing

$D(\mathbf{B})$ then the *part of* $\mathbf{B}$ *in* $\mathbf{Y}$ is the operator $\mathbf{B_Y}$ defined by

$$D(\mathbf{B_Y}) := \{y \in D(\mathbf{B}) : \mathbf{B}(y) \in \mathbf{Y}\};$$
$$\mathbf{B_Y}(y) := \mathbf{B}y,\ y \in D(\mathbf{B_Y}).$$

2.1.3 Function spaces

For $1 \leq p < \infty$, let $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ be the space of all Bochner measurable functions $f : \mathbb{R} \longrightarrow \mathbf{X}$ such that

$$\|f\|_p := \left(\int_{\mathbb{R}} \|f(t)\|^p \, dt\right)^{\frac{1}{p}} < \infty.$$

Let $\mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$ be the space of all Bochner measurable functions $f : \mathbb{R} \longrightarrow \mathbf{X}$ such that

$$\|f\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}} \|f(t)\| < \infty.$$

The conjugate index for $p, 1 \leq p < \infty$, shall be denoted by p' so that $\frac{1}{p} + \frac{1}{p'} = 1$.

The space of locally integrable functions $\mathrm{L}^1_{loc}(\mathbb{R}, \mathbf{X})$ is given by

$$\mathrm{L}^1_{loc}(\mathbb{R}, \mathbf{X}) = \Big\{ f : \mathbb{R} \longrightarrow \mathbf{X} \text{ such that } f \text{ is Bochner measurable and}$$
$$\text{for every compact } K \subset \mathbb{R}, \int_K \|f(t)\| \, dt < \infty \Big\}.$$

We denote by $C(\mathbb{R}, \mathbf{X})$ the vector space of all continuous functions $f : \mathbb{R} \longrightarrow \mathbf{X}$. For $k \in \mathbb{N}$, $C^k(\mathbb{R}, \mathbf{X})$ will be the space of all k-times differentiable functions with continuous kth derivative and $C^\infty(\mathbb{R}, \mathbf{X}) := \bigcap_{k=1}^\infty C^k(\mathbb{R}, \mathbf{X})$. $BUC(\mathbb{R}, \mathbf{X})$ will be the space of all bounded, uniformly continuous functions defined on $\mathbb{R}$ and taking values in $\mathbf{X}$.

$C_c(\mathbb{R}, \mathbf{X})$ and $C_c^\infty(\mathbb{R}, \mathbf{X})$ shall denote the space of all functions with compact support in $C(\mathbb{R}, \mathbf{X})$ and $C^\infty(\mathbb{R}, \mathbf{X})$, respectively. The space of functions in $C(\mathbb{R}, \mathbf{X})$ vanishing at infinity will be denoted by $C_0(\mathbb{R}, \mathbf{X})$. Further, $\mathcal{S}(\mathbb{R}, \mathbf{X})$ shall denote the Schwartz space of functions in $C^\infty(\mathbb{R}, \mathbf{X})$ which are rapidly decreasing. Then $C_c^\infty(\mathbb{R}, \mathbf{X}) \subset \mathcal{S}(\mathbb{R}, \mathbf{X}) \subset \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and $C_c^\infty(\mathbb{R}, \mathbf{X})$ is dense in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ for $1 \leq p < \infty$.

If $\mathbf{X} = \mathbb{C}$, then we shall write $\mathcal{S}(\mathbb{R})$ in place of $\mathcal{S}(\mathbb{R}, \mathbb{C}), C^\infty(\mathbb{R})$ instead of $C^\infty(\mathbb{R}, \mathbf{X})$ and so on.

A function $f : \mathbb{R}_+ \to \mathbf{X}$ may be considered to be defined on the whole of $\mathbb{R}$ by setting $f = 0$ on $(-\infty, 0)$. We shall denote by $\mathrm{L}^p(\mathbb{R}_+, \mathbf{X})$ the subspace of $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ consisting of functions which take the value 0 on $(-\infty, 0)$. $C(\mathbb{R}_+, \mathbf{X})$ shall denote the space of all continuous functions $f : \mathbb{R}_+ \to \mathbf{X}$.

2.2 Vector-valued functions

2.2.1 Exponential growth bound

The *exponential growth bound* of $f : \mathbb{R}_+ \longrightarrow \mathbf{X}$ is given by

$$\omega_0(f) = \inf\{w \in \mathbb{R} : \sup_{t\geq 0} \|e^{-wt} f(t)\| < \infty\}.$$

In this, and other similar definitions throughout the thesis, we allow the values ∞ and $-\infty$ according to the usual conventions. We say that f is *exponentially bounded* if $\omega_0(f) < \infty$.

A function $g : \Sigma_\theta \longrightarrow \mathbf{X}$ is said to be *exponentially bounded* if there exist constants M, w such that $\|g(z)\| \leq Me^{w|z|}$ $(z \in \Sigma_\theta)$. The restriction of g to $(0, \infty)$ may be exponentially bounded even if g is not exponentially bounded on Σ_θ. However, by $\omega_0(g)$ we shall always mean the exponential growth bound of the restriction of g to $\mathbb{R}_+$ with $g(0) = 0$.

2.2.2 The Laplace transform

For $f \in \mathrm{L}^1_{loc}(\mathbb{R}_+, \mathbf{X})$ we define the *abscissas of absolute convergence* and *convergence* of the Laplace transform of f [2, Section 1.4] by :

$$\begin{aligned} \mathrm{abs}(\|f\|) &= \inf\{\omega \in \mathbb{R} : \int_0^\infty e^{-\omega t}\|f(t)\|\, dt < \infty\}; \\ \mathrm{abs}(f) &= \inf\{\mathrm{Re}\,\lambda : \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} f(t)\, dt \text{ exists}\}. \end{aligned}$$

It is clear that

$$\mathrm{abs}(f) \leq \mathrm{abs}(\|f\|) \leq \omega_0(f).$$

We say that f is *Laplace transformable* if $\mathrm{abs}(f) < \infty$ and define the *Laplace transform* of f by $\hat{f}$ where

$$\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t)\, dt := \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} f(t)\, dt$$

whenever this limit exists. If $\int_0^\infty e^{-\lambda t} f(t)\, dt$ exists as a Bochner integral, then by the dominated convergence theorem, it agrees with the definition above. We now record some well known facts about the Laplace transform of a locally integrable function f with $\mathrm{abs}(f) < \infty$. For the proof of these we refer to [2, Sections 1.4 and 1.5]. :

1. The Laplace integral $\hat{f}(\lambda)$ converges if $\mathrm{Re}\,\lambda > \mathrm{abs}(f)$ and diverges if $\mathrm{Re}\,\lambda < \mathrm{abs}(f)$ [2, Proposition 1.4.1].

2. $\mathrm{abs}(f) = \omega_0(F - F_\infty)$, where $F(t) = \int_0^t f(s)\, ds$, $F_\infty = \lim_{t\to\infty} F(t)$ if the limit exists and $F_\infty = 0$ otherwise [2, Theorem 1.4.3].

3. If $\operatorname{Re}\lambda > \max(\operatorname{abs}(f), 0)$, then
$$\hat{F}(\lambda) = \frac{\hat{f}(\lambda)}{\lambda}.$$
[2, Corollary 1.6.5].

4. $\lambda \mapsto \hat{f}(\lambda)$ defines a holomorphic function from $H_{\operatorname{abs}(f)}$ into $\mathbf{X}$ and for $n \in \mathbb{N} \cup \{0\}$, $\operatorname{Re}\lambda > \operatorname{abs}(f)$,
$$\hat{f}^{(n)}(\lambda) = \int_0^\infty e^{-\lambda t}(-t)^n f(t)\, dt.$$
[2, Theorem 1.5.1].

5. If $\operatorname{abs}(\|f\|) < \infty$ then $\hat{f}$ is bounded on H_w whenever $w > \operatorname{abs}(\|f\|)$; indeed, $\sup_{\lambda\in H_w} \|\hat{f}(\lambda)\| \leq \int_0^\infty e^{-wt}\|f(t)\|\, dt < \infty$.

In general, $\hat{f}$ may have a holomorphic extension to a bigger region than $H_{\operatorname{abs}(f)}$. We shall denote the extension of $\hat{f}$ by the same symbol. We now define the largest such region with which we shall work :

$$\begin{aligned} D(\hat{f}) \;:=\; & \{\lambda = \alpha + i\eta : \hat{f} \text{ has a holomorphic extension to} \\ & H_{\operatorname{abs}(f)} \cup \{\beta + is : \alpha - \epsilon < \beta, |s - \eta| < \epsilon\} \text{ for some } \epsilon > 0\} \end{aligned}$$

Then $D(\hat{f})$ is a connected open set which is a union of horizontal line-segments extending infinitely to the right, and $\hat{f}$ has a unique holomorphic extension (also denoted by $\hat{f}$) to $D(\hat{f})$. Moreover, $D(\hat{f})$ is the largest such set with these properties. Next we define some abscissas of holomorphy and boundedness:

$$\operatorname{hol}(\hat{f}) = \inf\left\{\omega \in \mathbb{R} : H_\omega \subset D(\hat{f})\right\}; \tag{2.1}$$

$$\operatorname{hol}_0(\hat{f}) = \inf\left\{\omega \in \mathbb{R} : H_\omega \subset D(\hat{f}) \text{ and } \sup_{\operatorname{Re}\lambda>\omega} \|\hat{f}(\lambda)\| < \infty\right\}; \tag{2.2}$$

$$\operatorname{hol}^\infty(\hat{f}) = \inf\left\{\omega \in \mathbb{R} : Q_{\omega,b} \subset D(\hat{f}) \text{ for some } b \geq 0\right\}; \tag{2.3}$$

$$\operatorname{hol}_0^\infty(\hat{f}) = \inf\left\{\omega \in \mathbb{R} : Q_{\omega,b} \subset D(\hat{f}) \text{ and } \sup_{\lambda\in Q_{\omega,b}} \|\hat{f}(\lambda)\| < \infty \text{ for some } b \geq 0\right\}; \tag{2.4}$$

$$\operatorname{hol}_{\mathrm{n}}(\hat{f}) = \inf\left\{\omega \in \mathbb{R} : H_\omega \subset D(\hat{f}) \text{ and } \sup_{\operatorname{Re}\lambda>\omega} \frac{\|\hat{f}(\lambda)\|}{(1+|\lambda|)^n} < \infty\right\}; \tag{2.5}$$

$$\operatorname{hol}_{\mathrm{n}}^\infty(\hat{f}) = \inf\left\{\omega \in \mathbb{R} : Q_{\omega,b} \subset D(\hat{f}) \text{ and } \sup_{\lambda\in Q_{\omega,b}} \frac{\|\hat{f}(\lambda)\|}{(1+|\lambda|)^n} < \infty \text{ for some } b \geq 0\right\}. \tag{2.6}$$

for $n \in \mathbb{N}$.

Thus, $\mathrm{hol}(\hat{f})$ and $\mathrm{hol}_0(\hat{f})$ are the *abscissas of holomorphy* and *boundedness* of $\hat{f}$ [2, Section 1.5] and $\mathrm{hol}^\infty(\hat{f})$ and $\mathrm{hol}_0^\infty(\hat{f})$ are analogues which ignore horizontal strips in $\mathbb{C}$. For $n \in \mathbb{N}$, $\mathrm{hol}_\mathrm{n}(\hat{f})$ gives the minimal abscissa for the half-plane where $\hat{f}$ grows along vertical lines not faster than the n-th power and $\mathrm{hol}_\mathrm{n}^\infty(\hat{f})$ is the corresponding analogue ignoring horizontal strips.

It is clear from the definitions and the properties of the Laplace transform above, that for a Laplace transformable function $f : \mathbb{R}_+ \to \mathbf{X}$,

$$\begin{aligned} \mathrm{hol}^\infty(\hat{f}) &\leq \mathrm{hol}(\hat{f}) \leq \mathrm{abs}(f); && (2.7)\\ \mathrm{hol}(\hat{f}) &\leq \mathrm{hol}_0(\hat{f}) \leq \mathrm{abs}(\|f\|); && (2.8)\\ \mathrm{hol}^\infty(\hat{f}) &\leq \mathrm{hol}_0^\infty(\hat{f}) \leq \mathrm{hol}_0(\hat{f}); && (2.9)\\ \mathrm{hol}^\infty(\hat{f}) &\leq \mathrm{hol}_\mathrm{n}^\infty(\hat{f}) \leq \mathrm{hol}_0^\infty(\hat{f}). && (2.10) \end{aligned}$$

For an exponentially bounded function f we have

$$\mathrm{abs}(f) \leq \mathrm{hol}_0(\hat{f}) \tag{2.11}$$

[5]. However, (2.11) is false for some Laplace transformable functions [12]. If f is exponentially bounded, then it is clear that for $w > \omega_0(f)$, there is a constant M such that

$$\|\hat{f}(\lambda)\| \leq \frac{M}{(\mathrm{Re}\,\lambda - w)} \qquad (\mathrm{Re}\,\lambda > w). \tag{2.12}$$

For a Laplace transformable function the following holds:

Lemma 2.2.1. *If $w > \mathrm{abs}(f)$, and $0 < \theta < \frac{\pi}{2}$, then $\hat{f}$ is bounded on $H_w \cap \Sigma_\theta$. Further,*

$$\begin{aligned} \mathrm{hol}_0(\hat{f}) &= \max\left(\mathrm{hol}_0^\infty(\hat{f}), \mathrm{hol}(\hat{f})\right);\\ \mathrm{hol}_\mathrm{n}(\hat{f}) &= \max\left(\mathrm{hol}_\mathrm{n}^\infty(\hat{f}), \mathrm{hol}(\hat{f})\right),\ n \in \mathbb{N}. \end{aligned}$$

Proof. Let $0 < \theta < \frac{\pi}{2}$ and $w > \max(0, \mathrm{abs}(f))$. Since $\hat{f}(\lambda) = \lambda\hat{F}(\lambda)$ and $\mathrm{abs}(f) = \omega_0(F - F_\infty)$, (2.12) applied to $F - F_\infty$ yields a constant M such that

$$\|\hat{f}(\lambda)\| \leq \frac{M|\lambda|}{\mathrm{Re}\,\lambda - w} + \|F_\infty\|.$$

Choosing $\epsilon > 0$ such that $0 < \epsilon < \cos\theta$, we have

$$\|\hat{f}(\lambda)\| \leq K, \text{ for all } \lambda \in H_{\frac{w}{\epsilon}} \cap \Sigma_\theta,$$

where $K = (\cos\theta - \epsilon)^{-1} + \|F_\infty\|$ is a constant. Thus, the first statement follows on observing that $\hat{f}$ is bounded on compact subsets of $\mathbb{C}$.

From the inequalities (2.8) and (2.9) it follows that

$$\mathrm{hol}_0(\hat{f}) \geq \max(\mathrm{hol}_0^\infty(\hat{f}), \mathrm{hol}(\hat{f})).$$

Suppose $a \in \mathbb{R}$ is such that $\max(\mathrm{hol}_0^\infty(\hat{f}), \mathrm{hol}(\hat{f})) < a$. Then $\hat{f}$ has a holomorphic extension to H_a which is bounded on $Q_{a,b}$ for some $b > 0$. Let

$$S_0 = \{\lambda \in \mathbb{C} : a \leq \mathrm{Re}\,\lambda \leq \omega, |\,\mathrm{Im}\,\lambda| \leq b\},$$

where $\omega > \max(0, \mathrm{abs}(f))$. Then $\hat{f}$ is holomorphic on the compact set S_0 and therefore bounded. That $\hat{f}$ is bounded on $\{\mathrm{Re}\,\lambda > \omega, |\,\mathrm{Im}\,\lambda| \leq b\}$ follows from the first part. Therefore, $\sup_{\lambda \in S_0 \cup Q_{a,b}} \|\hat{f}(\lambda)\| < \infty$. Thus, $\mathrm{hol}_0(\hat{f}) \leq a$. Hence, $\mathrm{hol}_0(\hat{f}) \leq \max\big(\mathrm{hol}_0^\infty(\hat{f}), \mathrm{hol}(\hat{f})\big)$. The corresponding result for $\mathrm{hol}_\mathrm{n}(\hat{f})$ follows similarly. □

We note here that for a Laplace transformable function f, $\sup_{\lambda \in Q_{a,b}} \|\hat{f}(\lambda)\| < \infty$ actually implies, on using Lemma 2.2.1, that

$$\sup_{\lambda \in H_w \cup Q_{a,b}} \|\hat{f}(\lambda)\| < \infty,$$

where $w \in \mathbb{R}$ is sufficiently large. We shall often use this fact without mention.

2.2.3 Convolutions and the Fourier transform

Given $f : \mathbb{R} \to \mathbf{X}$ and $g : \mathbb{R} \to \mathbb{C}$, the *convolution* $g * f$ is defined by

$$(g * f)(t) = \int_{-\infty}^{\infty} g(t-s) f(s)\, ds$$

whenever this integral exists as a Bochner integral. From Young's inequality [2, Proposition 1.3.2] we have that if $g \in \mathrm{L}^p(\mathbb{R})$ and $f \in \mathrm{L}^q(\mathbb{R}, \mathbf{X})$, then $g * f \in \mathrm{L}^r(\mathbb{R}, \mathbf{X})$ and

$$\|g * f\|_r \leq \|g\|_p \|f\|_q,$$

where $1 \leq p, q, r \leq \infty$ and $1/p + 1/q = 1 + 1/r$.

By a *mollifier* we shall mean a sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathrm{L}^1(\mathbb{R})$ of the following form: $g_1 \in \mathrm{L}^1(\mathbb{R})$ satisfies $\int_{\mathbb{R}} g_1(t)\, dt = 1$, and $g_n \in \mathrm{L}^1(\mathbb{R})$ is given by $g_n(t) = n g_1(nt)$, for all $t \in \mathbb{R}$ and $n \in \mathbb{N}$. It is often convenient to choose such a sequence (g_n) in $C_c^\infty(\mathbb{R})$ with $g_n \geq 0$ for all $n \in \mathbb{N}$. For such a mollifier (g_n),

$$\lim_{n \to \infty} \|g_n * f - f\|_p = 0,$$

for $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X}), 1 \leq p < \infty$.

For $f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$, the *Fourier transform* $\mathcal{F}f$, is defined by

$$(\mathcal{F}f)(s) := \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt,$$

and the conjugate Fourier transform, $\bar{\mathcal{F}}f$ is given by

$$(\bar{\mathcal{F}}f)(s) := \int_{-\infty}^{\infty} e^{ist} f(t)\, dt = (\mathcal{F}f)(-s).$$

We quote here some of the properties of the vector-valued Fourier transform that shall be used frequently in the sequel. We refer to [2, Section 1.8] for the details. For $f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ and $g \in \mathrm{L}^1(\mathbb{R})$ we have

1. $\mathcal{F}(g * f)(s) = (\mathcal{F}g)(s)(\mathcal{F}f)(s)$.
2. $\displaystyle\int_{-\infty}^{\infty} g(t)(\mathcal{F}f)(t)\, dt = \int_{-\infty}^{\infty} (\mathcal{F}g)(t) f(t)\, dt$.
3. (Riemann-Lebesgue Lemma) $\mathcal{F}f \in C_0(\mathbb{R}, \mathbf{X})$.
4. (Inversion Theorem) If $\mathcal{F}f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$, then $f = \dfrac{1}{2\pi}\bar{\mathcal{F}}(\mathcal{F}f)$ a.e.
5. If $\mathbf{X}$ is a Hilbert space then we have Plancherel's theorem that $\frac{1}{\sqrt{2\pi}}\mathcal{F} : \mathrm{L}^2(\mathbb{R}, \mathbf{X}) \to \mathrm{L}^2(\mathbb{R}, \mathbf{X})$ is a unitary operator.

The next result makes use of the Riemann-Lebesgue Lemma to describe the behaviour of the Laplace transform of an exponentially bounded function f along vertical lines to the right of $\mathrm{hol}_0^\infty(\hat{f})$.

Lemma 2.2.2. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be measurable and exponentially bounded. Then*

$$\lim_{s \to \pm\infty} \hat{f}(\alpha + is) = 0,$$

for all $\alpha > \mathrm{hol}_0^\infty(\hat{f})$.

Proof. Let $\alpha > \beta > \mathrm{hol}_0^\infty(\hat{f})$. Then there exists $b > 0$ such that $Q_{\beta,b} \subset D(\hat{f})$. Let (s_n) be any sequence such that $s_n \geq b$ for all n and $s_n \to \infty$ as $n \to \infty$. Let $g_n(z) = \hat{f}(z + is_n)$, $z \in Q_{\beta,0}$. For $\mathrm{Re}\, z > \omega_0(f)$, let $h(t) = e^{-zt} f(t), t \in \mathbb{R}_+$. Then $h \in \mathrm{L}^1(\mathbb{R}_+, \mathbf{X})$ and it follows from the Riemann-Lebesgue Lemma that

$$\lim_{s \to \infty} \mathcal{F}h(s) = \lim_{s \to \infty} \hat{f}(z + is) = 0.$$

Thus, if $\mathrm{Re}\, z > \omega_0(f)$, then $g_n(z) \longrightarrow 0$ as $n \to \infty$. Now (g_n) is uniformly bounded on $\{\mathrm{Re}\, z > \beta, \mathrm{Im}\, z > 0\}$. By Vitali's Theorem [2, Theorem A.5], $\lim_{n\to\infty} g_n(z) = 0$ for all $z \in Q^o_{\beta,0}$. In particular,

$$0 = \lim_{n \to \infty} g_n(\alpha) = \lim_{n \to \infty} \hat{f}(\alpha + is_n).$$

Since (s_n) is arbitrary, we conclude that $\lim_{s\to\infty} \hat{f}(\alpha + is) = 0$. Similarly, we can show that as $s \to -\infty$, $\hat{f}(\alpha + is) \to 0$. □

2.3 Operator-valued functions

Let $\mathbf{T} : \mathbb{R}_+ \longrightarrow \mathcal{L}(\mathbf{X})$. We shall say that $\mathbf{T}$ is strongly continuous if it is continuous in the strong operator topology, that is, the map $t \mapsto \mathbf{T}(t)x$ from $\mathbb{R}_+$ to $\mathbf{X}$ is continuous for each $x \in \mathbf{X}$. If $\mathbf{T}$ is strongly continuous, then by the Uniform Boundedness Principle, it is also locally bounded. However, it is not necessarily Bochner measurable; indeed, $\mathbf{T}$ is Bochner measurable if and only if it is almost separably-valued in the norm topology, by Pettis's Theorem [2, Theorem 1.1.1]. Clearly, if $\mathbf{T}$ is uniformly continuous (that is, continuous with respect to the norm topology) then it is measurable and also strongly continuous and strong continuity of $\mathbf{T}$ implies continuity in the weak operator topology. On the other hand, if Ω is an open set in $\mathbb{C}$, then a function $\mathbf{S} : \Omega \longrightarrow \mathcal{L}(\mathbf{X})$ is holomorphic if and only if it is holomorphic in the weak operator topology (that is, $\langle \mathbf{S}(\cdot)x, x^* \rangle$ is holomorphic for all $x \in \mathbf{X}$ and $x^* \in \mathbf{X}^*$ [2, Proposition A.3]).

In what follows, to say $\mathbf{S} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ *converges uniformly, strongly and weakly* as $t \to \infty$ will refer to convergence in, respectively, the norm, strong operator and weak operator topology.

2.3.1 Laplace and Fourier transforms for operator-valued functions

Next, we recall the formulation of the definitions and results of Section 2.2 when f is replaced by a strongly continuous operator-valued function $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$. For such a function $\mathbf{T}$, let $\displaystyle\int_0^t e^{-\lambda s}\mathbf{T}(s)\,ds$ denote the bounded operator

$$x \mapsto \int_0^t e^{-\lambda s}\mathbf{T}(s)x\,ds.$$

Then

$$\begin{aligned}\operatorname{abs}(\mathbf{T}) &:= \inf\left\{\operatorname{Re}\lambda : \int_0^t e^{-\lambda s}\mathbf{T}(s)\ ds \text{ converges strongly as } t \to \infty\right\}\\ &= \sup\{\operatorname{abs}(\mathbf{T}(\cdot)x) : x \in \mathbf{X}\}\\ &= \inf\{\omega_0(\mathbf{S} - \mathbf{S}_0) : \mathbf{S}_0 \in \mathcal{L}(\mathbf{X})\}\end{aligned}$$

where $\mathbf{S}(t)x = \displaystyle\int_0^t \mathbf{T}(s)x\,ds$. We refer the reader to [2, Section 1.4] for the proof of the above equalities as well as the other results that follow. Whenever $\operatorname{Re}\lambda > \operatorname{abs}(\mathbf{T})$, the limit $\lim_{t\to\infty}\int_0^t e^{-\lambda s}\mathbf{T}(s)\,ds$ exists in operator norm. If $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ is strongly continuous and $\operatorname{abs}(\mathbf{T}) < \infty$, the *Laplace integral* of $\mathbf{T}$ is defined by

$$\hat{\mathbf{T}}(\lambda) := \int_0^\infty e^{-\lambda s}\mathbf{T}(s)\,ds := \lim_{t\to\infty}\int_0^t e^{-\lambda s}\mathbf{T}(s)\,ds \quad (\operatorname{Re}\lambda > \operatorname{abs}(\mathbf{T})).$$

Then $\hat{\mathbf{T}} : H_{\mathrm{abs}(\mathbf{T})} \to \mathcal{L}(\mathbf{X})$ is holomorphic and all the results mentioned in sub-section 2.2.2 for $\hat{f}$ hold for $\hat{\mathbf{T}}$ also, with $\mathrm{hol}(\hat{\mathbf{T}}), \mathrm{hol}_0(\hat{\mathbf{T}})$, $\mathrm{hol}^\infty(\hat{\mathbf{T}}), \mathrm{hol}_0^\infty(\hat{\mathbf{T}}), \mathrm{hol}_{\mathrm{n}}(\hat{\mathbf{T}})$ and $\mathrm{hol}_{\mathrm{n}}^\infty(\hat{\mathbf{T}})$ defined as in (2.1), (2.2), (2.3), (2.4), (2.5) and (2.6) respectively.

The Fourier transform of an operator valued function $\mathbf{S} : \mathbb{R} \to \mathcal{L}(\mathbf{X})$, is defined in a similar manner.

2.3.2 C_0-semigroups

By a C_0-semigroup defined on the Banach space $\mathbf{X}$ we shall mean a function $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ satisfying the following properties:

1. $\mathbf{T}$ is strongly continuous;
2. $\mathbf{T}(0) = \mathbf{I}$;
3. $\mathbf{T}(t+s) = \mathbf{T}(t)\mathbf{T}(s)$ for $t, s \in \mathbb{R}_+$.

For background information on C_0-semigroups we refer to the books [20], [42], [17] and [2].

The infinitesimal generator $\mathbf{A}$ of $\mathbf{T}$ is given by

$$\begin{aligned} D(\mathbf{A}) &= \left\{ x \in \mathbf{X} : \lim_{t \downarrow 0} \frac{\mathbf{T}(t)x - x}{t} \text{ exists } \right\} \\ \mathbf{A}x &= \lim_{t \downarrow 0} \frac{\mathbf{T}(t)x - x}{t} \quad (x \in D(\mathbf{A})). \end{aligned}$$

$\mathbf{A}$ is a closed, densely defined operator and $\overline{\bigcap_{n=1}^\infty D(\mathbf{A}^n)} = \mathbf{X}$. The resolvent and spectrum of $\mathbf{A}$ shall be denoted by $\rho(\mathbf{A})$ and $\sigma(\mathbf{A})$. For $\lambda \in \rho(\mathbf{A}), \mathbf{R}(\lambda, \mathbf{A}) = (\lambda - \mathbf{A})^{-1}$ shall denote the resolvent operator. From the definition of the resolvent, it is easy to deduce the very useful *resolvent identity* :

$$\mathbf{R}(\lambda, \mathbf{A}) - \mathbf{R}(\mu, \mathbf{A}) = (\mu - \lambda)\mathbf{R}(\lambda, \mathbf{A})\mathbf{R}(\mu, \mathbf{A}) \quad (\lambda, \mu \in \rho(\mathbf{A})).$$

Any C_0-semigroup is exponentially bounded [20, Proposition 5.5]. For $\mathrm{Re}\,\lambda > \omega_0(\mathbf{T})$,

$$\mathbf{R}(\lambda, \mathbf{A})x = \hat{\mathbf{T}}(\lambda)x \quad (x \in \mathbf{X}).$$

In fact, C_0-semigroups are exactly those strongly continuous operator-valued functions whose Laplace transforms are resolvents [2, Theorem 3.1.7].

2.3.3 Norm continuity and the critical growth bound

Let $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ be strongly continuous and exponentially bounded. A growth bound $\delta(\mathbf{T})$ measuring the absence of norm continuity of $\mathbf{T}$ has been introduced in [10], [11]:

$$\begin{aligned}\delta(\mathbf{T}) &:= \inf\{\omega : \text{there exists a norm continuous function } \mathbf{T}_1 : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X}) \\ &\qquad \text{such that } \omega_0(\mathbf{T} - \mathbf{T}_1) < \omega\} \\ &= \inf\{\omega : \text{there exists an infinitely differentiable function } \mathbf{T}_1 : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X}) \\ &\qquad \text{such that } \omega_0(\mathbf{T} - \mathbf{T}_1) < \omega\}.\end{aligned} \tag{2.13}$$

This is equal to the *growth bound of local variation* of $\mathbf{T}$, which is defined to be $\omega_0(f_{\mathbf{T}})$ where

$$f_{\mathbf{T}}(t) = \limsup_{h \downarrow 0} \|\mathbf{T}(t+h) - \mathbf{T}(t)\|.$$

[10, Theorem 2.3.7] (see also [11]). It is immediate from the definition that if $\mathbf{T}$ is norm continuous on $\mathbb{R}_+$ or norm continuous on (α, ∞) for some $\alpha > 0$ then $\delta(\mathbf{T}) = -\infty$.

A C_0-semigroup $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ is said to be *eventually norm continuous* [20, Definition II.4.17]) if there exists $\alpha \in \mathbb{R}_+$ such that $\mathbf{T} : (\alpha, \infty) \to \mathcal{L}(\mathbf{X})$ is norm continuous. If α may be chosen to be 0 then $\mathbf{T}$ is called *immediately norm continuous.* If $\mathbf{A}$ is the generator of an immediately norm continuous semigroup $\mathbf{T}$ then

$$\lim_{s \to \pm\infty} \|\mathbf{R}(a + is, \mathbf{A})\| = 0 \tag{2.14}$$

for all $a > \omega_0(\mathbf{T})$ ([20, Theorem II.4.18]). This condition is sufficient for immediate norm continuity if $\mathbf{X}$ is a Hilbert space [20, Theorem II.4.20],[52] or if $\mathbf{T}$ is a positive semigroup defined on $\mathrm{L}^p(\mathbb{R}), 1 < p < \infty$ [23]. It is not known whether (2.14) implies immediate norm continuity for arbitrary C_0-semigroups or not.

The semigroup $\mathbf{T}$ is said to be eventually compact (respectively, immediately compact) if there is an $\alpha > 0$ such that $\mathbf{T}(\alpha)$ is compact (respectively, $\mathbf{T}(t)$ is compact for all $t > 0$). $\mathbf{T}$ is eventually compact if and only $\mathbf{T}$ is eventually norm continuous and its generator has compact resolvent [20, Theorem II.4.29].

$\mathbf{T}$ is called *eventually differentiable* [20, Definition II.4.13]), [42, Section 2.4] if there exists an $\alpha \geq 0$ such that the map $t \mapsto \mathbf{T}(t)x$ is differentiable on (α, ∞) for every $x \in \mathbf{X}$.

A C_0-semigroup $\mathbf{T}$ is called *analytic* of angle $\theta \in (0, \frac{\pi}{2})$ ([20, Definition II.4.5]) if $\mathbf{T}$ has a holomorphic extension to Σ_θ (also denoted by $\mathbf{T}$) satisfying

- $\mathbf{T}(z_1 + z_2) = \mathbf{T}(z_1)\mathbf{T}(z_2)$ for all $z_1, z_2 \in \Sigma_\theta$;
- $\lim_{\Sigma_\beta \ni z \to 0} \mathbf{T}(z)x = x$ for all $x \in \mathbf{X}$ and $0 < \beta < \theta$.

All the classes of C_0-semigroups mentioned above satisfy $\delta(\mathbf{T}) = -\infty$. In [11, Definition 3.4] a class of C_0-semigroups has been introduced which includes all C_0-semigroups with finite growth bound falling in any of the above classes: $\mathbf{T}$ is said to be *asymptotically norm continuous* if $\delta(\mathbf{T}) < \omega_0(\mathbf{T})$. Such a semigroup has been called *norm continuous at infinity* in [37].

With $\Gamma_t = \{\lambda \in \mathbb{C} : e^{\delta(\mathbf{T})t} < |\lambda|\}$ the following version of the spectral mapping Theorem holds for asymptotically norm continuous semigroups [10, Theorem 4.4.1], [11, Theorem 3.6].

$$\sigma\left(\mathbf{T}(t)\right) \cap \Gamma_t = e^{t\sigma(\mathbf{A})} \cap \Gamma_t, \quad (t > 0). \tag{2.15}$$

In particular, $\omega_0(\mathbf{T}) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathbf{A})\}$. Since $\delta(\mathbf{T}) = -\infty$ for eventually norm continuous semigroups, (2.15) re-asserts the well known fact [20, Theorem IV.3.10] that the spectral mapping theorem

$$\sigma(\mathbf{T}(t)) \setminus \{0\} = e^{t\sigma(\mathbf{A})} \quad (t > 0) \tag{2.16}$$

holds for such semigroups.

For general C_0-semigroups, (2.16) fails to hold. However, in [38], the *critical spectrum* $\sigma_{crit}(\mathbf{T}(t))$ for a C_0-semigroup $\mathbf{T}$ has been introduced, which yields a spectral mapping theorem of the form

$$\sigma(\mathbf{T}(t)) \setminus \{0\} = e^{t\sigma(\mathbf{A})} \cup \sigma_{crit}(\mathbf{T}(t)) \setminus \{0\} \quad (t > 0),$$

for all C_0- semigroups. We recall the definition of the critical spectrum of a C_0-semigroup $\mathbf{T}$ [38, Definition 2.3]. Let $\ell^\infty(\mathbf{X})$ be the Banach space of all bounded sequences in $\mathbf{X}$, endowed with the sup-norm $\|(x_n)_{n\in\mathbb{N}}\| := \sup_{n\in\mathbb{N}} \|x_n\|$ and let $\tilde{\mathbf{T}}$ be the extension of $\mathbf{T}$ to $\ell^\infty(\mathbf{X})$, given by

$$\tilde{\mathbf{T}}(t)\left((x_n)_{n\in\mathbb{N}}\right) := \left(\mathbf{T}(t)x_n\right)_{n\in\mathbb{N}} \quad (t \geq 0).$$

Then the space of strong continuity $\ell^\infty_{\mathbf{T}}(\mathbf{X})$, of $\mathbf{T}$ given by

$$\ell^\infty_{\mathbf{T}}(\mathbf{X}) := \left\{(x_n)_{n\in\mathbf{N}} : \lim_{t\to 0} \sup_{n\in\mathbb{N}} \|\mathbf{T}(t)x_n - x_n\| = 0\right\}$$

is a closed and $\tilde{\mathbf{T}}$-invariant subspace of $\ell^\infty(\mathbf{X})$. On the quotient space $\check{\mathbf{X}} = \dfrac{\ell^\infty(\mathbf{X})}{\ell^\infty_{\mathbf{T}}(\mathbf{X})}$ define the semigroup of bounded operators $\check{\mathbf{T}}$ by

$$\check{\mathbf{T}}(t)\check{x} := \left(\mathbf{T}(t)x_n\right)_{n\in\mathbb{N}} + \ell^\infty_{\mathbf{T}}(\mathbf{X}), \text{ for } \check{x} := (x_n)_{n\in\mathbb{N}} + \ell^\infty_{\mathbf{T}}(\mathbf{X}).$$

Then the *critical spectrum* of the C_0-semigroup $\mathbf{T}$ is defined by

$$\sigma_{crit}(\mathbf{T}(t)) := \sigma(\check{\mathbf{T}}(t)) \quad (t > 0),$$

and the *critical growth bound* $\omega_{crit}(\mathbf{T})$ of $\mathbf{T}$ is defined as

$$\omega_{crit}(\mathbf{T}) = \omega_0(\check{\mathbf{T}}).$$

It has been shown in [38, Proposition 4.6] that the growth bound of local variation and the critical growth bound coincide for a C_0-semigroup, that is

$$\delta(\mathbf{T}) = \omega_{crit}(\mathbf{T}).$$

Due to this equality, henceforth we shall call the growth bound of local variation, $\delta(\mathbf{T})$, of any exponentially bounded operator-valued function $\mathbf{T}$ the critical growth bound of $\mathbf{T}$ and use the equivalent characterisations of this bound without mention.

2.3.4 Adjoint semigroups

We recall some standard definitions and facts concerning the adjoint of a C_0-semigroup. The details may be found in [39, Chapter 1, Chapter 2].

Let $\mathbf{A}$ be the generator of the C_0-semigroup $\mathbf{T}$. The adjoint semigroup $\mathbf{T}^*$ on $\mathbf{X}^*$, given by $\mathbf{T}^*(t) = \mathbf{T}(t)^*$ is weak*-continuous with weak*-generator $\mathbf{A}^*$, but is not necessarily strongly continuous.

The *semigroup dual of* $\mathbf{X}$ *with respect to* $\mathbf{T}$, denoted by $\mathbf{X}^\odot$ is defined as the linear subspace of $\mathbf{X}^*$ on which the adjoint semigroup $\mathbf{T}^*$ acts in a strongly continuous way; i.e.

$$\mathbf{X}^\odot = \{x^* \ \in \ \mathbf{X}^* \ : \lim_{t \downarrow 0} \| \ \mathbf{T}^*(t)x^* - x^* \ \| = 0\}.$$

Then $\mathbf{X}^\odot$ is a closed, weak*-dense, $\mathbf{T}^*(t)$-invariant linear subspace of $\mathbf{X}^*$ and $\mathbf{X}^\odot = \overline{D(\mathbf{A}^*)}$. We denote by $\mathbf{T}^\odot(t)$ the restriction of $\mathbf{T}^*(t)$ to $\mathbf{X}^\odot$. Then $(\mathbf{T}^\odot(t))_{t\geq 0}$ defines a C_0-semigroup on $\mathbf{X}^\odot$ whose generator $\mathbf{A}^\odot$ is the part of $\mathbf{A}^*$ in $\mathbf{X}^\odot$.

Starting with the C_0-semigroup $\mathbf{T}^\odot$, the duality construction can be repeated. We define $\mathbf{T}^{\odot *}$ to be the adjoint of $\mathbf{T}^\odot$ and write $\mathbf{X}^{\odot\odot}$ for $(\mathbf{X}^\odot)^\odot$. $\mathbf{T}^{\odot\odot}$ and $\mathbf{A}^{\odot\odot}$ are defined analogously.

The norm $\|\cdot\|'$ defined on $\mathbf{X}$ by

$$\|x\|' := \sup_{x^\odot \in B_{\mathbf{X}^\odot}} \big|\langle x^\odot, x\rangle\big|,$$

where $B_{\mathbf{X}^\odot}$ is the closed unit ball of $\mathbf{X}^\odot$, is an equivalent norm. In fact,

$$\|x\|' \leq \|x\| \leq M\|x\|',$$

with $M = \limsup_{t\downarrow 0} \|\mathbf{T}(t)\|$. Define the map $j : \mathbf{X} \longrightarrow \mathbf{X}^{\odot *}$ by

$$\langle jx, x^\odot\rangle \ := \ \langle x^\odot, x\rangle.$$

Then $j\mathbf{X} \subset \mathbf{X}^{\odot\odot}$; in fact j is an embedding, with $M^{-1} \leq \|j\| \leq 1$. We can, therefore, identify $\mathbf{X}$ isomorphically with the closed subspace $j\mathbf{X}$ of $\mathbf{X}^{\odot\odot}$. Thus, $\mathbf{T}^{\odot\odot}(t)$ is an extension of $\mathbf{T}(t)$, $\mathbf{A}^{\odot\odot}$ is an extension of $\mathbf{A}$ and $D(\mathbf{A}) = D(\mathbf{A}^{\odot\odot}) \cap \mathbf{X}$.

$\mathbf{X}$ is said to be $\odot$-*reflexive* or *sun-reflexive* with respect to $\mathbf{T}$ if and only if $j(\mathbf{X}) = \mathbf{X}^{\odot\odot}$.

Chapter 3

A non-analytic growth bound for Laplace transforms and semigroups of operators

Analytic C_0-semigroups play an extremely important role in the theory of evolution equations. In this chapter, we study a growth bound which measures the non-analyticity of C_0-semigroups and more generally, of any vector-valued, exponentially bounded measurable function. It turns out that the non-analytic behaviour of a vector-valued, exponentially bounded measurable function is closely related to the integrability of some derivative of the Laplace transform of the function along vertical lines.

In [10, Theorem 2.3.3] it is shown that integrability along a vertical line of some derivative of the Laplace transform of a strongly continuous and exponentially bounded function $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ is sufficient for norm continuity of $\mathbf{T}$ for $t > 0$. This idea has been used to define a growth bound for $\mathbf{T}$ which measures the non-integrability of the Laplace transform of $\mathbf{T}$. We recall the relevant definitions and results from [10, Section 2.4.2], where these are stated in the context of operator-valued functions $\mathbf{T}$, but remain valid for general vector-valued functions also. We state these for the general case.

A Bochner measurable function $f : \mathbb{R}_+ \to \mathbf{X}$ is said to have an L^1-*Laplace transform,* (see [10]) if there exist $r > 0$, $N \in \mathbb{N}$ and $C > 0$ such that $\hat{f}$ has a holomorphic extension to $Q_{0,r}$ and

$$\sup_{\omega \geq 0} \int_{|s| \geq r} \left\| \hat{f}^{(N)}(\omega + is) \right\| ds \leq C.$$

For an exponentially bounded, Bochner measurable function $f : \mathbb{R}_+ \to \mathbf{X}$, the *growth*

bound for non-integrability of f is defined by:

$$\zeta(f) = \inf\big\{w \in \mathbb{R} : \text{ there exists } f_1 : \mathbb{R}_+ \to \mathbf{X} \text{ such that } e^{-w\cdot} f_1(\cdot) \text{ has } \mathrm{L}^1\text{-Laplace transform and } \omega_0(f - f_1) \leq w\big\}. \tag{3.1}$$

The next theorem gives a very useful property of $\zeta(f)$.

Theorem 3.0.1. *([10, Proposition 2.4.6]) Let $f : \mathbb{R}_+ \to \mathbf{X}$ be Bochner measurable. Suppose that $\zeta(f) < 0$, so that there exist f_1, $f_2 : \mathbb{R}_+ \to \mathbf{X}$ and positive N, r and ϵ such that $f = f_1 + f_2$, $\omega_0(f_2) < -\epsilon$ and*

$$\sup_{w \geq -\epsilon} \int_{|s|>r} \big\| \hat{f}_1{}^{(N)}(w + is) \big\| \, ds < \infty.$$

Let $w > \omega_0(f)$ and Γ be the path consisting of line segments joining $-\epsilon - ir$, $w - ir$, $w + ir$ and $-\epsilon + ir$ in that order, and define

$$g_1(t) := \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \hat{f}_1(\lambda) \, d\lambda,$$
$$g_2(t) := f(t) - g_1(t).$$

Then $\omega_0(g_2) \leq -\epsilon$.

3.1 Introducing the non-analytic growth bound

The critical growth bound of a strongly measurable, exponentially bounded function $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ measures the growth bound of $\mathbf{T}$ modulo $\mathcal{L}(\mathbf{X})$-valued functions on $\mathbb{R}_+$ that are norm continuous for $t > 0$ and equivalently, modulo $\mathcal{L}(\mathbf{X})$-valued functions on $\mathbb{R}_+$ which are infinitely differentiable. A natural question that arises in this context is whether or not the growth bound of $\mathbf{T}$ modulo holomorphic, exponentially bounded $\mathcal{L}(\mathbf{X})$-valued functions is also equal to the critical growth bound of $\mathbf{T}$. This motivates us to define a new growth bound $\eta(\mathbf{T})$ for $\mathbf{T}$. As we shall see later, this definition extends in a useful way to the case of arbitrary vector-valued functions, unlike the definition of the critical growth bound. We make the definition in the most general setting useful for us.

Definition 3.1.1. Let $f : \mathbb{R}_+ \to \mathbf{X}$ be Laplace transformable. We define

$$\eta(f) := \inf\big\{w \in \mathbb{R} : \text{ there exist } \theta > 0 \text{ and an exponentially bounded, holomorphic function } g : \Sigma_\theta \to \mathbf{X} \text{ such that } \omega_0(f - g) < w\big\}.$$

It is clear from the definition that the growth bound $\eta(f)$ measures how well the function f can be approximated by exponentially bounded functions that are holomorphic on some sector Σ_θ. However, there are other, relatively smaller classes of approximating functions which can be considered, as will be shown in the next proposition. We shall need the following definition:

Definition 3.1.2. Let $f : \mathbb{R}_+ \to \mathbf{X}$ be a Laplace transformable function. If $Q_{\alpha,b} \subset D(\hat{f})$ we define for $t \geq 0$,

$$f_{\alpha,b}(t) := \frac{1}{2\pi i} \int_{\Gamma_{\alpha,b}} e^{t\lambda} \hat{f}(\lambda) d\lambda,$$

where $\Gamma_{\alpha,b}$ is any path in $D(\hat{f})$ from $\alpha - ib$ to $\alpha + ib$.

We immediately obtain, from the above definition, an estimate for the growth bound of the functions $f_{\alpha,b}$.

Lemma 3.1.3. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be Laplace transformable and $\alpha \in \mathbb{R}$, $b \geq 0$ be such that $Q_{\alpha,b} \subset D(\hat{f})$. Then,*

$$\begin{aligned} \omega_0\left(f_{\alpha,b}\right) &\leq \max\left(\alpha, \mathrm{hol}(\hat{f})\right), \text{ and} \\ \hat{f}_{\alpha,b}(\mu) &= \frac{1}{2\pi i} \int_{\Gamma_{\alpha,b}} \frac{\hat{f}(\lambda)}{\mu - \lambda}\, d\lambda, \end{aligned}$$

where $\Gamma_{\alpha,b}$ is any path in $D(\hat{f})$ from $\alpha - ib$ to $\alpha + ib$ and $\operatorname{Re}\mu$ lies to the right of $\Gamma_{\alpha,b}$. Further, if α_1, $b_1 \in \mathbb{R}$, $b_1 \geq 0$ satisfy $Q_{\alpha_1,b_1} \subset D(\hat{f})$ then

$$\mathrm{abs}\big(f_{\alpha,b} - f_{\alpha_1,b_1}\big) \leq \omega_0\big(f_{\alpha,b} - f_{\alpha_1,b_1}\big) \leq \max(\alpha, \alpha_1).$$

Proof. Let $\Gamma = [\alpha - ib, \gamma - ib] \cup [\gamma - ib, \gamma + ib] \cup [\gamma + ib, \alpha + ib]$, where $\gamma > \max\left(\alpha, \mathrm{hol}(\hat{f})\right)$. Then,

$$f_{\alpha,b}(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \hat{f}(\lambda)\, d\lambda,$$

and straightforward calculations show that

$$\left\| f_{\alpha,b}(t) \right\| \leq C_\gamma e^{\gamma t}$$

for some constant C_γ. Consequently, $\omega_0\left(f_{\alpha,b}\right) \leq \gamma$, for all $\gamma > \max\left(\alpha, \mathrm{hol}((\hat{f})\right)$. Therefore, it follows that $\omega_0(f_{\alpha,b}) \leq \max\left(\alpha, \mathrm{hol}((\hat{f})\right)$. The claim concerning $\hat{f}_{\alpha,b}$ follows directly from the definition of the Laplace transform of a function, on interchanging the order of integration. We note that for α_1, b_1 as in the hypothesis,

$$\hat{f}_{\alpha,b}(t) - \hat{f}_{\alpha_1,b_1}(t) = \frac{1}{2\pi i} \int_{\Gamma_2} e^{\lambda t} \hat{f}(\lambda) d\lambda,$$

where Γ_2 consists of two paths in $\{\lambda \in D(\hat{f}) : \operatorname{Re}\lambda \leq \max(\alpha, \alpha_1)\}$ joining $\alpha \pm ib$ to $\alpha_1 \pm ib_1$. Therefore,

$$\left\| f_{\alpha,b}(t) - f_{\alpha_1,b_1}(t) \right\| \leq C \exp(t \max(\alpha, \alpha_1))$$

for some constant C, so that $\omega_0(f_{\alpha,b} - f_{\alpha_1,b_1}) \leq \max(\alpha, \alpha_1)$. □

Observe that in the terminology introduced in Definition 3.1.2, Theorem 3.0.1 shows that if f, $f_1 : \mathbb{R}_+ \to \mathbf{X}$ are exponentially bounded, Bochner measurable functions with $\omega_0(f - f_1) < -\epsilon$ for some $\epsilon > 0$ and the function $t \mapsto e^{-\epsilon t} f_1(t)$ has L^1-Laplace transform, then $\omega_0\left(f - (f_1)_{-\epsilon,r}\right) \leq -\epsilon$, where r is as in the statement of the Theorem. The proof of implication (2)⇒(4) in the following proposition depends mainly on this fact.

Proposition 3.1.4. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be Laplace transformable and let $\omega \in \mathbb{C}$. The following are equivalent:*

1. *There exist $\theta > 0$ and an exponentially bounded, holomorphic function $g : \Sigma_\theta \to \mathbb{C}$ such that $\omega_0(f - g) < \omega$.*

2. *There is an exponentially bounded, measurable function $g : \mathbb{R}_+ \to \mathbf{X}$ such that, for each $\alpha < \omega$, there exists $b \geq 0$ such that*

   (a) $Q_{\alpha,b} \subset D(\hat{g})$;

   (b) $\sup_{\lambda \in Q_{\alpha,b}} \|\hat{g}(\lambda)\| < \infty$;

   (c) $\sup_{\gamma \geq \alpha} \int_{|s| \geq b} \left\|\hat{g}^{(N)}(\gamma + is)\right\| ds < \infty \quad (N = 1, 2...)$;

   (d) $\omega_0(f - g) < \omega$.

3. *There exist an exponentially bounded, measurable function $g : \mathbb{R}_+ \to \mathbf{X}$ and $\alpha < \omega$, $b \geq 0$, $N \in \mathbb{N}$ such that (2a), (2c) and (2d) hold.*

4. *There exist $\alpha < \omega$ and $b \geq 0$ such that*

   (a) $Q_{\alpha,b} \subset D(\hat{f})$;

   (b) $\sup_{\lambda \in Q_{\alpha,b}} \|\hat{f}(\lambda)\| < \infty$;

   (c) $\omega_0(f - f_{\alpha,b}) < \omega$.

5. *There exist $\alpha < \omega$ and $b \geq 0$ such that (4a) and (4c) hold.*

6. *There is an exponentially bounded, entire function $g : \mathbb{C} \to \mathbf{X}$ such that $\omega_0(f - g) < \omega$ and $\omega_0(g) \leq \max\left(\omega, \mathrm{hol}(\hat{f})\right)$.*

Proof. *(1)* $\Longrightarrow$ *(2)*: Suppose that $g : \Sigma_\theta \to \mathbf{X}$ is holomorphic and there exists $w' \in \mathbb{R}$ such that $\|g(z)\| \leq M e^{w'|z|} \quad (z \in \Sigma_\theta)$. We may assume without loss of generality, that $\omega < w'$. By considering a smaller sector if required, we may also assume that $0 < \theta \leq \frac{\pi}{2}$. Then $w' + \Sigma_{\theta+(\pi/2)} \subset D(\hat{g})$ and

$$\|\hat{g}(\lambda)\| \leq \frac{C}{|\lambda - w'|}$$

for some constant C [2, Theorem 2.6.1]. In particular, *(2a)* and *(2b)* hold if $b > (\omega - \alpha)\cot(\theta/2)$. Moreover, Cauchy's integral formula for derivatives gives

$$\begin{aligned} \left\|\hat{g}^{(N)}(\gamma + is)\right\| &= \left\|\frac{N!}{2\pi i}\int_{|\lambda-(\gamma+is)|=\epsilon s} \frac{\hat{g}(\lambda)}{(\lambda-(\gamma+is))^{N+1}}\,d\lambda\right\| \\ &\leq \frac{N!}{\epsilon^N |s|^N}\frac{C}{|s|(1-\epsilon)} \end{aligned}$$

whenever $\gamma \geq \alpha$, $|s| \geq b > (\omega-\alpha)\cot(\theta/2)$, $N \in \mathbb{N}$, $0 < \epsilon < 1$ and ϵ is sufficiently small so that the disc of radius ϵb and centre $\alpha + ib$ is contained in $\omega + \Sigma_{\theta+\pi/2}$. This implies *(2c)*.

(2) $\Longrightarrow$ *(3),(4)* $\Longrightarrow$ *(5)*: This is trivial.

(2) $\Longrightarrow$ *(4),(3)* $\Longrightarrow$ *(5)*: Suppose that $g : \mathbb{R}_+ \to \mathbb{C}$ is exponentially bounded and measurable, and $\alpha < \omega, b \geq 0$ and $N \geq 0$ are such that *(2a)*, *(2c)* and *(2d)* hold. Take β such that $\max(\alpha, \omega(f-g)) < \beta < \omega$. Then $Q_{\beta,b} \subset D(\widehat{f-g}) \cap D(\hat{g})$, and hence $Q_{\beta,b} \subset D(\hat{f})$. If *(2b)* holds then $\hat{f}$ is bounded on $Q_{\beta,b}$. By Lemma 3.1.3,

$$\omega_0\big(f_{\beta,b} - g_{\beta,b}\big) = \omega_0\big((f-g)_{\beta,b}\big) \leq \beta.$$

The assumptions on g imply that $t \mapsto e^{-\beta t}g(t)$ has L^1-Laplace transform and $\omega_0(f-g) < \beta$. Therefore, from Theorem 3.0.1, it follows that $\omega_0(f - g_{\beta,b}) \leq \beta$. Since

$$\omega_0(f - f_{\beta,b}) \leq \max\left(\omega_0(f-g_{\beta,b}), \omega_0(g_{\beta,b} - f_{\beta,b})\right),$$

it follows that $\omega_0(f - f_{\beta,b}) \leq \beta < \omega$.

(5) $\Longrightarrow$ *(6)*: For $z \in \mathbb{C}$, let

$$g(z) = \frac{1}{2\pi i}\int_{\Gamma_{\alpha,b}} e^{\lambda z}\hat{f}(\lambda)d\lambda.$$

Then g is entire and exponentially bounded, with $g(t) = f_{\alpha,b}(t)$ $(t \geq 0)$. So *(6)* follows from Lemma 3.1.3 and *(4c)*.

(6) $\Longrightarrow$ *(1)* : This is trivial. □

Remark 3.1.5.

1. If the conditions of Proposition 3.1.4 hold, then *(4c)* holds for every $\alpha < \omega$ and $b \geq 0$ satisfying *(4a)*. This follows from Lemma 3.1.3.

2. If *(4a)* holds then $\alpha \geq \mathrm{hol}^\infty(\hat{f})$; if *(4b)* also holds, then $\alpha \geq \mathrm{hol}_0^\infty(\hat{f})$. Conversely, if $\alpha > \mathrm{hol}^\infty(\hat{f})$ then there exists $b \geq 0$ such that *(4a)* holds; if $\alpha > \mathrm{hol}_0^\infty(\hat{f})$ and *(4a)* holds, then *(4b)* holds.

3. Proposition 3.1.4 remains valid if $\omega_0(f-g)$ is replaced by $\mathrm{abs}(\|f-g\|)$ in conditions *(1)*, *(2)*, *(3)* and *(6)*, and $\omega_0(f - f_{\alpha,b})$ is replaced by $\mathrm{abs}(\|f - f_{\alpha,b}\|)$ in conditions *(4)* and *(5)*.

4. Similarly, the equivalence of (*1*), (*3*), (*5*) and (*6*) remains valid if $\omega_0(f-g)$ is replaced by $\mathrm{abs}(f-g)$ throughout. In both these cases, the corresponding analogue of (*4c*) holds for every $\alpha < \omega$ and $b \geq 0$.

From Proposition 3.1.4, we conclude

Corollary 3.1.6. *The non-analytic growth bound and the growth bound for non-integrability for any Laplace transformable function $f : \mathbb{R}_+ \to \mathbf{X}$ coincide, that is, $\zeta(f) = \eta(f)$.*

Given a Laplace transformable function $f : \mathbb{R}_+ \to \mathbf{X}$, we shall call $\zeta(f)$ the *non-analytic growth bound* for f and use Definition 3.1.1 to describe this growth bound. It is clear from the definition that $\zeta(f) < \infty$ if and only if f is exponentially bounded.

Analogous to the non-analytic growth bound of a function f, we may define the *non-analytic abscissa of convergence* and the *non-analytic abscissa of absolute convergence* of $\hat{f}$ in the following manner:

Definition 3.1.7. Let $f : \mathbb{R}_+ \to \mathbf{X}$ be Laplace transformable. Let

$$\begin{aligned} \zeta_1(f) &:= \inf\{\omega \in \mathbb{R} : \text{ there exists } \theta > 0 \text{ and an exponentially bounded,} \\ &\qquad \text{holomorphic function } g : \Sigma_\theta \to \mathbf{X}, \text{ with } \mathrm{abs}(f-g) < \omega\} \\ \kappa(f) &:= \inf\{\omega \in \mathbb{R} : \text{ there exists } \theta > 0 \text{ and an exponentially bounded,} \\ &\qquad \text{holomorphic function } g : \Sigma_\theta \to \mathbf{X}, \text{ with } \mathrm{abs}(\|f-g\|) < \omega\} \end{aligned}$$

$\zeta_1(f)$ is called the *non-analytic abscissa of convergence* of $\hat{f}$ while $\kappa(f)$ is the *non-analytic abscissa of absolute convergence* of $\hat{f}$.

From Remark 3.1.5 we get equivalent characterisations for $\zeta_1(f)$ and $\kappa(f)$. It is immediate from the definitions that

$$\zeta_1(f) \leq \kappa(f) \leq \zeta(f) \leq \omega_0(f). \tag{3.2}$$

In the following Proposition, we note some basic properties of these bounds. (3.3) has been proven, for exponentially bounded, operator-valued functions $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$, in [10, Theorem 2.4.8], using the non-integrability growth bound description of $\zeta(\mathbf{T})$. The proof we give here for a general vector-valued function f uses the characterisation of $\zeta(f)$ as the non-analytic growth bound of f.

Proposition 3.1.8. *Let $f : \mathbb{R}_+ \to \mathbb{C}$ be Laplace transformable. Then*

1. $\zeta(f) \geq \kappa(f) \geq \mathrm{hol}_0^\infty(\hat{f})$ *and* $\kappa(f) \geq \zeta_1(f) \geq \mathrm{hol}^\infty(\hat{f})$;

2. Suppose $\zeta(f) < \infty$. Then

$$\begin{aligned} \omega_0(f) &= \max\big(\zeta(f), \mathrm{hol}(\hat{f})\big); && (3.3)\\ \mathrm{abs}(\|f\|) &= \max\big(\kappa(f), \mathrm{hol}(\hat{f})\big); && (3.4)\\ \mathrm{abs}(f) &= \max\big(\zeta_1(f), \mathrm{hol}(\hat{f})\big). && (3.5) \end{aligned}$$

3. Let $\tau \geq 0$, $\alpha \in \mathbb{R}$ and $f_\tau(t) = f(t+\tau), g(t) = e^{-\alpha t} f(t) \quad (t \geq 0)$. Then

$$\begin{aligned} &\zeta(f_\tau) = \zeta(f), && \kappa(f_\tau) = \kappa(f), && \zeta_1(f_\tau) = \zeta_1(f),\\ &\zeta(g) = -\alpha + \zeta(f), && \kappa(g) = -\alpha + \kappa(f), && \zeta_1(g) = -\alpha + \zeta_1(f). \end{aligned}$$

Proof. (*1*): From Remark 3.1.5 (*3*), it follows that $\mathrm{hol}_0^\infty(\hat{f}) \leq \kappa(f)$. $\zeta_1(f) \geq \mathrm{hol}^\infty(\hat{f})$ follows from Remark 3.1.5 (*4*). The other two inequalities are clear.

(*2*): That $\max(\zeta(f), \mathrm{hol}(\hat{f})) \leq \omega_0(f)$ is clear. Suppose $\max(\zeta(f), \mathrm{hol}(\hat{f})) < a$. From Proposition 3.1.4 (*6*), it follows that there is an exponentially bounded, entire function $g : \mathbb{C} \to \mathbf{X}$ such that $\omega_0(f-g) < a$ and $\omega_0(g) \leq a$. Since $\omega_0(f) \leq \max(\omega_0(f-g), \omega_0(g))$, it follows that $\omega_0(f) \leq a$. Thus, $\omega_0(f) = \max(\zeta(f), \mathrm{hol}(\hat{f}))$. The other equalities are similar.

(*3*): This follows from Proposition 3.1.4 (*6*). □

The inequalities in Proposition 3.1.8, (*1*) can all be strict.

Corollary 3.1.9. *1. If $\mathrm{hol}_0(\hat{f}) < \omega_0(f)$, then $\mathrm{hol}_0^\infty(\hat{f}) < \zeta(f)$.*

2. If $\mathrm{hol}_0(\hat{f}) < \mathrm{abs}(\|f\|)$, then $\mathrm{hol}_0^\infty(\hat{f}) < \kappa(f)$.

3. If $\mathrm{hol}_0(\hat{f}) < \mathrm{abs}(f)$, then $\mathrm{hol}_0^\infty(\hat{f}) < \zeta_1(f)$.

Proof. If $\mathrm{hol}_0(\hat{f}) < \omega_0(f)$, then from (2.8) and (3.3) it follows that $\omega_0(f) = \zeta(f)$. Therefore, if $\mathrm{hol}_0^\infty(\hat{f}) = \zeta(f)$ this would imply $\mathrm{hol}_0(\hat{f}) < \mathrm{hol}_0^\infty(\hat{f})$, contradicting Lemma 2.2.1. The other implications follow similarly. □

Examples 3.1.10.

1. Let $f(t) = e^t \sin e^t \quad (t \geq 0)$. Then $\omega_0(f) = 1 = \mathrm{abs}(|f|), \mathrm{abs}(f) = 0 = \mathrm{hol}_0(\hat{f})$ and $\mathrm{hol}(\hat{f}) = -\infty$, [2, Example 1.5.2]. Therefore, $\mathrm{hol}_0^\infty(\hat{f}) = 0$ and from Proposition 3.1.8 it follows that $\zeta(f) = \kappa(f) = 1$ while $\zeta_1(f) = 0$.

2. Bloch [12] has given an example of a Laplace transformable function $f : \mathbb{R}_+ \to \mathbb{C}$ such that $\mathrm{hol}_0(\hat{f}) = -\infty$ and $\mathrm{abs}(f) = \mathrm{abs}(|f|) = 0$. Then $\mathrm{hol}_0^\infty(\hat{f}) = \mathrm{hol}(\hat{f}) = -\infty$, and $\kappa(f) = \zeta_1(f) = 0$ by Proposition 3.1.8 (*2*). However, a function f satisfying $\mathrm{abs}(f) > \mathrm{hol}_0(\hat{f})$ cannot be exponentially bounded [2, Theorem 4.4.13]. Therefore, we have $\zeta(f) = \omega_0(f) = \infty$.

3. An example of an exponentially bounded function $f : \mathbb{R}_+ \to \mathbb{C}$ with $\mathrm{hol}_0(\hat{f}) < \omega_0(f)$, and therefore with $\mathrm{hol}_0^\infty(\hat{f}) < \zeta(f)$, can be obtained by taking f as in the proof of [4, Proposition 2.1] with the choices $k_m = e^{2m}$ and $q_m = 2m$. Explicitly,

$$f(t) = \begin{cases} e^{-m}\Phi_{e^{2m+1}}(t-2m) & \text{if } 2m \le t < 2m+1, m = 1, 2, \ldots; \\ 0 & \text{otherwise ,} \end{cases}$$

where, for $k > 2$, $0 \le t < 1$, Φ_k is given by

$$\Phi_k(t) = \begin{cases} \dfrac{2}{1-2t} & \text{if } 0 \le t \le \dfrac{k-2}{2k-2} \text{ or } \dfrac{k}{2k-2} \le t < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $|\Phi_k(t)| \le 2(k-1)$ and

$$\left| \int_0^1 \Phi_k(t) e^{-\lambda t}\, dt \right| \le C e^{\frac{\mathrm{Re}\,\lambda}{2k-2}} \qquad (\mathrm{Re}\,\lambda > 0)$$

[4, Proof of Proposition 2.1]. From these properties of Φ_k it is easy to deduce that $\omega_0(f) = 1/2$ and $\mathrm{hol}_0(\hat{f}) \le 0$, so that $\zeta(f) = 1/2$ and $\mathrm{hol}_0^\infty(\hat{f}) \le 0$.

Also, the function $g(t) = e^{2t} + f(t)$ gives an example for which the strict inequality $\mathrm{hol}_0^\infty(\hat{g}) < \zeta(g) < \omega_0(g)$ holds.

The bounds, $\zeta(f)$, $\kappa(f)$ and $\zeta_1(f)$ may be thought of as the non-analytic analogues of the bounds $\omega_0(f), \mathrm{abs}(\|f\|)$ and $\mathrm{abs}(f)$, respectively, of a Laplace transformable function f. *(1)* of the next theorem is an analogue of *(2)* on page 10, relating the non-analytic abscissa of convergence of the function f to the non-analytic growth bound of its primitive F in the same way as the abscissa $\mathrm{abs}(f)$ is related to the growth bound $\omega_0(F)$ of F.

Theorem 3.1.11. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be Laplace transformable and $F(t) = \int_0^t f(s)\, ds$. Then*

1. $\zeta(F) = \zeta_1(f)$;
2. $\mathrm{hol}_0^\infty(\hat{F}) = \mathrm{hol}_1^\infty(\hat{f})$.

Proof. *(1)*: Suppose first that $\zeta(F) < 0$. Then there exist $F_1, F_2 : \mathbb{R}_+ \to \mathbf{X}$ satisfying $\omega_0(F_2) < 0$, F_1 entire, and exponentially bounded with $\omega_0(F_1) \le \max(0, \mathrm{hol}(\hat{F}))$ and

$$F = F_1 + F_2. \tag{3.6}$$

Let $f_i = F_i'$, $i = 1, 2$. Then $\mathrm{abs}(f_i) = \omega_0(F_i - F_{i,\infty})$ where

$$F_{i,\infty} = \begin{cases} \lim_{t\to\infty} F_i(t) & \text{if the limit exists,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2$. Since $\omega_0(F_2) < 0$, $F_{2,\infty} = 0$, so that $\mathrm{abs}(f_2) = \omega_0(F_2) < 0$. From (3.6) it follows on differentiating, that $f = f_1 + f_2$, with f_1 holomorphic and exponentially bounded on $\mathbb{C}$. Therefore, $\zeta_1(f) \leq 0$.

Now suppose $\zeta(F) < \alpha$. Then, writing $G(t) = e^{-\alpha t}F(t)$ and using the first part we have that

$$G'(t) = g_1(t) + g_2(t), \quad (t \geq 0)$$

where g_1 is exponentially bounded in a sector and $\mathrm{abs}(g_2) < 0$. But

$$G'(t) = -\alpha e^{-\alpha t}F(t) + e^{-\alpha t}f(t). \tag{3.7}$$

Also, $\zeta(F) < \alpha$, so that there exist $F_1, F_2 : \mathbb{R}_+ \to \mathbf{X}$ satisfying $\omega_0(F_2) < \alpha$, F_1 exponentially bounded and holomorphic in a sector and $F = F_1 + F_2$. This together with (3.7) implies

$$f(t) = \big(\alpha F_1(t) + e^{\alpha t}g_1(t)\big) + \big(\alpha F_2(t) + e^{\alpha t}g_2(t)\big).$$

Since $\mathrm{abs}\big(t \mapsto \alpha F_2(t) + e^{\alpha t}g_2(t)\big) < \alpha$ and $t \mapsto \big(\alpha F_1(t) + e^{\alpha t}g_1(t)\big)$ is exponentially bounded in a sector, $\zeta_1(f) \leq \alpha$. Since $\alpha > \zeta(F)$ is arbitrary we conclude that $\zeta_1(f) \leq \zeta(F)$.

Conversely, suppose $\zeta_1(f) < \alpha$. Then by definition, $f = f_1 + f_2$, where f_1 is holomorphic and exponentially bounded in a sector and $\mathrm{abs}(f_2) < \alpha$. Let

$$F_i(t) = \int_0^t f_i(s)\, ds, \quad (t \geq 0)$$

for $i = 1, 2$. Then, $F(t) = F_1(t) + F_2(t), \quad t \geq 0$ and F_1 is holomorphic and exponentially bounded in a sector. Since we may write $F(t) = F_1(t) + F_{2,\infty} + F_2(t) - F_{2,\infty}$, it follows that $\zeta(F) \leq \alpha$ on noting that $\omega_0(F_2 - F_{2,\infty}) = \mathrm{abs}(f_2) < \alpha$ and $t \mapsto F_1(t) + F_{2,\infty}$ is holomorphic and exponentially bounded in a sector. We conclude, therefore that $\zeta(F) \leq \zeta_1(f)$. Hence, $\zeta_1(f) = \zeta(F)$.

(*2*): Recall that if $\hat{f}(\lambda)$ and $\hat{F}(\lambda)$ both exist and $\mathrm{Re}\,\lambda > 0$, then $\hat{f}(\lambda) = \lambda\hat{F}(\lambda)$. Suppose $\mathrm{hol}_0^\infty(\hat{F}) < \alpha$. Then, there exists $b > 0$ such that $Q_{\alpha,b} \subset D(\hat{F})$ and $\sup_{\lambda \in Q_{\alpha,b}} \|\hat{F}(\lambda)\| < \infty$. Therefore, the map $\lambda \mapsto \lambda\hat{F}(\lambda)$ is holomorphic in $Q_{\alpha,b}$. Therefore, $\hat{f}$ also has a holomorphic extension to $Q_{\alpha,b}$ which is equal to $\lambda\hat{F}(\lambda)$. Thus $Q_{\alpha,b} \subset D(\hat{f})$ and $\|\hat{f}(\lambda)\| = \|\lambda\hat{F}(\lambda)\|$ for all $\lambda \in Q_{\alpha,b}$. Therefore,

$$\sup_{\lambda \in Q_{\alpha,b}} \frac{\|\hat{f}(\lambda)\|}{(1 + |\lambda|)} \leq \sup_{\lambda \in Q_{\alpha,b}} \|\hat{F}(\lambda)\| < \infty.$$

Consequently, $\mathrm{hol}_1^\infty(\hat{f}) \leq \alpha$, so that $\mathrm{hol}_0^\infty(\hat{F}) \geq \mathrm{hol}_1^\infty(\hat{f})$. Similarly, since $\mathrm{hol}_1^\infty(\hat{f}) < \alpha$ implies that $Q_{\alpha,b} \subset D(\hat{f})$ for some $b > 0$ and $\sup\limits_{\lambda \in Q_{\alpha,b}} \dfrac{\|\hat{f}(\lambda)\|}{(1 + |\lambda|)} < \infty$, the map $\lambda \mapsto \dfrac{\hat{f}(\lambda)}{\lambda} = \hat{F}(\lambda)$ has a holomorphic extension to $Q_{\alpha,b}$ and $\sup_{\lambda \in Q_{\alpha,b}} \|\hat{F}(\lambda)\| < \infty$. Thus, $\mathrm{hol}_0^\infty(\hat{F}) \leq \mathrm{hol}_1^\infty(\hat{f})$.

□

Corollary 3.1.12. *For a Laplace transformable function $f : \mathbb{R}_+ \to \mathbf{X}$,*

$$\zeta_1(f) \geq \mathrm{hol}_1^\infty(\hat{f}).$$

Proof. Let $F(t) = \int_0^t f(s)\,ds \ \ (t \geq 0)$. Then from *(1)*, Proposition 3.1.8 and Theorem 3.1.11 it follows that $\zeta_1(f) = \zeta(F) \geq \mathrm{hol}_0^\infty(\hat{F}) = \mathrm{hol}_1^\infty(\hat{f})$. □

3.2 The non-analytic bounds for operator-valued functions

We shall now study the growth bounds introduced in Section 3.1 in the context of operator-valued, strongly continuous functions $\mathbf{T} : \mathbb{R}_+ :\to \mathcal{L}(\mathbf{X})$. In this case, $\zeta(\mathbf{T})$ is defined as

$$\begin{aligned}\zeta(\mathbf{T}) = \inf\big\{\omega_0(\mathbf{T}-\mathbf{S}) : \mathbf{S} \text{ is holomorphic and exponentially bounded} \\ \text{from } \Sigma_\theta \text{ to } \mathcal{L}(\mathbf{X}) \text{ for some } \theta > 0\big\}.\end{aligned} \tag{3.8}$$

$\zeta_1(\mathbf{T})$ and $\kappa(\mathbf{T})$ have analogous definitions. Since Proposition 3.1.4 remains valid when f is replaced by $\mathbf{T}$, we may use the equivalent descriptions of $\zeta(\mathbf{T})$ and the other two non-analytic bounds associated with $\mathbf{T}$ which it provides.

3.2.1 Reduction to the vector-valued case

Suppose that $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ is strongly continuous and Laplace transformable. From the uniform boundedness principle, and the definitions of the growth bound and abscissa of convergence it follows that

$$\begin{aligned}\omega_0(\mathbf{T}) &= \sup\{\omega_0(\mathbf{T}(\cdot)x) : x \in \mathbf{X}\} \\ &= \sup\{\omega_0(\langle \mathbf{T}(\cdot)x, x^*\rangle) : x \in \mathbf{X}, x^* \in \mathbf{X}^*\}, \qquad (3.9) \\ \mathrm{abs}(\mathbf{T}) &= \sup\{\mathrm{abs}(\mathbf{T}(\cdot)x) : x \in \mathbf{X}\} \\ &= \sup\{\mathrm{abs}(\langle \mathbf{T}(\cdot)x, x^*\rangle) : x \in \mathbf{X}, x^* \in \mathbf{X}^*\}. \qquad (3.10)\end{aligned}$$

Further, using arguments with Taylor series and the uniform boundedness principle it has been shown in [2, Proposition 1.5.5] that

$$\begin{aligned}\mathrm{hol}(\hat{\mathbf{T}}) &= \sup\big\{\mathrm{hol}(\hat{\mathbf{T}}(\cdot)x) : x \in \mathbf{X}\big\} \\ &= \sup\big\{\mathrm{hol}(\langle \hat{\mathbf{T}}(\cdot)x, x^*\rangle) : x \in \mathbf{X}, x^* \in \mathbf{X}^*\big\}.\end{aligned}$$

and a straightforward application of the uniform boundedness principle gives the corresponding result for $\mathrm{hol}_0(\hat{\mathbf{T}})$. It is not clear that $\mathrm{hol}^\infty(\hat{\mathbf{T}}) = \sup\big\{\mathrm{hol}^\infty(\hat{T}(\cdot)x) : x \in \mathbf{X}\big\}$. We now prove the corresponding fact for $\mathrm{hol}_0^\infty(\hat{\mathbf{T}})$, $\zeta(\mathbf{T})$, $\zeta_1(\mathbf{T})$ and $\mathrm{hol}_{\mathrm{n}}^\infty(\hat{\mathbf{T}})$. In particular, if $\mathbf{T}$ is a C_0-semigroup, we express $\zeta_1(\mathbf{T})$ in terms of the non-analytic growth bounds of certain orbit maps.

Theorem 3.2.1. *Let Ω be a non-empty, connected, open subset of $\mathbb{C}$ and $\mathbf{S} : \Omega \to \mathcal{L}(\mathbf{X})$ be a bounded holomorphic function. Let $(\Omega_n)_{n\geq 1}$ be a sequence of connected open subsets of $\mathbb{C}$ such that $\Omega_n \cap \Omega$ is non-empty for each n. Suppose that, for each $x \in \mathbf{X}$ there exists $n_x \geq 1$ and a bounded holomorphic function $\mathbf{H}_x : \Omega_{n_x} \to \mathbf{X}$ such that $\mathbf{H}_x(\lambda) = \mathbf{S}(\lambda)x$ for all $\lambda \in \Omega_n \cap \Omega$. Then there exists $N \geq 1$ and a bounded holomorphic function $\mathbf{U} : \Omega_N \to \mathcal{L}(\mathbf{X})$ such that $\mathbf{U}(\lambda) = \mathbf{S}(\lambda)$ for all $\lambda \in \Omega_N \cap \Omega$.*

Proof. For $n \geq 1$ and $k \geq 1$, let

$$X_{n,k} = \Big\{ x \in \mathbf{X} : \text{ there is a holomorphic function } \mathbf{H} : \Omega_n \to \mathbf{X} \text{ such that } \mathbf{H}(\lambda) = \mathbf{S}(\lambda)x, \lambda \in \Omega_n \cap \Omega \text{ and } \sup_{\lambda \in \Omega_n} \|\mathbf{H}(\lambda)\| \leq k \Big\}.$$

By assumption, $\mathbf{X} = \bigcup_{n,k=1}^{\infty} \mathbf{X}_{n,k}$. We show that each $\mathbf{X}_{n,k}$ is closed in $\mathbf{X}$. Let $(x_r)_{r\geq 1}$ be a sequence in $\mathbf{X}_{n,k}$ converging to some $x \in \mathbf{X}$. For each $r \geq 1$ there exists a holomorphic function $\mathbf{H}_r : \Omega_n \to \mathbf{X}$ such that

$$\|\mathbf{H}_r(\lambda)\| \leq k \quad (\lambda \in \Omega_n), \quad \mathbf{H}_r(\lambda) = \mathbf{S}(\lambda)x_r \quad (\lambda \in \Omega_n \cap \Omega).$$

Therefore, $\lim_{r\to\infty} \mathbf{H}_r(\lambda) = \mathbf{S}(\lambda)x$ $(\lambda \in \Omega_n \cap \Omega)$. Since $\{\mathbf{H}_r : r \geq 1\}$ is uniformly bounded, Vitali's Theorem implies that $\mathbf{H}(\lambda) := \lim_{r\to\infty} \mathbf{H}_r(\lambda)$ exists for all $\lambda \in \Omega_n$ and defines a holomorphic function $\mathbf{H} : \Omega_n \to \mathbf{X}$. Then $\|\mathbf{H}(\lambda)\| \leq k$ for all $\lambda \in \Omega_n$ and $\mathbf{H}(\lambda) = \mathbf{S}(\lambda)x$ $(\lambda \in \Omega_n \cap \Omega)$. Thus, $x \in \mathbf{X}_{n,k}$. Using Baire's Category Theorem, we find an $N \geq 1$, $k \geq 1$, $x_0 \in \mathbf{X}$ and an $\epsilon > 0$ such that $\|x - x_0\| < \epsilon$ implies $x \in \mathbf{X}_{n,k}$. So for each such x, there is a unique holomorphic function $\mathbf{H}_x : \Omega_N \to \mathbf{X}$ such that

$$\|\mathbf{H}_x(\lambda)\| \leq k \quad (\lambda \in \Omega_N) \text{ and } \mathbf{H}_x(\lambda) = \mathbf{S}(\lambda)x \quad (\lambda \in \Omega_N \cap \Omega).$$

Thus, if $\|x - x_0\| < \epsilon$, there is a holomorphic function $\mathbf{H}_{x-x_0} := \mathbf{H}_x - \mathbf{H}_{x_0} : \Omega_N \to \mathbf{X}$ such that

$$\|\mathbf{H}_{x-x_0}(\lambda)\| \leq 2k \quad (\lambda \in \Omega_N) \text{ and } \mathbf{H}_{x-x_0}(\lambda) = \mathbf{S}(\lambda)(x - x_0) \quad (\lambda \in \Omega \cap \Omega_N).$$

Therefore, for any $y \in \mathbf{X}$, we have that there is a unique holomorphic function, $\mathbf{H}_y : \Omega_N \to \mathbf{X}$ such that $\|\mathbf{H}_y(\lambda)\| \leq 4k\|y\|/\epsilon$ $(\lambda \in \Omega_N)$ and $\mathbf{H}_y(\lambda) = \mathbf{S}(\lambda)y$ $(\lambda \in \Omega_N \cap \Omega)$. So we may define

$$\mathbf{U}(\lambda)x = \mathbf{H}_x(\lambda) \quad (\lambda \in \Omega_N, x \in \mathbf{X}).$$

$\mathbf{U}(\lambda)$ is linear, by uniqueness of holomorphic extensions. This is the required bounded holomorphic function. □

Corollary 3.2.2. *Let $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ be strongly continuous and Laplace transformable. Then*

1. $\mathrm{hol}_0^\infty(\hat{\mathbf{T}}) = \sup\big\{\mathrm{hol}_0^\infty(\hat{\mathbf{T}}(\cdot)x) : x \in \mathbf{X}\big\}$*;*

2. $\zeta(\mathbf{T}) = \sup\big\{\zeta(\mathbf{T}(\cdot)x) : x \in \mathbf{X}\big\}$.

Proof. Clearly, $\mathrm{hol}_0^\infty(\hat{\mathbf{T}}) \geq \mathrm{hol}_0^\infty(\hat{\mathbf{T}}(\cdot)x)$ and $\zeta(\mathbf{T}) \geq \zeta(\mathbf{T}(\cdot)x)$ for all $x \in \mathbf{X}$. Suppose that $w > \sup\{\mathrm{hol}_0^\infty(\hat{\mathbf{T}}(\cdot)x) : x \in \mathbf{X}\}$. Let $\Omega = H_{\mathrm{abs}(\mathbf{T})}$ and $\Omega_n = Q_{w-n^{-1},n}$. By assumption, for each $x \in \mathbf{X}$, there exists $n_x \geq 1$ and a bounded holomorphic function on Ω_{n_x} which agrees with $\hat{\mathbf{T}}(\cdot)x$ on $\Omega_{n_x} \cap \Omega$. By Theorem 3.2.1, there exists an $N \geq 1$ and a bounded holomorphic function $\mathbf{U} : \Omega_N \to \mathcal{L}(\mathbf{X})$ agreeing with $\hat{\mathbf{T}}$ on $\Omega \cap \Omega_N$. Hence, $\mathrm{hol}_0^\infty(\hat{\mathbf{T}}) \leq w$.

Now, let $w > \sup\{\zeta(\mathbf{T}(\cdot)x) : x \in \mathbf{X}\}$. Since $\mathrm{hol}_0^\infty(\hat{\mathbf{T}}(\cdot)x) \leq \zeta(\mathbf{T}(\cdot)x)$ for all $x \in \mathbf{X}$ it follows from the first part that $w > \mathrm{hol}_0^\infty(\hat{\mathbf{T}})$. Take α such that $\mathrm{hol}_0^\infty(\hat{\mathbf{T}}) < \alpha < w$. Choose b such that $Q_{\alpha,b} \subset D(\hat{\mathbf{T}})$, so that $\mathbf{T}_{\alpha,b} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ is defined. Since $\zeta(\mathbf{T}(\cdot)x) < w$, we have from Proposition 3.1.4,

$$\sup_{t\geq 0} \big\|e^{-wt}(\mathbf{T}(t) - \mathbf{T}_{\alpha,b}(t))x\big\| < \infty$$

for each $x \in \mathbf{X}$. By the Uniform Boundedness Principle,

$$\sup_{t\geq 0} \big\|e^{-wt}(\mathbf{T}(t) - \mathbf{T}_{\alpha,b}(t))\big\| < \infty.$$

Thus, $\zeta(\mathbf{T}) \leq w$. This proves the second part. □

Corollary 3.2.3. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be Laplace transformable. Then*

1. $\mathrm{hol}_0^\infty(\hat{f}) = \sup\{\mathrm{hol}_0^\infty(x^* \circ \hat{f}) : x^* \in \mathbf{X}^*\}$;

2. $\zeta(f) = \sup\{\zeta(x^* \circ f) : x^* \in \mathbf{X}^*\}$.

Proof. These results follow by the same proofs as Theorem 3.2.1 and Corollary 3.2.2. □

Remark 3.2.4. Theorem 3.2.1 remains true if 'bounded holomorphic function' is replaced by 'linearly bounded holomorphic function' or more generally by 'polynomially bounded function'.

Therefore, the following generalisation of the above result also holds:

Corollary 3.2.5. *Let $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ be strongly continuous and Laplace transformable. Then*

1. $\mathrm{hol_n}^\infty(\hat{\mathbf{T}}) = \sup\{\mathrm{hol_n}^\infty(\hat{\mathbf{T}}(\cdot)x) : x \in \mathbf{X}\}$*;*

2. $\zeta_1(\mathbf{T}) = \sup\{\zeta_1(\mathbf{T}(\cdot)x) : x \in \mathbf{X}\}$;

Proof. The first part follows exactly along the lines of Corollary 3.2.2, on noting Remark 3.2.4. (*2*) follows from Theorem 3.1.11 (*1*) and Corollary 3.2.2 (*2*). □

3.2.2 The C_0-semigroup case

Let $\mathbf{T}$ be a C_0-semigroup on $\mathbf{X}$ with generator $\mathbf{A}$. We recall the definitions of the various spectral bounds of $\mathbf{A}$:

$$
\begin{aligned}
s(\mathbf{A}) &= \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathbf{A})\};\\
s_0(\mathbf{A}) &= \inf\Big\{\omega > s(\mathbf{A}) :\ \text{there exists } C_\omega \text{ such that } \|\mathbf{R}(\lambda,\mathbf{A})\| \le C_\omega\\
&\qquad\qquad \text{whenever } \operatorname{Re}\lambda > \omega\Big\};\\
s^\infty(\mathbf{A}) &= \inf\Big\{\omega \in \mathbb{R} : Q_{\omega,b} \subset \rho(\mathbf{A}) \text{ for some } b \ge 0\Big\};\\
s_0^\infty(\mathbf{A}) &= \inf\Big\{\omega \in \mathbb{R} : Q_{\omega,b} \subset \rho(\mathbf{A}) \text{ and } \sup_{\lambda\in Q_{\omega,b}} \|\mathbf{R}(\lambda,\mathbf{A})\| < \infty \text{ for some } b \ge 0\Big\};\\
s_n(\mathbf{A}) &= \inf\Big\{\omega \in \mathbb{R} : H_\omega \subset \rho(\mathbf{A}) \text{ and } \sup_{\operatorname{Re}\lambda>\omega} \frac{\|\mathbf{R}(\lambda,\mathbf{A})\|}{(1+|\lambda|)^n} < \infty\Big\};\\
s_n^\infty(\mathbf{A}) &= \inf\Big\{\omega \in \mathbb{R} : Q_{\omega,b} \subset \rho(\mathbf{A}) \text{ and } \sup_{\lambda\in Q_{\omega,b}} \frac{\|\mathbf{R}(\lambda,\mathbf{A})\|}{(1+|\lambda|)^n} < \infty \text{ for some } b \ge 0\Big\},
\end{aligned}
$$

for $n \in \mathbb{N}$.

The precise relation between these spectral bounds and the abscissas associated with the Laplace transform of $\mathbf{T}$ is given by the following equations:

$$
\begin{aligned}
s(\mathbf{A}) = \operatorname{hol}(\hat{\mathbf{T}}), \qquad s_0(\mathbf{A}) = \operatorname{hol}_0(\hat{\mathbf{T}}), \qquad s_n(\mathbf{A}) = \operatorname{hol}_{\mathrm{n}}(\hat{\mathbf{T}}),\\
s^\infty(\mathbf{A}) = \operatorname{hol}^\infty(\hat{\mathbf{T}}), \quad s_0^\infty(\mathbf{A}) = \operatorname{hol}_0^\infty(\hat{\mathbf{T}}), \quad s_n^\infty(\mathbf{A}) = \operatorname{hol}_{\mathrm{n}}^\infty(\hat{\mathbf{T}}).
\end{aligned}
$$

The first two of these equations follow from [2, Theorem 5.1.4] and the proof of the others works along similar lines.

From [2, Proposition 5.1.6] we also have

$$
\begin{aligned}
\operatorname{abs}(\mathbf{T}) = \omega_1(\mathbf{T}) &= \sup\big\{\omega(\mathbf{T}(\cdot)x) : x \in D(\mathbf{A})\big\}\\
&= \omega\left(\mathbf{T}(\cdot)\mathbf{R}(\lambda,\mathbf{A})\right),
\end{aligned}
\tag{3.11}
$$

where $\lambda \in \rho(\mathbf{A})$. Therefore, for the semigroup $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ the results in (3.3) of Proposition 3.1.8 read as

$$
\begin{aligned}
\omega_0(\mathbf{T}) &= \max(\zeta(\mathbf{T}), s(\mathbf{A})),\\
\omega_1(\mathbf{T}) &= \max(\zeta_1(\mathbf{T}), s(\mathbf{A})).
\end{aligned}
\tag{3.12}
$$

Moreover, we can write $\zeta_1(\mathbf{T})$ in terms of the non-analytic growth bounds of the orbit functions $u_x(t) = \mathbf{T}(t)(x)$, thus obtaining an analogue of (3.11).

Theorem 3.2.6. *Let $\mathbf{T} : \mathbb{R} \to \mathcal{L}(\mathbf{X})$ be a C_0-semigroup with generator $\mathbf{A}$. Then*

$$\begin{aligned}\zeta_1(\mathbf{T}) &= \zeta\big(\mathbf{T}(\cdot)\mathbf{R}(\lambda,\mathbf{A})\big)\\ &= \sup\big\{\zeta(\mathbf{T}(\cdot)x) : x \in D(\mathbf{A})\big\},\end{aligned}$$

where $\lambda \in \rho(\mathbf{A})$ satisfies $\operatorname{Re}\lambda > \omega_0(\mathbf{T})$.

Proof. Since $D(\mathbf{A}) = \operatorname{Ran}(\mathbf{R}(\lambda,\mathbf{A}))$, for all $\lambda \in \rho(\mathbf{A})$, we have using *(2)* of Corollary 3.2.2

$$\begin{aligned}\sup\big\{\zeta(\mathbf{T}(\cdot)x) : x \in D(\mathbf{A})\big\} &= \sup\big\{\zeta(\mathbf{T}(\cdot)\mathbf{R}(\lambda,\mathbf{A})\ ) : y \in \mathbf{X}\big\}\\ &= \zeta\big(\mathbf{T}(\cdot)\mathbf{R}(\lambda,\mathbf{A})\big).\end{aligned}$$

For $\operatorname{Re}\lambda > \omega_0$, and $x \in \mathbf{X}$,

$$\begin{aligned}\mathbf{T}(t)\mathbf{R}(\lambda,\mathbf{A})\ x &= \int_0^\infty e^{-\lambda s}\mathbf{T}(t+s)x\,ds\\ &= e^{\lambda t}\left(\mathbf{R}(\lambda,\mathbf{A})\ x - \int_0^t e^{-\lambda s}\mathbf{T}(s)x\,ds\right).\end{aligned}$$

Observing that $\zeta\big(t \mapsto e^{\lambda t}\mathbf{R}(\lambda,\mathbf{A})x\big) = -\infty$, we have

$$\begin{aligned}\zeta\big(\mathbf{T}(\cdot)\mathbf{R}(\lambda,\mathbf{A})x\big) &= \operatorname{Re}\lambda + \zeta\big(t \mapsto \int_0^t \mathbf{U}(s)x\,ds\big)\\ &= \operatorname{Re}\lambda + \zeta_1\big(\mathbf{U}(\cdot)x\big)\\ &= \zeta_1\big(\mathbf{T}(\cdot)x\big),\end{aligned}$$

where $\mathbf{U}(s) = e^{-\lambda s}\mathbf{T}(s)$ is the rescaled semigroup. So, it follows from Corollary 3.2.5 that

$$\begin{aligned}\zeta(\mathbf{T}(\cdot)\mathbf{R}(\lambda,\mathbf{A})\ ) &= \sup\{\zeta(\mathbf{T}(\cdot)\mathbf{R}(\lambda,\mathbf{A})\ y) : y \in \mathbf{X}\}\\ &= \sup\{\zeta_1(\mathbf{T}(\cdot)x) : x \in \mathbf{X}\}\\ &= \zeta_1(\mathbf{T}).\end{aligned}$$

□

In [11, Corollary 3.3] it has been shown that $s_0^\infty(\mathbf{A}) \le \delta(\mathbf{T})$ and strict inequality may hold ([11, Proposition 3.7]). Thus for a C_0-semigroup $\mathbf{T}$ on $\mathbf{X}$,

$$s_0^\infty(\mathbf{A}) \le \delta(\mathbf{T}) \le \zeta(\mathbf{T}). \tag{3.13}$$

If $\mathbf{X}$ is a Hilbert space then actually equality holds in (3.13). This follows from [11, Lemma 4.3] where it is shown that if $s_0^\infty(\mathbf{A}) < \alpha$ then there exists $b > 0$ such that $\omega_0(\mathbf{T} - \mathbf{T}_1) \le \alpha$, where

$$\mathbf{T}_1(t)x = \frac{1}{2\pi i}\int_{\Gamma_{\alpha,b}} e^{\mu t}\mathbf{R}(\mu,\mathbf{A})\ x\,d\mu,$$

$\Gamma_{\alpha,b}$ consists of line segments joining $\alpha - ib, \omega - ib, \omega + ib$ and $\alpha + ib$ successively, and $\omega > \max(\alpha, \omega_0(\mathbf{T}))$. Since $\mathbf{T}_1$ has a holomorphic, exponentially bounded extension to a sector Σ_θ, we have $\zeta(\mathbf{T}) \leq \alpha$. So, for a C_0-semigroup on a Hilbert space we have

$$s_0^\infty(\mathbf{A}) = \delta(\mathbf{T}) = \zeta(\mathbf{T}). \tag{3.14}$$

This result may be considered as the non-analytic analogue of the famous Gearhart-Prüss theorem [2, Theorem 5.2.1]:

Theorem 3.2.7. *(Gearhart-Prüss) For a C_0-semigroup $\mathbf{T}$ on a Hilbert space $\mathbf{X}$ with generator $\mathbf{A}$ the uniform exponential growth bound and the pseudo-spectral bound coincide, that is*

$$\omega_0(\mathbf{T}) = s_0(\mathbf{A}).$$

It is well known that such a theorem does not hold for individual semigroup orbits [2, Example 5.2.3] and in fact, the same is true of its non-analytic analogue. More precisely, an analogue of (3.14) does not hold for individual orbits of semigroups defined on Hilbert spaces. Indeed, if $\mathbf{T}$ is a C_0-semigroup on a Hilbert space with $\mathrm{hol}_0^\infty(\hat{u}_x) = \zeta(u_x)$, where $u_x(t) = \mathbf{T}(t)x$, then we must have

$$\mathrm{hol}_0(\hat{u}_x) = \omega_0(\hat{u}_x).$$

This follows from the inequality $\mathrm{hol}_0^\infty(\hat{u}_x) \leq \mathrm{hol}_0(\hat{u}_x) \leq \omega_0(u_x)$ and Proposition 3.1.8 *(2)*. This would imply that an analogue of Gearhart-Prüss Theorem [2, Theorem 5.2.1] holds for individual semigroup orbits, thus yielding a contradiction.

In [51] higher order analogues of the Gearhart-Prüss Theorem have been established, specifically, the equalities

$$\omega_n(\mathbf{T}) = s_n(\mathbf{A}) \quad (n \in \mathbb{N}),$$

where $\mathbf{T}$ is a C_0-semigroup with generator $\mathbf{A}$, defined on a Hilbert space $\mathbf{X}$. We shall obtain the corresponding result for the non-analytic abscissa of convergence $\zeta_1(\mathbf{T})$. For this, we need a lemma, with a proof very similar to [11, Lemma 4.2]. We state the lemma without proof.

Lemma 3.2.8. *Let $\mathbf{T}$ be a C_0-semigroup defined on the Hilbert space $\mathbf{X}$. Let $\alpha, w, b, K \in \mathbb{R}$, $\alpha < w$, $b > 0$ be such that $Q := Q_{\alpha,b} \cap \{\lambda : \mathrm{Re}\,\lambda \leq w\} \subset \rho(\mathbf{A})$ and*

$$\sup_{\lambda \in Q} \left\| \frac{\mathbf{R}(\lambda, \mathbf{A})}{(1 + |\lambda|)} \right\| \leq K.$$

Then for $x \in \mathbf{X}$ we have

$$\lim_{|s| \to \infty} \left\| \frac{\mathbf{R}(\mu + is, \mathbf{A})x}{\mu + is} \right\| = 0, \tag{3.15}$$

the limit being uniform with respect to $\mu \in [\alpha, w]$ *and*

$$\lim_{|s|\to\infty} \int_{\alpha+is}^{\omega+is} \frac{e^{\lambda t}\mathbf{R}(\lambda, \mathbf{A})x}{\lambda}\, d\lambda = 0. \tag{3.16}$$

Theorem 3.2.9. *For a* C_0*-semigroup* $\mathbf{T}$ *defined on a Hilbert space* $\mathbf{X}$*, with generator* $\mathbf{A}$*,*

$$\zeta_1(\mathbf{T}) = s_1^{\infty}(\mathbf{A}).$$

Proof. We first consider the case when $\omega_0(\mathbf{T}) < 0$. Let $s_1^{\infty}(\mathbf{A}) < \alpha$. We may assume here, without loss of generality, that $\alpha < 0$. By the definition of $s_1^{\infty}(\mathbf{A})$, there exist $b > 0$ and a constant K such that $Q_{\alpha,b} \subset \rho(\mathbf{A})$ and

$$\|\mathbf{R}(a + is, \mathbf{A})\| \leq K(1 + |s|), \quad a \geq \alpha,\ |s| \geq b.$$

Let $\omega > \max(\alpha, \omega_0(\mathbf{T}))$ and $\omega \neq 0$. From the Inversion theorem [20, Theorem III.5.14], we have

$$\int_0^t \mathbf{T}(s)x\, ds = \lim_{M\to\infty} \frac{1}{2\pi i} \int_{\omega-iM}^{\omega+iM} \frac{e^{\lambda t}}{\lambda}\mathbf{R}(\lambda, \mathbf{A})x\, d\lambda, \tag{3.17}$$

for $t \geq 0$, and all $x \in \mathbf{X}$. Define $S_{\alpha,b,\omega}(t)$ by

$$S_{\alpha,b,\omega}(t)x = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{\lambda t}\mathbf{R}(\lambda, \mathbf{A})x}{\lambda}\, d\lambda,$$

where Γ is the path consisting of line segments $[\alpha - ib, \omega - ib]$, $[\omega - ib, \omega + ib]$ and $[\omega + ib, \alpha + ib]$. Let $\phi \in \mathbf{X}^*$. Then

$$\begin{aligned}
\phi\left(\int_0^t \mathbf{T}(s)(x)ds\right) &= \lim_{M\to\infty} \frac{1}{2\pi i} \int_{\omega-iM}^{\omega+iM} \frac{e^{\lambda t}\phi(\mathbf{R}(\lambda, \mathbf{A})x)}{\lambda}\, d\lambda \\
&= \frac{1}{2\pi i} \int_{\omega-ib}^{\omega+ib} \frac{e^{\lambda t}\phi(\mathbf{R}(\lambda, \mathbf{A})x)}{\lambda}\, d\lambda \\
&\quad + \lim_{M\to\infty} \frac{1}{2\pi i} \left(\int_{\omega-iM}^{\omega-ib} + \int_{\omega+ib}^{\omega+iM}\right) \frac{e^{\lambda t}\phi(\mathbf{R}(\lambda, \mathbf{A})x)}{\lambda}\, d\lambda. \qquad (3.18)
\end{aligned}$$

For $M > b$, using Cauchy's Theorem, we have

$$\begin{aligned}
\int_{\omega+ib}^{\omega+iM} \frac{e^{\lambda t}\phi(\mathbf{R}(\lambda, \mathbf{A})x)}{\lambda}\, d\lambda &= \left(\int_{\omega+ib}^{\alpha+ib} + \int_{\alpha+ib}^{\alpha+iM}\right) \frac{e^{\lambda t}\phi(\mathbf{R}(\lambda, \mathbf{A})x)}{\lambda}\, d\lambda \\
&\quad + \int_{\alpha+iM}^{\omega+iM} \frac{e^{\lambda t}\phi(\mathbf{R}(\lambda, \mathbf{A})x)}{\lambda}\, d\lambda.
\end{aligned}$$

Using Lemma 3.2.8, we have that

$$\lim_{M\to\infty} \int_{\alpha+iM}^{\omega+iM} \frac{e^{\lambda t}\phi(\mathbf{R}(\lambda, \mathbf{A})x)}{\lambda}\, d\lambda = 0.$$

Consequently, (3.18) yields

$$\begin{aligned}
&\phi\left(\int_0^t \mathbf{T}(s)x\,ds - S_{\alpha,b,\omega}(t)x\right)\\
&\quad = \lim_{M\to\infty}\frac{1}{2\pi i}\left(\int_{\alpha+ib}^{\alpha+iM} + \int_{\alpha-iM}^{\alpha-ib}\right)\frac{e^{\lambda t}\phi(\mathbf{R}(\lambda,\mathbf{A})x)}{\lambda}\,d\lambda. \qquad (3.19)
\end{aligned}$$

Applying the resolvent identity, we have for $M > b$,

$$\begin{aligned}
&\left|\int_{\alpha+ib}^{\alpha+iM}\frac{e^{\lambda t}}{\lambda}\phi(\mathbf{R}(\lambda,\mathbf{A})x)\,d\lambda\right|\\
&= \left|\int_b^M \frac{e^{(\alpha+is)t}\phi(\mathbf{R}(\omega+is,\mathbf{A})x + (\omega-\alpha)\mathbf{R}(\omega+is,\mathbf{A})\mathbf{R}(\alpha+is,\mathbf{A})x)}{\alpha+is}\,ds\right|\\
&\le e^{\alpha t}\int_{-\infty}^{\infty}\left|\frac{\phi(\mathbf{R}(\omega+is,\mathbf{A})x)}{\alpha+is}\right|ds\\
&\qquad +(\omega-\alpha)e^{\alpha\, t}\int_{|s|>b}\frac{\|\mathbf{R}(\omega+is,\mathbf{A})^*\phi\|\|\mathbf{R}(\alpha+is,\mathbf{A})x\|}{|\alpha+is|}\,ds\\
&\le e^{\alpha t}\left(\int_{-\infty}^{\infty}|\phi(\mathbf{R}(\omega+is,\mathbf{A})x)|^2\,ds\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\frac{1}{|\alpha+is|^2}\,ds\right)^{\frac{1}{2}}\\
&\qquad +(\omega-\alpha)e^{\alpha t}\left(\int_{-\infty}^{\infty}\|\mathbf{R}(\omega+is,\mathbf{A})^*\phi\|^2\,ds\right)^{\frac{1}{2}}\left(\int_{|s|>b}\frac{\|\mathbf{R}(\alpha+is,\mathbf{A})x\|^2}{|\alpha+is|^2}\,ds\right)^{\frac{1}{2}}\\
&\le Ce^{\alpha t}(1+\omega-\alpha)\|\phi\|\,\|x\|, \qquad (3.20)
\end{aligned}$$

where C is a constant depending on α and b. The final step depends on the following observations. By Plancherel's Theorem, for $\omega > \omega_0(\mathbf{T})$,

$$\begin{aligned}
\int_{-\infty}^{\infty}\|\mathbf{R}(\omega+is,\mathbf{A})x\|^2\,ds &= 2\pi\int_0^{\infty}\|e^{-\omega t}\mathbf{T}(t)x\|^2\,dt\\
&\le C'\|x\|^2,
\end{aligned}$$

where C' is a constant and a similar estimate holds for $\displaystyle\int_{-\infty}^{\infty}\|\mathbf{R}(\omega+is,\mathbf{A})^*\phi\|^2\,ds$. Moreover,

$$\int_{|s|>b}\frac{\|\mathbf{R}(\omega+is,\mathbf{A})x\|^2}{|\alpha+is|^2}\,ds \le C'\frac{\|x\|^2}{b^2}$$

while

$$\begin{aligned}
&\int_{|s|>b}\frac{\|\mathbf{R}(\omega+is,\mathbf{A})\mathbf{R}(\alpha+is,\mathbf{A})x\|^2}{|\alpha+is|^2}\,ds\\
&\le \int_{|s|>b}\frac{K^2(1+|s|)^2}{\alpha^2+s^2}\|\mathbf{R}(\omega+is,\mathbf{A})x\|^2\,ds\\
&\le C''\|x\|^2,
\end{aligned}$$

where C'' is a constant depending on α and b.

It follows then from the resolvent identity $\mathbf{R}(\alpha+is,\mathbf{A})x = \mathbf{R}(\omega+is,\mathbf{A})x+(\omega-\alpha)\mathbf{R}(\omega+is,\mathbf{A})\mathbf{R}(\alpha+is,\mathbf{A})x$, that

$$\int_{|s|>b} \frac{\|\mathbf{R}(\alpha+is,\mathbf{A})x\|^2}{|\alpha+is|^2}\,ds \le C''\|x\|^2.$$

We obtain estimates similar to (3.20) for the other integral on the right hand side of (3.19), to arrive at

$$\left|\phi\left(\int_0^t \mathbf{T}(s)(x)\,ds - S_{\alpha,b,\omega}(t)x\right)\right| \le Ke^{\alpha t}\|\phi\|\|x\|, \tag{3.21}$$

for $x \in \mathbf{X}$. Since (3.21) holds for all $\phi \in \mathbf{X}^*$ and $x \in \mathbf{X}$, we have

$$\left\|\int_0^t \mathbf{T}(s)\,ds - S_{\alpha,b,\omega}(t)\right\| \;\le\; Ke^{\alpha t}. \tag{3.22}$$

For $x \in \mathbf{X}$ let

$$\mathbf{S}_2(t)x = \int_0^t \mathbf{T}(s)x\,ds - S_{\alpha,b,\omega}(t)x. \tag{3.23}$$

Then (3.22) implies that $\omega_0(\mathbf{S}_2) \le \alpha$. Further, differentiating (3.23) with respect to t yields

$$\mathbf{T}(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\mathbf{R}(\lambda,\mathbf{A})x\,d\lambda + \mathbf{T}_2(t)x,$$

where $\mathbf{T}_2(t)x$ is the derivative of $\mathbf{S}_2(t)x$, so that

$$\mathbf{S}_2(t)x = \int_0^t \mathbf{T}_2(s)x\,ds + \mathbf{S}_2(0)x.$$

Therefore,

$$\mathrm{abs}(\mathbf{T}_2) \le \omega_0(\mathbf{S}_2) \le \alpha.$$

Setting $\mathbf{T}_1(t) = \dfrac{1}{2\pi i}\displaystyle\int_\Gamma e^{\lambda t}\mathbf{R}(\lambda,\mathbf{A})\,d\lambda$ we have that $\mathbf{T}_1$ is holomorphic and exponentially bounded in a sector and $\mathrm{abs}(\mathbf{T}-\mathbf{T}_1) \le \alpha$. Hence, $\zeta_1(\mathbf{T}) \le \alpha$. Thus $s_1^\infty(\mathbf{A}) \ge \zeta_1(\mathbf{T})$ and from Corollary 3.1.12 it follows that $\zeta_1(\mathbf{T}) = s_1^\infty(\mathbf{A})$ if $\omega_0(\mathbf{T}) < 0$. For the general case, the result follows by rescaling. □

Example 3.2.10. We consider Example 5.1.10 in [2]. Here $\mathbf{X}$ is the Hilbert space given by

$$\begin{aligned}\mathbf{X} &:= \left\{x = (x_n)_{n\ge1} : x_n \in \mathbb{C}^n, \sum_{n=1}^\infty \|x_n\|^2 < \infty\right\},\\ \|x\| &:= \left(\sum_{n=1}^\infty \|x_n\|^2\right)^{1/2},\end{aligned} \tag{3.24}$$

where the norm on $\mathbb{C}^n$ is the Euclidean norm. Let $B_n = (\beta_{i,j}^{(n)})_{1\leq i,j\leq n}$ be the $n \times n$ matrix with $\beta_{i,i+1}^{(n)} = 1$ for $1 \leq i < n$, $\beta_{i,j}^{(n)} = 0$ otherwise, and let $A_n = i2^n I_n + B_n$. Let $\mathbf{A}$ be the operator on $\mathbf{X}$ defined by

$$\begin{aligned} D(\mathbf{A}) &= \left\{ x \in \mathbf{X} : \sum_{n=1}^{\infty} 2^{2n} \|x_n\|^2 < \infty \right\}, \\ \mathbf{A}x &= (A_n x_n)_{n\geq 1}. \end{aligned}$$

Let $\mathbf{T}$ be the C_0-semigroup generated by $\mathbf{A}$. It is shown in [2], that

$$\omega_0(\mathbf{T}) = 1,\ \omega_1(\mathbf{T}) = \frac{1}{2},\ \sigma(\mathbf{A}) = \{i2^n : n \geq 1\}$$

and $s_0(\mathbf{A}) = 1$. Also, $s^\infty(\mathbf{A}) = s(\mathbf{A}) = 0$. Since $\mathbf{X}$ is a Hilbert space, $s_1(\mathbf{A}) = \omega_1(\mathbf{T})$ [51, Theorem 1.4]. From (3.12) it follows that $\zeta_1(\mathbf{T}) = \omega_1(\mathbf{T})$. Hence, $s(\mathbf{A}) = s^\infty(\mathbf{A}) = 0 < s_1^\infty(\mathbf{A}) = s_1(\mathbf{A}) = \zeta_1(\mathbf{T}) = \omega_1(\mathbf{T}) = \frac{1}{2} < s_0(\mathbf{A}) = \zeta(\mathbf{T}) = \omega(\mathbf{T}) = 1$. □

3.3 Essential holomorphy

Thieme [48, Definition 2.6] has introduced *essentially norm-continuous* and *essentially norm-measurable* C_0-semigroups, generalising both eventually norm-continuous semigroups and essentially compact semigroups. In [11, Definition 2.1] these definitions have been modified to make them applicable to the more general situation of exponentially bounded families $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$. Continuing in the same spirit, we introduce the concept of *essential holomorphy.* In the context of C_0-semigroups, this may be thought of as a generalisation of analytic C_0-semigroups.

Definition 3.3.1. Let $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ be exponentially bounded and strongly continuous and let $\beta > 0$. $\mathbf{T}$ is said to be *essentially holomorphic (of type β)* if for each α such that $0 < \alpha < \beta$ there exist an exponentially bounded, holomorphic function $\mathbf{T}_1 : \Sigma_\theta \to \mathcal{L}(\mathbf{X})$ for some $\theta > 0$ with $\omega_0(\mathbf{T} - \mathbf{T}_1) \leq \omega_0(\mathbf{T}) - \alpha$.

As is evident from the definition, there is a very close relation between essential holomorphy of the function $\mathbf{T}$ and its non-analytic growth bound $\zeta(\mathbf{T})$. Precisely, we have

Theorem 3.3.2. *Let $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ be exponentially bounded and strongly continuous. Then the following are equivalent:*

1. *$\mathbf{T}$ is essentially holomorphic of type β;*

2. *$\zeta(\mathbf{T}) \leq \omega_0(\mathbf{T}) - \beta$.*

By analogy with the definition of an *asymptotically norm-continuous C_0-semigroup* we can define a C_0-semigroup $\mathbf{T}$ to be *asymptotically holomorphic* if $\zeta(\mathbf{T}) < \omega_0(\mathbf{T})$. It is immediate that $\mathbf{T}$ is essentially holomorphic if and only if it is asymptotically holomorphic.

We recall here that a C_0-semigroup $\mathbf{T}$ is called essentially norm-measurable (respectively, essentially norm-continuous) if there exists a decomposition of $\mathbf{T}$ as in Definition 3.3.1 with $\mathbf{T}_1$ being norm-measurable (respectively, norm-continuous) instead of holomorphic and exponentially bounded in a sector and the corresponding condition on $\omega_0(\mathbf{T} - \mathbf{T}_1)$ is satisfied. It is obvious that

$$\text{essential holomorphy} \;\Rightarrow\; \text{essential norm-continuity} \;\Rightarrow\; \text{essential norm-measurability}. \tag{3.25}$$

These concepts are equivalent if the underlying space is a Hilbert space. That essential norm-measurability $\Rightarrow$ essential norm-continuity for Hilbert spaces has been shown in [10, Theorem 3.4.3] and essential norm-continuity $\Rightarrow$ essential holomorphy follows from the fact that $\delta(\mathbf{T}) = \zeta(\mathbf{T})$ for C_0-semigroups on Hilbert spaces. We do not know whether the converse implications in (3.25) hold in general.

Example 3.3.3. Let $\mathbf{T}_1$ be a norm-continuous semigroup on $\mathbf{X}$ with generator $\mathbf{A}_1$ and $\mathbf{T}_2$ be a C_0 -semigroup on a Banach space $\mathbf{Y}$ with generator $\mathbf{A}_2$ satisfying $\omega_0(\mathbf{T}_1) > \omega_0(\mathbf{T}_2) > -\infty$. Then the C_0-semigroup $\mathbf{U} = \mathbf{T}_1 \oplus \mathbf{T}_2$ defined on $\mathbf{X} \bigoplus \mathbf{Y}$ has growth bound $\omega_0(\mathbf{U}) = \omega_0(\mathbf{T}_1)$. Further, since $\zeta(\mathbf{T}_1) = -\infty, \zeta(\mathbf{U}) = \zeta(\mathbf{T}_2) \leq \omega_0(\mathbf{T}_2)$. Therefore, we have $-\infty < \zeta(\mathbf{U}) < \omega_0(\mathbf{U})$. Thus $\mathbf{U}$ is essentially holomorphic.

3.4 A comparison of the critical growth bound and the non-analytic growth bound

For an exponentially bounded, strongly continuous function $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ the non-analytic growth bound $\zeta(\mathbf{T})$ may be thought of as the growth bound of $\mathbf{T}$ modulo operator-valued functions which are exponentially bounded and holomorphic on some sector, whereas the critical growth bound $\delta(\mathbf{T})$ is the growth bound of $\mathbf{T}$ modulo exponentially bounded and norm continuous operator-valued functions defined on $\mathbb{R}_+$. It is clear from the definitions in (2.13) and (3.8) that

$$\delta(\mathbf{T}) \leq \zeta(\mathbf{T}) \leq \omega_0(\mathbf{T}).$$

None of the known characterisations of the critical growth bound extend in a meaningful way to general vector-valued functions $f : \mathbb{R}_+ \to \mathbf{X}$, unlike in the case of the non-analytic growth bound. In particular, any continuous function $f : \mathbb{R}_+ \to \mathbf{X}$ would have its critical growth bound as $-\infty$ while the various abscissas of holomorphy of $\hat{f}$ may or may not be

finite. As an example, consider the function $f : \mathbb{R}_+ \to \mathbb{C}$, given by

$$f(t) = \sum_{n=1}^{\infty} \frac{e^{(\alpha+in)t}}{n^2}, \quad \alpha < 0.$$

Then f is continuous so that its critical growth bound is $-\infty$, but $\mathrm{hol}_0^\infty(\hat{f}) = \mathrm{hol}(\hat{f}) = \alpha = \omega_0(f)$. Thus, the critical growth bound does not give any information about the abscissas of holomorphy. Such a situation cannot arise in the case of the non-analytic growth bound. In fact, Proposition 3.1.8, establishes definite relations between $\mathrm{hol}_0^\infty(\hat{f}), \mathrm{hol}(\hat{f})$ and $\zeta(f)$. For the example above, it follows that $\zeta(f) = \alpha$.

Even for general operator-valued functions $\mathbf{T} : \mathbb{R}_+ \to \mathcal{L}(\mathbf{X})$ the critical growth bound $\delta(\mathbf{T})$ may not determine the value of $\mathrm{hol}_0^\infty(\hat{\mathbf{T}})$. In fact, the following corollary of Theorem 3.1.11 and Theorem 3.2.9 shows that for certain once integrated semigroups $\mathbf{S}$ defined on a Hilbert space, there may be no relation between the critical growth bound and $\mathrm{hol}_0^\infty(\hat{\mathbf{S}})$.

Corollary 3.4.1. *Let $\mathbf{T}$ be a C_0-semigroup defined on a Hilbert space $\mathbf{X}$ with generator $\mathbf{A}$ satisfying $s^\infty(\mathbf{A}) > -\infty$. Let $\mathbf{S}$ denote the once integrated semigroup obtained from $\mathbf{T}$, that is $\mathbf{S}(t)x = \int_0^t \mathbf{T}(s)x\, ds \quad (x \in \mathbf{X})$. Then $\delta(\mathbf{S}) = -\infty$ while $-\infty < \mathrm{hol}_0^\infty(\hat{\mathbf{S}}) = \zeta(\mathbf{S})$.*

Proof. We first note that such a semigroup exists. Suppose if possible, $\delta(\mathbf{S}) = \zeta(\mathbf{S})$. Now $\delta(\mathbf{S})$ is equal to $-\infty$ since $\mathbf{S}$ is norm continuous. Further, by Theorem 3.1.11 and Theorem 3.2.9,

$$\zeta(\mathbf{S}) = \zeta_1(\mathbf{T}) = s_1^\infty(\mathbf{A}) = \mathrm{hol}^\infty{}_1(\hat{\mathbf{T}}) = \mathrm{hol}_0^\infty(\hat{\mathbf{S}}).$$

Moreover, $s_1^\infty(\mathbf{A}) \geq s^\infty(\mathbf{A}) > -\infty$. Thus $\zeta(\mathbf{S}) > -\infty$. □

As mentioned before, the equality $s_0^\infty(\mathbf{A}) = \delta(\mathbf{T}) = \zeta(\mathbf{T})$ holds for C_0-semigroups on Hilbert spaces. We do not know whether the equality $\delta(\mathbf{T}) = \zeta(\mathbf{T})$ holds for all C_0-semigroups defined on general Banach spaces. However, it is known that the two growth bounds and the spectral bound coincide for C_0-semigroups that fall under the categories mentioned below. We remark that in all these cases, actually the equality

$$s_0^\infty(\mathbf{A}) = \zeta(\mathbf{T}) = -\infty$$

holds. These classes are:

- ([10, Theorem 6.6.4]) Eventually differentiable semigroups.
- ([10, Theorem 6.6.3]) C_0-semigroups with an L^p-resolvent for some $p \in (1, \infty)$. A C_0-semigroup $\mathbf{T}$ with generator $\mathbf{A}$ is said to have an L^p-resolvent (see [7]) if there exists $w \in \mathbb{R}$ and $b \geq 0$ such that $Q_{w,b} \subset \rho(\mathbf{A})$ and

$$\int_{|s|\geq b} \|\mathbf{R}(w + is, \mathbf{A})\|^p\, ds < \infty.$$

- ([10, Proposition 4.5.7]) Eventually compact semigroups.

A description of the critical growth bound $\delta(\mathbf{T})$ of a C_0-semigroup $\mathbf{T}$ in terms of the spectral radius of certain operators has been obtained in [6] using Banach algebra techniques. It is similar to the description of $\zeta(\mathbf{T})$ given by (*4*) of Proposition 3.1.4, stating that given a C_0-semigroup $\mathbf{T}$, and $\omega \in \mathbb{R}$,

$$\delta(\mathbf{T}) < \omega \iff \text{for each } \alpha < \omega,\ b \geq 0 \text{ satisfying } Q_{\alpha,b} \subset \rho(\mathbf{A}),$$
$$\omega_0\left(r\left(\mathbf{T}(\cdot) - \mathbf{T}_{\alpha,b}(\cdot)\right)\right) < \omega.$$

Here $r(B)$ denotes the spectral radius of a bounded operator B on $\mathbf{X}$. From the above description of the critical growth bound and Proposition 3.1.4 it follows that $\delta(\mathbf{T}) = \zeta(\mathbf{T})$ for any C_0-semigroup $\mathbf{T}$ for which the spectral radius and the norm of the operators $t \mapsto \mathbf{T}(t) - \mathbf{T}_{\alpha,b}$ coincide whenever $\alpha \in \mathbb{R}, b \geq 0$ satisfy $Q_{\alpha,b} \subset \rho(\mathbf{A})$.

In particular, multiplier semigroups satisfy this condition. Therefore, we have

Theorem 3.4.2. *Let $\mathbf{T}(t)$ be a C_0-semigroup on $\mathbf{X} = C_0(\Omega)$, where Ω is a locally compact space, given by*

$$\mathbf{T}(t)(f)(s) = e^{q(s)} f(s) \quad (f \in \mathbf{X}, s \in \Omega),$$

where $q : \Omega \longrightarrow \mathbb{C}$ is a continuous function satisfying

$$\sup_{s \in \Omega} \operatorname{Re} q(s) < \infty.$$

Then $\zeta(\mathbf{T}) = \delta(\mathbf{T}) = s_0^{\infty}(\mathbf{A}) = s^{\infty}(\mathbf{A})$.

Using the next result, we can add another category of C_0-semigroups to the class for which the critical and non-analytic growth bounds coincide:

Theorem 3.4.3. *([10, Theorem 6.6.1]) Let $\mathbf{T}$ be a C_0-semigroup with generator $\mathbf{A}$. If $\delta(\mathbf{T}) < a < \omega_0(\mathbf{T})$ and $a + i\mathbb{R} \subset \rho(\mathbf{A})$, then $\zeta(\mathbf{T}) < a$.*

Theorem 3.4.4. *If $\mathbf{T}$ is a C_0-semigroup such that its generator $\mathbf{A}$ has compact resolvents then $\delta(\mathbf{T}) = \zeta(\mathbf{T})$. Also, if $\mathbf{B}$ is a bounded operator and $\mathbf{S}$ is the C_0-semigroup generated by $\mathbf{A} + \mathbf{B}$ then $\delta(\mathbf{S}) = \zeta(\mathbf{S})$.*

Proof. If $\delta(\mathbf{T}) = \omega_0(\mathbf{T})$, then the result is true anyway. So we suppose that $\delta(\mathbf{T}) < \omega_0(\mathbf{T})$. Let $\mathbf{A}$ denote the generator of the semigroup. Then, if $\mathbf{R}(\lambda, \mathbf{A})$ is compact for $\lambda \in \rho(\mathbf{A})$ we have that the spectrum of $\mathbf{R}(\lambda, \mathbf{A})$ consists of a countable number of eigenvalues with zero as the only possible limit point. Therefore, $\sigma(\mathbf{A})$ consists of only countably many eigenvalues, with infinity as the only possible limit point. So

$$\sigma(\mathbf{A}) \bigcap \{\lambda \in \mathbb{C} : \delta(\mathbf{T}) < \operatorname{Re} \lambda < \omega_0(\mathbf{T})\}$$

is countable. From Theorem 3.4.3 it follows that $\delta(\mathbf{T}) \geq \zeta(\mathbf{T})$. The resolvent of $\mathbf{A}+\mathbf{B}$ is compact if that of $\mathbf{A}$ is. Therefore, the second claim follows from the first part. □

The equality $s_0^\infty(\mathbf{A}) = \zeta(\mathbf{T})$ for a C_0-semigroup on a Hilbert space depends on the L^2-integrability of the maps $s \mapsto \mathbf{R}(\omega + is, \mathbf{A})x, \omega > \omega_0(\mathbf{T}), x \in \mathbf{X}$. Therefore, the following generalisation holds:

Theorem 3.4.5. *Let $\mathbf{A}$ be the generator of a C_0-semigroup $\mathbf{T}$ on $\mathbf{X}$ and suppose there exists $\omega > \omega_0(\mathbf{T})$ such that for each $x \in \mathbf{X}$ and each $x^* \in \mathbf{X}^*$, the following conditions are satisfied:*

$$\int_{-\infty}^{\infty} \|\mathbf{R}(\omega + is, \mathbf{A})x\|^2 \, ds < \infty;$$
$$\int_{-\infty}^{\infty} \|\mathbf{R}(\omega + is, \mathbf{A})^* x^*\|^2 \, ds < \infty.$$

Then $s_0^\infty(\mathbf{A}) = \delta(\mathbf{T}) = \zeta(\mathbf{T})$.

Proof. The proof works on exactly the same lines as [11, Lemma 4.3]. □

It is clear that for any Laplace transformable function f, the non-analytic abscissa of convergence $\zeta_1(f)$ relates to the non-analytic growth bound $\zeta(f)$ in the same way as the abscissa of convergence, $\mathrm{abs}(f)$ relates to the exponential growth bound $\omega_0(f)$. This is true, in particular for a C_0-semigroup $\mathbf{T}$. One would like to be able to define an abscissa $\delta_1(\mathbf{T})$ corresponding to the bound $\delta(\mathbf{T})$ in a manner similar to $\zeta_1(\mathbf{T})$ and $\zeta(\mathbf{T})$. However, such an attempt is unsuccessful. In fact, if we make the definition:

$$\delta_1(\mathbf{T}) = \inf\{\omega \in \mathbb{R} : \mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2, \text{ where } \mathbf{T}_1 \text{ is norm continuous and } \mathrm{abs}(\mathbf{T}_2) < \omega\},$$

then we have that $\delta_1(\mathbf{T}) = -\infty$ for every C_0-semigroup $\mathbf{T}$ on $\mathbf{X}$. To see this, let $\mathbf{T}$ be a C_0-semigroup and $\mathbf{S}$ its once integrated semigroup. Since $\mathbf{S}$ is norm continuous, $\delta(\mathbf{S}) = -\infty$. So for $n \in \mathbb{N}$, there exists $\mathbf{S}_1, \mathbf{S}_2 : \mathbb{R}_+ \longrightarrow \mathbf{X}$ such that $\mathbf{S}_1$ is norm differentiable and $\omega_0(\mathbf{S}_2) < -n$. Consequently, $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$ where $\frac{d}{dt}\mathbf{S}_i(t)(x) = \mathbf{T}_i(t)(x), i = 1, 2$. This means that $\delta_1(\mathbf{T}) < -n$ and since $n \in \mathbb{N}$ was arbitrary, it follows that $\delta_1(\mathbf{T}) = -\infty$.

Chapter 4

Fractional growth bounds

4.1 Convolutions and regularisations

In this section, we give some estimates for the non-analytic growth bound when a function f is regularised by convolution. If $f : \mathbb{R}_+ \to \mathbf{X}$ is exponentially bounded and measurable and $\phi : \mathbb{R}_+ \to \mathbb{C}$ is in $\mathrm{L}^1_{loc}(\mathbb{R}_+)$, then the convolution $\phi * f$ is Laplace transformable with

$$\begin{aligned} \operatorname{abs}(\phi * f) &\leq \max(\operatorname{abs}(\phi), \operatorname{abs}(\|f\|)); \\ \widehat{\phi * f}(\lambda) &= \hat{\phi}(\lambda)\hat{f}(\lambda), \end{aligned}$$

whenever $\operatorname{Re}\lambda > \max(\operatorname{abs}(\phi), \operatorname{abs}(\|f\|))$ [2, Proposition 1.6.4]. Therefore,

$$\operatorname{hol}_0^\infty(\widehat{\phi * f}) \leq \max\left(\operatorname{hol}_0^\infty(\hat{\phi}), \operatorname{hol}_0^\infty(\hat{f}\ )\right).$$

Also, trivial estimates show that

$$\omega_0(\phi * f) \leq \max(\omega_0(\phi), \omega_0(f)). \tag{4.1}$$

We shall obtain the corresponding estimate for $\zeta(\phi * f)$ and give some sharper estimates when ϕ is more regular. For this we require a basic estimate, derived in the next theorem by closely following the strategy of [8, Corollary 2.2], [2, Proposition 4.4.11].

Theorem 4.1.1. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be an exponentially bounded and measurable function such that $\hat{f}$ has a bounded, holomorphic extension to $Q^\circ_{0,b}$ for some $b \geq 0$. Suppose $\phi : \mathbb{R} \longrightarrow \mathbb{C}$ is a measurable function and the following conditions hold:*

1. $\displaystyle\int_0^\infty e^{\omega t}|\phi(t)|\,dt < \infty$, *for some* $\omega > \max(0, \omega_0(f))$;
2. $\displaystyle\int_{-\infty}^0 |\phi(t)|\,dt < \infty$;
3. $\mathcal{F}\phi \in \mathrm{L}^1(\mathbb{R})$.

Let $\Gamma_{0,b}$ be a path joining $-ib$ to ib in $\{\lambda \in D(\hat{f}\ ) : 0 \leq \operatorname{Re}\lambda \leq \omega\}$, and set

$$\tilde{\phi}(-\lambda) = \int_{-\infty}^{\infty} e^{\lambda t}\phi(t)\,dt \quad (0 \leq \operatorname{Re}\lambda \leq \omega).$$

Then

$$\left\| \int_0^{\infty} \phi(t)f(t)\,dt - \frac{1}{2\pi i}\int_{\Gamma_{0,b}} \hat{f}(\lambda)\tilde{\phi}(-\lambda)\,d\lambda \right\| \ \leq\ C\|\mathcal{F}\phi\|_1, \tag{4.2}$$

where $C = \frac{1}{2\pi}\sup\left\{\|\hat{f}(\lambda)\| : \lambda \in Q^{\circ}_{0,b}\right\}$.

Proof. We first assume that ϕ has compact support and $\mathcal{F}\phi \in \mathrm{L}^1(\mathbb{R})$. Suppose that $\omega > \max(0, \omega_0(f))$ satisfies condition *(1)* of the hypothesis and $0 < \alpha < \omega$ is fixed. Now $t \mapsto e^{-\omega t}f(t) \in \mathrm{L}^1(\mathbb{R}_+, \mathbf{X})$ and its Fourier transform is $s \mapsto \hat{f}\ (\omega + is)$. Let $\psi \in C_c^{\infty}(\mathbb{R})$ with

$$\psi(t) = e^{(\omega-\alpha)t} \quad (t \in \operatorname{supp}\phi).$$

Then $(\bar{\mathcal{F}}\phi) * (\bar{\mathcal{F}}\psi) \in \mathrm{L}^1(\mathbb{R})$ and

$$\begin{aligned} \mathcal{F}(\bar{\mathcal{F}}\psi * \bar{\mathcal{F}}\phi)(t) &= 4\pi^2\psi(t)\phi(t) \\ &= 4\pi^2 e^{(\omega-\alpha)t}\phi(t) \end{aligned}$$

for all $t \in \mathbb{R}$. Therefore, $\mathcal{F}(\bar{\mathcal{F}}\psi * \bar{\mathcal{F}}\phi) \in \mathrm{L}^1(\mathbb{R})$ and

$$\begin{aligned} (\bar{\mathcal{F}}\psi * \bar{\mathcal{F}}\phi)(s) &= \mathcal{F}^{-1}(4\pi^2 e^{(\omega-\alpha)\cdot}\phi)(s) \\ &= 2\pi\int_{-\infty}^{\infty} e^{(is+\omega-\alpha)t}\phi(t)\,dt \\ &= 2\pi\tilde{\phi}(-\omega+\alpha-is) \end{aligned}$$

for all $s \in \mathbb{R}$. Using [2, Theorem 1.8.1] we therefore obtain

$$\begin{aligned} \int_0^{\infty} e^{-\alpha t}f(t)\phi(t)\,dt &= \int_0^{\infty} e^{-\omega t}f(t)e^{(\omega-\alpha)t}\phi(t)\,dt \\ &= \frac{1}{4\pi^2}\int_0^{\infty} e^{-\omega t}f(t)\mathcal{F}(\bar{\mathcal{F}}\psi * \bar{\mathcal{F}}\phi)(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\ (\omega+is)\tilde{\phi}(\alpha-(\omega+is))\,ds. \end{aligned}$$

Now we consider the integral

$$\int_{\Gamma_1} \hat{f}\ (\lambda)\tilde{\phi}(\alpha-\lambda)\,d\lambda,$$

where Γ_1 is the rectangle with vertices $\alpha - ir, \omega - ir, \omega - ib, \alpha - ib$ and $r > b$. We have

$$\int_{\omega-ir}^{\omega-ib} \hat{f}(\lambda)\tilde{\phi}(\alpha-\lambda)\,d\lambda = \left(\int_{\omega-ir}^{\alpha-ir} + \int_{\alpha-ir}^{\alpha-ib} + \int_{\alpha-ib}^{\omega-ib}\right)(\hat{f}(\lambda)\tilde{\phi}(\alpha-\lambda))\,d\lambda.$$

Consider the first integral on the right above. It is given by

$$\int_{\omega}^{\alpha} \hat{f}(\eta - ir)\tilde{\phi}(\alpha - \eta + ir)\, d\eta.$$

For $\alpha < \eta < \omega$,

$$\begin{aligned}\tilde{\phi}(\alpha - \eta + ir) &= \int_{-\infty}^{\infty} e^{-(\alpha-\eta)t}\phi(t)e^{-irt}\, dt \\ &\to 0,\end{aligned}$$

as $r \to \infty$ by the Riemann-Lebesgue lemma, since $s \mapsto e^{-(\alpha-\eta)s}\phi(s) \in \mathrm{L}^1(\mathbb{R})$. Moreover,

$$|\tilde{\phi}(\alpha - \eta + ir)| \leq K,$$

whenever $r > b, \alpha < \eta < \omega$, where

$$K = \left(\int_{-\infty}^{0} |\phi(t)|\, dt + \int_{0}^{\infty} e^{\omega t}|\phi(t)|\, dt\right)$$

is a constant independent of η and r. Therefore, by the dominated convergence theorem it follows that

$$\lim_{r\to\infty} \int_{\alpha}^{\omega} \hat{f}(\eta - ir)\tilde{\phi}(\alpha - \eta + ir)\, d\eta = 0.$$

Thus,

$$\int_{\omega-i\infty}^{\omega-ib} \hat{f}\ (\lambda)\tilde{\phi}(\alpha - \lambda) d\,\lambda = \left(\int_{\alpha-i\infty}^{\alpha-ib} + \int_{\alpha-ib}^{\omega-ib}\right) \hat{f}\ (\lambda)\tilde{\phi}(\alpha - \lambda)\, d\lambda.$$

We can similarly deal with the integral over $[\omega + ib, \omega + i\infty)$ to obtain,

$$\begin{aligned}\int_{0}^{\infty} e^{-\alpha t} f(t)\phi(t)\, dt &= \frac{1}{2\pi}\int_{\mathrm{Re}\,\lambda=\omega} \hat{f}\ (\lambda)\tilde{\phi}(\alpha - \lambda)\, d\lambda \\ &= \left(\int_{\alpha-i\infty}^{\alpha-ib} + \int_{\Gamma_{\alpha,b}} + \int_{\alpha+ib}^{\alpha+i\infty}\right) (\hat{f}\ (\lambda)\tilde{\phi}(\alpha - \lambda))\, d\lambda. \qquad (4.3)\end{aligned}$$

Further,

$$\begin{aligned}&\left\|\int_{\alpha-i\infty}^{\alpha-ib} \hat{f}\ (\lambda)\tilde{\phi}(\alpha - \lambda)\, d\lambda\right\| + \left\|\int_{\alpha+ib}^{\alpha+i\infty} \hat{f}\ (\lambda)\tilde{\phi}(\alpha - \lambda)\, d\lambda\right\| \\ &\qquad \leq C\int_{\alpha-i\infty}^{\alpha-ib} |\tilde{\phi}(\alpha - \lambda)|\, d\lambda + C\int_{\alpha+ib}^{\alpha+i\infty} |\tilde{\phi}(\alpha - \lambda)|\, d\lambda \\ &\qquad = C\int_{-\infty}^{-b} |\tilde{\phi}(-is)|\, ds + C\int_{b}^{\infty} |\tilde{\phi}(-is)|\, ds \\ &\qquad \leq C\|\mathcal{F}\phi\|_1. \qquad (4.4)\end{aligned}$$

Therefore, combining (4.3) and (4.4), we obtain,

$$\left\|\int_{0}^{\infty} e^{-\alpha t} f(t)\phi(t)\, dt - \frac{1}{2\pi i}\int_{\Gamma_{\alpha,b}} \hat{f}\ (\lambda)\tilde{\phi}(\alpha - \lambda)\, d\lambda\right\| \leq \frac{C}{2\pi}\|\mathcal{F}\phi\|_1. \qquad (4.5)$$

Then using the Dominated Convergence Theorem we have that

$$\sup_{\lambda\in\Gamma_{0,b}} |\tilde{\phi}(\alpha-\lambda)-\tilde{\phi}(-\lambda)| \longrightarrow 0$$

as $\alpha \downarrow 0$. Taking the limit as $\alpha \downarrow 0$ in (4.5) we obtain

$$\left\| \int_0^\infty f(t)\phi(t)\,dt - \int_{\Gamma_{0,b}} \hat{f}(\lambda)\tilde{\phi}(-\lambda)d\lambda \right\| \leq \frac{C}{2\pi}\|\mathcal{F}\phi\|_1. \tag{4.6}$$

Now consider the case when ϕ is any function in $\mathrm{L}^1(\mathbb{R})$ with $\mathcal{F}\phi \in \mathrm{L}^1(\mathbb{R})$ and there exists an $\omega > \max(0,\omega_0(f))$ such that $t \mapsto e^{\omega t}\phi(t) \in \mathrm{L}^1(\mathbb{R})$. Let $\psi \in C_c^\infty(\mathbb{R})$ be any function satisfying $0 \leq \psi \leq 1$, $\psi(0) = 1$, and $\int_{-\infty}^{\infty} \psi(s)\,ds = 1$. Let $\psi_n(t) = \psi(\frac{t}{n})$ $(t \in \mathbb{R})$ and $\phi_n(t) = \phi(t)\psi_n(t)$. Then $(2\pi)^{-1}(\mathcal{F}\psi_n)(s) = (2\pi)^{-1}n(\mathcal{F}\psi)(ns)$, which forms a mollifier (see Subsection 2.2.3). Therefore, $\mathcal{F}\phi_n = (2\pi)^{-1}\mathcal{F}\phi * \mathcal{F}\psi_n \longrightarrow \mathcal{F}\phi$ in $\mathrm{L}^1(\mathbb{R})$ as $n \to \infty$. Applying (4.6) to the functions ϕ_n and noting that

$$\| \int_{\Gamma_{0,b}} \hat{f}(\lambda)\tilde{\phi}_n(-\lambda)\,d\lambda - \int_{\Gamma_{0,b}} \hat{f}(\lambda)\tilde{\phi}(-\lambda)\,d\lambda\| \longrightarrow 0$$

we get (4.6) for a general ϕ. □

Remark 4.1.2. We can make the following observations concerning the proof of the above theorem.

1. The conditions *(1)* and *(2)* together, in the hypotheses of Theorem 4.1.1 may be reformulated as $\phi \in \mathrm{L}^1(\mathbb{R})$ and $e^{\omega\cdot}\phi \in \mathrm{L}^1(\mathbb{R})$ for some $\omega > \max(0,\omega_0(f))$.

2. The above theorem also remains true if we replace $\|\mathcal{F}\phi\|_1$ by $\int_{|s|>b} |\mathcal{F}\phi(s)|\,ds$ in (4.2). This is due to the fact that (4.4) remains valid if $\|\mathcal{F}\phi\|_1$ is replaced by $\int_{|s|>b} |\mathcal{F}\phi(s)|\,ds$.

□

We now state our basic result in a form which allows regularisations by functions defined on $\mathbb{R}$.

Theorem 4.1.3. *Let $f : \mathbb{R} \longrightarrow \mathbf{X}$ be an exponentially bounded measurable function. Let $\phi : \mathbb{R} \longrightarrow \mathbf{X}$ be locally integrable, and suppose that there exist $C > 0$, $\omega > \mathrm{hol}_0^\infty(\hat{f}\,)$, $\gamma > \max(\omega,\omega_0(f))$ and $\alpha \in (0,1]$ such that*

1. $\displaystyle\int_0^\infty |\phi(s)|e^{-\omega s}\,ds < \infty;$

2. $\displaystyle\int_{-\infty}^0 |\phi(s)|e^{-\gamma s}\,ds < \infty;$

3. $\displaystyle\left|\int_{-\infty}^\infty \phi(s)e^{-(\omega+i\eta)s}\,ds\right| \leq \frac{C}{|\eta|^\alpha} \quad (\eta \in \mathbb{R});$

4. $\displaystyle\int_{-\infty}^{\infty} |\phi(s) - \phi(s-h)| e^{-\gamma s}\, ds \le Ch^{\alpha} \quad (0 < h < 1).$

Then $\phi * f$ *is defined on* $\mathbb{R}_+$ *and*

$$\zeta(\phi * f) \le (1-\alpha)\gamma + \alpha\omega.$$

Proof. The fact that $(\phi * f)(t)$ exists follows immediately from the assumption (2). First we assume that $\omega = 0$ and $\mathrm{hol}_0^{\infty}(\hat{f}\,) < 0$ so that $\hat{f}$ has a bounded holomorphic extension to $Q_{0,b}^{\circ}$ for some $b \ge 1$. Let $t \ge 0$ and $\delta \in (0,1]$. Consider the function $\phi_\delta : \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\phi_\delta(s) = \frac{1}{\delta}(\phi * \chi_{(0,\delta)})(t-s) = \frac{1}{\delta}\int_{t-s-\delta}^{t-s} \phi(r)\, dr.$$

Then

$$\int_0^{\infty} \phi_\delta(s) f(s)\, ds = \frac{1}{\delta}(\phi * \chi_{(0,\delta)} * f)(t)$$

and by an application of Fubini's theorem,

$$\begin{aligned}
\tilde{\phi_\delta}(-\lambda) &= \frac{1}{\delta}\int_{-\infty}^{\infty} e^{\lambda s}\int_{t-s-\delta}^{t-s} \phi(r)\, dr\, ds \\
&= \frac{1}{\delta\lambda}\int_{-\infty}^{\infty} (1 - e^{-\lambda\delta}) e^{\lambda(t-r)}\phi(r)\, dr \\
&= e^{\lambda t}\left(\frac{1 - e^{-\lambda\delta}}{\lambda\delta}\right)\tilde{\phi}(\lambda).
\end{aligned}$$

In particular,

$$\begin{aligned}
(\mathcal{F}\phi_\delta)(s) &= \tilde{\phi_\delta}(is) \\
&= e^{-ist} e^{\frac{i\delta s}{2}}\left(\frac{\sin(s\delta/2)}{\delta s/2}\right)(\mathcal{F}\phi)(-s).
\end{aligned}$$

If $\alpha \in (0,1)$, we therefore have from assumption (3) that

$$\begin{aligned}
\|\mathcal{F}\phi_\delta\|_1 &\le \int_{-\infty}^{\infty}\left|\frac{\sin(\delta s/2)}{\delta s/2}\right|\frac{C}{s^{\alpha}}\, ds \\
&= 2C\left(\frac{2}{\delta}\right)^{1-\alpha}\int_0^{\infty}\frac{|\sin s|}{s^{1+\alpha}}\, ds \\
&= \frac{C_1}{\delta^{1-\alpha}},
\end{aligned}$$

where C_1 is a constant depending on C and α only. If $\alpha = 1$, we have

$$\begin{aligned}
\int_{|s|\ge b} |(\mathcal{F}\phi_\delta)(s)|\, ds &\le 2\left(\int_1^{1/\delta}\frac{C}{s}\, ds + \int_{1/\delta}^{\infty}\frac{2C}{\delta s^2}\, ds\right) \\
&\le C_1(1 + |\log\delta|).
\end{aligned}$$

Consequently, for any $\alpha \in (0,1]$, Theorem 4.1.1 and Remark 4.1.2 give

$$\left\| \frac{1}{\delta}(\phi * \chi_{(0,\delta)} * f)(t) - \frac{1}{2\pi i}\int_{\Gamma_{0,b}} e^{\lambda t}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\tilde{\phi}(\lambda)\hat{f}(\lambda)\,d\lambda \right\| \;\le\; C_2\frac{1+|\log\delta|}{\delta^{1-\alpha}}, \quad (4.7)$$

for some constant C_2. Since $\gamma > \omega_0(f)$, there exists M such that $\|f(t)\| \le Me^{\gamma t}$ $(t \ge 0)$. Therefore,

$$\begin{aligned}
&\left\| \frac{1}{\delta}(\phi * \chi_{(0,\delta)} * f)(t) - (\phi * f)(t) \right\| \\
&= \left\| \int_{-\infty}^{t} f(t-s)\left(\frac{1}{\delta}\int_{s-\delta}^{s}\phi(r)\,dr - \phi(s)\right)ds \right\| \\
&\le \int_{-\infty}^{t} Me^{\gamma(t-s)}\frac{1}{\delta}\int_0^{\delta}|\phi(s-r)-\phi(s)|\,dr\,ds \\
&\le \frac{M}{\delta}\int_0^{\delta} e^{\gamma t}\left(\int_{-\infty}^{\infty} e^{-\gamma s}|\phi(s-r)-\phi(s)|\,ds\right)dr \\
&\le Me^{\gamma t}\frac{1}{\delta}\int_0^{\delta} Cr^{\alpha}\,dr \\
&= C_3 e^{\gamma t}\delta^{\alpha}, \qquad (4.8)
\end{aligned}$$

where C_3 is some constant and we have used assumption *(4)* to get the penultimate inequality. Further,

$$\begin{aligned}
&\left\| \int_{\Gamma_{0,b}} e^{\lambda t}\tilde{\phi}(\lambda)\hat{f}(\lambda)\,d\lambda - \int_{\Gamma_{0,b}} e^{\lambda t}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\tilde{\phi}(\lambda)\hat{f}(\lambda)\,d\lambda \right\| \\
&= \left\| \int_{\Gamma_{0,b}} e^{\lambda t}\left(\frac{e^{-\lambda\delta}-(1-\lambda\delta)}{\lambda\delta}\right)\tilde{\phi}(\lambda)\hat{f}(\lambda)\,d\lambda \right\| \\
&\le C_4 e^{\gamma t}\delta, \qquad (4.9)
\end{aligned}$$

where C_4 is some constant, since we may choose $\Gamma_{0,b}$ and c so that $\operatorname{Re}\lambda \le \gamma$ and

$$|e^{-\lambda\delta} - (1-\lambda\delta)| \le c|\lambda|\delta^2$$

for all $\lambda \in \Gamma_{0,b}$. Combining the estimates (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned}
&\left\| (\phi * f)(t) - \frac{1}{2\pi i}\int_{\Gamma_{0,b}} e^{\lambda t}\tilde{\phi}(\lambda)\hat{f}(\lambda)\,d\lambda \right\| \\
&\le \left\| \frac{1}{\delta}(\phi * \chi_{(0,\delta)} * f)(t) - (\phi * f)(t) \right\| \\
&\quad + \left\| \frac{1}{\delta}(\phi * \chi_{(0,\delta)} * f)(t) - \frac{1}{2\pi i}\int_{\Gamma_{0,b}} e^{\lambda t}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\tilde{\phi}(\lambda)\hat{f}(\lambda)\,d\lambda \right\| \\
&\quad + \frac{1}{2\pi}\left\| \int_{\Gamma_{0,b}} e^{\lambda t}\tilde{\phi}(\lambda)\hat{f}(\lambda)\,d\lambda - \int_{\Gamma_{0,b}} e^{\lambda t}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\tilde{\phi}(\lambda)\hat{f}(\lambda)\,d\lambda \right\| \\
&\le C_5\left(\delta^{\alpha}e^{\gamma t} + \frac{1+|\log\delta|}{\delta^{1-\alpha}} + \delta e^{\gamma t}\right)
\end{aligned}$$

for each $\delta \in (0,1]$ and $t \geq 0$. Choosing $\delta = e^{-\gamma t}$ in the above inequality gives

$$\left\| (\phi * f)(t) - \frac{1}{2\pi i} \int_{\Gamma_{0,b}} e^{\lambda t} \tilde{\phi}(\lambda) \hat{f}(\lambda)\, d\lambda \right\| \leq C_6(1+\gamma t) e^{(1-\alpha)\gamma t}.$$

Since the function $z \mapsto \displaystyle\int_{\Gamma_{0,b}} e^{\lambda z} \tilde{\phi}(\lambda) \hat{f}(\lambda)\, d\lambda$ is entire and exponentially bounded on $\mathbb{C}$ we have

$$\zeta(\phi * f) \leq (1-\alpha)\gamma.$$

For the general case, that is, when $\mathrm{hol}_0^\infty(\hat{f}\,) > \omega$ consider the functions

$$f_\omega(t) = e^{-\omega t} f(t), \quad \phi_\omega(t) = e^{-\omega t}\phi(t) \quad (t \geq 0).$$

Then f_ω and ϕ_ω satisfy the assumptions of the special case above, with γ replaced by $\gamma - \omega$. Hence,

$$\zeta(\phi_\omega * f_\omega) \leq (1-\alpha)(\gamma - \omega).$$

Since $(\phi_\omega * f_\omega)(t) = e^{-\omega t}(\phi * f)(t)$, we have using (*3*) of Proposition 3.1.8,

$$\zeta(\phi * f) \leq (1-\alpha)\gamma + \alpha\omega.$$

□

If $f : \mathbb{R}_+ \to \mathbf{X}$ is regularised by a complex-valued function which is supported on the half-line and satisfies certain conditions, then we have

Corollary 4.1.4. *Suppose that $f : \mathbb{R}_+ \to \mathbf{X}$ is exponentially bounded and measurable. Let $\phi : \mathbb{R}_+ \to \mathbb{C}$ be locally integrable, and suppose that there exists $\alpha \in (0,1]$ such that*

1. $\displaystyle\int_0^\infty |\phi(s)| e^{-\omega s}\, ds < \infty$ *for all* $\omega > \mathrm{hol}_0^\infty(\hat{f}\,)$;
2. *For each* $\omega > \mathrm{hol}_0^\infty(\hat{f}\,)$, *there exists* C_ω *such that*

$$\int_0^\infty |\phi(s) - \phi(s-h)| e^{-\omega s}\, ds \leq C_\omega h^\alpha \quad (0 < h < 1).$$

Then $\phi * f$ is defined on $\mathbb{R}_+$ and

$$\zeta(\phi * f) \leq (1-\alpha)\omega_0(f) + \alpha\, \mathrm{hol}_0^\infty(\hat{f}\,).$$

Proof. Let $\omega > \mathrm{hol}_0^\infty(\hat{f}\,)$ and $\gamma > \max(\omega, \omega_0(f))$. Given μ with $|\mu| > \pi$, let $h = \pi/|\mu|$. Then, if $\mu > 0$, we may write

$$\begin{aligned} \int_0^\infty \phi(s) e^{-(\omega + i\mu)s}\, ds &= -\int_0^\infty \phi(s) e^{-\omega s - i\mu(s+\pi/\mu)}\, ds \\ &= -\int_0^\infty \phi(s-h) e^{-\omega(s-h) - i\mu s}\, ds. \end{aligned}$$

If $\mu < 0$, we get a similar expression. Therefore,

$$\begin{aligned}
&\left|\int_0^\infty \phi(s)e^{-(\omega+i\mu)s}\,ds\right| \\
&= \frac{1}{2}\left|\int_0^\infty \big((\phi(s)-\phi(s-h))e^{-\omega s} + (e^{-\omega h}-1)\phi(s-h)e^{-\omega(s-h)}\big)e^{-i\mu s}\,ds\right| \\
&\leq \frac{1}{2}\left(C_\omega h^\alpha + \left|e^{-\omega h}-1\right|\int_0^\infty |\phi(s)|e^{-\omega s}\,ds\right) \\
&\leq \frac{1}{2}(C_\omega h^\alpha + C'_\omega h) \\
&\leq \frac{C}{|\mu|^\alpha},
\end{aligned}$$

where C is a constant depending on ω. Thus assumption (*3*) of Theorem 4.1.3 is satisfied. That the other three assumptions of Theorem 4.1.3 are satisfied is immediate from our hypotheses here. Therefore,

$$\zeta(\phi * f) \leq (1-\alpha)\gamma + \alpha\omega.$$

Letting $\omega \downarrow \operatorname{hol}_0^\infty(\hat{f}\,)$ and $\gamma \downarrow \omega_0(f)$ we obtain the required result. □

Corollary 4.1.5. *Suppose that $f : \mathbb{R}_+ \to \mathbf{X}$ is exponentially bounded and measurable. Let $\phi : \mathbb{R}_+ \to \mathbb{C}$ be locally integrable, and suppose that there exist $C > 0, \beta > \omega_0(f)$ and $\alpha \in (0,1]$ such that*

1. $\displaystyle\int_0^\infty |\phi(s)|e^{\beta s}\,ds < \infty$;
2. $\displaystyle\int_{-h}^\infty |\phi(s)-\phi(s+h)|e^{\beta s}\,ds \leq Ch^\alpha \quad (0 < h < 1)$.

Let $\displaystyle F_\phi(t) = \int_0^\infty \phi(s)f(s+t)\,ds \quad (t \geq 0)$. Then

$$\zeta(F_\phi) \leq (1-\alpha)\omega_0(f) + \alpha \operatorname{hol}_0^\infty(\hat{f}\,).$$

Proof. Let $\check{\phi}(t) = \phi(-t)$ $(t \leq 0)$ and $\check{\phi}(t) = 0$ $(t > 0)$. Choose ω and γ such that $\operatorname{hol}_0^\infty(\hat{f}\,) < \omega < \beta$ and $\max(\omega, \omega_0(f)) < \gamma < \beta$. Then

$$\int_{-\infty}^0 |\check{\phi}(s)|e^{-\gamma s}\,ds = \int_0^\infty |\phi(s)|e^{\gamma s}\,ds < \infty$$

so that $\check{\phi}$ satisfies assumptions (*1*) and (*2*) of Theorem 4.1.3. Since

$$\left|\int_{-\infty}^\infty e^{-(\omega+i\eta)s}\check{\phi}(s)\,ds\right| = \left|\int_0^\infty e^{(\omega+i\eta)s}\phi(s)\,ds\right|,$$

arguments similar to those in the proof of Corollary 4.1.4 show that $\check{\phi}$ satisfies assumption (*3*) of Theorem 4.1.3. The last assumption of this theorem is also valid for $\check{\phi}$ as

$$\int_{-\infty}^\infty |\check{\phi}(s)-\check{\phi}(s-h)|e^{-\gamma s}\,ds = \int_{-h}^\infty |\phi(s)-\phi(s+h)|e^{\gamma s}\,ds.$$

Consequently, we have from Theorem 4.1.3,

$$\zeta(\check{\phi} * f) \leq (1-\alpha)\gamma + \alpha\omega.$$

Since $F_\phi = \check{\phi} * f$ the required result is obtained by letting $\gamma \downarrow \omega_0(f)$ and $\omega \downarrow \mathrm{hol}_0^\infty(\hat{f}\,)$. □

The following is an analogue of (2.11). Example 3.1.10 (*2*), shows that this result fails for some Laplace transformable functions. Thus the assumption that f be exponentially bounded cannot be relaxed.

Theorem 4.1.6. *If $f : \mathbb{R}_+ \to \mathbf{X}$ is an exponentially bounded, measurable function, then*

$$\zeta_1(f) = \zeta(F) \leq \mathrm{hol}_0^\infty(f),$$

where $F(t) = \displaystyle\int_0^t f(s)\,ds$.

Proof. From Theorem 3.1.11, $\zeta_1(f) = \zeta(F)$. Let $\omega > \max\left(0, \mathrm{hol}_0^\infty(\hat{f}\,)\right)$ and $\phi = \chi_{\mathbb{R}_+}$. Then ϕ satisfies all the assumptions of Theorem 4.1.3, with $\alpha = 1$. Since $F(t) = (\phi * f)(t)$ $(t \geq 0)$, an application of Theorem 4.1.3 shows that $\zeta_1(f) = \zeta(F) \leq \omega$. Hence, $\zeta_1(f) \leq \max\left(0, \mathrm{hol}_0^\infty(\hat{f}\,)\right)$. Now take $\omega > \mathrm{hol}_0^\infty(\hat{f}\,)$ and let $f_\omega(t) = e^{-\omega t} f(t)$. Since $\mathrm{hol}_0^\infty(\hat{f_\omega}) = \mathrm{hol}_0^\infty(\hat{f}\,) - \omega < 0$, the above discussion shows that $\zeta_1(f_\omega) < 0$. Thus,

$$\zeta_1(f) = \omega + \zeta_1(f_\omega) \leq \omega.$$

Letting $\omega \downarrow \mathrm{hol}_0^\infty(\hat{f}\,)$ we get the required result. □

Since the non-analytic growth bound is in a sense a measure of non-analyticity, regularising a function $f : \mathbb{R}_+ \to \mathbf{X}$ by convolving with an analytic function should reduce $\zeta(f)$. Indeed, we have,

Theorem 4.1.7. *Let $\psi : \Sigma_\beta \to \mathbb{C}$ be exponentially bounded and holomorphic, where $0 < \beta \leq \frac{\pi}{2}$, and suppose that $\displaystyle\int_0^1 |\psi'(s)|\,ds < \infty$. If $f : \mathbb{R}_+ \to \mathbf{X}$ is measurable and exponentially bounded, then*

$$\zeta(\psi * f) \leq \mathrm{hol}_0^\infty(\hat{f}) \leq \zeta(f).$$

Proof. The assumptions imply that $\psi * f$ is continuously differentiable. Moreover,

$$(\widehat{(\psi * f)'})(\lambda) = \lambda\hat{\psi}(\lambda)\hat{f}(\lambda) \tag{4.10}$$

for $\mathrm{Re}\,\lambda$ sufficiently large. Suppose that $|\psi(z)| \leq Me^{\omega|z|}$ for all $z \in \Sigma_\beta$. From [2, Theorem 2.6.1] , we have that $\hat{\psi}(\lambda)$ has a holomorphic extension to $\omega + \Sigma_{\frac{\beta+\pi}{2}}$ and

$$\sup_{\lambda \in \omega + \Sigma_{\frac{\beta+\pi}{2}}} \|(\lambda - \omega)\hat{\psi}(\lambda)\| < \infty.$$

Therefore, $\lambda \mapsto \lambda\hat{\psi}(\lambda)$ has a bounded holomorphic extension to $\omega + \Sigma_{\frac{\beta+\pi}{2}}$. Thus, (4.10) implies that $\mathrm{hol}_0^\infty((\widehat{\psi * f})') \leq \mathrm{hol}_0^\infty(\hat{f}\,)$. Then an application of Theorem 4.1.6 yields

$$\zeta(\psi * f) = \zeta_1((\psi * f)') \leq \mathrm{hol}_0^\infty((\widehat{\psi * f})') \leq \mathrm{hol}_0^\infty(\hat{f}\,).$$

□

Remark 4.1.8. The following variation of Theorem 4.1.7 is valid with an identical proof: If $f : \Sigma_\beta \to \mathbf{X}$ is an exponentially bounded, holomorphic function with $\displaystyle\int_0^1 \|f'(t)\|\,dt < \infty$ and $\psi : \mathbb{R}_+ \to \mathbb{C}$ is exponentially bounded and measurable, then $\zeta(\psi * f) \leq \mathrm{hol}_0^\infty(\hat{\psi}) \leq \zeta(\psi)$.

As an analogue of (4.1) we have,

Theorem 4.1.9. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ and $\psi : \mathbb{R}_+ \to \mathbb{C}$ be exponentially bounded and measurable. Then*

$$\zeta(\psi * f) \leq \max(\zeta(\psi), \zeta(f)).$$

Proof. Let $g : \mathbb{C} \to \mathbf{X}$ and $\phi : \mathbb{C} \to \mathbb{C}$ be exponentially bounded, entire functions. We write

$$\psi * f = \phi * f + (\psi - \phi) * g + (\psi - \phi) * (f - g).$$

By Theorem 4.1.7, $\zeta(\phi * f) \leq \zeta(f) \leq \omega_0(f - g)$ and by Remark 4.1.8,

$$\zeta((\psi - \phi) * g) \leq \zeta(\psi - \phi) \leq \omega_0(\psi - \phi).$$

Further,

$$\zeta((\psi - \phi) * (f - g)) \leq \omega_0((\psi - \phi) * (f - g)) \leq \max(\omega_0(\psi - \phi), \omega_0(f - g)).$$

Therefore,

$$\zeta(\psi * f) \leq \max(\omega_0(\psi - \phi), \omega_0(f - g)).$$

Taking the infimum over all possible choices of ϕ and g yields the result. □

4.2 Boundedness of convolutions and non-resonance conditions

In [7], several results have been given showing that the convolution $\psi * f$ of two bounded measurable functions ψ and f is bounded if a certain non-resonance condition is satisfied and any of various supplementary assumptions hold. The *half-line spectrum* $\mathrm{sp}(f)$, ([2, Sections 4.4, 4.7], [7]), of a bounded measurable function $f : \mathbb{R}_+ \to \mathbf{X}$ is defined to be:

$$\mathrm{sp}(f) = \left\{\mu \in \mathbb{R} : i\mu \notin D(\hat{f}\,)\right\}.$$

The non-resonance condition in this context is that the intersection of the half-line spectra of f and ψ is empty. One of the supplementary conditions used in [7, Theorem 4.1] corresponds to f having an L^1 Laplace transform with $N \geq 2$ in Theorem 3.0.1. Explicitly, [7, Theorem 4.1] states that if for the function $f \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbf{X})$ there exist b, $k \geq 2$ such that $\mathrm{sp}(f) \subset (-b, b)$ and

$$\int_{|\eta| \geq b} \left\| \hat{f}^{(k)}(i\eta) \right\| \, d\eta < \infty, \tag{4.11}$$

and $\psi \in \mathrm{L}^\infty(\mathbb{R}_+)$ is such that $\mathrm{sp}(\psi) \cap \mathrm{sp}(f)$ is empty, then $\psi * f$ is bounded; and condition (4.11) corresponds to f having an L^1 Laplace transform with $N \geq 2$. Using this, in [10, Theorem 2.4.12], boundedness of the convolution $\psi * f$ when $f : \mathbb{R}_+ \to \mathbf{X}$ and $\psi : \mathbb{R}_+ \to \mathbb{C}$ satisfy the non-resonance condition is deduced under a simple assumption on the growth bound of non-integrability of f. In fact, it is enough to impose the assumption on the non-analytic abscissa $\kappa(f)$:

Theorem 4.2.1. *[10, Theorem 2.4.12] Let $f : \mathbb{R}_+ \to \mathbf{X}$ and $\psi : \mathbb{R}_+ \to \mathbb{C}$ be bounded measurable functions. Suppose that $\mathrm{sp}(f) \cap \mathrm{sp}(\psi)$ is empty and $\kappa(f) < 0$. Then the convolution $\psi * f$ is bounded.*

The same idea as in the proof of [10, Theorem 2.4.12] yields, on using Fubini's theorem,

Corollary 4.2.2. *Let $f \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbf{X})$ and $\psi : \mathbb{R}_+ \to \mathbb{C}$ be bounded and Lipschitz continuous. Suppose that $\mathrm{sp}(f) \cap \mathrm{sp}(\psi)$ is empty. If $\zeta_1(f) < 0$, then $\psi * f \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbf{X})$.*

Proof. Let $f \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbf{X})$ with $\zeta_1(f) < 0$. Then, there exist $\alpha < 0$ and $b \geq 0$ such that $Q_{\alpha,b} \subset D(\hat{f})$ and $f = f_{\alpha,b} + f_2$ with $\mathrm{abs}(f_2) < 0$. Suppose that ψ is bounded and Lipschitz and $\mathrm{sp}(f) \cap \mathrm{sp}(\psi)$ is empty. Recall from Proposition 3.1.4 that $f_{\alpha,b}$ is given by

$$f_{\alpha,b}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,b}} e^{\lambda t} \hat{f}(\lambda) \, d\lambda,$$

where $\Gamma_{\alpha,b}$ is any path in $D(\hat{f})$ from $\alpha - ib$ to $\alpha + ib$. From Lemma 3.1.2, we have

$$\hat{f}_{\alpha,b}(\mu) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,b}} \frac{\hat{f}(\lambda)}{\mu - \lambda} \, d\lambda \tag{4.12}$$

if $\mathrm{Re}\, \mu$ is sufficiently large. The right-hand side of (4.12) defines a holomorphic function of μ on $\mathbb{C} \setminus \Gamma_{\alpha,b}$. Given $i\eta \in D(\hat{f})$, we may assume that $\Gamma_{\alpha,b}$ does not pass to the right of $i\eta$. This shows that $i\eta \in D(\hat{f}_{\alpha,b})$. Hence, $\mathrm{sp}(f_{\alpha,b}) \subset \mathrm{sp}(f)$, and in particular, $\mathrm{sp}(f_{\alpha,b}) \cap \mathrm{sp}(\psi)$ is empty. Further,

$$\left| \hat{f}^{(2)}_{\alpha,b}(i\eta) \right| = \left| \frac{1}{\pi} \int_{\Gamma_{\alpha,b}} \frac{\hat{f}(\lambda)}{(i\eta - \lambda)^3} \, d\lambda \right| = O(|\eta|^{-3})$$

as $|\eta| \to \infty$. Thus, by [7, Theorem 4.1], $\psi * f_{\alpha,b}$ is bounded. Let $F_2(t) = \int_0^t f_2(s)\,ds$ and

$$F_\infty = \begin{cases} \lim_{t\to\infty} F(t) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\omega_0(F_2 - F_\infty) = \mathrm{abs}(f_2) < 0$, so $\psi' * (F_2 - F_\infty)$ is bounded. Since

$$(\psi' * F_2)(t) = (\psi * f_2)(t) - \psi(0)F_2(t) \quad (t \in \mathbb{R}_+),$$

by Fubini's Theorem, and

$$(\psi' * \mathbf{1})(t) = \psi(t) - \psi(0) \quad (t \in \mathbb{R}_+),$$

where $\mathbf{1}$ is the function taking the constant value 1, we conclude that

$$(\psi' * (F_2 - F_\infty))(t) = F_\infty(\psi(0) - \psi(t)) + (\psi * f_2)(t) - \psi(0)F_2(t),$$

for all $t \in \mathbb{R}_+$. Thus, $(\psi * f_2)$ is in $\mathrm{L}^\infty(\mathbb{R}_+, \mathbf{X})$. Therefore, we conclude that $\psi * f = \psi * f_{\alpha,b} + \psi * f_2$ is in $\mathrm{L}^\infty(\mathbb{R}_+, \mathbf{X})$. □

In [7, Theorem 5.1] it has been shown that if $f \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbf{X})$, $\phi \in \mathrm{L}^\infty(\mathbb{R}_+)$ satisfy the non-resonance condition, and $\psi \in \mathrm{L}^1(\mathbb{R})$ with $\mathcal{F}\psi \in C_c^2(\mathbb{R})$, then $\phi * \psi * f$ is bounded. We present a comparable result in the following corollary.

Corollary 4.2.3. *Let $f \in \mathrm{L}^\infty(\mathbb{R}_+, \mathbf{X})$, $\phi \in \mathrm{L}^\infty(\mathbb{R}_+)$ and suppose that $\mathrm{hol}_0^\infty(\hat{f}\,) < 0$ and $\mathrm{sp}(f) \cap \mathrm{sp}(\phi)$ is empty. Let $\psi : \mathbb{R}_+ \to \mathbb{C}$ be locally integrable, and suppose there exist $\omega < 0$, $C > 0$ and $\alpha \in (0, 1]$ such that*

1. $\int_0^\infty |\psi(s)| e^{-\omega s}\,ds < \infty$;
2. $\int_0^\infty |\psi(s) - \psi(s-h)| e^{-\omega s}\,ds \le Ch^\alpha \quad (0 < h < 1)$.

Then $\phi * \psi * f$ is bounded.

Proof. It follows from Theorem 4.1.3 and the proof of Corollary 4.1.4 that $\zeta(\psi * f) \le (1-\alpha)\gamma + \alpha\omega$ for every $\gamma > 0$. Choose $\epsilon > 0$ such that $\epsilon < -\alpha\omega$. For $\gamma = \dfrac{-\alpha\omega - \epsilon}{1-\alpha}$ this gives $\zeta(\psi * f) \le -\epsilon < 0$. From assumption *(1)*, $\mathrm{abs}(|\psi|) \le \omega < 0$. Therefore, $\mathrm{sp}(\psi * f) \subset \mathrm{sp}(f)$ and the result follows on applying Theorem 4.2.1 to $\psi * f$ and ϕ. □

4.3 Fractional integrals and non-analytic growth bounds

We apply the results estimating the non-analytic growth bound of functions of the type $\psi * f$ obtained in Section 4.1, to some special cases. In particular we are able to obtain estimates for the non-analytic growth bound of the classical Riemann-Liouville fractional integral of an exponentially bounded measurable function. Recall from [21, Chapter XIII] that the Riemann-Liouville fractional integral of order α of an exponentially bounded measurable function $f : \mathbb{R}_+ \to \mathbf{X}$ is given by

$$(\mathcal{R}_\alpha f)(t) := \int_0^t \Gamma(\alpha)^{-1}(t-s)^{\alpha-1} f(s)\, ds \quad (t \geq 0),$$

for $\alpha > 0$, where $\Gamma(\alpha)$ is the usual Γ function given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \quad (\operatorname{Re} z > 0).$$

The Weyl fractional integral of f of order α is given by

$$\begin{aligned}(\mathcal{W}_\alpha f)(t) &:= \Gamma(\alpha)^{-1} \int_t^\infty (s-t)^{\alpha-1} f(s)\, ds \\ &= \Gamma(\alpha)^{-1} \int_0^\infty s^{\alpha-1} f(s+t)\, ds \quad (\alpha > 0)\end{aligned}$$

whenever the integral exists.

For $\mu \in \mathbb{R}$ and $\alpha > 0$, define

$$\psi_{\alpha,\mu}(s) = \begin{cases} \Gamma(\alpha)^{-1} s^{\alpha-1} e^{-\mu s} & (s > 0), \\ 0 & (s \leq 0). \end{cases} \tag{4.13}$$

It is easy to see that for $\operatorname{Re}\lambda > -\mu$,

$$\hat{\psi}_{\alpha,\mu}(\lambda) = \frac{1}{(\lambda+\mu)^\alpha}.$$

Thus, for $\alpha > 0$, $\beta > 0$ and $\operatorname{Re}\lambda > -\mu$,

$$\begin{aligned}\widehat{(\psi_{\alpha,\mu} * \psi_{\beta,\mu})}(\lambda) &= \hat{\psi}_{\alpha,\mu}(\lambda)\hat{\psi}_{\beta,\mu}(\lambda) \\ &= \frac{1}{(\lambda+\mu)^{\alpha+\beta}} \\ &= \hat{\psi}_{\alpha+\beta,\mu}(\lambda).\end{aligned}$$

Therefore, for α, $\beta > 0$, $\mu \in \mathbb{R}$,

$$\psi_{\alpha,\mu} * \psi_{\beta,\mu} = \psi_{\alpha+\beta,\mu}. \tag{4.14}$$

For an exponentially bounded and measurable function $f : \mathbb{R}_+ \to \mathbf{X}$ the convolution $\psi_{\alpha,\mu} * f$, where $\psi_{\alpha,\mu}$ is as in (4.13), exists for all $\alpha > 0$ and $\mu \in \mathbb{R}$ and is given by

$$(\psi_{\alpha,\mu} * f)(t) = \Gamma(\alpha)^{-1} \int_0^t s^{\alpha-1} e^{-\mu s} f(t-s)\, ds \quad (t \geq 0).$$

If $\mu = 0$, we shall write $\psi_\alpha := \psi_{\alpha,0}$. Writing $f_\mu(t) = e^{\mu t} f(t)$, it is easy to see that

$$(\psi_{\alpha,\mu} * f)(t) = e^{-\mu t} \left(\mathcal{R}_\alpha f_\mu\right)(t).$$

If $\mu > \omega_0(f)$ then the convolution $\check{\psi}_{\alpha,\mu} * f$, where $\check{\psi}_{\alpha,\mu}(s) = \psi_{\alpha,\mu}(-s)$ is well defined and we have

$$\begin{aligned}(\check{\psi}_{\alpha,\mu} * f)(t) &= \Gamma(\alpha)^{-1} \int_0^\infty \psi_{\alpha,\mu}(s) f(t+s)\, ds \\ &= \Gamma(\alpha)^{-1} \int_0^\infty s^{\alpha-1} e^{-\mu s} f(t+s)\, ds \\ &= e^{\mu t} \left(\mathcal{W}_\alpha f_{-\mu}\right)(t).\end{aligned}$$

Thus, $\psi_\alpha * f$ and $\check{\psi}_\alpha * f$ (when the latter exists) are respectively the Riemann-Liouville and Weyl fractional integrals of the exponentially bounded, measurable function f.

Definition 4.3.1. Let $f : \mathbb{R}_+ \to \mathbf{X}$ be an exponentially bounded measurable function, $\mu > \omega_0(f)$ and $\alpha > 0$. We define the growth bounds $\omega_{\alpha,\mu}(f)$ and $\zeta_{\alpha,\mu}(f)$ by

$$\begin{aligned}\omega_{\alpha,\mu}(f) &:= \omega_0(\check{\psi}_{\alpha,\mu} * f) = \omega_0\left(t \mapsto \int_0^\infty \psi_{\alpha,\mu}(s) f(t+s)\, ds\right) \\ \zeta_{\alpha,\mu}(f) &:= \zeta(\check{\psi}_{\alpha,\mu} * f) = \zeta\left(t \mapsto \int_0^\infty \psi_{\alpha,\mu}(s) f(t+s)\, ds\right).\end{aligned}$$

It follows immediately from the definitions that

$$\zeta_{\alpha,\mu}(f) \leq \omega_{\alpha,\mu}(f) \leq \omega_0(f). \tag{4.15}$$

An application of Corollary 4.1.5 gives the following estimate of $\zeta_{\alpha,\mu}(f)$ in terms of $\omega_0(f)$ and $\mathrm{hol}_0^\infty(\hat{f}\,)$. An improvement in this estimate is obtained later, in Proposition 4.3.8 (see also Remark 4.3.4 and Remark 4.5.2).

Corollary 4.3.2. *If $f : \mathbb{R}_+ \to \mathbf{X}$ is exponentially bounded and measurable then*

$$\zeta_{\alpha,\mu}(f) \leq (1-\alpha)\omega_0(f) + \alpha\, \mathrm{hol}_0^\infty(\hat{f}\,),$$

for $\mu > \omega_0(f)$ and $\alpha \in (0, 1]$.

Proof. We check that $\psi_{\alpha,\mu}$ satisfies the assumptions of Corollary 4.1.5 for $\omega_0(f) < \beta < \mu$. For such a choice of β, $\int_0^\infty s^{\alpha-1} e^{-(\mu-\beta)s}\, ds < \infty$, so assumption *(1)* of Corollary 4.1.5 is valid. Clearly, if $\alpha = 1$, assumption *(2)* holds as well. Now suppose $\alpha \in (0, 1)$. Then, putting $\eta = \mu - \beta$ we can write

$$\Gamma(\alpha) \int_{-h}^\infty |\psi_{\alpha,\mu}(s) - \psi_{\alpha,\mu}(s+h)| e^{\beta s}\, ds \;\leq\; I_1 + I_2 + I_3 + I_4, \tag{4.16}$$

where $0 < h < 1$ and

$$\begin{aligned}
I_1 &= \int_h^\infty e^{-\eta s}(s+h)^{\alpha-1}|1-e^{-\mu h}|\,ds \\
&\leq |1-e^{-\mu h}|\int_h^\infty e^{-\eta s}s^{\alpha-1}\,ds \\
&\leq \frac{h^\alpha}{\eta}(e^{|\mu|}-1)e^{-\eta h} \\
&\leq C_\mu h^\alpha; \\
I_2 &= \int_h^\infty e^{-\eta s}|s^{\alpha-1}-(s+h)^{\alpha-1}|\,ds \\
&\leq h\int_h^\infty (1-\alpha)e^{-\eta s}s^{\alpha-2}\,ds \\
&\leq C'h^\alpha; \\
I_3 &= \int_0^h e^{\beta s}|s^{\alpha-1}e^{-\mu s}-(s+h)^{\alpha-1}e^{-\mu(s+h)}|\,ds \\
&\leq \tilde{C}\int_0^h s^{\alpha-1}e^{-\eta s}\,ds \\
&\leq C\frac{h^\alpha}{\alpha} \\
I_4 &= \int_{-h}^0 (s+h)^{\alpha-1}e^{-\mu(s+h)+\beta s}\,ds \\
&\leq e^{-\eta h}\int_0^h s^{\alpha-1}e^{-\eta s}\,ds \\
&\leq C''\frac{h^\alpha}{\alpha},
\end{aligned}$$

C_μ, C, C', C'' being constants. Therefore, assumption (*2*) of Corollary 4.1.5 remains valid for $\psi_{\alpha,\mu}$. This implies that

$$\zeta\left(t \mapsto \int_0^\infty \psi_{\alpha,\mu}(s)f(t+s)\,ds\right) \leq (1-\alpha)\omega_0(f) + \alpha\,\mathrm{hol}_0^\infty(\hat{f}\,).$$

□

Thus, for an exponentially bounded, measurable function f with $\omega_0(f) < 0$, we can obtain from the above corollary, an estimate for the non-analytic growth bound of $\check{\psi}_\alpha * f$, the Weyl fractional integral of f. We now obtain an estimate for the non-analytic growth bound of the Riemann-Liouville fractional integral, $\psi_\alpha * f$ of an exponentially bounded, measurable function f, similar to the estimate obtained in Theorem 4.3.2, in terms of the growth bound $\omega_0(f)$ and $\mathrm{hol}_0^\infty(\hat{f}\,)$.

Theorem 4.3.3. *Let f be an exponentially bounded, measurable function and $\alpha \in (0,1]$. Then*

$$\zeta(\psi_\alpha * f) \leq (1-\alpha)\omega_0(f) + \alpha\,\mathrm{hol}_0^\infty(\hat{f}\,). \tag{4.17}$$

Proof. If $\text{hol}_0^\infty(\hat{f}\ ) > 0$, then the assumptions of Corollary 4.1.4 are satisfied and therefore (4.17) holds in this case. For $\alpha = 1$, (4.17) follows from Theorem 4.1.6. Let $\alpha \in (0,1)$ and $\mu \in \mathbb{R}$. Then $\psi_\alpha - \psi_{\alpha,\mu}$ satisfies the conditions of Theorem 4.1.7. Therefore, $\zeta\left((\psi_\alpha - \psi_{\alpha,\mu}) * f\right) \leq \text{hol}_0^\infty(\hat{f}\ )$. Take $\omega > \max\left(-\mu, \text{hol}_0^\infty(\hat{f}\ )\right)$ and $\gamma > \max(\omega, \omega_0(f))$. Then calculations similar to those in the proof of Theorem 4.3.2 show that all the assumptions of Theorem 4.1.3 are satisfied by $\psi_{\alpha,\mu}$. Thus,

$$\zeta(\psi_{\alpha,\mu} * f) \leq (1-\alpha)\gamma + \alpha\omega_0(f).$$

Therefore,

$$\begin{aligned} \zeta(\psi_\alpha * f) &\leq \max\left(\zeta(\psi_{\alpha,\mu} * f), \text{hol}_0^\infty(\hat{f}\ )\right) \\ &\leq (1-\alpha)\gamma + \alpha\omega. \end{aligned} \tag{4.18}$$

Taking the infimum over all possible choices of μ, ω and γ yields (4.17). □

Remark 4.3.4. It is possible to further improve the estimates for $\zeta_{\alpha,\mu}(f)$ (μ sufficiently large) and $\zeta(\psi_\alpha * f)$ obtained in Theorem 4.3.2 and Theorem 4.3.3. First consider $\zeta(\psi_\alpha * f)$. Note that if g is holomorphic and exponentially bounded in a sector, then since

$$\widehat{(\psi_\alpha * g)}(\lambda) = \frac{\hat{g}(\lambda)}{\lambda^\alpha},$$

for $\text{Re}\,\lambda$ sufficiently large, it follows from [2, Theorem 2.6.1] that $\psi_\alpha * g$ is holomorphic and exponentially bounded in a sector. Therefore, $\zeta(\psi_\alpha * g) = -\infty$. Hence, if $f : \mathbb{R}_+ \to \mathbf{X}$ is an exponentially bounded and measurable function and g is holomorphic and exponentially bounded in a sector, we have on applying Theorem 4.3.3 to $f - g$,

$$\begin{aligned} \zeta(\psi_\alpha * f) &\leq \max\left(\zeta(\psi_\alpha * (f-g)), \zeta(\psi_\alpha * g)\right) \\ &= \zeta(\psi_\alpha * (f-g)) \\ &\leq (1-\alpha)\omega_0(f-g) + \alpha\,\text{hol}_0^\infty\left(\hat{f} - \hat{g}\right) \\ &\leq (1-\alpha)\omega_0(f-g) + \alpha\,\text{hol}_0^\infty(\hat{f}). \end{aligned} \tag{4.19}$$

Taking the infimum over all possible choices of g in (4.19) we obtain, for $0 < \alpha \leq 1$, and $f : \mathbb{R}_+ \to \mathbf{X}$ exponentially bounded and measurable,

$$\zeta(\psi_\alpha * f) \leq (1-\alpha)\zeta(f) + \alpha\,\text{hol}_0^\infty(\hat{f}). \tag{4.20}$$

Next we consider $\zeta_{\alpha,\mu}(f) = \zeta(\check{\psi}_{\alpha,\mu} * f)$, $\mu > \omega_0(f)$, $\alpha > 0$. Let g be entire and exponentially bounded in a sector, with $\omega_0(g) < \mu$. Then $\check{\psi}_{\alpha,\mu} * g$ is also exponentially bounded and entire. Thus, using Theorem 4.3.2, Proposition 3.1.4 (*6*) and arguments similar to those in the above paragraph,

$$\zeta_{\alpha,\mu}(f) \leq (1-\alpha)\zeta(f) + \alpha\,\text{hol}_0^\infty(\hat{f}). \tag{4.21}$$

□

Observe that

$$\zeta_{\alpha,\mu}(f) = \zeta(\psi_{\alpha,\mu} * f) \text{ for } \mu > \omega_0(f), \text{ and } \alpha = 1. \tag{4.22}$$

Indeed, for an exponentially bounded, measurable function $f : \mathbb{R}_+ \to \mathbf{X}$

$$\begin{aligned}
\zeta_{1,\mu}(f) &= \zeta\left(t \mapsto \int_0^\infty e^{-\mu s} f(s+t)\, ds\right) \\
&= \zeta\left(t \mapsto e^{\mu t} \int_t^\infty e^{-\mu s} f(s)\, ds\right) \\
&= \mu + \zeta\left(t \mapsto \left(\int_0^\infty e^{-\mu s} f(s)\, ds - \int_0^t e^{-\mu s} f(s)\, ds\right)\right) \\
&= \mu + \zeta\left(t \mapsto \int_0^t e^{-\mu s} f(s)\, ds\right) \\
&= \mu + \zeta_1(t \mapsto e^{-\mu t} f(t)) \\
&= \mu - \mu + \zeta_1(f) \\
&= \zeta(F),
\end{aligned}$$

where $F(t) = \int_0^t f(s)\, ds$ and we have used Theorem 4.1.6 to obtain the last equality. Moreover,

$$\begin{aligned}
\zeta(\psi_{1,\mu} * f) &= \zeta\left(t \mapsto \int_0^t e^{-\mu(t-s)} f(s)\, ds\right) \\
&= \zeta\left(t \mapsto e^{-\mu t} \int_0^t e^{\mu r} f(r)\, dr\right) \\
&= -\mu + \zeta_1(g) \\
&= \zeta_1(f)
\end{aligned}$$

where $g(t) = e^{\mu t} f(t)$. Thus, for all $\mu > \omega_0(f)$,

$$\zeta_{1,\mu}(f) = \zeta_1(f) = \zeta(\psi_{1,\mu} * f). \tag{4.23}$$

We do not know whether (4.22) is true for $\alpha \in (0,1)$ or not. Next, we prove that $\zeta(\psi_{\alpha,\mu} * f)$ is independent of the choice of $\mu \in \mathbb{R}$ for all $\alpha > 0$. For $\alpha = 1$, this has already been shown in (4.23).

Theorem 4.3.5. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be an exponentially bounded, measurable function, $\alpha > 0$ and $\mu_1, \mu_2 \in \mathbb{R}$. Then*

$$\zeta(\psi_{\alpha,\mu_1} * f) = \zeta(\psi_{\alpha,\mu_2} * f).$$

Proof. Let $g_i = \psi_{\alpha,\mu_i} * f$, $i = 1, 2$ and $e_i = \psi_{\alpha,\mu_i}$, $i = 1, 2$. We will show that there exists a function $h : \mathbb{R}_+ \to \mathbb{C}$, which extends to an entire, exponentially bounded function on $\mathbb{C}$, and satisfies the equation

$$e_1(t) - e_2(t) = (e_2 * h)(t) \quad (t \geq 0). \tag{4.24}$$

Then $g_1 - g_2 = h * g_2$, so that $\zeta(g_1) \leq \max(\zeta(g_2), \zeta(h * g_2))$. Since g_2 is an exponentially bounded, measurable function, it will follow from Theorem 4.1.7 that $\zeta(h * g_2) \leq \zeta(g_2)$. Therefore, $\zeta(g_1) \leq \zeta(g_2)$. Interchanging the roles of e_1 and e_2 then gives the required result. To establish the existence of such an h we observe that the Laplace transform of such a function must satisfy, for $\operatorname{Re}\lambda > \max(-\mu_1, -\mu_2, \mu_2 - 2\mu_1)$,

$$\begin{aligned}
\hat{h}(\lambda) &= \frac{\hat{e_1}(\lambda) - \hat{e_2}(\lambda)}{\hat{e_2}(\lambda)} \\
&= \left(\frac{\mu_2 + \lambda}{\mu_1 + \lambda}\right)^{\alpha} - 1 \\
&= \left(1 + \frac{C}{\mu_1 + \lambda}\right)^{\alpha} - 1, \ (C = \mu_2 - \mu_1) \\
&= \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)C^n}{n!(\mu_1+\lambda)^n}, \qquad (4.25)
\end{aligned}$$

the series on the right-hand side above being absolutely convergent. Let

$$h_n(t) = e^{-\mu_1 t}\frac{\alpha(\alpha-1)\dots(\alpha-n+1)C^n t^{n-1}}{n!(n-1)!} \quad (n = 1, 2...; \ t \geq 0).$$

Then the series $\sum_{n=1}^{\infty} h_n(t)$ is absolutely convergent for each $t \geq 0$. Set

$$h(t) = \sum_{n=1}^{\infty} h_n(t), \quad t \geq 0.$$

Then $\omega_0(h) \leq \max(-\mu_1, -\mu_2, \mu_2 - 2\mu_1) = -\mu_1 + |\mu_2 - \mu_1|$. Moreover, for $\operatorname{Re}\lambda > -\mu_1 + |\mu_2 - \mu_1|$,

$$\hat{h}(\lambda) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)C^n}{n!(\mu_1+\lambda)^n},$$

and h has an entire, exponentially bounded extension. Thus the proof is complete. □

We note here that even though the corresponding non-analytic growth bound does not depend on μ, the exponential growth bound $\omega_0(\psi_{\alpha,\mu} * f)$ is not necessarily independent of $\mu \in \mathbb{R}$. For example, if $f \equiv 1$, and $\alpha = 1$, then $(\psi_{\alpha,\mu} * f)(t) = \frac{1}{\mu}(1 - e^{-\mu t})$, for $\mu \in \mathbb{R}$.

In the proof of the next Theorem, given a function $g : \mathbb{R}_+ \to \mathbb{C}$, the function $\check{g}$ will be given by

$$\check{g}(s) = \begin{cases} g(-s) & (s \leq 0) \\ 0 & (s > 0). \end{cases}$$

Theorem 4.3.6. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be an exponentially bounded, measurable function and $\gamma > 0$. Then $\zeta\big(t \to \int_0^{\infty} \psi_{\gamma,\mu}(s) f(s+t)\, ds\big) = \zeta_{\gamma,\mu}(f)$ is independent of the choice of $\mu > \omega_0(f)$.*

Proof. First suppose that $\omega_0(f) = 0$. Let $\mu_1 > \mu_2/2 > 0$. Following the same notation as in Theorem 4.3.5, we have $\omega_0(h) \leq -\mu_1 + |\mu_2 - \mu_1| < 0$ and h satisfies all the conditions for the function ϕ in Corollary 4.1.5, with $\alpha = 1$, $\beta \in (0, -\omega_0(h))$ with respect to the function G_2, where

$$G_i(t) = \int_0^\infty \psi_{\gamma,\mu_i}(s) f(s+t)\, ds = (\check{e}_i * f)(t),$$

and $e_i = \psi_{\gamma,\mu_i}, i = 1, 2$. Moreover, (4.24) implies that $G_1(t) - G_2(t) = (e_2 * h)\check{} * f$. Since $(e_2 * h)\check{} = \check{e}_2 * \check{h}$

$$G_1(t) - G_2(t) = \int_0^\infty h(s) G_2(s+t)\, ds. \tag{4.26}$$

Therefore, applying Corollary 4.1.5 to G_2 and h, we obtain

$$\zeta\left(t \mapsto \int_0^\infty h(s) G_2(s+t)\, ds\right) \leq \mathrm{hol}_0^\infty(G_2) \leq \zeta(G_2).$$

Since

$$\zeta(G_1) \leq \max\left(\zeta(G_2), \zeta\left(t \mapsto \int_0^\infty h(s) G_2(s+t)\, ds\right)\right),$$

we may conclude that

$$\zeta_{\gamma,\mu_1}(f) \leq \zeta_{\gamma,\mu_2}(f), \quad \text{if } \mu_1 > \mu_2/2. \tag{4.27}$$

Thus, if $0 < \mu_1/2 < \mu_2 < 2\mu_1$,

$$\zeta_{\gamma,\mu_1}(f) = \zeta_{\gamma,\mu_2}(f). \tag{4.28}$$

Repeated application of this equality yields

$$\zeta_{\gamma,\mu_1}(f) = \zeta_{\gamma,\mu_2}(f), \quad \text{for all } \mu_1, \mu_2 > 0.$$

Now consider the case when $\omega_0(f)$ is arbitrary, but $\omega_0(f) > -\infty$. Setting $a = \omega_0(f)$ and applying the result obtained in the above paragraph to the function $e^{-a\cdot} f$ we have for $\mu_1, \mu_2 > 0$,

$$\zeta_{\gamma,\mu_1}(e^{-a\cdot} f) = \zeta_{\gamma,\mu_2}(e^{-a\cdot} f),$$

hence

$$\zeta_{\gamma,\mu_1+a}(f) = \zeta_{\gamma,\mu_2+a}(f), \quad \text{for all } \mu_1, \mu_2 > 0.$$

Therefore,

$$\zeta_{\gamma,\mu_1}(f) = \zeta_{\gamma,\mu_2}(f) \quad \text{for all } \mu_1, \mu_2 > a.$$

If $\omega_0(f) = -\infty$, then $\zeta_{\gamma,\mu_2}(f) = -\infty = \zeta_{\gamma,\mu_2}(f)$. □

Unlike $\omega_0(\psi_{\gamma,\mu} * f)$, $\omega_{\gamma,\mu}(f)$ is independent of the choice of $\mu > \omega_0(f)$ for any exponentially bounded, measurable function f. We prove this in the following, using the same strategy as employed in Theorem 4.3.6.

Theorem 4.3.7. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be exponentially bounded and measurable and $\alpha > 0$. Then $\omega_{\alpha,\mu}(f)$ is independent of the choice of $\mu > \omega_0(f)$.*

Proof. We shall use the same notation as in Theorem 4.3.6. First suppose $\omega_0(f) = 0$. Then, for $\mu_1 > \mu_2/2 > 0$, we have from (4.26),

$$\omega_0(G_1) \leq \max\left(\omega_0(G_2), \omega_0\left(t \mapsto \int_0^\infty h(s)G_2(t+s)\,ds\right)\right). \tag{4.29}$$

Since $\omega_0(h) \leq -\mu_1 + |\mu_2 - \mu_1|$, and $\omega_0(G_2) \leq 0$, it is easy to see that

$$\omega_0\left(t \mapsto \int_0^\infty h(s)G_2(t+s)\,ds\right) \leq \omega_0(G_2).$$

It follows then from (4.29) that

$$\omega_{\alpha,\mu_1}(f) = \omega_0(G_1) = \omega_0(G_2) = \omega_{\alpha,\mu_2}(f) \quad (\mu_1/2 < \mu_2 < 2\mu_1). \tag{4.30}$$

Using (4.30) repeatedly we therefore obtain $\omega_{\alpha,\mu_1}(f) = \omega_{\alpha,\mu_2}(f)$, $\mu_1, \mu_2 > 0$, if $\omega_0(f) = 0$. The other cases may be treated as in Theorem 4.3.6. □

We collect some properties of the the bounds $\zeta_{\alpha,\mu}(f)$ and $\zeta(\psi_{\alpha,\mu} * f)$ in the following proposition. Note that *(1)* of Proposition 4.3.8 improves the estimates obtained in Corollary 4.3.2 and Corollary 4.3.3.

Proposition 4.3.8. *Suppose $f : \mathbb{R}_+ \to \mathbf{X}$ is an exponentially bounded, measurable function. Let $\mu > \omega_0(f)$ and $\nu \in \mathbb{R}$. Then the following hold:*

1. *For $\alpha \in (0, 1]$,*

$$\zeta_{\alpha,\mu}(f) \leq (1-\alpha)\zeta(f) + \alpha\,\mathrm{hol}_0^\infty(\hat{f});$$
$$\zeta(\psi_{\alpha,\nu} * f) \leq (1-\alpha)\zeta(f) + \alpha\,\mathrm{hol}_0^\infty(\hat{f}).$$

2. *$\zeta_{\alpha,\mu}(f) \leq \zeta(f)$, and $\zeta(\psi_{\alpha,\nu} * f) \leq \zeta(f)$, for all $\alpha > 0$.*

3. *The maps $\alpha \mapsto \zeta(\psi_{\alpha,\nu} * f)$, $\alpha \mapsto \zeta_{\alpha,\mu}(f)$ and $\alpha \mapsto \omega_{\alpha,\mu}(f)$ are decreasing on $(0, \infty)$.*

Proof. *(1)*: Theorem 4.3.5 and Theorem 4.3.6 show that $\zeta(\psi_{\alpha,\nu} * f)$ and $\zeta_{\alpha,\mu}(f)$ do not depend on the choice of $\nu \in \mathbb{R}$ and $\mu > \omega_0(f)$, respectively. Therefore, *(1)* follows from (4.20) and (4.21).

(2): Since $\mathrm{hol}_0^\infty(\hat{f}) \leq \zeta(f)$, it follows immediately from *(1)* that $\zeta(\psi_{\alpha,\nu} * f) \leq \zeta(f)$ for $\alpha \in (0, 1]$. Let $\beta \in (1, 2]$. Then we can choose $\alpha_1 \in (0, 1]$ such that $\beta - \alpha_1 \in (0, 1]$. Using (4.14) we have,

$$\begin{aligned}\zeta(\psi_{\beta,\nu} * f) &= \zeta(\psi_{\beta-\alpha_1,\nu} * (\psi_{\alpha_1,\nu} * f)) \\ &\leq \zeta(\psi_{\alpha_1,\nu} * f) \\ &\leq \zeta(f).\end{aligned}$$

Iterating this process, we obtain

$$\zeta(\psi_{\alpha,\nu} * f) \leq \zeta(f) \text{ for all } \alpha > 0. \tag{4.31}$$

Similarly, we have from (*1*) that $\zeta_{\alpha,\mu}(f) \leq \zeta(f)$ for $\alpha \in (0,1]$. Moreover, straightforward calculations show that for $\alpha, \gamma > 0$,

$$\begin{aligned}(\check{\psi}_{\gamma,\mu} * (\check{\psi}_{\alpha,\mu} * f))(t) &= \int_t^\infty (\psi_{\gamma,\mu} * \psi_{\alpha,\mu})(s-t) f(s)\, ds \\ &= \int_t^\infty (\psi_{\gamma+\alpha,\mu})(s-t) f(s)\, ds \\ &= (\check{\psi}_{\gamma+\alpha,\mu} * f)(t). \end{aligned} \tag{4.32}$$

Therefore, arguments similar to those in the previous paragraph yield

$$\zeta_{\alpha,\mu}(f) \leq \zeta(f), \text{ for all } \alpha > 0. \tag{4.33}$$

(*3*): Let $\beta > \alpha > 0$. Using (4.14) and (4.31) we have

$$\begin{aligned}\zeta(\psi_{\beta,\nu} * f) &= \zeta(\psi_{\beta-\alpha,\nu} * (\psi_{\alpha,\nu} * f)) \\ &\leq \zeta(\psi_{\alpha,\nu} * f).\end{aligned}$$

Using (4.32) and (4.33), we have for $\beta > \alpha > 0$,

$$\begin{aligned}\zeta_{\beta,\mu}(f) &= \zeta(\check{\psi}_{\beta-\alpha+\alpha,\mu} * f) \\ &= \zeta(\check{\psi}_{\beta-\alpha,\mu} * (\check{\psi}_{\alpha,\mu} * f)) \\ &= \zeta_{\beta-\alpha,\mu}(\check{\psi}_{\alpha,\mu} * f) \\ &\leq \zeta(\check{\psi}_{\alpha,\mu} * f) \\ &= \zeta_{\alpha,\mu}(f).\end{aligned}$$

Similarly, since $\omega_{\alpha,\mu}(f) \leq \omega_0(f)$ for all $\alpha > 0$, $\mu > \omega_0(f)$ it follows that $\omega_{\beta,\mu}(f) \leq \omega_{\alpha,\mu}(f)$. □

Both $\zeta_{\alpha,\mu}(f)$ and $\alpha \to \zeta(\psi_{\alpha,\mu} * f)$ behave well under translations and rescaling. More specifically, we have,

Corollary 4.3.9. *Let $f : \mathbb{R} \to \mathbf{X}$ be an exponentially bounded, measurable function and $\alpha > 0$. Then the following hold:*

1. *For $\omega \in \mathbb{R}$ and $\mu > \max(\omega_0(f), -\omega + \omega_0(f))$,*

$$\zeta_{\alpha,\mu}\left(e^{-\omega \cdot} f\right) = -\omega + \zeta_{\alpha,\mu}(f);$$

2. *For* $\omega, \mu \in \mathbb{R}$,

$$\zeta\left(\psi_{\alpha,\mu} * e^{-\omega\cdot} f\right) = -\omega + \zeta(\psi_{\alpha,\mu} * f);$$

3. *For* $\tau \geq 0$, $\zeta_{\alpha,\mu}(f_\tau) = \zeta_{\alpha,\mu}(f) \quad (\mu > \omega_0(f))$ *and* $\zeta(\psi_{\alpha,\mu} * f_\tau) = \zeta(\psi_{\alpha,\mu} * f) \quad (\mu \in \mathbb{R})$, *where* $f_\tau(t) = f(t+\tau) \quad (t \geq 0$.

Proof. *(1)* : Let $\omega \in \mathbb{R}$ and $\mu > \max(\omega_0(f), -\omega, \omega_0(f))$. Then from the definition of $\zeta_{\alpha,\mu}$,

$$\begin{aligned}
\zeta_{\alpha,\mu}\left(e^{-\omega\cdot} f\right) &= \zeta\left(t \mapsto \int_0^\infty \psi_{\alpha,\mu}(s) e^{-\omega(s+t)} f(s+t)\, ds\right) \\
&= \zeta\left(t \mapsto e^{-\omega t} \int_0^\infty \psi_{\alpha,\mu+\omega}(s) f(s+t)\, ds\right) \\
&= -\omega + \zeta_{\alpha,\mu+\omega}(f) \\
&= -\omega + \zeta_{\alpha,\mu}(f),
\end{aligned}$$

where we have used Theorem 4.3.6 to obtain the last equality.

(2): This follows similarly from the definition, on making use of Theorem 4.3.5.

(3) : Writing $h_\tau(t) = h(t+\tau) \quad (t \geq 0)$, for any function $h : \mathbb{R}_+ \to \mathbf{X}$, we see that

$$\begin{aligned}
\zeta_{\alpha,\mu}(f_\tau) &= \zeta\left(t \mapsto \int_0^\infty \psi_{\alpha,\mu}(s) f(t+\tau+s)\, ds\right) \\
&= \zeta(\left(\check{\psi}_{\alpha,\mu} * f\right)_\tau) \\
&= \zeta(\check{\psi}_{\alpha,\mu} * f),
\end{aligned}$$

where the last equality follows from Proposition 3.1.8 *(3)*.

For $t \geq 0$, we may write

$$\begin{aligned}
(\psi_{\alpha,\mu} * f_\tau)(t) &= \int_0^t \psi_{\alpha,\mu}(s) f(t+\tau-s)\, ds \\
&= \int_0^{t+\tau} \psi_{\alpha,\mu}(s) f(t+\tau-s)\, ds - \int_t^{t+\tau} \psi_{\alpha,\mu}(s) f(t+\tau-s)\, ds \\
&= (\psi_{\alpha,\mu} * f)(t+\tau) - \int_0^\tau \psi_{\alpha,\mu}(t+\tau-s) f(s)\, ds \\
&= (\psi_{\alpha,\mu} * f)_\tau(t) - \int_0^\tau \psi_{\alpha,\mu}(t+\tau-s) f(s)\, ds.
\end{aligned}$$

Since the function $t \mapsto \int_0^\tau \psi_{\alpha,\mu}(t+\tau-s) f(s)\, ds$ extends to an exponentially bounded, holomorphic function on a sector, we conclude that

$$\zeta(\psi_{\alpha,\mu} * f_\tau) = \zeta((\psi_{\alpha,\mu} * f)_\tau) = \zeta(\psi_{\alpha,\mu} * f).$$

□

For an exponentially bounded, measurable function f, since both $\omega_{\alpha,\mu}(f)$ and $\zeta_{\alpha,\mu}(f)$ are independent of the choice of $\mu > \omega_0(f)$, we can make the following definitions:

Definition 4.3.10. Let $f : \mathbb{R}_+ \to \mathbf{X}$ be a measurable and exponentially bounded function and $\alpha > 0$. Fix $\mu > \omega_0(f)$. We define $\omega_\alpha(f) := \omega_{\alpha,\mu}(f)$ to be the *fractional growth bound* of f of order α and $\zeta_\alpha(f) := \zeta_{\alpha,\mu}(f)$ to be the *non-analytic fractional growth bound* of f of order α.

4.4 Fractional growth bounds for C_0-semigroups

In this section, we take a closer look at the quantity $\zeta_{\alpha,\mu}(\mathbf{T})$ for $\alpha > 0$ and $\mu \in \mathbb{R}$ in the particular case when $\mathbf{T}$ is a C_0-semigroup. Throughout this section $\mathbf{T}$ shall denote a C_0-semigroup defined on $\mathbf{X}$ with generator $\mathbf{A}$ unless otherwise stated. It turns out that these bounds are related to the non-analytic growth bound of $\mathbf{T}$ in much the same way as the fractional growth bounds $\omega_\alpha(\mathbf{T})$ of $\mathbf{T}$ are related to the growth bound of $\mathbf{T}$. We first recall some definitions:

Let $\mu > \omega_0(\mathbf{T})$. For $\alpha > 0$ the fractional powers $\mathbf{R}(\mu, \mathbf{A})^\alpha$ are given by

$$\begin{aligned} \mathbf{R}(\mu, \mathbf{A})^\alpha x &:= \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} \mathbf{R}(t + \mu, \mathbf{A}) x \, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-\mu t} \mathbf{T}(t) x \, dt, \end{aligned} \tag{4.34}$$

for all $x \in \mathbf{X}$. The integrals on the right-hand side are absolutely convergent for all $x \in \mathbf{X}$ and define injective bounded linear operators on $\mathbf{X}$. Then the operators $(\mu - \mathbf{A})^\alpha$, $\alpha > 0$ are defined as the inverse of $\mathbf{R}(\mu, \mathbf{A})^\alpha$ with domain $D((\mu - \mathbf{A})^\alpha) = \mathrm{Ran}(\mathbf{R}(\mu, \mathbf{A})^\alpha)$. For details of fractional powers see [41, Section 2], [2, Section 3.8] and [31]. Observe here that $\mathbf{R}(\mu, \mathbf{A})^\alpha x = (\mathcal{W}_\alpha(e^{-\mu \cdot}\mathbf{T}))(0)x$.

The *fractional growth bounds* $\omega_\alpha(\mathbf{T})$ associated with the semigroup $\mathbf{T}$ are defined by

$$\begin{aligned} \omega_\alpha(\mathbf{T}) &= \omega_0 \left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha\right) \\ &= \sup \left\{\omega_0 \left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha x\right) : x \in \mathbf{X}\right\} \\ &= \sup \left\{\omega_0(\mathbf{T}(\cdot)x) : x \in D((\mu - \mathbf{A})^\alpha)\right\}. \end{aligned} \tag{4.35}$$

It has been shown in [31] that for $\mu_1, \mu_2 > \omega_0(\mathbf{T})$,

$$D((\mu_1 - \mathbf{A})^\alpha) = D((\mu_2 - \mathbf{A})^\alpha), \tag{4.36}$$

for each $\alpha > 0$. It follows therefore, that the growth bounds $\omega_\alpha(\mathbf{T})$ are independent of the choice of $\mu > \omega_0(\mathbf{T})$.

From Theorem 4.3.7 we can deduce a proof for this fact without resorting to (4.36). Indeed, using (4.34) we obtain

$$\omega_0(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha) = \omega_{\alpha,\mu}(\mathbf{T}),$$

and Theorem 4.3.7 shows that $\omega_{\alpha,\mu}(\mathbf{T})$ is independent of $\mu > \omega_0(f)$.

Analogous to the description of $\omega_\alpha(\mathbf{T})$ in (4.35) we may consider the non-analytic growth bounds of the function $t \mapsto \mathbf{T}(t)\mathbf{R}(\mu, \mathbf{A})^\alpha$ for $\alpha > 0$ where $\mu > \omega_0(\mathbf{T})$. (*2*) of Corollary 3.2.2 yields

$$\zeta\left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha\right) = \sup\left\{\zeta\left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha x\right) : x \in \mathbf{X}\right\}. \tag{4.37}$$

For $x \in \mathbf{X}$, $\mu > \omega_0(f)$ and $\alpha > 0$ it follows from (4.34), for all $t \geq 0$,

$$\begin{aligned}
\mathbf{T}(t)\mathbf{R}(\mu, \mathbf{A})^\alpha x &= \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} e^{-\mu s} \mathbf{T}(t+s)x \, ds \\
&= \int_0^\infty \psi_{\alpha,\mu}(s)\mathbf{T}(t+s)x \, ds \\
&= (\check{\psi}_{\alpha,\mu} * \mathbf{T}(\cdot)x)(t).
\end{aligned}$$

Therefore, from Theorem 4.3.6 it follows that for each $x \in \mathbf{X}$, $\zeta\left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha x\right)$ and thus $\zeta\left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha\right)$ is independent of the choice of $\mu > \omega_0(\mathbf{T})$.

Another way of determining that $\zeta\left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha x\right)$ is independent of the choice of $\mu > \omega_0(\mathbf{T})$, is as follows: Let $\mu_1, \mu_2 > \omega_0(\mathbf{T})$ and $\alpha > 0$. Then $\mathrm{Ran}(\mathbf{R}(\mu_1, \mathbf{A})^\alpha) = \mathrm{Ran}(\mathbf{R}(\mu_2, \mathbf{A})^\alpha)$, in view of (4.36). Thus $\mathbf{C} := (\mu_2 - \mathbf{A})^\alpha \mathbf{R}(\mu_1, \mathbf{A})^\alpha$ is a closed operator, and by the Closed Graph Theorem, $\mathbf{C} \in \mathcal{L}(\mathbf{X})$. Further, $\mathbf{R}(\mu_1, \mathbf{A})^\alpha = \mathbf{C}\mathbf{R}(\mu_2, \mathbf{A})^\alpha$. Therefore, for $x \in \mathbf{X}$,

$$\begin{aligned}
\zeta(\mathbf{T}(\cdot)\mathbf{R}(\mu_1, \mathbf{A})^\alpha x) &= \zeta(\mathbf{C}\mathbf{T}(\cdot)\mathbf{R}(\mu_2, \mathbf{A})^\alpha x) \\
&\leq \zeta(\mathbf{T}(\cdot)\mathbf{R}(\mu_2, \mathbf{A})^\alpha x).
\end{aligned}$$

Interchanging the roles of μ_1 and μ_2 shows that for each $x \in \mathbf{X}, \zeta\left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha x\right)$ is independent of μ.

We define the *fractional non-analytic growth bound of order* α, $\zeta_\alpha(\mathbf{T})$, $\alpha > 0$ associated with a C_0-semigroup $\mathbf{T}$ by

$$\begin{aligned}
\zeta_\alpha(\mathbf{T}) &:= \zeta\left(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha\right) \\
&= \sup\left\{\zeta_{\alpha,\mu}(\mathbf{T}(\cdot)x) : x \in \mathbf{X}\right\} \\
&= \zeta_{\alpha,\mu}(\mathbf{T}),
\end{aligned}$$

where $\mu > \omega_0(\mathbf{T})$. It is clear from the above definition and (4.37) that for $\mu > \omega_0(\mathbf{T})$,

$$\zeta_\alpha(\mathbf{T}) = \sup\left\{\zeta\left(\mathbf{T}(\cdot)x\right) : x \in D((\mu - \mathbf{A})^\alpha)\right\}. \tag{4.38}$$

Since $D((\mu - \mathbf{A})^\alpha)$ does not depend on the choice of μ this gives another proof for the fact that $\zeta_\alpha(\mathbf{T})$ is independent of μ.

In the discussion so far, we have established that $\zeta_\alpha(\mathbf{T})$ and $\omega_\alpha(\mathbf{T})$ are respectively the non-analytic and exponential growth bounds of the operator-valued function obtained on

convolving the semigroup $\mathbf{T}$ with functions of the type $\check{\psi}_{\alpha,\mu}$. Therefore, Corollary 4.3.2 and (3.3) yield corresponding relations for $\zeta_\alpha(\mathbf{T})$ and $\omega_\alpha(\mathbf{T})$. These are recorded in the following proposition along with other basic properties of $\zeta_\alpha(\mathbf{T})$.

Proposition 4.4.1. *Let $\mathbf{T}$ be a C_0-semigroup with generator $\mathbf{A}$ and $\alpha \geq 0$. Then*

1. $\zeta_\alpha(\mathbf{T}) \leq \omega_\alpha(\mathbf{T}) \leq \omega_0(\mathbf{T})$;
2. $\alpha \mapsto \zeta_\alpha(\mathbf{T})$ *is decreasing on* $[0,\infty)$;
3. $s^\infty(\mathbf{A}) \leq \zeta_\alpha(\mathbf{T}) \leq \zeta(\mathbf{T})$;
4. $\zeta_\alpha(\mathbf{T}) \ \leq \ (1-\alpha)\zeta(\mathbf{T}) + \alpha s_0^\infty(\mathbf{A}) \quad (0 \leq \alpha \leq 1)$;
5. $\omega_\alpha(\mathbf{T}) = \max(\zeta_\alpha(\mathbf{T}), s(\mathbf{A}))$,

where $\zeta_0(\mathbf{T}) = \zeta(\mathbf{T})$.

Proof. *(1)*,*(2)* and *(4)*: These are special cases of the corresponding properties already discussed for exponentially bounded, measurable functions in Proposition 4.3.8.

(3): From *(1)* of Proposition 3.1.8 we have, for $\mu > \omega_0(\mathbf{T})$,

$$\zeta_\alpha(\mathbf{T}) = \zeta\left(\mathbf{T}(\cdot)\mathbf{R}(\mu,\mathbf{A})^\alpha\right) \geq \mathrm{hol}^\infty\left(\widehat{\mathbf{T}(\cdot)}\mathbf{R}(\mu,\mathbf{A})^\alpha\right) = s^\infty(\mathbf{A}).$$

(5): This follows from (3.3) of Proposition 3.1.8 on noting that $\mathrm{hol}\left(\widehat{\mathbf{T}(\cdot)}\mathbf{R}(\mu,\mathbf{A})^\alpha\right) = s(\mathbf{A})$ for all $\alpha > 0$. □

Weis and Wrobel [50] showed that the map $\alpha \mapsto \omega_\alpha(\mathbf{T})$ is convex on $[0,\infty)$ and continuous on $(0,\infty)$ and that $\omega_1(\mathbf{T}) \leq s_0(\mathbf{A})$. It follows then that

$$\omega_\alpha(\mathbf{T}) \leq (1-\alpha)\omega_0(\mathbf{T}) + \alpha s_0(\mathbf{A}) \quad (0 < \alpha \leq 1).$$

The following result gives a sharper inequality :

Theorem 4.4.2. *Let $\mathbf{T}$ be a C_0-semigroup on $\mathbf{X}$ with generator $\mathbf{A}$. Then,*

$$\omega_\alpha(\mathbf{T}) \leq \max\left((1-\alpha)\omega_0(\mathbf{T}) + \alpha s_0^\infty(\mathbf{A}), s(\mathbf{A})\right) \quad (0 < \alpha \leq 1).$$

In particular, if $s_0^\infty(\mathbf{A}) = -\infty$ then $\omega_\alpha(\mathbf{T}) = s(\mathbf{A})$, for all $\alpha \in \ (0,\infty)$.

Proof. This follows immediately on combining *(4)* and *(5)* of Proposition 4.4.1. □

Our results in Proposition 4.4.1 *(5)* and Theorem 4.4.2 improve those obtained in [10, Theorem 6.4.6 and Corollary 6.4.5] respectively where the techniques of [8] have been used to show that if $s_0^\infty(\mathbf{A}) = -\infty$, then $\omega_{\frac{1}{2}}(\mathbf{T}) = s(\mathbf{A})$ and that for $\omega > \omega_0(\mathbf{T})$ and $\alpha \in (\frac{1}{2}, 1]$,

$$\zeta(\mathbf{R}(\omega,\mathbf{A})^\alpha \mathbf{T}(\cdot)) \leq \omega_0(\mathbf{T}) + (1-2\alpha)(\omega_0(\mathbf{T}) - s_0^\infty(\mathbf{A})).$$

We remark that Theorem 4.4.2 cannot be extended to the interval $[0,\infty)$. In fact, [24, Example 4.2] gives a semigroup of positive operators satisfying $\omega_0(\mathbf{T}) = 0$ while $s_0^\infty(\mathbf{A}) = s(\mathbf{A}) = -\infty$ where the semigroup $\mathbf{T}$ is given by $\mathbf{T}(t)(f)(s) = f(t+s)$, $f \in \mathbf{X}_1 \cap \mathbf{X}_2$, $\mathbf{X}_1 = \mathrm{L}^q(\mathbb{R}_+)$, and $\mathbf{X}_2 = \mathrm{L}^p(\mathbb{R}_+, e^{pt^2}dt)$ with $1 < p < q < \infty$. The generator of this semigroup has empty spectrum.

Analogous to the convexity of $\omega_\alpha(\mathbf{T})$ in α we obtain a result concerning the convexity of the maps $\alpha \mapsto \zeta_\alpha(\mathbf{T})$. The strategy of the proof is similar to that of [49, Lemma 3.5].

Theorem 4.4.3. *Let $\mathbf{T}$ be a C_0-semigroup on $\mathbf{X}$ with generator $\mathbf{A}$. Then the map $\alpha \mapsto \zeta_\alpha(\mathbf{T})$ from $[0,\infty) \to \mathbb{R}$ is convex.*

Proof. Let $0 \le \alpha < \beta$ and $0 < \theta < 1$. Put $\gamma = (1-\theta)\alpha + \theta\beta$. Take $\mu > \omega_0(\mathbf{T})$. Let $a > \zeta_\alpha(\mathbf{T})$, $b > \zeta_\beta(\mathbf{T})$. From (*4*) of Proposition 4.4.1 it follows that $s^\infty(\mathbf{A}) < \min(a,b)$. Choose $l, m \ge 0$ such that and $Q_{l,m} \subset \rho(\mathbf{A})$. Then there is a constant M such that

$$\begin{aligned} \|\mathbf{T}(t)\mathbf{R}(\mu,\mathbf{A})^\alpha - \mathbf{T}_{l,m}(t)\mathbf{R}(\mu,\mathbf{A})^\alpha\| &\le Me^{at} \\ \left\|\mathbf{T}(t)\mathbf{R}(\mu,\mathbf{A})^\beta - \mathbf{T}_{l,m}(t)\mathbf{R}(\mu,\mathbf{A})^\beta\right\| &\le Me^{bt} \end{aligned}$$

for all $t \ge 0$.

For $x \in \mathbf{X}$, the convexity estimate [31, Theorem 8.1] gives

$$\begin{aligned} &\|\mathbf{T}(t)\mathbf{R}(\mu,\mathbf{A})^\gamma x - \mathbf{T}_{l,m}(t)\mathbf{R}(\mu,\mathbf{A})^\gamma x\| \\ &\quad\le K\left\|\mathbf{R}(\mu,\mathbf{A})^\alpha\left(\mathbf{T}(t) - \mathbf{T}_{l,m}(t)\right)x\right\|^{1-\theta}\left\|\mathbf{R}(\mu,\mathbf{A})^\beta\left(\mathbf{T}(t) - \mathbf{T}_{l,m}(t)\right)x\right\|^{\theta} \qquad (4.39) \\ &\quad\le KMe^{((1-\theta)a+\theta b)t}\|x\|, \end{aligned}$$

where K is a constant. Thus, $\zeta_\gamma(\mathbf{T}) \le (1-\theta)a + \theta b$ whenever $a > \zeta_\alpha(\mathbf{T})$ and $b > \zeta_\beta(\mathbf{T})$ and the result follows. □

Remark 4.4.4. From the proof of Theorem 4.4.3, in particular (4.39), it follows that the map $\alpha \mapsto \zeta_\alpha\left(\mathbf{T}(\cdot)x\right)$ is convex on $(0,\infty)$ for each $x \in \mathbf{X}$. □

4.5 Convexity and fractional bounds for vector-valued functions

From Theorem 4.4.3 we can deduce the corresponding result about convexity of the map $\alpha \mapsto \zeta_\alpha(f)$.

Theorem 4.5.1. *Let $f : \mathbb{R}_+ \to \mathbf{X}$ be exponentially bounded and measurable. Then $\alpha \mapsto \zeta_\alpha(f)$ is convex on $(0,\infty)$.*

Proof. Let $f : \mathbb{R}_+ \to \mathbf{X}$ be measurable, with $\omega_0(f) < \infty$ and choose $w > \omega_0(f)$. Suppose first that f is continuous. Define

$$C_w := \big\{g \in C(\mathbb{R}_+, \mathbf{X}) : \lim_{t\to\infty} e^{-wt}g(t) = 0\big\}.$$

Then C_w is a Banach space with norm given by $\|g\| = \sup_{t\geq 0} \|e^{-wt}g(t)\|$ and $f \in C_w$. Let $\mathbf{S} : \mathbb{R}_+ \to \mathcal{L}(C_w)$ denote the shift C_0-semigroup given by

$$(\mathbf{S}(t)g)(s) = g(s+t).$$

Then, $\zeta(g) = \zeta(\mathbf{S}(\cdot)g)$ for all $g \in C_w$. Indeed, if $\zeta(\mathbf{S}(\cdot)g) < a$, then there exists $b > 0$ such that $Q_{a,b} \subset D(\widehat{(\mathbf{S}(\cdot)g)}$, $\sup_{\lambda\in Q_{a,b}} \|\widehat{(\mathbf{S}(\cdot)g)}(\lambda)\| < \infty$ and

$$\mathbf{S}(t)g = \frac{1}{2\pi}\int_{\Gamma_{a,b}} e^{\lambda t}\widehat{(\mathbf{S}(\cdot)g)}(\lambda)\,d\lambda + h_2(t) \quad (t \geq 0), \tag{4.40}$$

where $h_2 : \mathbb{R}_+ \to C_w$, with $\omega_0(h_2) < a$. If $\lambda \in D(\widehat{\mathbf{S}(\cdot)g})$, then $\lambda \in D(\hat{g})$ and $\widehat{(\mathbf{S}(\cdot)g)}(\lambda)(0) = \hat{g}(\lambda)$. Therefore, from (4.40) we have

$$\begin{aligned} g(t) &= (\mathbf{S}(t)g)(0) \\ &= \frac{1}{2\pi i}\int_{\Gamma_{a,b}} e^{\lambda t}\hat{g}(\lambda)\,d\lambda + h_2(t)(0) \quad (t \geq 0). \end{aligned}$$

The map $z \mapsto \frac{1}{2\pi i}\int_{\Gamma_{a,b}} e^{\lambda z}\hat{g}(\lambda)\,d\lambda$ is analytic and exponentially bounded in a sector $\Sigma_\theta, \theta > 0$, and $t \mapsto h_2(t)(0)$ is a map from $\mathbb{R}_+ \to \mathbf{X}$, satisfying

$$\|h_2(t)(0)\| \leq \|h_2(t)\| \leq Me^{at},$$

for some constant M. Thus, $\zeta(g) \leq a$.

Conversely, $\zeta(\mathbf{S}(\cdot)g) \leq \omega_0(\mathbf{S}) = w$. Suppose that $\zeta(\mathbf{S}(\cdot)g) < a < w$. Then there exists $b > 0$ such that $Q_{a,b} \subset D(\hat{g})$, $\sup_{\lambda\in Q_{a,b}} \|\hat{g}(\lambda)\| < \infty$, and

$$g(t) = \frac{1}{2\pi i}\int_{\Gamma_{a,b}} e^{\lambda t}\hat{g}(\lambda)\,d\lambda + g_2(t) \quad (t \geq 0),$$

with $\omega_0(g_2) < a$. The path $\Gamma_{a,b}$ may be chosen to be contained in the half-plane $\{\lambda : \operatorname{Re}\lambda < w\}$. Thus, for every $s \geq 0$, we may write, for $t \geq 0$,

$$(\mathbf{S}(t)g)(s) = \frac{1}{2\pi i}\int_{\Gamma_{a,b}} e^{\lambda(t+s)}\hat{g}(\lambda)\,d\lambda + (\mathbf{S}(t)g_2)(s). \tag{4.41}$$

Further,

$$\int_{\Gamma_{a,b}} e^{\lambda(t+s)}\hat{g}(\lambda)\,d\lambda = \int_{\Gamma_{a,b}} e^{\lambda t}e_\lambda(s)\hat{g}(\lambda)\,d\lambda,$$

where, $e_\lambda(s) := e^{\lambda s}$ $(s \in \mathbb{R}_+)$. Thus, from (4.41),

$$\mathbf{S}(t)g = \frac{1}{2\pi i}\int_{\Gamma_{a,b}} e^{\lambda t}e_\lambda\hat{g}(\lambda)\,d\lambda + \mathbf{S}(t)g_2.$$

For $z \in \mathbb{C}$, the integral $\frac{1}{2\pi i}\int_{\Gamma_{a,b}} e^{\lambda z} e_\lambda \hat{g}(\lambda)\, d\lambda$ exists as a Bochner integral in C_w, and this defines an exponentially bounded entire function with values in C_w. Moreover,

$$\begin{aligned}\|\mathbf{S}(t)g_2\| &= \sup_{s\geq 0} e^{-ws}\|g_2(s+t)\| \\ &= \sup_{s\geq 0} e^{(a-w)s}e^{at}e^{-a(s+t)}\|g_2(s+t)\| \\ &\leq e^{at}\sup_{s\geq 0} e^{-as}\|g_2(s)\|.\end{aligned}$$

Hence, $\omega_0(\mathbf{S}(t)g_2) \leq a$, so $\zeta(\mathbf{S}(\cdot)g) \leq a$. Thus $\zeta(g) = \zeta(\mathbf{S}(\cdot)g)$, for all $g \in C_w$.

Let $\alpha > 0$ and $\mu > \omega_0(f)$. Straightforward calculations show that

$$(\check{\psi}_{\alpha,\mu} * \mathbf{S}(\cdot)f)(t) = \mathbf{S}(t)(\check{\psi}_{\alpha,\mu} * f).$$

Noting that $\check{\psi}_{\alpha,\mu} * f \in C_w$, it follows that

$$\zeta_\alpha(f) = \zeta(\check{\psi}_{\alpha,\mu} * f) = \zeta(\mathbf{S}(\cdot)(\check{\psi}_{\alpha,\mu} * f)) = \zeta_\alpha(\mathbf{S}(\cdot)f).$$

Then the result is a consequence of Remark 4.4.4 for such an f. Now consider the general case. For any $\epsilon > 0$, and $\mu > \omega_0(f)$, $\check{\psi}_{\epsilon,\mu} * f$ is continuous (by the Dominated Convergence Theorem) and $\omega_0(\check{\psi}_{\epsilon,\mu} * f) < w$. Thus, $\check{\psi}_{\epsilon,\mu} * f \in C_w$. Moreover,

$$\check{\psi}_{\alpha,\mu} * (\check{\psi}_{\epsilon,\mu} * f) = \check{\psi}_{\alpha+\epsilon,\mu} * f, \quad \alpha > 0.$$

It follows from the first case that $\alpha \mapsto \check{\psi}_{\alpha+\epsilon,\mu} * f$ is convex on $(0,\infty)$ and the result follows. □

Remark 4.5.2. We can improve the estimate for $\zeta_{\alpha,\mu}(f)$ obtained in Corollary 4.3.2 using the result obtained above. We claim that for $f : \mathbb{R}_+ \to \mathbf{X}$ exponentially bounded and measurable, $\mu > \omega_0(f)$ and $\alpha \in (0,1]$,

$$\zeta_{\alpha,\mu}(f) \leq (1-\alpha)\zeta(f) + \alpha\zeta_1(f).$$

Indeed, due to convexity of $\alpha \mapsto \zeta_{\alpha,\mu}(f)$, we have for $\alpha \in (0,1]$,

$$\begin{aligned}\zeta_{\alpha,\mu}(f) &\leq (1-\alpha)\zeta_{0,\mu}(f) + \alpha\zeta_{1,\mu}(f) \\ &= (1-\alpha)\zeta(f) + \alpha\zeta_1(f) \\ &\leq (1-\alpha)\zeta(f) + \alpha\,\mathrm{hol}_0^\infty(\hat{f}).\end{aligned}$$

Here the last inequality follows from Theorem 4.1.6. Thus, we have an alternative proof of Corollary 4.3.2. □

Remark 4.5.3. The technique used in Theorem 4.5.1 to deduce convexity of $\alpha \mapsto \zeta_\alpha(f), \alpha > 0$ when f is an exponentially bounded, measurable function taking values in $\mathbf{X}$, from the corresponding result for orbits of C_0-semigroups, may also be employed to give another proof of the fact that $\zeta_{\alpha,\mu}(f)$ does not depend on $\mu > \omega_0(f)$ (see Theorem 4.3.6). Recall that on page 67, it has been shown that if $\mathbf{T}$ is a C_0-semigroup, then $\zeta(\mathbf{T}(\cdot)\mathbf{R}(\mu, \mathbf{A})^\alpha x)$ is independent of the choice of $\mu > \omega_0(\mathbf{T})$ using two different methods. One of these methods does not make use of the properties of $\zeta_{\alpha,\mu}(f)$.

Chapter 5

Fourier multipliers and the non-analytic growth bound

As we have mentioned before, if $\mathbf{T}$ is a C_0-semigroup defined on a Hilbert space $\mathbf{X}$ then $\omega_0(\mathbf{T}) < 0$ if and only if $s_0(\mathbf{A}) < 0$ ([2, Theorem 5.2.1]). This result does not extend to arbitrary Banach spaces. In [32] it has been shown that the above statement is true on arbitrary Banach spaces if the condition $s_0(\mathbf{A}) < 0$ is supplemented by the condition that the resolvent restricted to the imaginary axis is a Fourier multiplier for $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ for some $p \in [1, \infty)$. Analogously, as we have discussed in subsection 3.2.2, the assertion that $\zeta(\mathbf{T}) < 0$ if and only if $s_0^\infty(\mathbf{A}) < 0$ for a semigroup $\mathbf{T}$ defined on a Hilbert space may not hold for arbitrary Banach spaces. We look for supplementary Fourier multiplier-type conditions, similar to the ones in the case of the exponential growth bound, which together with $s_0^\infty(\mathbf{A}) < 0$ would ensure that $\zeta(\mathbf{T}) < 0$ for a C_0-semigroup defined on any Banach space.

Fourier multipliers have been used extensively for characterising stability and hyperbolicity of strongly continuous semigroups. M. Hieber [27] has given a characterisation of uniform stability in terms of these concepts while in [32], exponential growth bounds of higher orders for a C_0-semigroup have been described in terms of Fourier multiplier properties of the resolvent of its generator. Amongst the first results connecting the theory of Fourier multipliers to stability and hyperbolicity of C_0-semigroups were those by Kaashoek and S. Verduyn Lunel [30]. L. Weis in [49] used Fourier multiplier properties of the resolvent on Besov spaces to give alternative proofs of some stability results. An extensive study of the relation between Fourier multipliers and hyperbolicity has been done by Y. Latushkin and R. Shvydkoy [34]. Here we study the relation between the non-analytic growth bound of a semigroup and Fourier multipliers, drawing on ideas from [34] and [30]. The results obtained in the first section lead on to a characterisation of this growth bound in terms

of mild solutions of certain inhomogeneous Cauchy problems. This characterisation of the non-analytic growth bound is comparable to the characterisation of hyperbolicity obtained in [33], [16, Section 4.3], using the theory of evolution families.

We first recall some definitions and results. For $\psi_i \in \mathcal{S}(\mathbb{R})$ and $x_i \in \mathbf{X}, i = 1, 2, ..., n$, the element $\Psi = \sum_{i=1}^n \psi_i \otimes x_i$ in $\mathcal{S}(\mathbb{R}, \mathbf{X})$ will be given by

$$\Psi(t) = \sum_{i=1}^n (\psi_i \otimes x_i)(t) = \sum_{i=1}^n \psi_i(t) x_i \quad (t \in \mathbb{R}).$$

We define the space $\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ by

$$\begin{aligned} \mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X})) \quad = \quad & \{\mathbf{T} : \mathbb{R} \longrightarrow \mathcal{L}(\mathbf{X}) \text{ such that } \mathbf{T}(\cdot)x \text{ is Bochner measurable for all } x \in \mathbf{X} \\ & \text{and there exists } g \in \mathrm{L}^1(\mathbb{R}) \text{ with } \|\mathbf{T}(t)\| \leq g(t)\big\}, \end{aligned}$$

and the space $\mathcal{F}\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ by setting

$$\mathcal{F}\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X})) = \big\{\mathcal{F}\mathbf{T} : \mathbf{T} \in \mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))\big\}.$$

Here, the Fourier transform is taken in the strong operator topology. Thus, for $\mathbf{T} \in \mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$, $\mathcal{F}\mathbf{T}$ is given by

$$(\mathcal{F}\mathbf{T})(s)x = \int_{-\infty}^{\infty} e^{-ist}\mathbf{T}(t)x \, dt.$$

A function $\mathbf{m} \in \mathrm{L}^\infty(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ is called a *Fourier multiplier* on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, where $1 \leq p < \infty$, if for all $f \in \mathcal{S}(\mathbb{R}, \mathbf{X})$,

$$\mathbf{M}(f) = \mathcal{F}^{-1}(\mathbf{m}(\cdot)\mathcal{F}f) \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$$

and there exists a constant C such that

$$\|\mathbf{M}(f)\|_p \leq C\|f\|_p.$$

Here, $\mathcal{F}^{-1}f$ is to be interpreted as $\frac{1}{2\pi}\bar{\mathcal{F}}f$. Then $\mathbf{M}$ extends to a bounded linear operator on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. If $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$, it follows from the density of $\mathcal{S}(\mathbb{R}, \mathbf{X})$ in $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$, that

$$\mathcal{F}(\mathbf{M}(f)) = \mathbf{m}(\cdot)\mathcal{F}f \quad (f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})). \tag{5.1}$$

It is immediate then that

$$\operatorname{supp} \mathcal{F}(\mathbf{M}(f)) \subset \operatorname{supp} \mathcal{F}f.$$

For $1 \leq p < \infty$, we shall denote the space of all Fourier multipliers on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ by $\mathcal{M}_p(\mathbf{X})$ with the usual identification of functions which coincide a.e. We put

$$\|\mathbf{m}\|_{\mathcal{M}_p(\mathbf{X})} := \|\mathbf{M}\|, \tag{5.2}$$

where the norm of $\mathbf{M}$ refers to its norm as a bounded linear operator on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. The space $\mathcal{M}_p(\mathbf{X})$ equipped with the norm defined in (5.2) is a Banach algebra.

It is easy to see that if $\mathbf{m} \in \mathcal{F}\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ then $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ for all $1 \leq p < \infty$. Indeed, if $\mathbf{m} = \mathcal{F}\mathbf{S}$, for some $\mathbf{S} \in \mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ then, for any $f \in \mathcal{S}(\mathbb{R}, \mathbf{X})$ we find, on using the operator-valued version of Young's inequality (see [2, Remark 1.3.8]), that

$$\begin{aligned} \mathbf{M}(f) &= \mathcal{F}^{-1}(\mathcal{F}\mathbf{S}\mathcal{F}f) \\ &= \mathbf{S} * f \\ &\in \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \end{aligned} \tag{5.3}$$

and

$$\|\mathbf{S} * f\|_p \leq \|f\|_p \int_{-\infty}^{\infty} \|\mathbf{S}(t)\| \, dt, \tag{5.4}$$

for $1 \leq p < \infty$.

For the scalar-valued case, the well known Mikhlin's Theorem, [29, Theorem 7.9.5] states that if a function $\mathbf{m} \in C^1(\mathbb{R} \setminus \{0\})$ is bounded and the function $s \mapsto s\mathbf{m}'(s)$ is also bounded, then $\mathbf{m}$ is Fourier multiplier on $\mathrm{L}^p(\mathbb{R}), 1 < p < \infty$. For $p = 1$, a similar result for $\mathbf{m} \in C^1(\mathbb{R})$, involving the boundedness of $s \mapsto s^{j+\epsilon}\mathbf{m}^{(j)}(s), j = 0, 1$ for some $\epsilon > 0$ was given by Hieber[2, Prop. 8.2.3]. We shall need a related theorem for operators, due to H. Amann [1, Corollary 4.4] (see also [26]).

Theorem 5.0.4. *Let $\mathbf{m} \in C^2(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ satisfy the following conditions for some $\epsilon > 0$:*

$$\sup_{t \in \mathbb{R}} \|t^{\epsilon+j}\mathbf{m}^{(j)}(t)\| < \infty, \ j = 0, 1, 2.$$

Then $\mathbf{m} \in \mathcal{F}\mathrm{L}^1(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ and $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X}), 1 \leq p < \infty$.

If the underlying space $\mathbf{X}$ is a Hilbert space, then every $\mathbf{m} \in \mathrm{L}^\infty(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ is a Fourier multiplier on $\mathrm{L}^2(\mathbb{R}, \mathbf{X})$. This is a direct consequence of Plancherel's Theorem.

5.1 A characterisation for $\zeta(\mathbf{T})$

In this section we use ideas from [30] and [34] to study the non-analytic growth bound of a semigroup in terms of Fourier multiplier properties of the resolvent of its generator. In Lemmas 5.1.3, 5.1.4 and 5.1.6 we record some properties of the Fourier multiplier $\mathbf{m}$, on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and the associated bounded operator $\mathbf{M}$, when $\mathbf{m}$ is of a particular form, namely the resolvent multiplied by an appropriate smooth function. Lemma 5.1.4 and 5.1.6 are based on ideas used in [34, Theorem 2.7]. We first introduce some notation. Let $\mathbf{T}$ be a C_0-semigroup with generator $\mathbf{A}$. Given $\alpha > s_0^\infty(\mathbf{A})$, we shall say that $\phi \in C^\infty(\mathbb{R})$

satisfies condition (P_α) if there exists a bounded open subset U_α of $\mathbb{R}$ containing the set $\{s \in \mathbb{R} : \alpha + is \in \sigma(\mathbf{A})\}$ and such that

$$\phi(s) = \begin{cases} 0 & (s \in U_\alpha), \\ 1 & (|s| \text{ sufficiently large }). \end{cases} \tag{5.5}$$

Further, given $\alpha > s_0^\infty(\mathbf{A})$ and $b \geq 0$, such that $Q_{\alpha,b} \subset \rho(\mathbf{A})$, a function $\phi \in C^\infty(\mathbb{R})$ shall satisfy property $(P_{\alpha,b})$ if there exists $b_1 > b$ with

$$\phi(s) = \begin{cases} 0 & (|s| \leq b) \\ 1 & (|s| > b_1). \end{cases} \tag{5.6}$$

Clearly, if ϕ satisfies $(P_{\alpha,b})$ then it satisfies (P_α). Next, we describe the functions $\mathbf{m}_\alpha \in \mathrm{L}^1(\mathbb{R}, \mathcal{L}(\mathbf{X}))$. For $\alpha > s_0^\infty(\mathbf{A})$, and ϕ satisfying property (P_α) we set

$$\mathbf{m}_\alpha(s) = \phi(s)\mathbf{R}(\alpha + is, \mathbf{A}), \tag{5.7}$$

with the understanding that $\mathbf{m}_\alpha(s) = 0$ whenever $\phi(s) = 0$. Where the meaning is clear, we shall often write just $\mathbf{m}(s)$ instead of $\mathbf{m}_0(s)$.

We shall be investigating the links between the non-analytic growth bound $\zeta(\mathbf{T})$ and the Fourier multiplier properties of the functions $\mathbf{m}_\alpha$. It turns out that the behaviour of $\mathbf{m}_\alpha$ as a Fourier multiplier is independent of the choice of ϕ satisfying property (P_α). This fact will play an important role in the subsequent theory and we record it as

Remark 5.1.1. Let $\alpha > s_0^\infty(\mathbf{A})$ and ϕ_1, ϕ_2 satisfy (P_α). If $\mathbf{m}_\alpha(s) = \phi_1(s)\mathbf{R}(\alpha + is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X}), 1 \leq p < \infty$, then $\tilde{\mathbf{m}}_\alpha$, given by $\tilde{\mathbf{m}}_\alpha(s) = \phi_2(s)\mathbf{R}(\alpha + is, \mathbf{A})$ is also a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. Indeed, $\tilde{\mathbf{m}}_\alpha - \mathbf{m}_\alpha \in C_c^\infty(\mathbb{R}, \mathcal{L}(\mathbf{X})) \subset \mathcal{FS}(\mathbb{R}, \mathcal{L}(\mathbf{X})) \subset \mathcal{F}\mathrm{L}^1(\mathbb{R}, \mathcal{L}(\mathbf{X}))$, and therefore, is a Fourier multiplier. In fact, if $\mathbf{m}_\alpha$ is in $\mathcal{F}\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$, then so is $\tilde{\mathbf{m}}_\alpha$. □

In [34, Theorem 2.7], the case when $\phi \equiv 1$ has been dealt with. It is shown there that $\mathbf{T}$ is hyperbolic if and only if $i\mathbb{R} \subset \rho(\mathbf{A})$ and $\mathbf{m}(s) = \mathbf{R}(is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$.

The following lemma is technical in nature and is needed to prove the main result of this section. It reduces to Lemma 2.4 of [34], when $\phi(s) = 1$ for all $s \in \mathbb{R}$.

Lemma 5.1.2. *Suppose that there exists $b > 0$ such that $Q_{0,b} \subset \rho(\mathbf{A})$ and*

$$\sup_{\lambda \in Q_{0,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty.$$

Let $\phi \in C^\infty(\mathbb{R})$ satisfy $(P_{0,b})$. Let $x \in \mathbf{X}$, $\Phi \in \mathcal{S}(\mathbb{R})$, $t \in \mathbb{R}$ and $\tau > 0$. Then,

$$\begin{aligned}\int_{\mathbb{R}} e^{ist}\phi(s)\mathbf{T}(\tau)\mathbf{R}(is,\mathbf{A})x\mathcal{F}\Phi(s)\,ds = &\int_{\mathbb{R}} e^{is(t+\tau)}\phi(s)\mathbf{R}(is,\mathbf{A})x\mathcal{F}\Phi(s)\,ds\\ &- 2\pi\int_0^\tau \mathbf{T}(u)x\Phi(t+\tau-u)\,du\\ &+ 2\pi\int_0^\tau \mathbf{T}(u)x(\psi*\Phi)(t+\tau-u)\,du,\end{aligned}$$

where $\psi = \mathcal{F}^{-1}(1-\phi)$.

Proof. Since $\dfrac{d}{dt}(e^{-ist}\mathbf{T}(t)\mathbf{R}(is,\mathbf{A})x) = -e^{-ist}\mathbf{T}(t)x$, integrating with respect to t gives

$$e^{-is\tau}\mathbf{T}(\tau)\mathbf{R}(is,\mathbf{A})x = \mathbf{R}(is,\mathbf{A})x - \int_0^\tau e^{-ist}\mathbf{T}(t)x\,dt,$$

for $|s| > b$. Making use of the above we have,

$$\begin{aligned}\int_{\mathbb{R}} e^{ist}\phi(s)\mathbf{T}(\tau)\mathbf{R}(is,\mathbf{A})x\mathcal{F}\Phi(s)\,ds &= \int_{\mathbb{R}} e^{i(t+\tau)s}\phi(s)\mathbf{R}(is,\mathbf{A})x\mathcal{F}\Phi(s)\,ds\\ &\quad - \int_{\mathbb{R}} e^{i(t+\tau)s}\phi(s)\int_0^\tau e^{-isu}\mathbf{T}(u)x\mathcal{F}\Phi(s)\,du\,ds\\ &= \int_{\mathbb{R}} e^{i(t+\tau)s}\phi(s)\mathbf{R}(is,\mathbf{A})x\mathcal{F}\Phi(s)\,ds\\ &\quad - \int_{\mathbb{R}}\int_0^\tau e^{i(t+\tau-u)s}\mathbf{T}(u)x\mathcal{F}\Phi(s)\,du\,ds\\ &\quad + \int_{\mathbb{R}}\int_0^\tau e^{i(t+\tau-u)s}\mathbf{T}(u)x(1-\phi(s))\mathcal{F}\Phi(s)\,du\,ds\\ &= \int_{\mathbb{R}} e^{i(t+\tau)s}\phi(s)\mathbf{R}(is,\mathbf{A})x\mathcal{F}\Phi(s)\,ds\\ &\quad - 2\pi\int_0^\tau \mathbf{T}(u)x\Phi(t+\tau-u)\,du\\ &\quad + 2\pi\int_0^\tau \mathbf{T}(u)x(\psi*\Phi)(t+\tau-u)\,du,\end{aligned}$$

where $\psi = \mathcal{F}^{-1}(1-\phi)$. □

Lemma 5.1.3. *Let $1 \le p < \infty$. Suppose there exists $b > 0$ such that*

1. *$Q_{0,b} \subset \rho(\mathbf{A})$, $\sup_{\lambda\in Q_{0,b}} \|\mathbf{R}(\lambda,\mathbf{A})\| < \infty$ and*
2. *$\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R},\mathbf{X})$, where*

$$\mathbf{m}(s) = \phi(s)\mathbf{R}(is,\mathbf{A}) \quad (s \in \mathbb{R})$$

and $\phi \in C^\infty(\mathbb{R})$ satisfies $(P_{0,b})$.

Then, there exists an $\epsilon > 0$ *such that* $Q_{-\epsilon,b} \subset \rho(\mathbf{A})$, $\sup_{\lambda \in Q_{-\epsilon,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty$, *and whenever* $|\alpha| < \epsilon$, $\mathbf{m}_\alpha$ *is a Fourier multiplier on* $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, *where*

$$\mathbf{m}_\alpha(s) = \phi(s)\mathbf{R}(\alpha + is, \mathbf{A}) \quad (s \in \mathbb{R}).$$

Proof. Let $C_1 = \sup_{\lambda \in Q_{0,b}} \|\mathbf{R}(\lambda, \mathbf{A})\|$. The condition on the resolvent of $\mathbf{A}$ ensures that C_1 is finite. Set

$$\epsilon_1 = \min\left(\frac{1}{2C_1 \sup\{|\phi(s)| : s \in \mathbb{R}\}}, \frac{1}{2C_1}\right).$$

Then, $Q_{-\epsilon_1,b} \subset \rho(\mathbf{A})$ and $\sup_{\lambda \in Q_{-\epsilon_1,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty$. Further, for $|\alpha| < \epsilon_1$, the Neumann series

$$\sum_{n=0}^{\infty} (-1)^n \alpha^n \phi(s)^{n+1} \mathbf{R}(is, \mathbf{A})^{n+1} = \phi(s)\mathbf{R}(\alpha\phi(s) + is, \mathbf{A})$$

is uniformly convergent for all $s \in \mathbb{R}$, where both the sides of the above equality are taken to be zero when $\phi(s) = 0$. Since $s \mapsto \mathbf{m}(s) = \phi(s)\mathbf{R}(is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, so is $s \mapsto \phi(s)^{n+1}\mathbf{R}(is, \mathbf{A})^{n+1}$ and therefore so is

$$s \mapsto \sum_{n=0}^{\infty} (-1)^n \alpha^n \phi(s)^{n+1} \mathbf{R}(is, \mathbf{A})^{n+1}, \tag{5.8}$$

provided $|\alpha| < \epsilon$ where

$$\epsilon = \min\left(\frac{1}{\|\mathbf{m}\|}, \epsilon_1\right).$$

Here, $\|\mathbf{m}\| = \|\mathbf{m}\|_{\mathcal{M}_p(\mathbf{X})}$. Therefore we may conclude that $s \mapsto \phi(s)\mathbf{R}(\alpha\phi(s) + is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ for $|\alpha| < \epsilon$.

Define $\mathbf{F} : \mathbb{R} \longrightarrow \mathcal{L}(\mathbf{X})$ by

$$\begin{aligned}
\mathbf{F}(s) &= \phi(s)\mathbf{R}(\alpha\phi(s) + is, \mathbf{A}) - \phi(s)\mathbf{R}(\alpha + is, \mathbf{A}) \\
&= \begin{cases} 0 & (|s| < b) \\ \alpha\phi(s)(1 - \phi(s))\mathbf{R}(\alpha\phi(s) + is, \mathbf{A})\mathbf{R}(\alpha + is, \mathbf{A}) & (b \le |s| \le b_1) \\ 0 & (|s| > b_1). \end{cases}
\end{aligned}$$

So $\mathbf{F} \in C_c^\infty(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ and is therefore a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. Thus,

$$s \mapsto \phi(s)\mathbf{R}(\alpha + is, \mathbf{A}) = \phi(s)\mathbf{R}(\alpha\phi(s) + is, \mathbf{A}) - \mathbf{F}(s)$$

is also a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ for all $\alpha \in \mathbb{R}$ satisfying $|\alpha| < \epsilon$. □

Lemma 5.1.4. *Let* $1 \le p < \infty$. *Suppose there exists* $b > 0$ *such that conditions (1) and (2) of Lemma 5.1.3 hold. Then the bounded operator* $\mathbf{M}$ *on* $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ *associated with the Fourier multiplier* $\mathbf{m}$ *is a bounded map from* $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ *to* $\mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$.

Proof. Let $1 \leq p < \infty$. We have to show that the bounded map $\mathbf{M} : \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \longrightarrow \mathrm{L}^p(\mathbb{R}, \mathbf{X})$, $\mathbf{M}(f) = \mathcal{F}^{-1}(\mathbf{m}\mathcal{F}f)$, $f \in \mathcal{S}(\mathbb{R}, \mathbf{X})$ maps $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ to $\mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$. Let

$$\Phi = \sum_{i=1}^{n} \Phi_i \otimes x_i \quad (x_i \in \mathbf{X},\ \Phi_i \in \mathcal{S}(\mathbb{R})).$$

Then Φ is in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. Since $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, $\mathbf{M}(\Phi) \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and there is a constant K such that

$$\|\mathbf{M}(\Phi)\|_p \leq K\|\Phi\|_p.$$

Now $\mathbf{M}(\Phi) \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ implies that for each $n \in \mathbb{N}$, there exists $s \in [n, n+1]$ such that

$$\|\mathbf{M}(\Phi)(s)\|_{\mathbf{X}} \leq \|\mathbf{M}(\Phi)\|_p \leq K\|\Phi\|_p$$

i.e. for each $n \in \mathbb{N}$, there exists $s \in [n, n+1]$ such that

$$\|\mathcal{F}^{-1}(\phi\mathbf{R}(i\cdot, \mathbf{A})\mathcal{F}\Phi)(s)\|_{\mathbf{X}} \leq K\|\Phi\|_p.$$

Let $t \in [0, 2]$. Then,

$$\begin{aligned} \|\mathbf{T}(t)(\mathcal{F}^{-1}(\phi\mathbf{R}(i\cdot, \mathbf{A})\mathcal{F}\Phi)(s))\| &\leq \sup_{u\in[0,2]} \|\mathbf{T}(u)\|\|\mathcal{F}^{-1}(\phi\mathbf{R}(i\cdot, \mathbf{A})\mathcal{F}\Phi)(s)\| \\ &\leq C\|\Phi\|_p, \end{aligned} \tag{5.9}$$

where $C = \sup_{u\in[0,2]} \|\mathbf{T}(u)\|K$ is a constant. Using Lemma 5.1.2, we have

$$\begin{aligned} 2\pi\mathbf{T}(t)(\mathcal{F}^{-1}(\phi\mathbf{R}(i\cdot, \mathbf{A})\mathcal{F}\Phi)(s)) &= \int_{\mathbb{R}} e^{is\tau}\mathbf{T}(t)\phi(\tau)\mathbf{R}(i\tau, \mathbf{A})\mathcal{F}\Phi(\tau)\, d\tau \\ &= \mathbf{M}(\Phi)(s+t) - 2\pi\int_0^t \mathbf{T}(u)\Phi(s+t-u)\, du \\ &\quad + 2\pi\int_0^t \mathbf{T}(u)(\psi * \Phi)(s+t-u)\, du \end{aligned} \tag{5.10}$$

where $\psi = \mathcal{F}^{-1}(1-\phi)$. From (5.10) and (5.9) it follows that

$$\begin{aligned} \|\mathbf{M}(\Phi)(s+t)\| &\leq 2\pi\|\int_0^t \mathbf{T}(u)\Phi(s+t-u)du\| \\ &\quad + 2\pi\|\int_0^t \mathbf{T}(u)(\psi*\Phi)(s+t-u)du\| + 2\pi C\|\Phi\|_p \\ &\leq 2\pi\left(\int_0^2 \|\mathbf{T}(u)\|^{p'}du\right)^{\frac{1}{p'}}\left(\|\Phi\|_p + \|\psi*\Phi\|_p\right) + 2\pi C\|\Phi\|_p \\ &\leq \tilde{C}\|\Phi\|_p, \end{aligned}$$

where $\tilde{C}$ is a constant. We have used Hölder's inequality to obtain the second last estimate for $1 < p < \infty$ with p' such that $1/p + 1/p' = 1$, and for $p = 1$, $\left(\int_0^2 \|\mathbf{T}(u)\|^{p'}du\right)^{\frac{1}{p'}}$ is to be interpreted as $\sup_{t\in[0,2]} \|\mathbf{T}(u)\|$. By varying the choice of t, n we obtain :

$$\|\mathbf{M}(\Phi)(u)\| \leq \tilde{C}\|\Phi\|_p, \text{ for all } u \in \mathbb{R}.$$

Therefore, $\mathbf{M}(\Phi) \in \mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$. This holds for all such Φ and by a density argument it follows that $\mathbf{M}$ maps $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ into $\mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$ and is bounded. □

Remark 5.1.5. We observe here that if the operator $\mathbf{M}$ is bounded from $\mathrm{L}^r(\mathbb{R}, \mathbf{X})$ to $\mathrm{L}^q(\mathbb{R}, \mathbf{X})$ for some r, q satisfying $1 \leq r, q < \infty$ then arguments almost identical to those above yield that $\mathbf{M}$ maps $\mathrm{L}^r(\mathbb{R}, \mathbf{X})$ to $\mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$. □

Lemma 5.1.6. *Let* $\mathbf{A}$ *be the generator of a* C_0*-semigroup* $\mathbf{T}$ *and* $1 \leq p < \infty$*. Suppose there exists* $b > 0$ *satisfying conditions (1) and (2) of Lemma 5.1.3. Then the associated operator* $\mathbf{M}$ *on* $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ *is bounded from* $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$ *to* $\mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$*.*

Proof. For $p = 1$, this is immediate from Lemma 5.1.4. Assume that $1 < p < \infty$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Since $\mathbf{M} : \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \longrightarrow \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ is bounded, so is the adjoint map $\mathbf{M}^* : \mathrm{L}^p(\mathbb{R}, \mathbf{X})^* \longrightarrow \mathrm{L}^p(\mathbb{R}, \mathbf{X})^*$. Recall from Subsection 2.3.4 that $\mathbf{X}^\odot$ is the subspace of $\mathbf{X}^*$ on which the restriction $\mathbf{T}^\odot$ of the adjoint semigroup $\mathbf{T}^*$ is strongly continuous. There is a natural isometric embedding of $\mathrm{L}^q(\mathbb{R}, \mathbf{X}^*)$ and hence of $\mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot)$ in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})^*$, [19, Page 98, Chapter 4]. Similarly, $\mathrm{L}^p(\mathbb{R}, \mathbf{X}^{\odot *})$ is isometrically embedded in $\mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot)^*$. Since $\mathbf{X}$ is isomorphically embedded in $\mathbf{X}^{\odot *}$ (see Subsection 2.3.4) it follows that $\|\cdot\|'_p$, given by

$$\|f\|'_p = \sup\big\{|\langle f, g\rangle| : g \in \mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot), \|g\|_q = 1\big\}, \tag{5.11}$$

defines an equivalent norm on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. Further, $\mathbf{M}^\odot$, the restriction of $\mathbf{M}^*$ to the space $\mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot)$, is given by

$$\mathbf{M}^\odot(g) = \mathcal{F}(\phi(\cdot)\mathbf{R}(i\cdot, \mathbf{A}^\odot)\mathcal{F}^{-1}g),$$

for $g \in \mathcal{S}(\mathbb{R}, \mathbf{X}^\odot)$. Therefore, $\mathbf{M}^\odot$ is a bounded operator from $\mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot)$ to itself. Then Lemma 5.1.4 with $\mathbf{M}$ replaced by $\mathbf{M}^\odot$, $\mathbf{T}$ by the semigroup $\mathbf{T}^\odot$, $\mathbf{X}$ by $\mathbf{X}^\odot$ and p by q implies that $\mathbf{M}^\odot : \mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot) \longrightarrow \mathrm{L}^\infty(\mathbb{R}, \mathbf{X}^\odot)$ is bounded. Therefore, for any $f \in \mathcal{S}(\mathbb{R}, \mathbf{X})$, it follows on using (5.11) that

$$\begin{aligned}
\|\mathbf{M}(f)\|'_p &= \sup\big\{|\langle \mathbf{M}(f), g\rangle| : g \in \mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot), \|g\|_q = 1\big\} \\
&= \sup\big\{|\langle f, \mathbf{M}^\odot g\rangle| : g \in \mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot), \|g\|_q = 1\big\} \\
&\leq \|f\|_1 \sup\big\{\|\mathbf{M}^\odot g\|_\infty : g \in \mathrm{L}^q(\mathbb{R}, \mathbf{X}^\odot), \|g\|_q = 1\big\} \\
&\leq C\|f\|_1,
\end{aligned}$$

where C is a constant. Consequently, $\mathbf{M}$ is a bounded operator from $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$ to $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. Therefore, it follows from Remark 5.1.5 that $\mathbf{M} : \mathrm{L}^1(\mathbb{R}, \mathbf{X}) \longrightarrow \mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$ is bounded. □

We now state the main result of this section. It is comparable to [34, Theorem 2.7], where it has been shown that a strongly continuous semigroup is hyperbolic if and only if the map $s \mapsto \mathbf{R}(is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ for some/all $p, 1 \leq p < \infty$. We prove the corresponding result for the non-analytic growth bound.

Theorem 5.1.7. *Let $1 \leq p < \infty$. For a C_0-semigroup $\mathbf{T}$ with generator $\mathbf{A}$ the following are equivalent:*

1. $\zeta(\mathbf{T}) < 0$;

2. $\kappa(\mathbf{T}) < 0$;

3. *There exists $b \geq 0$ such that $Q_{0,b} \subset \rho(\mathbf{A})$, $\sup_{\lambda \in Q_{0,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty$ and $\mathbf{m} \in \mathcal{F}\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$, where*

$$\mathbf{m}(s) = \phi(s)\mathbf{R}(is, \mathbf{A}) \quad (s \in \mathbb{R}),$$

   for some $\phi \in C^\infty(\mathbb{R})$ satisfying (P_0);

4. *There exist $b \geq 0$ such that $Q_{0,b} \subset \rho(\mathbf{A})$, $\sup_{\lambda \in Q_{0,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty$ and $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, where*

$$\mathbf{m}(s) = \phi(s)\mathbf{R}(is, \mathbf{A}) \quad (s \in \mathbb{R}),$$

   for some $\phi \in C^\infty(\mathbb{R})$ satisfying (P_0);

5. *There exist $b > 0$ and $\epsilon' > 0$ such that $Q_{-\epsilon',b} \subset \rho(\mathbf{A})$, $\sup_{\lambda \in Q_{-\epsilon',b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty$ and for each $x \in \mathbf{X}, x^* \in \mathbf{X}^*$ and $\Phi \in \mathcal{S}(\mathbb{R})$*

$$|\langle r_\alpha, \Phi \rangle| \leq K_\alpha \|x\| \|x^*\| \|\mathcal{F}\Phi\|_1$$

   for each $\alpha, |\alpha| < \epsilon'$ and some constant K_α where

$$r_\alpha(s) = \begin{cases} \langle x^*, \phi(s)\mathbf{R}(\alpha + is, \mathbf{A})x \rangle & (|s| > b) \\ 0 & (|s| \leq b) \end{cases}$$

   for some $\phi \in C^\infty(\mathbb{R})$ satisfying $(P_{\alpha,b})$. Here,

$$\langle r_\alpha, \Phi \rangle = \int_{\mathbb{R}} r_\alpha(s)\Phi(s)\, ds.$$

Remark 5.1.8. We note here that if the equivalent conditions of Theorem 5.1.7 hold then conditions *(3)* and *(4)* in Theorem 5.1.7 hold for all ϕ in view of Remark 5.1.1.

Proof of Theorem 5.1.7 . (1) $\Longrightarrow$ (2): This follows immediately from the definitions of $\zeta(\mathbf{T})$ and $\kappa(\mathbf{T})$.

(2) $\Longrightarrow$ (3): Since $\kappa(\mathbf{T}) < 0$ there exist $\epsilon > 0$ and $b > 0$ such that $Q_{-\epsilon,b} \subset \rho(\mathbf{A})$, $\sup_{\lambda \in Q_{-\epsilon,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty$ and

$$\mathbf{T}(t) = \mathbf{T}_{-\epsilon,b}(t) + \mathbf{T}_2(t) \quad (t \geq 0),$$

with $\mathrm{abs}(\|\mathbf{T}_2\|) < 0$. We may choose b in such a way that $Q_{0,b} \subset \rho(\mathbf{A})$ and

$$\sup_{\lambda \in Q_{0,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty.$$

In view of Remark 5.1.1 it is enough to prove the assertion for $\phi \in C^\infty(\mathbb{R})$ satisfying $(P_{0,b})$. Suppose then that $\phi \in C^\infty(\mathbb{R})$ and there exists $b_1 > b$ such that (5.6) hold. For $x \in \mathbf{X}$ we can write,

$$\phi(s)\mathbf{R}(is, \mathbf{A})x = \phi(s)\hat{\mathbf{T}}_{-\epsilon,b}(is)x + \phi(s)\hat{\mathbf{T}}_2(is)x$$

where $\phi(s)\hat{\mathbf{T}}_{-\epsilon,b}(is)$ is taken to be zero whenever $\phi(s)$ is zero. Putting

$$\begin{aligned}
\mathbf{m}_1(s)x &= \begin{cases} \phi(s)\hat{\mathbf{T}}_{-\epsilon,b}(is)x & (|s| > b) \\ 0 & (|s| \leq b), \end{cases} \\
\mathbf{m}_2(s)x &= (1 - \phi(s))\hat{\mathbf{T}}_2(is)x, \\
\mathbf{m}_3(s)x &= \hat{\mathbf{T}}_2(is)x,
\end{aligned}$$

we have

$$\mathbf{m}(s) = \mathbf{m}_1(s) - \mathbf{m}_2(s) + \mathbf{m}_3(s).$$

Since the function $t \mapsto \mathbf{T}_{-\epsilon,b}(t)$ has an exponentially bounded, holomorphic extension to some sector $\Sigma_\beta, 0 < \beta \leq \frac{\pi}{2}$, there exists $w \in \mathbb{R}$ such that

$$\sup_{z \in \Sigma_\beta} \|e^{-wz}\mathbf{T}_{-\epsilon,b}(z)\| < \infty.$$

It follows then from [2, Theorem 2.6.1] that

$$\sup_{\lambda \in w + \Sigma_{\gamma + \frac{\pi}{2}}} \|(\lambda - w)^{j+1}\hat{\mathbf{T}}^{(j)}_{-\epsilon,b}(\lambda)\| < \infty, \quad (j = 0, 1, 2)$$

for $0 < \gamma < \beta$. Since b may be chosen such that $is \in w + \Sigma_{\gamma + \frac{\pi}{2}}$ for $|s| > b$, there is a constant C such that

$$\|(is - w)^{j+1}\hat{\mathbf{T}}^{(j)}_{-\epsilon,b}(is)\| \leq C \quad (j = 0, 1, 2), \tag{5.12}$$

for $|s| > b$. Since $\phi \in C^\infty(\mathbb{R})$, $\mathbf{m}_1 : \mathbb{R} \longrightarrow \mathcal{L}(\mathbf{X})$ is a smooth, operator-valued function. Further, for any μ, $0 < \mu < 1$ we have, using (5.12) for $j = 0$ and 1,

$$\begin{aligned}
\sup_{s \in \mathbb{R}} \|s^{1+\mu}\mathbf{m}_1^{(1)}(s)\| &\leq \sup_{b < |s| < b_1} |s^\mu \phi^{(1)}(s)| \sup_{|s| > b} \|s\hat{\mathbf{T}}_{-\epsilon,b}(is)\| \\
&\quad + \sup_{|s| > b} |\phi(s)| \sup_{|s| > b} \|s^{\mu+1}\hat{\mathbf{T}}^{(1)}_{-\epsilon,b}(is)\| \\
&\leq C_1 \sup_{|s| > b} \frac{|s|}{|is - w|} + C_2 \sup_{|s| > b} \frac{|s|^{1+\mu}}{|is - w|^2} \\
&< \infty,
\end{aligned}$$

where C_1, C_2 are constants. In fact, using similar arguments we find

$$\sup_{s\in\mathbb{R}} \|s^{j+\mu}\mathbf{m}_1^{(j)}(s)\| < \infty \quad (j = 0, 1, 2).$$

Therefore, Theorem 5.0.4 applied to $\mathbf{m}_1$ allows us to conclude that

$$\mathbf{m}_1 \in \mathcal{F}\mathrm{L}^1(\mathbb{R}, \mathcal{L}(\mathbf{X})).$$

Due to the particular choice of ϕ, $\mathbf{m}_2$ has compact support and is obviously smooth. Therefore, $\mathbf{m}_2 \in C_c^\infty(\mathbb{R}, \mathcal{L}(\mathbf{X})) \subset \mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$. Since $\mathcal{F}\mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X})) = \mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$, we conclude that $\mathbf{m}_2 \in \mathcal{F}\mathrm{L}^1(\mathbb{R}, \mathcal{L}(\mathbf{X}))$.

By the hypothesis, $\mathrm{abs}(\|\mathbf{T}_2\|) < 0$, so that $\mathbf{T}_2 \in \mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ and $\mathbf{m}_3(s) = (\mathcal{F}\mathbf{T}_2)(s)$, where $\mathbf{T}_2(t) = 0$ for $t < 0$. Therefore, $\mathbf{m}_3 \in \mathcal{F}\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$. Thus the linear combination

$$\mathbf{m} = \mathbf{m}_1 - \mathbf{m}_2 + \mathbf{m}_3 \ \in \ \mathcal{F}\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X})).$$

(3) $\Longrightarrow$ *(4)* : *(3)* implies that there exists $\mathbf{S} \in \mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ satisfying

$$\mathbf{m}(s) = \phi(s)\mathbf{R}(is, \mathbf{A}) = \mathcal{F}\mathbf{S}(s).$$

Therefore, it follows from (5.3) that $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$.

(4) $\Longrightarrow$ *(5)*: It follows from *(4)* that there is a bounded linear map $\mathbf{M} : \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \longrightarrow \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ with

$$\mathbf{M}(f) = \mathcal{F}^{-1}(\mathbf{m}\mathcal{F}f),$$

for $f \in \mathcal{S}(\mathbb{R}, \mathbf{X})$, where $\mathbf{m}(s) = \phi(s)\mathbf{R}(is, \mathbf{A}),\ (s \in \mathbb{R})$ and $\phi \in C^\infty(\mathbb{R})$ satisfies $(P_{0,b})$. From Lemma 5.1.6, it follows that $\mathbf{M}$ is a bounded map from $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$ to $\mathrm{L}^\infty(\mathbb{R}, \mathbf{X})$, that is, there exists $K > 0$ such that

$$\|\mathbf{M}(f)\|_\infty \leq K\|f\|_1. \tag{5.13}$$

Fix $x \in \mathbf{X}$. Let $\Phi \in \mathcal{S}(\mathbb{R})$. Then $\mathcal{F}^{-1}\Phi \otimes x \in \mathcal{S}(\mathbb{R}, \mathbf{X})$ and

$$\|\mathbf{M}(\mathcal{F}^{-1}\Phi \otimes x)\|_\infty \leq K\|\mathcal{F}^{-1}\Phi\|_1\|x\|.$$

Note that

$$\mathbf{M}(\mathcal{F}^{-1}\Phi \otimes x) = \mathcal{F}^{-1}(\phi\mathbf{R}(i\cdot, \mathbf{A})\Phi \otimes x), \tag{5.14}$$

which is continuous as $\phi\mathbf{R}(i\cdot, \mathbf{A})\Phi \otimes x \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$. Therefore,

$$\|\mathbf{M}(\mathcal{F}^{-1}\Phi \otimes x)(0)\| \leq K\|\mathcal{F}^{-1}\Phi\|_1\|x\|. \tag{5.15}$$

It follows from (5.14) and (5.15) that

$$\begin{aligned}\left\|\int_{\mathbb{R}} \phi(s)\mathbf{R}(is, \mathbf{A})x\Phi(s)\, ds\right\| &= 2\pi\|\mathbf{M}(\mathcal{F}^{-1}\Phi \otimes x)(0)\| \\ &\leq 2\pi K\|\mathcal{F}^{-1}\Phi\|_1\|x\| \\ &= K\|\mathcal{F}\Phi\|_1\|x\|,\end{aligned}$$

for any $x \in \mathbf{X}$. For $x^* \in \mathbf{X}^*$, and

$$r(s) = \langle x^*, \phi(s)\mathbf{R}(is, \mathbf{A})x\rangle,$$

we therefore have,

$$\left|\int_{\mathbb{R}} r(s)\Phi(s)\,ds\right| \leq C_1\|x\|\|x^*\|\|\mathcal{F}\Phi\|_1. \tag{5.16}$$

Since $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, using Lemma 5.1.3 we can find an $\epsilon > 0$ such that $Q_{-\epsilon,b} \subset \rho(\mathbf{A})$ and $\sup_{\lambda\in Q_{-\epsilon,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty$ and $\mathbf{m}_\alpha$, given by $\mathbf{m}_\alpha(s) = \phi(s)\mathbf{R}(\alpha + is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, whenever $|\alpha| < \epsilon$. Let α be such that $|\alpha| < \epsilon$ and consider the rescaled semigroup $e^{-\alpha\cdot}\mathbf{T}(\cdot)$. Since $\mathbf{m}_\alpha$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and $\mathbf{m}_\alpha(s) = \phi(s)\mathbf{R}(is, -\alpha + \mathbf{A})$, where $-\alpha + \mathbf{A}$ is the generator of the rescaled semigroup, the previous argument applied to $\mathbf{M}_\alpha$ and $e^{-\alpha\cdot}\mathbf{T}(\cdot)$ yields the required inequality for r_α.

(5) $\Longrightarrow$ *(1)*: From *(5)* we have that $s_0^\infty(\mathbf{A}) < 0$. So, for $x \in \mathbf{X}$,

$$\lim_{s\to\infty} \|\mathbf{R}(w + is, \mathbf{A})x\| = 0$$

uniformly for $w \in [0, a]$ for each $a > 0$ [40, Lemma 1.3.2]. Hence the complex inversion theorem for Laplace transforms and Cauchy's Theorem yield the following formula for $\mathbf{T}(t)x$ when $t \geq 0$:

$$\mathbf{T}(t)x = \mathbf{T}_{0,b}(t)x + \frac{1}{2\pi}(C, 1)\int_{|s|>b} e^{ist}\mathbf{R}(is, \mathbf{A})x\,ds,$$

where the Cesaro-convergence of the integral is uniform for $t \in [0, a]$ for each $a > 0$ (see [2, Theorem 2.3.4] and [40, Theorem 1.3.3]). Let

$$\begin{aligned}\mathbf{S}(t)x &= \mathbf{T}(t)x - \mathbf{T}_{0,b}(t)x - \frac{1}{2\pi}\int_{b\leq|s|\leq b_1} e^{ist}(1 - \phi(s))\mathbf{R}(is, \mathbf{A})x\,ds\\ &= \frac{1}{2\pi}(C, 1)\int_{|s|>b} e^{ist}\phi(s)\mathbf{R}(is, \mathbf{A})x\,ds.\end{aligned}$$

For $\Phi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\Phi \subset \mathbb{R}_+$, the uniformity of the Cesaro-convergence gives

$$\begin{aligned}\int_0^\infty \mathbf{S}(t)x\Phi(t)\,dt &= \lim_{N\longrightarrow\infty}\int_0^\infty \frac{1}{2\pi}\int_{-N}^N \left(1 - \frac{|s|}{N}\right)e^{ist}\phi(s)\mathbf{R}(is, \mathbf{A})x\,ds\,\Phi(t)\,dt\\ &= \lim_{N\longrightarrow\infty}\int_{-N}^N \left(1 - \frac{|s|}{N}\right)\phi(s)\mathbf{R}(is, \mathbf{A})x\mathcal{F}^{-1}\Phi(s)\,ds\\ &= \int_{-\infty}^\infty \phi(s)\mathbf{R}(is, \mathbf{A})x\mathcal{F}^{-1}\Phi(s)\,ds.\end{aligned} \tag{5.17}$$

The assumption *(5)* implies that for $x^* \in \mathbf{X}^*$,

$$\begin{aligned}\left|\int_0^\infty \langle x^*, \mathbf{S}(t)x\rangle\Phi(t)\,dt\right| &= |\langle r_0, \mathcal{F}^{-1}\Phi\rangle|\\ &\leq K_0\|x\|\|x^*\|\|\Phi\|_1\end{aligned}$$

for all $x \in \mathbf{X}$. Since $t \mapsto \langle x^*, \mathbf{S}(t)x\rangle$ is continuous, this implies that

$$\big|\langle x^*, \mathbf{S}(t)x\rangle\big| \leq K_0\|x\|\|x^*\|,$$

for all $t \geq 0$. Hence

$$\|\mathbf{S}(t)x\| \leq K_0\|x\|.$$

Further, since

$$\left\|\frac{1}{2\pi}\int_{b\leq|s|\leq b_1} e^{ist}(1-\phi(s))\mathbf{R}(is,\mathbf{A})x\,ds\right\| \leq C\|x\|$$

for some constant C, we conclude that

$$\|\mathbf{T}(t)x - \mathbf{T}_{0,b}(t)x\| \leq K_0\|x\|$$

for some constant K_0 and all $x \in \mathbf{X}$. It follows that

$$\|\mathbf{T}(t) - \mathbf{T}_{0,b}(t)\| \leq K_0$$

so that $\zeta(\mathbf{T}) \leq 0$.

Now, take $\alpha \in \mathbb{R} : -\epsilon < \alpha < 0$. Applying the above procedure to the rescaled semigroup $e^{-\alpha\cdot}\mathbf{T}(\cdot)$ and making use of the hypothesis on r_α we obtain $\zeta(\mathbf{T}) \leq \alpha < 0$. □

Remark 5.1.9. If the equivalent conditions of Theorem 5.1.7 hold then $\mathbf{m} = \mathcal{F}\mathbf{S}$ where $\mathbf{S} \in \mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ is given by

$$\mathbf{S}(t)x = \begin{cases} \mathbf{T}(t)x - \mathbf{T}_{0,b}(t)x - \dfrac{1}{2\pi i}\displaystyle\int_{b\leq|s|\leq b_1} e^{ist}(1-\phi(s))\mathbf{R}(is,\mathbf{A})x\,ds & (t \geq 0) \\ -\mathbf{T}_{0,b}(t)x - \dfrac{1}{2\pi i}\displaystyle\int_{b\leq|s|\leq b_1} e^{ist}(1-\phi(s))\mathbf{R}(is,\mathbf{A})x\,ds & (t < 0). \end{cases}$$

For $t \geq 0$ this follows from (5.17). For $t < 0$ one has

$$0 = \mathbf{T}_{0,b}(t)x + \frac{1}{2\pi}(C,1)\int_{|s|\geq b} e^{ist}\mathbf{R}(is,\mathbf{A})x\,dt$$

where the Cesaro-convergence is uniform on compact subsets of $(-\infty, 0)$, and a similar argument to the proof of *(5)* $\Longrightarrow$ *(1)* above leads to the conclusion. □

For a C_0-semigroup $\mathbf{T}$, it is trivial that $\omega_0(\mathbf{T}) = \mathrm{abs}(\|\mathbf{T}\|)$ (see e.g. [2, Prop. 5.1.1]). Theorem 5.1.7 gives the following analogue of this result for $\zeta(\mathbf{T})$. We do not know of any direct proof of this fact.

Corollary 5.1.10. *Let* $\mathbf{T}$ *be a* C_0*-semigroup on* $\mathbf{X}$*. Then*

$$\zeta(\mathbf{T}) = \kappa(\mathbf{T}).$$

As an immediate consequence of Theorem 5.1.7, we have

Corollary 5.1.11. *For a C_0-semigroup $\mathbf{T}$ with generator $\mathbf{A}$ defined on $\mathbf{X}$,*

$$s_0^\infty(\mathbf{A}) = \delta(\mathbf{T}) = \zeta(\mathbf{T}) \iff \textit{for each } \alpha > s_0^\infty(\mathbf{A}), s \mapsto \phi(s)\mathbf{R}(\alpha + is) \textit{ is a Fourier multiplier on } \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \textit{ for some } p \in [1, \infty), \textit{for some } \phi \in C^\infty(\mathbb{R}) \textit{ satisfying } (P_\alpha).$$

Remark 5.1.12. As already noted, if the equivalent conditions of Theorem 5.1.7 hold, then condition *(2)* holds whenever $\phi \in C^\infty(\mathbb{R})$, satisfies (P_0). Let $1 \leq p < \infty$. Assume now that $i\mathbb{R} \subset \rho(\mathbf{A})$. Then Theorem 5.1.7 gives

$$\zeta(\mathbf{T}) < 0 \iff \mathbf{R}(i\cdot, \mathbf{A}) \text{ is a Fourier multiplier on } \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \text{ and } s_0^\infty(\mathbf{A}) < 0. \tag{5.18}$$

Latushkin and Shvydkoy [34, Theorem 2.7] have shown that

$$\mathbf{T} \text{ is hyperbolic } \iff \mathbf{R}(i\cdot, \mathbf{A}) \text{ is a Fourier multiplier on } \mathrm{L}^p(\mathbb{R}, \mathbf{X}).$$

Therefore, (5.18) reduces to

$$\zeta(\mathbf{T}) < 0 \iff s_0^\infty(\mathbf{A}) < 0 \text{ and } \mathbf{T} \text{ is hyperbolic.}$$

However, using completely different methods it has been shown in [10, Theorem 3.5.2] that if $\mathbf{T}$ is hyperbolic, then $\zeta(\mathbf{T}) < 0 \iff s_0^\infty(\mathbf{A}) < 0$. Further, [10, Proposition 4.5.5] states that if $\mathbf{T}$ is a C_0-semigroup and $\delta(\mathbf{T}) < 0$ then $\mathbf{T}$ is hyperbolic if and only if $i\mathbb{R} \subset \rho(\mathbf{A})$. Thus (5.18) is just a reformulation of known results in the case when $i\mathbb{R} \subset \rho(\mathbf{A})$. □

Remark 5.1.13. From Theorem 5.1.7 we are able to obtain a different proof for the fact that $s_0^\infty(\mathbf{A}) = \delta(\mathbf{T}) = \zeta(\mathbf{T})$ for a strongly continuous semigroup on a Hilbert space. This analogue of the Gearhart-Prüss Theorem [see page 35] was first shown by M. Blake [10], [11, Lemma 4.3]. The alternative proof works in the following way: Suppose that $s_0^\infty(\mathbf{A}) < 0$. Then the first part of condition *(4)* of Theorem 5.1.7 is satisfied. Since $\mathbf{X}$ is a Hilbert space, the function $\mathbf{m}$ defined in hypothesis *(4)* of Theorem 5.1.7, being bounded, is a Fourier multiplier on $\mathrm{L}^2(\mathbb{R}, \mathbf{X})$. So it follows from Theorem 5.1.7 that $\zeta(\mathbf{T}) < 0$. A rescaling argument then implies that $s_0^\infty(\mathbf{A}) \geq \zeta(\mathbf{T})$. For any strongly continuous semigroup, $s_0^\infty(\mathbf{A}) \leq \delta(\mathbf{T}) \leq \zeta(\mathbf{T})$. Therefore, the required equality holds.

5.2 Perturbations

The technique used in Lemma 5.1.3 enables us to deduce how the non-analytic growth bound of a C_0-semigroup behaves under small bounded perturbations. We have,

Theorem 5.2.1. *$\zeta(\cdot)$ is upper semi-continuous with respect to bounded perturbations, that is, if $\mathbf{A}$ generates a C_0-semigroup $\mathbf{T}$ and $\epsilon > 0$ then there exists $\delta > 0$ such that $\zeta(\mathbf{S}) \leq \zeta(\mathbf{T}) + \epsilon$ whenever $\mathbf{S}$ is a C_0-semigroup generated by $\mathbf{A} + \mathbf{B}$ where $\mathbf{B}$ is a bounded operator such that $\|\mathbf{B}\| < \delta$.*

Proof. Let $\mathbf{A}$ be the generator of a C_0-semigroup $\mathbf{T}$. We note that it is enough to show that if $\zeta(\mathbf{T}) < 0$, then there exists a $\delta > 0$ such that for every bounded operator $\mathbf{B}$ with $\|\mathbf{B}\| < \delta$, $\zeta(\mathbf{S}) < 0$, $\mathbf{S}$ being the semigroup obtained on perturbing $\mathbf{A}$ by $\mathbf{B}$. So, suppose $\zeta(\mathbf{T}) < 0$. Then using Theorem 5.1.7 we can find a $b \geq 0$ and an $\epsilon_0 > 0$ such that $Q_{-\epsilon_0,b} \subset \rho(\mathbf{A})$, $\sup_{\lambda \in Q_{-\epsilon_0,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty$ and $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$ where

$$\mathbf{m}(s) = \phi(s)\mathbf{R}(is, \mathbf{A}) \quad (s \in \mathbb{R})$$

for some $\phi \in C^\infty(\mathbb{R})$ satisfying $(P_{0,b})$. We may also assume that $0 \leq \phi \leq 1$. Let $w > \omega_0(\mathbf{T})$,

$$K = \sup\{\|\mathbf{R}(\lambda, \mathbf{A})\| : \lambda \in Q_{-\epsilon_0,b} \cup \{\mu : \operatorname{Re}\mu > w\}\}$$

and $\mathbf{B}$ be a bounded operator on $\mathbf{X}$ with $\|\mathbf{B}\| \leq \frac{1}{2K}$. Let $\mathbf{S}$ be the C_0-semigroup generated by $\mathbf{A} + \mathbf{B}$. Then $Q_{-\epsilon_0,b} \cup \{\mu : \operatorname{Re}\mu > w\} \subset \rho(\mathbf{A} + \mathbf{B})$ and

$$\sup\{\|\mathbf{R}(\lambda, \mathbf{A} + \mathbf{B})\| : \lambda \in Q_{-\epsilon_0,b} \cup \{\mu : \operatorname{Re}\mu > w\}\} \leq 2K < \infty.$$

This is due to the fact that for any $\lambda \in Q_{-\epsilon_0,b} \cup \{\mu : \operatorname{Re}\mu > w\}$

$$\|\mathbf{R}(\lambda, \mathbf{A})\mathbf{B}\| \leq \frac{1}{2},$$

so that $(\mathbf{I} - \mathbf{R}(\lambda, \mathbf{A})\mathbf{B})$ is invertible (see [10, Lemma 5.2.1]). Further, for all such λ

$$\begin{aligned} \mathbf{R}(\lambda, \mathbf{A} + \mathbf{B}) &= (\mathbf{I} - \mathbf{R}(\lambda, \mathbf{A})\mathbf{B})^{-1}\mathbf{R}(\lambda, \mathbf{A}) \\ &= \sum_{n=0}^{\infty} (-1)^n (\mathbf{R}(\lambda, \mathbf{A})\mathbf{B})^n \mathbf{R}(\lambda, \mathbf{A}). \end{aligned} \tag{5.19}$$

Also, for any $s \in \mathbb{R}$ we have

$$\phi(s)\left(\mathbf{I} - \phi(s)\mathbf{R}(is, \mathbf{A})\mathbf{B}\right)^{-1}\mathbf{R}(is, \mathbf{A}) = \sum_{n=0}^{\infty} (-1)^n \phi(s)^{n+1} (\mathbf{R}(is, \mathbf{A})\mathbf{B})^n \mathbf{R}(is, \mathbf{A}), \tag{5.20}$$

where, both sides of the equality are zero for $|s| < b$. Since $s \mapsto \phi(s)\mathbf{R}(is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$, so is the map $s \mapsto \phi(s)\mathbf{R}(is, \mathbf{A})\mathbf{B}$. Therefore, for each n, the map $s \mapsto \phi(s)^n(\mathbf{R}(is, \mathbf{A})\mathbf{B})^n$ is also a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$. Thus,

$$s \mapsto \left(\mathbf{I} - \phi(s)\mathbf{R}(is, \mathbf{A})\mathbf{B}\right)^{-1}\phi(s)\mathbf{R}(is, \mathbf{A})$$

is a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$ provided $\|\phi(\cdot)\mathbf{R}(i\cdot, \mathbf{A})\mathbf{B}\|_{\mathcal{M}_1(\mathbf{X})} < 1$. Recall from Theorem 5.1.7 that under the present hypothesis, there is an operator $\mathbf{U} \in \mathcal{F}\mathrm{L}^1_s(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ such that $\phi(s)\mathbf{R}(is, \mathbf{A}) = (\mathcal{F}\mathbf{U})(s)$. Then it is easy to see that

$$\|\phi(\cdot)\mathbf{R}(i\cdot, \mathbf{A})\mathbf{B}\|_{\mathcal{M}_1(\mathbf{X})} \leq \|\mathbf{U}\|_1\|\mathbf{B}\|,$$

where $\|\mathbf{U}\|_1 = \displaystyle\int_{-\infty}^{\infty} \|\mathbf{U}(t)\|\, dt$. Let

$$\delta = \min\left(\frac{1}{2K}, \frac{1}{\|\mathbf{U}\|_1}\right).$$

Then, for $\|\mathbf{B}\| < \delta$, the map $s \mapsto (\mathbf{I} - \phi(s)\mathbf{R}(is, \mathbf{A})\mathbf{B})^{-1}\phi(s)\mathbf{R}(is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$. Define $\mathbf{F} : \mathbb{R} \longrightarrow \mathcal{L}(\mathbf{X})$ by

$$\begin{aligned}
\mathbf{F}(s) &= (\mathbf{I} - \phi(s)\mathbf{R}(is, \mathbf{A})\mathbf{B})^{-1}\phi(s)\mathbf{R}(is, \mathbf{A}) - \phi(s)\mathbf{R}(is, \mathbf{A} + \mathbf{B}) \\
&= \begin{cases} 0 & (|s| < b) \\ (\mathbf{I} - \phi(s)\mathbf{R}(is, \mathbf{A})\mathbf{B})^{-1}\phi(s)\mathbf{R}(is, \mathbf{A}) - \phi(s)\mathbf{R}(is, \mathbf{A} + \mathbf{B}) & (b \leq |s| \leq b_1) \\ 0 & (|s| > b_1). \end{cases}
\end{aligned}$$

So $\mathbf{F} \in C_c^\infty(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ and is therefore a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$. Thus, $s \mapsto \phi(s)\mathbf{R}(is, \mathbf{A} + \mathbf{B})$ is a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$ provided $\|\mathbf{B}\| < \delta$. From Theorem 5.1.7 it then follows that $\zeta(\mathbf{S}) < 0$ as required. □

Neither $\omega_0(\cdot)$ nor $\delta(\cdot)$ is lower semi-continuous under bounded perturbations of C_0-semigroups. Examples exhibiting this have been given by M. Blake [10, Example 5.3.4]. It turns out that $\zeta(\cdot)$ is not lower semi-continuous under bounded perturbations of C_0-semigroups either. In fact, [10, Example 5.3.4] works for this case also.

5.3 Inhomogeneous Cauchy problems

A characterisation of the non-analytic growth bound in terms of the existence of unique mild solutions of certain inhomogeneous Cauchy problems on $\mathbb{R}$, under some conditions, is presented in this section. By an inhomogeneous Cauchy problem on $\mathbb{R}$ we mean an equation

$$u'(t) = \mathbf{A}u(t) + f(t) \quad (t \in \mathbb{R}). \tag{ACP_f}$$

Here $\mathbf{A}$ is a closed operator on $\mathbf{X}$, and $f \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbf{X})$.

For $f \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbf{X})$, a function $u \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbf{X})$ is called a *mild solution* of (ACP_f) if

$$\int_s^t u(\tau)\, d\tau \in D(\mathbf{A}), \tag{5.21}$$

$$u(t) = u(s) + \mathbf{A}\int_s^t u(\tau)\, d\tau + \int_s^t f(\tau)\, d\tau, \tag{5.22}$$

for almost all t, s $(t \geq s)$. We observe here that then (5.22) is true for almost all $(s,t) \in \mathbb{R}^2$ (with the usual convention about $\int_a^b = -\int_b^a$). We define a *classical solution* as a C^1-function u such that $u(t) \in D(\mathbf{A})$, for all t and such that (ACP_f) is valid. For f continuous, a continuous mild solution u is a classical solution if and only if $u \in C^1(\mathbb{R}, \mathbf{X})$. In the special case when $\mathbf{A}$ is the generator of a C_0- semigroup, we have

Lemma 5.3.1. *Let u be a mild solution of (ACP_f) in $\mathrm{L}^1_{loc}(\mathbb{R}, \mathbf{X})$ and $f \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbf{X})$. If $\mathbf{A}$ is the generator of a C_0-semigroup then there exists a unique continuous function $\tilde{u}$, such that $\tilde{u} = u$ a.e. and*

$$\tilde{u}(t) = \mathbf{T}(t-s)\tilde{u}(s) + \int_s^t \mathbf{T}(t-r)f(r)\,dr \tag{5.23}$$

whenever $t \geq s$.

Proof. Let u be a mild solution of (ACP_f), where $f \in \mathrm{L}^1_{loc}(\mathbb{R}, \mathbf{X})$. First assume that both f, u are Laplace transformable. Then (5.21) and (5.22) hold for almost all $(t,s) \in \mathbb{R}^2$. Suppose first that $s = 0$ so that $\displaystyle\int_0^t u(\tau)\,d\tau \in D(\mathbf{A})$ and

$$u(t) = u(0) + \mathbf{A}\int_0^t u(\tau)\,d\tau + \int_0^t f(\tau)\,d\tau,$$

for almost all $t \geq 0$. Taking Laplace transforms, we have for $\operatorname{Re}\lambda$ sufficiently large,

$$\hat{u}(\lambda) = \frac{u(0)}{\lambda} + \frac{\mathbf{A}\hat{u}(\lambda)}{\lambda} + \frac{\hat{f}(\lambda)}{\lambda}$$
$$\hat{u}(\lambda) = \mathbf{R}(\lambda, \mathbf{A})\left(u(0) + \hat{f}(\lambda)\right).$$

Let $v_0(t) = \mathbf{T}(t)u(0) + \displaystyle\int_0^t \mathbf{T}(t-r)f(r)\,dr \quad (t \geq 0)$. Then for $\operatorname{Re}\lambda$ sufficiently large,

$$\hat{v}_0(\lambda) = \hat{u}(\lambda).$$

Therefore, by the Uniqueness Theorem, $u = v_0$ a.e., that is,

$$u(t) = \mathbf{T}(t)u(0) + \int_0^t \mathbf{T}(t-\tau)f(\tau)\,d\tau$$

for almost all $t \geq 0$. For s not necessarily 0, we obtain, on applying the previous case to appropriate translates of u and f,

$$u(t) = \mathbf{T}(t-s)u(s) + \int_s^t \mathbf{T}(t-\tau)f(\tau)\,d\tau$$

for almost all $s \in \mathbb{R}$ and for almost all $t \geq s$. Define, for $s \in \mathbb{R}$,

$$u_s(t) = \mathbf{T}(t-s)u(s) + \int_s^t \mathbf{T}(t-\tau)f(\tau)\,d\tau \quad (t \geq s).$$

Then u_s is continuous on $[s, \infty)$. Further, for almost all (s_1, s_2), $s_1 < s_2$,

$$u_{s_1}(t) = u_{s_2}(t) = u(t), \text{ for almost all } t \in [s_2, \infty).$$

By continuity, for almost all s_1, s_2, $u_{s_1}(t) = u_{s_2}(t)$ for all $t \in [s_2, \infty)$. Define

$$\tilde{u}(t) = u_s(t), \text{ for such } s.$$

This is independent of the choice of s (excluding a null set). Therefore, $\tilde{u}$ is continuous and $\tilde{u}(t) = u(t)$ a.e. Further,

$$\tilde{u}(t) = \mathbf{T}(t-s)\tilde{u}(s) + \int_s^t \mathbf{T}(t-\tau)f(\tau)\,d\tau$$

for almost all $t \geq s$, and therefore, by continuity for all $t \geq s$.

For the general case, fix $\tau > 0$. Replacing f, u by the functions g, v respectively, where

$$g(t) = \begin{cases} f(t) & (t < \tau) \\ 0 & (t \geq \tau); \end{cases}$$

$$v(t) = \begin{cases} u(t) & (t < \tau) \\ \mathbf{T}(t-\tau)u(\tau) & (t \geq \tau). \end{cases}$$

Then the previous case applies to g and v and we obtain the required result. □

In what follows, we shall be considering (ACP_f) with the function $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and $\mathbf{A}$ satisfying a condition of non-resonance. To make this precise, we recall the definition of the Carleman transform and Carleman spectrum of a function $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ (Sections 4.6 and 4.8 of [2]).

Let $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$, $1 \leq p \leq \infty$. The Carleman transform $\hat{f}$ of f is defined by

$$\hat{f}(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} f(t)\,dt & (\mathrm{Re}\,\lambda > 0) \\ \displaystyle -\int_0^\infty e^{\lambda t} f(-t)\,dt & (\mathrm{Re}\,\lambda < 0). \end{cases}$$

Then $\hat{f}$ is a holomorphic function defined on $\mathbb{C} \setminus i\mathbb{R}$. We use the same symbol for the Carleman transform and the Laplace transform. This will not lead to any confusion. A point $i\eta \in i\mathbb{R}$ is called *regular* for $\hat{f}$ if there exists an open neighbourhood V of $i\eta$ in $\mathbb{C}$ and a holomorphic function $h : V \longrightarrow \mathbf{X}$ such that $h(\lambda) = \hat{f}(\lambda)$ for all $\lambda \in V \setminus i\mathbb{R}$. The *Carleman spectrum* $\mathrm{sp}_c(f)$ of f, is defined by

$$\mathrm{sp}_c(f) = \{\eta \in \mathbb{R} : i\eta \text{ is not regular for } \hat{f}\}.$$

A closed operator $\mathbf{A}$ defined on $\mathbf{X}$ is said to have no *resonance* with $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ (or, $\mathbf{A}, f$ satisfy a condition of *non-resonance*) if $i\,\mathrm{sp}_c(f) \cap \sigma(\mathbf{A})$ is empty.

For $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ we define $\mathcal{F}f$ as a linear mapping from $\mathcal{S}(\mathbb{R})$ into $\mathbf{X}$ by

$$\langle \Phi, \mathcal{F}f \rangle = \int_{\mathbb{R}} f(t)(\mathcal{F}\Phi)(t)\, dt \quad (\Phi \in \mathcal{S}(\mathbb{R})).$$

The *support* of $\mathcal{F}f$ is defined by

$$\begin{aligned}\operatorname{supp} \mathcal{F}f = \big\{\eta \in \mathbb{R} :\ &\text{for all } \epsilon > 0,\ \text{there exists } \Phi \in \mathcal{S}(\mathbb{R}) \text{ such that} \\ &\operatorname{supp} \Phi \subset (\eta - \epsilon, \eta + \epsilon) \text{ and } \langle \Phi, \mathcal{F}f \rangle \neq 0\big\}. \end{aligned} \tag{5.24}$$

This is consistent with the definition of Fourier transforms of distributions. From [2, Theorem 4.8.1], we have for all $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$

$$\operatorname{sp}_c(f) = \operatorname{supp} \mathcal{F}f. \tag{5.25}$$

If $f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ then

$$\operatorname{sp}_c(f) = \operatorname{supp} \mathcal{F}f = \{s \in \mathbb{R} : (\mathcal{F}f)(s) \neq 0\}^-. \tag{5.26}$$

Here $\mathcal{F}f$ is the usual function. For $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and $\Phi \in \mathcal{S}(\mathbb{R})$ we have from [2, Remark 4.8.6],

$$\operatorname{sp}_c(f * \Phi) \subset \operatorname{sp}_c(f) \cap \operatorname{supp} \mathcal{F}\Phi. \tag{5.27}$$

Further, using the idea in the first part of the proof of [2, Theorem 4.8.1], it is easy to deduce the following.

Lemma 5.3.2. *If $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and $\Phi \in \mathcal{S}(\mathbb{R})$ are such that $\operatorname{sp}_c(f) \cap \operatorname{supp} \Phi$ is empty, then $\langle \Phi, \mathcal{F}f \rangle = 0$.*

It follows from Lemma 5.3.2, (5.24) and (5.25) that if E is a closed subset of $\mathbb{R}$ then

$$\begin{aligned}\operatorname{sp}_c(f) \subset E \iff\ &\Phi \in \mathcal{S}(\mathbb{R}), \operatorname{supp} \Phi \cap E = \emptyset \\ &\text{implies that } \langle \Phi, \mathcal{F}f \rangle = 0. \end{aligned} \tag{5.28}$$

Throughout this section, given two functions g, h defined on $\mathbb{R}$, we shall use $\langle g, h \rangle$ to denote

$$\int_{\mathbb{R}} g(s)h(s)\, ds$$

whenever this integral makes sense. In the expression $\langle \cdot, \cdot \rangle$ the first coordinate will usually be reserved for either a scalar or an operator-valued function.

Remark 5.3.3. Given $\Phi \in \mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$, and $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$, the integral $\displaystyle\int_{\mathbb{R}} (\mathcal{F}\Phi)(t) f(t)\, dt$ is well defined. Thus, given $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$, we may define the linear mapping Ff from $\mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ into $\mathbf{X}$ by

$$\langle \Phi, Ff \rangle = \int_{\mathbb{R}} (\mathcal{F}\Phi)(t) f(t)\, dt.$$

It is easy to see that Ff extends the definition of $\mathcal{F}f$ given above. In fact, we may define $\operatorname{supp} Ff$ analogously to the definition in (5.24). Further, a proof identical to that of [2, Theorem 4.8.1] shows that

$$\operatorname{supp} \mathcal{F}f = \operatorname{sp}_c(f) = \operatorname{supp} Ff,$$

for all $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$. The corresponding versions of Lemma 5.3.2 and (5.28) also hold. From now on we shall denote Ff also by $\mathcal{F}f$. This will not lead to any confusion since the choice of Φ in $\langle \Phi, \mathcal{F}f \rangle$ clearly indicates whether we are referring to the linear map from $\mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ or from $\mathcal{S}(\mathbb{R})$. □

If $\mathbf{m}$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, and $\mathbf{M}$ is the associated bounded operator on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, then the following analogue of (5.1) holds for all $\Phi \in \mathcal{S}(\mathbb{R})$

$$\langle \Phi, \mathcal{F}(\mathbf{M}f) \rangle = \langle \mathcal{F}(\Phi \mathbf{m}), f \rangle \quad (f \in \mathcal{S}(\mathbb{R}, \mathbf{X})). \tag{5.29}$$

Due to the density of $\mathcal{S}(\mathbb{R}, \mathbf{X})$ in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, (5.29) holds for all $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$, if $\mathbf{m}$ is smooth and has bounded derivatives. Then, from the definition of $\operatorname{supp} \mathcal{F}f (= \operatorname{supp} Ff)$, it follows that

$$\operatorname{supp} \mathcal{F}(\mathbf{M}f) \subset \operatorname{supp} \mathcal{F}f. \tag{5.30}$$

With this framework in the background, we proceed with our study of the relation between the existence of unique mild solutions of (ACP_f), $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X}), 1 \leq p < \infty$, and the non-analytic growth bound. We shall deal with the case $p = 1$ first because the proofs are a little simpler. In order to obtain our characterisation of $\zeta(\mathbf{T})$ in terms of mild solutions of (ACP_f) in this case, we need a result which may be considered as the analogue on $\mathbb{R}$ of the well known result characterising continuous mild solutions of (ACP_f) on $\mathbb{R}_+$ in terms of Laplace transforms [2, Theorem 3.1.3]. For the proof, we shall make use of the following analogue of [2, Proposition 1.7.6]. This is probably well known and we omit the proof which is very similar to [2, Proposition 1.7.6].

Lemma 5.3.4. *Let $\mathbf{A}$ be a closed linear operator on $\mathbf{X}$. For $f, g \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$, the following are equivalent:*

1. *$\mathcal{F}f(s) \in D(\mathbf{A})$ and $\mathbf{A}\mathcal{F}f(s) = \mathcal{F}g(s)$, for all $s \in \mathbb{R}$.*
2. *$f(t) \in D(\mathbf{A})$ and $\mathbf{A}f(t) = g(t)$ a.e. on $\mathbb{R}$.*

Lemma 5.3.5. *Let $f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ and consider the inhomogeneous Cauchy problem*

$$u'(t) = \mathbf{A}u(t) + f(t) \tag{ACP_f}$$

where $\mathbf{A}$ is a closed linear operator on $\mathbf{X}$. Suppose $u \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$. Then u is a mild solution of (ACP_f) if and only if $(\mathcal{F}u)(s) \in D(\mathbf{A})$ and for all $s \in \mathbb{R}$,

$$(is - \mathbf{A})\mathcal{F}u(s) = \mathcal{F}f(s). \tag{5.31}$$

Proof. Suppose first that $u, f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ and (5.31) holds. We show that u satisfies (5.21) and (5.22). Let (ρ_n) be a sequence of functions in $C_c^\infty(\mathbb{R})$ which forms a mollifier. Then

$$\begin{aligned} \lim_{n\to\infty} \|\rho_n * u - u\|_1 &= 0, \\ \lim_{n\to\infty} \|\rho_n * f - f\|_1 &= 0. \end{aligned}$$

From (5.31) it follows that

$$\begin{aligned} (is - \mathbf{A})\mathcal{F}(\rho_n * u)(s) &= (is - \mathbf{A})\mathcal{F}\rho_n(s)\mathcal{F}u(s) \\ &= \mathcal{F}\rho_n(s)\mathcal{F}f(s) \\ &= \mathcal{F}(\rho_n * f)(s), \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{5.32}$$

Since $\rho_n * u \in C^\infty(\mathbb{R}, \mathbf{X}) \cap \mathrm{L}^1(\mathbb{R}, \mathbf{X})$, and $(\rho_n * u)' \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$, (5.32) may be rewritten as

$$\mathcal{F}(\rho_n * u)'(s) - \mathbf{A}\mathcal{F}(\rho_n * u)(s) = \mathcal{F}(\rho_n * f)(s) \text{ for all } s \in \mathbb{R}.$$

Then Lemma 5.3.4 implies that for each $n \in \mathbb{N}, (\rho_n * u)(t) \in D(\mathbf{A})$ and

$$(\rho_n * u)'(t) - \mathbf{A}(\rho_n * u)(t) = (\rho_n * f)(t), \tag{5.33}$$

a.e. on $\mathbb{R}$. Since $\rho_n * u, (\rho_n * u)'$ and $\rho_n * f$ are continuous and $\mathbf{A}$ is closed, (5.33) is true for all $t \in \mathbb{R}$. So for each $n \in \mathbb{N}, \rho_n * u$ is a classical solution of (ACP_f). It is therefore also a mild solution of (ACP_f) so that

$$\int_s^t (\rho_n * u)(\tau)\, d\tau \in D(\mathbf{A}),$$

and whenever $t \geq s$,

$$(\rho_n * u)(t) = (\rho_n * u)(s) + \mathbf{A}\int_s^t (\rho_n * u)(\tau)\, d\tau + \int_s^t (\rho_n * f)(\tau)\, d\tau. \tag{5.34}$$

Since $\mathbf{A}$ is closed, letting $n \to \infty$ through a subsequence in (5.34) yields

$$\int_s^t u(\tau)\, d\tau \in D(\mathbf{A})$$

and

$$u(t) = u(s) + \mathbf{A}\int_s^t u(\tau)\, d\tau + \int_s^t f(\tau)\, d\tau,$$

for almost all t, s $(t \geq s)$ as required.

Conversely, suppose that u is a mild solution. Then, for almost all s, (5.21) and (5.22) hold for almost all t. Assuming first that this is true for $s = 0$, we have, for almost all t in $\mathbb{R}$,

$$u(t) = u(0) + \mathbf{A}\int_0^t u(\tau)\, d\tau + \int_0^t f(\tau)\, d\tau. \tag{5.35}$$

Let $u_+ = u \restriction_{\mathbb{R}_+}$, $f_+ = f \restriction_{\mathbb{R}_+}$ and $v(t) = u(-t)$, $g(t) = f(-t)$, for all $t \geq 0$. Then, using (5.35) and [2, Theorem 3.1.3] we have that for $\operatorname{Re}\lambda > 0$, $\hat{u}_+(\lambda) \in D(\mathbf{A})$ and

$$\hat{u}_+(\lambda) = \frac{u(0)}{\lambda} + \frac{\mathbf{A}\hat{u_+}(\lambda)}{\lambda} + \frac{\hat{f_+}(\lambda)}{\lambda}.$$

Letting $\operatorname{Re}\lambda \to 0$, we obtain that

$$\int_0^\infty e^{-i\tau t}u(t)\,dt \in D(\mathbf{A})$$

and

$$(i\tau - \mathbf{A})\int_0^\infty e^{-i\tau t}u(t)\,dt = u(0) + \int_0^\infty e^{-i\tau t}f(t)\,dt, \tag{5.36}$$

for all $\tau \in \mathbb{R}$. Also, (5.35) gives

$$v(t) = u(0) - \mathbf{A}\int_0^t v(\tau)\,d\tau - \int_0^t g(\tau)\,d\tau,$$

for almost all $t \geq 0$. Taking Laplace transforms, we have, as before, that

$$\int_0^\infty e^{i\tau t}v(t)\,dt \in D(\mathbf{A})$$

and

$$(i\tau - \mathbf{A})\int_0^\infty e^{i\tau t}v(t)\,dt = -u(0) + \int_0^\infty e^{i\tau t}g(t)\,dt, \tag{5.37}$$

for all $\tau \in \mathbb{R}$. Adding (5.36) and (5.37) yields $\mathcal{F}u(\tau) \in D(\mathbf{A})$ and

$$(i\tau - \mathbf{A})\mathcal{F}u(\tau) = \mathcal{F}f(\tau),$$

for all $\tau \in \mathbb{R}$.

If $s \neq 0$, then the previous case may be applied to $\tilde{u}(t) = u(t+s)$ and $\tilde{f}(t) = f(t+s)$. Then, the required result follows immediately . □

Theorem 5.3.6. *The following are equivalent for a C_0-semigroup $\mathbf{T}$:*

1. $\zeta(\mathbf{T}) < 0$;
2. $s_0^\infty(\mathbf{A}) < 0$ *and for all* $f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ *with* $i\,\mathrm{sp}_c(f) \cap \sigma(\mathbf{A})$ *empty, there exists a unique mild solution* $u \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ *of (ACP$_f$) such that* $i\,\mathrm{sp}_c(u) \cap \sigma(\mathbf{A})$ *is empty.*

Proof. *(1)* $\Longrightarrow$ *(2)* : From Theorem 5.1.7, *(1)* implies that $s_0^\infty(\mathbf{A}) < 0$. Let $f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$, with $i\,\mathrm{sp}_c(f) \cap \sigma(\mathbf{A})$ empty. Choose $\phi \in C^\infty(\mathbb{R})$ satisfying condition (P_0) and such that $\phi = 1$ near $\mathrm{sp}_c(f)$.

Then, from Theorem 5.1.7, and Remark 5.1.8 it follows that

$$\begin{aligned} \mathbf{M} : \mathrm{L}^1(\mathbb{R}, \mathbf{X}) &\longrightarrow \mathrm{L}^1(\mathbb{R}, \mathbf{X}) \\ \mathbf{M}(g) &= \mathcal{F}^{-1}(\phi(\cdot)\mathbf{R}(i\cdot, \mathbf{A})\mathcal{F}g) \quad (g \in \mathcal{S}(\mathbb{R}, \mathbf{X})) \end{aligned}$$

is a bounded operator. Let $u = \mathbf{M}(f)$. Using (5.1), we have that

$$\mathcal{F}u(s) = \phi(s)\mathbf{R}(is, \mathbf{A})\mathcal{F}f(s),$$

for all $s \in \mathbb{R}$, and $\operatorname{supp} \mathcal{F}u \subset \operatorname{supp} \mathcal{F}f$. Therefore,

$$i \operatorname{sp}_c(u) \cap \sigma(\mathbf{A}) \subset i \operatorname{sp}_c(f) \cap \sigma(\mathbf{A}) = \emptyset.$$

Further, for $s \in \operatorname{supp} \mathcal{F}f, \mathcal{F}u(s) = \mathbf{R}(is, \mathbf{A})\mathcal{F}f(s)$. Therefore, $\mathcal{F}u(s) \in D(\mathbf{A})$ and

$$(is - \mathbf{A})\mathcal{F}u(s) = \mathcal{F}f(s). \tag{5.38}$$

On the other hand, if $s \notin \operatorname{supp} \mathcal{F}f$, then $\mathcal{F}u(s) = 0 = \mathcal{F}f(s)$, so that (5.38) remains true. Therefore, for all $s \in \mathbb{R}$

$$(is - \mathbf{A})\mathcal{F}u(s) = \mathcal{F}f(s)$$

and Lemma 5.3.5 allows us to conclude that u is a mild solution of (ACP_f). If u_1 is another mild solution of (ACP_f) in $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$ with $i \operatorname{sp}_c(u_1) \cap \sigma(\mathbf{A})$ empty and $(is - \mathbf{A})\mathcal{F}u_1(s) = \mathcal{F}f(s)$ for all $s \in \mathbb{R}$, then $\mathcal{F}u = \mathcal{F}u_1$ on $\mathbb{R}$. It therefore follows that $u(t) = u_1(t)$ almost everywhere, i.e. $u = u_1$ in $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$.

(2) $\Longrightarrow$ (1) : Since $s_0^\infty(\mathbf{A}) < 0$ we can find $b > 0$ such that

$$Q_{0,b} \subset \rho(\mathbf{A}) \text{ and } \sup_{\lambda \in Q_{0,b}} \|\mathbf{R}(\lambda, \mathbf{A})\| < \infty.$$

Choose $\phi \in C^\infty(\mathbb{R})$ satisfying condition $(P_{0,b})$. Note that $i\mathbb{R} \cap \sigma(\mathbf{A}) \subset (-ib, ib)$. Let

$$\begin{aligned} \mathrm{Y} &= \big\{ f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X}) : \operatorname{sp}_c(f) \subset \mathbb{R} \setminus (-b, b) \big\} \\ &= \big\{ f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X}) : \mathcal{F}f = 0 \text{ on } (-b, b) \big\}. \end{aligned}$$

Then Y is closed in $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$. For $f \in \mathrm{Y}, i \operatorname{sp}_c(f) \cap \sigma(\mathbf{A})$ is empty. Therefore, hypothesis *(2)* implies the existence of a unique $u_f \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ with $i \operatorname{sp}_c(u_f) \cap \sigma(\mathbf{A})$ empty, which is a mild solution of (ACP_f). Further, from Lemma 5.3.5

$$(is - \mathbf{A})\mathcal{F}u_f(s) = \mathcal{F}f(s) \quad (s \in \mathbb{R}). \tag{5.39}$$

We show next that the linear map $\mathbf{B} : \mathrm{Y} \longrightarrow \mathrm{L}^1(\mathbb{R}, \mathbf{X})$, given by

$$\mathbf{B}(f) = u_f$$

is closed. Suppose $(f_n) \subset \mathrm{Y}$ converges to f in $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$ and $\mathbf{B}(f_n)$ converges to some $u \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$. Since for each n, $\mathbf{B}(f_n) = u_{f_n}$ is the unique mild solution of (ACP_{f_n}),

$$\int_s^t u_{f_n}(\tau)\, d\tau \in D(\mathbf{A})$$

and

$$u_{f_n}(t) = u_{f_n}(s) + \mathbf{A}\int_s^t u_{f_n}(\tau)\,d\tau + \int_s^t f_n(\tau)\,d\tau \tag{5.40}$$

for almost all $s, t \in \mathbb{R}$ and all $n \in \mathbb{N}$. Since $\mathbf{A}$ is closed, letting $n \to \infty$ in (5.40) through a subsequence, we have that $\displaystyle\int_s^t u(\tau)\,d\tau \in D(\mathbf{A})$ and

$$u(t) = u(s) + \mathbf{A}\int_s^t u(\tau)\,d\tau + \int_s^t f(\tau)\,d\tau$$

for almost all $s, t \in \mathbb{R}$. This implies that u is a mild solution of (ACP_f). If $\eta \in \mathrm{sp}_c(u_{f_n}) = \mathrm{supp}\,\mathcal{F}u_{f_n}$, then $i\eta \in \rho(\mathbf{A})$. Therefore, it follows from (5.39) that $i\,\mathrm{sp}_c(u_{f_n}) \subset i\,\mathrm{sp}_c(f_n) \subset \mathbb{R}\setminus(-b, b)$. Since u_{f_n} converges to u in $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$, it follows that $\mathrm{sp}_c(u) \subset \mathbb{R}\setminus(-b, b)$. Therefore, $i\,\mathrm{sp}_c(u) \cap \sigma(\mathbf{A})$ is empty. If v is any other mild solution of (ACP_f) with $i\,\mathrm{sp}_c(v) \cap \sigma(\mathbf{A})$ empty, then it follows from Lemma 5.3.5 that $\mathcal{F}u = \mathcal{F}v$. Therefore, $u = v$ in $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$. Thus, $u \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ is the unique mild solution corresponding to f given by *(2)* so that $\mathbf{B}(f) = u$. Since $\mathbf{B}$ is a closed linear map between two Banach spaces, it follows from the Closed Graph Theorem that $\mathbf{B}$ is bounded on Y, i.e. there is a constant K such that

$$\|\mathbf{B}(f)\|_1 \le K\|f\|_1 \quad (f \in \mathrm{Y}). \tag{5.41}$$

Now consider any $g \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$. Set $f_g = g - \mathcal{F}^{-1}(1-\phi) * g$. Then $\mathcal{F}f_g = \phi\mathcal{F}g$, so that $f_g \in \mathrm{Y}$. Thus, from the above discussion, it follows that there is a unique $u_{f_g} \in \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ with $i\,\mathrm{sp}_c(u_{f_g}) \cap \sigma(\mathbf{A})$ empty, which is a mild solution of (ACP_{f_g}), and satisfies

$$\begin{aligned}(is - \mathbf{A})\mathcal{F}u_{f_g}(s) &= \mathcal{F}f_g(s) \quad (s \in \mathbb{R}),\\ \|u_{f_g}\|_1 &\le K\|f_g\|_1.\end{aligned}$$

If $is \in \rho(\mathbf{A})$, then $\mathcal{F}u_{f_g}(s) = \phi(s)\mathbf{R}(is, \mathbf{A})\mathcal{F}g(s)$. On the other hand if $is \in \sigma(\mathbf{A})$, then $s \notin \mathrm{sp}_c(u_{f_g})$ so that $\mathcal{F}u_{f_g}(s) = 0$. Also, $\phi(s)\mathbf{R}(is, \mathbf{A})\mathcal{F}g(s) = 0$ since $s \in (-b, b)$. Therefore,

$$\mathcal{F}u_{f_g}(s) = \phi(s)\mathbf{R}(is, \mathbf{A})\mathcal{F}g(s), \text{for all } s \in \mathbb{R}.$$

If $g \in \mathcal{S}(\mathbb{R}, \mathbf{X})$, then the function $s \mapsto \phi(s)\mathbf{R}(is, \mathbf{A})\mathcal{F}g(s)$ is in $\mathcal{S}(\mathbb{R}, \mathbf{X})$ so that

$$u_{f_g} = \mathcal{F}^{-1}(\phi\mathbf{R}(i\cdot, \mathbf{A})\mathcal{F}g).$$

Further,

$$\begin{aligned}\|u_{f_g}\|_1 &\le K(\|g\|_1 + \|\mathcal{F}^{-1}(1-\phi) * g\|_1)\\ &\le K_1\|g\|_1.\end{aligned}$$

Setting $\mathbf{M}(g) = u_{f_g}$ one sees that $\mathbf{M} : \mathrm{L}^1(\mathbb{R}, \mathbf{X}) \longrightarrow \mathrm{L}^1(\mathbb{R}, \mathbf{X})$ is a bounded linear operator. Therefore, $\phi\mathbf{R}(i\cdot, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^1(\mathbb{R}, \mathbf{X})$. From Theorem 5.1.7 we have that $\zeta(\mathbf{T}) < 0$. □

Now we turn towards the analogue of Theorem 5.3.6 for $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, $(1 < p < \infty)$. The following Lemma corresponds to Lemma 5.3.5 with (5.31) replaced by a distributional equation.

Lemma 5.3.7. *Let* $\mathbf{A}$ *be a closed operator on* $\mathbf{X}$ *and* $u, f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$, $(1 \leq p < \infty)$. *Then*

1. *The following are equivalent:*

   (a) *u is a mild solution of (ACP_f);*

   (b) *For all $\Phi \in \mathcal{S}(\mathbb{R})$, and $\lambda \in \rho(\mathbf{A})$,*

   $$\langle \mathcal{F}(\Phi(i \cdot - \mathbf{A})\mathbf{R}(\lambda, \mathbf{A})), u\rangle = \langle \mathcal{F}\Phi, \mathbf{R}(\lambda, \mathbf{A})f\rangle; \tag{5.42}$$

   (c) *For all $\Phi \in \mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$, and $\lambda \in \rho(\mathbf{A})$, (5.42) holds.*

2. *If in addition, both $\mathrm{sp}_c(f) \cap \sigma(\mathbf{A})$ and $\mathrm{sp}_c(u) \cap \sigma(\mathbf{A})$ are empty, and $\sigma(\mathbf{A}) \cap i\mathbb{R}$ is compact then the conditions in (1) are also equivalent to :*

   $$\langle \mathcal{F}\Phi, u\rangle = \langle \mathcal{F}(\Phi \mathbf{R}(i\cdot, \mathbf{A})), f\rangle \tag{5.43}$$

   for $\Phi \in \mathcal{S}(\mathbb{R})$ (or in $\mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$) with $i \operatorname{supp} \Phi \subset \rho(\mathbf{A})$.

Proof. *(1).* *(1a)* $\Longrightarrow$ *(1b)* : Let u be a mild solution of (ACP_f). As in the proof of Lemma 5.3.5, we may assume without loss of generality, that

$$u(t) = u(0) + \mathbf{A}\int_0^t u(\tau)\, d\tau + \int_0^t f(\tau)\, d\tau,$$

for almost all $t \in \mathbb{R}$. Let $\Phi \in \mathcal{S}(\mathbb{R})$ and u_+, f_+ denote the restrictions of u, f to the non-negative reals, respectively. Further, set

$$\tilde{u}(t) = \int_0^t u_+(r)\, dr \text{ and } \tilde{f}(t) = \int_0^t f_+(r)\, dr \quad (t \geq 0).$$

Then, for a fixed $\lambda \in \rho(\mathbf{A})$ and almost all $t \geq 0$ we have,

$$\mathbf{R}(\lambda, \mathbf{A})u_+(t) = \mathbf{R}(\lambda, \mathbf{A})u(0) + \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})\tilde{u}(t) + \mathbf{R}(\lambda, \mathbf{A})\tilde{f}(t).$$

Therefore,

$$\begin{aligned}\langle \mathcal{F}\Phi, \mathbf{R}(\lambda, \mathbf{A})u_+\rangle = \int_0^\infty \mathcal{F}\Phi(s)\mathbf{R}(\lambda, \mathbf{A})u(0)\, ds + \langle \mathcal{F}\Phi, \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})\tilde{u}\rangle \\ + \langle \mathcal{F}\Phi, \mathbf{R}(\lambda, \mathbf{A})\tilde{f}\rangle.\end{aligned} \tag{5.44}$$

Using Fubini's theorem and the fact that $\Phi \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}
\langle \mathcal{F}(i\cdot\Phi), \mathbf{R}(\lambda,\mathbf{A})\tilde{u}\rangle &= \int_0^\infty \mathcal{F}(i\cdot\Phi)(s)\mathbf{R}(\lambda,\mathbf{A})\tilde{u}(s)\,ds \\
&= \int_0^\infty \int_0^s \mathcal{F}(i\cdot\Phi)(s)\mathbf{R}(\lambda,\mathbf{A})u_+(t)\,dt\,ds \\
&= \int_0^\infty \int_t^\infty \mathcal{F}(i\cdot\Phi)(s)\mathbf{R}(\lambda,\mathbf{A})u_+(t)\,ds\,dt \\
&= -\int_0^\infty \int_t^\infty (\mathcal{F}(\Phi))'(s)\mathbf{R}(\lambda,\mathbf{A})u_+(t)\,ds\,dt \\
&= \langle \mathcal{F}\Phi, \mathbf{R}(\lambda,\mathbf{A})u_+\rangle.
\end{aligned}$$

Similarly,

$$\langle \mathcal{F}(i\cdot\Phi), \mathbf{R}(\lambda,\mathbf{A})\tilde{f}\rangle = \langle \mathcal{F}\Phi, \mathbf{R}(\lambda,\mathbf{A})f_+\rangle.$$

The above remains true if $\mathbf{R}(\lambda,\mathbf{A})$ is replaced by $\mathbf{AR}(\lambda,\mathbf{A})$. On substituting $i\cdot\Phi$ in place of Φ in (5.44), one obtains

$$\begin{aligned}
\langle \mathcal{F}(i\cdot\Phi), \mathbf{R}(\lambda,\mathbf{A})u_+\rangle = \int_0^\infty \mathcal{F}(i\cdot\Phi)(s)\mathbf{R}(\lambda,\mathbf{A})u(0)\,ds + \langle \mathcal{F}\Phi, \mathbf{AR}(\lambda,\mathbf{A})u_+\rangle \\
+ \langle \mathcal{F}\Phi, \mathbf{R}(\lambda,\mathbf{A})f_+\rangle.
\end{aligned} \tag{5.45}$$

By considering similarly the restrictions u_- and f_- of u, f respectively, to the negative real axis we obtain

$$\begin{aligned}
\langle \mathcal{F}(i\cdot\Phi), \mathbf{R}(\lambda,\mathbf{A})u_-\rangle = \int_{-\infty}^0 \mathcal{F}(i\cdot\Phi)(s)\mathbf{R}(\lambda,\mathbf{A})u(0)\,ds + \langle \mathcal{F}\Phi, \mathbf{AR}(\lambda,\mathbf{A})u_-\rangle \\
+ \langle \mathcal{F}\Phi, \mathbf{R}(\lambda,\mathbf{A})f_-\rangle.
\end{aligned} \tag{5.46}$$

Adding (5.45) and (5.46) and noting that $\int_{\mathbb{R}} \mathcal{F}(i\cdot\Phi)(s)\,ds = 0$, yields

$$\langle \mathcal{F}(i\cdot\Phi), \mathbf{R}(\lambda,\mathbf{A})u\rangle = \langle \mathcal{F}\Phi, \mathbf{AR}(\lambda,\mathbf{A})u\rangle + \langle \mathcal{F}\Phi, \mathbf{R}(\lambda,\mathbf{A})f\rangle. \tag{5.47}$$

Since $\mathbf{AR}(\lambda,\mathbf{A})$ and $\mathbf{R}(\lambda,\mathbf{A})$ are bounded operators,

$$\langle \mathcal{F}\Phi, \mathbf{AR}(\lambda,\mathbf{A})u\rangle = \mathcal{F}(\Phi\mathbf{AR}(\lambda,\mathbf{A})), u\rangle$$

and

$$\langle \mathcal{F}(i\cdot\Phi), \mathbf{R}(\lambda,\mathbf{A})u\rangle = \langle \mathcal{F}(i\cdot\Phi\mathbf{R}(\lambda,\mathbf{A})), u\rangle.$$

Therefore, (5.47) may be rewritten as

$$\langle \mathcal{F}(\Phi(i\cdot -\mathbf{A})\mathbf{R}(\lambda,\mathbf{A})), u\rangle = \langle \mathcal{F}\Phi, \mathbf{R}(\lambda,\mathbf{A})f\rangle$$

as required.

(1b) $\Longrightarrow$(1a): Let $(g_n) \subset C_c^\infty(\mathbb{R})$ be a mollifier. Then $g_n * u \in \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \cap C^\infty(\mathbb{R}, \mathbf{X})$, for all n. For any $\Phi \in \mathcal{S}(\mathbb{R})$ and $h \in \mathcal{S}(\mathbb{R}, \mathbf{X})$, by Fubini's Theorem,

$$\langle \mathcal{F}\Phi, g_n * h\rangle = \langle \mathcal{F}(\mathcal{F}g_n\Phi), h\rangle. \tag{5.48}$$

Using the density of $\mathcal{S}(\mathbb{R}, \mathbf{X})$ in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and the Dominated Convergence Theorem we see that (5.48) in fact holds for all $h \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$. Let $\lambda \in \rho(\mathbf{A})$. Applying (5.42), with Φ replaced by $(\mathcal{F}g_n)\Phi$, and using (5.48) we arrive at

$$\begin{aligned}\langle \mathcal{F}(i\cdot(\mathcal{F}g_n)\Phi), \mathbf{R}(\lambda, \mathbf{A})u\rangle &= \langle \mathcal{F}(\mathcal{F}g_n\Phi), \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})u\rangle \\ &\quad + \langle \mathcal{F}(\mathcal{F}g_n\Phi), \mathbf{R}(\lambda, \mathbf{A})f\rangle \\ &= \langle \mathcal{F}\Phi, \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})(g_n * u)\rangle \\ &\quad + \langle \mathcal{F}\Phi, \mathbf{R}(\lambda, \mathbf{A})(g_n * f)\rangle.\end{aligned} \tag{5.49}$$

Also, making use of (5.48), we have

$$\begin{aligned}\langle \mathcal{F}\Phi, (g_n * u)'\rangle &= -\langle (\mathcal{F}\Phi)', g_n * u\rangle \\ &= -\langle \mathcal{F}(-i\cdot\Phi), g_n * u\rangle \\ &= \langle \mathcal{F}(i\cdot(\mathcal{F}g_n)\Phi), u\rangle.\end{aligned} \tag{5.50}$$

Therefore, from (5.50) it follows that

$$\langle \mathcal{F}\Phi, \mathbf{R}(\lambda, \mathbf{A})(g_n * u)'\rangle = \langle \mathcal{F}(i\cdot(\mathcal{F}g_n)\Phi), \mathbf{R}(\lambda, \mathbf{A})u\rangle.$$

Combining this with (5.49), we have

$$\langle \mathcal{F}\Phi, \mathbf{R}(\lambda, \mathbf{A})(g_n * u)'\rangle = \langle \mathcal{F}\Phi, \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})(g_n * u)\rangle + \langle \mathcal{F}\Phi, \mathbf{R}(\lambda, \mathbf{A})(g_n * f)\rangle. \tag{5.51}$$

Since $\mathcal{F}[\mathcal{S}(\mathbb{R})] = \mathcal{S}(\mathbb{R})$, and all the terms are continuous functions, we have

$$\mathbf{R}(\lambda, \mathbf{A})(g_n * u)'(s) = \mathbf{R}(\lambda, \mathbf{A})(g_n * f)(s) + \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})(g_n * u)(s)$$

for all $s \in \mathbb{R}$. As the first two terms of the above equation lie in $D(\mathbf{A})$, $\mathbf{A}\mathbf{R}(\lambda, \mathbf{A})(g_n * u)(s)$ is in $D(\mathbf{A})$ so that $(g_n * u)(s) \in D(\mathbf{A})$ and $\mathbf{A}\mathbf{R}(\lambda, \mathbf{A})(g_n * u)(s) = \mathbf{R}(\lambda, \mathbf{A})\mathbf{A}(g_n * u)(s)$. Using the injectivity of $\mathbf{R}(\lambda, \mathbf{A})$, we conclude that

$$(g_n * u)'(s) = \mathbf{A}(g_n * u)(s) + (g_n * f)(s) \text{ for all } s \in \mathbb{R}.$$

So, $g_n * u$ is a classical solution of (ACP_f). Therefore, it is also a mild solution and satisfies

$$\int_s^t (g_n * u)(\tau)\, d\tau \in D(\mathbf{A})$$

and

$$(g_n * u)(t) = (g_n * u)(s) + \mathbf{A}\int_s^t (g_n * u)(\tau)\,d\tau + \int_s^t (g_n * f)(\tau)\,d\tau \tag{5.52}$$

for all $t, s \in \mathbb{R}$. Since $g_n * u \longrightarrow u$ in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and $g_n * f \longrightarrow f$ in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$, letting $n \longrightarrow \infty$ in (5.52) through a subsequence, we have $\displaystyle\int_s^t u(\tau)\,d\tau \in D(\mathbf{A})$ and

$$u(t) = u(s) + \mathbf{A}\int_s^t u(\tau)\,d\tau + \int_s^t f(\tau)\,d\tau \text{ for almost all } t, s \in \mathbb{R}.$$

Therefore, u is a mild solution of (ACP_f).

(1b) $\Longrightarrow$ *(1c)*: First suppose that $\Phi = \Phi_0 \otimes U$, where $\Phi_0 \in \mathcal{S}(\mathbb{R}), U \in \mathcal{L}(\mathbf{X})$. Since (5.42) holds for Φ_0, it also holds for this Φ. Therefore, by linearity, (5.42) holds for $\Phi = \Sigma_{i=1}^n \Phi_i \otimes U_i$, $\Phi_i \in \mathcal{S}(\mathbb{R})$, $U_i \in \mathcal{L}(\mathbf{X})$, $i = 1, 2, ..., n$. Given $\Phi \in \mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$, we can thus find a sequence Φ_n of functions of this form such that

$$\begin{aligned} \|\mathcal{F}(\Phi_n - \Phi)\|_{\mathrm{L}^{p'}(\mathbb{R}, \mathcal{L}(\mathbf{X}))} &\longrightarrow 0 \text{ as } n \longrightarrow \infty; \\ \|\mathcal{F}(i \cdot \Phi_n - i \cdot \Phi)\|_{\mathrm{L}^{p'}(\mathbb{R}, \mathcal{L}(\mathbf{X}))} &\longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned}$$

Note that (5.42) may be rewritten as

$$\langle \mathcal{F}(i \cdot \Phi), \mathbf{R}(\lambda, \mathbf{A})u\rangle - \langle \mathcal{F}\Phi, \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})u\rangle = \langle \mathcal{F}\Phi, \mathbf{R}(\lambda, \mathbf{A})u\rangle.$$

Since both sides of the above equation are continuous in the $\mathrm{L}^{p'}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ norm, (5.42) holds for all $\Phi \in \mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$.

(1c) $\Longrightarrow$ *(1b)*: This is immediate.

(2): Let $i\,\mathrm{sp}_c(f) \cap \sigma(\mathbf{A})$ and $i\,\mathrm{sp}_c(u) \cap \sigma(\mathbf{A})$ be empty. First suppose that (5.43) holds for all $\Phi \in \mathcal{S}(\mathbb{R})$ with $i\,\mathrm{supp}\,\Phi \subset \rho(\mathbf{A})$. Let $\Psi \in \mathcal{S}(\mathbb{R})$ with $i\,\mathrm{supp}\,\Psi \subset \rho(\mathbf{A})$. Applying (5.43) with $\Phi = i \cdot \Psi$, we have for $\lambda \in \rho(\mathbf{A})$,

$$\begin{aligned} \langle \mathcal{F}(i \cdot \Psi), \mathbf{R}(\lambda, \mathbf{A})u\rangle &= \langle \mathcal{F}(i \cdot \Psi\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{R}(\lambda, \mathbf{A})f\rangle \\ &= \langle \mathcal{F}(\Psi(\mathbf{Id} + \mathbf{A}\mathbf{R}(i\cdot, \mathbf{A}))), \mathbf{R}(\lambda, \mathbf{A})f\rangle \\ &= \langle \mathcal{F}\Psi, \mathbf{R}(\lambda, \mathbf{A})f\rangle + \mathcal{F}(\Psi\mathbf{A}\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{R}(\lambda, \mathbf{A})f\rangle \\ &= \langle \mathcal{F}\Psi, \mathbf{R}(\lambda, \mathbf{A})f\rangle + \langle \mathcal{F}(\Psi\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})f\rangle \\ &= \langle \mathcal{F}\Psi, \mathbf{R}(\lambda, \mathbf{A})f\rangle + \langle \mathcal{F}\Psi, \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})u\rangle, \end{aligned} \tag{5.53}$$

where the last equality is obtained by applying the bounded operator $\mathbf{A}\mathbf{R}(\lambda, \mathbf{A})$ on both sides of equation (5.43). For $\Psi \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}\,\Psi \cap (\mathrm{sp}_c(f) \cup \mathrm{sp}_c(u))$ empty, we have on using Lemma 5.3.2,

$$\begin{aligned} \langle \mathcal{F}\Psi, \mathbf{R}(\lambda, \mathbf{A})f\rangle &= 0; \\ \langle \mathcal{F}(i \cdot \Psi), \mathbf{R}(\lambda, \mathbf{A})u\rangle &= 0; \\ \langle \mathcal{F}\Psi, \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})u\rangle &= 0. \end{aligned}$$

Since $i(\mathrm{sp}_c(f) \cup \mathrm{sp}_c(u)) \cap \sigma(\mathbf{A})$ is empty, there exist disjoint open sets U_1, U_2 in $\mathbb{R}$ such that $\mathrm{sp}_c(f) \cup \mathrm{sp}_c(u) \subset U_1$ and $\sigma(\mathbf{A}) \cap i\mathbb{R} \subset iU_2$. Let $g \in C^\infty(\mathbb{R})$, $0 \le g \le 1$ satisfy

$$g(s) = \begin{cases} 0 & (s \text{ near } \mathrm{sp}_c(f) \cup \mathrm{sp}_c(u) \text{ or near } \pm\infty), \\ 1 & (s \in \mathbb{R} \setminus U_1). \end{cases}$$

For $\Psi \in \mathcal{S}(\mathbb{R})$, writing $\Psi = g\Psi + (1-g)\Psi$, we see that $\mathrm{supp}(g\Psi) \subset \mathbb{R} \setminus (\mathrm{sp}_c(f) \cup \mathrm{sp}_c(u))$ and $i\,\mathrm{supp}((1-g)\Psi) \subset iU_1 \subset \rho(\mathbf{A}) \cap i\mathbb{R}$. Thus every $\Psi \in \mathcal{S}(\mathbb{R})$ may be written as $\Psi_1, \Psi_2 \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}\,\Psi_1 \cap (\mathrm{sp}_c(f) \cup \mathrm{sp}_c(u))$ empty and $i\,\mathrm{supp}\,\Psi_2 \subset \rho(\mathbf{A})$. Therefore, (5.53) holds for all $\Psi \in \mathcal{S}(\mathbb{R})$, so that (1b) is true.

Next suppose that the equivalent conditions of (1) hold. Let $\Psi \in \mathcal{S}(\mathbb{R})$, with $i\,\mathrm{supp}\,\Psi \subset \rho(\mathbf{A})$. Applying (1c) with $\Phi = \Psi\mathbf{R}(i\cdot, \mathbf{A})$ and using

$$is\Psi(s)\mathbf{R}(is, \mathbf{A}) = \Psi(s) + \Psi(s)\mathbf{A}\mathbf{R}(is, \mathbf{A})$$

we have,

$$\langle \mathcal{F}\Psi, \mathbf{R}(\lambda, \mathbf{A})u\rangle + \langle \mathcal{F}(\Psi\mathbf{A}\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{R}(\lambda, \mathbf{A})u\rangle = \langle \mathcal{F}(\Psi\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})u\rangle + \langle \mathcal{F}(\Psi\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{R}(\lambda, \mathbf{A})f\rangle.$$

Since $\langle \mathcal{F}(\Psi\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{A}\mathbf{R}(\lambda, \mathbf{A})u\rangle = \langle \mathcal{F}(\Psi\mathbf{A}\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{R}(\lambda, \mathbf{A})u\rangle$,

$$\langle \mathcal{F}\Psi, \mathbf{R}(\lambda, \mathbf{A})u\rangle = \langle \mathcal{F}(\Psi\mathbf{R}(i\cdot, \mathbf{A})), \mathbf{R}(\lambda, \mathbf{A})f\rangle.$$

The injectivity of $\mathbf{R}(\lambda, \mathbf{A})$ implies

$$\langle \mathcal{F}\Psi, u\rangle = \langle \mathcal{F}(\Psi\mathbf{R}(i\cdot, \mathbf{A})), f\rangle.$$

The claim for $\Psi \in \mathcal{S}(\mathbb{R}, \mathcal{L}(\mathbf{X}))$ follows along the same lines as *(1b)* $\Longrightarrow$ *(1c)*. □

We are now in a position to prove the L^p-version of Theorem 5.3.6.

Theorem 5.3.8. *Let $1 \le p < \infty$. The following are equivalent for a C_0-semigroup $\mathbf{T}$:*

1. $\zeta(\mathbf{T}) < 0$;
2. $s_0^\infty(\mathbf{A}) < 0$ *and for all $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ with $i\,\mathrm{sp}_c(f) \cap \sigma(\mathbf{A})$ empty, there exists a unique mild solution $u \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ of (ACP_f) such that $i\,\mathrm{sp}_c(u) \cap \sigma(\mathbf{A})$ is empty.*

Proof. *(1)* $\Longrightarrow$ *(2)* : Let $f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ with $i\,\mathrm{sp}_c(f) \cap \sigma(\mathbf{A})$ empty. Since $\zeta(\mathbf{T}) < 0, s_0^\infty(\mathbf{A}) < 0$. Choose $\phi \in C^\infty(\mathbb{R})$ satisfying (P_0) such that $\phi = 1$ near $\mathrm{sp}_c(f)$. Then from Remark 5.1.8 it follows that $\mathbf{M} : \mathcal{S}(\mathbb{R}, \mathbf{X}) \longrightarrow \mathcal{S}(\mathbb{R}, \mathbf{X})$ given by

$$\mathbf{M}(g) = \mathcal{F}^{-1}(\phi(\cdot)\mathbf{R}(i\cdot, \mathbf{A})\mathcal{F}g) \quad (g \in \mathcal{S}(\mathbb{R}, \mathbf{X})),$$

extends to a bounded linear operator on $\mathrm{L}^p(\mathbb{R},\mathbf{X})$. Let $\mathbf{M}(f)=u_f, f\in\mathrm{L}^p(\mathbb{R},\mathbf{X})$. Then, using (5.29) we have, for all $\Phi\in\mathcal{S}(\mathbb{R})$,

$$\langle\mathcal{F}\Phi,u_f\rangle=\langle\mathcal{F}(\phi\mathbf{R}(i\cdot,\mathbf{A})\Phi),f\rangle \tag{5.54}$$

and $\operatorname{supp}\mathcal{F}u_f\subset\operatorname{supp}\mathcal{F}f$. Thus $\operatorname{sp}_c(u_f)\subset\operatorname{sp}_c(f)$ so that $i\operatorname{sp}_c(u_f)\cap\sigma(\mathbf{A})$ is empty. Further, for $\Psi\in\mathcal{S}(\mathbb{R})$ with $i\operatorname{supp}\Psi\subset\rho(\mathbf{A})$,

$$\begin{aligned}\langle\mathcal{F}\Psi,u_f\rangle&=\langle\Psi\phi\mathbf{R}(i\cdot,\mathbf{A}),\mathcal{F}f\rangle\\&=\langle\Psi\mathbf{R}(i\cdot,\mathbf{A}),\mathcal{F}f\rangle-\langle\Psi\mathbf{R}(i\cdot,\mathbf{A}),\mathcal{F}(\mathcal{F}^{-1}(1-\phi)*f)\rangle\\&=\langle\Psi\mathbf{R}(i\cdot,\mathbf{A}),\mathcal{F}f\rangle \qquad (5.55)\\&=\langle\mathcal{F}(\Psi(\cdot)\mathbf{R}(i\cdot,\mathbf{A})),f\rangle. \qquad (5.56)\end{aligned}$$

Here we have used the following to obtain the second last inequality: From (5.27), it follows that $\operatorname{sp}_c\left(\mathcal{F}^{-1}(1-\phi)*f\right)\subset\operatorname{sp}_c(f)\cap\operatorname{supp}(1-\phi)$ which is empty due to the particular choice of ϕ. Thus, from [2, Theorem 4.8.2] we have $\mathcal{F}^{-1}(1-\phi)*f=0$. It follows then that $\langle\Psi\mathbf{R}(i\cdot,\mathbf{A}),\mathcal{F}(\mathcal{F}^{-1}(1-\phi)*f)\rangle=0$.

An application of Lemma 5.3.7 then shows that u_f is a mild solution of (ACP_f). If v_f is another mild solution of (ACP_f) with $i\operatorname{sp}_c(v_f)\cap\sigma(\mathbf{A})$ empty then from Lemma 5.3.7 we have that

$$\begin{aligned}\langle\mathcal{F}\Psi,u_f\rangle&=\langle\mathcal{F}(\Psi(\cdot)\mathbf{R}(i\cdot,\mathbf{A})),f\rangle\\&=\langle\mathcal{F}\Psi,v_f\rangle\end{aligned} \tag{5.57}$$

for all $\Psi\in\mathcal{S}(\mathbb{R})$ with $i\operatorname{supp}\Psi\subset\rho(\mathbf{A})$. On the other hand, if $\Psi\in\mathcal{S}(\mathbb{R})$ is such that $\operatorname{supp}\Psi\cap(\operatorname{sp}_c(u_f)\cup\operatorname{sp}_c(v_f))$ is empty, then the sets $\operatorname{sp}_c(u_f)\cap\operatorname{supp}\Psi$, and $\operatorname{sp}_c(v_f)\cap\operatorname{supp}\Psi$ are both empty, so that

$$\langle\mathcal{F}\Psi,u_f\rangle=0=\langle\mathcal{F}\Psi,v_f\rangle,$$

by Lemma 5.3.2. Writing $\Psi\in\mathcal{S}(\mathbb{R})$ as $\Psi_1+\Psi_2$, where $\Psi_1\in\mathcal{S}(\mathbb{R}),\operatorname{supp}\Psi_1\subset\rho(\mathbf{A})$ and $\Psi_2\in\mathcal{S}(\mathbb{R})$ is such that $\operatorname{supp}\Psi_2\cap(\operatorname{sp}_c(u_f)\cup\operatorname{sp}_c(v_f))$ is empty, we have that $\langle\Psi,u_f\rangle=\langle\Psi,v_f\rangle$ for all $\Psi\in\mathcal{S}(\mathbb{R})$. Therefore, we conclude that $u_f=v_f$.

(2) $\Longrightarrow$ *(1)*: From *(2)* it follows that there is a $b>0$ such that

$$Q_{0,b}\subset\rho(\mathbf{A})\text{ and }\sup_{\lambda\in Q_{0,b}}\|\mathbf{R}(\lambda,\mathbf{A})\|<\infty.$$

Then $\sigma(\mathbf{A})\cap i\mathbb{R}\subset(-ib,ib)$. Let $\phi\in C^\infty(\mathbb{R})$ satisfy $(P_{0,b})$. Let

$$\mathrm{Y}=\left\{f\in\mathrm{L}^p(\mathbb{R},\mathbf{X}):\operatorname{sp}_c(f)\subset\mathbb{R}\setminus(-b,b)\right\}.$$

Then Y is closed in $\mathrm{L}^p(\mathbb{R},\mathbf{X})$. As in the proof of Theorem 5.3.6, we have a linear map $\mathbf{B}:\mathrm{Y}\longrightarrow\mathrm{L}^p(\mathbb{R},\mathbf{X})$ given by

$$\mathbf{B}(f)=u_f,$$

where u_f is the unique mild solution of (ACP_f) such that $i\,\mathrm{sp}_c(u_f) \cap \sigma(\mathbf{A})$ is empty. From Lemma 5.3.7 we have, for all $\Psi \in \mathcal{S}(\mathbb{R})$ with $i\,\mathrm{supp}\,\Psi \subset \rho(\mathbf{A})$

$$\langle \mathcal{F}\Psi, u_f \rangle = \langle \mathcal{F}(\Psi \mathbf{R}(i\cdot, \mathbf{A})), f \rangle. \tag{5.58}$$

Note that (5.58) implies that $\mathrm{sp}_c(u_f) \subset \mathrm{sp}_c(f)$, $f \in \mathrm{Y}$. Indeed, if $i\,\mathrm{supp}\,\Psi \cap (\sigma(\mathbf{A}) \cup i\,\mathrm{sp}_c(f))$ is empty, then $\langle \mathcal{F}(\Psi \mathbf{R}(i\cdot, \mathbf{A})), f \rangle = 0$, by the operator-valued version of Lemma 5.3.2 (see Remark 5.3.3). Therefore (5.58) implies that $\langle \mathcal{F}\Psi, u_f \rangle = 0$. This implies that $i\,\mathrm{sp}_c(u_f) \subset \sigma(\mathbf{A}) \cup i\,\mathrm{sp}_c(f)$ (see (5.28)), and hence $\mathrm{sp}_c(u_f) \subset \mathrm{sp}_c(f)$.

We shall now show that $\mathbf{B}$ is closed. Then it will follow from the Closed Graph Theorem that $\mathbf{B}$ is bounded. Suppose $(f_n) \subset \mathrm{Y}$ converges to f in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ and $\mathbf{B}(f_n) = u_{f_n}$ converges to some u in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. Using the same arguments as in the proof of *(2)* $\Longrightarrow$ *(1)* of Theorem 5.3.6 we conclude that u is a mild solution of (ACP_f).

Now $\mathrm{sp}_c(f_n) \subset \mathbb{R} \setminus (-b, b)$, so that from Lemma 5.3.2 it follows that $\langle \mathcal{F}\Psi, f_n \rangle = 0$ for all $\Psi \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}\,\Psi \subset (-b, b)$. Since $\langle \mathcal{F}\Psi, f_n \rangle$ converges to $\langle \mathcal{F}\Psi, f \rangle$ as $n \to \infty$, $\Psi \in \mathcal{S}(\mathbb{R})$, it follows that $\langle \mathcal{F}\Psi, f \rangle = 0$ for all $\Psi \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}\,\Psi \subset (-b, b)$. It follows then from (5.28) that $\mathrm{sp}_c(f) \subset \mathbb{R} \setminus (-b, b)$. Therefore, $f \in \mathrm{Y}$.

Similarly, since $\mathrm{sp}_c(u_{f_n}) \subset \mathrm{sp}_c(f_n) \subset \mathbb{R} \setminus (-b, b)$ and u_{f_n} converges to u in $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$ it follows that $\mathrm{sp}_c(u) \subset \mathbb{R} \setminus (-b, b)$. Therefore, $i\,\mathrm{sp}_c(u) \cap \sigma(\mathbf{A})$ is empty. The uniqueness of u follows from arguments similar to those used in the proof of *(1)* $\Longrightarrow$ *(2)* above. Thus $\mathbf{B}(f) = u$.

Let $g \in \mathrm{L}^p(\mathbb{R}, \mathbf{X})$ be arbitrary. Then $f_g = g - \mathcal{F}^{-1}(1 - \phi) * g \in \mathrm{Y}$. Indeed, for $\Psi \in \mathcal{S}(\mathbb{R})$, with $\mathrm{supp}\,\Psi \subset (-b, b)$ we have on applying (5.48) (with $g_n = \mathcal{F}^{-1}(1 - \phi)$ and $h = g$),

$$\begin{aligned} \langle \mathcal{F}\Psi, \mathcal{F}^{-1}(1 - \phi) * g \rangle &= \langle \mathcal{F}((1 - \phi)\Psi), g \rangle \\ &= \langle \mathcal{F}\Psi, g \rangle. \end{aligned}$$

Thus, $\langle \mathcal{F}\Psi, f_g \rangle = 0$ for all $\Psi \in \mathcal{S}(\mathbb{R})$, with $\mathrm{supp}\,\Psi \subset (-b, b)$. From (5.28) it follows that $\mathrm{sp}_c(f_g) \subset \mathbb{R} \setminus (-b, b)$ so that $f_g \subset \mathrm{Y}$. Therefore, there is a unique mild solution $u_{f_g} = \mathbf{B}(f_g)$ of (ACP_f) with $i\,\mathrm{sp}_c(u_{f_g}) \cap \sigma(\mathbf{A})$ empty, which satisfies (5.58). Set $\mathbf{M}(g) = u_{f_g}$. Then $\mathbf{M}$ is a well defined linear map on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. Since $\mathbf{B}$ is bounded on Y, there exist constants K and K_1 such that

$$\begin{aligned} \|\mathbf{M}(g)\|_p &= \|u_{f_g}\|_p \\ &= \|\mathbf{B}(f_g)\| \\ &\leq K\|f\|_p \\ &= K\|g - \mathcal{F}^{-1}(1 - \phi) * g\|_p \\ &\leq K_1\|g\|_p. \end{aligned}$$

Suppose that $g \in \mathcal{S}(\mathbb{R}, \mathbf{X})$. Then, for $\Psi \in \mathcal{S}(\mathbb{R})$ with $i \operatorname{supp} \Psi \subset \rho(\mathbf{A})$

$$\begin{aligned}\langle \mathcal{F}\Psi, \mathbf{M}(g)\rangle &= \langle \mathcal{F}(\Psi \mathbf{R}(i\cdot, \mathbf{A})), f_g\rangle \\ &= \langle \Psi \mathbf{R}(i\cdot, \mathbf{A}), \mathcal{F} f_g\rangle \\ &= \langle \Psi \mathbf{R}(i\cdot, \mathbf{A}), \phi \mathcal{F} g\rangle \\ &= \langle \Psi, \phi \mathbf{R}(i\cdot, \mathbf{A}) \mathcal{F} g\rangle \\ &= \langle \mathcal{F}\Psi, \mathcal{F}^{-1}(\phi \mathbf{R}(i\cdot, \mathbf{A}) \mathcal{F} g)\rangle.\end{aligned}$$

Further, if $\Psi \in \mathcal{S}(\mathbb{R})$ is such that $\operatorname{supp} \Psi \subset [-b, b]$ then

$$\langle \mathcal{F}\Psi, \mathbf{M}(g)\rangle = 0 = \langle \mathcal{F}\Psi, \mathcal{F}^{-1}(\phi \mathbf{R}(i\cdot, \mathbf{A}) \mathcal{F} g)\rangle.$$

The first equality in the above follows from Lemma 5.3.2 on noting that $\operatorname{sp}_c(\mathbf{M}(g)) \subset \operatorname{sp}_c(f_g) \subset \mathbb{R} \setminus (-b, b)$, so that $\operatorname{supp} \Psi \cap \operatorname{sp}_c(\mathbf{M}(g)) = \emptyset$. Since any $\Psi \in \mathcal{S}(\mathbb{R})$ may be expressed as a sum of $\Psi_1, \Psi_2 \in \mathcal{S}(\mathbb{R})$, with $i \operatorname{supp} \Psi_1 \subset \rho(\mathbf{A})$ and $\operatorname{supp} \Psi_2 \subset [-b, b]$, it follows that

$$\langle \mathcal{F}\Psi, \mathbf{M}(g)\rangle = \langle \mathcal{F}\Psi, \mathcal{F}^{-1}(\phi \mathbf{R}(i\cdot, \mathbf{A}) \mathcal{F} g)\rangle$$

for all $\Psi \in \mathcal{S}(\mathbb{R})$. We conclude therefore that

$$\mathbf{M}(g) = \mathcal{F}^{-1}(\phi \mathbf{R}(i\cdot, \mathbf{A}) \mathcal{F} g)$$

for all $g \in \mathcal{S}(\mathbb{R}, \mathbf{X})$. Thus $s \mapsto \phi(s)\mathbf{R}(is, \mathbf{A})$ is a Fourier multiplier on $\mathrm{L}^p(\mathbb{R}, \mathbf{X})$. An application of Theorem 5.1.7 yields $\zeta(\mathbf{T}) < 0$.

□

We have used (*2*) of Lemma 5.3.7 to prove the above result. However, Theorem 5.3.8 can be proved, alternatively, by just making use of (*1*) of Lemma 5.3.7. We do not include the details here. The basic strategy of both the proofs, of course, is the same.

Remark 5.3.9. Let $1 \leq p < \infty$ and $\mathbf{T}$ be a C_0-semigroup with generator $\mathbf{A}$ such that $\sigma(\mathbf{A}) \cap i\mathbb{R}$ is empty. Then Theorem 5.3.8 gives

$$\begin{aligned}\zeta(\mathbf{T}) < 0 \iff s_0^\infty(\mathbf{A}) < 0 \text{ and for all } f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \text{ there exists} \\ \text{a unique mild solution of } (ACP_f) \text{ in } \mathrm{L}^p(\mathbb{R}, \mathbf{X}).\end{aligned}$$

Latushkin and Shvydkoy's result [34, Theorem 2.7] relating Fourier multiplier properties of the resolvent and hyperbolicity of the semigroup, (see Remark 5.1.12) and the proof of Theorem 5.3.8 also shows that

$$\begin{aligned}\mathbf{T} \text{ is hyperbolic } \iff \text{ For all } f \in \mathrm{L}^p(\mathbb{R}, \mathbf{X}) \text{ there exists a unique} \\ \text{mild solution of } (ACP_f) \text{ in } \mathrm{L}^p(\mathbb{R}, \mathbf{X}).\end{aligned} \tag{5.59}$$

(5.59) has been proven by Latushkin, Randolph and Schnaubelt ([33, Theorem 2.1], [16, Theorem 4.33]) in the more general setting of evolution families. A result similar to (5.59) was first established by Prüss [44]. The precise result there was proven under the assumption that the function f and the corresponding mild solution of (ACP_f) are both continuous. $\square$

Chapter 6

Weak compactness, sun-reflexivity and approximations of the identity

We examine in this chapter the relation between the existence of a certain type of approximation of the identity on a Banach space $\mathbf{X}$ and the sun-reflexivity of $\mathbf{X}$ with respect to some C_0-semigroup. Sauvageot [46] has shown that if $\mathbf{X}$ is a C^*-algebra, a weakly compact approximation of the identity exists if and only if $\mathbf{X}$ admits a C_0-semigroup $\mathbf{T}$, such that the canonical extension of $\mathbf{T}(t)$ maps $\mathbf{X}^{**}$ into the multiplier algebra of $\mathbf{X}$. We study these concepts for general Banach spaces and relate them to sun-reflexivity of the space with respect to some semigroup.

Before starting on the main theme, we record a small observation concerning the non-analytic growth bound and the adjoint of a C_0-semigroup:

Remark 6.0.10. For a C_0-semigroup $\mathbf{T}$, with generator $\mathbf{A}$, we have

1. $\zeta(\mathbf{T}) = \zeta(\mathbf{T}^{\odot})$;
2. $\delta(\mathbf{T}) = \delta(\mathbf{T}^{\odot})$;
3. $s_0^{\infty}(\mathbf{A}) = s_0^{\infty}(\mathbf{A}^{\odot})$.

For the adjoint C_0-semigroup $\mathbf{T}^{\odot}$ with generator $\mathbf{A}^{\odot}$ the following hold [40, Theorem 1.4.2]: $\rho(\mathbf{A}) = \rho(\mathbf{A}^{\odot})$ and $\mathbf{R}(\lambda, \mathbf{A})^{\odot} = \mathbf{R}(\lambda, \mathbf{A}^{\odot})$ for all $\lambda \in \rho(\mathbf{A})$. Also, $\|\mathbf{R}(\lambda, \mathbf{A})\| \leq \|\mathbf{R}(\lambda, \mathbf{A}^{\odot\odot})\| \leq \|\mathbf{R}(\lambda, \mathbf{A}^{\odot})\|$. Thus, if $\zeta(\mathbf{T}) < \beta$, then there exist $\alpha < \beta$ and $b > 0$ such that for all $t \geq 0$, $\mathbf{T}(t) = \mathbf{T}_{\alpha,b}(t) + \mathbf{T}_2(t)$, with $\omega_0(\mathbf{T}_2) < \beta$. It follows on taking adjoints, that

$$\mathbf{T}^{\odot}(t) = \mathbf{T}_{\alpha,b}^{\odot}(t) + \mathbf{T}_2{}^{\odot}(t) \quad (t \geq 0).$$

It is easy to see that

$$\mathbf{T}_{\alpha,b}^{\odot}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,b}} e^{\lambda t} \mathbf{R}(\lambda, \mathbf{A}^{\odot})\, d\lambda.$$

Since $\omega_0(\mathbf{T}_2^{\odot}) \leq \omega_0(\mathbf{T}_2)$, it follows that $\zeta(\mathbf{T}^{\odot}) < \beta$. On the other hand, if $\zeta(\mathbf{T}^{\odot}) < \beta$ then applying the above argument to the C_0-semigroup $\mathbf{T}^{\odot}$ instead of $\mathbf{T}$, we have that $\zeta(\mathbf{T}^{\odot\odot}) < \beta$ so that there exist $\alpha < \beta$ and $b > 0$ such that

$$\mathbf{T}^{\odot\odot}(t) = \int_{\Gamma_{\alpha,b}} e^{\lambda t}\mathbf{R}(\lambda, \mathbf{A}^{\odot\odot})\, d\lambda \ + \mathbf{S}(t),$$

with $\omega_0(\mathbf{S}) < \beta$. Since $\mathbf{T}^{\odot\odot}, \mathbf{A}^{\odot\odot}$ may be considered as extensions of $\mathbf{T}$ and $\mathbf{A}$ respectively, taking restrictions of the operators in the above equation to $\mathbf{X}$, we have that $\zeta(\mathbf{T}) < \beta$. The other two equalities also follow easily from the definitions. □

6.1 Weak compactness and sun-reflexivity

We recall some standard definitions and facts concerning the adjoint of a C_0-semigroup and weakly compact operators. The details may be found in [39, Chapter 1, Chapter 2].

A strongly continuous semigroup $\mathbf{T}$ on $\mathbf{X}$ is called *weakly compact* if $\mathbf{T}(t)$ is weakly compact for all $t > 0$, that is, $\mathbf{T}(t)$ maps bounded subsets of $\mathbf{X}$ to relatively weakly compact subsets of $\mathbf{X}$. Gantmacher's theorem asserts that a bounded operator $\mathbf{S}$ is weakly compact on $\mathbf{X}$ if and only if $\mathbf{S}^{**}\mathbf{X}^{**} \subset \mathbf{X}$.

Given a C_0-semigroup $\mathbf{T}$ on $\mathbf{X}$, the locally convex topology on $\mathbf{X}$ generated by the semi-norms

$$\left\{p_{x^{\odot}} : \ x^{\odot} \in \ \mathbf{X}^{\odot}\right\}, \text{where } p_{x^{\odot}}(x) = |\langle x^{\odot}, x\rangle| \quad (x \in \mathbf{X}),$$

is denoted by $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$.

A useful way of identifying $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$-continuous operators is given by [39, Proposition 2.4.3]:

Proposition 6.1.1. *If* $\mathbf{S}$ *commutes with* $\mathbf{T}(t)$ *for each* $t > 0$ *then* $\mathbf{S}$ *is* $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$*-continuous.*

Analogous to Gantmacher's Theorem for weak compactness of a bounded operator on $\mathbf{X}$, we have the following characterisation of $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$-compactness of a bounded operator on $\mathbf{X}$ [39, Theorem 2.4.2] :

Theorem 6.1.2. *A* $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$*-continuous operator* $\mathbf{S}$ *on* $\mathbf{X}$ *is* $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$*-compact if and only if* $\mathbf{S}^{\odot *}\mathbf{X}^{\odot *} \subset j\mathbf{X}$.

We quote now some results from [39] which bring out the relation between $\odot$-reflexivity of the space with respect to a given C_0-semigroup and the compactness of the semigroup and resolvents of the generator with respect to the different topologies. The following theorem gives a very useful characterisation of $\odot$-reflexivity [39, Theorem 2.5.2].

Theorem 6.1.3. *Let* $(\mathbf{T}(t))_{t \geq 0}$ *be a* C_0*-semigroup of operators on* $\mathbf{X}$ *with generator* $\mathbf{A}$*. Then the following are equivalent:*

1. $\mathbf{X}$ *is* $\odot$*-reflexive with respect to* $\mathbf{T}$*;*

2. $\mathbf{R}(\lambda, \mathbf{A})$ *is* $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$*-compact for some* $\lambda \in \rho(\mathbf{A})$*;*

3. $\mathbf{R}(\lambda, \mathbf{A})$ *is weakly compact for some* $\lambda \in \rho(\mathbf{A})$*.*

From the resolvent identity it follows that if $\mathbf{R}(\lambda, \mathbf{A})$ is weakly compact for one $\lambda \in \rho(\mathbf{A})$, then $\mathbf{R}(\lambda, \mathbf{A})$ is weakly compact for all $\lambda \in \rho(\mathbf{A})$ and the same is true for $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$-compactness of $\mathbf{R}(\lambda, \mathbf{A})$.

A Banach space $\mathbf{X}$ is said to have the *Dunford-Pettis Property* (DPP) if every weakly compact operator $\mathbf{B} : \mathbf{X} \longrightarrow \mathbf{Y}$, $\mathbf{Y}$ being any Banach space, maps relatively weakly compact subsets to relatively compact subsets. It is easy to see that if $\mathbf{B}$ is a weakly compact operator on a space with (DPP) then $\mathbf{B}^2$ is compact. A use of the resolvent identity yields the following corollary [39, Corollary 2.5.4]:

Corollary 6.1.4. *If* $\mathbf{X}$ *has the Dunford-Pettis property, then* $\mathbf{X}$ *is* $\odot$*- reflexive if and only if* $\mathbf{R}(\lambda, \mathbf{A})$ *is compact for some/ all* $\lambda \in \rho(\mathbf{A})$*.*

A semigroup $\mathbf{T}$ is said to be *strongly continuous for* $t > 0$ or $C_{>0}$ if

$$\lim_{t \downarrow 0} \|\mathbf{T}(s+t)x - \mathbf{T}(s)x\| = 0$$

holds for all $s > 0$ and $x \in \mathbf{X}$.

If $\mathbf{T}$ is weakly compact, then $\mathbf{R}(\lambda, \mathbf{A})$ is weakly compact for each $\lambda \in \rho(\mathbf{A})$ and it follows from Theorem 6.1.3 that $\mathbf{X}$ is $\odot$-reflexive. The converse is not always true, but we have from [39, Corollary 5.2.9]

Theorem 6.1.5. *Let* $(\mathbf{T}(t))_{t \geq 0}$ *be a* C_0*-semigroup of bounded linear operators with generator* $\mathbf{A}$*. Then, the following are equivalent:*

1. $\mathbf{T}$ *is weakly compact;*

2. $\mathbf{X}$ *is* $\odot$*-reflexive and* $\mathbf{T}^{**}$ *is* $C_{>0}$*.*

To the above set of results we can add an analogue of Theorem 6.1.5 for semigroups that are $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$-compact:

Theorem 6.1.6. *Let* $(\mathbf{T}(t))_{t \geq 0}$ *be a* C_0*-semigroup of bounded linear operators on* $\mathbf{X}$*. The following are equivalent:*

1. $\mathbf{T}(t)$ *is* $\sigma(\mathbf{X}, \mathbf{X}^{\odot})$*- compact for all* $t > 0$*;*

2. $\mathbf{X}$ *is* $\odot$*-reflexive and* $\mathbf{T}^{\odot *}$ *is* $C_{>0}$*.*

Proof. Let $(\mathbf{T}(t))_{t\,\geq\,0}$ be a C_0-semigroup on $\mathbf{X}$ which is $\sigma(\mathbf{X},\mathbf{X}^{\odot})$-compact. We show that then $\mathbf{R}(\lambda,\mathbf{A})$ is $\sigma(\mathbf{X},\mathbf{X}^{\odot})$-compact. In view of Theorem 6.1.2 and Proposition 6.1.1 it is enough to show that $\mathbf{R}(\lambda,\mathbf{A})^{\odot *}\mathbf{X}^{\odot *}\subset j\mathbf{X}$. We note here that the map $s\mapsto \mathbf{T}^{\odot *}(s)x^{\odot *}$ is continuous on $(0,\infty)$ for each $x^{\odot *}\in\mathbf{X}^{\odot *}$. This is because $\sigma(\mathbf{X},\mathbf{X}^{\odot})$-compactness of $\mathbf{T}$ implies, by Theorem 6.1.2, that $\mathbf{T}^{\odot *}(s)\mathbf{X}^{\odot *}\subset j\mathbf{X}\ \subset\ \mathbf{X}^{\odot\odot}$ for all $s>0$. So, for $s>0$,

$$\lim_{t\downarrow 0}\parallel\ \mathbf{T}^{\odot *}(t)(\mathbf{T}^{\odot *}(s)x^{\odot *})\ -\ \mathbf{T}^{\odot *}(s)x^{\odot *}\parallel=0.$$

This means that for all $s>0$,

$$\lim_{t\downarrow 0}\|\mathbf{T}^{\odot *}(t+s)x^{\odot *}-\mathbf{T}^{\odot *}(s)x^{\odot *}\|=0. \tag{6.1}$$

From (6.1), it also follows that the map $s\mapsto\mathbf{T}^{\odot *}(s)x^{\odot *}$ is measurable on $(0,\infty)$. Thus, for $\operatorname{Re}\lambda>\omega_0(\mathbf{T})$ and $y^{\odot}\in\mathbf{X}^{\odot}$, we have

$$\begin{aligned}\langle\mathbf{R}(\lambda,\mathbf{A})^{\odot *}x^{\odot *},y^{\odot}\rangle &= \langle x^{\odot *},\mathbf{R}(\lambda,\mathbf{A})^{\odot}y^{\odot}\rangle\\ &= \langle x^{\odot *},\mathbf{R}(\lambda,\mathbf{A}^{\odot})y^{\odot}\rangle\\ &= \langle x^{\odot *},\int_0^{\infty}e^{-\lambda s}\mathbf{T}^{\odot}(s)y^{\odot}\,ds\rangle\\ &= \langle\int_0^{\infty}e^{-\lambda s}\mathbf{T}^{\odot *}(s)x^{\odot *}\,ds,y^{\odot}\rangle.\end{aligned}$$

Since $\mathbf{T}^{\odot *}(s)x^{\odot *}\in\ j\mathbf{X}$ for all $s>0$ we have that $\mathbf{R}(\lambda,\mathbf{A})^{\odot *}\mathbf{X}^{\odot *}\subset j\mathbf{X}$. Thus $\mathbf{R}(\lambda,\mathbf{A})$ is $\sigma(\mathbf{X},\mathbf{X}^{\odot})$-compact and it follows from Theorem 6.1.3 that $\mathbf{X}$ is $\odot$-reflexive with respect to the semigroup $\mathbf{T}$. Further, (6.1) says precisely that $\mathbf{T}^{\odot *}$ is $C_{>0}$.

Next assume that $\mathbf{X}$ is $\odot$-reflexive, i.e. $j\mathbf{X}=\mathbf{X}^{\odot\odot}$, and $\mathbf{T}^{\odot *}$ is $C_{>0}$. To show $\mathbf{T}$ is $\sigma(\mathbf{X},\mathbf{X}^{\odot})$-compact, it is enough to show that $\mathbf{T}^{\odot *}(s)\mathbf{X}^{\odot *}\subset j\mathbf{X}=\mathbf{X}^{\odot\odot}$ for all $s>0$. Let $\mathbf{T}^{\odot *}(s)x^{\odot *}\in\ \mathbf{T}^{\odot *}(s)\mathbf{X}^{\odot *}$. Since $\mathbf{T}^{\odot *}$ is $C_{>0}$ we have that

$$\parallel\mathbf{T}^{\odot *}(t+s)x^{\odot *}-\mathbf{T}^{\odot *}(s)x^{\odot *}\parallel\longrightarrow 0\text{ as }t\to 0,$$

for all $s>0$ and $x^{\odot *}\in\mathbf{X}^{\odot *}$. In other words, we have, for all $s>0$,

$$\parallel\mathbf{T}^{\odot *}(t)(\mathbf{T}^{\odot *}(s)x^{\odot *})-\mathbf{T}^{\odot *}(s)x^{\odot *}\parallel\longrightarrow 0\text{ as }t\to 0.$$

Therefore, $\mathbf{T}^{\odot *}(s)x^{\odot *}\in\mathbf{X}^{\odot\odot}$, for all $s>0$. □

6.2 Approximations of the identity

We bring out in this section some connections between the existence of a certain type of approximations of the identity on a Banach space and the existence of a semigroup with respect to which the space is $\odot$-reflexive. The results obtained in this process show the

existence of separable Banach spaces which cannot be sun-reflexive with respect to any strongly continuous semigroup.

The ideas for some of the proofs are inspired by those in [46], where the existence of approximations of the identity are studied in the context of C^*-algebras admitting 'Strong Feller semigroups'. A bounded operator $\mathbf{B}$ defined on a C^*-algebra $\mathbf{X}$ is said to have the *strong Feller property* [46, Definition 3.2] if

$$\mathbf{B}^{**}(\mathbf{X}^{**}) \subset \mathcal{M}(\mathbf{X}) := \{x \in \mathbf{X}^{**} : xy \in \mathbf{X}, yx \in \mathbf{X}, \text{ for all } y \in \mathbf{X}\}$$

and a C_0-semigroup $\mathbf{T}$ on a C^*-algebra $\mathbf{X}$ is called a strong Feller semigroup if for each $t > 0$, $\mathbf{T}(t)$ has the strong Feller property. Since $\mathbf{X} \subset \mathcal{M}(\mathbf{X})$ for any C^*-algebra $\mathbf{X}$, from Gantmacher's Theorem it follows that every weakly compact operator on a C^*-algebra has the strong Feller property.

Recall (see e.g. [36, Section 1.g]) that a Banach space $\mathbf{X}$ is said to have the *compact approximation property* or *CAP* if for every compact set $\mathbf{K} \subset \mathbf{X}$ and $\epsilon > 0$, there exists a compact operator $\mathbf{B} \in \mathcal{L}(\mathbf{X})$ satisfying

$$\|\mathbf{B}x - x\| \leq \epsilon, \text{ for all } x \in \mathbf{K}. \tag{6.2}$$

If the compact operators above may be chosen to have norm less than or equal to 1, then $\mathbf{X}$ is said to have the *metric compact approximation property* or the metric CAP. Analogously, we may define the *weakly compact approximation property* or *WCAP* for a Banach space. We shall say that a Banach space $\mathbf{X}$ has the WCAP if for every $\epsilon > 0$ and every compact subset $\mathbf{K}$ of $\mathbf{X}$ there exists a weakly compact operator $\mathbf{B} \in \mathcal{L}(\mathbf{X})$ satisfying (6.2). If the weakly compact operators may be chosen to have norm less than or equal to one, then $\mathbf{X}$ is said to have the metric WCAP. The weakly compact approximation property has been considered by Grønbæk and Willis [25] and Lima and Nygaard [35] among others.

We shall call a net $(\mathbf{B}_\alpha)$ of bounded linear operators on $\mathbf{X}$ an *approximation of the identity* if

$$\lim_\alpha \|\mathbf{B}_\alpha x - x\| = 0, \text{ for all } x \in \mathbf{X}.$$

It follows immediately from the definitions that if $\mathbf{X}$ admits a uniformly bounded compact (respectively, weakly compact) approximation of the identity, then $\mathbf{X}$ has the CAP (respectively, WCAP).

The next Lemma, taken from [46], is technical in nature. It is integral to the proof of the main result of this section and therefore, for the sake of completeness, we include the proof of the Lemma here.

Lemma 6.2.1. *There exists a universal constant K, such that if $\mathbf{X}$ is a Banach space and $\mathbf{B}$ is a bounded linear operator on $\mathbf{X}$ with $\| \mathbf{B} \| \leq 1$ then we have*

$$\| (\mathbf{B} - \mathbf{I})e^{t(\mathbf{B}-\mathbf{I})} \| \leq Kt^{-1/2} \quad (t \in \mathbb{R}_+).$$

Proof. Let $t > 0$ be fixed and let $n = n_t$ denote the integer part of t. Then we have

$$(\mathbf{I} - \mathbf{B})e^{t\mathbf{B}} = \sum_{k=0}^{n}(1 - \frac{k}{t})\frac{t^k\mathbf{B}^k}{k!} - \sum_{k=n+1}^{\infty}(\frac{k}{t} - 1)\frac{t^k\mathbf{B}^k}{k!}.$$

With $\mathbf{B} = \mathbf{I}$ in the above formula, we obtain

$$\sum_{k=0}^{n}(1 - \frac{k}{t})\frac{t^k}{k!} = \sum_{k=n+1}^{\infty}(\frac{k}{t} - 1)\frac{t^k}{k!}.$$

So, for any $\mathbf{B}$ with $\|\mathbf{B}\| \leq 1$,

$$\begin{aligned}\|(\mathbf{I} - \mathbf{B})e^{t\mathbf{B}}\| &\leq \sum_{k=0}^{n}(1 - \frac{k}{t})\frac{t^k}{k!} + \sum_{k=n+1}^{\infty}(\frac{k}{t} - 1)\frac{t^k}{k!} \\ &= 2\sum_{k=0}^{n}(1 - \frac{k}{t})\frac{t^k}{k!} \\ &= 2\frac{t^n}{n!}.\end{aligned}$$

Therefore,

$$\begin{aligned}\| (\mathbf{B} - \mathbf{I})e^{t(\mathbf{B}-\mathbf{I})} \| &\leq 2e^{-t}\frac{t^n}{n!} \\ &\leq Kt^{-1/2},\end{aligned}$$

where the estimate in the last line of the above, has been obtained as follows: By Stirling's formula, there exists $t_0 > 1$ such that, for $t \geq t_0$

$$n! \geq \sqrt{2\pi n}e^{-n-1}n^n$$

where $n = n_t$. Setting $t = n + \theta$, $0 < \theta < 1$, we therefore have for $t \geq t_0$, $n = n_t$

$$\begin{aligned}2e^{-t}\frac{t^n}{n!} &\leq 2\frac{e^{-(n+\theta)}(n+\theta)^n}{\sqrt{2\pi n}e^{-n-1}n^n} \\ &= 2\frac{e^{-\theta+1}}{\sqrt{2\pi n}}\left(1 + \frac{\theta}{n}\right)^n \\ &\leq 2\frac{e}{\sqrt{2\pi n}}\left(1 + \frac{1}{n}\right)^n \\ &\leq \frac{2e^2}{\sqrt{2\pi n}} \\ &\leq \frac{2e^2}{\sqrt{2\pi(t-1)}}.\end{aligned}$$

Thus we can choose K such that $\frac{2e^2}{\sqrt{2\pi(t-1)}} \leq Kt^{-1/2}$ for $t \geq t_0$ and $2e^{2t} \leq Kt^{-1/2}$ for $t < t_0$.

□

Let $\left(e^{t\mathbf{B}_k}\right)_{t\geq 0}$ $(k \in \mathbb{N})$, be a family of commuting contraction semigroups. In [3], the conditions under which the infinite product $\prod_{k=1}^{\infty} e^{t\mathbf{B}_k}$ converges and defines a C_0-semigroup have been investigated. We quote [3, Proposition 2.7]:

Theorem 6.2.2. *Let $\{\mathbf{T}_k : k \in \mathbb{N}\}$ be a commuting family of contraction semigroups (or groups) on $\mathbf{X}$ with generator $\mathbf{B}_k$, $(k \in \mathbb{N})$. Assume that the space*

$$D_1 := \left\{ x \in \bigcap_{k\in\mathbb{N}} D(\mathbf{B}_k) : \sum_{k=1}^{\infty} \|\mathbf{B}_k x\| < \infty \right\}$$

is dense in $\mathbf{X}$. Then the semigroup (respectively, group) product $\prod_{k=1}^{\infty} \mathbf{T}_k$ exists, in the sense that for each $x \in \mathbf{X}$,

$$\left(\prod_{k=1}^{\infty} \mathbf{T}_k(t)\right) x := \lim_{n\to\infty} \prod_{k=1}^{n} \mathbf{T}_k(t)x$$

converges uniformly on compact subsets of $[0,\infty)$. Define $\mathbf{B}$ on D_1 by $\mathbf{B}x = \sum_{k=1}^{\infty} \mathbf{B}_k x$. Then $\mathbf{B}$ is closable and $\overline{\mathbf{B}}$ is the generator of the product semigroup.

Taking $\mathbf{B}_k = \mathbf{A}_k - \mathbf{I}$, $k \in \mathbb{N}$ where $\mathbf{A}_k$ are as below, yields the first part of the next result. However, we present a proof for it which is essentially similar to that of [46, Lemma 2.3], where the result has been proven for C^*-algebras.

Lemma 6.2.3. *Let $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ be a sequence of contractions on $\mathbf{X}$ and suppose that*

1. *$\{\mathbf{A}_n\}$ is a commuting family and*
2. *the subspace $D_1 = \{x \in \mathbf{X} : \sum_{n=1}^{\infty} \| x - \mathbf{A}_n(x) \| < \infty\}$ is dense in $\mathbf{X}$.*

Then there exists a contractive semigroup $(\mathbf{T}(t))_{t\geq 0}$, on $\mathbf{X}$ such that, for any $t \in \mathbf{R}_+$, the sequence $\{\mathbf{T}_n(t)\}$ given by

$$\mathbf{T}_n(t) = e^{t(\mathbf{A}_1 + \mathbf{A}_2 + \ldots \mathbf{A}_n - n\mathbf{I})} \quad (n \in \mathbb{N}),$$

converges to $\mathbf{T}(t)$ in the strong operator topology . Moreover, for any $t > 0$ we have,

$$\lim_{n\to\infty} \left\| \left(\mathbf{I} - \frac{(\mathbf{A}_1 + \ldots + \mathbf{A}_n)}{n} \right) \circ \mathbf{T}(t) \right\| = 0.$$

Proof. For $n, m \in \mathbb{N}$, $m \geq n$, set

$$\Delta_{nm} = (m-n)\mathbf{I} - (\mathbf{A}_{n+1} + \ldots + \mathbf{A}_m).$$

Then we may write

$$\begin{aligned} \mathbf{T}_m(t) &= e^{t(\mathbf{A}_1+\mathbf{A}_2+\ldots\mathbf{A}_n-n\mathbf{I})+t(\mathbf{A}_{n+1}\ldots+\mathbf{A}_m)-t(m-n)\mathbf{I}} \\ &= \mathbf{T}_n(t) \circ \exp(-t\Delta_{nm}). \end{aligned}$$

Note that both $\mathbf{T}_n(t)$ and $\exp(-t\Delta_{nm})$ are contractive on $\mathbf{X}$. For $x \in D_1$, we have

$$\begin{aligned}
\|\mathbf{T}_m(t)x - \mathbf{T}_n(t)x\| &= \|\mathbf{T}_n(t)(\exp(-t\Delta_{nm})x - x)\,\| \\
&\leq \|e^{-t\Delta_{nm}}x - x\| \\
&= \left\| \int_0^t \frac{d}{ds} e^{-s\Delta_{nm}} x\, ds \right\| \\
&\leq \int_0^t \left\| \frac{d}{ds} e^{-s\Delta_{nm}} x \right\| ds \\
&\leq \int_0^t \|\Delta_{nm}x\|\, ds \\
&= t\|\Delta_{nm}x\|. \qquad (6.3)
\end{aligned}$$

By the definition of D_1, $\|\Delta_{nm}x\|$ tends to 0 as $n \to \infty$, uniformly in m. Therefore, the sequence $\{\mathbf{T}_n(t)x\}_{n\in\mathbb{N}}$ is Cauchy for any $x \in D_1$. By the norm density of D_1, this property extends to all $x \in \mathbf{X}$. Therefore, we can define the pointwise norm limit semigroup $(\mathbf{T}(t))_{t\geq 0}$ by

$$\mathbf{T}(t)x = \lim_{n\to\infty} \mathbf{T}_n(t)x \quad (x \in \mathbf{X}, t \geq 0).$$

Further, applying the inequality obtained in (6.3) above, with $n = 0$ we have for $x \in D_1$,

$$\begin{aligned}
\|\ \mathbf{T}(t)x - x\| &= \lim_m \|\ \mathbf{T}_m(t)x - x\| \\
&= \lim_m \|e^{t(\mathbf{A}_1\ +\mathbf{A}_2\ +\ldots\mathbf{A}_m - m\mathbf{I})}x - x\| \\
&\leq t \lim_m \|\ (\mathbf{A}_1 + \ldots + \mathbf{A}_m - m\mathbf{I})x\| \\
&\to 0 \text{ as } t \to 0.
\end{aligned}$$

Therefore, the semigroup $\mathbf{T}$ is strongly continuous.

Now, $\mathbf{T}(t)$ can be written as

$$\mathbf{T}(t)x = \lim_m \mathbf{T}_n(t) \circ \left(e^{-t\Delta_{nm}}x\right).$$

Therefore, from Lemma 6.2.1 it follows that

$$\begin{aligned}
\left\| \left(\mathbf{I} - \frac{1}{n}(\mathbf{A}_1 + \ldots + \mathbf{A}_n) \right) \circ \mathbf{T}(t) \right\| &\leq \left\| \left(\mathbf{I} - \frac{1}{n}(\mathbf{A}_1 + \ldots + \mathbf{A}_n) \right) \circ \mathbf{T}_n(t) \right\| \\
&= \left\| \left(\mathbf{I} - \frac{1}{n}(\mathbf{A}_1 + \ldots + \mathbf{A}_n) \right) \circ e^{nt(\frac{1}{n}(\mathbf{A}_1 + \ldots + \mathbf{A}_n)) - \mathbf{I})} \right\| \\
&\leq K(nt)^{-1/2} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

□

Corollary 6.2.4. *Using the same notation as in Lemma 6.2.3 we have, for $x \in \mathbf{X}$, and* $\operatorname{Re}\lambda > 0$,

$$\begin{aligned}\mathbf{R}(\lambda, \mathbf{A})x &= \int_0^\infty e^{-\lambda s}\mathbf{T}(s)x\, ds \\ &= \lim_{n\to\infty}[\lambda\mathbf{I} - (\mathbf{A}_1 + \mathbf{A}_2 + ...\mathbf{A}_n - n\mathbf{I})]^{-1}(x), \\ &= \lim_{n\longrightarrow\infty}\mathbf{R}(\lambda, \mathbf{B}_n)(x)\end{aligned}$$

where $\mathbf{A}$ is the generator of the C_0-semigroup $\mathbf{T}$ and $\mathbf{B}_n = \mathbf{A}_1 + \mathbf{A}_2 + ... + \mathbf{A}_n - n\mathbf{I}, n \in \mathbb{N}$.

Proof. This follows immediately from the first Trotter-Kato Approximation Theorem [20, Theorem III.4.8], on noting, from the proof of Lemma 6.2.3, that the convergence

$$\mathbf{T}_n(t)x \longrightarrow \mathbf{T}(t)x, x \in \mathbf{X},$$

is uniform on compact subsets of $[0, \infty)$. □

The following lemma is an alteration of [46, Lemma 2.4]. The proof of [46, Lemma 2.4] does not seem to work with the particular choice of the finite dimensional subspaces cited in the statement. We present the result here with the necessary modifications and in the context of Banach spaces.

Lemma 6.2.5. *Let $\mathbf{A}_n$ be a sequence of contractions on the separable Banach space $\mathbf{X}$ and $\{x_n\}_{n\in\mathbb{N}}$ be a dense set in $\mathbf{X}$. Suppose that for each $n \in \mathbb{N}$,*

$$\|(\mathbf{I} - \mathbf{A}_n)(x)\| \leq 2^{-n}\|x\|,$$

for any $x \in D_n$, where D_n is the finite dimensional subspace of $\mathbf{X}$ generated by the set

$$\left\{\left(\prod_{i=1}^{k}\mathbf{A}_{r_i}\right)(x_l) : 0 \leq r_i \leq n(1 \leq i \leq k), 0 \leq l \leq n, 0 \leq k \leq n^2\right\}$$

and $\mathbf{A}_0 = \mathbf{I}$. For $n \in \mathbb{N}$, let $\mathbf{B}_n = \mathbf{A}_1 + \mathbf{A}_2 + ... + \mathbf{A}_n - n\mathbf{I}$. Then the sequence $\left\{\left(e^{t\mathbf{B}_n}\right)_{t\geq 0}\right\}_{n\in\mathbb{N}}$ of C_0-semigroups on $\mathbf{X}$ converges in the resolvent sense towards a C_0-semigroup $(\mathbf{T}(t))_{t\geq 0}$, that is,

$$\lim_{n\to\infty}\mathbf{R}(\mu, \mathbf{B}_n)(x) = \mathbf{R}(\mu, \mathbf{A})(x), \tag{6.4}$$

for any $x \in \mathbf{X}$ and $\mu \in (0, \infty)$, where $\mathbf{A}$ is the generator of $\mathbf{T}$.

Proof. Let $\Theta_n = \frac{1}{n}(\mathbf{A}_1 + \mathbf{A}_2 + ... + \mathbf{A}_n)$. Then, $n(\mathbf{I} - \Theta_n) = -\mathbf{B}_n$. For $n, p > 0, 0 < l < n$, and $\alpha > 0$ we have

$$\begin{aligned}(\mathbf{I} - \mathbf{A}_{n+p})(\mathbf{I} - \alpha\mathbf{B}_n)^{-1}(x_l) &= (\mathbf{I} - \mathbf{A}_{n+p})\frac{1}{1+n\alpha}\sum_{k=0}^{\infty}\left(\frac{n\alpha}{1+n\alpha}\right)^k\Theta_n^k(x_l) \\ &= E_1 + E_2,\end{aligned}$$

where,

$$E_1 = (\mathbf{I} - \mathbf{A}_{n+p}) \frac{1}{1+n\alpha} \sum_{k=0}^{(n+p)^2} \left(\frac{n\alpha}{1+n\alpha}\right)^k \Theta_n^k(x_l)$$
$$= \frac{1}{1+n\alpha} \sum_{k=0}^{(n+p)^2} \left(\frac{n\alpha}{1+n\alpha}\right)^k (\mathbf{I} - \mathbf{A}_{n+p}) \Theta_n^k(x_l),$$
$$E_2 = (\mathbf{I} - \mathbf{A}_{n+p}) \frac{1}{1+n\alpha} \sum_{k=(n+p)^2+1}^{\infty} \left(\frac{n\alpha}{1+n\alpha}\right)^k \Theta_n^k(x_l).$$

For $0 \leq k \leq (n+p)^2$, each of the terms $\Theta_n^k(x_l) \in D_{n+p}$. Therefore, the assumptions imply that

$$\|E_1\| \leq \frac{1}{1+n\alpha} \sum_{k=0}^{(n+p)^2} \left(\frac{n\alpha}{1+n\alpha}\right)^k 2^{-n-p}\|x_l\|$$
$$\leq 2^{-n-p+1}\|x_l\|.$$

Further, since $\|\Theta_n\| \leq 1$ for each $n \in \mathbb{N}$,

$$\|E_2\| \leq \frac{2}{1+n\alpha}\|x_l\| \sum_{k=(n+p)^2+1}^{\infty} \left(\frac{n\alpha}{1+n\alpha}\right)^k$$
$$\leq 2\left(\frac{n\alpha}{1+n\alpha}\right)^{(n+p)^2+1} \|x_l\|.$$

Fix $\alpha_0 > 0$ and $\alpha \in (0, \alpha_0]$. Since $\lim_{n\to\infty} \left(\frac{n\alpha}{1+n\alpha}\right)^n = e^{-1/\alpha}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, one has

$$\left(\frac{n\alpha}{1+n\alpha}\right)^n \leq e^{-1/2\alpha_0}$$

and

$$\left(\frac{n\alpha}{1+n\alpha}\right)^{(n+p)^2+1} \leq \left(\frac{n\alpha}{1+n\alpha}\right)^{n(n+p)} \leq e^{-(n+p)/2\alpha_0}.$$

Thus,

$$\left\|(\mathbf{I} - \mathbf{A}_{n+p}) (\mathbf{I} - \alpha\mathbf{B}_n)^{-1} (x_l)\right\| \leq 2\|x_l\| \left(2^{-n-p} + e^{-(n+p)/2\alpha_0}\right)$$

For fixed l and $m > n > \max(n_0, l)$, we therefore have

$$\left\|(\mathbf{I} - \alpha\mathbf{B}_m) (\mathbf{I} - \alpha\mathbf{B}_n)^{-1} (x_l) - x_l\right\| = \alpha \left\|\sum_{p=1}^{m-n} (\mathbf{I} - \mathbf{A}_{n+p}) (\mathbf{I} - \alpha\mathbf{B}_n)^{-1} (x_l)\right\|$$
$$\leq C\left(2^{-n} + e^{-n/2\alpha_0}\right),$$

where C is a constant depending only on α and $\|x_l\|$. Since $\mathbf{B}_m$ generates a contractive C_0-semigroup, $(\mathbf{I} - \alpha\mathbf{B}_m)^{-1}$ is a contraction. Therefore,

$$\left\|(\mathbf{I} - \alpha\mathbf{B}_m)^{-1}(x_l) - (\mathbf{I} - \alpha\mathbf{B}_n)^{-1}(x_l)\right\| \leq C\left(2^{-n} + e^{-n/2\alpha_0}\right), \tag{6.5}$$

so that $n \to (\mathbf{I} - \alpha \mathbf{B}_n)^{-1}(x_l)$ is a Cauchy sequence for any $l \in \mathbb{N}$. It follows by density that $n \to (\mathbf{I} - \alpha \mathbf{B}_n)^{-1}(x)$ is Cauchy for each $x \in \mathbf{X}$. Thus,

$$\mathbf{R}_\alpha(x) = \lim_{n \to \infty} (\mathbf{I} - \alpha \mathbf{B}_n)^{-1}(x) \tag{6.6}$$

exists for each $x \in \mathbf{X}$. Further, for $\alpha > 0$ and fixed $n \in \mathbb{N}$, the equality

$$(\mathbf{I} - \alpha \mathbf{B}_n)^{-1}(y) = \alpha (\mathbf{I} - \alpha \mathbf{B}_n)^{-1} \mathbf{B}_n y + y \quad (y \in D(\mathbf{B}_n)),$$

together with density of $D(\mathbf{B}_n)$ in $\mathbf{X}$ yields, for fixed $n \in \mathbb{N}$

$$\lim_{\alpha \downarrow 0} (\mathbf{I} - \alpha \mathbf{B}_n)^{-1}(x) = x \quad (x \in \mathbf{X}). \tag{6.7}$$

From (6.5), it is clear that the convergence in (6.6) is uniform for $\alpha \in (0, \alpha_0)$ for fixed $x = x_l$, $l \in \mathbb{N}$. Hence,

$$\lim_{\alpha \to 0} \mathbf{R}_\alpha(x_l) = x_l. \tag{6.8}$$

By density, (6.8) is true for all $x \in \mathbf{X}$. Further, the family $\{\mathbf{R}_\alpha\}_{\alpha > 0}$ satisfies

$$\alpha \mathbf{R}_\alpha - \beta \mathbf{R}_\beta = (\alpha - \beta) \mathbf{R}_\alpha \mathbf{R}_\beta \quad (\alpha, \beta > 0).$$

It follows that

$$\mathrm{Ker}(\mathbf{R}_\alpha) = \mathrm{Ker}(\mathbf{R}_\beta)$$
$$\mathrm{Ran}(\mathbf{R}_\alpha) = \mathrm{Ran}(\mathbf{R}_\beta)$$

for all $\alpha, \beta > 0$. From (6.8) it then follows that $\mathbf{R}_\alpha$ is one to one and $D = \mathrm{Ran}(\mathbf{R}_\alpha)$ is dense in $\mathbf{X}$ for all $\alpha > 0$. Then $\mathbf{A} = \dfrac{1}{\alpha}\left(\mathbf{I} - \mathbf{R}_\alpha^{-1}\right)$ is a closed, densely defined operator, independent of α. Further, for $\mu > 0, \mu \mathbf{R}\left(\mu, \mathbf{A}\right) = \mathbf{R}_{\frac{1}{\mu}}$, which is a contraction. Therefore, the Hille-Yosida Theorem [20, II.3.5] implies that $\mathbf{A}$ generates a C_0-semigroup $\mathbf{T}$ of contractions. Then (6.6) yields (6.4). □

We combine the $\odot$-reflexivity results quoted previously with some of the ideas from [46, Lemma 4.2] to obtain :

Theorem 6.2.6. *The following are equivalent for a separable Banach space* $\mathbf{X}$*:*

1. *There exists a contractive* C_0*-semigroup on* $\mathbf{X}$*, with respect to which* $\mathbf{X}$ *is* $\odot$*-reflexive;*
2. *There exists a contractive* C_0*-semigroup on* $\mathbf{X}$ *whose generator has weakly compact resolvents;*
3. *There exists a weakly compact, commuting, contractive approximation of the identity on* $\mathbf{X}$*;*

4. There exists a weakly compact, contractive approximation of the identity on $\mathbf{X}$;

5. There exists a weakly compact, contractive C_0*-semigroup on* $\mathbf{X}$.

Proof. *(1)* $\Longleftrightarrow$ *(2)*: This follows immediately from Theorem 6.1.3.

(2) $\Longrightarrow$ *(3)* : Let $\mathbf{A}$ be the generator of the weakly compact contractive C_0-semigroup $\mathbf{T}$. Then, by the Hille-Yosida Generation Theorem [20, Theorem II.3.5] $(0,\infty) \subset \rho(\mathbf{A})$ and $\|\lambda \mathbf{R}(\lambda, \mathbf{A})\| \leq 1$ for all $\lambda \in (0,\infty)$. Using the same arguments as in (6.7), we have for $y \in D(\mathbf{A})$,

$$\|n\mathbf{R}(n,\mathbf{A})y - y\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Since $\|n\mathbf{R}(n,\mathbf{A})\|$ is uniformly bounded, it follows that $\|n\mathbf{R}(n,\mathbf{A})y - y\| \longrightarrow 0$ for all $y \in \mathbf{X}$. Thus, $\mathbf{A}_n := n\mathbf{R}(n,\mathbf{A})$ is a commuting, contractive, weakly compact approximation of the identity.

(3) $\Longrightarrow$ *(5)* : Let $\{\mathbf{A}_n\}$ be a weakly compact, commuting, approximation of the identity, and suppose that $\|\mathbf{A}_n\| \leq 1$, for all $n \in \mathbb{N}$. By replacing, if required, the given sequence by a subsequence, one may assume that $\mathbf{A}_n$ is such that

$$D_1 = \{x \in \mathbf{X} : \sum_{n=1}^{\infty} \|x - \mathbf{A}_n(x)\| < \infty\} \tag{6.9}$$

is dense in $\mathbf{X}$. To arrive at such a subsequence, we reason as follows: Let $\{a_1, a_2, a_3 ...\}$ be a countable dense set in $\mathbf{X}$. Since $\{\mathbf{A}_n\}$ form an approximation of the identity on $\mathbf{X}$, we can find $n_1 \in \mathbb{N}$ such that

$$\|a_1 - \mathbf{A}_{n_1}(a_1)\| < \frac{1}{2}$$

We continue inductively, choosing at the k^{th} step, $n_k > n_{k-1}$ such that

$$\|a_j - \mathbf{A}_{n_k}(a_j)\| \leq \frac{1}{2^k} \quad (j = 1, 2, ..., k).$$

Then $\{\mathbf{A}_{n_k}\}$ forms a subsequence satisfying (6.9).

By Lemma 6.2.3 there exists a semigroup $(\mathbf{T}(t))_{t\geq 0}$ which is strongly continuous, and

$$\lim_{n\to\infty} \left\| \left(\mathbf{I} - \frac{(\mathbf{A}_1 + ... + \mathbf{A}_n)}{n} \right) \circ \mathbf{T}(t) \right\| = 0,$$

for each $t > 0$. So, for each positive real number t, $\mathbf{T}(t)$ is the norm limit of a sequence of weakly compact operators. Hence, it is itself weakly compact. Also, one notes from Lemma 6.2.3 that $\|\mathbf{T}(t)\| \leq 1$ for all $t \geq 0$. So $(\mathbf{T}(t))_{t\geq 0}$ is a contractive, weakly compact semigroup defined on $\mathbf{X}$.

(5) $\Longrightarrow$ *(1)*: From Theorem 6.1.5, it follows that if $\mathbf{T}$ is a weakly compact semigroup on $\mathbf{X}$ then $\mathbf{X}$ is $\odot$-reflexive with respect to it.

(3) $\Longrightarrow$ *(4)* : This is trivial.

(4) $\Longrightarrow$ *(2)*: Let $\{\mathbf{A}_n\}$ be a sequence of weakly compact contractions which form an approximation of the identity of $\mathbf{X}$. Suppose that $\{x_l\}_{l\in\mathbb{N}}$ is a countable dense subset of $\mathbf{X}$ and let D_n, $n \in \mathbb{N}$, be the finite dimensional subspaces of $\mathbf{X}$ defined in the statement of Lemma 6.2.5. Then for each $k \in \mathbb{N}$, the convergence $\|\mathbf{A}_n(x) - x\| \to 0$ as $n \to \infty$ is uniform for all unit vectors $x \in D_k$. Thus, we may assume, by replacing $\{\mathbf{A}_n\}$ by an appropriate subsequence if required, that the sequence $\{\mathbf{A}_n\}$ satisfies the assumptions of Lemma 6.2.5. It follows from that Lemma that there exists a C_0-semigroup $\mathbf{T}$ of contractions, with generator $\mathbf{A}$ such that for each $\mu > 0$,

$$\lim_{n\to\infty} (\mu\mathbf{I} - (\mathbf{A}_1 + \mathbf{A}_2 + ... + \mathbf{A}_n - n\mathbf{I}))^{-1}(x) = \mathbf{R}(\mu, \mathbf{A})x$$

for any $x \in \mathbf{X}$. Now let $\mathbf{B}_n$ and Θ_n be as in the proof of Lemma 6.2.5 and $\Delta_{nm} = \mathbf{B}_m - \mathbf{B}_n$, $m > n$. Fix $\beta > 0$. For $x \in \mathbf{X}$, define

$$\begin{aligned} x_{nm} &= \left(\mathbf{I} - \frac{1}{\beta}\mathbf{B}_n\right)^{-1}\left(\mathbf{I} - \frac{1}{n+\beta}\Delta_{nm}\right)^{-1}(x) \\ &= \beta(n+\beta)\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \Delta_{nm})x. \end{aligned}$$

Then, straightforward computations show that

$$\begin{aligned} x &= \left(\mathbf{I} - \frac{1}{n+\beta}\Delta_{nm}\right)\left(\mathbf{I} - \frac{1}{\beta}\mathbf{B}_n\right)x_{nm} \\ &= \left(\mathbf{I} - \frac{\mathbf{B}_m}{\beta} + \frac{n}{n\beta+\beta^2}\Delta_{nm}\Theta_n\right)x_{nm}, \end{aligned}$$

so that

$$\begin{aligned} \mathbf{R}(\beta, \mathbf{B}_m)x &= \frac{1}{\beta}\left(x_{nm} + \mathbf{R}(\beta, \mathbf{B}_m)\frac{n}{n+\beta}\Delta_{nm}\Theta_n x_{nm}\right) \\ &= (n+\beta)\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \Delta_{nm})x \\ &\quad + n\mathbf{R}(\beta, \mathbf{B}_m)\Delta_{nm}\Theta_n\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \Delta_{nm})x \end{aligned}$$

and

$$\begin{aligned} &\mathbf{R}(\beta, \mathbf{B}_m)x - (n+\beta)\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \Delta_{nm})x && (6.10) \\ &= n\mathbf{R}(\beta, \mathbf{B}_m)\left(\mathbf{B}_m - \mathbf{B}_n\right)\mathbf{R}(\beta, \mathbf{B}_n)\Theta_n\mathbf{R}(n+\beta, \Delta_{nm})x \\ &= n\left(\mathbf{R}(\beta, \mathbf{B}_m) - \mathbf{R}(\beta, \mathbf{B}_n)\right)\Theta_n\mathbf{R}(n+\beta, \Delta_{nm})x, && (6.11) \end{aligned}$$

for all $x \in \mathbf{X}$. For each $n \in \mathbb{N}$, applying Lemma 6.2.5 to the sequence $\{\mathbf{A}_{n+1}, \mathbf{A}_{n+2}, ...\}$, we obtain a contractive C_0-semigroup $\mathbf{T}_n$ with generator $\mathbf{G}_n$ such that

$$\lim_{m\to\infty} \mathbf{R}(\alpha, \Delta_{nm})y = \mathbf{R}(\alpha, \mathbf{G}_n)y,$$

for each $y \in \mathbf{X}$ and $\alpha > 0$. In particular, this is true for $\alpha = n+\beta, \beta > 0$. It follows therefore, on letting $m \to \infty$ in (6.10) that

$$\mathbf{R}(\beta, \mathbf{A}) - (n+\beta)\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \mathbf{G}_n) \tag{6.12}$$

$$= \quad n\left(\mathbf{R}(\beta, \mathbf{A}) - \mathbf{R}(\beta, \mathbf{B}_n)\right)\Theta_n\mathbf{R}(n+\beta, \mathbf{G}_n). \tag{6.13}$$

Since $\|\beta\mathbf{R}(\beta, \mathbf{B}_n)\| \leq 1$ for $\beta > 0$, we have

$$\|(\mathbf{I} - \Theta_n)\mathbf{R}(\beta, \mathbf{B}_n)\| = \frac{1}{n}\|\mathbf{B}_n\mathbf{R}(\beta, \mathbf{B}_n)\| \leq \frac{2}{n}.$$

Using (6.12) we therefore obtain

$$\begin{aligned}
&\big\|\mathbf{R}(\beta, \mathbf{A}) - n\left(\mathbf{R}(\beta, \mathbf{A}) - \mathbf{R}(\beta, \mathbf{B}_n)\right)\Theta_n\mathbf{R}(n+\beta, \mathbf{G}_n) \\
&\quad - \quad (n+\beta)\Theta_n\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \mathbf{G}_n)\big\| \\
&\quad = \quad \|(n+\beta)\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \mathbf{G}_n) - (n+\beta)\Theta_n\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \mathbf{G}_n)\| \\
&\quad \leq \quad \|(\mathbf{I} - \Theta_n)\mathbf{R}(\beta, \mathbf{B}_n)\|\|(n+\beta)\mathbf{R}(n+\beta, \mathbf{G}_n)\| \\
&\quad \leq \quad \frac{2}{n}.
\end{aligned}$$

It follows that $\mathbf{R}(\beta, \mathbf{A})$ is the limit in norm of the sequence of bounded operators $\{\Phi_n\}_{n\in\mathbb{N}}$ where, for $n \in \mathbb{N}$,

$$\Phi_n = n\left(\mathbf{R}(\beta, \mathbf{A}) - \mathbf{R}(\beta, \mathbf{B}_n)\right)\Theta_n\mathbf{R}(n+\beta, \mathbf{G}_n) - (n+\beta)\Theta_n\mathbf{R}(\beta, \mathbf{B}_n)\mathbf{R}(n+\beta, \mathbf{G}_n).$$

Since Θ_n is weakly compact for each $n \in \mathbb{N}$, so is Φ_n. This implies that $\mathbf{R}(\beta, \mathbf{A})$ is weakly compact, being the limit in norm of weakly compact operators. Thus, we have obtained a C_0-semigroup $\mathbf{T}$ whose generator $\mathbf{A}$ has weakly compact resolvents, and (*2*) is true. □

It is possible to obtain a version of Theorem 6.2.6 for C_0-semigroups which are not necessarily contractive. The price to pay for this is a more complicated condition concerning the existence of an approximation of the identity. We have,

Corollary 6.2.7. *Let* $\mathbf{X}$ *be a separable Banach space. Then the following are equivalent:*

1. *There exists a* C_0*-semigroup on* $\mathbf{X}$*, with respect to which* $\mathbf{X}$ *is* $\odot$*-reflexive;*
2. *There exists a* C_0*-semigroup on* $\mathbf{X}$ *whose generator has weakly compact resolvents;*
3. *There exists a weakly compact, commuting, approximation of the identity,* $\{\mathbf{A}_n\}$ *on* $\mathbf{X}$*, such that*

$$\{\mathbf{A}_{n_1}\mathbf{A}_{n_2}...\mathbf{A}_{n_k} : k \in \mathbb{N}, n_i \in \mathbb{N}\} \tag{6.14}$$

is bounded;

4. There exists a weakly compact, approximation of the identity $\{\mathbf{A}_n\}$ *on* $\mathbf{X}$ *such that*

$$\{\mathbf{A}_{n_1}\mathbf{A}_{n_2}...\mathbf{A}_{n_k} : k \in \mathbb{N}, n_i \in \mathbb{N}\}$$

is bounded;

5. There exists a weakly compact C_0*-semigroup on* $\mathbf{X}$*.*

Proof. If any of (*1*), (*2*) or (*5*) is true for a C_0-semigroup $\mathbf{T}$, then by considering the rescaled C_0-semigroup $e^{-\omega\cdot}\mathbf{T}(\cdot)$, $\omega > \omega_0(\mathbf{T})$ on the Banach space $\mathbf{X}$ equipped with the equivalent norm $|||\cdot|||$, given by

$$|||x||| = \sup_{t\geq 0} \|e^{-\omega t}\mathbf{T}(t)x\|, \tag{6.15}$$

we obtain the corresponding condition for a contractive C_0-semigroup. Then it follows from Theorem 6.2.6 and rescaling and renorming in the reverse direction, that (*1*), (*2*) and (*5*) are equivalent.

(*2*) $\Longrightarrow$ (*3*) : Let $\mathbf{T}$ be a C_0-semigroup whose generator $\mathbf{A}$ has weakly compact resolvents. Let $\mathbf{S}(t) = e^{-\omega t}\mathbf{T}(t), \omega > \omega_0(\mathbf{T})$. Then $\mathbf{S}$ is a contractive C_0-semigroup on $(\mathbf{X}, |||\cdot|||)$, where $|||\cdot|||$ is as in (6.15), with generator $\mathbf{B} = \mathbf{A} - \omega$, and $\mathbf{B}$ has weakly compact resolvents. From the proof of (*2*) $\Longrightarrow$ (*3*) in Theorem 6.2.6 there is a weakly compact approximation of the identity $\{\mathbf{A}_n\}$ such that $|||\mathbf{A}_n||| \leq 1$, and then $\{\mathbf{A}_{n_1}\mathbf{A}_{n_2}...\mathbf{A}_{n_k} : k \in \mathbb{N}, n_1 \in \mathbb{N}\}$ is bounded with respect to $\|\cdot\|$. Thus (*3*) holds.

(*3*) $\Longrightarrow$ (*4*) : This is obvious.

(*4*) $\Longrightarrow$ (*5*) : Let $\{\mathbf{A}_n\}$ be a weakly compact approximation of the identity on $\mathbf{X}$ satisfying the assumption in (*4*). We define a new norm $\|\cdot\|_b$ on $\mathbf{X}$ by setting,

$$\|x\|_b := \sup\{\|\mathbf{A}_{n_1}\mathbf{A}_{n_2}...\mathbf{A}_{n_k}(x)\| : k \in \mathbb{N},\, n_i \in \mathbb{N},\, i = 1, 2, ..., k\}.$$

It is easy to see that $\|\cdot\|_b$ defines an equivalent norm on $\mathbf{X}$ and then $\{\mathbf{A}_n\}$ forms a weakly compact contractive approximation of the identity of $\mathbf{X}$. It follows from Theorem 6.2.6 that there exists a C_0-semigroup on $\mathbf{X}$ which is weakly compact. □

We note here that in Theorem 6.2.6, (*5*) $\Longrightarrow$ (*2*) may be proven directly in the following manner: For $\operatorname{Re}\lambda > \omega_0(\mathbf{T})$,

$$\mathbf{R}(\lambda, \mathbf{A})^2 = \int_0^\infty e^{-\lambda t}\mathbf{T}(t)\mathbf{R}(\lambda, \mathbf{A})\, dt,$$

the integral existing as a $\mathcal{L}(\mathbf{X})$-valued Bochner integral. Since weakly compact operators form a norm closed ideal of $\mathcal{L}(\mathbf{X})$, weak compactness of $\mathbf{T}$ implies that of $\mathbf{R}(\lambda, \mathbf{A})^2$. For any $\mu \in \rho(\mathbf{A})$, we have, on making use of the resolvent identity (see proof of [39, Corollary 2.5.4], that

$$\lim_{\lambda\to\infty} \|(\lambda\mathbf{R}(\lambda, \mathbf{A}))^2\mathbf{R}(\mu, \mathbf{A}) - \mathbf{R}(\mu, \mathbf{A})\| = 0.$$

Therefore, $\mathbf{R}(\mu, \mathbf{A})$ is also weakly compact, being the limit in norm of weakly compact operators. Thus, in proving the implication $(5) \implies (2)$ we have just used the fact that weakly compact operators form a norm closed two sided ideal of $\mathcal{L}(\mathbf{X})$. In fact, in the proof of Theorem 6.2.6, this is the only property of weakly compact operators that has been used, except for the implications $(5) \implies (1)$. Therefore, we may deduce the following from Corollary 6.2.7:

Corollary 6.2.8. *Let $\mathbf{X}$ be a separable Banach space and $\mathcal{B} \subset \mathcal{L}(\mathbf{X})$ be a norm closed, two sided ideal of $\mathcal{L}(\mathbf{X})$. Then the following are equivalent:*

1. *There exists a C_0-semigroup on $\mathbf{X}$ whose generator has resolvents in $\mathcal{B}$;*
2. *There exists a commuting approximation of the identity, $\{\mathbf{A}_n\}$ on $\mathbf{X}$, such that $\mathbf{A}_n \in \mathcal{B}$ for each $n \in \mathbb{N}$ and*
$$\{\mathbf{A}_{n_1}\mathbf{A}_{n_2}...\mathbf{A}_{n_k} : k \in \mathbb{N}, n_i \in \mathbb{N}\} \tag{6.16}$$
is bounded.
3. *There exists an approximation of the identity $\{\mathbf{A}_n\}$ on $\mathbf{X}$ such that $\mathbf{A}_n \in \mathcal{B}$ for each $n \in \mathbb{N}$ and*
$$\{\mathbf{A}_{n_1}\mathbf{A}_{n_2}...\mathbf{A}_{n_k} : k \in \mathbb{N}, n_i \in \mathbb{N}\}$$
is bounded.
4. *There exists a C_0-semigroup $\mathbf{T}$ on $\mathbf{X}$ such that $\mathbf{T}(t) \in \mathcal{B}$ for all $t > 0$.*

Remark 6.2.9. We collect here some observations concerning the results obtained so far in this section.

1. In [46], Sauvageot has proven an analogue of Corollary 6.2.8 for strong Feller, completely positive contractions on a C^*-algebra. More precisely, it has been shown [46, Proposition 4.1] that on a separable C^*-algebra the following properties are equivalent:
   - There exists a strong Feller, completely positive approximation of the identity of $\mathbf{X}$;
   - There exists a strong Feller C_0-semigroup on $\mathbf{X}$ of completely positive contractions.

   Note that strong Feller, completely positive contractions defined on a C^*- algebra are not, in general, associated with a closed two sided ideal of $\mathcal{L}(\mathbf{X})$. Therefore, in [46], additional C^*-algebraic methods had to employed to prove the required results. The basic strategy, however, was the same.

2. As a particular case of Corollary 6.2.8, we may take the ideal of compact operators on $\mathbf{X}$. Then the equivalent conditions of Corollary 6.2.8 imply that $\mathbf{X}$ is $\odot$-reflexive with respect to a C_0-semigroup. If additionally, the separable Banach space $\mathbf{X}$ has (DPP), then the reverse implication also holds.

3. In general it is not true that $\mathbf{X}$ is $\odot$-reflexive if and only if $\mathbf{T}$ is weakly compact. As an example, one may consider the rotation group $\mathbf{T}$ defined on $C(\Gamma)$, Γ the unit circle, defined by

$$\mathbf{T}(t)f(e^{i\theta}) = f(e^{i(\theta+t)}).$$

Then $C(\Gamma)^{\odot} = \mathrm{L}^1(\Gamma)$ and $C(\Gamma)^{\odot\odot} = C(\Gamma)$, so that $C(\Gamma)$ is $\odot$-reflexive with respect to $\mathbf{T}$ [39, Example 1.3.9], but $\mathbf{T}$ is not weakly compact. However, in view of Theorem 6.2.6, we may conclude that if $\mathbf{X}$ is $\odot$-reflexive with respect to a C_0-semigroup then it at least admits a weakly compact C_0-semigroup.

□

We recall here that a Banach space $\mathbf{X}$ with (DPP) is said to have the *hereditary Dunford Pettis property* (DPh) if every closed subspace of $\mathbf{X}$ also has (DPP).

Theorem 6.2.10. *There exist separable Banach spaces which do not admit bounded, weakly compact, approximations of the identity and therefore, there exist separable Banach spaces which are not sun-reflexive with respect to any C_0-semigroup.*

Proof. It is known that ℓ^1 has (DPh), [18, Page 26] and that there exists a closed subspace $\mathbf{X}$ of ℓ^1 which does not have the compact approximation property [36, Theorem 1.g.4]. Suppose $\{\mathbf{B}_n\}$ is a bounded, weakly compact, approximation of the identity on $\mathbf{X}$. Then $\{\mathbf{B}_n^2\}$ is also an approximation of the identity on $\mathbf{X}$. Further, since $\mathbf{X}$ has (DPP), the operators $\mathbf{B}_n^2$ are compact for each $n \in \mathbb{N}$. Thus, $\mathbf{X}$ has CAP, which is a contradiction.

Now suppose that there exists a C_0-semigroup on $\mathbf{X}$ with respect to which $\mathbf{X}$ is sun-reflexive. Since $\mathbf{X}$ is separable, by Corollary 6.2.7 there exists a weakly compact approximation of the identity on $\mathbf{X}$ satisfying (6.14); in particular, this weakly compact approximation of the identity is bounded. The discussion in the previous paragraph shows that this is not possible. Thus, we may conclude that $\mathbf{X}$ cannot be sun-reflexive with respect to any C_0-semigroup. □

Bibliography

[1] H. Amann, Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications, Math. Nachr. **186** (1997), 5 – 56.

[2] W. Arendt, C.J.K. Batty, M. Hieber, and F. Neubrander, Vector valued Laplace transforms and Cauchy problems, Monographs in Mathematics, vol. 96, Birkhäuser, 2001.

[3] W. Arendt, A. Driouich, and O. El-Mennaoui, On the infinite product of C_0-semigroups, J. Functional Analysis **160** (1998), 524–542.

[4] C.J.K. Batty, Bounded Laplace transforms, primitives and semigroup orbits, Arch. Math. (Basel) (to appear).

[5] C.J.K. Batty and M. Blake, Convergence of Laplace integrals, C.R. Acad. Sci. Paris Ser.I **330** (2000), 71–75.

[6] C.J.K. Batty, M. Blake, and S. Srivastava, A non-analytic growth bound for Laplace transforms and semigroups of operators, Integral Equations and Operator Theory (to appear).

[7] C.J.K Batty and R. Chill, Bounded convolutions and solutions of inhomogeneous Cauchy problems, Forum Math. (1999), no. 11, 253–277.

[8] C.J.K. Batty, R. Chill, and J.M.A.M. van Neerven, Asymptotic behaviour of C_0-semigroups with bounded local resolvents, Math. Nachr. **219** (2000), 65–83.

[9] M. Blake, S. Brendle, and R. Nagel, On the structure of the critical spectrum of strongly continuous semigroups, Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Lecture Notes in Pure and Appl. Math., vol. 215, Dekker, New York, 2001, pp. 55–65.

[10] M.D. Blake, Asymptotically norm continuous semigroups of operators, D.Phil thesis, University of Oxford, 1999.

[11] M.D. Blake, A spectral bound for asymptotically norm-continuous semigroups, J. Operator Theory **45** (2001), 111–130.

[12] P.H. Bloch, Über den Zusammenhang zwischen den Konvergenzabszissen, der Holomorpie-und der Beschränktheitsabszisse bei der Laplace-Transformation, Comment. Math. Helv. **22** (1949), 34–47.

[13] S. Brendle, On the asymptotic behaviour of perturbed strongly continuous semigroups, Math. Nachr. **226** (2001), 35 – 47.

[14] S. Brendle and R. Nagel, Partial functional differential equations with nonautonomous past, pre-print.

[15] S. Brendle, R. Nagel, and J. Poland, On the spectral mapping theorem for perturbed strongly continuous semigroups, Arch. Math. (Basel) (2000), no. 74, 365 – 378.

[16] C. Chicone and Y. Latushkin, Evolution semigroups in dynamical systems and differential equations, Mathematical Surveys and Monographs, vol. 70, AMS, Providence, 1999.

[17] E.B. Davies, One-parameter semigroups, Academic Press, London, 1980.

[18] J. Diestel, A survey of results related to the Dunford-Pettis property, Proceedings of the Conference on Integration, Topology, and Geometry in Linear Spaces (Univ. North Carolina, Chapel Hill, N.C. 1979), Contemp. Math., vol. 2, Amer. Math. Soc., Providence,R.I., 1980, pp. 15 – 60.

[19] J. Diestel and J.J. Uhl, Vector measures, Mathematical Surveys, vol. 15, Amer. Math. Soc., Providence,R.I., 1977.

[20] K.J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, Graduate Texts in Mathematics, Springer-Verlag, 1999.

[21] A. Erdélyi et al, Tables of integral transforms, vol. 2, McGraw-Hill, New York, 1954.

[22] L. Gearhart, Spectral theory for contraction semigroups on Hilbert space, Trans. Amer. Math. Soc. **236** (1978), 385 – 394.

[23] V. Goersmeyer and L. Weis, Norm continuity of C_0-semigroups, Studia Math. **134** (1999), no. 2, 169–178.

[24] G. Greiner, J. Voigt, and M. Wolff, On the spectral bound of the generator of semigroups of positive operators, J. Operator Theory **5** (1981), 245–256.

[25] N. Grønbæk and G.A. Willis, Approximate identities in Banach algebras of compact operators, Canadian Mathematical Bulletin **36** (1993), 45 – 53.

[26] M. Hieber, Operator valued Fourier multipliers, Progress in Nonlinear Differential Equations and their Applications, vol. 35, Birkhäuser-Verlag, 1999, pp. 363–380.

[27] M. Hieber, A characterisation of the growth bound of a semigroup via Fourier multipliers, Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Lecture Notes in Pure and Appl. Math., vol. 215, Dekker, New York, 2001, pp. 121 – 124.

[28] E. Hille and R.S. Phillips, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., vol. 31, Providence, R.I., 1957.

[29] L. Hörmander, The analysis of linear partial differential operator i. Distribution theory anf Fourier analysis, Springer-Verlag, 1990.

[30] M.A. Kaashoek and S.M.Verduyn Lunel, An integrability condition on the resolvent for hyperbolicity of the semigroup, J. Differential Equations **112** (1994), 374–406.

[31] H. Komatsu, Fractional powers of operators, Pacific J. Math. **19** (1966), 285–346.

[32] Y. Latushkin and F. Räbiger, Fourier multipliers in stability and control theory, preprint, 2001.

[33] Y. Latushkin, T. Randolph, and R. Schnaubelt, Exponential dichotomy and mild solutions of nonautonomous equations in Banach spaces, J. Dynamical Differential Equations **10** (1998), no. 3, 489–510.

[34] Y. Latushkin and R. Shvydkoy, Hyperbolicity of semigroups and Fourier multipliers, preprint, 2001.

[35] A. Lima and O. Nygaard, Isometric factorization of weakly compact operators and the approximation property, Israel J. Math. **119** (2000), 325 – 348.

[36] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces, vol. 2, Springer Verlag, 1979.

[37] J. Martinez and J.M. Mazon, C_0-semigroups norm continuous at infinity, Semigroup Forum **52** (1996), 213–224.

[38] R. Nagel and J. Poland, The critical spectrum of a strongly continuous semigroup, Adv. Math. **152** (2000), 120–133.

[39] J.M.A.M.van Neerven, The adjoint of a semigroup of linear operators, Lecture Notes in Mathematics, vol. 1529, Springer Verlag, 1992.

[40] J.M.A.M.van Neerven, The asymptotic behaviour of a semigroup of linear operators, Birkhauser, 1996.

[41] J.M.A.M.van Neerven, B. Straub, and L. Weis, On the asymptotic behaviour of linear operators, Indag. Math **N.S. 6** (1995), 453–476.

[42] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Appl. Math. Sci., vol. 44, Springer Verlag, 1983.

[43] Vu Quoc Phong and E. Schüler, The operator equation $AX-XB = C$, admissibilty, and asymptotic behaviour of differential equations, J. Differential Equations **145** (1998), 394–419.

[44] J. Prüss, On the spectrum of C_0-semigroups, Trans. Amer. Math. Soc. **284** (1984), 847–857.

[45] W.M. Ruess and Vu Quoc Phong, Asymptotically almost periodic solutions of evolution equations in Banach spaces, J. Differential equations **122** (1995), 282 – 301.

[46] J.L. Sauvageot, Strong Feller semigroups on C^*-algebras, J. Operator Theory **42** (1999), 83–102.

[47] M. Slemrod, Asymptotic behaviour of C_0-semigroups as determined by the spectrum of the generator, Indiana Univ. Math. J. **25** (1976), 783 – 792.

[48] H.R. Thieme, Balanced exponential growth of operator semigroups, J. Math. Anal. Appl. **223** (1998), 30–49.

[49] L. Weis, Stability theorems for semigroups via multiplier theorems, Tübinger Berichte **6** (1996 /97).

[50] L. Weis and V. Wrobel, Asymptotic behaviour of C_0-semigroups in Banach spaces, Proc. Amer. Math. Soc. **124** (1996), 3663–3671.

[51] G. Weiss, The resolvent growth assumption for semigroups on Hilbert spaces, J. Math. Analysis and Applications **145** (1990), 154 –171.

[52] P. H. You, Characteristic conditions for a C_0-semigroup with continuity in the uniform operator topology for $t > 0$, Proc. Amer. Math. Soc. **116** (1992), 991 – 997.