Real Recurrence Sets

by

Leonard J. Konrad

(Trinity College)

Thesis for the degree

MSc by Research

Hilary 2015

Supervised by

Prof. Charles J.K. Batty

University of Oxford

Mathematical Institute

Abstract

Classical theorems such as the Poincaré recurrence theorem, the van der Corput theorem about equidistribution of sequences or the F. and M. Riesz theorem about measures on the torus show properties of the set $\mathbb{N}$. However, In particular the Poincaré recurrence easily allows to extend the recurrence result to sets such as $m\mathbb{N}$ for any $m \in \mathbb{N}$.

This motivates the notion of recurrence sets, i.e. sets $\mathcal{D} \subseteq \mathbb{N}$ (or $\mathcal{D} \subseteq \mathbb{Z}$) which are "strong" enough to force certain recurrence properties, and a thorough study of these sets and their relations with each other has been undertaken since the late 1970s.

This thesis deals with real recurrence sets $\mathcal{D} \subseteq \mathbb{R}$. Our first result shows that integer properties and most associated implications can be transferred to the real setting and allow a similar treatment.

For a set $\mathcal{D} \subseteq \mathbb{Z}$, the integer and real recurrence property coincide for many properties. This gives non-trivial recurrence examples from the integer theory, but also yields some counterexamples showing that some recurrence properties are distinct.

Using continuity and appropriate product systems, we show that we can reduce recurrence sets such as Poincaré or operator recurrence sets, in particular, every such recurrence set has a countable subset $\tilde{D} \subseteq \bigcup_{|n|>N} ([sn - \epsilon, sn + \epsilon] \cap \mathcal{D})$ for arbitrary small $\epsilon > 0$ and arbitrarily large $N \in \mathbb{N}$ having the same recurrence property.

We finally indicate how to further extend this topic by discussing topological dynamical systems, a quantitative analysis of recurrence sets and the use of locally compact abelian groups instead of $\mathbb{Z}$ and $\mathbb{R}$.

Acknowledgement

I am grateful to my supervisor Prof. Charles J.K. Batty. I am thankful to the Mathematical Institute for providing the inspiring environment and the Andrew-Mullins Award as well as to EPSRC for the fees award. I thank everyone in my college for the support and the company. I say a big thank you to my family, in particular, to my parents, who have supported me all the way. Last, but not not least, I give special thanks to Ani Hakobyan. Thank you for always being by my side.

Contents

1 Introduction **1**

1.1 Introduction 1

1.2 Notation 7

1.3 Overview and Main Results 9

2 Preliminaries **16**

2.1 Measure Preserving Systems and the Mean Ergodic Theorem 16

2.2 Fourier Transform and Bochner-Herglotz 20

2.3 The Strong Law of Large Numbers 22

2.4 The Sets $l^\infty_*(\mathbb{R})$, $l^\infty_c(\mathbb{R})$ and $l^\infty_{cc}(\mathbb{R})$ 24

2.5 Lemmata 27

3 Recurrence Properties **31**

3.1 Definition of Main Recurrence Properties 31

3.1.1 Poincaré 32

3.1.2 Operator Recurrence 33

3.1.3 Strong Recurrence 34

3.1.4 Strong Operator Recurrence 34

3.2 Characterisations 35

3.2.1 Operator Recurrence 36

3.2.2 Poincaré and Koopman Recurrence 43

3.2.3 Kamae, Mendès France 47

3.2.4 Spectral Measures 49

3.2.5 Correlativity 53

3.2.6 Van der Corput 60

3.2.7 Fürstenberg Correspondence Principle 68

3.2.8 Intersectivity and Combinatorial Recurrence 73

4 Examples **77**

4.1 First Examples and Classical Theorems 77

4.1.1 FMRiesz 78

4.1.2 Operator Recurrence and Poincaré 81

4.1.3 Van der Corput and Correlativity 84

4.2 A Rationally Linearly Independent Sequence 89

4.3 Integer and Real Recurrence Properties for $\mathcal{D} \subseteq \mathbb{Z}$ 91

4.3.1 Operator Recurrence and Poincaré 92

4.3.2 FMRiesz, FC+ and KMF 95

4.3.3 Intersectivity 96

4.3.4 Correlativity 99

4.3.5 Classical Examples 104

4.4 The Sets $\{\frac{1}{\log p}, p \text{ prime}\}$ and $\{\log p, p \text{ prime}\}$ 107

5 Reducing Recurrence Sets **108**

5.1 Reducing Results 108

5.2 Ramsey Property 113

5.3 Remarks 117

6 Alon-Peres Characterisation of Bourgain's Example **119**

7 Extending the theory **125**

7.1 Topological Recurrence 125

7.2 Quantitave Analysis 129

7.3 Locally compact abelian groups 133

7.3.1 Finite Groups 134

7.3.2 Setting for Locally Compact Abelian Groups 135

7.3.3 Example: Operator Recurrence and FMRiesz 139

Bibliography **142**

Appendix **152**

Chapter 1

Introduction

1.1 Introduction

In his work about the three body problem, Poincaré showed in 1890 that every (measure preserving) dynamical system is recurrent ([64, Theorem I]).
Caratheodory ([24]) first presented Poincaré's result in modern terminology, and it can be stated as follows.

Theorem 1 (Poincaré)**.** *Let* $(\Omega, \Sigma, \mu; \phi)$ *be a measure preserving system. Then for any* $A \in \Sigma$ *with* $\mu(A) > 0$*, there exists* $n \in \mathbb{N}$ *such that*

$$\mu(\phi^n(A) \cap A) > 0.$$

We call a measure preserving system $(\Omega, \Sigma, \mu; \phi)$ with a set A of positive measure **recurrent** if it satisfies the statement of Theorem 1, i.e. if there exists $n \in \mathbb{N}$ such that $\mu(\phi^n(A) \cap A) > 0$. We call it **infinitely recurrent** if the set $\{n \in \mathbb{N} : \mu(A \cap \phi^n(A)) > 0\}$ is infinite.
It turns out that not all natural numbers are required to obtain recurrence of a given measure preserving system $(\Omega, \Sigma, \mu; \phi)$ with a set A of positive measure. We also note that the restriction to natural numbers is not essential when discussing recurrence of invertible measure preserving systems. Using the invertibility of

$(\Omega, \Sigma, \mu; \phi)$, recurrence can immediately be extended to integers. This motivates the following definition.

Definition 1.1 (Poincaré Recurrence). *A set $\mathcal{D} \subseteq \mathbb{Z}$ is **Poincaré recurrent** if, given an invertible measure preserving system $(\Omega, \Sigma, \mu; \phi)$ and $A \in \Sigma$ with $\mu(A) > 0$, there exists $d \in \mathcal{D}$ such that*

$$\mu(A \cap \phi^d(A)) > 0.$$

This means that a set $\mathcal{D} \subseteq \mathbb{Z}$ is Poincaré if it is "large" enough to force recurrence of an invertible measure preserving system $(\Omega, \Sigma, \mu; \phi)$ with a set A of positive measure. It is easy to see that the set $m\mathbb{N}$ for some $m \in \mathbb{N}$ has this property simply by considering $(\Omega, \Sigma, \mu; \phi^m)$ and Theorem 1.

Using a similar argument, this also shows that a given invertible measure preserving system $(\Omega, \Sigma, \mu; \phi)$ with a set A of positive measure is not only once, but infinitely recurrent. Inductively, we obtain an infinite set $\{d_n : n \in \mathbb{N}\}$ such that $\mu(A \cap \phi^{d_n}(A)) > 0$ for all $n \in \mathbb{N}$ by considering $(\Omega, \Sigma, \mu; \phi^m)$ where $m > d_1, \ldots, d_n$ and $d_{n+1} := \min\{nm \in \mathbb{N} : \mu(A \cap \phi^{nm}(A)) > 0\}$.

More interesting examples are given by the squares $\{n^2 : n \in \mathbb{N}\}$, the set $\{p(n) : n \in \mathbb{N}\} \backslash \{0\}$ with a polynomial $p \neq 0$ having integer coefficients and satisfying $p(0) = 0$ ([36, Theorem 3.16], [75]), the set of differences $\{n - m; n > m \in I\}$ with any infinite subset of integers I ([42]), or the sets $\{p - 1 : p \text{ prime}\}$ and $\{p + 1 : p \text{ prime}\}$ ([42]).

Birkhoff ([19], Section VII) introduced and formalised the language of dynamical systems. He also considered analogous statements for topological systems ([19, Subsections VII.7-8], Section 7.1). Koopman ([47]) transferred the topic of measure preserving systems into a Hilbert space setting and allowed the use of functional analytic methods.

The study of these systems led to the mean ergodic theorems which deal with the convergence of the Cesàro averages

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} T^i f. \tag{1.1}$$

In 1931, von Neumann ([59]) showed that this limit exists in L^2 for the Koopman operator $T_\phi f := f \circ \phi$ of a measure preserving system $(\Omega, \Sigma, \mu; \phi)$ and equals the projection onto the fixed space of T_ϕ. Birkhoff ([20]) subsequently showed that the same statement holds for almost everywhere convergence and $f \in L^1(\Omega, \Sigma, \mu)$. Latterly, the concept of the convergence in (1.1) has been generalised in many directions by allowing more general operators, spaces or different dynamics such as strongly continuous semigroups or representation of certain groups. The convergence of these Cesàro averages is the central tool of this thesis connecting recurrence properties and it is a recurrent theme throughout all chapters.
A simple consequence of the mean ergodic theorem, i.e the convergence of $\frac{1}{N}\sum\limits_{n=0}^{N-1} U^n$ for a (suitable) operator U to the projection P onto $\operatorname{Fix}(U)$ along $\overline{\operatorname{ran}}(\operatorname{Id} - U)$, is the following result which also implies Theorem 1. This shows that the orbit of a unitary operator and certain vectors cannot be completely orthogonal to the spanning vector.

Theorem 2. *Let H be a Hilbert space, U a unitary operator on H with mean ergodic projection $P := \lim_{N\to\infty} \frac{1}{N}\sum\limits_{n=0}^{N-1} U^n$ and $x \in H$ with $Px \neq 0$. Then there exists $n \in \mathbb{N}$ such that*

$$\langle U^n x, x\rangle \neq 0.$$

As with Poincaré recurrence, we do not need all natural numbers to obtain the statement of Theorem 2. This hence motivates the following definition of operator recurrence.

Definition 1.2 (Operator Recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{Z}$ is* ***operator recurrent*** *if, given a Hilbert space H, a unitary operator U on H and x with $\|x\| = 1$ and*

$Px \neq 0$, there exists $d \in \mathcal{D}$ such that

$$\langle U^d x, x\rangle \neq 0.$$

Using the Koopman representation of an invertible measure preserving system as a unitary operator on $L^2(\Omega, \Sigma, \mu)$, it is straightforward that every operator recurrent set $\mathcal{D}$ is also Poincaré recurrent.

Fürstenberg ([37]) introduced a correspondence principle connecting number theoretical results with measure preserving systems (see, e.g. [48] for an overview). Among other results, this shows that every Poincaré set $\mathcal{D} \subseteq \mathbb{N}$ is **intersective**, i.e. for all $E \subseteq \mathbb{Z}$ with $\overline{d}_{\mathbb{Z}}(E) := \limsup_{n\to\infty} \frac{|E\cap\{-n,\ldots,n\}|}{2n+1} > 0$, there exists some $d \in \mathcal{D}$ such that $E \cap (E + d) \neq \emptyset$.

Theorem 3 (Fürstenberg)**.** *Let $E \subseteq \mathbb{Z}$ with $\overline{d}_{\mathbb{Z}}(E) > 0$. Then there exists an invertible measure preserving system $(\Omega, \Sigma, \mu; \phi)$ and $A \in \Sigma$ such that*

$$\mu(A) = \overline{d}_{\mathbb{Z}}(E)$$

and

$$\mu(A \cap \phi^d(A)) \leq \overline{d}_{\mathbb{Z}}(E \cap (E + d))$$

for all $d \in \mathbb{Z}$.

In 1931, van der Corput ([26]) showed an interesting property of the natural numbers concerning the equidistribution mod 1 of sequences. We call hereby a sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers **equidistributed mod 1** if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{[a,b]}(u_n) = b - a$$

for all $0 \leq a < b \leq 1$.

Theorem 4 (Van der Corput)**.** *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence such that the differences $(u_{n+d} - u_n)_{n\in\mathbb{N}}$ are equidistributed mod 1 for all $d \in \mathbb{N}$. Then $(u_n)_{n\in\mathbb{N}}$ is equidistributed mod 1.*

As in the case of Theorems 1 and 2, not all natural numbers are required for the statement of Theorem 4. A set $\mathcal{D} \subseteq \mathbb{N}$ is said to be **van der Corput** if, given a real sequence $(u_n)_{n\in\mathbb{N}}$, the equidistribution mod 1 of $(u_{n+d} - u_n)_{n\in\mathbb{N}}$ for all $d \in \mathcal{D}$ implies the equidistribution mod 1 of $(u_n)_{n\in\mathbb{N}}$. Many examples have been found, in particular the above mentioned Poincaré sets satisfy this van der Corput property. Equidistribution however is a rather probabilistic topic. Weyl ([83]) had introduced a criterion to transfer the question of equidistribution mod 1 into analytic terminology.

Theorem 5 (Weyl)**.** *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence. Then $(u_n)_{n\in\mathbb{N}}$ is equidistributed mod 1 if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i l u_k} = 0$$

holds for all $0 \neq l \in \mathbb{Z}$.

In view of Weyl's theorem, van der Corput's theorem now reads as follows.

Theorem 6 (Van der Corput (Revised))**.** *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i l u_{k+d}} \overline{e^{2\pi i l u_k}} = 0$$

holds for all $0 \neq l \in \mathbb{Z}$ and for all $d \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i l u_k} = 0$$

holds for all $0 \neq l \in \mathbb{Z}$.

Another seemingly unrelated result of the natural numbers is the following theorem ([66]).

Theorem 7 (F. and M. Riesz)**.** *Let μ be a positive measure on $\mathbb{T}$ such that*

$$\widehat{\mu}(n) = \int_{\mathbb{T}} x^{-n} \, d\mu(x) = 0$$

for all $n \in \mathbb{N}$, then $\mu(\{1\}) = 0$.

For example, the Lebesgue measure λ on $\mathbb{T}$ satisfies $\widehat{\lambda}(n) = 0$ for all $n \in \mathbb{N}$ and indeed $\lambda(\{1\}) = 0$. F. and M. Riesz actually showed a much stronger result in [66], i.e. that if the Fourier coefficients of a complex measure on $\mathbb{T}$ vanish, then it is absolutely continuous, i.e. $\mu(A) = 0$ for all Lebesgue null sets A and in particular $\mu(\{x\}) = 0$ for all $x \in \mathbb{T}$. However, we consider only positive measures and continuity at 1 in this thesis to set the corresponding FMRiesz set in relation with other recurrence sets, especially with operator recurrence (using L^2-spaces) and with correlativity (through Lemma 2.17).

The connection of Theorem 7 with Theorem 6 becomes apparent by using the positive-definiteness of $d \mapsto \gamma(d) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi i l u_{k+d}} \overline{e^{2\pi i l u_k}}$ and the Bochner-Herglotz theorem which yields a measure μ on $\mathbb{T}$ such that $\widehat{\mu}(n) = \gamma(n)$ for all $n \in \mathbb{Z}$.

The connection of Theorem 6 to harmonic analysis and Theorem 7 was introduced by Wiener ([84]) and then studied in more depth in France (see for example [4], [5], [18], [65], [25]). This topic was studied under the name “pseudo-aléatoires” (pseudo-random) sequences or functions and the van der Corput theorem was reproven by harmonic analytic methods (for example [18]). This gives the inspiration for the equivalence of correlativity and FMRiesz sets.

It is interesting to note that Theorem 2, Theorem 4, Theorem 6 and Theorem 7 share a close relation with each other although they look rather different at first glance. It is one goal of this thesis to analyse and establish these relations in a broader context.

The study of recurrence properties and their relations to each other was in particular started by Kamae and Mendès France ([42]) and Ruzsa ([70], [71]). Later results ([2], [17], [23], [33], [55], [57], [60], [63]) extended these relations and Bergelson and Lesigne gave a good account of recurrence sets in $\mathbb{Z}^k$ ([14]).

The motivation of this thesis comes from the study of recurrence sets $\mathcal{D} \subseteq \mathbb{R}$. All properties of integer Poincaré or van der Corput sets can be restated in terms of real numbers and do not rely on the discrete integer structure. In fact, it turns out that rather the mean ergodic theorem and the interplay between a group and its dual group are the key element for the characterisations to hold. This is in particular the case when dealing with FMRiesz and its connection to other recurrence properties, and hence with the Fourier transform and the Bochner-Herglotz theorem. In the integer situation, the duality of $\mathbb{Z}$ and $\mathbb{T}$ may not be expressed explicitly, but the proofs are based on the same principles and methods such as the Fourier transform.

An extensive study of such general recurrence sets $\mathcal{D} \subseteq \mathbb{R}$ has not yet been done to our knowledge, the most general studies are concerned with van der Corput sets $\mathcal{D} \subseteq \mathbb{Z}^k$ ([6], [8], [14], [16]) as well as with the set $\mathbb{R}$ itself in the topic of pseudo-random functions ([4], [5], [18]). Additionally, there are some results about the connection of Poincaré recurrence and combinatorial recurrence (e.g. [11], [82, Section 6], [38, Section 2]).

The results in this thesis are almost all formally new. However, in particular the results in Chapter 3 and 6 follow with minor modifications from the existing integer proofs. We indicate this by giving a reference "compare [X]" between the statement and the proof. On the other hand, Theorem 10 with the equivalence of integer and real operator recurrence for a set $\mathcal{D} \subseteq \mathbb{Z}$ and Theorem 11 about the reduction of recurrence sets have no precedent.

1.2 Notation

In this thesis, we deal with various notions of recurrence which are defined in detail in Chapter 3 (particularly in Section 3.1). We call a set $\mathcal{D}$ **recurrent** without further specification if it satisfies one of the properties listed in Theorems 8 or 9 and we similarly use the term **recurrence property** to indicate one of these properties without further specification. This name is motivated by properties such as Poincaré recurrence and combinatorial recurrence, but also applies to the other properties due to the close relationship between each other, and we use it in particular to talk about general behaviour of these properties.

When we deal with recurrence sets, two types of recurrence phenomena appear: Asymptotic recurrence and recurrence due to continuity at 0. We say that a set $\mathcal{D}$ satisfies a **recurrence property asymptotically** or **"at ∞"** if it is recurrent and if there exists $\epsilon > 0$ such that $\mathcal{D}\backslash(-\epsilon, \epsilon)$ satisfies the same recurrence property. We show in Proposition 5.4 for strong operator recurrence, strong recurrence, operator recurrence and Poincaré that we can choose ϵ arbitrarily large if 0 is not a limit point. When we consider $\mathbb{N}$ or $\mathbb{Z}$, only such asymptotic recurrence sets appear.

However, with the extension to the real numbers, we have to consider continuity at 0. We say that a set $\mathcal{D}$ satisfies a **"bounded" recurrence property** or **"around 0"** if it is recurrent and if there is $M > 0$ such that $\mathcal{D} \cap (-M, M)$ satisfies the same recurrence property. Bounded recurrence sets of recurrence properties with continuity are identified as the ones having 0 as a limit point (Section 4.1) and we can then choose M arbitrarily small.

Particularly in Chapter 5, we consider recurrence properties with continuity and we sometimes need to exclude 0 as a limit point of $\mathcal{D}$. In this case, there exists $\tilde{\epsilon} > 0$ for each $\mathcal{D}$ such that $\mathcal{D} \cap (-\tilde{\epsilon}, \tilde{\epsilon}) = \emptyset$. We write $\mathcal{D} \subseteq \mathbb{R}_*$ where $\mathbb{R}_* := \mathbb{R}\backslash(-\tilde{\epsilon}, \tilde{\epsilon})$ to indicate that 0 is not a limit point of $\mathcal{D}$ and we assume hereby that $\tilde{\epsilon}$ is implicitly given by $\mathcal{D}$ such that $\mathcal{D} \cap (-\tilde{\epsilon}, \tilde{\epsilon}) = \emptyset$. We can choose $\tilde{\epsilon}$ arbitrarily small, so for simplicity, we set $0 < \tilde{\epsilon} < 1$ to avoid the consideration of extra cases when the connection with integer properties comes in the play.

When dealing with a locally compact abelian group, we use the (possibly non-finite) **Haar measure** which is the unique (up to a positive factor) rotation invariant positive measure defined on the Borel sets of G (see also Definition 7.23) and denote it by λ. When the Haar measure on a group G is finite (for example on the Torus $\mathbb{T} = \{e^{2\pi ix} : x \in [0,1)\}$), we assume it to be normalised, i.e $\lambda(G) = 1$. All measures except for λ are assumed to be positive and finite unless noted otherwise (see also the comment after Theorem 7). We assume throughout the thesis that sets are measurable with respect to the corresponding σ-algebras and we assume it to be the Borel σ-algebra if not stated otherwise. We note that the assumption of measurability for recurrence sets $\mathcal{D} \subseteq \mathbb{R}$ can be dropped if we do not consider the property KMF.

All operators are assumed to be linear and bounded if not noted otherwise. We call an operator T **contractive** if $\|T\| \leq 1$. An operator on a Hilbert space H is **unitary** if $TT^* = T^*T = \mathrm{Id}$ and a **lattice isomorphism** on $L^2(\Omega, \Sigma, \mu; \mathbb{R})$ if it is additionally positive. A **(semi)group** of operators on a Hilbert space H is given through a family of operators $(T_t)_{t\in\mathbb{R}}$ or $(T_t)_{t\geq 0}$ such that $T_tT_s = T_{t+s}$ and $T_0 = \mathrm{Id}$. If $t \mapsto T_tx$ is continuous for all $x \in H$, we call it **strongly continuous** and **strongly measurable** if $t \mapsto T_tx$ is measurable ([3, Subsection A.1.1]) for all $x \in H$.

We follow the common misuse of notation and consider functions $f \in L^p(\Omega, \Sigma\,\mu)$ instead of equivalence classes. Hence, the corresponding equations are only to be understood almost everywhere.

1.3 Overview and Main Results

The main focus of this thesis is real recurrence in various forms. Our main motivation and focus hereby is in particular the treatment of operator recurrence and Poincaré recurrence as well as strong operator recurrence and strong recurrence.

Definition 1.3 (Poincaré Recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is* ***Poincaré*** *or* ***Poincaré recurrent*** *if, given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$, there* *exists $d \in \mathcal{D}$ such that*

$$\mu(A \cap \phi_d(A)) > 0.$$

Definition 1.4 (Operator Recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is* ***operator recurrent*** *if, given a Hilbert space H, a strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that*

$$\langle T_d x, x\rangle \neq 0.$$

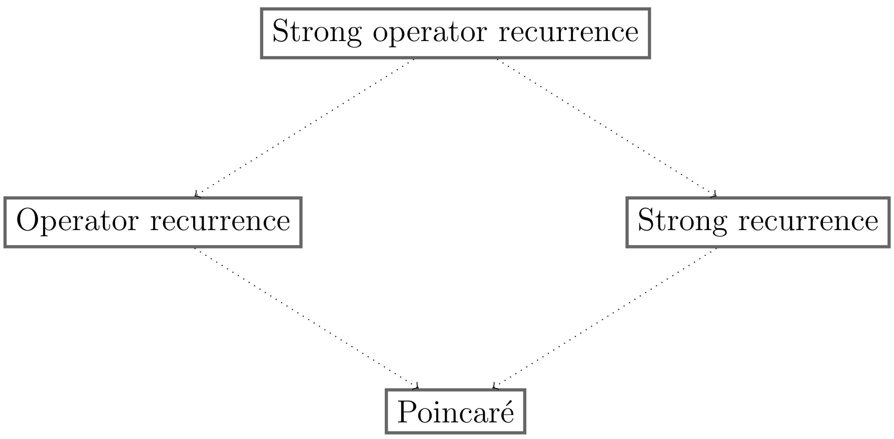
Definition 1.5 (Strong Recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is* ***strongly recurrent*** *if, given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$, we have*

$$\limsup_{|d|\to\infty, d\in\mathcal{D}} \mu(\phi_d(A) \cap A) > 0.$$

Definition 1.6 (Strong operator recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is* ***strongly operator recurrent*** *if, given a Hilbert space H, a strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, we have*

$$\limsup_{|d|\to\infty, d\in\mathcal{D}} |\langle T_d x, x\rangle| > 0.$$

The following implications follow easily, both for integers and reals, but they also give a framework for all other properties which are defined in detail in Chapter 3. Whenever we think of recurrence sets, we have one of these four properties in mind. This is motivated by the results for integer recurrence where every recurrence property is equivalent to one of these four (see Theorem 8 below).



In Chapter 2, we introduce notation and preliminary results. We discuss topics which are requisite in the later proofs, in particular, we discuss the Fourier transform, the Bochner-Herglotz theorem, measure preserving systems and their operator theoretical analogues, mean ergodicity, equidistribution and a version of the strong law of large numbers. We also include some lemmata which are used throughout the thesis.

In Chapter 3, we introduce recurrence properties and their relations to each other. In Section 3.1, we define the main recurrence properties for a set $\mathcal{D} \subseteq \mathbb{R}$ in a precise way. We focus hereby on those properties which we consider in more detail in the subsequent chapters. In Section 3.2, we show how these properties are linked with each other and introduce additional properties. The following theorem gives an overview over the integer results ([14], [17], [42], [57], [60], [63], [70], [71]).

Theorem 8. *Let $\mathcal{D} \subseteq \mathbb{Z}$. Then all integer properties within one of the following groups are equivalent to each other.*

(i) Poincaré, spectral Poincaré, combinatorial recurrence, intersectivity, real correlativity.

(ii) Operator recurrence, FMRiesz, Kamae and Mendès France, van der Corput, correlativity.

(iii) Strong recurrence, strong combinatorial recurrence, strong correlativity.

(iv) Strong operator recurrence, FC+, enhanced van der Corput.

We have to distinguish two types of real recurrence properties, recurrence with continuity assumptions (e.g. using strongly continuous groups in operator recurrence) and recurrence without continuity, but with corresponding measurability assumptions (e.g. using strongly measurable groups in operator recurrence).

Most of the relations in Theorem 8 are true for real recurrence sets without continuity, but in particular the proof of the equivalence of operator recurrence and correlativity uses FMRiesz as intermediate step which contains a natural continu-

ity assumption due to the continuity of $\widehat{\mu}$. Hence, it is not clear if correlativity implies operator recurrence.
For real recurrence with continuity, we still have the equivalences in (ii) and (iv) as in Theorem 8 and the implications

$$\text{real correlativity} \Rightarrow \text{Poincaré} \Rightarrow \text{continuous combinatorial recurrence},$$

but it is unclear if any of the reverse implications hold. Summarising, the final result of Section 3.2 is the following theorem.

Theorem 9. *Let $\mathcal{D} \subseteq \mathbb{R}$. Then all properties within the following groups are equivalent to each other.*
(i) Poincaré without continuity, combinatorial recurrence, real correlativity without continuity.
(ii) Van der Corput without continuity, correlativity without continuity.
(iia) Operator recurrence, FMRiesz, van der Corput, correlativity.
(iii) Strong recurrence without continuity, strong combinatorial recurrence, strong correlativity without continuity.
(iv) Strong operator recurrence, FC+, enhanced van der Corput.
In addition, we have the implications

$$\begin{aligned} \textit{combinatorial recurrence} &\Rightarrow \textit{intersectivity}, \\ \textit{real correlativity} &\Rightarrow \textit{Poincaré} \\ &\Rightarrow \textit{continuous combinatorial recurrence} \\ &\Rightarrow \textit{continuous intersectivity}, \\ \textit{operator recurrence without continuity} &\Rightarrow \textit{correlativity without continuity}, \end{aligned}$$

$$\begin{aligned} \textit{strong correlativity} &\Rightarrow \textit{strong recurrence} \\ &\Rightarrow \textit{strong continuous combinatorial recurrence}, \end{aligned}$$

strong operator recurrence without continuity

$\Rightarrow$ *enhanced van der Corput without continuity.*

Up to that point, only the relation between recurrence sets is analysed. In Chapter 4, we focus on examples of recurrence sets. In Section 4.1, we consider rather trivial examples such as $\mathbb{N}$, $\mathbb{R}\backslash(-\epsilon, \epsilon)$ or the characterisation of bounded recurrence sets. These examples yield also some classical theorems such as the van der Corput theorem.

In Section 4.3, we discuss how real recurrence and integer recurrence are related for a set $\mathcal{D} \subseteq \mathbb{Z}$. It is rather straightforward that every $\mathbb{Z}$-recurrent set is also $\mathbb{R}$-recurrent. We show that we even have equivalence for strong operator recurrence and operator recurrence (and hence for all properties which are equivalent to them).

Theorem 10. *Let $\mathcal{D} \subseteq \mathbb{Z}$. Then $\mathbb{Z}$-recurrence and $\mathbb{R}$-recurrence coincide for strong operator recurrence and operator recurrence.*

We also show that we obtain the equivalence of $\mathbb{Z}$-recurrence and $\mathbb{R}$-recurrence if we consider recurrence properties without continuity assumptions.

Using these results, we obtain further interesting examples of recurrence sets using integer results. We conclude the chapter by discussing the sets $\{\frac{1}{\log p} : p \text{ prime}\}$ and $\{\log p : p \text{ prime}\}$. These examples differ from the previous examples as they do not come directly from an integer set.

It is immediate that we can always enlarge a recurrence set without losing its recurrence property. The converse is less obvious. In Chapter 5, we discuss several ways to reduce a recurrence set. We show hereby the following theorem.

Theorem 11. *Let $\mathcal{D} \subseteq \mathbb{R}_*$ be strongly operator recurrent (operator recurrent, strongly recurrent, Poincaré), and let $N \in \mathbb{N}$, $s > 0$ and ϵ with $0 < \epsilon < \frac{s}{2}$ be given.*

Then there exists a countable set

$$\tilde{D} \subseteq \bigcup_{|n|>N} ([sn - \epsilon, sn + \epsilon] \cap \mathcal{D})$$

which is still strongly operator recurrent (operator recurrent, strongly recurrent, Poincaré). If $\mathcal{D}$ is strongly operator recurrent (strongly recurrent), then there exists a set $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ which additionally satisfies that $\tilde{\mathcal{D}} \cap [-N, N]$ is finite for all $N > 0$ and which is still strongly operator recurrent (strongly recurrent).

In Chapter 6, we discuss the problem if there exists a set $\mathcal{D} \subseteq \mathbb{R}$ which is intersective, but not FMRiesz. For the integer setting, Bourgain ([23]) constructed such a set. We extend this result to the real setting by using our results from Section 4.3 and we also show a corresponding result for intersectivity and strong combinatorial recurrence based on an integer example of Forrest ([33]). We also characterise the existence of a set which is intersective and not FMRiesz in terms of certain families of vectors in a Hilbert space.

Theorem 12. *There exists a set $\mathcal{D} \subset \mathbb{R}_*$ which is intersective, but not FMRiesz. There exists a set $\mathcal{D} \subset \mathbb{R}_*$ which is intersective, but not strongly combinatorially recurrent.*

In Chapter 7, we indicate how to further extend the topic of recurrence sets. We firstly note that we restrict our considerations in this thesis on four groups of recurrence properties (strong operator recurrence, operator recurrence, strong recurrence and Poincaré recurrence with their variants). One can obtain new recurrence properties by varying the type of recurrence and, as an example, we consider topological recurrence in Section 7.1.
We restrict ourselves to the study of the qualitative analysis of these sets. However, a quantitative analysis can be done in a similar way as for the integers by introducing certain "measures" corresponding to a recurrence property. In Section

7.2, we introduce the quantitative analysis of recurrence sets and give an example of a quantitative relation.

We lastly note that most proofs in Section 7.3 rely essentially on dual groups and the mean ergodic theorem. It is therefore straightforward to extend the topic of recurrence sets to locally compact abelian groups and in particular to $\mathbb{R}^k$ and $\mathbb{Z}^k$. While the setting for $\mathbb{R}^k$ and $\mathbb{Z}^k$ is similar to $\mathbb{R}$ and $\mathbb{Z}$, it requires some more background details for a general locally compact abelian group. In Section 7.3, we introduce the setting for locally compact abelian groups and, as an example, we prove the equivalence of operator recurrence and FMRiesz.

Chapter 2

Preliminaries

2.1 Measure Preserving Systems and the Mean Ergodic Theorem

A recurrent topic in this thesis are measure preserving systems and the mean ergodic theorem. They build the basis for strong operator recurrence, operator recurrence, strong recurrence and Poincaré recurrence.

Definition 2.1 (Measure Preserving System). *A* ***measure preserving system*** *(mps) is a probability space (Ω, Σ, μ) with a group of measure-preserving invertible transformations $(\phi^n)_{n\in\mathbb{Z}}$ or $(\phi_t)_{t\in\mathbb{R}}$ on (Ω, Σ, μ) such that $\phi_t\phi_s = \phi_{t+s} = \phi_s\phi_t$ for all $t, s \in \mathbb{R}$ and where $t \mapsto \phi_t$ is strongly measurable, i.e. $f \circ \phi_t$ is measurable for all $f \in L^2(\Omega, \Sigma, \mu)$, or such that $\phi^n\phi^m = \phi^{n+m} = \phi^m\phi^n$ for all $n, m \in \mathbb{Z}$, respectively.*

Remark 2.2. Every mps induces an operator T_ϕ, called the Koopman operator, or a group $(T_t^\phi)_{t\in\mathbb{R}}$ on the Hilbert space $L^2(\Omega, \Sigma, \mu)$ which are defined by $T_\phi f := f \circ \phi$ and $T_t^\phi f := f \circ \phi_t$ (see [29, Section 2.4]). The operators T_ϕ and T_t^ϕ are unitary and positive, and they reflect the behaviour of the original mps, but also allow the use of linear operator theory.

Definition 2.3 (Strongly Continuous Measure Preserving System)**.** *A **strongly continuous measure preserving system** (scmps) is a measure preserving system $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ such that its Koopman representation $(L^2(\Omega, \Sigma, \mu); (T_t^\phi)_{t\in\mathbb{R}})$ is strongly continuous or, equivalently, such that $\lim_{t\to s} \mu(\phi_t(A) \triangle \phi_s(A)) = 0$ for all $t, s \in \mathbb{R}$ and $A \in \Sigma$.*

Using the Koopman representation and density arguments, we obtain the following characterisation of measure preserving.

Lemma 2.4. *A scmps is measure preserving if and only if $\int_\Omega T_t^\phi f d\mu = \int_\Omega f d\mu$ for all $f \in L^2(\Omega, \Sigma, \mu)$ or a dense subspace thereof.*

We set

$$\fint_a^b f(t)\ d\lambda(t) := \frac{1}{b-a} \int_a^b f(t)\ d\lambda(t)$$

and

$$\fint_{\mathbb{R}} f(t)\ d\lambda(t) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\ d\lambda(t). \tag{2.1}$$

whenever the limit exists. If f is vector-valued, then the integral on the right-hand side is to be understood as a Bochner integral ([3, Section 1.1]). The function f will usually be given by $(T_t x)_{t\in\mathbb{R}}$, $(\mu(\phi_t(A) \cap A))_{t\in\mathbb{R}}$ or $\widehat{\mu}$, so is often in particular continuous, and the Bochner integral then coincides with the Riemann integral ([40]).

The next result gives an estimate about the averaged recurrence (compare [29, Theorem 3.1.7]).

Theorem 2.5 (Khintchine)**.** *Let $(\Omega, \Sigma, \mu; \phi)$ be a mps and $\mu(A) > 0$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu(\phi^n(A) \cap A) \geq (\mu(A))^2.$$

Let $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ be a scmps and $\mu(A) > 0$. Then

$$\fint_{\mathbb{R}} \mu(\phi_t(A) \cap A)\ d\lambda(t) \geq (\mu(A))^2.$$

The convergence of (2.1) plays an important role for recurrence set as it connects all properties with each other. The mean ergodic theorems 2.7 and 2.8 yield the convergence of (2.1) towards the orthogonal projection onto the corresponding fixed space for a strongly continuous group of contractions (see [27, Proposition Y.2 and Theorem Y.4], [49, Theorem 6.4.1], [29, Corollary 8.15]).

We occasionally write $P_{\mathbb{N}}$ and $P_{\mathbb{R}}$ instead of P to denote the mean ergodic projection corresponding to a single operator or a strongly continuous group as well as other appropriate indications if the context is not clear.

Definition 2.6 (Mean Ergodic)**.** *We call an operator T on a Hilbert space H* ***mean ergodic*** *if the corresponding mean ergodic projection*

$$P_{\mathbb{N}}x := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n x \tag{2.2}$$

onto $\mathrm{Fix}(T)$ *along* $\overline{\mathrm{ran}}(\mathrm{Id} - T)$ *exists in* $\|\cdot\|$.

We call a strongly continuous group $\mathcal{T} = (T_t)_{t\in\mathbb{R}} \subset \mathcal{L}(H)$ on a Hilbert space H ***mean ergodic*** *if the corresponding mean ergodic projection*

$$P_{\mathbb{R}}x := \lim_{T\to\infty} ⨍_{-T}^{T} T_t x \, d\lambda(t) \tag{2.3}$$

onto $\mathrm{Fix}(\mathcal{T})$ *along* $\overline{\mathrm{lin}}\{(\mathrm{Id} - T_t)x : x \in H, t \in \mathbb{R}\}$ *exists for all $x \in H$ in* $\|\cdot\|$.

We note that a mean ergodic operator T induces the decomposition

$$H = \mathrm{Fix}(T) \oplus \overline{\mathrm{ran}}(\mathrm{Id} - T), \tag{2.4}$$

and similarly a strongly continuous group of operators on H induces the decomposition

$$H = \mathrm{Fix}((T_t)_{t\in\mathbb{R}}) \oplus \overline{\mathrm{lin}}\{(\mathrm{Id} - T_t)x : x \in X, t \in \mathbb{R}\},$$

Theorem 2.7. *Let $(T_t)_{t\in\mathbb{R}} \subset \mathcal{L}(H)$ with a Hilbert space H be a strongly continuous group of contractions on H. Then $(T_t)_{t\in\mathbb{R}}$ is mean ergodic. If $H = L^2(\Omega, \Sigma, \mu)$ with the Koopman representation of a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$, then the convergence of (2.3) is also almost everywhere.*

The convergence of its discrete variant, $Px := \lim_{n\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} T^n x$, was proven for the L^2-norm by von Neumann ([59]) and almost everywhere by Birkhoff ([20]).

Theorem 2.8. *Let T be a contraction on a Hilbert space H. Then T is mean ergodic. If $H = L^2(\Omega, \Sigma, \mu)$ with the Koopman representation of a mps $(\Omega, \Sigma, \mu; \phi)$, then the convergence of (2.2) is also almost everywhere.*

Remark 2.9. We call an scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ **ergodic** if $\mu(\phi_t(A)) = \mu(A)$ for all $t \in \mathbb{R}$ implies $\mu(A) \in \{0, 1\}$. We then have

$$\oint_{\mathbb{R}} (T_t^\phi f)(\omega) d\lambda(t) = \int_\Omega f \, d\mu$$

for almost every $\omega \in \Omega$.

Lemma 2.10. *Let $\mathcal{T} = (T_t)_{t\in\mathbb{R}}, \mathcal{S} = (S_t)_{t\in\mathbb{R}}$ be two mean ergodic groups on a Hilbert space H with mean ergodic projections $P_\mathcal{T}$ and $P_\mathcal{S}$ such that* $\mathrm{Fix}(\mathcal{T}) \subseteq \mathrm{Fix}(\mathcal{S})$. *Then $P_\mathcal{T} x \neq 0$ implies $P_\mathcal{S} x \neq 0$.*

Proof. We note that $Px = 0$ if and only if $\langle x, y\rangle = 0$ for all $y \in \mathrm{Fix}(\mathcal{T})$ by the mean ergodic decomposition in Theorems 2.7 and 2.8. As $\langle x, y\rangle = 0$ for all $y \in \mathrm{Fix}(\mathcal{S})$ would imply $\langle x, y\rangle = 0$ for all $y \in \mathrm{Fix}(\mathcal{T})$ and hence $P_\mathcal{T} x = 0$, we obtain $P_\mathcal{S} x \neq 0$. □

Lemma 2.11. *Let $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ be a scmps and $E \in \Sigma$ with $\mu(E) > 0$. Then $P\mathbb{1}_E \neq 0$.*

Proof. For a contradiction, let us assume $P\mathbb{1}_E = 0$. Then $\langle \mathbb{1}_E, x\rangle = 0$ for all $x \in \mathrm{Fix}((T_t^\phi)_{t\in\mathbb{R}})$. However, $\mathbb{1} \in \mathrm{Fix}((T_t^\phi)_{t\in\mathbb{R}})$, hence, $0 < \mu(E) = \langle \mathbb{1}_E, \mathbb{1}\rangle = 0$, yielding a contradiction. □

Definition 2.12 (Upper Density)**.** *We define the **upper density** of a set A as follows.*

For $A \subseteq \mathbb{N}$, we set

$$\overline{d}_\mathbb{N}(A) := \limsup_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}.$$

For $A \subseteq \mathbb{Z}$, we set

$$\overline{d}_{\mathbb{Z}}(A) := \limsup_{n\to\infty} \frac{|A \cap \{-n, \ldots, n\}|}{2n+1}.$$

For $A \subseteq \mathbb{R}$, we set

$$\overline{d}_{\mathbb{R}}(A) := \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{1}_A(t)\ d\lambda(t).$$

Similarly, we define the ***densities*** $d_{\mathbb{N}}$, $d_{\mathbb{Z}}$ *and* $d_{\mathbb{R}}$ *as the corresponding limits whenever they exists.*

2.2 Fourier Transform and Bochner-Herglotz

Definition 2.13 (Fourier Transform of a Measure)**.** *Let μ be a measure on $\mathbb{R}$. Then its* ***Fourier transform*** $\widehat{\mu}$ *is defined by*

$$\widehat{\mu}(t) := \int_{\mathbb{R}} e^{-2\pi i t x}\ d\mu(x)$$

for $t \in \mathbb{R}$.

Remark 2.14. We have

$$\int_{\mathbb{R}} \left|e^{-2\pi i t x}\right|\ d\mu(x) = \mu(\mathbb{R}) < \infty$$

for all $t \in \mathbb{R}$. Hence the Fourier transform $\widehat{\mu}$ of a finite measure μ always exists. The Fourier transform $\widehat{\mu}$ of a finite measure μ is continuous ([67, 1.3.2]) and satisfies $\widehat{\mu}(0) = \mu(\mathbb{R})$ and, given $0 \neq \mu$, we have $\widehat{\mu}(t) \neq 0$ for a neighbourhood around 0.

Definition 2.15 (Positive-Definite)**.** *A function $f : \mathbb{R} \to \mathbb{C}$ is* ***positive-definite*** *if $f(-t) = \overline{f(t)}$ for all $t \in \mathbb{R}$ and*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} f(t_i - t_j) \geq 0$$

for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $t_1, \ldots, t_n \in \mathbb{R}$.

The following Bochner-Herglotz theorem ([67, 1.4.3]) yields the connection of Fourier transforms and positive-definite functions.

Theorem 2.16 (Bochner-Herglotz)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be continuous and positive-definite. Then there exists a measure μ on $\mathbb{R}$ such that*

$$\widehat{\mu}(t) = f(t)$$

for all $t \in \mathbb{R}$.

Lemma 2.17. *Let μ be the Bochner-Herglotz measure associated to the positive-definite and continuous function f. Then*

$$\mu(\{0\}) = \oint_{\mathbb{R}} f(t)\ d\lambda(t).$$

Proof. Let $x = 0$. Then

$$\oint_{\mathbb{R}} e^{-2\pi itx}\ d\lambda(t) = \oint_{\mathbb{R}} \mathbb{1}(t)\ d\lambda(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{1}(t)\ d\lambda(t) = \lim_{T\to\infty} \frac{2T}{2T} = 1.$$

Let $x > 0$ and $T = \frac{n}{x} + r$ with $n \in \mathbb{N}$ and $0 \leq r < \frac{1}{x}$. Then

$$\begin{aligned}
&\left|\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-2\pi itx}\ d\lambda(t)\right| \\
&\leq \lim_{T\to\infty} \frac{1}{2T} \left|\int_{-\frac{n}{x}}^{\frac{n}{x}} e^{-2\pi itx}\ d\lambda(t)\right| + \lim_{T\to\infty} \frac{1}{2T} \left|\int_{\frac{n}{x}}^{T} e^{-2\pi itx}\ d\lambda(t)\right| \\
&\quad + \lim_{T\to\infty} \frac{1}{2T} \left|\int_{-T}^{\frac{-n}{x}} e^{-2\pi itx}\ d\lambda(t)\right| \\
&\leq \lim_{T\to\infty} \frac{1}{2T} \left[\frac{i}{2\pi x} e^{-2\pi itx}\right]_{-\frac{n}{x}}^{\frac{n}{x}} + 2 \lim_{T\to\infty} \frac{r}{2T} = 0,
\end{aligned}$$

and similarly for $x < 0$, hence,

$$\oint_{\mathbb{R}} e^{-2\pi itx}\ d\lambda(t) = \mathbb{1}_{\{0\}}(x). \tag{2.5}$$

We further have

$$\begin{aligned}
\mu(\{0\}) &= \int_{\mathbb{R}} \mathbb{1}_{\{0\}}(x)\ d\mu(x) \overset{(2.5)}{=} \int_{\mathbb{R}} \oint_{\mathbb{R}} e^{-2\pi itx}\ d\lambda(t)\ d\mu(x) \\
&\overset{2.31}{=} \oint_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi itx}\ d\mu(x)\ d\lambda(t) = \oint_{\mathbb{R}} \widehat{\mu}(t)\ d\lambda(t) = \oint_{\mathbb{R}} f(t)\ d\lambda(t).
\end{aligned}$$

□

2.3 The Strong Law of Large Numbers

We prove a version of the strong law of large numbers for continuous parameters in this section. We follow hereby the approach of [54] for integers.
We consider a random variable given through the integral $\int_0^T X_t \, d\lambda(t)$ of a suitable bounded family $(X_t)_{0 \leq t \leq T}$ of random variables. This integral is to be understood pathwise, i.e. we consider

$$\omega \mapsto \int_0^T X_t(\omega) \, d\lambda(t)$$

for (almost) every $\omega \in \Omega$.
Without loss of generality, we assume $T \in \mathbb{N}$ as the fractional part of T yields a term tending to zero in the strong law of large numbers. We use the following lemmata ([28], [54, Lemma 2-3]).

Lemma 2.18. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of nonnegative numbers with*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty.$$

Then there exists a subsequence $(a_{n_k})_{k \in \mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k} < \infty$ and $\frac{n_{k+1}}{n_k} \to 1$.

Lemma 2.19. *Let $(Y_n)_{n \in \mathbb{N}}$ be a sequence of random variables with*

$$\sum_{n=1}^{\infty} \mathbb{E}\left[|Y_n|^2\right] < \infty.$$

Then $Y_n \to 0$ holds almost surely.

Theorem 2.20. *Let $(X_t)_{t \geq 0}$ be a family of random variables with mean 0 which is uniformly bounded by $M > 0$, has measurable paths and satisfies*

$$\sum_{T=1}^{\infty} \frac{1}{T} \mathbb{E}\left[\left|\frac{1}{T}\int_0^T X_t \, d\lambda(t)\right|^2\right] < \infty.$$

Then the strong law of large numbers holds, i.e.

$$⨍_0^{\infty} X_t \, d\lambda(t) = 0$$

almost surely.

Proof. We define new random variables $Y_T := \oint_0^T X_t \, d\lambda(t)$. Then by assumption, we have $\sum_{T=1}^{\infty} \frac{\mathbb{E}\left[|Y_T|^2\right]}{T} < \infty$, hence, by Lemma 2.18, there exists a subsequence $(T_k)_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} \mathbb{E}\left[|Y_{T_k}|^2\right] < \infty$ and $\frac{T_{k+1}}{T_k} \to 1$. Lemma 2.19 implies $Y_{T_k} \to 0$ almost surely, i.e. $\oint_0^{T_k} X_t \, d\lambda(t) \to 0$ almost surely. We further have

$$\begin{aligned} &\max_{0\leq s\leq T_{k+1}-T_k} \left| \frac{1}{T_k} \int_{T_k}^{T_k+s} X_t \, d\lambda(t) \right| \\ &\leq M \max_{0\leq s\leq T_{k+1}-T_k} \frac{T_k + s - T_k}{T_k} \leq M \frac{T_{k+1} - T_k}{T_k} \overset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

Let $T_{k+1} \geq T \geq T_k$. Then we have

$$\left| \frac{1}{T} \int_0^T X_t \, d\lambda(t) \right| \leq \left| \frac{1}{T_k} \int_0^{T_k} X_t \, d\lambda(t) \right| + \max_{0\leq s\leq T_{k+1}-T_k} \left| \frac{1}{T_k} \int_{T_k}^{T_k+s} X_t \, d\lambda(t) \right| \overset{k\to\infty}{\longrightarrow} 0$$

almost surely. □

By considering

$$\begin{aligned} \oint_{\mathbb{R}} X_t \, d\lambda(t) &= \lim_{T\to\infty} \oint_{-T}^{T} X_t \, d\lambda(t) = \lim_{T\to\infty} 2 \oint_{-T}^{0} X_t \, d\lambda(t) + 2 \lim_{T\to\infty} \oint_0^T X_t \, d\lambda(t) \\ &= 2 \oint_{-\infty}^{0} X_t \, d\lambda(t) + 2 \oint_0^{\infty} X_t \, d\lambda(t), \end{aligned}$$

we obtain the following result.

Theorem 2.21. *Let $(X_t)_{t\in\mathbb{R}}$ be a family of random variables with mean 0 which is uniformly bounded by $M > 0$, has measurable paths and satisfies*

$$\sum_{0\neq T=-\infty}^{\infty} \frac{1}{|T|} \mathbb{E}\left[\left| \frac{1}{|T|} \int_0^T X_t \, d\lambda(t) \right|^2 \right] < \infty. \tag{2.6}$$

Then the strong law of large numbers holds, i.e.

$$\oint_{\mathbb{R}} X_t \, d\lambda(t) = 0$$

almost surely.

Remark 2.22. Given a family of uniformly bounded random variables $(X_t)_{t\in\mathbb{R}}$ with possibly $\mathbb{E}[X_t] \neq 0$, where $t \mapsto \mathbb{E}[X_t]$ is measurable and such that

$$\oint_{\mathbb{R}} \mathbb{E}[X_t] \, d\lambda(t)$$

exists, we consider $(X_t - \mathbb{E}[X_t])_{t\in\mathbb{R}}$. If the family $(X_t - \mathbb{E}[X_t])_{t\in\mathbb{R}}$ satisfies Condition (2.6), then the statement of Theorem 2.21 still holds and we have

$$\oint_{\mathbb{R}} (X_t - \mathbb{E}[X_t])\ d\lambda(t) = 0$$

almost surely, hence,

$$\oint_{\mathbb{R}} X_t\ d\lambda(t) = \oint_{\mathbb{R}} \mathbb{E}[X_t]\ d\lambda(t)$$

almost surely.

2.4 The Sets $l_*^\infty(\mathbb{R})$, $l_c^\infty(\mathbb{R})$ and $l_{cc}^\infty(\mathbb{R})$

In order to work with the van der Corput and correlativity property, we introduce the space $l_*^\infty(\mathbb{R})$ as well as the sets $l_c^\infty(\mathbb{R})$ and $l_{cc}^\infty(\mathbb{R})$.
We set $l_*^\infty(\mathbb{R})$ as the space of all measurable and bounded functions on $\mathbb{R}$ which coincides with the usual $\mathcal{L}^\infty(\mathbb{R})$ (i.e. without identifying of functions which coincide outside a null set). We introduce the set $l_c^\infty(\mathbb{R})$ as it provides the right setting to consider a continuous version of correlativity on $\mathbb{R}$. We assume for $f \in l_c^\infty(\mathbb{R})$ that f is measurable and bounded such that

$$\oint f(t)\ d\lambda(t)$$

exists and where

$$\gamma(s) := \oint_{\mathbb{R}} f(t+s)\overline{f(t)}\ d\lambda(t)$$

exists for all $s \in \mathbb{R}$ and is continuous.
We define $l_{cc}^\infty(\mathbb{R}, \mathbb{R})$ as the set of all measurable, bounded and $\mathbb{R}$-valued functions f on $\mathbb{R}$ such that $e^{2\pi i k f} \in l_c^\infty(\mathbb{R}, \mathbb{T})$ for all $k \in \mathbb{Z}$. For a given compact abelian group K, we set $l_{cc}^\infty(\mathbb{R}, K)$ as the set of all measurable, bounded and K-valued functions f on $\mathbb{R}$ such that $\chi \circ f \in l_c^\infty(\mathbb{R})$ for all $\chi \in K^*$ (see Section 7.3 for more details on

the dual group K^*).

It is a priori not clear that the sets $l_c^\infty(\mathbb{R})$ and $l_{cc}^\infty(\mathbb{R})$ are non-trivial nor that they are distinct from $l_*^\infty(\mathbb{R})$. The set $l_c^\infty(\mathbb{R})$ is non-trivial since it contains for example f_1 defined by $f_1(t) := \lfloor t \rfloor$ satisfying $\oint_{\mathbb{R}} f_1(t)\, d\lambda(t) = \frac{1}{2}$ and $\oint_{\mathbb{R}} f_1(t+d)\overline{f_1(t)}\, d\lambda(t) = \frac{1}{3} - \frac{1}{2}\lfloor d \rfloor + \frac{1}{2}\lfloor d \rfloor^2$ for all $d \in \mathbb{R}$ which is continuous.

However, it is distinct from $l_*^\infty(\mathbb{R}, \mathbb{R})$. Consider $g \in l^\infty(\mathbb{Z})$ with $g(n) := (-1)^{m-1}$ for $2^m \leq |n| < 2^{m+1}$ and $g(0) = 1$. We define f_2 recursively by setting $f_2(t) := 1$ for $t \in [0, 1)$, $f_2(t) := \frac{g(n-1)}{f_2(t-1)}$ for $n > 0$ and $t \in [n, n+1)$ and $f_2(t) := \frac{g(n)}{f_2(t+1)}$ for $n < 0$ and $t \in [n, n+1)$. Then

$$\oint_{-N}^{N} f_2(t+1)\overline{f_2(t)}\, d\lambda(t) = \frac{1}{2N}\sum_{n=-N}^{N-1} f_2(n+1)\overline{f_2(n)} = \frac{1}{2N}\sum_{k=-N}^{N-1} g(n)$$

does not converge as $N \to \infty$, and hence, we have $f_2 \in l_*^\infty(\mathbb{R})$, but $f_2 \notin l_c^\infty(\mathbb{R})$.

The set $l_{cc}^\infty(\mathbb{R}, \mathbb{R})$ is non-trivial and clearly distinct from $l_c^\infty(\mathbb{R}, \mathbb{R})$ as it contains f_2 since $e^{2\pi i k f_2(t)} = 1$ for all $k \in \mathbb{Z}$. However it is also distinct from $l_*^\infty(\mathbb{R}, \mathbb{R})$. To see this, consider f_3 defined by $f_3(t) := \frac{1}{4} + \frac{1}{4} f_2(t)$. Then $f_3 \in l_*^\infty(\mathbb{R})$, but

$$\oint_{-N}^{N} e^{2\pi i f_3(t+1)} e^{-2\pi i f_3(t)}\, d\lambda(t) = \oint_{-N}^{N} f_2(t+1)\overline{f_2(t)}\, d\lambda(t)$$

does not converge as $N \to \infty$, hence, $f_3 \notin l_{cc}^\infty(\mathbb{R}, \mathbb{R})$.

For $l_{cc}^\infty(\mathbb{R}, K)$ and $K = \mathbb{T}$, consider $f_4(t) = e^{2\pi i t}$. Then

$$\begin{aligned}\oint_{\mathbb{R}} \chi_k(f_4(t+d))\overline{\chi_k(f_4(t))}\, d\lambda(t) &= \oint_{\mathbb{R}} \left(e^{2\pi i(t+d)}\right)^k \left(e^{-2\pi i t}\right)^k\, d\lambda(t) \\ &= \oint_{\mathbb{R}} e^{2\pi i k d}\, d\lambda(t) = e^{2\pi i k d}\end{aligned}$$

which is continuous, and $\oint_{\mathbb{R}} \chi_k(f_4(t))\, d\lambda(t) = 0$ holds for all $k \in \mathbb{Z}$, hence, $f_4 \in l_{cc}^\infty(\mathbb{R}, \mathbb{T})$. For $k = \mathbb{Z}_3$, consider f_5 defined by

$$f_5(t) := \begin{cases} 0, & \lfloor t \rfloor \in [0, \frac{1}{3}), \\ 1, & \lfloor t \rfloor \in [\frac{1}{3}, \frac{2}{3}), \\ 2, & \lfloor t \rfloor \in [\frac{2}{3}, 1) \end{cases}$$

which satisfies $\oint_{\mathbb{R}} f_5(t)\ d\lambda(t) = 0$ and

$$\oint_{\mathbb{R}} f_5(t+d)\overline{f_5(t)}\ d\lambda(t) = \begin{cases} 1 - 3d + 3d\chi(2), & 0 \leq \lfloor d \rfloor < \frac{1}{3}, \\ (1-3d)\chi(2) + 3d\chi(1), & \frac{1}{3} \leq \lfloor d \rfloor < \frac{2}{3}, \\ (1-3d)\chi(1) + 3d, & \frac{2}{3} \leq \lfloor d \rfloor < 1 \end{cases}$$

for all $\chi \in \mathbb{Z}_3^*$, hence, $f_5 \in l_{cc}^\infty(\mathbb{R}, \mathbb{Z}_3)$.

We need these continuity assumptions on $l_c^\infty(\mathbb{R})$ and $l_{cc}^\infty(\mathbb{R})$ in particular to relate correlativity and van der Corput to properties such as FMRiesz as it includes a natural continuity assumption. If we consider recurrence properties without continuity, we consider $l_*^\infty(\mathbb{R})$ instead.

These sets have not yet been introduced to our knowledge, but are based on the definition of "pseudo-random" functions or sequences as discussed for example by Bass ([4], [5]) and Bertrandias ([18]). A pseudo-random function as defined by Bass ([4, Section 2], [5, Section 6]) is a function $f \in l_c^\infty(\mathbb{R}, \mathbb{C})$ such that $f|_{(-\infty,0)} = 0$, $\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0$, $\gamma(0) = 0$ and $\lim_{|d|\to\infty} \gamma(d) = 0$. This definition combines the set $l_c^\infty(\mathbb{R}, \mathbb{C})$ with a variant of correlativity for functions with values in $\mathbb{C}$ instead of $\mathbb{T}$ (compare Definition 3.54). Such a correlative set $\mathcal{D}$ gives a criterion for f to be pseudo-random. A non-trivial function $f \in l_c^\infty(\mathbb{R}, \mathbb{C})$ with $f|_{(-\infty,0)} = 0$ is hence pseudo-random if it satisfies $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$.

There is a translation invariant Banach limit L on $l_*^\infty(\mathbb{R})$ extending $\oint_{\mathbb{R}} \cdot\ d\lambda(t)$ ([41, Example 0.3-0.4]) and we define an inner product on $l_*^\infty(\mathbb{R})$ by

$$\langle f, g \rangle := L(f\overline{g})$$

for $f, g \in l_*^\infty(\mathbb{R})$. We factor out the Kernel and we obtain a pre-Hilbert space with completion H. For $f, g \in l_c^\infty(\mathbb{R})$, we have

$$\langle f, g \rangle = \oint_{\mathbb{R}} f(t)\overline{g(t)}\ d\lambda(t)$$

Lemma 2.23. *We have $f \in l_c^\infty(\mathbb{R})$ if and only if $T_t f$ is continuous (continuous at 0) with respect to $\|\cdot\|$ for the extension of the shift group $(T_t)_{t\in\mathbb{R}}$ on H and*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t)$$

exists.

Proof. Let $t \mapsto T_t f$ be continuous. Then

$$\begin{aligned} |\gamma(d) - \gamma(s)| &= \left| \oint_{\mathbb{R}} (f(t+d) - f(t+s))\, \overline{f(t)}\ d\lambda(t) \right| = |\langle T_d f - T_s f, f\rangle| \\ &\overset{CS}{\leq} \|T_d f - T_s f\| \|f\| \overset{d\to s}{\longrightarrow} 0. \end{aligned}$$

Now let $f \in l_c^\infty(\mathbb{R})$ and $s \geq t$. We first note that the shift group is unitary on H since the mean is rotation invariant. Then

$$\begin{aligned} \|T_t f - T_s f\|^2 &= \|T_{t-s} f - f\|^2 = \langle T_{t-s} f, T_{t-s} f\rangle + \langle f, f\rangle - 2\Re\langle T_{t-s} f, f\rangle \\ &= 2\|f\|^2 - 2\Re\langle T_{t-s} f, f\rangle = 2\|f\|^2 - 2\Re \oint_{\mathbb{R}} f(g+t-s)\overline{f(g)}\ d\lambda(g) \overset{t-s\to 0}{\longrightarrow} 0. \end{aligned}$$

□

The van der Corput property deals with equidistributed functions. We define equidistribution mod 1 by one of the equivalent statements in Definition 2.24 (Compare [29, Section 4.4], [51, Section 1.9]).

Definition 2.24. *A function $f \in l_*^\infty(\mathbb{R}, [0,1])$ is* ***equidistributed mod*** 1 *if one of the following equivalent criteria hold.*

(i) Equidistribution mod 1: $\oint_{\mathbb{R}} (\mathbb{1}_{[a,b]} \circ f)(t)\ d\lambda(t) = b - a$ *for all* $0 \leq a < b \leq 1$.

(ii) Weyl criterion: $\oint_{\mathbb{R}} e^{2\pi i k f(t)}\ d\lambda(t) = 0$ *for all* $0 \neq k \in \mathbb{Z}$.

2.5 Lemmata

We collect some lemmata here which are used throughout the thesis.

Lemma 2.25. *Let* $b_T \to 1$. *Then* $\limsup_{T\to\infty} a_T b_T = \limsup_{T\to\infty} a_T$.

Lemma 2.26. *Let* $b_T \to 0$. *Then* $\limsup_{T\to\infty}(a_T + b_T) = \limsup_{T\to\infty} a_T$.

Lemma 2.27. *Let* H *be a Hilbert space,* $(T_t)_{t\in\mathbb{R}}$ *be a strongly continuous unitary group on* H *and* $x \in H$. *Then* $f_x(t) := \langle T_t x, x\rangle$ *is uniformly continuous and positive-definite.*

Proof. Let $\epsilon > 0$ be given and $\delta_\epsilon > 0$ be such that $\|x - T_t x\| < \epsilon$ for all $|t| < \delta_\epsilon$ by the strong continuity of $(T_t)_{t\in\mathbb{R}}$ and, without loss of generality, let $\|x\| = 1$. Let $|t - s| < \delta_\epsilon$. Then we have

$$\begin{aligned}|f_x(t) - f_x(s)| &= |\langle T_t x, x\rangle - \langle T_s x, x\rangle| = |\langle T_t x - T_s x, x\rangle| \leq \|T_t x - T_s x\| \cdot \|x\| \\ &= \|T_t x - T_s x\| \leq \|T_t\| \cdot \|x - T_{s-t} x\| = \|x - T_{s-t} x\| \leq \epsilon,\end{aligned}$$

the map $t \mapsto \langle T_t x, x\rangle$ is hence uniformly continuous. We further have

$$\begin{aligned}\sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} f(t_i - t_j) &= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} \langle T_{t_i - t_j} x, x\rangle = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} \langle T_{t_i} x, T_{t_j} x\rangle \\ &= \langle \sum_{i=1}^n \alpha_i T_{t_i} x, \sum_{j=1}^n \alpha_j T_{t_j} x\rangle \geq 0\end{aligned}$$

for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $t_1, \ldots, t_n \in \mathbb{R}$ and

$$f_x(t) = \langle T_t x, x\rangle = \overline{\langle x, T_t x\rangle} = \overline{\langle T_{-t} x, x\rangle} = \overline{f_x(-t)}.$$

□

Lemma 2.28. *Let* H *be a Hilbert space,* $(T_t)_{t\in\mathbb{R}}$ *be a strongly continuous unitary group on* H, $x \in H$ *and* $f_x(t) := \langle T_t x, x\rangle$. *Let* $|f_x(t_0)| > \tilde{M}$ *for some* $t_0 \in \mathbb{R}$ *and some* $\tilde{M} > 0$. *Then there exists* $M > 0$ *and* $\delta_M > 0$ *such that* $|f_x(t)| > M$ *for all* $t \in (t_0 - \delta_M, t_0 + \delta_M)$.

Proof. The function f_x is uniformly continuous by Lemma 2.27, hence, given $\epsilon > 0$, there exists $\delta > 0$, such that $|f_x(t) - f_x(t_0)| < \epsilon$ for all $t \in (t_0 - \delta, t_0 + \delta)$. Now we choose $M := \frac{\tilde{M}}{2}$ and $\epsilon := M$. Hence, we obtain $\delta > 0$ such that

$$|f_x(t)| > |f_x(t_0)| - \epsilon > M$$

for all $t \in (t_0 - \delta, t_0 + \delta)$. □

As corollaries, we obtain the following lemmata.

Lemma 2.29. *Let* $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *be a scmps and* $\mu(A) > 0$. *Then*

$$f_A(t) := \mu(\phi_t(A) \cap A)$$

is uniformly continuous and positive-definite.

Lemma 2.30. *Let* $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *be a scmps,* $A \in \Sigma$ *with* $\mu(A) > 0$ *and* $f_A(t) := \mu(\phi_t(A) \cap A)$. *Let* $|f_A(t_0)| > 0$ *for some* $t_0 \in \mathbb{R}$. *Then there exists* $M > 0$ *and* $\delta_M > 0$ *such that* $|f_A(t)| > M$ *for all* $t \in (t_0 - \delta_M, t_0 + \delta_M)$.

Lemma 2.31. *Let* μ *be a probability measure on* $\mathbb{R}$ *and* $f(t, x)$ *be bounded and measurable. Then*

$$\int_{\mathbb{R}} \oint_{\mathbb{R}} f(t,x)\ d\lambda(t)\ d\mu(x) = \oint_{\mathbb{R}} \int_{\mathbb{R}} f(t,x)\ d\mu(x)\ d\lambda(t)$$

whenever the corresponding limits exist.

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} &|\frac{1}{2T}\mathbb{1}_{[-T,T]\times\mathbb{R}}(t,x) f(t,x)|\ d(\lambda\times\mu)(t,x) \\ &\leq \|f\|_\infty \cdot \int_{\mathbb{R}\times\mathbb{R}} \frac{1}{2T}\mathbb{1}_{[-T,T]\times\mathbb{R}}(t,x)\ d(\lambda\times\mu)(t,x) = \|f\|_\infty\end{aligned}$$

independently of T, hence, Fubini's theorem implies

$$\int_{\mathbb{R}} \frac{1}{2T} \int_{-T}^{T} f(t,x)\ d\lambda(t)\ d\mu(x) = \frac{1}{2T} \int_{-T}^{T} \int_{\mathbb{R}} f(t,x)\ d\mu(x)\ d\lambda(t)$$

and the dominated convergence theorem allows taking the limit, yielding

$$\int_{\mathbb{R}} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t,x)\ d\lambda(t)\ d\mu(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{\mathbb{R}} f(t,x)\ d\mu(x)\ d\lambda(t).$$

□

The definition of the Banach space valued Riemann integral ([40]) and the use of linearity and continuity of the scalar product yield the following Lemma.

Lemma 2.32. *Let $t \mapsto x_t$ be continuous from $\mathbb{R}$ to a Hilbert space H. Then we have*

$$\oint_{\mathbb{R}} \langle x_t, y \rangle \; d\lambda(t) = \langle \oint_{\mathbb{R}} x_t \; d\lambda(t), y \rangle$$

whenever the corresponding limits exist.

Chapter 3

Recurrence Properties

This chapter gives the overview of recurrence properties. We first introduce and define the main versions of recurrence in Section 3.1. In particular, these are the properties which we discuss in Chapters 4-6 in more detail. We then prove the implications between them in Section 3.2 and also introduce additional properties.

3.1 Definition of Main Recurrence Properties

There are four main classes of properties which we consider, and all properties in one class are equivalent to each other in the integer setting (see Theorem 8). However, fewer implications are true for the real setting, e.g. the Fürstenberg correspondence principle (Subsection 3.2.7) fails to hold without further assumptions (see Remark 3.82 and Proposition 4.33).

We keep the traditional names where they exist and introduce appropriate ones where not. We mainly consider real properties and note that associated integer properties can be defined similarly (see for example [14], [23], [42], [57], [70], [71]). Occasionally however (in particular in Section 4.3), we also deal with integer recurrence sets $\mathcal{D} \subseteq \mathbb{N}$ or $\mathcal{D} \subseteq \mathbb{Z}$. We indicate this difference by writing $\mathbb{N}$-FMRiesz, $\mathbb{Z}$-FMRiesz or $\mathbb{R}$-FMRiesz instead of FMRiesz if it is not clear from the context.

Real recurrence properties rely often on continuity assumptions in contrast to integer recurrence properties where continuity is automatically given with the discrete topology. In the same way, we can define analogous real properties by dropping or adding these continuity assumptions and replace it with corresponding measurability assumptions (e.g. strong measurability instead of strong continuity or the use of $l_*^\infty(\mathbb{R})$ instead of $l_c^\infty(\mathbb{R})$ or $l_{cc}^\infty(\mathbb{R})$). We call these recurrence properties **recurrent with continuity (assumptions)** or **recurrent without continuity (assumptions)** and denote this with a subscript $*$ (e.g. rCor$_*$) if we drop a continuity assumption or a prefix c (e.g. cIS) if we add one.

We note that properties without continuity assumptions always imply the corresponding properties with continuity assumption (e.g. rCor$_*$ $\Rightarrow$ rCor). However, the converse is false at least for Poincaré recurrence (Proposition 4.33).

3.1.1 Poincaré

Definition 3.1 (Poincaré Recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **Poincaré** or **Poincaré recurrent** (P) if, given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$, there exists $d \in \mathcal{D}$ such that*

$$\mu(A \cap \phi_d(A)) > 0.$$

Remark 3.2. Poincaré recurrence (and similarly all other properties) can be restated as follows.

> A set $\mathcal{D} \subseteq \mathbb{R}$ is **Poincaré** if, given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ such that $\mu(A \cap \phi_d(A)) = 0$ for all $d \in \mathcal{D}$, we have $\mu(A) = 0$.

Definition 3.3 (Intersectivity)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **intersective** (IS) if, given $E \subseteq \mathbb{R}$ with $\overline{d}_{\mathbb{R}}(E) > 0$, we have*

$$(E - E) \cap \mathcal{D} \neq \emptyset.$$

Definition 3.4 (Combinatorial Recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **combinatorially recurrent** (CR) if, given $E \subseteq \mathbb{R}$ with $\overline{d}_{\mathbb{R}}(E) > 0$, there exists $d \in \mathcal{D}$ such that*

$$\overline{d}_{\mathbb{R}}(E \cap (E - d)) > 0.$$

Definition 3.5 (Real Correlativity)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **real correlative** (rCor) if, given $0 \leq f \in l_c^\infty(\mathbb{R}, \mathbb{R})$, $\oint_{\mathbb{R}} f(t+d)f(t)\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

3.1.2 Operator Recurrence

Definition 3.6 (Operator Recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **operator recurrent** (OR) if, given a Hilbert space H, a strongly continuous unitary group $(T_t)_{t \in \mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that*

$$\langle T_d x, x \rangle \neq 0.$$

Definition 3.7 (F. and M. Riesz)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **FMRiesz** if every probability measure on $\mathbb{R}$ with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ satisfies*

$$\mu(\{0\}) = 0.$$

Definition 3.8 (Correlativity)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **correlative** (Cor) if, given $f \in l_c^\infty(\mathbb{R}, \mathbb{T})$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

Definition 3.9 (Van der Corput)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **van der Corput** (vdC) if, given $f \in l_{cc}^\infty(\mathbb{R}, \mathbb{R})$, the equidistribution mod 1 of $f(\cdot + d) - f(\cdot)$ for all $d \in \mathcal{D}$ implies the equidistribution of f mod 1.*

3.1.3 Strong Recurrence

Strong recurrence sets and their variants, as well as FC+ sets with their variants are strong versions of the properties around Poincaré and operator recurrence sets. Instead of satisfying some condition on $\mathcal{D}$, they satisfy asymptotic properties along $\mathcal{D}$.

Definition 3.10 (Strong Recurrence). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **strongly recurrent** (SR) if, given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$, we have*

$$\limsup_{|d|\to\infty, d\in\mathcal{D}} \mu(\phi_d(A) \cap A) > 0.$$

Definition 3.11 (Strong Combinatorial Recurrence). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **strongly combinatorially recurrent** (SCR) if, given $E \subseteq \mathbb{R}$ with $\overline{d}_{\mathbb{R}}(E) > 0$, we have*

$$\limsup_{|d|_{\mathbb{R}}\to\infty, d\in\mathcal{D}} \overline{d}_{\mathbb{R}}(E \cap (E - d)) > 0.$$

Definition 3.12 (Strong Correlativity). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **strongly correlative** (SCor) if, given $0 \leq f \in l_c^{\infty}(\mathbb{R}, \mathbb{R})$, $\lim_{|d|\to\infty, d\in\mathcal{D}} \oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

3.1.4 Strong Operator Recurrence

Definition 3.13 (Strong operator recurrence). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **strongly operator recurrent** (SOR) if, given a Hilbert space H, a strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, we have*

$$\limsup_{|d|\to\infty, d\in\mathcal{D}} |\langle T_d x, x\rangle| > 0.$$

Definition 3.14 (FC+). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **FC+** if every probability measure on $\mathbb{R}$ with $\lim_{|d|\to\infty, d\in\mathcal{D}} \widehat{\mu}(d) = 0$ satisfies*

$$\mu(\{0\}) = 0.$$

Definition 3.15 (Enhanced van der Corput). *A set $\mathcal{D} \subseteq \mathbb{R}$ is* ***enhanced van der Corput*** *(EvdC) if, given $f \in l_c^\infty(\mathbb{R}, \mathbb{T})$, $\lim_{|d|\to\infty, d\in\mathcal{D}} \oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

3.2 Characterisations

In this section, we discuss the relationship among the properties introduced in Section 3.1 (apart from some counterexamples which are included in Chapters 4 and 6). The characterisations do not hold in the extent of Theorem 8 for integers, but the same spirit lies behind real recurrence. Our final result is the following theorem (see the appendix for a detailed diagram).

Theorem 3.16. *Let $\mathcal{D} \subseteq \mathbb{R}$. Then all properties within the following groups are equivalent to each other.*
(i) Poincaré without continuity, combinatorial recurrence, intersectivity, real correlativity without continuity.
(ii) Van der Corput without continuity, correlativity without continuity.
(iia) Operator recurrence, FMRiesz, van der Corput, correlativity.
(iii) Strong recurrence without continuity, strong combinatorial recurrence, strong correlativity without continuity.
(iv) Strong operator recurrence, FC+, enhanced van der Corput.
In addition, we have the implications

$$\begin{aligned} \textit{combinatorial recurrence} &\Rightarrow \textit{intersectivity}, \\ \textit{real correlativity} &\Rightarrow \textit{Poincaré} \\ &\Rightarrow \textit{continuous combinatorial recurrence} \\ &\Rightarrow \textit{continuous intersectivity}, \end{aligned}$$

operator recurrence without continuity $\Rightarrow$ *correlativity without continuity,*

strong correlativity $\Rightarrow$ *strong recurrence*

$\Rightarrow$ *strong continuous combinatorial recurrence,*

strong operator recurrence without continuity

$\Rightarrow$ *enhanced van der Corput without continuity.*

3.2.1 Operator Recurrence

The following proposition gives the equivalence of FMRiesz and operator recurrence. We postpone the proof to Subsection 7.3.3 where we prove this result for locally compact abelian groups.

Proposition 3.17 (FMRiesz $\Leftrightarrow$ OR)**.** *The following are equivalent.*
(i) Every probability measure on $\mathbb{R}$ *with* $\widehat{\mu}(d) = 0$ *for all* $d \in \mathcal{D}$ *satisfies*

$$\mu(\{0\}) = 0.$$

(ii) Given a Hilbert space H*, a strongly continuous unitary group* $(T_t)_{t\in\mathbb{R}}$ *on* H *and* $x \in H$ *with* $\|x\| = 1$ *and* $Px \neq 0$*, there exists* $d \in \mathcal{D}$ *such that*

$$\langle T_d x, x\rangle \neq 0.$$

The next results show that it is not necessary for $(T_t)_{t\in\mathbb{R}}$ to be unitary in the definition of operator recurrence and strong operator recurrence since operator recurrence and its variants are invariant under unitary equivalence and dilations. We show that an equivalent characterisation with positive-definite families of commuting operators holds by using unitary dilations as well as for unitary multiplicator groups.

We call a family of operators $(T_t)_{t\in\mathbb{R}}$ on a Hilbert space **positive-definite** if

$T_{-t} = T_t^*$ for all $t \in \mathbb{R}$ and

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \langle T_{t_i - t_j} x_i, x_j \rangle \geq 0$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in H$ and $t_1, \ldots, t_n \in \mathbb{R}$.

Definition 3.18 (Operator Recurrence(1)). *A set $\mathcal{D} \subseteq \mathbb{R}$ is called **operator recurrent(1)** (OR(1)) if, given a Hilbert space H, a strongly continuous positive-definite commuting family of operators $(T_t)_{t \in \mathbb{R}}$ on H with $T_0 = \mathrm{Id}$ and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that*

$$\langle T_d x, x \rangle \neq 0.$$

Definition 3.19 (Strong Operator Recurrence(1)). *A set $\mathcal{D} \subseteq \mathbb{R}$ is called **strongly operator recurrent(1)** (SOR(1)) if, given a Hilbert space H, a strongly continuous positive-definite commuting family of operators $(T_t)_{t \in \mathbb{R}}$ on H with $T_0 = \mathrm{Id}$ and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, we have*

$$\lim_{|d| \to \infty, d \in \mathcal{D}} |\langle T_d x, x \rangle| > 0.$$

Definition 3.20 (Unitary Dilation). *Let H be a Hilbert space and $(T_t)_{t \in \mathbb{R}}$ a strongly continuous commuting family of operators on H. Then we call $(\widehat{H}, (\widehat{T_t})_{t \in \mathbb{R}}, J, Q)$ a **unitary dilation** of $(H, (T_t)_{t \in \mathbb{R}})$ if $\widehat{H}$ is a Hilbert space, $(\widehat{T_t})_{t \in \mathbb{R}}$ a strongly continuous unitary group on $\widehat{H}$, $J : H \to \widehat{H}$ an isometric embedding with corresponding contraction $Q = J^* : \widehat{H} \to H$ such that*

$$T_t = Q \widehat{T}_t J$$

holds for all $t \in \mathbb{R}$.

We use the following dilation theorem ([78, Theorem I.7.1]). We note that all strongly continuous contractive semigroups $(T_t)_{t \geq 0}$ can be extended to $\mathbb{R}_-$ to obtain a strongly continuous positive-definite commuting family ([78, Section I.8]) by setting $T_{-t} := T_t^*$.

Theorem 3.21 (Sz.-Nagy)**.** *Let H be a Hilbert space and $(T_t)_{t\in\mathbb{R}}$ a strongly continuous positive-definite commuting family of operators on H with $T_0 = \mathrm{Id}$. Then there exists a unitary dilation $(\widehat{H}, (\widehat{T}_t)_{t\in\mathbb{R}}, J, Q)$.*

The following Lemma extends the mean ergodic theorem 2.7, i.e. the norm convergence of $\oint_{\mathbb{R}} T_t x \; d\lambda(t)$, to strongly continuous commuting positive-definite families $(T_t)_{t\in\mathbb{R}}$ of operators on H.

Lemma 3.22. *Let H be a Hilbert space and $(T_t)_{t\in\mathbb{R}}$ a strongly continuous positive-definite commuting family of operators on H with unitary dilation $(\widehat{H}, (\widehat{T}_t)_{t\in\mathbb{R}}, J, Q)$. Then we have*

$$Q\widehat{P}Jx = Px.$$

In particular, the mean ergodic projection P exists.

Proof. We have

$$\begin{aligned}\langle Q\widehat{P}Jx, y\rangle &= \langle \widehat{P}Jx, Jy\rangle = \langle \oint_{\mathbb{R}} \widehat{T}_t Jx \; d\lambda(t), Jy\rangle \overset{2.32}{=} \oint_{\mathbb{R}} \langle \widehat{T}_t Jx, Jy\rangle \; d\lambda(t) \\ &= \oint_{\mathbb{R}} \langle Q\widehat{T}_t Jx, y\rangle \; d\lambda(t) = \oint_{\mathbb{R}} \langle T_t x, y\rangle \; d\lambda(t) \overset{2.32}{=} \langle \oint_{\mathbb{R}} T_t x \; d\lambda(t), y\rangle = \langle Px, y\rangle\end{aligned}$$

for all $y \in H$. □

Proposition 3.23 (OR $\Leftrightarrow$ OR(1))**.** *The following are equivalent.*

(i) Given a Hilbert space H, a strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that

$$\langle T_d x, x\rangle \neq 0.$$

(ii) Given a Hilbert space H, a strongly continuous positive-definite commuting family of operators $(T_t)_{t\in\mathbb{R}}$ on H with $T_0 = \mathrm{Id}$ and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that

$$\langle T_d x, x\rangle \neq 0.$$

Proof. Every strongly continuous unitary group is in particular positive-definite since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\langle T_{t_i-t_j}x_i, x_j\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle T_{t_i}x_i, T_{t_j}x_j\rangle = \langle \sum_{i=1}^{n} T_{t_i}x_i, \sum_{j=1}^{n} T_{t_j}x_j\rangle \geq 0$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in H$ and $t_1, \ldots, t_n \in \mathbb{R}$, so operator recurrence implies operator recurrence(1).

For the converse, let a Hilbert space H, a strongly continuous positive-definite commuting family of operators $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$ be given. By Theorem 3.21, there exists a unitary dilation $(\widehat{H}, (\widehat{T}_t)_{t\in\mathbb{R}}, J, Q)$. Then we have $\|Jx\| = \|x\| = 1$ and

$$\begin{aligned} \|\widehat{P}Jx\|^2 = \langle \widehat{P}Jx, \widehat{P}Jx\rangle = \langle \widehat{P}Jx, Jx\rangle = \langle Q\widehat{P}Jx, x\rangle \\ \overset{3.22}{=} \langle Px, x\rangle = \langle Px, Px\rangle = \|Px\|^2 > 0. \end{aligned}$$

Operator recurrence implies that there exists $d \in \mathcal{D}$ such that $\langle \widehat{T}_d Jx, Jx\rangle \neq 0$, hence

$$\langle T_d x, x\rangle = \langle Q\widehat{T}_d Jx, x\rangle = \langle \widehat{T}_d Jx, Jx\rangle \neq 0.$$

□

We have defined operator recurrence for unitary groups and extended it to positive-definite families of operators. However, it suffices to consider only unitary multiplicator groups by using the spectral theorem. We also consider the associated $\mathbb{Z}$-properties as we use them to show the equivalence of $\mathbb{Z}$-operator recurrence and $\mathbb{Z}$-operator recurrence for $\mathcal{D} \subseteq \mathbb{Z}$ in Subsection 4.3.1.

Definition 3.24 (Multiplicator Operator Recurrence)**.** *A set $\mathcal{D} \subset \mathbb{Z}$ is called **$\mathbb{Z}$-multiplicator operator recurrent** ($\mathbb{Z}$-OR-M) if, given a Hilbert space $H = L^2(\Omega, \Sigma, \mu)$ (with a not necessarily finite measure μ), a multiplicator M on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that*

$$\langle M^d x, x\rangle \neq 0.$$

A set $\mathcal{D} \subset \mathbb{R}$ is called $\mathbb{R}$***-multiplicator operator recurrent*** *($\mathbb{R}$-OR-M) if, given a Hilbert space $H = L^2(\Omega, \Sigma, \mu)$ (with a not necessarily finite measure μ), a strongly continuous unitary multiplicator group $(M_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that*

$$\langle M_d x, x\rangle \neq 0.$$

The next theorem is due to Stone ([77]).

Theorem 3.25 (Stone)**.** *$(T_t)_{t\in\mathbb{R}}$ is a strongly continuous unitary group on a Hilbert space H if and only if there exists a (not necessarily bounded) self-adjoint operator A on H such that $T_t = e^{itA}$ for all $t \in \mathbb{R}$.*

Theorem 3.26. *T is a unitary operator on a Hilbert space H if and only if there exists a (not necessarily bounded) self-adjoint operator A on H such that $T^n = e^{inA}$ for all $n \in \mathbb{Z}$.*

Proof. As a unitary operator, $(T^n)_{n\in\mathbb{Z}}$ can be embedded into a strongly continuous unitary group ([30, Proposition 4.1]), and we can apply Theorem 3.25. □

The next theorem is the spectral theorem ([3, Theorem B.14]).

Theorem 3.27 (Spectral Theorem)**.** *Let A be a (not necessarily bounded) self-adjoint operator on a Hilbert space H. Then it is unitarily equivalent to a self-adjoint multiplicator on some $L^2(\Omega, \Sigma, \mu)$ (with a not necessarily finite measure μ), i.e.*

$$A = U^* M U$$

for some unitary $U : H \to L^2(\Omega, \Sigma, \mu)$.

Theorem 3.28. *Let $(T_t)_{t\in\mathbb{R}}$ be a strongly continuous unitary group on a Hilbert space H. Then it is unitarily equivalent to a strongly continuous unitary multiplicator group.*

Proof. By Stone's theorem 3.25, $T_t = e^{itA}$ holds for all $t \in \mathbb{R}$ with some self-adjoint A. By Theorem 3.27, there exists a self-adjoint multiplicator M on some $L^2(\Omega, \Sigma, \mu)$ (with a not necessarily finite measure μ) and a unitary $U : H \to L^2(\Omega, \Sigma, \mu)$ such that $A = U^*MU$. By Stone's theorem 3.25, iM generates a strongly continuous unitary multiplicator group $(M_t)_{t\in\mathbb{R}}$.
Let $x \in D(A^\infty) = \bigcap_{n\in\mathbb{N}} D(A^n)$ which is dense in H ([31, Proposition II.1.8]). Then we have

$$\begin{aligned} T_t x = e^{itA}x = \sum_{n=0}^{\infty} \frac{(itA)^n x}{n!} &= \sum_{n=0}^{\infty} \frac{(it)^n (U^*MU)^n x}{n!} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} U^* M^n U x \\ = U \sum_{n=0}^{\infty} \frac{(it)^n}{n!} M^n (U^* x) &= U e^{itM} U^* x = U M_t U^* x \end{aligned}$$

and we extend it to H by continuity. □

Proposition 3.29 ($\mathbb{R}$-OR $\Leftrightarrow$ $\mathbb{R}$-OR-M)**.** *Let $\mathcal{D} \subseteq \mathbb{R}$. The following are equivalent*
(i) Given a Hilbert space H, a strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $P_{T_t}x \neq 0$, there exists $d \in \mathcal{D}$ such that

$$\langle T_d x, x\rangle \neq 0$$

(ii) Given a Hilbert space $H = L^2(\Omega, \Sigma, \mu)$ (with a not necessarily finite measure μ), a strongly continuous unitary multiplicator group $(M_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $P_{M_t}x \neq 0$, there exists $d \in \mathcal{D}$ such that

$$\langle M_d x, x\rangle \neq 0.$$

Proof. As every strongly continuous unitary multiplicator group $(T_t)_{t\in\mathbb{R}}$ is in particular a strongly continuous unitary group, we obtain $\mathbb{R}$-OR $\Rightarrow$ $\mathbb{R}$-OR-M.
Now let a strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on a Hilbert space H and $x \in H$ with $P_{T_t}x \neq 0$ be given. By Theorem 3.28, we have a multiplicator group $(M_t)_{t\in\mathbb{R}}$ on some $L^2(\Omega, \Sigma, \mu)$ (with a not necessarily finite measure μ) and

$T_t = U^* M_t U$ for all $t \in \mathbb{R}$ with a unitary $U : H \to L^2(\Omega, \Sigma, \mu)$. We further note

$$\begin{aligned}\|P_{M_t}Ux\|^2 &= \langle P_{M_t}Ux, P_{M_t}Ux\rangle = \langle P_{M_t}Ux, Ux\rangle = \langle \oint_{\mathbb{R}} M_t Ux \; d\lambda(t), Ux\rangle \\ &\overset{2.32}{=} \oint_{\mathbb{R}} \langle M_t Ux, Ux\rangle \; d\lambda(t) = \oint_{\mathbb{R}} \langle UT_tU^*Ux, Ux\rangle \; d\lambda(t) = \oint_{\mathbb{R}} \langle T_t x, x\rangle \; d\lambda(t) \\ &\overset{2.32}{=} \langle \oint_{\mathbb{R}} T_t x \; d\lambda(t), x\rangle = \langle P_{T_t}x, x\rangle = \langle P_{T_t}x, P_{T_t}x\rangle = \|P_{T_t}x\|^2 > 0.\end{aligned}$$

By $\mathbb{R}$-multiplicator operator recurrence and since $\|Ux\| = \|x\| = 1$, there exists $d \in \mathcal{D}$ with $\langle M_d Ux, Ux\rangle \neq 0$, hence,

$$\langle T_d x, x\rangle = \langle U^* M_d Ux, x\rangle = \langle M_d Ux, Ux\rangle \neq 0.$$

□

Remark 3.30. Similarly, we obtain the result for integers as well as their strong variants, i.e.

$$\begin{aligned}\mathbb{Z}\text{-operator recurrence} &\Leftrightarrow \mathbb{Z}\text{-multiplicator operator recurrence}, \\ \mathbb{Z}\text{-strong operator recurrence} &\Leftrightarrow \mathbb{Z}\text{-strong multiplicator operator recurrence} \\ &\Leftrightarrow \mathbb{Z}\text{-strong operator recurrence}(1), \\ \mathbb{R}\text{-strong operator recurrence} &\Leftrightarrow \mathbb{R}\text{-strong multiplicator operator recurrence} \\ &\Leftrightarrow \mathbb{R}\text{-strong operator recurrence}(1).\end{aligned}$$

It is a priori not clear that the mean ergodic projection exists for for a unitary group $(T_t)_{t\in\mathbb{R}}$ on a Hilbert space H if strong continuity is replaced by strong measurability. The following theorem yields the existence by using the group structure of $(T_t)_{t\in\mathbb{R}}$ and the two-sided variant of the discrete mean ergodic Theorem 2.8 (compare [13, Theorem 2.1]).

Theorem 3.31. *Let $f : \mathbb{R} \to H$ be measurable and bounded such that*

$$A_t = \lim_{T\to\infty} \frac{1}{2T} \sum_{-T}^{T} f(T + t)$$

exists for almost every $t \in [0,1]$. *Then* $\lim_{T\to\infty} \oint_{-T}^{T} f(t)\, d\lambda(t) = \int_0^1 A_t \, d\lambda(t)$ *exists for all* $y \in H$ *and*

$$\oint_{\mathbb{R}} f(t)\, d\lambda(t) = \int_0^1 A_t \, d\lambda(t).$$

In particular, the mean ergodic projection P_{T_t} *exists in the strong topology.*

Remark 3.32. Similarly, we obtain the same implications for the properties by replacing strong continuity with weak measurability, i.e.

$$\begin{aligned}
\text{operator recurrence}_* &\Leftrightarrow \text{operator recurrence(1)}_* \\
&\Leftrightarrow \text{multiplicator operator recurrence}_*, \\
\text{strong operator recurrence}_* &\Leftrightarrow \text{strong operator recurrence(1)}_* \\
&\Leftrightarrow \text{strong multiplicator operator recurrence}_*, \\
\text{strong operator recurrence}_* &\Rightarrow \text{operator recurrence}_*.
\end{aligned}$$

3.2.2 Poincaré and Koopman Recurrence

We discuss several characterisations of Poincaré recurrence connecting it with operator theory. Having the Koopman representation of a scmps in mind, these results are not surprising.

Definition 3.33 (Koopman Recurrence)**.** *A set* $\mathcal{D} \subseteq \mathbb{R}$ *is* ***Koopman recurrent*** *(KR) if, given a scmps* $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *and* $0 < f \in L^2(\Omega, \Sigma, \mu)$*, there exists* $d \in \mathcal{D}$ *such that*

$$\langle T_d^{\phi} f, f \rangle > 0.$$

Definition 3.34 (Strong Koopman Recurrence)**.** *A set* $\mathcal{D} \subseteq \mathbb{R}$ *is* ***strongly Koopman recurrent*** *(SKR) if, given a scmps* $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *and* $0 < f \in L^2(\Omega, \Sigma, \mu)$*, we have*

$$\lim_{|d|\to\infty, d\in\mathcal{D}} \langle T_d^{\phi} f, f \rangle > 0.$$

Proposition 3.35 (P $\Leftrightarrow$ KR)**.** *Let $\mathcal{D} \subseteq \mathbb{R}$. The following are equivalent for any given scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$.*
(i) Given $A \in \Sigma$ with $\mu(A) > 0$, there exists $d \in \mathcal{D}$ such that $\mu(\phi_d(A) \cap A) > 0$.
(ii) Given $0 < f \in L^2(\Omega, \Sigma, \mu)$, there exists $d \in \mathcal{D}$ such that $\langle T_d^\phi f, f\rangle > 0$.

Proof. Let a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ be given. For $A \in \Sigma$ with $\mu(A) > 0$, we have

$$\mu(\phi_d(A) \cap A) = \langle T_d^\phi \mathbb{1}_A, \mathbb{1}_A\rangle > 0$$

for some $d \in \mathcal{D}$ by Remark 2.2 and Koopman recurrence. On the other hand, let $0 < f \in L^2(\Omega, \Sigma, \mu)$ be given. Then there exists $A \in \Sigma$ and $a > 0$ such that $a\mathbb{1}_A \leq f$. Using positivity, we obtain

$$\langle T_d^\phi f, f\rangle \geq a^2\langle T_d^\phi \mathbb{1}_A, \mathbb{1}_A\rangle = a^2\mu(\phi_d(A) \cap A) > 0$$

for some $d \in \mathcal{D}$ by Poincaré recurrence. □

Proposition 3.36 (OR $\Rightarrow$ KR)**.** *Let us assume that, given a Hilbert space H, a strongly continuous unitary group representation $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that*

$$\langle T_d x, x\rangle \neq 0.$$

Then, given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $0 < f \in L^2(\Omega, \Sigma, \mu)$, there exists $d \in \mathcal{D}$ such that

$$\langle T_d^\phi f, f\rangle > 0.$$

Proof. The Koopman representation $(L^2(\Omega, \Sigma, \mu), (T_t^\phi)_{t\in\mathbb{R}})$ for a given scmps is a strongly continuous unitary group (Remark 2.2) and similarly $(L^2(\Omega, \Sigma, \frac{\mu}{\mu(A)}),$ $(T_t^\phi)_{t\in\mathbb{R}})$ which satisfies $\|\mathbb{1}_A\| = 1$. We note $P\mathbb{1}_A \neq 0$ by Lemma 2.11, and hence operator recurrence and the positivity of T_d^ϕ and f hence imply that there exists $d \in \mathcal{D}$ such that $\langle T_d^\phi \mathbb{1}_A, \mathbb{1}_A\rangle > 0$. □

We note with the following proposition that it is sufficient to consider only ergodic scmps for Poincaré recurrence using a certain dilation construction. A **lattice dilation** is defined as the unitary dilation on $L^2(\Omega, \Sigma, \mu)$ in Definition 3.20, but with the additional assumption of positivity for all operators (see [45], [46]), and we proceed similarly to Proposition 3.23.

Proposition 3.37. *Let $\mathcal{D} \subseteq \mathbb{R}$. The following are equivalent.*
(i) Given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$, there exists $d \in \mathcal{D}$ such that

$$\mu(A \cap \phi_d(A)) > 0.$$

(ii) Given an ergodic scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$, there exists $d \in \mathcal{D}$ such that

$$\mu(A \cap \phi_d(A)) > 0.$$

Proof. We clearly have (i) $\Rightarrow$ (ii). For the converse, let a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$ be given. We consider its Koopman representation which satisfies $T_t\mathbb{1} = \mathbb{1} = T_t'\mathbb{1}$ for all $t \in \mathbb{R}$. The lattice dilation $(L^2(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu}), (\widehat{T}_t)_{t\in\mathbb{R}}, J, Q)$ through the Kolmogoroff-Daniell construction ([45], [46], [81]) yields the Koopman representation $(\widehat{T}_t)_{t\in\mathbb{R}}$ of an ergodic scmps $(\widehat{\Omega}, \widehat{\Sigma}, \widehat{\mu}; (\widehat{\phi}_t)_{t\in\mathbb{R}})$ and $\widehat{A} \in \widehat{\Sigma}$ with $\widehat{\mu}(\widehat{A}) > 0$ and $\mathbb{1}_{\widehat{A}} = J\mathbb{1}_A$.
Statement (ii) implies that there exists $d \in \mathcal{D}$ such that

$$\begin{aligned} 0 < \widehat{\mu}(\widehat{A} \cap \widehat{\phi}_d(\widehat{A})) &= \langle \widehat{T}_d J\mathbb{1}_A, J\mathbb{1}_A\rangle \\ &= \langle Q\widehat{T}_d J\mathbb{1}_A, \mathbb{1}_A\rangle = \langle T_d\mathbb{1}_A, \mathbb{1}_A\rangle = \mu(A \cap \phi_d(A)). \end{aligned}$$

□

Definition 3.38 (Lattice Operator Recurrent)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is* ***lattice operator recurrent*** *(LOR) if, given $L^2(\Omega, \Sigma, \mu)$, a strongly continuous bistochastic lattice* *semigroup* $(T_t)_{t\geq 0}$ *on* $L^2(\Omega, \Sigma, \mu)$ *and* $0 < f \in L^2(\Omega, \Sigma, \mu)$*, there exists* $d \in \mathcal{D}$ *such that*

$$\langle T_{|d|} f, f \rangle > 0.$$

We can extend a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $L^2(\Omega, \Sigma, \mu)$ to a strongly continuous family $(T_t)_{t\in\mathbb{R}}$ by setting $T_{-t} = T_t^*$. We note that it suffices to consider semigroups in Definition 3.38 since $\langle T_d f, f \rangle = \langle f, T_{-d}^* f \rangle$.

With the approach of Proposition 3.37 and by using the Koopman representation (Remark 2.2), we obtain the following result.

Proposition 3.39 (LOR ⇔ KR)**.** *Let* $\mathcal{D} \subseteq \mathbb{R}$*. The following are equivalent.*

(i) Given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *and* $0 < f \in L^2(\Omega, \Sigma, \mu)$*, there exists* $d \in \mathcal{D}$ *such that*

$$\langle T_d^\phi f, f \rangle > 0.$$

(ii) Given $L^2(\Omega, \Sigma, \mu)$*, a strongly continuous bistochastic lattice semigroup* $(T_t)_{t\geq 0}$ *on* $L^2(\Omega, \Sigma, \mu)$ *and* $0 < f \in L^2(\Omega, \Sigma, \mu)$*, there exists* $d \in \mathcal{D}$ *such that*

$$\langle T_{|d|} f, f \rangle > 0.$$

Remark 3.40. It has been observed that Poincaré's theorem 4.11 does not require countable additivity of the measure (see for example [9]), i.e. it is sufficient to allow contents instead of measures. It is possible to extend Poincaré recurrence in some way to more general systems (e.g. by using premeasures instead of measures through the usual construction of an outer measure). However, it is not clear how far this approach can go.

Remark 3.41. Similarly, we obtain the implications

$$\begin{aligned} \text{strong operator recurrence} &\Rightarrow \text{strong Koopman recurrence} \\ &\Leftrightarrow \text{strong recurrence} \\ &\Leftrightarrow \text{strong lattice operator recurrence,} \end{aligned}$$

$$\begin{aligned}\text{operator recurrence}_* &\Rightarrow \text{Koopman recurrence}_*\\ &\Leftrightarrow \text{Poincaré}_*\\ &\Leftrightarrow \text{lattice operator recurrence}_*,\\ \text{strong operator recurrence}_* &\Rightarrow \text{strong Koopman recurrence}_*\\ &\Leftrightarrow \text{strong Poincaré}_*\\ &\Leftrightarrow \text{strong lattice operator recurrence}_*.\end{aligned}$$

3.2.3 Kamae, Mendès France

We give a criterion for a set $\mathcal{D}$ to be FMRiesz based on a recurrence property by Kamae and Mendès France ([42]). For integers, KMF is equivalent to FMRiesz, but the proof of FMRiesz $\Rightarrow$ KMF relies on the periodicity of exponential polynomials (see, e.g. [14, Proposition 1.18] and [57, Theorem 2.3]) which is not given for the functions considered in Definitions 3.42 and Remark 3.44. We note that we write $\Re\left(e^{-2\pi i t x}\right)$ instead of $\cos(2\pi t x)$ to stress the connection with the dual group and to indicate how to deal with KMF for a locally compact abelian group (see Section 7.3).

Definition 3.42 (Kamae, Mendès France)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **KMF** if, for all $\epsilon > 0$, there exists a nonzero real function*

$$p_\epsilon(x) = \int_{\mathcal{D}} \Re\left(e^{-2\pi i t x}\right)\, d\nu_\epsilon(t)$$

satisfying $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ where ν_ϵ is a finite measure on $\mathcal{D}$.

Proposition 3.43 (KMF $\Rightarrow$ FMRiesz)**.** *Let us assume that, for all $\epsilon > 0$, there exists a nonzero real function*

$$p_\epsilon(x) = \int_{\mathcal{D}} \Re\left(e^{-2\pi i t x}\right)\, d\nu_\epsilon(t)$$

satisfying $p_\epsilon(0) = 1$ *and* $p_\epsilon \geq -\epsilon$ *where* ν_ϵ *is a finite measure on* $\mathcal{D}$.
Then every measure on $\mathbb{R}$ *with* $\widehat{\mu}(d) = 0$ *for all* $d \in \mathcal{D}$ *satisfies*

$$\mu(\{0\}) = 0.$$

Compare [14, Proposition 1.18], see also [57, Theorem 2.3].

Proof. Let μ be a measure on $\mathbb{R}$ with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ and, for $\epsilon > 0$, let a nonzero real function

$$p_\epsilon(x) = \int_{\mathcal{D}} \Re\left(e^{-2\pi i t x}\right)\ d\nu_\epsilon(t)$$

satisfying $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ with a finite measure ν_ϵ on $\mathcal{D}$ be given. Then we have

$$\begin{aligned} \int_{\mathbb{R}} p_\epsilon(x)\ d\mu(x) &= \int_{\mathbb{R}} \int_{\mathcal{D}} \Re\left(e^{-2\pi i t x}\right)\ d\nu_\epsilon(t)\ d\mu(x) \\ &= \Re\left(\int_{\mathcal{D}} \int_{\mathbb{R}} e^{-2\pi i t x}\ d\mu(x)\ d\nu_\epsilon(t)\right) = \Re\left(\int_{\mathcal{D}} \widehat{\mu}(t)\ d\nu_\epsilon(t)\right) = 0 \end{aligned}$$

But from $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ we deduce

$$0 = \int_{\mathbb{R}} p_\epsilon(x)\ d\mu(x) = \int_{\{0\}} p_\epsilon(x)\ d\mu(x) + \int_{\mathbb{R}\backslash\{0\}} p_\epsilon(x)\ d\mu(x) \geq \mu(\{0\}) - \epsilon\mu(\mathbb{R}\backslash\{0\}).$$

As $\epsilon > 0$ was arbitrary, we obtain $\mu(\{0\}) = 0$. □

Remark 3.44. Definition 3.42 is in particular a generalisation of the following properties.

(i) A set $\mathcal{D} \subseteq \mathbb{R}$ is **KMF(1)** if, for all $\epsilon > 0$, there exists a real trigonometric polynomial

$$p_\epsilon(x) = \sum_{t \in \mathcal{D}} a_t^\epsilon \Re\left(e^{-2\pi i t x}\right)$$

satisfying $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ with $a_t^\epsilon \in \mathbb{R}$ and where the set $\{t : a_t^\epsilon \neq 0\} \subset \mathcal{D}$ is nonempty and finite.

(ii) A set $\mathcal{D} \subseteq \mathbb{R}$ is **KMF(2)** if, for all $\epsilon > 0$, there exists a nonzero real function

$$p_\epsilon(x) = \int_{\mathcal{D}} f^\epsilon(t) \Re\left(e^{-2\pi i t x}\right) \, d\lambda(t)$$

satisfying $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ with $[f^\epsilon \neq 0] \subset \mathcal{D} \cap K$ for a compact set K and $f^\epsilon \in L^\infty(\mathbb{R}, \mathbb{R})$.

Analogously to Proposition 3.43, every KMF(1) and KMF(2) set is FMRiesz. We note however that KMF(2) requires $\mathcal{D}$ to have positive measure. However, there are $\mathbb{R}$-FMRiesz sets $\mathcal{D} \subseteq \mathbb{Z}$ and every $\mathbb{R}$-FMRiesz set has a countable $\mathbb{R}$-FMRiesz subset (Corollary 5.3), both having $\lambda(\mathcal{D}) = 0$. The implication $\mathbb{R}$-FMRiesz $\Rightarrow$ $\mathbb{R}$-KMF(2) hence cannot hold. On the other hand, it is not clear if a complete discrete treatment as in KMF(1) is sufficient (compare Chapter 5).

3.2.4 Spectral Measures

We discuss a characterisation of Poincaré recurrence using spectral measures. Since the family $(\int_\Omega f \circ \phi_t \cdot f d\mu)_{t \in \mathbb{R}}$ is positive-definite and continuous (Lemma 2.27), it is straightforward to consider the corresponding Bochner-Herglotz measures (Theorem 2.16).

Definition 3.45 (Spectral measure)**.** *Let $(\Omega, \Sigma, \mu; (\phi_t)_{t \in \mathbb{R}})$ be a scmps. For $A \in \Sigma$, its* ***spectral measure*** *σ_A is defined through*

$$\widehat{\sigma}(t) = \mu(A \cap \phi_t(A))$$

for $t \in \mathbb{R}$. For $0 \leq f \in L^2(\Omega, \Sigma, \mu; \mathbb{R})$, its ***spectral measure*** *σ_f is defined through*

$$\widehat{\sigma}_f(t) = \int_\Omega f \circ \phi_t \cdot f d\mu$$

for $t \in \mathbb{R}$.

Definition 3.46 (Spectral Poincaré)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is* ***spectral Poincaré*** *(spP) if one of the following equivalent statements holds for the spectral measure of every given ergodic scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$, $A \in \Sigma$ with $\mu(A) > 0$ and $0 < f \in L^2(\Omega, \Sigma, \mu)$.*

(i) If $\widehat{\sigma_A}(d) = 0$ for all $d \in \mathcal{D}$ with a spectral measure σ_A, then $\sigma_A = 0$.

(ii) If $\widehat{\sigma_f}(d) = 0$ for all $d \in \mathcal{D}$ with a spectral measure σ_f, then $\sigma_f = 0$.

Definition 3.47 (Strong Spectral Poincaré)**.** *A set $\mathcal{D} \subseteq \mathbb{R}_*$ is* ***strong spectral Poincaré*** *(SspP) if one of the following equivalent statements holds for the spectral measure of every given ergodic scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$, $A \in \Sigma$ with $\mu(A) > 0$ and $0 \leq f \in L^2(\Omega, \Sigma, \mu)$.*

(i) If $\lim_{|d|\to\infty, d\in\mathcal{D}} \widehat{\sigma_A}(d) = 0$ with a spectral measure σ_A, then $\sigma_A = 0$.

(ii) If $\lim_{|d|\to\infty, d\in\mathcal{D}} \widehat{\sigma_f}(d) = 0$ with a spectral measure σ_f, then $\sigma_f = 0$.

Proposition 3.48. *The two statements in Definition 3.46 are equivalent.*

Proof. Let an ergodic scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ with Koopman representation $(T_t^\phi)_{t\in\mathbb{R}}$ be given.

Let $A \in \Sigma$ with $\mu(A) > 0$ be given and its associated spectral measure σ_A satisfy $\widehat{\sigma_A}(d) = \mu(\phi_d(A) \cap A) = 0$. We set $f = \mathbb{1}_A \in L^2(\Omega, \Sigma, \mu)$, then $\sigma_f = \sigma_A$ holds by $\widehat{\sigma_f} = \widehat{\sigma_A}$ and the uniqueness of the Fourier transform ([67, Subsection 1.3.6]), and we have

$$\widehat{\sigma_f}(d) = \langle T_d^\phi f, f\rangle = \mu(\phi_d(A) \cap A) = 0$$

for all $d \in \mathcal{D}$. Property (ii) then implies $\sigma_A = 0$.

For the converse, let $0 < f \in L^2(\Omega, \Sigma, \mu)$ be given. There exists $A \in \Sigma$ with $\mu(A) > 0$ and $0 < c < \infty$ such that $c\mathbb{1}_A \leq f$ with $\sigma_A(\mathbb{R}) = \mu(A) > 0$. Statement (i) implies that there exists $d \in \mathcal{D}$ such that $\widehat{\sigma_A}(d) > 0$. Using positivity, we have

$$0 < c^2\widehat{\sigma_A}(d) = c^2\langle T_d^\phi \mathbb{1}_A, \mathbb{1}_A\rangle \leq \langle T_d^\phi f, f\rangle = \widehat{\sigma_f}(d).$$

□

Proposition 3.49 (rCor $\Rightarrow$ spP)**.** *Let us assume that, given* $0 \leq f \in l_c^\infty(\mathbb{R}, \mathbb{R})$, $\oint_\mathbb{R} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ *for all* $d \in \mathcal{D}$ *implies*

$$\oint_\mathbb{R} f(t)\ d\lambda(t) = 0.$$

Then, given an ergodic scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *and* $0 < f \in L^\infty(\Omega, \Sigma, \mu)$, $\widehat{\sigma_f}(d) = 0$ *for all* $d \in \mathcal{D}$ *with the spectral measure* σ_f *implies*

$$\sigma_f = 0.$$

Compare [14, Proposition 3.2]

Proof. Let an ergodic scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $0 < f \in L^2(\Omega, \Sigma, \mu)$ be given, and let σ_f be its associated spectral measure.
We assume $\widehat{\sigma_f}(d) = 0$ for all $d \in \mathcal{D}$. Hence for all $d \in \mathcal{D}$, we have

$$\begin{aligned} 0 = \widehat{\sigma_f}(d) &= \int_\Omega f \cdot f \circ \phi_d\ d\mu \overset{2.9}{=} \oint_\mathbb{R} T_t^\phi(f \cdot f \circ \phi_d)(\omega)\ d\lambda(t) \\ &= \oint_\mathbb{R} (T_t^\phi f)(\omega) \cdot T_t^\phi(f \circ \phi_d)(\omega)\ d\lambda(t) = \oint_\mathbb{R} (f \circ \phi_t)(\omega) \cdot (f \circ \phi_{t+d})(\omega)\ d\lambda(t) \\ &= \oint_\mathbb{R} g_\omega(t) g_\omega(t+d)\ d\lambda(t) \end{aligned}$$

for $g_\omega(t) := (f \circ \phi_t)(\omega)$ and almost every $\omega \in \Omega$. For every such $\omega \in \Omega$, we have $g_\omega \in l_c^\infty(\mathbb{R}, \mathbb{R})$ by the corresponding properties of $\widehat{\sigma}_f$. Real correlativity then yields

$$0 = \oint_\mathbb{R} g_\omega(t)\ d\lambda(t) = \oint_\mathbb{R} (f \circ \phi_t)(\omega)\ d\lambda(t)$$

for almost every $\omega \in \Omega$. Remark 2.9 yields

$$0 = \oint_\mathbb{R} (f \circ \phi_t)(\omega)\ d\lambda(t) = \int_\Omega f\ d\mu.$$

Hence, we have $f = 0$ since f is positive and therefore

$$0 = \int_\Omega f^2\ d\mu \overset{3.45}{=} \widehat{\sigma_f}(0) \overset{2.14}{=} \sigma_f(\mathbb{R}).$$

□

Proposition 3.50 (spP $\Leftrightarrow$ KR)**.** *The following are equivalent.*

(i) Given an ergodic scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *and* $0 \leq f \in L^2(\Omega, \Sigma, \mu)$ *with spectral measure* σ_f, $\widehat{\sigma_f}(d) = 0$ *for all* $d \in \mathcal{D}$ *implies*

$$\sigma_f = 0.$$

(ii) Given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *and* $0 < f \in L^2(\Omega, \Sigma, \mu)$*, there exists* $d \in \mathcal{D}$ *such that*

$$\langle T_d^\phi f, f\rangle > 0.$$

Proof. Let a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $0 < f \in L^2(\Omega, \Sigma, \mu)$ be given. Without loss of generality, we can assume that $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ is ergodic (Proposition 3.37).

We note $\sigma_f(\mathbb{R}) = \|f\|^2 > 0$ (Remark 2.14). Spectral Poincaré therefore implies that there exists $d \in \mathcal{D}$ such that

$$0 < \widehat{\sigma_f}(d) = \langle T_d^\phi f, f\rangle.$$

Conversely, given a spectral measure σ_f of an ergodic scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $0 < f \in L^2(\Omega, \Sigma, \mu)$, there exists $d \in \mathcal{D}$ such that

$$0 < \langle T_d^\phi f, f\rangle = \widehat{\sigma_f}(d)$$

by Koopman recurrence. □

Remark 3.51. Similarly, we obtain the implications for the corresponding strong properties as well as without continuity assumptions. In particular, the two statements in Definition 3.47 are equivalent, and we have the implications

$$\begin{aligned}
\text{strong correlativity} &\Rightarrow \text{strong spectral Poincaré},\\
&\Leftrightarrow \text{strong recurrence},\\
\text{real correlativity}_* &\Rightarrow \text{Poincaré}_*,\\
\text{strong correlativity}_* &\Rightarrow \text{strong recurrence}_*.
\end{aligned}$$

3.2.5 Correlativity

In this subsection, we discuss correlativity and connect it with FMRiesz. Closely connected to real correlativity and combinatorial recurrence is the following correlativity variant.

Definition 3.52 (Real Correlativity$_*^{0,1}$)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **real correlative**$_*^{0,1}$ (rCor$_*^{0,1}$) if, given $f \in l_*^\infty(\mathbb{R}, \{0,1\})$, $\oint_\mathbb{R} f(t+d)f(t)\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies*

$$\oint_\mathbb{R} f(t)\ d\lambda(t) = 0.$$

Proposition 3.53 (rCor$_*^{0,1}$ $\Leftrightarrow$ rCor$_*$)**.** *Let $\mathcal{D} \subseteq \mathbb{R}$. The following are equivalent.*
(i) Given $f \in l_*^\infty(\mathbb{R}, \{0,1\})$, $\oint_\mathbb{R} f(t+d)f(t)\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies

$$\oint_\mathbb{R} f(t)\ d\lambda(t) = 0.$$

(ii) Given $0 \le f \in l_*^\infty(\mathbb{R}, \mathbb{R})$, $\oint_\mathbb{R} f(t+d)f(t)\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies

$$\oint_\mathbb{R} f(t)\ d\lambda(t) = 0.$$

Proof. As $l_*^\infty(\mathbb{R}, \{0,1\}) \subset l_*^\infty(\mathbb{R}, \mathbb{R})$, we clearly have (ii) $\Rightarrow$ (i). On the other hand, let $0 \le f \in l_*^\infty(\mathbb{R}, \mathbb{R})$ satisfy $\oint_\mathbb{R} f(t)\ d\lambda(t) = \epsilon > 0$. We can approximate f by positive simple functions in L^∞, i.e. there exists $f_N := \sum_{n=1}^N a_n^N \mathbb{1}_{A_n^N} \nearrow f$ with $a_n^N > 0$ and convergence in L^∞-norm.
Let N be such that $\|f - f_N\|_\infty \le \frac{\epsilon}{2}$. Then $\oint_\mathbb{R} f_N(t)\ d\lambda(t) \ge \frac{\epsilon}{2}$, hence, there exists $1 \le n \le N$ such that

$$\oint_\mathbb{R} \mathbb{1}_{A_n^N}(t)\ d\lambda(t) > 0.$$

Using (i) and positivity, this implies

$$0 < \limsup_{T\to\infty} (a_n^N)^2 \oint_{-T}^{T} \mathbb{1}_{A_n^N}(t+d)\mathbb{1}_{A_n^N}(t)\ d\lambda(t) \le \limsup_{T\to\infty} \oint_{-T}^{T} f_N(t+d)f_N(t)\ d\lambda(t)$$
$$\le \limsup_{T\to\infty} \oint_{-T}^{T} f(t+d)f(t)\ d\lambda(t) = \oint_\mathbb{R} f(t+d)f(t)\ d\lambda(t)$$

for some $d \in \mathcal{D}$. $\square$

Definition 3.54 (Correlativity$_*^{\mathbb{C}}$). *A set $\mathcal{D} \subseteq \mathbb{R}$ is* ***correlative$_*^{\mathbb{C}}$*** *(Cor$_*^{\mathbb{C}}$) if, given $f \in l_*^\infty(\mathbb{R}, \mathbb{C})$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

We similarly obtain the following result (as well as the strong variants for enhanced van der Corput and strong real correlativity).

Proposition 3.55 (Cor$_*$ $\Leftrightarrow$ Cor$_*^{\mathbb{C}}$). *Let $\mathcal{D} \subset \mathbb{R}$. The following are equivalent.*
(i) Given $f \in l_*^\infty(\mathbb{R}, \mathbb{T})$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

(ii) Given $f \in l_*^\infty(\mathbb{R}, \mathbb{C})$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

Remark 3.56. Obviously, we have rCor $\Rightarrow$ rCor^{0,1} and Cor$^{\mathbb{C}}$ $\Rightarrow$ Cor. However, it is not clear if Propositions 3.53 and 3.55 (as well as their strong variants) are also true with the additional continuity assumption. If they were, it would yield the equivalence of Poincaré recurrence, continuous combinatorial recurrence and real correlativity.

Lemma 3.57. *Let $f \in l_c^\infty(\mathbb{R}, \mathbb{C})$. Then*

$$\gamma(d) := \oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t)$$

is positive-definite.

Proof. We have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\gamma(t_i-t_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\oint_{\mathbb{R}} f(t+t_i-t_j)\overline{f(t)}\ d\lambda(t)\\
&\overset{\tilde{t}:=g-t_j}{=} \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\overline{\alpha_j}\lim_{T\to\infty}\frac{1}{2T}\int_{-T-t_j}^{T-t_j} f(t+t_i)\overline{f(t+t_j)}\ d\lambda(t)\\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T-t_j}^{T-t_j}\left(\sum_{i=1}^{n}\alpha_i f(t+t_i)\right)\cdot\left(\sum_{j=1}^{n}\overline{\alpha_j f(t+t_j)}\right)\ d\lambda(t)\\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T-t_j}^{T-t_j}\left|\sum_{i=1}^{n}\alpha_i f(t+t_i)\right|^2\ d\lambda(t)\geq 0
\end{aligned}$$

for all $n\in\mathbb{N}$, $\alpha_1,\dots,\alpha_n\in\mathbb{C}$, $t_1,\dots,t_n\in\mathbb{R}$ as well as

$$\overline{\gamma(-d)} = \overline{\oint_{\mathbb{R}} f(t-d)\overline{f(t)}\ d\lambda(t)} = \oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = \gamma(d).$$

□

Lemma 3.58. *Let* $f\in l_c^\infty(\mathbb{R},\mathbb{C})$ *and* $m:=\oint_{\mathbb{R}} f(t)\ d\lambda(t)$ *as well as* $g(t):=f(t)-m$ *and* $\tilde{\gamma}(d):=\oint_{\mathbb{R}} g(t+d)\overline{g(t)}\ d\lambda(t)$ *be given. Then* $\gamma(d)=\tilde{\gamma}(d)+|m|^2$ *for all* $d\in\mathbb{R}$, $g\in l_c^\infty(\mathbb{R},\mathbb{C})$ *and* $\oint_{\mathbb{R}}\tilde{\gamma}(d)\ d\lambda(d)\geq 0$.

Proof. We have

$$\oint_{\mathbb{R}} g(t)\ d\lambda(t) = \oint_{\mathbb{R}}(f(t)-m)\ d\lambda(t) = \oint_{\mathbb{R}} f(t)\ d\lambda(t) - m$$

and

$$\begin{aligned}
\oint_{\mathbb{R}} f(t)\ d\lambda(t) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)\ d\lambda(t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T+d}^{T+d} f(t)\ d\lambda(t)\\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t+d)\ d\lambda(t) = \oint_{\mathbb{R}} f(t+d)\ d\lambda(t)
\end{aligned}$$

for all $d\in\mathbb{R}$. Therefore, we have

$$\begin{aligned}
\tilde{\gamma}(d) &= \oint_{\mathbb{R}} g(t+d)\overline{g(t)}\ d\lambda(t) = \oint_{\mathbb{R}}(f(t+d)-m)(\overline{f(t)}-\overline{m})\ d\lambda(t)\\
&= \oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) + |m|^2 - m\overline{\oint_{\mathbb{R}} f(t)\ d\lambda(t)} - \overline{m}\oint_{\mathbb{R}} f(t+d)\ d\lambda(t)\\
&= \oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) - |m|^2 = \gamma(d)-|m|^2.
\end{aligned}$$

Hence, γ and $\tilde{\gamma}$ differ only by a constant, so $g \in l_c^\infty(\mathbb{R}, \mathbb{C})$. In particular, $\tilde{\gamma}$ is continuous and also positive-definite by Lemma 3.57. Hence, there exists a measure ν on $\mathbb{R}$ such that $\widehat{\nu}(d) = \tilde{\gamma}(d)$. Lemma 2.17 implies

$$0 \leq \nu(\{0\}) = \oint_{\mathbb{R}} \tilde{\gamma}(d)\ d\lambda(d).$$

□

Proposition 3.59 (FMRiesz ⇒ Cor)**.** *Let us assume that every probability measure on* $\mathbb{R}$ *with* $\widehat{\mu}(d) = 0$ *for all* $d \in \mathcal{D}$ *satisfies*

$$\mu(\{0\}) = 0.$$

Then, given $f \in l_c^\infty(\mathbb{R}, \mathbb{T})$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ *for all* $d \in \mathcal{D}$ *implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

Compare [4, Theorems 1-3] and [14, Theorem 1.8].

Proof. Let $f \in l_c^\infty(\mathbb{R}, \mathbb{T})$ be given with $\gamma(d) = \oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ and such that $s \mapsto \gamma(s)$ is continuous.

By Lemma 3.57, γ is positive-definite, hence, by the Bochner Herglotz theorem 2.16, there exists a measure μ on $\mathbb{R}$ such that $\mu(\mathbb{R}) = \gamma(0)$ and $\widehat{\mu}(t) = \gamma(t)$ for all $t \in \mathcal{D}$.

We set $\tilde{\mu} := \frac{\mu}{\mu(\mathbb{R})}$ to obtain a probability measure. We have $\widehat{\mu}(d) = \frac{1}{\mu(\mathbb{R})}\gamma(d) = 0$ for all $d \in \mathcal{D}$, hence,

$$\mu(\{0\}) = \tilde{\mu}(\{0\}) \cdot \mu(\mathbb{R}) = 0$$

by FMRiesz. We set $m := \oint_{\mathbb{R}} f(t)\ d\lambda(t)$ as well as $g(t) := f(t) - m$ and

$$\tilde{\gamma}(d) := \oint_{\mathbb{R}} g(t+d)\overline{g(t)}\ d\lambda(t).$$

We finally conclude

$$\mu(\{0\}) \overset{2.17}{=} \oint_{\mathbb{R}} \gamma(t)\ d\lambda(t) \overset{3.58}{=} |m|^2 + \oint_{\mathbb{R}} \tilde{\gamma}(t)\ d\lambda(t) \overset{3.58}{\geq} |m|^2 = \left| \oint_{\mathbb{R}} f(t)\ d\lambda(t) \right|^2$$

and

$$0 \leq \left| \oint_{\mathbb{R}} f(t)\ d\lambda(t) \right| \leq \sqrt{\mu(\{0\})} = 0,$$

and $\mathcal{D}$ is therefore correlative. □

Similarly, we obtain the following result.

Proposition 3.60 (FMRiesz $\Rightarrow$ rCor)**.** *Let us assume that probability measure on $\mathbb{R}$ with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ satisfies*

$$\mu(\{0\}) = 0.$$

Then, given $0 \leq f \in l_c^\infty(\mathbb{R}, \mathbb{R})$, $\oint_{\mathbb{R}} f(t+d)f(t)\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

Remark 3.61. Proposition 3.60 suggests the following property:

> A set $\mathcal{D} \subseteq \mathbb{R}$ is symmetric FMRiesz if every probability measure on $\mathbb{R}$ with $\widehat{\mu}(-t) = \widehat{\mu}(t) = \overline{\widehat{\mu}(-t)}$ for all $t \in \mathbb{R}$ and with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ satisfies
>
> $$\mu(\{0\}) = 0.$$

We obviously have FMRiesz $\Rightarrow$ symmetric FMRiesz $\Rightarrow$ real correlativity, but at least one of the converse implications is false (Remark 6.5).

Lemma 3.62. *Let $T = m^2 + \rho$ and $t = n^2 + r$ with $0 \leq \rho < 2m+1$ and $0 \leq r < 2n+1$, $m, n \in \mathbb{N}$. Then*

$$\lim_{T\to\infty} \frac{1}{2T} \int_0^T e^{2\pi i r x}\ d\lambda(t) = \frac{1}{2}\mathbb{1}_{\{0\}}(x).$$

Proof. Let $x = 0$. Then

$$\lim_{T\to\infty} \frac{1}{2T} \int_0^T e^{2\pi irx}\ d\lambda(t) = \lim_{T\to\infty} \frac{1}{2T} \int_0^T \mathbb{1}\ d\lambda(t) = \lim_{T\to\infty} \frac{T}{2T} = \frac{1}{2}.$$

Let $x \neq 0$. Then

$$\left| \int_0^T e^{2\pi irx}\ d\lambda(t) \right| \leq \sum_{j=0}^{m-1} \left| \int_{j^2}^{(j+1)^2} e^{2\pi irx}\ d\lambda(t) \right| + \left| \int_{m^2}^{m^2+\rho} e^{2\pi irx}\ d\lambda(t) \right| \leq 2m + \rho,$$

hence,

$$\lim_{T\to\infty} \frac{1}{2T} \left| \int_0^T e^{2\pi irx}\ d\lambda(t) \right| \leq \lim_{T\to\infty} \frac{2m+\rho}{2m^2+\rho} = 0.$$

□

Proposition 3.63 (Cor $\Rightarrow$ FMRiesz)**.** *Let us assume that, given $f \in l_c^\infty(\mathbb{R}, \mathbb{T})$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

Then every probability measure on $\mathbb{R}$ with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ satisfies

$$\mu(\{0\}) = 0.$$

Compare [14, Theorem 1.8].

Proof. Let μ be a probability measure on $\mathbb{R}$ with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$. Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. random variables distributed by law μ. We define a new family of random variables $(Y_t)_{t\in\mathbb{R}}$ with values in $\mathbb{T}$ by

$$Y(t) := \begin{cases} e^{2\pi irX_m}, & t \geq 0, \\ 0, & t < 0, \end{cases}$$

where $t = m^2 + r$ with $m \in \mathbb{N}_0$ and $0 \leq r < 2m + 1$ (such a decomposition is unique).

We note that the paths $(Y_t(\omega))_{t\geq 0}$ are piecewise continuous for almost every $\omega \in \Omega$, hence, $\oint_{\mathbb{R}} Y_t \, d\lambda(t)$ is almost surely well-defined, and that Y_t and Y_s are independent if $t > s + 2m_t + 1$. In particular, given $T = m^2 + r > 0$ with $m \in \mathbb{N}_0$ and $0 \leq r < 2m + 1$, the random variables Y_s and Y_t are independent for $t, s \leq T$ if $|t - s| > 2\sqrt{T} + 1$.

For $Z_t := Y_t - \mathbb{E}[Y_t]$ with uniform bound $M = 2$, we hence have

$$\begin{aligned}
&\left| \sum_{0 \neq T = -\infty}^{\infty} \frac{1}{|T|} \mathbb{E}\left[\left| \frac{1}{|T|} \int_0^T Z_t \, d\lambda(t) \right|^2 \right] \right| \\
&= \left| \sum_{T=1}^{\infty} \frac{1}{T^3} \mathbb{E}\left[\left(\int_0^T Z_t \, d\lambda(t) \right) \overline{\left(\int_0^T Z_s \, d\lambda(s) \right)} \right] \right| \\
&= \left| \sum_{T=1}^{\infty} \frac{1}{T^3} \mathbb{E}\left[\int_0^T \int_0^T Z_t \overline{Z_s} \, d\lambda(t) \, d\lambda(s) \right] \right| \\
&\leq \sum_{T=1}^{\infty} \frac{1}{T^3} \int_0^T \int_0^T \left| \mathbb{E}\left[Z_t \overline{Z_s} \right] \right| \, d\lambda(t) \, d\lambda(s) \\
&\leq \sum_{T=1}^{\infty} \frac{1}{T^3} \cdot (2\sqrt{T} + 1) \cdot \sqrt{2} T \cdot M^2 < \infty
\end{aligned}$$

since $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ for independent random variables X, Y and $\mathbb{E}[Z_t] = 0$. Condition (2.6) in the strong law of large numbers 2.21 is hence satisfied. Then, almost surely, we have

$$\begin{aligned}
\oint_{\mathbb{R}} Y_t \, d\lambda(t) &\overset{2.21}{=} \oint_{\mathbb{R}} \mathbb{E}[Y_t] \, d\lambda(t) = \lim_{T \to \infty} \frac{1}{2T} \int_0^T \int_{\mathbb{R}} e^{2\pi i r x} \, d\mu(x) \, d\lambda(t) \\
&\overset{2.31}{=} \int_{\mathbb{R}} \lim_{T \to \infty} \frac{1}{2T} \int_0^T e^{2\pi i r x} \, d\lambda(t) \, d\mu(x) \overset{3.62}{=} \int_{\mathbb{R}} \frac{1}{2} \mathbb{1}_{\{0\}}(x) \, d\mu(x) = \frac{1}{2} \mu(\{0\}).
\end{aligned}$$

We note

$$Y(t+h)\overline{Y(t)} = e^{2\pi i (r+h) X_m} e^{-2\pi i r X_m} = e^{2\pi i h X_m} \tag{3.1}$$

for $0 \leq h + r < 2m + 1$ and $t + h > 0$. However, for fixed $h \in \mathbb{R}$, we have

$$\frac{\lambda(\{t = l^2 + r : 0 \leq h + r < 2l + 1\} \cap [-T, T])}{T} \overset{T \to \infty}{\longrightarrow} 1$$

For fixed $d \in \mathbb{R}$, we define $E_d := \{t = m^2 + r : 0 \leq d + r < 2m + 1\}$ and note $\oint_{\mathbb{R}} \mathbb{1}_{E_d^c} \, d\lambda(t) = 0$. Similarly as for $(Y_t - \mathbb{E}[Y_t])_{t \in \mathbb{R}}$, Condition (2.6) is satisfied for the family of random variables $(Y_{t+d}Y_t - \mathbb{E}[Y_{t+d}Y_t])_{t\in\mathbb{R}}$. We therefore have

$$\begin{aligned}
&\oint_{\mathbb{R}} Y(t+d)\overline{Y(t)}\ d\lambda(t) \overset{2.21}{=} \oint_{\mathbb{R}} \mathbb{E}[Y(t+d)\overline{Y(t)}]\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T} \oint_{[-T,T]\cap E_d} \mathbb{E}[Y(t+d)\overline{Y(t)}]\ d\lambda(t) \\
&\qquad + \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E_d^c} \mathbb{E}[Y(t+d)\overline{Y(t)}]\ d\lambda(t) \\
&\overset{(3.1)}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E_d} \mathbb{E}[e^{2\pi i d X_m}]\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{[-T,T]\cap E_d} \int_{\mathbb{R}} e^{2\pi i d x}\ d\mu(x)\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T} \int\limits_{[-T,T]\cap E_d} \overline{\widehat{\mu}(d)}\ d\lambda(t) = \frac{1}{2}\overline{\widehat{\mu}(d)}
\end{aligned}$$

almost surely for all $d \in \mathbb{R}$. Since $\widehat{\mu}$ is uniformly continuous (Remark 2.14) and since $\oint_{\mathbb{R}} Y(t)\ d\lambda(t) = \frac{1}{2}\mu(\{0\})$ exists almost surely as shown before, we obtain $Y \in l_c^\infty(\mathbb{R}, \mathbb{T})$ almost surely. Therefore,

$$\oint_{\mathbb{R}} Y(t+d)\overline{Y(t)}\ d\lambda(t) = \frac{1}{2}\overline{\widehat{\mu}(d)} = 0$$

for all $d \in \mathcal{D}$ almost surely implies

$$0 = 2 \cdot \oint_{\mathbb{R}} Y(t)\ d\lambda(t) = \mu(\{0\})$$

by correlativity, and $\mathcal{D}$ is hence FMRiesz. □

Remark 3.64. Similarly, we obtain the corresponding strong equivalence

enhanced van der Corput $\Leftrightarrow$ FC+.

3.2.6 Van der Corput

We now prove the equivalence of van der Corput and correlative sets. A major component is the Weyl equidistribution criterion Proposition 2.24.

Proposition 3.65 (Cor $\Rightarrow$ vdC)**.** *Let us assume that, given $f \in l_c^\infty(\mathbb{R},\mathbb{T})$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

Then, given $f \in l_{cc}^\infty(\mathbb{R},\mathbb{R})$, the equidistribution of $f(\cdot + d) - f(\cdot)$ mod 1 for all $d \in \mathcal{D}$ implies the equidistribution mod 1 of f.

Proof. Let $f \in l_{cc}^\infty(\mathbb{R},\mathbb{R})$ be given such that $f(\cdot + d) - f(\cdot)$ is equidistributed mod 1 for all $d \in \mathcal{D}$. Then, by the Weyl equidistribution criterion Proposition 2.24, this implies

$$0 = \oint_{\mathbb{R}} e^{2\pi i k(f(t+d)-f(t))}\ d\lambda(t) = \oint_{\mathbb{R}} e^{2\pi i k(f(t+d))} e^{-2\pi i k(f(t))}\ d\lambda(t)$$

for all $0 \neq k \in \mathbb{Z}$. Since $e^{2\pi i k(f(t))} \in l_c^\infty(\mathbb{R},\mathbb{T})$ for all $0 \neq k \in \mathbb{R}$ by assumption on f, correlativity implies

$$0 = \oint_{\mathbb{R}} e^{2\pi i k f(t)}\ d\lambda(t)$$

for all $0 \neq k \in \mathbb{Z}$, hence, the Weyl equidistribution criterion Proposition 2.24 implies the equidistribution mod 1 of f. $\square$

Let G be a locally compact abelian group (we refer to Section 7.3 for more details and references on (locally) compact abelian groups). Then there exists a (nonunique) rotation invariant mean L on $l^\infty(G)$, in particular on $l^\infty(\mathbb{R})$, which is given as a Banach limit extension of Cesàro averages (compare [62, Example 0.3-0.4]).

Given a compact abelian group K with its dual group K^*, we define a generalised equidistribution as follows. It is a straightforward observation that this definition coincides with Definition 2.24 when considering $K = \mathbb{T}$ (interpreting $\mathbb{T}$ as $[0,1]$).

Definition 3.66 (K-Equidistribution)**.** *Let K be a compact abelian group and L a rotation invariant mean given as Banach limit extension of Cesàro averages on*

$l^\infty_*(\mathbb{R})$. *Then* $f \in l^\infty_*(\mathbb{R}, K)$ *is* K***-equidistributed*** *if* $L(\chi \circ f) = \int_H \chi \, d\lambda = 0$ *holds for all* $e^* \neq \chi \in K^*$.

Definition 3.67 (K-van der Corput)**.** *Let a compact abelian group* K *be given. A set* $\mathcal{D} \subseteq \mathbb{R}$ *is* K***-van der Corput*** *(*K*-vdC) if, given* $f \in l^\infty_{cc}(\mathbb{R}, K)$*, the* K*-equidistribution of* $f(\cdot + d) - f(\cdot)$ *implies the* K*-equidistribution of* f.

Definition 3.68 (Generalised van der Corput)**.** *A set* $\mathcal{D} \subseteq \mathbb{R}$ *is* ***generalised van der Corput*** *(gvdC) if, given a compact abelian group* K *and* $f \in l^\infty_{cc}(\mathbb{R}, K)$*, the* K*-equidistribution of* $f(\cdot + d) - f(\cdot)$ *implies the* K*-equidistribution of* f.

Peres ([63, Theorem 4.2]) proved the following result for integers. We extend this result with a different approach to obtain a general van der Corput result for the corresponding real properties.

Theorem 3.69 (Peres)**.** *Let* $K = \mathbb{T} \oplus K_0$ *with a compact abelian metrisable group* K_0 *be given. Then the following are equivalent.*
(i) Given $(u_n)_{n\in\mathbb{N}} \subset K$*, the* K*-equidistribution of* $u_\cdot - u_{\cdot+d}$ *implies the* K*-equidistribution of* $(u_n)_{n\in\mathbb{N}}$.
(ii) Given $(u_n)_{n\in\mathbb{N}} \subset \mathbb{R}$*, the equidistribution mod 1 of* $u_\cdot - u_{\cdot+d}$ *implies the equidistribution mod 1 of* $(u_n)_{n\in\mathbb{N}}$.
In other words, a set $\mathcal{D} \subseteq \mathbb{N}$ *is van der Corput if and only if it is* K*-van der Corput.*

Proposition 3.70 (OR $\Rightarrow$ K-vdC)**.** *Let us assume that, given a Hilbert space* H*, a strongly continuous unitary group* $(T_t)_{t\in\mathbb{R}}$ *on* H *and* $x \in H$ *with* $\|x\| = 1$ *and* $Px \neq 0$*, there exists* $d \in \mathcal{D}$ *such that*

$$\langle T_d x, x \rangle \neq 0.$$

Then, given $f \in l^\infty_{cc}(\mathbb{R}, K)$*, the* K*-equidistribution of* $f(\cdot + d) - f(\cdot)$ *implies the* K*-equidistribution of* f.

Conpare [63, Theorem 3.3].

Proof. Using the scalar product induced by the Banach limit L on $l^\infty(\mathbb{R},\mathbb{R})$ (see Subsection 2.4), the K-equidistribution of $f(\cdot+d)-f(\cdot)$ rewrites as

$$L\left(\chi\circ(f(\cdot+d)-f(\cdot))\right) = L\left((\chi\circ(f(\cdot+d)))(\overline{\chi\circ f})\right) = \langle T_d(\chi\circ f), \chi\circ f\rangle$$

for all $d\in\mathcal{D}$ with the strongly continuous rotation group $(T_t)_{t\in\mathbb{R}}$ on H (Lemma 2.23). Operator recurrence hence implies $Px=0$. Since $\mathbb{1}\in\operatorname{Fix}((T_t)_{t\in\mathbb{R}})$ and $x\perp\operatorname{Fix}((T_t)_{t\in\mathbb{R}})$, we therefore have

$$0 = L\left(\mathbb{1}\cdot(\chi\circ f)\right) = L(\chi\circ f),$$

and f is K-equidistributed. □

Similarly, we obtain the following result.

Proposition 3.71 (OR ⇒ gvdC). *Let us assume that, given a Hilbert space H, a strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x\in H$ with $\|x\|=1$ and $Px\neq 0$, there exists $d\in\mathcal{D}$ such that*

$$\langle T_d x, x\rangle \neq 0.$$

Then, given a compact abelian group K and $f\in l^\infty_{cc}(\mathbb{R},K)$, the K-equidistribution of $f(\cdot+d)-f(\cdot)$ implies the K-equidistribution of f.

It is easy to see that any set $\mathcal{D}\subseteq\mathbb{R}$ is trivially K-van der Corput for the compact group $K=\{0\}$ since both $f\in l^\infty_{cc}(\mathbb{R},K)$ and $f(\cdot+d)-f(\cdot)$ for all $d\in\mathbb{R}$ are always K-equidistributed. We therefore exclude $K=\{0\}$ from the following considerations.

We first note the following lemmata which are helpful for the later integrations. They follow in the same way as the well-known half angle identities for sin and cos.

Lemma 3.72. *We have*

$$(\Re(\chi(y))^2 = \frac{1}{2}(\Re(\chi(2y)) + 1),$$
$$\Im(\chi(2y)) = 2\Im(\chi(y))\Re(\chi(y)).$$

Lemma 3.73. *Let $e^* \neq \chi, \chi^2 \in H^*$. Then*

$$\int_H \chi(h)\ d\lambda(h) = 0 = \int_H \chi(2h)\ d\lambda(h).$$

Proof. Using the rotation invariance of λ, we have

$$\chi(g) \int_H \chi(h)\ d\lambda(h) = \int_H \chi(h+g)\ d\lambda(h) = \int_H \chi(h)\ d\lambda(h)$$

for $\chi(g) \neq 1$. Hence, $\int_H \chi(h)\ d\lambda(h) = 0$. The same result holds for $\chi^2 \neq e^*$, i.e.

$$\int_H \chi(2h)\ d\lambda(h) = \int_H \chi^2(h)\ d\lambda(h) = 0$$

□

Proposition 3.53 shows that the use of $0 \leq f \in l_*^\infty(\mathbb{R}, \mathbb{R})$ and $f \in l_*^\infty(\mathbb{R}, \{0, 1\})$ yields the same real correlativity$_*$ sets. This indicates that we can extend the implication van der Corput $\Rightarrow$ correlativity to generalised van der Corput sets even with "small" groups such as $\mathbb{Z}_3$ or $\mathbb{Z}_5$.
However, we have to exclude some groups since $\chi^2 = e^*$ yields $\int_H \chi^2(h)\ d\lambda(h) = 1$ and hence

$$\mathbb{E}[X_t(\chi)] = \Re(f(t))$$

in the proof of Proposition 3.74. We therefore allow only compact abelian groups such that $\chi^2 \neq e^*$ for all $e^* \neq \chi \in K^*$. This includes for example $\mathbb{Z}_p$ for p prime and $\mathbb{T}$.
The same approach (for all compact abelian groups $K \neq \{0\}$) yields

K-van der Corput $\Rightarrow$ real correlativity.

Proposition 3.74 (K-vdC $\Rightarrow$ Cor). *Let $K \neq \{0\}$ be a compact abelian group such that $\chi^2 \neq e^*$ for all $e^* \neq \chi \in K^*$. Let us assume that, given $f \in l^\infty_{cc}(\mathbb{R}, K)$, the K-equidistribution of $f(\cdot + d) - f(\cdot)$ implies the K-equidistribution of f.*
Then, given $f \in l^\infty_c(\mathbb{R}, K)$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$, we have

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

Compare [71, Section 6].

Proof. Bounded K-van der Corput sets and correlative sets are characterised by having the limit point 0 (Propositions 4.24 and 4.25). Therefore, the equivalence of K-van der Corput and correlativity for a set $\mathcal{D}$ follows trivially if $\mathcal{D}$ has the limit point 0. Hence, we only have to consider the case when 0 is not a limit point of $\mathcal{D}$ and we therefore assume without loss of generality that $\mathcal{D} \cap (-\delta, \delta) = \emptyset$ for some $\delta > 0$.
Let $f = f_1 + i f_2 \in l^\infty_c(\mathbb{R}, \mathbb{T})$ satisfy $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$. For $e^* \neq \chi \in H^*$, let $(Y_t(\chi))_{t\in\mathbb{R}}$ be a family of random variables such that $Y_t(\chi)$ and $Y_s(\chi)$ are independent if $|t-s| \geq \delta$, with measurable paths as well as with values in H and distributed with law $\mathbb{1}(y) + \Re(f(t)\overline{\chi(y)})\ d\lambda(y)$. This is indeed a density function since $\Re(f(t)\overline{\chi(y)}) \geq -1$ and

$$\int_H \mathbb{1}(y) + \Re(f(t)\overline{\chi(y)})\ d\lambda(y) = \int_H \mathbb{1}(y)\ d\lambda(y) + \Re\left(f(t)\int_H \overline{\chi(y)}\ d\lambda(y)\right) = 1.$$

Let $(X_t(\chi) := \chi(Y_t(\chi)))_{t\in\mathbb{R}}$ be a family of random variable with values in $\mathbb{T}$ and measurable paths and where $X_t(\chi)$ and $X_s(\chi)$ are independent if $|t-s| \geq \delta$. Then

$$\begin{aligned}
&\mathbb{E}[X_t(\chi)] = \int_H \chi(y)(\mathbb{1}(y) + \Re(f(t)\overline{\chi(y)}))\ d\lambda(y) \\
&= \int_H \chi(y)\ d\lambda(y) \\
&\quad + \int_H (\Re(\chi(y)) + i\Im(\chi(y)))\,(\Re\,((f_1(t) + i f_2(t))(\Re(\chi(y)) - i\Im(\chi(y)))\ d\lambda(y)
\end{aligned}$$

$$
\begin{aligned}
&\overset{3.73}{=} f_1(t)\int_H (\Re(\chi(y)))^2\ d\lambda(y) + i f_1(t)\int_H \Im(\chi(y))\Re(\chi(y))\ d\lambda(y)\\
&\quad + f_2(t)\int_H \Im(\chi(y))\Re(\chi(y))\ d\lambda(y) + i f_2(t)\int_H (\Im(\chi(y)))^2\ d\lambda(y)\\
&\overset{3.72}{=} \frac{1}{2} f_1(t)\int_H (\Re(\chi(2y)) + \mathbb{1}(y))\, d\lambda(y) + \frac{1}{2} i f_1(t)\int_H \Im(\chi(2y))\ d\lambda(y)\\
&\quad + \frac{1}{2} f_2(t)\int_H \Im(\chi(2y))\ d\lambda(y) + i f_2(t)\int_H \left(\mathbb{1}(y) - (\Re(\chi(y)))^2\right)\ d\lambda(y)\\
&= \frac{1}{2} f_1(t)\left(\Re\left(\int_H (\chi(2y))\ d\lambda(y)\right) + \int_H \mathbb{1}(y)\ d\lambda(y)\right)\\
&\quad + \frac{1}{2} i f_1(t)\Im\left(\int_H \chi(2y)\ d\lambda(y)\right) + \frac{1}{2} f_2(t)\Im\left(\int_H \chi(2y)\ d\lambda(y)\right)\\
&\quad + i f_2(t)\left(\int_H \mathbb{1}(y) d\lambda(y) - \frac{1}{2}\int_H (\Re(\chi(2y)) + \mathbb{1}(y))\ d\lambda(y)\right)\\
&\overset{3.73}{=} \frac{1}{2}(f_1(t) + i f_2(t)) = \frac{1}{2} f(t)
\end{aligned}
$$

for all $t \in \mathbb{R}$ independently of $e^* \neq \chi \in H^*$. Condition (2.6) for $(X_t(\chi) - \mathbb{E}[X_t(\chi)])_{t\in\mathbb{R}}$ and $(X_{t+d}(\chi)\overline{X_t(\chi)} - \mathbb{E}[X_{t+d}(\chi)\overline{X_t(\chi)}])_{t\in\mathbb{R}}$ is satisfied similarly as in the proof of Proposition 3.63. The strong law of large numbers 2.21 hence implies

$$
\oint_{\mathbb{R}} X_t(\chi)\ d\lambda(t) = \oint_{\mathbb{R}} \mathbb{E}[X_t(\chi)]\ d\lambda(t) = \frac{1}{2}\oint_{\mathbb{R}} f(t)\ d\lambda(t)
$$

almost surely and independently of $e^* \neq \chi \in H^*$. We further have

$$
\begin{aligned}
\mathbb{E}[\oint X_{t+d}(\chi)\overline{X_t(\chi)}\ d\lambda(t)] &= \oint_{\mathbb{R}} \mathbb{E}[X_{t+d}(\chi)\overline{\chi(X_t(\chi))}]\ d\lambda(t)\\
= \oint_{\mathbb{R}} \mathbb{E}[X_{t+d}(\chi)]\overline{\mathbb{E}[X_t(\chi)]}\ d\lambda(t) &= \frac{1}{4}\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0
\end{aligned}
$$

for all $d \in \mathcal{D}$ and $j \neq 0$ and independently of $e^* \neq \chi \in H^*$ since $X_{t+d}(\chi)$ and $X_t(\chi)$ are independent. Hence,

$$
\begin{aligned}
\oint_{\mathbb{R}} X_{t+d}(\chi)\overline{X_t(\chi)}\ d\lambda(t) &\overset{2.21}{=} \oint_{\mathbb{R}} \mathbb{E}[X_{t+d}(\chi)\overline{X_t}(\chi)]\ d\lambda(t)\\
&\overset{2.31}{=} \mathbb{E}[\oint_{\mathbb{R}} X_{t+d}(\chi)\overline{X_t}(\chi)\ d\lambda(t)] = 0
\end{aligned}
$$

almost surely independently of $e^* \neq \chi \in H^*$ for all $d \in \mathcal{D}$. By the equidistribution criterion Definition 3.66, $X(\cdot + d) - X(\cdot)$ is almost surely equidistributed mod 1 and

$X(\cdot) \in l^\infty_{cc}(\mathbb{R}, \mathbb{R})$ almost surely by assumption on $f \in l^\infty_c(\mathbb{R}, K)$, hence, Property K-van der Corput implies

$$0 = \oint_{\mathbb{R}} X_t(\chi)\ d\lambda(t) = \frac{1}{2} \cdot \oint_{\mathbb{R}} f(t)\ d\lambda(t).$$

□

The next result is a special case of Proposition 3.74 with $K = \mathbb{T}$.

Proposition 3.75 (vdC $\Rightarrow$ Cor)**.** *Let us assume that, given $f \in l^\infty_{cc}(\mathbb{R}, [0,1])$, the equidistribution mod 1 of $f(\cdot + d) - f(\cdot)$ for all $d \in \mathcal{D}$ implies the equidistribution mod 1 of f.*
Then, given $f \in l^\infty_c(\mathbb{R}, \mathbb{T})$, $\oint_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$ implies

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

We clearly have the implication

$$\text{generalised van der Corput} \Rightarrow \text{van der Corput}$$

since $\mathbb{T}$ is a compact abelian group. Therefore, using Propositions 3.75, 3.71 and 3.74 as well as the equivalence of correlativity and operator recurrence, we obtain the following result.

Proposition 3.76. *Let $\mathcal{D} \subseteq \mathbb{R}$ and let a compact abelian group $K \neq \{0\}$ be given such that $\chi^2 \neq e^*$ for all $e^* \neq \chi \in K^*$. The following are equivalent.*
(i) $\mathcal{D}$ is operator recurrent.
(ii) $\mathcal{D}$ is van der Corput.
(iii) $\mathcal{D}$ is K-van der Corput.
(iv) $\mathcal{D}$ is generalised van der Corput.

Remark 3.77. Let $\mathcal{D} \subseteq \mathbb{R}$. Similarly, given $K \neq \{0\}$ such that $\chi^2 \neq e^*$ for all

$e^* \neq \chi \in K^*$, we obtain some implications without continuity, i.e. we have

$$\begin{aligned} \text{operator recurrence}_* &\Rightarrow \text{generalised van der Corput}_* \\ &\Rightarrow \text{K-van der Corput}_* \\ &\Rightarrow \text{correlativity}_*. \end{aligned}$$

However, it is not clear if the implication correlativity$_*$ $\Rightarrow$ operator recurrence$_*$ holds as we cannot use FMRiesz due to its natural continuity property.

3.2.7 Fürstenberg Correspondence Principle

In this subsection, we discuss the Fürstenberg correspondence principle and prove its variant for real systems. Fürstenberg used this correspondence principle to translate Szemeredi's theorem about arbitrarily long arithmetic progressions ([79]) into the language of ergodic theory and to prove it by using a multiple recurrence variant of Poincaré's theorem ([37]). This principle is used to prove number theoretical results in a similar way by using recurrence results, e.g. that Poincaré recurrence implies combinatorial recurrence.

The converse, i.e. that combinatorial recurrence and intersectivity imply Poincaré recurrence for $\mathbb{Z}$ was shown by Bertrand-Mathis ([17]) and Bergelson ([7, Theorem 1.5], [6, Section 2]). However, their proofs rely on countability, hence, they cannot be adapted for $\mathbb{R}$, but, without continuity assumptions, combinatorial recurrence implies Poincaré recurrence by using real correlativity (see Remark 3.82).

Definition 3.78 (Net)**.** *Let I be a directed set, then $(a_i)_{i\in I}$ is called a* ***net****. A net $(a_i)_{i\in I}$ with values in a topological space $(X, \mathcal{T})$ is called* ***convergent*** *to a if for all $U \in \mathcal{T}$ with $a \in U$ there exists $\beta \in I$ such that for all $\alpha > \beta$, we have $a_\alpha \in U$. We denote the limit a of a converging net $(a_j)_{j\in J}$ by $a = \lim_J a_j$.*

Let I, J be directed sets. Then $(b_j)_{j\in J}$ is a ***subnet*** *of $(a_i)_{i\in I}$ if there exists a monotone function $h : J \to I$ such that $\forall i \in I\ \exists j \in J : i \leq h(j)$ and $b_j = a_{h(j)}$.*

The following proposition is a Bolzano-Weierstraß type result ([58, Section 3.3]).

Proposition 3.79. *A space X is compact if and only if every net with values in X has a convergent subnet.*

We refer to [58] for more details on nets.

Lemma 3.80. *Let a compact Hausdorff space X, $x_* \in X$, a group action of continuous maps $(\phi_t)_{t\in\mathbb{R}}$ on X and a sequence $(T_S)_{S\in\mathbb{N}} \subset \mathbb{R}$ with $T_S \overset{S\to\infty}{\longrightarrow} \infty$ be given. For $T > 0$ and open $A \subseteq X$, let μ_T be the measure defined on X by*

$$\mu_T(A) = \frac{1}{2T}\int_{-T}^{T} \delta_{\phi_t(x_*)}(A)\ d\lambda(t) = \frac{1}{2T}\int_{-T}^{T} \mathbb{1}_A(\phi_t(x_*))\ d\lambda(t)$$

with corresponding integral

$$\int_X f d\mu_T = \frac{1}{2T}\int_{-T}^{T}\left(\int_X f\ d\delta_{\phi_t(x_*)}\right)\ d\lambda(t).$$

Then there exists a weak-cluster point μ of $(\mu_{T_S})_{S\in\mathbb{N}}$ and every weak* cluster point is invariant under $(\phi_t)_{t\in\mathbb{R}}$.*

Compare [29, Theorem 4.1].

Proof. We first note that $\mu_T(\mathbb{R}) = 1$ for all $T \in \mathbb{R}$, hence, by weak*-compactness of $\mathcal{M}_1$ (as consequence of the Banach-Alaoglu theorem ([68, Theorem 3.15])) there exists a subnet $(\mu_j)_{j\in J}$ and a weak*-limit μ such that $\lim_J \mu_j = \mu$.
Let $\gamma \in \mathbb{R}$ and $f \in C(X)$. Without loss of generality, let $\gamma \geq 0$ to avoid extra cases in (+). We have

$$\left|\int_X (f\circ\phi_\gamma)(x)\ d\mu_T(x) - \int_X f(x)\ d\mu_T(x)\right| = \left|\int_X (f\circ\phi_\gamma - f)(x)\ d\mu_T(x)\right|$$
$$= \left|\frac{1}{2T}\int_{-T}^{T}\int_X (f\circ\phi_\gamma - f)(x)\ d\delta_{\phi_t(x_*)}(x)\ d\lambda(t)\right|$$

$$
\begin{aligned}
&= \left| \frac{1}{2T} \int_{-T}^{T} \int_X (f \circ \phi_{\gamma+t} - f \circ \phi_t)(x)\ d\delta_{x_*}(x)\ d\lambda(t) \right| \\
&= \left| \frac{1}{2T} \int_X \int_{-T}^{T} (f \circ \phi_{\gamma+t} - f \circ \phi_t)(x)\ d\lambda(t)\ d\delta_{x_*}(x) \right| \\
&= \left| \frac{1}{2T} \int_X \left(\int_{-T}^{T} (f \circ \phi_{\gamma+t})(x)\ d\lambda(t) - \int_{-T}^{T} (f \circ \phi_t)(x)\ d\lambda(t) \right)\ d\delta_{x_*}(x) \right| \\
&= \left| \frac{1}{2T} \int_X \left(\int_{-T}^{T} (f \circ \phi_{\gamma+t})(x)\ d\lambda(t) - \int_{-T-\gamma}^{T-\gamma} (f \circ \phi_{\gamma+t})(x)\ d\lambda(t) \right)\ d\delta_{x_*}(x) \right| \\
&\overset{(\pm)}{=} \left| \frac{1}{2T} \int_X (\int_{-T}^{T-\gamma} (f \circ \phi_{\gamma+t})(x)\ d\lambda(t) - \int_{-T}^{T-\gamma} (f \circ \phi_{\gamma+t})(x)\ d\lambda(t) \right. \\
&\qquad \left. + \int_{T-\gamma}^{T} (f \circ \phi_{\gamma+t})(x)\ d\lambda(t) - \int_{-T-\gamma}^{-T} (f \circ \phi_{\gamma+t})(x)\ d\lambda(t))\ d\delta_{x_*}(x) \right| \\
&\leq \frac{1}{2T} \int_X 2\gamma \|f\|_\infty\ d\delta_{x_*}(x) = \frac{\gamma \|f\|_\infty}{T} \overset{T\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

so in particular, by uniqueness of limits, we have

$$
\begin{aligned}
&\left| \int_X (f \circ \phi_\gamma)(x)\ d\mu(x) - \int_X f(x)\ d\mu(x) \right| \\
&= \lim_J \left| \int_X (f \circ \phi_\gamma)(x)\ d\mu_j(x) - \int_X f(x)\ d\mu_j(x) \right| = 0
\end{aligned}
$$

for any weak* limit μ of $(\mu_{T_S})_{S\in\mathbb{N}}$, and Lemma 2.4 implies that μ is $(\phi_t)_{t\in\mathbb{R}}$ invariant.

□

Proposition 3.81 (Fürstenberg Correspondence Principle)**.** *Let $E \subseteq \mathbb{R}$ with $\overline{d}_{\mathbb{R}}(E) > 0$ be given. Then there exists a mps $(\Omega, \Sigma, \mu; (\tau_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ such that*

$$\mu(A) = \overline{d}_{\mathbb{R}}(E)$$

and

$$\mu(A \cap \tau_d(A)) \leq \overline{d}_{\mathbb{R}}(E \cap (E - d))$$

for all $d \in \mathbb{R}$.

Compare [29, Section 7.3] (see also [82, section 6] for a similar approach for $\mathbb{R}^2$).

Proof. We define $X_0 := \{0,1\}^{\mathbb{R}}$ which is a compact Hausdorff space by Tychonoff's theorem ([68, Theorem A3]). We define $\tilde{\tau}_s((x_t)_{t\in\mathbb{R}}) := (x_{t+s})_{t\in\mathbb{R}}$ as a shift on X_0. We define $x_E := \mathbb{1}_E$ and $\Omega := X := \overline{\{\tilde{\tau}_t(x_E) : t \in \mathbb{R}\}}$ with the σ-algebra Σ generated by the open sets in the subspace topology of X and $\tau_t := \tilde{\tau}_t|_X$.
We set $A := \{x \in X : x(0) = 1\} = X \cap \{x \in X_0 : x(0) = 1\}$ which is clopen and we note

$$\tau_t(\mathbb{1}_E) = \tau_t(x_E) \in A \Longleftrightarrow \mathbb{1}_E(t) = 1 \Longleftrightarrow t \in E.$$

Let $(T_S)_{S\in\mathbb{N}}$ be a sequence with $T_S \overset{S\to\infty}{\longrightarrow} \infty$ such that

$$\frac{1}{2T_S}\int_{-T_S}^{T_S} \mathbb{1}_E(t)\ d\lambda(t) \longrightarrow \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{1}_E(t)\ d\lambda(t) = \overline{d}_{\mathbb{R}}(E).$$

We define

$$\mu_{T_S}(\cdot) := \frac{1}{2T_S}\int_{-T_S}^{T_S} \delta_{\tau_t(x_E)}(\cdot)\ d\lambda(t).$$

Lemma 3.80 implies that there exists a weak*-cluster point μ of $(\mu_{T_S})_{S\in\mathbb{N}}$ with corresponding subnet $(\mu_j)_{j\in J}$ of $(\mu_{T_S})_{S\in\mathbb{N}}$ which is invariant under the mps $(\Omega, \Sigma, \mu; (\tau_t)_{t\in\mathbb{R}})$ and which satisfies

$$\begin{aligned}\mu(A) &= \lim_J \mu_j(A) = \lim_{S\to\infty} \mu_{T_S}(A) = \lim_{S\to\infty} \frac{1}{2T_S}\int_{-T_S}^{T_S} \delta_{\tau_t(x_E)}(A)\ d\lambda(t)\\ &= \lim_{S\to\infty} \frac{1}{2T_S}\int_{-T_S}^{T_S} \mathbb{1}_E(t)\ d\lambda(t) = \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{1}_E(t)\ d\lambda(t) = \overline{d}(E)\end{aligned}$$

as the limit $\lim_{S\to\infty} \mu_{T_S}(A)$ exists by construction. Moreover,

$$\begin{aligned}\mu(A \cap (\tau_d(A)) &= \lim_J \mu_j(A \cap (\tau_d(A)) \leq \limsup_{S\to\infty} \mu_{T_S}(A \cap (\tau_d(A))\\ &\leq \limsup_{T\to\infty} \mu_T(A \cap (\tau_d(A)) = \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \delta_{\tau_t(x_E)}(A \cap \tau_d(A))\ d\lambda(t)\\ &= \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{1}_E(t)\mathbb{1}_{E-d}(t)\ d\lambda(t) = \overline{d}_{\mathbb{R}}(E \cap (E-d))\end{aligned}$$

holds for all $d \in \mathbb{R}$. □

Remark 3.82. The Fürstenberg correspondence principle is used in the classical setting of $\mathbb{N}$ to show that every Poincaré set is also combinatorially recurrent.
For real recurrence sets, however, we would additionally require strong continuity of the system obtained in Proposition 3.81 to show that Poincaré recurrence implies combinatorial recurrence. Bergelson, Boshernitzan and Bourgain ([11]) showed that the Fürstenberg correspondence principle indeed fails for $\mathbb{R}$ in general (see Theorem 4.27 and Proposition 4.33).
This failure comes from the fact that combinatorial recurrence is a property without continuity assumptions, and merely a combinatorial property. If we drop the continuity assumptions in our framework, then the Fürstenberg correspondence principle holds again (see, e.g. [82, Proposition 6.6] and Remark 3.90). Using also other results from this chapter, this then yields the equivalence of Poincaré recurrence, combinatorial recurrence and real correlativity as in the classical setting.
On the other hand, one can impose a continuity assumption on intersectivity and combinatorial recurrence, and the Fürstenberg correspondence principle then yields the implication Poincaré recurrence $\Rightarrow$ continuous combinatorial recurrence (see Subsection 3.2.8).

The following result has been suggested by [11].

Proposition 3.83. *Let $E \subseteq \mathbb{R}$ in Proposition 3.81 satisfy $\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t)\triangle E) = 0$. Then $(\mathbb{R}, E)$ admits the Fürstenberg correspondence principle, i.e. the mps $(\Omega, \Sigma, \mu; (\tau_t)_{t\in\mathbb{R}})$ from Proposition 3.81 is strongly continuous.*

Proof. The open sets $A = \{x \in X : x(0) = 1\}$ and $A^c = \{x \in X : x(0) = 0\}$ with their translates generate the topology on X since

$$X \cap \{x \in \{0,1\}^{\mathbb{R}} : x(t_1) = i_1, \dots, x(t_n) = i_n\} = \tau_{t_1}(B_1) \cap \dots \cap \tau_{t_n}(B_n)$$

for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in \mathbb{R}$, $i_1, \dots, i_n \in \{0,1\}$ and $B_1, \dots, B_n \in \{A, A^c\}$. We further note $A \triangle \phi_t(A) = A^c \triangle (\phi_t(A))^c$. Hence, it is sufficient to check strong continuity on A. We have

$$\begin{aligned}\mu(\tau_t(A) \triangle A) &= \lim_J \mu_j(\tau_t(A) \triangle A) \le \limsup_{S\to\infty} \mu_{T_S}(\tau_t(A) \triangle A) \\ &\le \limsup_{T\to\infty} \mu_T(\tau_t(A) \triangle A) = \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \delta_{\tau_t(x_E)}(\tau_t(A) \triangle A)\ d\lambda(t) \\ &= \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(\mathbb{1}_{E+t}(t) + \mathbb{1}_E(t) - \mathbb{1}_{E+t}(t)\mathbb{1}_E(t)\right)\ d\lambda(t) \\ &= \overline{d}_{\mathbb{R}}((E+t) \triangle E) \overset{t\to 0}{\longrightarrow} 0\end{aligned}$$

and $(\Omega, \Sigma, \mu; (\tau_t)_{t\in\mathbb{R}})$ from Proposition 3.81 is therefore strongly continuous. □

3.2.8 Intersectivity and Combinatorial Recurrence

Proposition 3.83 allows a continuous version of intersectivity and combinatorial recurrence. This yields the implication Poincaré recurrence ⇒ continuous combinatorial recurrence. We further connect combinatorially recurrent sets with real correlative sets.

Definition 3.84 (Continuous Intersectivity). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **continuously intersective** (cIS) if, given $E \subseteq \mathbb{R}$ with $\overline{d}_{\mathbb{R}}(E) > 0$ and $\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t) \triangle E) = 0$ such that $d_{\mathbb{R}}(E \cap (E-d))$ exists for all $d \in \mathbb{R}$, we have*

$$(E - E) \cap \mathcal{D} \neq \emptyset.$$

Definition 3.85 (Continuous Combinatorial recurrence). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **continuously combinatorially recurrent** (cCR) if, given $E \subseteq \mathbb{R}$ with $\overline{d}_{\mathbb{R}}(E) > 0$ and $\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t) \triangle E) = 0$ such that $d_{\mathbb{R}}(E \cap (E-d))$ exists for all $d \in \mathbb{R}$, there exists $d \in \mathcal{D}$ such that*

$$\overline{d}_{\mathbb{R}}((E-d) \cap E) > 0.$$

Definition 3.86 (Continuous Strong Combinatorial Recurrence). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **continuously strongly combinatorially recurrent** (cSCR) if, given $E \subseteq \mathbb{R}$*

with $\overline{d}_{\mathbb{R}}(E) > 0$ *and* $\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t) \triangle E) = 0$ *such that* $d_{\mathbb{R}}(E \cap (E-d))$ *exists for all* $d \in \mathbb{R}$*, we have*

$$\limsup_{|d|\to\infty, d\in\mathcal{D}} \overline{d}_{\mathbb{R}}((E-d) \cap E) > 0.$$

Remark 3.87. We need the additional requirement that $d_{\mathbb{R}}(E\cap(E-d))$ exists for all $d \in \mathbb{R}$ to connect combinatorial recurrence with real correlativity (Proposition 3.89).

We also note that it is equivalent to consider $d_{\mathbb{R}}(E)$ instead of $\overline{d}_{\mathbb{R}}(E)$ and $d_{\mathbb{R}}((E-d) \cap E) > 0$ instead of $\overline{d}_{\mathbb{R}}((E-d) \cap E) > 0$ in Definitions 3.84, 3.85 and 3.86.

We obviously have

$$\begin{aligned}
\text{intersectivity} &\Rightarrow \text{continuous intersectivity},\\
\text{combinatorial recurrence} &\Rightarrow \text{continuous combinatorial recurrence}\\
&\Rightarrow \text{continuous intersectivity},\\
\text{strong combinatorial recurrence} &\Rightarrow \text{continuous strong combinatorial recurrence}.
\end{aligned}$$

But the Fürstenberg correspondence principle yields more.

Proposition 3.88 (P $\Rightarrow$ cCR)**.** *Let us assume that, given a scmps* $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *and* $A \in \Sigma$ *with* $\mu(A) > 0$*, there exists* $d \in \mathcal{D}$ *such that*

$$\mu(A \cap \phi_d(A)) > 0.$$

Then, given $E \subseteq \mathbb{R}$ *with* $\overline{d}_{\mathbb{R}}(E) > 0$ *and* $\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t) \triangle E) = 0$ *such that* $d_{\mathbb{R}}(E \cap (E-d))$ *exists for all* $d \in \mathbb{R}$*, there exists* $d \in \mathcal{D}$ *such that*

$$\overline{d}_{\mathbb{R}}((E-d) \cap E) > 0.$$

Proof. Let $E \subseteq \mathbb{R}$ with $\overline{d}_{\mathbb{R}}(E) > 0$ and $\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t) \triangle E) = 0$ be given. Proposition 3.81 yields a mps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ with $\mu(A) = \overline{d}_{\mathbb{R}}(E)$ and $\mu(A \cap$

$\tau_d(A)) \le \overline{d}_{\mathbb{R}}(E \cap (E-d))$ for all $d \in \mathbb{R}$. The mps is also strongly continuous by Proposition 3.83. Since $\mathcal{D}$ is Poincaré, there exists $d \in \mathcal{D}$ such that

$$0 < \mu(A \cap \tau_d(A)) \le \overline{d}_{\mathbb{R}}(E \cap (E-d))$$

and $\mathcal{D}$ is hence continuously combinatorially recurrent. □

Proposition 3.89 (cCR $\Leftrightarrow$ rCor^{0,1})**.** *The following are equivalent.*

(i) Given $E \subseteq \mathbb{R}$ *with* $\overline{d}_{\mathbb{R}}(E) > 0$ *and* $\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t) \triangle E) = 0$ *such that* $d_{\mathbb{R}}(E \cap (E-d))$ *exists for all* $d \in \mathbb{R}$*, there exists* $d \in \mathcal{D}$ *such that*

$$\overline{d}_{\mathbb{R}}((E-d) \cap E) > 0.$$

(ii) Given $f \in l_c^\infty(\mathbb{R}, \{0,1\})$*,* $\oint_{\mathbb{R}} f(t+d) f(t)\ d\lambda(t) = 0$ *for all* $d \in \mathcal{D}$ *implies*

$$\oint_{\mathbb{R}} f(t)\ d\lambda(t) = 0.$$

Proof. Let $f = \mathbb{1}_E \in l_c^\infty(\mathbb{R}, \{0,1\})$ be given such that $\oint_{\mathbb{R}} f(t+d) f(t)\ d\lambda(t) = 0$ for all $d \in \mathcal{D}$. Then

$$0 = \oint_{\mathbb{R}} f(t+d) f(t)\ d\lambda(t) = \oint_{\mathbb{R}} \mathbb{1}_{E-d}(t) \mathbb{1}_E(t)\ d\lambda(t) = d_{\mathbb{R}}((E-d) \cap E)$$

for all $d \in \mathcal{D}$. Since $f \in l_c^\infty(\mathbb{R}, \{0,1\})$, $d_{\mathbb{R}}(E \cap (E-d)) = \oint_{\mathbb{R}} f(t+d) f(t)\ d\lambda(t)$ exists for all $d \in \mathbb{R}$ and we have

$$\begin{aligned}
\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t) \triangle E) &= \lim_{t\to 0} \limsup_{T\to\infty} \oint_{-T}^{T} (\mathbb{1}_{E+t}(s) + \mathbb{1}_E(s) - 2\mathbb{1}_{E+t}(s)\mathbb{1}_E(s))\ d\lambda(s) \\
&= \lim_{t\to 0} \limsup_{T\to\infty} \oint_{-T}^{T} (f(s-t) + f(s) - 2f(s-t)f(s))\ d\lambda(s) \\
&= 2 \lim_{t\to 0} \lim_{T\to\infty} \oint_{-T}^{T} f(s)\ d\lambda(s) - 2 \lim_{t\to 0} \lim_{T\to\infty} \oint_{-T}^{T} f(s-t) f(s)\ d\lambda(s) \\
&= 2 \oint_{\mathbb{R}} f(s)\ d\lambda(s) - 2 \lim_{t\to 0} \oint_{\mathbb{R}} f(s-t) f(s)\ d\lambda(t) = 0.
\end{aligned}$$

Continuous combinatorial recurrence then implies

$$0 = \overline{d}_{\mathbb{R}}(E) = d_{\mathbb{R}}(E) = \oint_{\mathbb{R}} \mathbb{1}_E(t)\ d\lambda(t) = \oint_{\mathbb{R}} f(t)\ d\lambda(t).$$

On the other hand, let $E \subseteq \mathbb{R}$ be given such that

$$\begin{aligned} 0 &= \overline{d}_{\mathbb{R}}(E \cap (E-d)) = d_{\mathbb{R}}(E \cap (E-d)) \\ &= \oint_{\mathbb{R}} \mathbb{1}_{E\cap(E-d)}(t)\ d\lambda(t) = \oint_{\mathbb{R}} \mathbb{1}_E(t+d)\mathbb{1}_A(t)\ d\lambda(t). \end{aligned}$$

Since $d_{\mathbb{R}}(E \cap (E-d))$ exists for all $d \in \mathbb{R}$ and $\lim_{t\to 0} \overline{d}_{\mathbb{R}}((E+t) \triangle E) = 0$, property $\mathrm{rCor}^{0,1}$ then implies

$$0 = \oint_{\mathbb{R}} \mathbb{1}_A(t)\ d\lambda(t) = d_{\mathbb{R}}(E) = \overline{d}_{\mathbb{R}}(E).$$

□

Remark 3.90. Similarly, we obtain the same implications for the properties without continuity assumptions, i.e.

$$\begin{aligned} \text{Poincaré}_* &\Rightarrow \text{combinatorial recurrence}, \\ \text{strong recurrence} &\Rightarrow \text{continuous strong combinatorial recurrence}, \\ \text{strong recurrence}_* &\Rightarrow \text{strong combinatorial recurrence}, \\ \text{combinatorial recurrence} &\Leftrightarrow \text{real correlativity}_*^{0,1}, \\ \text{strong correlativity}^{0,1} &\Leftrightarrow \text{continuous strong combinatorial recurrence}, \\ \text{strong correlativity}_*^{0,1} &\Leftrightarrow \text{strong combinatorial recurrence}. \end{aligned}$$

Chapter 4

Examples

In this chapter, we present examples of recurrence sets. We first start with rather trivial examples like $\mathbb{N}$, $\mathbb{R}\backslash(-\epsilon, \epsilon)$ or the characterisation of bounded recurrence sets. We continue with a rationally linearly independent sequence which gives a first nontrivial example. The main part of this chapter is the equivalence of real and integer recurrence for many properties, and these results yield many more nontrivial examples by using classical integer recurrence sets. We finally discuss the sets $\{\log p, p \text{ prime}\}$ and $\{\frac{1}{\log p}, p \text{ prime}\}$ which are proper real recurrence sets and which do not come directly from integer sets.

4.1 First Examples and Classical Theorems

We consider standard examples in this section, in particular, we discuss the sets $\mathbb{R}\backslash(-\epsilon, \epsilon)$, $\mathbb{N}$ as well as bounded recurrence sets. The results for $\mathbb{N}$ are often classical theorems from the early 20th century such as the van der Corput theorem. Some of the results in this section follow from each other by using the characterisations from Section 3.2, but they are interesting enough on their own to be treated independently.

4.1.1 FMRiesz

Wiener's Lemma ([44, Corollary 1.7.13]) yields a deep connection of the Fourier transform $\widehat{\mu}$ of a complex measure μ with the atoms of μ.

Theorem 4.1 (Wiener's Lemma)**.** *Let μ be a complex measure on $\mathbb{T}$. Then*

$$\sum_{\tau\in\mathbb{T}} |\mu(\{\tau\})|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\widehat{\mu}(n)|^2$$

The following proposition hence can also be obtained as a corollary of Theorem 4.1. We note that it is not essential for the proof of Proposition 4.2 that μ is positive or a probability measure, but we require these assumptions to connect FMRiesz with operator recurrence and correlativity.

Proposition 4.2. *Let μ be a probability measure on $\mathbb{T}$ such that*

$$\widehat{\mu}(n) = \int_{\mathbb{T}} x^{-n} \, d\mu(x) = 0$$

for all $n \in \mathbb{N}$. Then

$$\mu(\{1\}) = 0.$$

In particular, $\mathbb{N}$ is $\mathbb{Z}$-FMRiesz.

Proof. Let $x = 1$. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} x^{-j} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} 1 = \lim_{n\to\infty} \frac{n}{n} = 1.$$

Let $x = e^{2\pi i t} \in \mathbb{T}\backslash\{1\}$. Then

$$\left|\sum_{j=0}^{n-1} (x)^{-j}\right| = \left|\sum_{j=0}^{n-1} \left(e^{2\pi i t}\right)^{-j}\right| = \left|\frac{1-e^{-2\pi i t n}}{1-e^{-2\pi i t}}\right| \leq \frac{2}{|1-e^{-2\pi i t}|}$$

for all $n \in \mathbb{N}$, hence,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} x^{-j} = \mathbb{1}_{\{1\}}(x). \tag{4.1}$$

Then we have

$$\mu(\{1\}) = \int_{\mathbb{T}} \mathbb{1}_{\{1\}}(x)\ d\mu(x) \overset{(4.1)}{=} \int_{\mathbb{T}} \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} x^{-j}\ d\mu(x)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_{\mathbb{T}} x^{-j}\ d\mu(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \widehat{\mu}(j) = 0.$$

□

Proposition 4.3. *Let an infinite set $A \subseteq \mathbb{R}$ be given. Then every probability measure μ on $\mathbb{R}$ with $\widehat{\mu}(a-b) = 0$ for all $a > b \in A$ satisfies*

$$\mu(\{0\}) = 0.$$

In other words, $\mathcal{D} := \{a - b : a > b \in A\}$ is $\mathbb{R}$-FMRiesz.

Compare [14, Corollary 1.13].

Proof. Let μ be a probability measure on $\mathbb{R}$ such that $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ and we note $\widehat{\mu}(-d) = \overline{\widehat{\mu}(d)} = 0$ for all $d \in \mathcal{D}$. We set

$$\tilde{A} := \{f_a := e^{2\pi i a x}, a \in A\} \subset L^2(\mathbb{R}, \mu)$$

which satisfies

$$\|f_a\|^2 = \int_{\mathbb{R}} |e^{2\pi i a x}|^2\ d\mu(x) = \int_{\mathbb{R}} \mathbb{1}(x)\ d\mu(x) = 1$$

for all $a \in A$ and

$$\langle f_{a_1}, f_{a_2} \rangle = \int_{\mathbb{R}} e^{2\pi i a_1 x} e^{-2\pi i a_2 x}\ d\mu(x) = \int_{\mathbb{R}} e^{-2\pi i (a_2 - a_1) x}\ d\mu(x) = \widehat{\mu}(a_2 - a_1) = 0$$

for all $a_1 \neq a_2 \in A$. $\tilde{A}$ hence forms an orthonormal family in $L^2(\mathbb{R}, \mu)$. Now let $J \subset A$ be a finite subset with $|J|$ elements. Then we have

$$|J|^2 \mu(\{0\}) = \mu(\{0\}) \left| \sum_{j \in J} e^{2\pi i j 0} \right|^2 \leq \int_{\mathbb{R}} \left| \sum_{j \in J} e^{2\pi i j x} \right|^2\ d\mu(x)$$
$$= \int_{\mathbb{R}} \sum_{j \in J} e^{2\pi i j x} \sum_{l \in J} e^{-2\pi i l x}\ d\mu(x) = \sum_{j \in J} \sum_{l \in J} \int_{\mathbb{R}} e^{2\pi i j x} e^{-2\pi i l x}\ d\mu(x)$$
$$= \sum_{j \in J} \sum_{l \in J} \langle f_j, f_l \rangle = |J|.$$

Dividing by $|J|^2$ and letting $|J| \to \infty$, we obtain $\mu(\{0\}) = 0$, and $\mathcal{D}$ is hence $\mathbb{R}$-FMRiesz. $\square$

Proposition 4.4. *Let $\epsilon > 0$ be given and let μ be a probability measure on $\mathbb{R}$ such that $\widehat{\mu}(t) = 0$ for all $|t| \geq \epsilon$. Then*

$$\mu(\{0\}) = 0.$$

In other words, $\mathbb{R}\backslash(\epsilon, \epsilon)$ is $\mathbb{R}$-FMRiesz.

Proof. The Fourier transform $\widehat{\mu}$ is uniformly continuous (Remark 2.14) and in particular bounded on $[-\epsilon, \epsilon]$, hence $\|\widehat{\mu}\|_\infty < \infty$. We conclude

$$|\mu(\{0\})| \overset{2.17}{=} \left| \oint_{\mathbb{R}} \widehat{\mu}(t)\ d\lambda(t) \right| = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{\epsilon}^{\epsilon} \widehat{\mu}(t)\ d\lambda(t) \right| \leq \lim_{T\to\infty} \frac{1}{2T} \cdot 2\epsilon \cdot \|\widehat{\mu}\|_\infty = 0.$$

$\square$

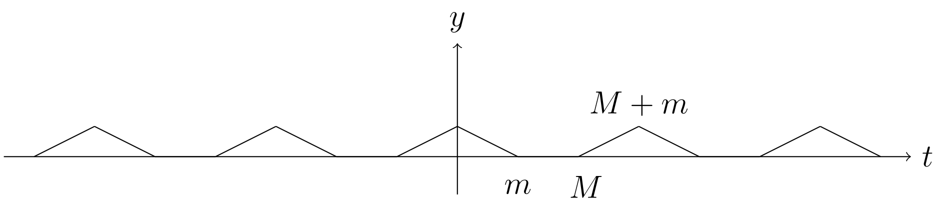
Proposition 4.5. *A set $\mathcal{D} \subset \mathbb{R}$ is boundedly $\mathbb{R}$-FMRiesz if and only if 0 is a limit point of $\mathcal{D}$.*

Proof. Let 0 be a limit point of $\mathcal{D}$, so there exists a sequence $(d_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ with $d_n \to 0$. Let us assume that there exists a probability measure μ on $\mathbb{R}$ such that $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$. Then by continuity of $\widehat{\mu}$ and Remark 2.14, we have

$$0 = \lim_{n\to\infty} \widehat{\mu}(d_n) = \widehat{\mu}(\lim_{n\to\infty} d_n) = \widehat{\mu}(0) \overset{2.14}{=} \mu(\mathbb{R}) = 1,$$

yielding a contradiction. The set $\mathcal{D}$ is therefore trivially $\mathbb{R}$-FMRiesz as there exists no such probability measure.

If 0 is not a limit point of $\mathcal{D}$, then $\mathcal{D} \cap (-m, m) = \emptyset$ for some $m > 0$ and we can assume that $\mathcal{D} \subseteq [-M, -m] \cup [m, M]$ for some $0 < m, M < \infty$. Consider the rotation $(\tau_t)_{t\in\mathbb{R}}$ on $(\mathbb{T}, \lambda)$ with period $M + m$ and $E = e^{2\pi i[0,m]}$. Then $f_E(t) := \lambda(\tau_t(E) \cap E)$ is continuous and positive-definite by Lemma 2.29. Hence, there exists a measure μ on $\mathbb{R}$ such that $\widehat{\mu}(t) = \lambda(\tau_t(E) \cap E)$ for all $t \in \mathbb{R}$ and with normalised probability measure $\tilde{\mu} := \frac{\mu}{\mu(\mathbb{R})}$ and $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$.



Graph of f_E.

However, we have

$$\tilde{\mu}(\{0\}) \cdot \mu(\mathbb{R}) = \mu(\{0\}) \overset{2.17}{=} \oint_{\mathbb{R}} f_E(t)\, d\lambda(t) = \frac{m}{M+m} > 0,$$

and $\mathcal{D}$ is therefore not $\mathbb{R}$-FMRiesz.

□

4.1.2 Operator Recurrence and Poincaré

Proposition 4.6. *Given a Hilbert space H, a unitary operator T on H and $x \in H$ with $P_{\mathbb{N}} x \neq 0$, there exists $n \in \mathbb{N}$ such that*

$$\langle T^n x, x \rangle \neq 0.$$

In other words, $\mathbb{N}$ is $\mathbb{N}$-operator recurrent.

Proof. We have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \langle T^i x, x \rangle = \langle \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} T^i x, x \rangle = \langle P_{\mathbb{N}} x, x \rangle = \|P_{\mathbb{N}} x\|^2 > 0$$

and hence, there must exist (even infinitely many) $n \in \mathbb{N}$ such that $\langle T^n x, x \rangle \neq 0$. □

Similarly, we obtain the following result.

Proposition 4.7. *Let $\epsilon > 0$ be given. Then, given a Hilbert space H, a strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $P_{\mathbb{R}} x \neq 0$, there exists d with $|d| \geq \epsilon$ such that*

$$\langle T_d x, x \rangle \neq 0$$

In other words, $\mathbb{R}\backslash(\epsilon, \epsilon)$ *is* $\mathbb{R}$*-operator recurrent.*

Remark 4.8. Propositions 4.6 and 4.7 indicate that the requirement $Px \neq 0$ is essential (see [60]). To illustrate this, consider $H = l^2(\mathbb{N})$ and $H = L^2(\mathbb{R}, \lambda)$, respectively, with the corresponding shift operators and $0 \neq x = \mathbb{1}_{\{0\}}$ and $0 \neq y = \frac{1}{\sqrt{\tilde{\epsilon}}}\mathbb{1}_{(0,\tilde{\epsilon})}$ with $\|x\| = 1 = \|y\|$. However, $PH = \{0\}$, so $P_{\mathbb{N}}x = 0 = P_{\mathbb{R}}y$, and we indeed have $\langle T^n x, x\rangle = 0$ for all $n \in \mathbb{N}$ and $\langle T_t y, y\rangle = 0$ for all $|t| \geq \tilde{\epsilon}$.

Similarly as in Proposition 4.5, we obtain the following result. We note that the assumption $Px \neq 0$ is not necessary in this case since the the proof is based merely on continuity and $\|x\| > 0$.

Proposition 4.9. *A set* $\mathcal{D} \subset \mathbb{R}$ *is boundedly* $\mathbb{R}$*-operator recurrent (Poincaré) if and only if* 0 *is a limit point of* $\mathcal{D}$*.*

Proposition 4.10. *Let an infinite set* $E := \{a_i : i \in I\} \subseteq \mathbb{R}$*, the difference set* $\mathcal{D} := E - E := \{a_i - a_j : a_i > a_j \in E\}$*, a scmps* $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ *and* $A \in \Sigma$ *with* $\mu(A) > 0$ *be given. Then there exists* $d = a - b \in \mathcal{D}$ *such that*

$$\mu(\phi_{a-b}(A) \cap A) > 0. \tag{4.2}$$

In other words, $\mathcal{D} := \{a - b : a > b \in E\}$ *is* $\mathbb{R}$*-Poincaré.*

Compare [35 , Theorem 3.1].

Proof. Let a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$ be given. For a contradiction, let us assume that $\mu(\phi_d(A) \cap A) = 0$ for all $d \in \mathcal{D}$. Then

$$0 = \mu(\phi_d(A) \cap A) = \mu(\phi_{a_i - a_j}(A) \cap A) = \mu(\phi_{a_i}(A) \cap \phi_{a_j}(A))$$

for all $a_i > a_j \in E$. Hence, the (infinite) family $(\phi_{a_i}(A))_{i\in I}$ has no pairwise intersections of positive measure while each member has measure $\mu(\phi_{a_i}(A)) = \mu(A) > 0$ since ϕ_t is measure preserving. However, μ is a probability measure where this is not possible, yielding a contradiction. Hence, there exists $a_i > a_j \in E$ such that

$$0 < \mu(\phi_{a_i}(A) \cap \phi_{a_j}(A)) = \mu(\phi_{a_i - a_j}(A) \cap A)$$

and $a_i - a_j \in \mathcal{D}$ satisfies Requirement (4.2) and $\mathcal{D}$ is hence $\mathbb{R}$-Poincaré. $\square$

Similarly, we obtain Poincaré's recurrence theorem.

Corollary 4.11 (Poincaré)**.** *Given a mps $(\Omega, \Sigma, \mu; \phi)$ and $A \in \Sigma$ with $\mu(A) > 0$, there exists $n \in \mathbb{N}$ such that*

$$\mu(\phi^n(A) \cap A) > 0.$$

In other words, $\mathbb{N}$ is $\mathbb{N}$-Poincaré.

Remark 4.12. Similarly, $\mathbb{N}$ is $\mathbb{N}$-strongly operator recurrent and $\mathbb{N}$-strongly recurrent. Since $\mathbb{N} \subset \mathbb{R} \backslash (\epsilon, \epsilon)$ for $0 \leq \epsilon < 1$, the set $\mathbb{R} \backslash (\epsilon, \epsilon)$ is in particular $\mathbb{R}$-strongly operator recurrent, $\mathbb{R}$-strongly recurrent and $\mathbb{R}$-Poincaré. The same holds for $\epsilon \geq 1$ (Proposition 5.4).

Remark 4.13. Obviously, a bounded set cannot have any $\mathbb{R}$-strong operator recurrence or $\mathbb{R}$-strong recurrence properties as defined in Definitions 3.10 and 3.13. However, we can modify this definition by using $\limsup_{|d| \to 0, d \in \mathcal{D}}$ instead of $\limsup_{|d| \to \infty, d \in \mathcal{D}}$. We therefore call a set $\mathcal{D} \subset \mathbb{R}$ **boundedly $\mathbb{R}$-strongly recurrent** if $\limsup_{|d| \to 0, d \in \mathcal{D}}$ is well-defined and if

$$\limsup_{|d| \to 0, d \in \mathcal{D}} \mu(\phi_d(A) \cap A) > 0$$

holds for any given scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t \in \mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$, and we call $\mathcal{D} \subset \mathbb{R}$ **boundedly $\mathbb{R}$-strongly operator recurrent** if

$$\limsup_{|d| \to 0, d \in \mathcal{D}} |\langle T_d x, x \rangle| \neq 0$$

holds for any given Hilbert space H, strongly continuous unitary group $(T_t)_{t \in \mathbb{R}}$ on H and $x \in H$ with $P_{\mathbb{R}} x \neq 0$.

We immediately obtain the following.

Proposition 4.14. *A set $\mathcal{D}$ is boundedly $\mathbb{R}$-strongly operator recurrent or boundedly $\mathbb{R}$-strongly recurrent if and only if* 0 *is a limit point.*

4.1.3 Van der Corput and Correlativity

We prove the van der Corput inequality and use this inequality to deduce van der Corput and correlativity properties of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}_*$ and to prove the van der Corput theorem.

Proposition 4.15 (Van der Corput Inequality)**.** *Let* $0 < S, T < \infty$ *and* $f \in l^\infty_*([-T,T]) \subseteq l^\infty_*(\mathbb{R})$ *be given. Then*

$$\left|\int_{-T}^{T} f(t)\ d\lambda(t)\right|^2 \leq \frac{T+S}{S}\int_{-2S}^{2S}\left|\int_{[-T,T]\cap[-T-d,T-d]} f(t+d)\overline{f(t)}\ d\lambda(t)\right|\ d\lambda(d).$$

Compare [14, Proposition 1.5].

Proof. We have

$$\left|\int_{-T}^{T} f(t)\ d\lambda(t)\right|^2 = \left|\int_{-T}^{T} \frac{1}{2S}\int_{-S}^{S} f(t)\ d\lambda(d)\ d\lambda(t)\right|^2 = \left|\frac{1}{2S}\int_{-S}^{S}\int_{-T}^{T} f(t)\ d\lambda(t)\ d\lambda(d)\right|^2$$

$$= \left|\frac{1}{2S}\int_{-S}^{S}\int_{-T-d}^{T-d} f(t+d)\ d\lambda(t)\ d\lambda(d)\right|^2$$

$$= \left|\frac{1}{2S}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{[-T-d,T-d]\times[-S,S]}(t,d)f(t+d)\ d\lambda(t)\ d\lambda(d)\right|^2$$

$$\overset{(+)}{=} \left|\frac{1}{2S}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{[-T-S,T+S]\times[-S,S]}(t,d)f(t+d)\ d\lambda(d)\ d\lambda(t)\right|^2$$

$$= \frac{1}{4S^2}\left|\int_{\mathbb{R}} \mathbb{1}_{[-T-S,T+S]}(t)\int_{-S}^{S} f(t+d)\ d\lambda(d)\ d\lambda(t)\right|^2$$

$$\overset{CS}{\leq} \frac{1}{4S^2}\int_{\mathbb{R}} \mathbb{1}_{[-T-S,T+S]}(t)\ d\lambda(t)\cdot\int_{\mathbb{R}}\left|\int_{-S}^{S} f(t+d)\ d\lambda(d)\right|^2\ d\lambda(t)$$

$$= \frac{T+S}{2S^2}\int_{\mathbb{R}}\left|\int_{-S}^{S} f(t+d)\ d\lambda(d)\right|^2\ d\lambda(t)$$

$$
\begin{aligned}
&= \frac{T+S}{2S^2} \int_{\mathbb{R}} \int_{-S}^{S} f(t+d)\ d\lambda(d) \int_{-S}^{S} \overline{f(t+s)}\ d\lambda(s)\ d\lambda(t) \\
&= \frac{T+S}{2S^2} \int_{-S}^{S} \int_{\mathbb{R}} \int_{-S}^{S} f(t+d)\overline{f(t+s)}\ d\lambda(d)\ d\lambda(t)\ d\lambda(s) \\
&\overset{\tilde{d}:=d-s}{=} \frac{T+S}{2S^2} \int_{-S}^{S} \int_{\mathbb{R}} \int_{-S-s}^{S-s} f(t+\tilde{d}+s)\overline{f(t+s)}\ d\lambda(\tilde{d})\ d\lambda(t)\ d\lambda(s) \\
&\overset{\tilde{t}:=t+s}{=} \frac{T+S}{2S^2} \int_{-S}^{S} \int_{-S-s}^{S-s} \int_{\mathbb{R}} f(\tilde{t}+\tilde{d})\overline{f(\tilde{t})}\ d\lambda(\tilde{t})\ d\lambda(\tilde{d})\ d\lambda(s) \\
&\le \frac{T+S}{2S^2} \int_{-S}^{S} \int_{-S-s}^{S-s} \left| \int_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) \right|\ d\lambda(d)\ d\lambda(s) \\
&\le \frac{T+S}{2S^2} \int_{-S}^{S} \int_{-2S}^{2S} \left| \int_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) \right|\ d\lambda(d)\ d\lambda(s) \\
&= \frac{T+S}{S} \int_{-2S}^{2S} \left| \int_{\mathbb{R}} f(t+d)\overline{f(t)}\ d\lambda(t) \right|\ d\lambda(d) \\
&= \frac{T+S}{S} \int_{-2S}^{2S} \left| \int_{[-T,T]\cap[-T-d,T-d]} f(t+d)\overline{f(t)}\ d\lambda(t) \right|\ d\lambda(d)
\end{aligned}
$$

and we note that we have used in (+) that $f(t) = 0$ for $|t| > T$, hence,

$$
\mathbb{1}_{[-T-d,T-d]\times[-S,S]}(t,d)f(t+d) = \mathbb{1}_{[-T-S,T+S]\times[-S,S]}(t,d)f(t+d).
$$

□

An integer version of Proposition 4.15 is proven similarly (a special case of [14, Proposition 1.5]).

Proposition 4.16 (Van der Corput Inequality). *Let $0 < m, n < \infty$ and $f \in l_*^\infty(\{-n, \ldots, n\})$ be given. Then*

$$
\left| \sum_{k=-n}^{n} f(k) \right|^2 \le \frac{m+n}{m} \sum_{d=-2m}^{2m} \left| \sum_{-n \le k, k+d \le n} f(k+d)\overline{f(k)} \right|.
$$

Proposition 4.17. *Let $\tilde{\epsilon} > 0$ and $f \in l_c^\infty(\mathbb{R},\mathbb{R})$ be given. Let $f(\cdot + d) - f(\cdot)$ be equidistributed mod 1 for all $|d| \ge \tilde{\epsilon}$. Then f is equidistributed mod 1. In other words, $\mathbb{R}\backslash(-\tilde{\epsilon},\tilde{\epsilon})$ is $\mathbb{R}$-van der Corput.*

Proof. By the Weyl Criterion Theorem 2.24, the equidistribution mod 1 of $f(\cdot + d) - f(\cdot)$ for all $d \in \mathbb{R}_*$ is equivalent to

$$\oint_{\mathbb{R}} e^{2\pi i k f(t+d)} e^{-2\pi i k f(t)} \, d\lambda(t) = 0$$

for all $0 \neq k \in \mathbb{Z}$ and all $d \in \mathbb{R}_*$.We note

$$\left| \oint_{\mathbb{R}} e^{2\pi i k f(t+d)} e^{-2\pi i k f(t)} \, d\lambda(t) \right| \leq \oint_{\mathbb{R}} \left| e^{2\pi i k f(t+d)} \right| \left| e^{-2\pi i k f(t)} \right| \, d\lambda(t) = \oint_{\mathbb{R}} \mathbb{1} \, d\lambda(t) = 1$$

for $|d| \leq \tilde{\epsilon}$. Now fix $0 \neq k \in \mathbb{Z}$. Proposition 4.15 yields

$$\begin{aligned} &\left| \int_{-T}^{T} e^{2\pi i k f(t)} \, d\lambda(t) \right|^2 \\ &\leq \frac{T+S}{S} \int_{-2S}^{2S} \left| \int_{[-T,T] \cap [-T-d, T-d]} e^{2\pi i k f(t+d)} e^{-2\pi i k f(t)} \, d\lambda(t) \right| \, d\lambda(d). \end{aligned}$$

Dividing by $(2T)^2$ and taking the limit $T \to \infty$ yields

$$\begin{aligned} &\left| \oint_{\mathbb{R}} e^{2\pi i k f(t)} \, d\lambda(t) \right|^2 \\ &\leq \lim_{T \to \infty} \frac{T+S}{2TS} \cdot \lim_{T \to \infty} \int_{-2S}^{2S} \left| \frac{1}{2T} \int_{[-T,T] \cap [-T-d, T-d]} e^{2\pi i k f(t+d)} e^{-2\pi i k f(t)} \, d\lambda(t) \right| \, d\lambda(d) \\ &= \frac{1}{2S} \int_{-2S}^{2S} \left| \oint_{\mathbb{R}} e^{2\pi i k f(t+d)} e^{-2\pi i k f(t)} \, d\lambda(t) \right| \, d\lambda(d) \\ &\leq \frac{1}{2S} \int_{-2S}^{2S} \mathbb{1}_{(-\tilde{\epsilon}, \tilde{\epsilon})} \, d\lambda(d) \leq \frac{\tilde{\epsilon}}{S} \overset{S \to \infty}{\longrightarrow} 0. \end{aligned}$$

And f is hence equidistributed mod 1 by Theorem 2.24. □

Similarly, we obtain the following result by using Proposition 4.16.

Proposition 4.18. *Let $f \in l_*^\infty(\mathbb{Z}, \mathbb{R})$. Let $f(\cdot + d) - f(\cdot)$ be equidistributed mod 1 for all $d \in \mathbb{Z} \backslash \{0\}$. Then f is equidistributed mod 1. In other words, $\mathbb{Z} \backslash \{0\}$ is $\mathbb{Z}$-van der Corput.*

Remark 4.19. We exclude 0 in Proposition 4.18 since $f(\cdot + 0) - f(\cdot) = 0$ is never equidistributed mod 1, and $\mathbb{Z}$ is therefore trivially $\mathbb{Z}$-van der Corput

We note

$$\begin{aligned} &(f(n+d)-f(n))_{n\in\mathbb{Z}} \text{ is equidistributed mod 1} \\ &\Leftrightarrow (f(n)-f(n-d))_{n\in\mathbb{Z}} \text{ is equidistributed mod 1} \\ &\Leftrightarrow (f(n-d)-f(n))_{n\in\mathbb{Z}} \text{ is equidistributed mod 1.} \end{aligned}$$

Hence the equidistribution mod 1 of $(f(n+d)-f(n))_{n\in\mathbb{Z}}$ for $d \in \mathbb{N}$ implies the equidistribution mod 1 of $(f(n+(-d))-f(n))_{n\in\mathbb{Z}}$.

Proposition 4.20. *Let $f \in l_*^\infty(\mathbb{Z},\mathbb{R})$. Let $f(\cdot+d)-f(\cdot)$ be equidistributed mod 1 for all $d \in \mathbb{N}$. Then f is equidistributed mod 1. In other words, $\mathbb{N}$ is $\mathbb{Z}$-van der Corput.*

Similarly, since finitely many entries do not matter, the equidistribution of $(f(n-d)-f(n))_{n\in\mathbb{N}}$ for $d<0$ implies the equidistribution mod 1 of $(f(n+(-d))-f(n))_{n\in\mathbb{N}}$. Proposition 4.16 for $f \in l_*^\infty(\{1,\dots,n\}) \subset l^\infty(\{-n,\dots,n\})$, hence implies the classical van der Corput theorem.

Proposition 4.21 (Van der Corput)**.** *Let $f \in l_*^\infty(\mathbb{N},\mathbb{R})$. Let $f(\cdot+d)-f(\cdot)$ be equidistributed mod 1 for all $d \in \mathbb{N}$. Then f is equidistributed mod 1. In other words, $\mathbb{N}$ is $\mathbb{N}$-van der Corput.*

Similarly, we obtain the following results.

Proposition 4.22. *Let $\tilde{\epsilon} > 0$ be given and let $f \in l_c^\infty(\mathbb{R},\mathbb{T})$. Then $⨏_{\mathbb{R}} f(t+d)\overline{f(t)}\, d\lambda(t) = 0$ for all $|d| \geq \tilde{\epsilon}$ implies*

$$⨏_{\mathbb{R}} f(t)\, d\lambda(t) = 0.$$

In other words, $\mathbb{R}\backslash(-\tilde{\epsilon},\epsilon)$ is $\mathbb{R}$-correlative.

Proposition 4.23. *Let $f \in l_*^\infty(\mathbb{Z},\mathbb{T})$. Then $\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n} f(k+d)\overline{f(k)} = 0$ for all $0 \neq d \in \mathbb{Z}$ implies*

$$\lim_{n\to\infty}\frac{1}{2n+1}\sum_{k=-n}^{n} f(k) = 0$$

In other words, $\mathbb{Z}\backslash\{0\}$ is $\mathbb{Z}$-correlative.

Proposition 4.24. *A set $\mathcal{D} \subset \mathbb{R}$ is boundedly $\mathbb{R}$-correlative if and only if* 0 *is a limit point.*

Proof. Let 0 be a limit point of $\mathcal{D}$, so there exists a sequence $(d_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ with $d_n \to 0$. Let us assume that there exists $f \in l_c^\infty(\mathbb{R},\mathbb{T})$ such that $⨍_{\mathbb{R}} f(t+d)\overline{f(t)}\, d\lambda(t) = 0$ for all $d \in \mathcal{D}$. Then by continuity of $d \mapsto ⨍_{\mathbb{R}} f(t+d)\overline{f(t)}\, d\lambda(t)$, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} ⨍_{\mathbb{R}} f(t+d_n)\overline{f(t)}\, d\lambda(t) = ⨍_{\mathbb{R}} f(t+\lim_{n\to\infty} d_n)\overline{f(t)}\, d\lambda(t) \\ &= ⨍_{\mathbb{R}} |f(t)|^2\, d\lambda(t) = 1, \end{aligned}$$

yielding a contradiction. The set $\mathcal{D}$ is therefore trivially $\mathbb{R}$-correlative as there exists no such $f \in l_c^\infty(\mathbb{R},\mathbb{T})$.
If 0 is not a limit point, then $\mathcal{D} \subseteq [-M,-m] \cup [m,M]$ for some $0 < m, M < \infty$. We use the function f_E from the proof of Proposition 4.5. We note $f_E \in l_c^\infty(\mathbb{R},\mathbb{C})$ as a consequence of Lemma 2.29 and the Bochner Herglotz theorem 2.16. Then

$$⨍_{\mathbb{R}} f_E(t+d)\overline{f_E(t)}\, d\lambda(t) = ⨍_{\mathbb{R}} 0\, d\lambda(t) = 0,$$

for all $d \in \mathcal{D}$, but

$$⨍_{\mathbb{R}} f_E(t)\, d\lambda(t) \overset{2.5}{\geq} (\mu(E))^2 > 0.$$

□

Similarly, we obtain the following result.

Proposition 4.25. *A set $\mathcal{D} \subset \mathbb{R}$ is boundedly $\mathbb{R}$-strongly correlative (enhanced van der Corput, real correlative, van der Corput, K-van der Corput, generalised van der Corput) if and only if* 0 *is a limit point.*

Corollary 4.26. *Let $\mathcal{D} \subset \mathbb{R}$. Then the bounded recurrence properties of $\mathcal{D}$ for the following properties coincide: $\mathbb{R}$-Poincaré, $\mathbb{R}$-operator recurrence, $\mathbb{R}$-strong recurrence, $\mathbb{R}$-strong operator recurrence, $\mathbb{R}$-real correlativity, $\mathbb{R}$-correlativity, $\mathbb{R}$-strong* *correlativity, $\mathbb{R}$-enhanced van der Corput, $\mathbb{R}$-van der Corput, $\mathbb{R}$-K-van der Corput, $\mathbb{R}$-generalised van der Corput.*

4.2 A Rationally Linearly Independent Sequence

In this subsection, we construct a rationally linearly independent sequence which is strongly operator recurrent. Bergelson, Boshernitzan and Bourgain ([11, Theorem D]) however have shown that such a sequence cannot be combinatorially recurrent. These results show that Poincaré recurrence and combinatorial recurrence (and hence Poincaré and Poincaré without continuity) are distinct.

We call a set $\{p_i : i \in I\}$ rationally linearly independent if for all finite subsets $J \subseteq I$ and $q_j \in \mathbb{Q}$, we have

$$\sum_{j \in J} q_j p_j = 0 \implies \forall j \in J : q_j = 0.$$

Theorem 4.27 (Bergelson, Boshernitzan, Bourgain)**.** *Let $\mathcal{D} = (d_n)_{n \in \mathbb{N}} \subset \mathbb{R}$ be a rationally linearly independent sequence. Then there exists a measurable set $E \subset \mathbb{R}$ such that*

$$d_{\mathbb{R}}(E) = \lim_{T \to \infty} \frac{\lambda(E \cap [-T, T])}{2T} = \frac{1}{4}$$

and

$$d_{\mathbb{R}}(E \cap (E - d_n)) = 0 \ \forall n \in \mathbb{N}.$$

In other words, $\mathcal{D}$ is not $\mathbb{R}$-combinatorially recurrent.

Lemma 4.28. *The set $\{\log p_n : p_n$ prime$\}$ is rationally linearly independent.*

Proof. Let $n \in \mathbb{N}$. Let further $p_1, \dots, p_n$ be distinct primes and let $q_1, \dots, q_n \in \mathbb{Z}$, $r_1, \dots, r_n \in \mathbb{N}$ be given such that

$$0 = \frac{q_1}{r_1} \log p_1 + \dots \frac{q_n}{r_n} \log p_n.$$

Then $0 = \log(p_1^{\frac{q_1}{r_1}} \cdots p_n^{\frac{q_n}{r_n}})$ and $1 = p_1^{\frac{q_1}{r_1}} \cdots p_n^{\frac{q_n}{r_n}}$. Expontiating the equation by $\operatorname{lcm}(r_1, \ldots, r_n)$, we obtain integers $l_1 = \frac{q_1 \cdot lcm(r_1,\ldots,r_n)}{r_1}, \ldots, l_n = \frac{q_n \cdot lcm(r_1,\ldots,r_n)}{r_n}$ such that $1 = p_1^{l_1} \cdots p_n^{l_n}$.

Multiplying this equation by the factors with negative exponent, we obtain

$$p_{t_1}^{l_{t_1}} \ldots p_{t_m}^{l_{t_m}} = p_{s_1}^{l_{s_1}} \ldots p_{s_k}^{l_{s_k}}.$$

The unique prime factorisation yields $l_1 = \ldots = l_n = 0$ and therefore $q_1 = \ldots = q_n = 0$, and the set is rationally linearly independent. □

Proposition 4.29. *There exists a rationally linearly independent sequence which is $\mathbb{R}$-strongly recurrent ($\mathbb{R}$-Poincaré) and which does not have 0 as a limit point.*

Proof. We note that $\mathbb{N}$ is strongly recurrent (Remark 4.12). We take a rationally linearly independent sequence such as

$$\mathcal{P} := \{\log p_i : p_i \text{ prime}, i \in \mathbb{N}\}$$

as shown in Lemma 4.28. We define a set $\mathcal{D}$ by $d_i := q_i p_i$ with $q_i \in \mathbb{Q}$ such that $q_i p_i \in (i, i + \frac{1}{i})$. Then $\mathcal{D} := \{d_n : n \in \mathbb{N}\}$ is a rationally linearly independent sequence by the rationally linear independence of $\mathcal{P}$, and we note $|d_n - n| \overset{n\to\infty}{\longrightarrow} 0$ by construction.

Now let $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ be a scmps and $A \in \Sigma$ with $\mu(A) > 0$. Since $\mathbb{N}$ is $\mathbb{N}$-strongly recurrent, there exists $\tilde{M} > 0$ such that $\limsup_{n\in\mathbb{N}} \mu(\phi_1^n(A) \cap A) = \tilde{M}$ and there exist also $M > 0$ and a subsequence $(m_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\mu(\phi_1^{m_n}(A) \cap A) > M$ for all $n \in \mathbb{N}$. Lemma 2.30 implies that there exist $\delta_M, N > 0$ such that $\mu(\phi_t(A) \cap A) > N$ for all $t \in \cup_{n\in\mathbb{N}}(m_n - \delta_N, m_n + \delta_N)$.

Using $|d_n - n| \overset{n\to\infty}{\longrightarrow} 0$, there exists i_0 such that $d(d_i, i) < \delta_N$ for all $i > i_0$, hence, every d_{m_n} with $m_n > i_0$ satisfies $\mu(\phi_{d_{m_n}}(A) \cap A) > N$ and therefore

$$\limsup_{|d|\to\infty, d\in\mathcal{D}} \mu(\phi_d(A) \cap A) \geq N$$

and $\mathcal{D}$ is hence $\mathbb{R}$-strongly recurrent. □

As in Proposition 4.29, we obtain the following results.

Proposition 4.30. *There exists a rationally linearly independent sequence which is $\mathbb{R}$-strongly operator recurrent ($\mathbb{R}$-operator recurrent) and which does not have 0 as a limit point.*

Proposition 4.31. *Let $D \subseteq \mathbb{N}$ be a $\mathbb{N}$-strong recurrence set and $(a_n)_{n\in\mathbb{N}}$ a sequence with $a_n \to 0$. Then $\mathcal{D} := \{d + a_d : d \in D\}$ is $\mathbb{R}$-strongly operator recurrent ($\mathbb{R}$-strongly recurrent, $\mathbb{R}$-operator recurrent, $\mathbb{R}$-Poincaré).*

Proposition 4.32. *There exists a rationally linearly independent sequence which is boundedly $\mathbb{R}$-strong operator recurrent (boundedly $\mathbb{R}$-strong recurrent, boundedly $\mathbb{R}$-operator recurrent, boundedly $\mathbb{R}$-Poincaré).*

Since we have chosen a rationally linearly independent set in Proposition 4.29, Theorem 4.27 implies the following.

Proposition 4.33. *There exists a set $\mathcal{D} \subset \mathbb{R}$ which is $\mathbb{R}$-Poincaré, but not $\mathbb{R}$-combinatorially recurrent and not $\mathbb{R}$-Poincaré$_*$.*

4.3 Integer and Real Recurrence Properties for $\mathcal{D} \subseteq \mathbb{Z}$

We now consider integer sets and observe that the real and integer version of many recurrence properties for integer sets coincide. This yields many non-trivial examples of recurrence sets coming from the classical integer theory.

We note that recurrence properties are symmetric, i.e., $\mathcal{D}$ is a recurrence set if and only if $\{|d| : d \in \mathcal{D}\}$ is one. For example, consider $\langle T_d x, x\rangle = \overline{\langle T_{-d} x, x\rangle}$, $A \cap (A - d) = (A + d) \cap A$ or $⨏_{\mathbb{R}} f(t + d)\overline{f(t)}\, d\lambda(t) = \overline{⨏_{\mathbb{R}} f(t - d)\overline{f(t)}\, d\lambda(t)}$ to see that d and $-d$ have the same effect on recurrence (compare also the comment before Proposition 4.21 for the van der Corput property). It hence does not matter if we consider $\mathcal{D} \subseteq \mathbb{N}$ or $\mathcal{D} \subseteq \mathbb{Z}$, and all results in this subsection are true for both cases. However, in order to simplify the arguments and notation, we restrict our discussion to sets $\mathcal{D} \subseteq \mathbb{Z}$ and note the corresponding $\mathbb{N}$-results without further mentioning.

Using also implications from Section 3.2, our final result of this section is the following theorem.

Theorem 4.34. *Let $\mathcal{D} \subseteq \mathbb{Z}$. Then $\mathbb{Z}$-recurrence and $\mathbb{R}$-recurrence coincide for strong operator recurrence and operator recurrence as well as for all properties of Theorem 9 without continuity assumptions.*

4.3.1 Operator Recurrence and Poincaré

We have shown in Subsection 3.2.1 using Stone's theorem and the spectral theorem that it is equivalent to consider only unitary multiplicators instead of unitary operators when dealing with operator recurrence. This can be used to show that $\mathbb{Z}$-operator recurrence and $\mathbb{R}$-operator recurrence for a set $\mathcal{D} \subseteq \mathbb{Z}$ are equivalent.

Proposition 4.35. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{Z}$-(multiplicator) operator recurrent. Then it is $\mathbb{R}$-(multiplicator) operator recurrent.*

Proof. Let a Hilbert space H, a strongly continuous (multiplicator) unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $P_{\mathbb{R}}x \neq 0$ be given.

We have $\mathrm{Fix}((T_t)_{t\in\mathbb{R}}) \subseteq \mathrm{Fix}(T_1)$, hence, $P_{\mathbb{Z}}x \neq 0$ by Lemma 2.10. As $\mathcal{D}$ is $\mathbb{Z}$-(multiplicator) operator recurrent and as T_1 is a unitary (multiplicator) operator, there exists $d \in \mathcal{D}$ such that

$$\langle T_d x, x\rangle = \langle T_1^d x, x\rangle \neq 0.$$

□

The equivalence of operator recurrence and multiplicator recurrence is shown in Proposition 3.29 and 3.30. We use this fact to show that an $\mathbb{R}$-operator recurrence set $\mathcal{D} \subseteq \mathbb{Z}$ is also $\mathbb{Z}$-operator recurrent.
For simplicity, we set $[m = 0] := \{\omega \in \Omega : m(\omega) = 0\}$ and similarly $[m = 2n\pi, n \in \mathbb{Z}] := \{\omega \in \Omega : m(\omega) = 2n\pi$ for some $n \in \mathbb{Z}\}$ and $[f \neq 0] := \{\omega \in \Omega : f(\omega) \neq 0\}$.

Proposition 4.36. *Let $(M_t)_{t\in\mathbb{R}} = (e^{itm})_{t\in\mathbb{R}}$ be a strongly continuous multiplicator group on $L^2(\Omega, \Sigma, \mu)$ with a not necessarily finite measure μ. Then*

$$\mathrm{Fix}((M_t)_{t\in\mathbb{R}}) = L^2([m = 0], \Sigma, \mu) \subseteq L^2(\Omega, \Sigma, \mu).$$

Proof. Let $f \in L^2([m = 0], \Sigma, \mu)$. Then $M_t f = e^{itm} f = f$ as $e^{itm} = 1$ on $[m = 0]$. Let $[f \neq 0] \nsubseteq [m = 0]$. Then there exists $t \in \mathbb{R}$ such that $M_t f = e^{itm} f \neq f$. If $[f \neq 0] \subseteq [m = 2n\pi, n \in \mathbb{Z}]$, choose irrational t, otherwise $t = 1$ works. □

Lemma 4.37. *Let M be a unitary multiplicator on $L^2(\Omega, \Sigma, \mu)$ with a not necessarily finite measure μ. Then there exists m such that $0 \leq m(\omega) < 2\pi$ for all $\omega \in \Omega$ and $M = e^{im}$.*

Proof. Every unitary multiplicator can be written as e^{ik} for some real-valued k. Define $m(\omega) := k(\omega) \bmod 2\pi$. Then $e^{ik} = e^{im}$ and $0 \leq m(\omega) < 2\pi$. □

Proposition 4.38. *Let $M = e^{im}$ be a unitary multiplicator on $L^2(\Omega, \Sigma, \mu)$ with a not necessarily finite measure μ such that $0 \leq m(\omega) < 2\pi$ for all $\omega \in \Omega$. Then*

$$\mathrm{Fix}((M^n, n \in \mathbb{N})) = L^2([m = 0], \Sigma, \mu) \subseteq L^2(\Omega, \Sigma, \mu).$$

Proof. Let $f \in L^2([m = 0], \Sigma, \mu)$. Then $Mf = e^{im} f = f$ as $e^{im} = 1$ on $[m = 0]$. Let $[f \neq 0] \nsubseteq [m = 0]$. Then $e^{im} f|_{[m\neq 0]} \neq f|_{[m\neq 0]}$, hence $f \notin \mathrm{Fix}((M^n, n \in \mathbb{N}))$. □

Corollary 4.39. *Let $m \in L^2(\Omega, \Sigma, \mu; \mathbb{R})$ satisfy $0 \leq m(\omega) < 2\pi$ for almost all $\omega \in \Omega$. Then the associated strongly continuous multiplicator group and multiplicator operator have the same fixed space.*

Proposition 4.40. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{R}$-multiplicator operator recurrent. Then it is $\mathbb{Z}$-multiplicator operator recurrent.*

Proof. Let M be a unitary multiplicator on $L^2(\Omega, \Sigma, \mu)$ with a not necessarily finite measure μ and $x \in L^2(\Omega, \Sigma, \mu)$ with

$$\langle M^d x, x \rangle = 0 \tag{4.3}$$

for all $d \in \mathcal{D}$. Lemma 4.37 yields $M = e^{im}$ for some $m \in L^2(\Omega, \Sigma, \mu; \mathbb{R})$ satisfying $0 \leq m(\omega) < 2\pi$ for all almost $\omega \in \Omega$.
Let $(M_t)_{t\in\mathbb{R}} = (e^{itm})_{t\in\mathbb{R}}$ be the associated strongly continuous unitary multiplicator group. $\mathbb{R}$-multiplicator operator recurrence and (4.3) imply $P_{M_t}x = 0$, hence, $P_M x = 0$ by Corollary 4.39 as their fixed spaces coincide and $P_{M_t} = P_M$. The set $\mathcal{D}$ is therefore $\mathbb{Z}$-multiplicator operator recurrent. □

Corollary 4.41. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{R}$-operator recurrent. Then it is $\mathbb{Z}$-operator recurrent*

Proof. This follows from the equivalence of operator recurrence and multiplicator operator recurrence established in Propositions 3.29 and 3.30 . □

We similarly obtain the corresponding strong results, i.e.

$$\begin{aligned} &\mathbb{Z}\text{-strong (multiplicator) operator recurrence} \\ &\Leftrightarrow \mathbb{R}\text{-strong (multiplicator) operator recurrence} \end{aligned}$$

for $\mathcal{D} \subseteq \mathbb{Z}$.

Proposition 4.42. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{Z}$-Poincaré. Then it is $\mathbb{R}$-Poincaré.*

Proof. Let $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ be a scmps and $A \in \Sigma$ with $\mu(A) > 0$. Then there exists $d \in \mathcal{D}$ such that

$$0 < \mu(\phi_1^d(A) \cap A) = \mu(\phi_d(A) \cap A).$$

□

Remark 4.43. It is not clear if the converse of Proposition 4.42 is true. Even more than with operator recurrence in Proposition 4.40, this question is related to the problem of embedding a discrete mps into a continuous one.
Ornstein ([61]) gave a counterexample, showing that such an embedding is not possible in general. But for mps on a standard Borel space (or equivalently on $([0,1],\lambda)$), it was shown that the generic mps can be embedded into a continuous one ([1], [69]).
However, without continuity assumption and using the implications

$$\begin{aligned}\mathbb{R}\text{-Poincaré}_* &\Rightarrow \mathbb{R}\text{-real correlativity}_*^{0,1} \Rightarrow \mathbb{Z}\text{-real correlativity}^{0,1}\\ &\Rightarrow \mathbb{Z}\text{-real correlativity} \Rightarrow \mathbb{Z}\text{-Poincaré},\end{aligned}$$

where $\mathbb{R}$-real correlativity$_*^{0,1}$ is the correlativity version for functions $f \in l_*^\infty(\mathbb{R}, \{0,1\})$ (Definition 3.52), we obtain the equivalence

$$\mathbb{R}\text{-Poincaré}_* \Leftrightarrow \mathbb{Z}\text{-Poincaré}$$

for $\mathcal{D} \subseteq \mathbb{Z}$ and similarly

$$\mathbb{R}\text{-strong recurrence}_* \Leftrightarrow \mathbb{Z}\text{-strong recurrence}.$$

4.3.2 FMRiesz, FC+ and KMF

Using the results from Subsection 4.3.1, we immediately obtain the corresponding results for FMRiesz and FC+.

Proposition 4.44. *A set $\mathcal{D} \subseteq \mathbb{Z}$ is $\mathbb{Z}$-FMRiesz ($\mathbb{Z}$-FC+) if and only if it is $\mathbb{R}$-FMRiesz ($\mathbb{R}$-FC+).*

Proof. We note the equivalence of FMRiesz and operator recurrence both for $\mathbb{Z}$ and $\mathbb{R}$ (Proposition 3.17 for $\mathbb{R}$ and [60] for $\mathbb{Z}$) and that a set $\mathcal{D} \subseteq \mathbb{Z}$ is $\mathbb{Z}$-operator recurrent if and only if it is $\mathbb{R}$-operator recurrent (Proposition 3.29, Remark 4.41). □

Proposition 4.45. *A set $\mathcal{D} \subseteq \mathbb{Z}$ is $\mathbb{Z}$-KMF if and only if it is $\mathbb{R}$-KMF.*

Proof. Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{Z}$-KMF and $\epsilon > 0$. Then there

exist a real trigonometric polynomial $p_\epsilon(x) = \sum_{d\in\mathcal{D}} a_d^\epsilon \Re\left(e^{-2\pi idx}\right)$ satisfying $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ with $a_d^\epsilon \in \mathbb{R}$ and where the set $\{d : a_d^\epsilon \neq 0\} \subset \mathcal{D}$ is nonempty and finite.

The measure $\nu_\epsilon = \sum_{d\in\mathcal{D}} a_d^\epsilon \delta_d$ yields a nonzero real function

$$p_\epsilon(x) = \int_{\mathcal{D}} \Re\left(e^{-2\pi itx}\right)\, d\nu_\epsilon(t)$$

satisfying $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ where ν_ϵ is a finite measure on $\mathcal{D}$. Hence, $\mathcal{D}$ is $\mathbb{R}$-KMF.

Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{R}$-KMF. Then it is $\mathbb{R}$-FMRiesz by Proposition 3.43, hence $\mathbb{Z}$-FMRiesz by Proposition 4.44. By the integer characterisation (Theorem 8), it is therefore $\mathbb{Z}$-KMF. □

4.3.3 Intersectivity

Proposition 4.46. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be a $\mathbb{R}$-intersective set. Then it is $\mathbb{Z}$-intersective.*

Proof. Let $E \subseteq \mathbb{Z}$ be given with $\overline{d}_{\mathbb{Z}}(E) > 0$ as well as $T = n + \frac{1}{10} + R$ with $0 \leq R < 1$. Consider

$$\tilde{E} := \bigcup_{n\in E} [n - \frac{1}{10}, n + \frac{1}{10}]$$

which satisfies

$$\begin{aligned}
\overline{d}_{\mathbb{R}}(\tilde{E}) &= \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{1}_{\tilde{E}}(t)\ d\lambda(t) \\
&= \limsup_{T\to\infty} \frac{1}{2T}\int_{-n-\frac{1}{10}}^{n+\frac{1}{10}} \mathbb{1}_{\tilde{E}}(t)\ d\lambda(t) + \frac{1}{2T}\int_{n+\frac{1}{10}}^{T} \mathbb{1}_{\tilde{E}}(t)\ d\lambda(t) \\
&\quad + \frac{1}{2T}\int_{-T}^{-n-\frac{1}{10}} \mathbb{1}_{\tilde{E}}(t)\ d\lambda(t) \\
&\overset{2.26}{=} \limsup_{T\to\infty} \frac{1}{2T}\int_{-n-\frac{1}{10}}^{n+\frac{1}{10}} \mathbb{1}_{\tilde{E}}(t)\ d\lambda(t) = \limsup_{T\to\infty} \frac{1}{5}\cdot\frac{|E\cap\{-n,...,n\}|}{2T} \\
&\overset{2.25}{=} \frac{1}{5}\overline{d}_{\mathbb{Z}}(E) > 0.
\end{aligned}$$

By $\mathbb{R}$-intersectivity, there exists $d \in \mathcal{D}$ and $\tilde{a}, \tilde{b} \in \tilde{E}$ such that

$$\tilde{a} - \tilde{b} = d \in \mathcal{D} \subseteq \mathbb{Z}.$$

Since the difference is an integer, the fractional parts of $\tilde{a}$ and $\tilde{b}$ are the same. Let $n \in E$ be such that $\tilde{a} \in [n - \frac{1}{10}, n + \frac{1}{10}]$, and let $m = \tilde{b} - (\tilde{a} - n) \in \mathbb{Z}$. Since $\tilde{b} \in \tilde{E}$ and $|\tilde{b} - m| \leq \frac{1}{10}$, we have $m \in E$. Moreover,

$$n - m = \tilde{a} - \tilde{b} = d \in \mathcal{D},$$

hence, $\mathcal{D}$ is $\mathbb{Z}$-intersective. □

Similarly, we obtain the following result.

Proposition 4.47. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{R}$-(strongly) combinatorially recurrent. Then it is $\mathbb{Z}$-(strongly) combinatorially recurrent.*

Lemma 4.48. *Let $E \subseteq \mathbb{R}$ with $\overline{d}_{\mathbb{R}}(E) > 0$ be given. Then there exists $x \in [0, 1)$ such that*

$$\overline{d}_{\mathbb{N}}(\{n \in \mathbb{Z} : x + n \in E\}) > 0.$$

Proof. For a contradiction, let us assume the contrary, i.e. for all $x \in [0,1)$, we have

$$\overline{d}_{\mathbb{Z}}(\{n \in \mathbb{Z} : x + n \in E\}) = 0.$$

We define sets $E_n := E \cap [n, n+1)$ for $n \in \mathbb{N}$ which satisfy $E = \bigcup_{-\infty}^{\infty} E_n$. We also use the Reverse Fatou Lemma

$$\limsup_{T\to\infty} \int_E f_T(t)\ d\lambda(t) \leq \int_E \limsup_{T\to\infty} f_T(t)\ d\lambda(t), \tag{4.4}$$

given an integrable g on E such that $f_T \leq g$ for all T, which is satisfied in our considerations for $E = [0,1]$ and $g = \mathbb{1}$. For simplicity, we consider $T \in \mathbb{N}$ as for $T \notin \mathbb{N}$, we only obtain an additional term tending to zero (Lemma 2.26). We set $\langle y \rangle$ as the fractional part of y. Since $n \leq x < n+1$ for all $x \in E_n$, the expression $\langle E_n \rangle$ is meaningful.

$$\begin{aligned}
&\overline{d}_{\mathbb{R}}(E) = \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathbb{1}_E(t)\ d\lambda(t) = \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{n=-\infty}^{\infty} \mathbb{1}_{E_n}(t)\ d\lambda(t) \\
&\overset{MCT}{=} \limsup_{T\to\infty} \frac{1}{2T} \sum_{n=-\infty}^{\infty} \int_{-T}^{T} \mathbb{1}_{E_n}(t)\ d\lambda(t) = \limsup_{T\to\infty} \frac{1}{2T} \sum_{n=-T}^{T-1} \int_{n}^{n+1} \mathbb{1}_{E_n}(t)\ d\lambda(t) \\
&= \limsup_{T\to\infty} \frac{1}{2T} \sum_{n=-T}^{T-1} \int_0^1 \mathbb{1}_{\langle E_n \rangle}(t)\ d\lambda(t) = \limsup_{T\to\infty} \int_0^1 \frac{1}{2T} \sum_{n=-T}^{T-1} \mathbb{1}_{\langle E_n \rangle}(t)\ d\lambda(t) \\
&\overset{(4.4)}{\leq} \int_0^1 \limsup_{T\to\infty} \frac{1}{2T} \sum_{n=-T}^{T-1} \mathbb{1}_{\langle E_n \rangle}(t)\ d\lambda(t) = \int_0^1 \overline{d}_{\mathbb{Z}}(\{n \in \mathbb{Z} : t + n \in E_n\})\ d\lambda(t) \\
&= \int_0^1 0\ d\lambda(t) = 0,
\end{aligned}$$

contradicting $\overline{d}_{\mathbb{R}}(E) > 0$. □

Remark 4.49. We actually even get a set with positive measure on $[0,1)$ in Lemma 4.48 such that $\overline{d}_{\mathbb{Z}}(\{n \in \mathbb{Z} : x + n \in E\} > 0$ for all $x \in E$, not just a single point $x \in [0,1)$.

Proposition 4.50. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be a $\mathbb{Z}$-intersective set. Then it is $\mathbb{R}$-intersective.*

Proof. By Lemma 4.48, there exists $x \in [0,1)$ such that

$$\overline{d}_{\mathbb{Z}}(\{n \in \mathbb{Z} : x + n \in E\}) > 0.$$

We set $\tilde{E} := \{n \in \mathbb{Z} : x + n \in E\}$ which satisfies $\overline{d}_{\mathbb{Z}}(\tilde{E}) > 0$. Since $\mathcal{D}$ is $\mathbb{Z}$-intersective, there exist $m > n \in \tilde{E}$ and $d \in \mathcal{D}$ such that $m - n = d$, hence, $(x + m) - (x + n) = m - n = d$, but $x + m, x + n \in E$, and $\mathcal{D}$ is hence $\mathbb{R}$-intersective. □

Remark 4.51. For $\mathcal{D} \subseteq \mathbb{Z}$, we also obtain the implication

$$\mathbb{Z}\text{-combinatorial recurrence} \Rightarrow \mathbb{R}\text{-combinatorial recurrence}$$

by using the equivalences

$$\begin{aligned}\mathbb{Z}\text{-combinatorial recurrence} &\Leftrightarrow \mathbb{Z}\text{-real correlativity}\\ &\Leftrightarrow \mathbb{R}\text{-real correlativity}_*\end{aligned}$$

and similarly the implication

$$\mathbb{Z}\text{-strong combinatorial recurrence} \Rightarrow \mathbb{R}\text{-strong combinatorial recurrence.}$$

4.3.4 Correlativity

Proposition 4.52. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{R}$-correlative$_*$. Then it is $\mathbb{Z}$-correlative.*

Proof. Let $f \in l^\infty_*(\mathbb{Z}, \mathbb{T})$ be such that $\lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} f(j+d)\overline{f(j)} = 0$ for all $d \in \mathcal{D}$. We define a function $\tilde{f}$ on $\mathbb{R}$ as follows.

$$\tilde{f}(t) := \sum_{j\in\mathbb{Z}} f(j)\mathbb{1}_{(j-1,j]}(t).$$

with $f(j) \in \mathbb{T}$ for all $j \in \mathbb{Z}$. Without loss of generality, we only consider $T \in \mathbb{N}$ as the fractional part of T yields a term which vanishes for $T \to \infty$ (Lemma 2.26).

Then

$$\begin{aligned}
\oint_{\mathbb{R}} \tilde{f}(t+d)\tilde{f}(t)\ d\lambda(t) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \tilde{f}(t+d)\tilde{f}(t)\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(\sum_{j\in\mathbb{Z}} f(j)\mathbb{1}_{(j-1,j]}(t+d)\right)\left(\sum_{i\in\mathbb{Z}} \overline{f(i)}\mathbb{1}_{(i-1,i]}(t)\right)\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\int_{-T}^{T} f(j)\mathbb{1}_{(j-1,j]}(t+d)\overline{f(i)}\mathbb{1}_{(i-1,i]}(t)\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\sum_{k=-T+1}^{T}\int_{k-1}^{k} f(j)\mathbb{1}_{(j-1,j]}(t+d)\overline{f(i)}\mathbb{1}_{(i-1,i]}(t)\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}\sum_{k=-T+1}^{T} f(j)\mathbb{1}_{\{j\}}(k+d)\overline{f(i)}\mathbb{1}_{\{i\}}(k) \\
&= \lim_{T\to\infty} \frac{1}{2T}\sum_{k=-T+1}^{T}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} f(j)\mathbb{1}_{\{j\}}(k+d)\overline{f(i)}\mathbb{1}_{\{i\}}(k) \\
&= \lim_{T\to\infty} \frac{1}{2T}\sum_{k=-T+1}^{T}\left(\sum_{j\in\mathbb{Z}} f(j)\mathbb{1}_{\{j\}}(k+d)\right)\left(\sum_{i\in\mathbb{Z}} \overline{f(i)}\mathbb{1}_{\{i\}}(k)\right) \\
&= \lim_{T\to\infty} \frac{1}{2T+1}\sum_{k=-T}^{T} f(k+d)\overline{f(k)} = 0
\end{aligned}$$

for all $d \in \mathcal{D}$. $\mathbb{R}$-Correlativity$_*$ then implies

$$\oint_{\mathbb{R}} \tilde{f}(t)\ d\lambda(t) = 0.$$

We hence obtain

$$\begin{aligned}
0 = \oint_{\mathbb{R}} \tilde{f}(t)\ d\lambda(t) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \tilde{f}(t)\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\sum_{j\in\mathbb{Z}} f(j)\mathbb{1}_{(j-1,j]}(t)\ d\lambda(t) = \lim_{T\to\infty} \frac{1}{2T}\sum_{j\in\mathbb{Z}}\int_{-T}^{T} f(j)\mathbb{1}_{(j-1,j]}(t)\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T}\sum_{j\in\mathbb{Z}}\sum_{k=-T+1}^{T}\int_{k-1}^{k} f(j)\mathbb{1}_{(j-1,j]}(t)\ d\lambda(t) \\
&= \lim_{T\to\infty} \frac{1}{2T}\sum_{j\in\mathbb{Z}}\sum_{k=-T+1}^{T} f(j)\mathbb{1}_{\{j\}}(k) = \lim_{T\to\infty} \frac{1}{2T}\sum_{k=-T+1}^{T}\sum_{j\in\mathbb{Z}} f(j)\mathbb{1}_{\{j\}}(k) \\
&= \lim_{T\to\infty} \frac{1}{2T}\sum_{k=-T+1}^{T}\sum_{j\in\mathbb{Z}} f(j)\mathbb{1}_{\{j\}}(k) = \lim_{T\to\infty} \frac{1}{2T+1}\sum_{k=-T}^{T} f(k).
\end{aligned}$$

□

Similarly, we obtain the following result.

Proposition 4.53. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{R}$-real correlative$_*$ ($\mathbb{R}$-enhanced van der Corput$_*$, $\mathbb{R}$-strongly correlative$_*$). Then it is $\mathbb{Z}$-real correlative ($\mathbb{Z}$-enhanced van der Corput, $\mathbb{Z}$-strongly correlative).*

Remark 4.54. As $\tilde{f}(\cdot + s)\overline{\tilde{f}(\cdot)}$ is a step function and $\tilde{f}$ is a step function with interval length 1 in the proof of Proposition 4.52, we even have

$$\lim_{s\to r} \oint_{\mathbb{R}} \tilde{f}(t+s)\overline{\tilde{f}(t)}\, d\lambda(t) = \lim_{s\to r} \oint_{\mathbb{R}} \tilde{f}(t)\overline{\tilde{f}(t)}\, d\lambda(t) + \lim_{s\to r}\lim_{T\to\infty} \frac{|s-r|}{2T} \sum_{k=-T}^{T} a_k$$
$$= \oint_{\mathbb{R}} \tilde{f}(t)\overline{\tilde{f}(t)}\, d\lambda(t)$$

with $|a_k| \leq 2$ whenever the corresponding averaged integrals exist, i.e. we have continuity. However,

$$\oint_{\mathbb{R}} f(t+d)f(t)\, d\lambda(t) = \lim_{T\to\infty} \frac{1}{2T+1} \sum_{k=-T}^{T} f(k+d)\overline{f(k)} \tag{4.5}$$

does not need to exist for all $d \in \mathbb{R}$, for example, using g with $g(n) := (-1)^{m-1}$ for $2^m \leq |n| < 2^{m+1}$ and $g(0) = 1$, we define f recursively by setting $f(0) := 1$, $f(n) := \frac{g(n-1)}{f(n-1)}$ for $n > 0$ and $f(n) := \frac{g(n)}{f(n+1)}$ for $n < 0$ and

$$\frac{1}{2N+1} \sum_{n=-N}^{N} f(n+1)\overline{f(n)} = \frac{1}{2N+1} \sum_{k=-N}^{N} g(n)$$

does not converge, and hence, we have $f \in l^\infty_*(\mathbb{Z}, \mathbb{T})$, but not $\tilde{f} \in l^\infty_c(\mathbb{R}, \mathbb{T})$. A similar problem appears for enhanced van der Corput, strong correlativity, real correlativity, real correlativity^{0,1} as well as for continuous intersectivity and continuous combinatorial recurrence in Propositions 4.46 and 4.47 although the implications

$\mathbb{R}$-enhanced van der Corput $\Leftrightarrow$ $\mathbb{Z}$-enhanced van der Corput,

$\mathbb{R}$-correlativity $\Leftrightarrow$ $\mathbb{Z}$-correlativity can be obtained by using operator recurrence (Propositions 4.35 and 4.41).

One could adapt the requirements and assume the existence of (4.5) for $f \in l^\infty_*(\mathbb{R})$ and $f \in l^\infty_*(\mathbb{Z})$ as it is done for the continuous versions. In order to keep the existing implications, other properties have to be adapted accordingly, also the integer variants, such as the following.

> A set $\mathcal{D} \subseteq \mathbb{R}$ is combinatorially recurrent if, given $E \subseteq \mathbb{R}$ with $d_\mathbb{R}(E) > 0$ such that $d_\mathbb{R}(E \cap E - t)$ exists for all $t \in \mathbb{R}$, there exists $d \in \mathcal{D}$ such that $d_\mathbb{R}(E \cap (E - d)) > 0$.

This would yield the equivalence of integer and real recurrence properties for a set $\mathcal{D} \subseteq \mathbb{Z}$ to hold for all properties in Theorem 9. However, the new constraints for combinatorial recurrence and intersectivity are stronger then for the classical recurrence properties and the equivalences

$$\text{real correlativity}^{0,1}_* \Leftrightarrow \text{real correlativity}_*,$$

$$\text{strong correlativity}^{0,1}_* \Leftrightarrow \text{strong correlativity}_*$$

become uncertain, hence, also the equivalences of the properties around Poincaré$_*$ and strong recurrence$_*$ in Theorem 9.

Proposition 4.55. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{Z}$-real correlative. Then it is $\mathbb{R}$-real correlative.*

Proof. Let $0 \leq f \in l^\infty_c(\mathbb{R}, \mathbb{R})$ with $\limsup_{T\to\infty} ⨏_{-T}^{T} f(t)\ d\lambda(t) > 0$ be given. Then we have

$$\begin{aligned} 0 < \limsup_{T\to\infty} ⨏_{-T}^{T} f(t)\ d\lambda(t) &= \limsup_{T\to\infty} \frac{1}{2T} \sum_{k=-T}^{T-1} \int_k^{k+1} f(t)\ d\lambda(t) \\ = \limsup_{T\to\infty} \int_0^1 \frac{1}{2T} \sum_{k=-T}^{T-1} f(t+k)\ d\lambda(t) &\leq \int_0^1 \frac{1}{2T} \limsup_{T\to\infty} \sum_{k=-T}^{T-1} f(t+k)\ d\lambda(t) \end{aligned}$$

by the Reverse Fatou Lemma (4.4). Hence, there exists a set $\Omega_p \subseteq [0, 1)$ with positive measure such that

$$\limsup_{T\to\infty} \frac{1}{2T} \sum_{k=-T}^{T-1} f(t+k) > 0$$

for all $t \in \Omega_p$. Hence, by $\mathbb{Z}$-real correlativity, for each $t \in \Omega_p$, there exists $d_t \in \mathcal{D}$ such that

$$0 < \limsup_{T\to\infty} \frac{1}{2T} \sum_{k=-T}^{T-1} f(t + d_t + k) f(t + k).$$

In particular, using countability of $\mathcal{D}$ and

$$\bigcup_{d\in\mathcal{D}} \left\{ t \in \Omega_p : \limsup_{T\to\infty} \frac{1}{2T} \sum_{k=-T}^{T-1} f(t + d + k) f(t + k) > 0 \right\} = \Omega_p,$$

there exists $d \in \mathcal{D}$ and $\Omega_d \subseteq \Omega_p$ with positive measure such that

$$0 < \limsup_{T\to\infty} \frac{1}{2T} \sum_{k=-T}^{T-1} f(t + d + k) f(t + k).$$

for all $t \in \Omega_d$. But this implies

$$\begin{aligned} 0 &< \limsup_{T\to\infty} \int_{\Omega_d} \frac{1}{2T} \sum_{k=-T}^{T-1} f(t + d + k) f(t + k) \; d\lambda(t) \\ &\leq \limsup_{T\to\infty} \int_0^1 \frac{1}{2T} \sum_{k=-T}^{T-1} f(t + d + k) f(t + k) \; d\lambda(t) \\ &= \limsup_{T\to\infty} ⨍_{-T}^{T} f(t + d) f(t) \; d\lambda(t) \end{aligned}$$

and $\mathcal{D}$ is hence $\mathbb{R}$-real correlative. □

Similarly, we obtain the following result.

Proposition 4.56. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be $\mathbb{Z}$-strongly correlative set. Then it is $\mathbb{R}$-strongly correlative.*

Remark 4.57. We note that we obtain

$$\mathbb{Z}\text{-real correlativity}^{0,1} \Leftrightarrow \mathbb{R}\text{-real correlativity}_*^{0,1},$$

$$\mathbb{Z}\text{-strong correlativity}^{0,1} \Leftrightarrow \mathbb{R}\text{-strong correlativity}_*^{0,1}$$

as in Propositions 4.52 and 4.55 for $\mathcal{D} \subseteq \mathbb{Z}$ (see Definition 3.52 for the definition of real correlativity$_*^{0,1}$).

4.3.5 Classical Examples

The results in this section show that classical examples of sets having properties such as intersectivity or operator recurrence also give examples for the associated real properties. We state some of these examples to present more interesting recurrence sets.

We state the examples with properties which are most convenient for us (we refer to the established classic characterisations in Theorem 8). A first example is given by the perfect squares ([34, Theorem 1.2], [73], [74]).

Theorem 4.58 (Fürstenberg, Sárközy)**.** *The set* $\{n^2 : n \in \mathbb{N}\}$ *is* $\mathbb{Z}$*-intersective and* $\mathbb{Z}$*-Poincaré.*

Corollary 4.59. *The set* $\{n^2 : n \in \mathbb{N}\}$ *is* $\mathbb{R}$*-intersective and* $\mathbb{R}$*-Poincaré.*

Theorem 4.58 can be generalised to polynomials with integer coefficients ([36, Theorem 3.16] and [75], see also [14, Theorem 0.1]).

Theorem 4.60 (Fürstenberg, Sárközy)**.** *Let* $0 \neq f$ *be a polynomial with integer coefficients and* $f(0) = 0$*. Then*

$$\{f(n) : n \in \mathbb{N}\} \backslash \{0\}$$

is $\mathbb{Z}$*-intersective and* $\mathbb{Z}$*-Poincaré.*

Corollary 4.61. *Let* f *be a polynomial with integer coefficients and* $f(0) = 0$*. Then* $\{f(n) : n \in \mathbb{N}\}$ *is* $\mathbb{R}$*-intersective and* $\mathbb{R}$*-Poincaré.*

Kamae and Mendès France have given necessary and sufficient conditions for a polynomial to satisfy the condition in Theorem 4.60 ([14, Proposition 1.20], [42, Example 3]).

Theorem 4.62 (Kamae, Mendès France)**.** *Let* $f \neq 0$ *be a polynomial with integer coefficients. The set* $\{f(n) : n \in \mathbb{N}\}$ *is* $\mathbb{Z}$*-intersective (*$\mathbb{Z}$*-operator recurrent) if and* *only if*

$$f(\mathbb{N}) \cap a\mathbb{Z} \neq \emptyset \tag{4.6}$$

for all $a \in \mathbb{N}$.

The condition $f(0) = 0$ obviously satisfies (4.6) since f can be written as $f(n) = nq(n)$ with another integer polynomial q.

Corollary 4.63. *Let* $f \neq 0$ *be a polynomial with integer coefficients. The set* $\{f(n) : n \in \mathbb{N}\}$ *is* $\mathbb{R}$*-intersective (*$\mathbb{R}$*-operator recurrent) if and only if* $f(\mathbb{N}) \cap a\mathbb{Z} \neq \emptyset$ *for all* $a \in \mathbb{N}$.

Another interesting area is recurrence along primes. The set $\{p : p \text{ prime}\}$ cannot be any set of recurrence which a rotation with period 4 easily shows. However, the sets $\{f(p+1) : p \text{ prime}\}$ and $\{f(p-1) : p \text{ prime}\}$ do exhibit such properties for a suitable polynomial f ([42, Example 3], [14, Proposition 1.22 and Corollary 2.13]).

Theorem 4.64 (Kamae, Mendès France)**.** *Let* $f \neq 0$ *be a polynomial with integer coefficients and* $f(0) = 0$*. Then the sets* $\{f(p+1) : p \text{ prime}\}$ *and* $\{f(p-1) : p \text{ prime}\}$ *are* $\mathbb{N}$*-strongly operator recurrent. In particular, the sets* $\{p+1 : p \text{ prime}\}$ *and* $\{p-1 : p \text{ prime}\}$ *are* $\mathbb{N}$*-strongly operator recurrent (*$\mathbb{N}$*-operator recurrent,* $\mathbb{N}$*-intersective).*

Corollary 4.65. *Let* $f \neq 0$ *be a polynomial with integer coefficients and* $f(0) = 0$*. Then the sets* $\{f(p+1) : p \text{ prime}\}$ *and* $\{f(p-1) : p \text{ prime}\}$ *are* $\mathbb{R}$*-strongly operator recurrent (*$\mathbb{R}$*-operator recurrent,* $\mathbb{R}$*-intersective).*

Measure preserving systems and Cesàro convergence have a special importance for recurrence sets. It is therefore not surprising that ergodic sequences yield further examples ([14, Proposition 2.10], [82, Lemma 5.5], [39, Lemma 1.5]).

Theorem 4.66 (Bergelson)**.** *Let $(d_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ be an ergodic sequence, i.e. given an ergodic scmps $(\Omega, \Sigma, \mu; \phi)$ and $f \in L^2(\Omega, \Sigma, \mu)$, we have*

$$\frac{1}{N}\sum_{n=1}^{N} T_\phi^{d_n} f \xrightarrow{L^2} \int_\Omega f \ d\mu.$$

Then $\{d_n : n \in \mathbb{N}\}$ is $\mathbb{Z}$-strongly operator recurrent ($\mathbb{Z}$-operator recurrent).

Corollary 4.67. *Let $(d_n)_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ be an ergodic sequence. Then $\{d_n : n \in \mathbb{N}\}$ is $\mathbb{R}$-strongly operator recurrent ($\mathbb{R}$-operator recurrent).*

Remark 4.68. The sequences $\{\lfloor bn^c \rfloor : n \in \mathbb{N}\}$ with $b \neq 0$ and irrational $c > 1$ and $\{\lfloor bn^c + d(\log n)^a \rfloor : n \in \mathbb{N}\}$ with $b, d \neq 0$, $c \geq 1$ and $a > 1$ are ergodic sequences ([14, Proposition 2.10] with reference to [22]), hence, $\mathbb{R}$-strongly operator recurrent ($\mathbb{R}$-operator recurrent).

Bourgain ([23]) gave a result of a different flavour, showing that the generic density condition for strongly operator recurrent, operator recurrent, strongly recurrent and Poincaré sets coincide, i.e. if we choose a set $\mathcal{D}$ randomly with some given pattern, then $\mathcal{D}$ has almost surely either all these properties or none.

Theorem 4.69 (Bourgain)**.** *Let $\mathbb{N} = \cup_{k=1}^{\infty} I_k$ with $I_k := [2^{2^k}, 2^{2^{k+1}})$ be a partition of the integers in intervals. Choose for each k a random subset $\mathcal{D}_k$ with $N_k = |\mathcal{D}_k|$ elements, assigning to each element of I_k the same probability δ_k. Let $\mathcal{D} = \cup_{k=1}^{\infty} \mathcal{D}_k$. Then the following holds almost surely.*
(i) If $\limsup_{k\to\infty} 2^{-k} N_k < \infty$, then $\mathcal{D}$ is not $\mathbb{N}$-Poincaré.
(ii) If $\limsup_{k\to\infty} 2^{-k} N_k = \infty$, then $\mathcal{D}$ is $\mathbb{N}$-strongly operator recurrent.

Corollary 4.70. *Let $\mathcal{D} \subseteq \mathbb{N}$ be chosen as in Theorem 4.69. Then it is either $\mathbb{R}$-strongly recurrent (hence, operator recurrent, strongly recurrent, Poincaré) or not $\mathbb{R}$-intersective.*

4.4 The Sets $\{\frac{1}{\log p}, p$ prime$\}$ and $\{\log p, p$ prime$\}$

We first consider the set $\{\frac{1}{\log p_n} : p_n \text{ prime}\}$ where its recurrence properties are directly characterised by the limit point 0.

Proposition 4.71. *The set $\{\frac{1}{\log p_n} : p_n$ prime$\}$ is strongly operator recurrent around 0, hence, operator recurrent, strongly recurrent and Poincaré.*

Proof. We note that $\log p_n \to \infty$ as $n \to \infty$, hence 0 is a limit point of $\{\frac{1}{\log p_n} : p_n \text{ prime}\}$ and the recurrence properties follow by Proposition 4.9 and Corollary 4.26. □

The set $\{\log p_n : p_n \text{ prime}\}$ is more interesting than $\{\frac{1}{\log p_n} : p_n \text{ prime}\}$ as it does not only depend on $\log p_n \to \infty$. The main tool for the proof of Proposition 4.72 is a version version of the prime number theorem by Erdös ([32]) showing that

$$\frac{p_{n+1}}{p_n} \overset{n\to\infty}{\longrightarrow} 1,$$

and hence

$$\log p_{n+1} - \log p_n \overset{n\to\infty}{\longrightarrow} 0.$$

Using uniform continuity of $\langle T_t x, x\rangle$ for a given Hilbert space H, unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, this implies the following result.

Proposition 4.72. *The set $\{\log p_n : p_n$ prime$\}$ is strongly operator recurrent, hence, operator recurrent, strongly recurrent and Poincaré.*

Since the set $\{\log p_n : p_n \text{ prime}\}$ is rationally linearly independent by Lemma 4.28, it also yields a further counterexample to the equivalence of Poincaré and combinatorial recurrence since it is not combinatorially recurrent by Theorem 4.27.

Proposition 4.73. *The set $\{\log p_n : p_n$ prime$\}$ is not combinatorially recurrent, hence, not strongly combinatorially recurrent.*

Chapter 5

Reducing Recurrence Sets

We note that we can arbitrarily enlarge a given recurrence set without loosing its recurrence property. In this chapter, we discuss several ways to reduce a given set while keeping its recurrence property. As the main tool is continuity, we restrict ourselves to the consideration of properties such as Poincaré and operator recurrence and their strong variants.

Since bounded recurrence properties are characterised by the limit point 0 (Propositions 4.9 and 4.14), we can reduce a set $\mathcal{D}$ with these properties as long as 0 is still a limit point of the reduced set $\tilde{\mathcal{D}}$. In particular, every such set has a countable subset with the same property. Our main focus hence is on asymptotic recurrence sets.

5.1 Reducing Results

Proposition 5.1. *Let $\mathcal{D} \subseteq \mathbb{R}$. Then the following are equivalent.*

(i) $\tilde{\mathcal{D}}$ is operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré) where $\tilde{\mathcal{D}}$ denotes an arbitrary dense subset of $\mathcal{D}$.

(ii) $\mathcal{D}$ is operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré).

(iii) $\overline{\mathcal{D}}$ *is operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré).*

Proof. We only have to show (iii) $\Rightarrow$ (i) since (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follows trivially.
Let the Hilbert space H, the strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$ be given. We note that $f_x(t) := \langle T_t x, x\rangle$ is uniformly continuous by Lemma 2.27.
Since $\overline{\mathcal{D}}$ is an operator recurrence set, there exists $d \in \overline{\mathcal{D}}$ such that $|\langle T_d x, x\rangle| = \epsilon \neq 0$. By Lemma 2.28, there exists $\delta_\epsilon > 0$ such that $\langle T_t, x\rangle \neq 0$ for all $|t-d| < \delta_\epsilon$. Since $\tilde{\mathcal{D}}$ is dense in $\overline{\mathcal{D}}$, there exists $\tilde{d} \in \tilde{\mathcal{D}}$ with $|d - \tilde{d}| < \delta_\epsilon$, hence $|\langle T_{\tilde{d}} x, x\rangle| > 0$ by Lemma 2.28. $\square$

Since any subspace of a separable metric space is itself separable ([58, Section III.4]), we obtain the following lemma.

Lemma 5.2. *Let* $\mathcal{D} \subseteq \mathbb{R}$. *Then there exists a countable set* $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ *which is dense in* $\mathcal{D}$.

Proposition 5.1 and Lemma 5.2 yield the following corollary.

Corollary 5.3. *Let* $\mathcal{D} \subseteq \mathbb{R}$ *be operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré). Then there exists a countable subset* $\tilde{\mathcal{D}}$ *which is still operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré).*

Proposition 5.4. *Let* $\mathcal{D} \subseteq \mathbb{R}_*$ *be operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré). Then*

$$\tilde{\mathcal{D}} := \mathcal{D} \backslash [-M, M]$$

is still operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré).

Proof. Let the Hilbert space H, the strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$ be given. Without loss of generality, we assume $M > \tilde{\epsilon}$, where $\tilde{\epsilon}$ is taken from the definition of $\mathbb{R}_*$.
By the operator recurrence property, there exists $d \in \mathcal{D}$ such that $\langle T_d x, x\rangle \neq 0$. For a contradiction, let us assume that

$$\langle T_d x, x\rangle = 0 \tag{5.1}$$

for all $d \in \mathcal{D}\backslash[-M, M]$.
Let $(\mathbb{T}, \lambda; (\tau_t)_{t\in\mathbb{R}})$ be the rotation on the torus with period $M + \frac{1}{2}\tilde{\epsilon}$ and let $E = e^{2\pi i[0,\frac{1}{2}\tilde{\epsilon}]}$ with $\tilde{\epsilon}$ from the definition of $\mathbb{R}_*$. We note that $P_\tau \mathbb{1}_E \neq 0$ by Lemma 2.11 and $\langle T_t^\tau \mathbb{1}_E, \mathbb{1}_E\rangle = 0$ for all $t \in \mathbb{R}_* \cap [-M, M]$.
Then $(H \otimes L^2(\mathbb{T}, \lambda); (T_t \otimes T_t^\tau)_{t\in\mathbb{R}})$ is a strongly continuous unitary group, and we have $\mathrm{Fix}((T_t)_{t\in\mathbb{R}}) \otimes \mathrm{Fix}((T_t^\tau)_{t\in\mathbb{R}}) \subseteq \mathrm{Fix}((T_t \otimes T_t^\tau)_{t\in\mathbb{R}})$ since

$$(T_t \otimes T_t^\tau)(x \otimes y) = (T_t x) \otimes (T_t^\tau y) = x \otimes y$$

for all $t \in \mathbb{R}$ and $x \otimes y \in \mathrm{Fix}((T_t)_{t\in\mathbb{R}}) \otimes \mathrm{Fix}((T_t^\tau)_{t\in\mathbb{R}})$, hence, $(P_T \otimes P_\tau)(x \otimes \mathbb{1}_E) \neq 0$ implies

$$P(x \otimes \mathbb{1}_E) \neq 0$$

as in Lemma 2.10. However, we have

$$\langle (T_d \otimes T_d^\tau)(x \otimes \mathbb{1}_E), x \otimes \mathbb{1}_E\rangle = \langle T_d x, x\rangle \cdot \langle T_d^\tau \mathbb{1}_E, \mathbb{1}_E\rangle = 0$$

for all $d \in \mathcal{D}$, for $\mathcal{D}\backslash[-M, M]$ by Assumption (5.1) and for $\mathcal{D} \cap [-M, M]$ by the choice of $(\mathbb{T}, \lambda; (\tau_t)_{t\in\mathbb{R}})$. This yields a contradiction to the operator recurrence property of $\mathcal{D}$. □

Corollary 5.5. *Every operator recurrence (strongly operator recurrent, strongly recurrent, Poincaré) set $\mathcal{D} \subseteq \mathbb{R}_*$ is unbounded.*

Proof. For a contradiction, let us assume that $|d| < M < \infty$ for all $d \in \mathcal{D}$ and for some $M > 0$. Then $\mathcal{D}\backslash[-M, M] = \emptyset$ is operator recurrent by Proposition 5.4, yielding a contradiction. $\square$

Proposition 5.6. *Let $\mathcal{D} \subseteq \mathbb{R}_*$ be operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré), and let $s > 0$ and ϵ with $0 < \epsilon < \frac{s}{2}$ be given. Then*

$$\tilde{\mathcal{D}} := \mathcal{D} \cap \left(\bigcup_{n \in \mathbb{Z}} [sn - \epsilon, sn + \epsilon] \right)$$

is still operator recurrent (strongly operator recurrent, strongly recurrent, Poincaré).

Proof. Let the Hilbert space H, the strongly continuous unitary group $(T_t)_{t \in \mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$ be given.
For a contradiction, let us assume that

$$\langle T_d x, x \rangle = 0 \tag{5.2}$$

for all $d \in \mathcal{D} \cap (\cup_{n \in \mathbb{Z}} [sn - \epsilon, sn + \epsilon])$.
Let $(\mathbb{T}, \lambda; (\tau_t)_{t \in \mathbb{R}})$ be the rotation on the torus with period s and let $E = e^{2\pi i [0, \frac{\epsilon}{s}]}$. We note that $P_\tau \mathbb{1}_E \neq 0$ by Lemma 2.11 and $\langle T_t^\tau \mathbb{1}_E, \mathbb{1}_E \rangle = 0$ for all $t \notin \cup_{n \in \mathbb{Z}} [sn - \epsilon, sn + \epsilon]$. Then $(H \otimes L^2(\mathbb{T}, \lambda); (T_t \otimes T_t^\tau)_{t \in \mathbb{R}})$ is a strongly continuous unitary group and we have

$$P(x \otimes \mathbb{1}_E) \neq 0$$

as in Proposition 5.4. However, we have

$$\langle (T_d \otimes T_d^\tau)(x \otimes \mathbb{1}_E), x \otimes \mathbb{1}_E \rangle = \langle T_d x, x \rangle \cdot \langle T_d^\tau \mathbb{1}_E, \mathbb{1}_E \rangle = 0$$

for all $d \in \mathcal{D}$, for $\mathcal{D} \cap (\cup_{n \in \mathbb{Z}} [sn - \epsilon, sn + \epsilon])$ by Assumption (5.2) and for $\mathcal{D} \notin \cup_{n \in \mathbb{Z}} [sn - \epsilon, sn + \epsilon]$ by the choice of $(\mathbb{T}, \lambda; (\tau_t)_{t \in \mathbb{R}})$. This yields a contradiction to the operator recurrence property of $\mathcal{D}$. $\square$

In contrast to Poincaré and operator recurrence sets where we obtained countable, but possibly still at least somewhere dense subsets, we obtain a stronger reduction result for strongly operator recurrent and strongly recurrent sets.
Without loss of generality, let $\mathcal{D} \subseteq \mathbb{R}_*$ be a strong operator recurrence or strong recurrence set of the form

$$\bigcup_{n\in\mathbb{Z}} ([sn-\epsilon, sn+\epsilon] \cap \mathcal{D})$$

for some $s, 0 < \epsilon < \frac{s}{2}$ (Proposition 5.6). We define a set $\tilde{\mathcal{D}}$ as follows. Let

$$d_n^i := \text{“one of the closest points to” } sn + \frac{i}{2^{|n|}}\epsilon$$

be defined for $n \in \mathbb{Z}$ and $-n \leq i \leq n$, i.e. we choose d_n^i such that

$$\left| |d_n^i - (sn + \frac{i}{2^{|n|}}\epsilon)| - \inf\{|d - (sn + \frac{i}{2^{|n|}}\epsilon)| : d \in \mathcal{D}\} \right| < \frac{1}{100 \cdot 2^{|n|}}\epsilon.$$

We note that the same $d \in \mathcal{D}$ may be chosen multiple times for example if the set $\mathcal{D}$ has a gap around some $sn + \frac{i}{2^{|n|}}\epsilon$. However, this does not matter. The set $\tilde{\mathcal{D}}$ is finally given by

$$\tilde{\mathcal{D}} := \{d_n^i : n \in \mathbb{Z}, -n \leq i \leq n\}.$$

Remark 5.7. For all $d \in [sn-\epsilon, sn+\epsilon] \cap \mathcal{D}$, there exists $\tilde{d} \in \tilde{\mathcal{D}}$ (not necessarily distinct from d) such that $|d - \tilde{d}| \leq \frac{1}{2^{|n|}}\epsilon$. This is clearly true if $d \in \tilde{\mathcal{D}}$ (as it may happen if d is an isolated point in $\mathcal{D}$). On the other hand, if there did not exist such a $\tilde{d} \neq d \in (d - \frac{1}{2^{|n|}}\epsilon, d + \frac{1}{2^{|n|}}\epsilon)$, then by the above procedure, we would have chosen d as one of the d_n^i.

Remark 5.8. By construction, the set $\tilde{\mathcal{D}}$ is countable and discrete, i.e. $\tilde{\mathcal{D}} \cap [-N, N]$ is finite for all $N > 0$.

Proposition 5.9. *Let $\mathcal{D} \subseteq \mathbb{R}_*$ be strongly operator recurrent (strongly recurrent). Then $\tilde{\mathcal{D}}$ is strongly operator recurrent (strongly recurrent).*

Proof. Let the Hilbert space H, the strongly continuous unitary group $(T_t)_{t\in\mathbb{R}}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$ with $\limsup_{|d|\to\infty, d\in\mathcal{D}} |\langle T_d x, x\rangle| > M$ be given.

By strong operator recurrence, there exists $\tilde{M} > 0$ and arbitrarily large $d \in \mathcal{D}$ such that $|\langle T_d x, x\rangle| > \tilde{M}$. By Lemma 2.28 (using the uniform continuity of $(\langle T_t x, x\rangle)_{t\in\mathbb{R}}$), there exist $M, \delta > 0$ such that $|\langle T_t x, x\rangle| > M$ for all $t \in (d-\delta, d+\delta)$ whenever $|\langle T_d x, x\rangle| > \tilde{M}$.

Choose $n \in \mathbb{N}$ such that $\frac{1}{2^{|n|}}\epsilon \leq \frac{1}{100}\delta$. Now let $|n| < m \in \mathbb{Z}$ be such that there exists d in the interval $[sm-\epsilon, sm+\epsilon]$ with $|\langle T_d x, x\rangle| > M$, and we note that there are infinitely many such m since $\limsup_{|d|\to\infty, d\in\mathcal{D}} |\langle T_d x, x\rangle| > \tilde{M}$.

For each such m, we either obtain $d \in \tilde{\mathcal{D}}$ or there exists another $d_* \neq d \in \tilde{\mathcal{D}}$ with distance $|d_* - d| \leq \frac{1}{2^{|n|}}\epsilon < \delta$ by Remark 5.7, hence $|\langle T_{d_*} x, x\rangle| > M$.

We finally conclude

$$\limsup_{|d|\to\infty, d\in\tilde{\mathcal{D}}} |\langle T_d x, x\rangle| > M > 0$$

and $\tilde{D}$ is strongly operator recurrent. □

5.2 Ramsey Property

We now discuss the Ramsey property of some recurrence properties, i.e. every finite decomposition of such a recurrent set $\mathcal{D}$ contains at least one set still having the same recurrence property.

Proposition 5.10. *Let $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \subseteq \mathbb{R}$ be Poincaré (strongly recurrent). Then at least one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is also Poincaré (strongly recurrent).*

Proof. Let us assume that neither $\mathcal{D}_1$ nor $\mathcal{D}_2$ are Poincaré. Then there exist scmps $(\Omega_1, \Sigma_1, \mu_1; (\phi_t)_{t\in\mathbb{R}})$, $A_1 \in \Sigma_1$ with $\mu_1(A_1) > 0$ and $(\Omega_1, \Sigma_1, \mu_1; (\psi_t)_{t\in\mathbb{R}})$, $A_2 \in \Sigma_2$

with $\mu_2(A_2) > 0$ such that

$$\mu_1(\phi_t(A_1) \cap A_1) = 0, \quad \mu_2(\phi_t(A_2) \cap A_2) = 0$$

for all $s \in \mathcal{D}_1, t \in \mathcal{D}_2$. But then the scmps $(\Omega_1 \times \Omega_2, \sigma(\Sigma_1, \Sigma_2), \mu_1 \times \mu_2; (\phi_t \times \psi_t)_{t\in\mathbb{R}})$ and the set $A = A_1 \times A_2$ with $(\mu_1 \times \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$ satisfy

$$\begin{aligned}(\mu_1 \times \mu_2)\left((\phi_d \times \psi_d)(A_1 \times A_2) \cap (A_1 \times A_2)\right)&\\ = \mu_1(\phi_t(A_1) \cap A_1) \cdot \mu_2(\phi_t(A_2) \cap A_2) = 0&\end{aligned}$$

for all $d \in \mathcal{D}$, contradicting the Poincaré property of $\mathcal{D}$. □

Iterating inductively yields the following corollary.

Corollary 5.11. *Let $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_n \subseteq \mathbb{R}$ be Poincaré (strongly recurrent). Then at least one $\mathcal{D}_i$ is also Poincaré (strongly recurrent).*

A similar approach Proposition 5.10 yields the Ramsey property for strong operator recurrence and operator recurrence. We use the convolution of measures (see [67, Subsection 1.3.1]) to show the Ramsey property for FMRiesz sets (and therefore for operator recurrence sets by Subsection 3.2.1). The convolution satisfies

$$\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$$

for measures μ and ν.

Proposition 5.12. *Let $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \subseteq \mathbb{R}$ be FMRiesz (FC+). Then at least one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is also FMRiesz (FC+).*

Compare [14, Corollary 1.12].

Proof. For a contradiction, let us assume that neither $\mathcal{D}_1$ nor $\mathcal{D}_2$ are FMRiesz. Then there exist probability measures μ_1 and μ_2 on $\mathbb{R}$ such that $\widehat{\mu_i}(d) = 0$ for all

$d \in \mathcal{D}_i$ and $\mu_i(\{0\}) > 0$. Consider the probability measure $\mu := \mu_1 * \mu_2$ on $\mathbb{R}$. Then we have

$$\widehat{\mu}(d) = \widehat{\mu_1 * \mu_2}(d) = \widehat{\mu_1}(d) \cdot \widehat{\mu_2}(d) = 0$$

for all $d \in \mathcal{D}$. Property FMRiesz for $\mathcal{D}$ then implies

$$\begin{aligned} 0 = (\mu_1 * \mu_2)(\{0\}) &= \int_{\mathbb{R}} \mathbb{1}_{\{0\}}(x)\ d(\mu_1 * \mu_2)(x) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{\{0\}}(x+y)\ d\mu_1(x)\ d\mu_2(y) \geq \mu_1(\{0\}) \cdot \mu_2(\{0\}) \geq 0, \end{aligned}$$

hence, $\mu_1(\{0\}) = 0$ or $\mu_2(\{0\}) = 0$, yielding a contradiction. □

Iterating inductively yields the following corollary.

Corollary 5.13. *Let $\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_n \subseteq \mathbb{R}$ be FMRiesz (FC+). Then at least one $\mathcal{D}_i$ is also FMRiesz (FC+).*

Corollary 5.14. *Let $\mathcal{D} \subseteq \mathbb{R}_*$ be strong operator recurrent (operator recurrent, strong recurrent, Poincaré) and $V \subset \mathcal{D}$ be a finite or bounded set. Then*

$$\tilde{\mathcal{D}} := \mathcal{D} \backslash V$$

is still strong operator recurrent (operator recurrent, strong recurrent, Poincaré).

Proof. A rotation on $\mathbb{T}$ with a sufficiently large period shows that the set V is not operator recurrent (compare Proposition 5.4), and we further have $\mathcal{D} = \tilde{\mathcal{D}} \cup V$. Propositions 5.10 and 5.12 yield the result. □

Corollary 5.15. *Let $\mathcal{D} \subseteq \mathbb{R}_*$ be strongly operator recurrent (operator recurrent, strongly recurrent, Poincaré), and let $s \in \mathbb{R}$ and ϵ with $0 < \epsilon < \frac{s}{2}$ be given. Then at least one of $\tilde{\mathcal{D}}_1 := \mathcal{D} \cap \left(\bigcup_{n\in\mathbb{Z}}[sn, sn+\epsilon]\right)$ and $\tilde{\mathcal{D}}_2 := \mathcal{D} \cap \left(\bigcup_{n\in\mathbb{Z}}[sn-\epsilon, sn)\right)$ is strong operator recurrent (operator recurrent, strong recurrent, Poincaré).*

Proof. By Proposition 5.6, the set

$$\tilde{\mathcal{D}} := \mathcal{D} \cap \left(\bigcup_{n \in \mathbb{Z}} [sn - \epsilon, sn + \epsilon] \right)$$

is an operator recurrence set. We have $\mathcal{D} = \tilde{\mathcal{D}}_1 \cup \tilde{\mathcal{D}}_2$, and Proposition 5.10 yields the result. □

Another interesting Ramsey-like decomposition result holds for integer KMF sets. We note the equivalence of $\mathbb{Z}$-KMF and $\mathbb{Z}$-operator recurrence (Theorem 8), hence, the following integer variant of Corollary 5.14 can be used ([71, Corollary 1]).

Lemma 5.16. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be KMF and $V \subset \mathcal{D}$ be finite. Then $\tilde{\mathcal{D}} := \mathcal{D} \backslash V$ is still KMF.*

Since it is neither clear if the equivalence of KMF and operator recurrence holds for the reals nor if a real KMF set has the Ramsey property or allows a result as in Corollary 5.14, we cannot extend the proof of Proposition 5.17 to the real setting.

Proposition 5.17. *Let $\mathcal{D} \subseteq \mathbb{Z}$ be a $\mathbb{Z}$-KMF set. Then there exists a countably infinite partition $\bigcup_{i \in I} \mathcal{D}_i \subseteq \mathcal{D}$ of disjoint $\mathbb{Z}$-KMF sets.*

Compare [71, Corollary 3] and [14, Corollary 1.24].

Proof. Inductively, we obtain a partition $(I_k)_{k \in \mathbb{N}}$ of finite subsets $I_k \subset \mathcal{D}$ in the following way.

Take $\mathcal{D} \backslash (I_1 \cup \ldots \cup I_{k-1})$ which is a $\mathbb{Z}$-KMF set by Lemma 5.16. Using the definition of $\mathbb{Z}$-KMF for $\epsilon = \frac{1}{k}$, there exists a real trigonometric polynomial $p_k(x) = \sum_{t \in \mathcal{D}} a_t^k \Re\left(e^{2\pi i t x}\right)$ satisfying $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ with $a_t^k \in \mathbb{R}$ and where the set $\{t : a_t^k \neq 0\} \subset \mathcal{D}$ is nonempty and finite. We set $I_k := \{t \in \mathcal{D} : a_t^k \neq 0\} \subset \mathcal{D} \backslash (I_1 \cup \ldots \cup I_{k-1})$.

Let $(A_i)_{i \in J}$ be a (finite or countable) partition of $\mathbb{N}$ consisting of infinite sets. We define

$$\mathcal{D}_i := \bigcup_{k \in A_i} I_k.$$

Then each $\mathcal{D}_i$ is $\mathbb{Z}$-KMF. To see this, let $\epsilon > 0$ be given and choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$ and $I_k \in \mathcal{D}_i$. By construction, there exists a real trigonometric polynomial, p_k, satisfying $p_k(0) = 1$ and $p_k \geq \frac{1}{k} > -\epsilon$ with $a_t^k \in \mathbb{R}$ and where the set $\{t : a_t^k \neq 0\} \subset \mathcal{D}$ is nonempty and finite. □

5.3 Remarks

Summarising, we obtain the following result.

Theorem 5.18. *Let $\mathcal{D} \subseteq \mathbb{R}_*$ be strongly operator recurrent (operator recurrent, strongly recurrent, Poincaré), and let $N \in \mathbb{N}$, $s > 0$ and ϵ with $0 < \epsilon < \frac{s}{2}$ be given. Then there exists a countable set*

$$\tilde{D} \subseteq \bigcup_{|n|>N} ([sn - \epsilon, sn + \epsilon] \cap \mathcal{D}) \tag{5.3}$$

which is still strongly operator recurrent (operator recurrent, strongly recurrent, Poincaré). If $\mathcal{D}$ is strongly operator recurrent (strongly recurrent), then there exists a set $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ which additionally satisfies that $\tilde{\mathcal{D}} \cap [-N, N]$ is finite for all $N > 0$ and which is still strongly operator recurrent (strongly recurrent).

Theorem 5.18 can be interpreted in such a way that every strong operator recurrence, operator recurrence, strong recurrence or Poincaré set $\mathcal{D} \subseteq \mathbb{R}_*$ is "almost" an integer set, i.e. every such set contains a subset in an arbitrarily thin tunnel around $\mathbb{Z}$ which still has the same properties.
Further reductions can be made using the Ramsey property (Section 5.2) such as in Corollaries 5.14 and 5.15 or going to dense subsets (Proposition 5.1). Corollary 5.3 as well as Proposition 5.9 state that all such sets have countable subsets with

the same properties, even discrete subsets in the case of strong operator recurrence and strong recurrence sets.

However, all the methods applied in this chapter are only of finite nature (as straightforward counterexamples immediately show). We can apply a decomposition using the Ramsey property only finitely many times and we have to choose a strictly positive ϵ for (5.3). A step forward is Proposition 5.9 where we obtain a discrete subset although the gaps within $\tilde{\mathcal{D}}$ are not bounded below. However, we conjecture the following result which stresses the connection to the integer sets.

Conjecture 5.19. *Let $\mathcal{D} \subseteq \mathbb{R}_*$ be strongly operator recurrent, operator recurrent, strongly recurrent or Poincaré. Then it contains a discrete subset with the same recurrence property such that* $\inf_{t,s\in\mathcal{D}} |t-s| > 0$.

Chapter 6

Alon-Peres Characterisation of Bourgain's Example

Bourgain ([23]) showed that there exists a $\mathbb{Z}$-intersective set $\mathcal{D} \subset \mathbb{N}$ which is not $\mathbb{Z}$-FMRiesz. We extend Bourgain's result for sets of the reals and discuss a characterisation of Bourgain's problem in this section, i.e. the existence of an intersective set which is not FMRiesz in terms of stationary families in Hilbert spaces.

Theorem 6.1 (Bourgain)**.** *There exists a set $\mathcal{D} \subset \mathbb{N}$ which is $\mathbb{Z}$-intersective, but not $\mathbb{Z}$-FMRiesz.*

Using the results from Section 4.3, we extend Bourgain's integer result to the reals.

Theorem 6.2. *There exists a set $\mathcal{D} \subset \mathbb{R}_*$ which is $\mathbb{R}$-intersective, but not $\mathbb{R}$-FMRiesz.*

Proof. Theorem 6.1 gives a set $\mathcal{D} \subset \mathbb{N}$ which is $\mathbb{Z}$-intersective, but not $\mathbb{Z}$-FMRiesz. By Proposition 4.50, $\mathcal{D}$ is also $\mathbb{R}$-intersective while it is not $\mathbb{R}$-FMRiesz by Proposition 4.44. □

A similar result holds for strong combinatorial recurrence by using a counterexample of Forrest ([33]).

Theorem 6.3 (Forrest)**.** *There exists a set $\mathcal{D} \subset \mathbb{N}$ which is $\mathbb{Z}$-intersective, but not $\mathbb{Z}$-strongly combinatorially recurrent.*

Theorem 6.4. *There exists a set $\mathcal{D} \subset \mathbb{R}_*$ which is $\mathbb{R}$-intersective, but not $\mathbb{R}$-strongly combinatorially recurrent.*

Proof. Theorem 6.3 gives a set $\mathcal{D} \subset \mathbb{N}$ which is $\mathbb{Z}$-intersective, but not $\mathbb{Z}$-strongly combinatorially recurrent. By Proposition 4.50, $\mathcal{D}$ is also $\mathbb{R}$-Poincaré while it is not a $\mathbb{R}$-strongly combinatorial recurrent by Proposition 4.47. □

Remark 6.5. Using different results from Chapters 3 and 4 as well as the integer characterisations in Theorem 8, we obtain many variants of Theorems 6.2 and 6.4, e.g. the existence of a $\mathbb{R}$-real correlative set $\mathcal{D}$ which is not $\mathbb{R}$-correlative or $\mathbb{R}$-operator recurrent.
This shows that the class of properties around Poincaré recurrence is in particular distinct from the class of properties around operator recurrence and also $\mathbb{R}$-strong recurrence differs from $\mathbb{R}$-Poincaré recurrence at least without continuity. On the other hand, it is not clear if $\mathbb{R}$-strong operator recurrence is indeed a stronger property than $\mathbb{R}$-strong recurrence or $\mathbb{R}$-operator recurrence (not even for the integer properties).

Let a Hilbert space H, $v \in H$ with $\|v\| = 1$ and $0 < c < 1$ be given. Then we define the set $H_{c,v}$ by

$$H_{c,v} := \{h \in H : \langle v, h \rangle = c, \|h\| = 1\}.$$

Definition 6.6. *Let a Hilbert space H be given. We call $(h_t)_{t\in\mathbb{R}} \subseteq H$ a* ***stationary family*** *if*

$$\langle h_{i+t}, h_{j+t} \rangle = \langle h_i, h_j \rangle$$

holds for all $i, j, t \in \mathbb{R}$ *and if* $t \mapsto h_t$ *is continuous. Similarly, we call* $(h_n)_{n\in\mathbb{N}} \subseteq H$ *a* ***stationary sequence*** *if*

$$\langle h_{i+t}, h_{j+t}\rangle = \langle h_i, h_j\rangle$$

holds for all $i, j, t \in \mathbb{N}$.

Alon and Peres ([2]) gave a characterisation of Bourgain's theorem 6.1 by translating the statement into the framework of stationary sequences in Hilbert spaces.

Theorem 6.7 (Alon, Peres)**.** *There exist* $0 < c < 1$*, a Hilbert space* H*,* $v \in H$ *with* $\|v\| = 1$ *and a stationary sequence* $(h_n)_{n\in\mathbb{N}} \subseteq H_{c,v}$ *such that we have* $\langle h_i, h_j\rangle = 0$ *for any* $S \subseteq \mathbb{N}$ *with* $\overline{d}_{\mathbb{N}}(S) > 0$ *and for some* $i, j \in S$.

Theorem 6.8 (Alon, Peres)**.** *Each choice of* $0 < c < 1$*,* H*,* $v \in H$ *and* $(h_n)_{n\in\mathbb{N}} \subseteq H_{c,v}$ *as in Theorem 6.7 leads to a set* $\mathcal{D} \subset \mathbb{N}$ *which is* $\mathbb{N}$*-intersective, but not* $\mathbb{N}$*-FMRiesz.*

We now prove the real variants of Theorems 6.7 and 6.8.

Theorem 6.9. *There exist* $0 < c < 1$*, a Hilbert space* H*,* $v \in H$ *with* $\|v\| = 1$ *and a stationary family* $(h_t)_{t\in\mathbb{R}} \subseteq H_{c,v}$ *such that we have* $\langle h_i, h_j\rangle = 0$ *for any* $S \subseteq \mathbb{R}$ *with* $\overline{d}_{\mathbb{R}}(S) > 0$ *and some* $i, j \in S$.

Compare [2].

Proof. Let $\mathcal{D} \subset \mathbb{R}$ be a set which is $\mathbb{R}$-intersective, but not $\mathbb{R}$-FMRiesz. Then there exists a probability measure μ on $\mathbb{R}$ with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ and $\mu(\{0\}) = c^2 > 0$ for some $0 < c < 1$.
We define the Hilbert space $H := L^2(\mathbb{R}, \mu)$ and elements $h_t := e^{2\pi i t x}$ for $t \in \mathbb{R}$ and $v(x) := \frac{1}{c}\mathbb{1}_{\{0\}}$. They all have norm 1 since

$$\|h_t\|^2 = \int_{\mathbb{R}} |h_t|^2 \, d\mu = \int_{\mathbb{R}} \mathbb{1} \, d\mu = \mu(\mathbb{R}) = 1$$

for $t \in \mathbb{R}$ and

$$\|v\|^2 = \int_{\mathbb{R}} |v|^2 \, d\mu = \frac{1}{c^2}\mu(\{0\}) = 1.$$

The family $(h_t)_{t\in\mathbb{R}}$ is stationary since

$$\begin{aligned}\langle h_{l+t}, h_{j+t}\rangle &= \int_{\mathbb{R}} e^{2\pi i(l+t)x} e^{-2\pi i(j+t)x} \, d\mu(x) = \int_{\mathbb{R}} e^{2\pi i(l+t-j-t)x} \, d\mu(x) \\ &= \int_{\mathbb{R}} e^{2\pi i l x} e^{-2\pi i j x} \, d\mu(x) = \langle h_l, h_j\rangle,\end{aligned}$$

and the map $t \mapsto h_t$ is continuous since

$$\lim_{t\to s}\|h_t - h_s\|^2 = \lim_{t\to s}\int_{\mathbb{R}} (h_t(x) - h_s(x)) \cdot (\overline{h_t(x)} - \overline{h_s(x)}) \, d\mu(x)$$
$$= \lim_{t\to s}\int_{\mathbb{R}} h_t(x)\overline{h_t(x)} \, d\mu(x) + \lim_{t\to s}\int_{\mathbb{R}} h_s(x)\overline{h_s(x)} \, d\mu(x) - \lim_{t\to s} 2\Re\int_{\mathbb{R}} h_s(x)\overline{h_t(x)} \, d\mu(x)$$
$$= 2 - 2\Re\int_{\mathbb{R}} \lim_{t\to s}(h_s(x)\overline{h_t(x)} \, d\mu(x) = 0.$$

We further have

$$\langle v, h_t\rangle = \int_{\mathbb{R}} v \cdot h_t \, d\mu = v(0) \cdot h_t(0) \cdot \mu(\{0\}) = \frac{1}{c} \cdot 1 \cdot c^2 = c.$$

Finally, let $S \subseteq \mathbb{R}$ be with $\overline{d}_{\mathbb{R}}(S) > 0$. Using intersectivity of $\mathcal{D}$, we have some $s, t \in S$ and $d \in \mathcal{D}$ such that $s - t = d$, hence

$$0 = \widehat{\mu}(d) = \widehat{\mu}(s-t) = \int_{\mathbb{R}} e^{-2\pi i(s-t)x} \, d\mu(x) = \int_{\mathbb{R}} e^{2\pi i t x} e^{-2\pi i s x} \, d\mu(x) = \langle h_t, h_s\rangle.$$

We have therefore found c, H, v, $(h_t)_{t\in\mathbb{R}}$ as required. □

Lemma 6.10. *Let $(h_t)_{t\in\mathbb{R}} \in H$ be a stationary sequence. Then $(\langle h_t, h_0\rangle)_{t\in\mathbb{R}}$ is positive-definite and continuous.*

Proof. We have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\overline{\alpha_j}\langle h_{t_i - t_j}, h_0\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\overline{\alpha_j}\langle h_{t_i}, h_{t_j}\rangle = \langle \sum_{i=1}^{n} \alpha_i h_{t_i}, \sum_{j=1}^{n} \alpha_j h_{t_j}\rangle \geq 0$$

for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $t_1, \ldots, t_n \in \mathbb{R}$ and

$$|\langle h_t, h_0\rangle - \langle h_s, h_0\rangle| = |\langle h_t - h_s, h_0\rangle| \leq \|h_t - h_s\|^2 \cdot \|h_0\|^2 \overset{t\to s}{=\!\!\longrightarrow} 0.$$

□

Theorem 6.11. *Each choice of $0 < c < 1$, H, $v \in H$ and $(h_t)_{t\in\mathbb{R}} \subseteq H_{c,v}$ as in Theorem 6.9 leads to a set $\mathcal{D} \subset \mathbb{R}$ which is $\mathbb{R}$-intersective, but not $\mathbb{R}$-FMRiesz.*

Compare [2].

Proof. Let us assume the existence of c, H, v, $(h_t)_{t\in\mathbb{R}}$ as stated in Theorem 6.9. By Lemma 6.10, $(\langle h_t, h_0\rangle)_{t\in\mathbb{R}}$ forms a positive-definite and continuous function, so by the Bochner-Herglotz Theorem 2.16, there exists a measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) = \langle h_0, h_0\rangle = 1$ and $\widehat{\mu}(t) = \langle h_t, h_0\rangle$ By Lemma 6.10, $(\langle h_t, h_0\rangle)_{t\in\mathbb{R}}$ forms a positive-definite and continuous function, so by the Bochner-Herglotz Theorem 2.16, there exists a measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) = \langle h_0, h_0\rangle = 1$ and $\widehat{\mu}(t) = \langle h_t, h_0\rangle$ for all $t \in \mathbb{R}$.
Now consider the family $(h_t - cv)_{t\in\mathbb{R}}$ which is stationary since

$$\begin{aligned}&\langle h_{t+\gamma} - cv, h_{s+\gamma} - cv\rangle = \langle h_{t+\gamma}, h_{s+\gamma}\rangle + c^2\langle v, v\rangle - c\langle h_{t+\gamma}, v\rangle - c\langle v, h_{s+\gamma}\rangle\\ &= \langle h_t, h_s\rangle + c^2\langle v, v\rangle - c^2 - c^2 = \langle h_t, h_s\rangle + c^2\langle v, v\rangle - c\langle h_t, v\rangle - c\langle h_s, v\rangle\\ &= \langle h_t - cv, h + s - cv\rangle\end{aligned}$$

for all $t, s, \gamma \in \mathbb{R}$ and continuous since

$$(h_t - cv) - (h_s - cv) = h_t - h_s \xrightarrow{t\to s} 0$$

by assumption on $(h_t)_{t\in\mathbb{R}}$. By Lemma 6.10, $(\langle h_t - cv, h_0 - cv\rangle)_{t\in\mathbb{R}}$ forms a positive-definite and continuous functions, hence, by the Bochner-Herglotz theorem 2.16, there exists a measure ν on $\mathbb{R}$ with $\widehat{\nu}(t) = \langle h_t - cv, h_0 - cv\rangle$. We conclude

$$\begin{aligned}&0 < \nu(\{0\}) + c^2 \overset{2.17}{=} \oint_{\mathbb{R}} \langle h_t - cv, h_0 - cv\rangle \; d\lambda(t) + c^2\\ &= \oint_{\mathbb{R}} \big(\langle h_t, h_0\rangle + c^2 - c\langle h_t, v\rangle - c\langle v, h_0\rangle\big) \; d\lambda(t) + c^2\\ &= \oint_{\mathbb{R}} \langle h_t, h_0\rangle \; d\lambda(t) - \oint_{\mathbb{R}} c^2 d\lambda(t) + c^2 = \oint_{\mathbb{R}} \langle h_t, h_0\rangle \; d\lambda(t) \overset{2.17}{=} \mu(\{0\}).\end{aligned}$$

The set

$$\mathcal{D} := \{t \in \mathbb{R} : \widehat{\mu}(t) = 0\}$$

is hence not $\mathbb{R}$-FMRiesz as we have found a probability measure with $\mu(\{0\}) > 0$ and $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$. However, it is intersective. To see this, let $S \subseteq \mathbb{R}$ be with $\overline{d}_{\mathbb{R}}(S) > 0$. Then

$$\widehat{\mu}(t-s) = \langle h_{t-s}, h_0 \rangle = \langle h_t, h_s \rangle = 0$$

for some $t, s \in S$ by assumption, hence, $t - s \in \mathcal{D}$ and

$$(S - S) \cap \mathcal{D} \neq \emptyset.$$

We therefore have found the set $\mathcal{D}$ which is $\mathbb{R}$-intersective, but not $\mathbb{R}$-FMRiesz. $\square$

Chapter 7

Extending the theory

In this chapter, we discuss some ways how to extend the theory of recurrence sets. Firstly, one can consider more properties than we have done so far since we have focused on the properties around strong operator, operator, strong and Poincaré recurrence. Then we point out how to do a quantitative analysis of the recurrence properties. Finally, we extend the theory from $\mathbb{Z}$ and $\mathbb{R}$ to locally compact abelian groups and discuss the corresponding framework of recurrence sets.

7.1 Topological Recurrence

In this thesis, we focus on the real variants of recurrence properties which form the four most prominent classes of properties for integers, strong operator recurrence, strong recurrence, operator recurrence and Poincaré recurrence (compare Theorems 8 and 9).

More properties can be considered by using different convergence methods of the Cesàro averages, e.g. by using different Følner sequences (see [14, Section 4]) or different recurrence strengths, e.g. "nice" combinatorial recurrence sets which satisfy $\lim_{|d|\to\infty, d\in\mathcal{D}} \overline{d}((E+d)\cap E) \geq (\overline{d}(E))^2$ for any given set E (see [14, Subsection 3.5], [55, Section 2]).

These properties extend the list of implications in Theorem 9 in many ways (see, e.g. [76, Figure 4.1]). However, it is not clear, not even for integers, how these properties are related to each other apart from the obvious implications such as nice combinatorial recurrence ⇒ strong combinatorial recurrence ⇒ combinatorial recurrence. There are examples showing that at least some of these sets indeed form new classes of properties. Nice combinatorial recurrence for integers is strictly stronger than strong combinatorial recurrence ([55, Theorem 2.5]) and we discuss topological recurrence ⇏ Poincaré below in Theorem 7.9 and its corollaries (see [50]).

In the following, we shall discuss recurrence of topological dynamical systems and the corresponding recurrence sets in more detail.

In analogy to measure preserving systems, one can consider topological dynamical systems by replacing the invertible measure preserving map ϕ on the measure space (Ω, Σ, μ) with an invertible homeomorphic map ψ on a compact metric space X.

Definition 7.1 (Topological Dynamical System)**.** *A **topological dynamical system** (tds) is a compact metric space X) with a group of invertible homeomorphisms $(\phi^n)_{n\in\mathbb{Z}}$ or $(\phi_t)_{t\in\mathbb{R}}$ on X such that $\phi_t\phi_s = \phi_{t+s}$ for all $t, s \in \mathbb{R}$ or $\phi^n\phi^m = \phi^{n+m}$ for all $n, m \in \mathbb{Z}$, respectively. A tds $(X; (\phi_t)_{t\in\mathbb{R}})$ is **strongly continuous** (sctds) if $t \mapsto f \circ \phi_t$ is continuous for all $f \in C(X)$.*

Theorem 7.2, originally due to Birkhoff, is the topological analogue of Poincaré's recurrence theorem 4.11 for measure preserving systems (see [36, Subsection I.1.4]).

Theorem 7.2 (Birkhoff)**.** *Let $(X; \phi)$ be a tds, and let open $\emptyset \neq \mathcal{O} \subseteq X$ be given. Then there exists $d \in \mathbb{N}$ such that $\phi^d(\mathcal{O}) \cap (\mathcal{O}) \neq \emptyset$.*
Let $(X; (\phi_t)_{t\in\mathbb{R}})$ be a sctds and $\emptyset \neq \mathcal{O} \subseteq X$ open. Then there exists $d \in \mathbb{R}_*$ such that $\phi_d(\mathcal{O}) \cap (\mathcal{O}) \neq \emptyset$.

Definition 7.3 (Topological Recurrence)**.** *A set $\mathcal{D} \subseteq \mathbb{R}$ is **topologically recurrent** (TR) if, given a sctds $(X; (\phi_t)_{t\in\mathbb{R}})$ and $\emptyset \neq \mathcal{O} \subseteq X$ open, there exists $d \in \mathcal{D}$*

such that

$$\phi_d(\mathcal{O}) \cap \mathcal{O} \neq \emptyset.$$

Definition 7.4 (r-Intersectivity). *A set $\mathcal{D} \subseteq \mathbb{R}$ is r**-intersective** if, given a finite partition $\mathbb{R} = C_1 \cup \ldots \cup C_r$, there exists $1 \leq i \leq r$ such that*

$$(C_i - C_i) \cap \mathcal{D} \neq \emptyset.$$

Definition 7.5 (Chromatic Intersectivity). *A set $\mathcal{D} \subseteq \mathbb{R}$ is **chromatically intersective** (CI) if it is r-intersective for all $r \in \mathbb{N}$.*

Proposition 7.6 (P $\Rightarrow$ TR). *Let us assume that, given a scmps $(\Omega, \Sigma, \mu; (\phi_t)_{t\in\mathbb{R}})$ and $A \in \Sigma$ with $\mu(A) > 0$, there exists $d \in \mathcal{D}$ such that*

$$\mu(A \cap \phi_d(A)) > 0.$$

Then, given a sctds $(X; (\phi_t)_{t\in\mathbb{R}})$ and open $\emptyset \neq \mathcal{O} \subseteq X$, there exists $d \in \mathcal{D}$ such that

$$\phi_d(\mathcal{O}) \cap \mathcal{O} \neq \emptyset.$$

Proof. Let the sctds $(X; (\phi_t)_{t\in\mathbb{R}})$ and open $\emptyset \neq \mathcal{O} \subseteq X$ be given. Without loss of generality, let $X = X_0 := \overline{\cup_{t\in\mathbb{R}}\phi_t(\mathcal{O})}$ (if not, we set $\mu(X \backslash X_0) := 0$).
There exists an invariant probability measure μ on X as in Lemma 3.80. Since X is compact, there exists $t_1, \ldots, t_n \in \mathbb{R}$ such that $X = \phi_{t_1}(\mathcal{O}) \cup \ldots \cup \phi_{t_n}(\mathcal{O})$ and since $\mu(\mathcal{O}) = \mu(\phi_{t_1}(\mathcal{O})) = \ldots = \mu(\phi_{t_n}(\mathcal{O}))$ by the invariance of μ, we deduce $\mu(\mathcal{O}) > 0$.
Using strong continuity with respect to $\|\cdot\|_\infty$ on the dense subspace $C(X) \subseteq L^2(X, \mu)$ (compare [31, Example 5.4]), we obtain a scmps $(X, \mu; (\phi_t)_{t\in\mathbb{R}})$. Hence, there exists $d \in \mathcal{D}$ such that $\mu(\phi_d(\mathcal{O}) \cap \mathcal{O}) > 0$, and therefore $\phi_d(\mathcal{O}) \cap \mathcal{O} \neq \emptyset$. $\square$

The next result gives a relationship between topological recurrence and chromatic intersectivity (see also [43], [82, Chapter 5] for a similar characterisation of topological recurrence) and we note that both properties are equivalent when considering the corresponding integer properties ([55, Proposition 0.12]).

Proposition 7.7 (CI $\Rightarrow$ TR)**.** *Let us assume that, given $r \in \mathbb{N}$ and a finite partition $\mathbb{R} = C_1 \cup \ldots \cup C_r$, there exists $1 \leq i \leq r$ such that*

$$(C_i - C_i) \cap \mathcal{D} \neq \emptyset.$$

Then, given a sctds $(X; (\phi_t)_{t\in\mathbb{R}})$ and open $\emptyset \neq \mathcal{O} \subseteq X$, there exists $d \in \mathcal{D}$ such that

$$\phi_d(\mathcal{O}) \cap \mathcal{O} \neq \emptyset.$$

Compare [55, Proposition 0.12].

Proof. Let a sctds $(X; (\phi_t)_{t\in\mathbb{R}})$ and open $\emptyset \neq \mathcal{O} \subseteq X$ be given. Without loss of generality, we assume $X = \overline{\cup_{t\in\mathbb{R}}\phi_t(\mathcal{O})}$. Using compactness, we obtain $t_1, \ldots, t_n$ such that $X = \cup_{i=1}^{n}\phi_{t_i}(\mathcal{O})$.
Let $x \in \mathcal{O}$. We choose a partition $\mathbb{R} = C_1 \cup \ldots \cup C_n$ such that

$$t \in C_i \Rightarrow \phi_t(x) \in \phi_{t_i}(\mathcal{O}).$$

Chromatic intersectivity implies that there exists $1 \leq i \leq n$, $d \in \mathcal{D}$ and $s \in C_i$ with $s + d \in C_i$, i.e. we have $\phi_s(x), \phi_{s+d}(x) \in \phi_{t_i}(\mathcal{O})$ and therefore $\phi_s(x) \in \phi_{t_i}(\mathcal{O}) \cap \phi_{t_i - d}(\mathcal{O})$. But this yields $\phi_{s-t_i+d}(x) \in \mathcal{O} \cap \phi_d(\mathcal{O})$ and $\mathcal{O} \cap \phi_d(\mathcal{O}) \neq \emptyset$. $\square$

Similarly as in Section 4.3.3, we obtain the following.

Proposition 7.8. *A set $\mathcal{D} \subseteq \mathbb{Z}$ is $\mathbb{Z}$-chromatically intersective if and only if it is $\mathbb{R}$-chromatically intersective.*

Křížž [50] showed that topological recurrence or chromatic intersectivity is strictly weaker than Poincaré recurrence (see also [56, Theorem 3.3.5]).

Theorem 7.9 (Kříž)**.** *There exists a set $\mathcal{D} \subset \mathbb{Z}$ which is $\mathbb{Z}$-chromatically intersective, but not $\mathbb{Z}$-intersective.*

The characterisations for $\mathbb{Z}$ (Theorem 8, [55, Proposition 0.12]) and the results from Section 4.3 yield the following corollaries.

Corollary 7.10. *There exists a set $\mathcal{D} \subset \mathbb{Z}$ which is $\mathbb{Z}$-topologically recurrent, but not $\mathbb{Z}$-Poincaré.*

Corollary 7.11. *There exists a set $\mathcal{D} \subset \mathbb{Z}$ which is $\mathbb{R}$-chromatically intersective, but not $\mathbb{R}$-intersective.*

Proof. Theorem 7.9 yields a set $\mathcal{D} \subset \mathbb{Z}$ which is $\mathbb{Z}$-chromatically intersective, but not $\mathbb{Z}$-intersective. By Proposition 7.8, $\mathcal{D}$ is $\mathbb{R}$-chromatically intersective, but not $\mathbb{R}$-intersective by Proposition 4.46. □

Remark 7.12. The main tool for reducing a recurrence set in Chapter 5 was the use of rotations on $\mathbb{T}$ and a suitable choice of product systems. The same approach can be applied for topological recurrence and we obtain analogous reducing and Ramsey results, i.e. for a given topologically recurrent set $\mathcal{D}$ and for given $s > 0$ and ϵ with $0 < \epsilon < \frac{s}{2}$,

$$\bigcup_{n \in \mathbb{Z}} ([sn - \epsilon, sn + \epsilon] \cap \mathcal{D})$$

is still topologically recurrent, and for a decomposition $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ of a topologically recurrent set, at least one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is still topologically recurrent.

7.2 Quantitave Analysis

Until this point, the thesis was concerned with a qualitative analysis of the relation between recurrence properties, i.e. in particular, with the implications given in Theorem 9. In a similar way, a quantitative analysis can be done. For each recurrence property, we can define corresponding "measures" (Definition 7.13). This has been thoroughly studied for $\mathbb{N}$ ([57, Chapter 2], [60], [63], [71], [72]), but it can be extended analogously for $\mathbb{R}$. We note that these measures are not measures in the usual mathematical sense, but we keep this term due to its use in the integer setting.

We introduce the measures corresponding to the classical measures and we note that similar measures can be obtained for all properties considered in this thesis, in particular, by adding or removing corresponding continuity assumptions (denoted by c or $*$, respectively). We conclude this section by proving the quantitative relation between correlativity and continuous intersectivity.

We define the **discrepancy mod 1** of a function $f \in l^\infty_*([-T,T],[0,1])$ by

$$D_T(f) = \sup_{0 \le a < b \le 1} \left| \int_{-T}^{T} (\mathbb{1}_{[a,b]} \circ f)(t) \, d\lambda(t) - 2T(b-a) \right|$$

Definition 7.13. *The* ***measure*** $\eta_{\mathbb{R}}$ ***corresponding to*** $\mathbb{R}$***-FMRiesz*** *for* $A \subseteq \mathbb{R}$ *is given by*

$$\eta_{\mathbb{R}}(A) := \sup \mu(\{0\})$$

where the supremum is taken over all measures on $\mathbb{R}$ *satisfying* $\widehat{\mu}(d) = 0$ *for all* $d \in A$.

The ***measure*** $\alpha_{\mathbb{R}}$ ***corresponding to*** $\mathbb{R}$***-van der Corput*** *for* $A \subseteq \mathbb{R}$ *is given by*

$$\alpha_{\mathbb{R}}(A) := \sup \limsup_{T\to\infty} \frac{1}{2T} D_T(f)$$

where the supremum is taken over all $f \in l^\infty_c(\mathbb{R},[0,1])$ *such that* $f(\cdot + d) - f(\cdot)$ *is equidistributed mod 1 for all* $d \in A$.

The ***measure*** $\beta_{\mathbb{R}}$ ***corresponding to*** $\mathbb{R}$***-correlativity*** *for* $A \subseteq \mathbb{R}$ *is given by*

$$\beta_{\mathbb{R}}(A) := \sup \limsup_{T\to\infty} \left| \fint_{-T}^{T} f(t) \, d\lambda(t) \right|$$

where the supremum is taken over all $f \in l_c^\infty(\mathbb{R}, \mathbb{C})$ such that $\limsup_{T\to\infty} \fint_{-T}^{T} |f(t)|^2 \, d\lambda(t) \leq 1$ *and* $\fint_{\mathbb{R}} f(t+d)\overline{f(t)} \, d\lambda(t) = 0$ *for all $d \in \mathcal{D}$.*

*The **measure $\gamma_\mathbb{R}$ corresponding to $\mathbb{R}$-KMF** for $A \subseteq \mathbb{R}$ is given by*

$$\gamma_\mathbb{R}(A) := \inf p_\epsilon(0)$$

where the infimum is taken over all $\epsilon > 0$ and all nonzero real functions

$$p_\epsilon(x) = \int_\mathcal{D} \Re\left(e^{-2\pi itx}\right) \, d\nu_\epsilon(t)$$

satisfying $p_\epsilon(0) = 1$ and $p_\epsilon \geq -\epsilon$ where ν_ϵ is a finite measure on A.

*The **measure $\delta_\mathbb{R}^c$ corresponding to $\mathbb{R}$-continuous intersectivity** for $A \subseteq \mathbb{R}$ is given by*

$$\delta_\mathbb{R}^c(A) := \sup \overline{d}_\mathbb{R}(E)$$

where the supremum is taken over all sets $E \subseteq \mathbb{R}$ with $(E - E) \cap A \neq \emptyset$ and $\lim_{t\to 0} \overline{d}_\mathbb{R}((E+t) \triangle E) = 0$ and such that $d(E \cap (E-t))$ exists for all $t \in \mathbb{R}$.

*The **measure $\zeta_\mathbb{R}$ corresponding to $\mathbb{R}$-operator recurrence** for $A \subseteq \mathbb{R}$ is given by*

$$\zeta_\mathbb{R}(A) := \sup|\langle x, z\rangle|$$

where the supremum is taken over all Hilbert spaces H, all strongly continuous unitary groups $(T_t)_{t\in T}$, all $x \in H$ with $\langle T_d x, x\rangle = 0$ for all $d \in A$, $P_\mathbb{R} z = z$ and $\|x\| = 1 = \|z\|$.

Remark 7.14. It is clear that a set $\mathcal{D}$ has a property whenever the associated measure vanishes. Theorem 9 shows that $\eta_\mathbb{R}$, $\beta_\mathbb{R}$, $\alpha_\mathbb{R}$ and $\zeta_\mathbb{R}$ vanish at the same time and that their vanishing of a set A implies $\delta_\mathbb{R}^c(A) = 0$.

For $\mathbb{N}$, the following quantitative results hold ([42], [57, Theorem 2.2-2.6], [63, Theorem 1.3], [70, Theorem 1], [71, Theorem 2]). The measure β^∞ is defined as β with the additional assumption $\|f\|_\infty \leq 1$.

Theorem 7.15 (Ruzsa)**.** *We have*

$$\delta_{\mathbb{N}} \leq (\beta_{\mathbb{N}})^2 = \gamma_{\mathbb{N}} = \zeta_{\mathbb{N}} = \eta_{\mathbb{N}} \leq \beta_{\mathbb{N}}^{\infty} \leq \beta_{\mathbb{N}},$$
$$\frac{\beta_{\mathbb{N}}^{\infty}}{2} \leq \alpha_{\mathbb{N}} \leq K\beta_{\mathbb{N}}^{\infty} \log(\frac{2}{\beta_{\mathbb{N}}^{\infty}}),$$
$$\delta_{\mathbb{N}} \leq \min(\alpha_{\mathbb{N}}, (\beta_{\mathbb{N}})^2)$$

for some $K > 0$.

Proposition 7.16. *We have $\delta_{\mathbb{R}}^c \leq (\beta_{\mathbb{R}})^2$. In particular, we have*

$$\mathbb{R}\text{-}correlativity \Rightarrow \mathbb{R}\text{-}continuous\ intersectivity.$$

Compare [71, Section 3] and [57, Theorem 2.8].

Proof. Fix $A \subseteq \mathbb{R}$ and $\epsilon > 0$. Let $E \subset \mathbb{R}$ be such that $\overline{d}_{\mathbb{R}}(E) > \delta_{\mathbb{R}}^c(A) - \epsilon$, $(E - E) \cap A = \emptyset$ and $\lim_{t \to 0} \overline{d}_{\mathbb{R}}((E + t) \triangle E) = 0$ and where $d(E \cap (E - t))$ exists for all $t \in \mathbb{R}$. Such an E exists by the approximation property of the supremum. Consider the function

$$\tilde{f}(t) := \left(\sqrt{\overline{d}_{\mathbb{R}}(E)}\right)^{-1} \mathbb{1}_E(t).$$

Then

$$\limsup_{T\to\infty} \oint_{-T}^{T} |\tilde{f}(t)|^2 \, d\lambda(t) = (\overline{d}_{\mathbb{R}}(E))^{-1} \limsup_{T\to\infty} \oint_{-T}^{T} \mathbb{1}_E(t) \, d\lambda(t) = 1$$

and

$$\oint_{\mathbb{R}} f(t + d)\overline{\tilde{f}(t)} \, d\lambda(t) = (\overline{d}_{\mathbb{R}}(E))^{-1} \limsup_{T\to\infty} \oint_{-T}^{T} \mathbb{1}_E(t + d)\mathbb{1}_E(t) \, d\lambda(t) = 0$$

for all $d \in A$ since $(E - E) \cap A = \emptyset$. By assumption on E (compare Proposition 3.89), we have $\tilde{f} \in l_c^{\infty}(\mathbb{R}, \mathbb{C})$. Hence, $\tilde{f}$ is admissible for $\beta_{\mathbb{R}}(A)$. We therefore have

$$\begin{aligned}
\beta_{\mathbb{R}}(A) &= \sup \limsup_{T\to\infty} \left| \oint_{-T}^{T} f(t) \, d\lambda(t) \right| \geq \limsup_{T\to\infty} \left| \oint_{-T}^{T} \tilde{f}(t) \, d\lambda(t) \right| \\
&= (\sqrt{\overline{d}_{\mathbb{R}}(E)})^{-1} \limsup_{T\to\infty} \left| \oint_{-T}^{T} \mathbb{1}_E(t) \, d\lambda(t) \right| = \left(\sqrt{\overline{d}_{\mathbb{R}}(E)}\right)^{-1} \overline{d}_{\mathbb{R}}(E) \\
&= \sqrt{\overline{d}_{\mathbb{R}}(E)} > \sqrt{\delta_{\mathbb{R}}^c(A) - \epsilon}.
\end{aligned}$$

Letting $\epsilon \to 0$ yields

$$\beta_{\mathbb{R}}(A) \geq \sqrt{\delta^c_{\mathbb{R}}(A)}.$$

□

Similarly, we obtain the result without continuity assumptions.

Corollary 7.17. *We have $\delta_{\mathbb{R}} \leq (\beta^*_{\mathbb{R}})^2$. In particular, we have*

$$\mathbb{R}\text{-}correlativity_* \Rightarrow \mathbb{R}\text{-}intersectivity.$$

7.3 Locally compact abelian groups

Another approach to generalise the classical theory is to consider a multidimensional setting with $\mathbb{Z}^k$ or $\mathbb{R}^k$ instead of $\mathbb{Z}$ or $\mathbb{R}$. For $\mathbb{Z}$ this was in particular done in [14] and it can be extended to a general countable discrete group in a similar way. The multidimensional extension of the theory for $\mathbb{R}$ leads to the treatment of locally compact abelian groups where $\mathbb{R}^k$ and $\mathbb{Z}^k$ are the most natural examples. Special cases for more general (semi)groups have been discussed, in particular intersectivity, combinatorial recurrence and Poincaré recurrence (see for example [10, Subsection 5.2], [12, Theorem 3.3], [15, Theorem 2.2]).

In contrast to $\mathbb{R}$ where the whole setup is straightforward, we have to go into more details if we consider locally compact abelian groups. For our purposes, this will in particular lead to amenability and the notion of dual groups. For the treatment of recurrence sets, we assume that G is a locally compact abelian regular Hausdorff group with Haar measure λ and dual group G^* (although some more restriction may be required by the use of pointwise convergence results). However, we note that some implications can be proven for even more general groups.

7.3.1 Finite Groups

In this subsection, we discuss some examples to illustrate that it is not interesting to consider recurrence of finite abelian groups. Most recurrence properties exhibit a similarly trivial behaviour for a compact group. We therefore assume afterwards that the groups are infinite and non-compact.

Remark 7.18. By the classification theorem for finite abelian groups ([52, Theorem 2.1.3]), we can assume that a finite abelian group is given by

$$G = \bigoplus_{k=1}^{n} \mathbb{Z}_{p_k}.$$

with powers p_k of primes.

Proposition 7.19. *Let G be a finite abelian group. Then $\mathcal{D} \subseteq G$ is Poincaré or operator recurrent if and only if $e \in \mathcal{D}$.*

Proof. Consider the mps $(\mathbb{T}^n, \lambda; (\tau_{g_1} \times \ldots \times \tau_{g_n})_{(g_1,\ldots,g_n)\in\oplus_{k=1}^n \mathbb{Z}_{p_k}})$, where τ_{g_k} is the rotation on $\mathbb{T}$ by angle $\frac{2\pi}{p_k}$ and $E = e^{2\pi i[0,\frac{1}{p_1}]} \times \ldots \times e^{2\pi i[0,\frac{1}{p_n}]}$ with measure $\lambda(E) = \frac{1}{p_1\cdots p_n} > 0$. Then $\lambda(E \cap \tau_g(E)) = 0$ for all $g \neq e$. Operator recurrence follows in the same way by using the Koopman representation. □

Proposition 7.20. *Let G be a finite abelian group. Then $\mathcal{D} \subseteq G$ is intersective if and only if $e \in \mathcal{D}$.*

Proof. Clearly we have $e \in E - E$ for all $E \neq \emptyset$. We further have $\overline{d}(\{e\}) > 0$ and $\{e\} - \{e\} = \{e\}$, hence, $\{e\} - \{e\} \cap \mathcal{D} = \emptyset$ if $e \notin \mathcal{D}$. □

Remark 7.21. For a finite group, $f(g+d) - f(g)$ is never equidistributed mod 1 since it only has finitely many values and therefore

$$\oint_G \mathbb{1}_{[a,b]}([f(g+d) - f(g)]) \, d\lambda(g) = \frac{1}{|G|} \sum_{g\in G} \mathbb{1}_{[a,b]} \left([f(g+d) - f(g)]\right) = 0 \neq b - a$$

for some $0 \leq a < b \leq 1$ where $[f(t)] := f(t) \bmod 1$. The van der Corput property is hence trivially satisfied for all $\mathcal{D} \subseteq G$.

7.3.2 Setting for Locally Compact Abelian Groups

In this subsection, we introduce the relevant definitions and results for groups and discuss which restrictions we have to apply and which groups we actually consider in this section (see [41] and [67] for more details).

General Group Assumptions

We usually denote a group by G, its neutral element by e and its elements by g. We usually choose addition to denote the group operation and multiplication for its dual group.
We equip G with a topology such that $g \mapsto -g$ and $g \times h \mapsto g+h$ is continuous. We assume without further mention that the topology on G is regular and Hausdorff. We noted in Section 1.2 that there arise two recurrence phenoma for the real numbers, bounded recurrence or recurrence around 0 due to continuity at 0, which was characterised by the limit point 0 (see Section 4.1), and asymptotical recurrence. The same phenomena may appear for locally compact abelian groups and we can define bounded recurrence and asymptotic recurrence of a set $\mathcal{D}$ similarly by requiring that $\mathcal{D} \cap U_e$ or $\mathcal{D} \backslash U_e$, respectively, has the same recurrence property where U_e is a neighbourhood of the identity with $U_e \subseteq K$ for some compact set K with $\lambda(K) > 0$.

Definition 7.22 (Locally Compact Group)**.** *A group G is **locally compact** if e has a compact neighbourhood. It is called σ-locally compact if there is a countable cover of G with compact sets.*

Definition 7.23 (Haar Measure)**.** *Let G be a locally compact abelian group. A rotation invariant measure on the σ-algebra generated by all open subsets is called the **Haar measure**.*

The Haar measure always exists for a locally compact abelian group and it is unique up to a multiplicative constant and every compact subset of G has finite Haar measure ([41, Chapter 3 and 4]).

Følner Condition and the Mean Ergodic Theorem

We first note that every locally compact abelian group is amenable ([62, Proposition 0.15]). By the Følner condition, a locally compact group G is amenable if and only if there exists a Følner net $(E_i)_{i\in I}$ ([62, Theorem 4.13]).

Definition 7.24 (Følner Sequence (Net))**.** *An increasing sequence $(E_n)_{N\in\mathbb{N}}$ (an increasing net $(E_i)_{i\in I}$) of compact subsets of G is called a **Følner sequence** (**Følner net**) if for any compact $K \subseteq G$ and $\delta > 0$, there exists $N \in \mathbb{N}$ ($J \in I$) such that*

$$\lambda(E_n \bigtriangleup KE_n) \leq \delta\lambda(E_n)$$

for all $n \geq N$ ($n \geq J$).

We always consider one fixed Følner net for each group in consideration. If there exists a Følner sequence, then we choose the sequence instead of the net for simplicity.

Given a Følner net, we define the limit $\lim_{E\to G}$ as the limit over the Følner net $(E_i)_{i\in I}$ whenever it exists. Similarly, we define $\limsup_{E\to G} := \lim_j \sup_{i\geq j}$.

Since we often deal with averages, we introduce the notation $⨍_G$ by

$$⨍_E f(g)\ d\lambda(g) := \frac{1}{\lambda(E)}\int_E f(g)\ d\lambda(g)$$

whenever $\lambda(E) < \infty$ and

$$⨍_G f(g)\ d\lambda(g) := \lim_{E\to G}\frac{1}{\lambda(E)}\int_E f(g)\ d\lambda(g)$$

whenever the limit exists. We define the **upper density** $\overline{d}_G$ of a set $S \subseteq G$ by

$$\overline{d}_G(S) := \limsup_{E\to G}\frac{1}{\lambda(E)}\lambda(S\cap E) = \limsup_{E\to G} ⨍_E \mathbb{1}_S(g)\ d\lambda(g)$$

and similarly the **density** d_G whenever it exists. It is not obvious a priori how to extend the strong properties to a locally compact abelian group as $\lim_{|d|\to\infty, d\in\mathcal{D}}$ is not well-defined anymore. We suggest to define $\lim_{d\to\infty, d\in\mathcal{D}}$ by setting $\lim_{d\to\infty, d\in\mathcal{D}} f(g) = y$ if

$$\forall\epsilon > 0 \ \ \exists \text{ compact } K \subseteq G \ \ \forall d \in \mathcal{D} \cap (G\backslash K): \quad |f(d) - y| < \epsilon.$$

We note that this definition coincides with the usual definition of $\lim_{|d|\to\infty, d\in\mathcal{D}}$ for $G = \mathbb{R}$ and $G = \mathbb{Z}$.
As for $\mathbb{R}$ and $\mathbb{N}$, the main connection between recurrence properties is the following mean ergodic theorem ([29, Theorem 8.13], [62, (5.7)], [80, Section 3.3]).

Theorem 7.25 (Mean Ergodic Theorem)**.** *Let $\mathcal{T} = (T_g)_{g\in G} \subset \mathcal{L}(H)$ with a Hilbert space H be a strongly continuous group of contractions on H. Then $\mathcal{T}$ is* ***mean ergodic****, i.e.*

$$Px := \oint_{\mathbb{R}} T_t x \; d\lambda(t)$$

exists for all $x \in H$ in $\|\cdot\|$ where P is the mean ergodic projection onto $\mathrm{Fix}(\mathcal{T})$ *along* $\overline{\mathrm{lin}}\{(\mathrm{Id} - T_g)x : x \in H, g \in G\}$ *and we have the ergodic decomposition*

$$H = \mathrm{Fix}(\mathcal{T}) \oplus \overline{\mathrm{lin}}\{(\mathrm{Id} - T_g)x : x \in X, g \in G\}.$$

In Section 3.2, we occasionally use pointwise convergence results such as the pointwise ergodic theorem and the strong law of large numbers. As the pointwise ergodic theorem does not hold in the generality of Theorem 7.25 ([53, Section 1]), this suggests that we may have to restrict the generality of the group such as to second countable or σ-locally compact groups as well as to allow only certain Følner nets where some corresponding pointwise results hold ([29, Subsection 8.6.2], [53, [62, (5.20) and (5.21)], [80, Section 5.6]).
We define $l^\infty_*(G)$, $l^\infty_c(G)$ and $l^\infty_{cc}(G)$ as in Section 2.4. As in Definition 2.24, we define equidistribution mod 1.

Definition 7.26. *A function $f \in l^\infty_*(G,[0,1])$ is* ***equidistributed mod 1*** *if*

$$\oint_G e^{2\pi i k f(g)}\, d\lambda(g) = 0$$

holds for all $0 \neq k \in \mathbb{Z}$.

Dual Groups, Fourier transforms and the Bochner-Herglotz Theorem

An important element in the theory of abstract harmonic analysis is the notion of the dual group (see [67, Section 1.2]).

Definition 7.27 (Dual Group)**.** *Let G be a locally compact group. A* ***character*** *γ is a continuous group homomorphism $\gamma : G \to \mathbb{T}$. The* ***dual group*** *G^* is the set of all characters equipped with the "uniform on compact subsets"-topology. The group operation is given by pointwise multiplication.*

The dual group of a locally compact abelian group is again locally compact ([67, Subsection 1.2.6]). The following shows that the bidual group $(G^*)^*$ is in fact the original group G ([67, Theorem 1.7.2]).

Theorem 7.28. *Let G be a locally compact abelian group. Then G and $(G^*)^*$ are algebraically isomorphic.*

We usually denote characters by χ and note that Theorem 7.28 allows to interchange $g(\chi)$ and $\chi(g)$.
We note that the work with the Fourier transform requires abelian groups ([67, Chapter 1]). In this case, the Fourier transform of a measure on G^* is defined as for measures on $\mathbb{R}$ and $\mathbb{T}$.

Definition 7.29 (Fourier Transform)**.** *Let G be a locally compact abelian group and μ a measure on G^*. Then the* ***Fourier transform*** *$\widehat{\mu}$ is defined by*

$$\widehat{\mu}(g) = \int_{G^*} \overline{\chi(g)}\, d\mu(\chi).$$

The Bochner-Herglotz theorem ([67, 1.4.3]) yields the correspondence of Fourier transforms and positive-definite functions on G. A function $f : G \to \mathbb{C}$ is **positive-definite** if $f(-g) = \overline{f(g)}$ for all $g \in G$ and if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} f(g_i - g_j) \geq 0$$

for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $g_1, \ldots, g_n \in G$.

Theorem 7.30 (Bochner-Herglotz)**.** *Let $f : G \to \mathbb{C}$ be a continuous and positive-definite function. Then there exists a measure μ on G^* such that*

$$\widehat{\mu}(g) = f(g)$$

for all $g \in G$.

7.3.3 Example: Operator Recurrence and FMRiesz

As an example of how to deal with recurrence properties in locally compact abelian groups, we prove the equivalence of operator recurrence and FMRiesz (see also Subsection 3.2.1).

Definition 7.31 (FMRiesz)**.** *A set $\mathcal{D} \subseteq G$ is **FMRiesz** if every probability measure μ on G^* with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ satisfies*

$$\mu(\{e^*\}) = 0.$$

Definition 7.32 (Operator Recurrence)**.** *A set $\mathcal{D} \subseteq G$ is **operator recurrent** (OR) if, given a Hilbert space H, a strongly continuous unitary group $(T_g)_{g \in G}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that*

$$\langle T_d x, x \rangle \neq 0.$$

As in Lemmata 2.17, 2.27 and 2.32, we obtain the following lemmata for locally compact abelian groups.

Lemma 7.33. *Let H be a Hilbert space, $(T_g)_{g\in G}$ be a strongly continuous unitary group on H and $x \in H$. Then $f_x(g) := \langle T_g x, x\rangle$ is uniformly continuous and positive-definite.*

Lemma 7.34. *Let $g \mapsto x_g \in H$ be continuous from G to a Hilbert space H. Then we have*

$$\oint_G \langle x_g, y\rangle \ d\lambda(g) = \langle \oint_G x_g \ d\lambda(g), y\rangle$$

whenever the corresponding limits exist.

Lemma 7.35. *Let μ be the Bochner-Herglotz measure associated to the positive-definite and continuous function f. Then*

$$\mu(\{e^*\}) = \oint_G f(g) \ d\lambda(g).$$

Proposition 7.36 (FMRiesz $\Rightarrow$ OR)**.** *Let us assume that every probability measure μ on G^* with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ satisfies*

$$\mu(\{e^*\}) = 0.$$

Then, given a Hilbert space H, a strongly continuous unitary group $(T_g)_{g\in G}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that

$$\langle T_d x, x\rangle \neq 0.$$

Compare [60].

Proof. For a contradiction, let us assume that $\mathcal{D}$ is not operator recurrent, i.e. there exists a Hilbert space H, a strongly continuous unitary group $(T_g)_{g\in G}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$ such that $\langle T_d x, x\rangle = 0$ for all $d \in \mathcal{D}$.
The function f_x given by $f_x(g) := \langle T_g x, x\rangle$ is positive-definite and continuous by Lemma 7.33, hence, by the Bochner-Herglotz theorem 7.30, there exists a measure

μ on G^* such that $\widehat{\mu}(g) = f_x(g) = \langle T_g x, x\rangle$ and $\mu(G^*) = f(e) = \langle x, x\rangle = 1$.
Then we have $\widehat{\mu}(d) = \langle T_d x, x\rangle = 0$ for all $d \in \mathcal{D}$ by assumption and

$$\begin{aligned}
\mu(\{e^*\}) &= \int_{G^*} \overline{\mathbb{1}_{\{e^*\}}(\chi)}\, d\mu(\chi) \overset{7.35}{=} \int_{G^*} \oint_G \overline{\chi(g)}\, d\lambda(g)\, d\mu(\chi) \\
&= \oint_G \int_{G^*} \overline{\chi(g)}\, d\mu(\chi)\, d\lambda(g) = \oint_G \widehat{\mu}(g)\, d\lambda(g) = \oint_G \langle T_g x, x\rangle\, d\lambda(g) \\
&\overset{7.34}{=} \langle \oint_G T_g x\, d\lambda(g), x\rangle = \langle Px, x\rangle = \langle Px, Px\rangle = \|Px\|^2 > 0.
\end{aligned}$$

Hence, $\mathcal{D}$ is not FMRiesz, yielding a contradiction. □

Proposition 7.37 (OR $\Rightarrow$ FMRiesz)**.** *Let us assume that, given a Hilbert space H, a strongly continuous unitary group $(T_g)_{g\in G}$ on H and $x \in H$ with $\|x\| = 1$ and $Px \neq 0$, there exists $d \in \mathcal{D}$ such that*

$$\langle T_d x, x\rangle \neq 0.$$

Then every probability measure μ on G^* with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ satisfies

$$\mu(\{e^*\}) = 0.$$

Compare [60].

Proof. For a contradiction, let us assume that $\mathcal{D}$ is not FMRiesz, i.e. there exists a probability measure μ on G^* with $\widehat{\mu}(d) = 0$ for all $d \in \mathcal{D}$ and $\mu(\{e^*\}) > 0$.
We define $h_g(\chi) := g(\chi)$ with

$$\|h_g\|^2 = \int_{G^*} |h_g(\chi)|^2\, d\mu(\chi) = \int_{G^*} \mathbb{1}(\chi)\, d\mu(\chi) = \mu(G^*) = 1 < \infty$$

for all $g \in G$ and we note

$$\langle h_g, h_\gamma\rangle = \int_{G^*} h_g(\chi)\overline{h_\gamma(\chi)}\, d\mu(\chi) = \int_{G^*} h_{g-\gamma}(\chi)\, d\mu(\chi) \overset{3.73}{=} 0$$

for all $g \neq \gamma \in G$. We set $H := \overline{\text{lin}}\{h_g : g \in G\} \subseteq L^2(G^*, \mu)$ and define $T_\gamma : H \to H$. by $T_\gamma h_g := h_{g+\gamma}$ for $g, \gamma \in G$. This is a group representation since

$$(T_\delta T_\gamma)(h_g) = T_\delta(T_\gamma h_g) = T_\delta(h_{g+\gamma}) = h_{g+\gamma+\delta} = T_{\gamma+\delta} h_g$$

for all $g, \gamma, \delta \in G$. Moreover, T_γ is unitary for all $\gamma \in G$ since

$$\begin{aligned}\langle T_\gamma h_f, T_\gamma h_g\rangle &= \langle h_{\gamma+f}, h_{\gamma+g}\rangle = \int_{G^*} \chi(\gamma+g)\overline{\chi(\gamma+f)}\ d\mu(\chi)\\ &= \int_{G^*} \chi(\gamma+g)\chi(-\gamma-f)\ d\mu(\chi) = \int_{G^*} \chi(g-f)\ d\mu(\chi)\\ &= \int_{G^*} \chi(g)\overline{\chi(f)}\ d\mu(\chi) = \langle h_g, h_f\rangle\end{aligned}$$

and $T_\gamma T_{-\gamma} = \mathrm{Id} = T_{-\gamma}T_\gamma$, and it is strongly continuous since

$$\begin{aligned}&\lim_{\gamma\to\delta}\|T_\gamma h_g - T_\delta g\|^2 = \lim_{\gamma\to\delta}\|h_{g+\gamma} - h_{g+\delta}\|^2\\ &= \lim_{\gamma\to\delta}\int_{G^*}(h_{g+\gamma}(\chi) - h_{g+s}(\chi))(\overline{h_{g+\gamma}(\chi)} - \overline{h_{g+\delta}(\chi)})\ d\mu(\chi)\\ &= \lim_{\gamma\to\delta}\left(\|h_{g+\gamma}\|^2 + \|h_{g+\delta}\|^2 - 2\Re\left(\int_{G^*}\chi(g+\delta)\overline{\chi(g+\gamma)}d\mu(\chi)\right)\right)\\ &= 2 - 2\Re\left(\int_{G^*}\lim_{\gamma\to\delta}\left(\chi(g+\delta)\overline{\chi(g+\gamma)}\right)\ d\mu(\chi)\right) = 2 - 2\Re(\|h_{g+\delta}\|^2) = 0.\end{aligned}$$

We also have

$$\overline{\widehat{\mu}(g)} = \int_{G^*}\chi(g)\ d\mu(\chi) = \int_{G^*} h_g(\chi)\cdot\overline{h_e(\chi)}\ d\mu(\chi) = \langle h_g, h_e\rangle = \langle T_g h_e, h_e\rangle$$

and in particular

$$\langle T_d h_e, h_e\rangle = \overline{\widehat{\mu}(d)} = 0$$

for all $d \in \mathcal{D}$. However, we have

$$0 < \mu(\{e^*\}) = \|Ph_e\|^2,$$

as in the proof of Proposition 7.36 for $x = h_e$. Hence, we have a Hilbert space H, a strongly continuous unitary group $(T_g)_{g\in G}$ on H and $x \in H$ with $\|x\| = 1$ and $\langle T_d x, x\rangle = 0$ for all $d \in \mathcal{D}$ but $Px \neq 0$. The set $\mathcal{D}$ is therefore not operator recurrent, yielding a contradiction. □

Bibliography

[1] O.N. Ageev - *On extensions of typical group actions*. arXiv:1212.2660.

[2] N. Alon and Y. Peres - *A note on Euclidean Ramsey theory and a construction of Bourgain*. Acta Math. Hungar. **57** (1991), 61-64.

[3] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander - *Vector-valued Laplace transforms and Cauchy problems*. Monographs in Mathematics **96** (2001), Birkhäuser Verlag, Basel.

[4] J. Bass - *Suites uniformeént denses, moyennes, trigonométriques, fonctions pseudo-aléatoires*. Bull. Soc. Math. France **87** (1959), 1-64.

[5] J. Bass - *Les fonctions pseudo-aléatoires*. Mémor. Sci. Math., Fasc. CLIII Gauthier-Villars, Éditeur-Imprimeur-Libraire, Paris, 1962.

[6] V. Bergelson - *Sets of recurrence of $\mathbb{Z}^m$-actions and properties of sets of differences in Z^m*. J. London Math. Soc. **31** (1985), 295-304.

[7] V. Bergelson - *Ergodic Ramsey theory*. Contemp. Math. **65** (1987), Amer. Math. Soc., Providence, RI, 63-87.

[8] V. Bergelson - *Ergodic Ramsey theory - an update*. Ergodic Theory of $\mathbb{Z}^d$ Actions. London Math. Soc. Lecture Note Ser. **228** (1996), Cambridge Univ. Press, Cambridge, 1-61.

[9] V. Bergelson - *The multifarious Poincaré recurrence theorem*. London Math. Soc. Lecture Note Ser. **277** (2000), Cambridge Univ. Press, Cambridge, 31–57.

[10] V. Bergelson - *Combinatorial and Diophantine applications of ergodic theory.* Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, 745–869.

[11] V. Bergelson, M. Boshernitzan and J. Bourgain - *Some results on nonlinear recurrence*. J. Anal. Math. **62** (1994), 29-46.

[12] V. Bergelson and N. Hindman - *Quotient sets and density recurrent sets*. Preprint.

[13] V. Bergelson, A. Leibman and C.G. Moreira - *From discrete- to continuous-time ergodic theorems*. Ergodic Theory and Dynamical Systems **32** (2012), 383-426.

[14] V. Bergelson and E. Lesigne - *Van der Corput sets in $\mathbb{Z}^d$*. Colloq. Math. **110** (2008), 1-49.

[15] V. Bergelson and R. McCutcheon - *Recurrence for semigroup actions and a non-commutative Schur theorem*. Contemp. Math. **215** (1998), 205–222.

[16] V. Bergelson and I. Ruzsa - *Sumsets in difference sets*. Israel J. Math. **174** (2009), 1-18.

[17] A. Bertrand-Mathis - *Ensembles intersectifs et récurrence de Poincaré*. Israel J. Math. **55** (1986), 184-198.

[18] J.-P. Bertrandias - *Suites pseudo-aléatoires et critères d'équirépartition modulo un*. Compositio Math. **16** (1964), 23-28.

[19] G.D. Birkhoff - *Dynamical systems.* American Mathematical Society, New York, 1927.

[20] G.D. Birkhoff - *Proof of the ergodic theorem.* Proc. Nat. Acad. Sci. U.S.A. **17** (1931), 656–660.

[21] M. Boshernitzan and E. Glasner - *On two recurrence problems.* Fund. Math. **206** (2009), 113–130.

[22] M. Boshernitzan, G. Kolesnik, A. Quas and M. Wierdl - *Ergodic averaging sequences.* J. Anal. Math. **95** (2005), 63–103.

[23] J. Bourgain - *Ruzsa's problem on sets of recurrence.* Israel J. Math. **59** (1987), 150–166.

[24] C. Caratheodory - *Über den Wiederkehrsatz von Poincaré.* S. B. Preuss. Akad. Wiss (1919), 580-584.

[25] J. Coquet, T. Kamae and M. Mendès France - *Sur la mesure spectrale de certaines suites arithmétiques.* Bull. Soc. Math. France **105** (1977), 369-384.

[26] J. van der Corput - *Diophantische Ungleichungen. I. Zur Gleichverteilung Modulo Eins.* Acta Mathematica **56** (1931), 373–456.

[27] R. Derndinger, R. Nagel and G. Palm - *Ergodic theory in the perspective of functional analysis, 13 lectures.* Unpublished manuscript.

[28] A. Dvoretzky - *On the strong stability of a sequence of events.* Ann. Math. Statistics **20** (1949), 296–299.

[29] M. Einsiedler and T. Ward - *Ergodic theory with a view towards number theory.* Graduate Texts in Mathematics **259** (2011), Springer-Verlag, London.

[30] T. Eisner - *Embedding operators into strongly continuous semigroups.* Arch. Math. (Basel) **92** (2009), 51–460.

[31] K.-J. Engel and R. Nagel - *One-parameter semigroups for linear evolution equations.* Graduate Texts in Mathematics **194** (2000), Springer-Verlag, New York.

[32] P. Erdös - *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem.* Proc. Nat. Acad. Sci. U.S.A. 35 **1949**, 374-384.

[33] A.H. Forrest - *The construction of a set of recurrence which is not a set of strong recurrence.* Israel J. Math. **76** (1991), 215-228.

[34] H. Fürstenberg - *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions.* J. Analyse Math. **31** (1977), 204–256.

[35] H. Fürstenberg - *Poincaré recurrence and number theory.* Bull. Amer. Math. Soc. (N.S.) **5** (1981), 211-234.

[36] H. Fürstenberg - *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981.

[37] H. Fürstenberg, Y. Katznelson and D. Ornstein - *The ergodic theoretical proof of Szemerédi's theorem.* Bull. Amer. Math. Soc. (N.S.) **7** (1982), 527–552.

[38] H. Fürstenberg, Y. Katznelson and B. Weiss - *Ergodic theory and configurations in sets of positive density.* Mathematics of Ramsey theory, Algorithms Combin. **5** (1990), Springer-Verlag, Berlin, 184–198.

[39] E. Glasner and B. Weiss - *On the interplay between measurable and topological dynamics*. Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, 597–648.

[40] R. Gordon - *Riemann integration in Banach spaces*. Rocky Mountain J. Math. **21** (1991), 923–949.

[41] E. Hewitt and K.A. Ross - *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*. Die Grundlehren der mathematischen Wissenschaften **115** (1963), Springer-Verlag, Berlin, 1963.

[42] T. Kamae, M. Mendès France - *Van der Corput's difference theorem*. Israel J. Math. **31** (1977), 335–341.

[43] Y. Katznelson - *Chromatic numbers of Cayley graphs on $\mathbb{Z}$ and recurrence*. Combinatorica **21** (2001), 211–219.

[44] Y. Katznelson - *An introduction to harmonic analysis. Third edition*. Cambridge University Press, Cambridge, 2004.

[45] M. Kern, R. Nagel and G. Palm - *Dilations of positive operators: construction and ergodic theory*. Math. Z. **156** (1977), 265–277.

[46] L.J. Konrad - *Dilations of bistochastic semigroups*. To appear.

[47] B.O. Koopman - *Hamiltonian systems and transformation in Hilbert space*. Proc. Nat. Acad. Sci. U.S.A. **17** (1931), 315-318.

[48] B. Kra - *Poincaré recurrence and number theory: 30 years later*. Bull. Amer. Math. Soc. (N.S.) **48** (2011), 497–501.

[49] U. Krengel - *Ergodic theorems*. de Gruyter Studies in Mathematics **6** (1985). Walter de Gruyter & Co., Berlin.

[50] I. Kříž - *Large independent sets in shift-invariant graphs: solution of Bergelson's problem*. Graphs Combin. **3** (1987), 145–158.

[51] L. Kuipers and H. Niederreiter - *Uniform distribution of sequences*. Wiley-Interscience, New York, 1974.

[52] H. Kurzweil and B. Stellmacher - *The theory of finite groups. An introduction*. Springer-Verlag, New York, 2004.

[53] E. Lindenstrauss - *Pointwise theorems for amenable groups*. Invent. Math. **146** (2001), 259–295.

[54] R. Lyons - *Strong laws of large numbers for weakly correlated random variables*. Michigan Math. J. **35** (1988), 353–359.

[55] R. McCutcheon - *Three results in recurrence*. London Math. Soc. Lecture Note Ser. **205** (1995), Cambridge Univ. Press, Cambridge, 349–358.

[56] R. McCutcheon - *Elemental methods in ergodic Ramsey theory*. Springer Lecture Notes in Mathematics **1722** (1999). Springer-Verlag, Berlin.

[57] J.L. Montgomery - *Ten lectures on the interface between analytic number theory and harmonic analysis*. CBMS Regional Conference Series in Mathematics **84** (1994), American Mathematical Society, Providence, RI.

[58] J. Nagata - *Modern general topology*. North-Holland Publishing Co., Amsterdam, 1985.

[59] J. von Neumann - *Proof of the quasi-ergodic hypothesis*. Proc. Nat. Acad. Sci. U.S.A. **18** (1932), 70–82.

[60] M. Nincevic, B. Rabar and S. Slijepcevic - *Ergodic characterization of van der Corput sets*. Arch. Math. (Basel) **98** (2012), 355-360.

[61] D.S. Ornstein - *A K-automorphism with no square root and Pinsker's conjecture.* Advances in Math. **10** (1973), 89-102.

[62] A.L.T. Paterson - *Amenability.* Mathematical Surveys and Monographs **29** (1988). American Mathematical Society, Providence, RI.

[63] Y. Peres - *Applications of Banach limits to the study of sets of integers.* Israel J. Math **62** (1988), 17–31.

[64] H. Poincaré - *Sur le problème des trois corps et les equations de la dynamique.* Acta Math. **13** (1890), 1-270.

[65] M. Queffelec - *Mesure spectrales associées à certaines suites arithmétiques.* Bull. Soc. Math. France **107** (1979), 385-421.

[66] F. and M. Riesz - *Über Randwerte einer analytischen Funktion.* Compte rendu du quatrième congrès des mathématiciens scandinaves, Almqvist & Wiksells, Uppsala, 1920, 27-44.

[67] W. Rudin - *Fourier analysis on groups.* Interscience Tracts in Pure and Applied Mathematics **12** (1962) Interscience Publishers, London, 1962.

[68] W. Rudin - *Functional Analysis.* McGraw-Hill Book Co., New York, 1973.

[69] T. de la Rue and J. de Sam Lazaro - *Une transformation générique peut être insérée dans un flot.* Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), 121–134.

[70] I.Z. Ruzsa - *Uniform distribution, positive trigonometric polynomials and difference sets.* Seminar on Number Theory, 1981/1982, Exp. No. 18, Univ. Bordeaux I, Talence, 1982.

[71] I.Z. Ruzsa - *Connections between the uniform distribution of a sequence and its differences.* Topics in classical number theory, Vol. I, II, Colloq. Math.

Soc. János Bolyai **34** (1984), North-Holland Publishing Co., Amsterdam, 1419–1443.

[72] I.Z. Ruzsa - *On measures of intersectivity.* Acta Math. Hungar. **43** (1984), 335–340.

[73] A. Sárközy - *On difference sets of sequences of integers I.* Acta Math. Acad. Sci. Hungar. **31** (1978), 125–149.

[74] A. Sárközy - *On difference sets of sequences of integers II.* Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **21** (1978), 45-53.

[75] A. Sárközy - *On difference sets of sequences of integers III.* Acta Math. Acad. Sci. Hungar. **31** (1978), 355-386.

[76] Y. Son - *Some results on joint ergodicity, sets of recurrence and substitution and tiling systems.* PhD thesis, 2013.

[77] M.H. Stone - *On one-parameter unitary groups in Hilbert Space.* Ann. of Math. (2) **33** (1932), 643–648.

[78] B. Sz.-Nagy and C. Foias - *Harmonic analysis of operators on Hilbert space.* North-Holland Publishing Co., Amsterdam, 1970.

[79] E. Szemerédi - *On sets of integers containing no k elements in arithmetic progression.* Acta Arith. **27** (1975), 199–245.

[80] A. Tempelman - *Ergodic theorems for group actions. Informational and thermodynamical aspects.* Mathematics and its Applications **78** (1992), Kluwer Academic Publishers Group, Dordrecht.

[81] G.F. Vincent-Smith - *Dilation of a dissipative quantum dynamical system to a quantum Markov process.* Proc. London Math. Soc. **49** (1984), 58–72.

[82] B. Weiss - *Single orbit dynamics*. CBMS Regional Conference Series in Mathematics **95** (2000), American Mathematical Society, Providence, RI.

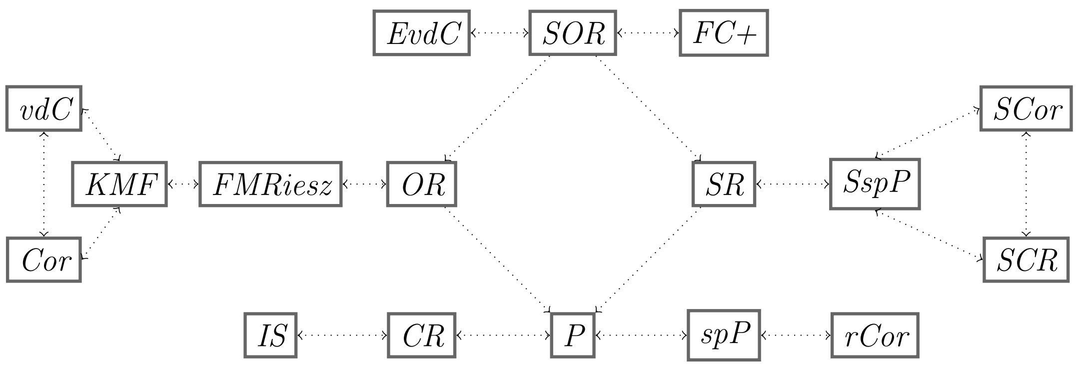
[83] H. Weyl - *Über die Gleichverteilung von Zahlen mod. Eins*. Math. Ann. **77** (1916), 313–352.

[84] N. Wiener - *The Fourier integral and certain of its applications*. Cambridge University Press, Cambridge, 1933.

Appendix

The following diagrams show the detailed implications of the recurrence properties which are summarised in Theorems 8 and 9.

Theorem 1. *Let $\mathcal{D} \subseteq \mathbb{Z}$. Then the following implications for integer recurrence hold.*



Theorem 2. *Let $\mathcal{D} \subseteq \mathbb{R}$. Then the following implications for real recurrence hold.*