Fluid flows on many scales

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A single equation can describe how fluids flow across a wide range of length scales, from ocean currents to swimming algae. The difference merely lies in the Reynold's number, says Julia Yeomans.

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{f} + \mathcal{R}^{-1}\, \nabla^2\, \boldsymbol{u}$$

The Navier–Stokes equation, written down by Claude-Louis Navier in 1822, is both beautiful and useful. It describes how simple fluids like water or air move, predicting how their velocity, $\boldsymbol{u}$, changes in space and time. The Navier-Stokes equation is vital in designing aerodynamic cars and aeroplanes and in extracting oil from underground reservoirs. And hidden in the mathematics is also an understanding of how bacteria swim, how rivers flow, and of the motion of planetary atmospheres. But the equation remains mysterious, despite being nearly two centuries old.

In the Navier-Stokes equation $\boldsymbol{f}$ is a force per unit volume, which gets the fluid moving. The expression on the left-hand side is the inertial term which describes how momentum moves around in the fluid. But fluids do not flow forever without being pushed, and the final term encodes how fluid viscosity dissipates the momentum. When a canoeist stops paddling, they will glide for a while (inertia) but eventually come to a halt (viscosity and friction).

The $\mathcal{R}$ in the Navier-Stokes equation is the Reynolds number, a dimensionless number that measures the relative importance of inertia and viscosity. The Reynolds number describes fluid flows across a wide range of length scales: the energetic fin whale moving through water has a Reynolds number about 10^8, and for blood flowing through a capillary, the Reynolds number would be 10^{-3}. 'Everyday' Reynolds numbers tend to lie between these values: stirring a cup of tea corresponds to a Reynolds number of 10^3, and a person swimming to 10^6.

As a rule of thumb, slow and small regimes correspond to low Reynolds numbers, where the flow is dominated by viscosity. Flows are laminar, calm and ordered — like a thread of golden syrup falling from a spoon. Streams of fluid move in straight lines, and one of the biggest problems in microfluidics is getting different liquids to mix.

Tiny microorganisms, for example swimming bacteria, move at low Reynolds number. In this limit, it is possible to forget about inertia, leaving just the right-hand side of the Navier–Stokes equation, the Stokes equation. Stokesian dynamics does not include any information about the direction of time, which presents a problem for the bacteria. If their swimming stroke is reversible in time, meaning it looks the same in a movie played backwards or forwards, their net movement will be zero over each complete stroke [1]. Bacteria have had to evolve strategies to allow them to move in their Stokesian world. These include flagella, long whip-like tails, where a wave moving along the tail defines a direction in time. We would face similar challenges if we tried to swim in treacle.

High Reynolds numbers, where the flow is dominated by inertia, correspond to large and fast scales — a regime of chaotic, turbulent flow [3]. The velocity field incorporates eddies, or vortices, of all sizes and energy cascades from larger vortices to smaller ones before eventually being dissipated as heat. Turbulent flow is epitomised in liquids by waterfalls or breaking waves. In gases, the viscosity is so small that turbulent flow is the norm and atmospheric turbulence near the earth's surface resulting from wind shear, convection and friction is an important factor in air pollution patterns.

These two regimes of fluid flow were investigated by Osborne Reynolds, and the Reynolds number bears his name in recognition of his experiments flowing water stained with red dye through a transparent pipe. In his words [2], “When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube.” At higher velocities Reynolds identified turbulence; the dye formed eddies that “curled and whirled about in the manner so familiar with smoke.” The transition between the two regimes takes place at Reynolds numbers around 2300, but the value depends on the geometry of the pipe and is changed by tiny perturbations such as roughness of the pipe walls or any vibration of the apparatus.

Although we have gained a lot of insight into turbulence through experiments and computer simulations, there is no predictive mathematical theory yet. The non-linear part of the inertial term in the Navier–Stokes equation is of relevance, but how it leads to an energy cascade is not understood. The Navier–Stokes equation will celebrate its 200th anniversary next year, but turbulence still guards its secrets.

References:

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3. P.A. Davidson, *Turbulence: An Introduction for Scientists and Engineers*. 2nd ed. Oxford University Press (2015).

Image:

https://www.alamy.com/stock-photo-wine-glass-filled-with-water-and-blue-drops-of-dye-173831869.html

Competing interests:

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