

A NONABELIAN BRUNN–MINKOWSKI INEQUALITY

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Abstract. Henstock and Macbeath asked in 1953 whether the Brunn–Minkowski inequality can be generalized to nonabelian locally compact groups; questions along the same line were also asked by Hrushovski, McCrudden, and Tao. We obtain here such an inequality and prove that it is sharp for helix-free locally compact groups, which includes real linear algebraic groups, Nash groups, semisimple Lie groups with finite center, solvable Lie groups, etc. The proof follows an induction on dimension strategy; new ingredients include an understanding of the role played by maximal compact subgroups of Lie groups, a necessary modified form of the inequality which is also applicable to nonunimodular locally compact groups, and a proportionated averaging trick.

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1 Introduction

1.1 Background. Let μ be the usual Lebesgue measure on \mathbb{R}^d , let X and Y be nonempty and compact subsets of \mathbb{R}^d , and set $X + Y := \{x + y : x \in X, y \in Y\}$. The Brunn–Minkowski inequality says that

$$\mu(X + Y)^{1/d} \geq \mu(X)^{1/d} + \mu(Y)^{1/d}. \quad (1.1)$$

For fixed $\mu(X)$ and $\mu(Y)$, the inequality provides us with the minimum value of $\mu(X + Y)$ which is obtained, for example, when X , Y , and $X + Y$ are d -dimensional hypercubes with side length $\mu(X)^{1/d}$, $\mu(Y)^{1/d}$, and $\mu(X)^{1/d} + \mu(Y)^{1/d}$, respectively.

Under the further assumption that X and Y are convex, the inequality in an equivalent form was proven by Brunn [Bru87] in 1887. In the celebrated *Geometrie der Zahlen* (Geometry of Numbers) [Min96] published in 1896, Minkowski introduced the current form of the inequality and established that the equality occurs in (1.1) if and only if X and Y are homothetic convex sets. Lyusternik [Lyu35] removed the convexity assumption in 1935. However, his proof that the same condition for equality still holds contained some errors, a situation eventually corrected by Hensstock and Macbeath [HM53] in 1953. The Brunn–Minkowski inequality is widely considered a cornerstone of convex geometry. See [Gar02] for an excellent survey on its numerous generalizations and applications.

In this paper, we consider the problem of generalizing the Brunn–Minkowski inequality to a locally compact group G . Here, up to positive constant factors, we have a unique left Haar measure μ generalizing the Lebesgue measure in \mathbb{R}^d ; see Appendix B for the precise definitions.

We temporarily further assume that μ is also invariant under the right translations. Such G is called *unimodular*. This assumption holds when $G = \mathbb{R}^d$ and in many other situations (e.g., when G is compact, discrete, a nilpotent Lie group, a semisimple Lie group, etc). Set $XY = \{xy : x \in X, y \in Y\}$ for nonempty compact $X, Y \subseteq G$. The translation invariance property of μ implies that

$$\mu(XY) \geq \max\{\mu(X), \mu(Y)\}$$

and should intuitively be even larger, hinting at a meaningful generalization of the Brunn–Minkowski inequality to this setting. This will be shown to be the case.

For an arbitrary locally compact group G , μ might no longer be right invariant. Hence, we still have $\mu(XY) \geq \mu(Y)$, but we might have $\mu(XY) < \mu(X)$. By a result of Macbeath [Mac60] in 1960, the trivial inequality $\mu(XY) \geq \mu(Y)$ for nonunimodular G is already sharp in the sense that for any $\alpha, \beta, \varepsilon > 0$, there are nonempty compact $X, Y \subseteq G$ with

$$\mu(X) = \alpha, \mu(Y) = \beta, \text{ and } \mu(XY) < \mu(Y) + \varepsilon.$$

We will later see in this paper that there is still a meaningful generalization of the Brunn–Minkowski inequality involving both μ and a right Haar measure ν . Surprisingly, it turns out that if one only cares about unimodular cases, the nonunimodular

cases are still needed for our proof. We will keep the setting and notation of this paragraph throughout the rest of the paper.

The problem of generalizing the Brunn–Minkowski inequality was proposed in 1953 by Henstock and Macbeath [HM53]; different variations of this problem were also later suggested by Hrushovski [Hru], by McCrudden [McC69], and by Tao [Tao11]. In the direction of the intuition described earlier, Kemperman [Kem64] showed in 1964 that $\mu(XY) \geq \mu(X) + \mu(Y)$ when G is connected, unimodular and noncompact. Even more important for us is the generalization (1.2), also in the same paper [Kem64], to all connected noncompact locally compact groups:

$$\frac{\nu(X)}{\nu(XY)} + \frac{\mu(Y)}{\mu(XY)} \leq 1. \quad (1.2)$$

While applicable to all locally compact groups, Kemperman’s inequalities are not sharp: even for \mathbb{R}^2 , they give a weaker conclusion than the Brunn–Minkowski inequality. The most definite result toward the correct lower bound was obtained by McCrudden [McC69] in 1969. In effect, he showed that when G is a unimodular solvable Lie group of dimension d , and m is the dimension of the maximal compact subgroup, we have

$$\mu(XY)^{1/(d-m)} \geq \mu(X)^{1/(d-m)} + \mu(Y)^{1/(d-m)}. \quad (1.3)$$

The above differs from McCrudden’s original statement in that m was defined using an inductive idea in [McC69]; the form in (1.3) is more suitable to get the later generalizations (Theorems 1.1 and 1.2) and to show that it is indeed sharp (Theorem 1.3). A number of special cases of this result were rediscovered by Gromov [Gro03], by Hrushovski [Hru12], by Leonardi and Mansou [LM05], and by Tao [Tao11]. Sharpness for nilpotent groups was essentially proven by Monti [Mon03]; see also Tao [Tao11].

1.2 Statement of main results. Suppose G is a Lie group with connected component G_0 . Following Levi decomposition (Fact E.5), we have an exact sequence of Lie groups

$$1 \rightarrow Q \rightarrow G_0 \rightarrow S \rightarrow 1$$

where Q is solvable and S is semisimple. It is known that the center $Z(S)$ is a finitely generated abelian group of rank h ; see Lemma 2.2 and Facts E.9, E.10. We call h the **helix dimension** of G . As an example, $\mathrm{SL}_2(\mathbb{R})$ has helix dimension 0 while its universal cover has helix dimension 1. If $h = 0$, equivalently S has finite center, we say that G is **helix-free**. Real linear algebraic groups and more generally, Nash groups (equivalently, semialgebraic Lie groups or groups definable in the field of real numbers) are helix free; see [BJO12, Lemma 4.5] and the subsequent discussion in the same paper. Our first main result is a generalization of Brunn–Minkowski inequality to Lie groups whose exponent will be seen to be sharp for helix-free Lie groups:

Theorem 1.1. *Suppose G is a Lie group, μ is a left Haar measure, ν is a right Haar measure, the dimension of G is d , the maximal dimension of a compact subgroup of G is m , the helix dimension of G is h , and X, Y are compact subsets of G with positive measure. Then*

$$\frac{\nu(X)^{1/(d-m-h)}}{\nu(XY)^{1/(d-m-h)}} + \frac{\mu(Y)^{1/(d-m-h)}}{\mu(XY)^{1/(d-m-h)}} \leq 1; \quad (1.4)$$

the left-hand-side is interpreted as $\max\{\nu(X)/\nu(XY), \mu(Y)/\mu(XY)\}$ if $d-m-h = 0$. In particular, if G is unimodular, then $\mu(XY)^{\frac{1}{d-m-h}} \geq \mu(X)^{\frac{1}{d-m-h}} + \mu(Y)^{\frac{1}{d-m-h}}$.

We require that X and Y be compact in our results since there are measurable sets X and Y such that XY is not measurable. By using the regularity property of Haar measure, the conclusions in our main theorems still hold for measurable X and Y if we replace $\mu(XY)$ and $\nu(XY)$ with inner Haar measures.

Now consider an arbitrary locally compact group G . Using the Gleason–Yamabe Theorem (Fact C.2), one can choose an open subgroup G' of G and a normal compact subgroup H of G' such that G'/H is a Lie group. It is shown in Proposition 2.8 that

$$n = \dim(G'/H) - \max\{\dim(K) : K \text{ is a compact subgroup of } G'/H\}$$

is independent of the choice of G' and H satisfying the above properties. We call n the **noncompact Lie dimension** of G . Let Q be the radical (i.e., the maximal connected closed solvable normal subgroup, see Fact E.4) of G'/H . Note that $(G'/H)_0/Q$ has discrete center $Z((G'/H)_0/Q)$ by Facts E.9 and E.10. We call

$$h = \text{rank}(Z((G'/H)_0/Q))$$

the **helix dimension** of G . We will also show that the helix dimension h of G'/H is independent of the choice of G' and H in Proposition 2.8. Here is our second main result.

Theorem 1.2. *Suppose G is a locally compact group with noncompact Lie dimension n and helix dimension h , μ is a left Haar measure, ν is a right Haar measure, and X, Y are compact subsets of G with positive measure. Then*

$$\frac{\nu(X)^{1/(n-h)}}{\nu(XY)^{1/(n-h)}} + \frac{\mu(Y)^{1/(n-h)}}{\mu(XY)^{1/(n-h)}} \leq 1; \quad (1.5)$$

the left-hand-side is interpreted as $\max\{\nu(X)/\nu(XY), \mu(Y)/\mu(XY)\}$ when $n-h = 0$. In particular, if G is unimodular, then $\mu(XY)^{\frac{1}{n-h}} \geq \mu(X)^{\frac{1}{n-h}} + \mu(Y)^{\frac{1}{n-h}}$.

When G is as in Theorem 1.1, the noncompact Lie dimension n is simply $d-m$, so Theorem 1.2 is a generalization of Theorem 1.1.

Our last main result tells us that when G is helix-free, the exponent $1/(n-h) = 1/n$ in Theorem 1.1 and Theorem 1.2 are sharp even when we assume further that $X = Y$. As usual in the current setting, we write X^k for the k -fold product of X .

Theorem 1.3. *Suppose G is a locally compact group with noncompact Lie dimension n , μ is a left Haar measure, and ν is a right Haar measure. Then*

- (1) *When $n = 0$, there is a compact set X with positive left and right measure in G such that $\mu(X^2) = \mu(X)$ and $\nu(X^2) = \nu(X)$.*
- (2) *When $n > 0$, for every $\varepsilon > 0$, there is a compact set X with positive left and right measure in G such that*

$$\frac{\nu(X)^{\frac{1}{n}-\varepsilon}}{\nu(X^2)^{\frac{1}{n}-\varepsilon}} + \frac{\mu(X)^{\frac{1}{n}-\varepsilon}}{\mu(X^2)^{\frac{1}{n}-\varepsilon}} > 1. \quad (1.6)$$

As a corollary, if G is unimodular with $n > 0$, for every $\varepsilon' > 0$, there is a compact set X in G such that $\mu(X^2) < (2^n + \varepsilon')\mu(X)$.

The upper bound given in Theorem 1.3 matches the lower bound given in Theorem 1.2 when the group is helix-free, that is a group has helix dimension 0, which essentially means the semisimple part of the group has finite center. Hence, for these groups, our theorems resolve the problem of generalizing the Brunn–Minkowski inequality.

We believe that the exponent in Theorem 1.3 should be correct for all locally compact groups, which is made precise by the following conjecture:

CONJECTURE 1.4 (Nonabelian Brunn–Minkowski Conjecture). *Suppose G is a locally compact group with noncompact Lie dimension n , μ is a left Haar measure, ν is a right Haar measure, and X, Y are compact subsets of G with positive measure. Then*

$$\frac{\nu(X)^{1/n}}{\nu(XY)^{1/n}} + \frac{\mu(Y)^{1/n}}{\mu(XY)^{1/n}} \leq 1;$$

the left-hand-side is interpreted as $\max\{\nu(X)/\nu(XY), \mu(Y)/\mu(XY)\}$ when $n = 0$.

While we do not know whether the exponent in Theorem 1.2 is sharp, it does have the correct order of magnitude. This is because the helix dimension h of G is always at most $n/3$, where n is the noncompact Lie dimension of G ; see Corollary 2.15. In particular, if there is a compact subset X of a connected noncompact unimodular group G with small $\mu(X^2)/\mu(X)$, Theorem 1.2 still allows us to deduce that G has a compact subgroup of small codimension, which constrains the structure of G . In an upcoming paper [AJTZ21] by An and the authors, we apply Theorem 1.2 to resolve the last open case of the Kemperman Inverse problem, also answering the connected case of a question by Tao. Recently, Fanlo and Hrushovski similarly use Theorem 1.2 to obtain a structural result for metric approximate groups [Fan21, FH21].

The next result shows that one can reduce Conjecture 1.4 to simply connected simple Lie groups. As a consequence of this, the only remaining cases are what one might regard initially as the simplest cases.

Theorem 1.5. *Suppose the nonabelian Brunn–Minkowski conjecture holds for all simply connected simple Lie groups, then it holds for all locally compact groups.*

1.3 Further remarks. The results of this paper continue a line of work by the first two authors [JT20] on small measure expansions in locally compact groups. Through classifying groups G and compact subsets X and Y of G with nearly minimal expansion, it is shown there that when G is a compact semisimple Lie group and $\mu(X)$ sufficiently small,

$$\mu(X^2) > (2 + c)\mu(X)$$

for a positive constant c independent of the choice of G . It is conjectured by Breuilard and Green [Green, Problem 78] that $\mu(X^2) > 3.99\mu(X)$ when $G = \mathrm{SO}_3(\mathbb{R})$ and X sufficiently small. The following corollary of Theorem 1.1 resolves the counterpart of the aforementioned conjecture for all noncompact simple Lie group G .

COROLLARY 1.6. *Let G be a noncompact semisimple Lie group, and let $X \subseteq G$ be compact. Then $\mu(X^2) \geq 4\mu(X)$.*

Equality can occur in (1.4) and (1.5) of Theorems 1.1 and 1.2, for example, when $G = \mathbb{R}^d$ for $d \geq 1$ as described earlier. In fact, the classical Brunn–Minkowski inequality in \mathbb{R}^d comes along with a complete classification of when equality occurs. For connected locally compact abelian group G with noncompact Lie dimension 1 (automatically has helix dimension 0 by Corollary 2.15), it follows from the work of Kneser [Kne56] that equality occurs in (1.5) exactly when there is a continuous surjective group homomorphism $\chi : G \rightarrow \mathbb{R}$ with the compact kernel, and X, Y are preimages of compact intervals in \mathbb{R} . The same conclusion is recently obtained by An and the authors [AJTZ21] for general locally compact groups, using the results of the current paper as ingredients.

On the other hand, a result by McCrudden [McC70] shows that when G is the Heisenberg group, equality cannot occur. It would be interesting to determine those groups with helix dimension 0 and compact subsets inside them where equality can occur. This question can also be asked without the assumption of helix dimension 0 once Conjecture 1.4 is resolved. It would also be interesting to understand when equality nearly happens and develop a theory similar to that of Christ, Figalli, and Jerison [Chr12, FJ15, FJ17] for \mathbb{R}^d .

Like the Brunn–Minkowski inequality for \mathbb{R}^d , our results do not rely on the normalization of Haar measures. However, by fixing a Haar measure μ on a unimodular group G , it would be interesting to determine the value of

$$\min\{\mu(XY) : X, Y \subseteq G \text{ are compact, } \mu(X) = \alpha, \mu(Y) = \beta\},$$

for given $\alpha, \beta \in \mathbb{R}^{>0}$, and to classify the situations where equality happens. We do not pursue this question here.

1.4 Overview of the proof. In this subsection, we discuss the idea of the proof of the main results and the organization of the paper. For expository purposes, we restrict our attention to helix-free locally compact groups, where we can fully prove

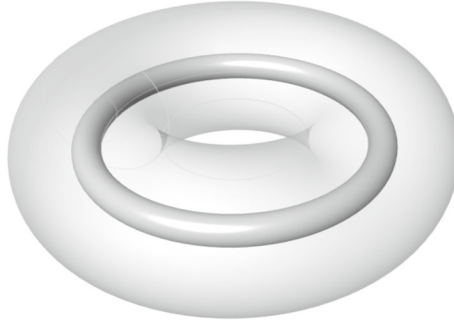


Figure 1: Let $G = \mathrm{SL}(2, \mathbb{R})$ (the open region bounded by the outer torus), and let $K = \mathrm{SO}(2, \mathbb{R})$ be the maximal compact subgroup of G . If we take X to be a small closed neighborhood of K (closed region bounded by the shaded inner torus), Theorem 1.3 says when X is sufficiently small, $\mu_G(X^2)$ will be very close to $4\mu_G(X)$ instead of $8\mu_G(X)$, although G has topological dimension 3.

Conjecture 1.4. The proof of the full versions of Theorems 1.1 and 1.2 requires a more involved discussion of the helix dimension, which is developed in Sect. 2.

In each of our results, the exponent of the inequalities is controlled by the non-compact Lie dimension n of G instead of just its topological dimension d as in the simpler versions for \mathbb{R}^d . Recall that, for a Lie group G , $n = d - m$ where m is the maximum dimension of a compact subgroup of G . The proof of Theorem 1.3 explains the critical role of m : our construction is essentially a small neighborhood of a compact subgroup of G having maximal dimension, see Fig. 1. One may conjecture that the above construction is optimal. Theorems 1.1 and 1.2 confirm this intuition for helix-free groups.

To motivate our proofs of Theorems 1.1 and 1.2, we first recall some proofs of the known cases of the Brunn–Minkowski inequality. Over \mathbb{R}^d , the usual strategy is to induct on dimensions. This is generalized by McCrudden [McC69, Theorem 1.2] to obtain the following “unimodular exponent splitting” theorem: Given an exact sequence of *unimodular* locally compact groups

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1,$$

if H and G/H satisfy Brunn–Minkowski inequalities with exponents $1/n_1$ and $1/n_2$, respectively, then the group G satisfies a Brunn–Minkowski inequality with exponent $1/(n_1 + n_2)$. McCrudden applied his unimodular exponent splitting theorem to obtain the Brunn–Minkowski inequality for unimodular solvable groups with sharp exponents. A simpler proof of his theorem is given in Sect. 4 for completeness. In the proof of our main theorems, one important ingredient will be an exponent splitting result (that is a generalization of his).

We now sketch a proof of McCrudden’s theorem and introduce a “spillover” argument which we will also use later on; some of the ideas presented here are also available in McCrudden’s original approach. For each g in G , we call $X \cap gH$ a

fiber of X , and refer to the Haar measure of $g^{-1}X \cap H$ in H as its *length*. Let $\pi : G \rightarrow G/H$ be the quotient map. We now partition X and Y each into N parts. Suppose $X = \bigcup_{i=1}^N X_i$ and $Y = \bigcup_{i=1}^N Y_i$, we require that the images under π of the X_i 's are pairwise disjoint, the shortest fiber-length in each X_i is at least the longest fiber-length in X_{i-1} , and likewise for the Y_i 's.

The induction hypotheses, i.e., the Brunn–Minkowski inequalities, in H and G/H give us a lower bound l_N on fiber-lengths in $X_N Y_N$ and a lower bound w_N on the size of $\pi(X_N Y_N)$ in G/H . Their product $l_N w_N$ is a lower bound for $\mu(X_N Y_N)$. Next we consider $(X_{N-1} \cup X_N)(Y_{N-1} \cup Y_N)$. Again a lower bound l_{N-1} on fiber lengths in this set and a lower bound w_{N-1} on the size of its image under π can be obtained from the induction hypotheses on H and G/H . From our method, we have $l_{N-1} \leq l_N$ and $w_{N-1} \geq w_N$. Then $l_{N-1} w_{N-1}$ will be a weak lower bound for $\mu((X_{N-1} \cup X_N)(Y_{N-1} \cup Y_N))$ since the fibers in $X_N Y_N$ are “exceptionally long”. Taking all of these into account, a stronger lower bound is

$$l_N w_N + l_{N-1}(w_{N-1} - w_N).$$

Repeating the above process, we get that $\mu(XY) = \mu((\bigcup_{i=1}^N X_i)(\bigcup_{i=1}^N Y_i))$ is at least

$$l_N w_N + \sum_{i=1}^{N-1} l_{N-i}(w_{N-i} - w_{N-i+1}).$$

By choosing X_i and Y_i “proportionately” to their fiber lengths, applying suitable Hölder’s inequalities, and taking the limit $N \rightarrow \infty$, we have the “spillover” argument which yields McCrudden’s theorem.

Induction via the exact sequence method completely stops working when one is looking to prove Brunn–Minkowski for simple groups since there is no nontrivial closed normal subgroup to induct from. Next, we explain how we overcome this main difficulty. Our method turns out to work also for semisimple groups in the same way and we will explain it in this more general setting.

Let us assume G is a connected semisimple Lie group with the finite center (hence helix-free and automatically unimodular) and think about how we can prove the special case of Theorem 1.1 for it. One can consider the Iwasawa decomposition $G = KAN$ where K has a compact Lie algebra and $Q = AN$ is solvable and try to prove the Brunn–Minkowski inequality for G from a suitable Brunn–Minkowski-type inequality for Q . However, Q may not be unimodular in general, so the classical form of the Brunn–Minkowski inequality does not apply to Q . Let Δ_Q be the modular function on G . One can compromise by choosing $Q' = \ker(\Delta_Q)$, which is unimodular, and try to use the known Brunn–Minkowski inequality for Q' to prove the Brunn–Minkowski inequality on G . This is indeed a good direction to go but along this direction one inevitably gives up on the sharp exponent $1/n$ and can at best prove a weaker inequality with the worse exponent $1/(n-1)$, where n is the noncompact Lie dimension of G .

Because of this, we need an analog of (1.1) for nonunimodular groups. We propose the inequality (1.4), which seems to be new in the literature. To prove (1.4) for AN , we need a nonunimodular exponent splitting result for the exact sequence $1 \rightarrow \ker(\Delta_G) \rightarrow G \rightarrow G/\ker(\Delta_G) \rightarrow 1$ coming out from the modular function $\Delta_G : G \rightarrow \mathbb{R}^{>0}$. It turns out that the spillover method can also be used to reduce the problem to the case where the modular function is almost constant on X and Y . We work this out in Lemma 5.1. In the next more involved step in the same section, we obtain an approximate version of McCrudden's theorem (Proposition 5.2), which involves another use of the spillover method, to finish off the proof.

In the next crucial step, we prove that the Brunn–Minkowski for a semisimple G follows from (1.4) for the solvable AN . Our method was motivated by a recent paper [JT20] by the first two authors, which characterizes nearly minimal expansion sets. Over there, a key idea is to choose a fiber f uniformly at random in Y and use Xf to estimate XY . For our current proof, we also choose two fibers f_X and f_Y randomly from X and Y , but with respect to two carefully chosen probability measures p_X and p_Y that are in general nonuniform. We show that by constructing p_X and p_Y based on the structure of X and Y , $\mu(XY)$ can be estimated by the expected size of $f_X f_Y$ in AN , and the latter is well controlled by the Brunn–Minkowski inequality (1.4) for AN . This part is done in Sect. 6. It is worth noting that in this case, our inequality matches the upper bound construction when the semisimple group has a finite center.

With the above preparation, we now explain how we prove Brunn–Minkowski for a general helix-free Lie group G . By Proposition 6.3, we can replace G with its identity component and thus assume G is connected. Next, we use Proposition 5.2 to replace G with $\ker(\Delta_G)$ and reduce the problem to the case where G is unimodular. Such G can be decomposed into a semi-direct product of a unimodular solvable group Q and a semisimple group S via the Levi decomposition. We already know how to handle S from the discussion in Sect. 6. McCrudden's theorem can then be used to deal with Q and to deduce the desired inequality for G .

In many of our reductions, we have an exact sequence of groups $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ and want to deduce the Brunn–Minkowski for G from the Brunn–Minkowski for H and G/H . One tricky issue is that this inductive method only gives sharp results if the sum of the noncompact Lie dimensions and helix dimensions of H and G/H is equal to the noncompact Lie dimension of G . Unfortunately, this is not always true (see the examples after the proof of Lemma 2.11). With this warning in mind, we must ensure the above property is always satisfied in the whole reduction. Our discussion in Sect. 2 guarantees this.

In the remaining part, we discuss some new challenges in the proof of Theorem 1.2 for a helix-free locally compact group G . The Gleason–Yamabe Theorem tells us that G contains an open subgroup G' that has a Lie quotient G'/H with H compact. For the start, we need to handle the nonuniqueness in the choice of G' and H in order to show that noncompact Lie dimension and helix dimension are well-defined for

locally compact groups. This is also done in Sect. 2, making heavy use of Lie theory and the Gleason–Yamabe Theorem.

Unlike the Lie group case discussed earlier, we cannot directly replace G by its identity component, as the latter may not be open. We must take the alternating path of deducing the Brunn–Minkowski inequality for G from its open subgroup G' given by the Gleason–Yamabe theorem and its Lie quotient G'/H . When G is unimodular, this can be achieved using Proposition 6.4 and Proposition 4.2. When G is not unimodular, we encounter an extra technical difficulty as we do not find an easy way to generalize Proposition 6.4. Instead, we reduce the problem to the unimodular case using Proposition 5.2, Proposition 6.3, and Lemma 7.2 to replace G by $\ker(\Delta_G)$ which is unimodular.

2 Noncompact Lie dimension and helix dimension

In this section, we show that noncompact Lie dimensions and helix dimensions are well-defined in locally compact groups and that they behave well in many exact sequences. The latter is the underlying reason that the lower bound in Theorem 1.1 and Theorem 1.2 matches the upper bound in Theorem 1.3 for helix-free locally compact groups.

Throughout this section, all groups are locally compact, and we will use various definitions and facts from Appendices C, D, and E. The following lemma discusses the behavior of Iwasawa decomposition undertaking a quotient by a compact normal subgroup.

LEMMA 2.1. *Suppose G is a connected semisimple Lie group, H is a (not necessarily connected) compact subgroup of G . Then we have the following.*

- (1) *There is an Iwasawa decomposition $G = KAN$ such that $H \leq K$.*
- (2) *Assume further that H is a normal subgroup of G , $G = KAN$ is an Iwasawa decomposition such that $H \leq K$, $G' = G/H$, and $\pi : G \rightarrow G'$ is the quotient map. Then there is an Iwasawa decomposition $G' = K'A'N'$ such that $\pi(K) = K'$.*

Proof. We first prove (1). Let $Z(G)$ be the center of G , $G' = G/Z(G)$ and $\rho : G \rightarrow G'$ be the quotient map, and $H' = \rho(H)$. By Facts E.9 and E.10, ρ is a covering map and G' is centerless. Let \mathfrak{g} be the common Lie algebra of G and G' , and $\exp : \mathfrak{g} \rightarrow G$ and $\exp' : \mathfrak{g} \rightarrow G'$ be the exponential maps. Using Fact E.14.2, it suffices to construct a Cartan involution τ of \mathfrak{g} such that if \mathfrak{k} is the subalgebra of \mathfrak{g} fixed by τ and $\exp(\mathfrak{k}) = K$, then $H \leq K$. Take a maximal compact subgroup K' of G' that contains H' . Let τ_0 be an arbitrary Cartan involution of \mathfrak{g} (this exists because of Fact E.13). Let \mathfrak{k}_0 be the subalgebra of \mathfrak{g} fixed by τ_0 , and $K'_0 = \exp(\mathfrak{k}_0)$ in G' . Then by Fact E.14.2 about Iwasawa decomposition and the earlier observation that G' is centerless, K'_0 is a maximal compact subgroup of G' . By Fact D.4.1 and the assumption that G is

connected, there is an automorphism σ' of G' such that $\sigma'(K'_0) = K'$. Let α be the automorphism of \mathfrak{g} obtained by taking the tangent map of σ' , and let

$$\tau = \alpha\tau_0\alpha^{-1} \text{ and } \mathfrak{k} = \alpha(\mathfrak{k}_0)$$

As every Cartan–Killing form is invariant under automorphisms of \mathfrak{g} , we get that τ is a Cartan involution. It is also easy to check that \mathfrak{k} is the subalgebra of \mathfrak{g} fixed by τ . Using the functoriality of the exponential function (Fact E.2), we get $K' = \exp'(\mathfrak{k})$. Now set $K = \exp(\mathfrak{k})$. By Fact E.14, we get an Iwasawa decomposition $G = KAN$. Therefore, by the functoriality of the exponential function (Fact E.2), $K' = \rho(K)$. Now as $H' \leq K'$, every element of H is in $Z(G)K$. By Fact E.14.2, we have $Z(G) \subseteq K$, so $H \leq K$ as desired.

We now prove (2). Set $K' = \pi(K)$. Let \mathfrak{g} , \mathfrak{h} , and \mathfrak{k} be the Lie algebras of G , H , and K , and let $\kappa_{\mathfrak{g}}$, $\kappa_{\mathfrak{h}}$, $\kappa_{\mathfrak{k}}$ be the Cartan–Killing form of \mathfrak{g} , \mathfrak{h} , and \mathfrak{k} . Then, $\mathfrak{g}' = \mathfrak{g}/\mathfrak{h}$ is the Lie algebra of G' , and $\mathfrak{k}' = \mathfrak{k}/\mathfrak{h}$ is the Lie algebra of K' by Fact E.1. Let τ be a Cartan involution of \mathfrak{g} that fixes \mathfrak{k} . We will construct from this a Cartan involution τ' of \mathfrak{g}' which fixes \mathfrak{k}' . If we have done so, then using Fact E.14, we obtain A' and N' such that $G' = K'A'N'$ is an Iwasawa decomposition, which completes the proof.

Now we construct τ' as described earlier. As \mathfrak{g} is semisimple, the Lie algebras \mathfrak{h} and \mathfrak{k} are also semisimple. With \mathfrak{q} the orthogonal complement of \mathfrak{k} in \mathfrak{g} with respect to $\kappa_{\mathfrak{g}}$ and \mathfrak{c} the orthogonal complement of \mathfrak{h} in \mathfrak{k} with respect to $\kappa_{\mathfrak{k}}$, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$ and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{c}$ by Fact E.7. By the same fact, with $\kappa_{\mathfrak{q}}$ and $\kappa_{\mathfrak{c}}$ the Cartan–Killing forms of \mathfrak{q} and \mathfrak{c} , we have $\kappa_{\mathfrak{g}} = \kappa_{\mathfrak{k}} \oplus \kappa_{\mathfrak{q}}$ and $\kappa_{\mathfrak{k}} = \kappa_{\mathfrak{h}} \oplus \kappa_{\mathfrak{c}}$. It is then easy to see that every element of $\mathfrak{c} \oplus \mathfrak{q}$ is orthogonal to \mathfrak{h} with respect to $\kappa_{\mathfrak{g}}$. A dimension comparison gives us $\mathfrak{c} \oplus \mathfrak{q} = \mathfrak{d}$ with \mathfrak{d} the orthogonal complement of \mathfrak{h} in \mathfrak{g} . In summary, we have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q} = \mathfrak{h} \oplus \mathfrak{c} \oplus \mathfrak{q} = \mathfrak{h} \oplus \mathfrak{d} \quad \text{and} \quad \kappa_{\mathfrak{g}} = \kappa_{\mathfrak{k}} \oplus \kappa_{\mathfrak{q}} = \kappa_{\mathfrak{h}} \oplus \kappa_{\mathfrak{c}} \oplus \kappa_{\mathfrak{q}} = \kappa_{\mathfrak{h}} \oplus \kappa_{\mathfrak{d}}. \tag{2.1}$$

As a particular consequence, the quotient map from \mathfrak{g} to \mathfrak{g}' restricts to isomorphisms of Lie algebras from \mathfrak{d} to $\mathfrak{g}' = \mathfrak{g}/\mathfrak{h}$ and from \mathfrak{c} to $\mathfrak{k}' = \mathfrak{k}/\mathfrak{h}$. Since \mathfrak{h} is a subalgebra of \mathfrak{k} , τ fixes \mathfrak{h} . As Cartan–Killing forms are invariant under automorphisms, τ restricts to an endomorphism of \mathfrak{d} , which is the orthogonal complement of \mathfrak{h} in \mathfrak{g} under $\kappa_{\mathfrak{g}}$. Therefore, $\tau|_{\mathfrak{d}}$ is an involution of \mathfrak{d} . The bilinear form

$$\mathfrak{d} \times \mathfrak{d} : (x, y) \mapsto -\kappa_{\mathfrak{d}}(x, \tau|_{\mathfrak{d}}(y))$$

is positive definite as it is simply the restriction to \mathfrak{d} of the positive definite bilinear form $\mathfrak{g} \times \mathfrak{g} : (x, y) \mapsto -\kappa_{\mathfrak{g}}(x, \tau(y))$. Hence, $\tau|_{\mathfrak{d}}$ is a Cartan involution of \mathfrak{d} . Recall that the subalgebra of \mathfrak{g} fixed by τ is \mathfrak{k} . By the first part of (2.1), the subalgebra of \mathfrak{d} fixed by $\tau|_{\mathfrak{d}}$ is \mathfrak{c} . Finally, let τ' be the pushforward of $\tau|_{\mathfrak{d}}$ under the quotient map from \mathfrak{g} to \mathfrak{g}' . It is easy to see that τ' satisfies the desired requirement. \square

The following lemma allows us to compute the noncompact Lie dimension of the universal cover of a compact Lie group. Recall that a finitely generated abelian group G is isomorphic to a direct sum $\mathbb{Z}^h \oplus G^{\text{tor}}$, with $h \geq 0$ and G^{tor} finite; the integer h is called the *rank* of G .

LEMMA 2.2. *Let K be a covering group of a compact Lie group K' with covering map $\rho : K \rightarrow K'$, and assume that K and K' are connected. Suppose $\ker(\rho)$ is a finitely generated abelian group of rank h , and m is the maximum dimension of a compact subgroup of K . Then $h = \dim(K) - m$.*

Proof. We first consider the case when K is a solvable group. Then $K' \cong \mathbb{T}^k$ where k is the dimension of K by Fact D.5.2. Recall that K is a quotient of the universal cover of K' , which is \mathbb{R}^k . Hence, $K \cong \mathbb{R}^h \times \mathbb{T}^{k-h}$. It is easy to see that the maximum dimension of a compact subgroup of K is $k - h$, which gives us the desired conclusion in this case.

We now prove the statement of the Lemma when K is not solvable. Let Q_K be the radical of K , $Q_{K'}$ the radical of K' , $S_K = K/Q_K$, and $S_{K'} = K'/Q_{K'}$. Note that K and K' have the same Lie algebra \mathfrak{k} . By Fact E.4, Q_K and $Q_{K'}$ have the same Lie algebra \mathfrak{q} , which is the radical of \mathfrak{k} . Moreover, by the functoriality of the exponential function (Fact E.2), ρ restricts to a covering map from Q_K to $Q_{K'}$ with kernel $\ker \rho \cap Q_K$. By Fact E.1, the Lie algebras of S_K and $S_{K'}$ are both isomorphic to $\mathfrak{k}/\mathfrak{q}$. Hence, S_K is a connected semisimple Lie group with compact Lie algebra. Using Fact E.11, we get S_K is compact with finite center $Z(S_K)$. Let $\pi : K \rightarrow S_K$ be the quotient map. Note that $\ker \rho$ is a subgroup of the center of K by Fact D.6. Hence, the image of $\pi|_{\ker \rho}$ is a subset of $Z(S_K)$, which is finite. As a consequence, $\ker \rho \cap Q_K$, which is the kernel of $\pi|_{\ker \rho}$, has the same rank h as $\ker \rho$. Let m_1 and m_2 be the maximum dimensions of a compact subgroup of Q_K and of S_K respectively. Then $m = m_1 + m_2$ by Fact D.4.2. By the special case for the solvable group K proven earlier, $h + m_1 = \dim Q_K$. As S_K is compact, $m_2 = \dim S_K$. Thus, $h + m = h + m_1 + m_2 = \dim(Q_K) + \dim(S_K)$ as desired. \square

The following proposition links the noncompact Lie dimension and the helix dimension.

PROPOSITION 2.3. *Suppose G is a connected semisimple Lie group of dimension d , m is the maximal dimension of a compact subgroup of G , h is the helix dimension of G , and $G = KAN$ is an Iwasawa decomposition of G . Then $h = \dim K - m$, or equivalently, $d - m - h = \dim(AN)$.*

Proof. Let $Z(G)$ be the center of G . Then $Z(G)$ has rank h by definition. By Fact E.14.2, we have $Z(G)$ is a subset of K . Let $G' = G/Z(G)$, and $K' = K/Z(G)$. Using Lemma 2.1.2, we obtain A' and N' such that $G' = K'A'N'$ is an Iwasawa decomposition. Let $\rho : G \rightarrow G'$ be the quotient map. The group $Z(G)$ is discrete by Fact E.9, so ρ and $\rho|_K$ are covering maps.

Now, the maximum dimension of a compact subgroup of G is the same as that of K by Lemma 2.1.1. Applying Lemma 2.2 to K , we have that $h = \dim K - m$. Note that $d = \dim(K) + \dim(AN)$ by Fact E.14, so we also get $d - m - h = \dim(AN)$. \square

The next lemma discusses the noncompact Lie dimensions and the helix dimensions of a Lie group and its open subgroups.

LEMMA 2.4. *Suppose G is a Lie group, and G' is an open subgroup of G . Then G and G' have the same dimension, the same maximum dimension of a compact subgroup, and the same helix dimension.*

Proof. It is clear that G and G' have the same dimension. Any compact subgroup of G' is a compact subgroup of G . If K is a compact subgroup of G , then $K \cap G'$ is an open subgroup of K , hence $K \cap G'$ has the same dimension as K . Therefore the maximum dimension of a compact subgroup of G is the same as that of G' . Finally, note that G and G' have the same identity component G_0 , and the helix dimension is defined using G_0 . Thus, G and G' have the same helix dimension. \square

The following lemma tells us the behavior of the radical under quotient by a compact normal subgroup.

LEMMA 2.5. *Suppose G is a Lie group, H is a compact normal subgroup of G , $G' = G/H$, and $\pi : G \rightarrow G'$ is the quotient map. Let Q be the radical of G , and let $S = G/Q$. Then we have the following:*

- (1) *with $Q' = \pi(Q)$, and $S' = G'/Q'$, we have HQ is closed in G , $Q' = HQ/H$, and $S' = G'/(HQ/H) = (G/H)/(HQ/H)$ is canonically isomorphic as a topological group to both G/HQ and $(G/Q)/(HQ/Q) = S/(HQ/Q)$;*
- (2) *Q' is the radical of G' .*

Proof. We prove (1). As H is compact, we get that HQ is closed in G by Lemma A.4. Then $Q' = HQ/H$, and $S' = G'/(HQ/H) = (G/H)/(HQ/H)$. The remaining part of (1) is a consequence of the third isomorphism theorem (Fact A.1.3).

We next prove (2). As Q' is a quotient of the solvable group Q , it is solvable. Moreover, Q' is a connected closed normal subgroup of G' as Q is a connected closed normal subgroup of G . By (1), G'/Q' is a quotient of the semisimple group S . Hence, G'/Q' is semisimple. Therefore, Q' is the maximal connected solvable closed normal subgroup of G' . In other words, Q' is the radical of G' . \square

The next lemma says in a Lie group, taking a quotient by a normal compact subgroup does not change the helix dimension. Doing so also does not change the difference between the dimension and the dimension of a maximal compact subgroup.

LEMMA 2.6. *Suppose G is a Lie group, H is a compact normal subgroup of G , and $G' = G/H$. Let d, m , and h be the dimension, the maximal dimension of a compact subgroup, and the helix dimension of G , respectively. Define d', m' , and h' likewise for G' . Then:*

- (1) $d = d' + \dim(H)$ and $m = m' + \dim(H)$;
- (2) $h = h'$.

Proof. We prove (1). Clearly, $d = d' + \dim(H)$. If K is a compact subgroup of G and $K' = \pi(K)$, then K' is a compact subgroup of G' , then

$\dim(K') + \dim(H) = \dim(K)$. Conversely, if K' is a compact subgroup of G' , then $K = \pi^{-1}(K')$ is a compact subgroup of G by Lemma A.5, and Lemma 2.2 implies that $\dim(K) = \dim(K') + \dim(H)$. Therefore, $m = m' + \dim(H)$.

We now prove (2). First further assume that both G and G' are semisimple. Let $\pi : G \rightarrow G'$ be the quotient map. Using Lemma 2.1.1, we obtain an Iwasawa decomposition $G = KAN$ of G such that $H \subseteq K$. By Lemma 2.1.2, we obtain an Iwasawa decomposition $G' = K'A'N'$ with $K' = \pi(K)$, $A' = \pi(A)$, and $N' = \pi(N)$. Let m_K be the maximum dimension of a compact subgroup of K , and m'_K be the maximum dimension of a compact subgroup of K' . By Proposition 2.3, $m_K + h = \dim(K)$, and $m_{K'} + h' = \dim(K')$. Now, by (1) applied to K , we have $m_K = m'_K + \dim(H)$. Therefore, we get $h = h'$.

Next, consider the case where G is connected. Let Q be the radical of G , $S = G/Q$, $Q' = \pi(Q)$, and $S' = G'/Q'$. Then by Lemma 2.5.2, Q' is the radical of G' . Hence, it suffices to show that S and S' have the same helix dimension. By Lemma 2.5.1, S' is isomorphic as a topological group to $S'/(HQ'/Q')$. Note that HQ'/Q' is isomorphic as a topological group to $H/(H \cap Q)$ by the second isomorphism theorem for Lie groups (Fact D.3.2). In particular, HQ'/Q' is compact, and S' is the quotient of S' by a compact group. Applying the known case for semisimple groups, we get the desired conclusion.

Finally, we address the general case. Let G_0 be the identity component of G . Then G_0 is open by Fact D.2, and G_0H/H is an open subgroup of $G' = G/H$. Hence, by Lemma 2.4, G has the same helix dimension as G_0 , and G' has the same helix dimension as G_0H/H . By the second isomorphism theorem (Fact A.1.2), G_0H/H is isomorphic as a topological group to $G_0/(G_0 \cap H)$, which is a quotient of G_0 by a compact subgroup. Thus, we get the desired conclusion for the general case from the known case discussed above for connected groups. \square

LEMMA 2.7. *Suppose G is an almost-Lie group, H_1 and H_2 are closed normal subgroups of G such that G/H_1 and G/H_2 are Lie groups, and $H = H_1 \cap H_2$. Then G/H is a Lie group.*

Proof. By Fact C.1, G/H is an almost-Lie group. In light of Fact C.2.2, we want to construct an open neighborhood U of the identity in G/H that contains no nontrivial compact subgroup. Let $\pi : G \rightarrow G/H$, $\pi_1 : G \rightarrow G/H_1$, and $\pi_2 : G \rightarrow G/H_2$ be the quotient maps. Using Fact A.1.3, we get continuous surjective group homomorphisms $p_1 : G/H \rightarrow G/H_1$ and $p_2 : G/H \rightarrow G/H_2$ such that

$$\pi_1 = p_1 \circ \pi \quad \text{and} \quad \pi_2 = p_2 \circ \pi.$$

As G/H_1 is a Lie group, we can use Fact C.2.2 to choose an open neighborhood U_1 of the identity in G/H_1 such that U_1 contains no nontrivial compact subgroup of G/H_1 . Choose an open neighborhood U_2 of the identity in G/H_2 likewise, and set

$$U = p_1^{-1}(U_1) \cap p_2^{-1}(U_2).$$

If $K \subseteq U$ is a compact subgroup of G/H , then $p_1(K)$ is a compact subgroup of U_1 . By our choice of U_1 , $p_1(K) = \{\text{id}_{G/H_1}\}$, which implies that $\pi_1^{-1}(p_1(K)) = \pi^{-1}(K)$ is a subgroup of H_1 . A similar argument yields that $\pi_2^{-1}(p_2(K)) = \pi^{-1}(K)$ is a subgroup of H_2 . Hence, $\pi^{-1}(K)$ must be a subgroup of $H = H_1 \cap H_2$. It follows that $K = \{\text{id}_{G/H}\}$, which is the desired conclusion. \square

Proposition 2.8 below ensures us the notion of noncompact Lie dimension and helix dimension of a locally compact group as described in the introduction are well defined.

PROPOSITION 2.8. *Suppose G' is an open subgroup of a locally compact group G , and $H \triangleleft G'$ is compact such that G'/H is a Lie group with dimension d , with m the maximum dimension of a compact subgroup, and helix dimension h . Then $d - m$ and h are independent of the choice of G' and H .*

Proof. We first prove a simpler statement: if G' is an almost-Lie subgroup of G , H is a compact subgroup of G' , and we define d , m , and h as in the statement of the proposition, then $d - m$ and h are independent of the choice of H . Let H_1 and H_2 be compact normal subgroups of G such that both G/H_1 and G/H_2 are Lie groups. Then by Lemma 2.7, $G/(H_1 \cap H_2)$ is also a Lie group. Note that G/H_1 and G/H_2 are quotients of $G/(H_1 \cap H_2)$ by compact subgroups by the third isomorphism theorem (Fact A.1.3). Hence, it follows from Lemma 2.6 that G/H_1 and $G/(H_1 \cap H_2)$ have the same difference between the dimension and the maximum dimension of a compact subgroup, and the same helix dimension. A similar statement holds for G/H_2 and $G/(H_1 \cap H_2)$. This completes the proof of the simpler statement.

Now we prove the proposition in general. Let G'_1 and G'_2 be open subgroups of G , and let H_1 and H_2 be compact normal subgroups of G'_1 and G'_2 respectively such that G'_1/H_1 and G'_2/H_2 are Lie groups. Using the Gleason–Yamabe Theorem (Fact C.2), we get an open subgroup G' of $G'_1 \cap G'_2$ which is an almost-Lie group. Then G' is an open subgroup of G . Note that $G' \cap H_1$ and $G' \cap H_2$ are compact subgroups of G' . Then $G'/(G' \cap H_1)$ is an open subgroup of G'_1/H_1 . It follows from Lemma 2.6 that G'/H_1 and $G'/(G' \cap H_1)$ have the same difference between the dimension and the maximum dimension of a compact subgroup and the same helix dimension. A similar statement holds for G'/H_2 and $G'/(G' \cap H_2)$. Thus, from the simpler statement we proved in the preceding paragraph, G'_1/H_1 and G'_2/H_2 have the same noncompact dimension and the same helix dimension. \square

We have the following two corollaries.

COROLLARY 2.9. *If H is an open subgroup of G , then H has the same noncompact Lie dimension and helix dimension as G .*

Proof. Proposition 2.8 implies that the noncompact Lie dimension and helix dimension of a locally compact group is the same as those of its open almost-Lie subgroups, if those exist. Hence, it suffices to show that there is a common almost-Lie open

subgroup of G and H . This is an immediate consequence of the Gleason–Yamabe Theorem (Fact C.2.1). \square

COROLLARY 2.10. *If H is a compact normal subgroup of G , then G/H has the same noncompact Lie dimension and helix dimension as G .*

Proof. Let π be the projection from G to G/H . If G/H is a Lie group, then from the definitions, G has the same noncompact Lie dimension and helix dimension as G/H . Hence, the conclusion holds in this special case.

Suppose there is a compact $K \triangleleft G/H$ such that $(G/H)/K$ is a Lie group, then $(G/H)/K$ is isomorphic as a topological group to $G/\pi^{-1}(K)$ by the third isomorphism theorem (Fact A.1.3). By Lemma A.5, $\pi^{-1}(K)$ is compact. Hence $(G/H)/K$ is a quotient of G by a compact normal subgroup, and we can use the previous case to get the desired conclusion.

Now we treat the general situation. By the Gleason–Yamabe Theorem, we get an almost-Lie open subgroup G' of G . Then $G'H$ is an open subgroup of G and hence has the same noncompact Lie dimension and helix dimension as G by Corollary 2.9. By the second isomorphism theorem (Fact A.1.2), we get that $G'/(G' \cap H)$ is isomorphic to $G'H/H$ which is an open subgroup of G/H . In particular, $G'/(G' \cap H)$ has the same noncompact Lie dimension and helix dimension as G/H by Corollary 2.9. Note that $G'/(G' \cap H)$ is an almost-Lie group by Fact C.1. Hence, we can find a compact subgroup K such that $(G'/(G' \cap H))/K$ is a Lie group. We are back to the earlier known situation in the second paragraph. \square

We have the following lemma about Iwasawa decompositions.

LEMMA 2.11. *Suppose $1 \rightarrow H \rightarrow G \xrightarrow{\pi} G/H \rightarrow 1$ is an exact sequence of connected semisimple Lie groups. Then there are Iwasawa decompositions $G = KAN$, $H = K_1A_1N_1$, and $G/H = K_2A_2N_2$ such that $K_1 = (K \cap H)_0$, and $K_2 = \pi(K)$.*

Proof. Let \mathfrak{g} and \mathfrak{h} be the Lie algebras of G and H , and let $\kappa_{\mathfrak{g}}$ and $\kappa_{\mathfrak{h}}$ be the Cartan–Killing form of \mathfrak{g} and \mathfrak{h} . Then $\mathfrak{g}/\mathfrak{h}$ is the Lie algebra of G/H , and $\mathfrak{g}, \mathfrak{h}$, and $\mathfrak{g}/\mathfrak{h}$ are semisimple. By Fact E.7,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{c} \text{ and } \kappa_{\mathfrak{g}} = \kappa_{\mathfrak{h}} \oplus \kappa_{\mathfrak{c}}$$

where $\kappa_{\mathfrak{c}}$ is the orthogonal complement of $\kappa_{\mathfrak{h}}$ with respect to $\kappa_{\mathfrak{g}}$, and $\kappa_{\mathfrak{c}}$ is the Cartan–Killing form of \mathfrak{c} . Therefore, the quotient map from \mathfrak{g} to $\mathfrak{g}/\mathfrak{h}$ induces an isomorphism from \mathfrak{c} to $\mathfrak{g}/\mathfrak{h}$, so we can identify $\mathfrak{g}/\mathfrak{h}$ with \mathfrak{c} . Let τ_1 and τ_2 be Cartan involutions of \mathfrak{h} and \mathfrak{c} . Then $\tau = \tau_1 \oplus \tau_2$ is an involution of \mathfrak{g} . As τ_1 and τ_2 are Cartan involutions, the bilinear forms $\mathfrak{h} \times \mathfrak{h} : (x_1, y_1) \mapsto -\kappa_{\mathfrak{h}}(x_1, \tau_1(y_1))$ and $\mathfrak{c} \times \mathfrak{c} : (x_2, y_2) \mapsto -\kappa_{\mathfrak{c}}(x_2, \tau_2(y_2))$ are positive definite. Hence, the bilinear form $\mathfrak{g} \times \mathfrak{g} : (x, y) \mapsto -\kappa_{\mathfrak{g}}(x, \tau(y))$ is also positive definite. Therefore, τ is a Cartan involution of \mathfrak{g} . Let \mathfrak{k} , \mathfrak{k}_1 , and \mathfrak{k}_2 be the Lie subalgebras of \mathfrak{g} , \mathfrak{h} , and \mathfrak{c} fixed by τ , τ_1 , and τ_2 respectively. It is easy to see that

$\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$. Let $\exp : \mathfrak{g} \rightarrow G$, $\exp_1 : \mathfrak{h} \rightarrow H$, and $\exp_2 : \mathfrak{c} \rightarrow G/H$ be the exponential maps, and set

$$K = \exp(\mathfrak{k}), K_1 = \exp_1(\mathfrak{k}_1) \text{ and } K_2 = \exp(\mathfrak{k}_2).$$

From Fact E.14, we obtain Iwasawa decompositions $G = KAN$, $H = K_1A_1N_1$, and $G/H = K_2A_2N_2$. By the functoriality of the exponential function (Fact E.2), we get $K_1 \leq K \cap H$ and $K_2 = \pi(K)$. Since K_1 is connected, by a dimension calculation we have $K_1 = (K \cap H)_0$. □

In a short exact sequence of locally compact groups, one may hope that the noncompact Lie dimension and the helix dimension of the middle term is the sum of those of the outer terms. This is not true in general. For instance, in the exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 1,$$

the noncompact Lie dimension of \mathbb{R} is 1, while both \mathbb{Z} and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has noncompact Lie dimension 0. Another example is the following. Let H be the universal cover of $SL(2, \mathbb{R})$. We identify the kernel of the covering map $\rho : H \rightarrow SL(2, \mathbb{R})$ with the additive group \mathbb{Z} of integers. Let $G = (H \times \mathbb{R})/\{(n, n) : n \in \mathbb{Z}\}$. Then we have the exact sequence

$$1 \rightarrow H \rightarrow G \rightarrow \mathbb{T} \rightarrow 1,$$

the helix dimension of H is 1, but the helix dimensions of G and \mathbb{T} are 0.

Nevertheless, we have the summability of noncompact Lie dimensions and helix dimensions in many short exact sequences of interest:

PROPOSITION 2.12. *Suppose $1 \rightarrow H \rightarrow G \xrightarrow{\pi} G/H \rightarrow 1$ is an exact sequence of connected Lie groups. Then we have the following:*

- (1) *If n , n_1 , and n_2 are the noncompact Lie dimensions of G , H , and G/H respectively, then $n = n_1 + n_2$.*
- (2) *If G is moreover semisimple, and h , h_1 , and h_2 are the helix dimensions of G , H , and G/H respectively, then $h = h_1 + h_2$.*

Proof. We first prove (1). Let m be the maximum dimension of a compact subgroup in G . As G is connected, m is also the dimension of an arbitrary maximal compact subgroup of G by Fact D.4.1. Defining m_1 and m_2 likewise for H and G/H , we get similar conclusions for them from the connectedness of H and G/H . Let K be a maximal compact subgroup of G . By Fact D.4.2, $K \cap H$ is a maximal compact subgroup in H , and $\pi(K)$ is a maximal compact subgroup in G/H . The kernel of $\pi|_K$ is $K \cap H$, and the image is $\pi(K)$. Hence, $m = m_1 + m_2$. This gives us (1) recalling that $m + n = \dim(G)$, $m_1 + n_1 = \dim(H)$, $m_2 + n_2 = \dim(G/H)$, and $\dim(G) = \dim(H) + \dim(G/H)$.

We now prove (2). Since $Z(G) \cap H \leq Z(H)$, and $\pi(Z(G)) \leq Z(G/H)$, we have $h \leq h_1 + h_2$. It remains to show $h \geq h_1 + h_2$. As G is semisimple, H and G/H are semisimple by Fact E.8. Take Iwasawa decompositions $G = KAN$, $H = K_1A_1N_1$, and $G/H = K_2A_2N_2$ as in Lemma 2.11. By the first isomorphism theorem for Lie groups (Fact D.3.1), $1 \rightarrow K \cap H \rightarrow K \rightarrow K_2 \rightarrow 1$ is an exact sequence of Lie groups. With $K'_2 = K/K_1$, we have an exact sequence

$$1 \rightarrow K_1 \rightarrow K \rightarrow K'_2 \rightarrow 1. \quad (2.2)$$

As $K_1 = (K \cap H)_0$, by the third isomorphism theorem, we have $K_2 = K/(K \cap H) = (K/K_1)/((K \cap H)/K_1) = K'_2/((K \cap H)/K_1)$. Since $(K \cap H)/K_1$ is discrete, K'_2 is a covering group of K_2 . Let $\phi : K'_2 \rightarrow K_2$ be the covering map. Note that ϕ has a discrete kernel, and K_2, K'_2 have the same dimension. Suppose S is a compact subgroup of K'_2 with the maximum dimension. Then $\phi(S)$ is a compact subgroup of K_2 , and S and $\phi(S)$ have the same dimension. This shows that the noncompact Lie dimension of K'_2 is at least the noncompact Lie dimension of K_2 . By (2.2) and Statement (1), the noncompact Lie dimension of K is the sum of noncompact Lie dimensions of K_1 and K'_2 , hence it is at least the sum of noncompact Lie dimensions of K_1 and K_2 . It then follows from Proposition 2.3 that $h \geq h_1 + h_2$. \square

LEMMA 2.13. *Suppose $1 \rightarrow H \rightarrow G \xrightarrow{\pi} (\mathbb{R}^{>0}, \times) \rightarrow 1$ is an exact sequence of Lie groups, then $H_0 = H \cap G_0$. In particular, if G is connected, then H is connected.*

Proof. As Lie groups are locally path connected, G_0 is open in G . As G_0 is an open connected subgroup of G , the map $\pi|_{G_0}$ is continuous and open. Hence, its image $\pi(G_0)$ is an open connected subgroup of $(\mathbb{R}^{>0}, \times)$. Therefore, $\pi(G_0) = (\mathbb{R}^{>0}, \times)$, and $\pi|_{G_0}$ is a quotient map by the first isomorphism theorem (Fact A.1.1). The kernel of $\pi|_{G_0}$ is $H \cap G_0$, so we get the exact sequence of Lie groups

$$1 \rightarrow H \cap G_0 \rightarrow G_0 \xrightarrow{\pi|_{G_0}} (\mathbb{R}^{>0}, \times) \rightarrow 1.$$

We now prove that $H_0 = H \cap G_0$. The forward inclusion is immediate by definition. By the third isomorphism theorem (Fact A.1.3), we get the exact sequence of Lie groups

$$1 \rightarrow (H \cap G_0)/H_0 \rightarrow G_0/H_0 \rightarrow (\mathbb{R}^{>0}, \times) \rightarrow 1.$$

The group $(H \cap G_0)/H_0$ is discrete. Hence, G_0/H_0 is a Lie group with dimension 1. As G_0 is connected, the Lie group G_0/H_0 is also connected. By Fact D.5, G_0/H_0 is either isomorphic to \mathbb{R} or \mathbb{T} . But since G_0/H_0 has $(\mathbb{R}^{>0}, \times)$ as a quotient, it cannot be compact, and therefore must be isomorphic to \mathbb{R} . This implies that $(H \cap G_0)/H_0$ is trivial, and hence $H_0 = H \cap G_0$. \square

The next proposition gives us a summability result of noncompact Lie dimensions along a short exact sequence of locally compact groups when the quotient group is $(\mathbb{R}^{>0}, \times)$.

PROPOSITION 2.14. *Suppose $1 \rightarrow H \rightarrow G \xrightarrow{\pi} (\mathbb{R}^{>0}, \times) \rightarrow 1$ is an exact sequence of locally compact groups. Then we have the following:*

- (1) *If n , n_1 , and n_2 are the noncompact Lie dimensions of G , H , and $(\mathbb{R}^{>0}, \times)$ respectively, then $n = n_1 + n_2 = n_1 + 1$.*
- (2) *G and H have the same helix dimension.*

Proof. First, we consider the case when G is a connected Lie group. Then by Lemma 2.13, H is also connected. Hence, (1) for this case is a consequence of Proposition 2.12.1.

We prove (2) for this special case. Let Q be the radical of G . We claim that $QH = G$, or equivalently, that $\pi(Q) = (\mathbb{R}^{>0}, \times)$. Suppose this is not true. Then $\pi(Q)$ is a connected subgroup of $(\mathbb{R}^{>0}, \times)$, so it must be $\{1\}$. Hence, $Q \subseteq H$. Then $(\mathbb{R}^{>0}, \times) = G/H$ which is isomorphic as a topological group to $(G/Q)/(H/Q)$ by the third isomorphism theorem (Fact A.1.3). This is a contradiction, because $(G/Q)/(H/Q)$ is semisimple as a quotient of the semisimple group G/Q , while $(\mathbb{R}^{>0}, \times)$ is solvable.

We next show that $Q \cap H$ is the radical of H . The radical of H is a characteristic closed subgroup of H (by Fact E.4), hence a connected solvable closed normal subgroup of G . Thus, the radical of H is a subgroup of $Q \cap H$. It is straightforward that $Q \cap H$ is solvable. We also have that $Q \cap H$ is second countable as both Q and H are second countable. From the preceding paragraph, $\pi(Q) = (\mathbb{R}^{>0}, \times)$. Using the first isomorphism theorem for Lie groups (Fact D.3.1), we have the exact sequence

$$1 \rightarrow Q \cap H \rightarrow Q \rightarrow (\mathbb{R}^{>0}, \times) \rightarrow 1.$$

Applying Lemma 2.13, we learn that $Q \cap H$ is connected. This completes the proof that $Q \cap H$ is the radical of H .

Note that $QH = G$ and Q is a closed subgroup of G . Hence, by the second isomorphism theorem for Lie groups (Fact D.3.2), $H/(Q \cap H)$ is isomorphic as a topological group to $HQ/Q = G/Q$. Therefore G and H have the same helix dimension.

Next, we address the slightly more general case where G is a Lie group but not necessarily connected. Let G_0 be the connected component of G . Then $\pi(G_0)$ is an open subgroup of $(\mathbb{R}^{>0}, \times)$, so $\pi(G_0) = (\mathbb{R}^{>0}, \times)$. By the first isomorphism theorem for Lie groups (Fact D.3.1), we have the exact sequence

$$1 \rightarrow G_0 \cap H \rightarrow G_0 \rightarrow (\mathbb{R}^{>0}, \times) \rightarrow 1.$$

Applying Lemma 6.3 and the known case of the current lemma where the middle term of the exact sequence is a connected Lie group, we obtain both (1) and (2) for this more general case.

Using the Gleason–Yamabe theorem and a similar argument as in the preceding paragraph, we can reduce (1) and (2) for general locally compact groups to the case where we assume that G is an almost-Lie group. Hence, there is a compact normal subgroup K of G such that G/K is a Lie group. As K is compact, $\pi(K)$ is a compact subgroup of $(\mathbb{R}^{>0}, \times)$, so $\pi(K) = \{1\}$. Hence $K \triangleleft H$. By the third

isomorphism theorem (Fact A.1.3), we have the exact sequence $1 \rightarrow H/K \rightarrow G/K \rightarrow (\mathbb{R}^{>0}, \times) \rightarrow 1$. Applying Lemma 2.6 and the known case of the current lemma where the middle term of the exact sequence is a Lie group, we obtain both (1) and (2) for this remaining case. \square

We discuss the relationship between the noncompact Lie dimension and helix dimension of a locally compact group G .

COROLLARY 2.15. *Suppose G has noncompact Lie dimension n and helix dimension h . Then we have $h \leq n/3$.*

Proof. We first check the result for simple Lie groups. If $h = 0$, then the statement holds vacuously. Hence, using Fact E.12, it suffices to consider the case where $h = 1$. Let $G = KAN$ be an Iwasawa decomposition. Then by Proposition 2.3, we have $n - 1 = \dim(AN) \geq 0$. Hence, $n > 0$ and G is not compact. From Fact E.15, we have $\dim(AN) \geq 2$. Therefore $n \geq 3$. Hence, we get the desired conclusion for simple Lie groups.

Now consider the case where G is a connected semisimple Lie group which is not simple. Using induction on dimension, we can assume we have proven the statement for all connected semisimple Lie groups of smaller dimensions. Using Fact E.6, we get an exact sequence of semisimple Lie groups $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ with $0 < \dim(H) < \dim(G)$. Replacing H with its connected component if necessary, we can arrange that H is connected. The desired conclusion then follows from Proposition 2.12.2.

For a general locally compact group G , from Proposition 2.8, we may assume G is a Lie group. Corollary 2.9 and Fact D.2 allow us to reduce the problem to connected Lie groups. By Lemma 2.5 and Lemma 2.4, the radical of G only contributes to the noncompact Lie dimension of G . Using Fact E.5 and Proposition 2.12.1, we reduce the problem to connected semisimple Lie groups. \square

3 Proof of Theorem 1.3

Theorem 3.1 is a version of Theorem 1.3.2 with compact sets replaced by open sets. We will deduce Theorem 1.3 from Theorem 3.1 at the end of this section using the inner regularity of the Haar measure. It is convenient to prove Theorem 3.1 first because, intuitively, the constructed set X is an open neighborhood of the maximal compact subgroup of G . On the other hand, in Theorem 1.3, we want a compact set as traditionally stated in the Brunn–Minkowski inequalities.

Theorem 3.1. *Suppose G is a locally compact group with noncompact Lie dimension $n > 0$, μ_G is a left Haar measure, and ν_G is a right Haar measure. Then, for every $\varepsilon > 0$, there is an open set X with positive left and right measure in G such that*

$$\frac{\nu_G(X)^{\frac{1}{n}-\varepsilon}}{\nu_G(X^2)^{\frac{1}{n}-\varepsilon}} + \frac{\mu_G(X)^{\frac{1}{n}-\varepsilon}}{\mu_G(X^2)^{\frac{1}{n}-\varepsilon}} > 1.$$

As a corollary, if G is unimodular with $n > 0$, for every $\varepsilon' > 0$, there is an open set X in G such that $\mu_G(X^2) < (2^n + \varepsilon')\mu_G(X)$.

We first prove the theorem when G is a unimodular Lie group.

Proof of Theorem 3.1, unimodular Lie group case. Since G is unimodular, without loss of generality we assume that $\mu_G = \nu_G$. Let d be the dimension of G . Let K be a maximal compact subgroup of G and let $m = \dim K$. Hence $n = d - m$ is the noncompact Lie dimension of G .

Since K is closed, G/K is a homogeneous (and smooth) manifold. For $g \in G$, we let $[g]$ denote its quotient image in G/K . Fix an arbitrary G -invariant (smooth) Riemannian metric on G/K (such a metric exists by first finding a K -invariant Riemannian metric at $[\text{id}]$ and then extend it onto the whole G/K by the action of G). This metric induces a volume measure Vol on G/K .

Let π be the projection from G to G/K . For any Borel subset U of G/K , $\pi^{-1}(U)$ is also Borel and hence μ_G -measurable. For any $r > 0$, we use B_r to denote the (open) r -ball around $[\text{id}]$ on G/K under the chosen metric and use D_r to denote $\pi^{-1}(B_r)$. We claim that:

- (i) There exists a constant $b > 0$ only depending on the metric on G/K such that as Borel measures $\pi_*(\mu_G) = b \cdot \text{Vol}$, and
- (ii) For any $r > 0$, $D_r \cdot D_r \subseteq D_{2r}$.

We postpone the proofs of claims (i) and (ii) to the end of this proof and first show how they lead to Theorem 1.3. We can take X to be D_δ for a sufficiently small $\delta > 0$ (depending on ε) to be determined. Then by (i),

$$\mu_G(X) = \pi_*(\mu_G(B_\delta)) = b \cdot \text{Vol}(B_\delta).$$

And by (ii), $X^2 \subseteq D_{2\delta}$ and hence as before, we get $\mu_G(X^2) \leq \mu_G(D_{2\delta}) = b \cdot \text{Vol}(B_{2\delta})$. Note that the invariant metric on G/K is smooth and thus

$$\lim_{\delta \rightarrow 0} \frac{\text{Vol}(B_{2\delta})}{\text{Vol}(B_\delta)} = 2^n.$$

Hence a sufficiently small δ can guarantee $\frac{\mu_G(X)^{\frac{1}{n}-\varepsilon}}{\mu_G(X^2)^{\frac{1}{n}-\varepsilon}} > \frac{1}{2}$ and we have proved Theorem 1.3 in this special case.

It remains to prove claims (i) and (ii). To see claim (i), note that Vol is G -invariant. We also see that $\pi_*(\mu_G)$ is G -invariant because $\mu_G(\pi^{-1}(U)) = \mu_G(g\pi^{-1}(U)) = \mu_G(\pi^{-1}(gU))$ for any $g \in G$ and any Borel subset $U \subseteq G/K$. Since the G -invariant Borel measure on G/K is unique up to a scalar (see Theorem 8.36 in [Kna02]), Vol has to be a scalar multiple of $\pi_*(\mu_G)$.

Finally we verify claim (ii). Taking arbitrary $g_1, g_2 \in D_r$ and it suffices to show $g_1g_2 \in D_{2r}$. By definition, there is a piecewise smooth curve γ_j connecting $[\text{id}]$ and $[g_j]$ such that the length of γ_j is strictly smaller than r (for $j = 1, 2$). Note that by the invariance of the metric, $[g_1]\gamma_2$ must have the same length as γ_2 . Let γ be the

curve formed by $[g_1]\gamma_2$ after γ_1 . It is a curve connecting $[\text{id}]$ and $[g_1g_2]$ and by the reasoning above has two pieces and each of them has length strictly smaller than r . Hence γ has length shorter than $2r$ and thus by definition $g_1g_2 \in D_{2r}$. We have verified (ii). \square

The following will be proved similarly to the above special case of Theorem 3.1.

PROPOSITION 3.2. *Given any unimodular Lie group G , let n be its noncompact Lie dimension. Let $\tilde{\varepsilon} > 0$ be fixed. Then there exists precompact open subsets X and X_1 with $\mu_G(X) > 0$ such that the closure $\overline{X} \subseteq X_1$ and $\mu_G(X_1 \cdot X) < (2 + \tilde{\varepsilon})^n \mu_G(X)$.*

Proof. We continue to use the notation from the proof. Let $X = D_\delta$ and $X_1 = D_{\delta_1}$, where $0 < \delta < \delta_1$ are both to be determined.

We see that X and X_1 are open, being the preimage of open sets under π . Since B_δ and B_{δ_1} are both precompact, Lemma A.5 implies X and X_1 are also precompact. Moreover we have $\overline{X} \subseteq \pi^{-1}(\overline{B_\delta}) \subseteq \pi^{-1}(B_{\delta_1}) = X_1$. It remains to choose δ and δ_1 such that $\mu_G(X_1 \cdot X) < (2 + \tilde{\varepsilon})^n \mu_G(X)$.

By the same reasoning as in the previous proof of Theorem 1.3 (unimodular Lie case), we see that $X_1 \cdot X \subseteq D_{\delta_1 + \delta}$. Now,

$$\lim_{\delta \rightarrow 0} \frac{\text{Vol}(B_{2\delta})}{\text{Vol}(B_\delta)} = 2^n, \quad \text{and} \quad \lim_{\delta_1 \rightarrow \delta} \frac{\text{Vol}(B_{2\delta})}{\text{Vol}(B_{\delta_1 + \delta})} = 1.$$

Hence we can take δ sufficiently small, and then δ_1 sufficiently close to δ , such that

$$\frac{\mu_G(X_1 \cdot X)}{\mu_G(X)} \leq \frac{\mu_G(D_{\delta_1 + \delta})}{\mu_G(D_\delta)} = \frac{\text{Vol}(B_{\delta_1 + \delta})}{\text{Vol}(B_\delta)} < (2 + \tilde{\varepsilon})^n,$$

which proves the proposition. \square

Next we use Proposition 3.2 to prove Theorem 3.1 for general Lie groups.

Proof of Theorem 3.1, Lie group case. We have already proved the theorem when G is unimodular. In the rest of this proof, we assume G is nonunimodular. Let G_0 be the connected component of G . Since $\mu_G|_{G_0}$ is a left Haar measure on G_0 , and $\nu_G|_{G_0}$ is a right Haar measure on G_0 , we may assume without loss of generality that $G = G_0$. As the only connected subgroups of $(\mathbb{R}^{>0}, \times)$ are itself and $\{1\}$, and G is not unimodular, the modular function Δ_G must be surjective. Hence, Δ_G is a quotient map by the first isomorphism for Lie groups Fact D.3.1.

Let H be the kernel of the modular function on G . By Proposition 2.14.1, the noncompact Lie dimension of H is $n - 1$ where n is the noncompact Lie dimension of G . By Fact B.2.1, H is unimodular. To avoid confusion, we will always use μ_G and ν_G for μ and ν below and use $\mu_H = \nu_H$ to denote a fixed Haar measure on H .

In light of Fact B.3, we can fix a Haar measure dr on the multiplicative group $(\mathbb{R}^{>0}, \times) = G/H$ such that for any Borel function f on g ,

$$\int_G f(x) d\mu_G(x) = \int_{G/H} \int_H f(rh) d\mu_H(h) d\mu_{G/H}(rH). \quad (3.1)$$

Let \mathfrak{g} and \mathfrak{h} be the Lie algebras of G and H , respectively. We fix an element $Z \in \mathfrak{g}$ such that $Z \notin \mathfrak{h}$. Note that $t \mapsto \Delta(\exp(tZ))$ is a nontrivial continuous group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^{>0}, \times)$. As the only connected subsets of $(\mathbb{R}^{>0}, \times)$ are points and intervals, this map must be surjective, and hence an isomorphism by the first isomorphism for Lie groups (Fact D.3). In light of the quotient integral formula (3.1), we can choose an appropriate Haar measure dt on \mathbb{R} such that for any Borel subset A of G , we have the Fubini-type measure formula

$$\mu_G(A) = \int_{\mathbb{R}} \mu_H((\exp(-tZ)A) \cap H) dt. \tag{3.2}$$

Without loss of generality we assume dt is the standard Lebesgue measure (otherwise we multiply μ_G by a constant).

With the preliminary discussions above, we now construct X satisfying the inequality in Theorem 1.3.

Before going into details of the construction, we first describe the intuition behind it. We arrange our X to live very close to H so that μ and ν are almost proportional on X and X^2 . We then realize that it suffices to choose our X to be like a thickened copy of the almost sharp example of Theorem 1.3 for (the unimodular group) H .

More precisely, let $\tilde{\varepsilon} > 0$ be a small number (depending on ε) to be determined. let \tilde{X} and \tilde{X}_1 be the X and X_1 , respectively, in Proposition 3.2 where we replace G by H . We now take $X = \{\exp(tZ)h : t \in [0, \tilde{\varepsilon}], h \in \tilde{X}\}$ and will show that X^2 is reasonably small when $\tilde{\varepsilon}$ is small enough.

By (3.2), we have

$$\mu_G(X^2) = \int_{\mathbb{R}} \mu_H((\exp(-tZ)X^2) \cap H) dt. \tag{3.3}$$

Note that an arbitrary element in X^2 can be written as

$$\begin{aligned} &\exp(t_1Z)h_1\exp(t_2Z)h_2 \\ &= \exp((t_1 + t_2)Z)(\exp(-t_2Z)h_1\exp(t_2Z)h_2) \in \exp((t_1 + t_2)Z)H, \end{aligned}$$

where $t_1, t_2 \in [0, \tilde{\varepsilon}]$ and $h_1, h_2 \in H$. Hence (3.3) is reduced to

$$\mu_G(X^2) = \int_0^{2\tilde{\varepsilon}} \mu_H((\exp(-tZ)X^2) \cap H) dt \tag{3.4}$$

and moreover for any $0 \leq t_0 \leq 2\tilde{\varepsilon}$, we see from the above discussion that

$$(\exp(-t_0Z)X^2) \cap H = \bigcup_{0 \leq t_1, t_2 \leq \tilde{\varepsilon}, t_1+t_2=t_0} (\exp(-t_1Z)\tilde{X}\exp(t_1Z)) \cdot \tilde{X}.$$

By Lemma D.7 and Proposition 3.2, when $\tilde{\varepsilon}$ is sufficiently small, which we will always assume, we have the above union contained in $\tilde{X}_1 \cdot \tilde{X}$. Now by (3.4),

$$\mu_G(X^2) \leq \int_0^{2\tilde{\varepsilon}} \mu_H(\tilde{X}_1 \cdot \tilde{X}) dt = 2\tilde{\varepsilon}\mu_H(\tilde{X}_1 \cdot \tilde{X}). \tag{3.5}$$

On the other hand, by (3.2)

$$\mu_G(X) = \tilde{\varepsilon} \mu_H(\tilde{X}). \quad (3.6)$$

Combining (3.5) and (3.6) and using the measure properties of \tilde{X} and \tilde{X}_1 guaranteed by Proposition 3.2, we have

$$\frac{\mu_G(X)}{\mu_G(X^2)} \geq \frac{\mu_H(\tilde{X})}{2\mu_H(\tilde{X}_1 \cdot \tilde{X})} > \frac{1}{2(2 + \tilde{\varepsilon})^{n-1}}. \quad (3.7)$$

Recall that the support of X with respect to μ_G consists of $a \in X$ such that $\mu_G(X \cap U) > 0$ for all open neighborhoods of a . By Fact B.2.4, the support of X with respect to μ_G is the same as that with respect to ν_G , so we can refer to the support of X without ambiguity. Recall also that $\Delta(\exp(\cdot Z))$ is an isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^{>0}, \times)$. Hence there exists a constant $C > 0$ only depending on Z such that on the support of X we have $e^{-C\tilde{\varepsilon}} < \Delta < e^{C\tilde{\varepsilon}}$ and on the support of X^2 we have $e^{-2C\tilde{\varepsilon}} < \Delta < e^{2C\tilde{\varepsilon}}$. Thus by Fact B.2.4, we have

$$\frac{\nu_G(X)}{\mu_G(X)} > e^{-C\tilde{\varepsilon}}$$

and

$$\frac{\mu_G(X^2)}{\nu_G(X^2)} > e^{-2C\tilde{\varepsilon}}.$$

Combining the above inequalities with (3.7), we have

$$\frac{\nu_G(X)}{\nu_G(X^2)} > \frac{e^{-3C\tilde{\varepsilon}}}{2(2 + \tilde{\varepsilon})^{n-1}}. \quad (3.8)$$

Hence for the X we constructed,

$$\frac{\nu_G(X)^{\frac{1}{n}-\varepsilon}}{\nu_G(X^2)^{\frac{1}{n}-\varepsilon}} + \frac{\mu_G(X)^{\frac{1}{n}-\varepsilon}}{\mu_G(X^2)^{\frac{1}{n}-\varepsilon}} > (1 + e^{-3C\tilde{\varepsilon}(\frac{1}{n}-\varepsilon)})(2(2 + \tilde{\varepsilon})^{n-1})^{-\frac{1}{n}+\varepsilon}. \quad (3.9)$$

It suffices to take $\tilde{\varepsilon}$ small enough such that the right hand side of (3.9) is > 1 . \square

With the Gleason–Yamabe Theorem and the results developed in Sect. 2, we are able to pass our Lie group constructions to general locally compact groups.

Proof of Theorem 3.1. By Fact C.2, there is open subgroup G' of G which is almost-Lie. Since $\mu_G|_{G'}$ is a left Haar measure on G' , and same holds $\nu_G|_{G'}$, we may assume without loss of generality that G is almost-Lie.

With this assumption, there is a short exact sequence $1 \rightarrow H \rightarrow G \xrightarrow{\pi} G/H \rightarrow 1$ where H is a compact subgroup, and G/H is a Lie group. Let X be a subset of G/H such that

$$\frac{\nu_{G/H}(X)^{\frac{1}{n}-\varepsilon}}{\nu_{G/H}(X^2)^{\frac{1}{n}-\varepsilon}} + \frac{\mu_{G/H}(X)^{\frac{1}{n}-\varepsilon}}{\mu_{G/H}(X^2)^{\frac{1}{n}-\varepsilon}} > 1, \quad (3.10)$$

where n is the noncompact Lie dimension of G/H . Thus by the quotient integral formula, we have

$$\begin{aligned} \mu_G(\pi^{-1}(X)) &= \int_{G/H} \mu_H(g^{-1}(\pi^{-1}(X)) \cap H) \, d\mu_{G/H}(g) \\ &= \int_{G/H} \mathbb{1}_X(g) \, d\mu_{G/H}(g) = \mu_{G/H}(X), \end{aligned}$$

and similarly $\nu_G(\pi^{-1}(X)) = \nu_{G/H}(X)$. Observe that $\pi^{-1}(X^2) = \pi^{-1}(X) \cdot \pi^{-1}(X)$. Thus the desired conclusion follows from (3.10) and Propositions 2.8. \square

We now prove Theorem 1.3.

Proof of Theorem 1.3. We consider first the case when $n = 0$ and G is a Lie group. Then the identity component G_0 of G is compact. Taking $X = G_0$, we have $X = X^2$ has positive left and right measures. In particular, we have

$$\mu_G(X) = \mu_G(X^2) \text{ and } \nu_G(X) = \nu_G(X^2),$$

and Theorem 1.3 holds in this case.

Suppose $n = 0$ and G is a locally compact group. By Fact C.2, there is open subgroup G' of G which is almost-Lie. Since $\mu_G|_{G'}$ is a left Haar measure on G' , and same holds $\nu_G|_{G'}$, we may replace G with G' and assume without loss of generality that G is almost-Lie. Let H be a compact subgroup of G such that G/H is a Lie group, and $\pi : G \rightarrow G/H$ is the quotient map. Choose compact $Y \subseteq G/H$ such that $Y^2 = Y$ has positive left and right measure. Choosing $X = \pi^{-1}(Y)$, we have $X = X^2$ has positive left and right measures, and Theorem 1.3 holds in this case too.

In the rest of the proof, we assume $n > 0$. Using Theorem 3.1 we obtain an open subset Y of G such that

$$\frac{\nu_G(Y)^{\frac{1}{n}-\varepsilon/2}}{\nu_G(Y^2)^{\frac{1}{n}-\varepsilon/2}} + \frac{\mu_G(Y)^{\frac{1}{n}-\varepsilon/2}}{\mu_G(Y^2)^{\frac{1}{n}-\varepsilon/2}} > 1.$$

By the inner regularity of μ_G and ν_G , we can choose $X \subseteq Y$ such that $\mu_G(X) > \mu_G(Y) - \delta$ and $\nu_G(X) > \nu_G(Y) - \delta$ for any given $\delta > 0$. On the other hand, for such X , we have $\mu_G(X^2) \leq \mu_G(Y^2)$ and $\nu_G(X^2) \leq \nu_G(Y^2)$ as $X^2 \subseteq Y^2$. Hence, by choosing δ sufficiently small, we can arrange that X satisfies the desired inequality (1.6). \square

4 Reduction to outer terms of certain short exact sequences

For $n \in \mathbb{Z}^{\geq 0}$ and $(x, y) \in \mathbb{R}^2$, we set

$$\|(x, y)\|_{1/n} = \begin{cases} (|x|^{1/n} + |y|^{1/n})^n & \text{if } n \neq 0, \\ \max\{|x|, |y|\} & \text{if } n = 0. \end{cases}$$

We say that the group G satisfies the **Brunn–Minkowski inequality with exponent** n , abbreviated as $\text{BM}(n)$, if for all compact $X, Y \subseteq G$,

$$\left\| \left(\frac{\nu(X)}{\nu(XY)}, \frac{\mu(Y)}{\mu(XY)} \right) \right\|_{1/n} \leq 1.$$

When G is unimodular and $n \geq 1$, the above is equivalent to having the inequality $\mu(XY)^{1/n} \geq \mu(X)^{1/n} + \mu(Y)^{1/n}$. Note that $\frac{\nu(X)}{\nu(XY)} \leq 1$ and $\frac{\mu(Y)}{\mu(XY)} \leq 1$. Hence, every locally compact group G satisfies the Brunn–Minkowski inequality with exponent $n = 0$. Moreover, if $n < n'$ and G satisfies the Brunn–Minkowski inequality with exponent n' , then it satisfies the Brunn–Minkowski inequality with exponent n .

Given a function $f : \Omega \rightarrow \mathbb{R}$, for every $t \in \mathbb{R}$, define the *superlevel set* of f

$$L_f^+(t) := \{x \in \Omega : f(x) \geq t\}.$$

We will use this notation at various points in the later proofs. We use the following simple consequence of Fubini’s theorem concerning the superlevel sets [Rud87, Theorem 8.16]:

Fact 4.1. *Let μ be a positive measure on some σ -algebra in the set Ω . Suppose $f : \Omega \rightarrow [0, \infty]$ is a compactly supported measurable function. For every $r > 0$,*

$$\int_{\Omega} f^r(x) \, d\mu(x) = \int_{\mathbb{R}^{\geq 0}} r x^{r-1} \mu(L_f^+(x)) \, dx.$$

The next proposition is the main result of this section. The current statement of the proposition is proved by McCrudden as the main result in [McC69]. We give a simpler (but essentially the same) proof here for the sake of completeness.

PROPOSITION 4.2. *Assume G is a unimodular group, $n_1, n_2 \geq 0$ are integers, H is a closed normal subgroup of G satisfying $\text{BM}(n_1)$, and the quotient group G/H is unimodular satisfying $\text{BM}(n_2)$. Then G satisfies $\text{BM}(n_1 + n_2)$.*

Proof. Suppose Ω is a compact subset of G . Let the “fiber length function” $f_{\Omega} : G/H \rightarrow \mathbb{R}^{\geq 0}$ be a measurable function such that for every $gH \in G/H$, $f_{\Omega}(gH) = \mu_H(g^{-1}\Omega \cap H)$. The case when both $n_1 = n_2 = 0$ holds trivially.

Now we split the proof into three cases.

Case 1. *When $n_1 \geq 1$ and $n_1 + n_2 \geq 2$.*

By the quotient integral formula (Fact B.3), we have

$$\begin{aligned} \mu_G^{1/(n_1+n_2)}(\Omega) &= \left(\int_{G/H} f_{\Omega}(x) \, d\mu_{G/H}(x) \right)^{1/(n_1+n_2)} \\ &= \left(\int_{G/H} (f_{\Omega}^{1/n_1}(x))^{n_1} \, d\mu_{G/H}(x) \right)^{1/(n_1+n_2)}. \end{aligned}$$

Now by Fact 4.1 and the fact that $\mu_{G/H}(L_{f_\Omega}^{+1/n_1}(t)) = \mu_{G/H}(L_{f_\Omega}^+(t^{n_1}))$, we get

$$\mu_G^{1/(n_1+n_2)}(\Omega) = \left(\int_{\mathbb{R}^{>0}} n_1 t^{n_1-1} \mu_{G/H}(L_{f_\Omega}^+(t^{n_1})) dt \right)^{1/(n_1+n_2)}. \tag{4.1}$$

Set $\alpha = \frac{n_1-1}{n_1+n_2-1}$, $\beta = \frac{n_2}{n_1+n_2-1}$, $\gamma = n_1 + n_2 - 1$, and

$$F_\Omega(t) = t^\alpha \mu_{G/H}^{\beta/n_2} \left(L_{f_\Omega}^+(t^{n_1}) \right),$$

for compact set Ω in G and $t > 0$ (Note that F_Ω is well-defined when $n_2 = 0$). Then (4.1) can be rewritten as

$$n_1^{-1/(\gamma+1)} \mu_G^{1/(\gamma+1)}(\Omega) = \left(\int_{\mathbb{R}^{>0}} F_\Omega^\gamma(t) dt \right)^{1/(\gamma+1)}. \tag{4.2}$$

Fix nonempty compact sets $X, Y \subseteq G$. By (4.2), we need to show that

$$\left(\int_{\mathbb{R}^{>0}} F_{XY}^\gamma(t) dt \right)^{1/(\gamma+1)} \geq \left(\int_{\mathbb{R}^{>0}} F_X^\gamma(t) dt \right)^{1/(\gamma+1)} + \left(\int_{\mathbb{R}^{>0}} F_Y^\gamma(t) dt \right)^{1/(\gamma+1)} \tag{4.3}$$

We will do so in two steps. First, we will show the following convexity property

$$F_{XY}(t_1 + t_2) \geq F_X(t_1) + F_Y(t_2). \tag{4.4}$$

For every $t_1, t_2 \in \mathbb{R}^{>0}$, since H satisfies $\text{BM}(n_1)$, by definition we have

$$L_{f_X}^+(t_1^{n_1}) L_{f_Y}^+(t_2^{n_1}) \subseteq L_{f_{XY}}^+((t_1 + t_2)^{n_1}). \tag{4.5}$$

Also, since G/H satisfies $\text{BM}(n_2)$, we have

$$\mu_{G/H}^{1/n_2} \left(L_{f_X}^+(t_1^{n_1}) \right) + \mu_{G/H}^{1/n_2} \left(L_{f_Y}^+(t_2^{n_1}) \right) \leq \mu_{G/H}^{1/n_2} \left(L_{f_{XY}}^+((t_1 + t_2)^{n_1}) \right). \tag{4.6}$$

Now by (4.6),

$$\begin{aligned} & (t_1 + t_2)^{n_1-1} \left(\mu_{G/H}^{1/n_2} \left(L_{f_{XY}}^+((t_1 + t_2)^{n_1}) \right) \right)^{n_2} \\ & \geq (t_1 + t_2)^{n_1-1} \left(\mu_{G/H}^{1/n_2} \left(L_{f_X}^+(t_1^{n_1}) \right) + \mu_{G/H}^{1/n_2} \left(L_{f_Y}^+(t_2^{n_1}) \right) \right)^{n_2} \\ & = \left\| (t_1^\alpha, t_2^\alpha) \right\|_{1/\alpha}^\gamma \left\| \left(\mu_{G/H}^{\beta/n_2} \left(L_{f_X}^+(t_1^{n_1}) \right), \mu_{G/H}^{\beta/n_2} \left(L_{f_Y}^+(t_2^{n_1}) \right) \right) \right\|_{1/\beta}^\gamma, \end{aligned}$$

applying Hölder’s inequality with exponents $1/\alpha$ and $1/\beta$, and using the fact that $n_1 \geq 1$ and $n_1 + n_2 \geq 2$ (which would imply $1/\alpha, 1/\beta \in [1, \infty]$), we obtain

$$\begin{aligned} & (t_1 + t_2)^{n_1-1} \left(\mu_{G/H}^{1/n_2} \left(L_{f_{XY}}^+((t_1 + t_2)^{n_1}) \right) \right)^{n_2} \\ & \geq \left(t_1^\alpha \mu_{G/H}^{\beta/n_2} \left(L_{f_X}^+(t_1^{n_1}) \right) + t_2^\alpha \mu_{G/H}^{\beta/n_2} \left(L_{f_Y}^+(t_2^{n_1}) \right) \right)^\gamma. \end{aligned}$$

We remark that the above inequalities also make sense when $n_2 = 0$. In that case $\|(a, b)\|_{1/n_2}$ is to be understood as $\max\{a, b\}$ for every $a, b \in \mathbb{R}^{\geq 0}$. The first line of the above inequality is $F_{XY}^\gamma(t_1 + t_2)$ and the last line is $(F_X(t_1) + F_Y(t_2))^\gamma$. So we finished the first step.

We now prove (4.3). As a consequence of the above convexity property (4.4), when $a, b \in \mathbb{R}$ and $F_X(a) \geq s_1$, $F_Y(b) \geq s_2$, we have $F_{XY}(a + b) \geq s_1 + s_2$. This implies

$$L_{F_X}^+(s_1) + L_{F_Y}^+(s_2) \subseteq L_{F_{XY}}^+(s_1 + s_2).$$

Thus using Kneser's inequality [Kne56] for \mathbb{R} (i.e. the Brunn–Minkowski inequality for \mathbb{R}), we have

$$\mu_{\mathbb{R}}(L_{F_{XY}}^+(s_1 + s_2)) \geq \mu_{\mathbb{R}}(L_{F_X}^+(s_1)) + \mu_{\mathbb{R}}(L_{F_Y}^+(s_2)). \quad (4.7)$$

Let $M_X = \text{ess sup}_x F_X(x)$, $M_Y = \text{ess sup}_x F_Y(x)$. By Fact 4.1 we have

$$\begin{aligned} \int_{\mathbb{R}^{>0}} F_{XY}^\gamma(s) \, ds &\geq \int_0^{M_X + M_Y} \gamma s^{\gamma-1} \mu_{\mathbb{R}}(L_{F_{XY}}^+(s)) \, ds \\ &= (M_X + M_Y)^\gamma \int_0^1 \gamma s^{\gamma-1} \mu_{\mathbb{R}}(L_{F_{XY}}^+(M_X s + M_Y s)) \, ds. \end{aligned}$$

By (4.7), the above quantity is at least

$$\begin{aligned} &(M_X + M_Y)^\gamma \left(\int_0^1 \gamma s^{\gamma-1} \mu_{\mathbb{R}}(L_{F_X}^+(M_X s)) \, ds + \int_0^1 \gamma s^{\gamma-1} \mu_{\mathbb{R}}(L_{F_Y}^+(M_Y s)) \, ds \right) \\ &= (M_X + M_Y)^\gamma \left(\frac{1}{M_X^\gamma} \int_{\mathbb{R}^{>0}} F_X^\gamma(s) \, ds + \frac{1}{M_Y^\gamma} \int_{\mathbb{R}^{>0}} F_Y^\gamma(s) \, ds \right), \end{aligned} \quad (4.8)$$

and the latter is a change of variables.

Finally, by (4.1) and (4.8),

$$\begin{aligned} n_1^{-1/(\gamma+1)} \mu_G^{1/(\gamma+1)}(XY) &= \left(\int_{\mathbb{R}^{>0}} F_{XY}^\gamma(t) \, dt \right)^{1/(\gamma+1)} \\ &= \left((M_X + M_Y)^\gamma \left(\frac{1}{M_X^\gamma} \int_{\mathbb{R}^{>0}} F_X^\gamma(s) \, ds + \frac{1}{M_Y^\gamma} \int_{\mathbb{R}^{>0}} F_Y^\gamma(s) \, ds \right) \right)^{1/(\gamma+1)}. \end{aligned}$$

By the fact $n_1 + n_2 \geq 2$ we see that $\gamma \geq 1$. Applying Hölder's inequality with exponents $(\gamma + 1)/\gamma$ and $\gamma + 1$, the above quantity is

$$\begin{aligned} &\left\| \left(M_X^{\frac{\gamma}{\gamma+1}}, M_Y^{\frac{\gamma}{\gamma+1}} \right) \right\|_{\frac{\gamma+1}{\gamma}} \cdot \left\| \left(\left(\frac{1}{M_X^\gamma} \int_{\mathbb{R}^{>0}} F_X^\gamma(t) \, dt \right)^{\frac{1}{\gamma+1}}, \left(\frac{1}{M_Y^\gamma} \int_{\mathbb{R}^{>0}} F_Y^\gamma(t) \, dt \right)^{\frac{1}{\gamma+1}} \right) \right\|_{\gamma+1} \\ &\geq \left(\int_{\mathbb{R}^{>0}} F_X^\gamma(t) \, dt \right)^{1/(\gamma+1)} + \left(\int_{\mathbb{R}^{>0}} F_Y^\gamma(t) \, dt \right)^{1/(\gamma+1)} \\ &= n_1^{-1/(\gamma+1)} \mu_G^{1/(\gamma+1)}(X) + n_1^{-1/(\gamma+1)} \mu_G^{1/(\gamma+1)}(Y), \end{aligned}$$

this proves the case 1.

Case 2. When $n_1 = 1$ and $n_2 = 0$.

In this case, the conclusion can be derived from (4.1) directly. In particular, using the fact that G/H satisfies BM(0) and H satisfies BM(1), we have

$$\mu_{G/H} \left(L_{f_{XY}}^+ (t_1 + t_2) \right) \geq \max \left\{ \mu_{G/H} \left(L_{f_X}^+ (t_1) \right), \mu_{G/H} \left(L_{f_Y}^+ (t_2) \right) \right\}.$$

Let $N_X = \sup_t f_X(t)$ and $N_Y = \sup_t f_Y(t)$. Therefore, by Hölder’s inequality,

$$\begin{aligned} \mu_G(XY) &= \int_{\mathbb{R}^{>0}} \mu_{G/H}(L_{f_{XY}}^+(t)) dt \\ &= \int_{\mathbb{R}^{>0}} (N_X + N_Y) \mu_{G/H}(L_{f_{XY}}^+((N_X + N_Y)t)) dt \\ &\geq (N_X + N_Y) \max \left\{ \int_0^1 \mu_{G/H}(L_{f_X}^+(N_X t)) dt, \int_0^1 \mu_{G/H}(L_{f_Y}^+(N_Y t)) dt \right\} \\ &\geq N_X \int_0^1 \mu_{G/H}(L_{f_X}^+(N_X t)) dt + N_Y \int_0^1 \mu_{G/H}(L_{f_Y}^+(N_Y t)) dt \\ &= \mu_G(X) + \mu_G(Y). \end{aligned}$$

Thus G satisfies BM(1).

Case 3. When $n_1 = 0$ and $n_2 \geq 1$.

Applying the Brunn–Minkowski inequality with exponent 0 on H , and the fact that G/H satisfies BM(n_2), we obtain

$$\mu_{G/H}^{1/n_2} \left(L_{f_{XY}}^+ (\max\{t_1, t_2\}) \right) \geq \mu_{G/H}^{1/n_2} \left(L_{f_X}^+ (t_1) \right) + \mu_{G/H}^{1/n_2} \left(L_{f_Y}^+ (t_2) \right). \tag{4.9}$$

Given a compact set Ω in G , we define

$$E_\Omega(t) = \mu_{G/H}^{1/n_2}(L_{f_\Omega}^+(t)), \quad t > 0.$$

Clearly E_Ω is a non-increasing function. By (4.9), we have

$$E_{XY}(\max\{a_1, a_2\}) \geq E_X(a_1) + E_Y(a_2) \tag{4.10}$$

for all a_1, a_2 . This can be seen as a “convexity property” for E , but the maximum operator inside the function E prevents us from using the same argument as used in Case 1 for F . On the other hand, using (4.10), we see that if $a, b \in \mathbb{R}$ and $E_X(a) \geq s_1$ and $E_Y(b) \geq s_2$, then $E_{XY}(\max\{a, b\}) \geq s_1 + s_2$. Hence $L_{E_X}^+(s_1) \cup L_{E_Y}^+(s_2) \subseteq L_{E_{XY}}^+(s_1 + s_2)$, and thus

$$\mu_{\mathbb{R}}(L_{E_{XY}}^+(s_1 + s_2)) \geq \max\{\mu_{\mathbb{R}}(L_{E_X}^+(s_1)), \mu_{\mathbb{R}}(L_{E_Y}^+(s_2))\}. \tag{4.11}$$

Now we consider $\mu_G(XY)$. We have

$$\mu_G^{1/n_2}(XY) = \left(\int_{\mathbb{R}^{>0}} E_{XY}^{n_2}(s) ds \right)^{1/n_2} = \left(\int_{\mathbb{R}^{>0}} n_2 s^{n_2-1} \mu_{\mathbb{R}}(L_{E_{XY}}^+(s)) ds \right)^{1/n_2}. \tag{4.12}$$

Let $P_X = \text{ess sup}_t E_X(t)$ and $P_Y = \text{ess sup}_t E_Y(t)$. By (4.11) and (4.12) we see

$$\begin{aligned} & n_2^{-1/n_2} \mu_G^{1/n_2}(XY) \\ & \geq \left((P_X + P_Y)^{n_2} \max \left\{ \int_0^1 s^{n_2-1} \mu_{\mathbb{R}}(L_{E_X}^+(P_X s)) \, ds, \int_0^1 s^{n_2-1} \mu_{\mathbb{R}}(L_{E_Y}^+(P_Y s)) \, ds \right\} \right)^{1/n_2} \\ & \geq \left(P_X^{n_2} \int_0^1 s^{n_2-1} \mu_{\mathbb{R}}(L_{E_X}^+(P_X s)) \, ds \right)^{1/n_2} + \left(P_Y^{n_2} \int_0^1 s^{n_2-1} \mu_{\mathbb{R}}(L_{E_Y}^+(P_Y s)) \, ds \right)^{1/n_2} \\ & = n_2^{-1/n_2} \mu_G^{1/n_2}(X) + n_2^{-1/n_2} \mu_G^{1/n_2}(Y). \end{aligned}$$

This proves the case when $n_1 = 0$, and hence finishes the proof of the proposition. □

5 Reduction to unimodular subgroups

The main result of this section allows us to obtain a Brunn–Minkowski inequality for a nonunimodular group from a certain unimodular normal subgroup. We use $\mu_{\mathbb{R}^\times}$ to denote a Haar measure on the multiplicative group $(\mathbb{R}^{>0}, \times)$. The next lemma concerns the case when the modular function on X and on Y is “nearly constant”.

LEMMA 5.1. *Suppose G is a locally compact group, and the modular function $\Delta_G : G \rightarrow (\mathbb{R}^{>0}, \times)$ is a quotient map of topological groups. Let X, Y be compact subsets of G , and parameters $a, b, \varepsilon > 0$ and $n \geq 0$ an integer, such that for every $x \in X$, $\Delta_G(x) \in [a, a + \varepsilon)$ and for every $y \in Y$, $\Delta_G(y) \in [b, b + \varepsilon)$. Suppose $H = \ker(\Delta_G)$ satisfies $\text{BM}(n)$. Then*

$$\frac{\nu_G(X)^{1/(n+1)}}{\nu_G(XY)^{1/(n+1)}} + \frac{\mu_G(Y)^{1/(n+1)}}{\mu_G(XY)^{1/(n+1)}} \leq 1 + f(\varepsilon),$$

where $f(\varepsilon)$ is an explicit function depending only on a, b, n and ε , and $f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover, this convergence is uniform when n is fixed and a and b vary over compact sets.

Proof. We first consider the case when $n \geq 1$. Fix Haar measures $\mu_H, \mu_{\mathbb{R}^\times}$ on H and on $(\mathbb{R}^{>0}, \times)$, and these two measures will also uniquely determine a left Haar measure μ_G on G and a right Haar measure ν_G on G via the quotient integral formula. For every compact subset Ω of G , define two functions $\ell_\Omega, r_\Omega : (\mathbb{R}^{>0}, \times) \rightarrow \mathbb{R}^{\geq 0}$ such that

$$\ell_\Omega(g) = \mu_H(g^{-1}\Omega \cap H), \text{ and } r_\Omega(g) = \mu_H(\Omega g^{-1} \cap H).$$

Note that given g_1, g_2 in G , $(X \cap Hg_1)(Y \cap g_2H)$ lies in

$$Hg_1g_2H = (g_1g_2)(g_1g_2)^{-1}H(g_1g_2)H = H(g_1g_2)H(g_1g_2)^{-1}(g_1g_2) \tag{5.1}$$

since H is normal.

For every compact sets X_1, X_2 in H , and g_1, g_2 in G , by (5.1), $X_1g_1g_2X_2 \subseteq g_1g_2H$. By Fact B.4.2 and the fact that H satisfies BM(n), we have

$$\begin{aligned} \mu_H^{1/n}((g_1g_2)^{-1}X_1g_1g_2X_2) &\geq \mu_H^{1/n}((g_1g_2)^{-1}X_1g_1g_2) + \mu_H^{1/n}(X_2) \\ &= (\Delta_G(g_1)\Delta_G(g_2))^{-1/n}\mu_H^{1/n}(X_1) + \mu_H^{1/n}(X_2). \end{aligned} \tag{5.2}$$

In light of this, for every $t_1, t_2 \geq 0$ we have

$$L_{\ell_{XY}}^+ \left(\left(\inf_{x \in X, y \in Y} (\Delta_G(x)\Delta_G(y))^{-1/n}t_1 + t_2 \right)^n \right) \supseteq L_{r_X}^+(t_1^n) + L_{\ell_Y}^+(t_2^n).$$

Applying the Brunn–Minkowski inequality on $(\mathbb{R}^{>0}, \times)$, we get

$$\mu_{\mathbb{R}^\times} \left(L_{\ell_{XY}}^+ \left(\left(\inf_{x \in X, y \in Y} (\Delta_G(x)\Delta_G(y))^{-1/n}t_1 + t_2 \right)^n \right) \right) \geq \mu_{\mathbb{R}^\times}(L_{r_X}^+(t_1^n)) + \mu_{\mathbb{R}^\times}(L_{\ell_Y}^+(t_2^n)). \tag{5.3}$$

Similarly, for the right Haar measure on H , we have

$$\mu_{\mathbb{R}^\times} \left(L_{r_{XY}}^+ \left(\left(t_1 + \inf_{x \in X, y \in Y} (\Delta_G(x)\Delta_G(y))^{1/n}t_2 \right)^n \right) \right) \geq \mu_{\mathbb{R}^\times}(L_{r_X}^+(t_1^n)) + \mu_{\mathbb{R}^\times}(L_{\ell_Y}^+(t_2^n)). \tag{5.4}$$

Let $M_X = \sup_x \mu_{\mathbb{R}^\times}(L_{r_X}^+(x))$ and $M_Y = \sup_y \mu_{\mathbb{R}^\times}(L_{\ell_Y}^+(y))$. By a change of variables and Fact 4.1,

$$\mu_G(XY) = \int_{\mathbb{R}^\times} \mu_H(g^{-1}XY \cap H) d\mu_{\mathbb{R}^\times}(g) = \int_{\mathbb{R}^{>0}} nt^{n-1} \mu_{\mathbb{R}^\times}(L_{\ell_{XY}}^+(t^n)) dt.$$

By a change of variables the above quantity can be rewritten as

$$\begin{aligned} &\left\| \left(\frac{1}{(a + \varepsilon)(b + \varepsilon)} M_X, M_Y \right) \right\|_{1/n} \\ &\cdot \int_0^1 nt^{n-1} \mu_{\mathbb{R}^\times} \left(L_{\ell_{XY}}^+ \left(\left(\left(\frac{1}{(a + \varepsilon)(b + \varepsilon)} M_X \right)^{1/n} t + M_Y^{1/n} t \right)^n \right) \right) dt. \end{aligned}$$

Using (5.3), the above quantity is at least

$$\begin{aligned} &\left\| \left(\frac{1}{(a + \varepsilon)(b + \varepsilon)} M_X, M_Y \right) \right\|_{1/n} \\ &\cdot \left(\int_0^1 nt^{n-1} \mu_{\mathbb{R}^\times}(L_{r_X}^+(M_X t^n)) dt + \int_0^1 nt^{n-1} \mu_{\mathbb{R}^\times}(L_{\ell_Y}^+(M_Y t^n)) dt \right) \\ &= \left\| \left(\frac{1}{(a + \varepsilon)(b + \varepsilon)} M_X, M_Y \right) \right\|_{1/n} \left(\frac{1}{M_X} \nu_G(X) + \frac{1}{M_Y} \mu_G(Y) \right). \end{aligned}$$

Thus applying Hölder’s inequality with exponents $(n + 1)/n$ and $n + 1$, we get

$$\begin{aligned} &\mu_G^{1/(n+1)}(XY) \\ &\geq \left\| \left(\left(\frac{1}{(a + \varepsilon)(b + \varepsilon)} M_X \right)^{\frac{1}{n+1}}, M_Y^{\frac{1}{n+1}} \right) \right\|_{\frac{n+1}{n}} \left\| \left(\left(\frac{1}{M_X} \nu_G(X) \right)^{\frac{1}{n+1}}, \left(\frac{1}{M_Y} \mu_G(Y) \right)^{\frac{1}{n+1}} \right) \right\|_{n+1} \end{aligned}$$

$$\geq \left(\frac{1}{(a+\varepsilon)(b+\varepsilon)} \nu_G(X) \right)^{1/(n+1)} + \mu_G^{1/(n+1)}(Y). \quad (5.5)$$

Similarly, for $\nu_G(XY)$ by (5.4) and a change of variables we get

$$\begin{aligned} \nu_G(XY) &= \int_{\mathbb{R}^\times} \mu_H(XYg^{-1} \cap H) d\mu_{\mathbb{R}^\times}(g) \\ &= \int_{\mathbb{R}^{>0}} nt^{n-1} \mu_{\mathbb{R}^\times}(L_{r_{XY}}^+(t^n)) dt \\ &\geq \left\| (M_X, abM_Y) \right\|_{1/n} \left(\frac{1}{M_X} \nu_G(X) + \frac{1}{M_Y} \mu_G(Y) \right), \end{aligned}$$

and using Hölder's inequality in a similar way as in (5.5), we obtain

$$\nu_G^{1/(n+1)}(XY) \geq \nu_G^{1/(n+1)}(X) + (ab\mu_G(Y))^{1/(n+1)}. \quad (5.6)$$

Therefore, combining (5.5) and (5.6), we conclude

$$\begin{aligned} &\frac{\nu_G^{1/(n+1)}(X)}{\nu_G^{1/(n+1)}(XY)} + \frac{\mu_G^{1/(n+1)}(Y)}{\mu_G^{1/(n+1)}(XY)} \\ &\leq \frac{1}{1 + (Cab)^{1/(n+1)}} + \frac{1}{1 + \left(\frac{1}{C(a+\varepsilon)(b+\varepsilon)}\right)^{1/(n+1)}} \\ &\leq 1 + \frac{(C(ab + \varepsilon(a + b + \varepsilon)))^{1/(n+1)} - (Cab)^{1/(n+1)}}{(1 + (Cab)^{1/(n+1)})(1 + (C(a + \varepsilon)(b + \varepsilon))^{1/(n+1)})}. \end{aligned}$$

where $C = \mu_G(Y)/\nu_G(X)$.

Hence

$$\frac{\nu_G^{1/(n+1)}(X)}{\nu_G^{1/(n+1)}(XY)} + \frac{\mu_G^{1/(n+1)}(Y)}{\mu_G^{1/(n+1)}(XY)} \leq 1 + f(\varepsilon)$$

where

$$f(\varepsilon) = \sup_{r>0} \frac{(r(ab + \varepsilon(a + b + \varepsilon)))^{1/(n+1)} - (rab)^{1/(n+1)}}{(1 + (rab)^{1/(n+1)})(1 + (r(a + \varepsilon)(b + \varepsilon))^{1/(n+1)})}$$

depends only on a, b, n and ε and we see $\lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0$ uniformly when a, b takes values in a compact set by an elementary computation.

The remaining case is when $n = 0$. Note that in this case, inequality (5.2) becomes

$$\mu_H((g_1g_2)^{-1}X_1g_1g_2X_2) \geq \max\{(\Delta_G(g_1)\Delta_G(g_2))^{-1}\mu_H(X_1), \mu_H(X_2)\}.$$

This implies for every t_1, t_2 ,

$$\mu_{\mathbb{R}^\times} \left(L_{\ell_{XY}}^+ \max \left\{ \inf_{x \in X, y \in Y} (\Delta_G(x)\Delta_G(y))^{-1}t_1, t_2 \right\} \right) \geq \mu_{\mathbb{R}^\times}(L_{r_X}^+(t_1)) + \mu_{\mathbb{R}^\times}(L_{\ell_Y}^+(t_2)).$$

For any compact set Ω in G , define two functions $\Phi_\Omega, \Psi_\Omega : \mathbb{R} \rightarrow \mathbb{R}$, by

$$\Phi_\Omega(t) = \mu_{\mathbb{R}^\times}(L_{\ell_\Omega}^+(t)), \quad \text{and} \quad \Psi_\Omega(t) = \mu_{\mathbb{R}^\times}(L_{r_\Omega}^+(t)).$$

Thus we have

$$\mu_{\mathbb{R}}(L_{\Phi_{XY}}^+(t_1 + t_2)) \geq \max \left\{ \inf_{x \in X, y \in Y} (\Delta_G(x)\Delta_G(y))^{-1} \mu_{\mathbb{R}}(L_{\Psi_X}^+(t_1)), \mu_{\mathbb{R}}(L_{\Phi_Y}^+(t_2)) \right\}.$$

Let $N_X = \sup_x \mu_{\mathbb{R}}(L_{\Psi_X}^+(x))$ and $N_Y = \sup_y \mu_{\mathbb{R}}(L_{\Phi_Y}^+(y))$. By a change of variable, for $\mu_G(XY)$ we have

$$\begin{aligned} \mu_G(XY) &= \int_{\mathbb{R}^{>0}} \mu_{\mathbb{R}}(L_{\Phi_{XY}}^+(t)) \, dt \\ &\geq (N_X + N_Y) \max \left\{ \frac{1}{(a + \varepsilon)(b + \varepsilon)} \frac{\nu_G(X)}{N_X}, \frac{\mu_G(Y)}{N_Y} \right\} \\ &\geq \frac{1}{(a + \varepsilon)(b + \varepsilon)} \nu_G(X) + \mu_G(Y). \end{aligned} \tag{5.7}$$

Similarly, for every t_1, t_2 we also have

$$\mu_{\mathbb{R}^\times} \left(L_{r_{XY}}^+ \max \left\{ t_1, \inf_{x \in X, y \in Y} \Delta_G(x)\Delta_G(y)t_2 \right\} \right) \geq \mu_{\mathbb{R}^\times}(L_{r_X}^+(t_1)) + \mu_{\mathbb{R}^\times}(L_{\ell_Y}^+(t_2)),$$

which implies

$$\mu_{\mathbb{R}}(L_{\Psi_{XY}}^+(t_1 + t_2)) \geq \max \left\{ \mu_{\mathbb{R}}(L_{\Psi_X}^+(t_1)), \inf_{x \in X, y \in Y} \Delta_G(x)\Delta_G(y) \mu_{\mathbb{R}}(L_{\Phi_Y}^+(t_2)) \right\}.$$

Therefore, for $\nu_G(XY)$ we get

$$\nu_G(XY) \geq \nu_G(X) + ab\mu_G(Y).$$

Together with (5.7), as in the case when $n \geq 1$, we get

$$\begin{aligned} \frac{\nu_G(X)}{\nu_G(XY)} + \frac{\mu_G(Y)}{\mu_G(XY)} &\leq \frac{1}{1 + Cab} + \frac{1}{1 + \frac{1}{C(a+\varepsilon)(b+\varepsilon)}} \\ &\leq 1 + \varepsilon \frac{C(a + b + \varepsilon)}{(1 + Cab)(1 + C(a + \varepsilon)(b + \varepsilon))}, \end{aligned}$$

where $C = \mu_G(Y)/\nu_G(X)$. The conclusion follows by taking

$$f(\varepsilon) = \sup_{r>0} \frac{\varepsilon r(a + b + \varepsilon)}{(1 + rab)(1 + r(a + \varepsilon)(b + \varepsilon))},$$

and we can see that $f(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ uniformly when a, b taken values in a compact set by elementary computations. □

The next proposition is the main result of the section. As mentioned in the introduction, the proof uses a discretized “spillover” method. We remark that one can always make the proof continuous as we did in Sect. 4, but we give a discrete proof here since we believe this reflects our idea in a clearer way.

PROPOSITION 5.2. *Suppose G is a locally compact group with $H = \ker(\Delta_G)$ satisfying $\text{BM}(n)$. Suppose the map $\Delta_G : G \rightarrow (\mathbb{R}^{>0}, \times)$ is a quotient map of topological groups, then G satisfies $\text{BM}(n + 1)$.*

Proof. Since X and Y are compact, there are a_1, a_2, b_1 and $b_2 > 0$, such that

$$a_1 = \inf_{x \in X} \Delta_G(x), \quad a_2 = \sup_{x \in X} \Delta_G(x), \quad b_1 = \inf_{y \in Y} \Delta_G(y), \quad b_2 = \sup_{y \in Y} \Delta_G(y).$$

We fix μ_G and ν_G as in the proof of Lemma 5.1, and let $\varepsilon > 0$ be a sufficiently small number (depending on a_1, a_2, b_1 and b_2). Then by Fact B.3 and familiar properties of integrable functions on \mathbb{R} , there is an $N > 0$, such that we can partition $[a_1, a_2]$ and $[b_1, b_2]$ into N subintervals, that is

$$[a_1, a_2] = \bigcup_{i=1}^N A_i, \quad [b_1, b_2] = \bigcup_{i=1}^N B_i,$$

such that each subinterval has length at most ε , and the intersection of X with $\Delta_G^{-1}(A_i)$ has ν_G -measure $\nu_G(X)/N$, the intersection of Y with $\Delta_G^{-1}(B_i)$ has μ_G -measure $\mu_G(Y)/N$, for every $1 \leq i \leq N$.

Let $X_i = X \cap \Delta_G^{-1}(A_i)$ and let $Y_i = Y \cap \Delta_G^{-1}(B_i)$. Then $\nu_G(X) = \sum_{i=1}^N \nu_G(X_i)$ and $\mu_G(Y) = \sum_{i=1}^N \mu_G(Y_i)$. In particular, we have $\mu_G(XY) \geq \sum_{i=1}^N \mu_G(X_i Y_i)$ and $\nu_G(XY) \geq \sum_{i=1}^N \nu_G(X_i Y_i)$. Observe that given $1 \leq i, j \leq N$ and $i \neq j$, $X_i Y_i$ and $X_j Y_j$ are disjoint. Indeed, the modulus of every element in $X_i Y_i$ lies in $A_i B_i$ and the modulus of every element in $X_j Y_j$ lies in $A_j B_j$. But $A_i B_i$ and $A_j B_j$ are disjoint subsets of $\mathbb{R}^{>0}$ when $i \neq j$.

By Lemma 5.1, for every $1 \leq i \leq N$, there is a function $f_i(\varepsilon)$, such that $f_i(\varepsilon) \rightarrow 0$ when $\varepsilon \rightarrow 0$ uniformly, and

$$\frac{\nu_G^{1/(n+1)}(X_i)}{\nu_G^{1/(n+1)}(X_i Y_i)} + \frac{\mu_G^{1/(n+1)}(Y_i)}{\mu_G^{1/(n+1)}(X_i Y_i)} \leq 1 + f_i(\varepsilon).$$

Take $\tilde{f}(\varepsilon) = \sup_i f_i(\varepsilon)$, hence $\tilde{f}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, for every $1 \leq t \leq N$,

$$\frac{\nu_G^{1/(n+1)}(X)}{\nu_G^{1/(n+1)}(XY)} + \frac{\mu_G^{1/(n+1)}(Y)}{\mu_G^{1/(n+1)}(XY)} \leq \left(\frac{N \nu_G(X_t)}{\sum_{i=1}^N \nu_G(X_i Y_i)} \right)^{\frac{1}{n+1}} + \left(\frac{N \mu_G(Y_t)}{\sum_{i=1}^N \mu_G(X_i Y_i)} \right)^{\frac{1}{n+1}}. \tag{5.8}$$

Also by Hölder’s inequality with exponents $\frac{n+2}{n+1}$ and $n + 2$, we observe that for every t we have

$$\begin{aligned} & \left(\sum_{i=1}^N \left(\frac{\nu_G(X_i)}{\nu_G(X_i Y_i)} \right)^{\frac{1}{n+2} \cdot \frac{n+2}{n+1}} \right)^{\frac{n+1}{n+2}} \left(\sum_{i=1}^N \nu_G(X_i Y_i) \right)^{\frac{1}{n+2}} \\ & \geq \sum_{i=1}^N \left(\frac{\nu_G(X_i)}{\nu_G(X_i Y_i)} \right)^{\frac{1}{n+2}} \nu_G(X_i Y_i)^{\frac{1}{n+2}} = N \nu_G^{\frac{1}{n+2}}(X_t). \end{aligned}$$

After simplification, the above inequality can be rewritten as

$$N(N \nu_G(X_t))^{\frac{1}{n+1}} \leq \left(\sum_{i=1}^N \left(\frac{\nu_G(X_i)}{\nu_G(X_i Y_i)} \right)^{\frac{1}{n+1}} \right) \left(\sum_{i=1}^N \nu_G(X_i Y_i) \right)^{\frac{1}{n+1}}. \tag{5.9}$$

Averaging (5.8) over all t and using inequality (5.9), we have

$$\begin{aligned} & \frac{\nu_G^{1/(n+1)}(X)}{\nu_G^{1/(n+1)}(XY)} + \frac{\mu_G^{1/(n+1)}(Y)}{\mu_G^{1/(n+1)}(XY)} \\ & \leq \frac{1}{N} \sum_{i=1}^N \left(\frac{\nu_G(X_i)}{\nu_G(X_i Y_i)} \right)^{1/(n+1)} + \frac{1}{N} \sum_{i=1}^N \left(\frac{\mu_G(X_i)}{\mu_G(X_i Y_i)} \right)^{1/(n+1)} \leq 1 + \tilde{f}(\varepsilon). \end{aligned}$$

The desired conclusion follows by taking $\varepsilon \rightarrow 0$. □

6 Reduction to cocompact and codiscrete subgroups

The main results in this section will help us to reduce the problem to cocompact subgroups or open normal subgroups. We make use of the following integral formula, see [Kna86, Proposition 5.26, Consequence 1].

Fact 6.1. *Let G be a connected unimodular Lie group. Suppose S, T are closed subgroups of G , such that $G = ST$, and the intersection $S \cap T$ is compact. Then there is a left Haar measure μ_S on S and a right Haar measure ν_T on T , such that*

$$\int_G f(x) \, d\mu_G(x) = \int_{S \times T} f(st) \, d\mu_S(s) \, d\nu_T(t),$$

for every $f \in C_c(G)$.

The next proposition allows us to reduce the problem to closed cocompact subgroups with the same noncompact Lie dimension.

PROPOSITION 6.2. *Suppose G is a connected unimodular Lie group, H is a connected closed subgroup of G satisfying $\text{BM}(n)$, K is a connected unimodular subgroup of G , such that $G = KH$ and $K \cap H$ is compact. Then G satisfies $\text{BM}(n)$.*

Proof. We assume $n \geq 1$, otherwise the result is trivial. Note that both G and K are unimodular. In light of this we will not be using ν_G, ν_K , etc. and only use $\mu_G = \nu_G$ and $\mu_K = \nu_K$ below.

We fix a Haar measure μ_K on K , and a Haar measure μ_G on G . These measures will also uniquely determine a left Haar measure μ_H and a right Haar measure ν_H on H such that we have the integral formula in Fact 6.1 and another similar formula involving $d\mu_H(h) d\mu_K(k)$. For a compact subset Ω of G , we define two functions $r_\Omega, \ell_\Omega : K \rightarrow \mathbb{R}^{\geq 0}$, such that

$$r_\Omega(k) := \nu_H(k\Omega \cap H), \quad \ell_\Omega(k) := \mu_H(\Omega k \cap H),$$

for every $k \in K$. We also define two bivariate functions $R_\Omega, L_\Omega : K \times K \rightarrow \mathbb{R}^{\geq 0}$ that for every k_1, k_2 in K ,

$$R_\Omega(k_1, k_2) := \nu_H(k_1\Omega k_2 \cap H), \quad L_\Omega(k_1, k_2) := \mu_H(k_1\Omega k_2 \cap H).$$

Thus Fact 6.1 gives us

$$\mu_G(\Omega) = \int_K \nu_H(k^{-1}\Omega \cap H) d\mu_K(k) = \int_K \mu_H(\Omega k^{-1} \cap H) d\mu_K(k).$$

We define two probability measures p_X and p_Y on K in the following way:

$$dp_X = \frac{r_X d\mu_K}{\mu_G(X)}, \quad dp_Y = \frac{\ell_Y d\mu_K}{\mu_G(Y)}.$$

Now, we choose a left coset k_1H of H in G randomly with respect to the probability measure p_X , and choose a right coset Hk_2 of H in G randomly with respect to the probability measure p_Y . By the fact that H satisfies $\text{BM}(n)$, we get

$$\left(\frac{r_X(k_1)}{R_{XY}(k_1, k_2)} \right)^{1/n} + \left(\frac{\ell_Y(k_2)}{L_{XY}(k_1, k_2)} \right)^{1/n} \leq 1.$$

This implies

$$\mathbb{E}_{p_X(k_1)} \mathbb{E}_{p_Y(k_2)} \left[\left(\frac{r_X(k_1)}{R_{XY}(k_1, k_2)} \right)^{1/n} + \left(\frac{\ell_Y(k_2)}{L_{XY}(k_1, k_2)} \right)^{1/n} \right] \leq 1. \quad (6.1)$$

On the other hand, by the definition of p_X ,

$$\mathbb{E}_{p_X(k_1)} \left(\frac{r_X(k_1)}{R_{XY}(k_1, k_2)} \right)^{\frac{1}{n}} = \frac{1}{\mu_G(X)} \int_K r_X^{\frac{n+1}{n}}(k_1) R_{XY}^{-\frac{1}{n}}(k_1, k_2) d\mu_K(k_1). \quad (6.2)$$

Applying Hölder's inequality with exponents $(n+1)/n$ and $n+1$, we get

$$\begin{aligned} & \int_K r_X(k_1) d\mu_K(k_1) \\ & \leq \left(\int_K \left| r_X(k_1) R_{XY}^{-\frac{1}{n+1}}(k_1, k_2) \right|^{\frac{n+1}{n}} d\mu_K(k_1) \right)^{\frac{n}{n+1}} \left(\int_K \left| R_{XY}^{\frac{1}{n+1}}(k_1, k_2) \right|^{n+1} d\mu_K(k_1) \right)^{\frac{1}{n+1}}. \end{aligned}$$

Thus (6.2) gives us

$$\begin{aligned} & \mathbb{E}_{\mathbb{P}_X(k_1)} \left(\frac{r_X(k_1)}{R_{XY}(k_1, k_2)} \right)^{\frac{1}{n}} \\ & \geq \frac{1}{\mu_G(X)} \left(\int_K r_X(k_1) \, d\mu_K(k_1) \cdot \left(\int_K R_{XY}(k_1, k_2) \, d\mu_K(k_1) \right)^{-\frac{1}{n+1}} \right)^{\frac{n+1}{n}}. \end{aligned}$$

Finally, using Fact 6.1 and the fact that G is unimodular, we conclude that

$$\mathbb{E}_{\mathbb{P}_X(k_1)} \left(\frac{r_X(k_1)}{R_{XY}(k_1, k_2)} \right)^{\frac{1}{n}} \geq \left(\frac{\mu_G(X)}{\mu_G(XY)} \right)^{\frac{1}{n}}.$$

We have a similar inequality concerning $\mathbb{E}_{\mathbb{P}_Y(k_2)} \left(\frac{\ell_Y(k_2)}{L_{XY}(k_1, k_2)} \right)^{\frac{1}{n}}$. Combining both inequalities with (6.1), we get

$$\left(\frac{\mu_G(X)}{\mu_G(XY)} \right)^{\frac{1}{n}} + \left(\frac{\mu_G(Y)}{\mu_G(XY)} \right)^{\frac{1}{n}} \leq 1,$$

and hence G satisfies $\text{BM}(n)$. □

Using the proportionated averaging trick in a similar fashion, the next result allows us to reduce the problem to certain open subgroups.

PROPOSITION 6.3. *Let G be a locally compact group, and let G' be an open normal unimodular subgroup of G . Suppose G' satisfies $\text{BM}(n)$ for some integer $n \geq 1$. Then G satisfies $\text{BM}(n)$.*

Proof. Let $\mu_{G'}$ be a left (and hence right) Haar measure on G' . By Fact B.4.2, there is a left Haar measure μ_G and a right Haar measure ν_G on G , such that for every compact set Ω in G we have

$$\nu_G(\Omega) = \sum_{\alpha \in G/G'} \Delta_G(g_\alpha^{-1}) \mu_{G'}(g_\alpha^{-1} \Omega \cap G'), \quad \mu_G(\Omega) = \sum_{\beta \in G' \backslash G} \Delta_G(g_\beta) \mu_{G'}(\Omega g_\beta^{-1} \cap G'),$$

with g_α and g_β are representative elements of α and β respectively.

Now we fix two compact sets X, Y in G . For every $\alpha \in G/G'$, let $X_\alpha = g_\alpha^{-1} X \cap G'$, and we similarly define $Y_\beta = Y g_\beta^{-1} \cap G'$ for every $\beta \in G' \backslash G$. Since G' satisfies $\text{BM}(n)$, we have that

$$\left(\frac{\mu_{G'}(X_\alpha)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{1/n} + \left(\frac{\mu_{G'}(Y_\beta)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{1/n} \leq 1. \tag{6.3}$$

Now we choose α from G/G' randomly with probability $\mathbb{P}_X(\alpha) = \frac{\Delta_G(g_\alpha^{-1}) \mu_{G'}(X_\alpha)}{\nu_G(X)}$. Therefore by Hölder’s inequality,

$$\mathbb{E}_{\mathbb{P}_X(\alpha)} \left(\frac{\mu_{G'}(X_\alpha)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{\frac{1}{n}} = \frac{1}{\nu_G(X)} \sum_{\alpha \in G/G'} \frac{(\mu_{G'}(X_\alpha) \Delta_G(g_\alpha^{-1}))^{\frac{n+1}{n}}}{(\mu_{G'}(X_\alpha Y_\beta) \Delta_G(g_\alpha^{-1}))^{\frac{1}{n}}}$$

$$\geq \left(\frac{\nu_G(X)}{\nu_G(XYg_\beta^{-1})} \right)^{\frac{1}{n}} = \left(\frac{\nu_G(X)}{\nu_G(XY)} \right)^{\frac{1}{n}}.$$

Similarly, we choose β from $G' \setminus G$ randomly with probability $p_Y(\beta) = \frac{\Delta_G(g_\beta)\mu_{G'}(Y_\beta)}{\mu_G(Y)}$. Again using Hölder's inequality, we conclude that

$$\mathbb{E}_{p_Y(\beta)} \left(\frac{\mu_{G'}(Y_\beta)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{\frac{1}{n}} \geq \left(\frac{\mu_G(Y)}{\mu_G(XY)} \right)^{\frac{1}{n}}.$$

Hence by (6.4),

$$\begin{aligned} & \left(\frac{\nu_G(X)}{\nu_G(XY)} \right)^{\frac{1}{n}} + \left(\frac{\mu_G(Y)}{\mu_G(XY)} \right)^{\frac{1}{n}} \\ & \leq \mathbb{E}_{p_X(\alpha)} \mathbb{E}_{p_Y(\beta)} \left[\left(\frac{\mu_{G'}(X_\alpha)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{1/n} + \left(\frac{\mu_{G'}(Y_\beta)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{1/n} \right] \leq 1, \end{aligned}$$

and thus G also satisfies $\text{BM}(n)$. \square

The normality assumption in Proposition 6.3 is used to control the modular function via Fact B.4.2. Thus when the ambient group G is unimodular, the same proof of Proposition 6.3 will allow us to reduce the problem to a not necessarily normal open subgroup. We include the proof here for the sake of completeness.

PROPOSITION 6.4. *Let G be a unimodular group, and let G' be an open subgroup of G . Suppose G' satisfies $\text{BM}(n)$ for some integer $n \geq 0$, then G satisfies $\text{BM}(n)$.*

Proof. When $n = 0$, the conclusion follows from $\mu(XY) \geq \mu(Y)$. In the remainder of the proof we assume $n \geq 1$.

Let μ_G be a Haar measure on G , and let $\mu_{G'}$ be the restricted Haar measure of μ_G on G' . As G' is open, it is also unimodular. By Fact A.1.1, for every compact set Ω in G we have

$$\mu_G(\Omega) = \sum_{g\Omega \in G/G'} \mu_{G'}(g\Omega \cap G') = \sum_{g\Omega \in G \setminus G'} \mu_{G'}(\Omega g \cap G').$$

Now we fix two compact sets X, Y in G . For every $\alpha \in G/G'$, let $X_\alpha = g_\alpha^{-1}X \cap G'$, and we similarly define $Y_\beta = Yg_\beta^{-1} \cap G'$ for every $\beta \in G' \setminus G$. Since G' satisfies $\text{BM}(n)$, we have that

$$\left(\frac{\mu_{G'}(X_\alpha)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{1/n} + \left(\frac{\mu_{G'}(Y_\beta)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{1/n} \leq 1. \quad (6.4)$$

Now we choose α from G/G' randomly with probability $p_X(\alpha) = \frac{\mu_{G'}(X_\alpha)}{\mu_G(X)}$ and β from $G' \setminus G$ randomly with probability $p_Y(\beta) = \frac{\mu_{G'}(Y_\beta)}{\mu_G(Y)}$. Then Hölder's inequality

gives us,

$$\mathbb{E}_{\mathbb{P}_X(\alpha)} \left(\frac{\mu_{G'}(X_\alpha)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{\frac{1}{n}} \geq \left(\frac{\mu_G(X)}{\mu_G(XYg_\beta^{-1})} \right)^{\frac{1}{n}} = \left(\frac{\mu_G(X)}{\mu_G(XY)} \right)^{\frac{1}{n}},$$

and

$$\mathbb{E}_{\mathbb{P}_Y(\beta)} \left(\frac{\mu_{G'}(Y_\beta)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{\frac{1}{n}} \geq \left(\frac{\mu_G(X)}{\mu_G(g_\alpha^{-1}XY)} \right)^{\frac{1}{n}} = \left(\frac{\mu_G(Y)}{\mu_G(XY)} \right)^{\frac{1}{n}}.$$

Hence by (6.4),

$$\begin{aligned} & \left(\frac{\nu_G(X)}{\nu_G(XY)} \right)^{\frac{1}{n}} + \left(\frac{\mu_G(Y)}{\mu_G(XY)} \right)^{\frac{1}{n}} \\ & \leq \mathbb{E}_{\mathbb{P}_X(\alpha)} \mathbb{E}_{\mathbb{P}_Y(\beta)} \left[\left(\frac{\mu_{G'}(X_\alpha)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{1/n} + \left(\frac{\mu_{G'}(Y_\beta)}{\mu_{G'}(X_\alpha Y_\beta)} \right)^{1/n} \right] \leq 1, \end{aligned}$$

and thus G also satisfies BM(n). □

7 Proof of Theorems 1.1, 1.2, and 1.5 and Corollary 1.6

7.1 A dichotomy lemma. In this subsection, we prove a dichotomy result for the kernel of a continuous homomorphism to $(\mathbb{R}^{>0}, \times)$.

The following lemma records a fact on open maps between locally compact groups.

LEMMA 7.1. *Suppose G, H are locally compact groups, $\phi : G \rightarrow H$ is a continuous and surjective group homomorphism, and there is an open subgroup G' of G such that $\phi|_{G'}$ is open. Then $\phi : G \rightarrow H$ is a quotient map of locally compact groups.*

Proof. By the first isomorphism theorem (Fact A.1.1), it suffices to check that ϕ is open. Suppose U is an open subset of G . Then $U = \bigcup_{a \in G} U \cap aG'$. For each $a \in G$, we have

$$\phi(U \cap aG') = \phi(a)\phi|_{G'}(a^{-1}U \cap G').$$

As $\phi|_{G'}$ is open, $\phi(U \cap aG')$ is open for each $a \in G$. Hence, $\phi(U) = \bigcup_{a \in G} \phi(U \cap aG')$ is open in H , which is the desired conclusion. □

In the next lemma, we present our main dichotomy result.

LEMMA 7.2. *Suppose G is a locally compact group, and $\pi : G \rightarrow (\mathbb{R}^{>0}, \times)$ is a continuous group homomorphism. Then exactly one of the following holds:*

- (1) we have the short exact sequence of locally compact groups

$$1 \rightarrow \ker \pi \rightarrow G \xrightarrow{\pi} (\mathbb{R}^{>0}, \times) \rightarrow 1;$$

(2) $\ker \pi$ is an open subgroup of G .

Proof. It is easy to see that (1) and (2) are mutually exclusive, so we need to prove that we are always either in (1) or (2). Consider first the case when G is a Lie group. Let G_0 be the identity component of G . Then G_0 is open by Fact D.2. Hence $\pi(G_0)$ is a connected subgroup of $(\mathbb{R}^{>0}, \times)$. As the only connected subsets of $(\mathbb{R}^{>0}, \times)$ are points and intervals, we deduce that $\pi(G_0)$ can only be $\{1\}$ or $(\mathbb{R}^{>0}, \times)$. In the former case, $\ker \pi$ is open as a union of translates of G_0 . Now suppose $\pi(G_0) = (\mathbb{R}^{>0}, \times)$. Since G_0 is a connected Lie group. Using the first isomorphism theorem for Lie groups (Fact D.3.1), we get $\pi|_{G_0}$ is open. Applying Lemma 7.1, we get that π is a quotient map as desired.

We now deal with the general situation where G is locally compact. Using the Gleason–Yamabe Theorem (Fact C.2.1), we obtain an almost-Lie open subgroup G' of G . Since G' is open, the natural embedding of $i : G' \rightarrow G$ induces a continuous homomorphism $\pi|_{G'} : G' \rightarrow (\mathbb{R}^{>0}, \times)$. Note that there is a compact normal subgroup H of G' such that G'/H is a Lie group. Then $H \leq \ker(\pi|_{G'})$ since $\pi|_{G'}(H)$ is a compact subgroup of $(\mathbb{R}^{>0}, \times)$. Let $\phi : G' \rightarrow G'/H$ be the quotient map. Hence the homomorphisms induce a continuous group homomorphism ψ from G'/H to $(\mathbb{R}^{>0}, \times)$.

$$\begin{array}{ccc} G' & \xrightarrow{\phi} & G'/H \\ \downarrow i & \searrow \pi|_{G'} & \downarrow \psi \\ G & \xrightarrow{\pi} & (\mathbb{R}^{>0}, \times) \end{array}$$

Note that the above diagram commutes. By the proven special case for Lie groups, we then either have the exact sequence

$$1 \rightarrow \ker \psi \rightarrow G'/H \rightarrow (\mathbb{R}^{>0}, \times) \rightarrow 1$$

or $\ker \psi$ is open in G'/H . In the former case, $\pi|_{G'}$ is open as a composition of open maps. By Lemma 7.1, we conclude that π is a quotient map in this case. In the latter case, $\ker(\pi|_{G'})$ is open in G' . Therefore $\ker \pi$ is open in G because $\ker \pi$ is a union of translations of $\ker(\pi|_{G'})$. \square

The modular function $\Delta_G : G \rightarrow (\mathbb{R}^{>0}, \times)$ is a continuous group homomorphism by Fact B.2.2, but generally not a quotient map. It is easy to construct examples where $G/(\ker \Delta_G)$ is discrete. The above proposition claims that these are the only two possibilities, which will be used in the later proofs.

7.2 Proofs of the main theorems. In this subsection, we prove Theorems 1.1 and 1.2. For the reader's convenience, Proposition 7.3 gathers together all the induction steps we can do using the earlier results with the exception of Proposition 6.2, which will be used in the proof of Theorem 1.1 directly.

PROPOSITION 7.3. *Let G be a locally compact group with noncompact Lie dimension n and helix dimension h . Then G satisfies $\text{BM}(n - h)$ if one of the following assumptions holds:*

- (1) *Let $\Delta_G : G \rightarrow (\mathbb{R}^{>0}, \times)$ be the modular function of G . With n' and h' the noncompact Lie dimension and the helix dimension of the locally compact group $\ker \Delta_G$ respectively, the locally compact group $\ker \Delta_G$ satisfies $\text{BM}(n' - h')$.*
- (2) *G is unimodular, G' is an open subgroup of G , n' and h' are the noncompact Lie dimension and the helix dimension of G' respectively, and the locally compact group G' satisfies $\text{BM}(n' - h')$.*
- (3) *G is unimodular, H is a compact normal subgroup of G , n' and h' are the noncompact Lie dimension and the helix dimension of G/H respectively, and the locally compact group G/H satisfies $\text{BM}(n' - h')$.*
- (4) *There is an exact sequence of connected semisimple Lie groups*

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1,$$

n_1 and h_1 are the noncompact Lie dimension and the helix dimension of H respectively, n_2 and h_2 are the noncompact Lie dimension and the helix dimension of G/H respectively, and the locally compact groups H and G/H satisfy $\text{BM}(n_1 - h_1)$ and $\text{BM}(n_2 - h_2)$ respectively.

- (5) *There is an exact sequence of connected unimodular Lie groups*

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$$

n_1 and h_1 are the noncompact Lie dimension and the helix dimension of H respectively, n_2 and h_2 are the noncompact Lie dimension and the helix dimension of G/H respectively, $h_1 = 0$, $h_2 = h$, and the locally compact groups H and G/H satisfy $\text{BM}(n_1 - h_1)$ and $\text{BM}(n_2 - h_2)$ respectively.

Proof. We first prove (1). Note that by Fact B.2.1, $\ker \Delta_G$ is unimodular. By Lemma 7.2, we either have the exact sequence of locally compact groups

$$1 \rightarrow \ker \Delta_G \rightarrow G \rightarrow (\mathbb{R}^{>0}, \times) \rightarrow 1$$

or $\ker \Delta_G$ is open in G . In the former case, by Proposition 2.14, we have $n = n' + 1$ and $h = h'$. Hence, in this case G satisfies $\text{BM}(n - h)$ by Proposition 5.2. In the latter case, by Corollary 2.9, $n = n'$ and $h = h'$. Here, we have that G satisfies $\text{BM}(n - h)$ by Proposition 6.3.

Next we prove (2). By Corollary 2.9, we have $n = n'$ and $h = h'$. The desired conclusion then follows from Proposition 6.4.

We now prove (3). By Corollary 2.10, we have $n = n'$ and $h = h'$. Also by Corollary 2.10, the compact group H has noncompact Lie dimension and helix dimension 0. Hence, using Proposition 4.2, we obtain the conclusion that we want.

We prove (4). By Proposition 2.12.1 and Proposition 2.12.2 respectively, we have $n = n_1 + n_2$ and $h = h_1 + h_2$. Recall that semisimple groups are unimodular. Using Proposition 4.2, we learn that G satisfies $\text{BM}(n - h)$.

Finally, we prove (5). By Proposition 2.12.1, we have $n = n_1 + n_2$. Since the helix dimension of H is 0, and the helix dimension of G/H is h , by Proposition 4.2, G satisfies $\text{BM}(n - h)$. \square

The following corollary says that when G is a Lie group, we can further reduce the problem to connected unimodular groups.

COROLLARY 7.4. *Let G be a Lie group with noncompact Lie dimension n and helix dimension h . Let $\Delta_G : G \rightarrow (\mathbb{R}^{>0}, \times)$ be the modular function of G . Let $G' = (\ker \Delta_G)_0$ be the identity component of $\ker \Delta_G$ with noncompact Lie dimension n' and helix dimension h' . Then G' is connected and unimodular, and if G' satisfies $\text{BM}(n' - h')$, G satisfies $\text{BM}(n - h)$.*

Proof. Note that $(\ker \Delta_G)_0$ is open in $\ker \Delta_G$ by Fact D.2. The desired conclusion is then a consequence of Proposition 7.3.1 and Proposition 7.3.2. \square

Now we are able to prove the main inequality (1.4) for Lie groups. As mentioned earlier, the main strategy is induction on dimension.

Proof of Theorem 1.1. Consider first the case where G is a solvable Lie group. Using Corollary 7.4, we can also assume that G is connected and unimodular. Recall that d is the topological dimension of G . The case when $d = 0$ or 1 is trivial, as every group satisfies $\text{BM}(0)$, and the one dimensional solvable Lie group is either \mathbb{T} or \mathbb{R} by Fact D.5.1. If G is abelian, then it is isomorphic to $\mathbb{T}^m \times \mathbb{R}^{d-m}$. We get the desired conclusion by applying Proposition 7.3.5 repeatedly. Otherwise, from the solvability of G we get the exact sequence

$$1 \rightarrow \overline{[G, G]} \rightarrow G \rightarrow G/\overline{[G, G]} \rightarrow 1$$

with both $\overline{[G, G]}$ and $G/\overline{[G, G]}$ connected, solvable and having smaller dimensions than G . Note that $G/\overline{[G, G]}$ is abelian, and hence unimodular. Applying Proposition 7.3.5, and the statement of the theorem for abelian Lie groups, we get the desired conclusion for this case.

Consider next the case where G is connected and semisimple. We may further assume that G is a connected simple Lie group, otherwise by Fact E.6, we can always find a connected group $H \triangleleft G$ such that both H and G/H are connected semisimple Lie groups with lower dimension; by Proposition 7.3.4, the Brunn–Minkowski inequality on G can be obtained from the Brunn–Minkowski inequalities on H and G/H . Now we write $G = KAN$ as in Fact E.14. We first consider the case when G has a finite center, and then K is compact. Let n be the noncompact Lie dimension of G . Hence, n is the dimension of the solvable Lie group $Q = AN$. Note that A and N are simply connected by Fact E.14. Hence their noncompact Lie dimensions are

the same as their dimensions by Fact D.5.2. By Proposition 2.12.1 and Fact E.14, the noncompact Lie dimension of Q is n , and hence Q satisfies $\text{BM}(n)$ from the solvable Lie case. We obtain the desired conclusion for G by applying Proposition 6.2.

Suppose the connected simple Lie group G has a center of rank $h \geq 1$. We again apply Proposition 6.2 and obtain inequality (1.4) for G with exponent $\dim(AN)$. By Proposition 2.3, we have $\dim(AN) = n - h$. The desired conclusion for the connected semisimple Lie groups follows similarly from Fact E.6 and Proposition 7.3.4.

Finally, we prove the statement for an arbitrary Lie group G . Using Corollary 7.4 again, we can assume that G is connected and unimodular. Then by Fact E.5 we obtain an exact sequence

$$1 \rightarrow Q \rightarrow G \rightarrow S \rightarrow 1,$$

where Q is a connected unimodular solvable group and S is a connected semisimple Lie group. We then apply Proposition 7.3.5 and the earlier two cases to get the desired conclusion. \square

Finally, we prove the inequality (1.4) for all locally compact groups.

Proof of Theorem 1.2. By Proposition 7.3.1, replacing G by $\ker \Delta_G$ if necessary, we can assume that G is unimodular. By the Gleason–Yamabe Theorem (Fact C.2.1), G has an almost-Lie open subgroup. Now using Proposition 7.3.2, we can further assume that G is a unimodular almost-Lie group. Then we can choose a compact subgroup K of G such that G/K is a unimodular Lie group. The desired conclusion then follows from Theorem 1.1 and Proposition 7.3.3. \square

We briefly discuss Theorem 1.5, which is a consequence of the proof of Theorem 1.2.

Proof of Theorem 1.5. Repeating the arguments in the proofs of Proposition 7.3, Corollary 7.4, Theorem 1.1, Theorem 1.2, and Fact E.6 while ignoring the helix dimension, it suffices to prove the theorem when G is a simple Lie group.

From the hypothesis, we already have the desired conclusion under the further assumption that our simple Lie group G is also simply connected. We now consider the general case. If G has finite center, the result is a special case of Theorem 1.1. So suppose the center $Z(\tilde{G})$ of G is infinite. Let \tilde{G} be the universal cover of G , $Z(\tilde{G})$ its center, and $\rho : \tilde{G} \rightarrow G$ the covering map. Then $\ker \rho$ is a subgroup of $Z(\tilde{G})$ by Fact E.10. Using Fact E.12, the center $Z(\tilde{G})$ has rank at most 1. By the earlier assumption, the center $Z(G)$ also has rank at least 1. Hence, by Fact E.10, both $Z(\tilde{G})$ and $Z(G)$ must have rank 1, and $\ker \rho$ is finite. Therefore, the desired conclusion for G can be reduced to that of \tilde{G} by taking the inverse image under ρ , which we already know from the hypothesis. \square

Proof of Corollary 1.6. Let G be a noncompact semisimple Lie group. By Proposition 2.12.1, Fact E.14.3 and Fact E.14.4, and Fact E.15, the noncompact Lie dimension of G is at least 2. The conclusion then follows from Theorem 1.1. \square

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Appendix A. Some results about topological groups

This section gathers some facts about topological groups which are needed in the proof. We begin with the three isomorphism theorems of topological groups. Note that the third isomorphism theorem is almost the same as the familiar result for groups, whereas the first two isomorphism theorems require extra assumptions; see [Bou98, Proposition III.2.24], [Bou98, Proposition III.4.1], and [Bou98, Proposition III.2.22] for details. For Fact A.1, we do not need to assume that G is locally compact. The quotient G/H is equipped with the quotient topology (i.e., $X \subseteq G/H$ is open if and only if its inverse image under the quotient map is open).

Fact A.1. *Suppose H is a closed normal subgroup of G . Then we have the following:*

- (1) (First isomorphism theorem) *Suppose $\phi : G \rightarrow Q$ is a continuous surjective group homomorphism with $\ker \phi = H$. Then the exact sequence of groups*

$$1 \rightarrow H \rightarrow G \rightarrow Q \rightarrow 1$$

is an exact sequence of topological groups if and only if ϕ is open; the former condition is equivalent to saying that Q is canonically isomorphic to G/H as topological groups.

- (2) (Second isomorphism theorem) *Suppose S is a closed subgroup of G and H is compact. Then $S/(S \cap H)$ is canonically isomorphic to the image of SH/H as topological groups. This is equivalent to saying that we have the exact sequence of topological groups*

$$1 \rightarrow H \rightarrow SH \rightarrow S/(S \cap H) \rightarrow 1.$$

- (3) (Third isomorphism theorem) *Suppose $S \leq G$ is closed, and $H \leq S$. Then S/H is a closed subgroup of G/H . If $S \triangleleft G$ is normal, then S/H is a normal subgroup of G/H , and we have the exact sequence of topological groups*

$$1 \rightarrow S/H \rightarrow G/H \rightarrow G/S \rightarrow 1;$$

this is the same as saying that $(G/H)/(S/H)$ is canonically isomorphic to G/S as topological groups.

Recall that a topological space X is **completely regular** if points can be separated from closed sets via continuous real-valued functions. More precisely, for any closed set $C \subseteq X$ and any point $a \in X \setminus C$, there exists a real-valued continuous function $f : X \rightarrow \mathbb{R}$ such that $f(a) = 1$ and $f(c) = 0$ for all $c \in C$.

Fact A.2. *The underlying topological space of a topological group is completely regular.*

We also need the following simple property of locally compact groups [Fol95, Theorem 6.7].

Fact A.3. *Closed subgroups and quotient groups of locally compact groups are locally compact.*

The following lemma holds for all topological groups.

LEMMA A.4. *Suppose $X, Y \subseteq G$, X is compact and Y is closed. Then XY is closed.*

Proof. Let a be in $G \setminus XY$. Then $X^{-1}a$ is compact and $X^{-1}a \cap Y = \emptyset$. For each point $x \in X^{-1}a$, we choose an open neighborhood of identity U_x such that $xU_x^2 \cap Y = \emptyset$. Then $(xU_x)_{x \in X^{-1}a}$ is an open cover of $X^{-1}a$. Using the fact that $X^{-1}a$ is compact, we get a subcover $(U_i)_{i=1}^k$. Set $U = \bigcap_{i=1}^k U_i$. It is easy to check that $X^{-1}aU \cap Y = \emptyset$. Then $aU \cap XY = \emptyset$, which implies that XY is closed as a can be chosen arbitrarily. \square

The next lemma records a simple fact of compact subgroups.

LEMMA A.5. *If H is a compact subgroup of G , then the quotient map $\pi : G \rightarrow G/H$ is a proper map (i.e., the inverse images of compact subsets are compact).*

Proof. Let Ω be a compact subset of G/H . In particular Ω is closed. Hence, $\pi^{-1}(\Omega)$ is closed, so it suffices to find a compact set containing $\pi^{-1}(\Omega)$. Since G is locally compact, we can find an open covering $(U_i)_{i \in I}$ of $\pi^{-1}(\Omega)$ such that U_i has compact closure \bar{U}_i for each $i \in I$. Then $(\pi U_i)_{i \in I}$ is an open cover of Ω as π is open. Using the assumption that Ω is compact, we get a finite $I' \subseteq I$ such that $(\pi(U_i))_{i \in I'}$ is an open cover of Ω . Then $\bigcup_{i \in I'} \bar{U}_i H$ is a compact set containing $\pi^{-1}(\Omega)$. \square

Appendix B. Measures and the modular function

We say that a measure μ on the collection of Borel subsets of G is a *left Haar measure* [Fol95, Section 2.2] if it satisfies the following properties:

- (1) (nonzero) $\mu(X) > 0$ for all open $X \subseteq G$;
- (2) (left-translation-invariant) $\mu(X) = \mu(aX)$ for all $a \in G$ and all measurable sets $X \subseteq G$;
- (3) (inner regular for open sets) when X is open, $\mu(X) = \sup \mu(K)$ with K ranging over compact subsets of X ;

- (4) (outer regular for Borel sets) when X is Borel, $\mu(X) = \inf \mu(U)$ and U ranging over open subsets of G containing X ;
- (5) (compactly finite) μ takes finite measure on compact subsets of G .

The notion of a *right Haar measure* is obtained by making the obvious modifications to the above definition. The following classical result by Haar guarantees the existence of such a measure on every locally compact group.

Fact B.1. [Fol95, Theorem 2.20] *Up to multiplication by a positive constant, there is a unique left Haar measure of G . A similar statement holds for right Haar measure.*

Given a locally compact group G , and μ is a left Haar measure on G . For every $x \in G$, recall that $\Delta_G : x \mapsto \mu_x/\mu$ is the *modular function* of G , where μ_x is a left Haar measure on G defined by $\mu_x(A) = \mu(Ax)$, for every measurable set A . When the image of Δ_G is $\{1\}$, we say G is *unimodular*. In general, $\Delta_G(x)$ takes values in $\mathbb{R}^{>0}$. We use $(\mathbb{R}^{>0}, \times)$ to denote the multiplicative group of positive real numbers together with the usual Euclidean topology. The next fact records some basic properties of the modular function; see [Fol95, Section 2.4].

Fact B.2. *Let G be a locally compact group. Assume μ is a left Haar measure and ν is a right Haar measure.*

- (1) *Suppose H is a normal closed subgroup of G , then $\Delta_H = \Delta_G|_H$. In particular, if $H = \ker \Delta_G$, then H is unimodular.*
- (2) *The function $\Delta_G : G \rightarrow (\mathbb{R}^{>0}, \times)$ is a continuous homomorphism.*
- (3) *For every $x \in G$ and every measurable set A , we have $\mu(Ax) = \Delta_G(x)\mu(A)$, and $\nu(xA) = \Delta_G^{-1}(x)\nu(A)$.*
- (4) *There is a constant c such that $\int_G f \, d\mu = c \int_G f \Delta_G \, d\nu$ for every $f \in C_c(G)$.*

We use the following integral formula [Fol95, Theorem 2.49] in our proofs.

Fact B.3. (Quotient integral formula) *Let G be a locally compact group, and let H be a closed normal subgroup of G . Let μ_G and μ_H be left Haar measures on G and on H , respectively. Then there is a unique left Haar measure $\mu_{G/H}$ on G/H , such that for every $f \in C_c(G)$,*

$$\int_G f(x) \, d\mu_G(x) = \int_{G/H} \int_H f(xh) \, d\mu_H(h) \, d\mu_{G/H}(xH).$$

The following fact is a consequence of a result of Haar measure on closed subgroups and quotients [Bou04, Proposition VII. 2.7.10].

Fact B.4. *Suppose G is nonunimodular, and $\Delta_G : G \rightarrow (\mathbb{R}^{>0}, \times)$ is the modular function of G , then we have the following:*

- (1) *If $K \triangleleft G$ is a compact normal subgroup of G , $\Delta_{G/K}$ is the modular function of G/K , and $\pi : G \rightarrow G/K$ is the quotient map, then we have $\Delta_G = \Delta_{G/K} \circ \pi$.*
- (2) *If $H \triangleleft G$ is a closed unimodular group, and μ_H is a Haar measure on H . Suppose G/H is unimodular, and X is a compact subset of H . Then for every $g \in G$, $\mu_H(gXg^{-1}) = \Delta_G(g)\mu_H(X)$.*

Appendix C. Almost-Lie groups and the Gleason–Yamabe Theorem

In our proof we need the solution of Hilbert’s 5th problem, which is known as the Gleason–Yamabe Theorem [Gle52, Yam53], to reduce the problem into Lie groups. For convenience, we introduce the following terminology. A locally compact group G is an **almost-Lie group** if every open neighborhood U of the identity in G contains a compact $H \triangleleft G$ such that G/H is a Lie group.

LEMMA C.1. *Suppose G is an almost-Lie group. Then every open subgroup of G and every quotient of G by a closed normal subgroup is an almost-Lie group.*

Proof. We first show that every open subgroup of G is almost-Lie. Let S be an open subgroup of G , and U be an open neighborhood of identity in S . We need to find a compact subgroup K of S such that $K \subseteq U$ and S/K is a Lie group. Since U is also a neighborhood of identity in G , U contains a compact normal subgroup K of G such that G/K is a Lie group. Note that $K \triangleleft S$. As S is open, S/K is open in G/K and hence a Lie group as desired.

Next, suppose H is a closed normal subgroup of G , and $\pi : G \rightarrow G/H$ is the quotient map. If U is an open neighborhood of the identity in G/H , then $\pi^{-1}(U)$ is an open neighborhood of identity in G . Hence, we can get a normal compact subgroup K of G such that $K \subseteq \pi^{-1}(U)$ and that G/K is a Lie group. Then $\pi(K)$ is a compact subgroup of U . With $S = \pi^{-1}(\pi(K))$, we have $\pi(K) = S/H$. Since K is normal in G we have $\pi(K)$ is normal in G/H and thus S is normal in G . Whence by the third isomorphism theorem (Fact A.1.3), we conclude that $(G/H)/\pi(K) \cong G/S$. By the third isomorphism theorem again, we have $G/S \cong (G/K)/(S/K)$, thus G/S is a Lie group. \square

We use the following strong version of the Gleason–Yamabe Theorem.

Fact C.2. *We have the following:*

- (1) (Gleason–Yamabe Theorem) *Suppose G is a locally compact group. Then there is an open subgroup of G which is an almost-Lie group.*
- (2) *An almost-Lie group G is a Lie group if and only if there is an open neighborhood U of the identity in G that contains no nontrivial compact subgroup of G .*

Fact C.2.2 is not officially part of the Gleason–Yamabe Theorem. However, the forward direction is an easy fact about the no small subgroup property of Lie groups, and the backward direction is a direct consequence of Fact C.2.1.

Appendix D. Some results about Lie groups

In this section, we gather some facts and lemmas about Lie groups and Lie algebras. Throughout the paper, all Lie groups are finite dimensional second countable *real* Lie groups.

Fact D.1. *Closed subgroups and quotient groups of Lie groups are Lie groups.*

The *identity component* of a topological group G is the connected component containing the identity element. The identity component of a topological group G might not be open even if G is locally compact. For instance, there are nondiscrete totally disconnected locally compact groups. For these groups, the identity component only consists of the identity element, and it is not open because the topology is not discrete. Nevertheless, the following holds for Lie groups [HN12, Proposition 9.1.15].

Fact D.2. *If G is a Lie group, then the identity component of G is open and is contained in every open subgroup of G .*

In Fact A.1, we introduce the three isomorphism theorems of topological groups. When G is a Lie group, we can weaken the assumption required for the first two isomorphism theorems; see [Bou89, Proposition 3.11.2, Proposition 3.31].

Fact D.3. *Suppose G is a Lie group, and H is a closed normal subgroup of G . Then we have the following:*

- (1) (First isomorphism theorem for Lie groups) *Suppose Q is a Lie group, $\phi : G \rightarrow Q$ is a surjective and continuous group homomorphism, and G has countably many connected components. Then Q is isomorphic as a topological group to G/H .*
- (2) (Second isomorphism theorem for Lie groups) *Suppose G is a finite dimensional Lie group, S is a closed subgroup of G , and SH is a closed subgroup of G . Then $S/(S \cap H)$ is canonically isomorphic to the image of SH/H as Lie groups. This is also equivalent to saying that we have the exact sequence of Lie groups*

$$1 \rightarrow H \rightarrow SH \rightarrow S/(S \cap H) \rightarrow 1.$$

We also need the following fact about maximal compact subgroups consisting of Theorem 14.1.3 (iii) and Theorem 14.3.13 (i) (a) of [HN12]:

Fact D.4. *Suppose G is a Lie group with finitely many connected components. Then we have the following:*

- (1) *All maximal compact subgroups of G are conjugate.*
- (2) *If $0 \rightarrow H \rightarrow G \xrightarrow{\pi} G/H \rightarrow 0$ is an exact sequence of connected Lie groups, and K is a maximal compact subgroup of G , then $K \cap H$ is a maximal compact subgroup of H , and $\pi(K)$ is a maximal compact subgroup of G/H .*

We also use the following classification results for Lie groups. Fact D.5 can be found on [Vin94, p. 212]. Fact D.5(2) is from Corollary 3 of [Vin94, Theorem 2.3.1] and [Vin94, Proposition 4.1.2].

Fact D.5. *Let G be a connected Lie group.*

- (1) *If G has dimension 1, then it is isomorphic to either \mathbb{R} or \mathbb{T} as a topological group.*
- (2) *Suppose G is a solvable group with dimension d , and the maximal compact subgroups of G have dimension m . Then G is diffeomorphic to $\mathbb{T}^m \times \mathbb{R}^{d-m}$. Moreover, if G is compact, then $G \cong \mathbb{T}^d$.*

We say that a topological group G (not necessarily simply connected) is a covering group of a topological group G' with covering homomorphism ρ if $\rho : G \rightarrow G'$ is a topological group homomorphism which is also a covering map. The following is a consequence of [HN12, Theorem 9.5.4]:

Fact D.6. *Suppose that G and G' are connected Lie groups and that G is a covering group of G' with covering homomorphism ρ . Then $\ker \rho$ is a closed normal subgroup of the center $Z(G)$ of G .*

We end this section with a lemma about conjugate actions on compact sets in Lie groups.

LEMMA D.7. For a Lie group G and a closed normal subgroup H , if a precompact $A \subseteq H$ such that the closure of A is contained in B and B is a relative open subset in H , then the following holds: when $g \in G$ is sufficiently close to id_G , we have $gAg^{-1} \subseteq B$.

Proof. We prove the lemma by contradiction. Assuming there exist sequences $g_n \rightarrow \text{id}$ and $\{h_n\} \subseteq A$ such that $g_n h_n g_n^{-1} \notin B$. Since A is precompact we may assume $h_n \rightarrow h \in \bar{A}$. But then $g_n h_n g_n^{-1} \rightarrow h \in \bar{A}$. This contradicts the fact that each $g_n h_n g_n^{-1}$ is in the closed set $H \setminus B$ that does not meet \bar{A} . Hence the assumption is false and the conclusion holds. \square

Appendix E. Solvable and Semisimple Lie groups

From [HN12, Section 9.1], there is a functor \mathbf{L} from the category of Lie groups to the category of Lie algebras that assigns to each Lie group G its Lie algebra $\mathbf{L}(G)$ and to a Lie group morphism $\phi : G \rightarrow H$ its tangent morphism $\mathbf{L}(\phi) : \mathbf{L}(G) \rightarrow \mathbf{L}(H)$ of Lie algebras. We will adopt a more colloquial language in this paper, invoking this functor implicitly.

Fact E.1. Suppose G and H are Lie groups, and \mathfrak{g} and \mathfrak{h} are their Lie algebras. If H is a subgroup of G , then \mathfrak{h} is a subalgebra of \mathfrak{g} . If H is a normal subgroup of G , then \mathfrak{h} is an ideal in \mathfrak{g} , and $\mathfrak{g}/\mathfrak{h}$ is canonically isomorphic to the Lie algebra of G/H .

Suppose \mathfrak{g} is the Lie algebra of G . The exponential function $\exp : \mathfrak{g} \rightarrow G$ is defined as in [HN12, Section 9.2]. We will use the functoriality of the exponential function [HN12, Proposition 9.2.10]

Fact E.2. Suppose G and H are Lie groups, $\phi : G \rightarrow H$ is a homomorphism of Lie groups, \mathfrak{g} and \mathfrak{h} are the Lie algebras of G and H , $\alpha : \mathfrak{g} \rightarrow \mathfrak{h}$ is the tangent morphism of ϕ , and $\exp_G : \mathfrak{g} \rightarrow G$ and $\exp_H : \mathfrak{h} \rightarrow H$ are the exponential maps. Then $\exp_H \circ \alpha = \exp_G \circ \phi$. In other words, the following diagram commutes:

$$\begin{array}{ccc}
 G & \xrightarrow{\phi} & H \\
 \exp_G \uparrow & & \uparrow \exp_H \\
 \mathfrak{g} & \xrightarrow{\alpha} & \mathfrak{h}
 \end{array}$$

Suppose \mathfrak{g} is a Lie algebra. The *derived Lie algebra* $[\mathfrak{g}, \mathfrak{g}]$ of \mathfrak{g} is the subalgebra of \mathfrak{g} generated by the Lie brackets of the pairs of elements of \mathfrak{g} . We say that \mathfrak{g} is *solvable* if the derived sequence

$$\mathfrak{g} \geq [\mathfrak{g}, \mathfrak{g}] \geq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \geq \dots$$

eventually arrives at the trivial Lie algebra $\{0\}$. A Lie group is *solvable* if its Lie algebra is solvable. The following is a consequence of [HN12, Proposition 5.4.3]:

Fact E.3. Every subalgebra and quotient algebra of a solvable Lie algebra is solvable. Hence, every closed subgroup and quotient group of a solvable Lie group is solvable.

The following is another consequence of [HN12, Proposition 5.4.3, Theorem 5.6.6]:

Fact E.4. Suppose \mathfrak{g} is a Lie algebra. Then \mathfrak{g} has a largest solvable ideal \mathfrak{q} . If G is a Lie group with Lie algebra \mathfrak{g} and $\exp : \mathfrak{g} \rightarrow G$ is the exponential map, then $Q = \langle \exp(\mathfrak{q}) \rangle$ is the largest closed connected solvable normal subgroup of G . Hence, Q is a characteristic subgroup of G .

The subalgebra \mathfrak{q} as in Fact E.4 is called the *radical* of \mathfrak{g} , and the subgroup Q as in Fact E.4 is called the *radical* of G . A Lie algebra is *semisimple* if it has a trivial radical. A Lie group is *semisimple* if its Lie algebra is semisimple, or equivalently, if it has trivial radical. The following results follows from [HN12, Theorem 5.6.6, Corollary 5.6.14]:

Fact E.5. *Let G be a connected Lie group and let Q be its radical. Then $S = G/Q$ is a semisimple Lie group.*

A Lie algebra is *simple* if it is semisimple and contains no ideals other than itself and the trivial ideal $\{0\}$. A Lie group is *simple* if its Lie algebra is simple. Note that a simple Lie group is not necessarily simple as a group. We use the following fact for simple Lie groups.

Fact E.6. *A connected Lie group G is a simple Lie group if and only if all its normal proper subgroups are discrete, and contained in $Z(G)$.*

Suppose \mathfrak{g} is a finite dimensional Lie algebra. For $x \in \mathfrak{g}$, let $\text{adx} : \mathfrak{g} \rightarrow \mathfrak{g}$, $y \mapsto [x, y]$. Then ad is an endomorphism of \mathfrak{g} . The *Cartan–Killing form* of $\kappa_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is given by

$$\kappa_{\mathfrak{g}}(x, y) = \text{tr}(\text{adx} \text{ady}).$$

The Cartan–Killing form is invariant under an automorphism of \mathfrak{g} ; this follows from a direct computation and the fact that if ρ is an automorphism of \mathfrak{g} , then $\text{ad}\rho(x) = \rho \circ \text{adx} \circ \rho^{-1}$. The following fact is from [HN12, Lemma 5.5.8].

Fact E.7. *Suppose \mathfrak{g} is a Lie algebra, $\kappa_{\mathfrak{g}}$ is the Cartan–Killing form of \mathfrak{g} , and \mathfrak{h} is an ideal of \mathfrak{g} . Then the orthogonal space \mathfrak{h}^{\perp} of \mathfrak{h} with respect to $\kappa_{\mathfrak{g}}$ is also an ideal. If \mathfrak{g} is semisimple, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ and $\kappa_{\mathfrak{g}} = \kappa_{\mathfrak{h}} \oplus \kappa_{\mathfrak{h}^{\perp}}$ where $\kappa_{\mathfrak{h}}$ and $\kappa_{\mathfrak{h}^{\perp}}$ are the Cartan–Killing form of \mathfrak{h} and \mathfrak{h}^{\perp} .*

The following fact follows from [HN12, Lemma 5.5.13]. It is also a consequence of Fact E.7 and the alternative characterization of semisimple Lie algebras as those whose Cartan–Killing form is nondegenerate.

Fact E.8. *Every ideal and quotient algebra of a semisimple Lie algebra is semisimple. Hence, every closed normal subgroup and quotient group of a semisimple Lie group is semisimple.*

The first and second assertions in the following fact are immediate consequences of Facts E.4, E.3, E.8

Fact E.9. *If G is a connected semisimple Lie group, then its center $Z(G)$ is a finitely generated discrete group, the quotient map $\rho : G \rightarrow G/Z(G)$ is a covering map.*

The following fact is a consequence of [HN12, Proposition 9.5.2 and Theorem 9.5.4].

Fact E.10. *If G and G' are connected Lie groups, $\rho : G \rightarrow G'$ is covering map, $Z(G)$ and $Z(G')$ are the centers of G and G' . Then we have $\ker \rho \leq Z(G)$ and $Z(G') = Z(G)/\ker \rho$.*

The first assertion in the following fact is known as Weyl’s theorem on Lie groups with semisimple compact Lie algebra [HN12, Theorem 12.1.17].

Fact E.11. *If G is a connected semisimple Lie group with compact Lie algebra, then G is compact and $Z(G)$ is finite.*

The following fact is a consequence of Fact E.11 and the result in [Sir60]. This can also be proven directly using [HN12, Proposition 13.1.10 (ii)]; we thank Jinpeng An for pointing this out to us.

Fact E.12. *If G is a simply connected simple Lie group, then the center $Z(G)$ of G has rank at most 1.*

Suppose \mathfrak{g} is a finite dimensional Lie algebra with Cartan–Killing form $\kappa_{\mathfrak{g}}$. A Lie algebra automorphism τ of \mathfrak{g} is a *Cartan involution* if $\tau^2 = \text{id}_{\mathfrak{g}}$ and $(x, y) \mapsto -\kappa_{\mathfrak{g}}(x, \tau(y))$ is a positive definite bilinear form. The following fact is [HN12, Theorem 13.2.10]

Fact E.13. *Let \mathfrak{g} be a semisimple Lie algebra. Then \mathfrak{g} has a Cartan involution τ .*

We refer the reader to [Kna02, Section 6.4] for the full definition of Iwasawa decomposition; we will need the following fact which is a consequence of [Kna02, Theorem 6.31, Theorem 6.46] and [HN12, Theorem 13.3.8].

Fact E.14 (Iwasawa decomposition). *Suppose G is a connected semisimple Lie group with Lie algebra \mathfrak{g} , τ is a Cartan involution of \mathfrak{g} , \mathfrak{k} the subalgebra of \mathfrak{g} fixed by τ , and $\exp : \mathfrak{g} \rightarrow G$ is the exponential map. Then there is an Iwasawa decomposition $G = KAN$ such that the following holds:*

(1) *the multiplication map*

$$\Phi : K \times A \times N \rightarrow G : (k, a, n) \mapsto kan$$

is a diffeomorphism.

- (2) *$K = \exp(\mathfrak{k})$ is a connected closed subgroup of G , $Z(G) \subseteq K$, and K is a maximal compact subgroup of G if $Z(G)$ is finite.*
- (3) *A is an abelian closed subgroup of G , N is a nilpotent closed subgroup of G , and both A and N are simply connected.*
- (4) *$Q = AN$, we have that Q is a solvable closed subgroup of G , and $N \triangleleft Q$.*

The following fact is a consequence of the definition of Iwasawa decomposition in [Kna02, Section 6.4].

Fact E.15. *If G is a noncompact semisimple Lie group with Iwasawa decomposition $G = KAN$, then AN has dimension at least 2.*

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