

LOCALLY TOROIDAL POLYTOPES OF RANK 6 AND SPORADIC GROUPS

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ABSTRACT. We augment the list of finite universal locally toroidal regular polytopes of type $\{3, 3, 4, 3, 3\}$ due to P. McMullen and E. Schulte, adding as well as removing entries. This disproves a related long-standing conjecture. Our new universal polytope is related to a well-known Y -shaped presentation for the sporadic simple group Fi_{22} , and admits $S_4 \times O_8^+(2):S_3$ as the automorphism group. We also discuss further extensions of its quotients in the context of Y -shaped presentations. As well, we note that two known examples of finite universal polytopes of type $\{3, 3, 4, 3, 3\}$ are related to Y -shaped presentations of orthogonal groups over \mathbb{F}_2 . Mixing construction is used in a number of places to describe covers and 2-covers.

1. INTRODUCTION AND RESULTS

Presentations of finite sporadic simple groups as quotients of Coxeter groups with diagram $Y_{\alpha\beta\gamma}$, cf. Fig. 1(a), were discovered 30 years ago [6, 7, 23] and remain a subject of considerable interest, cf. e.g. [24, 8, 14, 3, 1, 2]. For instance, the

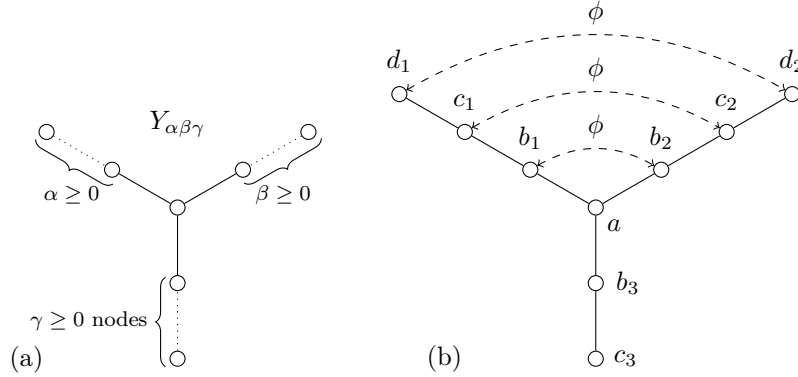
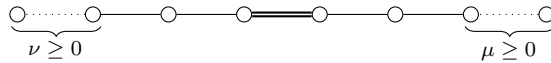


FIGURE 1. Y -diagrams; the general case and Y_{332} .

sporadic simple group Fi_{22} is a quotient of Y_{332} , cf. Fig. 1(b) modulo relations (2) below, cf. [6, p.233]. As explained in Section 3 below, these relations afford the automorphism ϕ of the diagram swapping b_1 , c_1 , and d_1 with, respectively b_2 , c_2 , and d_2 . Thus Fi_{22} has a subgroup isomorphic to a quotient of the Coxeter group with Schläfli symbol $[3, 3, 4, 3, 3]$, i.e. with the string diagram we will also denote by $[3^2, 4, 3^2]$, cf. Fig. 2.

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FIGURE 2. $[3^{1+\nu}, 4, 3^{1+\mu}]$ -diagram.

Groups of this kind arise in the study of *abstract regular polytopes*, for which the book [17] by P. McMullen and E. Schulte is a definite reference. These objects may be viewed as quotients (satisfying *intersection property* (1), cf. [17]) of Coxeter complexes of Coxeter groups with r -node string diagram; here r is called the *rank*. Abstract regular polytopes are classified according to topology of their vertex figures and facets; they are called *locally* X if the latter have topology of X . An up-to-date review on group-theoretic approaches to the problem of classification of abstract regular polytopes may be found e.g. in [5].

Following a problem posed by B. Grünbaum [12, p.196], particular attention has been paid to finite *locally toroidal* abstract regular polytopes, in this case the rank is bounded from above by 6, cf. [17, Lemma 10A.1].

One of the three rank 6 cases is the case corresponding to the group $[3^2, 4, 3^2]$, cf. [17, Sect. 12D] (or [16, Sect. 7]), where a conjecturally complete list of the finite universal examples is given, see [17, Table 12D1] (which is already in [16, Table V]) and [21, Problem 18]. The main results of the present paper give one more example, missing in that table, and remove erroneous infinite series of examples, of which only first terms actually exist.

In more detail, the facets $\{3, 3, 4, 3\}_s$ and vertex figures $\{3, 4, 3, 3\}_t$ here correspond to nontrivial finite quotients $[3, 3, 4, 3]_s$ and $[3, 4, 3, 3]_t$ of the affine Coxeter group \tilde{F}_4 , i.e. the groups $[3, 3, 4, 3]$ and $[3, 4, 3, 3]$, with normal Abelian subgroups either of the form q^4 or $q^2 \times (2q)^2$, with $q \geq 2$, where we follow notation from [6] to denote the direct product $(\mathbb{Z}/s\mathbb{Z})^k$ of k copies of the cyclic group of order s by s^k . The case q^4 is denoted in [16, 17] by $\{3, 3, 4, 3\}_{(q,0,0,0)}$, and the case $q^2 \times (2q)^2$ by $\{3, 3, 4, 3\}_{(q,q,0,0)}$ (and completely similarly for $\{3, 4, 3, 3\}$).

Theorem 1. Let Γ be a universal locally toroid rank 6 abstract regular polytope with vertex figures of type $\{3, 4, 3, 3\}_{(3000)}$ and facets of type $\{3, 3, 4, 3\}_{(2200)}$. Then Γ is the 24-fold cover of Γ_1 , where Γ_1 has $v = 11200 = 2^6 \cdot 5^2 \cdot 7$ vertices and $f = 14175 = 3^4 \cdot 5^2 \cdot 7$ facets, and the group $\Omega := O_8^+(2):S_3$, of order $g = 2^{13} \cdot 3^6 \cdot 5^2 \cdot 7$. The group of Γ is isomorphic to $S_4 \times \Omega$.

Here $\Gamma = \Gamma_4$ and Γ_1 are the biggest and the smallest member of the sequence of covers Γ_k , with $1 \leq k \leq 4$. Namely, Γ_k is a $k!$ -fold cover of Γ_1 and the group of Γ_k is isomorphic to $S_k \times \Omega$. More details on this are given in Section 3.

A number of comments on Theorem 1 are in order. The claim of finiteness of Γ depends upon coset enumeration, that was, for robustness purposes, carried out using two different implementations of the Todd-Coxeter algorithm: the built-in implementation of GAP system [10], and ACE implementation by G. Havas and C. Ramsay [13], also available as a GAP package [9]. Both computations in the case $t = (3000)$, $s = (2200)$ returned the index of the vertex stabiliser, the subgroup $F := 3^4:F_4$ in the quotient Θ of the Coxeter group $[3^2, 4, 3^2]$ modulo the relations on Fig. 5 equal to $268800 = 2^9 \cdot 3 \cdot 5^2 \cdot 7$. We explain below that on the other hand the identification of the group of Γ_1 does not need coset enumeration, and can be carried out by hand or with any software capable of multiplying 24×24 matrices

over \mathbb{F}_2 , the field of 2 elements. It is worth mentioning that both F and the facet stabiliser $(2^2.4^2):F_4$ are maximal subgroups in Ω , although only F remains maximal upon restricting to the simple subgroup $O_8^+(2)$ of Ω .

In Section 3 we sketch how to establish the existence of the group of Γ_1 as a subgroup in Fi_{22} . Apart from this, we discuss there other connections to the Y-shaped presentations already mentioned, and pose a number of related open problems. In Section 4 we give another proof of existence of Γ_1 and its covers Γ_k for $2 \leq k \leq 4$ by a computer-free argument. The latter is an example of an application of Proposition 3 below.

Fi_{22} is not the only quotient of the Coxeter group Y_{332} related to polytopes of type $\{3, 3, 4, 3, 3\}$. Namely, one also finds groups for such polytopes within the 2-modular quotient G of Y_{332} , which is isomorphic to $2^8:O_8^-(2):2$, as established by R. Griess in [11], where a general study of p -modular quotients of $Y_{\alpha\beta\gamma}$ is carried out; in case $p = 2$ one speaks of subgroups of $GL_{1+\alpha+\beta+\gamma}(2)$ generated by *transvections*—images of the generators of $Y_{\alpha\beta\gamma}$. One such polytope is described in the following result, and the other one is its 2-quotient. The group G also appears on [6, pp.232–233], as a quotient of X_{332} , a re-labelling of Y_{332} . See Section 5 for details.

Theorem 2. Let Γ be a locally toroid rank 6 abstract regular polytope with vertex figures of type $\{3, 4, 3, 3\}$ and facets of type $\{3, 3, 4, 3\}_{(2000)}$. Then its vertex figures are either of type $\{3, 4, 3, 3\}_{(2200)}$, and Γ has 128 facets, 32 vertices, and the group of order $2^{18} \cdot 3^2$, or its vertex figures are of type $\{3, 4, 3, 3\}_{(2000)}$, and Γ has 32 facets, 32 vertices, and the group of order $2^{15} \cdot 3^2$.

Basically, enumeration of the cosets of a facet stabiliser in the quotient of the Coxeter group $[3^2, 4, 3^2]$ by a relation forcing the facets $\{3, 3, 4, 3\}_{(2000)}$ shows that this group is finite, of order $2^{18} \cdot 3^2$. This *proves* that the vertex figures are of type $\{3, 4, 3, 3\}_{(2200)}$ (unless further relations are imposed). This identifies Γ with the second entry in [17, Table 12D1] (or [16, Table V]) for $t = 2$, i.e. $\mathbf{t} = (2200)$.

The quotient by a normal subgroup 2^3 produces the other case, $\mathbf{t} = (2000)$, the first entry in [loc.cit.] for $t = 2$. It shows that the cases with $t > 2$ in [loc.cit.] in fact do not arise. We will sketch alternative computer-free approaches to a proof of this part of the theorem in Section 5.

No novelty is claimed for the second part of Theorem 2 (e.g. it is already in [16]), it is added for the sake of completeness.

Currently known (up to taking duals) examples of universal finite polytopes of type $\{3, 3, 4, 3, 3\}$ are listed in Table 1. We also give there information on numbers of vertices v , facets f , groups, and whether there exists $\Gamma \in \langle \{3, 3, 4, 3\}_{\mathbf{s}}, \{3, 4, 3, 3\}_{\mathbf{t}} \rangle$ with group embedded as a subgroup in a “Y-overgroup“, i.e. in a quotient of Y_{332} over relations affording ϕ , as shown on Figure 1(b). The last example in the table is new, the rest already appear in [16].

Note that Lemma 13 gives a construction of a finite polytope (not known to be universal at present) with $\mathbf{s} = (6600)$, $\mathbf{t} = (3000)$ as the *mix*, of the last two entries in Table 1. The *mix* construction from [17, Sect. 7A] provides a useful technical tool for constructing covers, as well as 2-covers, of abstract polytopes, and, more generally, quotients of Coxeter complexes.

s	t	v	f	G	“Y-overgroup”
(2000)	(2000)	2^5	2^5	$[2^{15}3^2]$	$O_8^-(2):2$
(2000)	(2200)	2^5	2^7	$[2^{18}3^2]$	$2^8:O_8^-(2):2$
(2200)	(2200)	2^{11}	2^{11}	$[2^{24}3^2]$	—
(3000)	(3000)	$2^2 \cdot 3^2 \cdot 5 \cdot 13$	$2^2 \cdot 3^2 \cdot 5 \cdot 13$	$(3^2 \times L_4(3)).2^2$	—
(2200)	(3000)	$2^9 \cdot 3 \cdot 5^2 \cdot 7$	$2^3 \cdot 3^5 \cdot 5^2 \cdot 7$	$S_3 \times O_8^+(2):S_3$	Fi_{22}

TABLE 1. The known universal finite polytopes $\{\{3, 3, 4, 3\}_s, \{3, 4, 3, 3\}_t\}$, their groups, and (whenever they exist) “Y-overgroups”.

Proposition 3. (Mix construction [17]). Let $G = \langle g_1, \dots, g_r \rangle$ and $H = \langle h_1, \dots, h_r \rangle$ be two quotients of an arbitrary Coxeter group F with r generators γ_k , so that g_k and h_k are images of γ_k under the respective homomorphisms, $1 \leq k \leq r$. Let $G \diamond H := \langle g_1 h_1, g_2 h_2, \dots, g_r h_r \rangle \leq G \times H$. Then $G \diamond H$, called *mix* of G and H , is a quotient of F , with $g_k h_k$ the image of γ_k , for $1 \leq k \leq r$. \square

Note that in general $G \diamond H$ might fail the intersection property, despite it holding in G and H , cf. [17, Sect. 7A6]. Thus we will need extra tools to demonstrate the intersection property in $G \diamond H$, in particular [17, Lemma 12D4], which is Lemma 5 below.

Proposition 3 allows to mix two abstract polytopes with the same diagram \mathcal{D} , or, more generally, with the groups affording the Coxeter relations from \mathcal{D} (in the latter case the actual Coxeter diagram might be a “quotient”, in the same sense as Weyl group $A_2 \times A_2$ admits the relations of F_4). The corresponding algebraic system is a meet-semilattice, corresponding to the partially ordered set of certain normal subgroups in the Coxeter group with diagram \mathcal{D} . One application of this is to construct covers of abstract polytopes, used in Section 4 to analyse the group of Γ in Theorem 1. For the sake of completeness, in Section 6, cf. Proposition 11, we show how to use it to identify the group of the universal polytope $\{\{3, 3, 4, 3\}_{(3000)}, \{3, 4, 3, 3\}_{(3000)}\}$ with $(3^2 \times L_4(3)).2^2$, the example studied in a great detail in [18, Sect.4.2].

Another application of Proposition 3 is to construct certain 2-covers of abstract polytopes. For instance, one constructs a finite abstract polytope of type $\{\{3, 3, 4, 3\}_s, \{3, 4, 3, 3\}_t\}$ for a new tuple (s, t) of parameters, mixing a pair of examples from Table 1. More details on this are given in Section 6.

2. NOTATION AND PRELIMINARIES

Our notation for Coxeter groups and diagrams is standard. A Coxeter group G is generated by generators g_1, \dots, g_r , with the relations $g_i^2 = (g_i g_j)^{k_{ij}} = 1$, for $1 \leq i < j \leq r$ and $2 \leq k_{ij} < \infty$ (more generally, one may assume $k_{ij} = \infty$, but we will not use this here). Graphically this is drawn as an r -node graph (diagram) $\mathcal{D}(G)$ with $k_{ij} - 2$ edges joining vertices i and j (or, more generally, with labels k_{ij} on the edges for which $k_{ij} \geq 5$, but we will not use this here). Groups G for which $\mathcal{D}(G)$ is connected are called *irreducible*; it is a necessary (although not sufficient) condition for the irreducibility of the natural reflection representation of G with generators g_i being reflections in \mathbb{R}^r with respect to hyperplanes H_i , with angles

between H_i and H_j determined by k_{ij} . Subgroups $P_I := \langle g_i \mid i \in I \rangle \leq G$ for $I \subseteq \{1, \dots, r\}$ are called *parabolic* (called *special parabolics* in [17]).

For instance, the diagrams $Y_{\alpha,\beta,0}$, $Y_{\alpha,1,1}$, Y_{221} , Y_{321} , Y_{421} (cf. Fig. 1) correspond to irreducible finite Coxeter groups $A_{\alpha+\beta}$, $D_{\alpha+2}$, E_6 , E_7 , and E_8 , which are also the Weyl groups of the respective root systems so named. Note that the group A_n is isomorphic to the symmetric group S_{n+1} . If $\mathcal{D}(G)$ is an r -path, maybe with multiple edges, one talks about a *string diagram*, and encodes it as *Schläfli symbol* $[k_{12}, k_{23}, \dots, k_{r-1,r}]$ (assuming the g_i are ordered consecutively on the path $\mathcal{D}(G)$). We have already mentioned an example of a string diagram A_n ; its Schläfli symbol is $[3^{n-1}]$, where we abbreviated $n-1$ consecutive 3's as 3^{n-1} . Other examples we need here of irreducible finite Coxeter groups with string diagram are F_4 , with Schläfli symbol $[3, 4, 3]$, cf. Fig. 2, and B_n , with Schläfli symbol $[4, 3^{n-2}]$.

Further, we are interested in quotients of G that satisfy *intersection property*, that is

$$(1) \quad G_I \cap G_J = G_{I \cap J} \quad \text{for all } I, J \subseteq \{1, \dots, r\}.$$

For instance, finite quotients $p^6:E_6$, of the affine Coxeter group \tilde{E}_6 , i.e. the Coxeter group with diagram Y_{222} , and finite quotients $p^4:F_4$ of the affine Coxeter group \tilde{F}_4 , (the group with Schläfli symbol $[3, 3, 4, 3]$) satisfy the intersection property whenever $p \geq 2$.

Quotients of G with string diagram that satisfy (1) are called *string C-groups* in [16, 17], and they correspond to *abstract r -polytopes*. The order complexes of these polytopes are quotients of the *Coxeter complex* [25, Chapter 2], [20, 4] of G , a simplicial complex built from the cosets of the parabolic subgroups $G_I \leq G$, $I \subseteq \{1, \dots, r\}$, with respect to the respective normal subgroups (the latter might be trivial), cf. [17, Sect. 2C, 3D]. In particular for finite irreducible G they are classical polytopes: n -simplices for A_n , n -cubes for B_n , the 24-cell $\{3, 4, 3\}$ for F_4 , etc. Abstract regular polytopes are also known as satisfying the intersection property thin diagram geometries with string diagram, see e.g. [20, 4]. By $\langle \mathcal{P}_f, \mathcal{P}_v \rangle$ we denote the class of all polytopes with facets isomorphic to \mathcal{P}_f and vertex figures isomorphic to \mathcal{P}_v . This class contains unique polytope, called the *universal polytope* $\{\mathcal{P}_f, \mathcal{P}_v\}$, which covers each member of the class, cf. [17, Thm.4A2].

In the sequel we will need a classification of abstract polytopes of type $\{3, 3, 4, 3\}$ and, dually, $\{3, 4, 3, 3\}$. Due to duality, it suffices to deal with the former, corresponding to the quotients of the affine Coxeter group \tilde{F}_4 , see Fig. 3. (In our notation for *conjugation*, $x^y = y^{-1}xy$.)

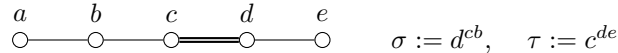


FIGURE 3. Affine Coxeter group \tilde{F}_4 , nodes labelled by generators.

F_4 acts on the lattice $\mathbb{Z}^4 \cup ((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^4)$ of \mathbb{R}^4 -vectors with coordinates being integer or half-integer, cf. e.g. [16, p.23], and one needs to classify the invariant sublattices of finite index. The latter may be generated (as a submodule) by a single vector, of the shape $\mathbf{s} := (s, 0, 0, 0)$, or $\mathbf{s}' := (s, s, 0, 0)$, with $2 \leq s \in \mathbb{Z}$. Combining [16, Theorems 3.4 and 3.5], one obtains the following.

Theorem 4. [16] Let Π be a finite abstract regular polytope $\{3, 3, 4, 3\}_{\mathbf{t}}$. Then for $\mathbf{t} = \mathbf{s}$ its group is isomorphic to $[3, 3, 4, 3]$ subject to the extra relation $(a\sigma\tau\sigma)^s = 1$, with τ and σ as on Fig. 3, whereas for $\mathbf{t} = \mathbf{s}'$ the extra relation is $(a\sigma\tau)^{2s} = 1$. \square

In what follows we denote the groups arising in Theorem 4 by $[3, 3, 4, 3]_{\mathbf{t}}$ (or, if appropriate, by $[3, 4, 3, 3]_{\mathbf{t}}$).

The mix construction in Proposition 3 needs extra tools to show that intersection property holds in our mixes. One of these is [17, Lemma 12D4] and some of its corollaries proved immediately after [loc.cit.] which we state here in our context.

Lemma 5. Let $H = [3, 3, 4, 3]_{(s,0,0,0)}$ with s an odd prime. Then the $[3, 4, 3]$ -parabolic is maximal in H . Consequently, let G be a finite quotient of $[3^2, 4, 3^2]$ such that the following holds for its parabolics: $G_{\{0,\dots,4\}} \cong [3, 3, 4, 3]_{(s,0,0,0)}$ and $G_{\{1,\dots,5\}} \cong [3, 4, 3, 3]_{(t,0,0,0)}$ (or $[3, 4, 3, 3]_{(t,t,0,0)}$), $t \geq 2$, and so that $G_{\{0,\dots,4\}}$ and $G_{\{1,\dots,5\}}$ are proper subgroups of G . Then (1) holds in G . \square

3. SPORADIC Y -PRESENTATIONS AND THEIR TWISTS

Our Theorem 1 will follow at once from the following.

Theorem 6. Let Γ be a locally toroid rank 6 abstract regular polytope with vertex figures of type $\{3, 4, 3, 3\}_{\mathbf{s}}$ and facets of type $\{3, 3, 4, 3\}_{\mathbf{t}}$. Let $\mathbf{s} = (3000)$ and $\mathbf{t} = (2200)$. Then Γ is finite and isomorphic to Γ_k , for some $1 \leq k \leq 4$, defined in and after Theorem 1. In particular, Γ is a quotient of Γ_4 , and the group of Γ_k is isomorphic to $S_k \times \Omega$.

According to [6], the sporadic simple group Fi_{22} has a presentation Y_{332} on Fig. 1(b) subject to extra relations $S = f_{12} = f_{21} = 1$, where

$$(2) \quad S = (ab_1c_1ab_2c_2ab_3c_3)^{10}, \quad f_{ij} = (ab_ib_jb_kc_ic_jd_i)^9, \quad \text{where } \{i, j, k\} = \{1, 2, 3\}.$$

The following observation was the starting point of this project.

Lemma 7. Let ϕ be the automorphism of the diagram Y_{332} from Fig. 1(b). Then S^ϕ and S generate the same normal subgroup, and $f_{12}^\phi = f_{21}$, $f_{21}^\phi = f_{12}$.

Proof. Note that $S, S^\phi \in Y_{222} = \tilde{E}_6$, and it is straightforward to check that they generate the same E_6 -invariant sublattice in the 6-dimensional E_6 -invariant lattice. The rest of the statement is obvious. \square

In view of Lemma 7, Fi_{22} contains a quotient $\Theta = \langle c_3, b_3, a, b_1b_2, c_1c_2, d_1d_2 \rangle$ of a Coxeter group with diagram $[3^2, 4, 3^2]$.

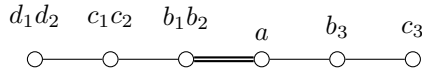


FIGURE 4. Coxeter group with diagram $[3^2, 4, 3^2]$ from ϕ .

One immediately identifies $\Theta_f = \langle b_3, a, b_1b_2, c_1c_2, d_1d_2 \rangle < \Theta$ with a subgroup in $2^6:Sp_6(2)$ presented by Y_{331} with extra relations $f_{12} = f_{21} = 1$, and $\Theta_v = \langle c_3, b_3, a, b_1b_2, c_1c_2 \rangle < \Theta$ with a subgroup in $3^5:E_6$ presented by Y_{222} with an extra relation $S = 1$, see [6, p.232-233].

Direct computations either by coset enumeration, or in appropriate E_6 - and E_7 -invariant lattices allow one to identify Θ_v and Θ_f as groups classified by Theorem 4. By Lemma 5, we have the following.

Lemma 8. The following isomorphisms hold: $\Theta_f = [3, 4, 3, 3]_{(2200)}$, and $\Theta_v = [3, 3, 4, 3]_{(3000)}$. The intersection property holds in Θ . \square

At this point we can already conclude that the example of $\{3, 3, 4, 3, 3\}$ -polytope provided by Θ does not appear in [16, Table V], as the pair of vectors $\mathbf{s} = (3000)$ and $\mathbf{t} = (2200)$ determining the vertex figure and the facet is not present there.

As ϕ on Fig. 1(b) induces an outer automorphism of Fi_{22} and centralises Θ , one concludes by inspecting the list of maximal subgroups of Fi_{22} that $\Theta \leq O_8^+(2):S_3$, and enumeration of cosets of Θ in Fi_{22} , or analysis of maximal subgroups of $O_8^+(2):S_3$ confirm that in fact $\Theta = O_8^+(2):S_3$, identifying it with the group of the example Γ_1 in our Theorem 6.

Remark 1. The problem of recognising groups presented by Y_{332} with similar extra relations, generalising ones on [6, pp.232-233], has been brought to the author's attention by Alexander A. Ivanov. Namely, one is interested in the general setting where subgroups $Y_{331} = \tilde{E}_7$ and $Y_{222} = \tilde{E}_6$ are made finite by factoring out the corresponding finite index invariant sublattices.

It is strikingly parallel to the problem of classifying the finite quotients of $[3, 3, 4, 3, 3]$ we are concerned with in this text. The simplest case, where Y_{222} becomes $2^6:E_6$, has been essentially settled in [11] (one gets orthogonal groups over \mathbb{F}_2 , in some cases acting on its natural \mathbb{F}_2 -module), but in general it is open. We remark that Proposition 3 allows to construct more examples, e.g. we can get a subgroup of $Fi_{22} \times 2^8:O_8^-(2)$ by applying it to the two known examples.

We hope to explore this topic further in another publication.

Problem 1. The question of understanding similarly constructed subgroups of the Monster-related groups with Y -presentations is largely open, although they ought to exist by arguments similar to the Y_{332} -case considered here.

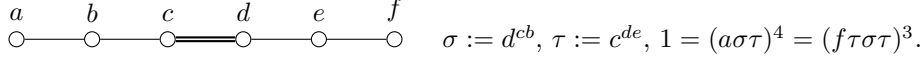
Preliminary experiments with enumerating cosets of one of our finite $[3^2, 4, 3^2]$ -quotients, namely the group of Γ_1 , isomorphic to $O_8^+(2):S_3$, in $[3^2, 4, 3^3]$, indicates that this gives the group $F_4(2)$, which is a subgroup of a quotient of Y_{333} isomorphic to ${}^2E_6(2)$. More precisely, the enumeration returns index 12673024, which is 4 times the index of $O_8^+(2):S_3$ in $F_4(2)$.

We should also mention that the $[3, 4, 3^3]$ -subgroup in this group, which corresponds to a locally toroidal polytope of the type considered in [17, Sect. 12C], is isomorphic to $L_4(3):2$, shown to be a subgroup in $F_4(2)$ by L. Soicher [24] by an argument involving a diagram automorphism similar to ϕ .

4. ANOTHER EXISTENCE PROOF OF Γ_1 , AND EXISTENCE OF ITS COVERS

Here we present another, direct, existence proof of Γ_1 , and complete the proof of Theorem 6 by constructing its covers.

We begin by writing down a presentation \mathcal{P} for Θ , the group of the *universal cover* of the examples from Theorem 6. In our context it simply means that in \mathcal{P} we do not impose any relations that involve all the generators. As Θ is the group of the (unique) universal polytope, we can conclude without loss in generality that \mathcal{P} is

FIGURE 5. The presentation \mathcal{P} for Θ , nodes labelled by generators.

Lemma 9. There is a surjective homomorphism $\hat{\cdot}: \Theta \rightarrow O_8^+(2):S_3$, with its image, the group of Γ_1 , satisfying the intersection property.

Proof. The embedding will be given by matrices in $GL_{24}(2)$, with all the elements, except images \hat{a} and \hat{f} of a and f , permutation matrices. Namely, $\langle b, c, d, e \rangle = F_4$ embeds as the permutation group in its natural action on the vertices of the 24-cell. The latter action has an imprimitivity system with three blocks of size 8, which will correspond to the three natural 8-dimensional submodules for $O_8^+(2):2$. We choose the generators so that the blocks are $\{1..8\}$, $\{9..16\}$, and $\{17..24\}$, and so that the centre of F_4 is the permutation $(1, 2)(3, 4) \dots (23, 24)$. Let us slightly abuse notation and view the following as specifications of 24×24 -permutation matrices over \mathbb{F}_2 .

$$\begin{aligned} \hat{b} &= \prod_{i \in \{0, 8, 16\}} (i+3, i+8)(i+4, i+7), & \hat{c} &= (3, 6)(4, 5)(11, 14)(12, 13)(17, 24)(18, 23), \\ \hat{d} &= (5, 6) \prod_{i=9}^{16} (i, i+8), & \hat{e} &= (21, 22) \prod_{i=1}^8 (i, i+8). \end{aligned}$$

In terms of the 24-cell, the size 8 imprimitivity block of a vertex v consists of the antipodal to v vertex of the 24-cell, and the vertices antipodal to v in each of the 6 octahedra (facets of the 24-cell) on v . The stabiliser of the block in F_4 is $\langle \hat{b}, \hat{c}, \hat{d}, \hat{e} \rangle$.

Define matrices over \mathbb{F}_2 , using I_k to denote $k \times k$ identity matrix, and E_{ij} to denote the matrix with 1 in the position ij and 0 elsewhere.

$$\begin{aligned} A &:= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, & F &:= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\ \hat{a} &:= I_3 \otimes A, & \hat{f} &:= \begin{pmatrix} I_8 + E_{21} & 0 & 0 \\ 0 & 0 & F \\ 0 & F & 0 \end{pmatrix}. \end{aligned}$$

Now it is routine to check that the relations in \mathcal{P} hold for just defined \hat{a}, \dots, \hat{f} . As well, the elements \hat{a}, \dots, \hat{f} leave invariant the quadratic form $\Phi := \sum_{j=1}^{24} x_j^2 + \sum_{i=1}^{12} x_{2i-1} x_{2i}$; this is immediate for the permutation matrices \hat{b}, \dots, \hat{e} , as they commute with the centre of $\langle \hat{b}, \hat{c}, \hat{d}, \hat{e} \rangle$, and a direct computation for \hat{a} and \hat{f} .

As this point we have $\hat{\Theta} \leq O_8^+(2):S_3$. To see that the latter is in fact equality, one can either inspect the list of the maximal subgroups of $O_8^+(2):S_3$, or investigate the action of the stabiliser $H := \langle \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{f} \rangle$ of the subspace spanned by the first 8 coordinates in $\hat{\Theta}$, and see that it acts transitively on the 135 nonzero Φ -isotropic

vectors there, by computing the orbit of the all-1 vector, and further identifying H with a rank 3 permutation group $O_8^+(2):2$ in its natural action on isotropic vectors.

Finally, the intersection property holds, by [16]. \square

Let $\psi : \Theta \rightarrow S_4$ be a map defined on generators of Θ , with $\psi(a) := (1, 2)$, $\psi(b) := (2, 3)$, $\psi(c) := (3, 4)$, and $\psi(d) = \psi(e) = \psi(f) = ()$. It turns out that it extends to a homomorphism onto. (Note that there is no symmetry here: if we try to send a, b, c to the identity instead, we get a trivial group).

Lemma 10. The map ψ is a homomorphism onto. Also, there is a homomorphism from Θ onto $S_4 \times O_8^+(2):S_3$, with its image the group of Γ_4 .

Proof. For the first part, observe that the Coxeter relations of Θ hold in $\psi(\Theta)$. Also, $\psi(\sigma) = ()$ and $\psi(\tau) = \psi(c) = (3, 4)$. It follows that the rest of the relations hold, as well, and we are done.

For the second part, we construct an embedding of Θ into $G := S_4 \times O_8^+(2):S_3 = \psi(\Theta) \times \hat{\Theta}$, as prescribed by Proposition 3. Consider $H := \psi(\Theta) \diamond \Theta \leq G$; recall that it is generated by $\psi(a)\hat{a}, \dots, \psi(f)\hat{f}$. Trivially, by checking the relations, H is a homomorphic image of Θ . Moreover, the intersection property holds in H by Lemma 5.

It remains to show $H = G$. Consider an element of $O_8^+(2)$ of order 5, as a word $\hat{w} := w(\hat{a}, \dots, \hat{f})$ in our generators, and the corresponding word $w := w(\psi(a)\hat{a}, \dots, \psi(f)\hat{f}) \in H$. Then $w = w(\psi(a), \dots, \psi(f))\hat{w}$. Thus $w^{12} = \hat{w}^2 \in \hat{\Theta}$ is of order 5. As $O := O_8^+(2)$ is generated by such elements, and as $\hat{\Theta}$ is generated by O and $\hat{d}, \hat{e}, \hat{f}$, we have $\hat{\Theta} \leq H$. Therefore $H = G$, as claimed. \square

5. THE CASE $\mathbf{s} = (2000)$

First, we prove Theorem 2. Let $\Gamma = \{\{3, 3, 4, 3\}_{(2000)}, \{3, 4, 3, 3\}\}$. Then its group Θ has presentation as follows.

$$\begin{array}{ccccccccc} a & & b & & c & & d & & e & & f \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array} \quad \sigma := d^{cb}, \tau := c^{de}, 1 = (a\sigma\tau\sigma)^2.$$

Enumeration of cosets of the facet stabiliser $\langle a, b, c, d, e \rangle$ in Θ shows that $|\Theta| = 2^{18} \cdot 3^2$. Working in the resulting permutation representation one computes the orders of $f\tau\sigma\tau$ and $f\tau\sigma\tau\sigma$ to be 4, thus identifying the stabiliser of the vertex with the one corresponding to the type $\{3, 4, 3, 3\}_{(2200)}$, as claimed.

The polytope Γ corresponds to the second entry in [17, Table 12D1] (or [16, Table V]) for $t = 2$. The quotient by a normal subgroup 2^3 in Θ produces the other case, $\mathbf{t} = (2000)$, the first entry in [loc.cit.] for $t = 2$. It shows that the cases with $t > 2$ in [loc.cit.] in fact do not arise. \square

More conceptually, one can notice that it is proved on [17, page 463] that the number of vertices in Γ equals 32. In particular, it means that the index of the subgroup $H := \langle b, c, d, e, f \rangle$ in Θ is 32. Let $\pi : \Theta \rightarrow \hat{\Theta}$ be the permutation representation of Θ on the cosets of H , and K its kernel. As H is a quotient of \tilde{F}_4 , and K is a normal subgroup there, we see that K is very small. It cannot contain a faithful 4-dimensional module for F_4 , as the latter would admit an action of a commuting with the action of $\langle d, e, f \rangle$, but still inducing a faithful action of $\langle a, b \rangle$.

(as the action of $\langle b, c \rangle$ would be faithful, and the latter is conjugate to $\langle a, b \rangle$). This implies that $|K| \leq 8$.

Another possibility is to carry out the computation of a presentation for H applying the Reidemeister–Schreier algorithm (which produces a presentation of a finite index subgroup of a finitely presented group) cf. e.g. [15], and find that $H = [3, 4, 3^2]_{(2200)}$.

5.1. Embedding of the group of Γ in Y -group. Consider the 2-modular quotient G of Y_{332} , which is isomorphic to $G := 2^8 : O_8^-(2) : 2$, as established by R. Griess in [11, pp.277–278]. Then Θ can be identified with its subgroup, as shown on Figure 4, under the automorphism ϕ of the Y -diagram on Figure 1(b). Specifically, the relation forcing the $\{3, 3, 4, 3\}_{(2000)}$ -facet turns the subgroup Y_{222} into $2^6 : U_4(2) : 2$, and, respectively, Y_{331} into $2^7 : Sp_6(2)$.

Adding further relation forcing the $\{3, 4, 3, 3\}_{(2000)}$ -vertex figure kills $O_2(G)$, and one obtains a quotient of $\{\{3, 3, 4, 3\}_{(2000)}, \{3, 4, 3, 3\}_{(2000)}\}$ by a subgroup of order 4.

The group G also features on [6, pp.232–233] as a quotient of X_{332} , a re-labelling of Y_{332} , with the node c_3 of the latter denoted by a_3 , by an explicitly given relation W . In complete analogy with the Fi_{22} situation, the relation W affords the automorphism ϕ , and allows one to establish the existence of Γ directly.

Remark 2. In his PhD thesis L. Soicher [22, Thm.A.2] proved that the group $[3^m, 4, 3^n]$, subject to a relation killing the centre of the $[3, 4, 3]$ -parabolic, is isomorphic to $2^{mn} : (S_{m+1} \times S_{n+1})$. Note that for $m = n = 2$ this example has the cover $\{\{3, 3, 4, 3\}_{(2000)}, \{3, 4, 3, 3\}_{(2000)}\}$ from the last part of Theorem 2 (see also the first entry in [16, Table V]).

Another “tower” of finite $[3^m, 4, 3^n]$ -examples with unbounded m and n may be constructed from 2-modular quotients of the group $Y_{\alpha\alpha\gamma}$ by twisting with ϕ , generalising the observation in the beginning of Section 5.1. There the $[3^2, 4, 3^2]$ -subgroups will correspond to $\{\{3, 3, 4, 3\}_{(2000)}, \{3, 4, 3, 3\}_{(2200)}\}$ from Theorem 2.

6. THE MIX CONSTRUCTION

Our first application of the mix construction in Proposition 3 was Lemma 10. Here we present more applications of the mix construction. The following constitutes another proof of a result in [18, Sect.4.2].

Lemma 11. The group of $\Gamma = \{\{3, 3, 4, 3\}_{(3000)}, \{3, 4, 3, 3\}_{(3000)}\}$ is isomorphic to $(3^2 \times L_4(3)).2^2$.

Proof. (Sketch) From coset enumeration, we know the order of the group in question. The group $\hat{\Theta} := L_4(3) : 2_2 \cong PGO_6^+(3)$ has a maximal subgroup isomorphic to $[3, 3, 4, 3]_{(3000)}$, according to [6]. As well, its central extension by the group of order 2, which is in fact split, is constructed in [18, Sect.4.2] as a quotient of the group Θ of Γ (more precisely, as the reduction modulo 3 of the natural 6-dimensional representation of the Coxeter group $[3, 3, 4, 3, 3]$).

Let Ω be the quotient of Θ obtained by imposed the extra relation that the generators (in the Coxeter diagram) of the “middle” dihedral subgroup of order 8 commute. It is easy to check that $\Omega = S_3 \times S_3$. Consider the mix $P := \hat{\Theta} \diamond \Omega$. One sees that the P contains a copy of the simple group $L_4(3)$, by considering words in P corresponding to elements of $\hat{\Theta}$ of order 13. Thus it also contains $O_3(\Omega) = 3^2$.

It remains to observe that the Abelian invariants of Ω and of $[3^2, 4, 3^2]$ are equal to $(2, 2)$, implying that the 2^2 acting on P is generated by P . Finally, P satisfies the intersection property by Lemma 5. \square

6.1. Constructing 2-covers. The following will be needed for rank 6 cases.

Proposition 12. Let $\mathbf{s} = (s, s', 0, 0) \in \{(s, s, 0, 0), (s, 0, 0, 0)\}$ and $\mathbf{t} = (t, t', 0, 0) \in \{(t, t, 0, 0), (t, 0, 0, 0)\}$. Denote $\ell := \text{lcm}(s, t)$. The mix $G \diamond H$ of $G = [3, 3, 4, 3]_{\mathbf{s}}$ and $H = [3, 3, 4, 3]_{\mathbf{t}}$ is $[3, 3, 4, 3]_{(\ell, \ell, 0, 0)}$ in the following cases

- (1) $s' = s, t' = t$;
- (2) $s' = s, t' = 0, 2\ell = \text{lcm}(2s, t)$.
- (3) $s' = 0, t' = t, 2\ell = \text{lcm}(s, 2t)$.

Otherwise $G \diamond H$ is $[3, 3, 4, 3]_{(\ell, 0, 0, 0)}$.

Proof. To establish the intersection property (1), we embed $G \diamond H$ “diagonally” into $G \times H$ considered as an affine matrix group over the suitable finite ring. This shows that $G \diamond H$ is isomorphic to some $[3, 3, 4, 3]_{\mathbf{u}}$ and so (1) holds. It remains to compute the relation \mathbf{u} implied by \mathbf{s} and \mathbf{t} .

Let \mathbf{s} and \mathbf{t} have the same “format”, i.e. either $s' = t' = 0$ or $s' = s, t' = t$. Then the only difference between the presentations of G and H is the exponent of the extra non-Coxeter relation in Theorem 4, and the claim follows.

It remains to deal with the case of different “formats”. Let $G = \langle a, b, c, d \rangle$, as in Theorem 4. Respectively, let $H = \langle a', b', c', d' \rangle$, and denote $\sigma' := d'c'b', \tau' := e'c'd'$.

Let $s' = s$ and $t' = 0$. Thus the orders of $a\sigma\tau$ and $a\sigma\tau\sigma$ equal to $2s$, while the orders of $a'\sigma'\tau'$ and $a'\sigma'\tau'\sigma'$ equal $2t$ and t , respectively. Then the order of $aa'\sigma\sigma'\tau\tau'$ equals 2ℓ , while the order of $aa'\sigma\sigma'\tau\tau'\sigma\sigma'$ equals $\text{lcm}(2s, t)$. We get the case $[3, 3, 4, 3]_{(\ell, \ell, 0, 0)}$ if and only if the two latter orders are equal. The case of $s' = 0, t' = t$ is dealt with in the same way. \square

This immediately implies

Lemma 13. There exist $\{\{3343\}_{\mathbf{s}}, \{3433\}_{\mathbf{t}}\}$ for $(\mathbf{s}, \mathbf{t}) = (3000, 6600)$.

Proof. Apply the mix construction to $\{\{3343\}_{(3000)}, \{3433\}_{(2200)}\}$ and $\{\{3343\}_{(3000)}, \{3433\}_{(3000)}\}$, compute \mathbf{s} and \mathbf{t} using Proposition 12, and note that the intersection property holds by Lemma 5. \square

Remark 3. One can also consider other mixes of examples in Table 1 to obtain, e.g. $(\mathbf{s}, \mathbf{t}) = (6000, 2200)$; however, we do not know whether the intersection property holds in this group and other similar examples, as Lemma 5 would not apply.

Remark 4. Analogously one can try to construct new examples of $\{\{3, 4, 3, 3\}_{\mathbf{s}}, \{4, 3, 3, 4\}_{\mathbf{t}}\}$. Here one needs to be able to compute in the meet-semilattice generated by the parameters \mathbf{t} of the groups $[4, 3, 3, 4]_{\mathbf{t}}$, for \mathbf{t} in the corresponding column of [16, Table VI] and in the examples from [19]. We leave this to another publication.

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