

# Polynomial bounds for chromatic number

## VI. Adding a four-vertex path

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## Abstract

A class of graphs is  $\chi$ -*bounded* if there is a function  $f$  such that every graph  $G$  in the class has chromatic number at most  $f(\omega(G))$ , where  $\omega(G)$  is the clique number of  $G$ ; the class is *polynomially*  $\chi$ -*bounded* if  $f$  can be taken to be a polynomial. The Gyárfás-Sumner conjecture asserts that, for every forest  $H$ , the class of  $H$ -free graphs (graphs with no induced copy of  $H$ ) is  $\chi$ -bounded. Let us say a forest  $H$  is *good* if it satisfies the stronger property that the class of  $H$ -free graphs is polynomially  $\chi$ -bounded.

Very few forests are known to be good: for example, it is open for the five-vertex path. Indeed, it is not even known that if every component of a forest  $H$  is good then  $H$  is good, and in particular, it was not known that the disjoint union of two four-vertex paths is good. Here we show the latter, and more generally, that if  $H$  is good then so is the disjoint union of  $H$  and a four-vertex path. We also prove a more general result: if every component of  $H_1$  is good, and  $H_2$  is any path (or broom) then the class of graphs that are both  $H_1$ -free and  $H_2$ -free is polynomially  $\chi$ -bounded.

# 1 Introduction

A class of graphs is  $\chi$ -bounded if there is a function  $f$  such that every graph  $G$  in the class has chromatic number at most  $f(\omega(G))$ , where  $\omega(G)$  is the clique number of  $G$ ; the class is *polynomially*  $\chi$ -bounded if  $f$  can be taken to be a polynomial. A graph is  $H$ -free if it has no induced subgraph isomorphic to  $H$ .

The Gyárfás-Sumner conjecture [4, 14] asserts:

**1.1 Conjecture:** *For every forest  $H$ , the class of  $H$ -free graphs is  $\chi$ -bounded.*

There has been a great deal of recent progress on  $\chi$ -bounded classes (see [9] for a survey), although the Gyárfás-Sumner conjecture remains open. In most cases, proofs of  $\chi$ -boundedness give fairly fast-growing functions, so it is interesting to ask: when do we get the stronger property of polynomial  $\chi$ -boundedness?

A provocative conjecture of Louis Esperet [3] asserted that every  $\chi$ -bounded hereditary class is polynomially  $\chi$ -bounded. But this was recently disproved by Briński, Davies and Walczak [1]. So the question now is: which hereditary classes are polynomially  $\chi$ -bounded? In particular, can 1.1 be strengthened to polynomial  $\chi$ -boundedness? Let us say a graph  $H$  is *good* if the class of  $H$ -free graphs is polynomially  $\chi$ -bounded. Perhaps every forest is good, but the only trees currently known to be good are those not containing the five-vertex path  $P_5$  [11]. It is not known whether  $P_5$  is good (although see [12] for the best current bounds for  $H = P_5$ ; and see [13] for the case when  $H$  is a general tree of radius two).

In the case of  $\chi$ -boundedness, it is not hard to show that a forest  $H$  satisfies the Gyárfás-Sumner conjecture if and only if all its components do. But it has *not* been shown that if every component of a forest  $H$  is good then  $H$  is good. Indeed, only some very restricted forests are known to be good [8, 10]. One outstanding case was when  $H$  is the disjoint union of two copies of the four-vertex path  $P_4$ ; and this was particularly annoying since the  $P_4$ -free graphs are very well-understood and rather trivial.

We will prove the following:

**1.2** *If  $H$  is a good forest, then the disjoint union of  $H$  and  $P_4$  is also good.*

In particular, the disjoint union of two or more copies of  $P_4$  is good. 1.2 is a consequence of the next result, about brooms. A  $(k, d)$ -broom is a tree obtained from a  $k$ -vertex path with one end  $v$  by adding  $d$  new vertices adjacent to  $v$ , and a broom is a tree that is a  $(k, d)$ -broom for some  $k, d$ . It is known that  $(3, d)$ -brooms are good [6, 11], but this is not known for larger brooms (all of which contain  $P_5$ ). We will show the following, which implies 1.2:

**1.3** *Let  $H_1$  be a forest such that every component of  $H_1$  is good, and let  $H_2$  be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial  $\phi$  such that  $\chi(G) \leq \phi(\omega(G))$  for every  $\{H_1, H_2\}$ -free graph  $G$ .*

( $\{H_1, H_2\}$ -free means both  $H_1$ -free and  $H_2$ -free.) To deduce 1.2 from 1.3, let  $H$  be a good forest, let  $H_1 = H_2$  be the disjoint union of  $H$  and  $P_4$ , and apply 1.3.

Some notation and terminology: if  $G$  is a graph and  $X \subseteq V(G)$ , we denote by  $G[X]$  the subgraph of  $G$  induced on  $X$ , and we sometimes write  $\chi(X)$  for  $\chi(G[X])$  and  $\omega(X)$  for  $\omega(G[X])$ . Two disjoint subsets  $A, B \subseteq V(G)$  are *complete* if every vertex in  $A$  is adjacent to every vertex of  $B$ , and *anti-complete* if there is no edge between  $A, B$ ; and we say a vertex  $v$  is *complete to*  $B$  if  $\{v\}$  is complete

to  $B$ , and so on. A graph  $G$  *contains* a graph  $H$  if some induced subgraph of  $G$  is isomorphic to  $H$ , and such a subgraph is a *copy* of  $H$ . The *cone* of a graph  $H$  is obtained from  $H$  by adding a new vertex adjacent to every vertex of  $H$ .

Let us say a graph is *0-bad* if it is good; and a graph  $J$  is  $\beta$ -bad, where  $\beta \geq 1$  is an integer, if either  $J$  is the disjoint union of two  $(\beta - 1)$ -bad graphs, or  $J$  is the cone of a  $(\beta - 1)$ -bad graph, or  $J$  is  $(\beta - 1)$ -bad. In general, cones are not forests, so they are not good. Nevertheless, we will prove the following strengthening of 1.3:

**1.4** *Let  $\beta \geq 0$ , let  $H_1$  be a  $\beta$ -bad graph, and let  $H_2$  be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial  $\phi$  such that  $\chi(G) \leq \phi(\omega(G))$  for every  $\{H_1, H_2\}$ -free graph  $G$ .*

This implies several results that were previously known. For instance, in [7] it is proved that:

**1.5** *Let  $H_1$  be either*

- *the disjoint union of a complete graph and a good graph, or*
- *the disjoint union of some complete graphs, or*
- *the cone of the disjoint union of some complete graphs.*

*Let  $H_2$  be a path. Then there is a polynomial  $\phi$  such that  $\chi(G) \leq \phi(\omega(G))$  for every  $\{H_1, H_2\}$ -free graph  $G$ .*

Some other results of [7, 8] are also special cases of 1.4.

## 2 Finding a disjoint union

Suppose that  $H$  is the disjoint union of good forests  $H_1, H_2$ . Choose  $c_1, c_2$  such that for  $i = 1, 2$ , every  $H_i$ -free graph  $G$  satisfies  $\chi(G) \leq \omega(G)^{c_i}$ . Thus, if  $G$  is  $H$ -free, we know that there do not exist disjoint, anticomplete subsets  $P, Q \subseteq V(G)$  with  $\chi(P) > \omega(P)^{c_1}$  and  $\chi(Q) > \omega(Q)^{c_2}$ ; because then  $G[P]$  is not  $H_1$ -free, and  $G[Q]$  is not  $H_2$ -free, and the union of a copy of  $H_1$  in  $G[P]$  and a copy of  $H_2$  in  $G[Q]$  gives a copy of  $H$ , which is impossible.

But we do not really need  $P, Q$  to be anticomplete. It is enough that  $\chi(P) > \omega(P)^{c_1}$ , and  $\chi(Q) > |H_1|r + \omega(Q)^{c_2}$ , where  $r$  denotes the maximum over  $v \in P$  of the chromatic number of the set of neighbours of  $v$  in  $Q$ ; because then if we choose a copy  $H'_1$  of  $H_1$  in  $G[P]$ , the chromatic number of the set of vertices in  $Q$  with no neighbours in  $V(H'_1)$  is at least  $\chi(Q) - |H_1|r > \omega(Q)^{c_2}$ , and so this set contains a copy of  $H_2$ , a contradiction. In the proof to come later in the paper, this is the only way we will ever use that  $G$  is  $H$ -free; and so we might as well prove a stronger theorem, replacing the hypothesis that  $G$  is  $H$ -free with the weaker hypothesis that there is no suitable pair  $P, Q$  in  $G$ .

Thus we will be excluding pairs of disjoint sets  $P, Q$  where  $\chi(P)$  is at least some power of  $\omega(P)$ , and for each vertex in  $P$ , its set of neighbours in  $Q$  has chromatic number at most some  $r$  that is small compared with the chromatic number of  $Q$ .

In our proof, it happens that when we find a suitable pair  $(P, Q)$ , it comes equipped with an extra vertex  $v$  that is complete to  $P$  and anticomplete to  $Q$ ; so we might as well prove that there is a “suitable triple”  $(v, P, Q)$ . Such a thing will also allow us to handle cones.

We denote the set of nonnegative integers by  $\mathbb{N}$ , and say a function  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  is *non-decreasing* if  $\phi(x) \leq \phi(x')$  for all  $x, x' \in \mathbb{N}$  with  $x \leq x'$ .

Let  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing, and let  $q \geq 0$  be an integer. We say a  $(\psi, q)$ -*scattering* in a graph  $G$  is a triple  $(v, P, Q)$  where:

- $P, Q$  are disjoint subsets of  $V(G)$ , and  $v \in V(G) \setminus (P \cup Q)$ ;
- $\{v\}$  is complete to  $P$  and anticomplete to  $Q$ ;
- $\chi(P) > \psi(\omega(P))$ ; and
- $\chi(Q) > qr + \psi(\omega(Q))$ , where  $r$  is the maximum, over  $u \in P$ , of the chromatic number of the set of neighbours of  $u$  in  $Q$ .

Thus we will replace the hypothesis in 1.4 that  $G$  is  $H_1$ -free and  $H_1$  is  $\beta$ -bad, with the hypothesis that  $G$  contains no  $(\psi, q)$ -scattering, for appropriate  $\psi, q$ . We will show:

**2.1** *Let  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing polynomial and let  $q \in \mathbb{N}$ . Let  $H_2$  be either a broom, or the disjoint union of a good forest and a number of paths. Then there is a polynomial  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that if  $\chi(G) > \phi(\omega(G))$  and  $G$  contains no  $(\psi, q)$ -scattering, then  $G$  contains  $H_2$ .*

**Proof of 1.4, assuming 2.1.** We proceed by induction on  $\beta$ . Let  $H_1$  be  $\beta$ -bad, and let  $H_2$  be either a broom, or the disjoint union of a good forest and a number of paths.

If  $H_1$  is good, the result is true, so we assume that  $H_1$  is not good, and therefore  $\beta \geq 1$ . Thus either  $H_1$  is the disjoint union of two  $(\beta - 1)$ -bad graphs  $J_1, J_2$ , or the cone of a  $(\beta - 1)$ -bad graph  $J_1$  (and in this case let  $J_2$  be the null graph). From the inductive hypothesis on  $\beta$ , for  $i = 1, 2$  there is a non-decreasing polynomial  $\phi_i$  such that if  $G$  is  $H_2$ -free and  $J_i$ -free then  $\chi(G) \leq \phi_i(\omega(G))$ , and by replacing  $\phi_1, \phi_2$  by  $\phi_1 + \phi_2$  we may assume that  $\phi_1 = \phi_2$ .

Let  $q = |J_1|$ . By 2.1, there is a non-decreasing polynomial  $\phi$  such that if  $\chi(G) > \phi(\omega(G))$  and contains no  $(\phi_1, q)$ -scattering, then  $G$  contains  $H_2$ . We claim that  $\phi$  satisfies 1.4.

Let  $G$  be  $\{H_1, H_2\}$ -free, and suppose that  $\chi(G) > \phi(\omega(G))$ . Since  $G$  is  $H_2$ -free, it follows from the choice of  $\phi$  that  $G$  contains a  $(\phi_1, q)$ -scattering  $(w, P, Q)$  say. Let  $r$  be the maximum, over  $v \in P$ , of the chromatic number of the set of neighbours of  $v$  in  $Q$ . Since  $\chi(P) > \phi_1(\omega(P))$ , there is an induced subgraph of  $G[P]$  isomorphic to  $J_1$ , say  $J'_1$ . Hence  $G$  contains the cone of  $J_1$ , so we may assume that  $H_1$  is the disjoint union of  $J_1, J_2$ . The set of vertices in  $Q$  with a neighbour in  $V(J'_1)$  has chromatic number at most  $r|J_1|$ , and since

$$\chi(Q) > |J_1|r + \phi_2(\omega(Q)),$$

it follows that the set (say  $Q'$ ) of vertices in  $Q$  that are anticomplete to  $J'_1$  has chromatic number more than  $\phi_2(\omega(Q))$ . From the choice of  $\phi_2$ , and since  $G$  is  $H_2$ -free, it follows that  $G[Q']$  is not  $J_2$ -free; but then, combining this copy of  $J_2$  with  $J'_1$ , we find a copy of  $H_1$  in  $G$ , a contradiction. This proves 1.4. ■

Let  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing function. We say a subgraph  $P$  of a graph  $G$  is  $\sigma$ -*nondominating* if there is a set  $X \subseteq V(G) \setminus V(P)$ , anticomplete to  $V(P)$ , with  $\chi(X) > \sigma(\omega(X))$ . Next we will show that to prove 2.1 it suffices to prove the following:

**2.2** Let  $\psi, \sigma : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing polynomials, and let  $q \geq 0$  an integer. Let  $H$  be a broom, and let  $J$  be a path. Then there is a non-decreasing polynomial  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that if  $G$  is a graph, and  $\chi(G) > \phi(\omega(G))$ , and  $G$  contains no  $(\psi, q)$ -scattering, then  $G$  contains  $H$  and a  $\sigma$ -nondominating copy of  $J$ .

**Proof of 2.1, assuming 2.2.** Let  $\psi, q, H_2$  be as in 2.1. If  $H_2$  is a broom, then 2.1 follows immediately from 2.2 (setting  $H = H_2$  and setting  $J$  to be some path, for instance the one-vertex path). Thus we assume that  $H_2$  is the disjoint union of a good forest  $J_1$  and a forest  $J_2$  that is a disjoint union of paths. Let  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing function such that every  $J_1$ -free graph  $G$  has chromatic number at most  $\sigma(\omega(G))$ ; and choose a path  $J$  such that  $J_2$  is an induced subgraph of  $J$ . By 2.2 (setting  $H$  to be some broom, for instance with one vertex) there is a non-decreasing polynomial  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that if  $\chi(G) > \phi(\omega(G))$  and  $G$  contains no  $(\psi, q)$ -scattering, then  $G$  contains a  $\sigma$ -nondominating copy  $J'$  of  $J$ .

We claim that  $\phi$  satisfies 2.1. Thus we must show that if  $G$  is  $H_2$ -free and contains no  $(\psi, q)$ -scattering then  $\chi(G) \leq \phi(\omega(G))$ . Suppose not. By the choice of  $f$ , and since  $G$  contains no  $(\psi, q)$ -scattering, it follows that  $G$  contains a copy  $J'$  of  $J$ , such that there is a set  $X \subseteq V(G)$  with  $\chi(X) > \sigma(\omega(X))$  anticomplete to  $V(J'_1)$ . But since  $\chi(X) > \sigma(\omega(X))$ , it follows that  $G[X]$  contains  $J_1$ , and since  $J$  contains  $J_2$ , and  $V(J)$  is anticomplete to  $X$ , it follows that  $G$  contains  $H_2$ . This proves 2.1. ■

We remark that there is an appealing possible strengthening of 2.2, that we could not prove:

**2.3 Conjecture:** Let  $\psi, \sigma : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing polynomials, let  $q \geq 0$  an integer, and let  $H$  be a broom. Then there is a non-decreasing polynomial  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that if  $G$  is a graph, and  $\chi(G) > \phi(\omega(G))$ , and  $G$  contains no  $(\psi, q)$ -scattering, then  $G$  a  $\sigma$ -nondominating copy of  $H$ .

Let us say a graph  $H$  is *self-isolating* if for every non-decreasing polynomial  $\psi : \mathbb{N} \rightarrow \mathbb{N}$ , there is a polynomial  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  with the following property: for every graph  $G$  with  $\chi(G) > \phi(\omega(G))$ , there exists  $A \subseteq V(G)$  with  $\chi(A) > \psi(\omega(A))$ , such that either

- $G[A]$  is  $H$ -free, or
- $G$  contains a copy  $H'$  of  $H$  such that  $V(H')$  is disjoint from and anticomplete to  $A$ .

Which graphs are self-isolating? It is proved in [10] that stars are self-isolating, and we will show in [2] that complete graphs and complete bipartite graphs are self-isolating. Let us observe that 2.2 implies that:

**2.4** Every path is self-isolating.

**Proof.** Let  $J$  be a path, and let  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  be a non-decreasing polynomial. Choose  $\phi$  satisfying 2.2 with  $H = J$  and  $\sigma = \psi$  and  $q = |J|$ , and let  $G$  be a graph with  $\chi(G) > \phi(\omega(G))$ . We claim that either there is a  $\psi$ -nondominating copy of  $J$  in  $G$ , or there exists  $A \subseteq V(G)$  with  $\chi(A) > \psi(\omega(A))$  such that  $G[A]$  is  $J$ -free. By 2.2 we may assume that there is a  $(\psi, q)$ -scattering  $(w, P, Q)$  in  $G$ . If  $G[P]$  is  $J$ -free, the claim holds, so we assume that there is a copy  $J'$  of  $J$  in  $G[P]$ . Thus  $|J'| = q$ . Let  $r$  be the maximum over  $v \in P$  of the chromatic number of the set of neighbours of  $v$  in  $Q$ . The set of vertices in  $Q$  with a neighbour in  $V(J')$  has chromatic number at most  $|J'|r = qr$ ; and  $\chi(Q) > \psi(\omega(Q)) + qr$  from the definition of a  $(\psi, q)$ -scattering. Consequently  $J'$  is  $\psi$ -nondominating, and hence  $J$  is self-isolating. This proves 2.4. ■

### 3 Constructing a horn

Let  $d \geq 0$  be an integer. If  $A, B \subseteq V(G)$  are disjoint, we say that  $A$  is  $d$ -dense to  $B$  if for every vertex  $v \in A$ , the set of non-neighbours of  $v$  in  $B$  has chromatic number at most  $d$ . Let us say a  $(d, z)$ -horn in a graph  $G$  is a triple  $(v, A, B)$  where

- $A, B$  are disjoint subsets of  $V(G)$ , and  $v \in V(G) \setminus (A \cup B)$ ;
- $v$  is complete to  $A$  and anticomplete to  $B$ ; and
- there is no  $Z \subseteq A \cup B$  with  $\chi(Z) \leq z$  such that  $A \setminus Z$  is  $d$ -dense to  $B \setminus Z$ .

We will need a  $(d, z)$ -horn  $(v, A, B)$  where  $z$  is at least some large function of the clique number of  $A \cup B$ , and this section produces such a horn. We show in 3.5 that if  $G$  has sufficiently large chromatic number (and, for convenience, all its proper induced subgraphs have smaller chromatic number), then either  $G$  contains both a  $(k, s)$ -broom and a  $\sigma$ -nondominating  $k$ -vertex path, or  $G$  contains a  $(d, z)$ -horn. To complete the proof of 2.2, it therefore suffices to handle graphs  $G$  that contain  $(d, z)$ -horns, for suitably chosen values of  $d, z$ , and we will do so in the next section.

We will use the following well-known version of Ramsey's theorem, proved (for instance) in [10] ( $|G|$  denotes the number of vertices of  $G$ ):

**3.1** *Let  $x \geq 2$  and  $y \geq 1$  be integers. For a graph  $G$ , if  $|G| \geq x^y$ , then  $G$  has either a clique of cardinality  $x + 1$ , or a stable set of cardinality  $y$ .*

If  $v \in V(G)$ , we denote by  $N(v)$  or  $N_G(v)$  the set of all neighbours of  $v$  in  $G$ . First, we need a result of Gyárfás [5] (we give the well-known proof, because it is so pretty.)

**3.2** *Let  $k \geq 1$  and  $x \geq 0$  be integers. Let  $G$  be a connected graph such that  $\chi(N(v)) \leq x$  for every vertex  $v$ . Let  $H$  be a connected induced subgraph of  $G$ , and let  $v \in V(G) \setminus V(H)$  with a neighbour in  $V(H)$ . If  $\chi(H) > (k - 2)x$ , there is an induced  $k$ -vertex path of  $G$  with one end  $v$  and all other vertices in  $V(H)$ .*

**Proof.** We proceed by induction on  $k$ . The result is clear for  $k \leq 2$ , so we assume that  $k \geq 3$ . Let  $J$  be obtained from  $H$  by deleting all vertices in  $N(v)$ ; thus  $\chi(J) > (k - 3)x > 0$ , and so there is a component  $H'$  of  $J$  with chromatic number more than  $(k - 3)x$ . Let  $v' \in N(v) \cap V(H)$  with a neighbour in  $V(H')$ . From the inductive hypothesis applied to  $v', H'$ , there is an induced  $(k - 1)$ -vertex path of  $G$  with one end  $v'$  and all other vertices in  $V(H')$ . Appending  $v$  to this path proves 3.2. ■

We deduce:

**3.3** *Let  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing, let  $k, x \geq 1$  be integers, and let  $G$  be a graph. If  $\chi(N(v)) \leq x$  for every  $v \in V(G)$ , and  $\chi(G) > kx + \sigma(\omega(G))$ , then there is a  $\sigma$ -nondominating  $k$ -vertex induced path  $P$  in  $G$ .*

**Proof.** We may assume that  $G$  is connected; choose  $v \in V(G)$ . Since  $\chi(G \setminus v) > kx - 1 \geq (k - 2)x$ , 3.2 (applied to  $v$  and to a component of  $G \setminus v$  of maximum chromatic number) implies that  $G$  contains a  $k$ -vertex induced path  $P$ . The set of vertices of  $G$  with a neighbour in  $V(P)$  has chromatic number at most  $kx$ , and the result follows. This proves 3.3. ■

The next result is also essentially due to Gyárfás (mentioned in [5]):

**3.4** *Let  $H$  be a  $(k, s)$ -broom, and suppose that  $G$  is  $H$ -free, and  $\chi(N(v)) \leq x$  for every  $v \in V(G)$ . Then*

$$\chi(G) \leq \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)x).$$

**Proof.** Suppose that  $\chi(G) > \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)x)$ . We may assume that  $G$  is connected. If every vertex of  $G$  has degree less than  $\omega(G)^{2s}$  then  $\chi(G) \leq \omega(G)^{2s}$ , a contradiction, so some vertex  $v$  has at least  $\omega(G)^{2s}$  neighbours. By 3.1 applied to  $G[N(v)]$ , there is a stable set  $S$  of neighbours of  $v$ , with  $|S| = 2s$ . Let  $M$  be the set of all vertices of  $G$  that do not belong to  $S \cup \{v\}$  and have a neighbour in  $S \cup \{v\}$ . Thus  $\chi(M) \leq (2s+1)x$ . Let  $H$  be a component of  $G \setminus (M \cup S \cup \{v\})$  of maximum chromatic number; then  $\chi(H) \geq \chi(G) - (2s+1)(x+1) > (k-2)x$ . Choose  $u \in M \cup S \cup \{v\}$  with a neighbour in  $V(H)$ . Since no vertex of  $S \cup \{v\}$  has a neighbour in  $V(H)$ , from the definition of  $M$ , it follows that  $u \in M$ . By 3.2 applied to  $u, H$ , there is an induced  $k$ -vertex path  $P$  of  $G$  with one end  $u$  and all other vertices in  $V(H)$ . Thus  $u$  is the only vertex of  $P$  with a neighbour in  $S \cup \{v\}$ . If  $u$  is adjacent to at least  $s$  vertices in  $S$ , then the subgraph induced on  $V(P)$  and some  $s$  of these neighbours is a  $(k, s)$ -broom, a contradiction. Thus there exists  $S' \subseteq S$  with  $|S'| = s$ , such that all vertices in  $S'$  are nonadjacent to  $u$ . If  $u$  is adjacent to  $v$ , the subgraph induced on  $V(P) \cup S \cup \{v\}$  is a  $(k+1, s)$ -broom, a contradiction. Thus  $u$  is adjacent to some  $w \in S \setminus S'$ , and nonadjacent to  $v$ . But then the subgraph induced on  $V(P) \cup S' \cup \{v, w\}$  is a  $(k+2, s)$ -broom, a contradiction. This proves 3.4. ■

**3.5** *Let  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing. Let  $k, s, d, z \geq 0$  and  $c \geq 2s$  be integers. Let  $G$  be a graph such that*

$$\begin{aligned} \chi(G) &> \omega(G)^c; \\ \chi(G') &\leq \omega(G')^c \text{ for every induced subgraph } G' \text{ of } G \text{ with } G' \neq G; \\ \omega(G)^c &\geq (\omega(G) - 1)^c + z + d\omega(G) + 2; \\ \omega(G)^c &\geq (2s+1)(z+1) + (k-2)z; \text{ and} \\ \omega(G)^c &\geq kz + \sigma(\omega(G)). \end{aligned}$$

Then either

- $G$  contains a  $(d, z)$ -horn; or
- $G$  contains a  $(k, s)$ -broom, and a  $\sigma$ -nondominating  $k$ -vertex path.

**Proof.** Suppose that  $\chi(N(v)) \leq z$  for every vertex  $v \in V(G)$ . By 3.4, and since

$$\chi(G) > \omega(G)^c \geq \max(\omega(G)^{2s}, (2s+1)(x+1) + (k-2)z)$$

(because  $c \geq 2s$ ), it follows that  $G$  contains a  $(k, s)$ -broom. By 3.3, since  $\chi(G) - kz > \sigma(\omega(G))$ , there is a  $\sigma$ -nondominating  $k$ -vertex induced path  $P$  in  $G$ , and so the second bullet holds.

Thus we assume that  $\chi(N(v)) > z$  for some vertex  $v$ . Let  $A$  be the set of neighbours of  $v$ , and  $B = V(G) \setminus (A \cup \{v\})$ . We claim that  $(v, A, B)$  is a  $(d, z)$ -horn. Suppose not; then there exists  $Z \subseteq A \cup B$  with  $\chi(Z) \leq z$ , such that  $A \setminus Z$  is  $d$ -dense to  $B \setminus Z$ . Let  $P \subseteq A \setminus Z$  be a clique with

cardinality  $p = \omega(A \setminus Z)$ . Then  $p \geq 1$ , since  $\chi(Z) \leq z < \chi(A)$ ; and  $p < \omega(G)$  since otherwise adding  $v$  would give a clique of cardinality  $\omega(G) + 1$ . For each  $u \in P$ , the set of vertices in  $B \setminus Z$  nonadjacent to  $u$  has chromatic number at most  $d$ , since  $A \setminus Z$  is  $d$ -dense to  $B \setminus Z$ ; and so the set of vertices in  $B$  with a non-neighbour in  $P$  has chromatic number at most  $pd \leq d\omega(G)$ . The set of vertices in  $B$  complete to  $P$  has clique number at most  $\omega(G) - p$  and so has chromatic number at most  $(\omega - p)^c$ . Hence  $\chi(B \setminus Z) \leq pd + (\omega(G) - p)^c$ , and so

$$\chi(G) \leq \chi(Z) + \chi(A \setminus Z) + \chi(B \setminus Z) + 1 \leq z + p^c + d\omega(G) + (\omega(G) - p)^c + 1.$$

Since  $1 \leq p \leq \omega(G) - 1$ ,  $p^c + (\omega(G) - p)^c \leq (\omega(G) - 1)^c + 1$ , and so

$$\omega(G)^c < \chi(G) \leq z + d\omega(G) + (\omega(G) - 1)^c + 2,$$

a contradiction. This proves 3.5. ■

## 4 Making taller horns

In this section we prove 2.2, and hence complete the proofs of 2.1, 1.4, 1.3, and therefore 1.2. Because of 3.5, we may assume that  $G$  contains a  $(d, z)$ -horn, for some suitable values of  $d, z$ ; and now we will show that, provided that  $G$  does not contain the proscribed scattering, we can use this horn to make a “ $k$ -tall”  $(d', z')$ -horn, which is a horn with a  $k$ -vertex path appended to its distinguished vertex. From such a horn, it is easy to obtain a  $(k, s)$ -broom and a  $\sigma$ -nondominating  $k$ -vertex path, to satisfy 2.2. The main step is therefore to convert an  $\ell$ -tall horn to an  $(\ell + 1)$ -tall horn, and for that we need the next result.

If  $d, z, \omega \geq 0$  are integers, a graph  $G$  is  $(d, z, \omega)$ -unsplittable if there is no partition  $(A, B, Z)$  of  $V(G)$  such that  $\chi(Z) \leq z$ , and  $\chi(A), \chi(B) > d\omega$ , and  $A$  is  $d$ -dense to  $B$ . We begin with:

**4.1** *If  $d, z \geq 0$  are integers, every graph  $G$  admits a partition  $(D_0, D_1, \dots, D_k)$  of its vertex set with  $k \leq \omega(G)$  such that  $\chi(D_0) \leq z\omega(G)$  and  $G[D_i]$  is  $(d, z, \omega(G))$ -unsplittable for  $1 \leq i \leq k$ .*

**Proof.** We may assume that  $G$  is not  $(d, z, \omega(G))$ -unsplittable, and so it admits a partition  $(D_0, D_1, D_2)$  such that  $\chi(D_0) \leq z$ ,  $\chi(D_1), \chi(D_2) > d\omega(G)$ , and  $D_1$  is  $d$ -dense to  $D_2$ . Hence we may choose  $k \geq 2$  maximum such that there is a sequence  $D_0, D_1, \dots, D_k$  of pairwise disjoint subsets of  $V(G)$  with union  $V(G)$ , and with the following properties:

- $\chi(D_0) \leq (k - 1)z$
- $D_i$  is  $d$ -dense to  $D_j$  for  $1 \leq i < j \leq k$ ; and
- $\chi(D_i) > d\omega(G)$  for  $1 \leq i \leq k$ .

We claim:

(1)  $k \leq \omega(G)$ .

Suppose that  $k > \omega(G)$ , and define  $d_i \in D_i$  for  $1 \leq i \leq \omega(G) + 1$  inductively as follows. Let

$1 \leq i \leq \omega(G) + 1$ , and suppose that  $d_1, \dots, d_{i-1}$  have been defined, all pairwise adjacent. The set of vertices in  $D_i$  that have a non-neighbour among  $d_1, \dots, d_{i-1}$  has chromatic number at most

$$(i-1)d \leq d\omega(G) < \chi(D_i),$$

and so some vertex  $d_i \in D_i$  is adjacent to all of  $d_1, \dots, d_{i-1}$ . This completes the inductive definition. But then  $\{d_1, \dots, d_{\omega(G)+1}\}$  is a clique of  $G$ , contradicting the definition of  $\omega(G)$ . This proves (1).

(2) For  $1 \leq i \leq k$ ,  $G[D_i]$  is  $(d, z, \omega(G))$ -unsplittable.

Suppose that  $(A, B, Z)$  is a partition of  $D_i$  such that  $\chi(Z) \leq z$ , and  $\chi(A), \chi(B) > d\omega(G)$ , and  $A$  is  $d$ -dense to  $B$ . Then the sequence

$$(D_0 \cup Z, D_1, \dots, D_{i-1}, A, B, D_{i+1}, \dots, D_k)$$

contradicts the maximality of  $k$ . This proves (2).

From (1), (2), this proves 4.1. ■

Let  $(v, A, B)$  be a  $(d, z)$ -horn in a graph  $G$ , and let  $k \geq 1$  be an integer. We say that  $(v, A, B)$  is  $k$ -tall if there is an induced path  $P$  in  $G$  with  $k$  vertices, with one end  $v$ , such that  $V(P) \setminus \{v\}$  is disjoint from and anticomplete to  $A \cup B$ . Thus every  $(d, z)$ -horn is 1-tall. We use 4.1 to prove a result which is the heart of the paper:

**4.2** Let  $G$  be a graph, let  $d, z, d', z', q \geq 0$  be integers, and let  $\psi : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing, satisfying:

$$\begin{aligned} z &\geq (2\psi(\omega(G)) + (1+q)z' + qd'\omega(G))\omega(G) \\ d &\geq (z' + d'\omega(G))\omega(G). \end{aligned}$$

Let  $(v, A, B)$  be an  $\ell$ -tall  $(d, z)$ -horn in a graph  $G$ , for some  $\ell \geq 1$ . Then either

- there exist  $P \subseteq A$  and  $Q \subseteq B$  such that  $(v, P, Q)$  is a  $(\psi, q)$ -scattering; or
- there exist  $v' \in A$  and disjoint subsets  $A', B'$  of  $B$  such that  $(v', A', B')$  is an  $(\ell+1)$ -tall  $(d', z')$ -horn.

**Proof.** Let  $p = \psi(\omega(G))$ . By 4.1,  $B$  admits a partition  $(D_0, D_1, \dots, D_k)$  with  $k \leq \omega(G)$  such that  $\chi(D_0) \leq z'\omega(G)$  and  $G[D_i]$  is  $(d', z', \omega(G))$ -unsplittable for  $1 \leq i \leq k$ . For  $1 \leq i \leq k$ , if  $\chi(D_i) \leq q(z' + d'\omega(G)) + p$  let  $P_i = \emptyset$ , and if  $\chi(D_i) > q(z' + d'\omega(G)) + p$  let  $P_i$  be the set of vertices  $a \in A$  such that  $\chi(U) \leq z' + d'\omega(G)$ , where  $U$  is the set of neighbours of  $a$  in  $D_i$ . Let  $P = P_1 \cup \dots \cup P_k$ .

Suppose that  $\chi(P_i) > p$ , for some  $i \in \{1, \dots, k\}$ . Consequently  $P_i \neq \emptyset$ , and so

$$\chi(D_i) > q(z' + d'\omega(G)) + p \geq q(z' + d'\omega(G)) + \psi(\omega(P_i));$$

and for each  $a \in P_i$ ,  $\chi(U) \leq z' + d'\omega(G)$ , where  $U$  is the set of neighbours of  $a$  in  $D_i$ . It follows that  $(v, P_i, D_i)$  is a  $(\psi, q)$ -scattering and the first bullet of the theorem holds. Thus we may assume that  $\chi(P_i) \leq p$  for  $1 \leq i \leq k$ , and consequently  $\chi(P) \leq p\omega(G)$ .

Let  $Z$  be the union of  $P, D_0$ , and all the sets  $D_i$  with  $1 \leq i \leq k$  such that

$$\chi(D_i) \leq q(z' + d'\omega(G)) + p.$$

Consequently

$$\chi(Z) \leq 2p\omega(G) + z'\omega(G) + q(z' + d'\omega(G))\omega(G) \leq z.$$

Since  $(v, A, B)$  is a  $(d, z)$ -horn, it follows that  $A \setminus Z$  is not  $d$ -dense to  $B \setminus Z$ ; and so there exists  $v' \in A \setminus P$  such that the set of vertices in  $B \setminus Z$  that are nonadjacent to  $v'$  has chromatic number more than  $d$ . Since  $B \setminus Z$  is the union of the sets  $D_i$  with  $\chi(D_i) > q(z' + d'\omega(G)) + p$ , there exists  $i \in \{1, \dots, k\}$  with  $\chi(D_i) \geq q(z' + d'\omega(G)) + p$  such that the set  $B'$  of vertices in  $D_i$  nonadjacent to  $v'$  has chromatic number more than  $d/\omega(G)$ . Since  $v' \notin P$ , the set  $A'$  of neighbours of  $v'$  in  $D_i$  has chromatic number more than  $d'\omega(G) + z'$ .

Let  $Z' \subseteq D_i$  with  $\chi(Z') \leq z'$ . Thus  $\chi(A' \setminus Z') \geq \chi(A') - \chi(Z') > d'\omega(G)$ ; and  $\chi(B' \setminus Z') > d/\omega(G) - z' \geq d'\omega(G)$ . Since  $G[D_i]$  is  $(d', z', \omega(G))$ -unsplittable, it follows that  $A' \setminus Z'$  is not  $d'$ -dense to  $B' \setminus Z'$ . This proves that  $(v', A', B')$  is a  $(d', z')$ -horn.

Since  $(v, A, B)$  is  $\ell$ -tall, there is an  $\ell$ -vertex induced path  $P$  of  $G$  with one end  $v$ , such that  $V(P) \setminus \{v\}$  is disjoint from and anticomplete to  $A \cup B$ . Then  $P' = G[V(P) \cup \{v'\}]$  is an  $(\ell + 1)$ -vertex path, and since  $V(P)$  is anticomplete to  $B$  and hence to  $A' \cup B'$ , it follows that  $(v', A', B')$  is  $(\ell + 1)$ -tall, and so the second bullet of the theorem holds. This proves 4.2.  $\blacksquare$

Now we prove 2.2, which we restate:

**4.3** *Let  $k, s \geq 1$  and  $q \geq 0$  be integers, and let  $\psi, \sigma : \mathbb{N} \rightarrow \mathbb{N}$  be non-decreasing polynomials. Then there exists an integer  $c \geq 0$  such that if  $G$  is a graph with  $\chi(G) > \omega(G)^c$ , and  $G$  contains no  $(\psi, q)$ -scattering, then  $G$  contains a  $(k, s)$ -broom and a  $\sigma$ -nondominating  $k$ -vertex path.*

**Proof.** Let  $\zeta_k : \mathbb{N} \rightarrow \mathbb{N}$  be the polynomial defined by  $\zeta_k(x) = \sigma(x) + x^s$ , and let  $\delta_k(x) = 0$ . For  $i = k - 1, \dots, 1$ , define polynomials  $\zeta_i, \delta_i : \mathbb{N} \rightarrow \mathbb{N}$  by

$$\begin{aligned}\zeta_i(x) &= 2x\psi(x) + (1 + q)x\zeta_{i+1}(x) + qx^2\delta_{i+1}(x) \\ \delta_i(x) &= x\zeta_{i+1}(x) + x^2\delta_{i+1}(x).\end{aligned}$$

Choose an integer  $c \geq 2s$  such that

$$\begin{aligned}x^c &\geq (x - 1)^c + \zeta_1(x) + x\delta_1(x) + 2 \\ x^c &\geq (2s + 1)(\zeta_1(x) + 1) + (k - 2)\zeta_1(x), \text{ and} \\ x^c &\geq k\zeta_1(x) + \sigma(x)\end{aligned}$$

for all integers  $x \geq 2$ . We claim that  $c$  satisfies 4.3. To see this, let  $G$  be a graph with  $\chi(G) > \omega(G)^c$ , and suppose that  $G$  contains no  $(\psi, q)$ -scattering. We must show that  $G$  contains a  $(k, s)$ -broom and a  $\sigma$ -nondominating  $k$ -vertex path. We show this by induction on  $|G|$ . If there is an induced subgraph  $G'$  of  $G$  with  $G' \neq G$  and  $\chi(G') > \omega(G')^c$ , then  $G'$  contains no  $(\psi, q)$ -scattering, and from the inductive hypothesis,  $G'$  contains a  $(k, s)$ -broom and a  $\sigma$ -nondominating  $k$ -vertex path, and hence so does  $G$ , as required. We may assume then that there is no such  $G'$ . Since  $\chi(G) > \omega(G)^c$ , it follows that  $\omega(G) \geq 2$ , and so the five displayed inequalities of 3.5 hold with  $z, d$  replaced by

$\zeta_1(\omega(G)), \delta_1(\omega(G))$  respectively. From 3.5, we may assume that  $G$  contains a  $(\delta_1(\omega(G)), \zeta_1(\omega(G)))$ -horn, which is therefore 1-tall.

From 4.2, it follows that for  $i = 2, \dots, k$ ,  $G$  contains an  $i$ -tall  $(\delta_i(\omega(G)), \zeta_i(\omega(G)))$ -horn, and so contains a  $k$ -tall  $(0, z)$ -horn  $(v, A, B)$  say, where  $z = \zeta_k(\omega(G))$ . Since this horn is  $k$ -tall, there is a  $k$ -vertex induced path  $P$  of  $G$  with one end  $v$ , such that  $V(P) \setminus \{v\}$  is disjoint from and anticomplete to  $A \cup B$ . From the definition of a  $(0, z)$ -horn,  $\chi(A), \chi(B) > z$ . Since  $\chi(A) > z \geq \omega(A)^s$ , 3.1 implies that there is a stable set  $S \subseteq A$  with  $|S| = s$ , and so  $G[V(P) \cup S]$  is a  $(k, s)$ -broom. Since  $\chi(B) > z > \sigma(\omega(B))$ , and  $V(P)$  is anticomplete to  $B$ ,  $P$  is  $\sigma$ -nondominating. This proves 4.3. ■

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