Teachers' Perceptions of Mathematical Tasks. An exploration of what teachers 'saw' in a published task, and how this changed as a result of experience of enactment of the same task

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Teachers' perception of mathematical tasks

An exploration of what teachers 'saw' in a published task, and how this changed as a result of experience of enactment of the same task.

Abstract

What do teachers perceive when presented with a mathematical task? This study reviewed the literature on tasks, teachers' use and relationship with tasks, perception and teachers' pedagogical decision making. Three teachers' submitted written planning notes, and interview responses were analysed to identify lesson images representing their intended use of a particular task. Teachers' beliefs, knowledge and goals were related to these lesson images, and the impact of the teachers' observation of an enactment of the task was discussed.

The three teachers had distinct plans for how they intended to use the same task, as shown in lesson images inferred from their written scripts and interviews. This difference in intended use reflected variation in how the task was perceived, and in how teachers' beliefs, knowledge and goals informed pedagogical decision making. Teachers adapted the task better to align with their own beliefs, knowledge and goals, but the design of the task also allowed and constrained teachers' decisions. Teachers formed a relationship with this particular task, which they either developed, challenged or acquiesced to, based on their differing perception of the task. Some teachers' experience of enactments of the task – observation of another teacher's lesson, and teaching their own lesson – led to changes in teacher's goals as reflected in their revised plans.

Useful revisions to teachers' plans were made after teaching using the task, which was too late for students in those lessons. An implication for future practice is to investigate ways in which teachers might refine lessons prospectively by more effectively anticipating student learning. The established Japanese practice of kyozai kenkyuu is offered as one such practice which offers this prospect.

1 Introduction, context and rationale

Mathematical tasks are a ubiquitous feature of mathematics teaching and learning, and teachers of mathematics have ready access to a wide range of published materials (Thompson & Watson, 2013:281). We might expect one given mathematical activity to result in one particular set of prescribed learning outcomes, but despite the mandates, recommendations and pleas of policy-makers and reformers seeking to establish a particular approach to the teaching and learning of mathematics, the literature suggests that this is not the case: tasks evolve as they are set up, enacted and implemented in actual lessons, and associated outcomes are not easily anticipated (Stein, Grover, & Henningsen, 1996). Amongst many contributory factors, Remillard (2005) found that teachers' interpretation of curriculum materials appeared to be key in accounting for the many destinations arising from one common starting point, despite each teacher's conviction that their approach was faithful to the published materials (Remillard, 2005:220). In her analysis of key concepts in research on teachers' use of mathematics curricula, Remillard (2005) went on to propose a model describing attributes of the teacher, the curriculum, and the participatory relationship between the two.

The context for this study is in my work as an advisory teacher for mathematics for a Local Authority, working with teachers in the mathematics department of a secondary (age 11-19) academy, the name given to state-maintained schools which are directly accountable to central rather than local government. The three-week research project was undertaken as part of an ongoing weekly professional development programme at the academy. One strand of that programme consisted of encouraging teachers to use particular suggested teaching materials in class in between sessions, aspects of which were subsequently discussed. An early example of a question used in a session is given in Figure 1:

65 + 37 = 67 + 35 True / False?

What is the purpose of asking students this question?

Figure 1 A question used in a teacher professional development session

Teachers' responses to this prompt included:

- Addition of 2-digit numbers
- Partitioning / Place value / structure of (base 10) number
- Adjustment / compensation strategies
- The meaning of the 'equals' symbol

There is opportunity for each of these aspects to be made available for learning, from a single starting point, and teachers differed in which aspects they 'saw' first. For individual teachers, some aspects were less visible than others, what Tahta (1980), referenced by Watson and Mason, referred to as inner tasks, where

> the task as specified or undertaken is the explicit or outer task. The implicit intention of the task, that is, what it affords of mathematical themes, concepts, theorems, connections to other topics and techniques, multiplicity of approaches, interpretations and re-presentations, has been referred to as the inner task (Watson & Mason, 2007:207).

Does what teachers see in a task affect how they plan to use a task? Teachers' initial appreciation or ignorance of what is available for learning changes what might happen in a lesson, because 'to steer instruction toward the mathematical point, the mathematical point must be known' (Sleep, 2012:4). Sullivan, Clarke and Clarke (2009) surveyed teachers involved in a project to design lessons from tasks, detailing the extent to which teachers' lessons had general or specific strategies related to the given task. After one year of engagement the authors found that a significant group were proposing 'somewhat vague lessons … [which] might suggest that teachers need to identify the purpose of the task before being able to design an appropriate lesson' (Sullivan, Clarke, & Clarke, 2009:100). The expected mathematical goal is an important aspect of how teachers set up a lesson, which for Joubert must involve 'careful analysis of the mathematics that might be provoked by the task, of the tools used to mediate the activity of the students and – crucially – the prior learning of the students' (Joubert, 2013:73).

The variation in teachers' initial appreciation of what was available in a given task was the starting point for this research. How does teacher perception of tasks vary – what do teachers 'see' – and what factors might account for that? How might this teacher perception change as a result of experience of enactment of the same task, either teaching their own lesson or observing a colleague's?

2 Literature Review

Rather than starting with the topic of this study, teachers' perception of mathematical tasks, this literature review starts with broad consideration of questions relating to mathematical tasks in the classroom, before focusing on the relationship of teachers and tasks, and the particular issue of tasks as a site of teacher learning. Acknowledging the variety of purposes for which tasks can be designed, I consider perception quite generally and then in the context of teachers perceiving mathematical tasks, finally relating this to teachers' pedagogical decision making.

2.1 Mathematical tasks in the classroom

How tasks are designed and how teachers relate to those tasks is important because of the prevalence of tasks in mathematics classrooms. In 2004 Sierpinska called on the mathematics education research community to refocus its attention on mathematical tasks, which she described as 'the very fabric of mathematics teaching and learning' (Sierpinska, 2004:25). This call was followed within a decade by an ICMI study into task design in mathematics education (Margolinas, 2013). In the introduction to the ICMI proceedings the authors described tasks as 'the mediating tools for teaching and learning mathematics' (Watson et al., 2013:12), whose pedagogic use and relation to learning are key issues, and most of which originate in collections of resources published as textbooks or online (Thompson & Watson, 2013:281). When not referring to the particular task which was used in this study, the meaning of 'task' in this literature review follows the definition used in the ICMI study, namely 'anything that a teacher uses to demonstrate mathematics, to pursue interactively with students, or to ask students to do something' (Margolinas, 2013:12). Terminologically, 'curriculum' in this literature review is considered as synonymous with 'task' other than in grain size; accounts of

teachers' use, consideration and relationship with curriculum are the same here as with task, which holds inasmuch as tasks constitute the curriculum.

Although curriculum might be considered as comprising tasks, how tasks relate to learning is a matter of debate. Doyle (1988) wrote influentially on academic tasks in the mathematics classroom, in research which was 'based on the premise that the tasks teachers assign determines [sic] how students come to understand a curriculum domain' (Doyle, 1988:167). 25 years later Ainley and Margolinas pointed out a tacit assumption that completion of tasks chosen by the teacher would result in students learning the intended mathematics, a view which persisted despite research findings which disputed such a simple relationship (Ainley & Margolinas, 2013:153). One such study which revealed this complexity was from US mathematics educators Stein, Grover and Henningsen (1996), who built on Doyle's finding that 'a central theme in academic-task research is the extent to which tasks can change their character once unleashed in real classroom settings' (Stein et al., 1996:460). In their model, shown in Figure 2, tasks are described as existing at three distinct levels: as represented in

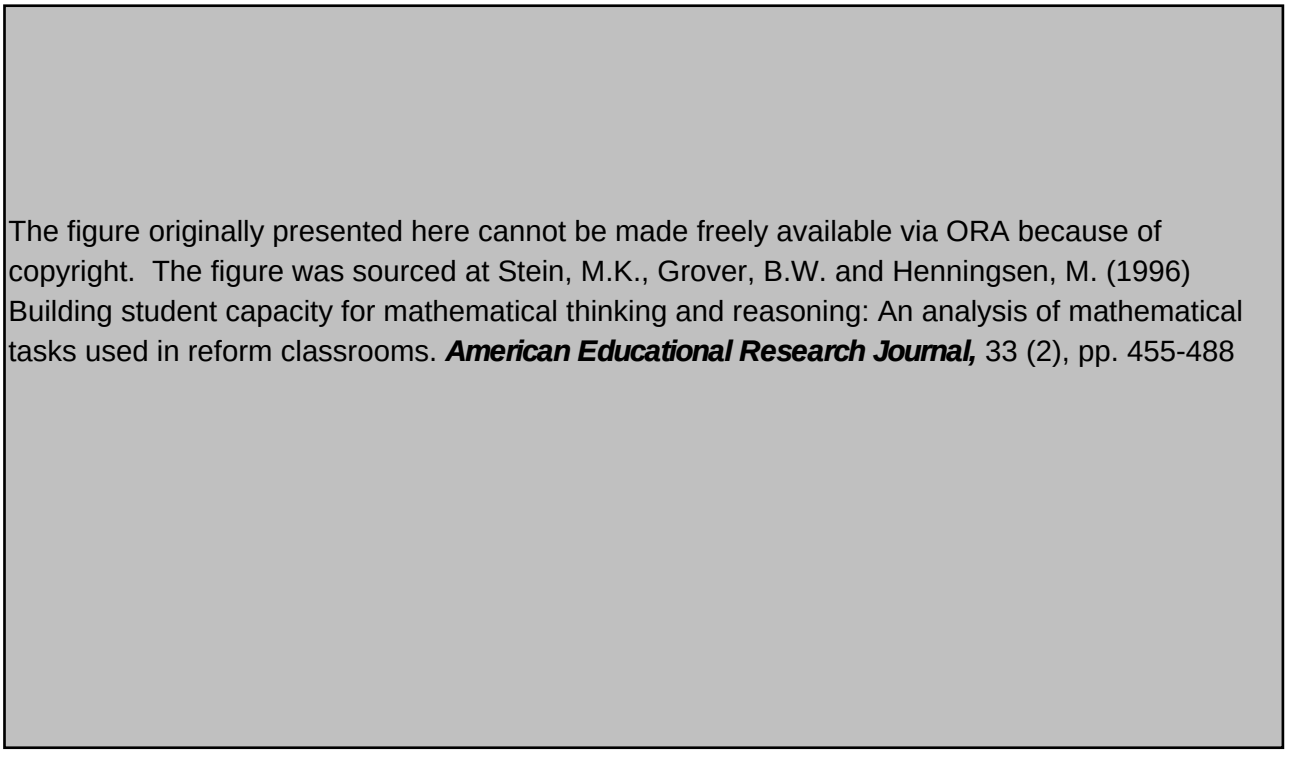


Figure 2 Relationship among various task-related variables and student learning.

(Stein, Grover & Henningsen, 1996:459)

instructional materials; as set up by the teacher; as implemented by students in the classroom. From each viewpoint, an apparently singular task can be distinct, because students' 'understanding of the objectives of the task can transform the task, [and] teachers also can wittingly (or unwittingly) change the nature of tasks by stressing less- or more-challenging aspects of the tasks or by altering the resources available to students' (Stein et al., 1996:460). These authors used Doyle's (1988) notion of task, incorporating the associated activity; methodologically, Stein et al.'s focus on objectives allowed for the identification of particular aspects of tasks which the authors identified as changing.

The shaded areas of the figure show the focus of the Stein et al. study as the relationship between the task as set up by a teacher and as implemented by students in the classroom, and explanatory factors. The authors reported that tasks demanding higher cognitive demand often resulted in a decline: the task as implemented and experienced by students was at a lower level than the teacher had intended, with teachers described as either succumbing to student pressure to reduce complexity, or resorting to 'telling' when students faced challenging aspect of the task (Stein et al., 1996:479-480).

Despite setting up the task for appropriately challenging learning, the lesson as enacted was different from – and reported as lower than – the teacher's original expectations. Other than the factors given in the first, unshaded, circle in Figure 2, the authors did not report on the differences between the task as given in instructional materials and the task as set up, maybe in part because 39% of the tasks used were created by teachers or otherwise from the school, rather than published curricular materials. This prevalence of unpublished curricular materials may be atypical more recently; introducing a theme in the 2013 ICMI study on task design in mathematics education, Thompson and Watson recognised that 'most teachers use textbooks and/or online packages of materials as their total or main source of tasks' (Thompson & Watson, 2013:281).

Stein et al.'s (1996) work was important in establishing a process of change which was seen to affect mathematical tasks as they were used by teachers and students, and in introducing a notion of perspective when analysing this change. However, they focused more on the interface between the

teacher's intention and what was enacted in the classroom rather than the interface between the teacher's intention and the task as given, which is closer to the focus of this study. In the next section I review further the literature on the relationship between teachers and tasks.

2.2 Participatory relationship between teachers and tasks

Stein et al.'s (1996) analysis of tasks took a particular stance regarding the role of the teacher, implicitly assuming that teachers were striving to meet certain expectations in line with a curriculum reform effort. Remillard (2005) identified and challenged the body of research holding this stance in her examination of key concepts in research on teachers' use of mathematics curricula. She found that 'many studies of curriculum material use take the text as the starting point and then consider the degree to which teachers follow or subvert it' (Remillard, 2005:216), constituting a research paradigm which sought to comment on the extent to which teachers were faithful to the curricular intentions of the given materials.

More teacher agency was implied by Shulman (1986) in his important categorisation of teacher knowledge when he asked: 'how do teachers take a piece of text and transform their understanding of it into instruction that their students can comprehend?' (Shulman, 1986:8), part of his conception of pedagogical content knowledge. Rowland also identified teachers' use of instructional materials as a critical feature of practice, adding this code within the 'transformation' dimension in a revision of his Knowledge Quartet (Rowland, 2013:21). This alternate view considers teachers' modification-in-use of tasks less as a possible cause of error, and more as an integral part of teaching, a perspective which for Remillard frames

> the teacher as interpreter of the written curriculum. This outlook holds to an interpretive view of text and assumes that fidelity between classroom action and written words in a teacher's guide is impossible, that teachers bring their own beliefs and experiences to their encounters with curriculum to create their own meanings, and that by using curriculum materials teachers interpret the intentions of the authors (Remillard, 2005:219-220).

In this view teachers have authority to make their own interpretation of tasks, which informs their decision making, described by Brown as a 'process of design in which teachers use curriculum materials in unique ways as they craft instructional episodes' (Brown, 2009:18), with teachers choosing to offload, adapt or improvise curriculum materials. Brown (2009) compared tasks and teaching to sheet music and concert performance, to make the point that just as one piece of written music is interpreted uniquely each time it is performed by a player, so a task is interpreted uniquely by a teacher in the enactment of a lesson (Brown, 2009:17). The curriculum materials themselves are 'static representations of abstract concepts and dynamic activities – a means for transmitting and producing activity, not the activity itself' (Brown, 2009:21). His Design Capacity for Enactment framework sought to account for variation in terms of features of curriculum resources and teacher knowledge and beliefs (Brown, 2009:26).

Building on her analysis in her extensive review of concepts used in research on teachers' curriculum use, Remillard (2005) proposed a subsuming model in which a participatory relationship between teacher and curriculum was central. Many factors accounted for how teachers affect the curriculum, including their knowledge, beliefs and capacity for pedagogical design. Specifically, Remillard called for further research on individual teachers' 'perception of and stance toward curriculum materials and the teacher's professional identity as it relates to the use of curriculum resources' (Remillard, 2005:237), because this perception has a direct effect on the relationship with the task. While these factors sought to account for teachers' impact on curricular materials, in Remillard's framework the teacher's role as an agent was not privileged: in effecting change on the curriculum, the teacher was changed. Remillard (2005) characterised curriculum as the representation of tasks and concepts – the actual task or concept is what is enacted rather than what is recorded for the purpose of communication and sharing, and this physical artifact will have a particular voice or look which affects how it is perceived. French mathematics educators Gueudet and Trouche (2009) similarly allowed for an impact of curricular materials on teachers, describing a process of instrumentation in which the

‘affordances and constraints of the artifact influence the subject's activity’ (Gueudet & Trouche, 2009:204), part of an ongoing cyclic process of documentational genesis in which teachers appropriate curricular resources for use in teaching.

For Remillard (2005) and Gueudet & Trouche (2009), teachers are both agents and subjects of change with respect to tasks and curriculum materials. Teachers can adapt given curriculum materials, but this agency is not without constraint. The context within which teachers operate, teachers’ knowledge, understanding and beliefs and orientation towards mathematics and the curriculum each have an impact as teachers transform (and are transformed by) curriculum materials for use in teaching.

With this conception of teachers as interpreters rather than conduits of the curriculum there is an associated change in expectations for teacher activity. Rather than seeking teachers’ compliance, a given task invites teacher interpretation and is an opportunity for a teacher further to develop their professional identity. This capacity for tasks to be the site of teacher development and learning is the focus of the next section of this literature review.

2.3 Educative curriculum materials

Given the need for teachers to interpret tasks and to have a capacity to make design decisions, writing twenty years ago Ball and Cohen (1996) called for greater attention to designing curriculum materials with teacher learning in mind, resulting in what have become known as educative curriculum materials, which do not make ‘heroic assumptions about each teacher's capacities as an original designer of curriculum’ (Ball & Cohen, 1996:7). This was acknowledged by curriculum designers such as Russell, who described the sufficiency of creating curriculum materials which were ‘good enough’ (Russell, 1997:251) for teachers to use as the basis for their own decision making. Rather than being constrained to faithful and compliant delivery of curriculum materials, ‘teachers work with their own understanding of the material, which shapes their interpretations of what the central ideas are, how they hear, evaluate, and respond to students' ideas, and how they decide how to focus and frame the

material for students' (Ball & Cohen, 1996:7).

Outlining design heuristics for written forms of such educative curriculum materials, Davis and Krajcik (2005) identified alignment of a teacher's beliefs with those of the curriculum among the factors affecting the materials' success in promoting teacher learning. They described this learning in terms of changes in a teacher's knowledge and practice, first noting how educative curriculum materials could help the teacher 'anticipate and interpret what learners may think about or do in response to instructional activities' (Davis & Krajcik, 2005:5). A teacher's practice, beliefs, knowledge, and capacity to contemplate learners' behaviour also featured in Clarke and Hollingsworth's interconnected model of professional growth (Clarke & Hollingsworth, 2002:951), although there a teacher's beliefs, considered together with knowledge and attitude as comprising the personal domain, were subject to change, rather than a fixed characteristic of which alignment with the curriculum is a mediating factor in teacher learning. Clarke and Hollingsworth (2002) described a causal chain, one of several routes through their model, which resulted in a change in this personal domain, after an external stimulus had led first to changes in the domains of practice and consequence. They gave the case of Robert, a teacher whose beliefs were unchanged after initial engagement with some novel teaching materials and approaches introduced in a training session; in Davis and Krajcik's (2005) terms we might say that his beliefs were misaligned with those of the curriculum. Actually using the materials (a change in the domain of practice) led to a change in what he considered salient outcomes (a change in the domain of consequence). His new-found appreciation of student learning in an investigative approach then did lead to change in his beliefs (the personal domain). The same original stimulus had not been been sufficiently convincing to lead directly to a change in this teacher's beliefs, but through a process of enactment and reflection from that starting point he experienced professional development as a change in several domains, despite his apparent initial reluctance.

Tasks can be intentionally designed to promote teacher learning, and educators have developed a

theoretical basis and expertise to this end. Thoughtful and rigorous design is not sufficient, however, for the task to have the intended effect, which also relies on the teacher's active participation both in enactment and reflection on the task.

2.4 Task purpose can be obscured

Imagine that instead of being presented with a printed task, or reading it online on a screen, a teacher was introduced to a task by the task designer who explained in person what was involved, similar to the situation described by Askew (2013) who himself led teachers' preparatory investigation of tasks he had designed (Askew & Canty, 2013:535). In the imaginary scenario, the two might be expected to set up a productive relationship; the task designer would stay and offer sufficient explanation for the teacher to be satisfactorily prepared to take the task and use it, potentially resolving any uncertainties or questions of intent on either side.

This thought experiment succinctly captures the two-way participatory relationship which exists at the interface of a published task and a teacher's setting up of that task ready to use in a lesson. It also demonstrates the complexity of this relationship, and the difficulty of capturing the many considerations in a printed representation, especially where teachers do not have any guidance relating to the task, or the time to read this, a situation which Davis and Krajcik describe as typical (2005:9).

Having access to curriculum guidance can make a difference to teaching, for example as reported in Sherin and Drake's (2009) research on elementary school teachers, which identified teachers' successful practice of first reading for big ideas, then returning to the guidance for details during the lesson itself. Stylianides (2008) pointed out that although explicit guidance might not lead to a lesson proceeding as the task designer intended, it at least gives an opportunity for the teacher to deepen their thinking (Stylianides, 2008:202). While students' aim may be to complete the task (Sierpinska, 2004), learning can go further if the teacher has established a mathematical point to guide instruction (Sleep, 2012), a pedagogical capacity which cannot be presumed but which can be developed (Sullivan

et al., 2009).

In the absence, however, of an unlikely personal visit from the task designer, nor an explanatory note, the intended aims of the task can be obscured. Several aims might lie hidden within the task as presented for publication: an effective context for students to learn mathematics; a particular stance with regard to mathematics and the teaching and learning of mathematics; and in the case of educative curriculum materials, an effective context to support teacher learning. When there is little or no explanatory guidance on the design and intended use of the task, the realisation of those aims is highly dependent on the teacher's interpretation of the published task. This interpretation serves as the basis for pedagogical decisions such as where to place emphasis and the time given to the task overall or to particular aspects of the task, each of which can serve to constrain students' opportunities to learn (Hiebert & Grouws, 2007:379).

Task design of mathematical tasks is a field of study in its own right, and teachers have access to and do use high-quality tasks which are the result of rigorous, theoretically driven processes to produce tasks which are effective for student and teacher learning (Margolinas, 2013:10). These carefully considered aims can remain implicit and hidden, however, either by miscommunication or design: we cannot read them like a book. In the common absence or lack of reference to clear and explicit guidance on the intended use of the task, the decisions teachers make on how to use the task can rely heavily on the teacher's perception of the task, which is the focus of this study and the following section of this literature review.

2.5 Teachers' perception of mathematical tasks

Because a published mathematical task is interpreted by the teacher as they design a lesson, teachers' interpretation of the task is significant. How and what do teachers perceive when they consider a task? Considering perception more generally, in his 1976 book Cognition and Reality the German-born US psychologist Ulric Neisser – a pioneer in the field of cognitive psychology – described perception as 'where cognition and reality meet' (Neisser, 1976:9). He was critical of a model which

considered perception as a linear process of the mental processing of external stimuli to form a conscious percept, and considered many demonstrative psychological research experiments as too contrived to merit validity outside of laboratory conditions. In his view, perception is better explained as a perceptual cycle, shown here in Figure 3, in which a schema – a structured mental representation of experience (Schoenfeld, 1998:24) – is modified in reaction to an object, which in turn directs the perceiver to seek more information from the external world. The perceiver is changed by the act of

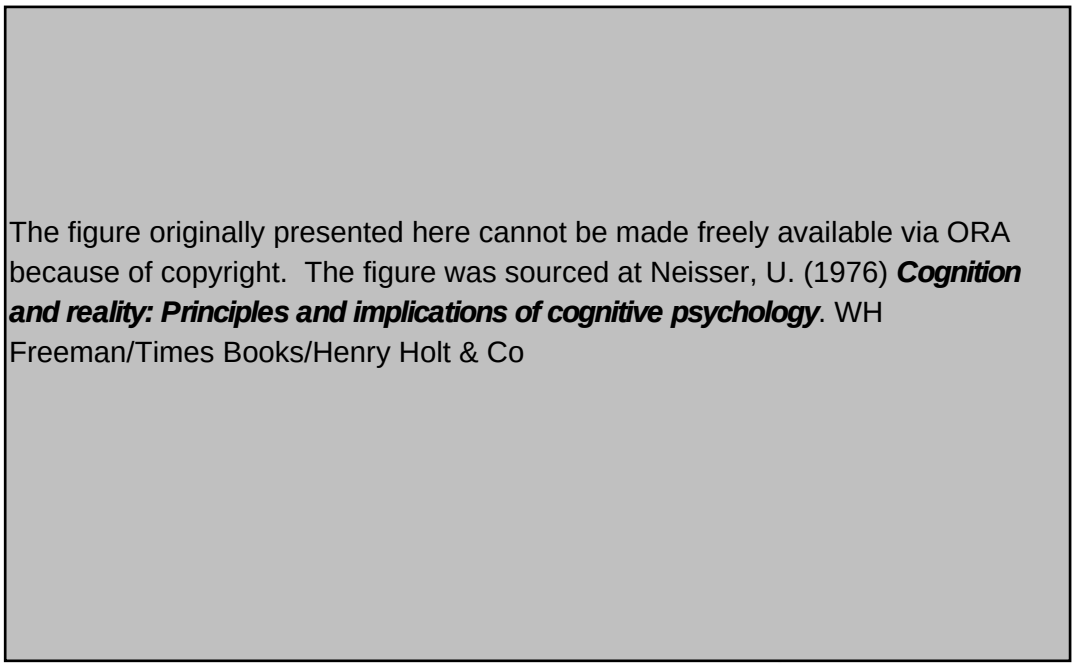


Figure 3 Neisser's perceptual cycle (Neisser, 1976:21)

perception, which 'depends on the skill and experience of the perceiver – on what he knows in advance' (Neisser, 1976:13).

In the context of a teacher preparing a lesson, the 'object' is initially the actual physical manifestation of the task. Consideration of this object stimulates and modifies the teacher's schema, akin to the teacher interpreting the task in the context of his or her historical experience, beliefs and understanding. This history suggests hypothetical pedagogical decisions for a new lesson which the teacher actively explores, anticipating student responses and considering how well they fit, now not against the original task but against an image of the lesson, the creation of which is the result of this act of perception. Given sufficient time and motivation, this cycle of perception ends when the

putative lesson image fits sufficiently well with the teacher's schema, so no more modification or exploration is warranted; the cycle may also be curtailed if the teacher is otherwise distracted. The term 'lesson image' was first proposed by Morine-Dershimer (1978) and developed by Schoenfeld who described it as the 'teacher's envisioning of the possibilities and contingencies related to a lesson' (Schoenfeld, 1998:17). My suggestion here is that Neisser's perceptual cycle (Neisser, 1976:21) provides a mechanism which helps us understand how a teacher forms a lesson image when stimulated to do so by being presented with a task.

Developing the field further, the Nobel-prize winning cognitive psychologist Kahneman (2011) established a large body of empirical work supporting a dual-process theory of mind. Decision-making is characterised by two distinct systems: System 1 is automatic, instinctive and inflexible; System 2 is more considered, rational and wilful, and can be called into action to moderate System 1 judgements. Although System 1 is very efficient – Kahneman suggests it is a trait of successful evolutionary progress (Kahneman, 2011:301) – it can also give rise to irrational decisions, described as biases, in identifiable circumstances which cause us uncontrollably to oversimplify, known as heuristics. A visual example is the well known Muller-Lyer illusion, shown in Figure 4. We can confirm by measurement that the horizontal lines are of equal length, yet despite 'knowing' this we cannot help but 'see' the top line as

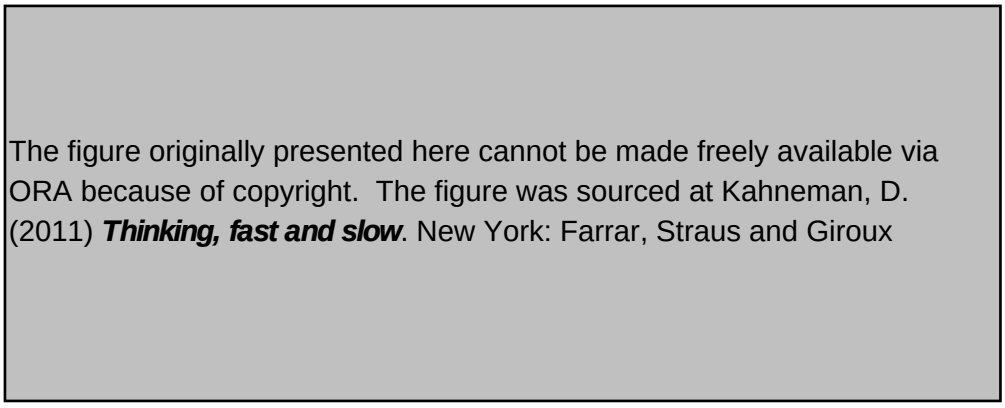


Figure 4 The Muller-Lyer illusion

(Kahneman, 2011)

shorter. From a heuristics-and-biases stance this demonstrates the extent to which an instinctive System 1 is beyond our control, even though we may ultimately use System 2 to make a reasoned

decision. A more socially constructed cognitive example is the framing effect, in which System 1 biases are attributed to how a question is asked, for example 'the statement that "the odds of survival one month after surgery are 90%" is more reassuring than the equivalent statement that "mortality within one month of surgery is 10%"' (Kahneman, 2011:88). With some statistical nous and a little application we can assert that the two statements are quantitavely identical, yet our gut reaction to each remains different. In the context again of teachers being presented with a mathematical task, this theory suggests we can expect teachers to make a snap judgement, then moderated by subsequent rational consideration. Factors affecting the initial perception include the teacher's experience, and how the task is framed. Teachers may have a learnt or otherwise intuitive reaction to a task, which takes effort and pedagogical skill to unpack to be made available for use in teaching.

According to Neisser (1976) and Kahneman (2011), perception is an internal construct, which is a difficulty in making it the subject of investigation, as in this study. These two authors believe that perception is automatic, and not necessarily consciously known by the perceiver at all. To investigate perception, the researcher relies on external characteristics such as self-reports or actions subsequent to the perception taking place, to be interpreted with reference to the teacher's experience, belief and understanding. In the following, final, section I review literature on teachers' decision making, with a view to relating teachers' demonstrable and explicit actions to implicit perception.

2.6 Teachers' decision making

This final section makes particular reference to the significant work of the Teacher Model Group at the University of California, Berkeley, which for the past three decades has developed a model of teaching-in-context, about which mathematics educator Alan Schoenfeld in particular has written extensively (Schoenfeld, 1998, 2010, 2011). In Schoenfeld's words, 'the model describes, at a level of mechanism, the ways in which the teacher's goals, beliefs, and knowledge interact, resulting in the teacher's moment-to-moment decision-making and actions' (Schoenfeld, 1998:1), aspects which are

interrelated as shown in Figure 5. The group's persuasive research methodology was to parse video-recorded episodes of teaching, interrogating these data to inform research questions about the

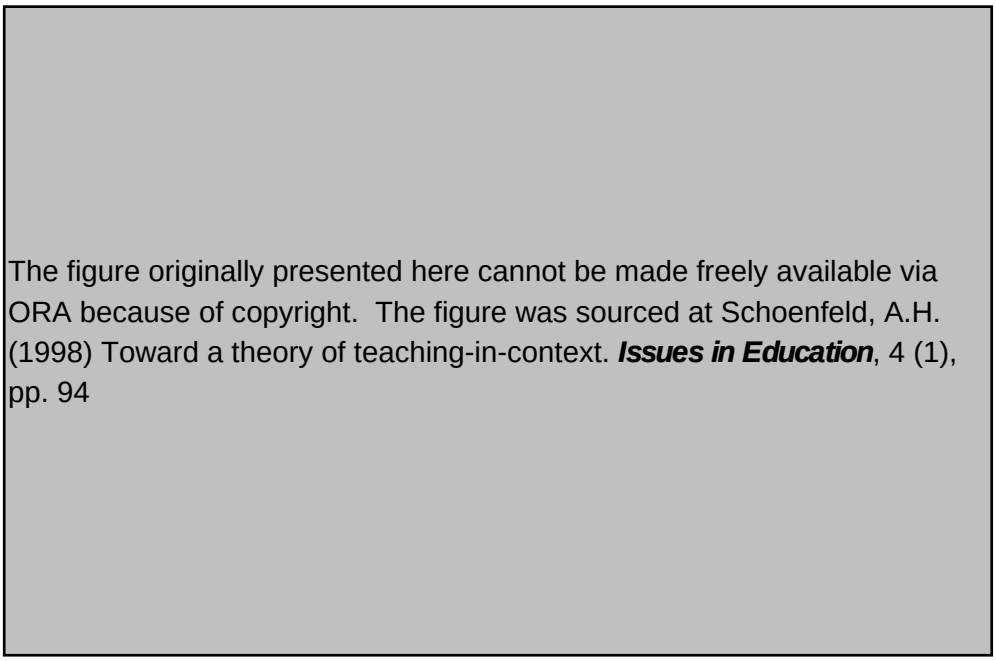


Figure 5 Teacher Model Group – Aspects of a model of teaching-in-context

(Schoenfeld, 1998:13)

interaction of teacher's goals, beliefs, and knowledge, for instance Aguirre and Speer (1999) documented how teachers' beliefs accounted for a change in goals in during unexpected contingent moments in teaching, and Zimmerlin and Nelson (1999) who used the same model to account for the teacher Nelson's decision making when the actual lesson veered away from his preconceived lesson image. The references cited here are useful starting points directing the reader to this group's work, which is too extensive for full coverage in this study, although the key terms do warrant further outlining here.

2.6.1 Teacher beliefs

There is a broad and varied literature on teacher beliefs, to which a significant early contribution came from Thompson (1984) who showed a link between teachers' practice and their conceptions of mathematics, teaching and learning. This central finding has been complemented by many studies,

leading to further, more nuanced investigations such as whether teachers can simultaneously conceive differently of school mathematics and the discipline of mathematics (Beswick, 2011) (they can). One famous account from Cohen (1990) related the case of Mrs Oublier, a teacher who agreed wholeheartedly with a particular effort to reform mathematics teaching, but whose practice seemed to be guided by other concerns. Schoenfeld (1998) described such differences as being between teachers' professed beliefs and those which might be attributed to them to account for observed practice (Schoenfeld, 1998:19). Speer (2008) recognised the increasing prevalence of researchers attributing beliefs to teachers, sometimes arising from a lack of shared understanding of terminology and meaning between teachers and researchers (Speer, 2008:361). A researcher attributing a belief which has not been professed by a teacher can lead to a delicate situation in which a teacher is either ignorant or in denial of a particular attributed belief, or there is an irreconcilable difference of opinion between the teacher and the researcher. Cases such as that given by Cohen (1990) also demonstrated that there may well be several pertinent beliefs, one of which may trump another in a given moment.

2.6.2 Teacher knowledge

In Schoenfeld's (2010) more recent account, teacher knowledge in the model is subsumed by the broader category of resources. As well as what is in the teacher's head, decision-making is affected by material and contextual resources available to the teacher (Schoenfeld, 2010:10).

What teachers of mathematics know and how this relates to instructional practice is another field which has been widely explored. Despite the variety of conceptions of teacher knowledge, there is a consensus that teaching involves knowledge other than disciplinary knowledge alone (e.g. Shulman, 1986; Gattegno, 1988; Rowland, 2013). A key difference from the mathematician is the teacher's need to 'psychologise the subject', to give regard to how the ideas will be formed in learners' minds, which Mason and Davis describe as an elaboration of Dewey's perspective given at the turn of the twentieth century (Mason & Davis, 2013: 185). Shulman (1986) proposed a taxonomy of teacher knowledge which has been influential. He distinguished general pedagogical knowledge, knowledge of learners,

and paid particular attention to content knowledge, subdivided into the subject matter per se, and pedagogical content knowledge, involving the representation of mathematics for the purpose of making it comprehensible to others (Shulman, 1986: 9). There was an international response to Shulman's call to establish a growing body of case literature to further explore the role of content knowledge for teaching (Shulman, 1986: 14), with some research based on observation (Ball, Thames, & Phelps, 2008; Rowland, Huckstep & Thwaites, 2005) and other researchers analysing actual teacher responses to hypothetical pedagogically-related situations (Ma, 1999; Stylianides & Stylianides, 2010; Zazkis, Liljedahl, & Sinclair, 2009). Despite the prevalence of this approach to categorise teacher knowledge, there is criticism that this has led to a conception of knowledge as something which teachers *have* rather than something they *do* with (Mason & Davis, 2013: 187), and some concern that this quasi-scientific dissection prejudices against a more holistic, connected view which is consistent with the nature of mathematics itself (Askew, 2008:29).

Building on Shulman's notion of pedagogical content knowledge (Shulman, 1986), Ball et al. (2008) described teachers' need to be able to 'unpack' school mathematics, part of what they termed specialized content knowledge, the 'mathematical knowledge and skill unique to teaching' (Ball et al., 2008: 400). Teachers transform curricular mathematics into a 'decompressed' form which is appropriate and useful to expose mathematical ideas to students as they develop understanding (Ball et al., 2008: 400).

Teachers are minded not just to being able to solve the task for themselves, but to use it for teaching, so have a need to be 'aware not only of the subject matter but of the pedagogical aspects of the subject matter' (Mason & Davis, 2013:189). More than supporting students to successfully complete a task, it is beneficial for teachers to be aware of how a task may represent an instantiation of a generality, how it relates to other 'mathematical themes, concepts, theorems, connections to other topics and techniques, multiplicity of approaches, interpretations and re-presentations' (Watson & Mason, 2007:207).

Experience, further to being structured by mental schemata as in Neisser's perceptual cycle (Neisser, 1976:21), is the accumulation of knowledge. Following many years teaching the same lesson on a University mathematics programme, Schoenfeld (1998) described that he knew 'before the class starts, how things are likely to unfold - down to an extremely fine level of detail. … there are few surprises.' (Schoenfeld, 1998:18). Experts have more knowledge and can use it differently, leading to an 'almost automatic perception and response' (Schoenfeld, 2010:28).

As was seen in the introduction here, responses to the example in Figure 1 indicated teachers' appreciation of a range of mathematical and pedagogical aspects which came quickly to mind when presented with a task, which they knew and which they knew were pertinent. In the close consideration given to knowledge in his model of teacher decision making, Schoenfeld considered not just the substance of teacher knowledge but also how it was accessed (Schoenfeld, 1998:29); what comes to mind more easily is more likely to feature in an action plan for a lesson using the task.

2.6.3 Teacher goals

Ball et al.'s, Mason & Davis' and Watson & Mason's suggestion is that one task may be represented from different perspectives for different purposes, which Schoenfeld (1998, 2010) also recognised and described as a range of goals which a teacher might face at any point in the course of a lesson, summarised but given here at length:

- Overarching content and social goals represent intentions that teachers have for the long term.
- More specific, medium-term content goals may be that the students grapple with the "big ideas" in a particular unit.
- Even more local goals may be that they come to understand a particular idea or learn a particular technique.
- The most fine-grained goals may have to do with simply demonstrating the steps of a technique or procedure, or providing a simple answer to a question a student has asked.

  Within just a few minutes in a lesson, a teacher may be working on goals at all of these levels

(summarised from Schoenfeld, 2010:72).

This categorisation of nested goals sits well alongside Watson and Mason's (2007) reported distinction between many coexistent perspectives within a task, namely the outer and inner tasks, following Tahta (1980), and 'a *meta task* consisting of opportunities to work on personal dispositions and propensities such as doing whatever first comes to mind, waiting to be told what to do, or diving into the first approach that comes to mind' (Watson & Mason, 2007:207, italics in original), which I relate to Schoenfeld's overarching content and social goals here.

Schoenfeld's view was that teachers simultaneously attend to goals at each level, of which some are foreseen and intended, and some are contingent and emerge unexpectedly during a lesson (Schoenfeld, 1998:21). Dependent on a teacher's beliefs and the pedagogical situation, one or more goals may become more important, a 'constellation of goals [which] shapes the actions taken by the teacher' (Schoenfeld, 1998:22).

2.6.4 Relating decision making to perception

This concludes the detailed explanation of Schoenfeld's model (Schoenfeld, 1998:13), but leaves one issue unaddressed: does the Teacher Model relate to a teacher's perception of a task, and if so, how? Schoenfeld's model (Schoenfeld, 1998:13) was tested in actual teaching situations, and shown to give a valid and reliable post hoc account of how teachers' beliefs, knowledge and goals related to the decisions teachers actually made during an enacted lesson (Aguirre & Speer, 1999; Zimmerlin & Nelson, 1999; Schoenfeld, 2010). This is similar to Stein, Grover and Henningsen (1996), whose research focused more on accounting for what happened during enacted lessons, less on the situation on the eve of the lesson, and the correspondence between a task as published and as set up by a teacher. Schoenfeld's model (Schoenfeld, 1998:13) also allowed for what came before the lesson to be analysed, both in terms of the history of the teacher's decision making, shown in the model, and in what was considered in anticipation, notably the teacher's lesson image and the constituent action plans. This group's literature gave an account of the situation before any teaching actually took place,

reporting a teacher's thought process – which could reasonably have started with the teacher being presented a task – and which was reported as resulting in the teacher's creation of a lesson image, a putative and idealised version of the pedagogical activity which the teacher intended. As Schoenfeld reported, 'the lesson image specifies action plans, and one can elaborate the rationales for them and the beliefs that underlie them' (Schoenfeld, 1998:29). This study aims to provide that elaboration, and therein to infer the relationship of the lesson image to a teacher's perception of a mathematical task.

2.7 Summary

In this review I have tried to report a coherent account of the literature with regard to mathematical tasks, their use by and for teachers, teaching and learning, and the role of teachers' perception of tasks, and in this summary I aim to recount the overall narrative structure and main points.

Tasks are ubiquitous in the mathematics classroom (Watson et al., 2013:12), but their effect in the classroom is complex: Stein et al. (1996) showed that activity based on a given task is not fixed, and a teacher's interpretation of the task is fundamental to how the task is used in a lesson (Remillard, 2005). In their participatory relationship the teacher and the task have an effect on each other (Remillard, 2005), and tasks are sometimes designed to also support teacher learning (Ball & Cohen, 1996). This guidance does not always come to bear or is otherwise ineffective, in which case the teacher's use of the task depends overwhelmingly on their perception of the task (Neisser, 1976), which involves their beliefs, knowledge and goals (Schoenfeld, 2010).

The work of the Teacher Model Group and in particular the interrelated aspects described in their model (Schoenfeld, 1998:13) are taken here as the theoretical framework and basis of the thematic analysis in this study. How this is used is described in full in subsection 3.12 of the section on methodology which follows.

3 Methodology

3.1 Context

This study was undertaken at a mixed secondary school (age 11 to 19), as part of my work as an advisory teacher for mathematics for a Local Authority, advising externally on the teaching and learning of mathematics. The mathematics department with whom this research was undertaken was situated in a state-maintained co-educational secondary academy with students from age 11 to 19. For the academic year in question, the department had committed one hour per week of directed (i.e. timetabled) time for professional development. The Head of Department and myself designed a Mathematics Professional Development Programme (MPDP), an intervention aimed at effecting change in teachers with the agreed aim of improving teaching for students' understanding and mathematical resilience. The group comprised 14 teachers, of which 4 were beginning teachers, 2 were newly qualified teachers (in their first year post-qualification), and 8 had more than five years' teaching experience. Timetabling restrictions meant that teachers were split into two groups (of nine and five) for the weekly MPDP sessions.

One aim of the programme was to encourage teachers' reflective practice, using tasks from a variety of sources in and between sessions (including but not restricted to: Bowland maths (2010b), Mathematics Assessment Project (Schoenfeld, 2014) and Improving Learning in Maths (Swan, 2005)), each of these three associated with the work of UK mathematics educator Malcolm Swan and the Shell Center for Mathematics Education based in the University of Nottingham, UK. Another aim was to introduce teachers to theory and research such as Skemp (1976), Cuoco et al. (1996), and Mason (2002). Teachers were invited and encouraged to share their experiences of using the materials, a cycle of enactment and reflection in line with the interconnected model of professional growth given by Clarke and Hollingsworth (2002:951).

In the first term of the year both of the groups (separately) observed me teach a 'lessonette', a 25 minute vignette of a lesson with around 15 students, half the typical class size, using a task which was

adapted from the published version especially to facilitate observation. Teachers were provided with a written plan a week in advance, and after the students had left the lessonette was immediately followed by a group discussion, focused on the differences and similarities between the lesson as planned and intended, and what had actually happened.

These two strands – working with novel materials, and observing and discussing a teaching episode representing an abridged lesson – were each involved in the intervention used for this research.

3.2 Research questions

This study originated in my interest in the variation of teachers' responses to a task, as outlined in Figure 1 and the introduction of this dissertation. It is common for teachers to access a wide range of readily available published tasks – what factors affect how they set it up? This study was designed to investigate the following research questions:

- What factors influence teachers' perception of mathematical tasks?
- How does experience of enactment of the same task affect teachers' perception of mathematical tasks?

3.3 Outline of the study

The three-week study started with the two groups of teachers (Group X and Group Y) being given an unfamiliar published resource and one hour to plan activity based on that task. A lesson planning pro forma was provided (Appendix 2) but it was made clear that teachers were free to submit their plan alternatively, and blank paper was also provided. The following week the groups observed and discussed a 'lessonette', collaboratively planned by several and taught by one of their colleagues, using the same activity as a starting point. The teachers involved in planning the lessonette had had several weeks to prepare, and were sworn to silence on which task was being used, so that observing teachers had not seen the task before the first week of the intervention. On the third week the pre-lessonette plans were returned to all teachers (including those who had planned the lessonette), who

had a further hour to revise their plans, which were resubmitted. In between the presentation of the task and the time given to revise their plans, some teachers chose to teach their own lessons using the task.

After reviewing teachers' original plans and revisions, four teachers were selected to complete a semi-structured interview, with questions designed to account for and investigate further the pre-lessonette plan and the changes made on the post-lessonette plan. One of the four teachers interviewed had planned and taught the lessonette with Group Y, the other three teachers from Group X had been observers only, two of whom also taught their own lesson using the activity in the period of the study. The common experience of these three teachers from Group X, who had all been observers only, and of the same lessonette, made them an appropriate group to report here. This dissertation reports only on those data relating to these three teachers.

The original intention had been to complete this cycle of activity two times, once in February and then again in July, giving an opportunity to make revisions to the process based on teachers' feedback, and an opportunity for a more longitudinal analysis of the data. This was scheduled, but the academy was unavoidably closed on the date of the second lessonette due to industrial action. The academy was extremely supportive and accommodating in this unforeseen circumstance, and scheduled an alternate date in the following week for the completion of this second cycle. However in the rearranged schedule some critical compromises had to be made regarding timing, with the abandonment of a discussion following the lessonette, and only around 20 minutes immediately following the lessonette for teachers to make revisions to their scripts. The effect of these changes meant that the data were not useable for this study, so data for this research was restricted to the first cycle only.

3.4 Ethics

Central University Research Ethics Committee (CUREC) procedures were followed, first approaching and gaining consent from the Principal to undertake the project in the academy. Before being offered

formal consent forms, teachers were briefed on what the research would involve. Some teachers had reservations on the proposed use of video recording of the lessonette, discussion and semi-structured interview, but were happy for these to be audio-recorded. Teachers' names were changed in any report of the study. Teachers were given an information booklet, a letter requesting their consent and a consent form, and it was clearly explained that at any point they could choose not to be involved, without the need to give an explanation. Teachers were also made aware that parental consent would be sought for the involvement of students, and that I would personally explain the project to students before inviting them, if interested, to seek consent to be involved. I presented the project to students in two classes [Year 8 and Year 9, ages 13-14], and offered an information booklet, a letter requesting their consent and a consent form, again with clear assurances in the case of non-involvement. All students who were involved in the two lessonettes submitted parental consent forms. All information and requests clearly outlined how data would be recorded, and assurances given that they would be securely stored, which was adhered to.

3.5 Data collection

This data collection schedule also provides a detailed account of the timeline of events in the study. The same schedule was followed for both of the teacher groups involved in the study; the information below shows the data collection schedule for each group.

Timing	Activity	Outputs, Data collected
January 2016	Lessonette teachers receive task for collaborative planning	Written plan for lessonette
February: week 1	Observing teachers receive task, and plan for one hour. Scripts submitted.	Field notes. Observing teachers' written pre-lessonette scripts.
February: week 2	Lessonette is taught and observed. All teachers discuss.	Field notes. Single-point audio recording of lessonette and discussion.

February: week 3	All teachers (including lessonette teachers) get back their pre-lessonette scripts, and have one hour to make revisions.	Field notes. All teachers' written post-lesssonette scripts.
June	Analysis of observing teachers' pre- and post-lessonette scripts, noting significant changes.	Record of significant changes noted in pairwise analysis of observing teachers' pre- and post-lessonette scripts.
July	Semi-structured interviews with 4 teachers	Audio recording of semi-structured interview.

Table 1 Timeline of key events in the study, and data collection schedule

The long delay between the final planning session and the analysis of the lesson plans was not scheduled, the result of unavoidable personal circumstances for the author, and the abandonment of a second cycle as explained in section 3.3. However, recall of their thinking at the time of the lessonette seemed to present little trouble to the four teachers selected for interview, especially when presented with their written post-lessonette plans.

3.6 The task

The task chosen for this study was *Taxi Cabs* (Bowland Maths, 2010) (Appendix 1 here) from Bowland Mathematics, a freely accessible website offering a set of tasks and associated professional development modules aimed at addressing a 'growing concern that school mathematics was not adequately equipping young people for an age where the ability to think in a mathematical way is becoming critically important' (Bowland Maths, 2014). Tasks from Bowland Maths are aimed at students in the age range of 11 – 14 years old, and are mostly set in realistic contexts. The tasks had been trialled and revised after classroom use, resulting in a pedagogical design which promoted both purpose and utility, which Ainley et al. (2006) identified as important if the realistic context of the task is to be more than superficially distinct from the mathematics involved (Ainley, Pratt, & Hansen, 2006:30).

Taxi Cabs was considered suitable for this study because it aligned well both with the department's focus on teaching for understanding and mathematical resilience, and with the growing importance of problem solving in school mathematics, one of three stated aims of the National Curriculum for mathematics in England (Department for Education, 2013). A further benefit for me as a researcher derived from the published task including guidance for teachers in the form of actual student answers, and a rubric to support assessment of students' written responses against specified problem solving processes (Bowland Maths, 2010). Stylianides (2016) described approaches researchers might use to gauge the fidelity of teachers' classroom use of curricular resources, and found that such teacher guidance provided the researcher with 'information to draw valid inferences about the intended purposes of tasks' (Stylianides, 2016:13). This teacher guidance was given to the teachers planning the lessonette, but was not shared with the other teachers, to whom only the task as designed for students' use was presented. This decision was made for two reasons: firstly, teachers had limited time in which to prepare an initial plan when first presented with the task; secondly, this study was interested to report on situations – considered typical – in which teachers referred to the published task without guidance, to describe the variation in their perception and decision making when considering how to plan activity based on that task.

3.6.1 Brief mathematical analysis of the task

This section provides more detail about the task, including references to the guidance given in the educative curriculum materials as published online (Bowland Maths, 2010), which teachers in this study were not given. The guidance advises teachers to check students' understanding of the context and what the problem is asking. There is also assessment guidance, showing progress on a rubric in each of four key processes, indicating the performance of four students, followed by these students' written work, to which some suggested prompts and feedback are added, written as if spoken by a teacher to students.

Each example of student work shows a different way to calculate how many small taxis are needed,

which is the rounded up answer to (75 – 7x6) / 4 . In the guidance all of the students could successfully complete this aspect, and calculate the total cost, which is useful to have completed as it gives the structure of a calculation which can be used more generally to investigate other combinations of large and small taxis. In the student examples given there is much greater variation in success at the second question. Two students are drawn to finding the cost for only small taxis or only large taxis, one student tries one other combination, and the student who is assessed as highest on the rubric is shown as having taken a more systematic approach, calculating the cost needed started with 10 then 9 then 8 large taxis, then giving the final correct answer of 9 large taxis and 3 small taxis, which holds exactly 75 people. However the student does not give any justification that this is the cheapest solution possible.

As educative curriculum materials, the prompts as recorded to be given to students can also be considered prompts for teachers. They include some aspects which do not feature explicitly in the task itself, such as listing possible combinations in an ordered way, and the notion of spare seats, which is critical to the structure of the problem as it results in students having to go beyond direct proportion. One key concept which is not mentioned is that of cost per person. The values are chosen such that a small taxi costs £10 per person, whereas a large taxi costs £9 per person. This aspect could be used to justify an approach which tried to use as many large taxis as possible to find the cheapest overall cost.

3.7 Pro forma

Teachers had the option to use a pro forma, a printed document on which to record their thinking about how to plan activity based on the task, which all but one of the eleven teachers did choose to use exclusively. This document included headings and space for sections which were suggestive of common features for lesson plans (Zazkis, Liljedahl, & Sinclair, 2009:42), namely: objectives (as a 'research question' and as a 'lesson objective'); phases of the lesson to describe the flow of the mathematical teaching and learning; materials that might be used; planned teacher-student

interaction; situation of the lesson with respect to prior and future learning in the broader curriculum. This study was an investigation into how teachers perceived a task, with a view to commenting on how teachers perceive tasks generally in a typical professional context. The titles and the design of the sections of the pro forma were suggestive, which might have biased teachers' use of the pro forma and affected their responses, but the design of the pro forma and what it intended to prompt were considered to be in keeping with a typical teaching situation. For transparency and for readers to be able to judge whether the design of the pro forma was inappropriately suggestive, the document is given in full in Appendix 2.

One less typical section for planned teacher-student interactions included a space to record 'anticipated student responses', which is a common feature of lesson planning in the Japanese context (Fujii, 2016) and which had been a consistent aspect of the ongoing professional development programme with these teachers. Three months before this study, all teachers had received a completed pro forma which was used to support their observation and discussion of a similar lessonette, which had been planned and taught by myself. Other than this prior experience with anticipated student responses and the format of the pro forma, no explicit guidance was given regarding the meaning or intention of the planning document. This was a purposeful decision in the design of this study, which was interested in inferring teachers' perception of a task by considering their interpretation of the task and how they intended to transform the task into classroom activity. Just as teachers might typically access no or minimal guidance in the intended use of a task, Zazkis et al. (2009) suggested that the planning templates used by teachers for lesson planning are no more than an 'empty shell that stands in the place of grounded theories of teaching practice' (Zazkis, Liljedahl, & Sinclair, 2009:40), making the lack of guidance in this study more consistent with typical practice. My choice not to give teachers explicit guidance in the use of the pro forma was informed by my interest in their open interpretation of the task, rather than in their understanding, compliance or fidelity to my own recommendations for planning lessons as embodied in the design of the pro

forma. Having seen the same pro forma used to document my own teaching of a lessonette, teachers knew that it represented and was informed by my own teaching approach, but I made it clear that the pro forma was offered for their own convenience, and they were under no obligation to use it as I had done previously, and that they were at liberty to record their intended activity in whatever form they chose.

In this dissertation I sometimes refer to teachers' work on this pro forma as a 'script', not in the sense of a highly detailed line-by-line account of their intended activity, but in the sense that the completed pro formae represented what was recorded in writing, evidence which was used to inform the discussion here.

Scripts were submitted after teachers had had one hour following their first sight of the task, and scanned copies retained for analysis. The original scripts were returned to teachers on the third week of the study, and using a differently coloured pen teachers were given a further hour to review their plan, making amendments, deletions and additions. At the end of this hour teachers were asked to record a summative comment explaining the overall changes which had been made.

3.8 Lessonette

The focus of this study is on teachers' perception of mathematical tasks, and how this changes as a result of experience of enactment of the same task. The scheduled enactment of the task was a collaborative activity which consisted of teachers jointly observing and discussing a short vignette of a lesson, designed to last for 25 minutes (half a lesson) with 15 students (half the typical class size), which in this study is referred to as a 'lessonette'. Planning teams comprising three of the eight teachers in Group X and two of the five teachers in Group Y were given several weeks to jointly plan a lessonette using the task and the associated teacher guidance, as described in Section 3.6 above, for the purpose of observation by the remaining teachers, with one of the planning team leading the lessonette on the day. Although these lessonette teachers did take part in the discussion following the lessonette, and also submitted revised plans in the following week, these teachers were not the

focus of this study, which was based on the experience of those teachers who had a shorter time to consider how they would use the task. For clarity I will consistently refer to this latter group as 'teachers', and those who planned the lessonette as 'lessonette teachers'. The only information and instruction given to these lessonette teachers was the intended size of the class, the duration of the lessonette, and the expectation that their colleagues would be observing. The same pro forma as was used with teachers was offered to the lessonette teachers, which in the event was used by one of the two planning teams.

The purpose of the lessonette was to give teachers an opportunity to directly observe student learning and to consider their colleagues' approach (presumed alternate) based on the same task. Although lessonette teachers had had longer to plan their activity, there was no expectation that the lessonette would be in any way exemplary or more accomplished than teachers' own thinking, rather it was an opportunity for the teachers observing the lessonette to reconsider what particular mathematics they intended to teach, and an opportunity to reflect on 'on their own and others' teaching activities in terms of impact on students' progress' (Tzur, 2010:50), which Tzur found was typical of long term professional development practices for mathematics teachers in Asian countries. Relieving teachers of the responsibility of teaching allowed a focus on student learning and the opportunity to identify salient features which might not otherwise be appreciated. In their interconnected model of teacher professional growth Clarke and Hollingsworth (2002:951) suggested that teachers' reflection on salient outcomes can cause teachers to experiment professionally, and Mason (2002) had it that the noticing, marking and recording of what is salient represented increasing sophistication as teachers research their own practice (2002:247). The intention was that by observing a contrasting lesson based on the same activity, the teachers would have an opportunity to consider the task with fresh eyes, and to clearly identify the decisions they had already made, ahead of an opportunity to review them the following week. As part of a briefing before the lessonette the observing teachers were reminded not to interact with the students during the lessonette in any way, and to take note of any

significant moments which might be discussed following the lessonette.

Once the lessonette had finished, students left the classroom, and all teachers present joined a discussion, with students' work made available in case it was helpful to corroborate a point of observation. Consistent with the advice given by Takahashi and McDougal (2015:520) in the context of their development of a form of lesson study called Collaborative Lesson Research, I acted as moderator for the discussion following the lessonette, prompting all teachers with prepared questions to encourage reports of evidence of student learning from their direct observation, to comment on what they thought was significant, and why.

Data were collected from a single point audio recording and field notes of the lessonette and discussion. The quality and clarity of the audio recording was not sufficient to allow for either full transcription or for its use for stimulated recall in semi-structured interview, which had been my original intention, but was nonetheless helpful in capturing teachers' discussion, provided some useful direct quotes, and alongside my field notes was valuable in my preparation of questions for interview.

3.9 Selection of teachers for semi-structured interview

The post-lessonette scripts were analysed to identify teachers for interview. The rationale for this analysis was informed by the research questions, repeated here for the reader's convenience:

- What factors influence teachers' perception of mathematical tasks?
- How does experience of enactment of the same task affect teachers' perception of mathematical tasks?

Some scripts were rejected on the basis that there was very little written, making it difficult to make any inference about those teachers' perception of the task at the time of planning activity. There is no necessary implication that these teachers' thinking was in any way deficient or wanting, as made clear by Schoenfeld (2010) who found that interviews with teachers revealed well formulated goals despite 'sketchy' lesson plans (Schoenfeld, 2010:72). However, the usefulness of teachers' scripts as

data in this study was reliant on there being sufficient detail to suggest teachers' thinking before and after the lessonette, then to be investigated further in interview, a criterion which ruled some scripts out at this stage in the analysis. Some teachers' mathematical analysis of the task was either obscure or hard to follow, and were rejected on this basis. The remaining scripts were reviewed and analysed, with scripts including distinctive approaches and/or significant changes before and after the lessonette considered more promising.

3.10 Semi-structured interviews

Three teachers and one lessonette teacher were selected for interview, to find out more about teachers' rationale for aspects of what had been submitted on their scripts. The interviews were semi-structured, following prepared questions asking teachers to account for features and changes in features within their scripts. This semi-structured format was chosen to facilitate the subsequent analysis, and because it allowed me to ask each teacher questions of the same type, yet still allowed sufficient flexibility to contingently explore any significant comments (Wilkinson & Birmingham, 2003:45). Having a fixed number of prepared questions also made the timing of interviews easier to manage, which was important as teachers made themselves available for interview during strictly limited non-teaching time. Care was taken to put interviewees at ease, with each interview taking place in the teacher's own classroom, sitting side-by-side at a table with the teacher's own script available as a physical reference to support the interview. Teachers had been assured of anonymity in any publication of these data inasmuch as names would be changed, but given the small size of the sample it was likely that teachers would be identifiable within their own school context. This was not explicitly addressed at the time of interview, which was a regrettable oversight. Further, the late changes in the schedule of this study (as explained in section 3.3) meant that interviews were conducted in the final week of the school year, making it impossible to share transcripts or an analysis of the interviews for approval from the teachers, which might have mitigated this potential compromise to their anonymity.

The interviews were audio-recorded, which freed me from a need to make comprehensive notes, although I did take some notes as the interview progressed, to highlight what seemed particularly emphatic points, and to record teacher comments for further questions later in the same interview.

Each interview started with a reminder of the purpose of the interview within the context of my own study and my interest in teachers' perception of mathematical tasks. With direct reference to the physical script which had been completed by that teacher, questions were focused on specific aspects which teachers had written. For each of two or three particular aspects which had been selected before interview, initial questions for all interviews were of the form:

> "Here you added/wrote/removed/changed [this aspect] in your plan – why did you make this change / write this? What caused you to add/write/remove/change it?"
>
> "How does this part relate to the intended learning?"

Once each of these aspects had been explored, including following up on teachers' contingent comments with ancillary questions, one final question regarding the whole script was:

> "Do you think your plan is in line with the original task designers' intentions?"

The duration of the four recorded interviews was 23, 35, 30 and 14 minutes, each of which was transcribed in full.

3.11 Thematic analysis

The data which were the subject of this analysis were teachers' post-lessonette scripts, single documents which included teachers' writing both before and after their observation of the lessonette, and the transcript of a semi-structured interview focused on accounting for aspects of what they had written on their scripts. The aim of analysis of the data in this study was twofold: to account for each teacher's intended lesson plan, expressed as a lesson image and including reference to their beliefs, knowledge and goals, and to identify changes in their plan, which might relate to their enactment of the same task, both with a view to inferring their perception of the task.

Braun and Clarke (2006) proposed thematic analysis as an appropriate and accessible approach for relatively inexperienced researchers, as opposed to more technically demanding practices such as grounded theory or discourse analysis in which themes are seen to emerge as the result of a systematic and iterative process of codification and interpretation (Braun & Clarke, 2006:81). Rather than generating a theory from source data, it is valid to use existing theory to analyse data so long as 'the theoretical framework and methods match what the researcher wants to know, and that they acknowledge these decisions, and recognize them as decisions' (Braun & Clarke, 2006:80). Rather than seeking to remove himself/herself in the creation of an objective account of the data, in ascribing qualities and motivations to the subjects of analysis the researcher must openly acknowledge his or her own role in the analytic process, recognising that 'goals are inferred by the analyst and attributed to the actors' (Schoenfeld, 2010:53). To mitigate this researcher bias, this report aims to offer transparent demonstration of how interpretations are achieved and direct quotations are used to add to the analysis' 'interpretive rigor' (Fereday & Muir-Cochrane, 2006:82).

3.12 Theoretical framework used for analysis in this study

In line with Braun and Clarke's advice (Braun & Clarke, 2006:80) in this subsection I detail the theoretical framework and methods used in this analysis.

The theoretical framework underpinning this analysis is the model developed by the Teacher Model Group, shown before in Figure 5 from Alan Schoenfeld's work (Schoenfeld, 1998, 2010). In that model, as discussed in the literature review here, a teacher's decision making is related to the teacher's beliefs, knowledge and goals. Before a lesson has been taught or otherwise enacted, a teacher is described as having formed a lesson image, which includes and encompasses an idealised and intended version of the lesson and which is created in preparation and readiness for teaching a lesson. Just as for a teacher's decision making in an actual lesson, the teacher's beliefs, knowledge and goals are involved in allowing for and constraining the teacher's lesson image.

The standard method of the Teacher Model Group is to conduct a fine-grained parsing of video-

recorded episodes, which is not appropriate in this study, focused as it is on accounting for teachers' thinking in anticipation of a lesson rather accounting for their decisions and actions in their enactment of a lesson. Instead, the method used in this study is to identify teachers' beliefs, knowledge and goals from their submitted scripts – synonymous with action plans in the Teacher Model (Schoenfeld, 1998:13) – and their commentary on those scripts as recorded in semi-structured interview.

The aim of this analysis is to characterise the decisions which teachers make when using a task to prepare a lesson, and to establish a rationale for the form and detail of this lesson image. In line with the framework, each teacher's beliefs, knowledge and goals are considered first in order to establish a reasonable explanatory account of the intended lesson image in terms of those aspects. In accordance with the framework, the major categorisations of knowledge are pedagogic knowledge, subject knowledge and pedagogical content knowledge; goals are identified as being long-term, local, social or content goals. As indicated by Schoenfeld (2010) and cautioned by Speer (2008), some care has to be taken to make it clear on what basis a belief might be attributed when it has not been explicitly professed. This issue extends further to the analysis and discussion of knowledge, goals and the lesson image. To mitigate the risk of unsubstantiated or biased attribution, the results and discussion include quotes from the teachers, given at length and often left in the original vernacular to indicate authenticity and to allow readers more opportunity to understand the inference made in this report, or to be better equipped to form their own view. Notwithstanding this decision, it is understood and acknowledged here that the selection of some data and the omission of others is likely to be partial and to represent my bias as I seek to justify my own arguments (Braun & Clarke, 2006:80).

Given the novel use of this framework in a modified context to that in which it was created and validated, it is possible that other considerations and aspects may be seen to constrain, allow or otherwise be involved in the process of a teacher forming a lesson image when presented with a task. These aspects are highlighted as they occur, and are related to the broader literature where possible.

4 Results

First, an account of the lessonette is presented, the short lesson based on the same task which was observed and discussed by the teachers in Group X, which included Katy, Diane and Mark, the three teachers who feature in this report. Together with the published task which had been presented a week beforehand, this experience of an enactment of a lesson using the task constituted the entirety of what was offered to teachers as a stimulus and basis for their thinking on planning a lesson based on the task. Based on a single-point audio recording and my field notes, the account of the lessonette is intentionally descriptive, with a view to it being a valid and reliable point of reference in the subsequent analysis of teachers' interviews. This lessonette and the published task served as the specific context for the study.

The descriptive account of the lessonette is followed by an analysis and some preliminary discussion of each of the three teachers' interviews, which seeks to relate findings to the theoretical framework wherever valid. This results section consisting of an analysis of individual teachers' accounts is followed by a section focused on discussion of all three, again relating findings to the literature, and drawing together evidence to answer the research questions.

4.1 The lessonette

The lessonette planned by teachers from Group X included only four 14-year old students who had given their consent to be involved, led by one lessonette teacher with eight (including the two other lessonette teachers who had planned the lesson) and the researcher observing. This unusual situation necessitated some changes to the teachers' plan (given here in full in Appendix 3), which had intended to include students moving between groups, and may have contributed to this small group of students being initially subdued, although they readily engaged in quiet discussion in two pairs. However, within the 23-minute duration of the lessonette the students did openly (and crucially here, observably) work with the task, successfully overcoming some initial difficulties with calculations, and culminating in a discussion of whether the large taxi was cheaper because it cost less per person, or

whether the small taxi was cheaper, because it cost less overall.

In the post-lesson discussion this notion of which taxi was cheaper was identified by Jane, the lessonette teacher, as a key feature of students' thinking, whose reasoning seemed to be 'because it's small it's cheaper'. Greg (one of the two other lessonette teachers who had planned the lesson with Jane) explained that the planning group had included some straightforward lowest common multiple questions at the start of the lesson with the intention that this would lead students to thinking in these terms, suggesting a particular method of trialling sums of multiples of 4 and 7 to arrive at a total of 75, but none of the teachers observed that the students had made that connection. Jane's expectation had been that students would adopt an unsystematic approach of calculating the cost of many different combinations of large and small taxis, then to compare the overall cost, but instead one student had reasoned that the cost would be cheapest if only small taxis were used, which had led to students' discussion of whether that was a reasonable assumption. Jane had not foreseen this issue of cost per person, but recognised it in the moment, and prompted students to explore this further, a contingent reaction which was itself noted by observers. Katy shared her view that the students hadn't really 'put themselves into the problem', which would have helped their understanding over and above a focus on the numbers involved, and may have helped them resolve the issue of which cab was 'cheaper'. Mark recounted how he had himself worked it out the week before, when he 'instantly knew – without working it out – that's how the world works: if you can get large things you do, because it's cheaper. It's obvious to us, but not to them', at which point Diane commented that 'for the kids it's different'.

Despite the unusual composition of the people in the classroom for the lessonette, the experience was not atypical to situations which might arise in a standard lesson. Students had been presented with a task, which they had found challenging, and with encouragement from their teacher they had followed their own reasoning, ultimately establishing a key context-specific question to be resolved. Students' own thinking and the teacher's management of the lessonette had revealed an issue of

conceptual understanding which neither the students nor the teacher had foreseen. As a result of their observations teachers reported, shared and discussed a range of salient features regarding student learning, teacher decision making and pedagogical design.

4.2 Three teachers' responses to planning for the task

Based on Katy's, Diane's and Mark's responses to questions about decisions they had made in their lesson plans before and after the lessonette, thematic content analysis (Braun & Clarke, 2006) was applied to transcripts from the three semi-structured interviews. Using Schoenfeld's model (Schoenfeld, 1998:13) as a guide, as shown in Figure 5 and discussed in section 3.12, appropriate keywords were used to code teachers' explanations and comments that related to their beliefs, knowledge and goals. Teachers' accounts are related to the literature wherever valid, and the analysis is used to infer teachers' perception of the task.

4.2.1 Katy's perception of the task

Katy was in her first year of teaching, rising to the challenge of becoming a teacher with diligent and determined efforts to experiment and reflect on her practice. After being given the task, and before the lessonette, she had used the task with her own high-attaining class. In her initial plan submitted for this study (Appendix 4) she had identified the concept of cost per person as important, but when she taught her own lesson she had struggled to find a way to bring this to students' attention. This revealed a tension for Katy, who had a strong idea of what she wanted students to consider, but didn't want to tell them directly. Recounting her experience of the lessonette, Katy recognised that she had a lower tolerance for discomfort than Jane, who had not said **'**what about *this*, you haven't thought about *this*', which is what I would say. I wouldn't necessarily give them the answer or give them the way to work it out, but I would say "but you haven't thought about this thing" '. After the lessonette Katy noted on her script that she should 'ensure prompting is as neutral as possible. [The] idea must come from students as this is the crux of the issue'. In interview Katy related her belief that the more a teacher directs and guides students, the less likely students are to understand what is being

discussed, similar to Watson and Mason's view that 'making the inner task explicit converts it into an outer task and runs the risk of directing attention so that students do not actually experience the purpose for themselves' (Watson & Mason, 2007).

Generalising about open, worded, problem solving questions which are 'quite bare' in their layout, such as Taxi Cabs, Katy made two observations: that 'lower ability kids can't access it very well' (leading to tasks such as these being often given only to higher-attaining students), and that 'it would take work for a teacher to break it down for lower ability kids'. Katy identified the usefulness for teachers to be able to identify constituent mathematical content within a task, a specialised content knowledge (Ball et al., 2008) which she had demonstrated in her own planning scripts. In interview she revealed that she had set herself the professional challenge to develop this capacity, especially for teaching lower-attaining students.

From her work with her own high-attaining class, Katy recounted an interesting episode where students described the need to minimise spare seats when speaking within the context, but had not realised that this could be applied to the mathematical solution to the problem. In the discussion following the lessonette she had opined that it was beneficial for students to have a strong sense of the realistic situation, and in her submitted lesson plan she devoted time to students understanding the context alone, before showing the rest of the task. Questioned about this disconnect between 'real-life' problems and working mathematically, she related both foundational beliefs and what Zimmerlin and Nelson (1999) would describe as an overarching goal, that her intended use of tasks such as Taxi Cabs was to pose students a challenge whose resolution was central to the mathematical understanding which she valued. I quote Katy verbatim at length on this significant point:

> It comes into everything, so being able to check your answers when you're doing simple numeracy, does that actually make sense, or what actually *is* a fraction, a percentage? From that basic number start can you actually understand it and can apply it to real life, and once you put that onto much bigger, more theoretical questions like a big ratio question, a big probability question, logistically

what am I thinking about here, how does that affect the maths? That's what we're aiming to get kids to do.

Katy perceived the task as an opportunity to identify and complete calculations and 'simple numeracy', but could 'see' more within the task that was available for students to learn. The task was not an end in itself, but a means to seek further understanding of mathematical questions which she believed were important. It is also notable that her account was subsequent to her teaching the task, and related to her explicit reflection on students' responses to the task, what Clarke and Hollingsworth (2002) described as salient outcomes, a change in the domain of consequence.

As well as this overarching goal, in her revised script Katy included additional local goals, such as a 'starter' question on per-unit amounts, 'to get them into the mindset' of considering cost per person in the Taxi Cab task, deliberately prompting use of the same later when this might be a useful concept involved in the task. Directing students' attention in this way to a particular way of thinking was highly suggestive, but Katy was explicitly aware of her tendency to lead students with her in-the-moment prompts, and this was an attempt to help her prospectively maintain self-control later in the lesson. Indeed, a more social goal which she noted on her script was to 'make sure process is very much *student led*' (Katy's emphasis), an intention she sought to realise by punctuating each phase of the activity with a 'summarising brief discussion of the previous phase'.

Katy made a significant addition at the end of the task, developing a discussion aimed at revealing the importance of minimising spare seats. She had used this with her own class, and reported not having enough time, which had led to her developing the idea. In her revised script, she added a purpose to this discussion, which was to 'ask students about methodology. ... what informed their decisions to try out different combinations?'. Asked whether she felt that this notion of comparing and contrasting various methods was implied in the original task, and likely the original task designers' intention, Katy explained that although this must have occurred to whoever designed the question, this aspect had been missed out in the published task, and that by adding this final episode she had

> probably expanded the potential of the question, because you could just give that question to kids and maybe not know how many kinds of approaches there will be, and assume that there'll only be one or two, and it's more about how they've laid it out on the page than it is about different ways of thinking.

Katy had made her own interpretation of the task, and had purposefully acted on it, with a clear rationale based on her beliefs about the subject, goals for student learning, and in terms of her own pedagogical understanding. Presented with the task, Katy had initially broken it down into constituent episodes and concepts, considered it as an example of a general class of 'real-life' problems, and had identified aspects of the task which she anticipated students would find problematic. Subsequent to her own use of the task in a lesson, and following the lessonette, Katy had new and further insight into student learning within the task, causing her to embellish and enhance certain aspects of her lesson better to accommodate the expected student responses, which was designed better to meet her more clearly expressed learning goal for students to be able to appreciate a multiplicity of methods. Katy had had an effect on the task, making her own interpretation of what might be intended, planning activity that aligned with her pre-existing beliefs and pedagogic style, and the task had had an effect on her, revealing new pedagogic realisations and presenting an opportunity for her to develop and clarify her beliefs about what is important in teaching. This mutual impact exemplified the participatory relationship between teacher and curriculum proposed by Remillard (2005).

4.2.2 Diane's perception of the task

Diane was a teacher with over 10 years' experience, who made significant revisions to her planning script as submitted before and after the lessonette, after which time – but before the semi-structured interview – she also used the task with her own class. In the 'title', 'research questions' and 'lesson objectives' fields of her revised plan, she removed the terms 'ratio/proportion', 'trial and improvement' and 'scaffolding', and added 'effective costing' and 'decision making'; her complete script is given here in Appendix 5. When asked to account for these changes, Diane described her

observation in both the lessonette and her own lesson, which she related in terms of students grappling with the problem within the realistic context, something she considered pedagogically valuable because 'from there they are able to build up how to do the question'. Having experienced enactments of the task and activity, Diane seemed to have noted a mismatch between her own and students' perception of the task, because

> as a teacher, even as I was planning the lesson, I was just thinking about the knowledge that I'm supposed to, the skills that the kids are supposed to learn. I was thinking about things like proportion, and ratio, multiples, so I'm just thinking of the proper maths, the proper numeracy. But then because of what the students said to me I started thinking in a different way, and to be truthful any time when I'm planning a lesson it will take time for me to be able to change it so that it can be adaptable to the kids, so that it can be what the kids are thinking.

Diane's account revealed a change in how she perceived activity associated with the task before and after observing students, initially considering in isolation the 'proper maths' students are 'supposed to learn', then focusing on 'what the kids are thinking'. This isn't to say that Diane was abandoning teaching these concepts; her original plan identified phases of the lesson for which she specified local goals, including the students' methodical laying out of work, and discussion of the efficiency of different combinations, goals which she chose to keep in her revised plan, and which she reported as being successful in her actual lesson. The change was in what she noticed and attended to each time she planned the lesson: before she was 'just thinking about the knowledge I'm supposed to', later she also attended to 'what the kids are thinking', which led to her emphatically removing her note-to-self to use 'scaffolding', to direct students to a particular method of solution.

It is not clear why what Diane noticed about her students had such an impact at precisely the occasion of this study – she did not account for this in interview – but it might not be a coincidence that her observation and discussion of students' learning in the lessonette was followed by her focus on observation in her own teaching using the same task: 'I see what they are doing, and as they carry on

doing what they are doing I am able to relate what they are doing and I am able to work with them'. Consequent to her observation of students engaging with the task, she made significant changes to the activity as planned in her revised script.

Rather than this being an epiphany for Diane, it seemed more that engagement in the study had served to remind her of her own long-term professional development. When prompted to explain her rationale for including 'decision making' in her revised script, Diane explained that she was

> thinking of the bigger picture. Ten years ago it was more of 'let's learn how to do calculations, let's learn BIDMAS, let's learn how to add' … The reason I put decision making is because when we look at the type of problems we have here, we have more of – I'm putting myself as a teacher more in the mind of the child – and I'm thinking 'when I go to the real world I need to know what are the best options for me?'

In thinking about a bigger picture, in her revised script Diane was recording her intention to attend to what Schoenfeld (2010:72) described as overarching long term social goals, which she included alongside more contextually bound fine-grained goals such as identifying the significance of spare seats.

In her revised activity design, Diane added structured opportunities for students to discuss their ongoing work, and at the foot of the page gave a 'reason: less teacher guidance and more student participation', which qualified as Diane's identification of a meta task in Watson and Mason's terms (Watson & Mason, 2007:207). Although her initial script had already included an opportunity at the end of the lesson for students to discuss their findings, this aspect of Diane's intended lesson was strongly amplified in her revised script, and a design element which she did not think was suggested by the task as published.

When prompted to consider whether her revised plan was in line with the task designers' intentions, Diane explained that she did not think that a lesson such as her own was something which this task's designers would have anticipated. Her perception of the task as published was as a cue for teachers

only to identify particular mathematical techniques which students might apply in finding a solution. She was not critical of the task as published, but did feel that she had made a bold interpretation which countered what she perceived as being expected, which was justifiable inasmuch as it could prompt useful student activity towards a solution to the problem, because 'either way the purpose is the same, because even though I do it the second way [with less teacher guidance] the thinking will be different but the application will be the same'. She was complying with what she perceived as the intended purpose of the task – to achieve a particular answer – but had confidently interpreted the task to include goals which she did not perceive as intended in the published resource. Diane's initial interpretation of the task was more focused on the identification of particular mathematical topics to cover, in which the realistic context was incidental: the learning goal was to know more about techniques such as trial and improvement, and topics such as ratio and proportion. Following observation of students engaging with the task, her interpretation of the task changed to include the learning goals of reasoning and decision making.

Diane's planned activity and the rationale for the revisions she makes gives some insight to her perception of the task. Diane had a strong sense of what the task was intending to achieve, and this had initially triggered what for her was a standard response to direct students' attention to the particular methods and topics which she understood to be involved. Over time and as a result of her experience of enactments of lessons based on the same task, however, she developed her relationship with the task (Remillard, 2005) and while maintaining fidelity to what she perceived as the intended purpose of the task, she set her own social and content goals (Schoenfeld, 1998), taking advantage of opportunities she had identified for student decision making and self-directed application of known mathematical topics in the realistic context of the question.

4.2.3 Mark's perception of the task

Mark was a teacher with over 20 years' experience, whose distinct interpretation of the task changed little over the course of the study. His pre- and post-lessonette scripts (shown in full in Appendix 6)

describe a particular method to reach a solution, which he used to solve the problem himself when it was initially presented.

Mark initially identified and did not revise 'methodical thought' as a lesson objective, and in interview he consistently described the importance of using logic to justify decisions in solving the task. This is an attributed rather than a professed belief (Schoenfeld, 1998), based on Mark's various unprompted references to 'look for some logic' in his own solution, 'the logic of setting things out', and 'the logic of looking through a task'. Commenting on what he thought was the learning intention of this and similar tasks from Bowland Maths, Mark expressed the view that 'you've got to read mathematically between the lines, … but if you don't spot that it's almost impossible to do the question'. In his script he recorded questions relating to the task such as 'why does price go up and then down?', and later: 'who pays for the empty seats?'. It is fair to presume that these were intended as prompts to use during a lesson, or maybe a note to himself for key underlying aspects to look out for during different episodes of his lesson. Zimmerlin and Nelson (1999), expressed the view that 'teachers plan down to the level of detail at which they feel they can comfortably manage improvisationally using established interactional skills and routines' (Zimmerlin & Nelson, 1999:267), which may have accounted for the Mark's relatively brief comments.

Mark's solution identified the use of the cost per person as a critical step which would be problematic for students, because whereas he as a teacher knew that 'generally if you buy things in larger quantities the unit price comes down, … students wouldn't have that experience'. This analysis has revealed that Mark made the same point during the discussion following the lessonette, and it is notable that he seemed to maintain this insight as an important feature of his intended plan. In his revised script, as Katy, he emphasised this particular concept by adding a 'starter' activity to address this, to direct students' attention early in the lesson to the concept of price per person, with the intention that students would realise the relevance when solving the problem. Whereas Katy included this starter activity as part of her effort to prompt students as neutrally as possible, Mark related this

adaptation to his observation of the lessonette, in which students had failed to appreciate the difference between overall cost and cost per person, which Mark described as the students 'doing comparison in what we would term "the wrong way" '.

Mark felt strongly that there was no need to make significant adaptations to the task ('I don't see the point of reinventing the wheel'), which he trusted was designed by 'somebody probably a lot more able than me at maths', with carefully chosen values to help reveal the appropriate mathematical thinking, so in his view it needed little preparation for use in class. Accounting for what he thought were the task designers' expectations, Mark explained that the teacher had a role in moving students from engagement in the practical situation 'to think[ing] about what they're doing in a deeper way'. Mark interpreted the task in terms of his own overarching goal (Schoenfeld, 2010) to develop students' methodical thinking, which cued him to instruct students in what he considered a general heuristic process for solving problems, namely to: extract key information; set work out logically and systematically; look for patterns; account for and explain what is noticed. His action plan represented an unpacking of the task (Ball et al., 2008: 400) into a series of episodes, each of which relate to relatively fine-grained content goals (Schoenfeld, 2010).

Analysis of Mark's pre- and post-lessonette scripts and his comments in interview revealed some of the thought processes in his consideration of his own lesson, from which we can tentatively infer his perception of the task. Mark consistently referred to logical thinking, which suggested his instinctive belief that mathematics was an expression of 'right', logical thinking, and that to learn mathematics is to become able to apply a rigorous, logical and systematic process to a given situation. His perception of the task rested on an assumption that the task had been designed with this same underpinning belief, and that there was little need to analyse or interrogate the task better to understand its purpose: he deferred to the expertise of the task designers, and prepared a lesson to faithfully realise their presumed aims. The task represented a contract, in which task designers had done their bit in creating an intriguing task, and in which he fulfilled his side of the bargain by designing an appropriate

lesson.

4.3 A metaphor

Lessons are complex orchestrations involving many interrelated factors. The analysis in the preceding section is not in the fine-grained detail of the work of the Teacher Model Group (Aguirre & Speer, 1999; Zimmerlin & Nelson, 1999; Schoenfeld, 2010), but does demonstrate the power of considering a teacher's beliefs, knowledge and goals to describe a lesson image. Having analytically deconstructed these three teachers' lesson images, I now aim to reconstruct them with a view to a more general discussion pointing out key similarities and differences which inform this study's research questions.

Before that discussion it will be helpful to summarise the preceding accounts of these three teachers' lessons. Rather than repeat the same terms, this section makes use of a metaphor shared by the Japanese-American mathematics educator Akihiko Takahashi (personal communication) of the teacher as a mountain guide, in which mathematics was a mountain terrain and the class a group ready to go on a trek. The task defined some place of interest in the mathematical terrain, to which there were many routes, with different sights along the way. Each teacher/mountain guide had made their way to that place once before, and had now made a plan for their group. Katy gave the group freedom to get to that place in their own way, but in good time because once they got there she wanted them to turn around and see from that vantage point where they had come from. Diane also gave the group freedom to get to that place in their own way, and once they had arrived that was the end of their trek. Mark planned to take the group along a particular route, with some sheer drops cautioned for anyone veering off course.

5 Discussion

This study was initially inspired by the different reactions of a group of teachers to one common task, and sought firstly to investigate what factors were involved and may be seen to affect teachers' perception of mathematical tasks. Secondly, this study sought to investigate the impact on this perception of teachers then experiencing an enactment of a lesson based on the same task.

There are similarities in these three teachers' lesson images, each of whom chose in their intended lessons to present the problem as published, although Katy appeared to take more care over establishing the meaningfulness of the context. All three teachers considered that there were opportunities for learning which were not given explicitly in the task itself, but each established different goals, and also differed in how they intended to steer the lesson to the purpose of the task, their chosen mathematical point (Sleep, 2012). This echoes Kullberg et al.'s (2013) accounts of three observably different lessons resulting from three teachers' treatment of the same task, leading to questions of whether sufficient use had been made of the 'possibilities that are inherent in the task' (Kullberg, Runesson, & Mårtensson, 2013:617).

5.1 Purpose of the task

Katy added a discussion of methodology to the end of her lesson, and having tried this with one class was now committed to spending longer on this aspect in her putative lesson; incrementally, since the initial presentation of the task and as a result of her experience of enactments of a lesson based on this task, the pursuit of this learning had become a goal-directed commitment on her action plan. For Katy, students arriving at a solution to the task was not the ultimate aim of the task: she intended to go on to draw students' attention to the whole problem-solving process itself. Katy perceived the task in terms of her own beliefs about the importance of the meaningfulness of mathematics, encouraging students to develop their own reasoning, and appropriated it as such.

Diane seemed initially to have an instinctive reaction to the task, similar to Kahneman's System 1 (2011), when she related that she was 'just thinking of the proper maths'. On first being presented

with the task, she had solved the problem herself, and having reached a solution she complied with what she perceived as the intention of the task: to scaffold student thinking, directing them to her own method of solution. Her experience of an enactment of a lesson based on the same task, either her own lesson, the lessonette, or both, led to a change in how she intended students to explore the task, but not in what she perceived as being the purpose of the task to arrive at a particular answer.

Mark perceived the task as a well crafted problem to solve, planning an intended lesson which involved the application of a general approach to solving problems in this particular situation. His view of the task and his associated lesson image changed very little as a result of his experience of the enacted lesson based on the same task, his enduring view seemed to be formed when he first engaged with the task.

5.2 Teachers' experience as a factor in perception of mathematical tasks

Presented with the same task, these three teachers perceived it differently, and prepared lesson plans with different intended outcomes and learning goals. What factors are seen to account for this different perception of the same one task? Neisser's perceptual cycle (Neisser, 1976:21) suggests that all that we perceive passes by our mental schemata, where experience comes to bear on our consideration of the task. The writer Anaïs Nin famously wrote that 'we do not see things as they are, we see them as we are' (Nin, 1961:124), which in our context can be interpreted as meaning that what teachers 'see' when presented with a task is not singular, dispassionate or objective, it is heavily influenced by the experience of the teacher who is perceiving it. In his book on how we think Schoenfeld (2010) related knowledge and perception when he said that 'what people perceive is a function of what they know; humans try to make new experiences fit with their current conceptions' (Schoenfeld, 2010:28). It is not unreasonable to speculate that we might identify a common style across a wide range of lessons prepared by one teacher despite their use of quite different tasks, in which case as the one common factor we would have to attribute this consistency to the teacher. Katy, Diane and Mark did each create lessons which reflected their own experience, and what they

perceived was a function of what they knew. The question then is whether teachers' perception is a function *only* of what they know: are there factors situated outside the perceiver?

5.3 Participatory relationship between teacher and task

As influential as experience was on how these teachers perceived mathematical tasks, this position unduly privileges the position of the teachers as the only agents in the teacher-task system, which contradicts the participatory relationship proposed by Remillard (2005). Considering Neisser's perceptual cycle (Neisser, 1976:21) again in our context, the task is the object which modifies the teacher's schema, and which is sampled as the teacher forms a lesson image. How the Taxi Cabs task was designed had an impact on these teachers, from the realistic context to the carefully orchestrated numbers to the voice and look of the resource as published, all characteristics noted as significant by Remillard (2005). For Katy the task exemplified and aligned with an aspect of mathematics teaching and learning which she had previously identified as important to her. Imagining the contrary situation, Katy would have a quite different experience were she to find herself working with a prescribed curriculum which did not align or was in some other way antithetical to her beliefs. For Katy this particular task was validating and allowed her to confidently develop her own teaching, to assume some of the authority associated with the task designers and the official status of a published resource. For Diane the task was initially something to be complied with, and then the site of her rediscovery and development of a more student-centred approach. For Mark the task was a professional challenge designed by respected experts, a tricky problem to be solved and an opportunity to revisit (rather than develop) his preferred approach to teaching towards a particular solution strategy.

Although the impact of the task on the teachers has to be inferred in terms of how it was interpreted – and different teachers think differently, as has been established – nonetheless we can consider that in these ways it was the task that made things happen, and so the task is an agent in a mutual relationship with the teacher, and features of its design are factors affecting teachers' perception of the task.

5.4 Research question 2: How does experience of enactment of the same task affect teachers' perception of mathematical tasks?

One research question in this study is whether and how teachers' experience of an enactment of a lesson based on the task caused a change in teachers' perception of the task. It is notable that Katy and Diane chose to use the task with their own classes, that this experience featured strongly in their account of their revised lesson image, and that Mark, who did not teach using the task himself, made minimal changes to his lesson; of course this may be correlation rather than causation.

Katy clarified an important 'big idea', which became the substantive mathematical goal of her lesson, as a result of her observation of her own students struggling to see beyond their own method, which lent credence and relevance to her own beliefs about the importance of making meaning when learning mathematically. Whereas she noticed students during her own lesson, the impact of the lessonette on Katy was in terms of her observation of the teacher, whose patience Katy noted, and which was reflected in her revisions to ensure that her own prompting was as neutral as possible.

Diane made significant changes to her lesson image, which she related explicitly to her observations of how students were thinking when using the task. Although she had said little in the discussion following the lessonette, what she did say revealed that she had been considering the task from the students' perspective, quite distinct from the perspective which she had adopted initially, as revealed in interview. It is also interesting to note from her own account that Diane used a similar approach to the lessonette teacher in her own lesson, both these teachers having maintained a focus on observing their students and trying to appreciate, successfully, what students were thinking as the lesson developed. Diane did not connect the change in her own practice to her experience observing the lessonette, but this may be something which she had not herself realised.

The experience of enactment of lessons based on the same task both plausibly and demonstrably made an impact on the goals which Katy and Diane subsequently included in their lesson image. This experience caused these two teachers to recognise new opportunities which they had not otherwise

seen, expanding their repertoire of 'possibilities and contingencies related to a lesson' (Schoenfeld, 1998:17) which constitute the teacher's lesson image. Seen in the changes to goals in their revised scripts, by focusing teachers' attention on observation these enactments changed what these teachers considered possible and likely to be involved in their own intended lesson, affecting their perception of the task.

Mark's account made little reference to the lessonette other than that he had noticed students making a mathematical mistake. The comment which he made during the discussion was the same as that made during the interview, which he reported to have first established when he initially solved the task. Mark's constancy suggested either that there were other factors involved, or that only teaching the lesson oneself caused a change in perception, and not observation of the lessonette.

Clarke and Hollingsworth (2002) describe teachers' professional development and growth as being mediated by processes of enactment and reflection. In these terms, the changes in salient outcomes and beliefs seen for Katy and Diane – and the lack of change for Mark – might be reasonably attributed to their own use of the task and subsequent reflection upon it during the study.

5.5 What factors influence teachers' perception of mathematical tasks?

The analysis has provided a number of answers to the central research question. Firstly: teacher beliefs. Both because this has been the principal result of this study, and because it has a strong impact, the importance of beliefs as a factor influencing teachers' perception of mathematical tasks is key. Everything that the teacher becomes consciously aware of, all goals which the teacher decides upon, and all actions which the teacher plans pass through the prism of the teacher's beliefs.

Secondly: the task itself. As these teachers perceived the task they inferred meaning which was unintended but which had an impact on their lesson images. My interpretation of the data and the analysis above, and a substantive finding of this study, is that each teacher ascribed some authority to the task, as seen in how they related to the task. Katy embraced the task, almost as 'someone' she was enthusiastic to work with. Diane and Mark, in different ways, both deferred to the authority

which they themselves ascribed to the task, variously complying with the presumed purpose. Remillard related both the importance of teachers as interpreters of the written curriculum (Remillard, 2005:237) and the need for further research on 'teacher's professional identity as it relates to the use of curriculum resources' (Remillard, 2005:237). In this analysis teachers appeared to have distinct professional identities, which revealed itself in the extent to which they were willing and able to actively interpret the task. This was unintentional, and a function of many latent features of the task, including: intrigue (a sense that there was something to find out about the task, something 'going on' in the choice, in this task, of these particular numbers); the officiality of the task (either as something which had been mandated, or in its associations with respected task designers); the relevance of the task (how it aligned with reform efforts more broadly and in the school); and in the physical aspect of the material artifact, from a sense of the quality of the product to its 'look and feel'. This is far from being an exhaustive list, but these have been suggested as important in this study. Each of these aspects of the task make an impact on teachers' perception via their beliefs, which makes it difficult for task designers to effect a particular intended aim. As well as making convincing arguments which may appeal to teachers' conscious, professional and considered view, there is a place here for tasks to be designed to affect teachers unconsciously, applying findings from cognitive psychology (e.g. Kahneman, 2011) as widely used in other persuasive domains such as politics and advertising. This could be the subject of future study.

Although it did not specifically emerge from this study, knowledge also plays a role in teachers' perceptions of mathematical tasks. Relating perception in terms of a lesson image which comes to mind when presented with a task, a novice who had never taught the relevant topic nor knew their students would differ from that of an expert who had been there dozen of times before, as related by Schoenfeld above (Section 2.6.2) for whom some familiar lessons presented few surprises (Schoenfeld, 1998:18). Due to the methods employed in this study, this has not become apparent as teachers who did not make relevant subject knowledge clear in their scripts were rejected at the stage

of selecting teachers for interview. Had these teachers been included, knowledge may well have emerged as a strong contributory factor to teachers' perception of mathematical tasks.

Schoenfeld (2010) considered teacher knowledge as a subset of the broader category of all of the resources which a teacher might put to use in their teaching. As well as the intellectual resources already accounted for, this category includes material and contextual resources. A teacher in a refugee camp (Watson, 2016) might well perceive a task differently from one of the teachers in this study, due to differences in the material resources available or appropriate, and contextual constraints which might be pertinent to the teacher's situation. Again, these aspects have not emerged as a result of methodological aspects of this study. All three teachers worked in the same department of the same school, and as such had similar material and contextual resources available to them, so this aspect did not emerge as a factor in this study.

There are other possible factors which may have contributed to the differences seen in the perception of the teachers included in this study, including teachers' sensitivity to students (Jaworski, 1992). Jaworski (1992) described sensitivity as involving a knowledge of students and their needs, and of an approach that works to meet those needs (Jaworski, 1992:8). This is reminiscent of Dewey's notion that there is a need for experience to *psychologized* (Dewey, 1902:22), because what concerns a teacher, rather than a subject specialist is 'how his own knowledge of the subject matter may assist in interpreting the child's needs and doings' (Dewey, 1902:23). Katy's anticipation of student responses and Diane's close and respectful observation of her students each attest to an empathetic dimension of their practice synonymous with sensitivity to students. As a teacher perceives a task, their sensitivity has an impact on the possibilities and contingencies to be considered as they plan their lesson.

Within the Teacher Group Model, this, and other factors, is bundled into the existing conception of teachers' knowledge, be it knowledge of students, or pedagogical content knowledge, or some other subdivision. Sensitivity has emerged as an explanatory factor which might account for the differences

seen in these teachers' perception of the same tasks, and raises the possibility of still more.

6 Conclusion and Implications for future practice

The findings of this study are that teachers' beliefs are an integral and unavoidable factor in teachers' perception of mathematical tasks. Tasks may be designed to achieve a particular intended learning outcome, which can take account of theories of student learning and the nature of mathematics, but the realisation of that task as activity is inevitably affected by the beliefs of each and every teacher who uses the task to teach.

It is neither desirable nor ethical to control what teachers believe, but teachers' beliefs are not fixed, so understanding how they can and do change is important. Changes in teachers' beliefs as an aspect of professional growth are associated with enactment and reflection on changes in teaching practice and in what teachers consider to be salient outcomes, and tasks have a critical role in stimulating this process (Clarke & Hollingsworth, 2002). Although it is possible for a task to directly affect teachers' knowledge and beliefs, Clarke and Hollingsworth (2002) proposed that an early focus on professional experimentation could successfully lead to changes in beliefs.

Teachers and tasks have a participatory relationship (Remillard, 2005), and this study has shown that how a teacher relates to a task affects their intentions for planned activity. Teachers unintentionally ascribe authority to tasks and relate to that authority according to their professional identity, their own beliefs and personal characteristics, which can act to constrain or allow the teacher's goal setting in their intended lesson plan.

Remillard (2005) proposed various features of tasks which have an impact on this relationship, and still more are suggested in the outcomes to this study. Task design has been the recent focus of concerted research efforts. The call for a 2013 ICMI study on task design in mathematics indicated that group's interest, amongst others, in 'how published tasks are appropriated by teachers for complex purposes' (Margolinas, 2013:10). Just as in Stein et al.'s (1996) seminal paper, reports in the

ICMI study focus more on the classroom implementation phase of the lesson and accounts of the difference between tasks as intended and enacted (e.g. Ding, Jones, & Pepin, 2013; Kullberg et al., 2013). Design features of tasks impact on the teacher-task relationship even before the lesson has been enacted, and further research on these features and the interface between the task as published and the task as intended to be used by the teacher would support our better understanding of the nature of these design features and the process by which they can constrain or allow teachers to make productive use of tasks, in how they appeal to teachers' conscious beliefs, knowledge and goals, and how they might affect the teacher unconsciously.

Programmes aimed at teachers' professional development and growth should support teachers' professional experimentation using rigorously designed tasks, with due consideration for the enabling processes of enactment and reflection and associated changes in beliefs and what teachers consider salient outcomes. From the outset, such initiatives should give regard to teachers' relationship with the task, considering issues of authority, permission to innovate, and alignment of the intended purpose of the task within the teacher's professional context.

Experience of enactments of the task did have an impact on two of the three teachers in this study. One issue with a teacher learning from reflection on their own use of a task is that this benefit is lost for those students who have already experienced the lesson; only the teacher's other classes and students in future years get to participate in the teacher's revised plan. It would be beneficial for a teacher to be able to make these revisions to their plans before teaching it, rather than as a result of teaching it. A Japanese approach called kyozai kenkyuu (Fujii, 2016), part of a cycle of a professional development practice called lesson study, addresses this problem. Fujii (2016) described how a group of teachers researched teaching materials and made pedagogical design decisions to effect successful outcomes in goals which ranged from overarching to fine-grained detail, by anticipating students' responses to tasks and making modifications to realise mathematical goals as seen from the students' perspective. In some detail the group of teachers hypothetically considered the mathematical flow of

an intended lesson, which enabled them to make key improvements before the lesson was actually taught.

Schoenfeld's model (Schoenfeld, 1998:13) was aimed at accounting for decisions which teachers made during lessons, and he and his group found it particularly revealing when teachers reacted to unanticipated moments. Rowland (2013) identified such contingent knowledge as one of four domains in his Knowledge Quartet, and Mason and Davis (2013) described the importance of teachers' mathematical awareness for in-the-moment pedagogy, which they applied both to teachers planning and while actually teaching (Mason & Davis, 2013). The changes seen in some teachers' lesson images in this study suggested how teachers developed their internalised working knowledge of the 'possibilities and contingencies' (Schoenfeld, 1998:17) associated with activity based on a task, but more research is needed to understand how else this might be supported and how it relates to their future practice.

Following their experience of enactments of the task in this study, Katy and Diane both made changes which related directly to considering learning from the students' perspective. This is reminiscent of the Japanese teachers in Fernandez and Cannon's (2005) investigation of Japanese and US teachers' practice in regard to planning mathematics lessons. They found that 'Japanese teachers' thinking was more focused on ensuring that students discover concepts and develop productive dispositions toward learning, whereas U.S. teachers' thinking tended to emphasize that students learn specific mathematical content' (Fernandez & Cannon, 2005:481). My study may have revealed and dispelled some aspects of this apparent cultural difference, but of course this is based on accounts of just three teachers, so also warrants further investigation.

In conclusion: how teachers perceive mathematical tasks is idiosyncratic, important in the setting of intended learning goals for a lesson, and is affected by many factors. Teachers' perception of mathematical tasks can change. The change in perception and the subsequent plans of Katy and Diane in this study raise the possibility of further questions: does the change in teachers' perception have

an enduring impact on teachers' initial perception of new tasks? What difference might a more refined lesson image make to the enacted lesson and student learning? How teachers perceive mathematical tasks is fundamental to how they teach and grow professionally.

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APPENDIX 1

The task: Taxi Cabs (Bowland Maths, 2010)

Taxi Cabs



Max and Liz are organising a trip to the airport for a party of 75 people.

They can use two types of taxi:

- A small taxi costs £40 for the trip and holds up to 4 people.
- A large taxi costs £63 for the trip and holds up to 7 people.

Max says they should order 6 large taxis

1. (a) So how many small taxis will they need?
   (b) How much will the total cost be?

Liz thinks that they can organise the journey more cheaply than this!

2. Is Liz right? How many taxis of each type should they order, to get the cost as low as possible?

APPENDIX 2
Blank pro forma offered to all teachers

Lessonette Planner

Name	
Research lessonette title:	
Phases within the lesson, purpose, and resolution	Research Questions: • • • Lesson objectives: • • • Phase A : Time: Phase B : Time: Phase C: Time:
Materials:	

APPENDIX 2 cont.

Blank pro forma offered to all teachers

Planned teacher – student interaction		
Key questions that the teacher will pose	Anticipated student responses	Teacher responses to student responses

APPENDIX 2 cont.

Blank pro forma offered to all teachers

Conceptual development – where this concept sits relative to prior and future learning, e.g. according to research, National Curriculum documents across Key Stages, connections with other concepts
Concept:
Prior learning This learning Future learning

Key discussion points	Decisions made / how points were resolved

Focus for observers:
• • •

APPENDIX 3

Group X lessonette plan

Taxi cabs

Lesson Plan

Starter (5 minutes) in whiteboards

Find the LCM of the following numbers

6 and 8

5 and 3

12 and 15

Main Lesson

The students will be in groups of 4 and they will have (1 leader in each group)

1) **Cubes (they won't be informed about the number of cubes – 75)**
2) **The problem in A4 (and on the board)**
3) **Post-it notes**
4) **Mini whiteboards and pens**
5) **A3 paper**
6) **Plain papers**

The students are expected to attempt and start solving the problem.

In 5 minutes the leader of each group will go to another group , take a feedback and return to their group.

In 5 minutes a feedback for the first part will be taken randomly.

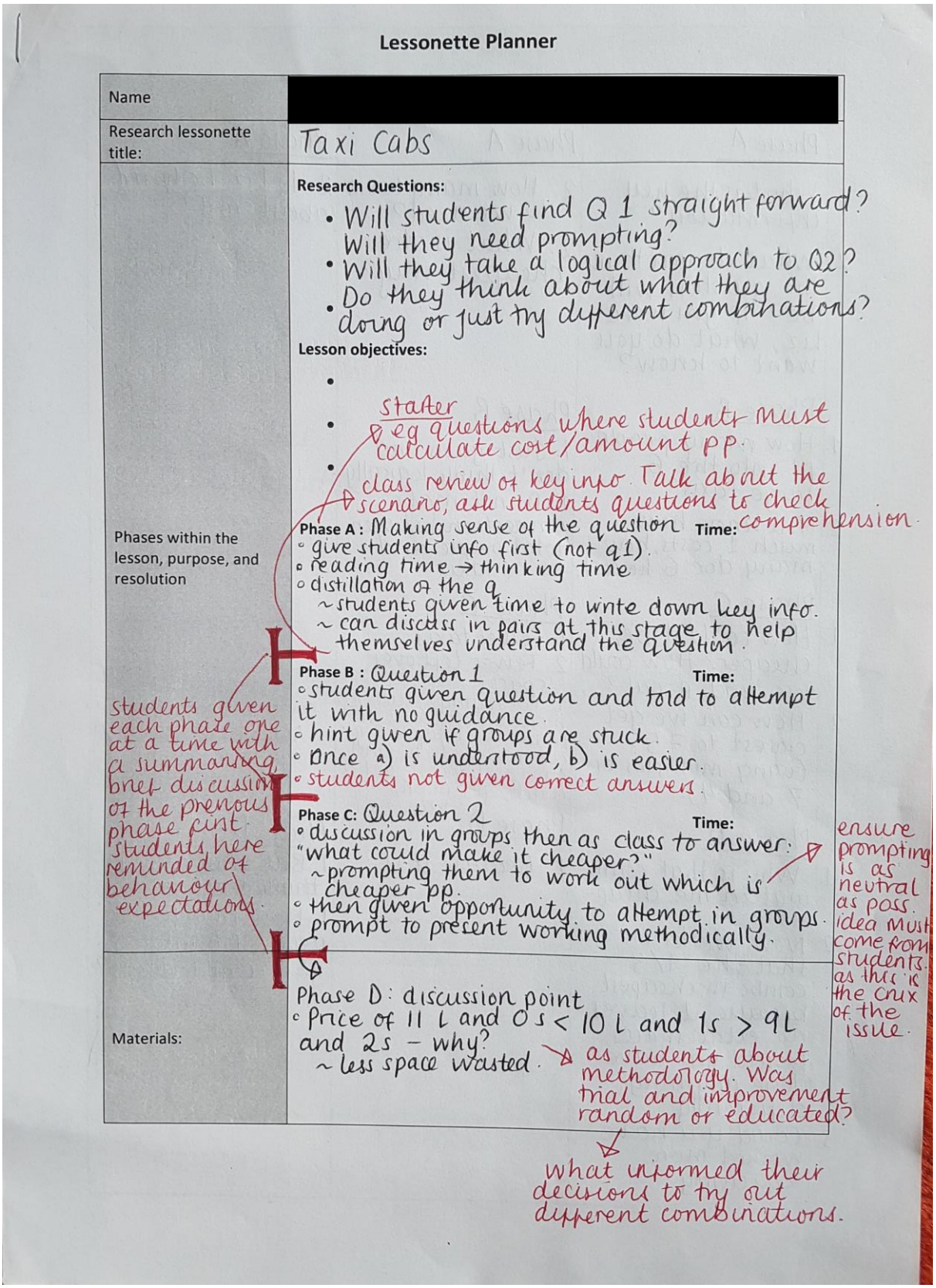
(Address any misconceptions so they can move on in the second part)

Students attempt the second question for 10 minutes,

Then a feedback and discussion will take place.

APPENDIX 4

Katy's script



Lessonette Planner

Name	[redacted]
Research lessonette title:	Taxi Cabs
Phases within the lesson, purpose, and resolution	**Research Questions:** • Will students find Q 1 straight forward? Will they need prompting? • Will they take a logical approach to Q2? • Do they think about what they are doing or just try different combinations? **Lesson objectives:** • • • **Phase A :** Making sense of the question **Time:** ◦ give students info first (not q1). ◦ reading time → thinking time ◦ distillation of the q ~ students given time to write down key info. ~ can discuss in pairs at this stage to help themselves understand the question. **Phase B :** Question 1 **Time:** ◦ students given question and told to attempt it with no guidance. ◦ hint given if groups are stuck. ◦ Once a) is understood, b) is easier. ◦ students not given correct answers. **Phase C:** Question 2 **Time:** ◦ discussion in groups then as class to answer: "what could make it cheaper?" ~ prompting them to work out which is cheaper pp. ◦ then given opportunity to attempt in groups. ◦ prompt to present working methodically.
Materials:	Phase D: discussion point ◦ Price of 1l L and 0s < 10 L and 1s > 9L and 2s – why? ~ less space wasted.

starter
eg questions where students must calculate cost/amount pp.

class review of key info. Talk about the scenario, ask students questions to check comprehension.

students given each phase one at a time with a summarising, brief discussion of the previous phase first. Students here reminded of behaviour expectations.

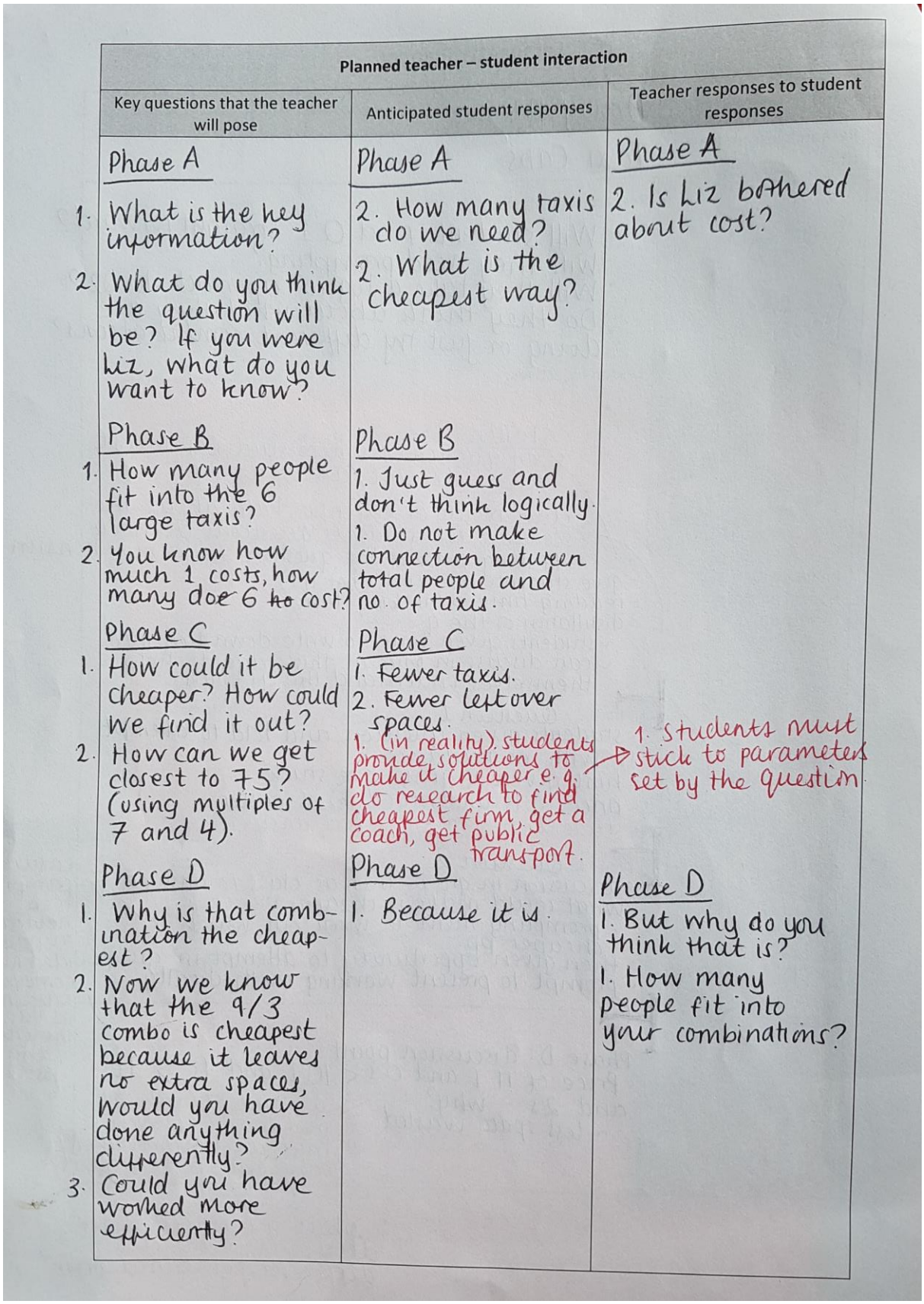
ensure prompting is as neutral as poss. idea must come from students as this is the crux of the issue.

as students about methodology. Was trial and improvement random or educated?

what informed their decisions to try out different combinations.

APPENDIX 4 cont.

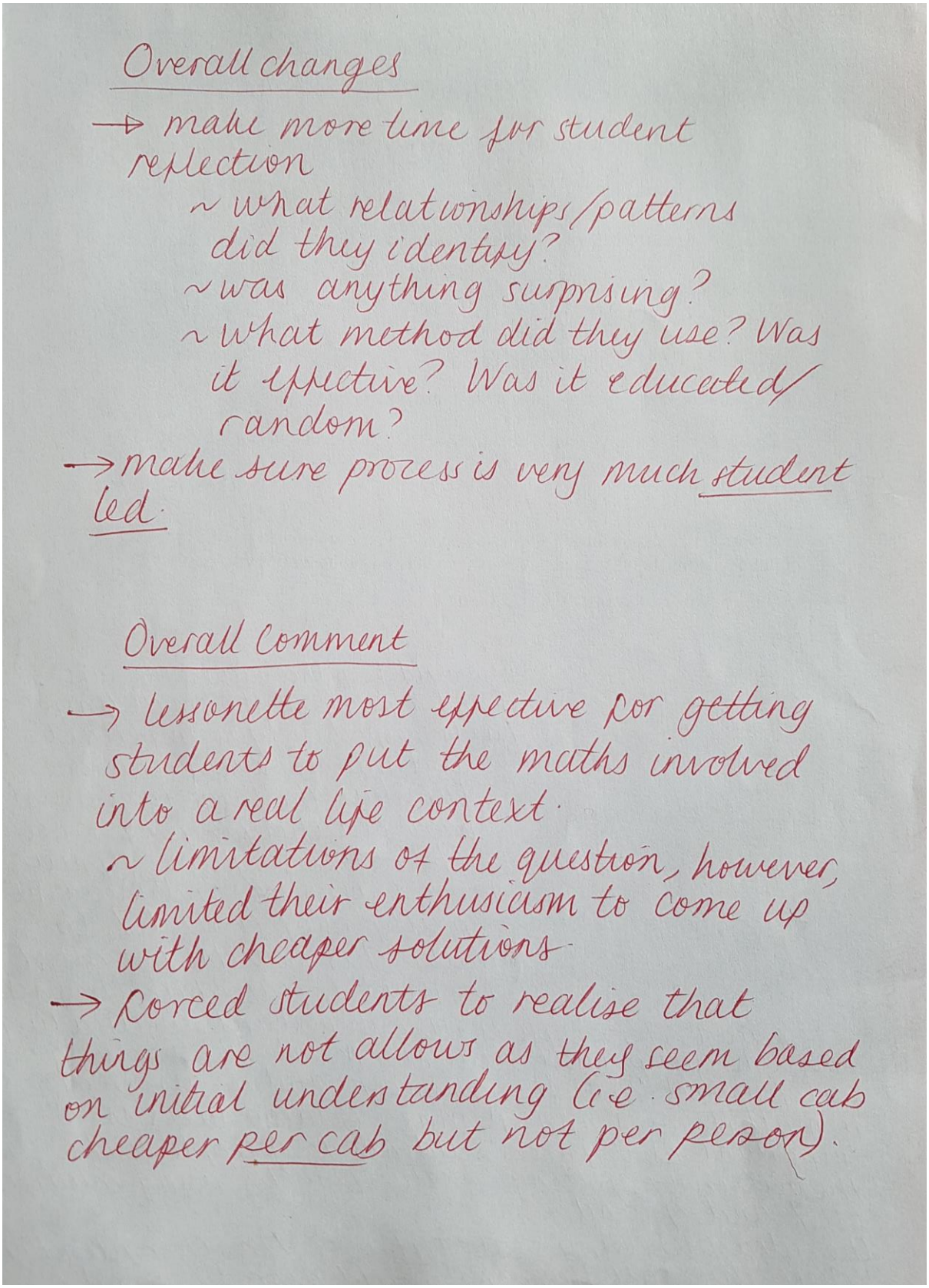
Katy's script



Planned teacher – student interaction		
Key questions that the teacher will pose	Anticipated student responses	Teacher responses to student responses
Phase A 1. What is the key information? 2. What do you think the question will be? If you were Liz, what do you want to know?	**Phase A** 2. How many taxis do we need? 2. What is the cheapest way?	**Phase A** 2. Is Liz bothered about cost?
Phase B 1. How many people fit into the 6 large taxis? 2. You know how much 1 costs, how many doe 6 ~~to~~ cost?	**Phase B** 1. Just guess and don't think logically. 1. Do not make connection between total people and no. of taxis.	
Phase C 1. How could it be cheaper? How could we find it out? 2. How can we get closest to 75? (using multiples of 7 and 4).	**Phase C** 1. Fewer taxis. 2. Fewer left over spaces. 1. (in reality) students provide solutions to make it cheaper e.g. do research to find cheapest firm, get a coach, get public transport.	1. Students must stick to parameters set by the question.
Phase D 1. Why is that combination the cheapest? 2. Now we know that the 9/3 combo is cheapest because it leaves no extra spaces, would you have done anything differently? 3. Could you have worked more efficiently?	**Phase D** 1. Because it is.	**Phase D** 1. But why do you think that is? 1. How many people fit into your combinations?

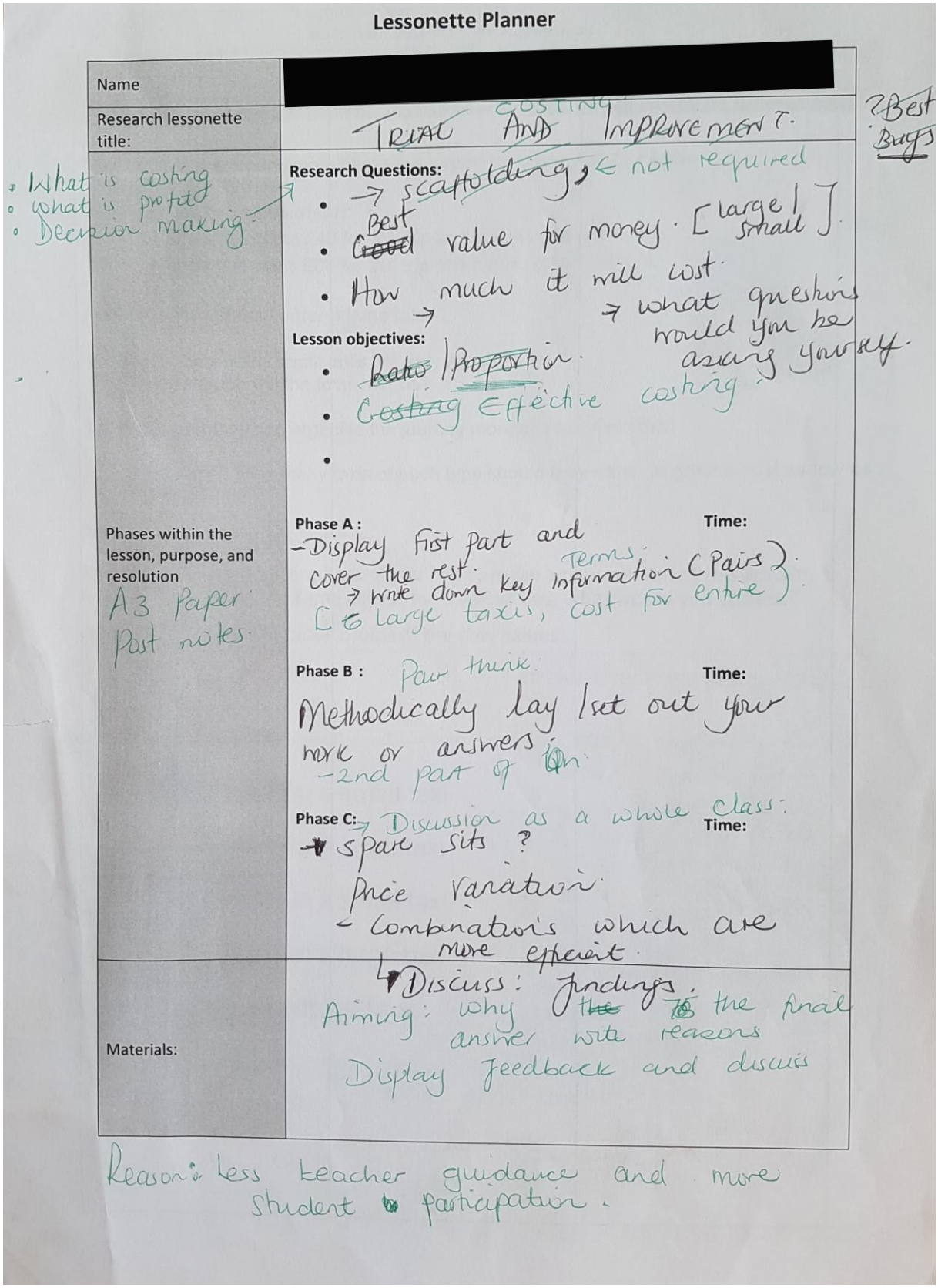
APPENDIX 4 cont.

Katy's script



APPENDIX 5

Diane's script



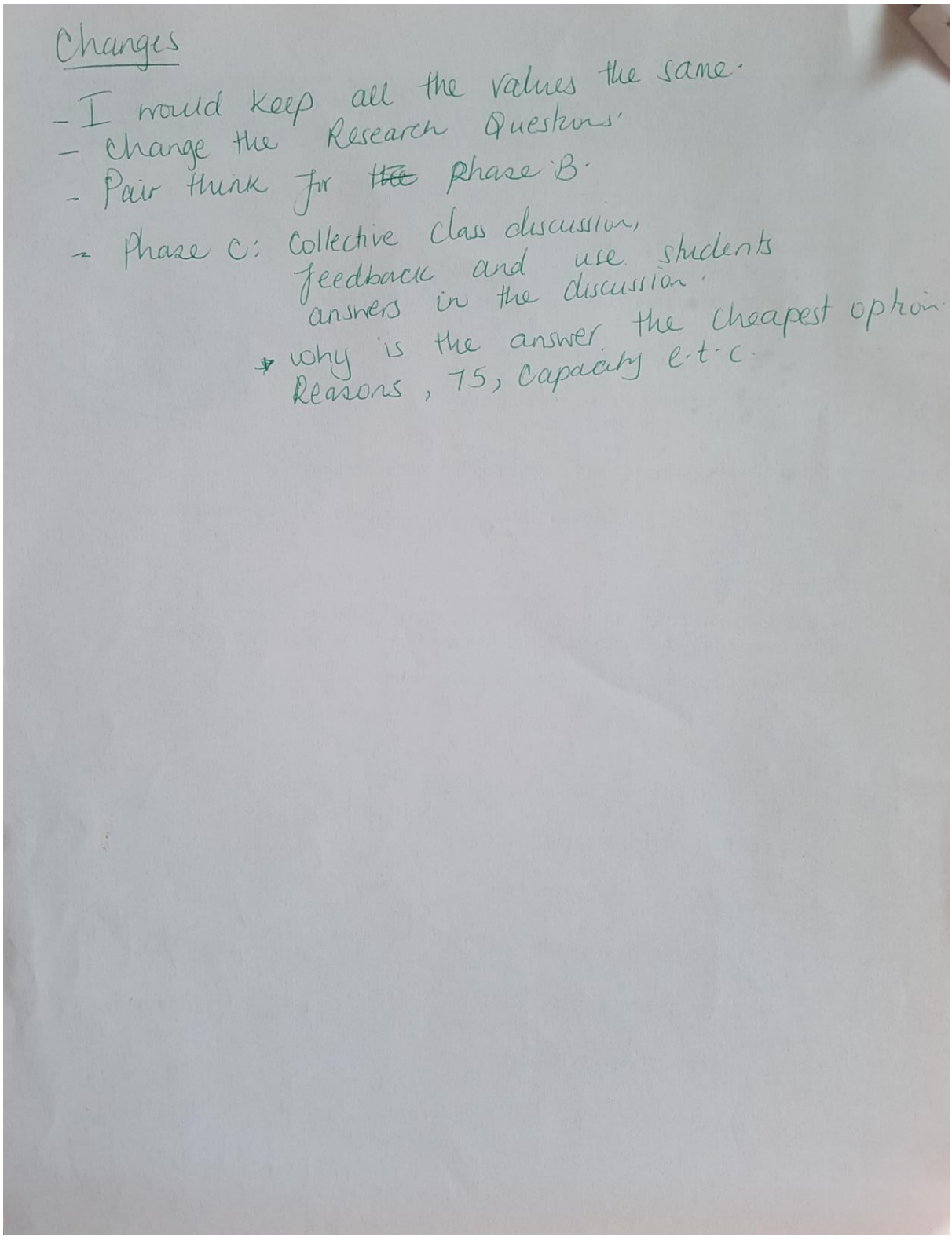
Lessonette Planner

Name	[redacted]
Research lessonette title:	TRIAL AND IMPROVEMENT. COSTING. ?Best Buys
	Research Questions: • → scaffolding, ← not required • Best value for money. [large/small] • How much it will cost. → what questions would you be asking yourself. **Lesson objectives:** • Ratio / Proportion. • Costing ← Effective costing •
Phases within the lesson, purpose, and resolution A3 Paper. Past notes.	**Phase A :** Time: - Display first part and cover the rest. → write down key information (Pairs). Terms. [← large taxis, cost for entire] **Phase B :** Pair think. Time: Methodically lay/set out your work or answers. - 2nd part of ⓐ **Phase C:** → Discussion as a whole class: Time: → spare sits ? price variation - combinations which are more efficient
Materials:	↳ Discuss: findings. Aiming: why the final answer with reasons Display feedback and discuss

Reason: less teacher guidance and more student participation.

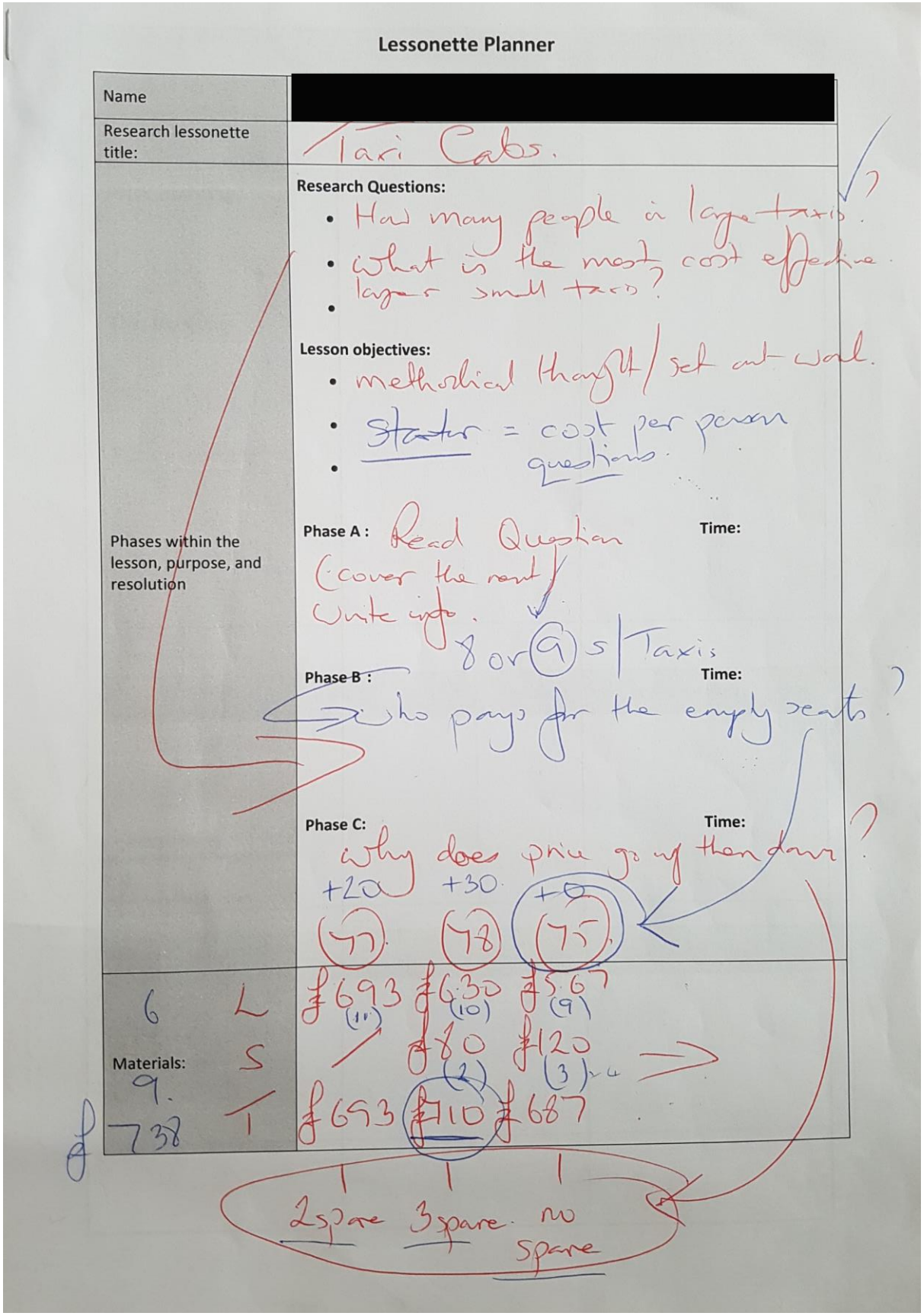
APPENDIX 5 cont.

Diane's script



APPENDIX 6

Mark's script



Lessonette Planner

Name	[redacted]
Research lessonette title:	Taxi Cabs.
Phases within the lesson, purpose, and resolution	**Research Questions:** • How many people in large taxis? • What is the most cost effective. larger small taxis? • **Lesson objectives:** • methodical thought/ set out work. • Starter = cost per person questions. • **Phase A :** Read Question (cover the rest) Write up. 8 or (9) s/Taxis **Time:** **Phase B :** who pays for the empty seats? **Time:** **Phase C:** why does price go up then down? **Time:** +20 +30 +0 (77) (78) (75)
Materials: 6 L 9. S 738 T	$693 $630 $567 (11) (10) (9) $80 $120 (2) (3) $693 $110 $687 2 spare 3 spare no spare

APPENDIX 6 cont.

Mark's script

