

Abstract

We show that large gaps between smooth numbers are infrequent. The key new tool is a novel mean value bound for a special type of Dirichlet polynomial.

The Differences Between Consecutive Smooth Numbers

D.R. Heath-Brown
Mathematical Institute, Oxford

In Celebration of the Seventy-Fifth Birthday of Robert Tijdeman

1 Introduction

If y is a positive real number, an integer $n \in \mathbb{N}$ is said to be y -smooth if all its prime factors $p \mid n$ satisfy $p \leq y$. We are interested in upper bounds for large gaps between consecutive y -smooth numbers. For example, if a_1, a_2, \dots is the sequence of y -smooth numbers in increasing order, one might ask for upper bounds for

$$\max\{a_{n+1} - a_n : a_n \leq x\}$$

in terms of x and y . In this paper we will primarily be interested in a measure for the frequency of large gaps, given by

$$\sum_{a_n \leq x} (a_{n+1} - a_n)^2.$$

The reader should note that there is a dependence on y which is not mentioned explicitly above. In order to assess bounds for this latter sum we will want to know how many y -free integers $a_n \leq x$ there are. The notation $\psi(x, y)$ is standard for this quantity. It has been extensively investigated, but for our purposes it will suffice to know that

$$x \ll_{\varepsilon} \psi(x, x^{\varepsilon}) \leq x$$

for any fixed $\varepsilon > 0$, and that

$$\psi(x, y) = x^{1+o(1)}$$

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if $y \geq (\log x)^{f(x)}$ for some function $f(x)$ tending to infinity with x (see Hildebrand and Tenenbaum [4, Corollary 1.3], for example). When $\psi(x, y) = x^{1+o(1)}$ one might hope that

$$\sum_{a_n \leq x} (a_{n+1} - a_n)^2 \ll_{\varepsilon} x^{1+\varepsilon} \quad (1)$$

for any fixed $\varepsilon > 0$.

If y is not too small compared to x it turns out that questions about y -smooth numbers are no harder, and sometimes easier, than the corresponding questions about primes. For example, recent work of Matomäki and Radziwiłł [7, Corollary 1] shows that for any $\varepsilon > 0$ the gaps between consecutive x^{ε} -smooth numbers up to x are at most $O_{\varepsilon}(x^{1/2})$. The corresponding result for primes is just out of reach, even on the Riemann hypothesis.

There has been much work on the sum

$$\sum_{p_n \leq x} (p_{n+1} - p_n)^2.$$

Under the Riemann hypothesis, work of Selberg [10] shows that

$$\sum_{p_n \leq x} (p_{n+1} - p_n)^2 \ll x(\log x)^3,$$

while Yu [12] obtains

$$\sum_{p_n \leq x} (p_{n+1} - p_n)^2 \ll_{\varepsilon} x^{1+\varepsilon},$$

subject only to the Lindelöf hypothesis. An unconditional bound of the same strength seems far out of reach, and the present paper is therefore designed to investigate the extent to which one can establish the corresponding bound (1) unconditionally. We shall content ourselves with an investigation of x^{ε} -smooth numbers. However our approach generalizes to y -smooth number in general, so long as $(\log x)/(\log y)$ is at most a small power of $\log x$.

Theorem 1 *Let a_n be the x^{ε} -smooth numbers, in increasing order. Then*

$$\sum_{\substack{a_n \leq x \\ a_{n+1} - a_n \geq x^{1/3+\varepsilon}}} (a_{n+1} - a_n)^2 \ll_{\varepsilon} x^{1+\varepsilon}.$$

Thus we achieve (1) for those gaps that have length at least $x^{1/3+\varepsilon}$. Unfortunately however our new method breaks down entirely for smaller gaps. To cover the remaining range we can use pre-existing methods, which lead to our next result.

Theorem 2 *Let a_n be the x^ε -smooth numbers, in increasing order. Then*

$$\sum_{a_n \leq x} (a_{n+1} - a_n)^{3/2} \ll_\varepsilon x^{1+\varepsilon}.$$

As a corollary we then obtain the following bound.

Theorem 3 *Let a_n be the x^ε -smooth numbers, in increasing order. Then*

$$\sum_{a_n \leq x} (a_{n+1} - a_n)^2 \ll_\varepsilon x^{7/6+\varepsilon}.$$

The key idea behind our proof of Theorem 1 is a new type of estimate for a certain mean value of Dirichlet polynomials. Let $T \geq 1$ and let \mathcal{M} be a set of distinct integers $m \in (0, T]$. We will write $R = \#\mathcal{M}$. For each $m \in \mathcal{M}$ let ε_m be a complex number of modulus at most 1. Suppose that N is a positive integer and that q_1, \dots, q_N are real coefficients in $[0, 1]$. Write

$$M(s) := \sum_{m \in \mathcal{M}} \varepsilon_m m^{-s} \quad \text{and} \quad Q(s) := \sum_{n \leq N} q_n n^{-s}.$$

The mean value in which we are interested is then

$$\mathcal{I}(\mathcal{M}, Q) := \int_0^T |M(it)Q(it)|^2 dt,$$

which will be related to

$$\mathcal{J}(\mathcal{M}, Q) := \sum_{\substack{(m_1, m_2, n_1, n_2) \in \mathcal{M}^2 \times \mathbb{N}^2 \\ |\log(m_1 n_1 / m_2 n_2)| \leq T^{-1}}} q_{n_1} q_{n_2}.$$

We then have the following results.

Theorem 4 (i) *We have*

$$\mathcal{I}(\mathcal{M}, Q) \ll T \mathcal{J}(\mathcal{M}, Q).$$

Moreover, if $\varepsilon_m = 1$ for every $m \in \mathcal{M}$ then

$$\mathcal{I}(\mathcal{M}, Q) \ll T \mathcal{J}(\mathcal{M}, Q) \ll \mathcal{I}(\mathcal{M}, Q).$$

(ii) *Under the Lindelöf Hypothesis, for any $\eta > 0$ and any $Q(s)$ we have*

$$\mathcal{I}(\mathcal{M}, Q) \ll_\eta N^2 R^2 + (NT)^\eta NRT. \quad (2)$$

(iii) Unconditionally, for any $\eta > 0$ and any $Q(s)$ we have

$$\mathcal{I}(\mathcal{M}, Q) \ll_{\eta} N^2 R^2 + (NT)^{\eta} \{NRT + NR^{7/4}T^{3/4}\}.$$

Moreover, (2) holds when either $N \geq T^{2/3}$ or $R \leq T^{1/3}$.

(iv) If

$$Q(s) = \frac{1}{k!} \left(\sum_{N^{1/k}/2 < p \leq N^{1/k}} p^{-s} \right)^k,$$

with p running over primes, then

$$\mathcal{I}(\mathcal{M}, Q) \ll_k N^2 R^2 + R^2 T + NRT.$$

Part (ii) is included here for motivation only. It is essentially Lemma 4 of Yu's work [12].

Part (iii) will not be used in this paper. However in later work we plan to explore the application of Theorem 4 to differences between consecutive primes. In particular we intend to use Theorem 4 to improve on Matomäki's bound [6]

$$\sum_{\substack{p_n \leq x \\ p_{n+1} - p_n \geq x^{1/2}}} p_{n+1} - p_n \ll x^{2/3}.$$

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2 Dirichlet Polynomials and Gaps

In this section we will describe a procedure for bounding

$$\#\{a_n \leq x : a_{n+1} - a_n \geq H\}$$

from above. It turns out that it is convenient to work on dyadic ranges $x < a_n \leq 2x$, and to work with gaps of length at least $2H$, rather than H . We will assume that $H \leq x^{3/4}$, and we define

$$\mathcal{N}(H, x) = \mathcal{N} = \#\{x < a_n \leq 2x : a_{n+1} - a_n \geq 2H\}.$$

Suppose that m is the smallest integer with $m > a_n/H$. Then $m > x/H$, and $m \leq a_n/H + 1 \leq 3x/H$. Moreover if

$$\delta_0 := H/(4x)$$

then

$$\begin{aligned} (1 + \delta_0)Hm &\leq \left(1 + \frac{H}{4x}\right) (a_n + H) = a_n + H + \frac{a_n}{4x}H + \frac{H}{4x}H \\ &\leq a_n + H + \frac{H}{2} + \frac{H}{4} < a_n + 2H \leq \min(4x, a_{n+1}), \end{aligned} \quad (3)$$

so that

$$[Hm, (1 + \delta_0)Hm] \subset (a_n, a_{n+1}) \quad (4)$$

in particular. For each a_n counted by \mathcal{N} there is a corresponding integer m , giving us a set $\mathcal{M} \subset (x/H, 4x/H)$ of such integers m , with $\#\mathcal{M} = \mathcal{N}$.

We now define

$$F(s) = \sum_{n \leq N} c_n n^{-s} = \frac{1}{k!} P(s)^{k-1} P_1(s), \quad (5)$$

with

$$P(s) := \sum_{x^{1/k}/2 < p \leq x^{1/k}} p^{-s} \quad \text{and} \quad P_1(s) := \sum_{x^{1/k} < p \leq 2^{k+2}x^{1/k}} p^{-s}.$$

Here we will have $N = 2^{k+2}x$. Clearly the coefficients c_n of $F(s)$ will be minorants for the characteristic function of the sequence $(a_j)_1^\infty$, at least when $n \in (x, 4x]$. Thus if $n \in (x, 4x]$ we will always have $c_n \leq 1$, and indeed we will have $c_n = 0$ unless n is a member of the sequence $(a_j)_1^\infty$. Under the above assumptions, if $m \in \mathcal{M}$ then

$$\sum_{Hm < n \leq (1+\delta_0)Hm} c_n = 0,$$

since there are no elements a_j in the interval $[Hm, (1 + \delta_0)Hm]$. We will compare the above sum with the average over a somewhat longer interval $(Hm, (1 + \delta_1)Hm]$, with

$$\delta_1 = \delta_1(x) = \exp(-\sqrt{\log x}) \geq \delta_0.$$

We now have

$$\begin{aligned} \sum_{Hm < n \leq (1+\delta_1)Hm} c_n &= \\ \frac{1}{k!} \sum_q \# \{x^{1/k} < p \leq 2^{k+1}x^{1/k} : p \text{ prime}, Hm/q < p \leq (1 + \delta_1)Hm/q\}, \end{aligned}$$

where q runs over products $q = p_1 \dots p_{k-1}$, counted according to multiplicity, with $x^{1/k}/2 < p_i \leq x^{1/k}$. Since $2^{1-k}x^{(k-1)/k} < q \leq x^{(k-1)/k}$ and

$$x < Hm < (1 + \delta_1)Hm < 2Hm < 2(1 + \delta_0)HM < 8x,$$

by (3), we see that the condition $Hm/q < p \leq (1 + \delta_1)Hm/q$ already implies that $x^{1/k} < p \leq 2^{k+1}x^{1/k}$. Moreover the Prime Number Theorem holds with a sufficiently good error term that we can deduce an asymptotic formula

$$\#\{Hm/q < p \leq (1 + \delta_1)Hm/q : p \text{ prime}\} \sim \frac{\delta_1 Hm}{q} \frac{k}{\log x}.$$

We then find that

$$\sum_{Hm < n \leq (1+\delta_1)Hm} c_n \gg_k \frac{\delta_1 x}{(\log x)^k}$$

so that

$$\sum_{Hm < n \leq (1+\delta_1)Hm} c_n \geq \delta_1 x (\log x)^{-k-1}, \quad (6)$$

say, for all $m \in (x/H, 4x/H)$ and all large enough x .

Thus

$$\delta_1^{-1} \sum_{Hm < n \leq (1+\delta_1)Hm} c_n - \delta_0^{-1} \sum_{Hm < n \leq (1+\delta_0)Hm} c_n \geq x (\log x)^{-k-1}$$

for every $m \in \mathcal{M}$, whence

$$\sum_{m \in \mathcal{M}} \left\{ \delta_1^{-1} \sum_{Hm < n \leq (1+\delta_1)Hm} c_n - \delta_0^{-1} \sum_{Hm < n \leq (1+\delta_0)Hm} c_n \right\} \geq x (\log x)^{-k-1} \mathcal{N}.$$

We now follow the usual analysis of Perron's formula, as in Titchmarsh [11, Sections 3.12 and 3.19] for example. If $0 < T \leq x$ we see that

$$\begin{aligned} \delta_0^{-1} \sum_{Hm < n \leq (1+\delta_0)Hm} c_n &= \frac{1}{2\pi i} \int_{-iT}^{iT} F(s) \frac{(1 + \delta_0)^s - 1}{\delta_0 s} (Hm)^s ds \\ &\quad + O(\delta_0^{-1} T^{-1} x \log x) \end{aligned}$$

and similarly

$$\begin{aligned} \delta_1^{-1} \sum_{Hm < n \leq (1+\delta_1)Hm} c_n &= \frac{1}{2\pi i} \int_{-iT}^{iT} F(s) \frac{(1 + \delta_1)^s - 1}{\delta_1 s} (Hm)^s ds \\ &\quad + O(\delta_1^{-1} T^{-1} x \log x). \end{aligned}$$

Since $\delta_1 \geq \delta_0$ we may conclude that

$$\begin{aligned} & \sum_{m \in \mathcal{M}} \left\{ \delta_1^{-1} \sum_{Hm < n \leq (1+\delta_1)Hm} c_n - \delta_0^{-1} \sum_{Hm < n \leq (1+\delta_0)Hm} c_n \right\} \\ &= \frac{1}{2\pi i} \int_{-iT}^{iT} F(s) \left\{ \frac{(1+\delta_1)^s - 1}{\delta_1 s} - \frac{(1+\delta_0)^s - 1}{\delta_0 s} \right\} H^s M(-s) ds \\ & \quad + O(\delta_0^{-1} T^{-1} x (\log x) \mathcal{N}), \end{aligned}$$

with

$$M(s) := \sum_{m \in \mathcal{M}} m^{-s}. \quad (7)$$

We pause to remark that it is the use of this Dirichlet polynomial $M(s)$ which is the most novel feature of the method introduced by Yu [12]. While the coefficients of $F(s)$ will have some useful arithmetic structure, those of $M(s)$ do not. None the less it is possible to use $M(s)$ in a non-trivial way in what follows.

We now insist that H satisfies

$$H \geq (\log x)^{k+3}.$$

Then the condition $T \leq x$ will be satisfied if we choose

$$T := (\log x)^{k+3} \frac{x}{H}. \quad (8)$$

Then

$$\delta_0^{-1} T^{-1} x \log x = o(x (\log x)^{-k-1}),$$

so that

$$\int_{-T}^T |F(it)M(it)| \cdot \left| \frac{(1+\delta_1)^{it} - 1}{\delta_1 t} - \frac{(1+\delta_0)^{it} - 1}{\delta_0 t} \right| dt \gg x (\log x)^{-k-1} \mathcal{N}.$$

Here we have

$$\frac{(1+\delta_1)^{it} - 1}{\delta_1 t} - \frac{(1+\delta_0)^{it} - 1}{\delta_0 t} \ll \min\{1, \delta_1(|t| + 1)\},$$

so that the integral for $|t| \leq \delta_1^{-1/8}$ contributes

$$\ll \delta_1^{3/4} \max_t |F(it)M(it)| \ll \delta_1^{3/4} x \mathcal{N}.$$

This is negligible compared to $x (\log x)^{-k-1} \mathcal{N}$ and we conclude that

$$\mathcal{N} \ll x^{-1} (\log x)^{k+1} \int_{\delta_1^{-1/8}}^T |F(it)M(it)| dt.$$

It is time to summarize our conclusions.

Lemma 1 Suppose that $x \ll N \ll x$ and $(\log x)^{k+3} \leq H \leq x^{3/4}$. Then

$$\mathcal{N} \ll x^{-1}(\log x)^{k+1} \int_{\delta_1^{-1/8}}^T |F(it)M(it)| dt.$$

We next show that the sum $P_1(it)$ must be relatively small in the range $\delta_1^{-1/8} \leq t \leq T$, and it is here that it is crucial that we have removed the region $|t| \leq \delta_1^{-1/8}$. We use a standard argument, see the proof of Lemma 19 in Heath-Brown [2], for example. In brief, the sum $P_1(it)$ is

$$\frac{1}{2\pi i} \int_{\nu-it/2}^{\nu+it/2} \log \zeta(s+it) x^{s/k} \frac{2^{(k+2)s} - 1}{s} ds + O(x^{1/(2k)}) + O(t^{-1} x^{1/k} \log x)$$

for $\nu = 1 + 1/\log t$, by Perron's formula. We can move the line of integration to $\operatorname{Re}(s) = 1 - (\log t)^{-3/4}$, say, and use the bound

$$\log \zeta(s+it) \ll \log t,$$

which is valid in the Vinogradov–Korobov zero-free region. Since

$$\exp(\tfrac{1}{8}\sqrt{\log x}) \leq t \leq T \leq x$$

we deduce that

$$P_1(it) \ll x^{1/k} \exp\{-(\log x)^{1/9}\}, \quad (9)$$

say. This small saving over the trivial bound will be enough for our purposes. It shows that

$$(\log x)^{k+1} |F(it)| \ll x^{1/k} \exp\{-(\log x)^{1/10}\} |P(it)|^{k-1}$$

for $\delta_1^{-1/8} \leq t \leq T$.

We now conclude as follows.

Lemma 2 We have

$$\mathcal{N} \ll x^{-(k-1)/k} \exp\{-(\log x)^{1/10}\} \int_0^T |P(it)^{k-1} M(it)| dt.$$

At this point we choose integers $a, b \geq 0$ with $a + b = k - 1$ and apply Cauchy's inequality to deduce that

$$\int_0^T |P(it)^{k-1} M(it)| dt \leq \left\{ \int_0^T |P(it)^b|^2 dt \right\}^{1/2} \left\{ \int_0^T |P(it)^a M(it)|^2 dt \right\}^{1/2}.$$

The standard mean-value estimate for Dirichlet polynomials (Montgomery [8, Theorem 6.1]) shows that

$$\int_0^T |P(it)|^{2b} dt \ll (T + x^{b/k}) x^{b/k}.$$

We will require that $x^{b/k} \geq T$, so that the above bound is $O(x^{2b/k})$. To handle the second integral we use part (iv) of Theorem 4, with $N = x^{a/k}$, which shows that

$$\int_0^T |P(it)^a M(it)|^2 dt \ll x^{2a/k} R^2 + R^2 T + x^{a/k} R T,$$

with $R = \#\mathcal{M} = \mathcal{N}$. We now require that $x^{a/k} \geq T^{1/2}$, in which case that above bound reduces to $O(x^{2a/k} R^2 + x^{a/k} R T)$.

Comparing our estimates we now see that

$$R \ll x^{-(k-1)/k} \exp\{-(\log x)^{1/10}\} x^{b/k} (x^{a/k} R + x^{a/2k} R^{1/2} T^{1/2}).$$

Since $a + b = k - 1$ one cannot have

$$R \ll x^{-(k-1)/k} \exp\{-(\log x)^{1/10}\} x^{b/k} . x^{a/k} R, \quad (10)$$

and we conclude that

$$R \ll x^{-(k-1)/k} x^{b/k} . x^{a/2k} R^{1/2} T^{1/2}. \quad (11)$$

We should stress at this point that unless (10) fails we can draw no conclusion whatsoever as to the size of R . It is for this reason that our approach to Theorem 1 breaks down entirely when $a_{n+1} - a_n \leq x^{1/3}$.

The bound (11) leads at once to the following result.

Lemma 3 *We have*

$$\mathcal{N} = R \ll x^{-a/k} T$$

provided that $T^{1/2} \leq x^{a/k} \leq x^{(k-1)/k} T^{-1}$.

A suitable integer a will exist provided that $T^{3/2} \leq x^{(k-2)/k}$, in which case we may choose a so that $x^{(k-2)/k} T^{-1} \leq x^{a/k} \leq x^{(k-1)/k} T^{-1}$. This shows that

$$\mathcal{N} \ll x^{-(k-2)/k} T^2.$$

Recalling our choice (8) we see that

$$\mathcal{N} \ll x^{1+2/k} H^{-2} (\log x)^{2k+6},$$

provided that $(x/H)^{3/2} \leq x^{(k-3)/k}$, say. However, we chose k to be an arbitrary fixed integer with $k > \varepsilon^{-1}$. By taking k suitably large we see that $\mathcal{N} \ll_{\varepsilon} x^{1+\varepsilon} H^{-2}$ as long as $H \geq x^{1/3+\varepsilon}$. Of course, as one decreases ε the x^{ε} -smooth numbers thin out, so that \mathcal{N} increases. Thus one gets a sharper bound, for a larger quantity, on a longer range of values H . Clearly Theorem 1 now follows via dyadic subdivision (for both the size of a_n , and the size of $a_{n+1} - a_n$).

3 Proof of Theorem 4

Our proof of part (i) of Theorem 4 follows ideas developed by Montgomery [9, Chapter 7, Theorem 3]. We begin with the following easy lemma.

Lemma 4 *Let $A(s) = \sum_{n \leq N} \alpha_n n^{-s}$, and set*

$$I_A(T) = \int_0^T |A(it)|^2 dt.$$

Then

$$\frac{4}{\pi^2} I_A(T) \leq T \sum_{\substack{m, n \leq N \\ |\log(m/n)| \leq T^{-1}}} |\alpha_m \alpha_n|,$$

and if the α_n are all non-negative real numbers, then

$$T \sum_{\substack{m, n \leq N \\ |\log(m/n)| \leq T^{-1}}} \alpha_m \alpha_n \leq \pi^2 I_A(T).$$

For the proof we use repeatedly the fact that the Fourier transform of

$$w(x) := \max(1 - |x|, 1)$$

is

$$\hat{w}(t) = \left(\frac{\sin(\pi t)}{\pi t} \right)^2.$$

Then

$$I_A(T) = \frac{1}{2} \int_{-T}^T |A(it)|^2 dt \leq \frac{1}{2} \left(\frac{\pi}{2} \right)^2 \int_{-\infty}^{\infty} \hat{w}(t/2T) |A(it)|^2 dt,$$

since $\sin(y)/y \geq 2/\pi$ for $|y| \leq \pi/2$. Thus

$$\begin{aligned}
I_A(T) &\leq \left(\frac{\pi}{2}\right)^2 T \int_{-\infty}^{\infty} \hat{w}(u) |A(2Tui)|^2 du \\
&= \left(\frac{\pi}{2}\right)^2 T \sum_{\substack{m,n \leq N \\ |\log(m/n)| \leq (2T)^{-1}}} \alpha_m \overline{\alpha_n} w(2T \log(m/n)) \\
&\leq \left(\frac{\pi}{2}\right)^2 T \sum_{\substack{m,n \leq N \\ |\log(m/n)| \leq T^{-1}}} |\alpha_m \alpha_n|,
\end{aligned}$$

from which the first inequality of the lemma follows.

Similarly, if the α_n are real and non-negative, then

$$\begin{aligned}
\sum_{\substack{m,n \leq N \\ |\log(m/n)| \leq T^{-1}}} \alpha_m \alpha_n &\leq \left(\frac{\pi}{2}\right)^2 \sum_{m,n \leq N} \alpha_m \alpha_n \hat{w}\left(\frac{1}{2}T \log(m/n)\right) \\
&= \left(\frac{\pi}{2}\right)^2 \int_{-\infty}^{\infty} w(u) |A(\tfrac{1}{2}Tui)|^2 du \\
&= \left(\frac{\pi}{2}\right)^2 \frac{2}{T} \int_{-\infty}^{\infty} w(2t/T) |A(it)|^2 dt \\
&\leq \left(\frac{\pi}{2}\right)^2 \frac{2}{T} \int_{-T}^T |A(it)|^2 dt \\
&= \pi^2 T^{-1} I_A(T),
\end{aligned}$$

as required.

Part (i) of Theorem 4 follows at once from Lemma 4 on taking $A(s) = M(s)Q(s)$.

We turn next to part (ii) of the theorem. By part (i) it suffices to handle the special case in which $Q(s)$ is

$$Z(s) := \sum_{n \leq N} n^{-s}.$$

Lemma 5 *Under the Lindelöf Hypothesis we have*

$$\mathcal{I}(\mathcal{M}, Z) \ll_{\eta} N^2 R^2 + (NT)^{\eta} NRT,$$

for any $\eta > 0$.

By Perron's formula we have

$$Z(it) = \frac{1}{2\pi i} \int_{3/2-iN}^{3/2+iN} \zeta(s+it) N^s \frac{ds}{s} + O(N^{1/2}).$$

Under the Lindelöf Hypothesis we can move the line of integration to $\operatorname{Re}(s) = \frac{1}{2}$ to show that

$$Z(it) \ll_{\eta} N^{1/2}(N + |t|)^{\eta} + N/(1 + |t|),$$

the second term coming from the pole at $s = 1 - it$ (if $|t| \leq N$).

We now use this bound to deduce that

$$\mathcal{I}(\mathcal{M}, Z) \ll_{\eta} N(N + T)^{2\eta} \int_0^T |M(it)|^2 dt + \sup_t |M(it)|^2 \int_0^T N^2/(1 + |t|)^2 dt.$$

By the usual mean value theorem (Montgomery [8, Theorem 6.1]) the first integral on the right is $O(RT)$, whence

$$\mathcal{I}(\mathcal{M}, Z) \ll_{\eta} N(N + T)^{2\eta} RT + R^2 N^2.$$

The lemma then follows on replacing η by $\eta/2$.

The remaining parts of Theorem 4 will require more effort. We begin with the following lemma.

Lemma 6 *Let N and N_0 be positive integers, and write*

$$Z(s) := \sum_{n \leq N} n^{-s} \quad \text{and} \quad Z_0(s) := \sum_{n \leq N_0} n^{-s}.$$

Then

$$\mathcal{J}(\mathcal{M}, Z) \ll N^2 R^2 T^{-1} + N N_0^{-1} R^2 + N N_0^{-1} \mathcal{J}(\mathcal{M}, Z_0). \quad (12)$$

In particular one sees that if $N_0 = T N^{-1}$ then

$$\mathcal{J}(\mathcal{M}, Z) \ll N^2 R^2 T^{-1} + N^2 T^{-1} \mathcal{J}(\mathcal{M}, Z_0).$$

This is exactly what one would expect from an examination of the integral $\mathcal{I}(\mathcal{M}, Z)$ if one were to apply the approximate functional equation to $Z(it)$, changing its length from N to T/N . However Lemma 6 covers more general values of N and N_0 . Moreover our proof can be adapted to handle Dirichlet polynomials $Q(s)$ where the approximate functional equation does not apply.

For the proof we begin by writing

$$\mathcal{J}(\mathcal{M}, Z) = \sum_{m_1, m_2 \in \mathcal{M}} \# \mathcal{N}(m_1, m_2; N, T),$$

with

$$\mathcal{N}(m_1, m_2; N, T) := \{(n_1, n_2) : n_1, n_2 \leq N, |\log(m_1 n_1 / m_2 n_2)| \leq T^{-1}\}.$$

Suppose that $m_1 \geq m_2$, say. Since $T \geq 1$ we have

$$\left| \frac{m_1 n_1}{m_2 n_2} - 1 \right| \leq 2T^{-1}$$

so that

$$|n_1 - n_2 m_2 / m_1| \leq 2NT^{-1}.$$

The set

$$\Lambda := \{(n_1, n_2, T(n_1 - n_2 m_2 / m_1)) : (n_1, n_2) \in \mathbb{Z}^2\}$$

is a 2-dimensional lattice, with

$$\det(\Lambda) = T \sqrt{1 + m_2^2 / m_1^2}.$$

According to part (iii) of Heath-Brown [3, Lemma 1] (which is based on Lemma 4 of Davenport [1]) the lattice has a basis \mathbf{e}_1 and \mathbf{e}_2 say, with

$$\det(\Lambda) \ll |\mathbf{e}_1| \cdot |\mathbf{e}_2| \ll \det(\Lambda),$$

and such that, if $\mathbf{e} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2$ then $\lambda_i \ll |\mathbf{e}| / |\mathbf{e}_i|$ for $i = 1, 2$. Without loss of generality we may assume that $|\mathbf{e}_1| \leq |\mathbf{e}_2|$.

The vectors \mathbf{e} produced from points of $\mathcal{N}(m_1, m_2; N, T)$ will have length $|\mathbf{e}| \leq N\sqrt{6}$ so that the corresponding coefficients λ_i satisfy $|\lambda_i| \leq c_0 N / |\mathbf{e}_i|$ for an appropriate numerical constant c_0 . In particular, if the pair m_1, m_2 is such that $|\mathbf{e}_2| > c_0 N$, then $\lambda_2 = 0$. We will call such pairs (m_1, m_2) “bad”. In this case every vector in Λ which arises from a pair $(n_1, n_2) \in \mathcal{N}(m_1, m_2; N, T)$ will be an integral scalar multiple of \mathbf{e}_1 .

For the remaining “good” pairs (m_1, m_2) we have $|\mathbf{e}_1|, |\mathbf{e}_2| \ll N$ so that there are $O(N/|\mathbf{e}_i|)$ choices for λ_i . This yields

$$\#\mathcal{N}(m_1, m_2; N, T) \ll \frac{N^2}{|\mathbf{e}_1| \cdot |\mathbf{e}_2|} \ll N^2 / T.$$

Since there are at most R^2 good pairs, the corresponding contribution to (12) is $O(N^2 R^2 / T)$, which is satisfactory.

Suppose on the other hand, that m_1, m_2 is a bad pair, with $m_1 \geq m_2$, and that $\mathbf{e}_1 = (u_1, u_2, u_3)$ is the shorter of the two basis vectors for Λ . If necessary we may replace \mathbf{e}_1 by $-\mathbf{e}_1$ so that $u_1 \geq 0$. If $\mathcal{N}(m_1, m_2; N, T)$ is non-empty, containing a pair $(n_1^{(0)}, n_2^{(0)})$ say, then we must have $(n_1^{(0)}, n_2^{(0)}) = \lambda^{(0)}(u_1, u_2)$ for some positive integer λ_0 . Moreover

$$|\log(m_1 n_1^{(0)} / m_2 n_2^{(0)})| \leq T^{-1}, \tag{13}$$

and hence

$$|\log(m_1 u_1 / m_2 u_2)| \leq T^{-1}.$$

On the other hand, if this latter condition is met, then

$$(n_1, n_2) \in \mathcal{N}(m_1, m_2; N, T)$$

if and only if $(n_1, n_2) = \lambda(u_1, u_2)$ for some integer λ satisfying

$$0 < \lambda u_1 \leq N \quad \text{and} \quad 0 < \lambda u_2 \leq N.$$

The condition on λ is that it should belong to a certain interval, $I = (0, L]$ say.

We stress that the lattice Λ , the basis $\mathbf{e}_1, \mathbf{e}_2$ and hence the interval I , all depend only on m_1 and m_2 , and not on N . However a pair (m_1, m_2) may be bad for some N and good for others.

For any interval $I = (0, L]$ and any real $\rho > 0$ one has

$$\#(\mathbb{Z} \cap I) \leq \text{meas}(I) = \rho^{-1} \text{meas}(\rho I) \leq \rho^{-1} (1 + \#(\mathbb{Z} \cap \rho I)).$$

Taking $\rho = N_0/N$ we therefore deduce that if (m_1, m_2) is bad for N then

$$\begin{aligned} & \# \mathcal{N}(m_1, m_2; N, T) \\ & \leq N N_0^{-1} (1 + \#\{\lambda \in \mathbb{Z} : 0 < \lambda u_i \leq N_0, (i = 1, 2)\}) \\ & \leq N N_0^{-1} (1 + \# \mathcal{N}(m_1, m_2; N_0, T)), \end{aligned}$$

since every pair $(n_1, n_2) = \lambda(u_1, u_2)$ produced above will satisfy (13). It follows that bad pairs (m_1, m_2) contribute a total

$$\ll N N_0^{-1} R^2 + N N_0^{-1} \mathcal{J}(\mathcal{M}, Z_0)$$

to (12), which suffices for the lemma.

We proceed to develop the above technique so as to apply to part (iv) of Theorem 4. We will use the following lemma.

Lemma 7 *Let N be a positive integer, and write*

$$Q(s) = \frac{1}{k!} \left(\sum_{p \leq N^{1/k}} p^{-s} \right)^k, \quad \text{and} \quad Q_h(s) = \frac{1}{h!} \left(\sum_{p \leq N^{1/k}} p^{-s} \right)^h,$$

with p running over primes. Then there exists a non-negative integer $h \leq k-1$ such that

$$\mathcal{J}(\mathcal{M}, Q) \ll_k N^2 R^2 T^{-1} + R^2 + N^{(k-h)/k} \mathcal{J}(\mathcal{M}, Q_h). \quad (14)$$

For the proof it will be convenient to write

$$\mathcal{Q}_h := \{q \in \mathbb{N} : q = p_1 \dots p_h : p_i \leq N^{1/k}\}.$$

We follow the same procedure as before, writing

$$\mathcal{J}(\mathcal{M}, Q) = \sum_{m_1, m_2 \in \mathcal{M}} \#\mathcal{N}(m_1, m_2; N, T),$$

where now

$$\mathcal{N}(m_1, m_2; N, T) := \{(q_1, q_2) : q_1, q_2 \in \mathcal{Q}_k, |\log(m_1 q_1 / m_2 q_2)| \leq T^{-1}\}.$$

As previously we have

$$\mathcal{N}(m_1, m_2; N, T) \ll N^2/T$$

for good pairs (m_1, m_2) , contributing $O(N^2 R^2/T)$ in Lemma 7.

For each bad pair (m_1, m_2) there will be a corresponding integer vector (u_1, u_2) such that every pair (q_1, q_2) belonging to $\mathcal{N}(m_1, m_2; N, T)$ takes the form $(q_1, q_2) = \lambda(u_1, u_2)$. Clearly we must have $u_1, u_2 \in \mathcal{Q}_h$ and $\lambda \in \mathcal{Q}_{k-h}$ for some integer h in the range $0 \leq h \leq k$. We will focus attention on the value of h which makes the largest contribution. If $h = k$ then λ must be 1, and since there are at most R^2 bad pairs (m_1, m_2) the overall contribution in Lemma 7 is $O(R^2)$. Otherwise we note that $\#\mathcal{Q}_{k-h} \leq N^{(k-h)/k}$, so that there are at most $N^{(k-h)/k}$ possibilities for λ . Moreover

$$\begin{aligned} & \#\{(m_1, m_2) \in \mathcal{M}^2 : (u_1, u_2) \in \mathcal{Q}_h^2, |\log(m_1 u_1 / m_2 u_2)| \leq T^{-1}\} \\ & \ll_h \mathcal{J}(\mathcal{M}, Q_h), \end{aligned}$$

so that the overall contribution to Lemma 7 is $O_k(N^{(k-h)/k} \mathcal{J}(\mathcal{M}, Q_h))$. This completes the proof.

4 Completing the Proof of Theorem 4

We have already dealt with parts (i) and (ii) of the theorem. For part (iii) we begin by applying Lemma 6 along with part (i) of Theorem 4. These yield

$$\mathcal{I}(\mathcal{M}, Z) \ll N^2 R^2 + N N_0^{-1} R^2 T + N N_0^{-1} \mathcal{I}(\mathcal{M}, Z_0).$$

However by Cauchy's inequality we obtain

$$\mathcal{I}(\mathcal{M}, Z_0) \leq \left\{ \int_0^T |M(it)|^2 dt \right\}^{1/2} \left\{ \int_0^T |M(it) Z_0(it)|^2 dt \right\}^{1/2}.$$

The first integral is $O(RT)$ by the usual mean value theorem (Montgomery [8, Theorem 6.1]). Moreover, if

$$D := \max\{d(n) : n \leq N_0^2\} \ll_\eta N_0^\eta,$$

then the Dirichlet series $D^{-1}Z_0(s)^2$ has coefficients bounded by 1, and supported in $(0, N]$. Thus, according to part (i) of Theorem 4, if

$$Z_1(s) := \sum_{n \leq N_0^2} n^{-s},$$

then

$$\mathcal{I}(\mathcal{M}, D^{-1}Z_0^2) \ll T\mathcal{J}(\mathcal{M}, D^{-1}Z_0^2) \leq T\mathcal{J}(\mathcal{M}, Z_1) \ll \mathcal{I}(\mathcal{M}, Z_1).$$

We may therefore deduce that

$$\mathcal{I}(\mathcal{M}, Z) \ll N^2R^2 + NN_0^{-1}R^2T + NN_0^{-1}(RT)^{1/2}\{D^2\mathcal{I}(\mathcal{M}, Z_1)\}^{1/2}. \quad (15)$$

If we take $N_0 = \sqrt{N}$ then $Z = Z_1$, and (15) yields

$$\mathcal{I}(\mathcal{M}, Z) \ll N^2R^2 + N^{1/2}R^2T + NRTD^2.$$

In particular, if $N \geq T^{2/3}$ we obtain

$$\mathcal{I}(\mathcal{M}, Z) \ll_\eta N^2R^2 + N^{1+\eta}RT, \quad (16)$$

as claimed. In the remaining case in which $N \leq T^{2/3}$ we use (15) a second time, taking $N_0 = \max(T^{1/3}, (RT)^{1/4})$. Then Z_1 has length at least $T^{2/3}$, so that (16) yields

$$\mathcal{I}(\mathcal{M}, Z_1) \ll_\eta N_0^4R^2 + N_0^{2+\eta}RT.$$

Inserting this into (15) produces

$$\begin{aligned} \mathcal{I}(\mathcal{M}, Z) &\ll_\eta N^2R^2 + NN_0^{-1}R^2T \\ &\quad + NN_0^{-1}(RT)^{1/2}\{T^{2\eta}(N_0^4R^2 + N_0^2RT)\}^{1/2} \\ &\ll_\eta N^2R^2 + NN_0^{-1}R^2T + NR^{3/2}T^{1/2+\eta}N_0 + NRT^{1+\eta}. \end{aligned}$$

However $NN_0^{-1}R^2T \leq NR^{7/4}T^{3/4}$, and

$$NR^{3/2}T^{1/2+\eta}N_0 \leq NR^{3/2}T^{5/6+\eta} + NR^{7/4}T^{3/4+\eta}.$$

For the first term on the right we have

$$NR^{3/2}T^{5/6} = (NRT)^{1/3}(NR^{7/4}T^{3/4})^{2/3} \leq \max(NRT, NR^{7/4}T^{3/4}).$$

We may therefore deduce that

$$\mathcal{I}(\mathcal{M}, Z) \ll_{\eta} N^2 R^2 + NRT^{1+\eta} + NR^{7/4}T^{3/4+\eta}$$

when $N \leq T^{2/3}$. In conjunction with (16) this suffices for part (iii) of Theorem 4, since $NR^{7/4}T^{3/4+\eta} \leq NRT^{1+\eta}$ when $R \leq T^{1/3}$.

Finally we deal with part (iv) of Theorem 4. By part (i) of the theorem (14) becomes

$$\mathcal{I}(\mathcal{M}, Q) \ll_k N^2 R^2 + R^2 T + N^{(k-h)/k} \mathcal{I}(\mathcal{M}, Q_h).$$

Moreover, Hölder's inequality shows that if

$$P(s) = \sum_{p \leq N^{1/k}} p^{-s}$$

then

$$\int_0^T |M(it)P(it)^h|^2 dt \leq \left(\int_0^T |M(it)|^2 dt \right)^{(k-h)/k} \left(\int_0^T |M(it)P(it)^k|^2 dt \right)^{h/k}.$$

As before, the first integral on the right is $O(RT)$, and we deduce that

$$\mathcal{I}(\mathcal{M}, Q) \ll_k N^2 R^2 + R^2 T + N^{(k-h)/k} (RT)^{(k-h)/k} \mathcal{I}(\mathcal{M}, Q)^{h/k}.$$

Since $0 \leq h < k$ it then follows that

$$\mathcal{I}(\mathcal{M}, Q) \ll_k N^2 R^2 + R^2 T + NRT,$$

as required.

5 Theorems 2 and 3

Our starting point for the proof of Theorem 2 will be Lemma 2. We shall assume that

$$H \geq T^{4/k}. \tag{17}$$

In the alternative case we have

$$\mathcal{N} \ll x \ll x^{1+6/k} H^{-3/2},$$

which is satisfactory for Theorem 2 when k is taken sufficiently large.

We break the range $(0, T]$ into subintervals $(h-1, h]$, and pick points β_h, γ_h from each interval at which $|P(it)|$ and $|M(it)|$ are maximal. We then set $b_j = \beta_{2j-1}, c_j = \gamma_{2j-1}$ if the odd values of h make the larger overall contribution, and otherwise we take $b_j = \beta_{2j}, c_j = \gamma_{2j}$. Thus

$$\int_0^T |P(it)^{k-1} M(it)| dt \ll \sum_j |P(ib_j)|^{k-1} |M(ib_j)|, \quad (18)$$

with points $b_j, c_j \in [0, T]$ satisfying $b_{j+1} - b_j \geq 1$ and $c_{j+1} - c_j \geq 1$ for every relevant index j .

Values of j for which $|P(ib_j)| \leq 1$ contribute $O(T\mathcal{N})$ to (18). We classify the remaining indices into $O(\log x)$ sets according to the dyadic range $(V, 2V]$ in which $|P(ib_j)|$ lies. Focusing on the value of V which makes the largest contribution, we re-label the relevant points as b_j, c_j for $1 \leq j \leq J$. We may then deduce that there is some V for which

$$\int_0^T |P(it)^{k-1} M(it)| dt \ll T\mathcal{N} + (\log x) V^{k-1} \sum_{j=1}^J |M(ic_j)|,$$

with $|P(ib_j)| \geq V$ for $1 \leq j \leq J$. If we insert this into Lemma 2 we see that we must have

$$\mathcal{N} \ll x^{-(k-1)/k} \exp\{-(\log x)^{1/11}\} V^{k-1} \sum_{j=1}^J |M(ic_j)|, \quad (19)$$

since $T \ll x^{(k-1)/k}$ under the assumption (17).

One way to use this bound is to apply Cauchy's inequality, noting that

$$\sum_{j=1}^J |M(ic_j)|^2 \ll T\mathcal{N}$$

by the well-known mean-value estimate of Montgomery [8, Theorem 7.3] (with $Q = 1, \chi = 1, \delta = 1$). This yields

$$\mathcal{N} \ll x^{-(k-1)/k} V^{k-1} (JT\mathcal{N})^{1/2},$$

and hence

$$\mathcal{N} \ll x^{-2(k-1)/k} V^{2k-2} JT. \quad (20)$$

We proceed to estimate J using the standard machinery of mean and large values of Dirichlet polynomials. Let $M \leq x$ be a parameter to be decided, and choose an integer $\leq k$ so that

$$x^{r/k} \leq M < x^{(r+1)/k}. \quad (21)$$

The Dirichlet polynomial $A(s) := P(s)^r$ then has coefficients which are $O_k(1)$ in size, and supported on integers up to $x^{r/k}$.

Suppose firstly that $V \leq x^{3/4k}$. We apply Montgomery's mean-value estimate to $P(s)^r$, which shows that

$$J \ll_k V^{-2r} (x^{r/k} + T) x^{r/k}.$$

Thus on taking $M = T$ we find that

$$J \ll_k V^{-2r} T^2,$$

whence the estimate (20) produces

$$\mathcal{N} \ll x^{-2(k-1)/k} V^{2k-2r} T^3.$$

Under our assumption that $V \leq x^{3/4k}$ this yields

$$\mathcal{N} \ll x^{-2(k-1)/k} x^{3/2-3r/2k} T^3.$$

Finally, recalling that we have chosen $M = T$, we see that (21) produces

$$\mathcal{N} \ll x^{-2(k-1)/k} x^{3/2} T^{-3/2} x^{3/2k} T^3 = T^{3/2} x^{-1/2+7/2k} \ll x^{1+4k} H^{-3/2},$$

under the assumption (17).

We turn now to the case in which $V \geq x^{3/4k}$, where we shall use the large values estimate of Huxley, [5, page 117] (with a trivial modification to handle our spacing condition on the b_k). In order to specify M we shall define σ by taking $V = x^{\sigma/k}$, whence $\frac{3}{4} \leq \sigma \leq 1$. We then set

$$M = (Tx^{2/k})^{1/(4\sigma-2)}. \quad (22)$$

In view of (8) this choice will satisfy $M \leq x$ provided that (17) holds and x is large enough.

Huxley's result now yields

$$\begin{aligned} J &\ll \{V^{-2r} x^{2r/k} + V^{-6r} T x^{4r/k}\} (\log x)^5 \\ &= \{x^{(2-2\sigma)r/k} + T x^{(4-6\sigma)r/k}\} (\log x)^5. \end{aligned}$$

However $Mx^{-1/k} \leq x^{r/k} \leq M$ by (21), whence

$$J \ll \{M^{2-2\sigma} + T M^{4-6\sigma} x^{2/k}\} (\log x)^5 \ll M^{2-2\sigma} (\log x)^5, \quad (23)$$

recalling our choice (22) for M .

We insert this bound into (19), and use the fact that $M(it) \ll \mathcal{N}$ to deduce that

$$\mathcal{N} \ll x^{-(k-1)/k} \exp\{-(\log x)^{1/11}\} V^{k-1} J\mathcal{N}.$$

We now apply the bound (23) together with the fact that $V = x^{\sigma/k}$ to deduce that

$$\mathcal{N} \ll x^{-(1-\sigma)(k-1)/k} \exp\{-(\log x)^{1/11}\} (\log x)^5 M^{2-2\sigma} \mathcal{N}.$$

Thus either $\mathcal{N} = 0$, in which case there is nothing to prove, or

$$(M^2 x^{-(k-1)/k})^{1-\sigma} \gg \exp\{(\log x)^{1/11}\} (\log x)^{-5}.$$

In particular, if $\mathcal{N} \neq 0$, we must have

$$M \geq x^{(k-1)/2k}$$

so that our definition (22) yields

$$x^{2\sigma-1} \leq T x^{(1+2\sigma)/k} \leq T x^{3/k}. \quad (24)$$

Finally we combine the estimates (20) and (23) to deduce that

$$\begin{aligned} \mathcal{N} &\ll T (\log x)^5 x^{-2(k-1)/k} V^{2k-2} M^{2-2\sigma} \\ &= T (\log x)^5 x^{-(2-2\sigma)(k-1)/k} (T x^{2/k})^{(1-\sigma)/(2\sigma-1)} \\ &= T (\log x)^5 x^{-(k-1)/(2k)} (T x^{2/k})^{1/2} \left(\frac{x^{(2\sigma-1)(k-1)/k}}{T x^{2/k}} \right)^{(4\sigma-3)/(4\sigma-2)}. \end{aligned}$$

According to (24) we have

$$\frac{x^{(2\sigma-1)(k-1)/k}}{T x^{2/k}} \leq \frac{x^{(2\sigma-1)}}{T x^{2/k}} \leq x^{3/k},$$

and since we are assuming that $\sigma \geq \frac{3}{4}$ we find that

$$\mathcal{N} \ll T (\log x)^5 x^{-(k-1)/(2k)} (T x^{2/k})^{1/2} x^{1/k} \ll T^{3/2} x^{-1/2+3/k} \ll x^{1+4/k} H^{-3/2}.$$

In every case we therefore have $\mathcal{N} \ll x^{1+6/k} H^{-3/2}$, and on taking k suitably large we see that Theorem 2 follows, by dyadic subdivision of the ranges for both a_n and $a_{n+1} - a_n$. It therefore remains to establish Theorem 3. However we have

$$\begin{aligned} \sum_{\substack{a_n \leq x \\ a_{n+1} - a_n \geq x^{1/3+\varepsilon}}} (a_{n+1} - a_n)^2 &\ll_{\varepsilon} x^{1/6+\varepsilon/2} \sum_{\substack{a_n \leq x \\ a_{n+1} - a_n \geq x^{1/3+\varepsilon}}} (a_{n+1} - a_n)^{3/2} \\ &\ll_{\varepsilon} x^{1/6+\varepsilon/2} \sum_{a_n \leq x} (a_{n+1} - a_n)^{3/2} \\ &\ll_{\varepsilon} x^{7/6+3\varepsilon/2}, \end{aligned}$$

by Theorem 2. Combining this with the estimate from Theorem 1 yields

$$\sum_{a_n \leq x} (a_{n+1} - a_n)^{3/2} \ll_{\varepsilon} x^{7/6+3\varepsilon/2}$$

which suffices for Theorem 3.

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Mathematical Institute,
Radcliffe Observatory Quarter,
Woodstock Road,
Oxford
OX2 6GG
UK

`rhb@maths.ox.ac.uk`