

# COMPACTIFICATIONS OF $\omega^* \setminus \{x\}$ AND $S_\kappa \setminus \{x\}$

MAX F. PITZ AND ROLF SUABEDISSEN

**ABSTRACT.** The title refers to the Stone-Čech remainder of the integers  $\omega^*$  and the Stone space  $S_\kappa$  of the  $\kappa$ -saturated Boolean algebra of cardinality  $\kappa$ . The latter space is characterised topologically as the unique  $\kappa$ -Parovičenko space of weight  $\kappa$ . It exists if and only if the consistent and independent equality  $\kappa = \kappa^{<\kappa}$  holds. The spaces  $S_\kappa$  are generalisations of the space  $\omega^*$ : under the Continuum Hypothesis,  $S_{\omega_1}$  is homeomorphic to  $\omega^*$ .

In this paper we investigate compactifications of spaces  $S_\kappa \setminus \{x\}$ , building on and extending corresponding results obtained by Fine & Gillman and Comfort & Negreponis for the space  $\omega^*$ .

We show that for every point  $x$  of  $S_\kappa$ , the Stone-Čech remainder of  $S_\kappa \setminus \{x\}$  is a  $\kappa^+$ -Parovičenko space of cardinality  $2^{2^\kappa}$  which admits a family of  $2^\kappa$  disjoint clopen sets. As a corollary we get that it is consistent with CH that the Stone-Čech remainders of  $\omega^* \setminus \{x\}$  are all homeomorphic.

## 1. INTRODUCTION

The purpose of this paper is to study compactifications of subspaces of the Stone-Čech remainder of the integers  $\omega^* = \beta\omega \setminus \omega$ , namely subspaces of the form  $\omega^* \setminus \{x\}$ . The questions discussed in this paper are motivated by two classical theorems about extensions of real-valued continuous functions on  $\omega^* \setminus \{x\}$ .

---

2010 *Mathematics Subject Classification.* Primary 54D40; Secondary 54D35, 54G05, 06E15.

*Key words and phrases.* Stone-Čech compactification, Parovičenko,  $\kappa$ -Parovičenko,  $\kappa$ -saturated Boolean algebra, monotone  $F$ -space.

The first author acknowledges support of the German Academic Exchange Service (DAAD).

**Theorem 1.1** (Fine and Gillman, 1960 [9, 11]). *Assuming the Continuum Hypothesis, for every point  $x \in \omega^*$  there are continuous bounded real-valued functions on  $\omega^* \setminus \{x\}$  that cannot be continuously extended to  $\omega^*$ .*

**Theorem 1.2** (van Douwen, Kunen and van Mill, 1989 [3]). *It is consistent with the continuum being  $\aleph_2$  that for every point  $x \in \omega^*$  all continuous real-valued functions on  $\omega^* \setminus \{x\}$  can be continuously extended to  $\omega^*$ .*

Phrased in the language of compactifications, Theorem 1.2 says that the Stone-Čech compactification of  $\omega^* \setminus \{x\}$  consistently coincides with  $\omega^*$ . Theorem 1.1, however, guarantees that under the Continuum Hypothesis (CH), the Stone-Čech compactification certainly adds more than a single point to  $\omega^* \setminus \{x\}$ .

This last observation gives rise to a variety of new, interesting questions. What are the precise mechanisms that increase the size of the Stone-Čech compactification of  $\omega^* \setminus \{x\}$  under CH? Can one determine its size or is this independent of ZFC+CH? Does the Stone-Čech remainder of  $\omega^* \setminus \{x\}$  reflect structural properties of  $\omega^*$  or  $\omega^* \setminus \{x\}$  and in particular, does this depend on which point  $x$  we remove?

What makes these problems particularly interesting is a surprising link between compactifications of  $\omega^* \setminus \{x\}$  and another research area in the vicinity of  $\omega^*$ : the theory of  $\kappa$ -Parovičenko spaces, developed by Negrepontis [16], Comfort and Negrepontis [2] and Dow [6].

A rigorous introduction to  $\kappa$ -Parovičenko spaces will follow in the next section. Intuitively, however,  $\kappa$ -Parovičenko spaces generalise crucial characteristics of  $\omega^*$  to spaces of (potentially larger) weight  $\kappa$ . Recall that under CH, or equivalently under  $\omega_1 = \omega_1^{<\omega_1}$ , the space  $\omega^*$  is topologically characterised as the unique Parovičenko space of weight  $\mathfrak{c}$ . Here, the Parovičenko properties entail a precise description of the behaviour of countable unions and intersections of clopen sets in  $\omega^*$ : disjoint pairs of such countable unions have disjoint closures, and every such non-empty intersection has non-empty interior. The  $\kappa$ -Parovičenko properties essentially consist of corresponding requirements for all  $\lambda$ -unions and  $\lambda$ -intersections of clopen sets for all  $\lambda < \kappa$ . And as for  $\omega^*$  under CH, for every cardinal  $\kappa$  with the property  $\kappa = \kappa^{<\kappa}$  there is a unique  $\kappa$ -Parovičenko space of weight  $\kappa$ , denoted by  $S_\kappa$ . In particular,  $S_{\omega_1}$  coincides with  $\omega^*$  under CH.

The main result of this paper is that under CH, the Stone-Čech remainder of  $\omega^* \setminus \{x\}$  is an  $\omega_2$ -Parovičenko space of cardinality  $2^{2^{\omega_1}}$  and contains a family of  $2^{\omega_1}$  disjoint open sets, regardless of the choice of  $x$ . In general, under  $\kappa = \kappa^{<\kappa}$ , the Stone-Čech remainder of  $S_\kappa \setminus \{x\}$  is a  $\kappa^+$ -Parovičenko space of cardinality  $2^{2^\kappa}$  and contains a family of  $2^\kappa$  disjoint clopen sets. Again, this holds for every point  $x$  in  $S_\kappa$ .

As a consequence, assuming  $2^c = \omega_2$ , the Stone-Čech remainder of any  $\omega^* \setminus \{x\}$  is homeomorphic to the unique  $\omega_2$ -Parovičenko space of weight  $\omega_2$ , independently of which point  $x$  gets removed. This is surprising, considering that subspaces  $\omega^* \setminus \{x\}$  and  $\omega^* \setminus \{y\}$  are typically very different.

Our main result also improves a theorem by A. Dow, who showed in 1985 that the remainder of  $S_\kappa \setminus \{p\}$  is a  $\kappa^+$ -Parovičenko space provided that  $p$  has a nested neighbourhood base in  $S_\kappa$ , [6]. Consequently, we now have a more comprehensive answer to a question by S. Negrepontis [16, p. 522]: under  $\kappa^+ = 2^\kappa$ , the space  $S_{\kappa^+}$  can be represented as the Stone-Čech remainder of any  $S_\kappa \setminus \{x\}$ , regardless of whether  $x$  has a nested neighbourhood base or not.

This paper is organised as follows. In Section 2 we recall the relevant background for the investigation of  $\omega^*$  and  $S_\kappa$ . Section 3 provides a short account of Fine and Gillman's result that under CH, no  $\omega^* \setminus \{x\}$  is  $C^*$ -embedded in  $\omega^*$ , and concludes with a characterisation of the clopen subsets of  $\omega^* \setminus \{x\}$ . Section 4 generalises these results to spaces  $S_\kappa$  for any  $\kappa$  with  $\kappa = \kappa^{<\kappa}$ . In Section 5 we show that the Stone-Čech compactification of  $S_\kappa \setminus \{x\}$  is a  $\kappa$ -Parovičenko space of weight  $2^\kappa$ .

The next two sections are concerned with the remainder of  $S_\kappa \setminus \{x\}$ . First we show in Section 6 that all remainders are  $\kappa^+$ -Parovičenko spaces and conclude that under additional set theoretic assumptions they are all homeomorphic. In Section 7 we answer some questions about cardinal invariants of these spaces and prove that the Stone-Čech remainder of  $S_\kappa \setminus \{x\}$  has cardinality  $2^{2^\kappa}$ . We present two proofs of this fact. The first exhibits a connection to the space of uniform ultrafilters on  $\kappa$ . The second builds on the observation that under CH, the space  $\omega^*$  is a *monotone  $F$ -space*, i.e. that it is possible to assign a separating clopen partition to pairs of disjoint open  $F_\sigma$ -sets, in the spirit of a monotone normality operator. We also give an explicit topological construction of a family of  $2^\kappa$  disjoint open sets in the Stone-Čech remainder of  $S_\kappa \setminus \{x\}$ .

Each of Section 5 to 7 builds only on Section 4 and can be read independently. Section 8 concludes the paper with some open questions.

The authors would like to thank Alan Dow for his help with the proof of Theorem 6.3 at the Spring Topology and Dynamics Conference 2014 in Richmond, Virginia.

## 2. BACKGROUND

We recall properties of  $\omega^*$  and  $S_\kappa$  which we use freely in later sections. All this and more can be found in [2, 12, 15]. The reader should note that our notion of a  $(\kappa)$ -Parovičenko space differs from the commonly adopted definition in the sense that it does not a priori include any assumptions regarding weight.

A subset  $Y$  of a space  $X$  is  *$C^*$ -embedded* if every bounded real-valued continuous function on  $Y$  can be continuously extended to all of  $X$ . For a Tychonoff space  $X$ , we write  $\beta X$  for its *Stone-Ćech compactification*, the unique compact Hausdorff space in which  $X$  is densely  $C^*$ -embedded, and we write  $X^* = \beta X \setminus X$  for the *remainder* of  $X$ . A space is *zero-dimensional* if it has a base of clopen (closed-and-open) sets. A space  $X$  is *strongly zero-dimensional* if its Stone-Ćech compactification  $\beta X$  is zero-dimensional. A Tychonoff space  $X$  is called  *$F$ -space* if each cozero-set is  $C^*$ -embedded in  $X$ . We list some facts about  $F$ -spaces [12, Ch. 14].

- (1)  $X$  is an  $F$ -space if and only if  $\beta X$  is an  $F$ -space.
- (2) A normal space is an  $F$ -space if and only if disjoint open  $F_\sigma$ -sets have disjoint closures.
- (3) Closed subspaces of normal  $F$ -spaces are again  $F$ -spaces.
- (4) Infinite closed subspaces of compact  $F$ -spaces contain a copy of  $\beta\omega$ . Therefore, compact  $F$ -spaces do not contain convergent sequences.

In compact zero-dimensional spaces, cozero-sets are countable unions of clopen sets. This motivates the following definition from [2, Ch. 14]. In a zero-dimensional space  $X$ , the *( $X$ )-type* of an open subset  $U$  of  $X$  is the least cardinal number  $\tau = \tau(U)$  such that  $U$  can be written as a union of  $\tau$ -many clopen subsets of  $X$ . A zero-dimensional space where open subsets of type less than  $\kappa$  are  $C^*$ -embedded is called  *$F_\kappa$ -space*. In zero-dimensional compact spaces the notions of  $F$ - and  $F_{\omega_1}$ -space coincide.

- (1')  $X$  is a strongly zero-dimensional  $F_\kappa$ -space if and only if  $\beta X$  is a strongly zero-dimensional  $F_\kappa$ -space. See Theorem 5.1.
- (2') In  $F_\kappa$ -spaces, disjoint open sets of types less than  $\kappa$  have disjoint closures, and in normal spaces both conditions are equivalent, [2, 6.5].

The space  $\omega^*$  is a compact zero-dimensional Hausdorff space without isolated points with the following two extra properties: it is an  $F$ -space in which each non-empty  $G_\delta$ -set has non-empty interior. Such a space is also called *Parovičenko space*.

We call a space a  *$G_\kappa$ -space* if every non-empty intersection of less than  $\kappa$ -many open sets has non-empty interior. Then  $\omega^*$  is a  $G_{\omega_1}$ -space.

- (5) The following are equivalent for a zero-dimensional space  $X$ :
  - (a)  $X$  is a  $G_\kappa$ -space,
  - (b) For every open subset  $U$  of  $X$  of type less than  $\kappa$ , no set  $H$  with  $U \subsetneq H \subseteq \overline{U}$  is open,
  - (c) For every open subset  $U$  of  $X$  such that  $1 < \tau(U) < \kappa$ , its closure  $\overline{U}$  is not open,
  - (d) No open subset  $U$  of  $X$  with  $1 < \tau(U) < \kappa$  is dense in  $X$ .

For proofs of the non-trivial implications  $(a) \Rightarrow (b)$  and  $(d) \Rightarrow (a)$  see [2, 14.5] and [2, 6.6] respectively.

We call a space a  $\kappa$ -Parovičenko space if it is a compact zero-dimensional  $F_\kappa$ - and  $G_\kappa$ -space without isolated points. This modifies the corresponding definitions of [4] and [6, 1.2], freeing  $\kappa$ -Parovičenko spaces from additional weight restrictions. Under our definition, the space  $\omega^*$  is an  $(\omega_1)$ -Parovičenko space of weight  $\mathfrak{c}$ .

It is not hard to prove that any  $\kappa$ -Parovičenko space has weight at least  $\kappa^{<\kappa}$ . Thus, the following two characterisation theorems say, loosely speaking, that the smallest possible  $(\kappa)$ -Parovičenko spaces, namely the ones of weight  $\kappa$ , are topologically unique, and larger ones are not.

**Theorem 2.1** (Parovičenko [17], van Douwen & van Mill [4]). *CH is equivalent to the assertion that every  $(\omega_1)$ -Parovičenko space of weight  $\mathfrak{c}$  is homeomorphic to  $\omega^*$ .*

**Theorem 2.2** (Negrepointis [16], Dow [6]). *The assumption  $\kappa = \kappa^{<\kappa}$  is equivalent to the assertion that all  $\kappa$ -Parovičenko spaces of weight  $\kappa^{<\kappa}$  are homeomorphic.*

If the condition  $\kappa = \kappa^{<\kappa}$  is satisfied then the unique  $\kappa$ -Parovičenko space of weight  $\kappa$  exists and is denoted by  $S_\kappa$  [2, 6.12]. Whenever the space  $S_{\omega_1}$  exists, it is homeomorphic to  $\omega^*$ . The existence of uncountable cardinals satisfying the equality  $\kappa = \kappa^{<\kappa}$  is independent of ZFC but an assumption like  $\kappa^+ = 2^\kappa$  implies the equality for  $\kappa^+$ . Also note that  $\kappa = \kappa^{<\kappa}$  implies that  $\kappa$  is regular.

A  $P_\kappa$ -point  $p$  is a point such that the intersection of less than  $\kappa$ -many neighbourhoods of  $p$  contains again an open neighbourhood of  $p$ . A  $P_{\omega_1}$ -point is simply called  $P$ -point. That  $P_\kappa$ -points exist in  $S_\kappa$  is well-known [2, 6.17]. A new proof of this fact is contained in Corollary 4.6. We list some facts about  $P_\kappa$ -points.

- (6) In  $S_\kappa$ ,  $p$  is a  $P_\kappa$ -point if and only if  $p$  has a nested neighbourhood base if and only if  $p$  is not contained in the boundary of any open set of type less than  $\kappa$ .
- (7) For every pair of  $P_\kappa$ -points in  $S_\kappa$  there exists an autohomeomorphism of  $S_\kappa$  mapping one  $P_\kappa$ -point to the other [2, 6.21].

In particular, the subspaces  $S_\kappa \setminus \{p\}$  are homeomorphic for all  $P_\kappa$ -points  $p$ . Corresponding results hold for  $P$ -points in  $\omega^*$  under CH.

### 3. THE BUTTERFLY LEMMA

We begin with the classic result that under CH,  $\omega^*$  does not occur as the Stone-Čech compactification of any of its dense subspaces.

A point  $x$  of a Hausdorff space  $X$  is called *strong butterfly point* if its complement  $X \setminus \{x\}$  can be partitioned into open sets  $A$  and  $B$  such that  $\overline{A} \cap \overline{B} = \{x\}$ . The sets  $A$  and  $B$  are called *wings* of the butterfly point  $x$ . Note that in  $X \setminus \{x\}$ , the wings  $A$  and  $B$  are clopen and non-compact.

The following lemma by Fine and Gillman [9, 11] states that under CH, every point in  $\omega^*$  is a strong butterfly point. We outline the proof, as later on we will use variations of this approach in Theorem 6.3 and Lemmas 6.7 and 7.4.

**Lemma 3.1** (Butterfly Lemma, Fine and Gillman). [CH]. *Every point in  $\omega^*$  is a strong butterfly point.*

*Proof.* Let  $x \in \omega^*$  and fix a neighbourhood base  $\{U_\alpha : \alpha < \omega_1\}$  of  $x$  consisting of clopen sets. By transfinite recursion we define families of clopen sets  $\{A_\alpha : \alpha < \omega_1\}$  and  $\{B_\alpha : \alpha < \omega_1\}$  not containing  $x$  such that all  $(A_\alpha, B_\beta)$ -pairs are disjoint and

$$X \setminus U_\alpha \subseteq A_\alpha \cup B_\alpha \quad \text{and} \quad A_\alpha \cap U_\alpha \neq \emptyset \neq B_\alpha \cap U_\alpha \quad \text{for all } \alpha < \omega_1.$$

Once the construction is completed, we define disjoint open sets

$$A = \bigcup_{\alpha < \omega_1} A_\alpha \quad \text{and} \quad B = \bigcup_{\alpha < \omega_1} B_\alpha.$$

Their union covers all of  $\omega^* \setminus \{x\}$  and both  $A$  and  $B$  limit onto  $x$ .

It remains to describe the recursive construction. Let  $\alpha < \omega_1$  and assume that  $A_\beta$  and  $B_\beta$  have been defined for all ordinals  $\beta < \alpha$ . Since countable unions of clopen sets are open  $F_\sigma$ -sets, by the  $F$ -space property there exist clopen sets  $C$  and  $D$  partitioning  $\omega^*$  and containing the disjoint sets  $\bigcup_{\beta < \alpha} A_\beta$  and  $\bigcup_{\beta < \alpha} B_\beta$  respectively.

The set  $U_\alpha \setminus \bigcup_{\beta < \alpha} (A_\beta \cup B_\beta)$  is a non-empty  $G_\delta$ -set of the Parovičenko space  $\omega^*$  as it contains  $x$  and thus, it has non-empty interior. Hence, inside this set we may find disjoint non-empty clopen sets  $C'$  and  $D'$  not containing  $x$ . By defining  $A_\alpha = (C \setminus U_\alpha) \cup C'$  and  $B_\alpha = (D \setminus U_\alpha) \cup D'$  we see that  $A_\alpha$  and  $B_\alpha$  are as required.  $\square$

We list some consequences of the Butterfly Lemma regarding  $\omega^*$  under CH. First must come the result for which the Butterfly Lemma was originally invented.

**Corollary 3.2** (Fine and Gillman). [CH]. *For every point  $x$  of  $\omega^*$  the subspace  $\omega^* \setminus \{x\}$  is not  $C^*$ -embedded in  $\omega^*$ .*  $\square$

The construction of the Butterfly Lemma can be used to build  $2^{\omega_1}$  many distinct clopen subsets of  $\omega^* \setminus \{x\}$ . Indeed, in the proof of Lemma 3.1, for every  $\alpha < \omega_1$  we have the choice of adding  $C'$  or  $D'$  to  $A_\alpha$ . The collection of all clopen subsets of a space  $X$  is denoted by  $\mathcal{CO}(X)$ .

**Corollary 3.3.** [CH]. *For all  $x$  in  $\omega^*$  we have  $|\mathcal{CO}(\omega^* \setminus \{x\})| = 2^{\omega_1}$ .  $\square$*

Compact clopen sets of  $\omega^* \setminus \{x\}$  are of course homeomorphic to  $\omega^*$ . The next lemma describes how the non-compact clopen sets look like.

**Lemma 3.4.** [CH]. *For every  $x$  in  $\omega^*$ , the one-point compactification of a clopen non-compact subset of  $\omega^* \setminus \{x\}$  is homeomorphic to  $\omega^*$ .*

*Proof.* Let  $A$  be a clopen non-compact subset of  $\omega^* \setminus \{x\}$ . Taking  $A \cup \{x\}$ , a closed subset of  $\omega^*$ , as representative of its one-point compactification, we see that it is a zero-dimensional compact  $F$ -space of weight  $\mathfrak{c}$  without isolated points.

For the  $G_{\omega_1}$ -space property, suppose that  $U \subseteq A \cup \{x\}$  is a non-empty  $G_\delta$ -set. If  $U$  has empty intersection with  $A$ , then the singleton  $U = \{x\}$  is a  $G_\delta$ -set, and hence has countable character in the compact Hausdorff space  $A \cup \{x\}$ . It follows that there is a non-trivial sequence in  $\omega^*$  converging to  $x$ , a contradiction. Thus,  $U$  intersects the open set  $A$  and their intersection is a non-empty  $G_\delta$ -set of  $\omega^*$  with non-empty interior.

An application of Parovičenko's theorem completes the proof.  $\square$

In absence of CH, the above proof still shows that the one-point compactification of any clopen non-compact subset of  $\omega^* \setminus \{x\}$  is a Parovičenko space of weight  $\mathfrak{c}$ .

It also follows that  $\omega^*$  contains  $P$ -points under CH, as the next lemma shows. Even more, we have another proof of [1, 2.3] that under CH for every point  $x$  of  $\omega^*$  there is a clopen copy of  $\omega^*$  contained in  $\omega^*$  such that  $x$  is a  $P$ -point with respect to that copy.

**Corollary 3.5.** [CH]. *For every point  $x$  of  $\omega^*$ , at least one of its wings together with  $x$  itself is a copy of  $\omega^*$  such that  $x$  is a  $P$ -point with respect to that copy.*

*Proof.* This follows from the last lemma together with the observation that if  $x$  was a non- $P$ -point with respect to both of its wings, it would be in the closure of two disjoint open  $F_\sigma$ -sets, contradicting the  $F$ -space property of  $\omega^*$ .  $\square$

#### 4. BUTTERFLIES IN $S_\kappa \setminus \{x\}$

In this section we generalise results from the previous section about  $\omega^* = S_{\omega_1}$  to general  $S_\kappa$ , assuming  $\kappa = \kappa^{<\kappa}$  throughout. The reader is encouraged to check that the Butterfly Lemma and its immediate consequences carry over nicely to  $S_\kappa$ .

**Lemma 4.1** (Butterfly Lemma). *Assume  $\kappa = \kappa^{<\kappa}$ . Every point of  $S_\kappa$  is a strong butterfly point.*  $\square$

**Corollary 4.2.** *Assume  $\kappa = \kappa^{<\kappa}$ . For every point  $x$  of  $S_\kappa$  the subspace  $S_\kappa \setminus \{x\}$  is not  $C^*$ -embedded in  $S_\kappa$ .*  $\square$

**Corollary 4.3.** *Assume  $\kappa = \kappa^{<\kappa}$ . For every point  $x$  in  $S_\kappa$  we have  $|\mathcal{CO}(S_\kappa \setminus \{x\})| = 2^\kappa$ .*  $\square$

The remaining part of this section is devoted to the proof of the following generalisation of Lemma 3.4.

**Lemma 4.4.** *Assume  $\kappa = \kappa^{<\kappa}$  and let  $x$  any point in  $S_\kappa$ . The one-point compactification of a clopen non-compact subset of  $S_\kappa \setminus \{x\}$  is homeomorphic to  $S_\kappa$ .*

The proof, however, is more delicate than in the case of  $\omega^*$ . The challenge lies in the fact that Lemma 3.4 used as corner stones two facts about  $F$ -spaces which do not carry through to general  $F_\kappa$ -spaces: In normal spaces, the  $F$ -space property is closed-hereditary and every infinite closed subset of  $S_{\omega_1}$  has the same cardinality as  $S_{\omega_1}$ . Both assertions do not hold for  $F_\kappa$ -spaces since  $S_\kappa$  contains, being an infinite compact  $F$ -space, closed copies of  $\beta\omega$ .

The following lemma is crucial in circumventing these obstacles.

**Lemma 4.5.** *Assume  $\kappa = \kappa^{<\kappa}$  and let  $x$  any point in  $S_\kappa$ . A clopen, non-compact subset of  $S_\kappa \setminus \{x\}$  is of  $S_\kappa$ -type  $\kappa$ .*

*Proof.* Suppose for a contradiction that there exists a clopen, non-compact subset  $A$  of  $S_\kappa \setminus \{x\}$  of  $S_\kappa$ -type  $\tau < \kappa$ . Find a representation

$$A = \bigcup_{\alpha < \tau} A_\alpha$$

where all  $A_\alpha$  are clopen subsets of  $S_\kappa$ . We first claim that there is a collection  $\{V_\alpha : \alpha < \tau\}$  of pairwise disjoint clopen sets of  $S_\kappa$  such that  $V_\alpha \subseteq A \setminus \bigcup_{\beta < \alpha} A_\beta$  for all  $\alpha < \tau$ .

We proceed by transfinite recursion. Choose a clopen subset  $V_0$  in the non-empty open set  $A \setminus A_0$ . Now consider  $\alpha < \tau$  and suppose that  $V_\beta$  have been defined for all  $\beta < \alpha$ . By minimality of  $\tau$ , the set  $U_\alpha = \bigcup_{\beta < \alpha} A_\beta \cup V_\beta$  is a proper subset of  $A$ , from which it follows by (5)b of Section 2 that  $U_\alpha$  is not dense in  $A$ . Thus, there is a non-empty clopen set  $V_\alpha$  in the interior of  $A \setminus U_\alpha$ . This completes the recursion and proves the claim.

Now, let  $f$  and  $g$  be disjoint cofinal subsets of  $\tau$ . We define disjoint open sets

$$V_f = \bigcup_{\alpha \in f} V_\alpha \quad \text{and} \quad V_g = \bigcup_{\alpha \in g} V_\alpha$$

of type at most  $\tau$  and claim that both sets limit onto  $x$ , contradicting the  $F_\kappa$ -space property of  $S_\kappa$ . Suppose the claim was false. Then  $\overline{V_f}$  is a



subset of  $A = \bigcup_{\alpha < \tau} A_\alpha$ . By compactness, there is a finite set  $F \subseteq \tau$  such that  $\overline{V_f} \subseteq \bigcup_{\beta \in F} A_\beta$ . But there are sets  $V_\alpha$  with arbitrarily large index contributing to  $V_f$ , a contradiction.  $\square$

*Proof of Lemma 4.4.* Let  $A$  be a clopen non-compact subset of  $S_\kappa \setminus \{x\}$ , and denote by  $X$  the closure of  $A$  in  $S_\kappa$ , i.e.  $X = A \cup \{x\} \subseteq S_\kappa$ . Then  $X$  is a compact zero-dimensional space of weight  $\kappa$  without isolated points. We check for the remaining  $\kappa$ -Parovičenko properties.

To show that  $X$  has the  $F_\kappa$ -space property, let  $U$  and  $V$  be disjoint open sets of  $X$  of type less than  $\kappa$ . By normality, it suffices to show that  $U$  and  $V$  have disjoint closures in  $X$ . First suppose that  $x$  belongs to  $U \cup V$ . Assume  $x \in U$ , so that  $x$  does not belong to the closure of  $V$ . The sets  $U \cap A$  and  $V \cap A$  are of  $A$ -type less than  $\kappa$ . And since  $A$  is an  $F_\kappa$ -space by [2, 14.1], they have disjoint closures in  $A$ , and therefore in  $X$ . Next, suppose that  $x$  does not belong to  $U \cup V$ . Then  $U$  and  $V$  are subsets of  $A$ , and consequently of  $S_\kappa$ -type less than  $\kappa$ . Thus,  $U$  and  $V$  have disjoint closures in  $S_\kappa$ , and hence in  $X$ . This establishes that  $X$  is an  $F_\kappa$ -space.

To show that  $X$  has the  $G_\kappa$ -space property, suppose that  $U = \bigcap_{\alpha < \lambda} U_\alpha$  is a non-empty set for  $\lambda < \kappa$  where all  $U_\alpha$  are clopen subsets of  $X = A \cup \{x\}$ . If  $U$  has empty intersection with  $A$ , then all  $X \setminus U_\alpha$  are clopen subsets of  $S_\kappa$ . It follows that  $A = \bigcup_{\alpha < \lambda} X \setminus U_\alpha$  is a clopen non-compact subspace  $S_\kappa \setminus \{x\}$  of type less than  $\kappa$ , contradicting Lemma 4.5. Thus,  $U$  intersects  $A$ , and their intersection has non-empty interior in  $S_\kappa$ .  $\square$

Following the proof of Corollary 3.5, we see that for every point  $x$  of  $S_\kappa$  there is a clopen copy of  $S_\kappa$  contained in  $S_\kappa$  such that  $x$  is a  $P_\kappa$ -point with respect to that copy. In particular,  $S_\kappa$  contains  $P_\kappa$ -points.

**Corollary 4.6.** *Assume  $\kappa = \kappa^{<\kappa}$ . For every point  $x$  of  $S_\kappa$  and for every butterfly around  $x$ , one of its wings together with  $x$  itself is a copy of  $S_\kappa$  such that  $x$  is a  $P_\kappa$ -point with respect to that copy.*  $\square$

## 5. THE STONE-ČECH COMPACTIFICATION OF $S_\kappa \setminus \{x\}$

In this section we show that  $\beta(S_\kappa \setminus \{x\})$  is a  $\kappa$ -Parovičenko space of large weight. The result is best possible in the sense that  $\beta(S_\kappa \setminus \{x\})$  cannot be a  $\kappa^+$ -Parovičenko space, for points in  $S_\kappa \setminus \{x\}$  continue to have character  $\kappa$ .

We first prove a theorem about  $F_\kappa$ -spaces which generalises the well-known corresponding theorem for  $F$ -spaces (compare (1) and (1') in Section 2). It is interesting to note that the  $F$ -space property is even hereditary with respect to  $C^*$ -embedded subspaces, but the same is not true for  $F_\kappa$ -spaces—see the remarks after Lemma 4.4.

**Theorem 5.1.** *A strongly zero-dimensional space is an  $F_\kappa$ -space if and only if its Stone-Ćech compactification is an  $F_\kappa$ -space.*

*Proof.* Suppose that  $X$  is an  $F_\kappa$ -space, and  $U$  an open set of  $\beta X$  of type less than  $\kappa$ . To show that  $U$  is  $C^*$ -embedded in  $\beta X$ , fix a continuous  $[0, 1]$ -valued function  $f$  defined on  $U$ . The set  $U \cap X$  is of type less than  $\kappa$  in  $X$ . By assumption,  $f|_{U \cap X}$  can be extended to a continuous  $[0, 1]$ -valued function  $F$  on  $X$  which again can be extended to a continuous  $[0, 1]$ -valued function  $\beta F$  on  $\beta X$ . Now  $\beta F|_U = f$ , as both functions agree on the dense subset  $U \cap X$ .

For the converse, assume that  $\beta X$  is an  $F_\kappa$ -space. We show by induction on  $\lambda$  that  $X$  is an  $F_\lambda$ -space for all  $\lambda \leq \kappa$ . For  $\lambda = \omega$  there is nothing to prove. So let  $\lambda \leq \kappa$  be uncountable and assume that  $X$  is an  $F_\mu$ -space for all  $\mu < \lambda$ . Let  $U$  be an open set of  $X$ -type  $\tau < \lambda$ . We aim to show that  $U$  is  $C^*$ -embedded in  $X$ .

There are  $X$ -clopen sets  $U_\alpha$  such that  $U = \bigcup_{\alpha < \tau} U_\alpha$ . Write  $V_\beta = \bigcup_{\alpha < \beta} U_\alpha$  and  $W_\beta = \bigcup_{\alpha < \beta} \overline{U_\alpha}$  where the closure is taken in  $\beta X$ . Note that all  $\overline{U_\alpha}$  are clopen subsets of  $\beta X$  and that  $V_\beta = W_\beta \cap X$ . Let  $f$  be a continuous  $[0, 1]$ -valued map on  $V_\tau = U$ . For each  $\beta < \tau$ , the set  $V_\beta$  is  $C^*$ -embedded in  $X$  by induction hypothesis, and hence,  $f|_{V_\beta}$  extends to  $X$  and then to  $\beta X$ . Let  $f_\beta$  be the restriction of this extension to  $W_\beta$ .

Since  $V_\beta = W_\beta \cap X$  is dense in  $W_\beta$  for all  $\beta < \tau$ , the function  $f_\tau = \bigcup_{\beta < \tau} f_\beta$  is well-defined on  $W_\tau$ . And since every  $\overline{U_\alpha}$  is a clopen subset of  $\beta X$ , it is not hard to check that  $f_\tau$  is continuous. Thus,  $f_\tau$  extends from  $W_\tau$  to  $\beta X$  by the  $F_\kappa$ -space property. The restriction of this extension to  $X$  is the required extension of  $f$ .  $\square$

**Theorem 5.2.** *Assume  $\kappa = \kappa^{<\kappa}$ . For every point  $x$  of  $S_\kappa$  the space  $\beta(S_\kappa \setminus \{x\})$  is a  $\kappa$ -Parovičenko space of weight  $2^\kappa$ .*

*Proof.* Let  $X = S_\kappa \setminus \{x\}$ . First we verify the  $\kappa$ -Parovičenko properties. Clearly,  $\beta X$  does not contain isolated points. By [2, 14.1], every open subspace of  $S_\kappa$  is a strongly zero-dimensional  $F_\kappa$ -space and hence so is  $\beta X$  by Theorem 5.1.

For the  $G_\kappa$ -space property, let  $U = \bigcap_{\alpha < \lambda} U_\alpha$  be a non-empty intersection of clopen sets in  $\beta X$  for some  $\lambda < \kappa$ . To prove that  $U$  has non-empty interior it suffices to show that it intersects  $X$ .

Assume for a contradiction that  $\lambda$  is minimal such that  $U$  has empty intersection with  $X$ . Consider the sets  $V_\beta = \bigcap_{\alpha < \beta} U_\alpha \cap X$ . Then  $\bigcap_{\beta < \lambda} V_\beta$  is empty, whereas, without loss of generality,  $V_\beta \setminus V_{\beta+1}$  is non-empty for all  $\beta < \lambda$ . This last set is a non-empty intersection of less than  $\kappa$ -many open sets in  $S_\kappa$  and therefore contains a compact clopen set  $W_\beta$ .

Let  $f$  and  $g$  be disjoint cofinal subsets of  $\lambda$ . We define disjoint open sets

$$W_f = \bigcup_{\alpha \in f} W_\alpha \quad \text{and} \quad W_g = \bigcup_{\alpha \in g} W_\alpha$$

of type at most  $\lambda < \kappa$  and claim that in  $S_\kappa$  both sets limit onto  $x$ , contradicting the  $F_\kappa$ -space property. Suppose the claim was false, e.g. that  $W_f$  does not limit onto  $x$ . Then the closure of  $W_f$  in  $S_\kappa$ , a compact set, would be contained in  $X = \bigcup_{\beta < \lambda} X \setminus V_\beta$ . But by construction of  $W_f$ , this open cover of  $\overline{W_f}$  does not have a finite subcover, giving the desired contradiction.

It remains to calculate the weight of  $\beta(S_\kappa \setminus \{x\})$ . By Theorem [2, 2.21] and Lemmas [2, 2.23(b) & 2.24], the weight of  $\beta X$  for a strongly zero-dimensional space  $X$  is equal to the cardinality of  $\mathcal{CO}(X)$ . Now apply Corollary 4.3.  $\square$

It is an interesting question whether it is a ZFC theorem that the Stone-Ćech compactification of  $\omega^* \setminus \{x\}$  is a Parovičenko space. A. Dow showed that the assertion that all open subspaces of  $\omega^*$  are strongly zero-dimensional  $F$ -spaces is equivalent to CH [5]. However, we are only interested in open subspaces of the form  $\omega^* \setminus \{x\}$ . And these may be strongly zero-dimensional  $F$ -spaces even under the negation of CH, for example when  $\omega^* \setminus \{x\}$  is  $C^*$ -embedded in  $\omega^*$ .

The result about the weight also follows from the fact that the remainder of  $S_\kappa \setminus \{x\}$  has weight  $2^\kappa$ , see Theorem 7.2.

In the case of  $\kappa = \omega_1$ , a tempting proof of Theorem 5.2 can be obtained by observing that  $\omega^* \setminus \{x\}$  is pseudocompact (as it cannot contain a closed copy of  $\omega$ ), implying that every  $G_\delta$ -set of  $\beta(\omega^* \setminus \{x\})$  intersects  $\omega^* \setminus \{x\}$ . But this approach does not seem to generalise to  $S_\kappa \setminus \{x\}$  without extra effort. At the same time, both approaches are somehow intertwined: the proof of Theorem 5.2 shows that  $S_\kappa \setminus \{x\}$  is  $\alpha$ -pseudocompact for all  $\alpha < \kappa$  [13, 2.2]. For more on  $\alpha$ -pseudocompactness see [13].

## 6. THE STRUCTURE OF $(S_\kappa \setminus \{x\})^*$

The previous section has shown that  $\beta(S_\kappa \setminus \{x\})$  is a  $\kappa$ -Parovičenko space of quite large weight. Thus, these spaces are too large for the  $\kappa$ -Parovičenko properties to provide meaningful topological restrictions on the variety of potential spaces of that size.

For a better understanding we therefore turn to an investigation of the remainders of these spaces. The main result of this section is that every space of the form  $(S_\kappa \setminus \{x\})^*$  is a  $\kappa^+$ -Parovičenko space of weight  $2^\kappa$ , regardless of the choice of  $x$ . It follows that under  $2^\kappa = \kappa^+$ , all such

remainders are homeomorphic to  $S_{\kappa^+}$ . In particular, it is consistent with CH that all remainders of spaces of the form  $\omega^* \setminus \{x\}$  are homeomorphic.

Note that by Theorem 5.2,  $(S_{\kappa} \setminus \{x\})^*$  is a compact zero-dimensional space. The next lemma is the first step for establishing the remaining Parovičenko properties.

**Lemma 6.1.** *Assume  $\kappa = \kappa^{<\kappa}$ . For every  $x \in S_{\kappa}$  the space  $(S_{\kappa} \setminus \{x\})^*$  has no isolated points.*

*Proof.* Suppose that  $z \in (S_{\kappa} \setminus \{x\})^*$  is isolated. Then there is a clopen non-compact subset  $A$  of  $S_{\kappa} \setminus \{x\}$  such that  $A \cup \{z\}$  is compact and  $A$  is  $C^*$ -embedded in  $A \cup \{z\}$ . By Lemma 4.4, the set  $A$  is homeomorphic to  $S_{\kappa} \setminus \{y\}$  for some  $y \in S_{\kappa}$ . However, this space does not have a one-point Stone-Čech compactification by Corollary 4.2, a contradiction.  $\square$

Our next observation is that for a  $P_{\kappa}$ -point  $p$ , the space  $S_{\kappa} \setminus \{p\}$  can be written as an *increasing* union of  $\kappa$ -many compact clopen sets, each homeomorphic to  $S_{\kappa}$ . And remainders of increasing unions of  $\kappa$ -Parovičenko spaces are well understood: the next theorem follows from a result by A. Dow from 1985 [6, 2.2].

**Theorem 6.2** (Dow). *Assume  $\kappa = \kappa^{<\kappa}$ . For a  $P_{\kappa}$ -point  $p$  of  $S_{\kappa}$  the space  $(S_{\kappa} \setminus \{p\})^*$  is a  $\kappa^+$ -Parovičenko space.*  $\square$

Our key result is the following strengthening of Theorem 6.2.

**Theorem 6.3.** *Assume  $\kappa = \kappa^{<\kappa}$ . For every point  $x$  of  $S_{\kappa}$  the space  $(S_{\kappa} \setminus \{x\})^*$  is a  $\kappa^+$ -Parovičenko space of weight  $2^{\kappa}$ .*

We remark that our proof of Theorem 6.3 makes essential use of Dow's original theorem, and does not handle all cases simultaneously. Before presenting the proof, we discuss some interesting corollaries.

**Corollary 6.4.** *Assume  $\kappa = \kappa^{<\kappa}$ . The Stone-Čech compactifications  $\beta(S_{\kappa} \setminus \{x\})$  and  $\beta(S_{\kappa} \setminus \{y\})$  are homeomorphic if and only if  $S_{\kappa} \setminus \{x\}$  and  $S_{\kappa} \setminus \{y\}$  are homeomorphic.*

*Proof.* By Theorem 6.3, points in the ground space can be distinguished from points in the remainder of  $\beta(S_{\kappa} \setminus \{x\})$  by their character. Thus, any homeomorphism between  $\beta(S_{\kappa} \setminus \{x\})$  and  $\beta(S_{\kappa} \setminus \{y\})$  restricts to a homeomorphism between  $S_{\kappa} \setminus \{x\}$  and  $S_{\kappa} \setminus \{y\}$ .  $\square$

**Corollary 6.5.** *Assume  $\kappa = \kappa^{<\kappa}$  and  $2^{\kappa} = \kappa^+$ . For every point  $x$  the space  $(S_{\kappa} \setminus \{x\})^*$  is homeomorphic to  $S_{\kappa^+}$ .*

*Proof.* Write  $\lambda = \kappa^+$ . The condition  $2^{\kappa} = \kappa^+$  implies  $\lambda = \lambda^{<\lambda}$ . Thus, for every  $x$  in  $S_{\kappa}$ , the space  $(S_{\kappa} \setminus \{x\})^*$  is a  $\lambda$ -Parovičenko space of weight  $\lambda = \lambda^{<\lambda}$  by Theorem 6.3, and hence, by Negrepontis' characterisation, homeomorphic to  $S_{\lambda}$ .  $\square$

**Corollary 6.6.** *Under the cardinal assumption  $2^c = \omega_2$ , the remainders of  $\omega^* \setminus \{x\}$  and  $\omega^* \setminus \{y\}$  are homeomorphic for all points  $x$  and  $y$  of  $\omega^*$ .  $\square$*

This is especially interesting when compared to the fact that there are  $2^c$  non-homeomorphic subspaces of the form  $\omega^* \setminus \{x\}$ , an observation that follows easily from Frolík's result [10] that there are  $2^c$  orbits in  $\omega^*$  under its autohomeomorphism group, i.e. that  $\omega^*$  is badly non-homogenous.

The remaining part of this section is devoted to the proof of Theorem 6.3. For this, we first need a lemma about separation of disjoint open sets of small type in  $S_\kappa \setminus \{x\}$ . Note that a clopen non-compact set of  $S_\kappa \setminus \{x\}$  is of  $S_\kappa$ -type  $\kappa$  by Lemma 4.5, but of  $(S_\kappa \setminus \{x\})$ -type 1.

**Lemma 6.7.** *Assume  $\kappa = \kappa^{<\kappa}$ . For any two disjoint open sets  $V$  and  $W$  of  $(S_\kappa \setminus \{x\})$ -type less than  $\kappa$ , there is a clopen set  $A$  of  $S_\kappa \setminus \{x\}$  such that  $V \subseteq A$  and  $W \cap A = \emptyset$ .*

*Proof.* The space  $S_\kappa \setminus \{x\}$  is an  $F_\kappa$ -space by [2, 14.1], so  $V$  and  $W$  have disjoint closures in  $S_\kappa \setminus \{x\}$ . If  $V$  and  $W$  have disjoint closures in  $S_\kappa$ , by compactness there is a clopen set  $A \subseteq S_\kappa$  separating  $V$  from  $W$ . Clearly, the intersection of  $A$  with  $S_\kappa \setminus \{x\}$  is as required.

So assume that both  $V$  and  $W$  limit onto  $x$ . We build a butterfly such that its wings separate  $V$  and  $W$ . Let  $\{U_\alpha : \alpha < \kappa\}$  be a clopen neighbourhood base for  $x$  in  $S_\kappa$ . By transfinite recursion we define  $S_\kappa$ -clopen sets  $A_\alpha$  and  $B_\alpha$  for  $\alpha < \kappa$  not containing  $x$  such that all  $(A_\beta, B_\gamma)$ -,  $(A_\beta, W)$ - and  $(V, B_\gamma)$ -pairs are disjoint and  $A_\beta \cup B_\beta$  covers  $S_\kappa \setminus U_\beta$  for all  $\beta, \gamma < \kappa$ .

Once the construction is completed, we define disjoint open sets  $A = \bigcup_{\alpha < \kappa} A_\alpha$  and  $B = \bigcup_{\alpha < \kappa} B_\alpha$ . Their union covers all of  $S_\kappa \setminus \{x\}$  and  $V \subseteq A$  and  $W \subseteq B$  as required.

It remains to complete the recursive construction. Let  $\alpha < \kappa$  and assume that  $A_\beta$  and  $B_\beta$  have been defined for all ordinals  $\beta < \alpha$  satisfying the inductive requirements. Since  $V$  is of  $(S_\kappa \setminus \{x\})$ -type less than  $\kappa$ , it follows easily that  $V \setminus U_\alpha$  is of  $S_\kappa$ -type less than  $\kappa$ . So the sets

$$(V \cup \bigcup_{\beta < \alpha} A_\beta) \setminus U_\alpha \quad \text{and} \quad (W \cup \bigcup_{\beta < \alpha} B_\beta) \setminus U_\alpha$$

are disjoint open sets of  $S_\kappa$ -type less than  $\kappa$ , and by the  $F_\kappa$ -space property there is a clopen partition  $(C, D)$  of  $S_\kappa$  separating them. We put  $A_\alpha = C \setminus U_\alpha$  and  $B_\alpha = D \setminus U_\alpha$ , preserving the inductive assumptions.  $\square$

*Proof of Theorem 6.3.* Because of Theorem 6.2, it suffices to prove the theorem for non- $P_\kappa$ -points  $x$ . So let us fix a non- $P_\kappa$ -point  $x$  of  $S_\kappa$  and an open subset  $V \subseteq S_\kappa$  of type less than  $\kappa$  that contains  $x$  in its boundary. By the  $F_\kappa$ -space property,  $V$  is  $C^*$ -embedded in  $S_\kappa$ . In particular, if we

write  $X = S_\kappa \setminus \{x\}$  then the closure of  $V$  in  $X$  has a one-point Stone-Ćech compactification. This means the set  $V$  limits onto precisely one point in the remainder of  $X$ . For the remaining parts of this proof, we denote this unique point in  $\overline{V}^{\beta X} \setminus X$  by  $\star$ .

**Claim 1.** *For every clopen non-empty set  $C$  of  $X^*$  not containing  $\star$  there is a clopen non-compact set  $D \subseteq X$  which misses  $V$  such that  $D^* = \overline{D} \setminus D = C$ . Moreover, every such  $D$  is homeomorphic to  $S_\kappa \setminus \{p\}$  for a  $P_\kappa$ -point  $p$ .*

To see that the claim holds, let  $C$  be a clopen subset of  $X^*$  not containing  $\star$  and find a clopen non-compact subset  $E$  of  $X$  with  $E^* = C$ . There is a clopen neighbourhood  $U$  of  $x$  in  $S_\kappa$  such that  $U \cap (E \cap V) = \emptyset$ ; otherwise, the closure of  $E \cap V$  in  $\beta X$  would grow into the remainder. But

$$\overline{E \cap V}^{\beta X} \setminus X \subseteq \overline{E}^{\beta X} \setminus X \cap \overline{V}^{\beta X} \setminus X = C \cap \{\star\} = \emptyset,$$

a contradiction. Hence, for some suitable  $U$ , the clopen non-compact set  $D = E \cap U$  of  $S_\kappa$  does not intersect  $V$ . And since the symmetric difference of  $D$  and  $E$  is compact, it follows from [2, 2.6d] that  $D^* = E^* = C$ , as claimed.

To see that  $D$  is homeomorphic to  $S_\kappa \setminus \{p\}$  for a  $P_\kappa$ -point  $p$ , note that  $D$  and  $X \setminus D$  form a butterfly around  $x$  in  $S_\kappa$  such that  $V$  witnesses that  $x$  is not a  $P_\kappa$ -point of  $S_\kappa \setminus D$ . Hence,  $D$  is homeomorphic to  $S_\kappa \setminus \{p\}$  for a  $P_\kappa$ -point  $p$  by Corollary 4.6, completing the proof of Claim 1.

**Claim 2.** *Every compact clopen set of  $X^* \setminus \{\star\}$  is a  $\kappa^+$ -Parovičenko space.*

By Claim 1, a compact clopen set of  $X^* \setminus \{\star\}$  is of the form  $(S_\kappa \setminus \{p\})^*$  for a  $P_\kappa$ -point  $p$  of  $S_\kappa$ . Claim 2 now follows from Theorem 6.2.

**Claim 3.** *The point  $\star$  is a  $P_{\kappa^+}$ -point of  $X^*$ .*

To prove Claim 3 we show that whenever  $\{C_\alpha : \alpha < \kappa\}$  is a collection of  $X^*$ -clopen sets not containing  $\star$ , there is a clopen set  $B^* \subseteq X^*$  not containing  $\star$  such that  $\bigcup_{\alpha < \kappa} C_\alpha \subseteq B^*$ .

From Claim 1, we know that for every  $C_\alpha$  there is a clopen non-compact subset  $D_\alpha \subseteq X$  such that  $D_\alpha \cap V = \emptyset$  and  $D_\alpha^* = C_\alpha$ . In  $X$ , we write  $F \subseteq^* G$  (read:  $F$  is *almost contained* in  $G$ ) if there is a clopen neighbourhood  $U$  of  $x$  in  $S_\kappa$ , such that  $F \cap U \subseteq G$ . Write  $F =^* G$  if  $F \subseteq^* G$  and  $G \subseteq^* F$ . We will use the well-known fact that  $C_\alpha \subseteq C_\beta$  in  $X^*$  if and only if  $D_\alpha \subseteq^* D_\beta$  in  $X$ .

Similar to the proof of Lemma 6.7, our aim is to build a butterfly in  $X$  with wings  $A$  and  $B$  such that  $V \subseteq A$  and  $D_\alpha \subseteq^* B$  for all  $\alpha < \kappa$ . Clearly then,  $B^*$  is as required.

To construct such a butterfly, fix a neighbourhood base  $\{U_\alpha : \alpha < \kappa\}$  of  $x$  in  $S_\kappa$  consisting of clopen sets. By recursion we will define families of  $S_\kappa$ -clopen sets  $\{A_\alpha : \alpha < \kappa\}$  and  $\{B_\alpha : \alpha < \kappa\}$  and a third family  $\{E_\alpha : \alpha < \kappa\}$  of  $X$ -clopen sets such that for all  $\alpha, \beta < \kappa$

- (1)  $A_\alpha \cap B_\beta = \emptyset$  and  $A_\alpha \cup B_\alpha = S_\kappa \setminus U_\alpha$ ,
- (2)  $A_\alpha \cap E_\beta = \emptyset$  and  $B_\alpha \cap V = \emptyset$ ,
- (3)  $E_\alpha \subseteq D_\alpha$  and  $E_\alpha =^* D_\alpha$ .

Once the construction is completed, it follows from (1) that  $A = \bigcup_{\alpha < \kappa} A_\alpha$  and  $B = \bigcup_{\alpha < \kappa} B_\alpha$  partition  $X$  into disjoint open sets. Condition (2) guarantees both  $V \subseteq A$  and  $E_\alpha \subseteq B$  and finally, condition (3) gives  $D_\alpha \subseteq^* B$  as desired.

It remains to complete the recursive construction. Let  $\alpha < \kappa$  and assume that  $A_\beta, B_\beta$  and  $E_\beta$  have been defined for all ordinals  $\beta < \alpha$  satisfying the inductive assumptions. The set  $S_\kappa \setminus \bigcup_{\beta < \alpha} A_\beta$  is an intersection of less than  $\kappa$ -many clopen sets in  $S_\kappa$  containing  $x$ . In particular, it is a non-empty intersection of less than  $\kappa$ -many clopen sets in  $D_\alpha \cup \{x\}$ . But by Claim 1, the point  $x$  is a  $P_\kappa$ -point with respect to  $D_\alpha \cup \{x\}$ , and hence there is a  $D_\alpha \cup \{x\}$ -clopen neighbourhood  $E'_\alpha$  of  $x$  in this space such that  $E'_\alpha \subseteq S_\kappa \setminus \bigcup_{\beta < \alpha} A_\beta$ . Now put  $E_\alpha = E'_\alpha \setminus \{x\}$  and note that  $E_\alpha =^* D_\alpha$ .

By Lemma 6.7 there exist  $X$ -clopen sets  $C$  and  $D$  partitioning  $X$  and containing the disjoint open sets

$$V \cup \bigcup_{\beta < \alpha} A_\beta \quad \text{and} \quad \bigcup_{\beta < \alpha} B_\beta \cup \bigcup_{\beta \leq \alpha} E_\beta$$

respectively. By defining  $A_\alpha = C \setminus U_\alpha$  and  $B_\alpha = D \setminus U_\alpha$  it is clear that  $A_\alpha, B_\alpha$  and  $E_\alpha$  satisfy the inductive assumptions (1)-(3). The proof of Claim 3 is complete.

**Claim 4.** *The space  $X^*$  is a  $\kappa^+$ -Parovičenko space.*

For this, note that  $X^*$  is a zero-dimensional compact space without isolated points by Lemma 6.1. Thus it only remains to check for the  $F_{\kappa^+}$ - and the  $G_{\kappa^+}$ -space property. So suppose one of these conditions fails. This is witnessed by some point  $x$ . By Claim 2,  $x$  must be  $\star$ , but this is a contradiction as  $\star$  is a  $P_{\kappa^+}$ -point by Claim 3. This proves Claim 4.

To complete the proof, recall that if a compact Hausdorff space is covered by two of its subspaces, not both subspaces can have strictly smaller weight [8, 3.1.20]. Since  $\beta X$  has weight  $2^\kappa$  by Theorem 5.2 and  $X$  has weight  $\kappa$ , the remainder  $X^*$  must have weight  $2^\kappa$  as well.  $\square$

## 7. CARDINAL INVARIANTS OF $(S_\kappa \setminus \{x\})^*$

The results from the previous section have shown that under  $2^\kappa = \kappa^+$ , all remainders of the form  $(S_\kappa \setminus \{x\})^*$  are homeomorphic to  $S_{\kappa^+}$ , which

settles all further topological questions regarding cardinality, weight and cellularity of these spaces. However, we do not yet know what happens in absence of  $2^\kappa = \kappa^+$ . In this section we therefore investigate cardinal invariants of  $(S_\kappa \setminus \{x\})^*$  without additional set-theoretic assumptions beyond its existence, i.e.  $\kappa = \kappa^{<\kappa}$ .

The next lemma says that for questions such as size, weight and cellularity of the remainder of  $S_\kappa \setminus \{x\}$ , it is enough to focus on the remainder of  $S_\kappa \setminus \{p\}$  for a  $P_\kappa$ -point  $p$ .

**Lemma 7.1.** *Assume  $\kappa = \kappa^{<\kappa}$ . For every  $x \in S_\kappa$  the space  $(S_\kappa \setminus \{x\})^*$  contains a clopen copy of  $(S_\kappa \setminus \{p\})^*$  for a  $P_\kappa$ -point  $p$ .*

*Proof.* By Corollary 4.6, the space  $S_\kappa \setminus \{x\}$  contains a clopen copy of  $S_\kappa \setminus \{p\}$  for a  $P_\kappa$ -point  $p$ .  $\square$

Every point of a  $\kappa$ -Parovičenko space has character at least  $\kappa$  and hence the whole space has, by [8, 3.12.11], cardinality at least  $2^\kappa$ . Thus, Theorem 6.2, Lemma 7.1 and [8, 3.5.3] give us the chain of inequalities

$$2^{\kappa^+} \leq |(S_\kappa \setminus \{p\})^*| \leq |(S_\kappa \setminus \{x\})^*| \leq 2^{2^\kappa}.$$

In particular, we see once again that under  $2^\kappa = \kappa^+$ , the cardinality of  $(S_\kappa \setminus \{x\})^*$  is clear. In the remaining part of this paper we show without additional set-theoretic assumptions that  $(S_\kappa \setminus \{p\})^*$ —and hence  $(S_\kappa \setminus \{x\})^*$  for all  $x$ —has maximal cardinality and cellularity.

We present two different proofs of these facts. Both approaches require interesting ideas which seem to be fundamentally different. The first proof was pointed out to us by the reviewer and treats the Stone-Čech compactification as the maximal compactification, which projects onto every other compactification. Identifying the right object to project on, however, requires some creativity. Our second proof views the Stone-Čech compactification in terms of  $C^*$ -embeddedness. This proof can be considered the elementary one, using nothing beyond the basic  $\kappa$ -Parovičenko properties of  $S_\kappa$ . It also gives an intrinsic reason why the remainders in question have large cellularity: we explicitly construct  $2^\kappa$  clopen sets in  $S_\kappa \setminus \{p\}$  that extend to disjoint clopen sets in the remainder.

For the first proof we let  $\kappa$  be the discrete space of cardinality  $\kappa$ , and identify  $\kappa^*$  with the free ultrafilters on  $\kappa$ . The space  $U(\kappa) \subseteq \kappa^*$  denotes the space of the *uniform* ultrafilters on  $\kappa$ , i.e. the ultrafilters whose members are subsets of  $\kappa$  of full cardinality.

Comfort and Negrepontis observed in [1, 2.4] that under CH we have  $\omega^* \setminus \{p\} \cong \omega_1^* \setminus U(\omega_1)$  for  $P$ -points  $p$ , implying that the remainder of  $\omega^* \setminus \{p\}$  has cardinality  $2^{2^{\omega_1}}$  and cellularity  $2^{\omega_1}$ . Even though the equality  $S_\kappa \setminus \{p\} \cong \kappa^* \setminus U(\kappa)$  does *not* generalise to larger cardinals  $\kappa > \omega_1$ , a similar approach can be used to compute the cardinality of  $(S_\kappa \setminus \{p\})^*$ .



**Theorem 7.2.** *Assume  $\kappa = \kappa^{<\kappa}$ . For every point  $x \in S_\kappa$  the remainder of  $S_\kappa \setminus \{x\}$  has size  $2^{2^\kappa}$  and contains a family of  $2^\kappa$  disjoint open sets.*

*Proof.* Let  $p$  be a  $P_\kappa$ -point of  $S_\kappa$ . By Lemma 7.1 it suffices to prove the theorem for  $(S_\kappa \setminus \{p\})^*$ .

For the cellularity, we observe that every  $\kappa^+$ -Parovičenko space contains a family of  $2^\kappa$  disjoint clopen sets: simply embed a binary tree of clopen sets of height  $\kappa + 1$  into the space, using compactness and the  $G_{\kappa^+}$ -property at limit stages. Now apply Theorem 6.2.

For the cardinality, note that  $S_\kappa \setminus \{p\}$  can be written as an increasing  $\kappa$ -union of clopen sets each homeomorphic to  $S_\kappa$ , and  $\kappa^* \setminus U(\kappa)$  can be written as an increasing  $\kappa$ -union of compact spaces of weight at most  $\kappa$ . By a transfinite recursion, using that  $S_\kappa$  surjects onto every compact Hausdorff space of weight at most  $\kappa$  [2, 6.14] at successor stages and the fact that every continuous map from an open set of type less than  $\kappa$  into a compact Hausdorff space extends to  $S_\kappa$  [2, 6.22] at limit stages, we may construct a continuous surjection  $f: S_\kappa \setminus \{p\} \rightarrow \kappa^* \setminus U(\kappa)$ .

The map  $f$  extends to a continuous surjection  $\beta f: \beta(S_\kappa \setminus \{p\}) \rightarrow \kappa^*$  such that  $U(\kappa) \subseteq \beta f((S_\kappa \setminus \{p\})^*)$ . Since  $U(\kappa)$  has cardinality  $2^{2^\kappa}$  [2, 7.8], the result follows.  $\square$

It is an interesting question whether the map  $f$  constructed in the previous theorem can be chosen to be perfect. If yes, [8, 3.7.16] would show the existence of a continuous surjection from  $(S_\kappa \setminus \{p\})^*$  onto  $U(\kappa)$ .

For the second proof we present an approach that is solely based on the Parovičenko properties.

The next lemma guarantees the existence of a *monotone cut operator* for all pairs of disjoint open sets of type less than  $\kappa$  in  $S_\kappa$ , in the spirit of a separation operator for monotone normality. We make this definition precise. For an ordered pair  $\langle A, B \rangle$  of disjoint open sets of  $S_\kappa$  of type less than  $\kappa$ , we define a *cut* between them to be a clopen set  $\mathcal{C}_{\langle A, B \rangle}$  such that

$$A \subseteq \mathcal{C}_{\langle A, B \rangle} \subseteq S_\kappa \setminus B.$$

Cuts in  $S_\kappa$  exist by the  $F_\kappa$ -space property. We write  $\langle A, B \rangle \leq \langle A', B' \rangle$  if  $A \subseteq A'$  and  $B \supseteq B'$ . A cut operator  $\mathcal{C}$  is called *monotone* if  $\langle A, B \rangle \leq \langle A', B' \rangle$  implies  $\mathcal{C}_{\langle A, B \rangle} \subseteq \mathcal{C}_{\langle A', B' \rangle}$ . The cut operator is called *symmetric* if  $\mathcal{C}_{\langle A, B \rangle} = S_\kappa \setminus \mathcal{C}_{\langle B, A \rangle}$ .

We need the following strengthening of the concept of a monotone cut operator. Let  $\{U_\alpha: \alpha < \kappa\}$  be a decreasing neighbourhood base of clopen sets of a  $P_\kappa$ -point  $p$  such that  $U_0 = S_\kappa$ . Call two subsets  $A$  and  $B$  of  $S_\kappa$   $\gamma$ -*equivalent* if  $A \cap U_\gamma = B \cap U_\gamma$  for some  $\gamma < \kappa$ . Also, call  $A$  a  $\gamma$ -*subset* of  $B$ , and write  $A \subseteq_\gamma B$ , if  $A \cap U_\gamma \subseteq B \cap U_\gamma$ . We extend this idea to capture “local monotonicity” and write  $\langle A, B \rangle \leq_\gamma \langle A', B' \rangle$  if  $A \subseteq_\gamma A'$  and

$B \supseteq_\gamma B'$ . Note that for  $\gamma \leq \delta < \kappa$ , if  $\langle A, B \rangle \leq_\gamma \langle A', B' \rangle$  holds then so does  $\langle A, B \rangle \leq_\delta \langle A', B' \rangle$ .

A cut operator  $\mathcal{C}$  with the property such that for all  $\gamma$ ,  $\langle A, B \rangle \leq_\gamma \langle A', B' \rangle$  implies  $\mathcal{C}_{\langle A, B \rangle} \subseteq_\gamma \mathcal{C}_{\langle A', B' \rangle}$  will be called a *strong monotone cut operator with respect to  $\{U_\alpha\}$* . Every strong monotone cut operator is also monotone.

**Lemma 7.3.** *Assume  $\kappa = \kappa^{<\kappa}$ . Let  $\mathcal{F}$  be the collection of all ordered pairs of disjoint open sets of  $S_\kappa$  of type less than  $\kappa$ . For every decreasing neighbourhood base of clopen sets  $\{U_\alpha: \alpha < \kappa\}$  of a  $P_\kappa$ -point in  $S_\kappa$  with  $U_0 = S_\kappa$ , there exists a symmetric strong monotone cut operator  $\mathcal{C}: \mathcal{F} \rightarrow \mathcal{CO}(S_\kappa)$  with respect to  $\{U_\alpha\}$ .*

*Proof.* By  $\kappa = \kappa^{<\kappa}$  we may list  $\mathcal{F} = \{\langle A_\alpha, B_\alpha \rangle: \alpha < \kappa\}$ , such that permuted pairs are next to each other. Let  $\{U_\alpha: \alpha < \kappa\}$  be a decreasing neighbourhood base of a  $P_\kappa$ -point  $p$  consisting of clopen sets such that  $U_0 = S_\kappa$ . In addition we define  $U_\kappa = \emptyset$ , obtaining the technical advantage that for all pairs of sets, one is a  $\gamma$ -subset of the other for some  $\gamma \leq \kappa$ .

We define an operator  $\mathcal{C}: \mathcal{F} \rightarrow \mathcal{CO}(S_\kappa)$  that satisfies for all ordinals  $\beta, \delta < \kappa$ :

$$\begin{aligned} (Cut) \quad & A_\beta \subseteq \mathcal{C}_{\langle A_\beta, B_\beta \rangle} \subseteq S_\kappa \setminus B_\beta, \\ (Sym) \quad & \mathcal{C}_{\langle A_\beta, B_\beta \rangle} = S_\kappa \setminus \mathcal{C}_{\langle B_\beta, A_\beta \rangle} \text{ and} \\ (Mon) \quad & \forall \gamma < \kappa (\langle A_\delta, B_\delta \rangle \leq_\gamma \langle A_\beta, B_\beta \rangle \Rightarrow \mathcal{C}_{\langle A_\delta, B_\delta \rangle} \subseteq_\gamma \mathcal{C}_{\langle A_\beta, B_\beta \rangle}). \end{aligned}$$

We proceed by transfinite recursion. Let  $\alpha < \kappa$  and suppose we have defined cuts  $\mathcal{C}_{\langle A_\beta, B_\beta \rangle}$  for all  $\beta < \alpha$  satisfying the inductive assumptions for all  $\beta, \delta < \alpha$ .

Consider  $\langle A_\alpha, B_\alpha \rangle$ . If  $\alpha$  is a successor and  $\langle A_\alpha, B_\alpha \rangle = \langle B_{\alpha-1}, A_{\alpha-1} \rangle$  we define  $\mathcal{C}_{\langle A_\alpha, B_\alpha \rangle} = S_\kappa \setminus \mathcal{C}_{\langle A_{\alpha-1}, B_{\alpha-1} \rangle}$ . This assignment takes care of *(Sym)* and a straightforward calculation shows that also *(Cut)* and *(Mon)* are satisfied.

Otherwise, for all  $\beta < \alpha$  we let  $\gamma_\beta^\downarrow$  and  $\gamma_\beta^\uparrow$  be the least ordinals such that  $\langle A_\beta, B_\beta \rangle \leq_{\gamma_\beta^\downarrow} \langle A_\alpha, B_\alpha \rangle$  and  $\langle A_\alpha, B_\alpha \rangle \leq_{\gamma_\beta^\uparrow} \langle A_\beta, B_\beta \rangle$ . This is well-defined, as these relations are satisfied for at least  $\kappa$ . Let

$$\mathcal{C}_\alpha^\downarrow = \left\{ U_{\gamma_\beta^\downarrow} \cap \mathcal{C}_{\langle A_\beta, B_\beta \rangle} : \beta < \alpha \right\} \text{ and } \mathcal{C}_\alpha^\uparrow = \left\{ U_{\gamma_\beta^\uparrow} \setminus \mathcal{C}_{\langle A_\beta, B_\beta \rangle} : \beta < \alpha \right\}.$$

The idea is that these sets contain all parts of the previously defined cuts we have to be aware of in order to make our operator respect *(Mon)*. Both sets have cardinality less than  $\kappa$  and consist of clopen sets.

We claim that the sets  $A_\alpha \cup (\bigcup \mathcal{C}_\alpha^\downarrow)$  and  $B_\alpha \cup (\bigcup \mathcal{C}_\alpha^\uparrow)$  are disjoint open sets of type less than  $\kappa$ . They are clearly open and of type less than  $\kappa$ .

We demonstrate only that  $\bigcup \mathcal{C}_\alpha^\downarrow$  and  $\bigcup \mathcal{C}_\alpha^\uparrow$  are disjoint, since the other cases are similar. For this we show that for any  $\beta, \delta < \alpha$ , the sets  $U_{\gamma_\beta^\downarrow} \cap \mathcal{C}_{\langle A_\beta, B_\beta \rangle} \in \mathcal{C}_\alpha^\downarrow$  and  $U_{\gamma_\delta^\uparrow} \setminus \mathcal{C}_{\langle A_\delta, B_\delta \rangle} \in \mathcal{C}_\alpha^\uparrow$  have empty intersection. By construction we have

$$\langle A_\beta, B_\beta \rangle \leq_{\gamma_\beta^\downarrow} \langle A_\alpha, B_\alpha \rangle \text{ and } \langle A_\alpha, B_\alpha \rangle \leq_{\gamma_\delta^\uparrow} \langle A_\delta, B_\delta \rangle.$$

With  $\gamma$  denoting the larger of  $\gamma_\beta^\downarrow$  and  $\gamma_\delta^\uparrow$  we may apply condition (Mon) to  $\langle A_\beta, B_\beta \rangle \leq_\gamma \langle A_\delta, B_\delta \rangle$  and obtain  $\mathcal{C}_{\langle A_\beta, B_\beta \rangle} \subseteq_\gamma \mathcal{C}_{\langle A_\delta, B_\delta \rangle}$ . In particular, the sets  $U_\gamma \cap \mathcal{C}_{\langle A_\beta, B_\beta \rangle}$  and  $U_\gamma \setminus \mathcal{C}_{\langle A_\delta, B_\delta \rangle}$  have empty intersection, and since  $U_\gamma = U_{\gamma_\beta^\downarrow} \cap U_{\gamma_\delta^\uparrow}$ , the result follows.

Now, since  $A_\alpha \cup (\bigcup \mathcal{C}_\alpha^\downarrow)$  and  $B_\alpha \cup (\bigcup \mathcal{C}_\alpha^\uparrow)$  are disjoint open sets of  $S_\kappa$  of type less than  $\kappa$ , there exist clopen sets containing the first set and not intersecting the second. Choose one and denote it by  $\mathcal{C}_{\langle A_\alpha, B_\alpha \rangle}$ .

This assignment clearly satisfies (Cut), so it remains to check for (Mon). Let  $\beta < \alpha$  and suppose  $\langle A_\beta, B_\beta \rangle \leq_\gamma \langle A_\alpha, B_\alpha \rangle$  for some  $\gamma < \kappa$ . Since we chose  $\gamma_\beta^\downarrow$  minimal, we have  $U_\gamma \subseteq U_{\gamma_\beta^\downarrow}$ . By construction we have  $U_{\gamma_\beta^\downarrow} \cap \mathcal{C}_{\langle A_\beta, B_\beta \rangle} \subseteq \bigcup \mathcal{C}_\alpha^\downarrow \subseteq \mathcal{C}_{\langle A_\alpha, B_\alpha \rangle}$  and therefore also  $U_\gamma \cap \mathcal{C}_{\langle A_\beta, B_\beta \rangle} \subseteq \mathcal{C}_{\langle A_\alpha, B_\alpha \rangle}$ . Thus,  $\mathcal{C}_{\langle A_\beta, B_\beta \rangle} \subseteq_\gamma \mathcal{C}_{\langle A_\alpha, B_\alpha \rangle}$ .

The case  $\langle A_\beta, B_\beta \rangle \geq_\gamma \langle A_\alpha, B_\alpha \rangle$  is similar and the proof is complete.  $\square$

We now consider a variation of the butterfly construction which is tailored to  $P_\kappa$ -points. Let  $\{U_\alpha : \alpha < \kappa\}$  be a decreasing neighbourhood base of a  $P_\kappa$ -point  $p$  of  $S_\kappa$  with  $U_0 = S_\kappa$ . We work through the ‘‘onion rings’’  $D_\alpha = U_\alpha \setminus U_{\alpha+1}$  and assign them either to the  $A$ - or the  $B$ -wing, following certain patterns. When compared to the original butterfly construction in Lemma 3.1, this adaptation has the advantage that one does not need to assign cuts at successor stages.

The *support* of a binary sequence  $f : \kappa \rightarrow 2$  is the set  $f^{-1}(\{1\})$ .

**Lemma 7.4.** *Assume  $\kappa = \kappa^{<\kappa}$  and let  $p$  be a  $P_\kappa$ -point of  $S_\kappa$ . There is a family  $\{A^f : f \in 2^\kappa\}$  of clopen subsets of  $S_\kappa \setminus \{p\}$  such that for all  $f, g \in 2^\kappa$*

- (1)  $A^f$  is non-empty whenever  $f$  has non-empty support,
- (2)  $A^f$  is non-compact whenever  $f$  has unbounded support,
- (3)  $A^{1-f} \cap A^f = \emptyset$ ,
- (4) if  $f \leq g$  (pointwise) then  $A^f \subseteq A^g$  and
- (5) if  $f = g$  eventually then there exists a clopen neighbourhood  $U$  of  $p$  such that  $A^f \cap U = A^g \cap U$ .

*Proof.* Let  $\{U_\alpha : \alpha < \kappa\}$  be a decreasing neighbourhood base of  $p$  with  $U_0 = S_\kappa$ , and let  $D_\alpha = U_\alpha \setminus U_{\alpha+1}$ . Let  $\mathcal{C}$  denote a fixed strong monotone

cut operator from Lemma 7.3 with respect to  $\{U_\alpha\}$ . For each sequence  $f \in 2^\kappa$  we build a butterfly with wings  $A^f$  and  $B^f$  around  $p$ . As always, this involves defining  $S_\kappa$ -clopen sets  $A_\alpha^f$  and  $B_\alpha^f$  for  $\alpha < \kappa$  such that all  $(A_\alpha^f, B_\beta^f)$ -pairs are disjoint and  $A_\alpha^f \cup B_\alpha^f$  covers  $S_\kappa \setminus U_\alpha$  for all  $\alpha$ . The rules for the recursive construction are: for all ordinals  $\alpha < \kappa$  set

$$A_{\alpha+1}^f = \begin{cases} A_\alpha^f \cup D_\alpha, & \text{if } f(\alpha) = 1, \\ A_\alpha^f, & \text{if } f(\alpha) = 0, \end{cases} \quad B_{\alpha+1}^f = \begin{cases} B_\alpha^f, & \text{if } f(\alpha) = 1, \\ B_\alpha^f \cup D_\alpha, & \text{if } f(\alpha) = 0, \end{cases}$$

and if  $\lambda < \kappa$  is a limit ordinal, put

$$A_\lambda^f = \mathcal{C} \left( \bigcup_{\beta < \lambda} A_\beta^f, \bigcup_{\beta < \lambda} B_\beta^f \right) \setminus U_\lambda \quad \text{and} \quad B_\lambda^f = (S_\kappa \setminus A_\lambda^f) \setminus U_\lambda.$$

The sets  $A^f = \bigcup_{\alpha < \kappa} A_\alpha^f$  and  $B^f = \bigcup_{\alpha < \kappa} B_\alpha^f$  are disjoint open and cover all of  $S_\kappa \setminus \{p\}$ . Thus, they define clopen subsets of  $S_\kappa \setminus \{p\}$ .

We claim the sets  $A^f$  satisfy assertions (1)-(5). It is clear that (1) is satisfied. Next, if  $f$  has unbounded support, then  $A^f$  limits onto  $p$ , i.e. is non-compact.

For (3) and (4), one shows by induction that  $A_\alpha^{1-f} = B_\alpha^f$  and  $A_\alpha^f \subseteq A_\alpha^g$  whenever  $f \leq g$ , using that the cut operator is symmetric and monotone, respectively.

For (5) suppose there exists an ordinal  $\delta < \kappa$  such that  $f(\alpha) = g(\alpha)$  for all  $\alpha \geq \delta$ . We show by induction that  $A_\alpha^f \cap U_\delta = A_\alpha^g \cap U_\delta$ . The claim is trivially true for  $\alpha < \delta$ . So let  $\alpha \geq \delta$  and assume the claim holds for all smaller ordinals. The situation is clear for successors, so assume that  $\alpha$  is a limit. By induction hypothesis, the pairs  $\langle \bigcup_{\beta < \alpha} A_\beta^f, \bigcup_{\beta < \alpha} B_\beta^f \rangle$  and  $\langle \bigcup_{\beta < \alpha} A_\beta^g, \bigcup_{\beta < \alpha} B_\beta^g \rangle$  are  $\delta$ -equivalent and hence it follows from the properties of the cut operator that

$$\begin{aligned} A_\alpha^f \cap U_\delta &= \mathcal{C} \left( \bigcup_{\beta < \alpha} A_\beta^f, \bigcup_{\beta < \alpha} B_\beta^f \right) \cap (U_\delta \setminus U_\alpha) \\ &= \mathcal{C} \left( \bigcup_{\beta < \alpha} A_\beta^g, \bigcup_{\beta < \alpha} B_\beta^g \right) \cap (U_\delta \setminus U_\alpha) = A_\alpha^g \cap U_\delta. \end{aligned}$$

This completes the induction step and the proof.  $\square$

We now show how to use a family with properties (1)-(5) of Lemma 7.4 to push ultrafilters and almost disjoint families from  $\kappa$  through to the space  $(S_\kappa \setminus \{p\})^*$ . For a subset  $U$  of  $\kappa$ , let  $\mathbb{1}_U \in 2^\kappa$  denote its characteristic function.

*Second proof of Theorem 7.2.* Let  $p$  be a  $P_\kappa$ -point of  $S_\kappa$ . Again, by Lemma 7.1, it suffices to prove the theorem for  $(S_\kappa \setminus \{p\})^*$ .

For the cardinality, we show that there are at least  $2^{2^\kappa}$ -many  $z$ -ultrafilters on  $S_\kappa \setminus \{p\}$ . Let  $\{A^f : f \in 2^\kappa\}$  be a family of clopen sets of  $S_\kappa \setminus \{p\}$  with properties (1), (3) and (4) of Lemma 7.4. For an ultrafilter  $\mathcal{U}$  on  $\kappa$ , consider the family

$$\Phi(\mathcal{U}) = \{A^{1_U} : U \in \mathcal{U}\}.$$

By (1) and (4),  $\Phi(\mathcal{U})$  is a filter base for some clopen filter on  $S_\kappa \setminus \{p\}$ , and it follows from (3) that whenever  $\mathcal{U}$  and  $\mathcal{U}'$  are distinct ultrafilters on  $\kappa$ , then  $\Phi(\mathcal{U})$  and  $\Phi(\mathcal{U}')$  can only be extended to distinct  $z$ -ultrafilters on  $S_\kappa \setminus \{p\}$ . As there are  $2^{2^\kappa}$  ultrafilters on  $\kappa$  the result follows.

For the cellularity, let  $\{A^f : f \in 2^\kappa\}$  be a family of clopen sets of  $S_\kappa \setminus \{p\}$  with properties (2)–(5) of Lemma 7.4. Recall that  $\kappa = \kappa^{<\kappa}$  if and only if  $\kappa$  is regular and  $\kappa = 2^{<\kappa}$  [2, 1.27]. In particular, from  $\kappa = 2^{<\kappa}$  we may conclude that there is an almost disjoint family  $\mathcal{E}$  of size  $2^\kappa$  on  $\kappa$ , i.e. a family whose members are subsets of  $\kappa$  of full cardinality such that the intersection of any two elements is of size less than  $\kappa$  [14, II.1.3]. By property (2), the family

$$\Phi(\mathcal{E}) = \{A^{1_E} : E \in \mathcal{E}\}$$

consists of non-compact clopen sets of  $S_\kappa \setminus \{p\}$ . We claim they have pairwise compact intersection. By regularity of  $\kappa$ , the intersection of distinct elements  $E$  and  $F$  of  $\mathcal{E}$  is bounded by an ordinal  $\delta < \kappa$ . The function

$$f : \kappa \rightarrow 2, \alpha \mapsto \begin{cases} 1_F(\alpha) & \text{if } \alpha \leq \delta, \\ 1 - 1_E(\alpha) & \text{if } \alpha > \delta, \end{cases}$$

satisfies  $1_F \leq f$ , and  $f = 1 - 1_E$  eventually. By property (5), there is a clopen neighbourhood  $U$  of  $p$  such that  $A^f \cap U = A^{1-1_E} \cap U$ . Then

$$A^{1_E} \cap A^{1_F} \cap U \subseteq A^{1_E} \cap A^f \cap U = A^{1_E} \cap A^{1-1_E} \cap U = \emptyset,$$

where the first inclusion follows from property (4) and the last equality from (3). Thus,  $A^{1_E} \cap A^{1_F}$  is a closed subset of the compact space  $S_\kappa \setminus U$ , hence compact.

It follows from [2, 2.6d] that whenever  $A$  and  $B$  are distinct elements in  $\Phi(\mathcal{E})$  then  $A^* = \overline{A} \setminus A$  and  $B^*$  are disjoint non-empty clopen subsets of  $(S_\kappa \setminus \{p\})^*$ . Since  $\Phi(\mathcal{E})$  has cardinality  $2^\kappa$  the result follows.  $\square$

In fact, if  $X$  is any compact zero-dimensional  $F_\kappa$ -space with the property  $\kappa = \kappa^{<\kappa}$ , and  $p \in X$  is a  $P_\kappa$ -point of character  $\kappa$  then the above methods show that  $(X \setminus \{p\})^*$  has cardinality at least  $2^{2^\kappa}$ , and contains a family of  $2^\kappa$  many disjoint clopen sets.

The two proofs presented in this section showing that  $(S_\kappa \setminus \{x\})^*$  has maximal cardinality seem to be of a very different flavour. Is there a

unifying argument showing that the two approaches presented are in some sense equivalent?

## 8. OPEN QUESTIONS

We list open questions that are motivated by results in this paper. Corollary 6.6 shows that it is consistent with CH that all remainders of the form  $(\omega^* \setminus \{x\})^*$  are homeomorphic. Is the negation of this result also consistent with CH?

**Question 8.1.** *Is it consistent with CH that for a  $P$ -point  $p$  and a non- $P$ -point  $x$  of  $\omega^*$  the remainders of  $\omega^* \setminus \{p\}$  and  $\omega^* \setminus \{x\}$  are non-homeomorphic?*

Theorem 1.2 shows that it is consistent with ZFC that every space  $\omega^* \setminus \{x\}$  has a one-point Stone-Čech remainder. Under CH, Theorem 7.2 shows that the remainder of  $\omega^* \setminus \{x\}$  has size  $2^{2^{\omega_1}}$  for every  $x$ .

**Question 8.2** (van Mill). *Which cardinalities, apart from 1 and  $2^{2^{\omega_1}}$ , can  $(\omega^* \setminus \{x\})^*$  consistently have?*

In connection with Question 8.2 note that A. Dow has constructed a model in which  $\omega^* \setminus \{x\}$  has a one-point Stone-Čech remainder if and only if  $x$  is not a  $P$ -point of  $\omega^*$  [7].

In Section 5 we mention the question whether one can prove without CH that  $\omega^* \setminus \{x\}$  is a strongly zero-dimensional  $F$ -space.

**Question 8.3.** *Is it a ZFC theorem that  $\beta(\omega^* \setminus \{x\})$  is a Parovičenko space?*

An affirmative answer to Question 8.3 proves that  $(\omega^* \setminus \{x\})^*$  is a compact  $F$ -space and hence rules out infinite cardinalities smaller than  $2^c$  in Question 8.2.

Lastly, Theorem 6.3 gives us a large class of topologically distinct spaces whose remainders are  $\kappa^+$ -Parovičenko spaces. Can we find a precise description of spaces with that behaviour?

**Question 8.4.** *Is there a characterisation for which spaces  $X$  its remainder  $X^*$  is a  $\kappa^+$ -Parovičenko space?*

## REFERENCES

- [1] W.W. Comfort and S. Negrepontis, *Homeomorphs of three subspaces of  $\beta\mathbb{N} \setminus \mathbb{N}$* , Math. Z. **107** (1968), 53–58.
- [2] W.W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer-Verlag, Berlin, 1974.
- [3] E.K. van Douwen, K. Kunen and J. van Mill, *There can be  $C^*$ -embedded dense proper subspaces in  $\beta\omega - \omega$* , Proc. Amer. Math. Soc. **105** (1989), no. 2, 462–470.

- [4] E.K. van Douwen and J. van Mill, *Parovičenko's characterization of  $\beta\omega - \omega$  implies CH*, Proc. Amer. Math. Soc. **72** (1978), no. 3, 539–541.
- [5] A. Dow, *CH and Open Subspaces of F-Spaces*, Proc. Amer. Math. Soc. **89** (1983), no. 2, 341–345.
- [6] A. Dow, *Saturated Boolean algebras and Stone spaces*, Topology Appl. **21** (1985), 193–207.
- [7] A. Dow, *Extending real-valued functions in  $\beta\kappa$* , Fund. Math. **152** (1997), no. 1, 21–41.
- [8] R. Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989.
- [9] N.J. Fine and L. Gillman, *Extension of continuous functions in  $\beta\mathbb{N}$* , Bull. Amer. Math. Soc. **66** (1960), 376–381.
- [10] Z. Frolík, *Sums of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87–91.
- [11] L. Gillman, *The space  $\beta\mathbb{N}$  and the continuum hypothesis*, in: General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the second Prague topological symposium. Praha, 1967. 144–146.
- [12] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York, 1976.
- [13] J.F. Kennison, *m-Pseudocompactness*, Trans. Amer. Math. Soc. **104** (1962), 436–442.
- [14] K. Kunen, *Set Theory*, Elsevier Science, North Holland, 1980.
- [15] J. van Mill, *An Introduction to  $\beta\omega$* , in: Handbook of Set-Theoretic Topology. Eds. K. Kunen and J.E. Vaughan. Elsevier Science, 1984. 503–567.
- [16] S. Negrepontis, *The Stone space of the saturated Boolean algebras*, Trans. Amer. Math. Soc. **141** (1969), 515–527.
- [17] I.I. Parovičenko, *A universal bicomact of weight  $\aleph$* , Soviet Math. Dokl. **4** (1963), 592–595.

(Pitz) MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD OX2 6GG,  
UNITED KINGDOM

*E-mail address*, Corresponding author: `pitz@maths.ox.ac.uk`

(Suabedissen) MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, OXFORD OX2  
6GG, UNITED KINGDOM

*E-mail address*: `suabedis@maths.ox.ac.uk`