

ATMENABILITY OF SOME GRAPHS OF GROUPS WITH CYCLIC EDGE GROUPS

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ABSTRACT. We show that certain graphs of groups with cyclic edge groups are aTmenable. In particular, this holds when each vertex group is either virtually special or acts properly and semisimply on \mathbb{H}^n .

1. INTRODUCTION

A group G is *aTmenable* or has the *Haagerup property* if G admits a metrically proper action on a Hilbert space by affine isometries. Gromov suggested the term aTmenable to highlight that aTmenability is both a generalization of amenability and a strong negation of Property (T) [Gro93]. Throughout this paper, we will use the following equivalent definition due to Cherix, Martin and Valette: a discrete group G is *aTmenable* if it acts metrically properly on a space with measured walls (see Section 2). For other equivalent definitions of aTmenability, we refer to [CCJ⁺01, CMV04, CDH10].

If a group G splits as a graph of groups with aTmenable vertex groups and finite edge groups then G is aTmenable [CCJ⁺01, Theorem 6.2.8]. In contrast, aTmenability can fail for a graph of groups whose vertex and edge groups are isomorphic to \mathbb{Z}^2 . For instance, as shown in [Car14], the following \mathbb{Z}^2 -by- F_2 group is not aTmenable:

$$G = \mathbb{Z}^2 \rtimes \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$$

The following problem was raised in [CCJ⁺01, 7.3.2].

Problem 1.1. Let G split as a graph of groups. Suppose each vertex group is aTmenable and each edge group is cyclic. Is G aTmenable?

Our main result, Theorem 5.3, solves a special case of this problem. As its statement is technical we highlight the following attractive consequence:

Theorem 1.2. *Let G split as a countable graph of groups where each edge group is infinite cyclic, and each vertex group either acts properly and semisimply on \mathbb{H}^n or is a virtually special group. Then G is aTmenable.*

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2. MEASURED WALLSPACES AND ATMENABILITY

Definition 2.1 (Measured Wallspace). Let X be a set. A *wall* is a partition of X into two disjoint sets. A wall $\Lambda = H \sqcup H^c$ *separates* x and y if $H \cap \{x, y\}$ is a singleton. Let \mathcal{W} be a set of walls in X . Let μ be a measure on a σ -algebra \mathcal{B} of subsets of \mathcal{W} . Let $\omega(x, y) \subseteq \mathcal{W}$ be the subset of walls separating x and y . Then $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a *measured wallspace* if $\omega(x, y) \in \mathcal{B}$ for each $x, y \in X$.

The function $x, y \mapsto \mu(\omega(x, y))$ is a pseudo-metric on X called the *wall metric* which we denote by $\#(x, y)$. For subsets $A, B \subseteq X$ a wall $\Lambda = H \sqcup H^c$ *separates* A and B if either $A \subseteq H$ and $B \subseteq H^c$, or $A \subseteq H^c$ and $B \subseteq H$. Let $\omega(A, B)$ denote the set of walls separating A and B . If A, B are countable subsets, then $\omega(A, B) \in \mathcal{B}$ since $\omega(A, B) = \bigcap_{a \in A, b \in B} \omega(a, b) \in \mathcal{B}$. Let $\#(A, B) = \mu(\omega(A, B))$.

Example 2.2 (\mathbb{R} as a measured wallspace). Let $\mathcal{W}_{\mathbb{R}}$ be the collection of walls of \mathbb{R} where each $W \in \mathcal{W}_{\mathbb{R}}$ is of the form $(-\infty, r] \sqcup (r, \infty)$ for some $r \in \mathbb{R}$. There is a bijection $\mathcal{W}_{\mathbb{R}} \rightarrow \mathbb{R}$ so we pull back the Lebesgue measure λ and Borel algebra $\mathcal{B}_{\mathbb{R}}$ to obtain a measured wallspace: $(\mathbb{R}, \mathcal{W}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}}, \lambda)$.

Let \mathbf{R} be the cube complex homeomorphic to \mathbb{R} with 0-skeleton \mathbb{Z} . The measured wallspace $(\mathbb{R}, \mathcal{W}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}}, \lambda)$ is isomorphic mod-zero (see Definition 4.1) to the continuous measured wallspace on $(\mathbf{R}, \mathcal{W}, \mathcal{B}, \mu)$ described in Example 2.3 and also with the measured wallspace structure on \mathbb{H}^n in Examples 2.4 with $\mathbb{R} = \mathbb{H}^1$.

Example 2.3 (Cube complexes as measured wallspaces). A CAT(0) cube complex \tilde{X} has two natural measured wallspace structures: the discrete and continuous. The walls of the *discrete* wallspace correspond to the hyperplanes in \tilde{X} , and we use the counting measure. This is a measured wallspace since any two points in \tilde{X} are separated by a finite number of hyperplanes.

The *continuous* measured wallspace structure will be more appropriate for our purposes. Each hyperplane Λ of \tilde{X} lies in a convex subcomplex called its *carrier*, consisting of all cubes intersecting Λ . The carrier is isometric to $\Lambda \times [-1, 1]$ where Λ corresponds to $\Lambda \times \{0\}$. For each hyperplane Λ , and $\alpha \in (-1, 1)$ we let each $\Lambda \times \{\alpha\}$ be a wall. We give the set of walls $\{\Lambda \times \alpha : \alpha \in (-1, 1)\}$ the Lebesgue measure on $(-1, 1)$. Note that the measure of the set of walls separating two 0-cubes is identical in the discrete and continuous wallspaces.

Example 2.4 (\mathbb{H}^n as a measured wallspace). We refer to [Rob98] for the full details of the construction outlined here. Let \mathbb{H}^n denote n -dimensional real hyperbolic space. Let \mathcal{H} denote the set of all totally geodesic codimension-1 hyperplanes. Let $\Lambda_0 \in \mathcal{H}$. Every totally geodesic codimension-1 hyperplane in \mathbb{H}^n is in the orbit $\text{Isom}(\mathbb{H}^n)\Lambda_0$. Identifying \mathcal{H} with the set of left cosets $\text{Isom}(\mathbb{H}^n)/\text{Stabilizer}(\Lambda_0)$, we can endow \mathcal{H} with the quotient topology. The Haar measure μ for $\text{Isom}(\mathbb{H}^n)$ passes to an invariant measure on the quotient since both $\text{Isom}(\mathbb{H}^n)$ and $\text{Stabilizer}(\Lambda_0)$ are unimodular. It is immediate that μ is defined on the Borel sets $\mathcal{B}_{\mathbb{H}}$ of \mathcal{H} . As shown in [Rob98, CMV04], after scaling μ we have:

$$(1) \quad \#(x, y) = d_{\mathbb{H}}(x, y)$$

Metric properness of a group acting on $(\mathbb{H}^n, \mathcal{H}, \mathcal{B}_{\mathbb{H}}, \mu)$ is then equivalent to metric properness of the action on \mathbb{H}^n itself.

Amenable groups may have quite different measured wallspace structures from the examples above, and we have not found a way to apply the method of this paper to arbitrary graphs of groups with amenable vertex groups and cyclic edge groups.

Example 2.5. Let G be a countable amenable group. Let $\{K_n\}_{n=1}^\infty$ be an increasing, exhaustive sequence of finite subsets of G . One way to define the amenability of G is to require that there is a sequence of *Følner* sets $\{A_n\}_{n=1}^\infty$ such that A_n is a finite subset of G , and such that for each $g \in K_n$ we have:

$$\frac{|gA_n \Delta A_n|}{|A_n|} < 2^{-n}$$

For each n , the partition $A_n \sqcup A_n^c$ is a wall of G that is assigned a weight $n/|A_n|$. A measured wallspace is obtained by including all G -translates of these partitions. This is a measured wallspace and G acts metrically properly on it [CMV04].

Remark 2.6. A primary difficulty in generalizing the results of this paper to arbitrary graphs of amenable groups with cyclic edge groups arises from the rich diversity of amenable structures for \mathbb{Z} . An interesting test case would be an amalgamated free product $A *_\mathbb{Z} B$ where the amenable structure for \mathbb{Z} on the left and right are highly unrelated. Other difficulties related to the “dispersal” property discussed in Section 3, and dealing with appropriate fundamental domains.

Definition 2.7. Let $(X_1, \mathcal{W}_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{W}_2, \mathcal{B}_2, \mu_2)$ be measured wallspaces, and let $p_i : X_1 \times X_2 \rightarrow X_i$ be the projection. Then we can define the *product* of these two wallspaces, as in [CMV04], with $X = X_1 \times X_2$, $\mathcal{W} = p_1^{-1}(\mathcal{W}_1) \sqcup p_2^{-1}(\mathcal{W}_2)$, \mathcal{B} the σ -algebra on \mathcal{W} given by $\mathcal{B}_1 \sqcup \mathcal{B}_2$, and μ the unique measure on \mathcal{W} such that the restriction to \mathcal{B}_i is μ_i .

The following is proven in [CMV04]:

Lemma 2.8. *Suppose G admits a metrically proper action on a measured wallspace. Then G is amenable.*

The following is proven in [Jol00, Prop 2.5(1)].

Lemma 2.9. *Suppose G is N -by- Q with N amenable and Q amenable. Then G is amenable.*

3. DISPERSED SUBGROUP RELATIVE TO AN ACTION

Definition 3.1 (Dispersed). Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a measured wallspace. Let $\{Y_i\}_{i \in \mathbb{N}}$ be a collection of countable subsets of X . Then $\{Y_i\}_{i \in \mathbb{N}}$ is *dispersed* if for each $d > 0$ there exists $n > 0$ such that there do not exist $i_1 < \dots < i_n$, such that $\#(Y_{i_p}, Y_{i_q}) \leq d$ for all $p \neq q$.

Let G act on the measured wallspace $(X, \mathcal{W}, \mathcal{B}, \mu)$. Let H be a subgroup of G , and let $\{g_1, g_2, \dots\}$ be an enumeration of left-coset representatives of H in G . A subgroup $H \leq G$ is *dispersed relative to $x \in X$* if $\{g_i H x\}_{i \in \mathbb{N}}$ is dispersed.

Remark 3.2. Dispersal of a subgroup is relative to a choice of basepoint $x \in X$, and it is unclear if dispersal is invariant of the choice.

Let $x, y \in X$ and suppose that H is dispersed relative to x . Let $\{g_i\}_{i \in \mathbb{N}}$ be a sequence of elements of G such that $g_i H$ is an enumeration of all the cosets of H . As H is dispersed relative to x , we know $\#(Hx, g_i Hx) \rightarrow \infty$ as $i \rightarrow \infty$. As $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a wallspace we know $\#(x, y) = d$ is finite, so we can then deduce that $\#(Hx, Hy) \leq d$ and $\#(g_i Hx, g_i Hy) \leq d$. Furthermore, $\#(y, g_i y) \geq \#(x, g_i x) - \#(x, y) - \#(g_i x, g_i y) \geq \#(x, g_i x) - 2d$.

The principal difficulty of showing invariance of the basepoint is that the triangle inequality cannot be applied to $\#(Hy, g_i Hy)$. Indeed, $\omega(Hx, g_i Hx)$ can be partitioned as $\Omega_1^i \sqcup \Omega_2^i \sqcup \Omega_3^i$ where:

$$\Omega_1^i = \{\Lambda \in \omega(Hy, g_i Hy) \cap \omega(Hx, g_i Hx)\}$$

$$\Omega_2^i = \{\Lambda \in \omega(Hx, g_i Hx) \mid Hy \cup g_i Hy \text{ is contained in a single halfspace of } \Lambda\}$$

$$\Omega_3^i = \{\Lambda \in \omega(Hx, g_i Hx) \mid \Lambda \text{ non-trivially partitions either } Hy \text{ or } g_i Hy\}$$

To show H is dispersed relative to y it suffices to show that $\mu(\Omega_1^i) \rightarrow \infty$ as $i \rightarrow \infty$. We know $\mu(\Omega_2^i)$ is bounded for all i since walls in Ω_2^i either separate $g_i y$ from $g_i x$, or separate x from y . Therefore, showing the dispersal of H relative to y can be viewed as a question of estimating $\mu(\Omega_3^i)$. An upper bound on $\mu(\Omega_3^i)$ for all i , would imply that $\mu(\Omega_1^i)$ diverges to infinity. Hence dispersal of H relative to x would imply dispersal relative to y .

In Proposition 3.4 we prove that Ω_3^i is bounded for CAT(0) cube complexes in our case of interest. We expect this is true for \mathbb{H}^n as well.

For a subcomplex $Y \subseteq X$ of a CAT(0) cube complex, let $\mathcal{N}_m(\tilde{Y})$ denote the m -neighborhood of \tilde{Y} . The *convex hull* of a subset S of a CAT(0) cube complex X is the smallest convex subcomplex $\text{hull}(S)$ containing S . We refer to [HW08, Lem 13.15] for the following:

Lemma 3.3. *Let Y be a convex subcomplex of a finite dimensional CAT(0) cube complex X . For each $r \geq 0$ there exists a convex subcomplex Y^{+r} such that $\mathcal{N}_r(Y) \subset Y^{+r} \subset \mathcal{N}_s(Y)$ for some $s = \dim(X)r$.*

We refer to the convex subcomplex Y^{+r} as the r -thickening of Y .

Proposition 3.4. *Suppose G acts properly and cocompactly on a CAT(0) cube complex X . Let $H \leq G$ be a subgroup that acts cocompactly on a nonempty convex subcomplex $Y \subseteq X$. If H is dispersed relative to a 0-cube $x \in X$, then H is dispersed relative to any 0-cube $x' \in X$.*

Proof. Without loss of generality, we may assume that $Y = \text{hull}(Hx)$. Indeed, we may assume that $x \in Y$ by replacing Y with $Y^{+\ell}$ for some ℓ , and thereafter that $Y = \text{hull}(Hx)$ by replacing Y with the convex hull of Hx in Y .

Let $Y' = \text{hull}(Hx')$. The hyperplanes intersecting $\text{hull}(Hx')$ are precisely the hyperplanes that non-trivially partition Hx' . The hyperplanes that separate Hx and gHx are precisely the hyperplanes that separate Y and gY , or equivalently the hyperplanes on a shortest geodesic between Y and gY . Following Remark 3.2, it suffices to bound Ω_3 ; that is to bound the number of hyperplanes separating Y, gY , but intersecting Y' or gY' . By symmetry it suffices to bound those intersecting Y' .

Observe that $Y' \subset Y^{+m}$ for some m . Choose a shortest geodesic σ between gY and Y^{+m} , and orient σ so that it ends at a point $v \in Y^{+m}$. Let $u \in Hx$ be a point with $v \in \mathcal{N}_m(u)$, and let σ' be a geodesic from v to u . Observe that any hyperplane Λ separating Y, gY and intersecting Y' must intersect σ' . Indeed, if Λ separates Y, gY then Λ must be crossed by the concatenation $\sigma\sigma'$. However, each hyperplane crossed by σ is disjoint from Y^{+m} and hence from Y' . Thus Λ must be crossed by σ' , and so the number of such hyperplanes is bounded by $|\sigma'| \leq m$. \square

Lemma 3.5. *Let $G_3 \leq G_2 \leq G_1$, and let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a measured wallspace with a G_1 action. If $G_2 \leq G_1$ is dispersed relative to $x \in X$, and $G_3 \leq G_2$ is dispersed relative to $x \in X$ then $G_3 \leq G_1$ is dispersed relative to $x \in X$.*

Conversely, if $G_3 \leq G_1$ is dispersed relative to $x \in X$, then $G_3 \leq G_2$ is dispersed relative to $x \in X$.

Proof. In the first case, suppose that G_3 is not a dispersed subgroup of G_1 relative to x . Then there exists a sequence $\{g_i\}_{i \in \mathbb{N}}$ such that $g_i G_3 \neq g_j G_3$ for $i \neq j$, and $\#(g_i G_3 x, G_3 x) < D$ for all $i \in \mathbb{N}$. If there are infinitely many $g_i G_3$ contained in a single left coset of G_2 , then this contradicts the dispersal of G_3 in G_2 relative to x . Otherwise, there must be infinitely many $g_i G_3$ contained in infinitely many distinct G_2 cosets, which contradicts the dispersal of G_2 in G_1 relative to x .

The converse case follows from the observation that the G_2 cosets of G_3 are a subset of the G_1 cosets of G_3 . \square

Lemma 3.6. *Let G act on a finite dimensional $CAT(0)$ cube complex X . Let $H \leq G$. Let $G' \leq G$ be a finite index subgroup, and let $H' = (G' \cap H)$. Suppose that $H' \leq G'$ is dispersed with respect to $x \in X$. Then $H \leq G$ is dispersed relative to $x \in X$.*

Proof. The claim follows by showing that for all $k > 0$ there exists $K > 0$ such that if $g_a H'$ and $g_b H'$ are distinct left cosets, then $\#(g_a H' x, g_b H' x) > K$ implies that $\#(g_a H x, g_b H x) > k$. Indeed, first observe by Lemma 3.5 that if $H' \leq G'$ is dispersed relative to some 0-cube, then so is $H' \leq G$. Now if $g_1 H, \dots, g_m H$ are distinct cosets with $m = m(K)$, then so are $g_1 H', \dots, g_m H'$, so there exists $\#(g_a H' x, g_b H' x) > K$, and hence $\#(g_a H x, g_b H x) > k$.

Suppose that $\#(g_a H' x, g_b H' x) = k$, for k sufficiently large. Let $Y_a = \text{hull}(g_a H' x)$, and let $Y_b = \text{hull}(g_b H' x)$. Note that Y_a and Y_b are separated by k walls.

There exists $r = r(H, H', x)$ such that $Hx \subseteq \mathcal{N}_r(H' x)$, and therefore $Hx \subseteq (Y_a)^{+r}$. By Lemma 3.3 we observe that $(Y_a)^{+r} \subseteq \mathcal{N}_s(Y_a)$ for $s = \dim(X)r$. We thus find that the number of walls separating $g_a Hx$ and $g_b Hx$ is at least $k - 2s$. Indeed, if σ is a combinatorial geodesic joining $(Y_a)^{+r}$ and $(Y_b)^{+r}$, then it has length at least $k - 2s$ and the walls intersecting σ separate the convex subcomplexes $(Y_a)^{+r}$ and $(Y_b)^{+r}$.

In conclusion, given $K > 0$, by letting $k = K + 2s = K + 2r \dim(X)$ we ensure $\#(g_a H' x, g_b H' x) > K$ implies $\#(g_a H x, g_b H x) > k$. \square

Although Lemma 3.6 only applies in the setting when the measured wallspace is a finite dimensional $CAT(0)$ cube complex, the following converse applies to general measured wallspaces.

Lemma 3.7. *Let G act on a measured wallspace $(X, \mathcal{W}, \mathcal{B}, \mu)$. Suppose that $H \leq G$ is dispersed relative to $x \in X$. Let $G' \leq G$, and $H' = H \cap G'$. Then $H' \leq G'$ is dispersed relative to $x \in X$.*

Proof. Suppose that $H' \leq G'$ is not dispersed relative to $x \in X$. Then there exists $D > 0$ and arbitrarily large collections of $g_1, \dots, g_m \in G'$ such that $g_i H' \neq g_j H'$ and $\#(g_i H' x, g_j H' x) < D$ for $1 \leq i < j \leq m$. Then $g_i H \neq g_j H$ for $i \neq j$, otherwise $g_j^{-1} g_i \in G' \cap H = H'$, which would imply that $g_i H' x = g_j H' x$. Thus there are arbitrarily large collections of left cosets $g_1 H, \dots, g_m H$ such that $\#(g_i H' x, g_j H' x) < D$ for $1 \leq i < j \leq m$. \square

Proposition 3.8. *Let G be a finitely generated abelian group, and let H be a subgroup of G . Then G acts on a $CAT(0)$ cube complex X such that H is dispersed relative to any 0-cube x in X .*

Proof. Let G be a rank n abelian group, and H a rank k subgroup. Fix a proper action of G on \mathbb{R}^n . Choose $g_{k+1}, \dots, g_n \in G$ such that $\langle H, g_{k+1}, \dots, g_n \rangle$ is commensurable with G . By Lemma 3.5, and the fact that finite index subgroups are always dispersed relative to any action, we can assume that $G = \langle H, g_{k+1}, \dots, g_n \rangle$.

Let $\Lambda_i \subseteq \mathbb{R}^n$ be the hyperplane stabilized by $\langle H, g_{k+1}, \dots, g_{i-1}, g_{i+1}, \dots, g_n \rangle$, for $k_1 \leq i \leq n$. As each Λ_i is a codimension-1 hyperplane, Λ_i defines a wall in \mathbb{R}^n . Taking the union \mathcal{W} of all Λ_i and their G -translates, we obtain a wallspace $(\mathbb{R}^n, \mathcal{W})$. Let X be the dual cube complex $C(\mathbb{R}^n, \mathcal{W})$. Observe that $C(\mathbb{R}^n, \mathcal{W}) = \mathbf{R}^{n-k}$, there is only one 0-cube orbit, and that the stabilizer in G of any 0-cube is precisely H . Therefore, the dispersal of H is equivalent to the properness of \mathbf{R}^{n-k} . \square

Lemma 3.9. *Let G act properly and cocompactly on a $CAT(0)$ cube complex X . Let H be a subgroup that acts properly and cocompactly on a convex subcomplex Y . Then H is dispersed relative to any 0-cube $y \in Y$.*

Proof. By [WW15, Lem 2.4] the subgroup H has bounded packing in G . Let $G/H = \{g_1 H, g_2 H, \dots\}$ be the collection of cosets of H . Bounded packing of H in G means that for all $d > 0$ there exists $n > 0$ such that there do not exist $i_1 < \dots < i_n$ with $d_G(g_{i_p} H, g_{i_q} H) \leq d$ for all $p \neq q$. Since G acts properly and cocompactly on X , and H acts cocompactly on Y , we deduce the analogous statement that: for all $d > 0$ there exists $n > 0$ such that there do not exist $i_1 < \dots < i_n$ with $d_X(g_{i_p} Y, g_{i_q} Y) \leq d$ for all $p \neq q$.

Suppose that $d_X(g_1 Y, g_2 Y) > 0$. Let γ be a combinatorial path in X with end-points on $g_1 Y, g_2 Y$, such that $|\gamma| = d_X(g_1 Y, g_2 Y)$. As Y is convex, a hyperplane dual to γ cannot intersect $g_1 Y$ or $g_2 Y$. Therefore, $\#(g_1 Y, g_2 Y) = d_X(g_1 Y, g_2 Y)$.

Thus, the bounded packing of H in G implies the dispersion of $\{g_i Y\}_{i \in \mathbb{N}}$, and therefore the dispersal of H relative to any 0-cube $y \in Y$. \square

Corollary 3.10. *Let G be a hyperbolic group that acts properly and cocompactly on a $CAT(0)$ cube complex X with either the continuous or discrete measured wallspace structure. Let H be a quasiconvex subgroup. Then H is dispersed relative to any 0-cube x in X .*

Proof. By [Hag06, SW15], there exists an H -cocompact convex subcomplex $Y \subset X$. We may thus apply Lemma 3.9 and Proposition 3.4. \square

Lemma 3.11. *Let $\gamma \subseteq \mathbb{H}^n$ be a biinfinite geodesic, and let $\theta \in (0, \pi/2)$. The set of hyperplanes intersecting γ at angle $< \theta$ is measurable.*

Let $[a, b]$ denote the set of points in the geodesic in \mathbb{H}^n joining a and b .

Proof. Observe that the set of all walls intersecting γ is measurable since it is equal to the ascending countable union of measurable sets $\omega(\gamma(-n), \gamma(n))$. Enumerate the dense, countable set $\{a_i\}_{i \in \mathbb{N}} \subseteq \gamma(\mathbb{Q})$. Let $a_i = \gamma(t_i)$. Let $\omega(\gamma(-\infty), a_i)$ denote the measurable set $\bigcup_{n < t_i} \omega(\gamma(n), a_i)$. Let $p_\gamma : \mathbb{H}^n \rightarrow \mathbb{H}^n$ be the projection map onto γ .

Let $x_i = \gamma(t_i + 1)$. Enumerate a countable subset of points $\{x_{ij}\}_{j \in \mathbb{N}} \subseteq p_\gamma^{-1}(x_i)$ that is dense in $p_\gamma^{-1}(x_i)$, and such that $d_{\mathcal{H}}(x_{ij}, x_i) \leq \tanh^{-1}(\tan(\theta) \sinh(1))$. The points a_i, x_i, x_{ij} form a right angle triangle in \mathbb{H}^n containing the unit segment $[a_i, x_i] \subseteq \gamma$, and hypotenuse $[a_i, x_{ij}]$ that meets γ at an angle less than θ . If $\Lambda \in \omega(\gamma(-\infty), a_i) \cap \omega(x_i, x_{ij})$, there is right angled triangle with vertices $\Lambda \cap \gamma, x_i$, and $[x_i, x_{ij}] \cap \Lambda$. Let ζ be the angle at $\Lambda \cap \gamma$. Since the side adjacent to ζ contains $[a_i, x_i]$, and the side opposite ζ is contained in $[x_i, x_{ij}]$, we can deduce that $\zeta < \theta$. Therefore, $\omega(\gamma(-\infty), a_i) \cap \omega(x_i, x_{ij})$ only contains walls that intersect $[\gamma(-\infty), a_i]$ at an angle of less than θ . Let

$$B_\theta^\infty = \bigcup_{i, j \in \mathbb{N}} \omega(\gamma(-\infty), a_i) \cap \omega(x_i, x_{ij}).$$

Since B_θ^∞ is the countable union of measurable sets, B_θ^∞ is a measurable set.

Let Λ be a hyperplane intersecting γ at an angle less than θ . Let $\alpha = \Lambda \cap \gamma$. Since $\{a_i\}_{i \in \mathbb{N}}$ is dense in γ there exists a subsequence $\{a_{i_k}\}_{k \in \mathbb{N}} \subseteq [\gamma(-\infty), \alpha]$ converging to α . Note that for k sufficiently large, Λ will non-trivially intersect $p^{-1}(x_{i_k})$. Since $\{x_{i_k j}\}_{j \in \mathbb{N}}$ are also dense in $p_\gamma^{-1}(x_{i_k})$, for k sufficiently large, there exist points a_{i_k} and $x_{i_k j}$ such that $\Lambda \in \omega(\gamma(-\infty), a_{i_k}) \cap \omega(x_{i_k j}, p_\gamma(x_{i_k j}))$. Therefore B_θ^∞ is precisely the set of all hyperplanes intersecting γ at an angle of less than θ . \square

Lemma 3.12. *Let G act discretely on \mathbb{H}^n . Consider the associated action on $(\mathbb{H}^n, \mathcal{H}, \mathbb{B}_{\mathbb{H}}, \mu)$. A cyclic subgroup H of G is dispersed relative to a point in the axis A of H .*

Proof. Let $\{g_i H\}_{i \in \mathbb{N}}$ be the left-cosets of H . To show that H is dispersed relative to $a \in A$, it suffices to show that $\{g_i A\}_{i \in \mathbb{N}}$ is dispersed. As H has bounded packing in G , for each d there exists m such that in any cardinality m subcollection $Q \subseteq \{g_i A\}_{i \in \mathbb{N}}$ there exists $g_p A, g_q A \in Q$ with $d_{\mathbb{H}}(g_p A, g_q A) > d$. It therefore suffices to show that there exists d_0 so that for biinfinite geodesics A_1, A_2 we have

$$(2) \quad \#(A_1, A_2) \geq \frac{1}{2} [d_{\mathbb{H}}(A_1, A_2) - 2d_0].$$

Let P be a geodesic between A_1 and A_2 , realizing $d_{\mathbb{H}}(A_1, A_2)$. We will first show that for any wall Λ cutting P , that does not separate A_1, A_2 either $\angle(\Lambda, P)$ is small or Λ intersects P close to A_1 or A_2 . We will secondly show that the measure of each of these subsets is a controlled part of the measure of the walls intersecting P .

We first consider how the angle at which Λ intersects P relates to the distance of $\Lambda \cap P$ from A_1 and A_2 . Note that P meets each A_i at a right angle. Consider an ideal hyperbolic triangle with angles $\theta_0, \frac{\pi}{2}, 0$, and let d_0 be the length of the finite

base side. Thus $\sinh(d_0) = \cot(\theta_0)$. Any finite hyperbolic triangle with angles $\theta, \frac{\pi}{2}$ meeting at a base of length $\geq d_0$, must satisfy $\theta < \theta_0$. Therefore, if $p \in P$ is at a distance more than d_0 from A_i , then a hyperplane Λ in \mathbb{H}^n intersecting p will not intersect A_i provided $\angle(P, \Lambda) \geq \theta_0$.

We now estimate the measures. Let ab be a length 1 geodesic segment in \mathbb{H}^n and $c \in ab$. Each hyperplane Λ with $\Lambda \cap ab = c$ has a unique geodesic perpendicular to Λ at c . Therefore the set of all such hyperplanes maps bijectively to the open ball of radius $\frac{\pi}{2}$ in \mathbb{R}^{n-1} , denoted $B_{\frac{\pi}{2}}$. Let $B_\theta \subset B_{\frac{\pi}{2}}$ denote the set of hyperplanes intersecting ab at an angle less than θ . We may thus identify the set of all hyperplanes transversely intersecting ab with $ab \times B_{\frac{\pi}{2}}$. By Equation (1) we have $\#(a, b) = d_{\mathbb{H}}(a, b)$. Note that $B_{\frac{\pi}{2}} = \omega(a, b)$ is a measurable set. By Lemma 3.11, we deduce that $B_{\frac{\pi}{2}} - B_\theta$ is also measurable. The measure of the set of hyperplanes intersecting the geodesic ab at an angle of at least $\theta \in (0, \frac{\pi}{2})$ is therefore equal to $\mu(ab \times (B_{\frac{\pi}{2}} - B_\theta))$. As $\mu(ab \times (B_{\frac{\pi}{2}} - B_\theta)) \rightarrow d_{\mathbb{H}}(a, b)$ as $\theta \rightarrow 0$, there exists $\theta_0 > 0$ such that $\mu(ab \times (B_{\frac{\pi}{2}} - B_{\theta_0})) \geq \frac{1}{2} d_{\mathbb{H}}(a, b) = \frac{1}{2}$.

Equation (2) follows by considering a maximal integral length subpath of P that is at least d_0 from each A_i . \square

A f.g. free abelian subgroup $A \leq G$ is *highest* if A does not have a finite index subgroup contained in a higher rank free abelian subgroup of G . Note that if G acts properly on a finite dimensional CAT(0) cube complex, then every free abelian subgroup of G is contained in a highest subgroup of G .

We proved the following in [WW15]:

Lemma 3.13. *Let X be a compact nonpositively curved cube complex. Let A be a highest abelian subgroup of $\pi_1 X$. There is a compact based nonpositively curved cube complex Y and a local isometry $Y \rightarrow R$ with $\pi_1 Y = A$.*

The subgroup $A_i \leq G$ is a *virtual retract* if there is a finite index subgroup $G_i \leq G$ and a retraction $G_i \rightarrow A'_i$ to a finite index subgroup of $A'_i \leq A_i$. G is *virtually special* if there is a finite index subgroup $G' \leq G$ such that G' is isomorphic to a subgroup of a right-angled Artin group (raag).

Proposition 3.14. *Let G be finitely generated and virtually special. Let Z_1, \dots, Z_n be infinite cyclic subgroups of G . Then G acts metrically properly on a CAT(0) cube complex X such that $Z_i \leq G$ is dispersed relative to any 0-cube in X .*

Proof. Let $G' \leq G$ be an index d subgroup such that G' is special and hence $G' \hookrightarrow \pi_1 R$, where R is a Salvetti complex for a right angled Artin group. Moreover, since G' is finitely generated we may assume that $\pi_1 R$ is finitely generated and hence R is compact. Let $Z'_i = Z_i \cap G'$. Let $A_i \leq \pi_1 R$ be a highest free abelian group containing Z'_i . By Lemma 3.13 there exists a based local isometry $Y_i \rightarrow R$ from compact non-positively curved cube complex Y_i such that $\pi_1 Y_i = A_i$. Applying canonical completion and retraction [HW08] there is a degree d_i cover $R_i \rightarrow R$ such that there is a retraction $r_i : \pi_1 R_i \rightarrow A_i$. By Lemma 3.8 A_i acts on a CAT(0) cube complex X_i such that $Z'_i \leq A_i$ is dispersed relative to any choice of 0-cube. Using the retraction r_i there is an action of $\pi_1 R_i$ on X_i that extends the action of A_i , and hence

an action of $\pi_1 R$ on $(X_i)^{d_i}$ such that $Z'_i \leq A_i$ is dispersed relative to any 0-cube in $(X_i)^{d_i}$. So G' acts on $(X_i)^{d_i}$, and by Lemma 3.5 we deduce that $Z'_i \leq (G' \cap A_i)$ is dispersed relative to any 0-cube in $(X_i)^{d_i}$. By Lemma 3.9 and Proposition 3.4 we know that $\pi_1 R$ acts metrically properly on \tilde{R} such that $A_i \leq \pi_1 R$ is dispersed relative to any 0-cube. Therefore Lemma 3.7 implies that G' acts metrically properly on \tilde{R} such that $G' \cap A_i \leq G'$ is dispersed relative to any 0-cube in \tilde{R} . Thus, G' acts diagonally on $\tilde{R} \times \prod_{i=1}^n (X_i)^{d_i}$ such that the action is metrically proper and by Lemma 3.5 $Z'_i \leq G'$ is dispersed relative to any 0-cube. Finally G acts metrically properly on $(\tilde{R} \times \prod_{i=1}^n (X_i)^{d_i})^d$ such that $Z'_i \leq G$ is dispersed relative to any 0-cube in $(\tilde{R} \times \prod_{i=1}^n (X_i)^{d_i})^d$, and therefore by Lemma 3.6 $Z_i \leq G$ is dispersed relative to any 0-cube in $(\tilde{R} \times \prod_{i=1}^n (X_i)^{d_i})^d$. \square

4. STANDARD PROBABILITY SPACES

Definition 4.1. An isomorphism $f : (\Omega_1, \mathcal{B}_1, \mu_1) \rightarrow (\Omega_2, \mathcal{B}_2, \mu_2)$ between measure spaces is an invertible map $f : \Omega_1 \rightarrow \Omega_2$ such that $f(U) \in \mathcal{B}_2$ if and only if $U \in \mathcal{B}_1$, and moreover $\mu_1(U) = \mu_2(f(U))$. We say (Ω_1, μ_1) and (Ω_2, μ_2) are isomorphic *mod-zero* if there is an isomorphism between (Ω'_1, μ'_1) and (Ω'_2, μ'_2) where each (Ω'_i, μ'_i) is obtained by removing a nullset.

A probability space $(\Omega, \mathcal{B}, \mu)$ is *standard* if it is isomorphic mod-zero to $([0, 1], \lambda)$. We caution that standard probability spaces are sometimes defined in a more general fashion that permits them to contain atoms. We refer to [dlR93] for the following:

Theorem 4.2 (Topological Characterization of Standard). *Suppose $(\Omega, \mathcal{B}, \mu)$ has no atoms. Then $(\Omega, \mathcal{B}, \mu)$ is standard if and only if there exists a topology τ on Ω such that the following hold:*

- (1) (Ω, τ) is metrizable.
- (2) \mathcal{B} is the completion of the σ -algebra generated by τ .
- (3) $\sup \mu(K) = 1$ where the supremum is taken over all compact sets K .

The following generalization of standard plays an important role in Section 5:

Definition 4.3 (\mathbb{Z} -standard measure space). Let (Ω, μ) be a measure space with a \mathbb{Z} action. We say (Ω, μ) is \mathbb{Z} -standard if there is a \mathbb{Z} -equivariant mod-zero isomorphism to $(\mathbb{R}, \varrho\lambda)$ where λ is the Lebesgue measure, and $\varrho > 0$.

Lemma 4.4 (Fundamental Domain). *Let (Ω, μ) be a measure space with a \mathbb{Z} -action. Suppose it has a measurable fundamental domain $D \subset \Omega$ that is standard, after scaling the measure. Then (Ω, μ) is \mathbb{Z} -standard.*

Proof. The mod-zero isomorphism $f : (D, \mu) \rightarrow ([0, \mu(D)], \lambda)$ extends to a \mathbb{Z} -equivariant mod-zero isomorphism $f : (\Omega, \mu) \rightarrow (\mathbb{R}, \lambda)$. \square

Example 4.5. The continuous measured wallspace $(\mathbf{R}, \mathcal{W}_{\mathbf{R}}, \mathcal{B}, \mu)$ of Example 2.2, with its natural \mathbb{Z} -action is \mathbb{Z} -standard with fundamental domain $\omega(0, 1)$.

Example 4.6. Let \mathbb{Z} act freely on a CAT(0) cube complex X , and let $(X, \mathcal{W}_X, \mathcal{B}_X, \mu_X)$ be the associated continuous measured wallspace of Example 2.3. Suppose \mathbb{Z} stabilizes a combinatorial axis A . Consider the measure space whose elements are

the walls intersecting A , and whose measurable subsets, and measure is induced from the measured wallspace structure on \tilde{X} . This will be isomorphic mod-zero to $(\mathbf{R}, \mathcal{W}_{\mathbf{R}}, \mathcal{B}_{\mathbf{R}}, \lambda)$. Then $(X, \mathcal{W}_X, \mathcal{B}_X, \mu_X)$ is \mathbb{Z} -standard.

Example 4.7. Recall the measured wallspace structure on \mathbb{H}^n in Example 2.4. In Lemma 3.12, we identified the subset of walls transversely intersecting a finite geodesic in \mathbb{H}^n with the points in $[a, b] \times B^{n-1}$, where B^{n-1} is the open ball of dimension n , but we did not determine the topology on the set of such walls. We will now show that the subspace of walls intersecting a bi-infinite geodesic is homeomorphic to $\mathbb{R} \times B^{n-1}$. Let $\gamma \subseteq \mathbb{H}^n$ be a bi-infinite geodesic, and let Λ be a hyperplane such that $\gamma \cap \Lambda = \{p\}$. There is a subspace $R \subset \text{Isom}(\mathbb{H}^n)$, consisting of rotations at p , with R homeomorphic to the open ball B^{n-1} such that $R\Lambda$ is the set of all hyperplanes intersecting γ transversely at p . Let $U_\gamma \subseteq \mathcal{H}$ be the subspace consisting of hyperplanes intersecting γ transversely. Each hyperplane in U_γ is obtained by rotating Λ by an element of R and then translating along γ , and conversely each such transformation gives a unique hyperplane in U_γ . Therefore, there is a subspace $V_\gamma \subseteq \text{Isom}(\mathbb{H}^n)$ that is homeomorphic to $\mathbb{R} \times B^{n-1}$, that injects in $\text{Isom}(\mathbb{H}^n)/\text{Stabilizer}(\Lambda)$ such that $V_\gamma\Lambda = U_\gamma$. As in Example 3.12, the subset $U_\gamma \subseteq \mathcal{H}$ can be identified with the set $\mathbb{R} \times B^{n-1}$, and is the image of V_γ in $\mathcal{H} = \text{Isom}(\mathbb{H}^n)/\text{Stabilizer}(\Lambda)$.

Note that $\text{Isom}(\mathbb{H}^n)/\text{Stabilizer}(\Lambda)$ is Hausdorff since $\text{Stabilizer}(\Lambda)$ is a closed subgroup of a Lie Group. Thus U_γ is Hausdorff. Hence, the restriction $V_\gamma \rightarrow U_\gamma$ is a topological embedding on each compact set. Since V_γ is locally compact, we see that $V_\gamma \rightarrow U_\gamma$ is also an open map, and we conclude it is a homeomorphism. We conclude that U_γ is homeomorphic to $\mathbb{R} \times B^{n-1}$.

Observe that there are no atoms in this measure space. Let $Z \subset \text{Isom}(\mathbb{H}^n)$ be a cyclic group acting freely on γ and hence on U_γ . Then Z preserves the product structure and acts freely on the \mathbb{R} -factor with some fundamental domain (a, b) , and hence the action of Z on U_γ has fundamental domain $(a, b) \times B^{n-1}$. As observed in Example 2.4, the Borel sets are measurable. Since $(a, b) \times B^{n-1}$ is metrizable, complete, and separable it is a standard probability space by Theorem 4.2. Consequently, $(U_\gamma, \mathcal{B}_{\mathbb{H}} \cap U_\gamma, \mu)$ is \mathbb{Z} -standard.

5. MAIN THEOREM

5.1. Graphs of Groups with Measured Wallspaces.

Definition 5.1 (Cut, Skim, Disjoint). Let $(X, \mathcal{W}, \mathcal{B}, \mu)$ be a measured wallspace with a G -action, for some group G . Let $x \in X$ and $\Lambda \in \mathcal{W}$. The wall Λ *cuts* Gx if each halfspace of Λ contains infinitely many elements of Gx . The wall Λ *skims* Gx if one halfspace contains a finite, non-empty subset of Gx . The wall Λ is *disjoint* from Gx if one halfspace is empty. Note that if G is countable, the sets of walls that separate, skim, and are disjoint from Gx belong to \mathcal{B} .

For $(X, \mathcal{W}, \mathcal{B}, \mu)$ and $\lambda > 0$, we define the *scaled measured wallspace* $(X, \mathcal{W}, \mathcal{B}, \lambda\mu)$ with property that its measure has been scaled by λ .

Definition 5.2. A splitting of a group G as a *graph of groups with measured wallspaces* consists of a simplicial graph of groups Γ such that:

- (1) Each vertex group G_v acts metrically properly on a measured wallspace $(X_v, \mathcal{W}_v, \mathcal{B}_v, \mu_v)$.
- (2) Each edge group G_e acts metrically properly on a measured wallspace $(X_e, \mathcal{W}_e, \mathcal{B}_e, \mu_e)$.
- (3) For each edge e incident to v , there is a chosen $x_v^e \in X_v$. Let $\mathcal{W}_v^e \subseteq \mathcal{W}_v$ denote the set of walls that cut $G_e x_v^e$. Let $\mathcal{B}_v^e \subset \mathcal{B}_v$ be the σ -algebra consisting of the subsets of \mathcal{W}_v^e .
- (4) The set of walls in \mathcal{W}_v that skim $G_e x_v^e$ have measure zero.
- (5) For each edge e adjacent to v , let $\varphi_v^e : G_e \rightarrow G_v$ denote the corresponding inclusion. There is $\varrho_v^e > 0$ and a G_e -equivariant mod-zero isomorphism $\phi_v^e : (\mathcal{W}_e, \mathcal{B}_e, \varrho_v^e \mu_e) \rightarrow (\mathcal{W}_v^e, \mathcal{B}_v^e, \mu_v)$.

Note that the requirement that Γ be simplicial is purely for notational convenience; if both endpoints of an edge were at the same vertex it would be inconvenient to specify the attaching maps. In any case, by possibly passing to an index 2 subgroup, one can always ensure simpliciality while maintaining the hypotheses of the following result which is our main theorem.

Theorem 5.3. *Let G split as a graph Γ of groups with measured wallspaces and:*

- (1) *Each edge group $G_e = \langle g_e \rangle$ is infinite cyclic.*
- (2) *$(\mathcal{W}_e, \mathcal{B}_e, \mu_e)$ has fundamental domain $\omega(x_e, g_e x_e)$.*
- (3) *The image $\phi_v^e(\omega(x_e, g_e x_e)) = \omega(x_v^e, \varphi_v^e(g_e) x_v^e)$ for all edges e incident to a vertex v .*
- (4) *The Bass-Serre tree T has countably many vertices.*
- (5) *The subgroup $G_e \leq G_v$ is dispersed relative to $x_v^e \in X_v$.*

Then G is aTmenable.

Remark 5.4. Condition (2) is equivalent to following condition: almost all Λ partition $G_e x_v^e$ as $\{g_e^n x_v^e\}_{n>M} \sqcup \{g_e^n x_v^e\}_{n \leq M}$ for some $M \in \mathbb{Z}$ depending on Λ, v, e .

Remark 5.5. Condition (3) can be replaced by the following more general condition which is not required for our applications: For any edge $e \in \Gamma$ with endpoints v_1, v_2 , we require that $\mu_e(L(v_1, e) \cap R(v_2, e)) < \infty$ and $\mu_e(L(v_2, e) \cap R(v_1, e)) < \infty$ where:

$$R(v, e) = \left\{ \Lambda \in \mathcal{W}_e \mid \phi_v^e(\Lambda) \in \bigcup_{m \in \{0, 1, 2, \dots\}} \omega(\varphi_v^e(g_e^m) x_v^e, \varphi_v^e(g_e^{m+1}) x_v^e) \right\}$$

$$L(v, e) = \left\{ \Lambda \in \mathcal{W}_e \mid \phi_v^e(\Lambda) \in \bigcup_{m \in \{0, 1, 2, \dots\}} \omega(\varphi_v^e(g_e^{-m}) x_v^e, \varphi_v^e(g_e^{-(m+1)}) x_v^e) \right\}$$

This is what would ensure that $\omega(x_{\tilde{v}_{i-1}}^{\tilde{e}_i}, x_{\tilde{v}_i}^{\tilde{e}_i})$ is a finite measurable set in Lemma 5.11.

Definition 5.6 (Monic). If a group G satisfies the hypotheses of Theorem 5.3, and has the additional property that each $\varrho_v^e = 1$, then we say that the splitting of G is *monic*. Much of our discussion will focus on the monic special case.

Before outlining the proof of Theorem 5.3, we show that it implies Theorem 1.2.

Proof of Theorem 1.2. Suppose that G splits as a graph of groups Γ . We will show that G satisfies the criterion of Theorem 5.3. Each edge group $G_e \cong \mathbb{Z}$ acts on $(\mathbf{R}, \mathcal{W}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}}, \mu)$ with fundamental domain $\omega(0, 1)$.

If G_v acts properly and semisimply on \mathbb{H}^n , then we let $(\mathbb{H}^n, \mathcal{H}, \mathcal{B}_{\mathbb{H}}, \mu)$ from Example 2.4 be the associated wallspace. If the edge e in Γ is incident to v , we let $x_v^e \in \mathbb{H}^n$ be a point in the geodesic axis of G_e .

Note that every wall in \mathcal{H} is either disjoint from, or intersects in precisely one point the geodesic containing x_v^e , hence Condition (2) holds in this case. By Lemma 3.12, the G_e is dispersed relative to x_v^e in their geodesic axis. If G_v is virtually special, then by Proposition 3.14 there is a CAT(0) cube complex X_v such that each edge group G_e is dispersed relative to a basepoint x_v^e in a combinatorial axis for G_e . Therefore, Condition (5) holds in the respective cases. And we let $(X_v, \mathcal{W}_v, \mathcal{B}_v, \mu_v)$ be the continuous wallspace of Example 2.3. Combinatorial axes are intersected by hyperplanes at most once. Hence, as x_v^e is contained in a combinatorial geodesic axis Condition (2) holds.

As described in Examples 4.6 and 4.7 the walls intersecting a geodesic axis stabilized by the edge group are \mathbb{Z} -standard, and have fundamental domains given by the walls separating two consecutive points in an orbit. Hence, for each edge e incident to a vertex v , there is a G_e -equivariant isomorphism ϕ_v^e that satisfies Condition (2). Moreover, we can choose ϕ_v^e such that the image $\phi_v^e(\omega(x_e, g_e x_e)) = \omega(x_v^e, \varphi_v^e(g_e) x_v^e)$, therefore satisfying Condition (3).

Thus G is aTmenable by Theorem 5.3. \square

5.2. Outline of Proof of Theorem 5.3. The proof of Theorem 5.3 is broken up into the following steps: We describe a homomorphism $G \rightarrow \mathbb{R}^*$ in Lemma 5.7. By Lemma 2.9 and the fact that subgroups of \mathbb{R}^* are amenable, it suffices to show that its kernel G' is aTmenable. By design, G' has the attractive property that its splitting is monic. We prove the monic special case of Theorem 5.3 by first constructing a measured wallspace $(X, \mathcal{W}, \mathcal{B}, \mu)$ that G acts on. Then we show that G acts metrically properly on $(X, \mathcal{W}, \mathcal{B}, \mu)$ in Proposition 5.13. The aTmenability of G then follows from Lemma 2.8.

5.3. The modular homomorphism. Our initial goal is the following result:

Lemma 5.7. *Let G split as a graph of groups with measured wallspaces satisfying the hypotheses of Theorem 5.3. There is a homomorphism $G \rightarrow \mathbb{R}^*$ whose kernel G' is monic.*

Proof. We orient the edges of Γ to obtain a directed graph. For an edge e from u to v we use the notation $\varrho_e^+ = \varrho_v^e$ and $\varrho_e^- = \varrho_u^e$. To each directed edge we associate the nonzero real weight $w(e) = \frac{\varrho_e^+}{\varrho_e^-}$, and let $w(e^{-1}) = w(e)^{-1}$. The modular homomorphism $f : \pi_1 \Gamma \rightarrow \mathbb{R}^*$ is induced by the function that maps a closed combinatorial path $e_{i_1} \cdots e_{i_n}$ to $w(e_{i_1}) \cdots w(e_{i_n})$. Let $\hat{\Gamma}$ denote the covering space of Γ associated to the kernel of f . Let $G' \leq G$ be the subgroup associated to $\hat{\Gamma}$.

Assign a copy of the measured wallspace $(X_v, \mathcal{W}_v, \mathcal{B}_v, \mu_v)$ to \hat{v} , where \hat{v} lies in the fiber of $v \in \Gamma$, and similarly for each edge space and edge space inclusion. For each edge \hat{e} incident to a vertex \hat{v} we let $x_{\hat{e}}$ be the point corresponding to $x_e \in X_v$, where \hat{e} lies in the fiber of e and \hat{v} in the fiber of v . After choosing a basepoint in $\hat{\Gamma}$ we can scale the measured wallspaces at each vertex so that $\varrho_{\hat{e}}^{\pm} = 1$ for all \hat{e} in $\hat{\Gamma}$. Each edge group remains infinite cyclic and dispersed in its respective vertex groups. \square

5.4. Constructing the Measured Wallspace. Assuming that the splitting of G is monic, we construct a measured wallspace $(X, \mathcal{W}, \mathcal{B}, \mu)$ that G acts on. Let T be the Bass-Serre tree of the splitting of G . For each $\tilde{v} \in T$, we let $(X_{\tilde{v}}, \mathcal{W}_{\tilde{v}}, \mathcal{B}_{\tilde{v}}, \mu_{\tilde{v}})$ be a copy of $(X_v, \mathcal{W}_v, \mathcal{B}_v, \mu_v)$ with the associated $G_{\tilde{v}}$ action. Let $X = \bigsqcup_{\tilde{v} \in V(T)} X_{\tilde{v}}$ be the disjoint union of these *vertex spaces*. Note that G acts on X , and thus on $\bigsqcup_{\tilde{v} \in V(T)} \mathcal{W}_{\tilde{v}}$, and $\bigsqcup_{\tilde{v} \in V(T)} \mathcal{B}_{\tilde{v}}$ such that $\mu_{\tilde{v}}(U) = \mu_{g\tilde{v}}(gU)$ for $U \in \mathcal{B}_{\tilde{v}}$ and $g \in G$.

For each edge e of Γ with endpoints u, v choose a lift \tilde{e}_o and let \tilde{u}_o, \tilde{v}_o denote its endpoints in T . Our identification of $X_{\tilde{u}_o}$ with X_u allows us to choose a point $\tilde{x}_{\tilde{u}_o}^{\tilde{e}_o} \in X_{\tilde{u}_o}$ corresponding to x_u^e . For each coset gG_e we fix the representative $g \in G$ and let $x_{g\tilde{u}_o}^{g\tilde{e}_o}$ equal $gx_{\tilde{u}_o}^{\tilde{e}_o}$. We likewise define each $x_{g\tilde{v}_o}^{g\tilde{e}_o}$. Having made these choices, for an edge \tilde{e} incident to a vertex \tilde{v} , Condition (5) ensures the orbits $\{gG_{\tilde{e}}x_{\tilde{v}}^{\tilde{e}}\}_{g \in G_{\tilde{v}}}$ are dispersed in $(X_{\tilde{v}}, \mathcal{W}_{\tilde{v}}, \mathcal{B}_{\tilde{v}}, \mu_{\tilde{v}})$.

In a similar vein, for each edge \tilde{e} in T projecting to an edge e in Γ we let $(X_{\tilde{e}}, \mathcal{W}_{\tilde{e}}, \mathcal{B}_{\tilde{e}}, \mu_{\tilde{e}})$ be a copy of $(X_e, \mathcal{W}_e, \mathcal{B}_e, \mu_e)$. If \tilde{e} is incident to \tilde{v} , then we let $\phi_{\tilde{v}}^{\tilde{e}}$ be a copy of ϕ_v^e mapping $(\mathcal{W}_{\tilde{e}}, \mathcal{B}_{\tilde{e}}, \mu_{\tilde{e}})$ isomorphically to $(\mathcal{W}_{\tilde{v}}^{\tilde{e}}, \mathcal{B}_{\tilde{v}}^{\tilde{e}}, \mu_{\tilde{v}})$. The action of G satisfies $g\phi_{\tilde{v}}^{\tilde{e}} = \phi_{g\tilde{v}}^{\tilde{e}}$.

Each edge \tilde{e} of T determines a *vertical wall* whose halfspaces are the sets of vertex spaces corresponding to the two sets of vertices separated by \tilde{e} .

We shall now define an equivalence relation \sim on $\bigsqcup_{\tilde{v} \in V(\Gamma)} \mathcal{W}_{\tilde{v}}$. Let \tilde{e} be an edge in T joining \tilde{u} to \tilde{v} , and let $\Lambda_1 \in \mathcal{W}_{\tilde{u}}$ and $\Lambda_2 \in \mathcal{W}_{\tilde{v}}$, then $\Lambda_1 \sim \Lambda_2$ if there is $\Lambda \in \mathcal{W}_{\tilde{e}}$ such that $\phi_{\tilde{u}}^{\tilde{e}}(\Lambda) = \Lambda_1$ and $\phi_{\tilde{v}}^{\tilde{e}}(\Lambda) = \Lambda_2$. More specifically there is a correspondence between the halfspaces of Λ_1 and the halfspaces of Λ_2 , determined by declaring two halfspace to be in correspondence precisely when they both contain all sufficiently large positive or negative translates $\varphi_{\tilde{v}}^{\tilde{e}}(g_{\tilde{e}}^m)x_{\tilde{v}}^{\tilde{e}}$ and $\varphi_{\tilde{u}}^{\tilde{e}}(g_{\tilde{e}}^m)x_{\tilde{u}}^{\tilde{e}}$. Any wall that belongs to the equivalence class of a wall in the null set excluded from the isomorphisms $\phi_{\tilde{u}}^{\tilde{e}}, \phi_{\tilde{v}}^{\tilde{e}}$ is discarded, and referred to as a *discarded wall*. We also discard any walls that skim an orbit $G_{\tilde{e}}x_{\tilde{v}}^{\tilde{e}}$. Condition (4) ensures that $V(T)$ is countable and so we deduce that the set of discarded walls in any $\mathcal{W}_{\tilde{v}}$ is a nullset. As T is a tree, each equivalence class contains at most one wall in each $\mathcal{W}_{\tilde{v}}$. Let T_{Λ} be the subtree of T spanned by the vertices \tilde{v} such that $X_{\tilde{v}}$ contains a wall in the equivalence class of Λ .

The *horizontal walls* are constructed from these equivalence classes. If Λ is a wall in $\mathcal{W}_{\tilde{v}}$ that has not been discarded, then the wall corresponding to its equivalence class partitions $X_{\tilde{v}}$ by taking the union of the corresponding equivalence classes of halfspaces for all $\tilde{v} \in V(T_{\Lambda})$. If $\tilde{v} \notin V(T_{\Lambda})$, then let \tilde{u} be the closest vertex in T_{Λ} to \tilde{v} , and \tilde{e} the edge separating them. Let $\Lambda' \in \mathcal{W}_{\tilde{u}}$ such that $\Lambda' \sim \Lambda$. Then $G_{\tilde{e}}x_{\tilde{v}}^{\tilde{e}} \subset X_{\tilde{u}}$ is disjoint from Λ' and hence is contained in either the left or right halfspace of $\Lambda' \in \mathcal{W}_{\tilde{u}}$. Accordingly, $X_{\tilde{v}}$ is added to the left or right halfspace.

Let \mathcal{W}^h and \mathcal{W}^v denote the horizontal and vertical walls. Let $\mathcal{W} = \mathcal{W}^h \sqcup \mathcal{W}^v$ denote the set of all walls. There is a natural map, modulo the nullset of discarded walls, $\mathcal{W}_{\tilde{v}} \hookrightarrow \mathcal{W}^h$ that takes a wall to the horizontal wall constructed from the equivalence class containing it.

Define the measurable subsets \mathcal{B}^h of \mathcal{W}^h to be the σ -algebra generated by the inclusion of the elements of $\mathcal{B}_{\tilde{v}}$ into \mathcal{W}^h . Define the measurable subsets \mathcal{B}^v of \mathcal{W}^v

to be the collection of all subsets of vertical walls. Let \mathcal{W}_v^h denote the image of the embedding of $\mathcal{W}_{\tilde{v}}$ in \mathcal{W}^h . Let \mathcal{B} be the σ -algebra generated by $\mathcal{B}^v \cup \mathcal{B}^h$.

Let $\{\tilde{e}_1, \tilde{e}_2, \dots\}$ be an enumeration of representatives of the edge orbits. If $\Lambda_{g\tilde{e}_i}$ is the vertical wall corresponding to $g\tilde{e}_i$ then let $\mu^v(\Lambda_{g\tilde{e}_i}) = i$.

The measure on \mathcal{B}^h is defined as follows: Choose an enumeration $\{\tilde{v}_1, \tilde{v}_2, \dots\}$ of the vertices of T . Given a measurable subset $U \in \mathcal{B}^h$, we partition it as

$$U = \bigsqcup_{i=1}^{\infty} U_i,$$

where we recursively define

$$U_i = \mathcal{W}_{v_i}^h \cap (U - \cup_{j=1}^{i-1} U_j).$$

Note each U_i is measurable because each $\mathcal{W}_{v_i}^h$ is measurable.

We define the measure μ^h as follows:

$$(3) \quad \mu^h(U) = \sum_{i=1}^{\infty} \mu_i(U_i)$$

Observe that μ^h is a measure because firstly $\mu^h(\emptyset) = 0$, and secondly if $\{A_i\}_{i \in \mathbb{N}}$ is a sequence of disjoint measurable sets, then $\mu^h(A) = \sum_{i=1}^{\infty} \mu^h(A_i)$ by rearranging the sum in Equation (3).

Finally, let $\mu = \mu^h + \mu^v$.

Lemma 5.8. *Let A be a measurable set of walls such that $A \subset \mathcal{W}_{\tilde{u}}^h$ and $A \subset \mathcal{W}_{\tilde{v}}^h$ for vertices \tilde{u}, \tilde{v} of T . Then $\mu_{\tilde{u}}(A) = \mu_{\tilde{v}}(A)$.*

Proof. Let $\tilde{u} = \tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_n = \tilde{v}$ be the sequence of vertices on a geodesic path $\tilde{e}_1 \tilde{e}_2 \dots \tilde{e}_n$ in T from \tilde{u} to \tilde{v} . Observe that $A \subset \mathcal{W}_{\tilde{v}_i}^h$ for each i . Then note that $\mu_{\tilde{v}_i}(A) = \mu_{\tilde{v}_{i+1}}(A)$ since the splitting of G is monic and hence $\varrho_{\tilde{e}_i}^{\pm} = 1$. \square

Lemma 5.9. *The measure μ^h is well-defined, in the sense that it does not depend on the enumeration of the vertices.*

Proof. Suppose $U = U_1 \sqcup U_2 \sqcup U_3 \dots$ is a decomposition of a measurable set U with respect to an enumeration $\{\tilde{u}_1, \tilde{u}_2, \dots\}$, and $U = V_1 \sqcup V_2 \sqcup V_3 \dots$ is a decomposition with respect to $\{\tilde{v}_1, \tilde{v}_2, \dots\}$. (For brevity we use the notation $\mu_i = \mu_{\tilde{u}_i}$ and $\mu_j = \mu_{\tilde{v}_j}$.) The claim that $\sum \mu_i(U_i) = \sum \mu_j(V_j)$ follows from the following equation where the second equality holds by Lemma 5.8.

$$\sum_i \mu_i(U_i) = \sum_i \sum_j \mu_i(U_i \cap V_j) = \sum_i \sum_j \mu_j(U_i \cap V_j) = \sum_j \sum_i \mu_j(U_i \cap V_j) = \sum_j \mu_j(V_j) \quad \square$$

Lemma 5.10. *The measure μ^h is G -invariant.*

Proof. This holds from the following equation. Its first equality follows from the definition with respect to an enumeration $\{\tilde{u}_1, \tilde{u}_2, \dots\}$. Its second equality follows

from the fact that $\mu_{\tilde{u}_i}(U_i) = \mu_{g\tilde{u}_i}(gU_i)$. Its final equality follows by defining a second enumeration $\tilde{v}_i = g\tilde{u}_i$ and applying Lemma 5.9.

$$\mu^h(U) = \sum_i \mu_{\tilde{u}_i}(U_i) = \sum_i \mu_{g\tilde{u}_i}(gU_i) = \mu^h(gU) \quad \square$$

Lemma 5.11. *Let $a, b \in X$, then $\omega(a, b) \in \mathcal{B}$ and $\mu(\omega(a, b)) < \infty$.*

Proof. If a and b are in the same vertex space then $\omega(a, b) \in \mathcal{B}$ by construction. Otherwise, consider the geodesic in T between the vertices \tilde{v}_0, \tilde{v}_n that a, b project to. Let $\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_n$ be the corresponding sequence of vertices in T , and let $\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n$ be the connecting edges, and assume for notational purposes that each \tilde{e}_i is directed from \tilde{v}_{i-1} to \tilde{v}_i . We thus have the following sequence of points where $x_{\tilde{v}_{i-1}}^{\tilde{e}_i} \in X_{\tilde{v}_{i-1}}$ and $x_{\tilde{v}_i}^{\tilde{e}_i} \in X_{\tilde{v}_i}$.

$$a, x_{\tilde{v}_0}^{\tilde{e}_1}, x_{\tilde{v}_1}^{\tilde{e}_1}, x_{\tilde{v}_1}^{\tilde{e}_2}, x_{\tilde{v}_2}^{\tilde{e}_2}, \dots, x_{\tilde{v}_{n-1}}^{\tilde{e}_n}, x_{\tilde{v}_n}^{\tilde{e}_n}, b$$

The set $\omega(a, b)$ consists of walls in \mathcal{W} separating an odd number of consecutive elements of the sequence. By construction, $\omega(a, x_{\tilde{v}_0}^{\tilde{e}_1}) \in \mathcal{B}$, and $\omega(x_{\tilde{v}_i}^{\tilde{e}_i}, x_{\tilde{v}_{i+1}}^{\tilde{e}_{i+1}}) \in \mathcal{B}$, and $\omega(x_{\tilde{v}_n}^{\tilde{e}_n}, b) \in \mathcal{B}$. We claim that $\omega(x_{\tilde{v}_{i-1}}^{\tilde{e}_i}, x_{\tilde{v}_i}^{\tilde{e}_i})$ consists of a single vertical wall. Indeed suppose $\omega(x_{\tilde{v}_{i-1}}^{\tilde{e}_i}, x_{\tilde{v}_i}^{\tilde{e}_i})$ contained a horizontal wall Λ . Then Λ would have been constructed an equivalence class containing walls $\Lambda_{i-1} \in \mathcal{W}_{\tilde{v}_{i-1}}$ and $\Lambda_i \in \mathcal{W}_{\tilde{v}_i}$ such that $x_{\tilde{v}_{i-1}}^{\tilde{e}_i}$ is in the left halfspace of Λ_{i-1} but $x_{\tilde{v}_i}^{\tilde{e}_i}$ is in the right halfspace of Λ_i (or vice versa). Thus $\Lambda_{i-1} \in \varphi_{\tilde{v}_{i-1}}^{\tilde{e}_i}(g_{\tilde{e}_i}^p)\omega(x_{\tilde{v}_{i-1}}^{\tilde{e}_i}, \varphi_{\tilde{v}_{i-1}}^{\tilde{e}_i}(g_{\tilde{e}_i})x_{\tilde{v}_{i-1}}^{\tilde{e}_i})$, and $\Lambda_i \in \varphi_{\tilde{v}_i}^{\tilde{e}_i}(g_{\tilde{e}_i}^q)\omega(x_{\tilde{v}_i}^{\tilde{e}_i}, \varphi_{\tilde{v}_i}^{\tilde{e}_i}(g_{\tilde{e}_i})x_{\tilde{v}_i}^{\tilde{e}_i})$ for some $q < 0 \leq p$ (or $p < 0 \leq q$). Hence $\Lambda_{i-1} \sim \Lambda_i$ would contradict Condition (3). \square

Corollary 5.12. *$(X, \mathcal{W}, \mathcal{B}, \mu)$ is a measured wallspace with an isometric G -action.*

Proof. Lemma 5.11 shows that $(X, \mathcal{W}, \mathcal{B}, \mu)$ is a measured wallspace. \square

Proposition 5.13. *The action of G on $(X, \mathcal{W}, \mathcal{B}, \mu)$ is metrically proper.*

Proof. Let $x \in X$. Suppose that for some infinite subset $J \subset G$ and some $d > 0$, we have $\#(g_i x, g_j x) < d$ for all $g_i, g_j \in J$. Let $S \subset T$ be the smallest subtree whose vertex spaces contain the points $\{gx : g \in J\}$. Note that S has diameter $\leq d$ since each edge has a vertical wall with measure ≥ 1 .

We now verify that S is locally finite. Indeed, if there were infinitely many edges at some $\tilde{v} \in S^0$, then either there would exist an edge e incident to \tilde{v} , whose $G_{\tilde{v}}$ -orbit contains infinitely many edges of S at \tilde{v} , or else there are infinitely many edges in S incident to \tilde{v} that belong to distinct $G_{\tilde{v}}$ -orbits.

In the former case let $\{g_1 \tilde{e}, g_2 \tilde{e}, \dots\}_{g_i \in G_{\tilde{v}}}$ be an enumeration of infinitely many edges in a $G_{\tilde{v}}$ -orbit. Letting $G_{\tilde{e}} x_{\tilde{e}}^{\tilde{v}} \subset X_{\tilde{v}}$ be the basepoint orbit corresponding to \tilde{e} , its translate $g_i G_{\tilde{e}} x_{\tilde{e}}^{\tilde{v}} \subset X_{\tilde{v}}$ is the basepoint orbit corresponding to $g_i \tilde{e}$. By Condition (5) of Theorem 5.3, the sets $\{g_i G_{\tilde{e}} x_{\tilde{e}}^{\tilde{v}}\}_{i \in \mathbb{N}}$ are dispersed in $(X_{\tilde{v}}, \mathcal{W}_{\tilde{v}}, \mathcal{B}_{\tilde{v}}, \mu_{\tilde{v}})$ and hence in $(X, \mathcal{W}, \mathcal{B}, \mu)$. Therefore, there exists $i, j \in \mathbb{N}$ such that $\#(g_i G_{\tilde{e}} x_{\tilde{e}}^{\tilde{v}}, g_j G_{\tilde{e}} x_{\tilde{e}}^{\tilde{v}}) > d$. By construction $g_i G_{\tilde{e}} x_{\tilde{e}}^{\tilde{v}} = G_{g_i \tilde{e}} x_{g_i \tilde{e}}^{\tilde{v}}$. Consequently, the elements of Jx lying in vertex spaces separated by $g_i \tilde{e}$ and $g_j \tilde{e}$ must be at distance at least $d + 2$ from each other.

In the latter case, by definition of the measure on the vertical walls there must exist some edge \tilde{e} in S , incident to \tilde{v} such that the corresponding vertical wall $\Lambda_{\tilde{e}}$ has $\mu(\Lambda_{\tilde{e}}) > d$. This would imply that the two orbit points $g_1\tilde{x}, g_2\tilde{x}$ lying in vertices separated by \tilde{e} have $\#(g_1\tilde{x}, g_2\tilde{x}) > d$.

Therefore, S is locally finite and $\text{diam}(S) < \infty$, hence S has finitely many vertices. Thus, there exists an infinite subset $J' \subset J$ such that the elements of $\{g_j\tilde{x} : j \in J'\}$ all lie in the same vertex space $X_{\tilde{v}}$. This contradicts that $G_{\tilde{v}}$ acts metrically properly on $(X_{\tilde{v}}, \mathcal{W}_{\tilde{v}}, \mathcal{B}_{\tilde{v}}, \mu_{\tilde{v}})$ since $\#_{\tilde{v}}(p, q) = \#(p, q)$ for $p, q \in X_{\tilde{v}}$ by construction. \square

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