Received: 11 July 2025 | Revised: 16 September 2025 | Accepted: 6 October 2025

DOI: 10.1112/jlms.70374

SPECIAL ISSUE: 100 years of the Journal of the London Mathematical Society

Journal of the London Mathematical Society

SURVEY ARTICLE

On the paper "Bundle gerbes" by Michael Murray

Nigel Hitchin

Mathematical Institute, University of Oxford, Oxford, UK

Correspondence
Nigel Hitchin, Mathematical Institute, University of Oxford, Oxford, OX2 6GG, UK.
Email: nigel.hitchin@maths.ox.ac.uk

Abstract
The article gives a brief survey of Murray's notion of bundle gerbes as introduced in his 1996 paper published in the *Journal of the London Mathematical Society*, together with some of its applications.

MSC 2020
53C08

1 | INTRODUCTION

Gerbes are higher order geometric objects, which make considerable demands on the intuition to appreciate, but more than that they require effective tools to perform calculations, and Michael Murray's original paper [13] introduced an approach which has been used successfully by many in the field since its publication 30 years ago. His viewpoint is an alternative to the more abstract notion of a sheaf of groupoids in Brylinski's book [2] which opened up the concept, and while today's mathematicians may be more familiar with those categorical aspects, for the working geometer the bundle approach to gerbes is of enduring value.

What does "higher order" mean in a geometric context? Consider the standard objects of differential geometry: differentiable manifolds, functions, vector fields. These are global concepts, manifolds such as spheres or tori, which we can relate by diffeomorphisms, immersions, submersions, and so on. To work with them we have to use local coordinates on open neighborhoods U_α of M homeomorphic to open sets in $\mathbf{R}^n$ which are related on twofold intersections by invertible smooth transformations. So, a global function on M is a collection of functions f_α on U_α (expressed in terms of the local coordinates) such that $f_\beta - f_\alpha = 0$ on $U_\alpha \cap U_\beta$. The open sets serve to *recognize* when a space is a manifold and enable us to systematically define a global

---

This article forms part of the "100 years of the Journal of the London Mathematical Society" special issue celebrating the founding of the Journal of the London Mathematical Society in 1926.

function or vector field in terms of these local functions or vector-valued functions satisfying compatibility conditions on twofold intersections. But the U_α are not part of the manifold, when we look at a sphere we do not see these open sets, we choose them as appropriate for the problem at hand. This may seem obvious nowadays but from Weyl's "The idea of a Riemann surface," through Hodge's "Harmonic integrals" many pages were dedicated over decades to transmitting the concept of a manifold in an acceptable form.

The postwar years introduced the notion of fiber bundles and in particular principal bundles. A principal circle bundle P over M is a manifold with a free action of the circle group S^1 whose quotient is M and over an open set U_α of a covering is a product $U_\alpha \times S^1$, where now over $U_\alpha \cap U_\beta$ the two product decompositions are related by a circle-valued function $g_{\alpha\beta}$ with $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$. This is one order higher than functions on U_α and there is a compatibility condition on threefold intersections $U_\alpha \cap U_\beta \cap U_\gamma$, namely $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$, rather than two. Note that the product $g_{\alpha\beta} g'_{\alpha\beta}$ for two principal bundles P, P' defines a tensor product operation $P \otimes P'$. This approach may be higher order than functions in this sense but we are still dealing with manifolds and can use the same techniques.

The next stage is where gerbes enter. We could consider circle-valued functions $g_{\alpha\beta\gamma}$ on threefold intersections with a compatibility on fourfold intersections. This can be done, but our intuition about fourfold intersections begins to fail. It is said that a newborn child has the concept of the number three but maybe even mathematicians find it hard to go beyond. Alternatively, we could consider principal circle bundles $P_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. We can form tensor products and inverses but now $P_{\beta\alpha}$ can only be *isomorphic* to the inverse, or dual, $P^*_{\alpha\beta}$ of $P_{\alpha\beta}$ rather than equal to it, so attention has to be paid to the actual isomorphism. This is the basis for the alternative definition in the paper.

The author of this article once pursued the open covering approach with his student Chatterjee [5, 9] but it was clear that there were limitations. In particular, how do you deal with a group action which does not preserve a covering? When does an invariant gerbe descend to a quotient? It seems that following discussions about gerbes with the author [14], Murray began his own individual approach and introduced his *bundle gerbes*.

2 | BUNDLE GERBES

Here is the definition, phrased in terms of a projection of manifolds $\pi : Y \to M$ and the fiber products $Y^{[p]} = \{(y_1, \dots, y_p) \in Y^p : \pi(y_1) = \cdots = \pi(y_p)\}$ with projections π_i to the ith factor.

Definition 1. A bundle gerbe over M is a surjective submersion $\pi : Y \to M$ and a principal circle bundle P over $Y^{[2]}$ together with:

(1) an isomorphism of principal bundles $m : \pi_3^* P \otimes \pi_1^* P \to \pi_2^* P$ over $Y^{[3]}$,
(2) the multiplication m is associative in the sense that there is a commutative diagram of isomorphisms over $Y^{[4]}$.

The definition implies that there are natural isomorphisms $P_{(y_1,y_2)} \cong P^*_{(y_2,y_1)}$ and $P_{(y,y)} \cong S^1$.

A circle bundle P on Y defines in a natural way a bundle $\delta(P)$ on $Y^{[2]}$ and this is the definition of a *trivial* bundle gerbe. Then two bundle gerbes (P, Y) and (Q, X) over M are defined to be *stably isomorphic* if $(P, Y)^* \otimes (Q, X)$ is trivial.

The advantage of this formalism is the mental picture, familiar from the geometer's experience with fibrations of manifolds of various types – we are dealing with manifolds and principal bundles satisfying a few extra conditions. This is slightly deceptive though, because Y could be taken to be the disjoint union of the open sets U_α in the more naive description above. Then $Y^{[2]}$ is the disjoint union of the ordered intersections $U_\alpha \cap U_\beta$, and so forth. This offers the opportunity to utilize the formalism of simplicial spaces, familiar to topologists.

There are alternative choices for Y, however. For example, suppose $\tilde{G}$ is a central extension of a Lie group G by the circle. A simple case is $\tilde{G} = U(n)$ and G is the projective unitary group $PU(n)$, the quotient by the scalar matrices. Then we can take Y to be a principal G-bundle over M and we have a map $g : Y^{[2]} \to G$ defined by $y_1 g(y_1, y_2) = y_2$. Since the homomorphism $\tilde{G} \to G$ is itself a principal circle bundle, pulling back by the map g defines a bundle gerbe. This is more natural than taking local lifts $\tilde{g}_{\alpha\beta}$ of transition matrices $g_{\alpha\beta}$ and taking the circle-valued function $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, but the two choices are stably equivalent.

The concept of bundle gerbe therefore avoids the intuitive problem of thinking of a gerbe as some kind of space, but instead assembles in a coherent way a set of defining data which is a space.

3 | THE COHOMOLOGY CLASS

Defining a gerbe by functions $g_{\alpha\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ for a suitable covering leads to a Čech cohomology class in the second cohomology group of the sheaf of circle-valued functions and hence from an exact sequence a characteristic class in $H^3(M, \mathbf{Z})$ which Murray calls the Dixmier–Douady class. This puts gerbes in a hierarchy beginning with functions $f : M \to S^1$ defining a class in $H^1(M, \mathbf{Z})$ and circle bundles with first Chern class in $H^2(M, \mathbf{Z})$. Then one can show that two bundle gerbes are stably isomorphic if and only if they have the same characteristic class, so this is a way of geometrically representing degree 3 classes.

The example of the group extension above gives a torsion class, because we can also write $PU(n) = SU(n)/\mathbf{Z}_n$ and in fact if M is simply connected and the fibers of a bundle gerbe $Y \to M$ are connected and finite-dimensional then the characteristic class is always torsion [15]. These "small bundle gerbes" nevertheless have applications in index theory, as for example [11]. This fact governs one's intuition about the manifold Y: if we want to represent a non-torsion class then Y is either disconnected or infinite-dimensional.

Non-torsion classes have curvature representatives in de Rham cohomology, as described by Brylinski [2] in terms of connective structures, which are 1-forms $A_{\alpha\beta}$ with

$$A_{\beta\gamma} - A_{\alpha\gamma} + A_{\alpha\beta} = g^{-1}_{\alpha\beta\gamma} d g_{\alpha\beta\gamma}$$

and 2-forms F_α called a curving such that $F_\beta - F_\alpha = dA_{\alpha\beta}$. Then $dF_\alpha = dF_\beta$ on $U_\alpha \cap U_\beta$ and defines a closed 3-form curvature H. From the bundle gerbe point of view the connective structure is a connection on the circle bundle P over $Y^{[2]}$ which respects the multiplication, and the curving is a further choice.

Any compact simple Lie group has a real degree 3 cohomology class given by the bi-invariant (and therefore closed) 3-form $B([X, Y], Z)$ on the Lie algebra where B is the Killing form. There are multiple approaches in the literature to realizing this with a gerbe [16]. One construction with $Y \to G$ having infinite-dimensional fiber consists of taking Y as the based loop group $\Omega(G)$ of all smooth maps $g : [0, 1] \to G$ with $g(0) = g(1) = 1$. This has a well-known central extension $\hat{\Omega}(G)$ [18] and the above method defines a gerbe over $M = G$ realizing the cohomology class. An alternative finite-dimensional approach, due to Meinrenken [12] also uses bundle gerbes. In the case of $G = SU(n)$ the space $Y = \{(g, z) \in G \times S^1 : \det(g - zI) \neq 0\}$ with the first factor the projection to G. Note that the fiber over a general point is disconnected, and consists of n disjoint open intervals in the circle.

4 | HOLONOMY

A connection on a principal circle bundle over a manifold defines by parallel translation an element of the circle, or $\mathbf{R}/\mathbf{Z}$, around any smooth closed curve (since the group is abelian this is independent of any base point). In the same way a connective structure on a bundle gerbe defines holonomy around a closed surface Σ.

In the language of bundle gerbes, the holonomy is defined as follows. Since Σ is 2-dimensional, the characteristic class vanishes and so the bundle gerbe is trivial, and induced by a circle bundle R on Σ. The bundle gerbe connection defines a 2-form curvature $F(= dA_{\alpha\beta})$ on $Y^{[2]}$, but given the trivialization one can choose a connection on R over Σ which induces the bundle gerbe connection and then the difference of the two curvatures is the pullback of a 2-form on Σ. The integral of this form is only defined modulo $2\pi\mathbf{Z}$ since a different choice of trivialization will be given by a line bundle with connection and its Chern class will contribute an integer. But then we get a well-defined $\mathbf{R}/\mathbf{Z}$ invariant.

5 | PHYSICAL APPLICATIONS

The notion of holonomy connects with the WZW (Wess–Zumino–Witten) term of the physicists, a real number modulo integers associated to a surface. This was traditionally defined by making the surface bound a three-manifold. The 2-dimensional analogue consists of integrating the curvature 2-form of a connection on a principal circle bundle over a 2-dimensional disc to obtain the holonomy around the boundary circle but the usual notion of holonomy for a connection means we do not need the disc. Likewise, the theory of gerbes gives an invariant purely in terms of the surface. The gerbe language then makes sense of “2-form potentials” and “B-fields” used extensively in the physics literature.

One adherent of gerbes in a physical context was Krzysztof Gawędzki and in the paper [6] he begins with a discussion using the language of open coverings but quickly moves to Murray’s bundle gerbes. He develops their use in further papers [7, 8], and in particular in the context of the celebrated work of Fu, Kane, and Mele on topological insulators. The author in [9] was also physically motivated to introduce gerbes and holonomy, in this case the holonomy of a flat gerbe on 3-dimensional tori in order to make mathematical sense of the SYZ (Strominger–Yau–Zaslow) approach to mirror symmetry.

6 | ONWARD AND UPWARD

In recent years, gerbes have made their presence felt in many areas, in generalized geometry [10], twisted K-theory [1], anomalies [4], and quantization [3]. With increasing sophistication in the community about higher order structures, more categorical approaches have yielded deeper results.

The formalism in terms of simplicial manifolds can clearly be carried forward to even higher structures. An outstanding area is what might be called 2-gerbes, where the characteristic class is in degree 4. This was part motivation for Brylinski's book. Work from the Murray school began the process [20], adapting the formalism of bundle gerbes. Now one takes a bundle gerbe on $Y^{[2]}$, a product over $Y^{[3]}$ and an *associator* over $Y^{[4]}$. The compatibility conditions become significantly more complicated commutative diagrams [21] and adapting the informal language which mathematicians use to communicate ideas becomes more challenging.

The main focus of 2-gerbes concerns the notion of a "string structure" on a manifold. A close cousin is the Hull–Strominger system, which has its origins in supergravity and involves an equation $dd^c\omega = \operatorname{tr} R^2 - \operatorname{tr} F^2$ where ω is a Hermitian form. This is saying that the difference of the degree 4 Chern-Weil representatives of characteristic classes of the manifold and a vector bundle over it differ by a specific exact form, which yields a nonclosed 3-form $d^c\omega$. This is a form of isomorphism between two 2-gerbes which should have a gerbe-like interpretation.

A major change in emphasis in pure mathematics over the last 30 years has been the influence of quantum field theory which forces us to express results in the language of category theory. Graeme Segal recognized this in his Presidential Address [19] to the Society, observing "we encounter mathematical objects which we feel intuitively are 'space', but whose space-like properties cannot be captured by the usual concept of a topological space," a gerbe is probably the simplest such object. Topological quantum field theory is the most popular representative in mathematics of this viewpoint. One associates an invariant, a complex number, to a manifold of dimension n and a 1-dimensional complex vector space to an $(n-1)$-dimensional one with relations for manifolds with smooth boundary. But one step further, for example, where the boundary has "corners," requires a concept which, in a family, is essentially a gerbe.

Work on gerbes is therefore nowadays more often framed in the language of higher categories and ideas from homotopy theory or algebraic geometry. For example, in [17], the concept of a trivial bundle gerbe with connection is described as a 2-category whose objects are 2-forms and the 2-category of bundle gerbes with connection is the application of Grothendieck's plus construction. This level of sophistication also allows one to discuss equivariance under a group action.

Further progress will almost inevitably be framed in these terms, but new generations of mathematicians can handle this. Just as the concept of a manifold, or a fiber bundle gradually became easily acceptable, so will these new structures, and Murray's paper helps to point the way.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Nigel Hitchin https://orcid.org/0000-0002-4484-7604

REFERENCES

1. P. Bouwknegt, A. L. Carey, V. Mathai, and M. K. Murray, and D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, Commun. Math. Phys. **228** (2002), 17–49.
2. J.-L. Brylinski, *Loop spaces, characteristic classes, and geometric quantization*, Progress in Mathematics, vol. 107, Birkhäuser, Boston, MA, 1993.

3. S. Bunk, *Gerbes in geometry, field theory and quantization*, Complex Manifolds **8** (2021), 150–182.
4. A. L. Carey and M. K. Murray, *Faddeev's anomaly and bundle gerbes*, Lett. Math. Phys. **37** (1996), 29–36.
5. D. S. Chatterjee, *On gerbs*, Ph.D. thesis, Cambridge, 1998.
6. K. Gawędzki and N. Reis, *WZW branes and gerbes*, Rev. Math. Phys. **14** (2002), 1281–1334.
7. K. Gawędzki, R. Suszek, and K. Waldorf, *Bundle gerbes for orientifold sigma models*, Adv. Theor. Math. Phys. **15** (2011), 621–688.
8. K. Gawędzki, R. Suszek, and K. Waldorf, *Square root of gerbe holonomy and invariants of time-reversal-symmetric topological insulators*, J. Geom. Phys. **120** (2017), 169–191.
9. N. J. Hitchin, *Lectures on special Lagrangian submanifolds*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds, C. Vafa and S.-T. Yau (eds.), Studies in Advanced Mathematics, vol. 23, Amer. Math. Soc./International Press, Providence, RI, 2001, pp. 151–182.
10. N. J. Hitchin, *Lectures on generalized geometry*, arXiv 1008.0973.
11. V. Mathai and R. B. Melrose, *Index and small bundle gerbes*, Adv. Math. **483** (2025), 110679.
12. E. Meinrenken, *The basic gerbe over a compact simple Lie group*, Enseign. Math. **49** (2003), 307–333.
13. M. K. Murray, *Bundle gerbes*, J. Lond. Math. Soc. **54** (1996), 209–416.
14. M. K. Murray, *An introduction to bundle gerbes*, The many facets of geometry, O. Garcia-Prada, et al. (eds.), Oxford Academic, Oxford, 2010, pp. 237–260.
15. M. K. Murray, *A note on bundle gerbes and infinite-dimensionality*, J. Aust. Math. Soc. **90** (2011), 81–92.
16. M. K. Murray and D. Stevenson, *The basic bundle gerbe on unitary groups*, J. Geom. Phys. **58** (2008), 1571–1590.
17. T. Nikolaus and C. Schweigert, *Equivariance in higher geometry*, Adv. Math. **226** (2011), 3367–3408.
18. A. Pressley and G. B. Segal, *Loop groups*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 1986.
19. G. B. Segal, *Space and spaces*, Bull. Lond. Math. Soc. **48** (2016), 1–11.
20. D. Stevenson, *Bundle 2-gerbes*, Proc. Lond. Math. Soc. **88** (2004), 405–435.
21. K. Waldorf, *String connections and Chern–Simons theory*, Trans. Amer. Math. Soc. **365** (2013), 4393–4432.