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ORIGINAL RESEARCH

Inferentialism and Uniformity

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Logical inferentialism is the view that the logical constants get their meanings from their roles in inference. It is an instance of the broader inferentialist project that sees all expressions as deriving their meaning from their inferential roles. Inferentialism stands in contrast to referentialism, the mainstream view which explains the semantic properties of linguistic expressions in terms of the referential relations they bear to worldly objects.[1]

By *core inferentialists*, we mean logical inferentialists who endorse the central commitments of the view just outlined. In this paper, we present a novel challenge to core inferentialists. We will argue that, when we examine the sorts of quantifiers the core inferentialist counts as logical, we find unattractive results. Since inferentialism is prima facie more plausible as a thesis about the logical constants than other parts of language, this is a challenge not just to logical inferentialists but inferentialists about language in general.[2]

Our challenge is that the core inferentialists' account is not *uniform*: roughly, their judgements about the logicality of quantifiers exhibit a strange pattern. The core inferentialist is happy to accept the familiar existential quantifier as logical, as well as other finite cardinality quantifiers such as 'there are at least 327'. They rule out the quantifier 'there are infinitely many' as logical. However, they should accept 'there are

[1] Three major inferentialists of recent times include Michael Dummett, Robert Brandom and Neil Tennant; see [6], [4] and [36] for representative works. Inferentialism is inspired by the 'meaning is use' dictum attributed to the late Wittgenstein. Murzi and Steinberger [16] is a useful survey.

[2] Inferentialism is less plausible when applied to expressions, such as proper names (e.g. 'Napoleon') or certain predicates (e.g. 'is red', 'is square', 'is a horse'), that connect more directly to the world than logical constants and therefore seem better suited to a referentialist treatment. As Brandom puts it, his strategy is to 'look to the contents of logical concepts as providing the key to understanding conceptual content generally' ([3], p. 653).

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uncountably many' as logical. If you agree with us that these verdicts seem bizarre, you'll see this as a reason to doubt logical inferentialism.

We first outline the uniformity challenge in §1. In §2, we spell out some assumptions that give rise to it. We argue briefly that standard forms of inferentialism endorse all but one of them, and that they are committed to this last one, despite this being rarely acknowledged. We then reply to potential inferentialist objections in §§3–4. Along the way, as well as posing a challenge to core inferentialism, we will explore some overlooked aspects of inferentialism's motivation and presentation, of interest in their own right.

1 The Uniformity Challenge

1.1 Uniformity: the Very Idea

Call an expression such as 'there are at least κ many', where κ is a term for a finite or infinite cardinal, a *cardinality quantifier*.[3] Which of these quantifiers you take to be logical will depend on what you think logic is.

Suppose, for example, that you are a committed first-orderist: you take first-order logic, with identity, to be the correct logic, so that the logical is anything definable in its terms.[4] In that case, you take the quantifiers 'there are at least n-many' to be logical for any finite n, as they are first-order definable, and the quantifiers 'there are at least κ-many' to be non-logical for any infinite κ, which are not first-order definable, as is well known.[5]

Alternatively, suppose you adopt a semantic account of the logical constants. This rival approach to inferentialism is usually based on isomorphism invariance.[6] Since isomorphisms preserve cardinality, on this approach all cardinality quantifiers—finite and infinite—are logical. Invariantism thus offers a simple and uniform account of which cardinality quantifiers are logical: they all are.

Whatever their other merits or demerits, at least first-orderism and invariantism offer uniform accounts of the cardinality quantifiers. The contrast is with conceptions whose patterns of logicality and non-logicality are haphazard. It would be utterly bizarre, for example, to suppose that 'there are at least 327' is non-logical, whereas 'there are at least 326' and 'there are at least 328' are logical. No such account is tenable.

Call the requirement of avoiding bizarre and haphazard patterns *uniformity*. How exactly to cash out the requirement of uniformity is not entirely clear. The following seems a reasonable criterion: an account of cardinality quantifiers is uniform if it either

[3] Expressions such as 'there are at most κ many' or 'there are exactly κ many' are also cardinality quantifiers. Our focus will be on quantifiers of the form 'there are at least κ many', but most of our morals carry over to the others.

[4] Meaning anything definable in the vocabulary common to all first-order languages. This was, famously, Quine's view (see his [20]).

[5] One way to see this is to use the Löwenheim-Skolem Theorems; see footnote 9 for an example of such an argument.

[6] See Sher [31], building on Tarski [35], as well as Griffiths and Paseau [10].

treats all quantifiers as logical, or if it treats just some initial portion of them as logical. An account that inserts a break in the scale of quantifiers, with the logical at one end and the non-logical at the other, is uniform. Whether or not this is the right way to spell out the requirement, what is clear is that the first-orderist's and invariantist's accounts are uniform, whereas the account in the previous paragraph is not.

The question is whether the logical inferentialist's account is uniform. We shall present a case that it is not. Core inferentialists all accept first-order logic with identity and therefore take all the quantifiers 'there are at least n-many' for finite n to be logical. The question is which other cardinality quantifiers, if any, they should accept as logical.

Our uniformity challenge for core inferentialists consists of three premises. The first is that, as well as the finite cardinality quantifiers, they should also accept the quantifier 'there are uncountably many' (Q) as logical. The second is that they should not accept 'there are infinitely many' (I) as logical. The third premise, which we take to be obvious and will not defend further, is that such a non-uniform account of quantifiers is incorrect.

Our uniformity challenge is, as far as we are aware, novel. Other objections in the literature share a similar structure; for example, some discussion has focused on the inferentialist treatment of modal operators.[7] If inferentialists must accept certain modal operators as logical but not others, there is some sense in which their treatment of modal operators is not uniform. But the sense of uniformity here is not exactly ours, since ours depends on a cardinality scale. Moreover, it is far less clear that all modal operators should be accepted as logical. If, for example, we think that S5 is the logic of *logical* necessity, which is a logical notion, then we might not worry if the operators of other logics fail to be logical. In contrast, the lack of uniformity we discuss in relation to quantifiers is more straightforwardly problematic.

1.2 First Premise: *Q* is Logical

Some of the key properties of the quantifier 'there exist uncountably many' were articulated in a long article by H.J. Keisler in [13]. Extend first-order logic (FOL) to the logic $\mathcal{L}_Q$ by adding the quantifier Q, interpreted as 'there are uncountably many'.[8] When the sentence ϕ semantically follows from a set of sentences Γ in this extended logic $\mathcal{L}_Q$, we write $\Gamma \vDash_{\mathcal{L}_Q} \phi$. Note that $\mathcal{L}_Q$ has more expressive power than FOL,

[7] See e.g. Read [23].

[8] More precisely, add the symbol Q to FOL's vocabulary; supplement FOL's formation rules with the stipulation that if ϕ is a formula and α is a variable then $Q\alpha\phi$ is a formula; and add to FOL's satisfaction clauses the following stipulation, where $\mathfrak{M}$ is a model and g and h assignments over it:

$(\mathfrak{M}, g) \vDash Q\alpha\phi$ iff $\{h : (\mathfrak{M}, h) \vDash \phi$ and assignment h agrees with assignment g on all variables other than $\alpha\}$ is uncountable.

since the former but not the latter can express the quantifier 'there exist uncountably many'.[9]

We also extend any sound and complete proof system for FOL with the following four rules for the quantifier Q:

$$\frac{Qx\phi}{Qy\phi[y/x]} \text{ if } y \text{ is not free in } \phi$$

$$\frac{}{\neg Qx(x = y \vee x = z)} \text{ where } y \text{ and } z \text{ are distinct variables from } x$$

$$\frac{\forall x(\phi \to \psi)}{Qx\phi \to Qx\psi}$$

$$\frac{\neg Qx\exists y\phi \wedge Qy\exists x\phi}{\exists x Qy\phi}$$

The first rule allows us to rename bound variables; it is trivially sound for the given $\mathcal{L}_Q$-semantics. The second rule—which features no premises—is sound for the $\mathcal{L}_Q$-semantics because it states that uncountably many things are not equal to one of two things (or just one thing). The third rule's soundness follows from the fact that if all ϕs are ψs and there are uncountably many ϕs then there are uncountably many ψs (i.e. the quantifier is monotone increasing). Finally, the fourth rule is sound because if a relation has countably many things in its domain and uncountably many in its codomain then some element of its domain must be related to uncountably many elements of its codomain. More succinctly: the countable union of countable sets is countable. If the sentence ϕ follows from a set of sentences Γ in this deductive system, we write $\Gamma \vdash_{\mathcal{L}_Q} \phi$.

The remarkable result Keisler proved (following an earlier result of Vaught's) is that the proof system just given is not merely sound, but also complete for countable sets of sentences. More precisely, if Γ is a countable set of $\mathcal{L}_Q$-sentences and ϕ any $\mathcal{L}_Q$-sentence, then

$$\Gamma \vDash_{\mathcal{L}_Q} \phi \text{ iff } \Gamma \vdash_{\mathcal{L}_Q} \phi.$$

Core inferentialists accept first-order logic as logical. We claim that this result shows that they should also accept $\mathcal{L}_Q$ as logical, and so accept Q ('there are uncountably many') as a logical constant.[10] In §3, we sharpen this up by considering further conditions that the inferentialist might impose for an expression to be logical.

1.3 Second Premise: *I* is not Logical

The second premise is that the cardinality quantifier 'there are infinitely many' is not logical according to the core inferentialist. Here, the argument proceeds by *reductio*.

[9] The downward Löwenheim-Skolem Theorem implies that if a sentence of FOL has an infinite model then it has a countably infinite model. Hence no FOL-sentence can be true in all and only models with an uncountable domain, unlike the $\mathcal{L}_Q$-sentence $Qx(x = x)$.

[10] We speak interchangeably of 'there exist uncountably many' and its formal representative Q being logical. Similarly, later, for 'there exist infinitely many' and I.

Write the formalisation of ‘there are infinitely many’ as I and give it the obvious syntax, identical to that of Q, and an analogous semantics to Q’s as well, save that I is interpreted as ‘there are infinitely many’. Call the system extending first-order logic in this way $\mathcal{L}_I$. In $\mathcal{L}_I$, we may express the informal argument, A,

$$\frac{\begin{array}{l}\text{There exists at least one pear}\\ \text{There exist at least two pears}\\ \quad\vdots\\ \text{There exist at least } n \text{ pears}\\ \quad\vdots\end{array}}{\text{There exist infinitely many pears}}$$

as the sequent

$(A')\ \exists_1, \cdots, \exists_n, \cdots \vDash_{\mathcal{L}_I} Ix(Px)$

where $\exists_j$ is the first-order sentence formalising ‘there are at least j pears’, for j finite, using the predicate ‘Px’ to formalise ‘is a pear’. This sequent’s premise set, though infinite, can be recursively specified.[11] If the quantifier I interpreted as ‘there are infinitely many’ were logical, then A would be logically valid. Furthermore, none of this argument’s finite subarguments are valid. To put this more formally:

$$\exists_{k_1}, \cdots, \exists_{k_n} \nvDash_{\mathcal{L}_I} Ix(Px)$$

for any finite n. It follows that the logic $\mathcal{L}_I$ is non-compact,[12] and therefore that it is not sound and complete either.[13] But any logic acceptable to a core inferentialist should be sound and complete. It would be against the spirit of inferentialism to adopt a logic with no sound and complete proof procedure, precisely because deductive rules are ultimately what validity is owed to on this picture. This contradicts the assumption that I is logical.

The first premise of the uniformity challenge was that Q is logical. The second premise is that I, in contrast to Q, is not logical. Hence the core inferentialist’s account of cardinality quantifiers is non-uniform.

2 Inferentialist Commitments

Our uniformity challenge assumes that core inferentialists share some commitments. Some are background commitments, which we will be very brief about. Like almost everyone else, inferentialists assume that first-order logic is part of logic or, if they are pluralists, part of some acceptable logics. This is an adequacy condition on

[11] So even constructivist inferentialists, who reject the completed infinite, should accept the above specification of the premise set.

[12] A compact logic, recall, is one in which any valid argument has a valid finite subargument. More precisely, if the argument with premise set Γ and conclusion ϕ is valid then the argument with premise set Γ^* for some finite subset Γ^* of Γ and conclusion ϕ is valid.

[13] Compactness follows from soundness and completeness since derivations must be finite.

any plausible inferentialist view. Similarly, inferentialists should accept model theory, on pain of rejecting large parts of mathematical logic, theoretical computer science, linguistics, etc. As discussed in §4, they take formal semantics to have an inferentialist foundation, so for them an applied model theory must ultimately answer to proof theory. But to abjure model theory altogether would be a step too far. The counterpart in the rival camp to an inferentialism that rejects model theory would be a referentialism that has no truck with proof-theoretic notions, even when they answer to model-theoretic ones, an equally implausible and uninstantiated position.[14]

Two commitments beyond these very standard ones are worthy of further comment. The first is the completeness of the proof system, and the second is the inferential characterisation of expressions other than via introduction and elimination rules. In this section, we consider why core inferentialists are committed to both. Briefly, the first is a standard part of most inferentialist views; and the second, although much less common, *should* be an inferentialist commitment. It would triple the length of this paper and take us too far afield to defend these two points in great detail. In this section, we summarise why we think a form of inferentialism with both commitments, to which our objection applies—and which we call core inferentialism—is an interesting one to consider.

2.1 Completeness

A standard version of inferentialism holds that an expression's meaning is determined by speakers' dispositions to infer according to its inference rules in our language. On this picture, as pointed out above, the kind of referentialist semantics that prevails in linguistics is in good standing and there is no question of relinquishing it. What the inferentialist does, rather, is invert the usual order of priority: ultimately, referentialist semantics is answerable to inference rules rather than the other way around. This is mirrored by the logic: the right proof system must be complete, since an adequate model-theoretic semantics must reflect it.

This version of inferentialism is standard. It is defended in, for example, Garson [7], who sees it as compatible with model-theoretic semantics. It is what Murzi and Steinberger in their survey of inferentialism call the orthodox account ([16], p. 200), citing Christopher Peacocke, Harold Hodes and John MacFarlane as adherents (2017, p. 214). Murzi and Steinberger characterise the difference between inferentialism and referentialism as residing at the metasemantic level, the locus of disagreement being

[14] It is also worth noting that even if they abjure model theory, inferentialists should still recognise that inference rules can be out of kilter with our informal grasp of a notion's meaning. And this might be enough even for them to appreciate our arguments in §§1.2–1.3. After all, the latter were mostly cast informally, and did not require a full-dress model-theoretic semantics. In particular, we provided the semantic clause for the quantifier Q in a footnote, which we could have omitted. As further evidence for this same point, Jon Barwise once playfully noted that the Q-rules seem sound for the informal reading of Q as 'many' ([1], p. 45). For the rules to be sound with respect to this reading, the main assumption is that a small number of a small number of things amount to no more than a small number, 'small' being the opposite of 'many'. That we can appreciate Barwise's quip in the absence of a model-theoretic semantics for 'small' shows that inference rules can be judged against an informal grasp of an expression's meaning.

whether inferential relations are explanatorily prior to referential relations or vice versa ([16], p. 199).[15]

Clearly, on this standard picture, soundness and completeness remain key. The meanings of expressions, as determined by the inference rules, must be reflected by the referentialist semantics. That said, the details of our challenge in this paper do not hinge on taking inferentialism as a fundamentally metasemantic thesis. It also applies to inferentialists, who adopt a semantic—as well as (or possibly instead of) a metasemantic—approach, an example being Incurvati and Schlöder [12].

While standard, we do not pretend that every inferentialist fits this mould. Inspired by McGee [14], Murzi and Topey [17] argue that the inferentialist's dispositions to infer should be open-ended, meaning not fixed to a particular language but ranging over potential extensions of the language. By this means, Murzi and Topey ([17], pp. 3410–11) vindicate the second-order universal and existential quantifiers as logical, because I- and E-rules for these can be given in terms of possible languages, any relation in the domain being nameable in some possible language. Yet, as is well known, second-order logic with standard semantics is not complete, so Murzi and Topey give up the requirement of completeness. Likewise, some recent work in proof-theoretic semantics considers its extension to second- and higher-order logic, e.g. Gheorghiu and Pym [8]. In such systems, completeness does not feature.[16]

It is not part of our aim in this paper to assess exceptions such as Murzi and Topey's open-ended inferentialism. We note simply that it is non-standard, since the great majority of inferentialists believe they can specify the meanings of expressions of *our* language in terms of *our* language. Inferences, or dispositions to infer, are standardly given in terms of our language, not any possible one.

We have seen that typical inferentialists are committed to completeness. In §4.1, we shall further argue that they should be. This is what we mean by saying that completeness is a core inferentialist commitment.

2.2 I- and E-rules

The Q-rules in §1.2 are not presented as introduction (I)-rules and elimination (E)-rules. I-rules specify the circumstances under which an assertion with that expression

[15] Murzi and Steinberger ([16], p. 214) interpret Williamson [37] as attacking inferentialism in the same spirit that motivates the objection examined in the present section. As they point out, accusing inferentialists of having an inferior account of how expressions contribute to the meanings of the sentences in which they feature involves a conflation of semantics with metasemantics. The inferentialist's thesis is metasemantic and need not involve a rejection of the usual semantics. Although Murzi and Steinberger are right about the general point, Williamson's article is a review of a book by Brandom and specifically targets Brandom. If we read Brandom as an inferentialist who is unwilling to accept standard truth-conditional semantics and does not confine himself to the metasemantic level, Williamson's criticism is valid. How to read Brandom correctly is not something we can go into here, although we do note that, as he puts it, the 'constructive task of semantic theories of inferentialist shape' is 'roughly, underwriting representational semantic characterizations in terms of inferential ones' ([3], p. 659). On the basis of this admittedly single sentence, even Brandom seems to be a standard inferentialist.

[16] The aim of proof-theoretic semantics is to give a semantics in terms of proofs. It defines the validity of formulas not with respect to structures as in model theory but with respect to proof-theoretically defined atomic bases; see Schroeder-Heister [28] for an early statement, [29] for a more detailed account, and [30] for a recent exposition. In this programme, the question of completeness makes sense and is a live question.

as main operator is warranted, and E-rules specify what follows from an assertion with that expression as main operator. It is common for inferentialists to present rules for the use of a given expression in terms of I- and E-rules. So one might worry that the Q-rules are not inferentially acceptable ones.

We recognise that inferentialists usually present meaning specifications as I- and E-rules. However, although common, we do not believe it is necessary. By inferentialists' own lights, they need not be committed to this; rather, they should be committed to I- and E-rules being just one of the ways in which one can fix the meaning of an expression. The second point is that I- and E-rules for a quantifier like Q should not be expected.

Our first point is essentially that the demand for I- and E-rules puts the cart before the horse. To explain why, it will be helpful briefly to consider the motivation for inferentialism.

Inferentialists hold that meanings, in some sense, come from their use in inference. Roughly, expressions don't get their meanings *externally* by their relationship to the world, but rather *internally* by their relationship to other expressions. An expression like 'and', for example, gets its meaning from the fact that a sentence in which it is the main operator, such as 'A and B', follows from 'A' and 'B', and that 'A and B' in turn implies each of 'A' and 'B'.

On the inferentialist picture, then, it is the use of an expression that bestows meaning on it. The role of the formalism is to represent this. For the logical inferentialist, the proof system is meant to codify this use. As Steinberger puts it, 'it is the practice represented, not the formalism as such, that confers meanings' ([33], p. 335). For this reason, the particular proof-theoretic articulation of the rules is less important than the practice that those rules reflect.

Now, although natural-deduction systems use I- and E-rules, many systems do not. The proof system for, say, first-order logic can be presented as a (Gentzen-style) natural-deduction system or a (Hilbert-style) axiomatic system. And, while the first usually involves I- and E-rules, the second usually does not. In a typical axiomatic system, the only propositional connectives are the conditional and negation and the only rule pertaining to these is modus ponens — so the system is lacking an I-rule for the conditional as well as both I- and E-rules for negation. Or consider that in typical deductive presentations of many relevance logics, many of the axioms and rules plainly do not take the form of I- and E-rules.

The question is whether a system using I- and E-rules is automatically better at reflecting meaning-constituting practice than any others. Is there a sound reason for the inferentialist to privilege natural-deduction systems for the job? There is a decent sense in which an axiomatic system is more difficult to use and less natural than a natural-deduction system, hence the latter's name. But we know from experience that students often find tree systems, in which only a subset of the rules are considered I- and E-rules for a particular connective, more intuitive and natural still.

Moreover, if natural-deduction systems serve the inferentialist's purposes better it is only because they implement the idea that the meaning of a connective is determined by its associated rules, one set of rules per connective. Nothing in this idea requires the associated rules to be I- and E-rules. Learning how to manipulate symbols is not always reducible to learning how to introduce or eliminate them.

Nor should the inferentialist put too much weight on the particular proof system. For consider the meanings of the logical constants captured by various different proof systems. Limiting our focus again to first-order logic, the inferentialist would want to say that axiomatic, natural-deduction, tree and sequent calculus presentations capture the same meaning of ‘and’. After all, these systems all deliver the same extensions of what is provable. So to say that one captures the meaning of ‘and’ better than another is to deny that meaning supervenes on inferential power. The inferential powers are the same in every case.

This is not to say that one system cannot be more helpful or useful than another, just that they all capture the same meanings. As the name suggests, inferentialists will have a preference for inference rules over axioms, given their thesis that inference is what determines meaning. But note that, in conformity with this preference, Q was characterised in §1.2 using inference rules, just not I- and E-rules.

It seems to us, therefore, that the demand for I- and E-rules puts the formal cart before the use-theoretic horse. Formalism is secondary to use, and use may not take the form of I- and E-rules. Since this issue has not been much discussed in the literature, we are doing work on our opponents’ behalf in exploring the potential motivation for I- and E-rules. But, it seems to us, there is little in the motivation for the view to insist on I- and E-rules. Other sorts of presentations can serve inferentialist goals just as well, including for example the way we presented the Q rules above.

In support of the same conclusion, but more briefly, remember that Q expresses ‘there are uncountably many’. It is akin to an existential quantifier. In standard natural-deduction presentations, I-rules for existential quantifiers introduce the existential generalisation from an instance. Cardinality quantifiers introduce existential generalisations from the appropriate number of distinct instances; for example, if we have 327 provably distinct instances of Fs, we can assert ‘there are at least 327 Fs’. Even if a proof system lacked identity, so that ‘there are at least 327’ could not be defined from the existential quantifier and identity, we could conjure up an introduction rule for this quantifier using 327 constants.[17]

So, if Q were given an I-rule, it would presumably follow this pattern: if we have uncountably many distinct instances, we can assert ‘there are uncountably many’. But any plausibly inferentialist system, such as countable FOL to which we added Q above, will have no more than countably many names. In any such system, we are not able to express the grounds for such an I-rule because we cannot list uncountably many provably distinct things. So it should come as no surprise that Q doesn’t have an I-rule of this kind.

To recap, inferences, or dispositions to infer, are what matter to the inferentialist. The rules for Q perfectly well capture its inferential role.[18] They are not I- and E-rules, but those are not essential and nor should they be expected for a quantifier such as Q. This, again, is all work on our opponents’ behalf, but we see no motivated reason to insist on I- and E-rules.

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[17] In the absence of identity, the distinctness of these 327 constants’ referents could be guaranteed by their having different properties.

[18] Although see §3.4 for a qualification.

2.3 Conclusion

We have motivated completeness as a standard inferentialist commitment. We have also seen that inference rules need not take the form of I- and E-rules. Core inferentialists, who are or should be committed to both, therefore ought to be concerned by our challenge.

We now turn to some objections to the first premise in §3 and to the second premise in §4, where we will revisit the issue of completeness. As already noted, it should be understood throughout that our target is core inferentialists, who share the commitments detailed in the present section.

3 Responses to the First Premise

It is crucial to our challenge that Q is a logical constant by inferentialist standards. Here, we immediately hit an obstacle: it is surprisingly difficult to find a clear articulation of the criteria of logicality in the inferentialist literature.[19] Two such criteria are *harmony* and *conservativeness*. We take them in turn. We cannot delve into all the details here, but the essential points can be summarised as follows.

3.1 Logicality and Harmony

The requirement of harmony asks that there be a balance of inferential powers between a logical constant's I- and E-rules. According to the *local reduction* or *levelling of local peaks* conception of harmony, an I-rule immediately followed by an E-rule for the same connective can both be eliminated, as can easily be done for conjunction for example. This ensures that the E-rule is not stronger than the I-rule. A symmetric requirement of *local expansion*, often omitted by inferentialists, must then be added to ensure that the E-rule is not weaker than the I-rule either.[20]

Some version of harmony could therefore be argued to be a criterion for logicality, as Tennant does ([36], p. 296, fn 24). The problem with this idea is threefold. First, harmony only applies to terms with I- and E-rules. If logicality requires harmony, expressions that cannot be given in terms of harmonious rules must be declared non-logical. Yet if the discussion in §2.2 is along the right lines, this should strike us as wrong-headed. If there is no good reason to restrict inferential rules to I- and E-rules, there is presumably no good reason for a criterion of logicality to apply only to connectives possessing I- and E-rules.

Second, as Rumfitt [26] notes, although harmonious rules are of great proof-theoretic interest, the philosophical motivation for harmony is wanting. Its main motivation stems from the controversial view of consequence as a matter of pre-

[19] This is in stark contrast to the model-theoretic literature on the logical constants, where invariance criteria of one sort or other are generally taken as the standard. See Part II of Griffiths and Paseau [10] for discussion, and Part III of that same book for a response to objections, which develops some of the points in the earlier [9] or [18].

[20] See Steinberger [34] for the many different versions of harmony discussed in the literature.

serving correct assertibility rather than truth (the latter being the usual view). And, as Rumfitt also points out ([26], p. 229), even if we take consequence to preserve correct assertibility, the chief argument for harmony assumes that whenever we have some ground for asserting a statement, we must have a *direct* ground for asserting it — yet we may well, for example, have no direct grounds for asserting either disjunct of a disjunction we have grounds for asserting.[21]

Third, harmony does not work on its own terms. The rules for classical negation are not harmonious, yet most inferentialists today would, or should, be loath to give up classical logic.[22] Whether it applies to identity and modal operators is at best moot.[23] On account of all these reasons, harmony is an inadequate criterion of logicality.

3.2 Logicality and Conservativeness

Other than various versions of harmony, the only other requirement discussed by inferentialists is that of *conservativeness*. We know from Prior's [19] infamous example of *tonk* that not just any rules succeed in bestowing a coherent meaning on a logical constant. Conservativeness is respected when the introduction of a new item of vocabulary, along with its rules or associated principles, does not extend the range of derivations in the old vocabulary. Belnap [2] offered conservativeness as a response to the challenge posed by *tonk*. Dummett [6] also discusses this requirement under the label of 'total harmony'. And global inferentialists such as Brandom ([4], pp. 68–69) have frequently appealed to conservativeness as a condition on a putative new logical constant or, better, a collection of constants.

Conservativeness is not motivated in the same problematic way harmony is, although exactly how best to motivate it remains an open question. Classical negation is not conservative over the usual fragments of propositional logic,[24] and the test for conservativeness also raises a question the one for harmony does not: relative to what system(s) must the rules for a new connective be conservative? Setting these points aside, the great advantage conservativeness enjoys over harmony is that it can apply to connectives lacking I- and E-rules. And indeed we can show that the Q-rules are conservative relative to countable first-order logic. Consider a sound and complete proof system $\vdash_{\mathsf{FOL}}$ for countable first-order logic (FOL), and the proof system $\vdash_{\mathcal{L}_Q}$ above for FOL augmented with the quantifier Q. The system $\vdash_{\mathcal{L}_Q}$ is conservative over $\vdash_{\mathsf{FOL}}$. Here is the simple argument, for Γ a countable set of FOL-sentences and ϕ an FOL-sentence:

(1) $\Gamma \vdash_{\mathcal{L}_Q} \phi$ Assumption

[21] See Rumfitt [26] for further arguments against other motivations.

[22] An exception is Neil Tennant; see e.g. Tennant [36]. As is well known, the late Michael Dummett pressed the requirement of harmony into service as an argument against classical logic; see Dummett [6]. Note that our 'should' does not assume classical-logic monism; it also encompasses logical pluralism, as long as the latter deems classical logic one of the acceptable logics.

[23] For reference and discussion, see Section 4.1 of Griffiths and Paseau [11].

[24] When the standard rules for classical negation are added to a propositional logic system that already includes the rules for the conditional (and, optionally, for conjunction and/or disjunction), Peirce's Law becomes derivable.

(2)	$\Gamma \vDash_{\mathcal{L}_Q} \phi$	from (1) by $\mathcal{L}_Q$-Soundness
(3)	$\Gamma \vDash_{\mathsf{FOL}} \phi$	from (2)
(4)	$\Gamma \vdash_{\mathsf{FOL}} \phi$	from (3) by FOL-Completeness

The inference from (2) to (3) is underwritten as follows. Let $\mathcal{M}$ be an FOL-model in which the set of FOL-sentences Γ is true and ϕ is false. Then $\mathcal{M}$ is also an $\mathcal{L}_Q$-model in which Γ is true and ϕ is false, since the additional Q-clauses don't affect the evaluation of these FOL-sentences. Hence it follows that if $\Gamma \nvDash_{\mathsf{FOL}} \phi$ then $\Gamma \nvDash_{\mathcal{L}_Q} \phi$.[25]

In sum, we don't think the requirement of harmony is an appropriate one to impose on Q (§3.1). That leaves conservativeness, which is also not problem-free, but at least applies to connectives without I- and E-rules. We saw that Q passes this test. You might say that neither harmony nor conservativeness is the right inferentialist test for logicality. But in that case, inferentialists are in even deeper trouble, since these are the main criteria that several decades of inferentialist-friendly literature has thrown up.

3.3 Impurity

A distinct but related worry that the inferentialist might raise about the Q-rules appeals to their *impurity*. Rules are pure when they make no use of other logical constants. By this standard, the Q-rules are plainly impure: identity, disjunction, conditionals and first-order quantifiers all make an appearance.

We will be brief on this response since, although some inferentialists such as Tennant ([36], pp. 314–5) demand purity, it lacks motivation. Milne, for example, discusses Tennant on purity but concludes that there is 'precious little in the way of argument' ([15], p. 524). Other inferentialists, such as Dummett, explicitly welcome impure rules. Dummett does note that there are problems in the vicinity:

> Certainly we do not want such a relation dependence to be cyclic; but there would be nothing in principle objectionable if we could so order the logical constants that the understanding of each depended only on the understanding of those preceding in the ordering. ([6], p. 257)

It does not strike us as certain that a dependence relation cannot be cyclic: perhaps a collection of interdependent constants are learnt together rather than individually. Be that as it may, the problem Dummett describes does not arise for the Q-rules. We can learn the meanings of the standard first-order logical constants first, and then the meaning of Q second. Indeed, this is surely the expected order. We see no reason to think that the logical constants should be learnable in any possible order, any more than we should be able to learn the meaning of 'vixen' before 'female' and 'fox'. So we find no reason to think impurity poses a problem.

[25] Although the language of FOL and that of $\mathcal{L}_Q$ are different, models for them may be considered to be the same. If you prefer to type models in such a way that no $\mathcal{L}_Q$-model is an FOL-model, reformulate the argument in the obvious way.

3.4 Unique Characterisation

A distinct line of response from the inferentialist runs as follows: the Q-rules do not uniquely determine the quantifier's meaning.

This response gets off the ground because the Q-rules are not just sound and complete for the quantifier Q as 'there are uncountably many', which we could express more formally as 'there are at least $\aleph_1$-many'. So-interpreted, the Q-rules are sound and complete for countable sets of sentences. It turns out, however, that the rules are likewise sound and complete for the quantifier 'there are at least $\aleph_2$-many', 'there are at least $\aleph_3$-many' and many other cardinality quantifiers.[26] Clearly, a great many readings of Q as a transfinite cardinality quantifier are compatible with the $\mathcal{L}_Q$-completeness theorem.

There is, however, a telling rejoinder. It is that categoricity can be restored if we specify that Q is the weakest cardinality quantifier for which the stated inferential rules are sound and complete. Informally, the idea is that 'there are at least $\aleph_1$-many As' is implied by 'there are at least $\aleph_\alpha$-many As' for $\alpha \geq 1$. So if Q^* is a cardinality quantifier specified as satisfying all and only the same rules as Q then $Q^*\phi$ implies $Q\phi$ for any formula ϕ. In short, 'there are at least $\aleph_1$-many' is the weakest in the class of cardinality quantifiers for which the Q-rules are sound and complete. If only cardinality quantifiers can be expressed using these rules then 'there are at least $\aleph_1$-many' can be singled out inferentially.

The acute reader will have spotted that the rules in §1.2 are also sound for the quantifier 'there are infinitely many'. However, they are not complete for this quantifier, as we saw in §1.3. This serves to distinguish 'there are infinitely many' from 'there are uncountably many'. The inferentialist can think of the Q-rules as being supplemented with a 'that's all' clause, which rules in 'there are uncountably many' and rules out 'there are infinitely many'.

We can, in short, inferentially distinguish 'there are at least $\aleph_1$-many' from stronger cardinality quantifiers such as 'there are at least $\aleph_2$-many' and from the weaker one 'there are at least $\aleph_0$-many'.

4 Responses to the Second Premise

4.1 Completeness Again

The argument for the non-logicality of I assumes that any logic acceptable to the inferentialist is sound and complete, hence compact. We saw in §2.1 that inferentialists typically accept this requirement. But should they resist it? We sketch some reasons that they should not. We begin by considering two different sorts of inferentialist approaches.

[26] Which ones exactly? The rules are sound for any quantifier 'there are $\aleph_\alpha$-many' for which $\aleph_\alpha$ is a regular infinite cardinal. Moreover, under the Generalised Continuum Hypothesis, the completeness theorem for countable sets of $\mathcal{L}_Q$-sentences holds for the interpretation of Q as 'there are κ^+-many', where κ^+ is the successor of a regular (infinite) cardinal κ. See Schmerl ([27], p. 183).

One approach is to understand logical consequence proof-theoretically, as the conclusion's derivability from the premise set. Now, this picture should make room for semantics if it is to be plausible overall (recall the discussion in §2). So there is no question of forgoing the model-theoretic apparatus that allows us to formalise A—our informal argument with infinitely many premises and conclusion 'There exist infinitely many pears'—as a valid sequent in $\mathcal{L}_I$, all on the assumption that I is logical. But on this picture, consequence is equated with deducibility, so semantics is secondary and merely shadows deductive inference. Soundness and completeness fall out of this shadowing. In particular, if a conclusion logically follows from a set of sentences then it must do so in the inferentialist's proof system. In other words, it must be deducible from a finite subset of these sentences, since this is a quasi-definitional feature of proof systems.

A different approach is to accept the mainstream semantic characterisation of logical consequence. Following the mainstream, inferentialists might cash out logical consequence in semantic terms, most likely by means of the standard model-theoretic account. What makes the conception distinctly inferentialist is that expressions' semantic values (and thus sentences' truth-conditions) are ultimately owed to their inferential roles. Ian Rumfitt call this sort of inferentialist approach *indirect*.

> ...the IRS [Inferential Role Semantics] theorist is free to take an indirect approach. He might take the rules that characterize a connective's inferential role as specifying its sense, but allow that it also has a reference, or a semantic value. The semantic value will be the contribution the connective makes to the truth-conditions of a formula in which it occurs. Once we have a specification of truth-conditions for formulae of the relevant language, we can apply the traditional account of consequence in terms of the preservation (or necessary preservation) of truth. (2017, p. 244)

Context makes it clear that this is Rumfitt's own view. Furthermore, he remarks that in his Dewey lectures of the early 2000s, Dummett retreated to this position; and he enlists Christopher Peacocke and Harold Hodes as further inferentialists of the indirect ilk. In these last two inferentialists' works, one finds a 'determination theory', that is, an account of how inferential roles determine semantic values.

On this second, indirect, sort of inferentialism, logical consequence is owed to semantic relations. But semantic relations are determined by the connectives' inferential roles. Logical consequence, therefore, cannot outstrip the inferential rules associated with the connectives. (Be they I- and E-rules or not, as discussed in §3.1.) Soundness and completeness fall out of the essential link between semantics and inferential role, i.e. the determination theory. It follows that on this approach as well, soundness and completeness is a safe assumption.

Whichever approach one takes, the first premise of the uniformity challenge, in §1.2, remains plausible.

As further evidence that inferentialists accept completeness and hence compactness, consider Carnap's categoricity problem. Most inferentialists regard it as a problem worth addressing.[27] Very briefly, Carnap [5] argued that standard proof-theoretic treat-

[27] See Griffiths and Paseau [11].

ments of many first-order logical constants are compatible with many interpretations of those logical constants. For example, the standard truth tables for $\neg$, $\vee$ and $\rightarrow$ cannot be recovered from the proof-theoretic treatments of these expressions. These inference rules' usual proof-theoretic codification has failed to determine these logical constants' meanings.

Carnap's categoricity problem is well-established in the literature and accepted as a genuine challenge by inferentialists, who have proposed a variety of responses.[28] We are using soundness and completeness in the same way as they are used to generate the categoricity problem. These results are reasonable tools to determine whether some rules have conferred a stable meaning on a term. If inferentialist proof systems were not held to the standard of completeness, Carnap's observation would not be a problem, since a connective's proof rules might underdetermine its meaning. Inferentialists regard it as a genuine problem precisely because they accept the requirement of completeness and the need to read off semantics from proof rules.

4.2 Logical Validity

Our *reductio* argument involved taking argument A to be logically valid, but taking none of its finite subarguments to be valid. The second point should be clear, since the conclusion that there are infinitely many pears, far from being a logical truth, is in fact false, and it is not entailed by any finite subset of the premises.

As for the logical validity of A, it's hard to see how we might resist this if I is assumed logical (this is the assumption which the argument in §1.3 aimed at reducing to absurdity). We established in §2 that an inferentialist should accept a semantic treatment, just as long as they ultimately see it as based on an inferential account of the constants. And it should be clear that in the logic $\mathcal{L}_I$, the conclusion of A', A's formalisation, is a consequence of its premises. Upon first encountering A, one might wonder whether its manifest validity is owed to logic or to mathematics, given the presence of the cardinality quantifier 'there are infinitely many', which seems to include some mathematical content. But any such doubt is surely removed when we are firmly told that this cardinality quantifier is logical, as we are for the purposes of the *reductio*. Since for the purposes of the *reductio* $\mathcal{L}_I$ is assumed to be just as logical as first-order logic, the arguments it sanctions as valid must be logically valid.

The final potential response is that we have put our finger on a problem for inferentialism but misdiagnosed its source. As we have seen, argument A, formalised as A' in $\mathcal{L}_I$, is valid but its validity cannot be respected by any compact logic. But logic, for a core inferentialist, must be compact if it is complete. Hence inferentialism is untenable, not because it suffers from a problem of uniformity but because it condones a compact logic that cannot respect the validity of A. The root of the problem is not uniformity but the inferentialist's commitment to a sound and complete, and hence compact, logic.

How should inferentialists think of argument A? Because it lacks the valid-finite-subargument property, the inferentialist will think that this sequent is *not* logical.

[28] They include bilateral systems, put forward in Rumfitt [25] elaborating on Smiley [32], as well as multiple-conclusion systems, discussed for example in Steinberger [33].

And the obvious culprit is the quantifier ‘there are infinitely many’, formalised as I. Because this quantifier is not logical, the inferentialist will say, you should not expect it to have use-rules that underwrite the argument’s validity, in the way that the Boolean connectives’ use-rules underwrite the validity of propositional sequents. So A is not logically valid, which is compatible with the core inferentialist’s commitment to a sound and complete and therefore compact logic.

This is not to deny the meaningulness of ‘there are infinitely many’ for the inferentialist. Sensible inferentialists must see the word ‘infinite’ and well-formed expressions containing it or its cognates as meaningful. And as they see it, these meanings must in some way bottom out in inference rules. Nothing we have said here precludes this. Our point has been that, however meaning is bestowed on it, ‘there are infinitely many’ is not plausibly logical, for the inferentialist, as argued in §1.3.

In short, we suggest that the inferentialist should see ‘there are infinitely many’ as non-logical, on pain of giving up compactness, which follows from the commitment of completeness discussed in §2. If they should see ‘there are uncountably many’ as logical, as we also suggested, their account of cardinality quantifiers is not uniform.

5 Summary

Suppose an inferentialist accepts the assumptions presented and motivated in this paper. Then, if they accept at least first-order logic with identity, they take the finite cardinality quantifiers to be logical, ‘there are infinitely many’ to be non-logical and ‘there are uncountably many’ to be logical. This inferentialist therefore fails to meet the requirement of uniformity: their judgements of logicality are haphazard. The challenge applies to logical inferentialism, the most plausible local version of the more general inferentialist project.[29]

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References

1. Barwise, J. (1977). An introduction to first-order logic. In J. Barwise (Ed.), *Handbook of Mathematical Logic* (pp. 5–46). Elsevier.
2. Belnap, N. (1962). Tonk, plonk and plink. *Analysis, 23*, 130–1.
3. Brandom, R. (2007). Inferentialism and some of its challenges. *Philosophy and Phenomenological Research, 74*, 651–676.
4. Brandom, R. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.
5. Carnap, R. (1943). *Formalization in logic*. Harvard University Press.
6. Dummett, M. (1991). *The logical basis of metaphysics*. Harvard University Press.
7. Garson, J. (2013). *What logics mean*. Cambridge University Press.
8. Gheorghiu, A. & Pym, D. (2025). Proof-theoretic semantics for second-order logic. arXiv:2508.07786.
9. Griffiths, O. & Paseau, A. C. (2016). Isomorphism invariance and overgeneration. *Bulletin of Symbolic Logic, 22*, 482–503.
10. Griffiths, O. & Paseau, A. C. (2022). *One true logic*. Oxford University Press.
11. Griffiths, O. & Paseau, A.C. (2026). Model-theoretic and proof-theoretic validity. In E. Brendel, M. Carrara, F. Ferrari, O. Hjortland, G. Sagi, & G. Sher (Eds.), *Oxford handbook of philosophy of logic* (pp. tbc). Oxford University Press.
12. Incurvati, L. & Schlöder, J. (2023). *Reasoning with attitude*. Oxford University Press.
13. Keisler, H. J. (1970). Logic with the quantifier "there exist uncountably many". *Annals of Mathematical Logic, 1*, 1–93.
14. McGee, V. (2000). Everything. In G. Sher & R. Tieszen (Eds.), *Between logic and intuition: Essays in honor of Charles Parsons* (pp. 54–78). Cambridge University Press.
15. Milne, P. (2002). Harmony, purity, simplicity and a "seemingly magical fact". *The Monist, 85*, 498–534.
16. Murzi, J. & Steinberger, F. (2017). Inferentialism. In B. Hale, C. Wright, & A. Miller (Eds.), *A companion to the philosophy of language* (pp. 197–224). Wiley Blackwell.
17. Murzi, J. & Topey, B. (2021). Categoricity by convention. *Philosophical Studies, 178*, 3391–3420.
18. Paseau, A. C. (2014). The overgeneration argument(s): A succinct refutation. *Analysis, 74*, 40–7.
19. Prior, A. (1960). The runabout inference-ticket. *Analysis, 21*, 38–9.
20. Quine, W. V. (1970). *Philosophy of logic*. Harvard University Press.
21. Read, S. (2000). Harmony and autonomy in classical logic. *Journal of Philosophical Logic, 29*, 123–154.
22. Read, S. (2004). Identity and harmony. *Analysis, 64*, 113–9.
23. Read, S. (2008). Harmony and modality. In C. Dégremont, L. Kieff, & H. Rückert (Eds.), *Dialogues, logics and other strong things: Essays in honour of Shalid Rahman* (pp. 285–303). College Publications.
24. Read, S. (2016). Harmonic inferentialism and the logic of identity. *Review of Symbolic Logic, 9*, 408–20.
25. Rumfitt, I. (2000). "Yes" and "No". *Mind, 109*, 781–823.
26. Rumfitt, I. (2017). Against Harmony. In B. Hale, C. Wright, & A. Miller (Eds.), *A companion to the philosophy of language* (pp. 225–249). Wiley Blackwell.
27. Schmerl, J. H. (1985). Transfer theorems and their applications to logics. In J. Barwise & S. Feferman (Eds.), *Model-theoretic logics* (pp. 177–209). Springer-Verlag.
28. Schroeder-Heister, P. (1991). Uniform proof-theoretic semantics for logical constants (abstract). *Journal of Symbolic Logic, 56*, 1142.
29. Schroeder-Heister, P. (2006). Validity concepts in proof-theoretic semantics. *Synthese, 148*, 525–571.
30. Schroeder-Heister, P. (2023). Proof-theoretic semantics. *Stanford encyclopedia of philosophy*.
31. Sher, G. (1991). *The bounds of logic*. The MIT Press.
32. Smiley, T. (1996). Rejection. *Analysis, 56*, 1–9.
33. Steinberger, F. (2011). Why conclusions should remain single. *Journal of Philosophical Logic, 40*, 333–55.
34. Steinberger, F. (2011). What harmony could and could not be. *Australasian Journal of Philosophy, 89*, 617–639.

35. Tarski, A. (1986). What are logical notions? *History and Philosophy of Logic*, *7*, 143–54.
36. Tennant, N. (1997). *The taming of the true*. Oxford University Press.
37. Williamson, T. (2010). Review of Robert Brandom, reason in philosophy: Animating ideas. *Times Literary Supplement, 5579*, 22–23.