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On two reversible cellular automata with two particle species

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Abstract

We introduce a pair of time-reversible models defined on the discrete space–time lattice with three states per site, specifically, a vacancy and a particle of two flavours (species). The local update rules reproduce the rule 54 reversible cellular automaton when only a single species of particles is present, and satisfy the requirements of flavour exchange (C), space-reversal (P), and time-reversal (T) symmetries. We find closed-form expressions for three local conserved charges and provide an explicit matrix product form of the grand canonical Gibbs states, which are identical for both models. For one of the models this family of Gibbs states seems to be a complete characterisation of equilibrium (i.e. space and time translation invariant) states, while for the other model we empirically find a sequence of local conserved charges, one for each support size larger than 2, hinting to its algebraic integrability. Finally, we numerically investigate the behaviour of spatio-temporal correlation functions of charge densities, and test the hydrodynamic prediction for the model with exactly three local charges. Surprisingly, the numerically observed ‘sound velocity’ does not match the hydrodynamic value. The deviations are either significant, or they decay extremely slowly with the simulation time, which leaves us with an open question for the mechanism of such a glassy behaviour in a deterministic locally interacting system.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Understanding dynamical phenomena in large systems of interacting particles is one of the central issues of statistical mechanics [1]. Although the microscopic laws of motion in physical systems are *deterministic*, one often simplifies the models by considering a stochastic microscopic description to obtain analytically tractable models; such as simple exclusion processes and reaction–diffusion processes [2]. Nevertheless, the ultimate goal should be to derive the emergence of universal macroscopic statistical laws, say the Fick’s or Fourier’s law of diffusive transport and precise conditions for various super- or sub-diffusive anomalies, from deterministic reversible laws of motion. Ideally, one would like to achieve this in typical Hamiltonian systems of interacting particles, such as Fermi–Pasta–Ulam–Tsingou (FPUT) chains [3, 4], but at the moment this seems out of reach within any rigorous framework. Arguably the most suitable systems for this purpose, which can be viewed as caricatures of Hamiltonian dynamics containing its essential features, are reversible cellular automata (RCA).

The simplest type of RCA, namely in one spatial dimension and with a three-site Margolus neighbourhood, have been completely classified in reference [5], where it has been pointed out that the reversible cellular automaton with the rule code 54 (RCA54)³ has all the features of an integrable interacting particle system with solitonic excitations and conserved currents. Indeed, in recent years, equilibrium and non-equilibrium statistical mechanics of RCA54 has been essentially completely solved (see reference [7] for a review), where probably the main achievement was a rigorous derivation of diffusive dynamical structure factor [8]. The main reason for the remarkable utility of RCA54 lies in the underlying algebraic structures which seem to go beyond Yang–Baxter integrability and allow for exact access of time-dependent statistical quantities, such as dynamical correlation functions of local observables. Nevertheless, the observations of anomalous (superdiffusive) transport in some Hamiltonian particle systems, like FPUT and related anharmonic oscillator chains [3, 4], or integrable chains with nonabelian symmetries [9–11], possess a natural question about the existence of an exactly solvable RCA featuring anomalous transport properties. The obvious playground to look for such models are multispecies RCA which would reduce to RCA54 under some limiting situations, say for dynamics restricted to configurations with single particle species or isolated solitonic excitations. Permutation symmetry among the species (or particle flavours) may then serve as a kind of discrete analog of $SU(2)$ symmetry which was the minimal requirement for observing anomalous spin transport of Kardar–Parisi–Zhang universality in integrable spin chains [9, 12].

In this paper we construct two one-dimensional RCA with three states per site, which naturally generalise RCA54 to two particle species. We show that, by requiring the update rules to be invariant under the species permutation (conjugation), space reversal and time reversal, the model is fixed up to one binary choice. One choice corresponds to the particle flavour conservation in the annihilation–creation process. For this model we find a numerical evidence of an extended number of conservation laws with local charge density, which hints to its integrability, but the Yang–Baxter integrability structures are still not known. By flipping the species conjugation in the particle annihilation/creation processes, we obtain the second model, which

³ Equivalent to rule code 250R of reference [6], which used a slightly more complicated setup.

has exactly the same local charges of support size 2, but lacks additional local conservation laws and is hence likely to be non-integrable. We provide a simple algebraic characterization of Gibbs equilibrium states for both models, and extend the result to a stationary generalized Gibbs state with one higher conservation law for the integrable model. Finally, we simulate spatio-temporal correlation functions of the three local charge densities. In the integrable case the situation seems complex and difficult to describe quantitatively due to the coupling among an extensive number of charges, while the non-integrable case with exactly three charges may be amenable to nonlinear fluctuating hydrodynamics which has been successfully applied to anharmonic (e.g. FPUT) particle chains [13]. We find a small but potentially significant deviation from the hydrodynamic prediction even for the sound velocity, which is determined by the standard Euler-scale hydrodynamics. The results are not inconsistent with the numerically determined velocity eventually converging to the value predicted by hydrodynamics at ‘astronomically’ long times, but the reason for existence of such long time scales in such a simple parameter free model remains obscure.

2. Definition of the dynamics

2.1. Time evolution of configurations

The main idea behind the construction of the dynamical rules is an attempt to generalize the dynamics of RCA54 to a model with two particle species. It is defined on the lattice of length $2n$, where each site can be either empty (denoted by 0) or occupied by a particle of one of two species (1 and 2). The configuration at time t is given by a ternary string $\underline{s}^t = (s_1^t, s_2^t, \dots, s_{2n}^t)$, $s_j^t \in \{0, 1, 2\}$, and time evolution is defined in two time-steps,

$$\underline{s}^{t+1} = \begin{cases} \chi_\alpha^e(\underline{s}^t), & t \equiv 0 \pmod{2}, \\ \chi_\alpha^o(\underline{s}^t), & t \equiv 1 \pmod{2}, \end{cases} \quad (1)$$

where the subscript $\alpha \in \{1, 2\}$ discriminates between the two different dynamical maps. At even time-steps the even sites change, while in the odd time-steps the values at the odd sites change,

$$\begin{aligned} \chi_\alpha^e(s_1, s_2, \dots, s_{2n}) &= (s_1, s'_2, s_3, s'_4, \dots, s'_{2n}), \\ \chi_\alpha^o(s_1, s_2, \dots, s_{2n}) &= (s'_1, s_2, s'_3, s_4, \dots, s_{2n}), \end{aligned} \quad (2)$$

where the updated values s'_j are given by a local three-site update rule

$$s'_j = \chi_\alpha(s_{j-1}, s_j, s_{j+1}). \quad (3)$$

Local maps χ_1 and χ_2 differ only in one three-site configuration,

$$\chi_1(0, s, 0) = s, \quad \chi_2(0, s, 0) = \bar{s}, \quad (4)$$

where $s \in \{1, 2\}$ and $\bar{\cdot}$ denotes the flip (exchange) of particle species, $\bar{s} \equiv (3 - s) \pmod{3}$. All other rules coincide. To determine them, we first require that they reproduce the RCA54 behaviour when only one particle is present, which implies the following

$$\begin{aligned} \chi_\alpha(0, 0, 0) &= 0, \quad \chi_\alpha(0, 0, s) = s, \quad \chi_\alpha(0, s, s) = 0, \quad \chi_\alpha(s, 0, 0) = s, \\ \chi_\alpha(s, 0, s) &= s, \quad \chi_\alpha(s, s, 0) = 0, \quad \chi_\alpha(s, s, s) = 0. \end{aligned} \quad (5)$$

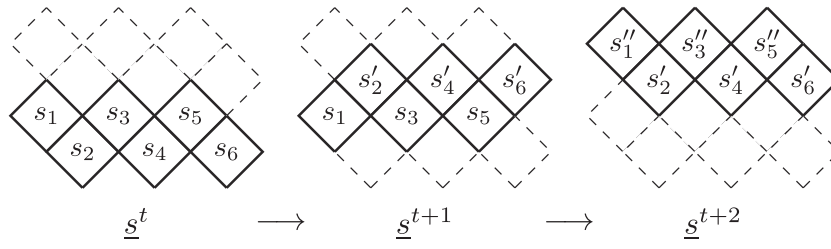


Figure 1. Graphical representation of the geometry of time evolution. In the first time-step the sites at the bottom of the zig-zag saw-shaped chain are updated according to the local time-evolution rules, $s'_j = \chi_\alpha(s_{j-1}, s_j, s_{j+1})$. This can be graphically expressed by removing the site with the old value s_j and add the updated value s'_j to the top of the chain, so that the sites that were previously on the top are now lying on the bottom of the updated chain. In the second time step, the procedure is repeated, $s''_j = \chi_\alpha(s'_{j-1}, s'_j, s'_{j+1})$, which completes the full period of time evolution.

Furthermore, the local map χ_α is required to be reversible, symmetric with respect to the left–right reflection, and invariant under the flip of particle species⁴ $1 \leftrightarrow 2$,

$$\begin{aligned} \chi_\alpha(s_1, \chi_\alpha(s_1, s_2, s_3), s_3) &= s_2, \\ \chi_\alpha(s_1, s_2, s_3) &= \chi_\alpha(s_3, s_2, s_1), \quad \chi_\alpha(\overline{s_1}, \overline{s_2}, \overline{s_3}) = \overline{\chi_\alpha(s_1, s_2, s_3)}, \end{aligned} \quad (6)$$

which, together with equation (5), fix the following update rules,

$$\chi_\alpha(0, s, \overline{s}) = s, \quad \chi_\alpha(s, \overline{s}, s) = \overline{s}, \quad \chi_\alpha(s, \overline{s}, 0) = \overline{s}, \quad \chi_\alpha(s, 0, \overline{s}) = 0. \quad (7)$$

Additionally, the remaining four transitions are determined by

$$\chi_\alpha(s, s, \overline{s}) = \overline{s}, \quad \chi_\alpha(s, \overline{s}, \overline{s}) = s, \quad (8)$$

resulting in stripes of alternating flavour configurations propagating freely (see figure 4 and the discussion at the end of the subsection). Thus the time evolution is completely determined by equations (4), (5), (7), and (8).

To visualize the dynamics the lattice can be drawn in a staggered zig-zag shape, where at each time the bottom of the chain corresponds to the sites that are being updated, while the top sites do not change, as is schematically shown in figure 1. Using this convention, the local time-evolution rules can be summarized by a collection of $27 = 3^3$ diagrams connecting the distinct three-site configurations at the bottom of 2×2 rhombic plaquettes with the updated site at the top, where the two particle flavours are represented by boxes of two distinct colours and vacant sites by empty boxes, as shown in figure 2. The collection of 27 update rule diagrams, as a subset of all $81 = 3^4$ rhombus configurations, is closed under separate actions of C (species flip), P (horizontal reflection), and T (vertical reflection).

In both models, analogously to the RCA54, a pair of consecutive sites with the same colour on the empty background behaves as a soliton that moves with velocity 1 either to the left or to the right. When two oppositely moving particles of the same colour meet they temporarily

⁴ We can argue that our dynamical laws have, respectively, T , P , and C , symmetries.

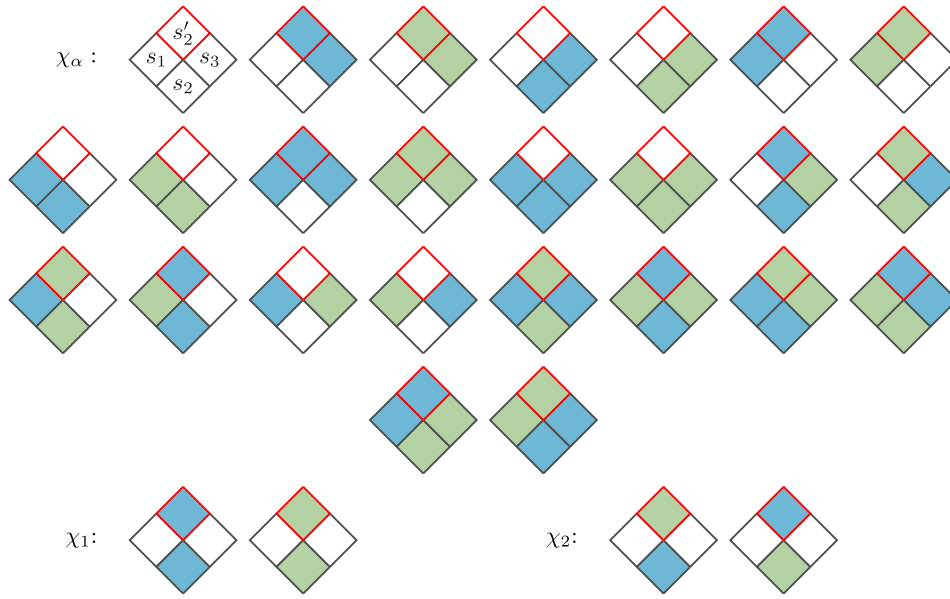


Figure 2. Graphical representation of local time-evolution maps χ_1 and χ_2 . The red-bordered rectangle at the top represents the new central value s'_2 , determined as $s'_2 = \chi_\alpha(s_1, s_2, s_3)$. Blue and green coloured squares correspond to $s_j = 1$ and $s_j = 2$ respectively, while white squares denote empty sites. Time evolution in the two automata differs only in the last two diagrams, $\chi_1(0, s, 0) \neq \chi_2(0, s, 0)$, $s \in \{1, 2\}$ (the bottom most row), while all the others are the same.

merge into one site and then reappear in the next time step, and thus effectively get delayed for one site. However, as is shown on the left part of figure 3, in one case (the automaton given by χ_1) the particles preserve the colour, while in the model given by χ_2 the colour of the particles gets exchanged when scattering. Similarly, the collision of the particles of opposite types differs in the two models: in one case the particles bounce off each other, while in the second system they pass through each other unperturbed. Another feature of the dynamics is the existence of longer composite quasiparticles, which consist of stripes of occupied sites of alternating colour moving at speed 1 or -1 . However, these excitations are not stable under scattering, as can be seen in an example of the time evolution starting from some typical (random) initial state in figure 4.

2.2. States and observables

Before continuing with the discussion of dynamical properties, let us first define the notion of statistical states and observables, and introduce notation used throughout the paper.

A *statistical* (also *macroscopic*) state is a probability distribution over the set of configurations and is completely determined by a 3^{2n} dimensional normalized vector with nonnegative components,

$$\mathbf{p} = [p_0 \quad p_1 \quad \dots \quad p_{3^{2n}-1}]^T, \quad p_s \geq 0, \quad \sum_{s=0}^{3^{2n}-1} p_s = 1. \quad (9)$$

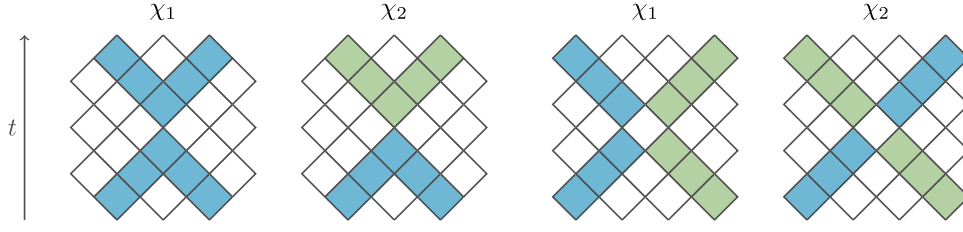


Figure 3. Scattering of simple particles. To understand the interactions between different types of solitons, we start with a configuration with two oppositely moving particles (at the bottom) and evolve it in time (upwards). When the two particles are of the same type, they merge into one site and in the next time step reappear, and effectively obtaining a delay of 1 site. For the model given by χ_1 the particles emerging after the scattering are of the same type as before, while in the case of χ_2 the particle type is changed. If two particles of opposite colours meet, they do not obtain any delay and they either change their directions (χ_1) or pass through each other undisturbed (χ_2).

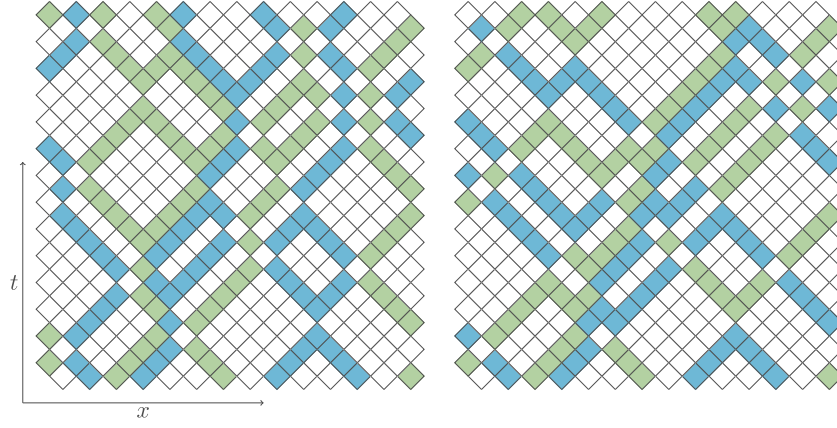


Figure 4. Example of dynamics induced by time-evolution rules. The initial configuration (at the bottom) is evolved in time using the maps χ_1 on the left, and χ_2 on the right.

Each coefficient p_s corresponds to the probability of the configuration given by the ternary representation of s ,

$$s \equiv \underline{s} = (s_1, s_2, s_3, \dots, s_{2n}), \quad s = \sum_{j=1}^{2n} 3^{2n-j} s_j. \quad (10)$$

Time evolution of macroscopic states is given in terms of a 27×27 local propagator with the following matrix elements,

$$U_{(s'_1, s'_2, s'_3), (s_1, s_2, s_3)}^\alpha = \delta_{s'_1, s_1} \delta_{s'_2, \chi_\alpha(s_1, s_2, s_3)} \delta_{s'_3, s_3}, \quad (11)$$

where α encodes the choice of the automaton. Time-evolution is given by two distinct operators $U^{\alpha(e)}$ and $U^{\alpha(o)}$,

$$U^{\alpha(e)} = U_{123}^\alpha U_{345}^\alpha \dots U_{2n-2n,1}^\alpha, \quad U^{\alpha(o)} = U_{234}^\alpha U_{456}^\alpha \dots U_{2n,1,2}^\alpha, \quad (12)$$

where $U_{j-1,j,j+1}^\alpha$ is a three-site operator that nontrivially acts on sites $(j-1, j, j+1)$,

$$U_{j-1,j,j+1}^\alpha = 1^{\otimes j-2} \otimes U^\alpha \otimes 1^{2n-j-1}, \quad 1 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}. \quad (13)$$

Time-evolution of the state \mathbf{p} can be therefore succinctly expressed as

$$\mathbf{p}^{t+1} = \begin{cases} U^{\alpha(e)} \mathbf{p}^t, & t \equiv 0 \pmod{2}, \\ U^{\alpha(o)} \mathbf{p}^t, & t \equiv 1 \pmod{2}. \end{cases} \quad (14)$$

Observables are real valued functions over the set of configurations, $a : \mathbb{Z}_3^{2n} \rightarrow \mathbb{R}$, that prescribe a unique value $a(\underline{s}) \in \mathbb{R}$ to each configuration \underline{s} . Expectation value of an observable a in a state \mathbf{p} is given by

$$\langle a \rangle_{\mathbf{p}} = \sum_{s=0}^{3^{2n}-1} a(s) p_s = \mathbf{a} \cdot \mathbf{p}, \quad \mathbf{a} = [a(0) \ a(1) \ \dots \ a(3^{2n}-1)], \quad (15)$$

where we identify $a(s) \equiv a(\underline{s})$, with \underline{s} being the ternary representation of s . The last equality implies that the space of observables can be understood as the vector space that is *dual* to the space of macroscopic states. Additionally, component-wise multiplication is well defined,

$$(a \cdot b)(\underline{s}) = a(\underline{s}) b(\underline{s}), \quad (16)$$

which makes the space of observables a commutative algebra. *Local* observables act nontrivially only on a finite subsection of the chain, and their space is spanned by the following convenient basis,

$$[q]_j(\underline{s}) = \delta_{s_j, q}, \quad q \in \{0, 1, 2\}. \quad (17)$$

The one-site basis element $[q]_j$ is occupation indicator (also referred to as *density*) of particles of flavour q (or density of *vacant sites* if $q = 0$) at the position j . Any local observable can be written as a linear combination of products of one-site observables. For convenience we introduce the following short-hand notation for a complete basis of observables acting on r consecutive sites,

$$[q_1 q_2 q_3 \dots q_r]_j = [q_1]_j \dots [q_2]_{j+1} \dots [q_r]_{j+r-1}. \quad (18)$$

Time-evolution of observables is defined via the relation (15),

$$\langle a \rangle_{\mathbf{p}^t} = \mathbf{a} \cdot \mathbf{p}^t = \mathbf{a}^t \cdot \mathbf{p} = \langle a^t \rangle_{\mathbf{p}}, \quad (19)$$

and is given in terms of the same time-evolution operators $U^{\alpha(e)}, U^{\alpha(o)}$,

$$\mathbf{a}^{2t} = (U^{\alpha(e)} U^{\alpha(o)})^t \mathbf{a}, \quad \mathbf{a}^{2t+1} = (U^{\alpha(e)} U^{\alpha(o)})^t U^{\alpha(e)} \mathbf{a}. \quad (20)$$

Note the difference between the definitions for even and odd times, which is a consequence of the staggering of time evolution. Time-evolution of local observables is particularly simple as one only has to consider the subsection of the chain on which the observables act nontrivially. This follows from the fact that the time-evolution operator preserves the identity observable, which implies the following,

$$U_{x-1,x,x+1}^\alpha [q_1 q_2 \dots q_r]_j = [q_1 q_2 \dots q_r]_j, \quad \text{for } x < j \text{ or } x > j+r-1. \quad (21)$$

3. Ergodicity and local conserved quantities

Besides the seemingly small differences in the collision rules for simple particles, the two models exhibit qualitatively very different dynamical features: the automaton given by χ_1 behaves as an integrable system, while χ_2 appears to be non-integrable. In this section we provide evidence supporting this characterization.

Let us start by considering the dynamics of a simple initial state, where half of the chain is initialized in a randomly chosen configuration and the second half is empty. Evolving this configuration in time, with the assumption of periodic boundaries, the trajectory in the case of χ_1 is very regular, with almost periodically appearing gaps, while in the trajectory of χ_2 this behaviour very quickly disappears, as is shown in figure 5. The dynamics of χ_1 therefore appears to be nonergodic, and suggests the existence of additional local conservation laws.

Local conserved quantities are observables Q preserved under time evolution,

$$U^{\alpha(o)} U^{\alpha(e)} Q = Q, \quad (22)$$

that can be expressed as a linear combination of strictly local terms q with finite maximum support $r > 0$,

$$Q = \sum_{j=1}^n \eta_{2j}(q), \quad \eta_x([s_1 s_2 \dots s_k]_j) = [s_1 s_2 \dots s_k]_{j+x}. \quad (23)$$

Note that due to the staggering of time-evolution we assume invariance under translations for *even* numbers of sites. The three most local conserved quantities can be found easily and are the same for both models,

$$\begin{aligned} Q_1 &= \sum_{j=1}^{2n} (-1)^j ([00]_j + [11]_j + [22]_j), \\ Q_2 &= \sum_{j=1}^{2n} [12]_j, \quad Q_3 = \sum_{j=1}^{2n} [21]_j. \end{aligned} \quad (24)$$

Despite the fact that the charges involve only nearest neighbour couplings, their densities are supported on three adjacent sites because of the staggering (23)

$$q_1 = \sum_{p,s=0}^2 ([pps] - [spp]), \quad q_2 = \sum_{s=0}^2 ([12s] + [s12]), \quad q_3 = \sum_{s=0}^2 ([21s] + [s21]). \quad (25)$$

The quantities Q_2 and Q_3 represent the total number of pairs of consecutive sites occupied by pairs 1, 2 and 2, 1 respectively, while Q_1 is measuring the difference between numbers of simple (unit width) solitons that move to the right and to the left.

However, these are not necessarily all of the local conservation laws. For larger (but finite) lengths of the support r , we can numerically search for conserved observables by diagonalizing the time-evolution operator reduced to the space of extensive local observables, as described in reference [14]. The numerical results reported in table 1, suggest that (24) is the complete set of the local conservation laws for χ_2 . However, in the case of χ_1 we observe (at least for accessible support lengths $r \leq 8$) that the number of conserved quantities increases linearly with the support. In particular, the system exhibits exactly one conserved quantity with

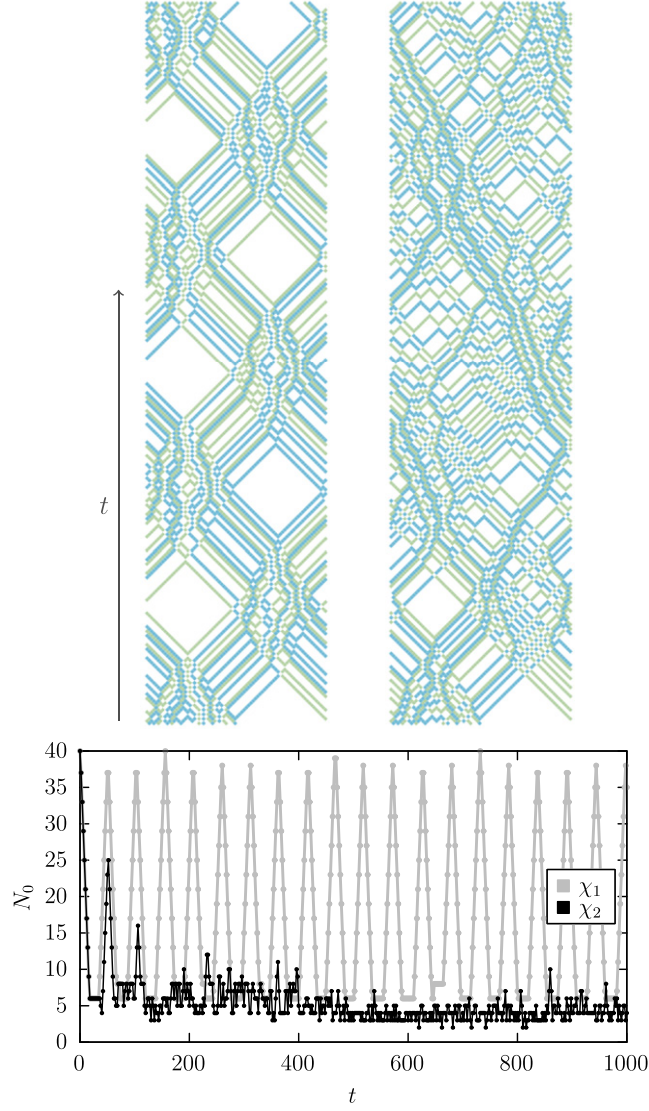


Figure 5. Time evolution of a half-empty configuration with periodic boundary conditions. The initial configuration consists of the left half initialized in a randomly selected configuration of 40 sites, and the right half that is initially empty. The system is then left to evolve according to χ_1 and χ_2 in the left and right top image respectively. The plot at the bottom shows the time dependence of the maximal number of consecutive 0 in a configuration, N_0 . This provides a quantitative description of the behaviour seen from the images of trajectories on larger time scales. In the case of χ_1 the quantity N_0 shows periodic behaviour that persists on time scales shown here, while for χ_2 the oscillations disappear very quickly. The initial configuration is the same in all three figures.

exact support $r > 3$ (see appendix A for the explicit form of the quantity with support $r = 4$). Note that the linear increase of the number of conservation laws with the support is typical

Table 1. Numbers of local conserved quantities. The reported values $\#_1(r)$ and $\#_2(r)$ are numbers of linearly independent local extensive observables with the support at most r that are invariant under the time evolution given by χ_1 and χ_2 , respectively.

r	3	4	5	6	7	8
$\#_1(r)$	3	4	5	6	7	8
$\#_2(r)$	3	3	3	3	3	3

of Bethe–Ansatz/Yang–Baxter integrable spin chains [15] and is very different to the situation observed in RCA54, where the number of conservation laws grows *exponentially* with the system size [16].

The automaton given by the dynamical rule χ_1 therefore appears to be integrable, while χ_2 is not. Even though we cannot provide a proof for this, we will occasionally discriminate between the two models by referring to them as (non)integrable.

4. Matrix product state representation of Gibbs states

Analogously to RCA54 and the deterministic PXP model (rule 201) [17, 18], the two-species automata exhibit a family of stationary states that can be efficiently expressed in terms of products of matrices satisfying a cubic algebraic relation.

A stationary state \mathbf{p}_α of the automaton determined by χ_α should be invariant under evolution for two time-steps,

$$\mathbf{p}_\alpha = U^{\alpha(o)} U^{\alpha(e)} \mathbf{p}_\alpha. \quad (26)$$

We introduce two versions of the state, \mathbf{p}_α and \mathbf{p}'_α , that are mapped into each other under the even and the odd time-evolution operator,

$$\mathbf{p}'_\alpha = U^{\alpha(e)} \mathbf{p}_\alpha, \quad \mathbf{p}_\alpha = U^{\alpha(o)} \mathbf{p}'_\alpha. \quad (27)$$

A class of states that satisfies this condition can be expressed as a *matrix product state* (MPS) by defining $\mathbf{W}_j(\xi, \omega, \lambda)$, $\mathbf{W}'_j(\xi, \omega, \lambda)$ as vectors in the physical space associated to the site $1 \leq j \leq 2n$,

$$\mathbf{W}_j(\xi, \omega, \lambda) = \begin{bmatrix} W_0(\xi, \omega, \lambda) \\ W_1(\xi, \omega, \lambda) \\ W_2(\xi, \omega, \lambda) \end{bmatrix}_j, \quad \mathbf{W}'_j(\xi, \omega, \lambda) = \begin{bmatrix} W'_0(\xi, \omega, \lambda) \\ W'_1(\xi, \omega, \lambda) \\ W'_2(\xi, \omega, \lambda) \end{bmatrix}_j, \quad (28)$$

while their components are the following 3×3 matrices,

$$\begin{aligned} W_0(\xi, \omega, \lambda) &= \begin{bmatrix} \xi & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & W_1(\xi, \omega, \lambda) &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & \xi & \omega \\ 0 & 0 & 0 \end{bmatrix}, \\ W_2(\xi, \omega, \lambda) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \lambda & \xi \end{bmatrix}, \\ W'_s(\xi, \omega, \lambda) &= W_s\left(\frac{1}{\xi}, \omega, \lambda\right), \quad s \in \{0, 1, 2\}. \end{aligned} \quad (29)$$

These matrices together with the update rules χ_1, χ_2 satisfy the following cubic algebraic relations,⁵

$$\begin{aligned} W_{s_1} W'_{\chi_1(s_1, s_2, s_3)} W_{s_3} &= W'_{s_1} W_{s_2} W_{s_3}, & W_{s_1} W'_{\chi_2(s_1, s_2, s_3)} W_{s_3} &= W'_{s_1} W_{s_2} W_{s_3}, \\ W_{s_1} W'_{\chi_1(s_1, s_2, s_3)} W'_{s_3} &= W'_{s_1} W_{s_2} W'_{s_3}, & W_{s_1} W'_{\chi_2(s_1, s_2, s_3)} W'_{s_3} &= W'_{s_1} W_{s_2} W'_{s_3}, \end{aligned} \quad (30)$$

for any combination of $s_1, s_2, s_3 \in \{0, 1, 2\}$. This can be compactly summarized in the vectorized form as

$$U_{123}^\alpha \mathbf{W}_1 \mathbf{W}_2 \mathbf{W}_3 = \mathbf{W}'_1 \mathbf{W}_2 \mathbf{W}_3, \quad U_{123}^\alpha \mathbf{W}_1 \mathbf{W}_2 \mathbf{W}'_3 = \mathbf{W}'_1 \mathbf{W}_2 \mathbf{W}'_3, \quad (31)$$

where $\alpha \in \{1, 2\}$ discriminates between the two models. Note that the second relation follows from the first one from $U^\alpha = (U^\alpha)^{-1}$ and the mapping between the primed and unprimed matrices, $W_s \xleftrightarrow{\xi \leftrightarrow \xi^{-1}} W'_s$.

The immediate consequence of the algebraic relation is that the following family of states is stationary in *both* models,

$$\begin{aligned} \mathbf{p}(\xi, \omega, \lambda) &= \frac{1}{Z} \text{tr}(\mathbf{W}_1 \mathbf{W}'_2 \mathbf{W}_3 \dots \mathbf{W}_{2n}), \\ \mathbf{p}'(\xi, \omega, \lambda) &= \frac{1}{Z} \text{tr}(\mathbf{W}'_1 \mathbf{W}_2 \mathbf{W}'_3 \dots \mathbf{W}_{2n}), \end{aligned} \quad (32)$$

since for both $\alpha = 1$ and $\alpha = 2$, the states \mathbf{p} and \mathbf{p}' satisfy the condition (27). The prefactor Z is determined by the normalization condition,

$$Z = \text{tr}((W_0 + W_1 + W_2)(W'_0 + W'_1 + W'_2))^n, \quad (33)$$

and the length of the chain is assumed to be $2n$.

The set of vectors $\mathbf{p}(\xi, \omega, \lambda)$ can be interpreted as a family of *Gibbs states*. To see that, we first note that the structure of the matrices W_s, W'_s implies the following,

$$\begin{aligned} W_s \hat{e}_b &= \hat{e}_s (\delta_{s,b}(\xi - 1) + \delta_{s,1} \delta_{b,2}(\omega - 1) + \delta_{s,2} \delta_{b,1}(\lambda - 1) + 1), \\ W'_s \hat{e}_b &= \hat{e}_s (\delta_{s,b}(\xi^{-1} - 1) + \delta_{s,1} \delta_{b,2}(\omega - 1) + \delta_{s,2} \delta_{b,1}(\lambda - 1) + 1), \end{aligned} \quad \hat{e}_s = \begin{bmatrix} \delta_{s,0} \\ \delta_{s,1} \\ \delta_{s,2} \end{bmatrix}, \quad (34)$$

where $\{\hat{e}_s\}_{s=0,1,2}$ is the canonical basis of the auxiliary space. Comparing the coefficients and components in these relations with the explicit form of conserved quantities (24), one immediately realizes that the components of the probability distribution $\mathbf{p}(\xi, \omega, \lambda)$ corresponds to the probabilities of configurations in the Gibbs ensemble,

$$p_{\underline{s}} = \frac{1}{Z} \text{tr}(W_{s_1} W'_{s_2} W_{s_3} \dots W'_{s_{2n}}) \propto e^{\mu_1 Q_1(\underline{s}) + \mu_2 Q_2(\underline{s}) + \mu_3 Q_3(\underline{s})}, \quad (35)$$

where we identify chemical potentials μ_1, μ_2, μ_3 as

$$\mu_1 = \log \xi, \quad \mu_2 = \log \omega, \quad \mu_3 = \log \lambda. \quad (36)$$

⁵ For compactness of notation, we will often suppress the explicit dependence on parameters ξ, ω, λ , i.e. $W_s^{(\prime)}(\xi, \omega, \lambda) \rightarrow W_s^{(\prime)}$.

This explains why the same state \mathbf{p} is stationary in both models: the first three conserved quantities are common to both automata and therefore they exhibit the same simple Gibbs state. Note that in the case of χ_2 , we expect this to be the complete family of equilibrium states, while for χ_1 one should include the (presumably) infinite set of local charges to obtain the most general family of generalised Gibbs states (GGE).

5. Stationary states with higher conservation laws

Since the integrable model exhibits more than three conservation laws, it is possible to find richer GGEs that take into account also some of the conserved quantities with larger support. Here we show an example with four chemical potentials: in addition to $\log \xi$, $\log \omega$ and $\log \lambda$ we introduce also $\log \mu$, which corresponds to the conserved charge with support 4. Throughout the section we mostly consider the automaton given by the set of rules χ_1 , and at the end we also discuss some related observations for χ_2 .

5.1. Patch state ansatz

The first form of the stationary state we consider is the *patch-state* ansatz, as introduced in [19]. We assume that the components of the translationally invariant steady state \mathbf{p} can be written in the following form,

$$P_{(s_1, s_2, \dots, s_{2n})} = \frac{1}{Z} t_{s_1 s_2 s_3 s_4} t_{s_3 s_4 s_5 s_6} \cdots t_{s_{2n-3} s_{2n-2} s_{2n-1} s_{2n}} t_{s_{2n-1} s_{2n} s_1 s_2}, \quad (37)$$

where Z is a normalization constant and t is a tensor with components $t_{s_1 s_2 s_3 s_4}$ that are determined so that the fixed-point condition (26) is satisfied. In particular, the form of the stationarity condition that is most convenient for us is

$$U^{1(e)} \mathbf{p} = U^{1(o)} \mathbf{p}, \quad (38)$$

which is in terms of the components rewritten as,

$$P_{(s_1 \chi(s_1, s_2, s_3) s_3 \chi(s_3, s_4, s_5) \dots \chi(s_{2n-1}, s_{2n}, s_1))} = P_{(\chi(s_{2n}, s_1, s_2) s_2 \chi(s_2, s_3, s_4) s_5 \dots s_{2n})}, \quad (39)$$

for each configuration $\underline{s} = (s_1, s_2, \dots, s_{2n}) \in \mathbb{Z}_3^{2n}$. Up to a gauge transformation

$$t_{s_1 s_2 s_3 s_4} \rightarrow f_{s_1 s_2} t_{s_1 s_2 s_3 s_4} f_{s_3 s_4}^{-1}, \quad (40)$$

we find a unique⁶ four-parameter family of solutions (to be proven below) that can be compactly summarized in terms of the 9×9 transfer matrix T with components, $T_{(s_1 s_2), (s'_1 s'_2)} = t_{s_1 s_2 s'_1 s'_2}$,

⁶ We assume all of the entries to be nonvanishing, which means that each configuration \underline{s} arises with a non-zero probability $p_{\underline{s}} > 0$.

$$T(\xi, \omega, \lambda, \mu) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \mu & \mu & 1 & \frac{1}{\xi^2} & \frac{\mu}{\xi^2} & \frac{1}{\xi^2} & \frac{\omega}{\xi} & \frac{\omega}{\xi} & \frac{\mu\omega}{\xi} \\ \mu & 1 & \mu & \frac{\lambda}{\xi} & \frac{\lambda\mu}{\xi} & \frac{\lambda}{\xi} & \frac{1}{\xi^2} & \frac{1}{\xi^2} & \frac{\mu}{\xi^2} \\ \mu & \mu & 1 & \mu & \mu & \mu & 1 & 1 & \mu \\ \xi^2 & \xi^2 & \xi^2 & 1 & \mu & 1 & \xi\omega & \xi\omega & \xi\omega \\ \mu\xi\omega & \xi\omega & \mu\xi\omega & \lambda\omega & \lambda\mu\omega & \lambda\omega & \frac{\omega}{\xi} & \frac{\omega}{\xi} & \frac{\mu\omega}{\xi} \\ \mu & 1 & \mu & 1 & \mu & 1 & \mu & \mu & \mu \\ \lambda\mu\xi & \lambda\mu\xi & \lambda\xi & \frac{\lambda}{\xi} & \frac{\lambda\mu}{\xi} & \frac{\lambda}{\xi} & \lambda\omega & \lambda\omega & \lambda\mu\omega \\ \xi^2 & \xi^2 & \xi^2 & \lambda\xi & \lambda\xi & \lambda\xi & 1 & 1 & \mu \end{bmatrix}. \quad (41)$$

As before, we can define two versions of the state, \mathbf{p} and \mathbf{p}' , that get mapped into each other under the evolution for one time-step (cf (27)). In our case the odd-time-step version of the stationary state \mathbf{p}' is \mathbf{p} shifted for one site with the components of the primed state explicitly given by

$$p'_{(s_1 s_2 \dots s_{2n})} = p_{(s_2 \dots s_{2n} s_1)} = t_{s_2 s_3 s_4 s_5} t_{s_4 s_5 s_6 s_7} \dots t_{s_{2n} s_1 s_2 s_3}. \quad (42)$$

To show that (27) really holds for \mathbf{p} and \mathbf{p}' defined above, we only need to verify that the tensor t satisfies a finite set of local relations. We start by explicitly rewriting the condition for a finite length of the chain, say $2n = 6$,

$$\begin{aligned} t_{s_1 s_2 s_3 s_4} t_{s_3 s_4 s_5 s_6} t_{s_5 s_6 s_1 s_2} &= t_{\chi_1(s_1, s_2, s_3) s_3 \chi_1(s_3, s_4, s_5) s_5} t_{\chi_1(s_3, s_4, s_5) s_5 \chi_1(s_5, s_6, s_1) s_1} t_{\chi_1(s_5, s_6, s_1) s_1 \chi_1(s_1, s_2, s_3) s_3} \\ &= t_{s_2 \chi_1(s_2, s_3, s_4) s_4 \chi_1(s_4, s_5, s_6) s_6} t_{s_4 \chi_1(s_4, s_5, s_6) s_6 \chi_1(s_6, s_1, s_2) s_2} t_{s_6 \chi_1(s_6, s_1, s_2) s_2 \chi_1(s_2, s_3, s_4) s_4}. \end{aligned} \quad (43)$$

The proof of (27) for the chain of length $2n = 6$ amounts to checking that the above relations are fulfilled for all 3^6 configurations. To extend the proof to an arbitrary length, we observe that the following set of compatibility conditions is also fulfilled,

$$\begin{aligned} \frac{t_{s_1 s_2 s_3 s_4} t_{s_3 s_4 s_5 s_6}}{t_{s_1 s_2 s_5 s_6}} &= \frac{t_{s_0 \chi_1(s_0, s_1, s_2) s_2 \chi_1(s_2, s_3, s_4) s_4} t_{s_2 \chi_1(s_2, s_3, s_4) s_4 \chi_1(s_4, s_5, s_6) s_6} t_{s_4 \chi_1(s_4, s_5, s_6) s_6 \chi_1(s_6, s_7, s_8) s_8}}{t_{s_0 \chi_1(s_0, s_1, s_2) s_2 \chi_1(s_2, s_5, s_6) s_6} t_{s_2 \chi_1(s_2, s_5, s_6) s_6 \chi_1(s_6, s_7, s_8) s_8}} \\ &= \frac{t_{\chi_1(s_{-1}, s_0, s_1) s_1 \chi_1(s_1, s_2, s_3) s_3} t_{\chi_1(s_1, s_2, s_3) s_3 \chi_1(s_3, s_4, s_5) s_5} t_{\chi_1(s_3, s_4, s_5) s_5 \chi_1(s_5, s_6, s_7) s_7}}{t_{\chi_1(s_{-1}, s_0, s_1) s_1 \chi_1(s_1, s_2, s_5) s_5} t_{\chi_1(s_1, s_2, s_5) s_5 \chi_1(s_5, s_6, s_7) s_7}}. \end{aligned} \quad (44)$$

Even though these two sets of relations are rather large and checking their validity cannot be easily performed by hand (as they involve $2 \times 3^9 + 2 \times 3^6 = 40\,824$ separate equations), the number of relevant relations is still finite and they can be easily verified using computer algebra systems. From here, equation (27) immediately follows and the pair of states $(\mathbf{p}, \mathbf{p}')$ is therefore invariant under time-evolution.

5.2. Equivalent MPS

Another, perhaps more convenient, form of the stationary state \mathbf{p} introduced above is an MPS, defined analogously to the Gibbs state from section 4,

$$\mathbf{p} = \frac{1}{Z} \text{tr} (\mathbf{W}_1 \mathbf{W}'_2 \dots \mathbf{W}'_{2n}), \quad \mathbf{p} = \frac{1}{Z} \text{tr} (\mathbf{W}_1 \mathbf{W}'_2 \dots \mathbf{W}'_{2n}). \quad (45)$$

However, since the stationary state is now richer, the matrices \mathbf{W}_s , \mathbf{W}'_s now have to be larger. In particular, we find a 7×7 dimensional representation, for which the state (45) exactly corresponds to the patch state ansatz from the previous subsection,

$$\begin{aligned} \mathbf{W}_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \mathbf{W}'_0 &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & \mu & 1 & \mu & \mu & 1 & \mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & 1 & \mu & 1 & \mu & \mu & \mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{W}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi\omega & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & \mathbf{W}'_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & \mu & 1 & \frac{1}{\xi^2} & \frac{\mu}{\xi^2} & \frac{\omega}{\xi} & \frac{\mu\omega}{\xi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi^2 & \xi^2 & \xi^2 & 1 & \mu & \xi\omega & \xi\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{W}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda\xi & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, & \mathbf{W}'_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & 1 & \mu & \frac{\lambda}{\xi} & \frac{\lambda\mu}{\xi} & \frac{1}{\xi^2} & \frac{\mu}{\xi^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi^2 & \xi^2 & \xi^2 & \lambda\xi & \lambda\xi & 1 & \mu \end{bmatrix}. \end{aligned} \quad (46)$$

To independently verify the stationarity of (45), we identify a set of cubic algebraic relations that generalises equation (31),

$$U_{123}^1 \mathbf{X}_1 \mathbf{W}_2 \mathbf{W}'_3 = \mathbf{W}_1 \mathbf{W}'_2 \mathbf{X}_3, \quad U_{123}^1 \mathbf{X}'_1 \mathbf{W}'_2 \mathbf{W}_3 = \mathbf{W}'_1 \mathbf{W}_2 \mathbf{X}'_3, \quad (47)$$

where the components of \mathbf{X} , \mathbf{X}' are 7×7 matrices given in appendix C. To prove that the MPS built out of these matrices is stationary for periodic boundaries, one needs to also take into account the following set of easily verifiable compatibility relations,

$$U_{123}^1 \mathbf{X}_1 \mathbf{W}_2 \mathbf{X}'_3 = \mathbf{W}_2 \mathbf{W}'_2 \mathbf{W}_3, \quad U_{123}^1 \mathbf{X}'_1 \mathbf{W}'_2 \mathbf{X}_3 = \mathbf{W}'_1 \mathbf{W}_2 \mathbf{W}'_3. \quad (48)$$

5.3. The second model

For the ‘non-integrable’ rule χ_2 , one needs to fix $\mu = 1$ and the state remains stationary. However, the precise form of local algebraic relations changes slightly. The identities on the right

of (47) and (48) remain valid,

$$\begin{aligned} U_{123}^2 \mathbf{X}'_1 \mathbf{W}'_2 \mathbf{W}_3|_{\mu \rightarrow 1} &= \mathbf{W}'_1 \mathbf{W}_2 \mathbf{X}'_3|_{\mu \rightarrow 1}, \\ U_{123}^2 \mathbf{X}'_1 \mathbf{W}'_2 \mathbf{X}'_3|_{\mu \rightarrow 1} &= \mathbf{W}'_1 \mathbf{W}_2 \mathbf{W}'_3|_{\mu \rightarrow 1}, \end{aligned} \quad (49)$$

while the other two identities do not hold directly, but have to be multiplied by W'_s from the left for any $s \in \{0, 1, 2\}$, so that the relation becomes effectively quartic rather than cubic,

$$\begin{aligned} U_{234}^2 \mathbf{W}'_1 \mathbf{X}_2 \mathbf{W}_3 \mathbf{W}'_4|_{\mu \rightarrow 1} &= \mathbf{W}'_1 \mathbf{W}_2 \mathbf{W}'_3 \mathbf{X}_4|_{\mu \rightarrow 1}, \\ U_{234}^2 \mathbf{W}'_1 \mathbf{X}_2 \mathbf{W}_3 \mathbf{X}'_4|_{\mu \rightarrow 1} &= \mathbf{W}'_1 \mathbf{W}_2 \mathbf{W}'_3 \mathbf{W}_4|_{\mu \rightarrow 1}. \end{aligned} \quad (50)$$

Using the modified set of identities, one can prove that \mathbf{W}' and \mathbf{W} in the case of $\mu = 1$ provide a stationary state also for the non-integrable model. One might wonder if there is another 7×7 representation, for which the limit $\mu \rightarrow 1$ gives directly the stationary state also for χ_2 without resorting to the modified algebra. At the moment such a representation has still not been found, and it is not clear whether it exists or not.

Nonetheless, since we expect that the case $\mu = 1$ corresponds to the Gibbs state, the matrices W_s , W'_s should in this limit reduce to the 3×3 representation introduced in section 4, which obeys a simple cubic algebra for *both* models. To prove this, we introduce the following set of 3×7 and 7×3 matrices,

$$\begin{aligned} S_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\xi} & 0 \\ 0 & 0 & \frac{1}{\xi} \\ 1 & 0 & 0 \\ 0 & \xi & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \xi \end{bmatrix}, & R_2 &= \begin{bmatrix} \frac{1}{2\xi} & 0 & 0 \\ \frac{1}{4\xi} & 0 & 0 \\ \frac{1}{4\xi} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \\ S_2 &= \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{\xi}{1-\xi^2} & 0 & 0 & \frac{\xi}{\xi^2-1} & 0 & 0 \\ 0 & 0 & \frac{\xi}{1-\xi^2} & 0 & 0 & 0 & \frac{\xi}{\xi^2-1} \end{bmatrix}, & R_1 &= \begin{bmatrix} \xi & \xi & \xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \end{aligned} \quad (51)$$

which for $\mu = 1$ satisfy the following set of relations for any pair of $s, s' \in \{0, 1, 2\}$,

$$W_s S_1 S_2 W'_{s'} = W_s W'_{s'}, \quad W'_s R_2 R_1 W_{s'} = W'_s W_{s'}. \quad (52)$$

This immediately implies that by the following transformation

$$R_1 W_s S_1|_{\mu \rightarrow 1} \mapsto W_s, \quad S_2 W'_s R_2|_{\mu \rightarrow 1} \mapsto W'_s, \quad (53)$$

we obtain precisely the set of 3×3 matrices (29).

6. Spatio-temporal correlation functions

Let us now consider dynamical correlation functions of local densities of conserved quantities. In particular, we define $C(x, t)$ as the following correlation function,

$$C(x, t) = \langle q_x^{(+t)} q_0^{(+)} \rangle_{\mathbf{p}} - \langle q_0^{(+)} \rangle_{\mathbf{p}}^2, \quad q_j^{(+)} = [12]_j + [21]_j, \quad (54)$$

where the state \mathbf{p} is assumed to be stationary (i.e. invariant under time evolution). For simplicity this is the only correlation function we discuss here, however the results do not strongly depend on the exact choice of the correlation function in question. Note that $q^{(+)}$ is the local density of the sum of Q_2 and Q_3 ,

$$\sum_{j=1}^{2n} q_j^{(+)} = Q_2 + Q_3. \quad (55)$$

In this section, the position label is bounded by $\pm t$ (and not 1 and $2n$ as elsewhere), and the full system size at time t is assumed to be much larger than $2t$. In this limit the expectation values of local observables in the Gibbs states can be expressed in terms of MPS defined on a finite length of the chain, as described in appendix B.

We obtain the correlation function $C(x, t)$ numerically by averaging over many trajectories starting from random initial configurations that are sampled according to the probability distribution \mathbf{p} . The simplest stationary distribution \mathbf{p} is a *maximum-entropy* (also *infinite-temperature*) state \mathbf{p}_∞ , for which every configuration is equally likely. The numerically computed correlation function profiles for $\mathbf{p} = \mathbf{p}_\infty$ are shown in figures 6 and 7. At small times both profiles exhibit a ‘heat’ peak centred around position $x = 0$ and two ‘sound’ peaks at the side that move to the left and to the right with fixed velocities. At longer times, however, we observe that in the integrable case each one of the side peaks splits into two separate contributions that move with very similar but different velocities. In the *non-integrable* model this does not appear to happen, which is consistent with the finite number (three) of local conservation laws which equals the number of peaks.

6.1. Hydrodynamic discrepancies on long time-scales

To obtain more insight into the structure of $C(x, t)$, let us consider the hydrodynamic predictions for it. We focus on the *non-integrable* automaton, since in this case we expect that the MPS parametrization introduced in section 4 gives the complete set of stationary states.

We start by generalizing the correlation function $C(x, t)$ to an arbitrary pair of charges q_α, q_β , $\alpha, \beta \in \{1, 2, 3\}$,

$$C_{\alpha,\beta}(x, t) = \langle q_{\alpha x}^t q_{\beta 0} \rangle_{\mathbf{p}} - \langle q_{\alpha 0} \rangle_{\mathbf{p}} \langle q_{\beta 0} \rangle_{\mathbf{p}}, \quad (56)$$

where $\mathbf{p} = \mathbf{p}(\xi, \omega, \lambda)$ is a Gibbs state. Note that the correlation function defined in (54) is a linear combination of $C_{\alpha,\beta}(x, t)$,

$$C(x, t) = C_{2,2}(x, t) + C_{2,3}(x, t) + C_{3,2}(x, t) + C_{3,3}(x, t). \quad (57)$$

At the hydrodynamic scale in the linear order the correlation function $C_{\alpha,\beta}(x, t)$ obeys a linear homogeneous partial differential equation (see e.g. the discussion in [20, section 2] and [13]),

$$\partial_t C_{\alpha,\beta}(x, t) + \partial_x \sum_{\gamma} A_{\alpha,\gamma} C_{\gamma,\beta}(x, t) = 0, \quad (58)$$

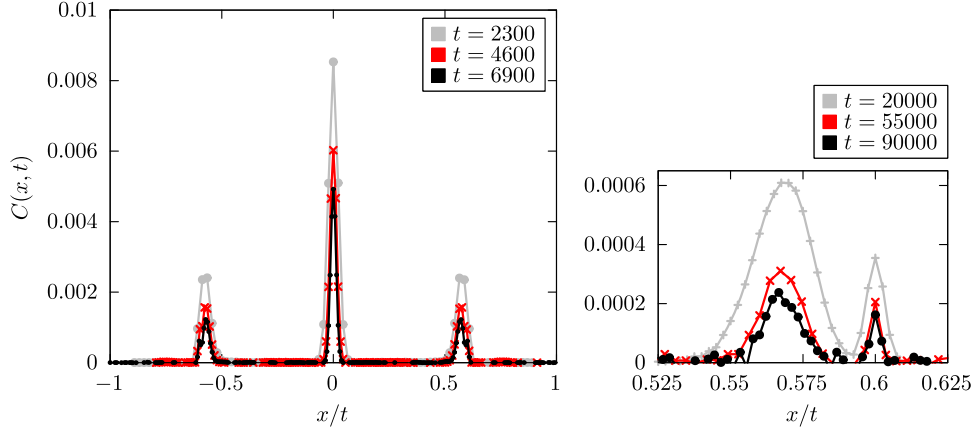


Figure 6. Dynamical correlation function $C(x, t)$ for the *integrable* case and $\mathbf{p} = \mathbf{p}_\infty$. At short times the correlation function appears to exhibit three peaks: a central one and two side peaks moving to the left and right. However, at longer times it becomes evident that each of the side peaks separates into two distinct peaks that move with very close but different velocities, as can be seen in the closeup on the right.

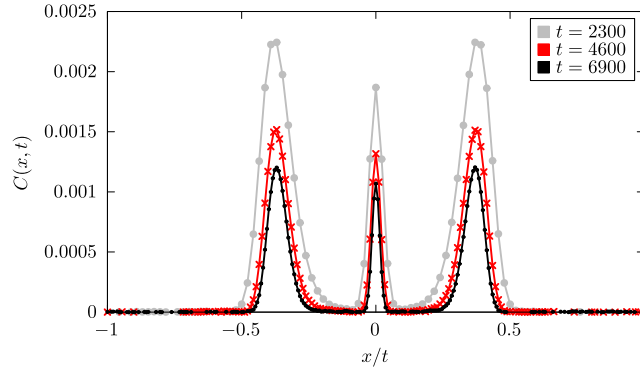


Figure 7. Dynamical correlation function $C(x, t)$ for the *non-integrable* case and $\mathbf{p} = \mathbf{p}_\infty$. The correlation function exhibits three peaks: a central one and two side ones that move to the left and right.

where A is a 3×3 matrix of derivatives of the expectation values of currents with respect to the densities of charges,

$$A_{\alpha,\beta} = \frac{\partial \langle j_\alpha 0 \rangle_{\mathbf{p}}}{\partial \langle q_\beta 0 \rangle_{\mathbf{p}}}. \quad (59)$$

Here the currents j_α ($\alpha = 1, 2, 3$) together with densities of conserved charges obey the following continuity equation,

$$q_{\alpha 2x}^{2t+2} - q_{\alpha 2x}^{2t} + j_{\alpha 2x+1}^{2t+1} - j_{\alpha 2x-1}^{2t+1} = 0. \quad (60)$$

Their explicit form is given in appendix D.1.

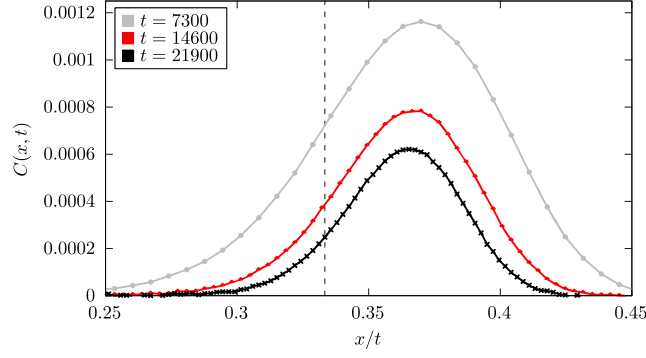


Figure 8. Closer look at the right-moving peak of $C(x, t)$ for the *non-integrable* case. The dashed vertical line denotes the hydrodynamic prediction for the velocity of the peak $v_3 = \frac{1}{3}$.

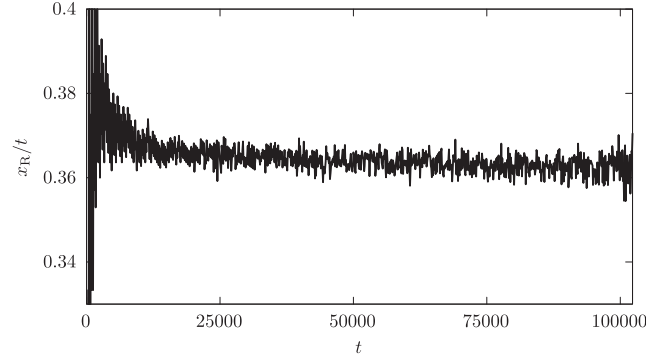


Figure 9. Perceived velocity of the right-moving peak of $C(x, t)$ as a function of time. The velocity is determined as the position of the right-moving peak x_R divided by the time t . Note that the hydrodynamic prediction for the velocity is $v_R = \frac{1}{3}$, which is not matched by the numerical results on very long timescales. However, a small drift in the velocity can be observed (see figure 10).

At $t = 0$, correlation function $C_{\alpha, \beta}(x, 0)$ has a peak at $x = 0$ with the tails decreasing exponentially in $|x|$. Therefore at time t in this regime we expect the correlation function to (in general) exhibit three peaks at positions $x = tv_\alpha$, where v_α , $\alpha = 1, 2, 3$ are the eigenvalues of A . In the case of the maximum entropy state $\mathbf{p} = \mathbf{p}_\infty$ the matrix A takes the following form (see appendix D.2 for the details),

$$A|_{\xi=\omega=\lambda=1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 \end{bmatrix} = R \begin{bmatrix} -\frac{1}{3} & & \\ & 0 & \\ & & \frac{1}{3} \end{bmatrix} R^{-1}, \quad (61)$$

for a suitably chosen similarity transformation R . Therefore for a correlation function between any linear combinations of densities of charges we expect the peaks to be located at $x = 0$, and $x = \pm \frac{1}{3}t$. As we can see from the closeup of the peak in figure 8, this does not seem to be

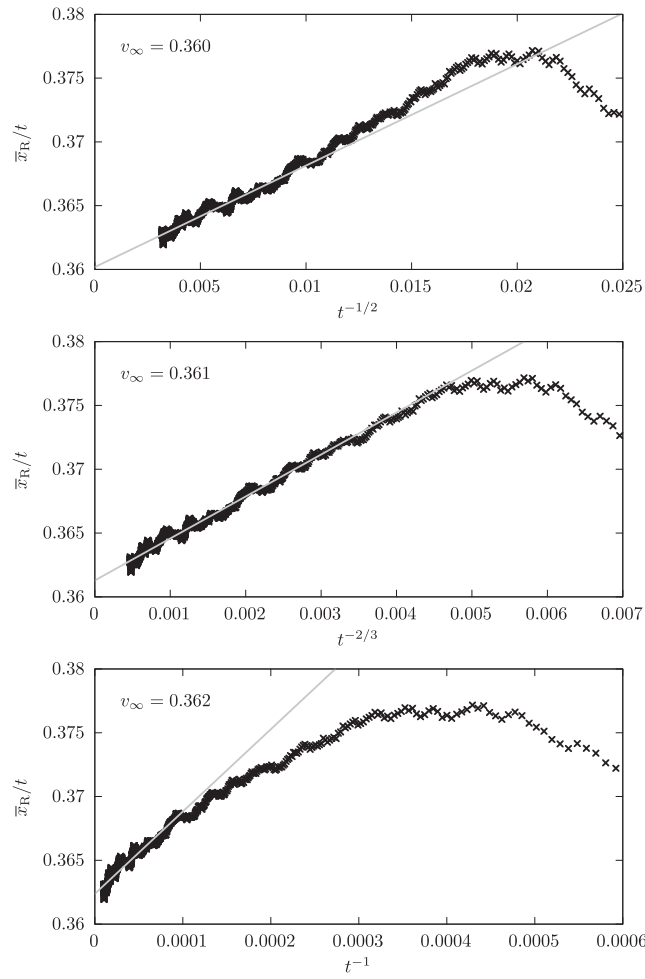


Figure 10. Perceived velocity of the right-moving peak of $C(x, t)$ as a function of different choices of the algebraic scaling of time $t^{-\gamma}$. Here instead of the position x_R we report the moving-time-averaged position $\bar{x}_R = \frac{1}{\Delta} \int_{t-\Delta/2}^{t+\Delta/2} x_R(t)$, with $\Delta = 5000$. The grey solid line corresponds to the best fit of the linear function and for each γ we extract the limit $v_\infty = \lim_{t \rightarrow \infty} \bar{x}_R/t$ from the fit. We do not have any analytical prediction for the scaling exponent γ , however the numerical fits reported above suggest γ takes a value inside the interval $(1/2, 1)$, and it is close to $2/3$. For all the reported cases the extracted asymptotic velocity v_∞ disagrees with the hydrodynamic prediction ($v_3 = \frac{1}{3}$).

the case: the velocity observed in numerics is closer to 0.36 than to $\frac{1}{3}$.

To investigate this issue further, in figure 9 we plot the ratio between the position of the right peak x_R and time t on longer time-scales. The numerically perceived velocity shows a very slow, almost insignificant drift away from 0.36. In figure 10 we plot the averaged values of velocities with respect to different choices of the scaling functions of time, $t^{-\gamma}$. The numerical data is well approximated by

$$\frac{\bar{x}_R}{t} - v_\infty(\gamma) \propto t^{-\gamma}, \quad (62)$$

for appropriately determined $v_\infty(\gamma)$ (extracted from the fit) and γ . The quality of scaling is not very sensitive to varying γ between $1/2$ and 1 , but perhaps the best scaling is seen for $\gamma = 2/3$ where the asymptotic velocity reads $v_\infty = 0.361$. Interestingly, the asymptotic velocity $v_\infty(\gamma)$ does *not* match the hydrodynamic prediction $v_3 = 1/3$. Even though we cannot exclude the possibility of x/t reaching v_3 by eventually breaking the scaling (62), the time-scales associated with this convergence should be extremely long (note that the longest times in our data are of the order 10^5). This suggests the existence of an additional *quasi-local conserved quantity* or an *almost conserved local quantity* in this dynamical system. The existence of such quantities in non-integrable locally interacting systems should be of general interest beyond the context of RCA.

7. Conclusions

We have proposed two RCA, which generalise the interacting particle-like dynamics of rule 54 [5, 7] to two particle species. In addition to simple one-species solitons that behave analogously to solitons in RCA54, the models exhibit more complicated multi-colour particles, which are, however, not stable under scattering. Empirical evidence suggests that one of the automata is an integrable system, while the other one possesses only three local integrals of motion. These three conservation laws are common to both models, therefore a class of GGE with nonzero chemical potentials corresponding to these observables is stationary for both automata. For periodic boundaries we find a staggered matrix-product formulation of this class of states, consisting of matrices that obey a cubic algebraic relation analogous to [17]. We extend this class to four-parameter stationary states of the integrable automaton, which we express both in a patch-state-ansatz (analogous to [19]) and matrix-product-state form. We numerically study the behaviour of spatio-temporal correlation functions in the non-integrable model. We find significant deviations from the hydrodynamic prediction obtained by assuming local thermalization in the presence of no other conservation laws apart from the three known ones.

The collection of results presented here leaves many open questions. First, it would be interesting to find Yang–Baxter formulation for the integrable model and thus rigorously establish the integrability of the first automaton. Perhaps recent ideas of [21, 22] could provide a good starting point. Moreover, it is still not understood which aspects of solvability of rule 54 carry on to these models. For instance, it would be interesting to find non-equilibrium stationary states of the boundary driven setup (analogous to [17, 19, 23]), explore how (non)integrability affects the form of stationary states, and whether the exact treatment of large-deviation statistics is feasible [24]. From a physical perspective, it would be interesting to understand the nature of transport in these models. In particular, preliminary analysis suggests anomalous broadening of sound peaks, while heat peaks exhibit diffusive spreading, but a more extensive numerical exploration remains to be done.

Probably the most urgent question concerns the explanation of the disagreement between the behaviour of correlation functions and the hydrodynamic prediction. Even though the numerics cannot rule out the eventual relaxation to the hydrodynamic values, the time-scales for that are surprisingly long and one should understand what the mechanism for this could be (see e.g. references [25–28] for some recent related results). Precisely because of mathematical and conceptual simplicity of the model we hope that further analytic progress will be possible in future.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Conserved quantities

A.1. Algorithm

The numerical procedure to search for local conserved quantities is a variant of the algorithm introduced in section 7 of [14], which can be understood as an exact diagonalization of the time-evolution operator projected to the space of extensive local observables.

We start by introducing a convenient orthogonal basis of local observables,

$$[\hat{0}] = [0] + [1] + [2], \quad [\hat{1}] = [1] - [2], \quad [\hat{2}] = -2[0] + [1] + [2], \quad (\text{A.1})$$

where $[\hat{0}]$ is the identity observable (i.e. its expectation value is 1 for each state) and it has the role of the unit element in the commutative algebra of local observables, while $[\hat{1}]$ and $[\hat{2}]$ are two other independent basis elements chosen so that the basis is orthogonal⁷. Note that by definition, the string of $\hat{0}$ in the definition of an observable with a larger support can be removed and replaced by a translation operator,

$$\underbrace{[\hat{0}\hat{0} \dots \hat{0}]}_m \hat{s}_1 \hat{s}_2 \dots \hat{s}_k = [\hat{s}_1 \hat{s}_2 \dots \hat{s}_k]_{m+1}, = \eta_m([\hat{s}_1 \hat{s}_2 \dots \hat{s}_k]), \quad (\text{A.2})$$

which follows from

$$a \cdot [\hat{0}]_x = [\hat{0}]_x \cdot a = a, \quad (\text{A.3})$$

for any local observable a .

We denote the local time-evolution operator written in this basis by $\hat{U}_{x-1, x+1}^\alpha$ and it is given by

$$\hat{U}_{123}^\alpha = (R \otimes R \otimes R)^{-1} U_{123}^\alpha (R \otimes R \otimes R), \quad (\text{A.4})$$

⁷ The inner product that is used here is the expectation value of the product of local observables in the maximum entropy state $a, b \mapsto \langle ab \rangle_{\mathbf{p}_\infty}$, with $\mathbf{p}_\infty \propto [1 \ 1 \ \dots \ 1]^T$.

with R being the local basis transformation,

$$R = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}. \quad (\text{A.5})$$

We proceed by defining the following projector to local observables with support r ,

$$\hat{P}_r \left[\underbrace{\hat{0}\hat{0}\dots\hat{0}}_{2m} \hat{s}_1 \hat{s}_2 \dots \hat{s}_k \right] \Big|_{s_1+s_2>0, s_k>0} = \begin{cases} [\hat{s}_1 \hat{s}_2 \dots \hat{s}_k], & k \leq r, \\ 0, & k > r, \end{cases} \quad (\text{A.6})$$

which maps all local observables with support $k < r$ to an equivalent observable that acts nontrivially on the section of the chain $[1, \dots, k]$. The important feature of this map is that it provides a unique way to express any extensive observable with (quasi)local density:

$$A = \sum_{j=1}^n \eta_{2j}(a) = \lim_{r \rightarrow \infty} \sum_{j=1}^n \eta_{2j}(\hat{P}_r a), \quad (\text{A.7})$$

where the left-hand side is not unique (i.e. there exist many different equivalent choices of a that give the same A), while the expression on the rhs is. Exploiting this uniqueness, we can now interpret the local densities of extensive conserved charges with support smaller than r to be the eigenvectors corresponding to the eigenvalue 1 of the reduced time-evolution operator \hat{U}_r defined as,

$$\hat{U}_r^\alpha = \hat{P}_r \hat{U}_{123}^\alpha \hat{U}_{345}^\alpha \dots \hat{U}_{k-2,k-1,k}^\alpha \eta_1 \hat{P}_r, \quad k = \begin{cases} r+1, & r \equiv 0 \pmod{2}, \\ r+2, & r \equiv 1 \pmod{2}, \end{cases} \quad (\text{A.8})$$

where $k = k(r)$ is either $r+1$ or $r+2$ (depending on the parity of r). Therefore, to find the number of local conservation laws with small support r one can just diagonalize the above operator, which is how table 1 was obtained.

A.2. Conservation law with support 4

Diagonalising the reduced time-evolution operator (A.8) one can also find the explicit form of conservation laws, and as a special example we can obtain the first higher conservation law that is conserved in the first automaton χ_1 , but not for χ_2 . Note, however, that not only is the local density not unique (as noted in equation (A.7)), but also any linear combination of local conservation laws with support smaller than r is also conserved and there are many different choices of a set of linearly independent conserved charges (unless we require orthogonality with respect to some inner product). One such choice of the new conservation law Q_4 ,

$$Q_4 = \sum_{j=1}^n \eta_{2j}(\tilde{q}_4), \quad (\text{A.9})$$

is determined by the following local density

$$\begin{aligned}
\tilde{q}_4 = & \frac{1}{6}[\hat{0}\hat{1}\hat{0}\hat{1}] + \frac{1}{18}[\hat{0}\hat{1}\hat{1}\hat{0}] + \frac{1}{36}[\hat{0}\hat{1}\hat{1}\hat{2}] - \frac{1}{12}[\hat{0}\hat{1}\hat{2}\hat{1}] + \frac{1}{54}[\hat{0}\hat{2}\hat{0}\hat{2}] + \frac{1}{12}[\hat{0}\hat{2}\hat{1}\hat{1}] \\
& - \frac{1}{54}[\hat{0}\hat{2}\hat{2}\hat{0}] + \frac{1}{108}[\hat{0}\hat{2}\hat{2}\hat{2}] + \frac{1}{6}[\hat{1}\hat{0}\hat{1}\hat{0}] + \frac{1}{6}[\hat{1}\hat{1}\hat{0}\hat{0}] - \frac{1}{8}[\hat{1}\hat{1}\hat{1}\hat{1}] + \frac{1}{12}[\hat{1}\hat{1}\hat{2}\hat{0}] \\
& - \frac{1}{24}[\hat{1}\hat{1}\hat{2}\hat{2}] - \frac{1}{12}[\hat{1}\hat{2}\hat{1}\hat{0}] + \frac{1}{24}[\hat{1}\hat{2}\hat{1}\hat{2}] + \frac{1}{8}[\hat{1}\hat{2}\hat{2}\hat{1}] + \frac{1}{54}[\hat{2}\hat{0}\hat{2}\hat{0}] + \frac{1}{36}[\hat{2}\hat{1}\hat{1}\hat{0}] \\
& + \frac{1}{72}[\hat{2}\hat{1}\hat{1}\hat{2}] + \frac{1}{24}[\hat{2}\hat{1}\hat{2}\hat{1}] - \frac{1}{54}[\hat{2}\hat{2}\hat{0}\hat{0}] - \frac{1}{24}[\hat{2}\hat{2}\hat{1}\hat{1}] + \frac{1}{108}[\hat{2}\hat{2}\hat{2}\hat{0}] - \frac{1}{72}[\hat{2}\hat{2}\hat{2}\hat{2}].
\end{aligned} \tag{A.10}$$

This choice of the linear combination of the first 4 conservation laws was chosen so that the state introduced in section 5 satisfies

$$\mathbf{p} \propto \xi^{Q_1} \omega^{Q_2} \lambda^{Q_3} \mu^{Q_4}. \tag{A.11}$$

To see that this really holds, we note that by definition the extensive observable given by the density q_4 defined in terms of patch-state-ansatz values $t_{s_1 s_2 s_3 s_4}$ (see subsection 5.1) as,

$$q_4 = \sum_{s_1, s_2, s_3, s_4 \in \{0,1,2\}} \left. \frac{\partial t_{s_1 s_2 s_3 s_4}}{\partial \mu} \right|_{\xi, \omega, \lambda, \mu \rightarrow 1} [s_1 s_2 s_3 s_4], \tag{A.12}$$

is conserved. Using this definition, we obtain

$$\begin{aligned}
q_4 = & [0100] + [0101] + [0111] + [0122] + [0200] + [0202] + [0211] + [0222] \\
& + [1000] + [1001] + [1010] + [1011] + [1012] + [1022] + [1111] + [1200] \\
& + [1202] + [1211] + [1222] + [2000] + [2002] + [2011] + [2020] + [2021] \\
& + [2022] + [2100] + [2101] + [2111] + [2122] + [2222].
\end{aligned} \tag{A.13}$$

To prove that \tilde{q}_4 and q_4 give the same extensive observable Q_4 (even though $\tilde{q}_4 \neq q_4$), one just needs to verify that following holds,

$$\hat{P}q_4 = \hat{P}\tilde{q}_4 = \tilde{q}_4, \tag{A.14}$$

which can be checked explicitly.

Appendix B. Asymptotic probabilities in Gibbs ensembles

Expectation value of a basis observable $[s_1 s_2 \dots s_{2m}]$ with support $2m$ in the Gibbs state \mathbf{p} is expressed as the following product of matrices,

$$\langle [s_1 s_2 \dots s_{2m}] \rangle_{\mathbf{p}} = \frac{1}{\text{tr } T^n} \text{tr} (W_{s_1} W'_{s_2} \dots W'_{s_{2m}} T^{n-m}), \tag{B.1}$$

where we introduced the *transfer matrix* T , defined as the sums of products of matrices W_s , W'_s on consecutive sites,

$$\begin{aligned}
T &= \sum_{s_1, s_2} W_{s_1} W'_{s_2} = \sum_{s_1, s_2} W'_{s_1} W_{s_2} \\
&= \begin{bmatrix} 3 & \lambda + \xi + \xi^{-1} & \omega + \xi + \xi^{-1} \\ \omega + \xi + \xi^{-1} & 2 + \lambda\omega & 1 + \omega(\xi + \xi^{-1}) \\ \lambda + \xi + \xi^{-1} & 1 + \lambda(\xi + \xi^{-1}) & 2 + \lambda\omega \end{bmatrix}.
\end{aligned} \tag{B.2}$$

When the system size is much bigger than the support of the observable in question, the expression simplifies into

$$\lim_{n \rightarrow \infty} \langle [s_1 s_2 \dots s_{2m}] \rangle_{\mathbf{p}} = \frac{\Lambda^{-m}}{\langle l | r \rangle} \langle l | W_{s_1} W'_{s_2} \dots W'_{s_{2m}} | r \rangle, \tag{B.3}$$

where Λ , $|r\rangle$ and $\langle l|$ are the leading eigenvalue and the corresponding right and left eigenvectors of the transfer matrix T ,

$$T|r\rangle = \Lambda|r\rangle, \quad \langle l|T = \Lambda\langle l|. \tag{B.4}$$

Equivalently, for the other time-step parity the expectation value can be expressed by simply exchanging $W_s \leftrightarrow W'_s$,

$$\lim_{n \rightarrow \infty} \langle [s_1 s_2 \dots s_{2m}] \rangle_{\mathbf{p}'} = \frac{\Lambda^{-m}}{\langle l | r \rangle} \langle l | W'_{s_1} W_{s_2} \dots W_{s_{2m}} | r \rangle. \tag{B.5}$$

Appendix C. Cubic algebra for higher stationary states

The components of \mathbf{X} , \mathbf{X}' that together with \mathbf{W} and \mathbf{W}' satisfy equations (47) and (48) take the following form,

$$X_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X'_0 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & \mu & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & 1 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{C.1}$$

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^2 & 1 & \mu & \xi\omega & 0 \\ \xi^2 & \xi^2 & 0 & 0 & 0 & 0 & \xi\omega \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X'_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\omega}{\xi} & \frac{1}{\xi^2} & \frac{\mu}{\xi^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi\omega & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{C.2}$$

and

$$X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & \lambda\xi & 0 & 1 & \mu \\ \xi^2 & 0 & \xi^2 & 0 & \lambda\xi & 0 & 0 \end{bmatrix}, \quad X'_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{\xi} & 0 & 0 & 0 & \frac{1}{\xi^2} & \frac{\mu}{\xi^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda\xi & 0 & 0 & 0 & 1 & 1 \end{bmatrix}. \quad (\text{C.3})$$

Appendix D. Euler scale hydrodynamics

D.1. Currents

The local density of the currents that satisfy the continuity equation (60) is

$$j_{1x} = \sum_{s=0}^2 ([ss]_x - [s0s]_x - [ss]_x - [ss]_{x+1}) \quad (\text{D.1})$$

$$j_{2x} = [112]_x - [122]_x, \quad j_{3x} = [221]_x - [211]_x.$$

D.2. Matrices of cross correlations

To express A , it is convenient to use the chain rule,

$$A = LK^{-1}, \quad (\text{D.2})$$

where K and L are 3×3 matrices with the matrix elements given by,

$$K_{\alpha,\beta} = \frac{\partial \langle q_{\alpha 0} \rangle_{\mathbf{p}}}{\partial \mu_{\beta}}, \quad L_{\alpha,\beta} = \frac{\partial \langle j_{\alpha 0} \rangle_{\mathbf{p}}}{\partial \mu_{\beta}}, \quad (\mu_1, \mu_2, \mu_3) = (\log \xi, \log \omega, \log \lambda). \quad (\text{D.3})$$

Here μ_{α} , $\alpha \in \{1, 2, 3\}$ are chemical potentials associated to the three conserved charges. A convenient way to evaluate the relevant expectation values is to use the matrix product from appendix B, which in the limit $\xi, \omega, \lambda \rightarrow 1$ gives the following form of K and L ,

$$K|_{\xi=\omega=\lambda=1} = \begin{bmatrix} \frac{4}{9} & 0 & 0 \\ 0 & \frac{4}{27} & \frac{2}{27} \\ 0 & \frac{2}{27} & \frac{4}{27} \end{bmatrix}, \quad L|_{\xi=\omega=\lambda=1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 \end{bmatrix}. \quad (\text{D.4})$$

Plugging these into (D.2), we obtain precisely A as given in (61).

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