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Combinatorial problems raised from 2-semilattices

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Abstract

The Constraint Satisfaction Problem (CSP) provides a general framework for many combinatorial problems. In [A.A. Bulatov, A.A. Krokhin, P.G. Jeavons, Classifying the complexity of constraints using finite algebras, SIAM J. Comput. 34 (3) (2005) 720–742; P.G. Jeavons, On the algebraic structure of combinatorial problems, Theoret. Comput. Sci. 200 (1998) 185–204] and then in [A.A. Bulatov, P.G. Jeavons, Algebraic structures in combinatorial problems, Technical Report MATH-AL-4-2001, Technische Universität Dresden, Dresden, Germany, 2001], a new approach to the study of the CSP has been developed which uses properties of universal algebras assigned to certain subclasses of the CSP such that the time complexity and other properties of subclasses can be derived from the properties of the assigned algebras. In this paper we briefly survey this approach, and then prove that problem classes corresponding to finite 2-semilattices, that is groupoids satisfying the identities $xx = x$, $xy = yx$, $x(xy) = (xx)y$, can be solved in polynomial time. Making use of this result we classify finite conservative groupoids, and 4-element algebras with minimal clone of term operations with respect to the complexity of the corresponding CSP-class.
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1. Introduction

The aim of a constraint satisfaction problem (CSP, for short) is to assign values to variables subject to constraints which are imposed on possible combinations of values of certain subsets of variables. A wide range of combinatorial problems can be expressed as CSPs. The standard examples include classical combinatorial problems such as propositional satisfiability

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problem [19], graph colorability, clique, scheduling problem, and many others [20]; and also problems from various areas of computer science and artificial intelligence, for example, database theory [47], temporal and spatial reasoning [43], machine vision [36], belief maintenance [16], technical design [38], natural language comprehension [1], and programming language analysis [37].

Throughout the paper we assume that $\mathrm{P} \neq \mathrm{NP}$. Recall that problems from P are said to be *tractable*. The general constraint satisfaction problem is known to be NP-complete, and hence, intractable [33,36]. However, certain restrictions on the form of constraints may affect the complexity of the corresponding problem class and give rise to a tractable subclass of the general CSP. There is, therefore, a fundamental research direction aiming to recognize tractable subclasses of the constraint satisfaction problem.

This goal has been achieved by Schaefer in 1978 [42] in the important special case of Boolean constraint satisfaction problems when variables are assigned by Boolean values. Schaefer identified six different families of tractable constraints for Boolean constraint satisfaction problems (which he called "Generalized Satisfiability Problems"), and proved that any problem involving constraints not contained in one of these six families is NP-complete. This important result is known as Schaefer's Dichotomy Theorem for Boolean relations.

In the same paper Schaefer raised the question of how this result could be generalized to domains with more than 2 elements. This problem is still unsolved in spite of some progress has been made recently. In particular, Feder and Vardi [18] used techniques from logic programming and group theory to identify three broad families of tractable constraints which include all of Schaefer's six classes. Another fruitful approach was suggested by Jeavons and co-authors, who have shown that all tractable constraint classes over finite domains can be characterized using invariance properties of relations [22–24,26].

The latter approach has been developed in [6,9], and allows one to parameterize subclasses of the CSP over a finite set of values by finite algebras such that the complexity of a subclass is completely determined by the properties of the corresponding algebra. An algebra is said to be *tractable* [*intractable, NP-complete*] if the corresponding CSP-class is tractable [intractable, NP-complete]. The tractability has been proved for many famous classes of algebras including finite semilattices [26], groups [18], quasiprimal [22] and paraprimal [14] algebras. Some types of algebras appear to be NP-complete. An archetype example of an NP-complete algebra is provided by G-sets. For several classes of finite algebras, the tractable algebras have been characterized: 2- and 3-element algebras [2,4,42], strictly simple algebras [9], finite semigroups [8]. The results obtained allow one to pose a plausible conjecture on a complete classification of finite algebras with respect to the complexity of the corresponding CSP-class [6].

In this paper we investigate the class of finite 2-semilattices, that is groupoids which satisfy all the semilattice identities involving at most 2 variables, or, equivalently, the identities $xx = x$ (idempotency), $xy = yx$ (commutativity), $x(xy) = (xx)y$ (restricted associativity). 2-semilattices provide a natural generalization of semilattices and inherit many of their properties. We prove that every finite 2-semilattice is tractable.

Another reason why 2-semilattices are of interest is that they play an important role in the study of algebras with minimal clone of term operations. Algebras whose clone of term operations is an atom in the clone lattice attract a special attention in both, clone theory and the study of the CSP, because they correspond to CSP-classes that are just smaller than the general CSP. We shall call such algebras *clone minimal algebras*. In [41], clone minimal algebras have been classified into 5 types. For 4 of them the tractability or intractability can be easily derived from the earlier results. For the remaining type of finite idempotent groupoids there are only some partial results. In [7], it is shown that every tractable idempotent groupoid with minimal clone of term operations has a binary commutative term operation, and this condition is sufficient for the tractability when the groupoid is 2- or 3-element. The latter result heavily uses the description of 3-element clone minimal algebras [12]. Such a description in the general case is still an open problem, though some achievements have been obtained [13,29,32,48]. In this paper we use the description of 4-element groupoids with minimal clone of term operations [44] that implies that if such a groupoid has a commutative binary term operation then it is term equivalent to a 2-semilattice, to show that the necessary condition for the tractability of clone minimal algebras stated in [7] is sufficient for 4-element algebras.

We also use our main result to classify, with respect to the tractability, another well-known class of groupoids, conservative groupoids [10,27,28,49]. Recall that a groupoid $\mathbb{G}$ is said to be *conservative* if $ab \in \{a, b\}$ for any $a, b \in \mathbb{G}$. We show that a finite conservative groupoid is tractable if and only if it is commutative. Otherwise, it is NP-complete.

The paper is organized as follows. In Section 2 we survey the connection between finite algebras and the complexity of the CSP. In Section 3.1 we introduce an important type of tractable problem classes, classes of finite width, that have a simple solving algorithm; in Section 3.2 we prove several auxiliary results; in Section 3.3 we study simple 2-semilattices, and show that problems related to simple 2-semilattices are of width 3. Finally, in Section 3.4 we prove this property in the general case, and in Section 4 we apply our main result to conservative groupoids and 4-element clone minimal algebras.

2. Preliminaries

2.1. Constraint satisfaction problem

By an n-ary relation on a collection of sets $\{A_i \mid i \in I\}$ we mean a subset of $A_{i_1} \times \cdots \times A_{i_n}$ where $i_1, \ldots, i_n \in I$ together with the tuple $(i_1, \ldots, i_n)$ called the *signature*.

The 'constraint satisfaction problem' was introduced by Montanari in 1974 [36] and has been widely studied [17,18,31,33–35,46].

Definition 2.1. The *constraint satisfaction problem* (CSP) is the combinatorial decision problem with

INSTANCE: a quadruple $(V; \mathcal{A}; \delta; \mathcal{C})$ where

- V is a set of *variables*;
- $\mathcal{A} = \{A_i \mid i \in I\}$ is a collection of sets of values [*domains*];
- δ is a mapping from V to I, called the *domain function*;
- $\mathcal{C}$ is a set of *constraints*;
- Each constraint $C \in \mathcal{C}$ is a pair $\langle s, \varrho\rangle$, where
  - $s = (v_1, \ldots, v_{m_C})$ is a tuple of variables of length m_C, called the *constraint scope*;
  - ϱ is an m_C-ary relation over $\mathcal{A}$ with signature $(\delta(v_1), \ldots, \delta(v_{m_C}))$, called the *constraint relation*.

Question: does there exist a *solution*, i.e. a function φ, from V to $\bigcup_{v\in V} A_{\delta(v)}$, such that, for each variable $v \in V$, $\varphi(v) \in A_{\delta(v)}$, and for each constraint $\langle s, \varrho\rangle \in \mathcal{C}$, with $s = (v_1, \ldots, v_m)$, the tuple $(\varphi(v_1), \ldots, \varphi(v_m))$ belongs to ϱ?

We shall consider constraint satisfaction problems over finite domains only. With this restriction, the size of a problem instance can be taken to be the length of a string containing domains for all variables, all constraint scopes, and all tuples of the constraint relations from the instance.

Often all the variables can be supposed to have a common set of values. To distinguish this important particular case, we will refer to the general case as to the *multi-sorted* CSP, while to the case with a common domain as the *one-sorted* CSP. A one-sorted problem instance is then written as $(V, A, \mathcal{C})$ where A is the common domain and $V, \mathcal{C}$ are as before.

The constraint satisfaction problem is NP-complete in general, as has been proved in [36] and will be seen from the examples below. However, some restrictions may affect the complexity. There are two ways to restrict the general CSP: first, to impose some restrictions on the possible form of constraint relations, second, to restrict the way in which the constraints may interact. In this paper we explore the first way.

Let Γ be a set of relations over a collection of sets $\mathcal{A} = \{A_i \mid i \in I\}$. Then CSP$(\Gamma)$ denotes the subclass of CSP defined by the property: for any instance $\mathcal{P} \in$ CSP(Γ), every variable has one of the sets from $\mathcal{A}$ as the domain, and every constraint relation of $\mathcal{P}$ belongs to Γ. The set Γ is said to be *tractable* if, for each finite subset $\Gamma' \subseteq \Gamma$, the class CSP(Γ') is tractable; Γ is said to be *NP-complete* if CSP(Γ') is NP-complete for certain finite subset $\Gamma' \subseteq \Gamma$. If there is a uniform polynomial time algorithm that solves CSP(Γ), then Γ is said to be *globally tractable*.

Example 2.1. The binary *disequality* relation, denoted by $\neq_D$, is defined as

$$\neq_D = \{(d_1, d_2) \in D^2 \mid d_1 \neq d_2\}.$$

Note that CSP$(\{\neq_D\})$ precisely corresponds to the GRAPH $|D|$-COLORABILITY problem [19]. Indeed, a problem instance of the GRAPH $|D|$-COLORABILITY with the input graph $G = (V, E)$ corresponds to the constraint satisfaction problem $(V; D; \{\langle (u, v), \neq_D\rangle \mid (u, v) \in E\})$, see Fig. 1. Clearly, the graph can be correctly colored if and only if all its vertices can be assigned values so that the values of adjacent vertices are different. Thus CSP$(\{\neq_D\})$ is tractable when $|D| = 2$ and NP-complete when $|D| \geqslant 3$.

Example 2.2. An instance of GRAPH UNREACHABILITY consists of a graph $G = (V, E)$ and a pair of vertices, $v, w \in V$. The question is whether there is no path in G from v to w. This can expressed as the CSP instance $(V, \{0, 1\}, \mathcal{C})$ where

$$\mathcal{C} = \{\langle e, \{(0,0), (1,1)\}\rangle \mid e \in E\} \cup \{\langle (v), \{(0)\}\rangle, \langle (w), \{(1)\}\rangle\}.$$

Indeed, this CSP instance has a solution if and only if one can label the vertices of G by 0's and 1's such that v is labeled 0, w is labeled 1, and adjacent vertices are assigned equal labels, see Fig. 2. Clearly, this is possible if and only if there is no path from v to w. Thus,

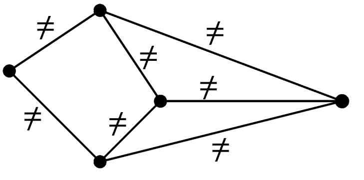

Fig. 1. GRAPH COLORABILITY as an instance of the CSP.

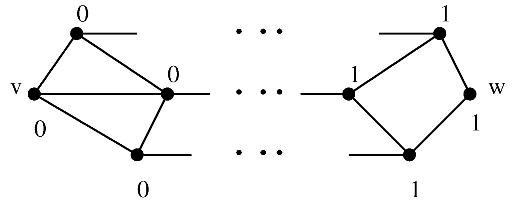

Fig. 2. GRAPH UNREACHABILITY as an instance of the CSP.

GRAPH UNREACHABILITY $\subseteq$ CSP($\{=_{\{0,1\}}, \{(0)\}, \{(1)\}\}$) where $=_{\{0,1\}}$ denotes the equality relation on $\{0, 1\}$.

Example 2.3. A system of linear equations over a field F can be expressed as the CSP instance $(V; F; \mathcal{C})$ where V is the set of variables of the system, and each constraint $\langle s, \varrho\rangle$ from $\mathcal{C}$ corresponds to an equation. Then s is the set of variables appearing in the equation, and ϱ is the solution space of the equation, that is a hyperplane.

For further examples including essentially multi-sorted CSPs the reader is referred to [5,6].

Up to now, no set of relations is known to be tractable but not globally tractable. Thus a natural conjecture is that any tractable set of relations is also globally tractable. However, this cannot be taken for granted and requires additional efforts in each particular case.

Conjecture 2.1 *(local–global conjecture). Any tractable set of relations is globally tractable.*

2.2. Constraint satisfaction problem and finite algebras

A connection between finite algebras and problem classes of the form CSP(Γ) has been established in [9,20] for the one-sorted CSP, and in [5,6] for the multi-sorted CSP. We start with the one-sorted case.

The ultimate goal in the study of CSP(Γ) is to find a way to distinguish between tractable and intractable sets of relations. In this connection, it is natural to ask how a tractable set of relations can be extended so that the extended set of relations remains tractable. In other words, which operations on relations do preserve tractability? An important class of such operations has been discovered in [20].

We make use of the standard correspondence between relations and logical predicates: to each m-ary relation $\varrho \subseteq A^m$ we assign the m-ary predicate on A which becomes true precisely on m-tuples from ϱ. Recall that an existential first order formula is said to be *primitive positive* if its quantifier-free part is a conjunction of predicates. These formulas can be used as operations that allow one to derive new relations from given ones.

Example 2.4. The standard product of two binary relations ϱ_1 and ϱ_2 may be defined by the following primitive positive formula:

$$(x, z) \in \varrho_1 \circ \varrho_2 \quad \Longleftrightarrow \quad (\exists y)\ (x, y) \in \varrho_1 \wedge (y, z) \in \varrho_2.$$

A set of relations on a set A closed with respect to all operations defined by primitive positive formulas is called a *relational clone*. Given a set Γ of relations, the smallest relational clone containing Γ is said to be *generated by* Γ and denoted by $\langle\Gamma\rangle$.

Theorem 2.1. [20] *A set of relations, Γ, is tractable if and only if $\langle\Gamma\rangle$ is tractable.*

In a sense, Theorem 2.1 amounts to say that, in the study CSP(Γ), a reasonable strategy is to concentrate on relational clones rather than arbitrary families of relations. In many cases, this strategy has proved to be successful: for example, in [20] it is shown that Schaefer's result [42] on the CSP over a two-element domain can easily be deduced from Theorem 2.1 and the classical description of clones on a two-element set [40].

An m-ary relation ϱ is said to be *invariant* with respect to an operation $f(x_1,\dots,x_n)$ [or f *preserves* ϱ] if, for any $(a_{11},\dots,a_{m1}),\dots,(a_{1n},\dots,a_{mn})\in\varrho$, we have

$$\begin{pmatrix} f(a_{11},a_{12},\dots,a_{1n}) \\ f(a_{21},a_{22},\dots,a_{2n}) \\ \vdots \\ f(a_{m1},a_{m2},\dots,a_{mn}) \end{pmatrix}\in\varrho.$$

This also can be expressed by saying that ϱ is a subalgebra of the mth direct power of the algebra $(A; f)$. Given a set of relations, Γ, the set of all operations preserving Γ is denoted by $\operatorname{Pol}\Gamma$. Analogously given a set of operations, C, on A, the set of all relations which preserved by operations from C is denoted by $\operatorname{Inv} C$.

It is well known [39,45] that, for any finite algebra $\mathbb{A}=(A; F)$, the set $\operatorname{Inv} F$, that is the set of all subalgebras of finite direct powers of $\mathbb{A}$, forms a relational clone. Moreover, for any relational clone Γ over a finite set A, there is a finite algebra $\mathbb{A}=(A; F)$ such that $\Gamma=\operatorname{Inv} F$. In fact, F can be chosen to be $\operatorname{Pol}\Gamma$.

Thus, every finite algebra $\mathbb{A}$ determines a problem class CSP($\operatorname{Inv} F$) which will also be denoted by CSP($\mathbb{A}$). The algebra $\mathbb{A}$ is said to be (*globally*) *tractable* [*NP-complete*] if the set $\operatorname{Inv} F$ is (globally) tractable [NP-complete]. Conversely, by Theorem 2.1, for any problem class of the form CSP(Γ), where Γ is a family of relations over a finite set A, there is the corresponding finite algebra $\mathbb{A}=(A;\operatorname{Pol}\Gamma)$, and Γ is (globally) tractable [NP-complete] if and only if $\mathbb{A}$ is (globally) tractable [NP-complete].

In the case of the multi-sorted CSP, the algebraic structure of problem classes can be introduced in a more straightforward way [5,6]. Let $\mathcal{A}$ be a class of similar finite algebras. The class of all subalgebras of all finite direct products of algebras from $\mathcal{A}$ is denoted by $\mathrm{SP_{fin}}\,\mathcal{A}$. By CSP($\mathcal{A}$) we mean the subclass CSP($\mathrm{SP_{fin}}\,\mathcal{A}$) of the multi-sorted CSP. The class $\mathcal{A}$ is said to be (*globally*) *tractable* [*NP-complete*] if $\mathrm{SP_{fin}}\,\mathcal{A}$ is (globally) tractable [NP-complete].

If $\mathcal{A}$ is finite and (globally) tractable, say $\mathcal{A}=\{\mathbb{A}_1,\dots,\mathbb{A}_n\}$, then $\mathbb{A}_1,\dots,\mathbb{A}_n$ are called *mutually* (*globally*) *tractable*. The tractability of a finite class of algebras is equivalent to the tractability of a single algebra.

Theorem 2.2. [6] *Let $\mathbb{A}_1,\dots,\mathbb{A}_n$ be similar finite algebras. Then $\mathbb{A}_1,\dots,\mathbb{A}_n$ are mutually* (*globally*) *tractable if and only if $\mathbb{A}_1\times\cdots\times\mathbb{A}_n$ is* (*globally*) *tractable.*

Theorem 2.2 makes it possible to define an algebraic structure of a CSP-class starting from a set of multi-sorted relations. Given a finite set of relations, Γ, over a collection of sets $\{A_1,\dots,A_n\}$, we may treat relations from Γ as relations over $A=A_1\times\cdots\times A_n$; then set $\mathbb{A}=(A;\operatorname{Pol}\Gamma')$ where Γ' is Γ together with the kernels of projections of A onto $A_1,\dots,A_n$; finally, $\mathbb{A}$ can be represented as $\mathbb{A}_1\times\cdots\times\mathbb{A}_n$, and $\Gamma\subseteq\mathrm{SP_{fin}}(\mathbb{A}_1,\dots,\mathbb{A}_n)$.

Any tractable class of algebras can be enlarged so that the enlarged class remains tractable.

Theorem 2.3. [6] *If a class of similar finite algebras $\mathcal{A}$ is tractable then the class of finite algebras from the variety it generates is also tractable.*

The theorem says that the tractability can be characterized by a specific term or a family of terms. Some terms guaranteeing the tractability have been identified earlier [3,6,9,21,22,24–26].

Proposition 2.1. *If a variety has one of the following terms: a constant term, a semilattice term, a near-unanimity term, a Mal'tsev term then the class of its finite algebras is globally tractable.*

Clearly, an algebra with a rich family of term operations has a small class of invariant relations. Therefore poor algebras give rise to hard problem classes. In particular, the NP-completeness of a wide class of algebras can be characterized just by presence of an algebra with poor set of term operations. Recall that a *G-set* is a set with group action on it, and algebras are said to be *term equivalent* if they have the same set of term operations.

Proposition 2.2. [9] *If the variety generated by a class of finite similar algebras, $\mathcal{A}$, contains an algebra term equivalent to a finite G-set, then $\mathcal{A}$ is NP-complete.*

Moreover, all known NP-complete classes of finite algebras satisfy the condition from Proposition 2.2.

Conjecture 2.2 *(dichotomy conjecture). A class of finite algebras is tractable if and only if the variety it generates contains no G-set. Otherwise it is NP-complete.*

Now we can formulate the main result of the paper.

Theorem 2.4. *The class of finite 2-semilattices is globally tractable.*

3. Proof of Theorem 2.4

3.1. Algorithm

First we introduce a property of a subclass of the CSP that provides a polynomial time solution algorithm. Throughout the paper we treat relations as subalgebras of direct products and use for them the same notation as for algebras.

For an n-ary relation $\mathbb{C}$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{C}$, and $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, by $\mathrm{pr}_I\, \mathbf{a}$ we denote the tuple $(a_{i_1}, \ldots, a_{i_k})$, and by $\mathrm{pr}_I\, \mathbb{C}$ the set $\{\mathrm{pr}_I\, \mathbf{b} \mid \mathbf{b} \in \mathbb{C}\}$. We sometimes will also write $\mathrm{pr}_{i_1,\ldots,i_k}\, \mathbb{C}$ instead of $\mathrm{pr}_I\, \mathbb{C}$.

Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be an instance of the multi-sorted CSP, and $W \subseteq V$. By $\mathcal{P}_W$ we denote the *restricted problem*, that is the problem instance defined as $(W; \mathcal{A}; \delta|_W; \mathcal{C}')$ where, for every $\langle s, \mathbb{C}\rangle \in \mathcal{C}$, there is $\langle s', \mathbb{C}'\rangle \in \mathcal{C}'$ with $s' = s \cap W$, and $\mathbb{C}' = \mathrm{pr}_{s'}\, \mathbb{C}$. Every solution of $\mathcal{P}_W$ is said to be a *partial solution* of $\mathcal{P}$ on W. Let us denote the set of all partial solutions on W by $\mathcal{S}_W$. Notice that $\mathcal{S}_W$ can be treated as a $|W|$-ary relation.

We call the problem instance $\mathcal{P}$ *k-minimal* if, for any k-element subset W of V, there is a constraint $\langle s, \mathbb{C}\rangle \in \mathcal{C}$ such that $W \subseteq s$, and for any $\langle s, \mathbb{C}\rangle \in \mathcal{C}$, we have $\mathrm{pr}_{W\cap s}\, \mathbb{C} = \mathrm{pr}_{W\cap s}\, \mathcal{S}_W$.

Any problem instance $\mathcal{P}$ can be transformed to an equivalent k-minimal instance $\mathcal{P}'$. To do this we employ the algorithm k-MINIMALITY, see Fig. 3. As is easily seen, the time complexity

Input. A problem instance $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$.

Output. A k-minimal problem instance $\mathcal{P}' = (V; \mathcal{A}; \delta; \mathcal{C}')$ equivalent to $\mathcal{P}$.

Step 1 /* initialization
Set $\mathcal{P}' := (V; \mathcal{A}; \delta; \mathcal{C}')$ where
$$\mathcal{C}' = \mathcal{C} \cup \{\langle W, \prod_{v\in W} \mathbb{A}_{\delta(v)}\rangle \mid W \subseteq V,\ |W| = k\};$$
and $\mathcal{P}'' := \mathcal{P}'$.
Step 2 **Do**
Step 3 **Set** $\mathcal{P}' := \mathcal{P}''$.
Step 4 **For each** $W \subseteq V$ with $|W| = k$ **do**
Step 5 **Solve** the restricted problem $\mathcal{P}'_W$.
Step 6 **For each** constraint $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}''$, **replace** C with $\langle s, \mathbb{C}'\rangle$ where $\mathbb{C}' = \{\mathbf{a} \in \mathbb{C} \mid \mathrm{pr}_{s\cap W}\, \mathbf{a} \in \mathrm{pr}_{s\cap W}\, \mathcal{S}'_W\}$.
Step 7 **Until** $\mathcal{P}'' = \mathcal{P}'$.
Step 8 **Output** $\mathcal{P}'$.

Fig. 3. Algorithm k-MINIMALITY.

of this algorithm is $O(m^3 n^k)$ where n is the number of variables and m is the total number of tuples in the constraint relations. Therefore, Theorem 2.4 can be deduced from the following theorem which will be proved in the next three subsections.

Theorem 3.1. *Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be an instance of the multi-sorted constraint satisfaction problem where $\mathcal{A}$ is a class of finite 2-semilattices. Then if $\mathcal{P}$ is 3-minimal and has no empty constraint relation, then $\mathcal{P}$ has a solution.*

Problem classes satisfying the property stated in Theorem 3.1 have been widely studied in computer science, see, for example, [11,15,18,22]. A subclass of the constraint satisfaction problem, **K**, is said to be of *relational width k* if, for any $\mathcal{P} \in \mathbf{K}$, the corresponding k-minimal problem $\mathcal{P}'$ is also in **K**, and $\mathcal{P}$ has a solution if and only if $\mathcal{P}'$ has no empty constraint relation. As is easily seen, any problem class of finite relational width is globally tractable. Thus Theorem 3.1 says that the class CSP($\mathcal{A}$), where $\mathcal{A}$ is the class of all finite 2-semilattices is of relational width 3, and therefore globally tractable.

By applying the algorithm 1-MINIMALITY we may reduce an arbitrary problem instance to a 1-minimal one. Thus, from now on we shall assume all problem instances to be 1-minimal, and all constraint relations to be subdirect products of their domains.

3.2. Preliminary results

Let $\mathbb{G} = (G, *)$ be a finite 2-semilattice. Consider the digraph (G, E) associated with $\mathbb{G}$ such that $(a, b) \in E$ if and only if $a * b = b * a = b$. For every $(a, b) \in E$ we also write $a \leqslant b$. Since $a * (a * b) = b * (a * b) = a * b$, for any $a, b \in \mathbb{G}$, we have $a, b \leqslant a * b$, and the graph (G, E) is connected. For strongly connected components A, B, we write $A \leqslant B$ iff there are $a \in A$, $b \in B$ such that $a \leqslant b$. Moreover, there exists a unique strongly connected component $\overline{\mathbb{G}}$ of $(G; E)$ such that, for any $a \in \mathbb{G}$, there is $b \in \overline{\mathbb{G}}$ with $a \leqslant b$. We call this component the *greatest strongly connected component* of $\mathbb{G}$. We say that $\mathbb{G}$ is *strongly connected* if (G, E) is strongly connected.

Lemma 3.1. *Let $\mathbb{D}$ be a strongly connected subgroupoid of a 2-semilattice $\mathbb{G}$. Then for every $a, b \in \mathbb{D}$ there exists $n \in \mathbb{N}$ such that the equation*

$$\left(\cdots\left((a * x_1) * x_2\right) * \cdots * x_{n-1}\right) * x_n = b$$

is solvable in $\mathbb{D}$.

Proof. The lemma follows straightforwardly from the definition of strongly connectedness. □

Lemma 3.2. *Let $\mathbb{D}$ be a subdirect product of 2-semilattices $\mathbb{D}_1, \ldots, \mathbb{D}_n$ and $I_1, \ldots, I_k \subseteq \{1, \ldots, n\}$. Let also $\overline{\mathbb{D}}_{I_t}$ denote the greatest strongly connected component of $\mathrm{pr}_{I_t} \mathbb{D}$. Then for any tuple $\mathbf{a}' = (a_i)_{i \in I_1} \in \overline{\mathbb{D}}_{I_1}$ there exist $a_j \in \mathbb{D}_j$, $j \in \{1, \ldots, n\} - I_1$, such that $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{D}$ and $\mathrm{pr}_{I_t} \mathbf{a} \in \overline{\mathbb{D}}_{I_t}$ for each $t \in \{2, \ldots, k\}$.*

Proof. Choose $\mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathbb{D}$ such that $\mathrm{pr}_{I_t} \mathbf{b}_t \in \overline{\mathbb{D}}_{I_t}$, and set $\mathbf{b} = (\cdots(\mathbf{b}_1 * \mathbf{b}_2) * \mathbf{b}_3 \cdots) * \mathbf{b}_k$. Obviously, $\mathrm{pr}_{I_t} \mathbf{b} \in \overline{\mathbb{D}}_{I_t}$ for each $t \in \{1, \ldots, k\}$. By Lemma 3.1, there are l and $\mathbf{c}_1, \ldots, \mathbf{c}_l \in \mathbb{D}$ such that $\mathrm{pr}_{I_1} \mathbf{c}_1, \ldots, \mathrm{pr}_{I_1} \mathbf{c}_l$ form a solution of the equation

$$\left(\cdots(\mathrm{pr}_{I_1} \mathbf{b} * x_1) * x_2 \cdots\right) * x_l = \mathbf{a}'.$$

Then the tuple

$$\left(\cdots(\mathbf{b} * \mathbf{c}_1) * \mathbf{c}_2 \cdots\right) * \mathbf{c}_l$$

satisfies the conditions of the lemma. □

Corollary 3.1. *Let $\mathbb{D}$ be a subdirect product of 2-semilattices $\mathbb{D}_1, \ldots, \mathbb{D}_n$, $I \subseteq \{1, \ldots, n\}$, and let $\overline{\mathbb{D}}$ be the greatest strongly connected component of $\mathbb{D}$. Then the greatest strongly connected component of $\mathrm{pr}_I \mathbb{D}$ equals $\mathrm{pr}_I \overline{\mathbb{D}}$.*

To prove Corollary 3.1 we just have to apply Lemma 3.2 where $I_1, \ldots, I_k$ are: $I_1 = I$, $I_2 = \{1, \ldots, n\}$.

3.3. Simple 2-semilattices

First, we prove Theorem 3.1 assuming that all 2-semilattices occurring in an instance are simple.

Lemma 3.3. *If $\mathbb{G}$ is simple then either $\mathbb{G}$ is strongly connected or $\mathbb{G}$ consists of 2 strongly connected components and the greatest one is one-element* (*see Fig.* 4).

Proof. If $|\overline{\mathbb{G}}| > 1$ and $G \neq \overline{G}$ then the equivalence relation θ, such that one class of θ is $\overline{\mathbb{G}}$ and the others are one-element, is a congruence of $\mathbb{G}$. Otherwise, if $|\overline{\mathbb{G}}| = 1$ but $\mathbb{G}$ has at least 3 strongly connected components, then let A be a strongly connected component such that $A \leqslant \overline{\mathbb{G}}$, and $A \leqslant B \leqslant \overline{\mathbb{G}}$ for no strongly connected component $B \neq A, \overline{\mathbb{G}}$. In this case the classes of the required non-trivial congruence are $\overline{\mathbb{G}} \cup A$ and one-element subsets. □

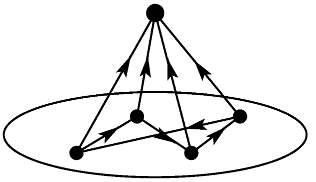

Fig. 4. A simple not strongly connected 2-semilattice.

Lemma 3.4. *Let $\mathbb{D}$ be a subdirect product of strongly connected* 2*-semilattices* $\mathbb{D}_1, \mathbb{D}_2$*, and there exists an element* $a \in \mathbb{D}_1$ *such that* $\{a\} \times \mathbb{D}_2 \subseteq \mathbb{D}$*. Then* $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$.

Proof. We prove by induction that $\{c\} \times \mathbb{D}_2 \subseteq \mathbb{D}$ for every $c \in \mathbb{D}_1$. The inclusion $\{a\} \times \mathbb{D}_2 \subseteq \mathbb{D}$ forms the base case of induction. Further, suppose that there is $d \in \mathbb{D}_1$ such that $d \leqslant c$ and $\{d\} \times \mathbb{D}_2 \subseteq \mathbb{D}$. Take an arbitrary $b \in \mathbb{D}_2$. For a certain $b_1 \in \mathbb{D}_2$, we have $\binom{c}{b_1} \in \mathbb{D}$; by Lemma 3.1, there exist $b_2, \ldots, b_l \in \mathbb{D}_2$ such that $(\cdots(b_1 * b_2) * \cdots) * b_l = b$. It is easy to see that

$$\left(\cdots\left(\begin{pmatrix} c \\ b_1 \end{pmatrix} * \begin{pmatrix} d \\ b_2 \end{pmatrix}\right) * \cdots\right) * \begin{pmatrix} d \\ b_l \end{pmatrix} = \begin{pmatrix} c \\ b \end{pmatrix} \in \mathbb{D}.$$

Since $\mathbb{D}_1$ is strongly connected, this proves the lemma. □

Our first goal is to show that a subdirect product of simple strongly connected 2-semilattices has a very restricted form. The *graph* of a mapping $\pi : \mathbb{D}_1 \to \mathbb{D}_2$ is the binary relation $G_\pi = \{(a, \pi(a)) \mid a \in \mathbb{D}_1\}$ over $\mathbb{D}_1, \mathbb{D}_2$.

Lemma 3.5. *Let $\mathbb{D}$ be a subdirect product of simple strongly connected* 2*-semilattices* $\mathbb{D}_1, \mathbb{D}_2$*. Then $\mathbb{D}$ is either the graph of a bijective mapping from* $\mathbb{D}_1$ *to* $\mathbb{D}_2$*, or* $\mathbb{D}_1 \times \mathbb{D}_2$.

Proof. Notice first, that if $\mathbb{D}$ is the graph of a mapping $\pi : \mathbb{D}_1 \to \mathbb{D}_2$, then the kernel of π is a congruence of $\mathbb{D}_1$ and, since $\mathbb{D}_1$ is simple, π is a bijection. The same holds if $\mathbb{D}$ is the graph of a mapping from $\mathbb{D}_2$ into $\mathbb{D}_1$.

Suppose that $\mathbb{D}$ is neither $\mathbb{D}_1 \times \mathbb{D}_2$ nor the graph of a bijective mapping, and that $|\mathbb{D}_1| + |\mathbb{D}_2|$ is the smallest number such that there exists a subdirect product of simple strongly connected 2-semilattices with this property. We show that there is $b \in \mathbb{D}_1$ [or $b \in \mathbb{D}_2$] such that $\{b\} \times \mathbb{D}_2 \subseteq \mathbb{D}$ [respectively, $\mathbb{D}_1 \times \{b\} \subseteq \mathbb{D}$].

For $a \in \mathbb{D}_1, b \in \mathbb{D}_2$ by B_a, C_b we denote the sets $\{c \mid (a, c) \in \mathbb{D}\}$, $\{c \mid (c, b) \in \mathbb{D}\}$, respectively.

Claim 1. *For any $A \subset \mathbb{D}_1$ [any $A \subset \mathbb{D}_2$], there is $a \in \mathbb{D}_2$ [respectively, $a \in \mathbb{D}_1$] and $b \in A$, $c \in \mathbb{D}_1 - A$ [respectively, $c \in \mathbb{D}_2 - A$] such that $(b, a), (c, a) \in \mathbb{D}$ [respectively, $(a, b), (a, c) \in \mathbb{D}$].*

We prove in the case $A \subset \mathbb{D}_1$. Since $\mathbb{D}$ is not the graph of a mapping from $\mathbb{D}_2$ to $\mathbb{D}_1$, there are $(d_1, e), (d_2, e) \in \mathbb{D}$ with $d_1 \neq d_2$. Consider the graph $(\mathbb{D}_1, F)$ where $(b, c) \in F$ if and only if there are unary polynomials f_1, f_2 of $\mathbb{D}$ such that $f_1\binom{d_1}{e} = \binom{b}{e'}$, $f_2\binom{d_2}{e} = \binom{c}{e'}$. Notice that, any pair $(b, c) \in F$ belongs to the congruence generated by (d_1, d_2) and there is $e' \in \mathbb{D}_2$ such that $(b, e'), (c, e') \in \mathbb{D}$. Finally, as $\mathbb{D}_1$ is simple, $(\mathbb{D}_1, F)$ is connected, that implies the required statement. The case $A \subset \mathbb{D}_2$ is quite analogous.

Take $a \in \mathbb{D}_1$ such that $|B_a| > 1$, set $E_1 = \{a\}$, and for each $i > 0$

$$E_{i+1} = \begin{cases} \bigcup_{b\in E_i} B_b & \text{if } i \text{ is odd,} \\ \bigcup_{b\in E_i} C_b & \text{if } i \text{ is even.} \end{cases}$$

By Claim 1, for each $i > 0$, $E_i \subset E_{i+2}$ unless $E_i = \mathbb{D}_1$ or $E_i = \mathbb{D}_2$. Therefore, for some $l > 1$, $E_l = \mathbb{D}_1$ or $E_l = \mathbb{D}_2$. Without loss of generality, suppose $E_l = \mathbb{D}_2$, and $E_{l-1} \neq \mathbb{D}_1$, $E_{l-2} \neq \mathbb{D}_2$.

Claim 2. *For each i, $1 \leqslant i \leqslant l$, E_i is a subalgebra of $\mathbb{D}_1$ or $\mathbb{D}_2$.*

We prove the claim by induction. In the base case of induction $E_1 = \{a\}$ is a subalgebra, because $\mathbb{D}_1$ is idempotent. If E_i is a subalgebra, and $E_i \subseteq \mathbb{D}_1$, then for any $a_2, b_2 \in E_{i+1}$ there are $a_1, b_1 \in E_i$ such that $(a_1, a_2), (b_1, b_2) \in \mathbb{D}$. Then $\binom{a_1*b_1}{a_2*b_2} = \binom{a_1}{a_2} * \binom{b_1}{b_2} \in \mathbb{D}$, $a_1 * b_1 \in E_i$, hence, $a_2 * b_2 \in E_{i+1}$. The proof in the case $E_i \subseteq \mathbb{D}_2$ is analogous.

Thus, E_{l-1} is a proper subgroupoid of $\mathbb{D}_1$ such that $\bigcup_{b\in E_{l-1}} B_b = \mathbb{D}_2$.

Define a sequence $\mathbb{B}_0, \mathbb{B}_1, \ldots, \mathbb{B}_k$ of groupoids and a sequence of congruences $\theta_0, \theta_1, \ldots, \theta_k$ where θ_i is a congruence of $\mathbb{B}_i$ through the following rules.

(1) $\mathbb{B}_0$ is the greatest strongly connected component of E_{l-1}.

(2) Suppose that $\mathbb{B}_i$ is already defined. Let θ_i be its maximal congruence or the identity relation if $\mathbb{B}_i$ is simple.

(3) If $\mathbb{B}_i$ is a singleton, then $k = i$ and the process stops. Otherwise set $\mathbb{B}_{i+1}$ to be the greatest strongly connected component of a class of θ_i containing an element b with $|B_b| > 1$ (as we shall prove later, such a class exists).

For a congruence η, by a/η we denote the η-class containing a. Further, set $\mathbb{B}'_i = \mathbb{B}_i/\theta_i$ and

$$\mathbb{D}^{(i)} = \bigl\{(a, b) \mid a = a'/\theta_i \text{ where } a' \in \mathbb{B}_i,\ (a', b) \in \mathbb{D}\bigr\} \subseteq \mathbb{B}'_i \times \mathbb{D}_2.$$

We prove that, for every i, (i) for any $b \in \mathbb{D}_2$ there exists $a \in \mathbb{B}_i$ such that $(a, b) \in \mathbb{D}$, (ii) there exists $b \in \mathbb{B}_i$ such that $|B_b| > 1$, and (iii) $\mathbb{D}^{(i)} = \mathbb{B}'_i \times \mathbb{D}_2$. If $i = 0$, then (i) holds by the choice of E_{l-1}, and Lemma 3.2. Furthermore, there is $a \in E_{l-1}$ such that $|B_a| > 1$, say, $b, c \in B_a$. Since $b, c \leqslant b * c$, we may assume that $b \leqslant c$. Take $d \in \mathbb{B}_0$ such that $\binom{d}{b} \in \mathbb{D}$. Then we have $\binom{d}{b} * \binom{a}{b} = \binom{d*a}{b} \in \mathbb{D}$, $\binom{d*a}{b} * \binom{a}{c} = \binom{d*a}{c} \in \mathbb{D}$, and $d * a \in \mathbb{B}_0$. Thus $|B_{d*a}| > 1$, that proves (ii). Therefore, $\mathbb{D}^{(0)}$ is not the graph of a mapping, and, since $\mathbb{B}'_0$ is simple and $|\mathbb{B}'_0| + |\mathbb{D}_2| < |\mathbb{D}_1| + |\mathbb{D}_2|$, we get $\mathbb{D}^{(0)} = \mathbb{B}'_0 \times \mathbb{D}_2$.

Suppose that for $i - 1$ properties (i), (ii), (iii) hold. Then, for any $a' \in \mathbb{B}'_{i-1}$ we have $\{a'\} \times \mathbb{D}_2 \subseteq \mathbb{D}^{(i-1)}$, that is, by Lemma 3.2, for every $b \in \mathbb{D}_2$ there exists $a \in \mathbb{B}_i$ such that $(a, b) \in \mathbb{D}$, that proves (i) for i. By (ii) for $i - 1$ the θ_{i-1}-class A containing $\mathbb{B}_i$ contains an element b such that $|B_b| > 1$. Arguing as in the previous paragraph, b can be chosen to be in $\mathbb{B}_i$. Finally, as $\mathbb{B}'_i$ is simple we have $\mathbb{D}^{(i)} = \mathbb{B}'_i \times \mathbb{D}_2$.

We have proved $\mathbb{D}^{(k)} = \mathbb{B}'_k \times \mathbb{D}_2$. Since $\mathbb{B}_k$ is a singleton, say, $\mathbb{B}_k = \{b\}$ this implies $\mathbb{B}_k = \mathbb{B}'_k$, that is $\{b\} \times \mathbb{D}_2 \subseteq \mathbb{D}$.

To complete the proof we just have to apply Lemma 3.4. □

Note that the second part of the proof is valid for a subdirect product of not only simple 2-semilattices.

Corollary 3.2. *Let $\mathbb{D}$ be a subdirect product of 2-semilattices $\mathbb{D}_1, \mathbb{D}_2$ where $\mathbb{D}_2$ is simple strongly connected and $\mathbb{D}$ is not the graph of any mapping $\pi : \mathbb{D}_1 \to \mathbb{D}_2$. Then there exists $a \in \mathbb{D}_1$ such that $\{a\} \times \mathbb{D}_2 \subseteq \mathbb{D}$.*

To prove this we should put $\mathbb{B}_0$ equal to $\overline{\mathbb{D}}_1$.

Corollary 3.3. *Let $\mathbb{D}$ be a subdirect product of strongly connected 2-semilattices $\mathbb{D}_1, \mathbb{D}_2$ where $\mathbb{D}_2$ is simple. Then either $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$, or there is a surjective mapping $\pi : \mathbb{D}_1 \to \mathbb{D}_2$ such that $\mathbb{D} = \{(a, \pi(a)) \mid a \in \mathbb{D}_1\}$.*

Proof. If $\mathbb{D}$ is not the graph of a mapping then we are in the conditions of Corollary 3.2. Therefore there exists $a \in \mathbb{D}_1$ such that $\{a\} \times \mathbb{D}_2 \subseteq \mathbb{D}$. Since $\mathbb{D}_1$ is strongly connected, by Lemma 3.4, we get $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$. □

Lemma 3.6. *Let $\mathbb{D}$ be a subdirect product of simple strongly connected 2-semilattices $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$. If $\mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{i,j}\,\mathbb{D}$ for every $i, j \in \{1, 2, 3\}$, then $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3$.*

Proof. Suppose without loss of generality that $|\mathbb{D}_1| \leqslant |\mathbb{D}_2| \leqslant |\mathbb{D}_3|$. For $a \in \mathbb{D}_1$ set

$$\mathbb{D}_a = \big\{(b_2, b_3) \mid (a, b_2, b_3) \in \mathbb{D}\big\}.$$

Notice that, for every $a \in \mathbb{D}_1$, $\mathbb{D}_a$ is a subgroupoid of $\mathrm{pr}_{2,3}\,\mathbb{D}$, and, since $\mathrm{pr}_{1,2}\,\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$, $\mathrm{pr}_{1,3}\,\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_3$, the groupoid $\mathbb{D}_a$ is a subdirect product of $\mathbb{D}_2, \mathbb{D}_3$. By Lemma 3.5, $\mathbb{D}_a$ is either the graph of a bijective mapping or $\mathbb{D}_2 \times \mathbb{D}_3$.

Let us assume $\mathbb{D}_a = \mathbb{D}_2 \times \mathbb{D}_3$ for some $a \in \mathbb{D}_1$. Then, as $\mathbb{D}_2 \times \mathbb{D}_3$ is strongly connected, we are in the conditions of Corollary 3.3 where $\mathbb{D}_1$ plays the role of $\mathbb{D}_2$, and $\mathbb{D}_2 \times \mathbb{D}_3$ the role of $\mathbb{D}_1$. Since $\mathbb{D}$ is not the graph of a mapping $\mathbb{D}_2 \times \mathbb{D}_3 \to \mathbb{D}_1$, we get $\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3$.

Now suppose that, for every $a \in \mathbb{D}_1$, the set $\mathbb{D}_a$ is the graph of a bijective mapping π_a: $\mathbb{D}_2 \to \mathbb{D}_3$. This immediately implies $|\mathbb{D}_2| = |\mathbb{D}_3|$, let us denote this number by k, and as $\mathrm{pr}_{2,3}\,\mathbb{D} = \mathbb{D}_2 \times \mathbb{D}_3$, there are at least k different relations of the form $\mathbb{D}_a$. Therefore, $|\mathbb{D}_1| = k$ and $|\mathbb{D}_a| = k$ for any $a \in \mathbb{D}_1$. Moreover, $|\mathrm{pr}_{2,3}\,\mathbb{D}| = k^2$, which means $\mathbb{D}_a \cap \mathbb{D}_{a'} = \emptyset$ whenever $a \neq a'$, $a, a' \in \mathbb{D}_1$. The equivalence relation $\sim$ on $\mathrm{pr}_{2,3}\,\mathbb{D}$ where $(a, b) \sim (c, d)$ iff $(a, b), (c, d) \in \mathbb{D}_e$ for some $e \in \mathbb{D}_1$, is a congruence of $\mathrm{pr}_{2,3}\,\mathbb{D} = \mathbb{D}_2 \times \mathbb{D}_3$.

Since $\mathbb{D}_2 \cong \mathbb{D}_3$, $\mathbb{D}_2 \times \mathbb{D}_3$ can be treated as the square of $\mathbb{D}_2$. Recall that an element a of an algebra $\mathbb{A}$ is said to be *absorbing* if whenever $t(x, y_1, \ldots, y_n)$ is an $(n+1)$-ary term operation of $\mathbb{A}$ such that t depends on x and $(b_1, \ldots, b_n) \in A^n$, then $t(a, b_1, \ldots, b_n) = a$. A congruence θ of $\mathbb{A}^2$ is said to be *skew* if it is the kernel of no projection mapping of $\mathbb{A}^2$ onto its factors. $\mathbb{D}_2$ is a simple idempotent algebra, therefore, by the results of [30] one of the following holds: (a) $\mathbb{D}_2$ is term equivalent to a module; (b) $\mathbb{D}_2$ has an absorbing element; or (c) $\mathbb{D}_2^2$ has no skew congruence. Case (a) is impossible, because $\mathbb{D}_2$ has a 2-element subalgebra term equivalent to a semilattice, but no module has such a subalgebra. If in case (b) a is an absorbing element, then $a * b = a$ for any $b \in \mathbb{D}_2$ that contradicts the strongly connectedness. Finally, case (c) is also impossible, because $\sim$ is a skew congruence.

Thus, our assumption that $\mathbb{D}_a$ is the graph of a mapping for all $a \in \mathbb{D}_1$ cannot be the case, and the lemma is proved. □

Lemma 3.7. *Let $\mathbb{D}$ be a subdirect product of strongly connected simple 2-semilattices $\mathbb{D}_1, \ldots, \mathbb{D}_n$. If $\mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{i,j}\,\mathbb{D}$ for every $i, j \in \underline{n}$, then $\mathbb{D} = \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$.*

Proof. We prove the lemma by induction. The base case of induction $n = 2, 3$ is proved in Lemmas 3.5, 3.6. Suppose that the lemma holds for each number less than n. Take $a \in \mathbb{D}_1$ and denote by $\mathbb{D}_a$ the set $\{(b_2, \ldots, b_n) \mid (a, b_2, \ldots, b_n) \in \mathbb{D}\}$. By Lemma 3.6, $\mathbb{D}_1 \times \mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{1,i,j}\, \mathbb{D}$ for any $2 \leqslant i, j \leqslant n$. Then $\mathbb{D}_i \times \mathbb{D}_j \subseteq \mathrm{pr}_{i,j}\, \mathbb{D}_a$, and by induction hypothesis $\mathbb{D}_a = \mathbb{D}_2 \times \cdots \times \mathbb{D}_n$. The lemma is proved. □

Definition 3.1. A relation $\mathbb{D} \subseteq \mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ is said to be *almost trivial* if there exists an equivalence relation θ on the set $\{1, \ldots, n\}$ with classes $I_1, \ldots, I_k$, such that

$$\mathbb{D} = \mathrm{pr}_{I_1}\, \mathbb{D} \times \cdots \times \mathrm{pr}_{I_k}\, \mathbb{D}$$

where $\mathrm{pr}_{I_j}\, \mathbb{D} = \{(a_{i_1}, \pi_{i_2}(a_{i_1}), \ldots, \pi_{i_l}(a_{i_1})) \mid a_{i_1} \in \mathbb{D}_{i_1}\}$, $I_j = \{i_1, \ldots, i_l\}$, for certain bijective mappings $\pi_{i_2} : \mathbb{D}_{i_1} \to \mathbb{D}_{i_2}, \ldots, \pi_{i_l} : \mathbb{D}_{i_1} \to \mathbb{D}_{i_l}$.

Proposition 3.1. *A subdirect product of simple strongly connected* 2*-semilattices is an almost trivial relation.*

Proof. Let $\mathbb{D}$ be a subdirect product of simple strongly connected 2-semilattices $\mathbb{D}_1, \ldots, \mathbb{D}_n$. We prove the proposition by induction on n. When $n = 1$ the result holds trivially.

We now prove the induction step. By Lemma 3.5, for any pair $i, j \in \underline{n}$ the projection $\mathrm{pr}_{i,j}\, \mathbb{D}$ is either $\mathbb{D}_i \times \mathbb{D}_j$, or the graph of a bijective mapping. Assume that there exist i, j such that $\mathrm{pr}_{i,j}\, \mathbb{D}$ is the graph of a mapping $\pi : \mathbb{D}_i \to \mathbb{D}_j$. By the inductive hypothesis $\mathrm{pr}_{\underline{n}-\{j\}}\, \mathbb{D}$ is almost trivial, and therefore can be represented in the form

$$\mathrm{pr}_{\underline{n}-\{j\}}\, \mathbb{D} = \mathrm{pr}_{I_1}\, \mathbb{D} \times \cdots \times \mathrm{pr}_{I_k}\, \mathbb{D}$$

where $I_1 \cup \cdots \cup I_k = \underline{n} - \{j\}$. Suppose, for simplicity, that i is the last coordinate position in I_1, that is,

$$\mathrm{pr}_{I_1}\, \mathbb{D} = \big\{(a_{i_1}, \ldots, a_{i_k}, a_i) \mid a_{i_1} \in \mathbb{D}_{i_1},\ a_{i_s} = \pi_{s1}(a_{i_1}) \\ \text{for } s \in \{2, \ldots, k\},\ a_i = \pi_i(a_{i_1})\big\}.$$

Then

$$\mathrm{pr}_{I_1 \cup \{j\}}\, \mathbb{D} = \big\{(a_{i_1}, \ldots, a_{i_k}, a_i, a_j) \mid a_{i_1} \in \mathbb{D}_{i_1},\ a_{i_s} = \pi_{s1}(a_{i_1}) \\ \text{for } s \in \{2, \ldots, k\},\ a_i = \pi_i(a_{i_1}),\ a_j = \pi\pi_i(a_{i_1})\big\},$$

and we have $\mathbb{D} = \mathrm{pr}_{I_1 \cup \{j\}}\, \mathbb{D} \times \cdots \times \mathrm{pr}_{I_k}\, \mathbb{D}$, as required.

Finally, if $\mathrm{pr}_{i,j}\, \mathbb{D} = \mathbb{D}_i \times \mathbb{D}_j$ for all $i, j \in \underline{n}$, then the result follows by Lemma 3.7. □

Corollary 3.4. *Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a* 3*-minimal problem instance, where $\mathcal{A}$ is a class of simple strongly connected* 2*-semilattices, and each constraint relation is a subdirect product of its domains. Then if none of the constraint relations of $\mathcal{P}$ is empty then $\mathcal{P}$ has a solution.*

Proof. By Proposition 3.1, all the constraint relations are almost trivial. For every constraint $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$, denote by θ_C the equivalence relation on s corresponding to the almost trivial relation $\mathbb{C}$, and by θ the least equivalence relation on V that contains θ_C for all constraints C. By 3-minimality, for any $u, v \in V$ and any constraint $C = \langle s, \mathbb{C}\rangle$, if $u, v \in s$ then $\mathrm{pr}_{u,v}\,\mathbb{C} = \mathcal{S}_{u,v}$. Therefore

$$\theta = \bigcup_{C \in \mathcal{C}} \theta_C = \bigl\{(u, v) \mid \mathcal{S}_{u,v} \text{ is the graph of a bijective mapping } \pi_{u,v} : \mathbb{A}_{\delta(u)} \to \mathbb{A}_{\delta(v)}\bigr\}.$$

Moreover, if $(u, v), (v, w) \in \theta$ then there is $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$ such that $u, v, w \in s$. The relations $\mathrm{pr}_{u,v}\,\mathbb{C} = \mathcal{S}_{u,v}$, $\mathrm{pr}_{v,w}\,\mathbb{C} = \mathcal{S}_{v,w}$ are the graphs of $\pi_{u,v}, \pi_{v,w}$, respectively. Therefore, $(u, w) \in \theta_C$ and $\pi_{u,w} = \pi_{u,v} \circ \pi_{v,w}$. Denote the classes of θ by $V_1, \ldots, V_k$.

Finally, it is not hard to see that any mapping ψ constructed through the following rules is a solution to $\mathcal{P}$:

– Choose a representative v_j from each V_j.
– Assign to v_j an arbitrary value $\psi(v_j) \in \mathbb{A}_{\delta(v_j)}$.
– Set $\psi(v) = \pi_{v_j,v}(\psi(v_j))$ for all $v \in V_j$ and $j \in \{1, \ldots, k\}$. □

3.4. General 2-semilattices

In this section we prove Theorem 3.1 in the case of multi-sorted problem instances over arbitrary finite 2-semilattices. Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a 3-minimal problem instance, and $\mathcal{A}$ a class of finite 2-semilattices. We show that $\mathcal{P}$ can be transformed to another 3-minimal problem instance which satisfies some additional conditions.

Proposition 3.2. *Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a 3-minimal problem instance without empty constraint relations where $\mathcal{A}$ is a class of finite 2-semilattices. Then the problem instance $\mathcal{P}' = (V; \mathcal{A}; \delta'; \mathcal{C}')$, where*

- *for every $u \in V$, $\mathbb{A}_{\delta'(u)}$ is the greatest strongly connected component of $\mathbb{A}_{\delta(u)}$;*
- *for each $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$ there is $C' = \langle s, \mathbb{C}'\rangle \in \mathcal{C}'$ where $\mathbb{C}'$ is the greatest strongly connected component of $\mathbb{C}$,*

satisfies the following conditions

- *$\mathcal{P}'$ is 3-minimal, and has no empty constraint relation*;
- *for any $u, v, w \in V$, $\mathcal{S}'_u, \mathcal{S}'_{u,v}, \mathcal{S}'_{u,v,w}$ are strongly connected*;
- *if $\mathcal{P}'$ has a solution, then $\mathcal{P}$ does.*

Proof. By Corollary 3.1, for any constraint $C' = \langle s, \mathbb{C}'\rangle \in \mathcal{C}'$ and any $u, v, w \in s$, $\mathrm{pr}_{u,v,w}\,\mathbb{C}'$ is the greatest strongly connected component of $\mathrm{pr}_{u,v,w}\,\mathbb{C}$; therefore $\mathcal{S}'_{u,v,w}$ is the greatest strongly connected component of $\mathcal{S}_{u,v,w}$. Thus the 3-minimality of $\mathcal{P}'$ follows from the 3-minimality of $\mathcal{P}$.

To prove the second condition we just notice that, for any $u, v, w \in V$, there is $\langle s, \mathbb{C}\rangle \in \mathcal{C}$ with $u, v, w \in s$. This means that $\mathcal{S}'_{u,v,w} = \mathrm{pr}_{u,v,w}\,\mathbb{C}'$ and therefore strongly connected. The third condition is obvious. □

We need another auxiliary lemma.

Lemma 3.8. *Let $\mathbb{D}$ be a subdirect product of strongly connected 2-semilattices $\mathbb{D}_1, \dots, \mathbb{D}_n$ where $\mathbb{D}_1$ is simple, $\mathrm{pr}_{2,\dots,n}\,\mathbb{D}$ is strongly connected, and $\mathrm{pr}_{1,i}\,\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_i$ for $i \in \{2, \dots, n\}$, Then $\mathbb{D} = \mathbb{D}_1 \times \mathrm{pr}_{2,\dots,n}\,\mathbb{D}$.*

Proof. We prove the lemma by induction on n. The case $n = 2$ is obvious. Consider the case $n = 3$. We use induction on $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$. The trivial case $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3| = 2$ gives the base case of induction. If all of $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$ are simple, then the result follows from Corollary 3.1. Otherwise, suppose that $\mathbb{D}_3$ is not simple. Take a maximal congruence θ of $\mathbb{D}_3$, fix a θ-class C and consider $\mathbb{D}' \subseteq \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3/\theta$, $\mathbb{D}'' \subseteq \mathbb{D}$ such that

$$\mathbb{D}' = \big\{(a, b, c/\theta) \mid (a, b, c) \in \mathbb{D}\big\},$$
$$\mathbb{D}'' = \big\{(a, b, c) \mid (a, b, c) \in \mathbb{D}, c \in C\big\}$$

and $\mathbb{D}''' \subseteq \mathbb{D}$, the greatest strongly connected component of $\mathbb{D}''$. Obviously, $\mathrm{pr}_{1,3}\,\mathbb{D}'' = \mathbb{D}_1 \times C$. Moreover, by Lemma 3.1, $\mathrm{pr}_{1,3}\,\mathbb{D}''' = \mathbb{D}_1 \times \overline{C}$, as $\mathbb{D}_1 \times \overline{C}$ is the greatest strongly connected component of $\mathbb{D}_1 \times C$. By Lemma 3.2, $\mathrm{pr}_{2,3}\,\mathbb{D}'$ is either the graph of a bijective mapping, or $\mathbb{D}_2 \times \mathbb{D}_3/\theta$.

Case 1. $\mathrm{pr}_{2,3}\,\mathbb{D}'$ is the graph of a bijective mapping $\pi : \mathbb{D}_2 \to \mathbb{D}_3/\theta$.

In this case $\overline{B} = \mathrm{pr}_2\,\mathbb{D}'''$ is the greatest strongly connected component of $B = \pi^{-1}(d)$. Since for each $(a, b) \in \mathbb{D}_1 \times B \subseteq \mathrm{pr}_{1,2}\,\mathbb{D}$ there is $c \in C$ with $(a, b, c) \in \mathbb{D}$, we have $\mathrm{pr}_{1,2}\,\mathbb{D}'' = \mathbb{D}_1 \times B$. Furthermore, $\mathbb{D}_1 \times \overline{B}$ is the greatest strongly connected component of $\mathbb{D}_1 \times B$, hence, $\mathrm{pr}_{1,2}\,\mathbb{D}''' = \mathbb{D}_1 \times \overline{B}$.

Since $|\mathbb{D}_1| + |\overline{B}| + |\overline{C}| < |\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$, and $\mathrm{pr}_{2,3}\,\mathbb{D}'''$ is strongly connected, inductive hypothesis implies $\mathbb{D}_1 \times \mathrm{pr}_{2,3}\,\mathbb{D}''' \subseteq \mathbb{D}'''$. In particular, there is $(a, b) \in \mathrm{pr}_{2,3}\,\mathbb{D}''' \subseteq \mathrm{pr}_{2,3}\,\mathbb{D}$ such that $\mathbb{D}_1 \times \{(a, b)\} \subseteq \mathbb{D}$. To finish the proof we just apply Lemma 3.4.

Case 2. $\mathrm{pr}_{2,3}\,\mathbb{D}' = \mathbb{D}_2 \times \mathbb{D}_3/\theta$.

Since $|\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3/\theta| < |\mathbb{D}_1| + |\mathbb{D}_2| + |\mathbb{D}_3|$, $\mathbb{D}_3/\theta$ is simple, and $\mathrm{pr}_{1,2}\,\mathbb{D} = \mathbb{D}_1 \times \mathbb{D}_2$, by inductive hypothesis, $\mathbb{D}' = \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}_3/\theta$. Therefore, $\mathrm{pr}_{1,2}\,\mathbb{D}'' = \mathbb{D}_1 \times \mathbb{D}_2$. By Lemma 3.1, $\mathrm{pr}_{1,2}\,\mathbb{D}''' = \mathbb{D}_1 \times \mathbb{D}_2$. Then we argue as in Case 1.

Let us assume that the lemma is proved for $n-1$. Then $\mathbb{D}_1 \times \mathrm{pr}_{3,\dots,n}\,\mathbb{D} \subseteq \mathrm{pr}_{1,3,\dots,n}\,\mathbb{D}$. Denoting $\mathrm{pr}_{3,\dots,n}\,\mathbb{D}$ by $\mathbb{D}'$ we have $\mathbb{D} \subseteq \mathbb{D}_1 \times \mathbb{D}_2 \times \mathbb{D}'$, and the conditions of the lemma hold for this subdirect product. Thus $\mathbb{D} = \mathbb{D}_1 \times \mathrm{pr}_{2,\dots,n}\,\mathbb{D}$ as required. □

Now we are in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\mathcal{P} = (V; \mathcal{A}; \delta; \mathcal{C})$ be a 3-minimal problem instance without empty constraint relations. We may assume that $\mathcal{P}$ satisfies the conditions of Proposition 3.2. We prove by induction on the number of elements in $\mathbb{A}_{\delta(v)}$, $v \in V$, that $\mathcal{P}$ has a solution.

The base case of induction. If all $\mathbb{A}_{\delta(v)}$, $v \in V$, are simple or 1-element, then the required result follows from the result of Section 3.3.

Induction step. Suppose that the theorem holds for all problem instances $\mathcal{P}' = (V; \mathcal{A}'; \delta'; \mathcal{C}')$ where $|\mathbb{A}'_{\delta'(v)}| \leqslant |\mathbb{A}_{\delta(v)}|$ for $v \in V$ and at least one inequality is strict.

Let us assume that, for a certain $u \in V$, $\mathbb{A}_{\delta(u)}$ is not simple and θ is a maximal congruence of $\mathbb{A}_{\delta(u)}$. By Corollary 3.3, for any $v \in V - \{u\}$, $\mathcal{S}_{u,v}/\theta = \{(a/\theta, b) \mid (a,b) \in \mathcal{S}_{u,v}\}$ is either the direct product $\mathbb{A}_{\delta(u)}/\theta \times \mathbb{A}_{\delta(v)}$, or the graph of a surjective mapping $\pi_v : \mathbb{A}_{\delta(v)} \to \mathbb{A}_{\delta(u)}/\theta$. Let W denote the set consisting of u and all $v \in V$ such that $\mathcal{S}_{u,v}/\theta$ is the graph of π_v, and

$$\theta_v = \begin{cases} \theta, & \text{if } v = u, \\ \ker \pi_v, & \text{if } v \in W, \\ =_v, & \text{otherwise,} \end{cases}$$

for $v \in V$ where $=_v$ denotes the equality relation on $\mathbb{A}_{\delta(v)}$. Consider the *factor problem* $\widetilde{\mathcal{P}} = (V; \widetilde{\mathcal{A}}; \widetilde{\delta}; \widetilde{\mathcal{C}})$ where $\widetilde{\mathbb{A}}_{\widetilde{\delta}(v)} = \mathbb{A}_{\delta(v)}/\theta_v$, $v \in V$, $\widetilde{\mathcal{A}}$ consists of $\widetilde{\mathbb{A}}_{\widetilde{\delta}(v)}$, and for each $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$, $s = (v_1, \dots, v_k)$, there is $\widetilde{C} = \langle s, \widetilde{\mathbb{C}}\rangle \in \widetilde{\mathcal{C}}$ such that

$$\widetilde{\mathbb{C}} = \{(a_{v_1}/\theta_{v_1}, \dots, a_{v_k}/\theta_{v_k}) \mid (a_{v_1}, \dots, a_{v_k}) \in \mathbb{C}\}.$$

As is easily seen, the factor problem is 3-minimal, therefore, by the induction hypothesis it has a solution. Let $(\mathfrak{a}_1, \dots, \mathfrak{a}_n)$ be a solution. Notice that if $v \in W$, then $\mathfrak{a}_v$ is a congruence block of $\mathbb{A}_{\delta(v)}$, that is, $\mathfrak{a}_v$ is a subset of $A_{\delta(v)}$ in this case. The 3-minimality of $\mathcal{P}$ implies that, for any constraint $\langle s, \mathbb{C}\rangle \in \mathcal{C}$, any $v, w \in s \cap W$, and any $\mathbf{a} \in \mathbb{C}$, if $a_v \in \mathfrak{a}_v$ then $a_w \in \mathfrak{a}_w$. Set $\mathcal{P}' = (V; \mathcal{A}'; \delta'; \mathcal{C}')$ where

$$\mathbb{A}'_{\delta'(v)} = \begin{cases} \mathfrak{a}_v, & \text{if } v \in W, \\ \mathbb{A}_{\delta(v)}, & \text{otherwise,} \end{cases}$$

$\mathcal{A}' = \mathcal{A} \cup \{\mathfrak{a}_v \mid v \in W\}$, and for each $C = \langle s, \mathbb{C}\rangle \in \mathcal{C}$ there is $C' = \langle s, \mathbb{C}'\rangle \in \mathcal{C}'$ with

$$\mathbf{a} \in \mathbb{C} \quad \text{if and only if} \quad \mathbf{a} \in \mathbb{C} \text{ and } a_v \in \mathfrak{a}_v \text{ for all } v \in W \cap s.$$

Since $|\mathfrak{a}_u| < |\mathbb{A}_{\delta(u)}|$, to complete the proof we just have to show that $\mathcal{P}'$ is 3-minimal. For $U = \{u_1, u_2, u_3\} \subseteq V$ set $S_U = \mathcal{S}_U \cap (S_1 \times S_2 \times S_3)$ where $S_i = \mathbb{A}'_{\delta'(u_i)}$. Clearly, for any $C' = \langle s, \mathbb{C}'\rangle \in \mathcal{C}'$, we have $\mathrm{pr}_{U\cap s} \mathbb{C}' \subseteq \mathrm{pr}_{U\cap s} S_U$. Therefore, if we prove the reverse inclusion then we get the equality $\mathrm{pr}_{U\cap s} \mathbb{C}' = \mathrm{pr}_{U\cap s} S_U$ which implies the 3-minimality of $\mathcal{P}'$.

Take $\mathbf{b} = (a_{u_1}, a_{u_2}, a_{u_3}) \in S_U$, $\langle s, \mathbb{C}\rangle \in \mathcal{C}$, and $\mathbf{a} \in \mathbb{C}$ such that $\mathrm{pr}_{U\cap s} \mathbf{a} = \mathrm{pr}_{U\cap s} \mathbf{b}$. If $U \cap W \cap s \neq \emptyset$ then, for any $v \in s \cap W$, $a_v \in \mathfrak{a}_v$, and therefore $\mathbf{a} \in \mathbb{C}'$. If $s \cap W = \emptyset$ then $\mathbb{C}' = \mathbb{C}$, and again $\mathbf{a} \in \mathbb{C}'$. Otherwise, consider the relation $\widetilde{\mathbb{C}}$. Choose $v \in s \cap W$ and set $\mathbb{D} = \mathrm{pr}_{(s-W)\cup\{v\}} \widetilde{\mathbb{C}}$. Since $\mathcal{P}$ satisfies the conditions of Proposition 3.2, $\mathrm{pr}_{s-W} \mathbb{C}$ is strongly connected. Then, by Lemma 3.8, $\mathbb{D} = \mathrm{pr}_v \mathbb{D} \times \mathrm{pr}_{s-W} \widetilde{\mathbb{C}} = \mathrm{pr}_v \mathbb{D} \times \mathrm{pr}_{s-W} \mathbb{C}$. This means that there is $\mathbf{c} \in \mathbb{C}$ such that $\mathrm{pr}_{s-W} \mathbf{c} = \mathrm{pr}_{s-W} \mathbf{a}$ and $c_v \in \mathfrak{a}_v$. Therefore, $c_w \in \mathfrak{a}_w$ for any $w \in s \cap W$, and hence $\mathbf{c} \in \mathbb{C}'$. Since $s \cap U \subseteq s - W$, we have $\mathrm{pr}_{s\cap U} \mathbf{c} = \mathrm{pr}_{s\cap U} \mathbf{b}$, as required. □

4. Applications

4.1. Conservative groupoids

To characterize tractable conservative groupoids, notice that every commutative conservative groupoid is a 2-semilattice. Therefore, the class of finite commutative conservative groupoids is globally tractable. Then, let $\mathbb{A} = (A, *)$ be a non-commutative conservative groupoid. Since the identity $x * x = x$ holds in $\mathbb{A}$, for any two-element subset B of $\mathbb{A}$, the operation $*|_B$ is either a projection or a semilattice operation. If for all two element subsets B, $*|_B$ is a semilattice operation, then $\mathbb{A}$ is commutative. Therefore, $*|_B$ is a projection for a certain B; by Proposition 2.2, $\mathbb{B} = (B; *|_B)$ is NP-complete. Since $\mathbb{B}$ is a subalgebra of $\mathbb{A}$ this implies the NP-completeness of $\mathbb{A}$. Thus, we have proved

Theorem 4.1. *A class of finite conservative groupoids is globally tractable if and only if every its member is commutative. Otherwise, it is NP-complete.*

4.2. The complexity of clone minimal algebras

Recall that a *clone* is a set of operations containing all the projection operations and closed with respect to superposition. A clone is said to be *minimal* if it is non-trivial and has no proper subclones. Algebras whose clone of term operations is minimal attract a special attention in both clone theory and the study of the CSP, because they correspond to relational clones that are just smaller than the relational clone of all relations, and therefore the corresponding CSP-classes are just smaller than the general CSP with a fixed domain.

In [41], algebras with minimal clone of term operations have been classified into 5 types. A ternary operation $m(x, y, z)$ is said to be a *majority operation* if it satisfies the identities $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$. An operation $f(x_1, \dots, x_n)$ is said to be a *semiprojection* if there is i from the range 1 to n such that $f(x_1, \dots, x_n) = x_i$ whenever $x_1, \dots, x_n$ are not all different.

Theorem 4.2 *(Rosenberg's Five Types Theorem [41]). If a finite algebra has a minimal clone of term operations then it is term equivalent to an algebra $(A; f)$ where f is one of the following types*:

(1) *a unary operation with $f^2 = f$*;
(2) *a binary idempotent operation*;
(3) *a majority operation*;
(4) *a semiprojection*;
(5) *$x + y + z$ where $+$ is an operation of an elementary 2-group.*

The tractability or NP-completeness of most of clone minimal algebras can be deduced from earlier results.

Theorem 4.3. [7] *Let $\mathbb{A} = (A; f)$ be a finite clone minimal algebra, and f of one of the five types listed in Theorem 4.2. Then*

(1) *if f is a unary constant operation, or a majority operation, or $x + y + z$ for a certain elementary Abelian 2-group, then $\mathbb{A}$ is tractable;*
(2) *if f is a unary non-constant operation, or a semiprojection, then $\mathbb{A}$ is NP-complete;*
(3) *if f is a binary idempotent operation and $\mathbb{A}$ is tractable, then there is a commutative binary idempotent operation g such that $\mathbb{A}$ and $(A; g)$ are term equivalent.*

Thus, idempotent groupoids are the only type of clone minimal algebras whose classification with respect to the complexity of the corresponding problem class is still unknown. There is a complete description of clone minimal groupoids of size 2, 3, 4, see [12,40,44]. Remarkably, in these cases every groupoid with minimal clone of term operations either have a factor term equivalent to a G-set, or is a 2-semilattice. In the former case it is NP-complete by Proposition 2.2, while in the latter case it is tractable by Theorem 2.4.

Corollary 4.1. *Let $\mathbb{A} = (A; f)$ be a clone minimal algebra, $|A| \leqslant 4$, and f of one of the five types listed in Theorem* 4.2. *If f is a unary constant operation, or a majority operation, or $x + y + z$ for a certain elementary Abelian* 2*-group, or has a commutative idempotent binary term operation, then $\mathbb{A}$ is tractable. Otherwise $\mathbb{A}$ is NP-complete.*

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