

RESEARCH ARTICLE

The logarithmic Bogomolov–Tian–Todorov theorem

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DFG, Grant/Award Number: RU 1629/4-1; JGU

Abstract

We prove that the log smooth deformations of a proper log smooth saturated log Calabi–Yau space are unobstructed.

MSC (2020)

14B12 (primary), 14A21, 14J32 (secondary)

1 | INTRODUCTION

The Bogomolov–Tian–Todorov (BTT) theorem [2, 33, 34] states that deformations of cohomologically Kähler manifolds with trivial canonical bundle are unobstructed. This was later proved with algebraic methods (T^1 -lifting criterion): If k is a field of characteristic 0 and X is a proper smooth k -variety with torsion canonical bundle, then the deformations of X are unobstructed [6, 22, 28]. This means that the functor

$$\mathrm{Def}_X : \mathbf{Art}_k \longrightarrow \mathbf{Set}$$

which parametrizes the deformations of $X \rightarrow \mathrm{Spec} k$ is smooth. We refer the reader to § 2.3 for the definition of smoothness of a functor. Here \mathbf{Art}_k is the category of Artinian local k -algebras with residue field k , and \mathbf{Set} is the category of sets. A crucial ingredient in these proofs is that the Hodge–de Rham spectral sequence degenerates.

In this paper, we generalize the BTT theorem to the context of logarithmic schemes. The setting is as follows. Fix a field k of characteristic 0 and a sharp toric monoid Q . Consider the monoid k -algebra $k[Q]$ and its completion $k[[Q]]$ at its unique monomial maximal ideal, that is, the ideal

generated by $Q \setminus \{0\}$. Consider the category $\mathbf{Art}_{k[[Q]]}$ of Artinian local $k[[Q]]$ -algebras with residue field k . In particular, $k[[Q]]$ (respectively, k) is the initial (respectively, terminal) object of $\mathbf{Art}_{k[[Q]]}$.

Every object A in $\mathbf{Art}_{k[[Q]]}$ gives rise to a log scheme $\mathrm{Spec}(Q \rightarrow A)$; this is the log scheme with underlying scheme $\mathrm{Spec} A$ and with log structure associated to the monoid homomorphism $Q \rightarrow A$ induced by the structure ring homomorphism $k[[Q]] \rightarrow A$.

We denote by S_0 the log scheme $\mathrm{Spec}(Q \rightarrow k)$ induced by k . Hence S_0 is the log scheme with underlying scheme $\mathrm{Spec} k$ and with log structure $Q \oplus k^* \rightarrow k$ given by

$$(q, a) \mapsto \begin{cases} a & \text{if } q = 0, \\ 0 & \text{if } q \neq 0. \end{cases}$$

Now we fix a log smooth saturated morphism $f_0 : X_0 \rightarrow S_0$ of log schemes. Kato has defined in [20] the functor of the log smooth deformations of f_0 , that is, the functor

$$\mathrm{LD}_{X_0/S_0} : \mathbf{Art}_{k[[Q]]} \longrightarrow \mathbf{Set}$$

which to every object A in $\mathbf{Art}_{k[[Q]]}$ associates the set of isomorphism classes of log smooth deformations $f : X \rightarrow \mathrm{Spec}(Q \rightarrow A)$ of $f_0 : X_0 \rightarrow S_0$.

In the logarithmic setting, the Calabi–Yau condition, which appears in the BTT theorem, is expressed by triviality of the log canonical bundle $\omega_{X_0/S_0} := \Omega_{X_0/S_0}^d := \bigwedge^d \Omega_{X_0/S_0}^1$; here d is the relative dimension of $f_0 : X_0 \rightarrow S_0$.

Our main result is:

Theorem 1.1. *Let k be a field of characteristic 0, let Q be a sharp toric monoid, and let $S_0 = \mathrm{Spec}(Q \rightarrow k)$ be the log scheme with underlying scheme $\mathrm{Spec} k$ and ghost sheaf Q . Let $f_0 : X_0 \rightarrow S_0$ be a proper log smooth saturated morphism of relative dimension d . If ω_{X_0/S_0} is the trivial line bundle, then the log smooth deformation functor*

$$\mathrm{LD}_{X_0/S_0} : \mathbf{Art}_{k[[Q]]} \longrightarrow \mathbf{Set}$$

is smooth.

The proof, which is presented in § 3.4, is divided in two cases. If $Q = 0$, we adapt the formalism of differential graded Lie algebras, which was used by Iacono and Manetti to give an algebraic proof of the BTT theorem [16–18], to the context of log schemes. If $Q \neq 0$, we use the recent formalism developed by Chan–Leung–Ma [5] (see also [4]) and by Felten–Filip–Ruddat [8] to construct smoothings of degenerate Calabi–Yau varieties, and we apply an algebraic result (Proposition 2.8).

1.1 | Applications

We explain two applications of the log BTT theorem.

Log Calabi–Yau pairs

Let X be a smooth proper variety over a field k of characteristic 0 and let D be an SNC effective divisor on X . Let X_0 be the log scheme given by X equipped with the divisorial log structure

associated to D . Let S_0 be the log scheme given by $\text{Spec } k$ with the trivial log structure. Then $X_0 \rightarrow S_0$ is log smooth and saturated. One has $\omega_{X_0/S_0} = \omega_X(D)$. Therefore Theorem 1.1 applies when D is an anticanonical divisor, that is, the pair (X, D) is log Calabi–Yau. If this is the case, then log smooth deformations of $X_0 \rightarrow S_0$ are unobstructed.

Log smooth deformations of $X_0 \rightarrow S_0$ are exactly *locally trivial* deformations of the pair (X, D) , that is, of the closed embedding $D \hookrightarrow X$. The log BTT theorem implies: If D is an SNC effective anticanonical divisor on X , then the functor of locally trivial deformations of (X, D) is smooth.

If D is a smooth divisor, then every deformation of (X, D) is locally trivial. Therefore, if D is a smooth anticanonical divisor on X , then deformations of the pair (X, D) are unobstructed: This recovers a result by Iacono [16], Sano [30], and Katzarkov–Kontsevich–Pantev [21] (see also [24, 25, 35]).

If D is a non-smooth SNC divisor on D , then there might be deformations of the pair (X, D) which are not locally trivial and hence not covered by our BTT theorem. By [9], there are — not only in characteristic 0 but over every field k with $\text{char}(k) \neq 2$ — indeed pairs (X, D) of a smooth projective variety X and an SNC effective anticanonical divisor $D \subset X$ such that (not necessarily locally trivial) deformations of (X, D) are obstructed.

Simple normal crossing schemes

Fix an algebraically closed field k . A *normal crossing scheme* over k is a scheme X of finite type over k such that every closed point $x \in X$ has an étale neighborhood $x \rightarrow U \rightarrow X$ which admits an étale map

$$U \rightarrow \text{Spec } k[x_1, \dots, x_m]/(x_1 \cdots x_r),$$

for some $0 \leq r \leq m$, such that x maps to the origin. A normal crossing scheme X is called *d-semistable* if the coherent sheaf $\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ is isomorphic to the structure sheaf of the singular locus of X (see [11] and [1]).

If X is a d -semistable normal crossing scheme of pure dimension d , then X has a log smooth log structure X_0 over $S_0 = \text{Spec}(\mathbb{N} \rightarrow k)$ such that the sheaf ω_{X_0/S_0} of log d -differentials is isomorphic to the dualizing sheaf ω_X . From Theorem 1.1, we deduce:

Corollary 1.2. *Let X be a d -semistable normal crossing scheme proper over an algebraically closed field of characteristic 0 such that $\omega_X \simeq \mathcal{O}_X$. Then the functor LD_{X_0/S_0} is smooth.*

This removes the assumptions $H^{d-1}(X, \mathcal{O}_X) = 0$ and $H^{d-2}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$, where d is the dimension of X and $\tilde{X} \rightarrow X$ its normalization, from [23, Theorem 4.2]. We remind the reader that, under the assumptions of Corollary 1.2, X is formally smoothable by [5, 8].

1.2 | Generalizations

In Theorem 1.1, we assume $f_0 : X_0 \rightarrow S_0$ log smooth. However, in many geometric situations which arise from the degeneration of varieties, the degenerate (log) space $f_0 : X_0 \rightarrow S_0$ has log singularities. The deformation theory of a general log toroidal family in the sense of [8] is not yet

well understood, but the special cases which arise in the Gross–Siebert program ([13–15]) are. For example, according to [29], if $f_0 : X_0 \rightarrow S_0 = \text{Spec}(\mathbb{N} \rightarrow \mathbb{C})$ is a simple toric log Calabi–Yau space, then the functor $\mathcal{D} : \mathbf{Art}_{\mathbb{C}[[t]]} \rightarrow \mathbf{Set}$ of divisorial deformations has a hull. Moreover, the theory in [8] shows that \mathcal{D} satisfies Theorem 3.7 below; thus, following our proof of Theorem 1.1 above, the functor \mathcal{D} of divisorial deformations is unobstructed. However, at least when $f_0 : X_0 \rightarrow S_0$ carries a polarization, this is not a new result. Namely, $f_0 : X_0 \rightarrow S_0$ is a fiber in a whole family $f : X \rightarrow \text{Spec}(A)$ of toric log Calabi–Yau spaces over an algebraic torus $\text{Spec}(A)$ according to [14]. An application of the Gross–Siebert algorithm in [12] — here we use the polarization — then yields a canonical formal family $\mathfrak{X} \rightarrow \text{Spf}(A[[t]])$, where $\text{Spf}(A[[t]])$ is the completion of $\text{Spec}(A) \subseteq \text{Spec}(A[t])$. According to [29], there is an analytic family $\mathcal{Y} \rightarrow U \times \mathbb{D}$ where $U \subseteq \text{Spec}(A)_{an}$ is a neighborhood of the point $a \in \text{Spec}(A)_{an}$ which corresponds to the space $f_0 : X_0 \rightarrow S_0$, and \mathbb{D} is a small disk; when we complete it in $(a, 0) \in U \times \mathbb{D}$, the completion is isomorphic to the completion of the canonical formal family $\mathfrak{X} \rightarrow \text{Spf}(A[[t]])$ in $a \in \text{Spf}(A[[t]])$. According to [29] as well, this completion of the canonical formal family is a versal family for the divisorial deformation functor \mathcal{D} . Since A is an algebraic torus, it is smooth, so the hull is a formally smooth $\mathbb{C}[[t]]$ -algebra, and \mathcal{D} is unobstructed.

Notation and conventions

Every ring is commutative with identity. The set of non-negative (respectively, positive) integers is denoted by \mathbb{N} (respectively, \mathbb{N}^+). Every monoid is commutative and denoted additively. A monoid is said to be *sharp* if 0 is the unique invertible element. A monoid Q is said to be *toric* if there exist an integer $n \geq 0$ and rational polyhedral cone σ in \mathbb{R}^n of dimension n such that Q is isomorphic to $\sigma \cap \mathbb{Z}^n$.

2 | VALUATIONS AND SMOOTH DEFORMATION FUNCTORS

2.1 | Valuations on Noetherian complete local domains

Following [3, VI, § 3], we define valuations on integral domains. On the disjoint union $\mathbb{N} \cup \{\infty\}$, we consider the extensions of the sum $+$ and the ordering \leq from \mathbb{N} defined as follows: $\infty + \infty = \infty$, $n + \infty = \infty + n = \infty$ and $n \leq \infty$ for every $n \in \mathbb{N}$.

Definition 2.1. A *non-trivial valuation* on a ring Λ is a function $\nu : \Lambda \rightarrow \mathbb{N} \cup \{\infty\}$ such that:

- (i) $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in \Lambda$;
- (ii) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ for all $a, b \in \Lambda$;
- (iii) $\nu^{-1}(\infty) = \{0\}$;
- (iv) $\nu(\Lambda) \setminus \{0, \infty\} \neq \emptyset$.

It follows that Λ is an integral domain and $\nu(1) = 0$.

Definition 2.2. Let (Λ, \mathfrak{m}) be a local domain. A non-trivial valuation ν on Λ is said to be *compatible with the \mathfrak{m} -adic topology* if the \mathfrak{m} -adic topology coincides with the linear topology induced by the following descending filtration of ideals: $\{a \in \Lambda \mid \nu(a) \geq n\}$ as $n \in \mathbb{N}$.

Remark 2.3. If a local domain (Λ, \mathfrak{m}) has a non-trivial valuation which is compatible with the \mathfrak{m} -adic topology, then Λ is *not* a field. Although in Lemma 3.1 below we construct such a valuation on $\Lambda = k[[Q]]$ for a sharp toric monoid $Q \neq 0$, there is no such valuation on $k = k[[0]]$.

Example 2.4. Fix an integer $m \geq 1$. Fix a field k and consider the power series ring $\Lambda = k[[t_1, \dots, t_m]]$ with maximal ideal \mathfrak{m} . Every element $w = (w_1, \dots, w_m) \in (\mathbb{N}^+)^m$ induces a non-trivial valuation ν_w on Λ which is compatible with the \mathfrak{m} -adic topology as follows: if

$$a = \sum_{i_1, \dots, i_m \geq 0} a_{i_1, \dots, i_m} t_1^{i_1} \cdots t_m^{i_m}$$

then

$$\nu_w(a) = \min \left\{ w_1 i_1 + \cdots + w_m i_m \mid i_1 \geq 0, \dots, i_m \geq 0, a_{i_1, \dots, i_m} \neq 0 \right\}.$$

If Λ is a Noetherian complete local ring, then every power series with coefficients in Λ and finitely many variables can be evaluated on tuples of elements of the maximal ideal of Λ . Now we give a sufficient criterion that ensures that the zero power series is the only one for which every evaluation is zero:

Lemma 2.5. *Let (Λ, \mathfrak{m}) be a Noetherian complete local domain, and let $f \in \Lambda[[x_1, \dots, x_n]]$ be a power series such that*

$$\forall a_1 \in \mathfrak{m}, \dots, \forall a_n \in \mathfrak{m}, \quad f(a_1, \dots, a_n) = 0.$$

If there exists a non-trivial valuation on Λ which is compatible with the \mathfrak{m} -adic topology, then $f = 0$.

Note that Lemma 2.5 does not hold for every Noetherian complete local ring: for example, $\Lambda = k[t]/(t^{r+1})$ and $f = t^r x \in \Lambda[[x]]$, for any integer $r \geq 0$.

In the proof of Lemma 2.5, we make use of the following:

Lemma 2.6 [3, VII, § 3, no. 7, Lemma 2]. *Let Λ be a ring, and fix a non-zero power series $f(x_1, \dots, x_n) \in \Lambda[[x_1, \dots, x_n]]$ with coefficients in Λ . Then there exist positive integers u_1, \dots, u_{n-1} such that the power series $f(x^{u_1}, \dots, x^{u_{n-1}}, x) \in \Lambda[[x]]$ is non-zero.*

Proof of Lemma 2.5. For a contradiction, assume $f \neq 0$. By Lemma 2.6, there exist $u_1, \dots, u_{n-1} \in \mathbb{N}^+$ such that the power series $g(x) := f(x^{u_1}, \dots, x^{u_{n-1}}, x) \in \Lambda[[x]]$ is non-zero. It is clear that $g(a) = f(a^{u_1}, \dots, a^{u_{n-1}}, a) = 0$ for every $a \in \mathfrak{m}$.

Let $\nu : \Lambda \rightarrow \mathbb{N} \cup \{\infty\}$ be a non-trivial valuation which is compatible with the \mathfrak{m} -adic topology. Set $g = \sum_{i \geq 0} b_i x^i$ with $b_i \in \Lambda$. Let $d \geq 0$ be the minimum index $i \geq 0$ such that $b_i \neq 0$.

By (3) in Definition 2.1, $\nu(b_d) \in \mathbb{N}$. By (4), there exists an element in Λ whose valuation is a positive integer. By taking a sufficient high power, since ν is compatible with the \mathfrak{m} -adic topology, we can find $a \in \mathfrak{m} \setminus \{0\}$ such that $\nu(a) > \nu(b_d)$.

For every $i \geq 1$, we have

$$\nu(b_{d+i} a^{d+i}) = \nu(b_{d+i}) + i\nu(a) + d\nu(a) \geq \nu(a) + d\nu(a) > \nu(b_d) + d\nu(a) = \nu(b_d a^d).$$

Therefore, by (2) in Definition 2.1, for every $r \geq 1$, we have

$$\nu\left(\sum_{i=1}^r b_{d+i} a^{d+i}\right) > \nu(b_d a^d).$$

By taking the limit for $r \rightarrow +\infty$, since $\nu : \Lambda \rightarrow \mathbb{N} \cup \{\infty\}$ is continuous by [3, VI, § 5, no. 1, Proposition 1], we get

$$\nu\left(\sum_{j>d} b_j a^j\right) > \nu(b_d a^d).$$

From

$$g(a) = b_d a^d + \left(\sum_{j>d} b_j a^j\right),$$

we deduce $\nu(g(a)) = \nu(b_d a^d) \neq \infty$, so $g(a) \neq 0$ which is a contradiction. □

2.2 | Formally smooth algebras

We fix a Noetherian complete local ring Λ ; let \mathfrak{m} be its maximal ideal and let $k = \Lambda/\mathfrak{m}$ be its residue field.

We denote by \mathbf{Comp}_Λ the category of Noetherian complete local Λ -algebras with residue field k . Arrows in \mathbf{Comp}_Λ are Λ -algebra homomorphisms which are compatible with the projection onto k . In particular, all arrows in \mathbf{Comp}_Λ are local homomorphisms. Let \mathbf{Art}_Λ be the full subcategory of \mathbf{Comp}_Λ consisting of Artinian rings. It is clear that \mathbf{Comp}_k (respectively, \mathbf{Art}_k) is a full subcategory of \mathbf{Comp}_Λ (respectively, \mathbf{Art}_Λ) by considering a k -algebra as a Λ -algebra via $\Lambda \twoheadrightarrow k$.

An object R in \mathbf{Comp}_Λ is called *formally smooth (over Λ)* if it is isomorphic to the power series ring $\Lambda\llbracket x_1, \dots, x_n \rrbracket$ for some non-negative integer n . We refer the reader to [32, Appendix B] for properties of formally smooth algebras.

Proposition 2.7. *Let $(\Lambda, \mathfrak{m}, k)$ be a Noetherian complete local domain with a non-trivial valuation which is compatible with the \mathfrak{m} -adic topology. Let R be an object in \mathbf{Comp}_Λ such that, for every integer $\ell \geq 1$, the function*

$$\mathrm{Hom}_{\mathbf{Comp}_\Lambda}(R, \Lambda/\mathfrak{m}^{\ell+1}) \longrightarrow \mathrm{Hom}_{\mathbf{Comp}_\Lambda}(R, \Lambda/\mathfrak{m}^\ell)$$

induced by $\Lambda/\mathfrak{m}^{\ell+1} \twoheadrightarrow \Lambda/\mathfrak{m}^\ell$ is surjective.

Then R is a formally smooth Λ -algebra.

Proof. Let \mathfrak{m}_R be the maximal ideal of R . Let n be the dimension of the k -vector space $T_\Lambda^\vee R = \mathfrak{m}_R/(\mathfrak{m}_R + \mathfrak{m}_R^2)$. By [32, Corollary B.6], there exists a surjection $\Lambda\llbracket x_1, \dots, x_n \rrbracket \twoheadrightarrow R$ in \mathbf{Comp}_Λ whose kernel I is contained in $\mathfrak{m}\llbracket x_1, \dots, x_n \rrbracket + (x_1, \dots, x_n)^2$. In other words, $R = \Lambda\llbracket x_1, \dots, x_n \rrbracket/I$

and every element in I is of the form

$$c_0 + c_1x_1 + \dots + c_nx_n + g \tag{2.1}$$

where $c_0, c_1, \dots, c_n \in \mathfrak{m}$ and g is a power series with order ≥ 2 .

Since Λ is \mathfrak{m} -adically complete, for every $a = (a_1, \dots, a_n) \in \mathfrak{m} \times \dots \times \mathfrak{m}$ we can consider the evaluation homomorphism

$$\text{ev}_a : \Lambda[[x_1, \dots, x_n]] \longrightarrow \Lambda$$

defined by $f \mapsto f(a)$. It is a surjective local homomorphism. Denote by J_a the image of I under ev_a ; J_a is an ideal of Λ which is contained in \mathfrak{m} . We want to show that $J_a = 0$. By contradiction, let us assume that $J_a \neq 0$. By the Krull intersection theorem, $\bigcap_{\ell \geq 0} \mathfrak{m}^\ell = 0$. Therefore, there exists an integer $\ell \geq 1$ such that $J_a \subseteq \mathfrak{m}^\ell$ and $J_a \not\subseteq \mathfrak{m}^{\ell+1}$. Consider the composition

$$\Lambda[[x_1, \dots, x_n]] \xrightarrow{\text{ev}_a} \Lambda \twoheadrightarrow \Lambda/J_a \twoheadrightarrow \Lambda/\mathfrak{m}^\ell;$$

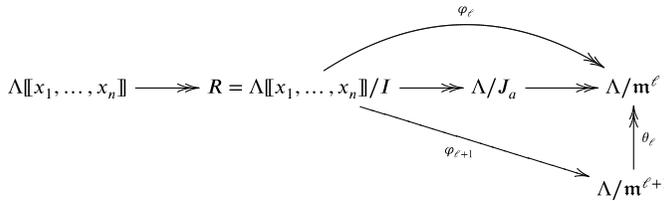
its kernel contains I , therefore we have a surjective homomorphism

$$\varphi_\ell : R = \Lambda[[x_1, \dots, x_n]]/I \longrightarrow \Lambda/\mathfrak{m}^\ell.$$

From the assumption, we deduce the existence of a homomorphism

$$\varphi_{\ell+1} : R \longrightarrow \Lambda/\mathfrak{m}^{\ell+1}$$

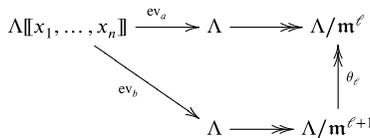
such that $\varphi_\ell = \theta_\ell \circ \varphi_{\ell+1}$, where $\theta_\ell : \Lambda/\mathfrak{m}^{\ell+1} \twoheadrightarrow \Lambda/\mathfrak{m}^\ell$ is the canonical projection map.



For $i = 1, \dots, n$, let $b_i \in \Lambda$ such that $b_i + \mathfrak{m}^{\ell+1} \in \Lambda/\mathfrak{m}^{\ell+1}$ is the image of $x_i + I \in R$ via $\varphi_{\ell+1}$. Set $b = (b_1, \dots, b_n) \in \mathfrak{m} \times \dots \times \mathfrak{m}$. In other words, $\varphi_{\ell+1}$ is induced by the evaluation on b . In particular, this implies

$$\forall f \in I, f(b) \in \mathfrak{m}^{\ell+1}. \tag{2.2}$$

We have a commutative diagram



where $\Lambda \rightarrow \Lambda/\mathfrak{m}^\ell$ and $\Lambda \rightarrow \Lambda/\mathfrak{m}^{\ell+1}$ are the canonical projection maps. For every index $1 \leq i \leq n$, it is clear that a_i and b_i have the same image in $\Lambda/\mathfrak{m}^\ell$, that is, $a_i - b_i \in \mathfrak{m}^\ell$. From the particular form of elements of I in (2.1), we deduce that

$$\forall f \in I, f(a) - f(b) \in \mathfrak{m}^{\ell+1}.$$

By (2.2), this implies $f(a) \in \mathfrak{m}^{\ell+1}$ for every $f \in I$. Hence $J_a \subseteq \mathfrak{m}^{\ell+1}$, which is a contradiction. Therefore, $J_a = 0$.

We have proved that

$$\forall a \in \mathfrak{m} \times \dots \times \mathfrak{m}, \forall f \in I, f(a) = 0.$$

By Lemma 2.5, we deduce $I = 0$. Hence $R = \Lambda[[x_1, \dots, x_n]]$. □

2.3 | Deformation functors

Here we briefly summarize some notions from [31]. We fix a Noetherian complete local ring Λ with maximal ideal \mathfrak{m} and residue field $k = \Lambda/\mathfrak{m}$. A *deformation functor* is a functor $F : \mathbf{Art}_\Lambda \rightarrow \mathbf{Set}$ such that $F(k)$ is a singleton and F satisfies Schlessinger’s conditions (H1) and (H2). If $R \in \mathbf{Comp}_\Lambda$ then $h_R := \text{Hom}_{\mathbf{Comp}_\Lambda}(R, -)$ is a deformation functor. If F is a deformation functor, then the set $F(k[t]/(t^2))$ has a natural structure as k -vector space, which is called the *tangent space* of F .

A natural transformation $F \rightarrow G$ of deformation functors is called *smooth* if for every surjection $B \rightarrow A$ in \mathbf{Art}_Λ the function $F(B) \rightarrow F(A) \times_{G(A)} G(B)$ is surjective. A deformation functor F is called *smooth* if the natural transformation $F \rightarrow h_\Lambda$ is smooth. For $R \in \mathbf{Comp}_\Lambda$, R is a formally smooth Λ -algebra if and only if h_R is smooth.

A *hull* for a deformation functor F is an object $R \in \mathbf{Comp}_\Lambda$ such that there exists a smooth natural transformation $h_R \rightarrow F$ which induces a bijection on tangent spaces. If a hull exists, it is unique.

Now we give a sufficient criterion for the smoothness of a deformation functor.

Proposition 2.8. *Let $(\Lambda, \mathfrak{m}, k)$ be a Noetherian complete local domain with a non-trivial valuation which is compatible with the \mathfrak{m} -adic topology. Let $F : \mathbf{Art}_\Lambda \rightarrow \mathbf{Set}$ be a deformation functor with finite-dimensional tangent space.*

If, for every integer $\ell \geq 1$, the function

$$F(\Lambda/\mathfrak{m}^{\ell+1}) \longrightarrow F(\Lambda/\mathfrak{m}^\ell)$$

induced by $\Lambda/\mathfrak{m}^{\ell+1} \twoheadrightarrow \Lambda/\mathfrak{m}^\ell$ is surjective, then F is smooth.

Proof. By [31, Theorem 2.11], the functor F has a hull R . Consider a smooth map $h_R \rightarrow F$.

Fix an arbitrary integer $\ell \geq 1$. Since $F(\Lambda/\mathfrak{m}^{\ell+1}) \rightarrow F(\Lambda/\mathfrak{m}^\ell)$ is surjective,

$$F(\Lambda/\mathfrak{m}^{\ell+1}) \times_{F(\Lambda/\mathfrak{m}^\ell)} h_R(\Lambda/\mathfrak{m}^\ell) \longrightarrow h_R(\Lambda/\mathfrak{m}^\ell) \tag{2.3}$$

is surjective. Since $h_R \rightarrow F$ is smooth, the function

$$h_R(\Lambda/\mathfrak{m}^{\ell+1}) \longrightarrow F(\Lambda/\mathfrak{m}^{\ell+1}) \times_{F(\Lambda/\mathfrak{m}^{\ell})} h_R(\Lambda/\mathfrak{m}^{\ell}) \tag{2.4}$$

is surjective. By composing (2.4) and (2.3), we obtain that $h_R(\Lambda/\mathfrak{m}^{\ell+1}) \rightarrow h_R(\Lambda/\mathfrak{m}^{\ell})$ is surjective. By Proposition 2.7, h_R is smooth.

We conclude with [31, Proposition 2.5(iii)]. □

Recall from § 2.2 that \mathbf{Art}_k is a full subcategory of \mathbf{Art}_{Λ} .

Lemma 2.9. *Let $F : \mathbf{Art}_{\Lambda} \rightarrow \mathbf{Set}$ be a deformation functor. Consider the restricted functor $F_0 := F|_{\mathbf{Art}_k} : \mathbf{Art}_k \rightarrow \mathbf{Set}$. Then:*

- (1) F_0 is a deformation functor;
- (2) if $R \in \mathbf{Comp}_{\Lambda}$ is the hull of F , then $R \otimes_{\Lambda} k$ is the hull of F_0 .

Proof. (1) Since the embedding $\mathbf{Art}_k \hookrightarrow \mathbf{Art}_{\Lambda}$ preserves pull-backs of Artinian rings, the restricted functor F_0 satisfies Schlessinger’s conditions (H1) and (H2).

(2) It is clear that the tangent space of F_0 coincides with the tangent space of F . Consider a smooth map $h_R \rightarrow F$ which induces a bijection on tangent spaces. It remains smooth after restriction to \mathbf{Art}_k , so we have a smooth map $(h_R)_0 \rightarrow F_0$ of deformation functors which induces a bijection on tangent spaces. The surjection $R \rightarrow R \otimes_{\Lambda} k$ induces an isomorphism

$$h_{R \otimes_{\Lambda} k} = \text{Hom}_k(R \otimes_{\Lambda} k, -) \rightarrow \text{Hom}_{\Lambda}(R, -) = (h_R)_0$$

of functors. □

3 | PROOFS

3.1 | Toric rings

Fix a field k and monoid Q . Consider the monoid k -algebra $k[Q]$ defined as follows: the underlying k -vector space of $k[Q]$ has a basis $\{z^q\}_{q \in Q}$ indexed by elements of Q and the product on $k[Q]$ is the k -linear extension of the rule $z^{q_1} \cdot z^{q_2} = z^{q_1+q_2}$ for all $q_1, q_2 \in Q$. If Q is finitely generated, then $k[Q]$ is of finite type over k .

If Q is sharp, then the ideal $(z^q \mid q \in Q \setminus \{0\})$ is a maximal ideal in $k[Q]$; we denote by $k[[Q]]$ the completion of $k[Q]$ with respect to this maximal ideal. Thus, if Q is finitely generated and sharp, then $k[[Q]]$ is a Noetherian complete local ring with residue field k .

Lemma 3.1. *If k is a field and Q is a non-zero sharp toric monoid, then $k[[Q]]$ admits a non-trivial valuation which is compatible with the adic topology of the maximal ideal of $k[[Q]]$.*

Proof. This is a generalization of Example 2.4. Set $M \simeq \mathbb{Z}^n$ and let σ be an n -dimensional rational polyhedral cone in $M_{\mathbb{R}}$ such that $Q = \sigma \cap M$. We consider the dual lattice $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ with

duality pairing $\langle -, - \rangle : M \times N \rightarrow \mathbb{Z}$ and the dual cone

$$\sigma^\vee = \{v \in N_{\mathbb{R}} \mid \forall q \in \sigma, \langle q, v \rangle \geq 0\}.$$

Pick $w \in N$ in the interior of σ^\vee , and consider the valuation $\nu_w : k[[Q]] \rightarrow \mathbb{N} \cup \{\infty\}$ defined by

$$\nu_w \left(\sum_{q \in Q} b_q z^q \right) = \min\{\langle q, w \rangle \mid q \in Q, b_q \neq 0\}.$$

This is compatible with the adic topology of the maximal ideal of $k[[Q]]$. □

3.2 | Log smooth deformation theory

We fix a field k (of arbitrary characteristic) and a sharp toric monoid Q . As explained in the introduction, every ring A in $\mathbf{Art}_{k[[Q]]}$ gives rise to a log scheme $\text{Spec}(Q \rightarrow A)$ with underlying scheme $\text{Spec} A$. In particular, k gives rise to $S_0 := \text{Spec}(Q \rightarrow k)$, which is the log scheme on the scheme $\text{Spec} k$ with ghost sheaf Q .

We fix a proper log smooth saturated morphism $f_0 : X_0 \rightarrow S_0$ of log schemes of relative dimension d . Let Ω_{X_0/S_0}^1 be the sheaf of log differentials of X_0 relative to S_0 ; let furthermore $\omega_{X_0/S_0} = \Omega_{X_0/S_0}^d$ be the log canonical line bundle on X_0 .

Kato defines in [20] the functor of log smooth deformations of $X_0 \rightarrow S_0$, that is, the functor

$$\text{LD}_{X_0/S_0} : \mathbf{Art}_{k[[Q]]} \longrightarrow \mathbf{Set}$$

which to every object A in $\mathbf{Art}_{k[[Q]]}$ associates the set of isomorphism classes of log smooth deformations $f : X \rightarrow \text{Spec}(Q \rightarrow A)$ of $f_0 : X_0 \rightarrow S_0$ (see [20, Definition 8.1]). This functor is a deformation functor and has a hull by [20, Theorem 8.7].

Now we assume furthermore that k has characteristic 0. In [7], the first author proves that LD_{X_0/S_0} is controlled by a $k[[Q]]$ -linear *predifferential* graded Lie algebra (pdgla, for short) $(L_{X_0/S_0}^\bullet, [-, -], d, \ell)$. More precisely, $(L_{X_0/S_0}^\bullet, [-, -])$ is a graded Lie algebra over $k[[Q]]$ endowed with a derivation d which need not be a differential but admits an element $\ell \in L_{X_0/S_0}^2$ such that $d^2 = [\ell, -]$. Via a modified Maurer–Cartan equation we associate a deformation functor $\mathbf{Art}_{k[[Q]]} \rightarrow \mathbf{Set}$, which is isomorphic to LD_{X_0/S_0} . Directly from the definitions it follows that the restricted functor $\text{LD}_{X_0/S_0} |_{\mathbf{Art}_k}$ is controlled by the k -linear (ordinary) dgla $L_{X_0/S_0}^\bullet \otimes_{k[[Q]]} k$. In particular, it has a hull; Lemma 2.9 shows that the hull of $\text{LD}_{X_0/S_0} |_{\mathbf{Art}_k}$ is $R \otimes_{k[[Q]]} k$ where R is the hull of LD_{X_0/S_0} .

3.3 | Homotopy abelianity

In this section, we prove that the restricted deformation functor $\text{LD}_{X_0/S_0} |_{\mathbf{Art}_k}$ is unobstructed; in particular, we obtain the log BTT theorem in the case $Q = 0$. We fix a field k of characteristic 0.

Recall that a differential graded Lie algebra (dgla, for short) L^\bullet is *abelian* if $[-, -] = 0$, and *homotopy abelian* if it is quasi-isomorphic to an abelian dgla. This is the case if and only if L^\bullet is

formal and $H^*(L^*)$ is abelian; in this case, the associated deformation functor $\text{Def}_{L^*} : \mathbf{Art}_k \rightarrow \mathbf{Set}$ is smooth. We say that a deformation functor $F : \mathbf{Art}_k \rightarrow \mathbf{Set}$ is *controlled* by the dgla L^* if it is isomorphic to Def_{L^*} . We refer the reader to [10, 26] for details.

Now we state Iacono’s abstract Bogomolov–Tian–Todorov theorem from [17] because it will be useful to us below. Recall that a *Cartan homotopy* $\mathbf{i} : L^* \rightarrow M^*$ of dglas is a homogeneous linear map of degree -1 such that

$$\mathbf{i}_{[a,b]} = [\mathbf{i}_a, d_M \mathbf{i}_b] \quad \text{and} \quad [\mathbf{i}_a, \mathbf{i}_b] = 0$$

are satisfied; here, $\mathbf{i}_a := \mathbf{i}(a) \in M^*$ is traditional notation since the elements of M^* are typically homomorphisms themselves — compare this with our choice for M^* below. The *Lie derivative* of \mathbf{i} is given by $\mathbf{l}_a = d_M \mathbf{i}_a + \mathbf{i}_{d_L a}$; it defines a homomorphism $\mathbf{L} : L^* \rightarrow M^*$ of complexes, which is homotopic to 0 via the homotopy \mathbf{i} .

Theorem 3.2 (Abstract Bogomolov–Tian–Todorov theorem [17, Theorem 3.3]). *Let L^*, M^* be dglas over a field k of characteristic 0, let $\mathbf{i} : L^* \rightarrow M^*$ be a Cartan homotopy, let $H^* \subseteq M^*$ be a sub-dgla, and assume that*

- (i) $\mathbf{l}_a \in H^*$ for every $a \in L^*$;
- (ii) $H^* \rightarrow M^*$ is injective in cohomology, that is, $H^*(H^*) \rightarrow H^*(M^*)$ is injective;
- (iii) the morphism $\mathbf{i} : L^* \rightarrow M^*/H^*[-1]$ of complexes is injective in cohomology.

Then L^* is homotopy abelian.

Now we move to the logarithmic setting. Recall that the restricted deformation functor $\text{LD}_{X_0/S_0} |_{\mathbf{Art}_k}$ is controlled by the k -linear dgla $L^*_{X_0/S_0} \otimes_{k[[Q]]} k$. We prove:

Theorem 3.3. *In the setting of Theorem 1.1, if ω_{X_0/S_0} is the trivial line bundle, then the k -linear dgla $L^*_{X_0/S_0} \otimes_{k[[Q]]} k$ is homotopy abelian. In particular, the restricted functor $\text{LD}_{X_0/S_0} |_{\mathbf{Art}_k}$ is smooth.*

In the remainder of this section, we explain the proof of Theorem 3.3.

The construction of the $k[[Q]]$ -pdgla $L^*_{X_0/S_0}$ in [7] relies on the Thom–Whitney resolution; we briefly recall its basic properties. For a more complete treatment, cf. [7, 18–27]. Given a sheaf \mathcal{F} of k -vector spaces on X_0 and an affine Zariski open cover $\mathcal{U} = \{U_i\}$ of X_0 , we obtain the semi-simplicial Čech resolution $\mathcal{F}(\mathcal{U})$, which is a semicosimplicial sheaf on X_0 . Similarly, when \mathcal{G}^* is a complex of sheaves, we obtain a semicosimplicial complex of sheaves $\mathcal{G}^*(\mathcal{U})$. Now the Thom–Whitney resolution $\text{TW}^{p,*}(\mathcal{G}^*(\mathcal{U}))$ is a double complex — in the sense that $d_1 d_2 + d_2 d_1 = 0$ — such that $\mathcal{G}^p \rightarrow \text{TW}^{p,*}(\mathcal{G}^*(\mathcal{U}))$ is a resolution of the sheaf \mathcal{G}^p . If \mathcal{G}^p is quasi-coherent, then the resolution is acyclic so that it computes the cohomology $H^q(X_0, \mathcal{G}^p)$. The Thom–Whitney resolution is more subtle than the usual Čech complex $\check{C}^*(\mathcal{U}; \mathcal{G}^*)$, which can be obtained by just taking alternating sums of boundary maps in $\mathcal{G}^*(\mathcal{U})$; this additional complexity allows to extend algebraic structures (like a Lie bracket or a product) from \mathcal{G}^* to the actual complex $\text{TW}^{p,*}(\mathcal{G}^*(\mathcal{U}))$, not only to its cohomology $H^*(X_0, \mathcal{G}^*)$ — cf. the construction of the cup product in cohomology. In the notation $\text{TW}^{p,q}(-)$ of [7], there is a switch of indices relative to the standard notation — that is, $\text{TW}^{p,q}(-) := C^{q,p}_{\text{TW}}(-)$. This does not affect the formation of the total complex $\text{Tot}_{\text{TW}}(-)$. The Thom–Whitney resolution does not naively commute with the shift functor; instead, we have an

isomorphism

$$s : \text{Tot}_{\text{TW}}(\mathcal{G}^*(\mathcal{U})) [m] \xrightarrow{\cong} \text{Tot}_{\text{TW}}(\mathcal{G}^*[m](\mathcal{U})),$$

which is given on $\text{TW}^{p,q}(-)$ by multiplication with $(-1)^{qm}$. Finally, note that our construction of the Thom–Whitney resolution follows the standard but differs from [18] by the order of the tensor factors; this makes a difference in some signs.

In [7], the Thom–Whitney resolution $\text{PV}_{X_0/S_0} := \text{TW}^{*,*}(\mathcal{G}^*_{X_0/S_0}(\mathcal{U}))$ of the Gerstenhaber algebra $\mathcal{G}^*_{X_0/S_0}$ of polyvector fields is considered; it gives rise to a Thom–Whitney resolution $\mathcal{G}^{-1}_{X_0/S_0} \rightarrow \text{PV}^{-1,*}_{X_0/S_0}$ of log derivations. It is clear from the construction in [7] that

$$L^*_{X_0/S_0} \otimes_{k[[Q]]} k \cong \Gamma(X_0, \text{PV}^{-1,*}_{X_0/S_0}) = \Gamma(X_0, \text{Tot}_{\text{TW}}(\Theta^1_{X_0/S_0}(\mathcal{U}))),$$

where $\Theta^1_{X_0/S_0}$ is considered as a complex concentrated in degree 0.

Proof of Theorem 3.3. In preparation for applying Theorem 3.2 in our situation, we apply the semicosimplicial Čech resolution to the de Rham complex $\Omega^*_{X_0/S_0}$; this yields the semicosimplicial Čech complex $\Omega^*_{X_0/S_0}(\mathcal{U})$. We denote the singly graded total complex of the Thom–Whitney resolution by $\text{T}\Omega^* := \Gamma(X_0, \text{Tot}_{\text{TW}}(\Omega^*_{X_0/S_0}(\mathcal{U})))$. Similarly, we write $\text{T}\Omega^i := \Gamma(X_0, \text{Tot}_{\text{TW}}(\Omega^i_{X_0/S_0}[-i](\mathcal{U})))$ for the resolutions of the individual sheaves of differential forms. Then we have injective graded maps $\text{T}\Omega^i \rightarrow \text{T}\Omega^*$ because the Thom–Whitney construction is exact. Namely, $\text{Tot}_{\text{TW}}(-(\mathcal{U}))$ transforms exact sequences of sheaves into exact sequences of sheaves by the construction in [7, 5.3] and the exactness result in [27, 2.4]; then $\Gamma(X_0, -)$ is only left exact, but $\text{Tot}_{\text{TW}}(\mathcal{F}(\mathcal{U}))$ is acyclic at least whenever \mathcal{F} is quasi-coherent by [7, 5.6]. However, only for $i = d$, the graded linear map $\text{T}\Omega^d \rightarrow \text{T}\Omega^*$ is a homomorphism of complexes since only $\Omega^d_{X_0/S_0}[-d] \rightarrow \Omega^*_{X_0/S_0}$ is compatible with the differential. Here, d is the relative dimension of $f_0 : X_0 \rightarrow S_0$. Moreover, the composed map $\text{T}\Omega^{d-1} \rightarrow \text{T}\Omega^*/\text{T}\Omega^d$ is a homomorphism of complexes since $\text{T}\Omega^*/\text{T}\Omega^d$ is the resolution of $\Omega^*_{X_0/S_0}/\Omega^d_{X_0/S_0}[-d]$ and $\Omega^{d-1}_{X_0/S_0} \rightarrow \Omega^*_{X_0/S_0}/\Omega^d_{X_0/S_0}[-d]$ is a homomorphism of complexes. The two homomorphisms of complexes are injective in cohomology by the argument in [18, Theorem 6.4] and the fact that the Hodge–de Rham spectral sequence $H^q(X_0, \Omega^p_{X_0/S_0}) \Rightarrow \mathbb{H}^{p+q}(X_0, \Omega^*_{X_0/S_0})$ degenerates at E_1 — this is proven in [19] as well as a special case of [8, Theorem 1.9].

We now set $L^* := L^*_{X_0/S_0} \otimes_{k[[Q]]} k$ and $M^* := \text{Hom}_k(\text{T}\Omega^*, \text{T}\Omega^*)$; the latter space is a dgla by the construction in [17, Example 2.2]. Using the Künneth formula, we compute its cohomology via the natural isomorphism

$$H^*(M^*) \xrightarrow{\cong} \text{Hom}^*_k(H^*(\text{T}\Omega^*), H^*(\text{T}\Omega^*)).$$

A contraction $\lrcorner : L^* \times \text{T}\Omega^* \rightarrow \text{T}\Omega^*$ is (by definition) a bilinear map of degree -1 such that the induced map $L^* \rightarrow M^* = \text{Hom}_k(\text{T}\Omega^*, \text{T}\Omega^*)$ is a Cartan homotopy. We construct such a contraction — and thus a Cartan homotopy — from a *semicosimplicial contraction*, that is, a system of contractions $\lrcorner_n : \Theta^1_{X_0/S_0}(\mathcal{U})_n \times \Omega^*_{X_0/S_0}(\mathcal{U})_n \rightarrow \Omega^*_{X_0/S_0}(\mathcal{U})_n$ which is compatible with the coface maps of the semicosimplicial complex — cf. [17]. Concretely, this contraction is given by contracting log

derivations with differential forms — that is, by the unique map $\dashv : \Theta^1_{X_0/S_0} \times \Omega^p_{X_0/S_0} \rightarrow \Omega^{p-1}_{X_0/S_0}$ which satisfies $\theta \dashv \omega = \langle \omega, \theta \rangle$ for $\omega \in \Omega^1_{X/S}$ and

$$\theta \dashv (\omega \wedge \eta) = (\theta \dashv \omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge (\theta \dashv \eta).$$

We set

$$H^* := \{m \in M^* \mid m(\mathrm{T}\Omega^d) \subset \mathrm{T}\Omega^d\} \subset M^*;$$

it is a sub-dgla whose embedding is injective in cohomology by [17, Example 2.17] because $\mathrm{T}\Omega^d \subset \mathrm{T}\Omega^*$ is a sub-dg-vector space whose embedding is injective in cohomology.

We show the condition $\mathbf{l}_\theta \in H$ by explicit computation. We have decompositions

$$L^* = \bigoplus_{\lambda} \Gamma(X_0, \mathrm{TW}^{0,\lambda}(\Theta^1_{X_0/S_0}(\mathcal{U}))) \quad \text{and} \quad \mathrm{T}\Omega^d = \bigoplus_{\mu} \Gamma(X_0, \mathrm{TW}^{d,\mu-d}(\Omega^*_{X_0/S_0}(\mathcal{U}))),$$

so let $\theta = (t_n \otimes \theta_n)_n \in L^*$ be of bidegree $(0, \lambda)$ and $\omega = (a_n \otimes \omega_n)_n \in \mathrm{T}\Omega^d$ be of bidegree $(d, \mu - d)$. Then

$$\begin{aligned} \mathbf{l}_\theta(\omega) &= d\mathbf{i}_\theta(\omega) + (-1)^\lambda \mathbf{i}_\theta(d\omega) + \mathbf{i}_{d\theta}(\omega) \\ &= ((t_n \wedge a_n) \otimes d(\theta_n \dashv \omega_n))_n \in \Gamma(X_0, \mathrm{TW}^{d,\mu-d+\lambda}(\Omega^*_{X_0/S_0}(\mathcal{U}))). \end{aligned}$$

To prove the final condition, let $\eta \in \Omega^d_{X_0/S_0}$ be a volume form. In the diagram

$$\begin{array}{ccc} L^* & \xrightarrow{\alpha} & \mathrm{Hom}_k^*(\mathrm{T}\Omega^d, \mathrm{T}\Omega^{d-1})[-1] \\ \downarrow \gamma & & \downarrow \mathrm{ev}_\eta \\ \mathrm{Tot}_{\mathrm{TW}}(\Omega^{d-1}_{X_0/S_0}(\mathcal{U}')) & \xleftarrow{s} & \mathrm{T}\Omega^{d-1}[d-1] \end{array}$$

the map α induced by contraction is a homomorphism of complexes: In a computation analogous to that of $\mathbf{l}_\theta(\omega)$, the differentials $d(\theta_n \dashv \omega_n)$ vanish, for $\Omega^{d-1}_{X_0/S_0}[-d+1]$ is concentrated in a single degree. The evaluation ev_Ω maps a morphism $\phi : \mathrm{T}\Omega^d \rightarrow \mathrm{T}\Omega^{d-1}$ to the image $(-1)^{d|\phi|} \phi(1 \otimes \Omega)$ where $1 \otimes \Omega \in \mathrm{T}\Omega^d$ is the element induced by Ω in degree d and $|\phi|$ is the degree of the original map (without shift in the Hom complex). It is a homomorphism of complexes because $d\Omega = 0$. The map s is the above mentioned comparison map for the two shifted versions of the Thom–Whitney resolution. When we evaluate the contraction at Ω , this gives an isomorphism $\Theta^1_{X_0/S_0} \cong \Omega^{d-1}_{X_0/S_0}$, and γ is the induced map of Thom–Whitney complexes. The diagram is commutative, so α is injective in cohomology (as γ is an isomorphism). The homomorphism α fits into a diagram

$$\begin{array}{ccc} L^* & \longrightarrow & M^*/H^*[-1] \\ \downarrow \alpha & & \downarrow \\ \mathrm{Hom}_k^*(\mathrm{T}\Omega^d, \mathrm{T}\Omega^{d-1})[-1] & \xrightarrow{\eta} & \mathrm{Hom}_k^*(\mathrm{T}\Omega^d, \mathrm{T}\Omega^*/\mathrm{T}\Omega^d)[-1]; \end{array}$$

because $T\Omega^{d-1} \rightarrow T\Omega^*/T\Omega^d$ is injective in cohomology, η is injective in cohomology by the Künneth theorem. Thus, $L^* \rightarrow M^*/H^*[-1]$ is injective in cohomology. \square

Remark 3.4. Theorem 3.3 remains true if we relax the Calabi–Yau condition in the sense that we only require that $\omega_{X_0/S_0}^{\otimes N}$ is trivial for some $N > 0$. In this case, the N -cyclic covering

$$\pi : Y_0 := X_0[\sqrt[N]{\omega_{X_0/S_0}}] \rightarrow X_0$$

is a finite étale covering; we endow Y_0 with the induced log structure from X_0 , thus we turn $Y_0 \rightarrow S_0$ into a proper and saturated log smooth morphism with $\omega_{Y_0/S_0} \cong \mathcal{O}_{Y_0}$. The canonical map $\Theta_{X_0/S_0}^1 \rightarrow \pi_* \Theta_{Y_0/S_0}^1$ induces a homomorphism

$$L_{X_0/S_0}^* \otimes_{k[[Q]]} k \rightarrow L_{Y_0/S_0}^* \otimes_{k[[Q]]} k$$

of dglas, which is injective in cohomology (because $H^*(X_0, \Theta_{X_0/S_0}^1) \rightarrow H^*(Y_0, \Theta_{Y_0/S_0}^1)$ is injective). Thus, $L_{X_0/S_0}^* \otimes_{k[[Q]]} k$ is homotopy abelian by [17, 2.11].

We do not know if Theorem 1.1 remains true in this situation because it is unclear if Theorem 3.7 below holds. The latter relies on the existence of a volume form, which is used to transport the de Rham differential to the polyvector fields and thus construct the Batalin–Vilkovisky operator; such a volume form is not given in the situation where the log canonical bundle is only a torsion line bundle but not trivial.

Remark 3.5. In case $Q \neq 0$, the smoothness of $\text{LD}_{X_0/S_0} |_{\mathbf{Art}_k}$ follows from Theorem 3.7 below as well. This second proof does not generalize to the torsion case discussed in the Remark above.

Remark 3.6. Theorem 3.3 holds as well for log toroidal families $f_0 : X_0 \rightarrow S_0$ in the sense of [8]; we do not need to construct a version of L_{X_0/S_0}^* in this case — the correct dglA is obtained by the Thom–Whitney resolution of the reflexive sheaf Θ_{X_0/S_0}^1 . This dglA then controls the functor of locally trivial log deformations — a notion which makes sense on the subcategory $\mathbf{Art}_k \subseteq \mathbf{Art}_{k[[Q]]}$ but not on the full category $\mathbf{Art}_{k[[Q]]}$. Theorem 3.3 holds because the Hodge–de Rham spectral sequence of the reflexive de Rham complex W_{X_0/S_0}^* degenerates at E_1 . As in the log smooth case, it suffices to assume that ω_{X_0/S_0} is a torsion line bundle; in fact, the cyclic covering $X_0[\sqrt[N]{\omega_{X_0/S_0}}] \rightarrow S_0$ of a log toroidal family is a log toroidal family as well.

3.4 | Proof of Theorem 1.1

We start by recalling a fundamental result about smoothings of log Calabi–Yau spaces:

Theorem 3.7 (Chan–Leung–Ma [5], Felten–Filip–Ruddat [8], Felten [7]). *Let k, Q , and $X_0 \rightarrow S_0$ be as in Theorem 1.1. Denote by \mathfrak{m} the maximal ideal of $k[[Q]]$. If ω_{X_0/S_0} is the trivial line bundle, then the function*

$$\text{LD}_{X_0/S_0}(k[[Q]]/\mathfrak{m}^{\ell+1}) \longrightarrow \text{LD}_{X_0/S_0}(k[[Q]]/\mathfrak{m}^\ell)$$

is surjective for every integer $\ell \geq 1$.

Proof. This follows from an algebraic version of the method in [5], which is also employed in [8]. Given a Maurer–Cartan solution in $L_{X_0/S_0}^\bullet \otimes_{k[[Q]]} (k[[Q]]/\mathfrak{m}^\ell)$, we must find a lifting in $L_{X_0/S_0}^\bullet \otimes_{k[[Q]]} (k[[Q]]/\mathfrak{m}^{\ell+1})$. Its existence follows from the analogue of [5, Theorem 5.5] once we algebraize the theory in [5] in the spirit of [7]. The crucial ingredient here is the fact that the Hodge–de Rham spectral sequence

$$H^q(X_0, \Omega_{X_0/S_0}^p) \Rightarrow \mathbb{H}^{p+q}(X_0, \Omega_{X_0/S_0}^\bullet)$$

degenerates at E_1 — see [19] or [8] — and that $\omega_{X_0/S_0} \cong \mathcal{O}_{X_0}$. The algebraization is rather straightforward; the first author will elaborate it in a separate paper, where he provides the technical machinery in more generality. \square

Proof of Theorem 1.1. If $Q = 0$, we use Theorem 3.3. If $Q \neq 0$, we combine Theorem 3.7, Lemma 3.1 and Proposition 2.8. \square

ACKNOWLEDGEMENTS

We thank Alessio Corti and Helge Ruddat for many useful conversations. The first author was funded by Helge Ruddat’s DFG grant RU 1629/4-1 and thanks JGU Mainz for its hospitality.

Open Access Funding provided by Universita degli Studi di Bologna within the CRUI-CARE Agreement.

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The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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