

The TAZRP speed process

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Received 25 November 2019; revised 15 July 2020; accepted 22 October 2020

Abstract. In (*Ann. Probab.* **39** (4) (2011) 1205–1242) Amir, Angel and Valkó studied a multi-type version of the totally asymmetric simple exclusion process (TASEP) and introduced the TASEP speed process, which allowed them to answer delicate questions about the joint distribution of the speed of several second-class particles in the TASEP rarefaction fan. In this paper we introduce the analogue of the TASEP speed process for the totally asymmetric zero-range process (TAZRP), and use it to obtain new results on the joint distribution of the speed of several second-class particles in the TAZRP with a reservoir. There is a close link from the speed process to questions about stationary distributions of multi-type versions of the TAZRP; for example we are able to give a precise description of the contents of a single site in equilibrium for a multi-type TAZRP with continuous labels.

Résumé. Dans (*Ann. Probab.* **39** (4) (2011) 1205–1242) Amir, Angel et Valkó ont étudié une version multi-type de processus d'exclusion simple totalement asymétrique (TASEP), et introduit le processus de vitesse du TASEP, ce qui leur avait permis de résoudre des questions délicates sur la loi jointe de la vitesse de plusieurs particules de seconde classes dans l'éventail de raréfaction du TASEP. Dans cet article, nous introduisons l'analogue du processus de vitesse du TASEP pour le processus de portée zéro totalement asymétrique (TAZRP), et nous l'utilisons pour obtenir de nouveaux résultats sur la loi jointe de la vitesse de plusieurs particules de seconde classe dans le TAZRP avec un réservoir. Il existe un lien étroit entre le processus de vitesse et la loi stationnaire de versions multi-types du TAZRP ; par exemple, nous sommes en mesure de donner une description précise du contenu d'un site donné à l'équilibre pour un TAZRP multi-type avec étiquettes continues.

MSC2020 subject classifications: 82C22; 60K35

Keywords: Speed of a second class particle; Zero range process; Speed process

1. Introduction

In the totally asymmetric simple exclusion process (TASEP), each site of $\mathbb{Z}$ contains either a particle or a hole. If a particle has a hole to its right, they exchange places at rate 1. In [10], Ferrari and Kipnis considered the TASEP with Riemannian initial data – that is, where there exists an asymptotic density of particles to the left of the origin, and also a (perhaps different) asymptotic density of particles to the right of the origin – and with a second-class particle placed at the origin. The second-class particle interacts with holes as if it was a particle, and with particles as if it was a hole.

As the configuration evolves, the position of the second-class particle, $X_2(t)$, changes, and a natural question is whether the limit

$$U = \lim_{t \rightarrow \infty} t^{-1} X_2(t), \quad (1)$$

exists, and if so, in what sense. Consider for example an initial condition with particles at all negative sites and holes at all positive sites (often referred to as the rarefaction initial condition). It was shown in [10] that the limit in (1) exists in distribution, and that

$$U \sim \text{U}[-1, 1], \quad (2)$$

The hydrodynamics of the TASEP are described by the inviscid Burgers equation; for the rarefaction initial condition, the equation displays an entire interval of characteristics emanating from the origin – the so-called “rarefaction fan” – and one has the interpretation that the speed of the second-class particle is distributed uniformly across the set of characteristics.

The natural question of whether the convergence in (1) can be strengthened to almost sure convergence was resolved in [20] by Mountford and Guiol, for the rarefaction initial condition, using large deviations for last-passage percolation and microscopic approximation of the Hamilton-Jacobi equation related to the TASEP hydrodynamics. A different proof was given by Ferrari and Pimentel [13] using a direct coupling between the path of the second-class particle and an interface in a two-type last-passage percolation model.

In [9] Ferrari, Gonçalves and Martin considered the TASEP process (and partially-asymmetric versions of it) starting from a configuration with two second-class particles P and Q at positions 0 and 1 respectively, with only first class particles to their left and only holes to their right. They showed, for example, that for the TASEP, the probability that P attempts a jump over Q at some time $t > 0$ is $\frac{2}{3}$.

In order to answer further questions about the joint distribution of the speed of several second-class particles in the rarefaction fan – such as, what is the probability that the two second-class particles develop the same speed? – Amir, Angel and Valkó [1] introduced the TASEP speed process. In this model one starts from an initial condition in which every site of $\mathbb{Z}$ contains a particle of a different type, with a hierarchy determined by their initial position. Each particle sees itself as a second-class particle viewing all particles to its left as first-class particles, and all particles to its right as holes. In this way, the particle positioned at any site $i \in \mathbb{Z}$ develops a speed almost surely, and one obtains the so-called TASEP speed process

$$\{U_i\}_{i \in \mathbb{Z}}, \quad (3)$$

a process indexed by $\mathbb{Z}$ which encodes the joint speed of all particles. This process proved to be a rich model encoding much information about the joint behaviour of second-class particles around the rarefaction fan. In the case of two second-class particles in the rarefaction fan, an explicit joint distribution of the speed was obtained, in particular, it was shown that with positive probability ($\frac{1}{6}$) the two particles develop the same speed. In fact, it was shown that with probability 1, the set of speeds attained is dense in $[-1, 1]$, and that for any speed v which is attained, there are in fact infinitely many particles, called a convoy, with speed v .

The TASEP speed process was also used in [6] and [5] by Coupier and Heinrich to show that in the last-passage percolation model, there are no three geodesics with the same direction. Results from [1] about the speed process of the TASEP, and about related questions concerning speeds of particles in partially asymmetric systems, were recently extended to models with inhomogeneity in space and time by Borodin and Bufetov [3].

A closely related and also widely-studied interacting particle system is the (constant-rate) Totally Asymmetric Zero-Range Process (TAZRP). In this process each site of $\mathbb{Z}$ can contain any finite number n of particles. Each site is equipped with a Poisson clock with rate 1, upon ringing, if there is a particles at site x it jumps to site $x + 1$. Note that for the TASEP the full rarefaction fan is obtained by taking the maximum density (1) to the left of the origin and minimum density (0) to the right of it. As for the TAZRP the number of particles at each site is unbounded, it seems that the analogue to the full rarefaction fan initial condition for the TASEP is the initial condition where to the left of the origin, the density is infinite and to the right it is zero. This initial condition can be modelled by setting a reservoir for the TAZRP at the left of the origin. The TAZRP with a reservoir is simply the TAZRP on $\{-1, 0, 1, \dots\}$ where at site -1 there are infinitely many particles. In this model the particles obey the dynamics of TAZRP on $\{0, 1, \dots\}$ while the reservoir itself is equipped with a Poisson clock of rate one, which whenever rings, a particle jumps from the reservoir to site 0. In [14] it was shown that the TAZRP with a reservoir has a hydrodynamic limit given by the function

$$h(x, t) = \begin{cases} \frac{1 - \sqrt{\frac{x}{t}}}{\sqrt{\frac{x}{t}}} & \frac{x}{t} \in (0, 1) \\ 0 & \frac{x}{t} > 1. \end{cases} \quad (4)$$

$h(x, t)$ can be understood as the density of particles around position xn at time tn , for large n . In [14] Gonçalves considered second-class particles positioned at time $t = 0$ at site 0 for the ZRP with general rate function g and independent Riemannian initial data. In particular, this includes the case with a reservoir at site -1 and with all sites to the right of the second-class particle empty. Extending a well known coupling between the TASEP and the TAZRP to configurations with second-class particle, it was shown that, in the case of the constant rate TAZRP, the second-class particle has speed V almost surely, and that $V = (\frac{1+U}{2})^2$ where $U \sim U[-1, 1]$. In [2], Balázs and Nagy obtained the distribution of the speed of a second-class particle at the rarefaction fan for a large set of models including the TASEP and ZRP using a signed measure on the configurations.

In this paper we continue the study of speeds of particles in the TAZRP, and of related questions concerning stationary distributions of multi-type versions of the process.

We consider an initial condition η^* in which each site of $\mathbb{Z}$ has an infinite column of particles (with a bottom particle but no top particle). Every particle has higher priority than all the particles above it, and also than all the particles at sites to its right. In this way, every particle sees itself as a second-class particle sitting on top of a finite stack of first-class particles at its own site, with an infinite reservoir of first-class particles to its left, and empty space to its right.

We show that every particle develops a speed with probability 1, leading to an array $U = \{U_{z,i}\}_{z \in \mathbb{Z}, i \in \mathbb{N}_0}$ where $U_{z,i}$ is the speed of the particle positioned at column z on top of i particles in η^* . Furthermore, the distribution of this “speed process” U is shown to be a stationary distribution for a multi-type version of the TAZRP, whose particles have types in $\mathbb{R}$. Indeed, all translation-invariant stationary distributions of such a system can be obtained via appropriate rescalings of the speed process. Although any individual speed is a continuous random variable, any pair of speeds have positive probability to be equal.

The properties above are analogous to ones known for the TASEP from [1]. However, in the case of the TAZRP we can go much further than has been possible for the TASEP in describing the joint distribution of several speeds. In particular, we give an explicit description of the joint distribution of the speeds of all the particles in a given column, and hence of the contents of a typical site in a stationary multi-type TAZRP.

Our approach begins with the coupling between configurations with second-class particles in TASEP and configurations with second-class particles in TAZRP, in particular the connection between the speed of a second-class particle in the TAZRP with the flux of holes seen by a second-class particle in the TASEP. This is combined with the results in [12] showing that the second-class particle in the TASEP starting from Riemann initial data has a speed with probability 1, and an expression of the flux of holes seen by a second-class particle as a function of this particle’s speed,

To get more precise information about the joint distribution of speeds, we then develop a new approach involving fixed points of multi-type queues. We can think of a site z of the multi-type TAZRP as a priority queue whose service process is a Poisson process of rate 1. When a service occurs, the highest-priority particle present leaves the queue, moving from z to $z + 1$. In a translation-invariant equilibrium, the distribution of the queue’s arrival process (the process of particles moving from $z - 1$ to z) is the same as the distribution of the queue’s departure process. Taking as a starting point results of Martin and Prabhakar [19], we are able to build up a detailed description of the possible distributions of the contents of the queue for systems with some finite number n of types; by taking appropriate limits, we can then pass to the full picture of multi-type equilibria.

The rest of the paper is organized as follows. In the next section we define the models and give the main results. In Section 3 we describe the coupling between the TASEP and the TAZRP, with and without second-class particles. In Section 4 we prove that distribution of the TAZRP speed process is stationary with respect to the TAZRP dynamics (Theorem 1) and start to obtain results on the distributions of the speeds. In Section 5 we study the fixed points of multi-type priority queues, and prove Theorem 2 describing the equilibrium distributions of a single column in the multi-type TAZRP with a finite number of types. In Section 6 we use the results of Section 5 to prove results about the TAZRP speed process (Theorem 3 and Theorem 4). In Section 7 we prove a result concerning overtaking between particles which have the same speed (Theorem 5).

2. Main results

The totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$ is a Markov process on $\mathcal{Y} = \{0, 1\}^{\mathbb{Z}}$ whose generator is defined for cylinder functions $f : \mathcal{Y} \rightarrow \mathbb{R}$ by

$$L^{\text{EP}} f(\xi) = \sum_{x \in \mathbb{Z}} \xi(x)(1 - \xi(x + 1))(f(\xi^{x,x+1}) - f(\xi)), \quad (5)$$

where

$$\xi^{x,x+1}(z) = \begin{cases} \xi(x + 1) & z = x \\ \xi(x) & z = x + 1 \\ \xi(z) & \text{otherwise.} \end{cases}$$

Define the measures $\{\nu_\alpha : 0 \leq \alpha \leq 1\}$ as the i.i.d. product measures on $\mathcal{Y}$ s.t. $\nu_\alpha(\xi(0) = 1) = \alpha$. It is well known that any stationary measure with respect to (5) that is also translation invariant is a convex combination of $\{\nu_\alpha : 0 \leq \alpha \leq 1\}$ (see [17]). Another way to describe the TASEP is through the so-called Harris construction. In the Harris construction, we attach to each bond connecting two adjacent sites x and $x + 1$ a Poisson clock $\mathcal{T}^{(x,x+1)}$ of rate one. The dynamics of the

process are as follows. At the ring of the clock $\mathcal{T}^{(x,x+1)}$ at time t , if there is a particle at site x and no particle at $x+1$ at time $t-$ then at time t the particle at site x jumps to site $x+1$; otherwise, there is no change in the configuration. This construction is well defined since on any finite time interval, a.s. the graph can be broken into finite subgraphs on which the dynamics depend only on its clocks (and not those of other subgraphs).

The totally asymmetric zero range process (TAZRP) on $\mathbb{Z}$ is a Markov process on $\mathcal{X} = \mathbb{N}_0^{\mathbb{Z}}$ whose generator is given by

$$L^{\text{ZR}} f(\eta) = \sum_{x \in \mathbb{Z}} g(\eta(x)) (f(\eta^{x,x+1}) - f(\eta)),$$

where $g : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ satisfies a Lipschitz condition and vanishes at 0, and where

$$\eta^{x,x+1}(z) = \begin{cases} \eta(x) - 1 & z = x \\ \eta(x+1) + 1 & z = x+1 \\ \eta(z) & \text{otherwise.} \end{cases}$$

We shall be interested in the case where $g \equiv 1$ (i.e. $g(x) = 1$ for all $x \geq 1$), also known as the constant-rate ZRP). As in the TASEP, the TAZRP with $g \equiv 1$ can be built through the Harris construction. On each $z \in \mathbb{Z}$ a finite number of particles are stacked one on top of the other. We attach a Poisson clock $\mathcal{T}^{(x,x+1)}$ to each pair of adjacent sites; upon ringing, if there is at least one particle at site x then the bottom particle at x makes a jump to the top of the stack at $x+1$, otherwise there is no change in the configuration. Alternatively the constant-rate TAZRP can be thought of as a system of M/M/1 queues in tandem, one at each site of $\mathbb{Z}$. Using the same arguments as before, one can show that the dynamics are well defined. The stationary and translation invariant distributions are well known for the TAZRP (see for example p.29 of [16]) and in the case where $g \equiv 1$ are given by $\{\mu_\rho : 0 \leq \rho < \infty\}$ where μ_ρ is product measure whose marginals are geometric with mean ρ , i.e.

$$\mu_\rho(\eta(x) = k) = \left(\frac{\rho}{1+\rho} \right)^k \frac{1}{1+\rho}, \quad k \in \mathbb{N}_0. \quad (6)$$

More generally, we can study the *multi-type TAZRP* on

$$\mathcal{Z} = \{\eta \in \mathbb{R}^{\mathbb{Z} \times \mathbb{N}_0} : \eta(z, i) \geq \eta(z, i+1)\}. \quad (7)$$

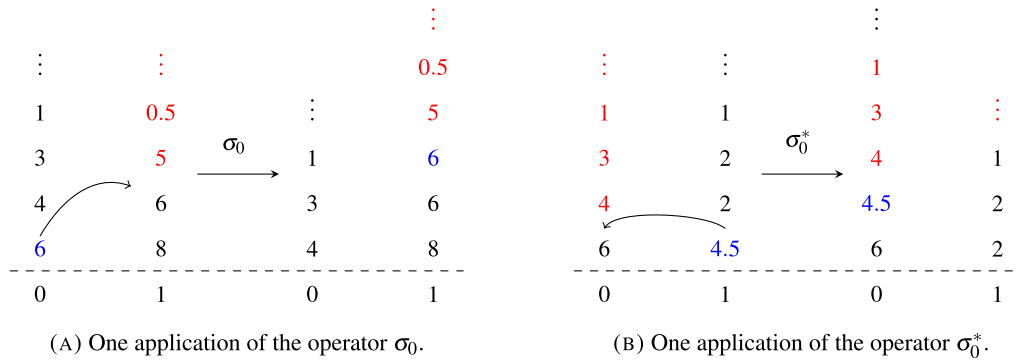
To each particle we assign a “class” in $\mathbb{R}$ and now the queues become priority queues with infinitely many customers. At each service the highest priority (greatest value) particle jumps to the next queue. We will think of the particles at each queue as sorted according to their class, with the strongest at the bottom. The value $\eta(z, i)$ represents the class of the i th strongest particle at site z . One can get a similar Harris construction using the same clocks $\mathcal{T}^{(x,x+1)}$ as before: when an adjacent pair rings the bottom particle at x jumps to $x+1$ and positions itself according to its priority. The multi-type TAZRP on $\mathcal{Z}$ can be defined through the generator

$$Lf(\eta) = \sum_{x \in \mathbb{Z}} (f(\sigma_x \eta) - f(\eta)), \quad (8)$$

where the operator σ_x is defined in the following way: let $i_{\text{sort}^x}(\alpha) = \min\{i : \eta(z, i) < \alpha\}$. In other words, i_{sort^x} is the lowest index for which $\eta(z, \cdot)$ is smaller than α . The operator σ_x is defined through

$$\sigma_x \eta(z, i) = \begin{cases} \eta(x, i+1) & z = x \\ \eta(x, 0) & z = x+1, i = i_{\text{sort}^{x+1}}(\eta(x, 0)) \\ \eta(x+1, i-1) & z = x+1, i > i_{\text{sort}^{x+1}}(\eta(x, 0)) \\ \eta(z, i) & \text{otherwise.} \end{cases} \quad (9)$$

In words, σ_x takes the lowest-positioned (and hence of highest value) particle in column x , whose type is $\eta(x, 0)$, and puts it at position $i_{\text{sort}^{x+1}}(\eta(x, 0))$ in column $x+1$ and shifts the position of all particles of value lower than that of $\eta(x, 0)$ upward by one (see Figure 1).

Fig. 1. The two operators σ and σ^* acting on columns 0 and 1.

Remark 1. At this point it is not clear why the dynamics in (8) are well defined on the set (7) as it could be the case that for some x

$$\eta(x, 0) < \inf_i \eta(x + 1, i). \quad (10)$$

Nevertheless, we shall point out where needed, why on the set of configurations in $\mathcal{Z}$ the dynamics are well defined.

We shall also need the operator σ_x^* on $\mathcal{Z}$, which takes $\eta(x + 1, 0)$ and puts it in the correct position in column x . More precisely, we define

$$\sigma_x^* \eta(z, i) = \begin{cases} \eta(x + 1, i + 1) & z = x + 1 \\ \eta(x + 1, 0) & z = x, i = i_{\text{sort}^x}(\eta(x + 1, 0)) \\ \eta(x, i - 1) & z = x, i > i_{\text{sort}^x}(\eta(x + 1, 0)) \\ \eta(z, i) & \text{otherwise.} \end{cases} \quad (11)$$

We would like to consider a process analogous to the TASEP speed process introduced in [1]. In [1], an ergodic process $\{U_i\}_{i \in \mathbb{Z}}$ was constructed where $U_0 \sim U[-1, 1]$. The marginal U_i , represents the speed of a second-class particle positioned between infinite first class particles to its left and infinite holes to its right, under the TASEP dynamics. The coupling between the different marginals is obtained by starting from an initial condition where there is a hierarchy between the particles. In this initial condition, each particle is stronger than its neighbour to its right. On the ring of the bell at the edge connecting sites x and $x + 1$, the particle at site x jumps to site $x + 1$ if and only if the particle at site $x + 1$ is weaker (i.e. higher class) than the particle at x .

In the ZRP the number of particles at each site is not bounded. We consider starting the dynamics from a configuration where each site has an infinite number of particles.

We denote by $p_{z,n}$ the particle sitting at site z with n other particles below it. Here too, we impose a full hierarchy (order relation) on the initial particles according to the lexicographical order: $p_{i,j}$ is stronger than $p_{k,l}$ (denoted $p_{i,j} > p_{k,l}$) if $i < k$, or if $i = k$ and $j < l$. (That is, each particle is stronger than those at sites to its right, or at the same site and directly above it).

Consider a specific particle in our initial configuration. If we only care about the dynamics of that particle, then we do not care about the hierarchy between the other particles. Thinking of that particle as second-class, we may consider all particles underneath it or at sites to its left as first-class particles, and all the particles above it or at sites to its right as holes. We will show in Section 3 that, under the multi-type TAZRP dynamics, this particle will develop a speed with probability 1. We record this speed in $U_{i,j}$, the (i, j) 'th element of the array $\{U_{i,j}\}_{i,j \in \mathbb{Z} \times \mathbb{N}_0}$, the TAZRP speed process.

Another way to visualize the configuration of particles denoted above by $p_{z,n}$ is by considering an array of numbers $\eta^* \in \mathcal{Z}$ with

$$\eta^*(z, i) < \eta^*(w, j) \quad \text{if and only if} \quad (w = z \text{ and } i > j) \quad \text{or} \quad (w < z) \quad (12)$$

where $\mathcal{Z}$ is the set defined in (7) (see Figure 2). Here, particle $p_{z,i}$ is identified with the number $\eta^*(z, i)$ and the index (z, i) throughout the dynamics. The number $\eta^*(z, i)$ plays the role of the class, or type of the particle which determines its interaction with other particles in the configuration in time. Note that stronger particles correspond to higher values, as opposed to the set-up in [1]. Between each pair of neighbouring columns in the array we assign a Poisson clock, where

	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$p_{-1,i}$	$p_{0,i}$	$p_{1,i}$	$p_{2,i}$
$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$p_{-1,1}$	$p_{0,1}$	$p_{1,1}$	$p_{2,1}$
	$p_{-1,0}$	$p_{0,0}$	$p_{1,0}$	$p_{2,0}$
	-1	0	1	2

Fig. 2. The initial configuration η^* .

upon ringing the largest number in the left column (sitting at the bottom of the column) makes a jump to the right column and positions itself on top of all the numbers that are strictly larger than itself. We shall go from the picture of the array of numbers to the array of particles often. We also use the words class and type interchangeably, so that

$$p_{z,i} \text{ has higher class than } p_{w,j} \iff \eta^*(z, i) > \eta^*(w, j). \quad (13)$$

Let $X_{z,i}(t)$ denote the position of the particle $p_{z,i}$ at time t , that is, the site (column) $X_{z,i}(t) \in \mathbb{Z}$ where the particle $p_{z,i}$ can be found at time t under the dynamics of (8) and the initial condition η^* , i.e. the multi-type TAZRP. Let $p_{z,i}$ and $p'_{w,i}$ be two particles in the configurations η and η' respectively. We say the particle $p_{z,i}$ sees the same environment as particle $p'_{w,i}$ if for every $l \in \mathbb{N}_0$ and $k \in \mathbb{Z}$

$$\begin{aligned} p_{z+k,l} \leq p_{z,i} &\iff p'_{w+k,l} \leq p'_{w,i}, \\ p_{z+k,l} \geq p_{z,i} &\iff p'_{w+k,l} \geq p'_{w,i}. \end{aligned}$$

In other words, two particles see the same environment if the relative order between them and other particles in the configuration is preserved relative to their position.

We say that $p_{z,i}$ has a speed if $\lim_{t \rightarrow \infty} t^{-1} X_{z,i}(t)$ exists and call the limit the speed of $p_{z,i}$. We have the following result, a proof of which we give in Section 3.

Lemma 1. *For every $z \in \mathbb{Z}$ and $i \in \mathbb{N}$ the particle $p_{z,i}$ has a strictly positive speed with probability one. That is, the following limit exists and is strictly positive a.s.*

$$U_{z,i} = \lim_{t \rightarrow \infty} t^{-1} X_{z,i}(t) > 0. \quad (14)$$

We are now in a position to define the TAZRP speed process.

Definition 1. The TAZRP speed process $U = \{U_{z,i}\}_{\mathbb{Z} \times \mathbb{N}_0}$ is given by

$$U_{z,i} = \lim_{t \rightarrow \infty} t^{-1} X_{z,i}(t) \quad \text{for } z \in \mathbb{Z}, i \in \mathbb{N}_0,$$

where the limit is with probability one. We define μ to be the distribution of the process U on $\mathcal{Z}$.

Note that the interaction between particles in the TAZRP depends only on the relative ordering of their types, and is insensitive to any relabelling which preserves that ordering; in particular, the distribution of the speed process is the same for any initial condition η^* which satisfies (12).

We also show the following property of the TAZRP speed process, which together with the previous Lemma show that $U \in \mathcal{Z}$ and that on the support of μ in the set $\mathcal{Z}$ the dynamics in (8) is well defined.

Lemma 2. $\sum_{i=0}^{\infty} \mathbb{E}[U_{z,i}] = 1$ and $\mathbb{P}(\inf_{0 \leq i} U_{z,i} = 0) = 1$.

Note that although, clearly, the measure μ is translation invariant, it may not be reflection invariant. Let π denote the reflection operator on $\mathcal{Z}$ defined by $\pi\eta(x) = \eta(-x)$. Then π operates on measures on $\mathcal{Z}$ in the usual way, and we define $\mu^\pi = \pi\mu$. We also denote by μ_0^π the distribution of the 0'th column of μ^π (and by stationarity the distribution of any column).

Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. For $\eta \in \mathcal{Z}$, we write $G(\eta)$ for the configuration $G(\eta)_{z,i} = G(\eta_{z,i})$. Note that $G(\eta) \in \mathcal{Z}$. An easy yet important observation is that the dynamics of the multi-type TAZRP (and likewise the TASEP) are conserved under a monotone relabelling of the types. (See Lemma 4 and Corollary 1 in Section 4 below.)

We can now state our first main result.

Theorem 1. *The distribution μ^π is an ergodic stationary distribution of the multi-type TAZRP. Any other translation-invariant ergodic stationary distribution is the distribution of $G(\eta)$ where $\eta \sim \mu^\pi$, for some non-decreasing function G from $\mathbb{R}$ to $\mathbb{R}$.*

Our next result is about the stationary measures for the n -type TAZRP. In the n -type TAZRP there are n different classes of particle which may be present at any site. The first-class particles have the highest priority, followed by the second-class, and so on. We may imagine the particles at a site (or column) ordered according to their type, with the highest-priority particles at the bottom. When the clock rings at site x , a particle of the highest priority jumps to site $x + 1$ and positions itself according to its class in column $x + 1$. The n -type TAZRP can be obtained by restricting the multi-type TAZRP to a subset of the set $\mathcal{Z}$ in (7). Let $\mathcal{Z}_n = \mathcal{Z} \cap \mathcal{R}_n^{\mathbb{Z} \times \mathbb{N}_0}$, where $\mathcal{R}_n = \{-1, \dots, -n, -n - 1\}$. Let $\eta(t)$ be the multi-type TAZRP on $\mathcal{Z}$. Then the set $\mathcal{Z}_n$ is closed under the dynamics of $\eta(t)$, that is, if $\eta(0) \in \mathcal{Z}_n$ then $\eta(t) \in \mathcal{Z}_n$ for all $t > 0$. We define the n -type TAZRP to be the multi-type TAZRP restricted to $\mathcal{Z}_n$.

The interpretation is that for $i = 1, 2, \dots, n$, a particle of type i has a label of type $-i \in \mathcal{R}_n$. If the total number of particles of types 1 up to n at a site x in a configuration $\eta \in \mathcal{Z}_n$ is k , then $\eta_{x,i} \geq -n$ for $i \leq k - 1$ and $\eta_{x,i} = -n - 1$ for $i \geq k$. We interpret the label $-n - 1$ as describing a “hole” or “absence” of a particle in the n -type TAZRP. The choice of $\mathcal{R}_n$ is not crucial – one could take any ordered set of size $n + 1$ – but using the set $\mathcal{R}_n$, one can read off the class of the particles by removing the minus sign from the particle label. The TAZRP is the n -type TAZRP for $n = 1$.

Let $\alpha, p \in (0, 1)$. We say a random variable X has geometric distribution with parameter α , denoted $X \sim \text{Geom}(\alpha)$, if $\mathbb{P}(X = k) = (1 - \alpha)\alpha^k$ for $k \geq 0$. We say that X has a Bernoulli–geometric distribution, denoted $X \sim \text{Ber}(p) \text{Geom}(\alpha)$, if

$$\mathbb{P}(X = k) = \begin{cases} (1 - p) & k = 0 \\ p(1 - \alpha)\alpha^{k-1} & k \geq 1. \end{cases}$$

Note that $\text{Ber}(\alpha) \text{Geom}(\alpha)$ is the same as $\text{Geom}(\alpha)$.

For $1 \leq i \leq n$ let us denote by Q_i the number of particles of class i in column 0 of configurations in $\mathcal{Z}_n$.

Theorem 2. *For any translation-invariant ergodic stationary distribution of the n -type TAZRP, with non-zero and finite density of particles of types $1, 2, \dots, n$, there are $\lambda_1, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i < 1$ such that the random variables Q_i are independent, and*

$$Q_i \sim \text{Ber}\left(\frac{\lambda_i}{1 - (\lambda_1 + \dots + \lambda_{i-1})}\right) \text{Geom}(\lambda_1 + \dots + \lambda_i), \quad i > 1 \quad (15)$$

Remark 2. In the case $i = 1$, the empty sum in the denominator in (15) is taken to be 0; as one should expect from the well-known geometric i.i.d. distribution of the TAZRP, or basic results on stationary distributions on stationary distributions of $M/M/1$ queues in series, the distribution of Q_1 is then geometric with parameter λ_1 . In general, the sum of a geometric and an independent Bernoulli–geometric is not geometric, but for particular values of the parameters such relations do hold, and in this case one obtains, also as expected, that $Q_1 + \dots + Q_i$ is geometric for each $1 \leq i \leq n$, with parameter $\lambda_1 + \dots + \lambda_i$. We may also interpret λ_i as the intensity at which particles of type i move from site 0 to site 1. Also λ_i is the probability that the highest-priority particle at site 0 is of type i .

Now that we have stated the result that the distribution of the speed process U is a stationary measure for the multi-type TAZRP dynamics (Theorem 1), we turn to investigating this measure. As each of the second-class particles has speed, the column of speeds $\{U_{0,i}\}_{i \in \mathbb{N}_0}$ can be thought of as a marked point process where the points are the set of speeds in $[0, 1]$ attained by the particles at column zero and the mark associated with the point $v \in [0, 1]$ is the number of particles attaining a specific speed. The following result characterizes the distribution of a column of the speed process.

Theorem 3. *Let U be the speed process. The distribution of $\{U_{0,i}\}_{i \in \mathbb{N}_0}$ is a marked Poisson process on $[0, 1]$, with intensity $\frac{1}{2x}$ and mark distribution $\text{Geom}(1 - \sqrt{x})$. In particular, for a fixed $j > 0$, the sequence of speeds $\{U_{0,i}\}_{i=j+1}^\infty$ conditioned on $U_{0,j}$ is independent of $\{U_{0,i}\}_{i=0}^{j-1}$.*

Theorem 3 shows that the set of values of $U_{0,\cdot}$ accumulates at 0. We also see that conditioning on some particle attaining the speed v , the probability of finding another particle with speed v is positive. Moreover, it gives a Markovian property for the column of speeds.

Note that a related marked-Poisson-process structure was recently found by Fan and Seppalainen [8] in the description of joint distributions of Busemann functions for the last-passage percolation model (see for example their Theorem 3.4).

Our next result states the joint distribution of two second-class particles starting at column 0.

Theorem 4. *Let U be the TAZRP speed process and let $f(x) = 1 - \sqrt{x}$. Then for $i < j$ and $x_1 > x_2$*

$$\mathbb{P}(x_1 \geq U_{0,i}, x_2 \geq U_{0,j}) = 1 - f(x_1)^{i+1} - f(x_2)^{j+1} \left(1 - \left(\frac{f(x_1)}{f(x_2)} \right)^{i+1} \right),$$

and

$$\mathbb{P}(U_{0,i} = U_{0,j} \in dx) = (i+1) \frac{f(x)^j}{2\sqrt{x}} dx.$$

Theorem 4 again says that there is a positive probability for two second-class particles at a column to have the same speed; we also see that conditional on having the same speed, the distribution of the speed has a density.

In general, obtaining results on the joint distribution of two columns is hard. We have the following result in this direction.

Proposition 1. *Let U be the speed process of the TAZRP, and let $f(x) = 1 - \sqrt{x}$ and $j, k \in \mathbb{N}$. Then,*

$$\begin{aligned} \mathbb{P}(U_{0,0} > x_1, U_{-1,j-1} > x_1 > U_{-1,j} > \cdots > U_{-1,j+k-1} > x_2) \\ = (f(x_2) - f(x_1)) f(x_1)^j f(x_2)^k. \end{aligned} \tag{16}$$

Take two particles, $p_{0,j}$ and $p_{i,k}$ where $0 < i$. Both particles develop speed v and u respectively. On the event that $v = u$, what is the probability that $p_{0,j}$ overtakes $p_{i,k}$? that is, what is the probability that $X_{0,j}(t) > X_{i,k}(t)$ for some $t > 0$? Our next result shows that overtaking occurs with probability 1.

Theorem 5. *Let U be the TAZRP speed process. Suppose $i > 0$ and condition on $U_{0,j} \geq U_{i,k}$. Then with probability 1, $p_{0,j}$ overtakes $p_{i,k}$.*

3. The coupling between TASEP and TAZRP

3.1. The basic coupling

We begin by describing a coupling between the exclusion and zero-range process on $\mathbb{Z}$. There are, in fact, two natural ways to define such a coupling, particle-hole and particle-particle, and both work for ASEP-AZRP as well. In the particle-hole coupling, each particle in the TASEP configuration will correspond to a column in the TAZRP, and consecutive holes between particles in the TASEP correspond to particles sitting in the same column (the column that corresponds to the first particle to their left). The advantage of this coupling is that clocks on the particles in the TASEP correspond naturally to clocks on the sites (columns) in the TAZRP, though the direction of movement is reversed. This coupling was originally introduced by Kipnis in [15], where he used it to relate several observables between TASEP and TAZRP, e.g. the position of a tagged particle at time t in the TASEP with the current through a bond up to time t in the TAZRP.

We will be more interested in the particle-particle coupling as it can be generalized to deal with second-class particles. In the particle-particle coupling, each hole in the TASEP configuration corresponds to a column in the TAZRP, and the particles between consecutive holes become particles sitting in the column corresponding to the first hole to their right. As this is the coupling we plan to use, we describe it more rigorously. Let ξ_t be a TASEP configuration. Denote by $\{y_i(0)\}_{i \in \mathbb{Z}}$ the positions of all the holes at time 0, ordered so that $\cdots < y_{-1}(0) < y_0(0) < y_1(0) < \cdots$, with $y_0(0)$ denoting the first hole in position > 0 at time 0. Let $y_i(t)$ denote the position of the i 'th hole at time t . We construct a configuration η_t from ξ_t by setting $\eta_t(i) = y_{i+1}(t) - y_i(t) - 1$. It is not hard to check that under this coupling η_t follows TAZRP dynamics, and that, in fact, the clocks on the columns correspond to the clocks of the particles in the TASEP that may indeed jump. We denote by $\Phi : \mathcal{Y} \rightarrow \mathcal{X}$ (Figure 3) the mapping between TASEP and TAZRP configurations described above.

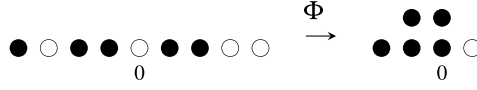
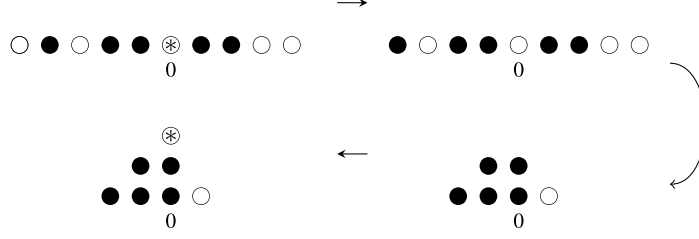
Fig. 3. The mapping Φ between the TASEP and the TAZRP.

Fig. 4. The four steps in mapping a TASEP configuration with a second-class particle to a TAZRP configuration with a second-class particle.

3.2. The coupling with second-class particle

In [14], Gonçalves generalized the coupling discussed in the preceding subsection and introduced a coupling between the TASEP and the TAZRP where the configurations in both dynamics have one second-class particle. Let $\xi_0 \in \mathcal{Y}$ be a TASEP configuration with a second-class particle q_2 , and $i \in \mathbb{Z}$ be the index of the first hole to the right of q_2 . Replace the second-class particle q_2 with a hole to obtain the configuration $\xi' \in \mathcal{Y}$, so that $\Phi(\xi') = \eta' \in \mathcal{X}$. Finally, put a second-class particle p_2 on top of the i 'th column in η' to obtain η_0 . The mapping (Figure 4) just defined is a bijection between TASEP and TAZRP configurations with second-class particles so that throughout the dynamics of ξ_t (TASEP starting from ξ_0) and η_t (TAZRP starting from η_0) the position of the second-class particle in the TAZRP can tell the flux of holes seen by the second-class particle in the TASEP. The position, and hence the speed of the second-class particle p_2 , can be found by considering the flux of holes passing across the second-class particle q_2 in the TASEP configuration. Let

$$H_2^{\text{tasep}}(t, \xi_0) = \inf\{i : \text{the hole } y_i \text{ is to the right of } q_2\} \quad (17)$$

be the number of holes that have crossed the second-class particle q_2 under TASEP dynamics starting from ξ_0 (here we assume that the q_2 was positioned at time $t = 0$ between y_{-1} and y_0). It is not hard to verify that $X_{p_2}(t) = H_2^{\text{tasep}}(t, \xi)$, i.e. the position of the particle p_2 equals the flux of holes crossing q_2 . Let ξ_0 be a Riemann initial data; that is, the limits

$$\rho := \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=-m}^{-1} \xi_0(k) \quad \text{and} \quad \lambda := \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{k=0}^m \xi_0(k) \quad (18)$$

exist. It was shown in [12, Proposition 2.2 and Theorem 3] that

$$\lim_{t \rightarrow +\infty} \frac{H_2^{\text{tasep}}(t, \xi_0)}{t} = \left(\frac{1 + \mathcal{U}}{2} \right)^2 \quad \text{almost surely,} \quad (19)$$

where $\mathcal{U}$ is the speed of the second-class particle q_2 . It now follows that the speed of the second-class particle p_2 starting from a Riemann initial condition equals $(\frac{1+\mathcal{U}}{2})^2$. In [12], it was also shown that when the initial configuration is i.i.d. on either side of the origin, with density λ to the left and density ρ to the right where $\rho < \lambda$, then $\mathcal{U}$ has uniform distribution on the interval $[1 - 2\lambda, 1 - 2\rho]$. In this paper, we are mostly interested in the case where $\lambda = 1$ and $\rho = 0$ which corresponds, by the coupling above, to the TAZRP starting with infinitely many particles at site -1 , a second-class particle at site 0 and holes on all positive sites. In this case, the distribution of the speed U of p_2 is given by

$$\mathbb{P}(U \leq v) = \sqrt{v} \quad v \in [0, 1]. \quad (20)$$

The coupling described above between configurations with second-class particles can be extended to configurations with finitely many second-class particles up to the point in time where one second-class particle attempts a jump to a site where there is another second-class particle. Let ξ be a TASEP configuration with m second-class particles such that between the positions of any two second-class particles there is at least one hole. This corresponds in the TAZRP to the case where each column has at most one second-class particle. First we register in $\{i_j\}_{j=1}^m$ the indices of the holes to the

right of each second-class particle. Then replace the second-class particles with holes, apply the mapping Φ on the new configuration and finally place second-class particles at the top of columns $i_1, \dots, i_m$. We will use this coupling in the proof of Theorem 5.

Remark 3. In [4], in the context of last-passage percolation, Cator and Pimentel obtained the distribution of the speed of a second-class particle in any Riemann initial condition. Using the coupling in Section 3.1 one can translate the results in [4] to results for the distribution of the speed of a second-class particle in the TAZRP starting from a larger set of initial conditions.

Before we turn to the proof of Lemma 1, we note that it is a straightforward consequence of (19). Nevertheless, for the sake of self-containment we give here a proof that uses only the results in [12], that the second-class particle positioned at the origin between all particles to the left and holes to its right has a speed with probability 1, and that this speed is > -1 a.s.

Proof of Lemma 1. *Step 1:* We first show that the particle $p_{0,0}$ (the particle located at the bottom of the 0'th column) has a speed. By calling $p_{0,0}$ a 2nd class particle, all particles to its left are 1st class compared to it as they are of higher class. Similarly all particles to its right or above it are seen as 3rd class (holes), and using the coupling of the TAZRP with the TASEP we get to the TASEP configuration

$$\dots 111123333 \dots$$

It was shown in [12,20] that particle 2 has speed $\mathcal{U}$ a.s., and as explained in Section 3.2, the speed $U_{0,0}$ equals

$$U_{0,0} = \left(\frac{1 + \mathcal{U}}{2} \right)^2. \quad (21)$$

And in particular it is strictly positive with probability 1.

Step 2: We now claim that particle $p_{0,i}$ also develops a strictly positive speed for all $i > 0$. Consider the event that $i - 1$ particles jump from column -1 to 0 before any other jump is made in columns -2 and 0. This event has positive probability, and if it happens we reach a configuration where at site 0 we have

$$\begin{array}{c} \vdots \\ p_{0,0} \\ p_{-1, -(i-1)} \\ \vdots \\ p_{-1,0} \end{array}$$

Under this event $p_{0,0}$ sees the same environment as $p_{0,i}$ sees at η^* (recall (12)), that is, all the particles below it or to its left are first class particles while all particles to its right or above it are holes. Thus if $p_{0,i}$ has positive probability not to have a strictly positive speed under η^* , then so does $p_{0,0}$, contradicting the fact that with probability 1 $p_{0,0}$ has a strictly positive speed. $\square$

We end this section with a proof of Lemma 2.

Proof of Lemma 2. We will use the mass transport principle (See e.g. [18] Chapter 8). For each $k \geq 0$ and any t define a mass-transport function $f_t^k : \mathbb{Z} \times \mathbb{Z} \rightarrow [0, \infty]$ by

$$f_t^k(z, y) = \begin{cases} \sum_{i=0}^k 1_{X_{z,i}(t) > y} & y \geq z \\ 0 & y < z. \end{cases}$$

One may think of this as each of the first k particles in every column sending a mass of 1 to each position they jump from up to time t . Define $Z_t^k = \frac{1}{t} \sum_{y=-\infty}^{\infty} f_t^k(0, y)$ and $Y_t^k = \frac{1}{t} \sum_{z=-\infty}^{\infty} f_t^k(z, 0)$. Y_t^k is just the averaged rate at which particles of (initial) height $\leq k$ jumped from 0, and therefore $\mathbb{E}[Y_t^k] \leq 1$ and $\lim_{k \rightarrow \infty} \mathbb{E}[Y_t^k] = 1$ for all t . On the other hand, $\lim_{t \rightarrow \infty} Z_t^k = \sum_{i=0}^k U_{0,i}$. Since the distribution of f_t^k is translation invariant, the mass-transport principle gives that $\mathbb{E}[Z_t^k] = \mathbb{E}[Y_t^k]$, and therefore $\sum_{i=0}^{\infty} \mathbb{E}[U_{i,0}] = 1$.

We now turn to $\inf_{i \geq 0} U_{z,i}$. Fix any $\epsilon > 0$. Since $U_{z,i} \geq 0$, by Markov's inequality $\mathbb{P}(U_{z,i} > \epsilon) < \frac{\mathbb{E}[U_{z,i}]}{\epsilon}$, and therefore $\sum_{i \geq 0} \mathbb{P}(U_{z,i} > \epsilon) < \infty$. By the Borel–Cantelli Lemma this a.s. happens only for finitely many i -s, and therefore $\mathbb{P}(\inf_{i \geq 0} U_{z,i} \leq \epsilon) = 1$. Since ϵ was arbitrary, we are done. $\square$

4. Stationarity of the TAZRP speed process

The key to understanding why the distribution of U (or more precisely, its reflection) gives a stationary distribution for the multi-type TAZRP is to understand the effect of a small change to the initial condition η^* on the speed process.

Specifically, in the next lemma we consider how the speed process starting from $\sigma_0 \eta^*$ is different from that starting from η^* , where σ_x is the operator given in (9) and η^* is the initial condition given in (12). More precisely, consider the two initial conditions $\eta(0) = \eta^*$ and $\eta'(0) = \sigma_0 \eta^*$. Let $\mathcal{T}$ be a Poisson process on $\mathbb{Z} \times \mathbb{R}_+$, representing the different clocks on the sites of $\mathbb{Z}$. Apply now the Harris construction with $\mathcal{T}$ to the two initial conditions $\eta(0)$ and $\eta'(0)$. Let U be the speed process associated with the process η as defined in Definition 1. We define the speed process U' to be the speed process associated with the process $\eta'(t)$, that is

$$U'_{z,i} = \lim_{t \rightarrow \infty} t^{-1} X_{z,i}(t), \quad (22)$$

where we assume the dynamics start from η' . It is important to note that particles in $\eta(0)$ and $\eta'(0)$ are the same particles indexed by $\mathbb{Z} \times \mathbb{N}$ according to their position in $\eta(0)$. More precisely, the particle associated with $\eta^*(0, 0)$ is identified with $(0, 0)$ and its class is the number $\eta^*(0, 0)$. The operation σ_0 will move the particle $(0, 0)$ and place it at the bottom of column 1 (this is due to the initial order imposed on η^* where every particle in the 0'th column is stronger than any particle in column 1). However it is important to note that we still identify this particle by its initial position in η^* , i.e. $(0, 0)$. This means that the position of the particle $(0, 0)$ in η^* is $X_{0,0}(0) = 0$ whereas its position in $\sigma_0 \eta^*$ is $X'_{0,0}(0) = 1$. Hence, the arrays $\{U_{z,i}\}_{(z,i) \in \mathbb{Z} \times \mathbb{N}_0}$ and $\{U'_{z,i}\}_{(z,i) \in \mathbb{Z} \times \mathbb{N}_0}$ register the speed of particle $(0, 0)$ at position $(0, 0)$ of the array, despite the fact that $X'_{0,0}(0) = 1$.

Lemma 3. *Let η and η' be two TAZRPs defined by the Harris construction with initial condition η^* and $\sigma_0 \eta^*$ respectively, and a Poisson process $\mathcal{T}$ on $\mathbb{Z} \times \mathbb{R}_+$. Let U and U' be the TAZRP speed processes associated with η and η' respectively, then*

$$\sigma_0^* U = U'. \quad (23)$$

In order to prove the lemma we shall make use of a process we call the sorting process. The configuration $\bar{\eta}$ of the sorting process compares two initial configurations η and ξ of the multi-type TAZRP. Applying the dynamics of the sorting process on the initial configuration keeps track of the development of the two initial configurations η and ξ when one applies on them the same Poisson clocks in the Harris construction.

For $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ we write $(x_2, y_2) \leq (x_1, y_1)$ whenever $x_1 \geq x_2$ and $y_1 \geq y_2$. Let

$$\mathcal{W} = \{\bar{\eta} \in (\mathbb{R}^2)^{\mathbb{Z} \times \mathbb{N}_0} : \bar{\eta}_{z,i} \leq \bar{\eta}_{z,j}, \text{ for all } j \leq i \text{ and } z\}. \quad (24)$$

Note that $\mathcal{W}$ is simply an array indexed by $\mathbb{Z} \times \mathbb{N}_0$ that contains pairs of real numbers. We say that $(x_1, y_1), (x_2, y_2)$ are ordered if either $(x_1, y_1) \leq (x_2, y_2)$ or $(x_1, y_1) \geq (x_2, y_2)$, otherwise we say that they are unordered. Let $\bar{\eta} \in \mathcal{W}$, and for $k \in \{1, 2\}$ let $\bar{\eta}_{z,i}^k$ denote the k 'th component of the pair $\bar{\eta}_{z,i}$. We attach independent Poisson clocks of rate 1 to each site (column) $x \in \mathbb{Z}$, at the ring of the clock of the column x , the largest (with respect to the order on pairs) pair sitting at the bottom of the column, jumps to the column to its right where the pairs rearrange into elementwise order. More precisely, if the pair $\bar{\eta}_{z,0}$ jumps to column $z + 1$, then we arrange the sets of numbers

$$\begin{aligned} A &= \bar{\eta}_{z,0}^1 \cup \{\bar{\eta}_{z+1,i}^1\}_{i \in \mathbb{N}_0} \\ B &= \bar{\eta}_{z,0}^2 \cup \{\bar{\eta}_{z+1,i}^2\}_{i \in \mathbb{N}_0}, \end{aligned} \quad (25)$$

according to their order to obtain the decreasing sequences $\{a_i\}_{i=0}^\infty$ and $\{b_i\}_{i=0}^\infty$. Then replace the column $\bar{\eta}_{z+1,\cdot}$ by a new column whose i 'th element is (a_i, b_i) . We call this process on $\mathcal{W}$ the *sorting process*. We say the pairs $(x_1, y_1) = \bar{\eta}_{z,0}(t-)$ and $(x_2, y_2) \in \bar{\eta}_{z+1,\cdot}(t-)$ interact if the jump of the pair (x_1, y_1) at time t to column $z + 1$ results in $(x_2, y_2) \notin \bar{\eta}_{z+1,\cdot}(t)$. Note that (x_1, y_1) interacts only with pairs in $\bar{\eta}_{z+1,\cdot}(t-)$ that are unordered with respect to itself (see Figure 5). We make the following observations:

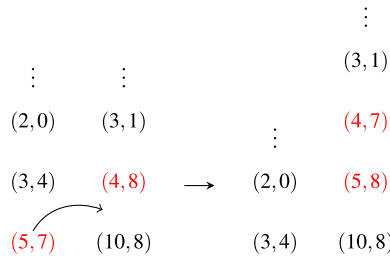


Fig. 5. Illustration of the sorting process dynamics. The particle (5, 7) interacts with the only particle that it is not ordered with respect to, (4, 8), to create two new particles - (4, 7) and (5, 8).

- (1) If (x, y) is a pair in $\bar{\eta}$ that is ordered with respect to all other pairs in $\bar{\eta}$, then (x, y) will not interact throughout the dynamics.
- (2) If $\eta, \xi \in \mathcal{Z}$, then $\bar{\eta}_{z,i} = (\eta(z, i), \xi(z, i)) \in \mathcal{W}$.

Proof of Lemma 3. Define $\bar{\eta}$ by $\bar{\eta}_{z,i} = (\eta(z, i)^*, \sigma_0 \eta(z, i)^*)$, where η^* is as in (12), and let $\bar{\eta}_{z,i}(t)$ be the sorting process starting from the initial condition $\bar{\eta}$. The idea of the proof is that the sorting process marginals $\bar{\eta}^1$ and $\bar{\eta}^2$ are the multi-type TAZRP with initial conditions η^* and $\sigma_0 \eta^*$ respectively, this allows us to compare the position of the same particle in the two processes. First note that all pairs in $\bar{\eta}(0)$ are ordered with respect to any other pair except the pairs in $\bar{\eta}_{0,\cdot}(0)$ and the pair $\bar{\eta}_{1,0}(0)$. It follows that pairs that are not in $\bar{\eta}_{0,\cdot}(0) \cup \bar{\eta}_{1,0}(0)$ do not interact throughout the dynamics. Let

$$A = \{i : \bar{\eta}_{0,i}(0) \text{ interacts with a pair } (x, y) \text{ s.t. } x = \eta^*(1, 0)\}, \quad (26)$$

and let

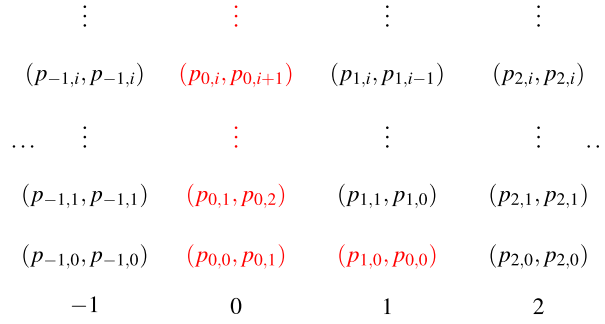
$$i_{\text{fast}} = \begin{cases} -1, & A = \emptyset \\ \sup A, & A \neq \emptyset. \end{cases}$$

Note that if $U_{0,i} < U_{1,0}$ then $i_{\text{fast}} < i$ as particle $p_{0,i}$ cannot overtake particle $p_{1,0}$ and so $\bar{\eta}_{0,i}(0)$ cannot interact with any pair of the form $(\eta_{1,0}^*, y)$ for some $y \in \eta_{0,\cdot}^*$. As $\lim_{i \rightarrow \infty} U_{0,i} = 0$ we conclude that $i_{\text{fast}} \neq \infty$. On the event that $i_{\text{fast}} \geq 0$, the pairs $\{\bar{\eta}_{0,i}\}_{i=0}^{i_{\text{fast}}}$ will interact with particles whose first coordinate is $p_{1,0}$ according to their order. Once the pair $\bar{\eta}_{0,0}$ has interacted with $\bar{\eta}_{1,0}$ at some time $t_0 > 0$, then the two pairs are ordered into two new pairs $(p_{0,0}, p_{0,0})$ and $(p_{1,0}, p_{0,1})$. The pair $(p_{0,0}, p_{0,0})$ is ordered w.r.t. all pairs in $\bar{\eta}(t_0)$ and therefore will not interact at later times $t > t_0$. The next interaction (if $i_{\text{fast}} > 0$) will be between the pairs $\bar{\eta}_{0,1} = (p_{0,1}, p_{0,2})$ and $(p_{1,0}, p_{0,1})$ at some time $t_1 > t_0$. The interaction will lead to the formation of the pairs $(p_{0,1}, p_{0,1})$ and $(p_{1,0}, p_{0,2})$ at time t_1 and we see that the pair $(p_{0,1}, p_{0,1})$ is ordered w.r.t. all other pairs and so will not interact again (see Figure 6). We continue in the same way until all pairs $\{(p_{0,i}, p_{0,i})\}_{i=0}^{i_{\text{fast}}}$ have formed by time $t_{i_{\text{fast}}}$ as well as the pair $(p_{1,0}, p_{0,i_{\text{fast}}+1})$. By the definition of i_{fast} , no interactions will occur at time $t > t_{i_{\text{fast}}}$. More precisely, if column z rings at time $t > t_{i_{\text{fast}}}$, then $\{\bar{\eta}_{z+1,i}(t)\}_{i \geq 0} = \{\bar{\eta}_{z+1,i}(t-)\}_{i \geq 0} \cup \bar{\eta}_{z,0}(t-)$. Now, let X_t and X'_t be the processes that keep track of the horizontal position of the different particles in η and η' respectively i.e.

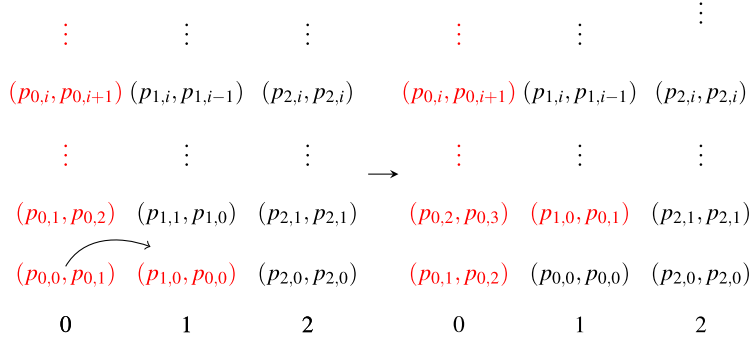
$$\begin{aligned} X_{z,i}(t) = n & \iff p_{z,i} \in \eta(n, \cdot)(t) \\ X'_{z,i}(t) = n & \iff p'_{z,i} \in \eta'(n, \cdot)(t). \end{aligned} \quad (27)$$

This implies that for $t > t_{i_{\text{fast}}}$

$$\begin{aligned} X_{z,i}(t) &= X'_{z,i}(t) \quad \text{if } z \notin \{0, 1\} \vee (z = 0, 0 \leq i \leq i_{\text{fast}}) \\ X_{z,i+1}(t) &= X'_{z,i}(t) \quad \text{if } z = 1, i > 0 \\ X_{0,i} &= X'_{0,i+1} \quad \text{if } i > i_{\text{fast}} + 1 \\ X_{0,i_{\text{fast}}+1}(t) &= X'_{1,0}(t). \end{aligned} \quad (28)$$



(A) The initial configuration $\bar{\eta}$. Only the pairs in red are not ordered. Any other couple in the configuration is ordered with respect to all other pairs.



(B) One step in the sorting process starting from $\bar{\eta}$. The pairs $(p_{0,0}, p_{0,1})$ and $(p_{1,0}, p_{0,0})$ interact and give rise to two new pairs in column 1 - $(p_{0,0}, p_{0,0})$, which is ordered with respect to any other particle in the configuration, and $(p_{1,0}, p_{0,1})$ which is unordered with respect to any pair in column 0.

Fig. 6. The sorting process.

Multiplying by t^{-1} and letting t go to infinity we obtain

$$\begin{aligned}
 U_{z,i} &= U'_{z,i} \quad \text{if } z \notin \{0, 1\} \vee (z = 0, 0 \leq i \leq i_{\text{fast}}) \\
 U_{z,i+1} &= U'_{z,i} \quad \text{if } z = 1, i > 0 \\
 U_{0,i} &= U'_{0,i+1} \quad \text{if } i > i_{\text{fast}} + 1 \\
 U_{0,i_{\text{fast}}+1} &= U'_{1,0}.
 \end{aligned} \tag{29}$$

One can now verify that the relations in (29) between U and U' as arrays indexed by $\mathbb{Z} \times \mathbb{N}_0$ are equivalent to (23) as configurations in $\mathcal{W}$, and the result is proved (That $i_{\text{fast}} + 1 = i_{\text{sort}^0}$ is a consequence of Theorem 5, but we do not need it here). $\square$

We are now ready for the proof of Theorem 1. We defer the proof of the uniqueness of μ^π to Section 6.

Proof of Theorem 1 without uniqueness. Let $\mathcal{T}_0$ be a Poisson process on $\mathbb{Z} \times \mathbb{R}$ with rate 1. One can think of a realization of $\mathcal{T}_0$ as a set in $\mathbb{Z} \times \mathbb{R}$, so that for every $x \in \mathbb{Z} \times \mathbb{R}$ and any realization of $\mathcal{T}_0$ the set

$$\mathcal{T}_0 + x$$

is well defined. Let $\mathcal{T}_s = \mathcal{T}_0 + 0 \times (0, s)$ be the translation of $\mathcal{T}_0$ by s units of time. Also define $\mathcal{T}_s^+ = \mathcal{T}_s \cap \mathbb{Z} \times \mathbb{R}_+$, the restriction of $\mathcal{T}_s$ to $\mathbb{Z} \times \mathbb{R}_+$. Define $U(s)$ to be the speed process constructed through the Harris construction with initial condition η^* and the Poisson process $\mathcal{T}_s^+$. For each $s > 0$, $U(s)$ has distribution μ , and it is enough to show that $U(s)$ satisfies the TAZRP dynamics in order to show stationarity of the measure μ^π . Starting from $U(0)$, adding an infinitesimal time s adds, at each site i , the operator σ_i at rate 1. According to Lemma 3 this should result in applying σ_i^* to $U(0)$ to obtain $U(s)$ at rate one. It is straightforward to see that $\pi \sigma_i^* \eta = \sigma_{-i-1} \pi \eta$ which implies that the process

$\pi U(s)$ is defined through the generator (8) and the initial condition $\pi U(0) = \mu^\pi$ which implies the stationarity of μ^π . To see that μ^π is ergodic, it is enough to note that μ^π is generated by applying some deterministic mapping F on the Poisson process $\mathcal{T}_0$, which is ergodic w.r.t. the translation operator τ , and that $\tau F(\mathcal{T}_0) = F(\tau \mathcal{T}_0)$. $\square$

Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. Let $\eta \in \mathcal{Z}$, we write $G(\eta)$ for the configuration $G(\eta)_{z,i} = G(\eta_{z,i})$. Note that $G(\eta) \in \mathcal{Z}$. An easy yet important observation is that the dynamics of the multi-type TAZRP (and likewise the TASEP) are conserved under a monotone relabelling of the types.

Lemma 4. *Let $G : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing function. Let $\eta \in \mathcal{Z}$, and let $\mathcal{T}$ be a Poisson process on $\mathbb{Z} \times \mathbb{R}_+$ and consider $\eta(t)$ and $\eta_G(t)$, the multi-type TAZRP defined through the Harris construction with $\mathcal{T}$ and the initial conditions η and $G(\eta)$ respectively. Then*

$$G(\eta(t)) = \eta_G(t) \quad \forall t \geq 0. \quad (30)$$

Proof. By the definition of η_G , (30) holds for $t = 0$. Now, following the Harris construction, it is enough to show, that

$$\sigma_i G(\eta) = G(\sigma_i \eta) \quad \text{for every } i \in \mathbb{Z}, \quad (31)$$

which is not hard to verify. $\square$

Corollary 1. *Let $G : \mathbb{R} \rightarrow \mathbb{R}_n$ be an increasing function. Then the distribution of the process $G(\pi U(\cdot))$ is a stationary and ergodic distribution for the n -type TAZRP.*

Proof. Since G is increasing, $G(\pi U(t)) \in \mathcal{Z}_n$ for every $t > 0$. By Lemma 4, the stationarity of $\pi U(\cdot)$ implies the stationarity of $G(\pi U(\cdot))$. Moreover, since $\tau G(\pi U(\cdot)) = G(\tau \pi U(\cdot))$ we see that the ergodicity of $U(\cdot)$ implies that of $G(\pi U(\cdot))$. $\square$

One can use the one-point marginals of the 1-type TAZRP along with Corollary 1 to obtain the one-point marginal of U .

Lemma 5. *Let U be the TAZRP speed process. Then, for every $j \in \mathbb{N}_0$*

$$\mathbb{P}(U_{0,j} \leq v) = 1 - (1 - \sqrt{v})^{j+1} \quad v \in [0, 1]. \quad (32)$$

Proof. Let

$$G_v(x) = \begin{cases} -1 & x > v \\ -2 & x \leq v. \end{cases} \quad (33)$$

Note that by Corollary 1, $G_v(U)$ is a stationary and ergodic measure of the 1-type TAZRP and that

$$\begin{aligned} \mathbb{P}(U_{0,0} \leq v) &= \mathbb{P}(\#\{i : G(U)_{0,i} = -1\} = 0) \\ &= \mathbb{P}_{\mu_\alpha}(\eta_0 = 0) \\ &= \frac{1}{1 + \alpha}, \end{aligned}$$

where in the second equality $\alpha > 0$ to be determined later and we used the well-known unique stationary ergodic measures for the TAZRP mentioned in (6). By (20) we see that $\mathbb{P}(U_{0,0} \leq v) = \sqrt{v}$, and therefore that

$$\alpha = \frac{1 - \sqrt{v}}{\sqrt{v}}. \quad (34)$$

Similarly, we see that

$$\begin{aligned} \mathbb{P}(U_{0,j} \leq v) &= \mathbb{P}_{\mu_\alpha}(\eta_0 \leq j) \\ &= 1 - \left(\frac{\alpha}{1 + \alpha} \right)^{j+1}. \end{aligned} \quad (35)$$

Plugging (34) in (35) we obtain the result. $\square$

Remark 4. Lemma 5 implies the result in [14, Theorem 2.1, case $\rho = \infty$]. Indeed, the equality there can be written with our notation and by using the monotonicity of the speeds of particles in one column, as

$$\lim_{t \rightarrow \infty} \sum_{i=0}^{\infty} \mathbb{P}(X_{0,i}(t) \geq ut) = \frac{1 - \sqrt{u}}{\sqrt{u}},$$

which follows easily by using (32).

5. Stationary measures for the n -type TAZRP

5.1. One-column distribution in stationarity

Our approach to investigating the n -type TAZRP is through thinking of each column of the n -type TAZRP as a queue. Such a queue has services at times of a Poisson process of rate 1, and its arrival process contains particles of types from 1 to n . The server attends to particles according to their class; when a service occurs, the particle with the highest priority is served (if any particle is present), and departs from the queue.

Let λ_i be the intensity of arrivals of type i . We are interested in the case where the behaviour of the queue is stationary in time and ergodic, with a finite average number of particles of each type present in the queue, and so we need

$$\sum_{i=1}^n \lambda_i < 1. \quad (36)$$

We wish to consider stationary distributions of the n -type TAZRP which are translation-invariant. In this case the departure process from the queue (say, the process of particles moving from site x to site $x + 1$) has the same distribution as the arrival process to the queue (say, the process of particles moving from site $x - 1$ to site x). In this sense we say that the distribution of the arrival process is a fixed point for the queueing server. Using a coupling approach analogous to that used by Mountford and Prabhakar [21], one can show that for any $\lambda_1, \dots, \lambda_n$ satisfying (36), there is a unique ergodic fixed point with intensity λ_i of arrivals of type i (see [19] for discussion). We denote this process by $F^{(n)}$, or $F_{\lambda_1, \dots, \lambda_n}^{(n)}$ when we need to emphasise the dependence on the arrival intensities.

Let us mention a few immediate properties of the processes $F^{(n)}$:

- By Burke's Theorem, the process $F_{\lambda_1}^{(1)}$ is a Poisson process of rate λ_1 .
- More generally, again by Burke's Theorem, for each i , the combined process of all points in $F_{\lambda_1, \dots, \lambda_n}^{(n)}$ of types $1, \dots, i$ is a Poisson process with rate $\sum_{j=1}^i \lambda_j$.
- The process $F_{\lambda_1, \dots, \lambda_n}^{(n)}$ restricted to types $1, \dots, n - 1$, i.e. removing the type- n points, gives the process $F_{\lambda_1, \dots, \lambda_{n-1}}^{(n-1)}$.

The following proposition, which is the starting-point of our analysis of n -type equilibrium distributions, shows that $F^{(n)}$ can be obtained by feeding $F^{(n-1)}$ into a queue with service rate $\sum_{j=1}^n \lambda_j$. It was shown as a by-product of the construction of the multi-type Hammersley process by Ferrari and Martin in [11], and more directly using interchangeability properties of queues by Martin and Prabhakar in [19].

Proposition 2. Consider an exponential server with rate $\sum_{i=1}^n \lambda_i$, and an arrival process with distribution $F^{(n-1)}$. Take the departure process and add to it a point of type n whenever the queue has an unused service. The resulting output process has distribution $F^{(n)}$.

For $0 < s \leq 1$ we write $\mathbb{P}_{\lambda_1, \dots, \lambda_n}^{(s)}$ for the distribution of the vector $(Q_1, \dots, Q_n)$, where Q_i is the number of particles of type i at some fixed time in the queue with arrival process $F_{\lambda_1, \dots, \lambda_n}^{(n)}$ and with an exponential server of rate s . Where there is no room for confusion we abbreviate by $\mathbb{P}^{(s)}$.

Remark 5. For a fixed $1 \leq i \leq n$, let $c = \sum_{j=1}^i \lambda_j$. Note that the distribution of $\mathbb{P}_{\lambda_1, \dots, \lambda_i}^{(c)}$ is equal to that of $\mathbb{P}_{\lambda_1/c, \dots, \lambda_n/c}^{(1)}$ restricted to $(Q_1, \dots, Q_i)$.

Proof of Theorem 2. To prove Theorem 2, we need to show that under $\mathbb{P}^{(1)}$, the distribution of $Q_1, \dots, Q_n$ is that given by (15). The proof of (15) is by induction on n and as it is a bit technical, we first prove the theorem for the case where $n = 2$. We then continue to prove the induction for general n .

As observed at Remark 2, the result for $n = 1$ is a well-known property of $M/M/1$ queues.

Fix $a \geq 0$ and $b > 0$. Define an event A_ϵ as follows: the process $F^{(2)}$ contains a 1's followed by b 2's within the time interval $(0, \epsilon)$. As ϵ gets small this event becomes unlikely; we will look at the dominant contribution to the probability computed in two different ways.

Firstly, by definition of $F^{(2)}$ as a fixed point, $F^{(2)}$ is the output process of a rate-1 server with arrival process also distributed as $F^{(2)}$, and hence with queue distributed as $\mathbb{P}_{\lambda_1, \lambda_2}^{(1)}$. If ϵ is very small, the dominant way to get the event A_ϵ is not to rely on any arrivals to the queue, but to suppose that the queue already contains precisely a 1's and at least b 2's at time 0, and then that we see $a + b$ services before time ϵ . The probability of this event will decay as ϵ^{a+b} and any other way of achieving it decays quicker. Since the rate of service is 1, we get

$$\mathbb{P}(A_\epsilon) \sim \mathbb{P}_{\lambda_1, \lambda_2}^{(1)}(Q_1 = a, Q_2 \geq b) \frac{\epsilon^{(a+b)}}{(a+b)!}, \quad (37)$$

where by $f(\epsilon) \sim g(\epsilon)$ we mean that $f(\epsilon)/g(\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0$.

Alternatively, by Proposition 2, $F^{(2)}$ is the output process of rate- $(\lambda_1 + \lambda_2)$ server fed by $F^{(1)}$ (which is just a Poisson process of rate λ_1), with unused services designated as type-2 departures. In terms of such a queue, the dominant way to get the event A_ϵ as $\epsilon \rightarrow 0$ is for the queue to contain precisely a 1's at time 0, and then to see $a + b$ services before time ϵ . Again this is better than relying on any new arrivals to the queue. In this case we get

$$\mathbb{P}(A_\epsilon) \sim \mathbb{P}_{\lambda_1}^{(\lambda_1 + \lambda_2)}(Q_1 = a) \frac{(\epsilon(\lambda_1 + \lambda_2))^{(a+b)}}{(a+b)!}. \quad (38)$$

Comparing (37) and (38) we get

$$\begin{aligned} \mathbb{P}^{(1)}(Q_1 = a, Q_2 \geq b) &= (\lambda_1 + \lambda_2)^{(a+b)} \mathbb{P}_{\lambda_1}^{(\lambda_1 + \lambda_2)}(Q_1 = a) \\ &= (\lambda_1 + \lambda_2)^{(a+b)} \mathbb{P}_{\lambda_1/(\lambda_1 + \lambda_2), \lambda_2/(\lambda_1 + \lambda_2)}^{(1)}(Q_1 = a) \\ &= (\lambda_1 + \lambda_2)^{(a+b)} \left(1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}\right) \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^a \\ &= (1 - \lambda_1) \lambda_1^a \frac{\lambda_2}{1 - \lambda_1} (\lambda_1 + \lambda_2)^{b-1} \\ &= \mathbb{P}^{(1)}(Q_1 = a) \frac{\lambda_2}{1 - \lambda_1} (\lambda_1 + \lambda_2)^{b-1}, \end{aligned}$$

where in the second equality we used Remark 5. From this it follows quickly that under $\mathbb{P}_{\lambda_1, \lambda_2}^{(1)}$, Q_1 and Q_2 are independent, and Q_2 has Bernoulli-geometric distribution with parameters $\lambda_2/(1 - \lambda_1)$ and $\lambda_1 + \lambda_2$ as claimed.

We now turn to the proof for general $n \in \mathbb{Z}$. As an initial part of the induction step for general n , it is useful to state a lemma relating $\mathbb{P}_{\lambda_1, \dots, \lambda_{n-1}}^{(\lambda_1 + \dots + \lambda_n)}$ and $\mathbb{P}_{\lambda_1, \dots, \lambda_n}^{(1)}$.

Lemma 6. Assume the induction hypothesis (15) for $n - 1$. Then for all $a_1, \dots, a_{n-1} \in \mathbb{Z}_+$,

$$\begin{aligned} &(\lambda_1 + \dots + \lambda_n) \sum_{j=1}^{n-1} a_j \mathbb{P}^{(\lambda_1 + \dots + \lambda_n)}(Q_1 = a_1, \dots, Q_{n-1} = a_{n-1}) \\ &= \frac{\lambda_n}{\lambda_1 + \dots + \lambda_n} \frac{1}{1 - (\lambda_1 + \dots + \lambda_{n-1})} \mathbb{P}^{(1)}(Q_1 = a_1, \dots, Q_{n-1} = a_{n-1}). \end{aligned} \quad (39)$$

Proof. Recall that we can move from $\mathbb{P}^{(1)}$ to $\mathbb{P}^{(\lambda_1 + \dots + \lambda_n)}$ by replacing λ_i by $\lambda_i/(\lambda_1 + \dots + \lambda_n)$ for each i . By the induction hypothesis, under $\mathbb{P}^{(1)}$, the Q_i , $1 \leq i \leq n - 1$ are independent with distribution given by (15). Hence they are also independent under $\mathbb{P}^{(\lambda_1 + \dots + \lambda_n)}$. It will be enough to show that for each i , for any $a_i \in \mathbb{Z}$,

$$(\lambda_1 + \dots + \lambda_n)^{a_i} \mathbb{P}^{(\lambda_1 + \dots + \lambda_n)}(Q_i = a_i) = \frac{\lambda_{i+1} + \dots + \lambda_n}{\lambda_i + \dots + \lambda_n} \frac{1 - (\lambda_1 + \dots + \lambda_{i-1})}{1 - (\lambda_1 + \dots + \lambda_i)} \mathbb{P}^{(1)}(Q_i = a_i). \quad (40)$$

Then the claim in (39) will follow by multiplying (40) over $i = 1, 2, \dots, n - 1$ and telescoping the products. The distribution of Q_i under $\mathbb{P}^{(1)}$ is given by (15), so to obtain the distribution of Q_i under $\mathbb{P}^{(\lambda_1 + \dots + \lambda_n)}$, we rescale the parameters

as above to get

$$Q_i \sim \text{Ber}\left(\frac{\lambda_i}{\lambda_1 + \dots + \lambda_n}\right) \text{Geom}\left(\frac{\lambda_1 + \dots + \lambda_i}{\lambda_1 + \dots + \lambda_n}\right)$$

Now one can check (40) directly for each value of a_i . There are essentially two cases, $a_i = 0$ and $a_i > 0$ (corresponding to the two forms of the probability for a Bernoulli–geometric random variable). $\square$

Now we can carry out the induction step. Following what we did in the case $n = 2$, fix $a_1, \dots, a_{n-1} \geq 0$ and $b \geq 1$. Let A_ϵ be the event that, during the time interval $(0, \epsilon)$, the process $F^{(n)}$ contains a_1 1's, then a_2 2's, and so on up to a_{n-1} points of type $(n-1)$, and then finally b points of type n . Again we let $\epsilon \rightarrow 0$ and look at two ways of approximating the probability of the event A_ϵ . First we look at $F^{(n)}$ as the output of a queue of rate 1 fed by an arrival process whose distribution is $F^{(n)}$. As ϵ becomes small, the dominant way for the event A_ϵ to occur is that at time 0 the queue already contains precisely a_i customers of type i for $1 \leq i \leq n-1$, and at least b customers of type n , and that then $a_1 + \dots + a_{n-1} + b$ services occur during the interval $(0, \epsilon)$. This gives

$$\mathbb{P}(A_\epsilon) \sim \mathbb{P}^{(1)}(Q_1 = a_1, \dots, Q_{n-1} = a_{n-1}, Q_n \geq b) \frac{\epsilon^{(b + \sum_{j=1}^{n-1} a_j)}}{(b + \sum_{j=1}^{n-1} a_j)!}. \quad (41)$$

On the other hand, look at $F^{(n)}$ as the output of a queue of rate $\lambda_1 + \dots + \lambda_n$, fed by an arrival process whose distribution is $F^{(n-1)}$, and with points of type n added at times of unused service. Then the dominant way for A_ϵ to occur for small ϵ is that at time 0 the queue already contains precisely a_i customers of type i for $1 \leq i \leq n-1$, and then $a_1 + \dots + a_{n-1} + b$ services occur during $(0, \epsilon)$. This leads to

$$\mathbb{P}(A_\epsilon) \sim \mathbb{P}^{(\lambda_1 + \dots + \lambda_n)}(Q_1 = a_1, \dots, Q_{n-1} = a_{n-1}) \frac{(\epsilon(\lambda_1 + \dots + \lambda_n))^{b + \sum_{j=1}^{n-1} a_j}}{(b + \sum_{j=1}^{n-1} a_j)!}. \quad (42)$$

Comparing (41) and (42) and continuing using (39), we get that for $b \geq 1$,

$$\begin{aligned} & \mathbb{P}^{(1)}(Q_1 = a_1, \dots, Q_{n-1} = a_{n-1}, Q_n \geq b) \\ &= (\lambda_1 + \dots + \lambda_n)^{(b + \sum_{j=1}^{n-1} a_j)} \mathbb{P}^{(\lambda_1 + \dots + \lambda_n)}(Q_1 = a_1, \dots, Q_{n-1} = a_{n-1}) \\ &= \frac{\lambda_n}{\lambda_1 + \dots + \lambda_n} \frac{1}{1 - (\lambda_1 + \dots + \lambda_{n-1})} \mathbb{P}^{(1)}(Q_1 = a_1, \dots, Q_{n-1} = a_{n-1}) (\lambda_1 + \dots + \lambda_n)^b \\ &= \mathbb{P}^{(1)}(Q_1 = a_1, \dots, Q_{n-1} = a_{n-1}) \frac{\lambda_n}{1 - (\lambda_1 + \dots + \lambda_{n-1})} (\lambda_1 + \dots + \lambda_n)^{(b-1)}. \end{aligned} \quad (43)$$

From (43) we see that Q_n is independent of $Q_1, \dots, Q_{n-1}$, and has the Bernoulli–geometric distribution of (15) as claimed. This completes the induction step and the proof. $\square$

5.2. Two-column distribution in stationarity

Using the same ideas as in the preceding proof we can also say something about two neighbouring queues. Denote by Q_i^z the number of particles of type i in the column z in equilibrium and define $\mathbb{P}^{2,(1)}$ as the joint probability of two queues $\mathbb{Q}^0, \mathbb{Q}^1$ (the joint distribution of $(Q_1^0, \dots, Q_n^0, Q_1^1, \dots, Q_n^1)$), where the departure of $\mathbb{Q}^0$ is the arrival process of $\mathbb{Q}^1$ and where the server process of both queues is of rate 1.

Lemma 7. *Let $\mathbb{P}^{2,(1)}$ be the distribution of two queues in tandem at stationarity, then*

$$\mathbb{P}^{2,(1)}(Q_1^0 \geq 1, Q_1^1 = a, Q_2^1 \geq b) = \lambda_2 \lambda_1^{a+1} (\lambda_1 + \lambda_2)^b.$$

Proof. We think of the process $F^{(2)}$ in two ways. First we think of it as the departure process of the queue $\mathbb{Q}^1$ fed by $F^{(2)}$ and served at rate 1. Let N^1 and N^0 two independent Poisson processes of rate 1 that are independent of $\mathbb{Q}^0$ and $\mathbb{Q}^1$.

Let A be the event where one sees in the departure process the sequence that begins with a 1's then b 2's and then one 1 in the time interval $[0, \epsilon)$. The probability of A is dominated by

$$\mathbb{P}^{2,(1)}(Q_1^0 \geq 1, Q_1^1 = a, Q_2^1 \geq b) \mathbb{P}(\text{In the interval } [0, \epsilon], N^1 \text{ has exactly } a + b \text{ epochs before } N^0 \text{ has its first epoch after which there is another epoch of } N^1). \quad (44)$$

To see that, note that we need to have at least one first class particle in $\mathbb{Q}^0$, a first class particles in $\mathbb{Q}^1$ and at least b second-class particles in $\mathbb{Q}^1$, then, in the time interval $[0, \epsilon)$ the following must happen in order:

- (1) $(a + b)$ customers must be served in $\mathbb{Q}^1$ before any customer is served in $\mathbb{Q}^0$;
- (2) one service in $\mathbb{Q}^0$;
- (3) one service in $\mathbb{Q}^1$.

Recall that if X is the sum of n i.i.d. exponential r.v's of rate λ then $X \sim \Gamma(n, \lambda)$, i.e.

$$\mathbb{P}(X \in dx) = f^{n,\lambda}(x) dx = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} dx. \quad (45)$$

We have

$$\begin{aligned} \mathbb{P}(A) &\sim \mathbb{P}^{2,(1)}(Q_1^0 \geq 1, Q_1^1 = a, Q_2^1 \geq b) \int_0^\epsilon f^{a+b,1}(r_1) \int_{r_1}^\epsilon f^{1,1}(r_2) (1 - e^{-(\epsilon-r_2)}) dr_2 dr_1 \\ &\sim \mathbb{P}^{2,(1)}(Q_1^0 \geq 1, Q_1^1 = a, Q_2^1 \geq b) \int_0^\epsilon \left[\frac{r_1^{a+b-1}}{(a+b-1)!} e^{-r_1} \right] \int_{r_1}^\epsilon e^{-(r_2-r_1)} (1 - e^{-(\epsilon-r_2)}) dr_2 dr_1 \\ &\sim \mathbb{P}^{2,(1)}(Q_1^0 \geq 1, Q_1^1 = a, Q_2^1 \geq b) \int_0^\epsilon \left[\frac{r_1^{a+b-1}}{(a+b-1)!} \right] \int_{r_1}^\epsilon e^{-r_2} (1 - e^{-(\epsilon-r_2)}) dr_2 dr_1 \\ &\sim \mathbb{P}^{2,(1)}(Q_1^0 \geq 1, Q_1^1 = a, Q_2^1 \geq b) \int_0^\epsilon \frac{r_1^{a+b-1}}{(a+b-1)!} \frac{(\epsilon-r_1)^2}{2} dr_1. \end{aligned} \quad (46)$$

Using integration by parts twice

$$\begin{aligned} \int_0^\epsilon \frac{r_1^{a+b-1}}{(a+b-1)!} (\epsilon-r_1)^2 dr_1 &= 2 \int_0^\epsilon \frac{r_1^{a+b+1}}{(a+b+1)!} dr_1 \\ &= \frac{2\epsilon^{a+b+2}}{(a+b+2)!}. \end{aligned} \quad (47)$$

Plugging (47) into (46)

$$\mathbb{P}(A) \sim \mathbb{P}^{2,(1)}(Q_1^0 \geq 1, Q_1^1 = a, Q_2^1 \geq b) \frac{2\epsilon^{a+b+2}}{(a+b+2)!}. \quad (48)$$

On the other hand, we can think of the two queues under $\mathbb{P}^{2,(\lambda_1+\lambda_2)}$, that is, having $F^{(1)}$ as their arrival process and served at rate $\lambda_1 + \lambda_2$. We can obtain $F^{(2)}$ by interpreting an unserved epoch in $\mathbb{Q}^1$ as a second-class particle. We thus have

$$\begin{aligned} \mathbb{P}(A) &\sim \mathbb{P}^{2,(\lambda_1+\lambda_2)}(Q_1^0 \geq 1, Q_1^1 = a) \mathbb{P}(N^1 \text{ has exactly } a + b \text{ epochs in the interval } [0, \epsilon] \text{ before } N^0 \text{ has its first epoch in the interval } [0, \epsilon] \text{ after which } N^1 \text{ has an epoch}) \\ &\sim \mathbb{P}^{2,(\lambda_1+\lambda_2)}(Q_1^0 \geq 1, Q_1^1 = a) \int_0^\epsilon f^{a+b,\lambda_1+\lambda_2}(r_1) \int_{r_1}^\epsilon f^{1,\lambda_1+\lambda_2}(r_2) (1 - e^{-(\lambda_1+\lambda_2)(\epsilon-r_2)}) dr_1 dr_2 \\ &\sim \mathbb{P}^{2,(\lambda_1+\lambda_2)}(Q_1^0 \geq 1, Q_1^1 = a) \int_0^\epsilon \left[\frac{(\lambda_1 + \lambda_2)^{a+b} r_1^{a+b-1}}{(a+b-1)!} \right] \int_{r_1}^\epsilon e^{-(\lambda_1+\lambda_2)r_2} (1 - e^{-(\lambda_1+\lambda_2)(\epsilon-r_2)}) dr_2 dr_1 \\ &\sim \mathbb{P}^{2,(\lambda_1+\lambda_2)}(Q_1^0 \geq 1, Q_1^1 = a) (\lambda_1 + \lambda_2)^{a+b+1} \int_0^\epsilon \frac{r_1^{a+b-1}}{(a+b-1)!} \frac{(\epsilon-r_1)^2}{2} dr_1 \end{aligned}$$

$$\sim \mathbb{P}^{2,(\lambda_1+\lambda_2)}(Q_1^0 \geq 1, Q_1^1 = a)(\lambda_1 + \lambda_2)^{a+b+1} \frac{\epsilon^{a+b+2}}{(a+b+2)!}. \quad (49)$$

Comparing (46) and (49) and letting ϵ go to zero, we conclude that

$$\begin{aligned} \mathbb{P}^{2,(1)}(Q_1^0 \geq 1, Q_1^1 = a, Q_2^1 \geq b) &= \mathbb{P}^{2,(\lambda_1+\lambda_2)}(Q_1^0 \geq 1, Q_1^1 = a)(\lambda_1 + \lambda_2)^{a+b+1} \\ &= \mathbb{P}^{2,(\lambda_1+\lambda_2)}(Q_1^0 \geq 1) \mathbb{P}^{2,(\lambda_1+\lambda_2)}(Q_1^1 = a)(\lambda_1 + \lambda_2)^{a+b+1} \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \left(1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}\right) \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^a (\lambda_1 + \lambda_2)^{a+b+1} \\ &= \lambda_2 \lambda_1^{a+1} (\lambda_1 + \lambda_2)^{b-1}, \end{aligned}$$

where in the second equality we used the independence of the number of first class particles across different columns and in the third equality the fact that the distribution of Q_1^i is geometric and Remark 5. $\square$

6. Marginals of the TAZRP speed process

In this section we apply the results in Section 5 to obtain more refined results on the speed process. We divide the results into two subsections. The first deals with the distribution of one column of the speed process, whereas the second deals with the distribution of two columns.

6.1. Distribution of the speeds at a single column

One may think of a column in the multi-type TAZRP in stationarity as a queue with a countable number of classes. For example, the column of the speed process, U_0 , can be thought of as a marked point process $\mathcal{P}$ on $[0, 1]$. Each realization of U_0 is a countable set of numbers in $[0, 1]$, where we label each number in that set (the speed $U_{z,i}$ for some i which is to be thought of as the class of a particle) with a number in $\mathbb{N}$ that denotes the number of particles in that class. For example, if $U_{0,i-1} > U_{0,i} = U_{0,i+1} = 1/2 > U_{0,i+2}$, then $(\frac{1}{2}, 2) \in \mathcal{P}$, that is, there are two particles of class $1/2$ in the column. In what follows we would like to show that the TAZRP speed process can be viewed as the continuum version of the stationary measure discussed in Section 5.2. In fact, we prove this by approximating U_0 by n -type TAZRP for large n .

For each $n \in \mathbb{N}$, fix $x_0 = 1 > x_1 > \dots > x_{n+1} = 0$, and define the function $G_{\mathbf{x}}^n : [0, 1] \rightarrow \mathcal{R}_n$ by

$$G_{\mathbf{x}}^n(x) = -\min\{i : x \geq x_i\}, \quad (50)$$

where $\mathbf{x} = (x_1, \dots, x_n)$. By Corollary 1 applying the map $G_{\mathbf{x}}^n$ on each element $\eta(z, i)$ of $\eta \in \mathcal{Z}$ gives an element of $\mathcal{Z}_n$ so that $G_{\mathbf{x}}^n(\pi U)$ is a stationary and ergodic distribution for the n -type TAZRP.

Proof of Theorem 3. Let $G_{\mathbf{x}}^n : [0, 1] \rightarrow \mathcal{R}_n$ be the function defined in (50) associated with $x_i = 1 - i(n+1)^{-1}$, for $1 \leq i \leq n+1$. Applying $G_{\mathbf{x}}^n$ on πU_0 we obtain an ergodic and stationary measure for the n -type TAZRP. By the uniqueness of the stationary and ergodic measures of the n -type TAZRP, we see that the queue $Q^n = (Q_1^n, \dots, Q_n^n)$ has a Bernoulli-geometric product distribution as in Theorem 2. The arrival rates to the queue Q^n are given by

$$\lambda_i = \sqrt{1 - (i-1)n^{-1}} - \sqrt{1 - in^{-1}} = \sqrt{x_{i-1}} - \sqrt{x_i}, \quad \text{for } 1 \leq i \leq n. \quad (51)$$

To see that, note that by stationarity of $G^{(n)}$, the arrival rate of customers of type i to the queue equals the rate of departure of customers of type i under $\mathbb{P}^{(1)}$. By the stationarity of $\mathbb{P}^{(1)}$, the rate of departure of particles of type i equals the probability that the i 'th customer is the first in the queue, that is, particle of type i is next to be served in the queue.

$$\begin{aligned} \lambda_i &= \mathbb{P}^{(1)}(Q_1 = 0, \dots, Q_{i-1} = 0, Q_i > 0) \\ &= \mathbb{P}(U_{0,0} \in (x_i, x_{i-1}]) = 1 - \sqrt{x_i} - (1 - \sqrt{x_{i-1}}), \end{aligned} \quad (52)$$

where in the third equality we used the marginal distribution of $U_{0,0}$ in (32). By Theorem 2 we see that $Q_1^n, \dots, Q_n^n$ are independent and that

$$Q_i^n \sim \text{Ber}\left(\frac{\sqrt{x_{i-1}} - \sqrt{x_i}}{\sqrt{x_{i-1}}}\right) \text{Geom}(1 - \sqrt{x_i})$$

$$= \text{Ber}\left(\frac{\sqrt{1 - (i-1)n^{-1}} - \sqrt{1 - in^{-1}}}{\sqrt{1 - (i-1)n^{-1}}}\right) \text{Geom}(1 - \sqrt{1 - in^{-1}}). \quad (53)$$

Fix $\epsilon > 0$ and let $i \in [1, n - \lfloor \epsilon n \rfloor]$. In what follows we use C to denote a constant that may depend on some variables and that changes from line to line. Note that by Taylor expansion of $\sqrt{x}$ around $1 - (i-1)n^{-1}$ there exists $C(i, \epsilon) > 0$ such that

$$\begin{aligned} \frac{\sqrt{1 - (i-1)n^{-1}} - \sqrt{1 - in^{-1}}}{\sqrt{1 - (i-1)n^{-1}}} &= \frac{\sqrt{1 - (i-1)n^{-1}}^{-1} n^{-1}}{2\sqrt{1 - (i-1)n^{-1}}} + Cn^{-2} \\ &= \frac{n^{-1}}{2(1 - (i-1)n^{-1})} + Cn^{-2} = \frac{n^{-1}}{2x_{i-1}} + Cn^{-2}, \end{aligned} \quad (54)$$

where for a fixed ϵ and every n , $C(\cdot, \epsilon)$ is bounded uniformly on $i \in [1, n - \lfloor \epsilon n \rfloor]$. Fix $y \in [\epsilon, 1)$, and let $i_n = \lceil ny \rceil$. Then, $y \in [x_{n-i_n}, x_{n-i_n-1})$, and if $y \in U_0$ then $Q_{n-i_n}^n \neq 0$. Plugging (54) in (53), there exists a $C(i, \epsilon)$ such that

$$Q_{n-i_n}^n \sim \text{Ber}\left(\frac{n^{-1}}{2x_{n-i_n-1}} + Cn^{-2}\right) \text{Geom}(1 - \sqrt{x_{n-i_n-1}} - Cn^{-1}). \quad (55)$$

As $|y - x_{n-i_n-1}| \leq n^{-1}$, plugging y into (55), there exists $C(y, \epsilon) > 0$ where $C(\cdot, \epsilon)$ is bounded on $[\epsilon, 1)$, such that

$$Q_{n-i_n}^n \sim \text{Ber}\left(\frac{n^{-1}}{2y} + Cn^{-2}\right) \text{Geom}(1 - \sqrt{y} + Cn^{-1}). \quad (56)$$

Now let $\mathcal{P}^n = \{(p_i, l_i)\}_{i=1}^n$ be the marked point process associated with the queue Q^n by

$$\begin{aligned} p_i &= x_{n-i} \quad 1 \leq i \leq n \quad (\text{points}) \\ l_i &= Q_{n-i}^n \quad 1 \leq i \leq n \quad (\text{marks}). \end{aligned} \quad (57)$$

Let $\mathcal{P}^{n,1} = \bigcup_{l_i \neq 0} p_i$. If $t_1, t_2, t_3 \in (\epsilon, 1]$ such that $t_1 < t_2 < t_3$, then by Theorem 2 and n large enough

$$\#\{\mathcal{P}^{n,1} \cap [t_1, t_2)\}, \#\{\mathcal{P}^{n,1} \cap (t_2, t_3]\} \quad (58)$$

are independent. By (56) we see that

$$\delta^{-1} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{P}^{n,1} \cap [y, y + \delta) \neq \emptyset) = \frac{1}{2y}, \quad (59)$$

which implies that $\mathcal{P}^{n,1}$ converges to an inhomogeneous Poisson process with intensity $\frac{1}{2x}$ on $[\epsilon, 1)$. Next, again by (56), we see that conditioned on the event

$$\mathcal{P}^{n,1} \cap [y, y + \delta) \neq \emptyset, \quad (60)$$

if $p_i \in \mathcal{P}^{n,1} \cap [y, y + \delta)$, $l_i \sim \text{Geom}(r(\delta, n))$ and

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} r(\delta, n) = 1 - \sqrt{y}, \quad (61)$$

which implies the result on $[\epsilon, 1)$. Taking $\epsilon \rightarrow 0$ concludes the proof. $\square$

Proof of Theorem 4. Fix $0 = x_3 < x_2 < x_1 < 1$ and consider the map given in (50) associated with x_1, x_2 . First note that U_0 and πU_0 have the same distribution and therefore, so do $G(\pi U)$ and $G(U)$ a fact we use in the computations below. Since $G(\pi U)$ is stationary w.r.t. the 2-type TAZRP with some λ_1 and λ_2 , we can relate x_1, x_2 to λ_1, λ_2 . By the definition of the projection $G(\pi U)$

$$\mathbb{P}((\pi U)_{0,0} < x_2) = \mathbb{P}(U_{0,0} < x_2) = \mathbb{P}^{(1)}(Q_1 = 0, Q_2 = 0)$$

$$\mathbb{P}((\pi U)_{0,0} < x_1) = \mathbb{P}(U_{0,0} < x_1) = \mathbb{P}^{(1)}(Q_1 = 0).$$

Using Lemma 5 and Theorem 2 we see that

$$\begin{aligned}\lambda_1 &= 1 - \sqrt{x_1} \\ \lambda_2 &= \sqrt{x_1} - \sqrt{x_2},\end{aligned}\tag{62}$$

and

$$\begin{aligned}\mathbb{P}^{(1)}(Q_1 = k) &= (1 - \lambda_1)\lambda_1^k \\ \mathbb{P}^{(1)}(Q_2 = k) &= \begin{cases} 1 - \frac{\lambda_2}{1 - \lambda_1} & k = 0 \\ \frac{\lambda_2}{1 - \lambda_1}(1 - (\lambda_1 + \lambda_2))(\lambda_1 + \lambda_2)^{k-1} & k > 0. \end{cases}\end{aligned}\tag{63}$$

For $i < j$

$$\begin{aligned}\mathbb{P}(x_1 \geq U_{0,i}, x_2 \geq U_{0,j}) &= \mathbb{P}^{(1)}(Q_1 \leq i, Q_1 + Q_2 \leq j) \\ &= \sum_{l=0}^i \mathbb{P}^{(1)}(Q_1 = l) \mathbb{P}^{(1)}(Q_2 \leq j - l).\end{aligned}\tag{64}$$

Using (63)

$$\begin{aligned}\mathbb{P}^{(1)}(Q_2 \leq m) &= \frac{1 - (\lambda_1 + \lambda_2)}{1 - \lambda_1} + \frac{\lambda_2(1 - (\lambda_1 + \lambda_2)^m)}{1 - \lambda_1} \\ &= 1 - \frac{\lambda_2(\lambda_1 + \lambda_2)^m}{1 - \lambda_1}.\end{aligned}\tag{65}$$

Plugging (65) into (64) and using (62)

$$\begin{aligned}\mathbb{P}(x_1 \geq U_{0,i}, x_2 \geq U_{0,j}) &= \sum_{l=0}^i (1 - \lambda_1)\lambda_1^l \left(1 - \frac{\lambda_2(\lambda_1 + \lambda_2)^{j-l}}{1 - \lambda_1}\right) \\ &= 1 - \lambda_1^{i+1} - (\lambda_1 + \lambda_2)^{j+1} \left(1 - \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^{i+1}\right) \\ &= 1 - (1 - \sqrt{x_1})^{i+1} - (1 - \sqrt{x_2})^{j+1} \left(1 - \left(\frac{1 - \sqrt{x_1}}{1 - \sqrt{x_2}}\right)^{i+1}\right),\end{aligned}\tag{66}$$

which is what we wanted.

Next we compute the joint distribution on the diagonal, that is, the probability that the i 'th and j 'th particles have the same speed.

$$\begin{aligned}\mathbb{P}(x_1 > U_{0,i} \geq U_{0,j} > x_2) &= \mathbb{P}^{(1)}(Q_1 \leq i, Q_1 + Q_2 \geq j + 1) \\ &= \sum_{l=0}^i \mathbb{P}^{(1)}(Q_1 = l) \mathbb{P}^{(1)}(Q_2 \geq j + 1 - l | Q_1 = l) \\ &= \sum_{l=0}^i [(1 - \lambda_1)\lambda_1^l] \left[\frac{\lambda_2(\lambda_1 + \lambda_2)^{j-l}}{1 - \lambda_1} \right] \\ &= \lambda_2(\lambda_1 + \lambda_2)^j \sum_{l=0}^i \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^l \\ &= (\lambda_1 + \lambda_2)^{j+1} \left(1 - \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^{i+1}\right),\end{aligned}\tag{67}$$

where we used the independence of Q_1 and Q_2 . Plugging (62) into (67) we obtain

$$\mathbb{P}(x_1 > U_{0,i} \geq U_{0,j} > x_2) = (1 - \sqrt{x_2})^{j+1} \left(1 - \left(\frac{1 - \sqrt{x_1}}{1 - \sqrt{x_2}} \right)^{i+1} \right).$$

Dividing by $x_1 - x_2$ and letting $x_1 \rightarrow x_2$ we conclude that

$$\mathbb{P}(U_{0,i} = U_{0,j} \in dx) = (i + 1) \frac{(1 - \sqrt{x})^j}{2\sqrt{x}} dx.$$

□

At this point we can also give the proof of the uniqueness statement in Theorem 1. We wish to show that if μ is the distribution of the speed process, so that μ^π is a stationary distribution of the multi-type TAZRP, then every translation-invariant ergodic stationary distribution of the multi-type TAZRP is of the form $G(\mu^\pi)$ for some non-decreasing function G .

Proof of the uniqueness statement in Theorem 1. The coupling method implemented by Mountford and Prabhakar [21], following an approach introduced by Ekhaus and Gray [7], shows that for given $\lambda_1, \dots, \lambda_n$, there is a unique translation-invariant ergodic stationary distribution of the n -type TAZRP $\mu_{\lambda_1, \dots, \lambda_n}$ such that the rate of jumps of particles of type i from site 0 to site 1 is λ_i . In fact, since in stationarity the rate of such jumps is just the probability that the highest-priority particle at site 0 has type i , these distributions are characterised by the distribution of the type of that particle; under $\mu_{\lambda_1, \dots, \lambda_n}$, the probability that the highest-priority particle at site 0 has type i is λ_i .

Any distribution ν on $\mathcal{Z}$ is characterised by the probabilities of cylinder events of the form

$$\{\eta(z, 1) \leq a_1, \dots, \eta(z, k) \leq a_k\}.$$

Hence in fact ν is characterised by its projections $G_{\mathbf{x}}^n(\nu)$ where $G_{\mathbf{x}}^n$ is a function of the form defined at (50).

If ν is a stationary distribution for the multi-type TAZRP, then we know that any such $G_{\mathbf{x}}^n(\nu)$ is stationary for the n -type TAZRP. Suppose that ν and $\tilde{\nu}$ are two translation-invariant ergodic stationary distributions for the multi-type TAZRP, such that the distribution of $\eta(0, 0)$ is the same under ν and $\tilde{\nu}$. Then, by the characterisation of the distributions $\mu_{\lambda_1, \dots, \lambda_n}$ above, the n -type stationary distributions $G_{\mathbf{x}}^n(\nu)$ and $G_{\mathbf{x}}^n(\tilde{\nu})$ are in fact the same for any such $\mathbf{x}$. Hence ν and $\tilde{\nu}$ are the same. So for any given distribution of $\eta(0, 0)$, there is at most one translation-invariant ergodic stationary distribution.

But under μ^π , the distribution of $\eta(0, 0)$ is non-atomic. So for any desired target distribution, we can find a non-decreasing function G with the desired distribution of $\eta(0, 0)$ under $G(\mu^\pi)$. Hence indeed all translation-invariant ergodic stationary distributions are of the form $G(\mu^\pi)$, as desired. □

6.2. Joint distribution of multiple columns

In this section we apply the results in Section 5.2 to Proposition 1.

Proof of Proposition 1. Let $\mathbb{Q}^0$ and $\mathbb{Q}^1$ be two queues with arrival process $F^{(2)}$ in stationarity, s.t the departure process of $\mathbb{Q}^0$ is the arrival process of $\mathbb{Q}^1$. It is not hard to see that

$$\begin{aligned} \mathbb{P}(U_{0,0} > x_1, U_{-1,j-1} > x_1 > U_{-1,j} > \dots > U_{-1,j+k-1} > x_2) \\ &= \mathbb{P}((\pi U)_{0,0}, (\pi U)_{1,j-1} > x_1 > (\pi U)_{1,j} > \dots > (\pi U)_{1,j+k-1} > x_2) \\ &= \mathbb{P}^{(1)}(Q_1^0 \geq 1, Q_1^1 = j, Q_2^1 \geq k). \end{aligned} \tag{68}$$

By Lemma 7 we see that

$$\mathbb{P}^{(1)}(Q_1^0 \geq 1, Q_1^1 = j, Q_2^1 \geq k) = \lambda_2 \lambda_1^j (\lambda_1 + \lambda_2)^{k-1}.$$

Using (62) we obtain the result. □

Remark 6. Fix $v \in (0, 1)$. Apply the function G_v^1 on the reflected speed process πU and define the random variables

$$A_i = \text{the number of particles in the column } G_v^1(\pi U)_i, \text{ whose speed exceeds } v, \quad i \in \mathbb{Z}. \tag{69}$$

As the distribution of $G_v^1(\pi U)$ is stationary with respect to the 1-type TAZRP we see that the events $\{A_i\}_{i \in \mathbb{Z}}$ are independent.

7. Overtaking

Consider the initial condition η^* . Let $i_1 \leq i_2$ and j_1, j_2 be such that $p_{i_1, j_1} > p_{i_2, j_2}$. We define their meeting time $T \in \mathbb{R}_+ \cup \infty$ as the first time that the particle p_{i_1, j_1} is at the same column as p_{i_2, j_2} . We say the particle p_{i_1, j_1} overtakes the particle p_{i_2, j_2} if $T < \infty$. Note that

$$X_{p_{i_1, j_1}}(t) < X_{p_{i_2, j_2}}(t) \quad t < T$$

$$X_{p_{i_1, j_1}}(t) \geq X_{p_{i_2, j_2}}(t) \quad t \geq T.$$

Proof of Theorem 5. The case where $U_{0, j} > U_{i, k}$ is clear and so we assume $U_{0, j} = U_{i, k}$. The proof will rely on the following observations:

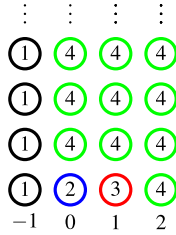
- (1) As we are concerned only with the positions of the particles $p_{0, j}$ and $p_{i, k}$, we may change the types of the particles in the configuration (even using a *non-monotone* relabelling) as long as the relative priority is preserved *with respect to the two particles*.
- (2) We may also ignore any part of the dynamics that does not affect the positions of the two particles.
- (3) We are only interested in the dynamics until the overtaking time T .

We will use these guidelines to simplify the TAZRP configuration. We will then use the coupling with the TASEP and results on the TASEP speed process to conclude that overtaking occurs.

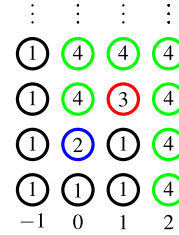
We divide the proof into several cases according to the values of i, j, k .

Case 1: First assume that $i = 1$ and $j = k = 0$, in other words, we assume the particle $p_{0,0}$ is at the bottom of the column 0 has the same speed as that of the particle at the bottom of column 1. As the particles above $p_{0,0}$ are weaker than $p_{0,0}$ we may consider them as holes with respect to both $p_{0,0}$ and $p_{1,0}$, as this would not affect the dynamics of the two particles until overtaking occurs. We now re-label the rest of the particles as follows (Figure 7):

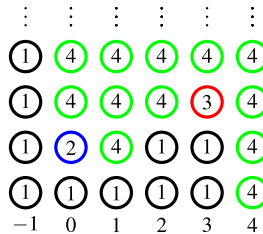
- $p_{0,0} = -2$ and $p_{1,0} = -3$.
- $p_{i,j} = -1$ for all $i < 0, j \in \mathbb{N}_0$.
- $p_{i,j} = -4$ for $i > 1$ and for $i = 1, j > 0$.



(A) Case 1. The particles above particles 2 and 3 are of lower class and do not affect the dynamics up to the meeting time.



(B) Case 2. Although the particles below particles 2 and 3 are of different class in the speed process, treating them as first class particles does not affect the dynamics between particles 2 and 3.



(C) Case 3 (2). Any particle in columns 0 – 3 whose speed is strictly greater than that of the red and blue particles are treated as first class particles while all particles with speed below are treated as holes.

Fig. 7. Illustration of the three configurations described in cases 1, 2 and 3 of the proof of Theorem 5, with the minus signs omitted for neater presentation. The blue particle will ultimately meet the red particle.

It is straightforward to check that this keeps the order of priority with respect to the two particles. Using the coupling of the TAZRP with the TASEP we see that this configuration translates to (for the TASEP we use the convention in [1] that stronger particles are those with smaller value)

$$\dots 1111234444\dots \quad (70)$$

Recall from Section 3.2 that the coupling of the TAZRP with several second-class particles holds until the first time that two second-class particles are in the same column, which in this case is up to time T . Since particle $p_{0,0}$ and $p_{1,0}$ have the same speed, so do the particles 2 and 3 in (70). In [1, Theorem 1.14] it was shown that with probability 1 particle 2 overtakes particle 3. By the coupling with the TAZRP we see that $p_{0,0}$ overtakes $p_{1,0}$.

Case 2: Next assume $i = 1$, $j \geq 0$, $k \geq 0$. (if $j = k = 0$ this degenerates back to the previous case). Here we label the particles in the same way as before except for $p_{m,l}$ where $m = 0$ and $0 \leq l < j$ or $m = 1$ and $0 \leq l < k$ which are labeled as first class particles, i.e. $p_{m,l} = -1$. Although the latter ($m = 1$ and $0 \leq l < k$) are of smaller value than the particle $p_{0,j}$, until time T there is no interaction between them and $p_{0,j}$ (as they are always strictly to the right of $p_{0,j}$) so the labelling is consistent with the dynamics up to the point of overtaking. This translates to the following multi-type TASEP configuration

$$\dots 111121 \underbrace{\dots 131}_{j} \underbrace{\dots 14444}_{k} \dots \quad (71)$$

We now claim that particle 2 overtakes particle 3 in (71). Assume it does not, then there is some positive probability $p > 0$ of reaching (71) from (70). This implies that starting from (70) with some positive probability particle 2 will not overtake particle 3 contradicting that starting from configuration (70) particle 2 a.s. overtakes particle 3.

Case 3 Finally we prove the theorem for $i > 1$. We use induction on i . Suppose $U_{0,j} = U_{i+1,k}$ for some $j, k \in \mathbb{N}_0$, and that our hypothesis holds for $1 \leq i' \leq i$. There are two possibilities:

- (1) There exists $1 \leq m \leq i$ and $l \in \mathbb{N}_0$ s.t $U_{0,j} = U_{m,l} = U_{i+1,k}$.
- (2) For every $1 \leq m \leq i$ and $l \in \mathbb{N}_0$ $U_{m,l} \neq U_{0,j} = U_{i+1,k}$.

For case 1 we use the induction hypothesis twice to conclude that particle $p_{m,l}$ overtakes particle $p_{i+1,k}$ and that particle $p_{0,j}$ overtakes particle $p_{m,l}$ which together implies that $p_{0,j}$ overtakes $p_{i+1,k}$. It remains therefore to deal with case 2. Note that by (32) we see that for every $m \in \mathbb{Z}$ w.p 1 we have

$$\lim_{l \rightarrow \infty} U_{m,l} = 0. \quad (72)$$

Equation (72) implies that for $1 \leq m \leq i$, column m has only a finite number of particles whose speed exceeds $U_{0,j}$ and all the speeds of all other particles in the column are strictly smaller than $U_{0,j}$. Particles located at column m for $1 \leq m \leq i$ and whose speed is smaller than $U_{i+1,k}$ cannot overtake particle $p_{i+1,k}$ and are of value smaller than that of $p_{0,j}$ and therefore will not change the dynamics of $p_{0,j}$ and $p_{i+1,k}$ and can be considered as holes by both particles. Particles located at column m for $1 \leq m \leq i$ and whose speed is larger than $U_{i+1,k}$ cannot be overtaken by particle $p_{0,j}$ (whose speed is smaller then theirs but whose value is greater) and we can therefore label them as first class particles. Since we are in case 2 all particles in columns $1 \leq m \leq i$ fall into one of the above two options. The particles at columns 0 and $i + 1$ are labelled as before. All together, we see that we can translate our TAZRP configuration to the following TASEP configuration

$$\dots 111121 \underbrace{\dots 14}_{j} \underbrace{1 \dots 1}_{N_1} \dots \underbrace{41 \dots 131}_{N_i} \underbrace{\dots 14444}_{k} \dots, \quad (73)$$

where for $1 \leq m \leq i$ N_m is the number of particles whose speed is greater than $U_{0,j}$. To see how one obtains (73), consider Figure 7c which illustrates case 3 (2) with $i = 3$, $j = 1$, $k = 2$. The speed of the particle $p_{0,j}$ is labeled as a particle of type 2 while the speed of particle $p_{i+1,k}$ is labeled as a particle of type 3. All particles in column 1 and 2 whose speed is greater than that of $p_{0,j}$ are labeled as first class particles (type 1), particles in column 1 and 2 whose speed is below that of $p_{0,j}$ are labeled as particles of type 4. Using the coupling between TAZRP and TASEP one arrives at

$$\dots 111121414113114444\dots \quad (74)$$

We now argue as before: assume particle 2 does not overtake particle 3, then there is some positive probability $p' > 0$ of reaching (73) from (70). This implies that starting from (70), with some positive probability particle 2 will not overtake particle 3 contradicting that starting from configuration (70) particle 2 a.s. overtakes particle 3. We have now proved the inductive step. As we already proved the hypothesis of the induction for $i = 0$ (particles 2 and 3 are located in two adjacent columns) the result follows. $\square$

Acknowledgments

We are grateful to two anonymous referees for their detailed and insightful comments.

Funding

G. Amir was supported by the Israel Science Foundation through grant 575/16 and by the German Israeli Foundation through grant I-1363-304.6/2016. O. Busani was supported by EPSRC's EP/R021449/1 Standard Grant. P. Gonçalves thanks FCT/Portugal for support through the project UID/MAT/04459/2013. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 715734).

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