

# AXISYMMETRIC BUCKLING OF A SPHERICAL SHELL EMBEDDED IN AN ELASTIC MEDIUM UNDER UNIAXIAL STRESS AT INFINITY

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## Summary

The problem of a thin spherical linearly-elastic shell, perfectly bonded to an infinite linearly-elastic medium is considered. A constant axisymmetric stress field is applied at infinity in the matrix, and the displacement and stress fields in the shell and matrix are evaluated by means of harmonic potential functions. In order to examine the stability of this solution, the buckling problem of a shell which experiences this deformation is considered. Using Koiter's nonlinear shallow shell theory, restricting buckling patterns to those which are axisymmetric, and using the Rayleigh–Ritz method by expanding the buckling patterns in an infinite series of Legendre functions, an eigenvalue problem for the coefficients in the infinite series is determined. This system is truncated and solved numerically in order to analyse the behaviour of the shell as it undergoes buckling, and to identify the critical buckling stress in two cases — namely where the shell is hollow and the stress at infinity is either uniaxial or radial.

## 1. Introduction

In underwater applications, *anechoic tiles* are often used to minimise the acoustic reflection coefficient of submerged structures. These tiles may consist of a rubber substrate containing a significant number of microscopic ( $\sim 20\text{ }\mu\text{m}$  radius) hollow elastic spheres. The thickness of these shells is around 2% of the radius. When submerged, the tiles are compressed uniaxially by the pressure of the water, causing the shells to buckle, which softens the material. As part of an investigation (1) into the static elastic properties of the tiles, we will in this article analyse the deformation and subsequent buckling of a single isolated spherical shell embedded in an elastic substrate with an applied stress field at infinity. We will assume that both the substrate and the shell are linearly-elastic materials and that there is perfect bonding between them.

While the buckling problem of complete spherical shells has been studied for many years and is well understood due to the contributions of, among others, Koiter (2), the buckling problem of shells which are embedded in an elastic medium has not received much attention. One of the few studies along these lines was made by Fok and Allwright (3, 4), who considered the buckling problem of infinite embedded cylindrical shells (in two dimensions) and complete embedded spherical shells (in three dimensions), in each case with an applied

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radial stress field applied at infinity in the substrate. However this analysis, as we shall see, involves a simplifying assumption which renders it invalid in many parameter regimes.

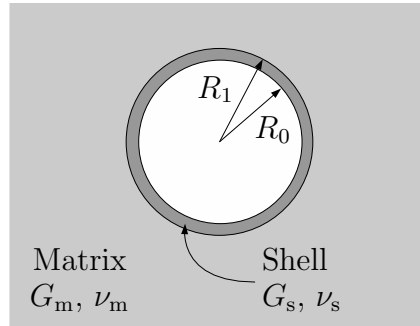
Our investigation into the buckling problem of the embedded shell will follow the spirit of Koiter's analysis (2). Following a description of the problem we will review the necessary elements of buckling theory and how they relate to our case, including in particular the use of energy criteria. We find that in order to proceed with the buckling analysis, a description of the state of stress in the shell prior to buckling is required. This will be found using a method presented by Love (5). Using this we subsequently obtain a functional of the shell displacements which is to be minimised. The minimisation will be carried out using the Rayleigh–Ritz method, which yields an infinite system of linear equations for coefficients of the basis functions in the Rayleigh–Ritz expansion. This system is solved numerically by truncation and the results are discussed in Sections 8 and 9.

## 2. Physical description of the problem

The configuration of the problem is shown in Figure 1. We consider a spherical shell with internal radius  $R_0$  and external radius  $R_1$  embedded in an isotropic linearly-elastic matrix. By setting

$$R_0 = \hat{R} - \frac{h}{2}, \quad R_1 = \hat{R} + \frac{h}{2},$$

we can alternatively say that the shell has a spherical mid-surface of radius  $\hat{R}$  and a constant thickness  $h$ . The matrix is characterised by its shear modulus  $G_m$  and Poisson ratio  $\nu_m$ , and likewise the shell is characterised by  $G_s$  and  $\nu_s$ .



**Fig. 1** Configuration of the physical problem.

We assume that the shell is perfectly bonded to the matrix, so that the displacement and traction at  $R = R_1$  are continuous. In addition, we impose that the state of stress in the shell is a superposition of two states: the response to a uniaxial stress  $\tau_{zz}|_\infty = -q_z$  at infinity, and the (purely radial) response to the applied stresses  $\tau_{RR}|_\infty = -q_R$  and  $\tau_{RR}|_{R_0} = -q_{in}$ . Finally, we assume that the buckling patterns in the shell will be axisymmetric. This assumption will greatly simplify the buckling analysis.

We adopt the convention that vectors and tensors indexed with numeric subscripts and superscripts are the covariant and contravariant components respectively. Of these

indices, Greek letters vary over 1, 2 and denote surface quantities; for the spherical polar coordinate system that we use in this article, ‘1’ corresponds to the (colatitudinal)  $\theta$ -direction and ‘2’ to the (azimuthal)  $\phi$ -direction. Latin letters vary over 1, 2, 3 and denote three-dimensional quantities. In the spherical polar case the corresponding directions are  $R, \theta, \phi$  respectively. Vectors and tensors indexed with a coordinate (in our article,  $x, y, z$  for Cartesian components and  $R, \theta, \phi$  for spherical polar components) such as  $\tau_{RR}$  above, are referred to *unit* vectors, so that these are the *physical* components of the relevant vector or tensor.

### 3. Conditions for change in stability

We will now review the conditions needed for the change in stability of a mechanical system. Suppose that a mechanical system is in a state I. To investigate the stability of this state, we superimpose a *virtual* displacement, to create a second state II. The state of the system is characterised by its potential energy.

Denoting this potential energy by  $W$ , we define the change in potential energy as  $\Delta W = W_{II} - W_I$ , where the subscripts refer to the state of the system. Suppose now that the virtual displacement is given by  $\mathbf{v}$ . We can split  $\Delta W$  into terms which are linear in  $\mathbf{v}$ , terms quadratic in  $\mathbf{v}$ , and terms of higher order:

$$\Delta W = \Delta W_1 + \Delta W_2 + \dots \quad (3.1)$$

The equilibrium state of the system can be found by setting  $\Delta W_1 = 0$  for all variations  $\mathbf{v}$  and using the calculus of variations to find the Euler equations for the system. For a given loading parameter, the equilibrium solution is stable if  $\Delta W$ , now containing quadratic terms and higher, is positive definite for all sufficiently small  $\mathbf{v}$  (*i.e.*  $\Delta W$  has a strict local minimum at the equilibrium state  $\mathbf{v} = \mathbf{0}$ ). Conversely if  $\Delta W < 0$  for some sufficiently small  $\mathbf{v}$ , the equilibrium state is unstable. The *critical* state occurs when, for  $\mathbf{v}$  small enough,  $\Delta W > 0$  with  $\Delta W = 0$  for at least one virtual displacement  $\mathbf{v}_c$ . This displacement is then known as the buckling displacement and the corresponding loading parameter is the critical load. This case implies that  $\Delta W$  is a minimum at  $\mathbf{v}_c$ , motivating the *Trefftz criterion* for buckling, which states that the buckling load can be found by searching for a stationary value of  $\Delta W$  (6). In fact, since we consider arbitrarily small virtual displacements, the Trefftz criterion is usually applied to the quadratic terms of  $\Delta W$ , so that we search for stationary values of  $\Delta W_2$ . This leads to linear buckling equations and thus a simpler problem. While it can be shown that  $\Delta W_2 > 0$  is a necessary condition for the positivity of  $\Delta W$  (and hence shell stability), the converse is not true. However in most practical situations the two statements appear to be equivalent. A fuller discussion of this can be found in Koiter’s article (7) and references therein.

We now consider how this theory applies to the shell embedded in a matrix. Firstly, in our analysis the virtual displacement  $\mathbf{v}$  will refer to the virtual shell displacement; the virtual displacement in the matrix will be found as a linear function of  $\mathbf{v}$ , as the matrix obeys linear elasticity. The total potential energy of the system will be given by the sum of the potential energy in the shell and the potential energy in the matrix. We will first consider the shell.

The potential energy density of a thin shell can be written (8) as

$$V = \frac{1}{2} h E^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta} \gamma_{\lambda\mu} + \frac{1}{24} h^3 E^{\alpha\beta\lambda\mu} \rho_{\alpha\beta} \rho_{\lambda\mu}, \quad (3.2)$$

so that the potential energy of the shell is the integral of this expression over the entire shell surface. Here  $\gamma_{\alpha\beta}$  is the middle-surface strain tensor,  $\rho_{\alpha\beta}$  is the tensor of changes of curvature, and

$$E^{\alpha\beta\lambda\mu} = G_s \left( a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu_s}{1-\nu_s} a^{\alpha\beta} a^{\lambda\mu} \right) \quad (3.3)$$

is the elasticity tensor for shells, where  $a^{\alpha\beta}$  is the contravariant metric tensor of the shell's middle-surface.

Denoting quantities pertaining to state I by a superscript (I), we suppose as before that the displacement in the shell in state II is given by  $\mathbf{v}^{(I)} + \mathbf{v}$ , where  $\mathbf{v}$  is the virtual displacement. Using the expression (3.2) for the potential energy density in the shell, the change in potential energy density, denoted  $\Delta V$ , is given by

$$\begin{aligned} \Delta V = & \frac{h}{2} E^{\alpha\beta\lambda\mu} \left( \gamma_{\alpha\beta}^{(I)} + \gamma_{\alpha\beta} \right) \left( \gamma_{\lambda\mu}^{(I)} + \gamma_{\lambda\mu} \right) + \frac{h^3}{24} E^{\alpha\beta\lambda\mu} \left( \rho_{\alpha\beta}^{(I)} + \rho_{\alpha\beta} \right) \left( \rho_{\lambda\mu}^{(I)} + \rho_{\lambda\mu} \right) \\ & - \frac{h}{2} E^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}^{(I)} \gamma_{\lambda\mu}^{(I)} - \frac{h^3}{24} E^{\alpha\beta\lambda\mu} \rho_{\alpha\beta}^{(I)} \rho_{\lambda\mu}^{(I)} \end{aligned} \quad (3.4)$$

$$= \frac{h}{2} E^{\alpha\beta\lambda\mu} \left( \gamma_{\alpha\beta}^{(I)} \gamma_{\lambda\mu} + \gamma_{\alpha\beta} \gamma_{\lambda\mu}^{(I)} + \gamma_{\alpha\beta} \gamma_{\lambda\mu} \right) + \frac{h^3}{24} E^{\alpha\beta\lambda\mu} \left( \rho_{\alpha\beta}^{(I)} \rho_{\lambda\mu} + \rho_{\alpha\beta} \rho_{\lambda\mu}^{(I)} + \rho_{\alpha\beta} \rho_{\lambda\mu} \right). \quad (3.5)$$

In this analysis we suppose that the in-surface displacement components  $v_\alpha$  of the shell are referred to the base vectors of the shell middle-surface, and that  $w$  is the normal displacement, so that  $\mathbf{v} = v_\alpha \mathbf{a}^\alpha + w \mathbf{a}^3$ . This is essentially the same notation as that of Green and Zerna (9) and of Koiter (10, 2).

We will determine the pre-buckled state I using the hypothesis of *linear* shell theory, so that terms which are of quadratic order or higher in the strain tensor are negligible. Thus we replace the middle-surface strain tensor for state I, defined  $\gamma_{\alpha\beta}^{(I)}$ , by its linearised counterpart  $\theta_{\alpha\beta}^{(I)}$ , which is defined by

$$\theta_{\alpha\beta} = \frac{1}{2} (v_\alpha|_\beta + v_\beta|_\alpha) - b_{\alpha\beta} w, \quad (3.6)$$

where the notation  $v_\alpha|_\beta$  denotes covariant differentiation and  $b_{\alpha\beta}$  is the second fundamental tensor of the surface. Likewise, we replace the tensor of changes of curvature  $\rho_{\alpha\beta}^{(I)}$  by its linearised counterpart  $\bar{\rho}_{\alpha\beta} = w|_{\alpha\beta}$ . Then we have

$$\Delta V = \frac{h}{2} E^{\alpha\beta\lambda\mu} \left( \theta_{\alpha\beta}^{(I)} \gamma_{\lambda\mu} + \gamma_{\alpha\beta} \theta_{\lambda\mu}^{(I)} + \gamma_{\alpha\beta} \gamma_{\lambda\mu} \right) + \frac{h^3}{24} E^{\alpha\beta\lambda\mu} \left( \bar{\rho}_{\alpha\beta}^{(I)} \rho_{\lambda\mu} + \rho_{\alpha\beta} \bar{\rho}_{\lambda\mu}^{(I)} + \rho_{\alpha\beta} \rho_{\lambda\mu} \right).$$

Furthermore, given that the pre-buckling displacements satisfy the *linear* theory of shells, then (9) we can relate the middle-surface strain tensor and the tensor of changes of curvature to the stress resultants  $n^{\alpha\beta}$  and stress couples  $m^{\alpha\beta}$  of the shell in state I by

$${}^{(I)}n^{\alpha\beta} = h E^{\alpha\beta\lambda\mu} \theta_{\lambda\mu}^{(I)}, \quad {}^{(I)}m^{\alpha\beta} = \frac{h^3}{12} E^{\alpha\beta\lambda\mu} \bar{\rho}_{\lambda\mu}^{(I)}.$$

Thus we can write

$$\Delta V = {}^{(I)}n^{\alpha\beta} \gamma_{\alpha\beta} + {}^{(I)}m^{\alpha\beta} \rho_{\alpha\beta} + \frac{h}{2} E^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta} \gamma_{\lambda\mu} + \frac{h^3}{24} E^{\alpha\beta\lambda\mu} \rho_{\alpha\beta} \rho_{\lambda\mu}.$$

We now assume that the response of the shell to the *virtual* displacement satisfies the *nonlinear shallow shell theory* as employed by Koiter (2). In this theory, the strain measures are given by

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}w_{,\alpha}w_{,\beta}, \quad \rho_{\alpha\beta} = \bar{\rho}_{\alpha\beta} = w|_{\alpha\beta}. \quad (3.7)$$

A nonlinear shell theory for the virtual displacements is essential for buckling problems, as we shall see shortly. The shallow shell approximation is the simplest nonlinear theory, given that it only contains one nonlinear term. We can now take  $\Delta V$  and split it into linear terms, quadratic terms, and terms of higher order. The linear terms are

$$\Delta V_1 = {}^{(I)}n^{\alpha\beta}\theta_{\alpha\beta} + {}^{(I)}m^{\alpha\beta}\rho_{\alpha\beta}$$

and the quadratic terms are

$$\Delta V_2 = \frac{1}{2}{}^{(I)}n^{\alpha\beta}w_{,\alpha}w_{,\beta} + \frac{h}{2}E^{\alpha\beta\lambda\mu}\theta_{\alpha\beta}\theta_{\lambda\mu} + \frac{h^3}{24}E^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}\rho_{\lambda\mu}.$$

The reason for choosing the nonlinear theory for the virtual displacement is now apparent, since if we had not included the nonlinear term  $\frac{1}{2}w_{,\alpha}w_{,\beta}$  in  $\gamma_{\alpha\beta}$ , then  $\Delta V_2$  would be independent of the pre-buckled state I.

Thus the change in potential energy of the shell is

$$\begin{aligned} \Delta W_s = & \iint_{\text{mid-shell surface}} \left( {}^{(I)}n^{\alpha\beta}\theta_{\alpha\beta} + {}^{(I)}m^{\alpha\beta}\rho_{\alpha\beta} \right) dS \\ & + \iint_{\text{mid-shell surface}} \left( \frac{1}{2}{}^{(I)}n^{\alpha\beta}w_{,\alpha}w_{,\beta} + \frac{h}{2}E^{\alpha\beta\lambda\mu}\theta_{\alpha\beta}\theta_{\lambda\mu} + \frac{h^3}{24}E^{\alpha\beta\lambda\mu}\rho_{\alpha\beta}\rho_{\lambda\mu} \right) dS \\ & + \text{higher order terms.} \end{aligned} \quad (3.8)$$

We now consider the matrix. There will be a virtual displacement  $\mathbf{u}$  which arises due to the virtual displacement  $\mathbf{v}$  of the shell, and in addition we need to consider the pre-buckling matrix displacement  $\mathbf{u}^{(I)}$ . These are superimposed on each other due to the assumption of linear elasticity in the matrix, giving a total displacement of  $\mathbf{u}^{(I)} + \mathbf{u}$ . Now, the potential energy density in the matrix is given by

$$V = \frac{1}{2}A^{ijkl}e_{ij}e_{kl}, \quad (3.9)$$

where

$$A^{ijkl} = G_m \left( g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu_m}{1-2\nu_m}g^{ij}g^{kl} \right).$$

In this expression  $g^{ij}$  are metric tensors for the particular coordinate system used and  $e_{ij}$  is the (covariant) strain tensor, given by  $e_{ij} = \frac{1}{2}(u_i|_j + u_j|_i)$ . From (3.9), we can see that the change in potential energy density between states I and II is

$$\Delta V = \frac{1}{2}A^{ijkl} \left( e_{ij}^{(I)}e_{kl} + e_{ij}e_{kl}^{(I)} + e_{ij}e_{kl} \right),$$

where  $e_{ij}^{(I)}$ ,  $e_{ij}$  are the strain tensors formed from  $\mathbf{u}^{(I)}$ ,  $\mathbf{u}$  respectively. The first two terms in

the expression above are linear in the virtual displacement  $\mathbf{u}$  (and hence linear in  $\mathbf{v}$ ). The linear and quadratic terms in the change of potential energy of the matrix thus become

$$\Delta W_m = \iiint_{R>R_1} \frac{1}{2} A^{ijkl} \left( e_{ij}^{(I)} e_{kl} + e_{ij} e_{kl}^{(I)} \right) dV + \iiint_{R>R_1} \frac{1}{2} A^{ijkl} e_{ij} e_{kl} dV. \quad (3.10)$$

Finally, we note that if a hydrostatic pressure  $q_{in}$  is applied to the inner surface of the shell, the potential energy of this loading will be given by

$$\Delta W_{in} = - \iint_{R=R_0} q_{in} w dS, \quad (3.11)$$

which is linear in the virtual displacement.

Therefore, the total change in potential energy is given by the sum of (3.8), (3.10) and (3.11), or  $\Delta W = \Delta W_s + \Delta W_m + \Delta W_{in}$ . However we write this as the sum of terms which are linear and quadratic in the virtual displacement, as in (3.1). We find that

$$\begin{aligned} \Delta W_1 = & \iint_{\text{mid-shell surface}} \left( {}^{(I)}n^{\alpha\beta} \theta_{\alpha\beta} + {}^{(I)}m^{\alpha\beta} \rho_{\alpha\beta} \right) dS \\ & + \iiint_{R>R_1} \frac{1}{2} A^{ijkl} \left( e_{ij}^{(I)} e_{kl} + e_{ij} e_{kl}^{(I)} \right) dV - \iint_{R=R_0} q_{in} w dS, \end{aligned} \quad (3.12)$$

and

$$\Delta W_2 = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3, \quad (3.13)$$

where

$$\mathcal{J}_1 = \iint_{\text{mid-shell surface}} \frac{1}{2} {}^{(I)}n^{\alpha\beta} w_{,\alpha} w_{,\beta} dS, \quad (3.14)$$

$$\mathcal{J}_2 = \iint_{\text{mid-shell surface}} \left( \frac{h}{2} E^{\alpha\beta\lambda\mu} \theta_{\alpha\beta} \theta_{\lambda\mu} + \frac{h^3}{24} E^{\alpha\beta\lambda\mu} \rho_{\alpha\beta} \rho_{\lambda\mu} \right) dS, \quad (3.15)$$

$$\mathcal{J}_3 = \iiint_{R>R_1} \frac{1}{2} A^{ijkl} e_{ij} e_{kl} dV. \quad (3.16)$$

It will be convenient to consider these three contributions to the energy integral separately. The critical buckling stress is found at a stationary point of  $\Delta W_2$ .

#### 4. Pre-buckled state of stress

In this section we will obtain the state of stress in the shell before buckling, which we denoted by (I). As stated in the previous section, this can be found by taking the functional (3.12) and solving the variational problem  $\Delta W_1 = 0$  for each possible variation  $\mathbf{v}$ . Ideally we would like a closed-form solution to this problem. However,  $\Delta W_1$  is a combination of shell strain measures and three-dimensional strains in the matrix, rendering this a difficult task.

Fortunately, however, shell theory is an approximation of full linear elasticity theory in the limit as the shell thickness ratio  $h/\widehat{R}$  tends to zero. Love (5) provided a method for solving simple elasticity problems in a spherical geometry, so a solution in closed form can be found before taking the limit  $h/\widehat{R} \rightarrow 0$ . In the following section we will calculate this full solution before extracting the stress resultant  ${}^{(I)}n^{\alpha\beta}$ , which is the only appearance of the pre-buckling solution in the integral  $\Delta W_2$ .

#### 4.1 Love's method

Love's method was used by Goodier (11) to solve the problem of a spherical elastic inclusion embedded in a dissimilar elastic matrix. This work was repeated by Liu and Nauman (12) and Bilgen and Insana (13). Mazzullo (14) has built on previous work in the case of a multi-layered inclusion, which is solved numerically due to the large system of equations that results from the analysis. In particular this work looked at the solution of a thick spherical shell surrounding a dissimilar material, all embedded in a matrix of a third material, undergoing uniaxial stress at infinity. On setting the stiffness of the innermost material to zero, we recover the solution to the uniaxial problem outlined in Section 2.

The theory behind Love's method is reviewed in a recent paper by Rahman and Michelitsch (15). It involves constructing displacement fields as a sum of elementary solutions, which take one of three forms — namely the  $\phi$ ,  $\omega$ , and  $\chi$  solutions. Let  $\phi_n$ ,  $\omega_n$ , and  $\chi_n$  be solid spherical harmonics of order  $n$ . Then the displacement fields arising from each solution are given (15) by

$$\begin{aligned}\mathbf{u} &= \nabla \phi_n, \\ \mathbf{u} &= R^2 \nabla \omega_n + \alpha_n R \omega_n \mathbf{e}_R, \\ \mathbf{u} &= \nabla \wedge (R \chi_n \mathbf{e}_R),\end{aligned}$$

respectively, where  $\mathbf{e}_R$  is the radial unit vector and

$$\alpha_n = \frac{-2(3n+1-2(2n+1)\nu)}{n+5-4\nu}. \quad (4.1)$$

The method involves writing the displacement in each material as a linear combination of these elementary solutions, and then finding the coefficients by considering the displacement and traction boundary conditions.

#### 4.2 Application to the embedded spherical shell

As stated in Section 2, the state of stress in the pre-buckled shell will be found by considering two states of applied stress at infinity and superposing the results. The first state is uniaxial compression at infinity, where the only applied stress component at infinity is  $\tau_{zz}|_\infty = -q_z$ . The second state is radial compression at infinity, which we will consider in due course. We will not write down the derivation of the solution to this problem; the interested reader may refer to (1).

Applying Love's method to the uniaxial problem, we prescribe the deformation in the matrix to be a superposition of the states arising from the following elementary solutions:

$$\phi_{-1} = \frac{c_1}{R} P_0(\mu), \quad \phi_{-3} = \frac{c_2}{R^3} P_2(\mu), \quad \omega_{-3} = \frac{c_3}{R^3} P_2(\mu)$$

(where  $c_i$  are undetermined constants,  $\mu = \cos \theta$  and  $P_n(\mu)$  are Legendre polynomials), together with a homogeneous field  $\mathbf{u}^\infty$  which is the displacement given by a constant stress field with only one component,  $\tau_{zz} = -q_z$ . In the shell, we suppose that the deformation is given by the superposition of the states arising from the following elementary solutions:

$$\begin{aligned}\phi_{-1} &= \frac{c_4}{R} P_0(\mu), & \phi_{-3} &= \frac{c_5}{R^3} P_2(\mu), & \omega_{-3} &= \frac{c_6}{R^3} P_2(\mu), \\ \phi_2 &= c_7 R^2 P_2(\mu), & \omega_2 &= c_8 R^2 P_2(\mu), & \omega_0 &= c_9 P_0(\mu),\end{aligned}$$

where again  $c_i$  are undetermined constants. One can thus obtain the displacement and stress in both materials, and solve for the constants by matching displacements and tractions at  $R = R_1$ , and setting the traction at  $R = R_0$  to zero. The resulting expressions for the constants  $c_i$  are somewhat unwieldy so we neglect to record them here. However, for future reference we note that the component  $\tau_{\theta\theta}$  of stress in the shell is given by

$$\begin{aligned} \tau_{\theta\theta} = 2G_s \left( -\frac{c_4}{R^3} + \alpha_0^{(s)} c_9 + \frac{c_5}{R^5} + \frac{c_6}{R^3} + c_7 + c_8 R^2 + \frac{3\nu_s \alpha_0^{(s)} c_9}{1 - 2\nu_s} \right) P_0(\mu) \\ + 2G_s \left[ -\frac{7c_5}{R^5} - 2c_7 + \frac{(-7 + \alpha_{-3}^{(s)})c_6}{R^3} + (-2 + \alpha_2^{(s)})R^2 c_8 \right. \\ \left. + \frac{\nu_s}{1 - 2\nu_s} \left( -\frac{6c_6}{R^3} + (4 + 5\alpha_2^{(s)})c_8 R^2 \right) \right] P_2(\mu). \end{aligned} \quad (4.2)$$

In this expression,  $\alpha_n^{(s)}$  is given by substituting the Poisson ratio of the shell,  $\nu_s$ , into (4.1).

We now consider the case where the state of stress is purely radial, with

$$\tau_{RR}|_{\infty} = -q_R, \quad \tau_{RR}|_{R_0} = -q_{in}. \quad (4.3)$$

Then, the pre-buckling displacement will also be purely radial, with

$$\mathbf{u} = \left( A_s R + \frac{B_s}{R^2} \right) \mathbf{e}_R, \quad \mathbf{u} = \left( A_m R + \frac{B_m}{R^2} \right) \mathbf{e}_R$$

in the shell and the matrix respectively (5). Again we can find the stresses in both materials, and match displacements and tractions between the two. We merely note here that we can find  $\tau_{\theta\theta}$  again, given in this case by

$$\tau_{\theta\theta} = \frac{2G_s(1 + \nu_s)}{1 - 2\nu_s} A_s \left( 1 + \frac{R_0^3}{2R^3} \right) + \frac{q_{in} R_0^3}{2R^3}, \quad (4.4)$$

where

$$A_s = \frac{\frac{q_{in} R_0^3}{4R_1^3} \left( \frac{1}{G_m} - \frac{1}{G_s} \right) - \frac{3q_R(1 - \nu_m)}{4G_m(1 + \nu_m)}}{\left\{ 1 + \frac{(1 + \nu_s)}{2(1 - 2\nu_s)} \left[ \frac{R_0^3}{R_1^3} + \frac{G_s}{G_m} \left( 1 - \frac{R_0^3}{R_1^3} \right) \right] \right\}}.$$

#### 4.3 Finding the stress resultant

We will now use the newly-found solution to the pre-buckled state in the shell to find the stress resultant  $^{(I)}n^{11}$ , which is the only component required for the buckling analysis. The reason for this can be seen from (3.14). As the buckling pattern will be assumed to be axisymmetric, we have  $w_{,2} = \partial w / \partial \phi = 0$ , so the only remaining term is  $^{(I)}n^{11}$ . This component is related to the stress component  $\tau_{\theta\theta}$  in the shell. This, as we have seen, is given by the sum of (4.2) and (4.4).

Next we will find the stress resultant  $n^{11}$  from this value of  $\tau_{\theta\theta}$ . Green and Zerna (9) state that

$$n^{11} = \int_{-h/2}^{h/2} \sigma^{11} d\theta_3,$$



where  $\sigma^{\alpha\beta}$  is a measure of stress related to  $\tau^{ij}$  and  $\theta_3$  is the coordinate which is directed normal to the shell. However,  $\theta_3 \in [-h/2, h/2]$  and  $h/\hat{R} \ll 1$ , so let  $\theta_3 = h\xi$ , then

$$n^{11} = h \int_{-1/2}^{1/2} \sigma^{11} d\xi.$$

We will further assume that

$$n^{11} \sim h \int_{-1/2}^{1/2} \left( \lim_{h/\hat{R} \rightarrow 0} \sigma^{11} \right) d\xi, \quad (4.5)$$

because we will only require the first term of  $n^{11}$  for the shell buckling problem; any terms of higher order being neglected. But from results in Green and Zerna (9) applied to spherical polar coordinates,

$$\sigma^{11} = \left( 1 + \frac{\theta_3}{\hat{R}} \right)^3 \tau^{22}.$$

Since  $\tau^{ij}$  is the contravariant stress tensor referred to the three-dimensional base vectors described in Section 2, the index '2' corresponds to the coordinate in the  $\theta$ -direction. Therefore  $\tau^{22} = \tau_{\theta\theta}/R^2$ , so that

$$\sigma^{11} = \frac{1}{\hat{R}^2} \left( 1 + \frac{h\xi}{\hat{R}} \right) \tau_{\theta\theta}, \quad \text{and} \quad \lim_{h/\hat{R} \rightarrow 0} \sigma^{11} = \frac{1}{\hat{R}^2} \lim_{h/\hat{R} \rightarrow 0} \tau_{\theta\theta}.$$

But  $\lim_{h/\hat{R} \rightarrow 0} \tau_{\theta\theta}$  is merely the expression formed from the sum of (4.2) and (4.4) evaluated at  $R = \hat{R}$ , with the constants  $c_i$  and  $A_s$  replaced by their values in the limit  $\hat{h}/\hat{R} \rightarrow 0$ . These quantities, distinguished by an overbar, are given by

$$\begin{aligned} \bar{c}_4 &= \frac{q_z \hat{R}^3}{12G_m} \frac{(1 + \nu_s)(1 - \nu_m)}{(1 - \nu_s)(1 + \nu_m)}, & \bar{c}_5 &= -\frac{q_z \hat{R}^5}{2G_m} \frac{(1 - \nu_m)}{(1 - \nu_s)(7 - 5\nu_m)}, \\ \bar{c}_6 &= -\frac{5q_z \hat{R}^3}{6G_m} \frac{(1 - \nu_m)(1 - 2\nu_s)}{(1 - \nu_s)(7 - 5\nu_m)}, & \bar{c}_7 &= -\frac{q_z}{6G_m} \frac{(1 - \nu_m)(7 - 5\nu_s)}{(1 - \nu_s)(7 - 5\nu_m)}, \\ \bar{c}_8 &= 0, & \bar{c}_9 &= \frac{q_z}{12G_m} \frac{(1 - \nu_m)(5 - 4\nu_s)}{(1 - \nu_s)(1 + \nu_m)}, \\ \bar{A}_s &= \frac{1 - 2\nu_s}{6G_s(1 - \nu_s)} \left[ q_{in} \left( \frac{G_s}{G_m} - 1 \right) - 3q_R \frac{G_s}{G_m} \frac{(1 - \nu_m)}{(1 + \nu_m)} \right]. \end{aligned}$$

Substituting these into (4.5), we find (as  $\lim_{h/\hat{R} \rightarrow 0} \tau_{\theta\theta}$  is constant),

$$n^{11} = \frac{h}{\hat{R}^2} \lim_{h/\hat{R} \rightarrow 0} \tau_{\theta\theta} = p_0 P_0(\mu) + p_2 P_2(\mu), \quad (4.6)$$

where

$$p_0 = \frac{q_z h G_s}{2 \hat{R}^2 G_m} \frac{(1 - \nu_m)(-5\nu_m + 15\nu_m \nu_s - 17 + 3\nu_s)}{(1 - \nu_s)(7 - 5\nu_m)(1 + \nu_m)} + \frac{h(1 + \nu_s)}{2 \hat{R}^2 (1 - \nu_s)} \left[ q_{in} \left( \frac{G_s}{G_m} - 1 \right) - 3q_R \frac{G_s}{G_m} \frac{(1 - \nu_m)}{(1 + \nu_m)} \right] + \frac{q_{in} h}{2 \hat{R}^2},$$

$$p_2 = \frac{10q_z h G_s (1 - \nu_m)}{\hat{R}^2 G_m (1 - \nu_s)(7 - 5\nu_m)}.$$

#### 4.4 Interpretation

Given the stress resultant calculated in the previous section, we now wish to find which regions of the shell are in compression and tension. We have

$$n^{11} = p_0 + \frac{p_2}{4} + \frac{3p_2}{4} \cos 2\theta.$$

At the transition point between tension and compression, we have  $n^{11} = 0$ , giving

$$\theta = \left\{ \frac{1}{2} \cos^{-1} \left[ -\frac{4}{3p_2} \left( p_0 + \frac{p_2}{4} \right) \right], \pi - \frac{1}{2} \cos^{-1} \left[ -\frac{4}{3p_2} \left( p_0 + \frac{p_2}{4} \right) \right] \right\}, \quad (4.7)$$

assuming that

$$\left| \frac{4}{3p_2} \left( p_0 + \frac{p_2}{4} \right) \right| < 1. \quad (4.8)$$

In the case that  $q_z > 0$  (compression at infinity), we have that  $n^{11} < 0$  in between the values in (4.7), assuming condition (4.8) still holds. (If  $q_z < 0$ ,  $n^{11} > 0$  in between the two values.)

Note in particular that if  $q_R = q_{in} = 0$ , the values in (4.7) depend only on the Poisson ratios  $\nu_m$  and  $\nu_s$ . For example, taking the values  $\nu_s = 0.35$ ,  $\nu_m = 0.45$  and setting  $q_R = q_{in} = 0$ , we find that we are in compression for  $\theta \in (0.582, 2.559)$ , *independently* of the shear moduli of the materials and the magnitude of the applied stress at infinity. The region of compression is shown as the thick curve in Figure 2 (the thin curve representing areas in tension).

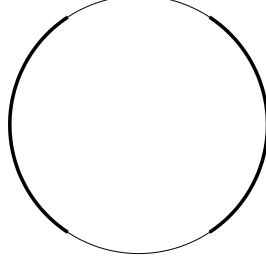
### 5. The Rayleigh–Ritz method

Before determining the deformation in the prebuckled state, we consider the problem of finding the critical load by finding the stationary point of (3.13), by the Trefftz criterion. We limit our consideration to axisymmetric buckling patterns, so that we can write  $\mathbf{v} = v_R \mathbf{e}_R + v_\theta \mathbf{e}_\theta$ , where  $v_R$ ,  $v_\theta$  are functions of  $R$  and  $\theta$  only. Note that these are the physical components of the shell displacement. These can be related to the components referred to base vectors by the relations

$$v_1 = \hat{R} v_\theta, \quad w = v_R.$$

To find the stationary value of (3.13), we will use the Rayleigh–Ritz approach (16) which involves writing the virtual displacement as an infinite series,

$$v_R = \sum_{n=0}^{\infty} \mathcal{U}_n P_n(\mu) \quad v_\theta = \sum_{n=1}^{\infty} \mathcal{V}_n P_n^1(\mu), \quad (5.1)$$



**Fig. 2** Areas of the spherical shell in compression (thick) and tension (thin).

or equivalently

$$w = \sum_{n=0}^{\infty} \mathcal{U}_n P_n(\mu), \quad v_1 = \sum_{n=1}^{\infty} \hat{R} \mathcal{V}_n P_n^1(\mu), \quad (5.2)$$

where  $P_n^1(\mu)$  is an associated Legendre function. These can be given by the formula

$$P_n^m(\mu) = (-1)^m (1 - \mu^2)^{m/2} \frac{d^m P_n}{d\mu^m}. \quad (5.3)$$

The coefficients are then found by solving

$$\frac{\partial}{\partial \mathcal{U}_n} \Delta W_2 = 0, \quad \frac{\partial}{\partial \mathcal{V}_n} \Delta W_2 = 0. \quad (5.4)$$

We will get an infinite system of linear equations whose determinant must be set to zero for a nonzero buckling deformation. The critical value for the applied stress at infinity will be found from this condition.

During the course of the analysis we will require a number of identities involving Legendre polynomials and the associated Legendre functions, which can be found, for example, in Lebedev's treatise (17). Chief among these identities is the orthogonality condition,

$$\int_{-1}^1 P_k^m(\mu) P_l^m(\mu) d\mu = \frac{2}{2k+1} \frac{(k+m)!}{(k-m)!} \delta_{kl}. \quad (5.5)$$

A number of other identities which will be used are

$$P'_{n+1}(\mu) - P'_{n-1}(\mu) = (2n+1)P_n(\mu), \quad (5.6)$$

$$\mu P_n^m(\mu) = \frac{n-m+1}{2n+1} P_{n+1}^m(\mu) + \frac{n+m}{2n+1} P_{n-1}^m(\mu), \quad (5.7)$$

$$(1 - \mu^2)P'_n(\mu) = nP_{n-1}(\mu) - n\mu P_n(\mu). \quad (5.8)$$

We also note that the differential equation satisfied by  $P_n^m(\mu)$  is

$$(1 - \mu^2) \frac{d^2 P_n^m}{d\mu^2} - 2\mu \frac{d P_n^m}{d\mu} + \left[ n(n+1) - \frac{m^2}{1 - \mu^2} \right] P_n^m(\mu) = 0. \quad (5.9)$$

## 6. Stationarity of the second variation $\Delta W_2$

We will now return to the expression for the change in potential energy derived at the end of Section 3. Recall that we had considered the linear terms  $\Delta W_1$  and the quadratic terms  $\Delta W_2$ . We stated that the equation  $\Delta W_1 = 0$  would give the equilibrium (pre-buckling) state. However, we have calculated this state already by the full equations of linear elasticity. It can be shown that by substituting the full linear solution obtained earlier into  $\Delta W_1$ , we have that  $\Delta W_1 = O(h^2/\widehat{R}^2)$ , which is small compared to the individual terms in the integral. Thus we need not consider  $\Delta W_1$  further.

Now consider the quadratic terms, given by (3.13). First we will consider  $\mathcal{J}_1$ . Given that  $w$  is independent of the coordinate  $\phi$ , from (3.14) we have that

$$\mathcal{J}_1 = \iint_{\text{shell}} \frac{{}^{(1)}n^{11}}{2} \left( \frac{dw}{d\theta} \right)^2 dS,$$

where we take as read that ‘shell’ means the mid-shell surface. The stress resultant will be given by (4.6), but for simplicity we will consider two cases only: firstly  $q_z = q_\infty$ ,  $q_R = 0$  and  $q_{\text{in}} = 0$ ; and secondly  $q_z = 0$ ,  $q_R = q_\infty$  and  $q_{\text{in}} = 0$ . Thus

$${}^{(1)}n^{11} = q_\infty (p_0 P_0(\mu) + p_2 P_2(\mu)), \quad (6.1)$$

where for the first case we redefine

$$p_0 = \frac{hG_s}{2\widehat{R}^2 G_m} \frac{(1 - \nu_m)(-5\nu_m + 15\nu_m\nu_s - 17 + 3\nu_s)}{(1 - \nu_s)(7 - 5\nu_m)(1 + \nu_m)}, \quad (6.2)$$

$$p_2 = \frac{10hG_s(1 - \nu_m)}{\widehat{R}^2 G_m(1 - \nu_s)(7 - 5\nu_m)}, \quad (6.3)$$

and for the second case

$$p_0 = -\frac{3hG_s(1 + \nu_s)(1 - \nu_m)}{2\widehat{R}^2 G_m(1 + \nu_m)(1 - \nu_s)}, \quad p_2 = 0. \quad (6.4)$$

Thus

$$\mathcal{J}_1 = \iint_{\text{shell}} \frac{q_\infty}{2} (p_0 + p_2 P_2(\mu)) \left( \frac{dw}{d\theta} \right)^2 dS.$$

We now consider  $\mathcal{J}_2$  from (3.15). In his analysis, Koiter (2) employed the van der Neut substitution,

$$v_\alpha = \psi_{,\alpha} + \varepsilon_{\alpha\lambda} a^{\lambda\mu} \chi_{,\mu}, \quad (6.5)$$

where  $\varepsilon_{\alpha\lambda}$  is the surface alternating tensor and  $\psi, \chi$  are functions to be determined. Under our assumption of axisymmetry, this simplifies to  $v_\alpha = \psi_{,\alpha}$ , where from (5.2)<sub>2</sub> we have

$$\psi = \sum_{n=0}^{\infty} \left( \widehat{R} \mathcal{V}_n \right) P_n(\mu), \quad (6.6)$$

with  $\mathcal{V}_0$  arbitrary (we will set it to zero without loss of generality). Then the linearised strain tensor  $\theta_{\alpha\beta}$  of (3.6) becomes

$$\theta_{\alpha\beta} = \psi|_{\alpha\beta} - \frac{w}{\widehat{R}} a_{\alpha\beta},$$

since in shallow buckling the order of covariant differentiation is irrelevant (2). Hence the integrand in  $\mathcal{J}_2$  becomes

$$\frac{h}{2} E^{\alpha\beta\lambda\mu} \frac{w^2}{\widehat{R}^2} a_{\alpha\beta} a_{\lambda\mu} - h E^{\alpha\beta\lambda\mu} \psi|_{\alpha\beta} \frac{w}{\widehat{R}} a_{\lambda\mu} + \frac{h}{2} E^{\alpha\beta\lambda\mu} \psi|_{\alpha\beta} \psi|_{\lambda\mu} + \frac{h^3}{24} E^{\alpha\beta\lambda\mu} w|_{\alpha\beta} w|_{\lambda\mu}, \quad (6.7)$$

using (3.7) to determine  $\rho_{\alpha\beta}$ . From (3.3), we find that the first and second terms become respectively

$$\frac{2hG_s(1+\nu_s)}{1-\nu_s} \frac{w^2}{\widehat{R}^2}, \quad -\frac{2hG_s(1+\nu_s)}{\widehat{R}(1-\nu_s)} w \nabla^2 \psi,$$

using the fact that

$$a^{\alpha\beta} \psi|_{\alpha\beta} = \nabla^2 \psi \quad (6.8)$$

is the surface Laplacian on the shell. The integrals of the other two terms are both of the form

$$\iint_{\text{shell}} E^{\alpha\beta\lambda\mu} w|_{\alpha\beta} w|_{\lambda\mu} dS.$$

Given that  $E^{\alpha\beta\lambda\mu}$  is composed entirely of  $a^{\alpha\beta}$  terms, we have  $E^{\alpha\beta\lambda\mu}|_{\rho} = 0$  since covariant differentiation of metric tensors gives zero, or  $a^{\alpha\beta}|_{\rho} = a_{\alpha\beta}|_{\rho} = 0$ . Hence the integral may be written

$$\iint_{\text{shell}} \left[ (E^{\alpha\beta\lambda\mu} w|_{\alpha\beta} w|_{\lambda})|_{\mu} - E^{\alpha\beta\lambda\mu} w|_{\alpha\beta\mu} w|_{\lambda} \right] dS,$$

where the first term disappears by the divergence theorem (as the shell is closed). Repeating this process we find that, after changing the order of covariant differentiation (allowable under the assumptions of shallow shell theory), the integral may be written

$$\iint_{\text{shell}} E^{\alpha\beta\lambda\mu} w|_{\alpha\beta\lambda\mu} w dS, \quad \text{or} \quad \frac{2G_s}{1-\nu_s} \iint_{\text{shell}} w \nabla^4 w dS.$$

Thus the integral corresponding to the fourth term in (6.7) becomes

$$\frac{2G_s}{1-\nu_s} \iint_{\text{shell}} (\nabla^2 w)^2 dS$$

on using Green's theorem. Using the same method on the third term in (6.7) we find finally that

$$\mathcal{J}_2 = \frac{hG_s}{1-\nu_s} \iint_{\text{shell}} \left[ \left( \nabla^2 \psi - (1+\nu_s) \frac{w}{\widehat{R}} \right)^2 + (1-\nu_s^2) \frac{w^2}{\widehat{R}^2} + \frac{h^2}{12} (\nabla^2 w)^2 \right] dS.$$

Now consider the integral  $\mathcal{J}_3$ , from (3.16). The stress tensor is given by  $\tau^{ij} = A^{ijkl} e_{kl}$ , so that

$$\mathcal{J}_3 = \frac{1}{2} \iiint_{R>R_1} \tau^{ij} u_i|_j dV$$

on using the symmetry of the stress tensor and the definition of the strain tensor. Then, by the divergence theorem and the equilibrium equation for the stress tensor, we have

$$\mathcal{J}_3 = \frac{1}{2} \iint_{\partial V} \tau^{ij} n_j u_i dS,$$

where  $\partial V$  is the internal boundary of the region,  $R = R_1$ , since both the stresses and displacements vanish at infinity. The normal vector points inwards, so that the only non-zero component is  $n_1 = -1$ . This gives

$$\mathcal{J}_3 = -\frac{1}{2} \iint_{\partial V} (\tau^{11} u_1 + \tau^{12} u_2) \, dS.$$

The displacement components  $u_1$  and  $u_2$  here correspond to the displacement of the outer surface of the shell, which (to leading order) can be replaced by the middle-surface displacement components. Thus, writing the stresses in physical components, we obtain

$$\mathcal{J}_3 = -\frac{1}{2} \iint_{R=R_1} (\tau_{RR} w + \tau_{R\theta} v_\theta) \, dS.$$

Thus, combining the integrals  $\mathcal{J}_1$ – $\mathcal{J}_3$ , we have

$$\begin{aligned} \Delta W_2 = \iint_{\text{shell}} \left\{ \frac{q_\infty}{2} (p_0 + p_2 P_2(\mu)) \left( \frac{dw}{d\theta} \right)^2 + \frac{hG_s}{1 - \nu_s} \left[ \left( \nabla^2 \psi - (1 + \nu_s) \frac{w}{R} \right)^2 \right. \right. \\ \left. \left. + (1 - \nu_s^2) \frac{w^2}{\widehat{R}^2} + \frac{h^2}{12} (\nabla^2 w)^2 \right] \right\} dS - \frac{1}{2} \iint_{R=R_1} (\tau_{RR} w + \tau_{R\theta} v_\theta) \, dS. \quad (6.9) \end{aligned}$$

### 6.1 The functional in terms of the Legendre coefficients

If  $F(\theta)$  is some function which is independent of  $\phi$ , then

$$\iint_{\text{shell}} F(\theta) \, dS = 2\pi \widehat{R}^2 \int_{-1}^1 F(\mu) \, d\mu. \quad (6.10)$$

We will now use this fact to simplify (6.9) for each term  $\mathcal{J}_i$  in turn. We know that  $P_2(\mu) = (3\mu^2 - 1)/2$ , which we can substitute together with (5.2)<sub>1</sub> and the result (6.10) into the first term in (6.9), denoted  $\mathcal{J}_1$ , to get

$$\begin{aligned} \mathcal{J}_1 = 2\pi q_\infty \left( p_0 - \frac{p_2}{2} \right) \widehat{R}^2 \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} \mathcal{U}_n^2 \\ + \frac{3\pi q_\infty p_2 \widehat{R}^2}{2} \sum_{n,m=1}^{\infty} \mathcal{U}_n \mathcal{U}_m \int_{-1}^1 \mu P_n^1(\mu) \cdot \mu P_m^1(\mu) \, d\mu, \end{aligned}$$

using the orthogonality condition (5.5) for Legendre functions.

Using relation (5.7) we find that the second term above is equal to

$$\begin{aligned} \frac{3\pi q_\infty p_2 \widehat{R}^2}{2} \sum_{n,m=1}^{\infty} \frac{\mathcal{U}_n}{2n+1} \frac{\mathcal{U}_m}{2m+1} \int_{-1}^1 (nP_{n+1}^1(\mu) + (n+1)P_{n-1}^1(\mu)) \\ \times (mP_{m+1}^1(\mu) + (m+1)P_{m-1}^1(\mu)) \, d\mu \end{aligned}$$

which, on using the orthogonality condition, gives us  $\mathcal{J}_1$  in the form

$$\begin{aligned} \mathcal{J}_1 = \pi q_\infty \hat{R}^2 \sum_{n=1}^{\infty} \left[ \left\{ (2p_0 - p_2) \frac{n(n+1)}{2n+1} \right. \right. \\ \left. \left. + 3p_2 \left( \frac{n^2(n+1)(n+2)}{(2n+1)^2(2n+3)} + \frac{(n-1)n(n+1)^2}{(2n-1)(2n+1)^2} \right) \right\} \mathcal{U}_n^2 \right. \\ \left. + \frac{6p_2 n(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)} \mathcal{U}_n \mathcal{U}_{n+2} \right]. \quad (6.11) \end{aligned}$$

Now consider the second term of (6.9), which we denoted  $\mathcal{J}_2$ . From (6.8) we note that

$$\nabla^2 \psi = \frac{1}{\hat{R}^2} \psi|_{11} + \frac{1}{\hat{R}^2 \sin^2 \theta} \psi|_{22}.$$

Using the fact that  $\psi_{,2} = 0$  and  $w_{,2} = 0$ , and the definitions of covariant differentiation in spherical polar coordinates, we obtain

$$\nabla^2 \psi = \frac{1}{\hat{R}^2} (\psi_{,11} + \cot \theta \psi_{,1}), \quad \nabla^2 w = \frac{1}{\hat{R}^2} (w_{,11} + \cot \theta w_{,1}).$$

From these, using (6.6), (5.2)<sub>1</sub> and (5.9), we find that

$$\nabla^2 \psi = - \sum_{n=0}^{\infty} \frac{n(n+1) \mathcal{V}_n}{\hat{R}} P_n(\mu), \quad \nabla^2 w = - \sum_{n=0}^{\infty} \frac{n(n+1) \mathcal{U}_n}{\hat{R}^2} P_n(\mu),$$

which, on substitution into the second term in (6.9) and using the orthogonality condition on Legendre functions, gives

$$\begin{aligned} \mathcal{J}_2 = \frac{4\pi h G_s}{\hat{R}^2 (1 - \nu_s)} \sum_{n=0}^{\infty} \frac{1}{2n+1} \left[ (n(n+1) \mathcal{V}_n + (1 + \nu_s) \mathcal{U}_n)^2 \hat{R}^2 \right. \\ \left. + (1 - \nu_s^2) \hat{R}^2 \mathcal{U}_n^2 + \frac{h^2 n^2 (n+1)^2 \mathcal{U}_n^2}{12} \right]. \quad (6.12) \end{aligned}$$

In order to calculate  $\mathcal{J}_3$ , the third term in (6.9), we need to determine the displacement field  $\mathbf{u}$  that is induced in the elastic matrix from the virtual deformation  $\mathbf{v}$  of the shell. This is found by solving the elasticity equations for  $\mathbf{u}$  with a displacement boundary condition  $\mathbf{u}|_{R=R_1} = \mathbf{v}$ , assuming that the displacement on the outer shell surface is approximately equal to the mid-surface displacement  $\mathbf{v}$ . We also state that the stress field vanishes as  $R \rightarrow \infty$ . This problem has been solved by Lur'e (18). Firstly the boundary displacement is decomposed into a series of homogeneous surface vector spherical harmonics,

$$\mathbf{v} = \sum_{n=0}^{\infty} \mathbf{Y}_n(\theta, \phi).$$

Then the vector

$$\mathbf{U}_{-n-1} = \left( \frac{R_1}{R} \right)^{n+1} \mathbf{Y}_n(\theta, \phi) \quad (6.13)$$

is formed, which gives us the final solution

$$\mathbf{u} = \sum_{n=0}^{\infty} \left[ \mathbf{U}_{-n-1} - \frac{1}{2}(R_1^2 - R^2) \frac{\nabla(\nabla \cdot \mathbf{U}_{-n-1})}{(3 - 4\nu_m)(n+1) + 2(1 - \nu_m)} \right]. \quad (6.14)$$

The first difficulty we find when trying to solve this problem is that the terms in the series in (5.1), taken together, are not homogeneous surface vector spherical harmonics. We need to write  $\mathbf{v}$  as a series of the following:

$$\mathbf{Y}_n = \alpha_n P_n^1(\mu)(\cos \phi \mathbf{e}_x + \sin \phi \mathbf{e}_y) + \beta_n P_n(\mu) \mathbf{e}_z, \quad (6.15)$$

where  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  is the Cartesian three-dimensional basis. On transforming to spherical coordinates,

$$\mathbf{Y}_n = \left( \alpha_n \sqrt{1 - \mu^2} P_n^1(\mu) + \beta_n \mu P_n(\mu) \right) \mathbf{e}_R + \left( \alpha_n \mu P_n^1(\mu) - \beta_n \sqrt{1 - \mu^2} P_n(\mu) \right) \mathbf{e}_\theta.$$

From (5.3) and (5.8) we find that

$$\mathbf{Y}_n = ((n\alpha_n + \beta_n)\mu P_n(\mu) - n\alpha_n P_{n-1}(\mu)) \mathbf{e}_R + (\alpha_n \mu P_n^1(\mu) - \beta_n \sqrt{1 - \mu^2} P_n(\mu)) \mathbf{e}_\theta.$$

Now, from (5.6),

$$\sqrt{1 - \mu^2} P_n(\mu) = \frac{\sqrt{1 - \mu^2}}{2n + 1} (P'_{n+1}(\mu) - P'_{n-1}(\mu)) = \frac{P_{n-1}^1(\mu) - P_{n+1}^1(\mu)}{2n + 1}.$$

Using this together with (5.7), we find that

$$\mathbf{Y}_n = (n(\gamma_n - \alpha_n)P_{n-1}(\mu) + (n+1)\gamma_n P_{n+1}(\mu)) \mathbf{e}_R + ((\alpha_n - \gamma_n)P_{n-1}^1(\mu) + \gamma_n P_{n+1}^1(\mu)) \mathbf{e}_\theta, \quad (6.16)$$

where  $\gamma_n = (n\alpha_n + \beta_n)/(2n + 1)$ .

Thus

$$\begin{aligned} \mathbf{v} &= \sum_{n=0}^{\infty} [n(\gamma_n - \alpha_n)P_{n-1} + (n+1)\gamma_n P_{n+1}] \mathbf{e}_R + \sum_{n=0}^{\infty} [(\alpha_n - \gamma_n)P_{n-1}^1 + \gamma_n P_{n+1}^1] \mathbf{e}_\theta \\ &= \sum_{m=0}^{\infty} [(m+1)(\gamma_{m+1} - \alpha_{m+1}) + m\gamma_{m-1}] P_m(\mu) \mathbf{e}_R \\ &\quad + \sum_{m=1}^{\infty} [\alpha_{m+1} - \gamma_{m+1} + \gamma_{m-1}] P_m^1(\mu) \mathbf{e}_\theta, \end{aligned}$$

on rearranging the indices of the terms in the sums. This we can compare with (5.1) to get

$$\mathcal{U}_n = (n+1)(\gamma_{n+1} - \alpha_{n+1}) + n\gamma_{n-1}, \quad \mathcal{V}_n = \alpha_{n+1} - \gamma_{n+1} + \gamma_{n-1},$$

or

$$\gamma_n - \alpha_n = \frac{\mathcal{U}_{n-1} - (n-1)\mathcal{V}_{n-1}}{2n-1}, \quad \gamma_n = \frac{\mathcal{U}_{n+1} + (n+2)\mathcal{V}_{n+1}}{2n+3}, \quad (6.17)$$



which we can substitute into the relations that we find for  $\alpha_n$  and  $\gamma_n$ .

Now, from (6.16) and (6.13) we find that

$$\begin{aligned} \mathbf{U}_{-n-1} = \left(\frac{R_1}{R}\right)^{n+1} \{ [n(\gamma_n - \alpha_n)P_{n-1}(\mu) + (n+1)\gamma_n P_{n+1}(\mu)] \mathbf{e}_R \\ + [(\alpha_n - \gamma_n)P_{n-1}^1(\mu) + \gamma_n P_{n+1}^1(\mu)] \mathbf{e}_\theta \}. \end{aligned}$$

On using various identities involving Legendre functions, we find that

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{U}_{-n-1}) = \frac{(n+2)}{R_1^2} \left(\frac{R_1}{R}\right)^{n+3} (2n+1)(n+1)\gamma_n P_{n+1}(\mu) \mathbf{e}_R \\ - \frac{1}{R_1^2} \left(\frac{R_1}{R}\right)^{n+3} (2n+1)(n+1)\gamma_n P_{n+1}^1(\mu) \mathbf{e}_\theta. \end{aligned}$$

Thus, from (6.14), we find that the radial and tangential components of the virtual displacement field in the matrix are given by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{e}_R = \sum_{n=0}^{\infty} \left\{ [n(\gamma_n - \alpha_n)P_{n-1}(\mu) + (n+1)\gamma_n P_{n+1}(\mu)] \left(\frac{R_1}{R}\right)^{n+1} \right. \\ \left. - \frac{n+2}{2} \left(1 - \frac{R^2}{R_1^2}\right) \left(\frac{R_1}{R}\right)^{n+3} \frac{(2n+1)(n+1)\gamma_n P_{n+1}(\mu)}{(3-4\nu_m)(n+1) + 2(1-\nu_m)} \right\}, \\ \mathbf{u} \cdot \mathbf{e}_\theta = \sum_{n=0}^{\infty} \left\{ [(\alpha_n - \gamma_n)P_{n-1}^1(\mu) + \gamma_n P_{n+1}^1(\mu)] \left(\frac{R_1}{R}\right)^{n+1} \right. \\ \left. + \frac{1}{2} \left(1 - \frac{R^2}{R_1^2}\right) \left(\frac{R_1}{R}\right)^{n+3} \frac{(2n+1)(n+1)\gamma_n P_{n+1}^1(\mu)}{(3-4\nu_m)(n+1) + 2(1-\nu_m)} \right\}. \end{aligned}$$

Rearranging the indices and using (6.17) gives

$$\begin{aligned} \mathbf{u} = \sum_{m=0}^{\infty} \left[ A_m \left(\frac{R_1}{R}\right)^{m+2} + B_m \left(\frac{R_1}{R}\right)^m \right] P_m(\mu) \mathbf{e}_R \\ + \sum_{m=1}^{\infty} \left[ C_m \left(\frac{R_1}{R}\right)^{m+2} + D_m \left(\frac{R_1}{R}\right)^m \right] P_m^1(\mu) \mathbf{e}_\theta, \quad (6.18) \end{aligned}$$

where

$$A_n + B_n = \mathcal{U}_n, \quad C_n + D_n = \mathcal{V}_n, \quad (6.19)$$

and

$$A_n = -(n+1)C_n, \quad (6.20)$$

$$C_n = \frac{1}{2n+1} \left\{ n\mathcal{V}_n - \mathcal{U}_n + \frac{n(2n-1)}{2} \left[ \frac{\mathcal{U}_n + (n+1)\mathcal{V}_n}{(3-4\nu_m)n + 2(1-\nu_m)} \right] \right\}. \quad (6.21)$$

Now we use the stress-strain relations in spherical polar coordinates to obtain the stress components from the displacement (6.18). On substituting these into the third term of (6.9), we get an integral involving squares of Legendre functions. We finally obtain, on using (6.19)–(6.20) to simplify the resulting expression,

$$\begin{aligned} \mathcal{J}_3 = 4\pi G_m R_1 \sum_{n=0}^{\infty} \left\{ \frac{\mathcal{U}_n}{2n+1} \left[ n\mathcal{U}_n - 2(n+1)C_n \right. \right. \\ \left. \left. + \frac{\nu_m}{1-2\nu_m} ((n-2)\mathcal{U}_n - 2(n+1)C_n + n(n+1)\mathcal{V}_n) \right] \right. \\ \left. + \frac{n(n+1)\mathcal{V}_n}{2(2n+1)} [(n+1)\mathcal{V}_n + 2C_n - \mathcal{U}_n] \right\}, \quad (6.22) \end{aligned}$$

where  $C_n$  is given by (6.21).

### 7. The eigenvalue problem

Now we use the results (6.11), (6.12) and (6.22), and apply conditions (5.4). From the second of these, we have

$$\frac{\partial}{\partial \mathcal{V}_n} (\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3) = 0. \quad (7.1)$$

Now,

$$\begin{aligned} \frac{\partial \mathcal{J}_1}{\partial \mathcal{V}_n} &= 0, \\ \frac{\partial \mathcal{J}_2}{\partial \mathcal{V}_n} &= \frac{8\pi h G_s n(n+1)(n(n+1)\mathcal{V}_n + (1+\nu_s)\mathcal{U}_n)}{(1-\nu_s)(2n+1)}, \\ \frac{\partial \mathcal{J}_3}{\partial \mathcal{V}_n} &= \frac{4\pi G_m R_1}{2n+1} \left\{ \mathcal{U}_n \left[ -2(n+1)E_n + \frac{\nu_m}{1-2\nu_m} (n(n+1) - 2(n+1)E_n) \right] \right. \\ &\quad \left. + \frac{1}{2} n(n+1)(2(n+1)\mathcal{V}_n + 2C_n - \mathcal{U}_n + 2E_n\mathcal{V}_n) \right\}, \end{aligned}$$

where  $C_n$  is given by (6.21) and we write

$$E_n = \frac{\partial C_n}{\partial \mathcal{V}_n} = \frac{1}{2n+1} \left[ n + \frac{n(n+1)(2n-1)}{2((3-4\nu_m)n + 2(1-\nu_m))} \right].$$

Substituting the above into (7.1) gives us

$$\mathcal{V}_n = \mu_n \mathcal{U}_n, \quad (7.2)$$

where

$$\begin{aligned} \mu_n = - \left\{ \frac{2hn(n+1)G_s}{R_1(1-\nu_s)G_m} + (n+1) + 2E_n \right\}^{-1} \left\{ \frac{2h(1+\nu_s)G_s}{R_1(1-\nu_s)G_m} - \frac{2E_n(1-\nu_m)}{n(1-2\nu_m)} \right. \\ \left. + \frac{1}{2n+1} \left( \frac{n(2n-1)}{2((3-4\nu_m)n + 2(1-\nu_m))} - 1 \right) - \frac{1}{2} + \frac{\nu_m}{1-2\nu_m} \right\} \quad (7.3) \end{aligned}$$

with  $\mu_0 = 0$ . Substituting (7.2) into the expressions for  $\mathcal{J}_1$  to  $\mathcal{J}_3$  gives us  $\Delta W_2$  in the form

$$\Delta W_2 = \sum_{n=0}^{\infty} [(a_n q_{\infty} + b_n) \mathcal{U}_n^2 + c_n q_{\infty} \mathcal{U}_n \mathcal{U}_{n+2}],$$

where the coefficients are given by

$$\begin{aligned} a_n &= \pi \hat{R}^2 \left\{ (2p_0 - p_2) \frac{n(n+1)}{2n+1} \right. \\ &\quad \left. + 3p_2 \left[ \frac{n^2(n+1)(n+2)}{(2n+1)^2(2n+3)} + \frac{(n-1)n(n+1)^2}{(2n-1)(2n+1)^2} \right] \right\}, \\ b_n &= \frac{4\pi h G_s}{\hat{R}^2(1-\nu_s)(2n+1)} \left[ (n(n+1)\mu_n + 1 + \nu_s)^2 \hat{R}^2 \right. \\ &\quad \left. + (1-\nu_s^2) \hat{R}^2 + h^2 n^2 (n+1)^2 / 12 \right] \\ &\quad + \frac{4\pi G_m R_1}{2n+1} \left[ n - 2(n+1)F_n + \frac{\nu_m}{1-2\nu_m} (n-2-2(n+1)F_n + n(n+1)\mu_n) \right. \\ &\quad \left. + \frac{n(n+1)\mu_n}{2} ((n+1)\mu_n + 2F_n - 1) \right], \\ c_n &= \frac{6\pi \hat{R}^2 p_2 n(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)}, \end{aligned} \quad (7.4)$$

where

$$F_n = \frac{C_n}{\mathcal{U}_n} = \frac{1}{2n+1} \left\{ n\mu_n - 1 + \frac{n(2n-1)}{2} \left[ \frac{1+(n+1)\mu_n}{(3-4\nu_m)n+2(1-\nu_m)} \right] \right\}.$$

Then from (5.4)<sub>1</sub> we have

$$2(a_n q_{\infty} + b_n) \mathcal{U}_n + c_{n-2} q_{\infty} \mathcal{U}_{n-2} + c_n q_{\infty} \mathcal{U}_{n+2} = 0, \quad (7.5)$$

where  $c_{-2}$  and  $c_{-1}$  are both zero. If we let  $\lambda = 1/q_{\infty}$ , then we have the system

$$\left( -\frac{a_n}{b_n} - \lambda \right) \mathcal{U}_n - \frac{c_{n-2}}{2b_n} \mathcal{U}_{n-2} - \frac{c_n}{2b_n} \mathcal{U}_{n+2} = 0.$$

We can now split up our consideration of the  $\mathcal{U}_n$  coefficients into odd and even  $n$ :

$$\left( -\frac{a_{2n-1}}{b_{2n-1}} - \lambda \right) \mathcal{U}_n^{\text{odd}} - \frac{c_{2n-3}}{2b_{2n-1}} \mathcal{U}_{n-1}^{\text{odd}} - \frac{c_{2n-1}}{2b_{2n-1}} \mathcal{U}_{n+1}^{\text{odd}} = 0 \quad (7.6)$$

$$\left( -\frac{a_{2n-2}}{b_{2n-2}} - \lambda \right) \mathcal{U}_n^{\text{even}} - \frac{c_{2n-4}}{2b_{2n-2}} \mathcal{U}_{n-1}^{\text{even}} - \frac{c_{2n-2}}{2b_{2n-2}} \mathcal{U}_{n+1}^{\text{even}} = 0 \quad (7.7)$$

for  $n = 1, 2, \dots$ , where  $\mathcal{U}_n^{\text{odd}} = \mathcal{U}_{2n-1}$  and  $\mathcal{U}_n^{\text{even}} = \mathcal{U}_{2n-2}$ . The expressions (7.6)–(7.7) comprise two eigenvalue problems for infinite tridiagonal matrices:

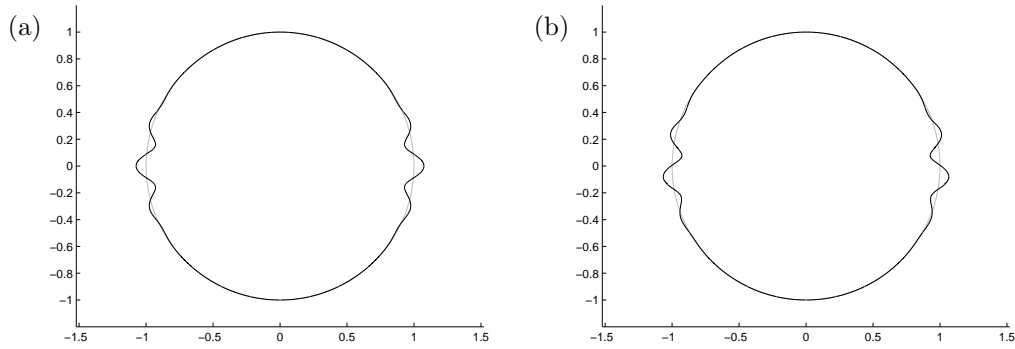
$$(\mathbf{A}^{\text{odd}} - \lambda \mathbf{I}) \mathcal{U}^{\text{odd}} = \mathbf{0}, \quad (\mathbf{A}^{\text{even}} - \lambda \mathbf{I}) \mathcal{U}^{\text{even}} = \mathbf{0},$$

which can be solved numerically.

## 8. Results

Recall that we planned to consider two modes of deformation only, namely uniaxial compression at infinity, where the stress field at infinity had only the component  $\tau_{zz} = -q_\infty$ , and a hydrostatic compression at infinity, for which  $\tau_{RR} = -q_\infty$ . We will consider the uniaxial case first.

The infinite systems of the previous section are truncated and solved numerically using (6.2) and (6.3) for  $p_0$  and  $p_2$ . We set  $G_m = 1$  since we can scale  $q_\infty$  with the true value of  $G_m$  without changing the problem mathematically; likewise  $\hat{R}$  can be set to 1 without loss of generality. We assume that  $G_s = 100 G_m$  and  $h = 0.02 \hat{R}$ , and that the Poisson ratios of the two materials are  $\nu_m = 0.45$ ,  $\nu_s = 0.35$ . The truncation point of the infinite system must be chosen carefully so as not to ‘lose’ any information from the system; we choose the point such that if the number of terms is doubled the solution remains the same to within a certain tolerance. We will be searching for the lowest positive value of  $q_\infty$ , in order to find the first point at which the equilibrium solution becomes unstable (considering a gradual quasisteady loading of the material). This corresponds to finding the *largest* possible eigenvalue  $\lambda$ . Considering even and odd buckling modes separately, the largest positive eigenvalue in both cases is  $\lambda = 18.18$ , giving a lowest critical compressive stress at infinity of  $q_\infty = 0.0550$ . The corresponding eigenvectors in each case give us the constants  $\mathcal{U}_n$ , and hence from (7.2) the constants  $\mathcal{V}_n$ . We substitute these values of the coefficients into (5.1) to determine the displacement components of the characteristic buckling pattern which would occur at the critical buckling stress. These buckling patterns are shown in Figure 3.



**Fig. 3** Buckling patterns for the largest positive eigenvalue: (a) even, (b) odd.

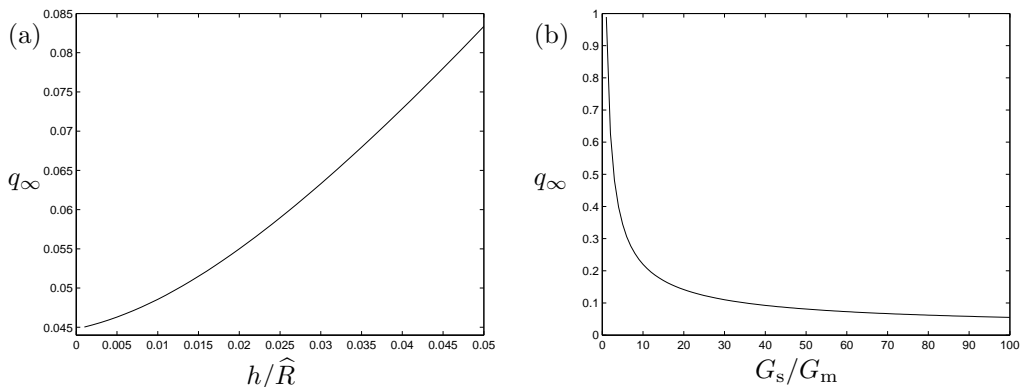
It may seem surprising that the spheres buckle around the equator. After all, by common experience if a spherical shell is placed between two flat plates and compressed, which is a superficially similar mode of deformation, the spheres tend to buckle at the poles. However, compressing spherical shells between flat plates induces a different pattern of stresses in the shell from embedding them in an elastic material. An embedded spherical shell, under uniaxial compression at infinity, is in compression (in the  $\tau_{\theta\theta}$  component) around the equator

while in tension around the poles (see Figure 2); this is why buckling occurs around the equator. Conversely, placing a shell between two flat plates and compressing it results in the region of highest compressive stress — and hence buckling — being around the poles.

One should therefore be wary of modelling the buckling of embedded shells by sandwiching them between flat plates, for the reasons stated above. This approach does have its uses, however: if the shells are *not* bonded to the elastic matrix, and the matrix is much more compliant than the shell (as is the case here, since  $G_s = 100 G_m$ ), then the shells will only be in contact with the matrix at the poles, mimicking the sandwiching approach. The likely true configuration of the spheres is *partial* bonding, which would require us to solve a coupled delamination problem for the shells.

It is interesting to note the behaviour of the solution to the eigenvalue problem as we vary the parameters. In particular, we can vary the thickness ratio  $h/\hat{R}$ . On keeping the other parameters fixed, the effect that changing this ratio has on the critical stress  $q_\infty$  can be seen in Figure 4(a). We see that as this ratio tends to zero the critical stress tends to a fixed value. We can also determine what effect this has on the buckling pattern. As shown in Figure 5, as  $h/\hat{R} \rightarrow 0$  the number of oscillations in the pattern increases while the region over which buckling occurs becomes ever smaller.

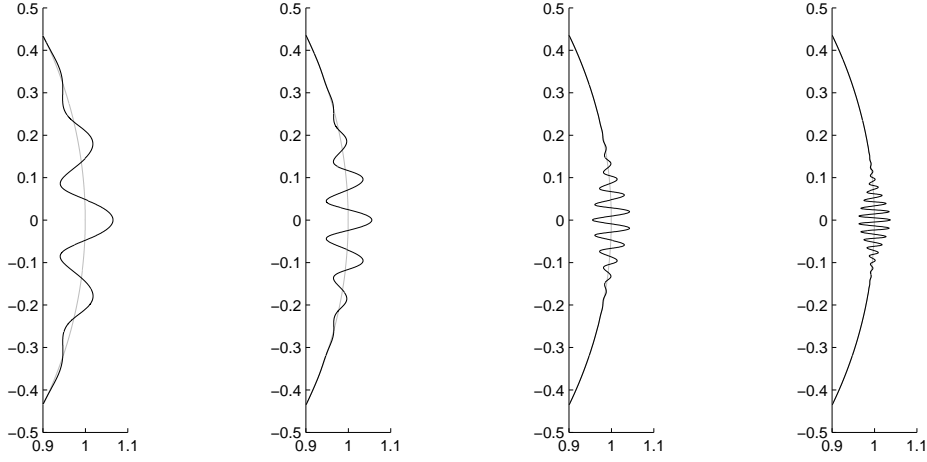
The second parameter that we consider is the ratio of shear moduli,  $G_s/G_m$ . In the limit as this quantity tends to zero, holding the other constants at the values chosen previously, we see from Figure 4(b) that the critical stress  $q_\infty$  blows up. The effect on the buckling pattern in this limit is similar to that seen in Figure 5.



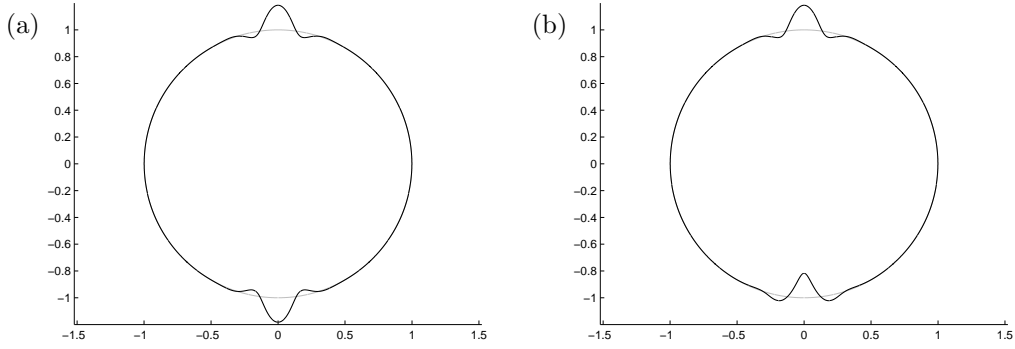
**Fig. 4** Dependence of critical stress  $q_\infty$  on parameters: (a)  $h/\hat{R}$ , (b)  $G_s/G_m$ .

Finally, we note that by solving the eigenvalue problem above, we are also able to find the largest negative eigenvalue. This corresponds to the lowest critical *tensile* stress at infinity for which the equilibrium configuration is unstable (restricting buckling patterns to axisymmetric deformations). In both odd and even cases we find the critical stress to be  $-q_\infty = 0.1328$  for the material constants given previously. The corresponding buckling patterns are given in Figure 6.

These results are physically unrealistic for the simple reason that we only consider



**Fig. 5** Buckling patterns for  $h/\hat{R} = 0.01, 0.005, 0.002, 0.001$ .



**Fig. 6** Buckling patterns for the largest negative eigenvalue: (a) even, (b) odd.

axisymmetric deformations. Had we considered buckling in the  $\phi$ -direction, we would have taken account of the fact that the stress component  $\tau_{\phi\phi}$  in the shell is in compression around the equator. Thus the most likely *lowest* critical stress for the case of tension at infinity would correspond to non-axisymmetric buckling around the equator. In the case of uniaxial compression at infinity the component  $\tau_{\phi\phi}$  is in compression around the poles. It is implausible that buckling in the  $\phi$ -direction would occur here at a lower critical stress than the axisymmetric buckling mode.

### 8.1 Hydrostatic stress at infinity

We will now calculate the lowest critical stress in the case where a hydrostatic stress at infinity is used. We will use (6.4), and substitute them into the linear system (7.5). Now,  $p_2 = 0$  so that  $c_n = 0$  for each  $n$ . This means that (7.5) becomes  $(a_n q_\infty + b_n) \mathcal{U}_n = 0$ .

Therefore each of the buckling modes are given from (5.1) by

$$v_R = \mathcal{U}_n P_n(\mu), \quad v_\theta = \mathcal{V}_n P_n^1(\mu) = \mu_n \mathcal{U}_n P_n^1(\mu)$$

for each  $n$ , with corresponding critical stress

$$q_\infty = -\frac{b_n}{a_n}. \quad (8.1)$$

The constants  $b_n$  in this expression are unchanged from (7.4), and substituting the relevant value of  $p_0$  gives

$$a_n = -\frac{3\pi h(1-\nu_m)(1+\nu_s)G_s}{(1-\nu_s)(1+\nu_m)G_m} \frac{n(n+1)}{(2n+1)}.$$

We require the lowest critical stress at infinity, which involves finding the minimum value of (8.1) over  $n = 0, 1, 2, \dots$ . For the parameter values given previously, we find that the lowest critical stress is  $q_\infty = 0.08072$ , found when  $n = 18$ .

This result is to be compared with that of Fok and Allwright (3), who considered the buckling of an embedded shell with a hydrostatic stress field at infinity, but having introduced a simplifying assumption that the shell was inextensible, or that  $\hat{R}\nabla^2\psi + 2w = 0$ . This assumption gives  $\mu_n = \frac{2}{n(n+1)}$ , which is perhaps an oversimplification when compared to our result (7.3).

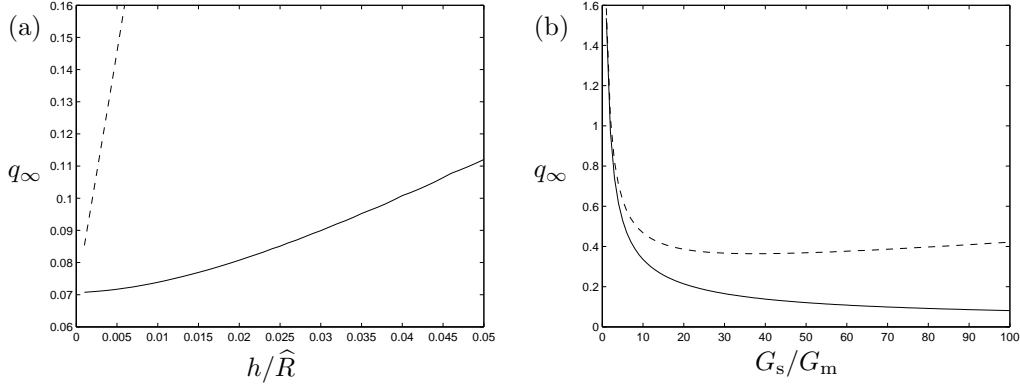
In our notation, Fok and Allwright found that the critical stress at infinity for each mode  $n$  satisfied

$$q_\infty = \frac{4G_s(1+\nu_s)(1+\nu_m)}{3(1-\nu_m)} \left[ 1 + \frac{G_m(1-\nu_s)\hat{R}}{G_s(1+\nu_s)h} \right] \times \left\{ \frac{[n(n+1) - (1-\nu_s)]}{12(1-\nu_s^2)} \frac{h^3}{\hat{R}^3} + \frac{2h}{\hat{R}(n-1)(n+2)(1+\nu_s)} + \frac{G_m[(2n^3 - n^2 + 3n + 2) - \nu_m(2n^3 - 3n^2 + 5n + 2)]}{G_s(1+\nu_s)(n-1)^2(n+2)[3n+2-2\nu_m(2n+1)]} \right\} \quad (8.2)$$

which, on minimising using our parameter values, gives the lowest critical stress as  $q_\infty = 0.4215$  when  $n = 18$ . The value of  $q_\infty$  compares quite badly with our result, indicating that the simplifying inextensibility assumption of the authors is not valid in our parameter regime (despite the fact that the order  $n = 18$  of the buckling pattern as calculated by Fok and Allwright agrees with the value arising from our theory). The results on letting the parameters  $h/\hat{R}$  and  $G_s/G_m$  tend to zero are displayed in Figure 7, and show a similar behaviour to the uniaxial case. The unbroken line corresponds to the results in our theory while the dashed line is the result of Fok and Allwright. Note that in the limit as both parameters tend to zero, the minimum  $q_\infty$  according to (8.2) becomes asymptotically closer to the value given by our theory.

## 9. Conclusions and discussion

In analysing the buckling behaviour of an embedded shell, we have obtained a number of new results. The first is the observation that in uniaxial compression the shell buckles in



**Fig. 7** Dependence of critical stress  $q_\infty$  in the radial problem on parameters: (a)  $h/\hat{R}$ , (b)  $G_s/G_m$ .

a region around its equator, as opposed to near its poles. This is justified since buckling only occurs where the shell is in compression, which holds in this equatorial region. The second result is an improved expression for the critical buckling stress when the stress field at infinity is purely hydrostatic, with confirmation that the result of Fok and Allwright are inaccurate when the thickness ratio  $h/\hat{R}$  is not small. In addition we have analysed the behaviour of the shell undergoing uniaxial compression as two ratios tend to zero: the thickness ratio and the ratio of shear moduli  $G_s/G_m$ . In both limits, the buckling pattern becomes more oscillatory, which leads us to consider an analysis in these limits by WKB theory. Those results will be published in the near future.

A few words should be said regarding the assumptions made during the analysis. The model chosen for the shell in order to analyse buckling was Koiter's shallow shell theory. This is the simplest nonlinear shell theory and the underlying assumption is that the wavelength of the buckling patterns is small compared to the radius of curvature of the shell. While this assumption seems to be confirmed *a posteriori* by the buckling patterns obtained, it may be worthwhile investigating more comprehensive nonlinear shell theories to verify that the observed buckling pattern justifies the use of the shallow shell assumption.

Another assumption made in the buckling problem is that the buckling patterns are *axisymmetric*. This assumption was not justified mathematically however, and it would be worthwhile to investigate more general buckling patterns which do not assume axisymmetry.

The main difficulty in implementing these two ideas is that the energy functional was greatly simplified by the use of the van der Neut substitution (6.5). For a spherical shell pressurised by a hydrostatic load, Koiter used the substitution to decouple the surface invariant  $\chi$  from the radial displacement  $w$  and the remaining surface invariant  $\psi$  in the energy functional. This greatly simplified the analysis. For our configuration, which has a non-uniform pre-buckling stress distribution in the shell and a complicated term due to the energy in the matrix, this substitution will not have the same effect (apart from the axisymmetric case, in which  $\chi$  is arbitrary and can be set to zero without loss of generality). This will also be the case if we consider additional terms in the shell energy functional due to the use of a different shell theory.



Finally, we note that *imperfections* in the shell cause it to buckle at a lower critical stress than a pristine shell would. Koiter analysed this effect for a shell undergoing hydrostatic pressure by including an additional term in the energy functional, but to follow this approach for the embedded shell would be more difficult for the reasons outlined above.

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