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On the many facets of the AdS/CFT correspondence – from bootstrap to supergravity

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To Claudia and to my wonderful family:

my home, my compass, my purpose.

Abstract

In this thesis we consider two distinct approaches to the study of the AdS/CFT correspondence. In the first we study correlation functions of local operators in holographic SCFTs, which from the bulk perspective represent the most natural generalization of flat-space scattering amplitudes to AdS backgrounds. For maximally supersymmetric CFTs, all four-point functions between single-trace half-BPS operators have been recently computed at tree level, using the idea of maximally R-symmetry violating (MRV) limit. In this thesis we illustrate how the MRV limit can be used to derive analogous results in theories with less than maximal supersymmetry, where the external operators that we consider are dual to gluons in AdS. We analyze our results in detail, finding several interesting features of (super) gluon scattering amplitudes in AdS, ranging from color-kinematics duality to Parisi-Sourlas supersymmetry and hidden conformal symmetry. We also revisit the results for maximally supersymmetric theories, where the half-BPS operators are dual to gravitons, in particular examining certain properties related to dimensional reductions: the existence of protected sectors and again Parisi-Sourlas supersymmetry.

In the second approach we consider the holographic realization of renormalization group (RG) flows between SCFTs living in different spacetime dimensions. From a string/M-theory perspective this is understood in terms of branes wrapping compact cycles in special holonomy manifolds. We consider in particular the case in which the compactification space is a *spindle* $\mathbb{\Sigma}$, an orbifold which is topologically a two-sphere with conical singularities at the poles. We describe solutions of the form AdS $\times$ $\mathbb{\Sigma}$ of gauged supergravities in various dimensions, that can be then uplifted to $D = 10/11$ supergravity and we argue that they describe holographically the compactification of certain SCFTs on $\mathbb{\Sigma}$. This interpretation is supported by a field theory computation based on anomaly polynomials and extremization principles in even dimensions. Spindles have several novel features with respect to other supergravity solutions describing RG flows across dimensions, most prominently the fact that supersymmetry is not realized with a topological twist, as we discuss in detail. In the case of four dimensions we also discuss the relation between spindles and accelerating black holes, whose conformal boundary is studied in detail to elucidate certain aspects of how spindles preserve supersymmetry.

Statement of originality

This thesis is based on a number of papers written in collaboration with various authors, whose work I gratefully acknowledge.

Specifically, **part I** of the thesis is based on the two papers

[1] Connor Behan, Pietro Ferrero and Xinan Zhou, *More on holographic correlators: twisted and dimensionally reduced structures*, *JHEP* 04 (2021) 008, [2101.04114].

[2] Luis Fernando Alday, Connor Behan, Pietro Ferrero and Xinan Zhou, *Gluon Scattering in AdS from CFT*, *JHEP* 06 (2021) 020, [2103.15830].

On the other hand, **part II** is based on

[3] Pietro Ferrero, Jerome P. Gauntlett, Juan Manuel Pérez Ipiña, Dario Martelli and James Sparks, *D3-branes wrapped on a spindle*, *Phys.Rev.Lett.* 126 (2021) 11, 111601, [2011.10579].

[4] Pietro Ferrero, Jerome P. Gauntlett, Juan Manuel Pérez Ipiña, Dario Martelli and James Sparks, *Accelerating black holes and spinning spindles*, *Phys. Rev. D* 104 (2021) 4, 046007, [2012.08530].

[5] Pietro Ferrero, Jerome P. Gauntlett, Dario Martelli and James Sparks, *M5-branes wrapped on a spindle*, *JHEP* 11 (2021) 002, [2105.13344].

[6] Pietro Ferrero, Matteo Inglese, Dario Martelli and James Sparks, *Multi-charge accelerating black holes and spinning spindles*, [2109.14625].

[7] Pietro Ferrero, Jerome P. Gauntlett and James Sparks, *Supersymmetric spindles*, *JHEP* 01 (2022) 102, [2112.01543].

Versions of [3, 4] have appeared in a previous DPhil thesis from the University of Oxford [8]. Due to space constraints, some papers written by the author of this thesis during his DPhil have not been included but only referred to in the main text [9–11]. The same applies to some work in preparation [12].

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List of abbrevations

AdS	anti-de Sitter
BCJ	Bern-Carrasco-Johansson
BPS	Bogomol'nyi–Prasad–Sommerfield
BTZ	Bañados-Teitelboim-Zanelli
CFT	conformal field theory
CY	Calabi-Yau
FT	field theory
GK	Gauntlett-Kim
IR	infra-red
KE	Kähler-Einstein
KK	Kaluza-Klein
MHV	maximally helicity violating
MRV	maximally R-symmetry violating
OPE	operator product expansion
QCD	quantum chromodynamics
QFT	quantum field theory
RG	renormalization group
SCFT	superconformal field theory
SCQM	superconformal quantum mechanics
SCWI	superconformal Ward identities
SE	Sasaki-Einstein
SYM	super Yang-Mills
UV	ultra-violet

Chapter 1

Introduction

1.1 The AdS/CFT correspondence

Since the work of 't Hooft [13], it was believed that the large-N expansion of gauge theories should be the starting point of their dual reformulation as theories of strings. After Polchinski's fundamental contribution to the understanding of the role of D-branes in string theory [14], Maldacena was led to formulate his conjecture [15] that certain conformal field theories (CFTs), arising as the world-volume theories of D- or M-branes, should be dual to superstring theory or M-theory on anti-de-Sitter (AdS) backgrounds. The work of [16, 17] complements the original formulation of the correspondence with a concrete prescription to relate observables in gravity and CFT. Maldacena's conjecture, now known as the AdS/CFT correspondence, can be seen as one of the most remarkable achievements of string theory. Among the several reasons for its success and importance, one aspect that we would like to emphasize in this thesis is that the AdS/CFT correspondence allows to address, in a unified framework, two of the most basic open problems in theoretical physics: quantum gravity and strongly-coupled quantum field theories (QFTs).

It was already understood by Einstein in 1916[1] that his theory of General Relativity would require modifications in order to be compatible with quantum mechanics, but despite several years of efforts there is no universal consensus, let alone experimental evidence, on what a theory of quantum gravity should look like. There are certain widely believed and well-supported expectations, such as the holographic principle [20] and the absence of global symmetries [21][2], as well as a vast amount of conjectures with varying degrees of evidence and rigour. These mostly arise from black hole physics or from the study of which low-energy effective field theories could possibly be coupled to quantum

[1] At p. 696 of [18] we read "*...it appears that the quantum theory would have to modify not only Maxwellian electrodynamics, but also the new theory of gravitation*" [19].

[2] Both of these properties of quantum gravity are naturally realized in the AdS/CFT correspondence. The holographic principle is fairly obvious since the CFT is supposed to describe a theory of gravity living in one dimension higher. The absence of global symmetries can be naively traced back to the fact that global symmetries in the CFT are dual to gauge symmetries in gravity, but see [22] for a more precise discussion.

gravity: the so-called *swampland* program [23] – see, *e.g.*, [24, 25] for reviews. A take-home message from these explorations and basic physical intuition seems to be that any theory of quantum gravity should be inherently different from a local QFT. On the contrary, the fundamental statement of the AdS/CFT correspondence is that quantum gravity on certain classes of spacetimes can be completely equivalent to certain QFTs[3]. Given our higher degree of understanding of QFT, the correspondence can then be used to give an independent definition of quantum gravity in AdS, via the dual QFT. This gives an additional motivation to develop tools that allow to access QFTs beyond the perturbative regime, as they could provide key insights on gravitational physics, in addition to being interest *per se*.

On the other hand, a paramount aspect of the correspondence is that the regime of parameters that is under the highest degree of control is the one in which string theory is weakly coupled and we focus on low-energy interactions, which is controlled by the supergravity approximation. In this regime, the dual CFT is typically strongly coupled and evades standard field theory tools such as perturbation theory. Therefore, the correspondence provides an alternative description of certain QFTs at strong coupling, where computations would otherwise be out of reach. Through various generalizations, AdS/CFT can be used to make quantitative or qualitative predictions for a variety of theories, ranging from QCD with a large number of colors [27] to condensed matter systems [28–30]. The situation is even more apparent for theories that do not admit a Lagrangian description at all, for which it is often the case that their holographic description is one of the few tools that allow us to compute certain observables, or even just to define the theory. Most famously, this is the case for $4d\ \mathcal{N}=3$ theories [31, 32] or $6d\ \mathcal{N}=(2,0)$ SCFTs [33, 34]. An interesting direction from this perspective is therefore that of finding generalizations of the correspondence by exploring solutions with AdS factors in theories of supergravity, as they could provide evidence for the existence of novel CFTs or a way to access strongly-coupled sectors of existing ones.

Among the many facets of the AdS/CFT correspondence, in this thesis we will consider two different and somewhat complementary approaches. In the first, mostly based on field-theory techniques, we shall adopt a bootstrap perspective for the study of four-point functions between single-trace operators in the supergravity approximation. The use of qualitative features of the supergravity description provides invaluable input for the bootstrap, allowing computations in strongly coupled CFTs, while on the other hand the use of CFT techniques makes it possible to open a window on the study of scattering amplitudes in AdS. In the second approach we will mostly focus on the study of

[3] For insightful comments on how locality in the bulk of AdS can be recovered from the dual CFT description, see [26].

supergravity solutions with AdS factors, which allow to describe interesting field theory phenomena geometrically, as well as unveiling novel strongly interacting CFTs. Here the study of the geometry of supergravity solutions provides a unique opportunity to access non-trivial field theories, where the holographic description is sometimes the only one available, while the existence of dual CFTs is a guiding principle in the search for, and understanding of, solutions in supergravity. All in all, our point of view will be that the correspondence provides a unique opportunity to gain a deeper understanding of two rather different but equally fascinating topics and our best chance to fully exploit its existence is the combination of tools and knowledge from both field theory and supergravity, to learn more about both simultaneously.

1.2 Maximally supersymmetric backgrounds

In order to set the notation and introduce some basic concepts related to the correspondence, let us briefly review the three simplest and most well-established examples, which were already discussed in [15]. On the gravity side, one has branes in flat space which preserve one half of the supersymmetry of type IIB or $D = 11$ supergravity, corresponding to sixteen supercharges. Near their horizon, the metric takes the form of a product between AdS and a sphere and there is a supersymmetry enhancement: sixteen additional supercharges are preserved. The dual supersymmetric CFTs (SCFTs) are the theories living on the world-volume of the branes and their supergravity description through the correspondence is only reliable in the regime where a large number (N) of branes is present: the large-N limit. Theories with thirty-two supercharges (sixteen Poincaré and sixteen superconformal partners) possess the maximal number of supersymmetries allowed for a non-trivial SCFT – see the discussion in section 5.1 of [35] – and therefore provide the most symmetric examples of local QFTs that one could consider.

1.2.1 $\mathbf{AdS_5 \times S^5}$

The simplest and most explored instance of the correspondence is that of $\text{AdS}_5 \times S^5$, which is the spacetime near a stack of N D3-branes in flat space. The dual CFT is $4d$ $\mathcal{N} = 4$ super Yang-Mills (SYM) with 't Hooft coupling $\lambda = g_{YM}^2 N$ and $SU(N)$ gauge group[4]. Its superconformal algebra is $\mathfrak{psu}(2,2|4)$, which contains an $\mathfrak{su}(2,2) \simeq \mathfrak{so}(2,4)$ subalgebra corresponding to conformal symmetry (dual to the isometries of AdS_5), an $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$ R-symmetry algebra (dual to the isometries of S^5)

[4]Note that one can consider other gauge groups by introducing orientifolds. In particular, one obtains orthogonal and symplectic gauge groups by replacing the internal S^5 with $\mathbb{RP}^5 = S^5/\mathbb{Z}_2$ [36].

and sixteen Poincaré supercharges, as well as their superconformal partners. For type IIB supergravity to provide a reliable approximation to string theory, one requires the string coupling constant g_s to be small and the AdS radius L to be larger than the string scale $\ell_s = \sqrt{\alpha'}$, where α' is the inverse string tension. Given the dictionary between the parameters in gravity and CFT,

$$\lambda = (L/\ell_s)^4 \,, \quad N = (4\pi\, g_s)^{-1} \,(L/\ell_s)^4 \,, \tag{1.1}$$

these conditions can be met only for large N and large λ, with small g_{YM} (related to g_s by $g_{YM}^2 = 4\pi\, g_s$). In other words, the string scale must be larger than the Planck scale $\ell_P = g_s^{1/4}\, \ell_s$, related to the ten-dimensional Newton's constant by $16\pi\, G_{(10)} = (2\pi)^7\, \ell_P^8$. The stress-tensor central charge of $\mathcal{N} = 4$ SYM is

$$C_{\mathcal{T}}^{(4d)} = 30\,(N^2 - 1)\,, \tag{1.2}$$

where we have used the conventions of [37] in which a single free boson has $C_{\mathcal{T}} = 1$. Thus, the large-N limit can be seen as a limit of large central charge, or large number of degrees of freedom.

Two pieces of information will be particularly useful in the body of this thesis. The first is that the elementary fields of the AdS_5 supergravity theory can be understood as Kaluza-Klein (KK) reductions of the fields of type IIB supergravity on the S^5. Since in type IIB there is only one (massless) supergravity multiplet, one obtains one massless[5] super-multiplet in five dimensions as well as an infinite tower of massive KK modes. The dual multiplets in the CFT are half-BPS (Bogomol'nyi–Prasad–Sommerfield), meaning that their superconformal primary is annihilated by one half of the Poincaré supercharges. They are labelled by an integer $k \geq 2$ and their super-primary is a Lorentz scalar transforming in the rank-k symmetric traceless representation of $\mathfrak{so}(6)$. The case $k = 2$ is the stress-tensor multiplet, dual to the massless graviton in five dimensions. A class of observables that we will focus our attention on **part I** of this thesis is four-point correlation functions of such half-BPS operators, which allow to study graviton scattering in AdS.

The other observation is related to how one can deform this rigid, highly supersymmetric setup to explore other AdS/CFT dual pairs. A first way is that of replacing the S^5 with a Sasaki-Einstein (SE) five-manifold: as was already observed in [39] this corresponds to placing a stack of D3 branes

[5] Actually the only massless fields in the multiplet are the graviton, the fifteen gauge fields gauging the $SO(6)$ isometry of S^5 and two scalars (singlets of $SO(6)$). The remaining modes are massive, but we still refer this as "massless" multiplet as the masses of all fields are fixed by $\mathcal{N} = 4$ supersymmetry and by the fact that the graviton is massless. A detailed account of the field content of this multiplet can be found, *e.g.*, in [38].

at the tip of a conical singularity in a Calabi-Yau (CY) three-fold, where the CY_3 is a cone over SE_5. This still provides a solution to the type IIB supergravity equations of motion but partially breaks supersymmetry. In **part II** of this thesis, we will consider another possibility, which is that of deforming the AdS_5 factor in the metric: black holes or other spacetimes that are only *asymptotically locally* AdS [40, 41] give access, holographically, to the study of deformations of the original $\mathcal{N} = 4$ SYM theory. While finding new solutions of the full type IIB supergravity is particularly complicated, it is useful to observe that the theory admits a *consistent truncation* on its $AdS_5 \times S^5$ vacuum where massive KK modes are neglected. This is $5d\ \mathcal{N} = 8\ SO(6)$ gauged supergravity, which has therefore the property that all its solutions can be uplifted on S^5 to solutions of type IIB supergravity supported by F_5 flux. We will exploit this fact and explore deformations of $\mathcal{N} = 4$ SYM by studying solutions to gauged supergravities in five dimensions.

1.2.2 $AdS_7 \times S^4$

The other two maximally supersymmetric cases pertain to the domain of M-theory, rather than string theory. One can consider solutions describing a stack of N M5-branes in flat space, whose near-horizon limit is $AdS_7 \times S^4$. The dual CFTs are the mysterious $6d\ \mathcal{N} = (2,0)$ SCFTs living on the world-volume of M5-branes, whose existence was first argued for in [33, 34][6]. The relevant superalgebra for this case is $\mathfrak{osp}(8^*|4)$, with an $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$ R symmetry. For supergravity to provide a reliable approximation one requires the AdS radius L to be larger than the eleven-dimensional Planck length ℓ_P, with

$$L/\ell_P = (8\,\pi\,N)^{1/3}\,, \tag{1.3}$$

so that once again we need large N, or large central charge

$$C_T^{(6d)} = 70\,(4N^3 - 3N - 3)\,. \tag{1.4}$$

Notice that, as opposed to the string theory case, there is no string scale, or correspondingly no 't Hooft coupling: this is related to the fact that all $6d\ \mathcal{N} = (2,0)$ theories do not admit exactly marginal deformations and no Lagrangian description is available.

Connecting to the points discussed above, KK modes of the $D = 11$ supergravity multiplet are

[6] More precisely, this is the case of A_N-type theories. One can also obtain theories of D_N-type by replacing the internal S^4 with $\mathbb{RP}^4 = S^4/\mathbb{Z}_2$ [42].

again arranged in half-BPS super-multiplets, whose super-primary is a scalar in the rank-k symmetric traceless representation of $\mathfrak{so}(5)$, $k = 2$ being the stress-tensor (describing the massless graviton). Interestingly, in this case the only four-manifolds that lead to supersymmetric solutions other than S^4 are its orbifolds S^4/Γ, with $\Gamma \subset SO(5)$ (see [43] for more details and constructions with $\mathcal{N} = (1,0)$ supersymmetry). On the other hand, there is again a consistent truncation to the massless KK modes in $D = 7$, which is a $7d$ $\mathcal{N} = 4$ $SO(5)$ gauged supergravity, whose solutions uplift to $D = 11$ on S^4.

1.2.3 $\mathbf{AdS_4 \times S^7}$

The third and final example is that of N M2-branes, whose near-horizon limit $\mathrm{AdS}_4 \times S^7$ defines a $3d$ $\mathcal{N} = 8$ SCFT. However, at the time of [15] it was not clear what the world-volume theory of M2-branes should be. It was only found in [44] that this is a matter-coupled Chern-Simons theory, now known as the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory (at level $k = 1$). The relevant superalgebra is again $\mathfrak{osp}(8|4)$ (a different real form compared to the one of the previous subsection), with R symmetry $\mathfrak{so}(8)$. Here the ratio between the AdS radius and the Planck length is

$$L/\ell_P = (\pi^2\, N/2)^{1/6}\,, \tag{1.5}$$

while the central charge has a complicated expression in terms of Airy functions [45, 46] which at large N gives

$$C_{\mathcal{T}}^{(3d)} \approx \frac{64\sqrt{2}}{3\pi}\, N^{3/2}\,. \tag{1.6}$$

Again, there are no marginal deformations: the only coupling constant is the Chern-Simons level, which is fixed to $k = 1$. It is also interesting to notice that the theory at level k is dual (at large N) to M2-branes on $\mathbb{R}^8/\mathbb{Z}_k$[7] and that the cases $k = 1, 2$ preserve $\mathcal{N} = 8$ supersymmetry, which is lowered to $\mathcal{N} = 6$ for $k \geq 3$. The ABJM theory is weakly coupled for large values of k, while small k leads to a strongly coupled theory.

The super-primaries of the half-BPS multiplets describing the KK modes of the graviton in the $3d$ SCFT are again Lorentz scalars, in the rank-k symmetric traceless representation of $\mathfrak{so}(8)$, $k = 2$ being the stress-tensor multiplet. If one is willing to give up some supersymmetry, in this case there

[7] The action of $\mathbb{Z}_k$ is best described by considering $\mathbb{R}^8 = \mathbb{C}^4$, with $\mathbb{Z}_k$ acting as a phase $\mathrm{e}^{2\pi \mathrm{i}/k}$ on the four complex coordinates of $\mathbb{C}^4$ [44].

is more freedom in the choice of internal geometry as the S^7 can be replaced by any SE_7 still leading to a supersymmetric solution of $D = 11$ supergravity. Moreover, a truncation to the massless KK modes of the $D = 11$ fields leads to a $4d$ $\mathcal{N} = 8$ $SO(8)$ gauged supergravity, whose solutions all uplift to $D = 11$ on S^7.

1.3 Bootstrapping holographic correlators

In this thesis we consider two complementary approaches to the study of the AdS/CFT correspondence. In the first perspective, which we shall adopt in **part I**, we consider well-established examples of the correspondence where one has a high amount of supersymmetry (either sixteen or eight Poincaré supercharges, as well as one example with six) that makes their structure relatively rigid. This allows to compute non-trivial observables such as correlation functions of half-BPS operators, that describe holographically the scattering of gravitons and gluons in AdS.

Correlation functions of local operators are the most basic dynamical observables in any CFT. When a weakly-coupled Lagrangian description is available, they can in principle be computed using the standard technique of Feynman diagrams as a perturbative expansion in the coupling constant. However, this is not possible outside the weakly-coupled regime, or where a Lagrangian for the theory is not available at all. In essentially all known examples of the AdS/CFT correspondence one has to deal with strongly-coupled CFTs: either because they do no admit a Lagrangian description at all, which is the case for the $6d$ $\mathcal{N} = (2, 0)$ SCFTs, or because we do not have much control on the stringy regime and we consider the regime of parameter space in which supergravity provides a good approximation, which is the case for $4d$ $\mathcal{N} = 4$ SYM. One might argue that it is holography itself to provide us with a Lagrangian that describes such theories in the large-N limit, but such Lagrangian is so complicated that it has been worked out explicitly only in the case of $\mathrm{AdS}_5 \times S^5$ (and only up to quartic vertices between KK modes of the ten-dimensional graviton), where the result is already dauntingly complicated [47]. Even in this simplest case using standard diagrammatic techniques is an incredibly complicated task, but more in general no Lagrangian is known beyond the sector of massless KK modes (described by gauged supergravity), which in any case could only be used for stress-tensor correlators and only at tree level.

Given this scenario, a possibility is that of resorting to so-called *bootstrap* methods, which (roughly speaking) consist in fully exploiting all symmetries and self-consistency conditions of a problem, possibly with some basic external input, to put constraints on a certain theory to the extent that, in

certain cases, it is possible to "solve" it. The idea that self-consistency conditions might be enough to solve certain physical problems first made its appearance in theoretical physics in the form of the *S-matrix bootstrap* [48, 49], followed by the introduction of the so-called *integrable bootstrap* [50] and *conformal bootstrap* [51, 52], both mostly focused (at least originally) on two-dimensional QFTs (or CFTs). It was only after several years and some foundational works on the structure of higher-dimensional CFTs [37, 53–55] that the conformal bootstrap was applied beyond the two-dimensional regime, where the infinite-dimensional Virasoro algebra provides a larger amount of symmetry. The modern incarnation of the conformal bootstrap is due to Rattazzi, Rychkov, Tonni and Vichi [56], who introduced a numerical technique to put bounds on conformal dimensions and operator product expansion (OPE) coefficients, based on crossing symmetry of four-point functions. Arguably one of the main successes of the numerical conformal bootstrap is the precise determination of certain dimensions and OPE coefficients in the $3d$ Ising model [57], see [58] for a review. These developments also led to a revival of the S-matrix bootstrap, starting with [59, 60] (see [61] for a recent review).

While numerical investigations can prove extremely powerful in certain cases, especially when the theories of interest sit at special points in exclusion plots (as it is famously the case for the $3d$ Ising model), an interesting question is whether, at least in certain regimes, it is possible to have analytic control on CFTs. Following some observations on the large-spin behaviour of the anomalous dimensions of certain operators in gauge theories [62] it was realized in [63, 64] that the spectrum of CFTs largely simplifies in the large spin regime, allowing for analytic results to be obtained. These observations were made systematic in a series of papers [65–69], where the frameworks of *light-cone bootstrap* and *large spin perturbation theory* were introduced, and finally put on firmer grounds with the discovery of the so-called *Lorentzian inversion formula* [70]. More generally, analytic bootstrap methods can be applied in CFTs where one has a point where the theory simplifies and an expansion in a small parameter around such point. This could of course be the coupling constant in the case of a Lagrangian theory, but analogous ideas were applied for instance in the ϵ-expansion, critical $O(N)$ models at large N, large charge and, of course, holography, which is the case we are interested in. For recent reviews of these developments, see [71, 72] and references therein.

A crucial difference between numerical and analytic methods is that the first allow to study CFTs non perturbatively, obtaining numerical determinations of certain CFT data valid in the full-fledged theory and charting the space of allowed CFTs. The second approach has the advantage to allow for the derivation of analytic results, which can be preferable in certain cases but comes at the expense that one has to work order by order in a small parameter and has to *assume* the existence of

a CFT, rather than *deriving* it from first principles and basic assumptions. Despite these shortcomings, the analytic conformal bootstrap provides an ideal framework where to compute the kind of observables we are interested in: four-point functions of single-trace operators in holographic CFTs, in a $1/N$ expansion (and $1/\lambda$ in the case of $\mathcal{N} = 4$ SYM). There are several reasons to motivate the interest in such objects. First, they provide a way to study quantum gravity in AdS (as well as in Minkowski space through the flat-space limit [73]) with CFT techniques, through fundamental dynamical observables such as the four-graviton scattering amplitude. This gives a window on full-fledged theories of quantum gravity that can be defined, order by order in $1/N$, via the dual CFTs, in such a way that quantum-mechanical effects of gravity can be studied without an independent definition. Moreover, an appealing prospect is that of understanding the mysterious M-theory from the study of ABJM theory: in this case one has a Lagrangian description allowing for localization computations [74], which provide a fundamental input for the bootstrap machinery. Exploring higher and higher orders in the $1/N$ expansion of ABJM four-point functions one can then study the quantum effective action of M-theory, which is not known beyond the first few terms [75–80]. A second motivation is given by the fact that holographic correlators are the natural analogue of flat-space scattering amplitudes, as we shall motivate in greater detail in the rest of this thesis. It is then interesting to see whether some of the beautiful properties satisfied by amplitudes in flat space [81, 82] are still valid on curved backgrounds, starting from AdS space which is maximally symmetric. For example, a concrete and fascinating question is whether the observed relations between scattering amplitudes in gravity and gauge theories (see [83, 84] for reviews) can teach us important lessons about their structure, beyond the relation between perturbative amplitudes in Minkowski space[8]. Starting from AdS, one can test the regime of validity of such relations and dig into the hidden structure of holographic correlators. Finally, a third reason for the interest in the study of holographic correlation functions is that, using only basic information from the dual AdS Lagrangians, they allow to access strongly-coupled sectors of CFTs, allowing for a determination of non-protected data such as conformal dimensions and OPE coefficients of operators in long multiplets, in a $1/N$ expansion. Even more basic observables such as three-point functions of protected operators are unknown in theories with less than maximal supersymmetry and bootstrap methods, combined with intuition from the bulk, have proven capable of giving an analytic determination, as shown in [2].

1.4 AdS solutions in supergravity

[8] See [10] for some work by the author of this thesis on the topic.

Another possible approach to the study of the AdS/CFT correspondence is to try and extend its regime of applicability by finding new examples or describing new physics on one of the two sides holographically. While certainly finding examples beyond the framework of string theory, or without supersymmetry, is an ambitious and interesting goal, it is fair to say that almost all reliable examples of the correspondence arise from supersymmetric solutions of $D = 10/11$ supergravity, where the stringy origin is essential to identify the dual field theories and supersymmetry guarantees the stability of the configuration. Studying solutions of such supergravities that contain AdS factors (at least in certain asymptotic regions of spacetime) is then essential for the development of the correspondence and, crucially, can lead to the discovery of novel CFTs that are strongly coupled and therefore hardly accessible otherwise, or to a gravitational and geometric description of certain field theory phenomena.

A first, natural modification of the maximally supersymmetric solutions described in section 1.2 is obtained by placing M2- or D3-branes at the tip of a CY conical singularity, rather than in flat space [39, 85–87]. The local description of such singularities is given by a metric for the CY written as a cone over a SE manifold [88, 89], which led to a renewed interest in such spaces from the physics community and to the discovery of new infinite families of SE metrics [90–94] as well as their CFT duals [95–97]. The non-trivial geometries of SE manifolds are then found to have deep relations to the properties of the dual field theories, whose properties can then be "geometrized": a striking example is the famous a-maximization in four-dimensional field theories [98], which was found to be dual to a minimization problem for the volume of SE manifolds in [99]. More in general, there is a number of field theory phenomena that can be described by supergravity solutions via the correspondence, in the large-N limit. They range from conformal manifolds [100–105] to RG flows between two CFTs [39, 106–122] and extend to non-conformal examples where one can study finite temperature physics, confinement and phase transitions [123–133]. On the other hand, the AdS/CFT correspondence has also provided a new stimulus to the study of the microscopic origin of black hole entropy. After the celebrated results of [134] in flat space, asymptotically AdS black holes and black branes have recently received a lot of attention [135–146] and the study of their thermodynamic properties made possible by field theory techniques such as extremization principles [147–152] and supersymmetric localization [74]. For a nice review of these developments, see [153].

Given the variety of physical phenomena that can be described with the holographic correspondence, an important problem is that of classifying all supersymmetric solutions to the equations of motion of $D = 10/11$ supergravity theories containing AdS factors. This was made possible by the

method introduced in [154, 155], where it was shown that the existence of Killing spinors poses, through the Killing spinor bilinears, certain constraints on the structure groups of the Lorentzian manifolds describing the spacetime: the so-called G-structures. In the special case of solutions with AdS factors, this translates into constraints on the geometry of the compact internal space, which then "geometrizes" properties of the dual field theory. Since no SCFTs exist in dimension $d > 6$ [156, 157], it is natural to consider spacetimes containing AdS_{d+1} factors with $1 \leq d \leq 6$ and classifications of supersymmetric solutions exist in all these cases: a non-exhaustive list of references classifying AdS_{d+1} solutions with $3 \leq d \leq 6$, with various assumptions and different amounts of preserved supersymmetry, can be found in table 1.1. When moving down to $d = 1, 2$, *i.e.* to AdS_2

	IIA	IIB	M
AdS_7	[158]	none	$\text{AdS}_7 \times S^4/\Gamma$
AdS_6	$\text{AdS}_6 \times HS^4$ [159], see also [160]	[161–164]	none
AdS_5	[165]	[166, 167]	[168, 169]
AdS_4	[170–173]	[174]	[175]

Table 1.1: Some references discussing the classification of solutions with AdS factors in type IIA and type IIB supergravity ($D = 10$), as well as $D = 11$ supergravity, or M-theory. In the table, $\Gamma \subset SO(5)$ is a finite subgroup of the isometry group of S^4 and more details about its action on spacetime coordinates can be found in [43].

and AdS_3, clearly there is more freedom in the geometry of the internal space and in the algebra of super-isometries, so a complete classification is arguably much harder to achieve. A partial list of references for the AdS_3 case is given in [176–198], with a recent overview of the state of the art given in the introduction of [199]. Similarly, some references regarding AdS_2 solutions are [178, 179, 192, 200–213]. It is also interesting to notice that AdS solutions realizing all existing superconformal algebras have been now found [214].

AdS_2 solutions of $D = 11$ supergravity with purely electric four-form flux and AdS_3 solutions of type IIB supergravity supported by five-form flux were classified in [200] and [177] respectively and later revisited by Gauntlett and Kim (GK) in [215], where their internal space geometry was termed GK geometry and bears several similarities with SE geometry, including interesting connections between volume extremization problems and extremization principles in the dual field theories [192], generalizing the results of [99]. Explicit examples of AdS_2 and AdS_3 solutions with GK geometry were found in [178, 179], but their dual field theories were not understood until [3], where it was shown that they are related to a physical phenomenon known as the renormalization group (RG) flow across dimensions, whose supergravity dual was first discussed in [113]. It consists of placing

a certain CFT on a spacetime which is a product of a lower-dimensional Minkowski space and a compact manifold, and flowing to the infra-red (IR) where, at small energies compared to the length scale of the compact manifolds, one finds a CFT in lower dimensions. The holographic dual is given by supergravity solutions that interpolate between AdS spacetimes of different dimensionality. A more detailed discussion of this will be given in chapter 6. Starting with [3, 4] it was realized that the solutions of [178, 179] describe the RG flow across dimensions in the case where the compact space is an orbifold known as *spindle*, which is topologically a two-dimensional weighted projective space. This presents several novelties with respect to the previously existing literature on the topic and will be the subject of **part II** of this thesis.

Part I

Gluons and gravitons in AdS

Chapter 2

Holographic correlators

In the first part of this thesis, we shall focus on the study of correlation functions of local operators in holographic CFTs, which are dual to correlators between boundary excitations in AdS. As already discussed in [20, 216–218], these have the natural interpretation of AdS scattering amplitudes and are among the most fundamental observable in the AdS/CFT correspondence. At least in principle, they can be computed in a perturbative expansion in the coupling constant of the bulk theory, which is assumed to be small, using Witten's diagrammatic rules [17], which are very reminiscent of flat space Feynman diagrams. Since the very early days of the correspondence, much attention has been devoted to the study of holographic four-point functions, in particular in the case of $\text{AdS}_5 \times S^5$ [16, 219–222]. Such cases are particularly interesting because the graviton and its KK modes on the sphere reside in half-BPS multiplets, whose superconformal primary is a scalar. Studying CFT correlation functions of scalars living in short multiplets then allows to access the scattering of gravitons in a theory of quantum gravity, which makes this program particularly appealing.

As we shall review momentarily, the use of modern techniques such as Mellin space [223, 224] and analytic bootstrap methods [225–229] has led to tremendous progress in the computation of holographic four-point functions, to the extent that all four-point functions between half-BPS operators are known at tree level for all maximally supersymmetric backgrounds [230, 231], with some of their properties analyzed in [1]. Analogous results are available for a large class of theories with less than maximal supersymmetry [2], where the half-BPS multiplets describe gluons, rather than gravitons.

2.1 Holographic CFTs and their spectrum

Let us focus momentarily on the simplest instance of the AdS/CFT correspondence, which relates $4d\,\mathcal{N}=4\,SU(N)$ SYM to type IIB superstring theory in $\text{AdS}_5\times S^5$. While there are reasons to believe that the correspondence should hold for any value of the string coupling g_s and string tension α' (or,

equivalently via (1.1), for any value of the rank N and the 't Hooft coupling λ), it is fair to say that the only regime on which we have computational control is that of large N and large λ[1]. On the string theory side, the limit of small g_s and α' reduces the full type IIB superstring theory to semi-classical type IIB supergravity, while in $\mathcal{N} = 4$ SYM this corresponds to the 't Hooft limit [13] where one naturally classifies operators according to the number of traces involved in their definition in terms of elementary fields. Single-trace operators are dual to elementary fields in AdS, while multi-trace ones are dual to multi-particle states[2].

The case of $\mathcal{N} = 4$ SYM is particularly lucky because one has a gauge theory with a Lagrangian description on the CFT side, but the picture above can be easily generalized to non-Lagrangian theories with AdS dual, obtaining a notion of "holographic CFT". The concept of large N is replaced by an expansion for large central charge, while single- and multi-trace operators are replaced with the dual of single- and multi-particle states in AdS. A common feature of CFTs with AdS duals is that their structure greatly simplifies in the limit of large central charge, for the dual supergravity description should be reliable. In particular, elementary fields in local supergravity theories in AdS can have spin and most two and are typically either a finite number, or a finite number of KK families. In other words, all operators that can detect the non-localities of string or M-theory (such as those dual to string excitations) should decouple in the holographic limit, acquiring parametrically large conformal dimensions. Such observations led to the conjecture of [26] that *any CFT that has a large-N expansion, and in which all single-trace operators of spin greater than two have parametrically large dimensions, has a local bulk dual.* This represents the great simplification that a dual supergravity description implies for holographic CFTs, making it possible to compute non-trivial dynamical observables such as four-point functions, in a large-N expansion.

After this necessary preamble, we are ready to see what are the implications for the observables of interest in this thesis: four-point functions of half-BPS operators in holographic SCFTs. In the maximally supersymmetric cases, all elementary fields (the $D = 10/11$ supergravity multiplet and its KK modes) are encompassed by half-BPS operators, so that all single-particle states are dual to such protected multiplets. In particular, no single-trace long operators are present[3]. From the super-

[1] See [232, 233] for some recent progress in understanding the string theory dual of large N, $\mathcal{N} = 4$ SYM at weak coupling.

[2] The identification is exact only in the strict large-N limit, but at subleading orders single-particle states in AdS are dual to mixtures of single- and multi-trace operators – see the discussion of [234] where the importance of this observation is stressed in the case of extremal three-point functions.

[3] A famous example of a long, single-trace operator in $\mathcal{N} = 4$ SYM is the Konishi operator, whose dimension grows with $\lambda^{1/4}$ at strong coupling and therefore it decouples from the spectrum. Thus, the Konishi operator is dual to a string excitation, rather than a supergravity one [16, 17, 235].

gravity perspective, all the operators that one could form are given by normal-ordered products of elementary fields, and this reflects in the fact that in the dual SCFTs the only operators in addition to the half-BPS ones are those multi-trace operators that can be built from products of half-BPS operators and can sit in either short or long multiplets. Moreover, the OPE coefficients between two single-trace and one multi-trace operator is suppressed by a power of N which increases with the number of traces, in such a way that in the supergravity approximation one can focus on double-trace operators only, ignoring the contribution of higher-trace ones (with three traces or more).

We will also consider a class of theories with less than maximal supersymmetry, where in the AdS supergravity one can have independently vector multiplets and gravity multiplets. In those cases, single-trace short operators in the CFT are dual to gluons (and their KK modes on the internal space, always an S^3) and gravitons (and their KK modes). The situation looks therefore more complicated, but it turns out that in all examples that we shall consider the three-point function of two gluons and one graviton is suppressed by a power of N compared to that of three gluons. Therefore, at leading order the structure of the OPE is exactly the same as that of maximally supersymmetric theories, just with gravitons replaced by gluons (and their super-partners). This can be seen as a consequence of the fact that the self-interaction of gluons is stronger than the gluon-graviton coupling at large N, so at first order one ends up with a theory of self-interacting gluons on a rigid AdS background.

2.2 The traditional method

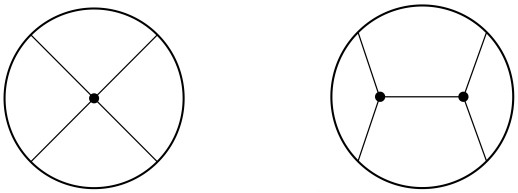
The traditional method for the computation of holographic correlators was first described in [17] and it follows from one of the most basic entries of the AdS/CFT dictionary, which relates the generating functional of correlation functions between boundary CFT operators $\mathcal{O}_i$, with sources $\bar{\phi}_i$, to the path integral of a dual AdS theory with fields ϕ_i that asymptote $\bar{\phi}_i$ at the boundary of AdS[4]

$$\langle \mathrm{e}^{\mathrm{i}\int_{\partial\mathrm{AdS}} \bar{\phi}_i \mathcal{O}_i} \rangle_{\mathrm{CFT}} = \int_{\mathrm{AdS}} \mathcal{D}\phi_i \ \mathrm{e}^{\mathrm{i}\mathcal{S}[\phi_i]}\Big|_{\phi_i(\partial\mathrm{AdS})=\bar{\phi}_i} . \tag{2.1}$$

The semiclassical expansion of the right-hand side (since at large N one has a weakly coupled Lagrangian description in supergravity) leads to an expansion in Witten diagrams such as those represented in figure 2.1, where the circle represents the conformal boundary of AdS while lines connecting the circle to the interior are known as *boundary-to-bulk* propagators and lines connecting

[4]Let z be a coordinate parametrizing the distance from the conformal boundary, located at $z = 0$. Then the correct statement is that $\lim_{z\to 0} \phi_i(z, \vec{x}) = z^{\#}\, \bar{\phi}_i(\vec{x})$, with $\vec{x}$ coordinates on ∂AdS.

points in the interior are known as *bulk-to-bulk* propagators. The most basic correlators that were



(a) A contact diagram.

(b) An exchange diagram.

Figure 2.1: Examples of tree-level quartic Witten diagrams.

computed in full-fledged holographic setups are three-point functions between half-BPS operators in maximally supersymmetric CFTs, in [236–239]. These are fundamental results since they give access to the OPE coefficients for the most supersymmetric operators, but do not give access to the non-protected sector of the theories. The simplest dynamical observables that make this possible are four-point functions of protected operators, but despite the great interest, results have been hard to obtain for several years, since the traditional method requires detailed knowledge of the supergravity Lagrangian in AdS. In practice, for the simplest case of $\text{AdS}_5 \times S^5$, this means expanding all the fields in KK modes on S^5, replacing this expansion in the type IIB supergravity Lagrangian and retaining all cubic and quartic interactions. This extremely challenging task has been completed in [47], resulting in a fifteen pages long expression. No such results are available for the other maximally supersymmetric backgrounds, let alone cases with less supersymmetry.

Another obstacle is that Witten diagrams naturally evaluate to functions that depend on the locations $\vec{x}_i \in \partial\text{AdS}$ of the boundary conditions $\bar{\phi}_i$ in (2.1). Because of conformal invariance the non-trivial dependence for a four-point function is only through two cross ratios, but it is expressed through complicated functions even for the simplest cases. Focusing on tree level, which will be of interest for this thesis, contact diagrams evaluate to so-called D-functions [220, 222], which admit a closed-form expression in terms of polylogarithms only when the external operators have integer dimensions, and are otherwise only known as infinite series in most other cases. For exchange diagrams the situation is even worse, with closed-form expressions available only when certain relations between the dimensions of external and exchanged operators are satisfied [240, 241]. At loop level the evaluation of individual diagrams in spacetime is even harder (see, *e.g.*, [242–247] for some approaches), while using bootstrap methods and suitable ansätze considerable progress has been made in the computation of the crossing-symmetric sum of all diagrams contributing to a certain correlator, possibly with supersymmetry [226, 248–251]. More progress in understanding the spacetime

structure of Witten diagrams has been made in the case of the AdS_2/CFT_1 correspondence, where four-point functions only depend on one cross ratio [9, 11, 12].

Given the complicated state of affairs, some technical simplifications are required to make progress beyond the few correlators that can be computed with the traditional method in $AdS_5 \times S^5$ [221, 222, 252–256] and the stress-tensor four-point function in $AdS_7 \times S^4$ [257], with no results for $AdS_4 \times S^7$.

2.3 Mellin space and bootstrap methods

A fundamental tool to understand the structure of holographic correlators was introduced by Mack in [223, 224], building on previous work [258, 259]: Mellin space, which can be thought of as the AdS analogue of Fourier space for flat-space scattering amplitudes. Consider a four-point function of identical operators $\mathcal{O}$ of dimension Δ, written as

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}(x_3)\,\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}\,x_{34}^{2\Delta}}\,\mathcal{G}(U,V)\,, \tag{2.2}$$

where we have introduced the cross ratios

$$U = \frac{x_{12}^2\,x_{34}^2}{x_{13}^2\,x_{24}^2}\,, \qquad V = \frac{x_{14}^2\,x_{23}^2}{x_{13}^2\,x_{24}^2}\,, \tag{2.3}$$

for a certain function G. Then the Mellin transform of $\mathcal{G}(U,V)$ is a $\mathcal{M}(s,t)$ such that

$$\mathcal{G}(U,V) = \int_{-i\,\infty}^{i\,\infty} \frac{ds\,dt}{(4\pi i)^2}\,U^{\frac{s}{2}}\,V^{\frac{t}{2}}\,\mathcal{M}(s,t)\,\Gamma\left[\Delta-\frac{s}{2}\right]^2\,\Gamma\left[\Delta-\frac{t}{2}\right]^2\,\Gamma\left[\Delta-\frac{u}{2}\right]^2\,, \tag{2.4}$$

where s,t,u are Mellin variables (analogue of Mandelstam variables), subject to the constraint $s+t+u=4\Delta$.

The Mellin representation of AdS scattering amplitudes has several advantages, to the extent that it is considered to be the most natural one [73, 260]. First of all, it has a simple analytic structure in terms of s and t, where poles are related to the exchange of operators in the OPE and the factorization properties are transparent [223, 261]. Moreover, it bears several concrete similarities with the representation of flat-space amplitudes in terms of Mandelstam variables, allowing for instance for the derivation of the analogue of Feynman rules for Witten diagrams [260, 262–264] and even to the extent that in the limit of large AdS radius Mellin amplitudes reduce to flat-space ones, where the Mellin variables s and t (taken to be large in the limit) are identified with Mandelstam variables:

this is the so-called *flat-space limit* [73]. Finally, Bose symmetry of CFT correlators translates into a symmetry of $\mathcal{M}$ under permutation of the Mellin variables s, t and u.

For the purposes of this thesis, we will mostly be concerned with the Mellin representation of contact and exchange quartic Witten diagrams, represented in figure 2.1. Much like flat-space amplitudes, Mellin amplitudes for contact interactions of the schematic type $\partial^{2k}\,\phi^4$ are given by polynomials in the Mellin variables of degree k. On the other hand, the exchange of an operator of dimension Δ and spin ℓ in the s-channel produces admits a Mellin representation

$$\mathcal{M}^{(s)}(s,t) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_{m,\ell}(t,u)}{s-\Delta+\ell-2m} + \mathcal{P}_{\ell-1}(s,t)\,, \tag{2.5}$$

where $\mathcal{Q}_{m,\ell}$ are Mack polynomials, of degree ℓ, with m labelling the descendants of the exchanged primary operator. $\mathcal{P}_{\ell-1}(s,t)$ is a polynomial of degree $\ell-1$, parametrizing the choice of (on-shell equivalent) cubic couplings which differ by contact terms with no more than $2(\ell-1)$ derivatives. Notice that the exchange of a "single-trace" operator in the OPE (therefore visible as an exchange Witten diagram 2.1b) produces a family of single poles in (2.5), while contact terms are completely regular as they only give rise to double-trace operators in the OPE, whose corresponding poles are already accounted for by the gamma functions present in (2.4).

While the Mellin representation certainly simplifies the structure of holographic correlators, it does not overcome the difficulty of obtaining a quartic Lagrangian for the graviton KK modes and would still require each diagram to be evaluated individually. The solution to this problem arises with the introduction of bootstrap methods, where one focuses only on certain qualitative aspects of the supergravity Lagrangian (such as selection rules and the number of derivatives in quartic vertices) and works with an ansatz on which crossing symmetry and supersymmetry are imposed. Focusing on tree-level correlators, the joint application of Mellin space and the bootstrap philosophy[5] led to a simple formula for the four-point functions between arbitrary half-BPS operators in $\text{AdS}_5 \times S^5$ [225, 227], the first results beyond the stress-tensor sector in $\text{AdS}_7 \times S^4$ [265], some isolated results for $\text{AdS}_4 \times S^7$ [267] and even some cases with half of the maximal supersymmetry [268]. A nice review of the application of the analytic bootstrap to the study of holographic correlators can be found in [71].

2.4 The MRV limit and hidden structures

[5] It is worth stressing that bootstrap methods also greatly simplify computations in position space, when an explicit expression for all relevant exchange diagrams is known [227, 265, 266].

Despite the great progress achieved with the use of Mellin space and bootstrap methods, a complete solution for general four-point functions between half-BPS operators was lacking even at tree level and in maximally supersymmetric backgrounds, except for $AdS_5 \times S^5$, until very recently. Even in the latter case, a general solution to the algebraic bootstrap problem in Mellin space has originally been found due to the particular simplicity of the result (for the reduced correlator), rather than to an actual understanding of its structure [225]. The difference with the scattering amplitudes program in ordinary Minkowski spacetime is striking: in that case, scattering amplitudes at tree level are known for any number of external points in a vast array of theories and possess several interesting properties that are not visible from their Lagrangian description, also allowing for high loops explorations, in particular in gravity and gauge theories (with or without supersymmetry) – see [81–83, 269] for reviews.

A fundamental step in the direction of understanding the structure of holographic correlators was made in [230, 231], where a systematic and constructive way to solve for all four-point functions between half-BPS operators in all maximally symmetric theories at tree level was found. The method relies on the possibility of choosing a special configuration in the R-symmetry polarizations of external operators, for which the correlators simplify drastically, due to the properties of the supermultiplets exchanged in the OPE. Such configurations were denoted with the term maximally R-symmetry violating (MRV), in analogy with the notion of maximally helicity violating (MHV) amplitudes in flat space (see for example the celebrated Parke-Taylor formula for gluon scattering [270]). The idea, which we shall describe in detail in the next chapters, roughly consists in the fact that supersymmetry prescribes the presence of certain zeroes and the absence of certain poles in the Mandelstam variables in MRV configurations. Imposing these properties on a Bose-symmetric ansatz in Mellin space, based on a combination of contact diagrams and exchange diagrams compatible with the selection rules, leads to a constructive solution for MRV amplitudes where no guesswork is involved. Restoring the full R-symmetry dependence is then completely trivial and led the authors of [230, 231] to a complete solution for holographic correlators in maximally supersymmetric backgrounds.

Interestingly, the idea of MRV limit turns out to be more than just a computational tool: rather, it is related to interesting physics and reveals beautiful properties of holographic correlators. Indeed, as already noticed in [230, 231], the MRV limit naturally gives a representation where AdS amplitudes are written as a sum of exchange diagrams with no contact terms, thus giving a concrete realization of the idea that in gauge theories and gravity it should be possible to reconstruct all amplitudes just from

the cubic couplings in the Lagrangian, which is well-known for tree-level amplitudes in flat space [271, 272]. Moreover, such exchange diagrams have the interesting property of having a particularly mild behaviour in the so-called Regge limit [273–276] and are related to the notion of Polyakov-Regge blocks [277, 278][6]. In addition, it was pointed out in [1] that the MRV limit connects to two beautiful structures related to dimensional reductions. First, it turns out that the representation of holographic correlators arising from the MRV limit is such that they can be written in terms of an operator acting on auxiliary amplitudes of a scalar (therefore non-supersymmetric) theory in an AdS space of four dimensions less than the original one. This is due to the structure observed in [241] for the dimensional reduction of generic Witten exchange diagrams and is very reminiscent of the so-called Parisi-Sourlas supersymmetry [280–283], which relates a supersymmetric CFT in $d+2$ dimensions to a non-supersymmetric one in d dimensions. Second, the MRV limit can be thought of as suppressing some redundant information, only retaining what is strictly necessary to reconstruct the full correlator. It turns out that the information which is retained is closely related to a protected sector of operators that is common to all maximally supersymmetric CFTs and has the form of a chiral algebra in six [284] and four [285] dimensions, and a topological quantum mechanics in three dimensions [286, 287].

The scope of the MRV limit is not limited to maximally supersymmetric theories. Rather, in [2] it has been found to be applicable to tree-level four-point functions in a large class of theories with less than maximal supersymmetry. The theories considered in [2], which range from three to six dimensions and all have either six or eight supercharges, share the common feature that their sector of half-BPS operators holographically describes gluons and their KK modes on a three-sphere, which is the common internal space to all the dual geometries. Thus, exploring theories with less than the maximal supersymmetry allows to access the study of gluon scattering in AdS and to explore the properties of the associated amplitudes. These include again Parisi-Sourlas supersymmetry and, in some cases, protected sectors, but also hidden conformal symmetry [288–290], color-kinematics duality [291] and the double copy [292].

[6] See also [9, 279] for their analogue in one-dimensional CFTs.

2.5 Outline of part I

The rest of **part I** of this thesis is organized as follows. In chapter 3 we will elucidate the idea of MRV limit in the case of four-point functions of half-BPS operators in CFTs with eight (or six, in one case) supercharges in all dimensions $3 \leq d \leq 6$. Their duals all share $\text{AdS}_{d+1} \times S^3$ geometry, whose string theory origin is given by certain $(d-1)$-dimensional branes where the field theory lives as well as some higher-dimensional branes. The half-BPS operators in the CFT holographically describe the KK reduction of gluons living on the higher-dimensional branes on S^3 and at first order in the $1/N$ expansion our results describe the scattering of such super gluons on a rigid AdS background. Some of the hidden properties of gluon scattering amplitudes in AdS are then investigated. This chapter is based on [2]. In chapter 4, based on [1], we switch the focus to the case of maximally supersymmetric CFTs, where all four-point functions between half-BPS operators were computed in [230, 231]. We consider certain structures related to dimensional reductions that are hidden in those results, in particular Parisi-Sourlas supersymmetry and the chiral algebra sector of SCFTs. We will conclude with some discussion and outlook in chapter 5.

Chapter 3

Gluon scattering in AdS

The findings of [230, 231] that, thanks to the idea of the MRV limit, one can obtain all holographic four-point functions between half-BPS operators in all maximally supersymmetric CFTs, provide strong motivations to explore more ambitious extensions. One exciting direction is to consider theories which are not maximally supersymmetric. There is a variety of SCFTs in diverse dimensions which preserve half of the maximal superconformal symmetry (*i.e.*, eight Poincaré supercharges) while still admitting a weakly coupled holographic dual. Constrained by less supersymmetry, such theories have richer structures which are otherwise forbidden, such as global symmetries, and therefore are expected to exhibit new interesting features. Moreover, in five dimensions eight-supercharge SCFTs are also in a sense "maximal", since the 5d $\mathcal{N} = 2$ supersymmetry algebra does not admit a superconformal extension. On the other hand, it is also extremely interesting to ask what happens when *gauge theories*, instead of supergravity, are placed in AdS. Gauge theories in AdS were first explored in the late '80s [293], where the original motivation was to use the negative curvature of AdS to bring the infrared behavior under analytic control without conflicting with gauge invariance. This problem was later revisited in *e.g.* [294, 295].[1] However, much remains *terra incognita* due to enormous computational difficulties in AdS.

In this chapter, we develop powerful techniques to explore both directions in an array of theories. More precisely, we will approach the problem by studying tree-level four-point functions of *super gluons*. These super gluons are the scalar super primaries of a family of superconformal short multiplets whose super descendants have at most Lorentz spin 1. A distinguished member of the family is the flavor current multiplet, which harbors spin-1 conserved currents associated with continuous global flavor symmetries. These conserved currents are dual to massless spin-1 gauge fields in AdS, while the spin-1 operators in other short multiplets are dual to massive vector bosons. The existence of such multiplets demands a certain amount of supersymmetry. The required amount of

[1] There is also an extensive list of works studying general quantum field theories in AdS. See [59, 296–299] for a sampling of results from various points of view.

supersymmetry is only half as much (at most) as in the maximally superconformal cases. However, as we will see, this amount of supersymmetry also makes the calculation tractable. By focusing on correlators of super gluons, we can avoid dealing with the complicated kinematics associated with spinning operators. On the other hand, correlators of spinning operators can be obtained from the super gluon correlators by using superconformal symmetry.

While we expect our techniques to be useful in other setups, in this chapter we will consider super gluon scattering amplitudes in two classes of theories. The first class arises from D-branes or M-branes near singularities, which preserve *eight* Poincaré supercharges. Theories in this class are full-fledged SCFTs which include 4d $\mathcal{N}=2$ SCFTs arising from D3-branes near F-theory singularities [300, 301], 5d F_4 Seiberg exceptional theories [302] engineered from D4-D8/O8 systems, as well as the 6d $\mathcal{N}=(1,0)$ E-string theory [303, 304] from M5-branes on an end-of-the-world M9-brane. These theories play important roles in the landscape of SCFTs. For example, the 6d E-string theory is related to various lower-dimensional conformal field theories. The circle compactification of the E-string theory in the presence of E_8 Wilson lines gives rise to the 5d Seiberg theories, while compactifying it on Riemann surfaces leads to 4d $\mathcal{N}=1$ theories. These theories share a common feature in their dual holographic descriptions: the background geometries all have a singular locus of the form $\text{AdS}_{d+1}\times S^3$ where d is the spacetime dimension of the SCFT. On this locus there is a vector multiplet with a certain gauge group (or flavor symmetry group from the boundary CFT perspective)[2], which upon reduction on S^3 gives rise to the aforementioned super gluons. Another important feature of these theories is that the self-coupling of super gluons is stronger than their coupling to super gravitons by powers of N, where N is the number of branes probing the singularities and is taken to be large. This hierarchy of couplings implies that the dominant contribution in tree-level processes is given by super gluon exchanges, and graviton exchanges will only appear at higher orders in $1/N$. Essentially, at leading order we are only dealing with supersymmetric gauge theories in AdS.

The second class of theories is a type of phenomenological model of holography in which flavors (fundamental matter) in the boundary supersymmetric gauge theories are included by adding probe D-branes. The two theories we will focus on are 4d $\mathcal{N}=4$ SYM and 3d $\mathcal{N}=6$ ABJM with fundamental matter. In the first case, D7-branes which fill the AdS_5 and wrap an S^3 inside S^5 are added [305]. In the second case, D6-branes which fill the AdS_4 and wrap an $\mathbb{RP}^3$ inside $\mathbb{CP}^3$ are introduced [306–308]. The number of flavor branes in both cases needs to be much smaller than the number

[2]This is because in AdS/CFT global symmetry currents in the boundary couple to gauge fields in the bulk. We will use the terms *flavor*, *gauge*, *color* interchangeably in this chapter depending on the perspective we want to take.

of D3- or M2-branes so that the theories remain conformal. The flavor branes can then be treated as probes and do not backreact to the geometry. However, the added flavor branes break half of the supersymmetry, to 4d $\mathcal{N} = 2$ and 3d $\mathcal{N} = 3$ respectively. The low energy degrees of freedom on the probe branes are again vector multiplets, and give rise to super gluons in AdS after Kaluza-Klein reduction.

The main result of this chapter is a complete list of all tree-level four-point functions of super gluons with *arbitrary* Kaluza-Klein levels in *all* the theories mentioned above. Our main tool is the maximally R-symmetry violating (MRV) method developed in [230, 231] together with constraints from the superconformal Ward identities. Using these ingredients, we develop a streamlined algorithm for computing general four-point super gluon amplitudes that can be applied to any spacetime dimension. By only inputting the spectrum and imposing superconformal symmetry, we show that four-point amplitudes in vast families of theories are completely fixed. The answer for the amplitudes takes the universal form as a sum over exchange contributions, with no additional contact terms

$$\mathcal{M} = \mathsf{c}_s M_s + \mathsf{c}_t M_t + \mathsf{c}_u M_u \ .$$

Here $\mathsf{c}_s = f^{I_1 I_2 J} f^{J I_3 I_4}$, $\mathsf{c}_t = f^{I_1 I_4 J} f^{J I_2 I_3}$, $\mathsf{c}_u = f^{I_1 I_3 J} f^{J I_4 I_2}$ are color structures associated with exchanging adjoint representations, and f^{IJK} are the structure constants of the gauge group. Remarkably, we find that the only dependence on the gauge group is through these color structures. The amplitude factors $M_{s,t,u}$ capture nontrivial dynamics, and will be explicitly given for each theory. These super gluon amplitudes encode a wealth of data in the dual SCFTs. In particular, our analysis outputs the *complete* three-point functions of all super primaries of these short multiplets (see section 3.3 and 3.4), which to our knowledge were not known in most of these theories. Note that the structure above is very reminiscent of that of flat space gluon amplitudes at tree level – *e.g.*, the dependence on gauge groups is exactly the same. In fact, the comparison can be sharpened by taking the flat space limit of our results. In this limit AdS amplitudes simplify drastically and reduce to

$$\mathcal{M}|_{\text{flat}} = \left(\frac{\mathsf{c}_s N_s}{s} + \frac{\mathsf{c}_t N_t}{t} + \frac{\mathsf{c}_u N_u}{u} \right) \times (\text{wavefunctions on } S^3),$$

with simple polynomials $N_{s,t,u}$ homogeneous in Mandelstam variables obeying $N_s + N_t + N_u = 0$. The information of different Kaluza-Klein levels of AdS super gluons in the flat space limit is factored out in an S^3 wavefunction factor. We will show that this result matches exactly with the flat space

four-gluon scattering amplitude after choosing appropriate polarizations. However, to appreciate the full richness of our results we need to go back to finite AdS curvature where more interesting mathematical structures become visible. Some of these structures have analogues in super graviton amplitudes in maximally supersymmetric theories, while other features are only possible with less supersymmetry and therefore are new. Let us highlight some of these structures below.

Parisi-Sourlas supersymmetry: As mentioned above, all super gluon amplitudes can be expressed as the sum of exchange amplitudes without additional contact terms. The exchange amplitudes receive contributions from only a finite set of super gluon multiplets due to selection rules. For each multiplet, we find its contribution can be written as a differential operator acting on the sum of two AdS_{d+1} scalar exchange Witten diagrams. However, a closer look at the sum reveals that it is identical to just a single scalar exchange Witten diagram which lives in a lower dimensional AdS_{d-1} space. This curious dimensional reduction phenomenon turns out to be related to a holographically realized Parisi-Sourlas supersymmetry [241], and is a feature shared by all the AdS supersymmetric gauge theories considered in this chapter. Similar dimensional reduction was observed in maximally supersymmetric super graviton amplitudes [309]. However, in that case the dimension of the AdS space is reduced by four instead of two.

Hidden conformal symmetry: For 4d $\mathcal{N} = 2$ theories, we also observe an *eight* dimensional hidden conformal symmetry. In position space, we can define a *reduced* correlator $H_{k_1k_2k_3k_4}$ to fully capture the dynamical information, where k_i labels the Kaluza-Klein levels of the external super gluons. We find that the reduced correlator with lowest Kaluza-Klein level $H_{2222}(x_{ij}^2)$, depending on the AdS_5 distances x_{ij}^2 only, can be promoted into a generating function $\mathbf{H}$. The promotion is done by replacing the five dimensional distances in $H_{2222}(x_{ij}^2)$ with the eight dimensional distances $x_{ij}^2 \to x_{ij}^2 - t_{ij}$ where t_{ij} are the distances on S^3. Since the $\text{AdS}_5 \times S^3$ background is conformally flat, they can also be viewed as the conformally invariant distances in $\mathbb{R}^{7,1}$ after a Weyl transformation. General correlators $H_{k_1k_2k_3k_4}$ can then be obtained by Taylor expanding the generating function $\mathbf{H}$ in t_{ij} and collecting all admissible monomials of t_{ij} in the reduced correlator. The same type of hidden conformal structures also made appearances in IIB supergravity in $\text{AdS}_5 \times S^5$ [288] and $\text{AdS}_3 \times S^3 \times K3$ [289, 290], where the symmetries are ten and six dimensional respectively. However, our new example makes it clear that such hidden structures are not unique to supergravity theories, but can be found in supersymmetric gauge theories as well. It also provides further evidence that these structures originate from the conformal flatness of the background geometry.

AdS color-kinematics duality: A fascinating property of flat space amplitudes is the color-kinematics duality [291]. A natural question is whether this structure extends to AdS. In this chapter, we provide an affirmative answer by finding an AdS version of the color-kinematics duality. This AdS color-kinematics duality holds for the massless super gluons, *i.e.*, the gauge fields in AdS, and is almost identical to the flat space relation. These massless four-point Mellin amplitudes have a form analogous to the flat space four-gluon amplitude

$$\mathcal{M} \sim \mathtt{c}_s\mathtt{n}_s\left(\frac{1}{s-d+2}+\dots\right)+\mathtt{c}_t\mathtt{n}_t\left(\frac{1}{t-d+2}+\dots\right)+\mathtt{c}_u\mathtt{n}_u\left(\frac{1}{u-d+2}+\dots\right)$$

The kinematic factors $\mathtt{n}_{s,t,u}$ are linear in the Mellin-Mandelstam variables, and the ... denote satellite poles at $d-2+2\mathbb{Z}_+$ which are completely fixed by symmetry. Thanks to the Jacobi identity, $\mathtt{c}_{s,t,u}$ satisfy $\mathtt{c}_s+\mathtt{c}_t+\mathtt{c}_u=0$. Remarkably, $\mathtt{n}_{s,t,u}$ satisfy the same identity $\mathtt{n}_s+\mathtt{n}_t+\mathtt{n}_u=0$, giving an AdS relation that completely parallels the flat space one. This structure for massless super gluons also admits an interesting modification to massive modes, as we will discuss later in this chapter.

The rest of the chapter is organized as follows. In section 3.1 we give a brief review of the theories which we will consider and provide some background. We also discuss various kinematical aspects and introduce the Mellin representation. In section 3.2 we outline the strategy of our computation. We introduce the MRV method for non-maximally superconformal theories, and explain how to fix the super gluon amplitudes by imposing symmetry constraints. sections 3.3 and 3.4 contain the main results of this chapter. In section 3.3 we give all four-point super gluon amplitudes in the three full-fledged SCFTs, and in section 3.4 we give the results for two models with flavor branes. In section 3.5 we study the flat space limit of AdS super gluon amplitudes and show how all amplitudes in this limit are reproduced from the flat space gluon scattering amplitudes. Using the flat space limit, we also outline an alternative method to derive our results. In section 3.6 we discuss in detail various interesting mathematical structures in these super gluon amplitudes: Parisi-Sourlas supersymmetry, hidden conformal symmetry and AdS color-kinematics duality.

3.1 Preliminaries

In this section we discuss a few preliminary ingredients for the rest of the chapter. In section 3.1.1 we describe the theories to be considered. In section 3.1.2 we specify the correlators we will consider and discuss their kinematic features. In section 3.1.3 we review the Mellin representation

and how superconformal constraints are implemented in Mellin space.

3.1.1 Theories

In this chapter we will discuss holographic correlators in two types of theories. The first type is a distinguished class of SCFTs in 4d, 5d and 6d, which arises from branes probing singularities. These theories preserve half of the maximal superconformal symmetry, *i.e.*, eight Poincaré supercharges, and are listed in table 3.1. Let us describe them in more detail. $\mathcal{N}=2$ *theories from F-theory singularities* [300, 301]: These theories arise from N D3-branes, with N large, near an F-theory 7-brane singularity. The resulting near horizon geometry has a metric similar to $\text{AdS}_5 \times S^5$, but with one of the angular coordinates of the compact space having a changed periodicity

Dimension	SCFT_d	holographic origin
$d=4$	$\mathcal{N}=2$ theories	D3 near F-theory singularities
$d=5$	Seiberg exceptional theories	D4-D8/O8 system
$d=6$	E-string theory	M5 on an end-of-the-world 9-brane

Table 3.1: A class of holographic theories with eight Poincaré supercharges and a supergravity limit. These theories arise in a similar fashion from branes probing singularities.

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\Omega_3^2 \,. \tag{3.1}$$

Here $d\Omega_3^2$ is the metric on S^3, $0 \le \theta \le \frac{\pi}{2}$, and ϕ is periodic with period $2\pi(1-\nu/2)$. Seven types of singularities give rise to a conformal field theory on the D3-branes and those correspond to[3]

$$\nu = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{4}{3}, \frac{3}{2}, \frac{5}{3} \,. \tag{3.2}$$

The 7-brane wraps the AdS_5 and is located at $\theta = 0$, which is an S^3 in the compact space. Therefore the $SO(6)$ isometry is broken to

$$SO(4) \times SO(2) \simeq SU(2)_R \times SU(2)_L \times U(1)_r \tag{3.3}$$

where the first factor $SU(2)_R$ becomes the $SU(2)$ R-symmetry of $\mathcal{N}=2$, and the other $SU(2)_L$ becomes a flavor symmetry. On the 7-brane, there is a $7+1$ dimensional $\mathcal{N}=1$ SYM theory with gauge group G_F. From the point of view of the four-dimensional theory living on the $D3$-brane G_F

[3]The allowed type of singularities (and the corresponding groups in (3.4)) come from the requirement that one has an F-theory background with a constant value of the axio-dilaton field which is invariant under $SL(2,\mathbb{Z})$ transformations [310].

plays the role of a global symmetry. The above choices for ν in (3.2) correspond to the following global symmetries

$$G_F = U(1)\,,\ SU(2)\,,\ SU(3)\,,\ SO(8)\,,\ E_6\,,\ E_7\,,\ E_8\ . \tag{3.4}$$

The simplest case corresponds to a $\mathbb{Z}_2$ orientifold singularity with $\nu = 1$. In this case the four dimensional theory is a $USp(2N)$ $\mathcal{N} = 2$ gauge theory with $SO(8)$ global symmetry and with one antisymmetric and four fundamental hypermultiplets.

Seiberg exceptional theories [302]: This class of theories comes from the UV fixed point of a $USp(2N)$ gauge theory coupled to $N_f \leq 7$ hypermultiplets in the fundamental representation, and a single hypermultiplet in the antisymmetric representation[4]. The flavor symmetry is enhanced to $G = E_{N_f+1} \times SU(2)_L$ at the fixed point.[5] These theories can also be constructed from Type IIA string theory (or Type I′ string theory) by a D4-D8/O8 setup. The dual geometry is a warped product of AdS_6 and a hemisphere HS^4. Therefore we are left with an $SO(4) \simeq SU(2)_R \times SU(2)_L$ isometry. The first factor becomes the $SU(2)$ R-symmetry of the 5d F_4 superconformal group, while the $SU(2)_L$ becomes a flavor symmetry. The boundary of the four-hemisphere gives an 8-brane that fills AdS_6. Similar to the above F-theory singularity case, there is a 9d $\mathcal{N} = 1$ vector multiplet propagating on this $\text{AdS}_6 \times S^3$ boundary with a gauge group E_{N_f+1}.

E-string theory [303, 304]: The E-string theory can be engineered by placing N M5-branes on top of an "end-of-the-world" 9-brane of the Hořava-Witten compactification of M-theory. The dual geometry is $\text{AdS}_7 \times S^4/\mathbb{Z}_2$, which has a $\mathbb{Z}_2$ fixed locus $\text{AdS}_7 \times S^3$. The $\mathbb{Z}_2$ quotient breaks the isometry from $SO(5)$ down to $SO(4) \simeq SU(2)_R \times SU(2)_L$. Again, the first factor is identified with the $SU(2)$ R-symmetry of the 6d $\mathcal{N} = (1, 0)$ superconformal group, and the other $SU(2)_L$ becomes part of the flavor symmetry. In addition, on the $\mathbb{Z}_2$ locus there is a 10d $\mathcal{N} = 1$ SYM multiplet with E_8 gauge group, which is a flavor group from the point of view of the six dimensional SCFT.

In these three families of theories, the Kaluza-Klein fields in AdS come from two sources: supergravity in the full 10d or 11d space, and degrees of freedom living on the singular locus. The spectrum of the supergravity modes in AdS_5 and AdS_7 can be analyzed following the method of [301] (or in the simpler orbifold cases, the spectrum can be obtained by projecting to the singlets

[4]Note that since the Yang-Mills coupling is irrelevant in five dimensions, gauge theories are non-renormalizable by naive power counting. The existence of CFTs in the UV is highly non trivial and provides a rare example of asymptotic safety.

[5]Note $E_1 = SU(2)$, $E_2 = SU(2) \times U(1)$, $E_3 = SU(3) \times SU(2)$, $E_4 = SU(5)$, $E_5 = SO(10)$.

[300, 311]). The analysis of the AdS_6 case is more involved due to the warped product structure [312]. The supergravity modes can be organized into superconformal multiplets, and contain both short and long types. On the other hand, analyzing the spectrum from the singular locus is much simpler. Because the vector super multiplets contain fields with at most spin 1, all the KK modes must belong to $\frac{1}{2}$-BPS multiplets. Their spectrum therefore is completely fixed by their R-symmetry representations [159, 301, 311], and there is no need to analyze the linearized equations of motion. In this chapter, we will focus on the latter type of AdS excitations, *i.e.*, field modes descending from the singular loci, and compute all four-point functions of their dual operators.

The second class of models shares a lot similarities with the theories discussed above. These are the theories arising from adding "flavors", *i.e.*, fundamental quarks, to supersymmetric gauge theories. We list these theories in table 3.2.

Dimension	$SCFT_d$	holographic origin
$d = 3$	$\mathcal{N} = 6$ ABJM with flavors	D6 wrapped over $AdS_4 \times \mathbb{RP}^3$
$d = 4$	$\mathcal{N} = 4$ SYM with flavors	D7 wrapped over $AdS_5 \times S^3$

Table 3.2: A class of holographic theories obtained by adding flavor branes. Note the number of preserved supersymmetries after adding flavor branes is halved, *i.e.*, $\mathcal{N} = 3$ for the flavored ABJM and $\mathcal{N} = 2$ for the flavored SYM.

Holographically, these models can be obtained by adding probe flavor branes. Such a construction was first considered for 4d $\mathcal{N} = 4$ SYM in [305], where $N_f \ll N$ D7-branes are added as probes to $AdS_5 \times S^5$.[6] The D7-branes fill AdS_5 and wrap an equatorial S^3 inside the S^5. The presence of the probe flavor branes breaks the $\mathcal{N} = 4$ superconformal symmetry to $\mathcal{N} = 2$. Similarly, D6-branes can also be added to $AdS_4 \times \mathbb{CP}^3$, which adds fundamental matter to the $\mathcal{N} = 6$ ABJM theory [306–308]. The D6-branes fill AdS_4 and wrap an $\mathbb{RP}^3$ inside $\mathbb{CP}^3$. Note that the flavored theory now has only $\mathcal{N} = 3$ superconformal symmetry, *i.e.*, six Poincaré supercharges. Nevertheless, we will see that it can be treated in the same fashion. We also note that the flavor branes are only probes and do not backreact to the bulk geometry, which differs from the situation for theories in table 3.1. Finally, using the same argument as above, we can see that the AdS Kaluza-Klein modes from the low energy degrees of freedom of the flavor branes contain only short multiplets [308, 313]. It is this sector of operators for which we will compute correlators.

[6] This is also known as the "quenched" approximation. The β-function of the theory remains zero at leading order in large N for $N_f \ll N$.

3.1.2 Kinematics of four-point functions

Operators

The primary focus of this chapter is the class of holographic SCFTs with eight Poincaré supercharges summarized in table 3.1 in the previous subsection. The examples include 4d $\mathcal{N}=2$ theories constructed from D3-branes near F-theory singularities [300, 301], 5d $F(4)$ theories constructed from D4-D8 systems in massive Type IIA supergravity [314], and 6d $(1,0)$ theories from M5-branes on an "end-of-the-world" M9-brane [315]. However, the same techniques will also apply to the theories with flavor branes summarized in table 3.2, even though the 3d theory has only six supercharges. As already mentioned in section 3.1.1, what these theories have in common is that there are branes that fill the AdS factors and an S^3 (or $\mathbb{RP}^3$) in the compact space. The presence of the branes breaks the compact space isometries as follows

$$\begin{aligned}
&\text{AdS}_7: && SO(5) \to SO(4) = SU(2)_R \times SU(2)_L \,, \\
&\text{AdS}_6: && SO(5) \to SO(4) = SU(2)_R \times SU(2)_L \,, \\
&\text{AdS}_5: && SO(6) \to SO(4) \times SO(2) = SU(2)_R \times SU(2)_L \times U(1) \,, \\
&\text{AdS}_4: && SO(6) \to SO(4) = SU(2)_R \times SU(2)_L
\end{aligned}$$

where $SO(4) = SU(2)_R \times SU(2)_L$ is the isometry of an S^3 (or $\mathbb{RP}^3$). One of the $SU(2)$ factors is identified with the $SU(2)_R$ R-symmetry of the superconformal group[7], while the other $SU(2)$ factor becomes part of the global symmetry. The $U(1)$ factor for the AdS_5 case is identified with the $U(1)_r$ symmetry for the 4d $\mathcal{N}=2$ superconformal symmetry.

The low-energy theories on these AdS-filling branes are SYM with a certain gauge group G_F. An important feature of these theories at large N is that the degrees of freedom localized on the branes interact more strongly with themselves than with the gravitational degrees of freedom in the bulk. This can be seen from the fact that the flavor symmetry central charge ($c_{\mathcal{J}}$) is larger than the stress tensor central charge ($c_{\mathcal{T}}$) by powers of N, and the exchange contributions are inversely proportional to the central charges. In particular, in all examples where the explicit results are known, one has

$$c_{\mathcal{J}} \sim N^{\epsilon} \,, \quad c_{\mathcal{T}} \sim N^{1+\epsilon} \,, \quad c_{\mathcal{T}}/c_{\mathcal{J}} \sim N \,, \tag{3.5}$$

[7]For concreteness, we list the superconformal groups. The 3d $\mathcal{N}=3$ superconformal group is $OSp(3|4)$; the 4d $\mathcal{N}=2$ superconformal group is $SU(2,2|2)$; in 5d the only superconformal group is F_4; in 6d the $\mathcal{N}=(1,0)$ superconformal group is $OSp(8^*|2)$.

for large N, where $\epsilon = (d-2)/2$ and d is the dimension of the spacetime where the CFT lives. While in general external brane degrees of freedom (*i.e.*, super-gluons) can exchange both supergravity modes and other super-gluon modes, at tree level the former exchanges are suppressed by one power of N and can therefore be neglected. The result is that at tree level the external brane degrees of freedom are decoupled from bulk supergravity and one ends up studying pure super Yang-Mills theory in a rigid AdS background, with no dynamical gravity. Metric fluctuations become important at higher orders in the large N expansion, but we shall not study such contributions in this paper.

Note that the vector multiplet on the branes contains fields with Lorentz spin at most 1. This condition guarantees that the Kaluza-Klein reduction leads to only short multiplets, which allows us to fix the spectrum from R-symmetry charges without analyzing the linearized equations of motion. Such short multiplets are $\frac{1}{2}$-BPS for $4 \leq d \leq 6$, and $\frac{1}{3}$-BPS for 3d $\mathcal{N} = 3$. This strategy to obtain the spectrum was used in [159, 301, 311], and the analyses led to the following results. The super multiplets are labelled by an integer Kaluza-Klein level $k = 2, 3, \ldots$, with $k = 2$ corresponding to the current multiplet. The super primaries are scalar operators which have spin-$\frac{k}{2}$ under the $SU(2)_R$ R-symmetry group and spin-$\frac{k-2}{2}$ under $SU(2)_L$. In the 4d $\mathcal{N} = 2$ case there is also a $U(1)_r$ R-symmetry. However, the super primaries are neutral under this $U(1)_r$. The super primaries have protected conformal dimensions

$$\Delta_k = \epsilon k \ , \quad \epsilon = \frac{d-2}{2} \tag{3.6}$$

where d is the spacetime dimension of the boundary theory. Finally, all components of the multiplet transform in the adjoint representation of the gauge group G_F, which is a flavor symmetry from the CFT point of view. We denote the super primaries as

$$\mathcal{O}^{I;\alpha_1,\ldots,\alpha_k;\bar{\alpha}_1,\ldots,\bar{\alpha}_{k-2}}(x) \tag{3.7}$$

where $\alpha_i = 1, 2$, $\bar{\alpha}_i = 1, 2$ are the $SU(2)_R$, $SU(2)_L$ indices respectively, and $I = 1, \ldots, \dim(G_F)$ is a flavor symmetry index.

Four-point functions

To handle the R-symmetry and $SU(2)_L$ flavor symmetry indices, it is convenient to multiply the

super primary operators with auxiliary $SU(2)$ spinors

$$\mathcal{O}_k^I(x,v,\bar{v}) = \mathcal{O}^{I;\alpha_1,\ldots,\alpha_k;\bar{\alpha}_1,\ldots,\bar{\alpha}_{k-2}}(x) v^{\beta_1}\ldots v^{\beta_k}\epsilon_{\alpha_1\beta_1}\ldots\epsilon_{\alpha_k\beta_k}\bar{v}^{\bar{\beta}_1}\ldots\bar{v}^{\bar{\beta}_{k-2}}\epsilon_{\bar{\alpha}_1\bar{\beta}_1}\ldots\epsilon_{\bar{\alpha}_{k-2}\bar{\beta}_{k-2}} \,. \tag{3.8}$$

The contraction with the spinors automatically projects the indices to the spin-$\frac{k}{2}$ and spin-$\frac{k-2}{2}$ representations of $SU(2)_R$ and $SU(2)_L$. The four-point correlation functions

$$G^{I_1I_2I_3I_4}(x_i;v_i,\bar{v}_i) = \langle \mathcal{O}_{k_1}^{I_1}(x_1,v_1,\bar{v}_1)\mathcal{O}_{k_2}^{I_2}(x_2,v_2,\bar{v}_2)\mathcal{O}_{k_3}^{I_3}(x_3,v_3,\bar{v}_3)\mathcal{O}_{k_4}^{I_4}(x_4,v_4,\bar{v}_4)\rangle \,, \tag{3.9}$$

now become a function not only of the spacetime coordinates x_i, but also internal coordinates v_i, $\bar{v}_i$. By exploiting the covariance under conformal symmetry, R-symmetry, and $SU(2)_L$ flavor symmetry, we can extract a kinematic factor such that the correlators depend only on the cross ratios

$$\begin{aligned} G^{I_1I_2I_3I_4}(x_i;v_i,\bar{v}_i) = &\prod_{i<j}\left(\frac{(v_i\cdot v_j)(\bar{v}_i\cdot\bar{v}_j)}{x_{ij}^{2\epsilon}}\right)^{\gamma_{ij}^0}\left(\frac{(v_1\cdot v_2)(v_3\cdot v_4)}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^{\mathcal{E}} \\ &\times((\bar{v}_1\cdot\bar{v}_2)(\bar{v}_3\cdot\bar{v}_4))^{\mathcal{E}-2}\,\mathcal{G}^{I_1I_2I_3I_4}(U,V;\alpha,\beta) \,, \end{aligned} \tag{3.10}$$

where

$$x_{ij} = x_i - x_j \,, \quad (v_i\cdot v_j) = v_i^\alpha v_j^\beta\epsilon_{\alpha\beta} \,, \quad (\bar{v}_i\cdot\bar{v}_j) = \bar{v}_i^{\bar{\alpha}}\bar{v}_j^{\bar{\beta}}\epsilon_{\bar{\alpha}\bar{\beta}} \,. \tag{3.11}$$

Here we have assumed without loss of generality that $k_1 \le k_2 \le k_3 \le k_4$ and distinguish between two different cases

$$k_1+k_4 \ge k_2+k_3 \;\text{ (case I)} \,, \qquad k_1+k_4 < k_2+k_3 \;\text{ (case II)} \,. \tag{3.12}$$

The exponents are given by

$$\begin{aligned} &\gamma_{12}^0 = \gamma_{13}^0 = 0 \,, \;\; \gamma_{34}^0 = \frac{\kappa_s}{2} \,, \;\; \gamma_{24}^0 = \frac{\kappa_u}{2} \,, \\ &\gamma_{14}^0 = \frac{\kappa_t}{2} \,, \;\; \gamma_{23}^0 = 0 \,, \;\text{(I)}, \quad \gamma_{14}^0 = 0 \,, \;\; \gamma_{23}^0 = \frac{\kappa_t}{2} \,, \;\text{(II)} \end{aligned} \tag{3.13}$$

where

$$\kappa_s \equiv |k_3+k_4-k_1-k_2| \,, \;\; \kappa_t \equiv |k_1+k_4-k_2-k_3| \,, \;\; \kappa_u \equiv |k_2+k_4-k_1-k_3| \,, \tag{3.14}$$

and

$$\mathcal{E} = \frac{k_1 + k_2 + k_3 - k_4}{2} \quad \text{(case I)} , \qquad \mathcal{E} = k_1 \quad \text{(case II)} \tag{3.15}$$

is the extremality. We have also defined the conformal cross ratios

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z} , \quad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}) , \tag{3.16}$$

the R-symmetry cross ratio

$$\alpha = \frac{(v_1 \cdot v_3)(v_2 \cdot v_4)}{(v_1 \cdot v_2)(v_3 \cdot v_4)} , \tag{3.17}$$

and the $SU(2)_L$ flavor symmetry cross ratio

$$\beta = \frac{(\bar{v}_1 \cdot \bar{v}_3)(\bar{v}_2 \cdot \bar{v}_4)}{(\bar{v}_1 \cdot \bar{v}_2)(\bar{v}_3 \cdot \bar{v}_4)} . \tag{3.18}$$

It is not difficult to see that the function $\mathcal{G}^{I_1 I_2 I_3 I_4}(U, V; \alpha, \beta)$ is a polynomial in α of degree $\mathcal{E}$ and a polynomial in β of degree $\mathcal{E} - 2$.

Superconformal Ward identities

The form (3.10) exploits only the bosonic part of the superconformal group. The fermionic generators impose further constraints in the form of superconformal Ward identities [316]

$$(z\partial_z - \epsilon\alpha\partial_\alpha)\mathcal{G}^{I_1 I_2 I_3 I_4}(z, \bar{z}; \alpha, \beta)\Big|_{\alpha=\frac{1}{z}} = 0 . \tag{3.19}$$

Another set of identities can be obtained by replacing z with $\bar{z}$.

It is useful to compare these superconformal Ward identities to the ones with maximal superconformal symmetry, which take the same form. In the latter case, the R-symmetry group is $SO(n)$ with $n = 5, 6, 8$ and therefore the four-point functions have two independent R-symmetry cross ratios α, $\bar{\alpha}$ which are on the same footing. Meanwhile, maximally superconformal theories do not have flavor symmetries[8] and there is no cross ratio β. The superconformal Ward identities in the maximal case can be viewed as replacing β with $\bar{\alpha}$ in (3.19) and omitting the flavor indices I_i

$$(z\partial_z - \epsilon\alpha\partial_\alpha)\mathcal{G}(z, \bar{z}; \alpha, \bar{\alpha})\Big|_{\alpha=\frac{1}{z}} = 0 \qquad \text{(maximally superconformal)} . \tag{3.20}$$

[8]This can be argued, for example, from the classification of supermultiplets for the relevant superconformal algebras [35].

There are in total four identities by interchanging z, $\bar{z}$, and α, $\bar{\alpha}$. While the non-maximal case (3.19) has naively half as many identities, the independent flavor structures give rise to more constraints, making the superconformal Ward Identities as powerful in the end.

Flavor projectors and crossing matrices

That external operators transform in the adjoint representation of the flavor group G_F brings extra complexities. To deal with the flavor symmetry indices, we introduce the following projectors

$$\mathrm{P}_a^{I_1I_2|I_3I_4}\ , \quad \mathrm{P}_a^{I_1I_4|I_3I_2}\ , \quad \mathrm{P}_a^{I_1I_3|I_2I_4}\ , \tag{3.21}$$

for s-, t-, and u-channels respectively. The index a runs over all irreducible representations in the tensor product of $\mathbf{adj}_{G_F} \times \mathbf{adj}_{G_F}$. The projector $\mathrm{P}_a^{I_1I_2|I_3I_4}$, for example, represents the flavor tensor structure that corresponds to exchanging the flavor symmetry representation a in the $12 \to 34$ channel. These projectors satisfy the following relations

$$\begin{aligned}
&\mathrm{P}_a^{I_1I_2|I_3I_4} = (-1)^{|\mathbf{R}_a|}\mathrm{P}_a^{I_2I_1|I_3I_4}\ ,\\
&\mathrm{P}_a^{I_1I_2|I_3I_4} = \mathrm{P}_a^{I_3I_4|I_1I_2}\ ,\\
&\mathrm{P}_a^{I_1I_2|I_3I_4}\mathrm{P}_b^{I_1I_2|I_3I_4} = \delta_{ab}\dim(\mathbf{R}_a)\ ,\\
&\mathrm{P}_a^{I_1I_2|I_3I_4}\mathrm{P}_b^{I_4I_3|I_5I_6} = \delta_{ab}\mathrm{P}_a^{I_1I_2|I_5I_6}\ ,
\end{aligned} \tag{3.22}$$

which follow from basic representation theory. The use of projectors makes it easy to talk about independent flavor structures. Note that projectors in different channels $\{\mathrm{P}_a^{I_1I_2|I_3I_4}\}$, $\{\mathrm{P}_a^{I_1I_4|I_3I_2}\}$, $\{\mathrm{P}_a^{I_1I_3|I_2I_4}\}$, with a running over all representations in $\mathbf{adj}_{G_F} \times \mathbf{adj}_{G_F}$, separately form a basis. Using this we can project, for example, the superconformal Ward identities (3.19) into the intermediate s-channel representations, and obtain independent constraints

$$(z\partial_z - \epsilon\alpha\partial_\alpha)\left(\mathrm{P}_a^{I_1I_2|I_3I_4}\mathcal{G}^{I_1I_2I_3I_4}(z,\bar{z};\alpha,\beta)\right)\Big|_{\alpha=\frac{1}{z}} = 0\ , \quad \mathbf{R}_a \in \mathbf{adj}_{G_F} \times \mathbf{adj}_{G_F}\ . \tag{3.23}$$

In applications it is also often convenient to consider the decomposition of t- and u-channel projectors into the s-channel. This is achieved by contracting the external adjoint indices of the projectors to form flavor crossing matrices

$$(\mathrm{F}_t)_a{}^b = \frac{1}{\dim(\mathbf{R}_a)}\mathrm{P}_a^{I_3I_2|I_1I_4}\mathrm{P}_b^{I_1I_2|I_3I_4}\ , \quad (\mathrm{F}_u)_a{}^b = \frac{1}{\dim(\mathbf{R}_a)}\mathrm{P}_a^{I_4I_2|I_3I_1}\mathrm{P}_b^{I_1I_2|I_3I_4}\ . \tag{3.24}$$

These matrices can be computed using the methods of [317], and explicit examples can be found in table 6 of [318]. Such flavor crossing matrices accompany the standard crossing equations for four-point functions

$$\begin{aligned}\sum_b (F_t)_a{}^b G^a(x_3,x_2,x_1,x_4) &= G^b(x_1,x_2,x_3,x_4)\ ,\\ \sum_b (F_u)_a{}^b G^a(x_4,x_2,x_3,x_1) &= G^b(x_1,x_2,x_3,x_4)\end{aligned} \tag{3.25}$$

where

$$G^{I_1I_2I_3I_4}(x_1,x_2,x_3,x_4) = \sum_a \mathrm{P}_a^{I_1I_2|I_3I_4} G^a(x_1,x_2,x_3,x_4)\ , \tag{3.26}$$

and we have suppressed R-symmetry and the other $SU(2)$ flavor symmetry.

3.1.3 Mellin representation

The amplitude interpretation of holographic correlators is manifest in the Mellin representation [73, 223]

$$\begin{aligned}\mathcal{G}^{I_1I_2I_3I_4}(U,V;\alpha,\beta) &= \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}-a_s} V^{\frac{t}{2}-a_t} \mathcal{M}^{I_1I_2I_3I_4}(s,t;\alpha,\beta)\,\Gamma^{(\epsilon)}_{\{k_i\}}\ ,\\ \Gamma^{(\epsilon)}_{\{k_i\}} &= \Gamma[\tfrac{\epsilon(k_1+k_2)-s}{2}]\Gamma[\tfrac{\epsilon(k_3+k_4)-s}{2}]\Gamma[\tfrac{\epsilon(k_1+k_4)-t}{2}]\Gamma[\tfrac{\epsilon(k_2+k_3)-t}{2}]\Gamma[\tfrac{\epsilon(k_1+k_3)-u}{2}]\Gamma[\tfrac{\epsilon(k_2+k_4)-u}{2}]\end{aligned} \tag{3.27}$$

where $\mathcal{M}^{I_1I_2I_3I_4}$, the Mellin amplitude, is naturally identified as the scattering amplitude in AdS, and $a_s = \frac{\epsilon}{2}(k_1+k_2) - \epsilon\mathcal{E}$, $a_t = \epsilon\mathcal{E} - \frac{\epsilon}{2}(k_1-k_4)$, $s+t+u = \epsilon\sum_{i=1}^4 k_i \equiv \epsilon\Sigma$. The advantage of this representation can already be seen at the level of individual Witten diagrams. For example, the Mellin amplitude of a contact diagram with $2L$ derivatives is a degree-L polynomial in Mellin variables. On other hand, an exchange Witten diagram with exchanged dimension Δ and spin ℓ has an amplitude that has a series of simple poles with polynomial residues

$$\mathcal{M}^{(s)}_{\Delta,\ell}(s,t) = \sum_{m=0}^{\infty} \frac{\mathcal{Q}_{m,\ell}(t,u)}{s-\Delta+\ell-2m} + \mathcal{P}_{\ell-1}(s,t)\ . \tag{3.28}$$

Here $\mathcal{Q}_{m,\ell}$ are degree-ℓ polynomials, and the regular piece $\mathcal{P}_{\ell-1}$ is a polynomial of degree $\ell-1$. Note that the regular piece is ambiguous and can be modified by adding contact diagrams with no more than $2(\ell-1)$ derivatives. This modification corresponds to choosing different cubic couplings which are equivalent only on-shell. Mellin amplitudes of exchange diagrams up to spin 2 are given for a specific (very convenient) choice of contact terms in appendix A of [2].

We now briefly review how to implement the superconformal Ward identity (3.23) in Mellin space. The main difficulty is that (3.23) is not symmetric in z and $\bar{z}$, which causes square roots to appear when expressed in terms of U and V. These square roots are difficult to interpret in Mellin space. However, as we will see below, by taking independent linear combinations of (3.23) and its counterpart with $z \leftrightarrow \bar{z}$, all the z, $\bar{z}$ dependence becomes polynomial in U and V. The latter is easy to exploit in Mellin space. The method was first developed in [267], and was further streamlined in [231]. Let us define

$$\mathcal{M}_a(s,t;\alpha,\beta) = \mathrm{P}_a^{I_1I_2|I_3I_4}\mathcal{M}^{I_1I_2I_3I_4}(s,t;\alpha,\beta) = \sum_{q=0}^{\mathcal{E}} \alpha^q \mathcal{M}_a^{(q)}(s,t;\beta) \ . \tag{3.29}$$

Starting from the basic identity

$$z\partial_z = U\partial_U - \frac{z}{1-z}V\partial_V \ , \tag{3.30}$$

and using the fact that $U\partial_U$, $V\partial_V$ act multiplicatively in the Mellin representation (3.27)

$$U\partial_U \to \left(\frac{s}{2} - a_s\right)\times \ , \quad V\partial_V \to \left(\frac{t}{2} - a_t\right)\times \ , \tag{3.31}$$

we find the Ward identity (3.23) takes the following form acting on the Mellin amplitude

$$\sum_{q=0}^{\mathcal{E}} \left((1-z)z^{\mathcal{E}-q}\left(\frac{s}{2} - a_s - q\right) - z^{\mathcal{E}-q+1}\left(\frac{t}{2} - a_t\right)\right)\mathcal{M}_a^{(q)}(s,t;\beta) = 0 \ . \tag{3.32}$$

Here we have made the inverse Mellin integrals implicit, with the understanding that the powers of z and $1-z$ multiply the integrals from outside. Replacing z with $\bar{z}$, we obtain another identity

$$\sum_{q=0}^{\mathcal{E}} \left((1-\bar{z})\bar{z}^{\mathcal{E}-q}\left(\frac{s}{2} - a_s - q\right) - \bar{z}^{\mathcal{E}-q+1}\left(\frac{t}{2} - a_t\right)\right)\mathcal{M}_a^{(q)}(s,t;\beta) = 0 \ . \tag{3.33}$$

We now consider two independent linear combinations of (3.32) and (3.33), and get

$$\sum_{q=0}^{\mathcal{E}} \left((\zeta_\pm^{\mathcal{E}-q} - \zeta_\pm^{(\mathcal{E}-q+1)})\left(\frac{s}{2} - a_s - q\right) - \zeta_\pm^{(\mathcal{E}-q+1)}\left(\frac{t}{2} - a_t\right)\right)\mathcal{M}_a^{(q)}(s,t;\beta) = 0 \tag{3.34}$$

where we have defined

$$\zeta_+^{(n)} = z^n + \bar{z}^n \ , \quad \zeta_-^{(n)} = \frac{z^n - \bar{z}^n}{z - \bar{z}} \ . \tag{3.35}$$

Crucially, $\zeta_{\pm}^{(n)}$ can be expressed in terms of U and V using

$$\begin{aligned}\zeta_{+}^{(n)} &= 2^{1-n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\Big((1+U-V)^2-4U\Big)^k\,(1+U-V)^{n-2k}\,,\\ \zeta_{-}^{(n)} &= 2^{1-n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k+1}\Big((1+U-V)^2-4U\Big)^k\,(1+U-V)^{n-2k-1}\,,\end{aligned} \tag{3.36}$$

where we remind the reader that polynomials in U and V act as difference operators in Mellin space. More precisely, each monomial U^mV^n multiplying the Mellin integral (3.27) can be absorbed by shifting s and t, and thus becomes a difference operator $\widehat{U^mV^n}$ acting as

$$\widehat{U^mV^n}\circ\mathcal{M}(s,t)=\frac{\Gamma^{(\epsilon)}_{\{k_i\}}(s-2m,t-2n)}{\Gamma^{(\epsilon)}_{\{k_i\}}(s,t)}\mathcal{M}(s-2m,t-2n)\,. \tag{3.37}$$

Replacing the polynomials in (3.34) with the difference operators gives the superconformal Ward identities in Mellin space.

3.2 MRV limit and bootstrap method

From holography, we expect the super gluon Mellin amplitudes to have the following general structure at tree level

$$\mathcal{M}^{I_1I_2I_3I_4}=\mathcal{M}_s^{I_1I_2I_3I_4}+\mathcal{M}_t^{I_1I_2I_3I_4}+\mathcal{M}_u^{I_1I_2I_3I_4}+\mathcal{M}_{\rm con}^{I_1I_2I_3I_4}\,. \tag{3.38}$$

Here $\mathcal{M}_{s,t,u}^{I_1I_2I_3I_4}$ are exchange contributions and $\mathcal{M}_{\rm con}^{I_1I_2I_3I_4}$ comes from contact terms. The contributions $\mathcal{M}_{t,u}^{I_1I_2I_3I_4}$ are related to $\mathcal{M}_s^{I_1I_2I_3I_4}$ by Bose symmetry as usual, but we need to pay additional attention to the flavor symmetry. It is useful to introduce the following flavor structures

$$\mathsf{c}_s=f^{I_1I_2J}f^{JI_3I_4}\,,\quad \mathsf{c}_t=f^{I_1I_4J}f^{JI_2I_3}\,,\quad \mathsf{c}_u=f^{I_1I_3J}f^{JI_4I_2} \tag{3.39}$$

where we have dropped the flavor indices in $\mathsf{c}_{s,t,u}$ to avoid cluttering the notation, and f^{IJK} are the structure constants of the flavor symmetry algebra. In this chapter we normalize the structure constants as

$$f^{IL}{}_Kf^{JK}{}_L=\psi^2h^\vee\delta^{IJ} \tag{3.40}$$

where $h^\vee$ is the dual Coxeter number and ψ is the length squared of the longest root of the flavor

group. Since all the exchanged fields are in the adjoint representation of the flavor group, the exchange contributions $\mathcal{M}_{s,t,u}^{I_1I_2I_3I_4}$ are respectively proportional to $\mathsf{c}_{s,t,u}$. These flavor structures can also be expressed in terms of the projectors introduced in section 3.1.2

$$\mathsf{c}_s = \psi^2 h^\vee \mathrm{P}_{\mathbf{adj}}^{I_1I_2|I_3I_4} \ , \quad \mathsf{c}_t = \psi^2 h^\vee \mathrm{P}_{\mathbf{adj}}^{I_1I_4|I_2I_3} \ , \quad \mathsf{c}_u = \psi^2 h^\vee \mathrm{P}_{\mathbf{adj}}^{I_1I_3|I_4I_2} \ . \tag{3.41}$$

Let us denote

$$\mathcal{M}_s^{I_1I_2I_3I_4} = \mathsf{c}_s M_s^{(k_1,k_2,k_3,k_4)}, \ \mathcal{M}_t^{I_1I_2I_3I_4} = \mathsf{c}_t M_t^{(k_1,k_2,k_3,k_4)}, \ \mathcal{M}_u^{I_1I_2I_3I_4} = \mathsf{c}_u M_u^{(k_1,k_2,k_3,k_4)} \tag{3.42}$$

where we have added the superscript (k_1, k_2, k_3, k_4) to manifest the Kaluza-Klein level associated with each external super gluon. Then Bose symmetry relates the t- and u-channel exchange contributions to the s-channel as the permutations of labels $(1, 2, 3, 4) \to (1, 4, 2, 3)$ and $(1, 2, 3, 4) \to (1, 3, 4, 2)$, and gives

$$\begin{aligned} M_t^{(k_1,k_2,k_3,k_4)} &= (\alpha-1)^{\mathcal{E}}(\beta-1)^{\mathcal{E}-2}\left(M_s^{(k_1,k_4,k_2,k_3)}\Big|^{\{s,t,u\}\to\{t,u,s\}}_{\{\alpha,\beta\}\to\{\frac{1}{1-\alpha},\frac{1}{1-\beta}\}}\right) , \\ M_u^{(k_1,k_2,k_3,k_4)} &= (-\alpha)^{\mathcal{E}}(-\beta)^{\mathcal{E}-2}\left(M_s^{(k_1,k_3,k_4,k_2)}\Big|^{\{s,t,u\}\to\{u,s,t\}}_{\{\alpha,\beta\}\to\{\frac{\alpha-1}{\alpha},\frac{\beta-1}{\beta}\}}\right) . \end{aligned} \tag{3.43}$$

Similarly, under crossing the full Mellin amplitude transforms as

$$\begin{aligned} \mathcal{M}^{I_1I_2I_3I_4}_{(k_1,k_2,k_3,k_4)} &= (\alpha-1)^{\mathcal{E}}(\beta-1)^{\mathcal{E}-2}\left(\mathcal{M}^{I_1I_4I_2I_3}_{(k_1,k_4,k_2,k_3)}\Big|^{\{s,t,u\}\to\{t,u,s\}}_{\{\alpha,\beta\}\to\{\frac{1}{1-\alpha},\frac{1}{1-\beta}\}}\right) , \\ \mathcal{M}^{I_1I_2I_3I_4}_{(k_1,k_2,k_3,k_4)} &= (-\alpha)^{\mathcal{E}}(-\beta)^{\mathcal{E}-2}\left(\mathcal{M}^{I_1I_3I_4I_2}_{(k_1,k_3,k_4,k_2)}\Big|^{\{s,t,u\}\to\{u,s,t\}}_{\{\alpha,\beta\}\to\{\frac{\alpha-1}{\alpha},\frac{\beta-1}{\beta}\}}\right) , \end{aligned} \tag{3.44}$$

or in terms of the flavor stripped amplitudes

$$\begin{aligned} \mathcal{M}^{a}_{(k_1,k_2,k_3,k_4)} &= \sum_b (-1)^{|\mathbf{R}_a|}(\mathrm{F}_u)_a{}^b(\alpha-1)^{\mathcal{E}}(\beta-1)^{\mathcal{E}-2}\left(\mathcal{M}^{b}_{(k_1,k_4,k_2,k_3)}\Big|^{\{s,t,u\}\to\{t,u,s\}}_{\{\alpha,\beta\}\to\{\frac{1}{1-\alpha},\frac{1}{1-\beta}\}}\right) , \\ \mathcal{M}^{a}_{(k_1,k_2,k_3,k_4)} &= \sum_b (-1)^{|\mathbf{R}_a|}(\mathrm{F}_t)_a{}^b(-\alpha)^{\mathcal{E}}(-\beta)^{\mathcal{E}-2}\left(\mathcal{M}^{b}_{(k_1,k_3,k_4,k_2)}\Big|^{\{s,t,u\}\to\{u,s,t\}}_{\{\alpha,\beta\}\to\{\frac{\alpha-1}{\alpha},\frac{\beta-1}{\beta}\}}\right) \end{aligned} \tag{3.45}$$

where we have projected the flavor representations into the s-channel and the flavor crossing matrices F_t, F_u were defined in section 3.1.2. Note that since the sum of the exchange contributions satisfies Bose symmetry, (3.44) and (3.45) must also separately hold for the contact contributions.

We now outline our general strategy to compute the Mellin amplitudes (3.38).

1. We fix the contribution of the bosonic components inside each super multiplet exchange by looking at the *maximally R-symmetry violating* (MRV) limit to be defined in the next subsection. In this limit, superconformal symmetry dictates the amplitude to exhibit a *zero* in a Mandelstam-Mellin variable, which allows us to determine the contribution of each component up to an overall factor.

2. We use R-symmetry to go away from the MRV limit and restore the general R-symmetry polarization dependence in the multiplet exchange amplitudes.

3. We write down an ansatz in terms of all possible multiplet exchange amplitudes with unfixed coefficients, as well as contact terms with arbitrary flavor structures. The contact terms should only arise from quartic vertices with zero derivatives in order for the result to be consistent with the flat space limit.

4. We use the superconformal Ward identities to uniquely solve the unknown parameters up to an overall factor. In fact, with a suitable prescription to restore R-symmetry in the second step, we find all contact terms in the ansatz vanish. Furthermore, these unknown overall factors can be reduced to just one which is related to the flavor central charge, by considering different mixed correlators.

The MRV notion was first introduced in [230, 231] for maximally superconformal theories. There it was also shown that the zeros dictated by superconformal symmetry fix the relative coefficients of component fields inside each multiplet. Unlike the situation in [230, 231], however, here we do not know the super primary three-point functions and therefore cannot use them to fix the overall coefficient of each multiplet. Therefore, we have to resort to the superconformal Ward identities in Mellin space, reviewed in section 3.1.3. The method of using Mellin space superconformal Ward identities to fix the exchange and contact contributions was first introduced and used in [267, 268]. In section 3.2.1 we will define and study the MRV amplitudes, and in section 3.2.2 we will show how to obtain multiplet exchange amplitudes with arbitrary R-symmetry polarizations. The last two steps of the above plan will be explained in detail in section 3.2.3.

3.2.1 MRV amplitudes from zeros

Let us begin by listing the fields that can appear in the exchange amplitudes. A quick look at the super descendant content of the short multiplets tells us that there are only three components allowed

inside each multiplet of $\mathcal{O}_p^I$.[9] The exchanged component fields are summarized in table 3.3. The s-

component field	s_p^I	$A_{p,\mu}^I$	r_p^I
Lorentz spin ℓ	0	1	0
conformal dimension Δ	ϵp	$\epsilon p+1$	$\epsilon p+2$
$SU(2)_R$ spin j_R	$\frac{p}{2}$	$\frac{p}{2}-1$	$\frac{p}{2}-2$
$SU(2)_L$ spin j_L	$\frac{p-2}{2}$	$\frac{p-2}{2}$	$\frac{p-2}{2}$

Table 3.3: Component fields of half-BPS multiplets that are exchanged in the OPE of two half-BPS operators.

channel multiplet exchange amplitude therefore can be written as the linear combination of the three fields

$$\mathcal{S}_p(s,t;\alpha)=\lambda_{s_p}\mathcal{Y}_p(\alpha)\mathcal{M}_{\epsilon p,0}(s,t)+\lambda_{A_p}\mathcal{Y}_{p-2}(\alpha)\mathcal{M}_{\epsilon p+1,1}(s,t)+\lambda_{r_p}\mathcal{Y}_{p-4}(\alpha)\mathcal{M}_{\epsilon p+2,0}(s,t)\ . \tag{3.46}$$

Here we have stripped away the color structure c_s. By $\mathcal{M}_{\Delta,\ell}$ we denote the s-channel exchange Mellin amplitudes for bosonic Witten diagrams, λ_{field} are coefficients to be fixed, and $\mathcal{Y}_p$ are the R-symmetry polynomials[10]

$$\mathcal{Y}_p(\alpha)=\frac{\Gamma\left[\frac{2p-\kappa_t-\kappa_u+4}{4}\right]\Gamma\left[\frac{2p+\kappa_t+\kappa_u+4}{4}\right]}{p!}P^{(\frac{\kappa_t}{2},\frac{\kappa_u}{2})}_{\frac{p}{2}-\frac{\kappa_t+\kappa_u}{4}}(2\alpha-1) \tag{3.47}$$

where $P_m^{(a,b)}(x)$ is the Jacobi polynomial and we assumed that we are in case I of (3.12).[11] Moreover, we have temporarily suppressed the $SU(2)_L$ dependence, since the whole multiplet transforms in the same representation as we can see from the above table. This $SU(2)_L$ dependence will only appear as a multiplicative polynomial in β, which we will restore later on.

Let us now define the u-channel MRV limit of Mellin amplitudes as the R-symmetry slice with $\alpha=0$, which corresponds to setting $v_1=v_3$.[12] We expect the MRV amplitudes

$$\mathbf{MRV}^{I_1I_2I_3I_4}(s,t)=\mathcal{M}^{I_1I_2I_3I_4}(s,t;\alpha=0) \tag{3.48}$$

[9] The 3d case may appear to contain more scalar super descendants of dimension $\epsilon p+1$ in the exchange amplitudes. However, these extra fields are not exchanged as we will explain in section 3.4.2.

[10] The normalization is chosen in such a way that in the s-channel OPE limit $\alpha\to\infty$, the leading term $\alpha^{\frac{1}{4}(2p-\kappa_t-\kappa_u)}$ appears with unit coefficient.

[11] Considering case I is sufficient to obtain the general result. The final expressions for the Mellin amplitudes are independent of which of the two cases we consider.

[12] Note that we have assumed the ordering $k_1\le k_2\le k_3\le k_4$. For arbitrary ordering, we should set $v_1=v_3$ if $k_1+k_3\le k_2+k_4$, and $v_2=v_4$ if $k_1+k_3>k_2+k_4$. The generalization of the MRV notion to other channels is obvious.

to have the following two features:

1. there are no poles in u;

2. there is a zero in u at $u = \epsilon \max\{k_1 + k_3, k_2 + k_4\}$.

Note that $v_1 = v_3$ implies that the only R-symmetry representation which can be exchanged in the u-channel has spin-$\frac{k_1+k_3}{2}$. The first property then follows from the fact that the R-symmetry polynomials associated with the u-channel exchanges vanish in the MRV limit, because all the cubic couplings associated with exchange Witten diagrams are non-extremal, *i.e.*, with $p < k_1 + k_3$. Going to the MRV limit suppresses these exchange contributions. The second property is the statement that long operators with twist $\epsilon \max\{k_1 + k_3, k_2 + k_4\}$ should decouple in the MRV limit.[13] The zero in u precisely offsets the double pole at $u = \epsilon \max\{k_1 + k_3, k_2 + k_4\}$ from the Gamma function factor, which would otherwise lead to logarithmic singularities upon evaluating the inverse Mellin transformation. Such logarithmic singularities are associated with anomalous dimensions, which are characteristic of long operators.[14] These conditions are in fact satisfied by each multiplet either in the s-channel or t-channel, and turn out to fix the multiplet exchange amplitudes uniquely up to an overall constant as we will see below.

The presence of zeros in the MRV limit fixes the contact term ambiguity for each bosonic amplitude $\mathcal{M}_{\Delta,\ell}$ in (3.46). This can be seen by examining the u-channel Regge limit which corresponds to $s \to \infty$, keeping u fixed. Since the process involves at most spin-1 fields, the residue at each pole is linear in Mandelstam variables. On the other hand, in the MRV limit we must produce a zero factor $(u - \epsilon \max\{k_1 + k_3, k_2 + k_4\})$, which implies the MRV amplitude can only be of the form

$$\textbf{MRV} \sim (u - \epsilon \max\{k_1 + k_3, k_2 + k_4\}) \left(\sum_i \frac{\mu_{s,i}}{s - \nu_{s,i}} + \sum_j \frac{\mu_{t,j}}{t - \nu_{t,j}} \right) . \tag{3.49}$$

Here $\mu_{s,i}$, $\nu_{s,i}$, *etc.*, are numbers. Note that this behavior of MRV amplitudes forbids us from adding

[13] In a long multiplet of which the super primary has $SU(2)_R$ spin j, the highest $SU(2)_R$ spin of the super descendants is $j + 2$. For the whole long multiplet (exchanged in the u-channel) to fit inside the correlator, we therefore must have $j + 2 \leq \frac{1}{2} \min\{k_1 + k_3, k_2 + k_4\}$. Meanwhile, for the multiplet to be visible in the u-channel MRV limit, the super primary must satisfy $j + 2 = \frac{1}{2} \min\{k_1 + k_3, k_2 + k_4\}$, and the visible component is the super descendant with $SU(2)_R$ spin $j + 2$. For 4d $\mathcal{N} = 2$, 5d F_4, and 6d $(1, 0)$, this super descendant is obtained from the super primary by acting with Q^4 (see, *e.g.*, [35]). The super primary with twist τ decouples in the MRV limit, while a super descendant with twist $\tau + 2$ remains. For 3d $\mathcal{N} = 3$, the spin $j + 2$ operator first appears in Q^2 acting on the super primary, and the minimal visible twist is $\tau + 1$.

[14] Strictly speaking, this argument applies when the Gamma function poles overlap, which is not always guaranteed when ϵ is an half integer. However, for these values of ϵ we can just assume $k_i \in 2\mathbb{Z}$ so that the poles overlap and we work out the coefficients. The coefficients are essentially the coefficients of bosonic conformal blocks in the superconformal block, and do not depend on the parity of k_i. Therefore the solution to these coefficients obtained with this assumption is in fact general.

additional contact terms – the only allowed zero-derivative contact terms correspond to constants, but these would be incompatible with the condition of having a zero in u. Therefore we can see that the s-channel part of the MRV amplitude behaves as s^{-1} in the u-channel Regge limit. Note that this Regge behavior is not satisfied by a generic exchange Mellin amplitude of a spin-ℓ field, which goes like $s^{\ell-1}$. However, we also note that a spin-ℓ exchange Witten diagram can absorb, via redefining cubic vertices, contact terms with up to $2(\ell-1)$ derivatives. These contact terms correspond to degree $\ell-1$ polynomials in Mellin space, and can be precisely used to improve the Regge behavior to s^{-1}. In practice, this improvement means we eliminate t in favor of u and m from the numerators in (3.28) and discard all regular terms

$$\mathcal{P}^{(s)}_{\Delta,\ell}(s,t)=\sum_{m=0}^{\infty}\frac{\mathcal{Q}_{m,\ell}(\sum_i\Delta_i-u-(\Delta-\ell+2m),u)}{s-\Delta+\ell-2m}\,. \tag{3.50}$$

These exchange Witten diagrams with improved u-channel Regge behavior were called the Polyakov-Regge blocks in [277, 278].

Using the Polyakov-Regge blocks in (3.46), we can now impose the u-channel zeros at $\alpha=0$. This condition must hold for each pole, and therefore imposes strong conditions. We find that λ_{A_p} and λ_{r_p} can be uniquely solved in terms of λ_{s_p}

$$\begin{aligned}\lambda_{A_p} &= -\frac{\mathcal{Y}_p(0)}{\mathcal{Y}_{p-2}(0)}\frac{\epsilon(k_1-k_2+p)(k_3-k_4+p)}{4p(p\epsilon+1)}\lambda_{s_p}\,, \\ \lambda_{r_p} &= \frac{\mathcal{Y}_p(0)}{\mathcal{Y}_{p-4}(0)}\frac{\epsilon^2(k_1-k_2+p-2)(k_1-k_2+p)(k_3-k_4+p-2)(k_3-k_4+p)}{16(p-2)(p-1)(p\epsilon+1)(p\epsilon-\epsilon+1)}\lambda_{s_p}\,.\end{aligned} \tag{3.51}$$

The MRV limit of $\mathcal{S}_p$ then reduces to

$$\mathcal{S}_p^{\text{MRV}}=-\lambda_{s_p}\mathcal{Y}_p(0)\sum_{m=0}^{\infty}\frac{2(p-1)p(k_2\epsilon+k_4\epsilon-u)f_{m,0}|_{\Delta_E=\epsilon p}}{(k_1-k_2-p)(-k_3+k_4+p)(m+p\epsilon-\epsilon)(s-\epsilon p-2m)}\,. \tag{3.52}$$

where $f_{m,0}|_{\Delta_E=\epsilon p}$ is defined in appendix A of [2].

With the solution (3.51) at hand, let us show that our ansatz (3.46) obeys the s-channel Bose symmetry up to contact terms. These contact terms are important, and will be discussed in the next subsection. Bose symmetry dictates that the contribution to a four-point function from each exchanged field in the s-channel must be separately invariant upon applying $1\leftrightarrow 2$ and $3\leftrightarrow 4$ to the external bosons. To determine the $1\leftrightarrow 2$ (or $3\leftrightarrow 4$) interchage parity of the fields allowed by our selection rules, we note that any $(\frac{p}{2}-n,\frac{p-2}{2})$ representation of $SU(2)_R\times SU(2)_L$ appearing

in the product of $(\frac{k_1}{2}, \frac{k_1-2}{2})$ and $(\frac{k_2}{2}, \frac{k_2-2}{2})$ has parity $(-1)^{n+1}$. Furthermore, under $1 \leftrightarrow 2$ the Lorentz spin of the exchanged field contributes another factor $(-1)^\ell$ in a spin-ℓ exchange Witten diagram (modulo contact terms). Combined with the fact that the adjoint representation of G_F is antisymmetric in the tensor product of two adjoints, this shows that every component in the table above (3.46) has positive total parity. It is therefore necessary for each component field to appear with symmetric three-point function coefficients. This can be easily checked to be the case after using the explicit expressions for $\mathcal{Y}_{p-2r}(0)$, since both lines of (3.51) depend only on $(k_1 - k_2)^2$ and $(k_3 - k_4)^2$.

3.2.2 Full multiplet exchange amplitudes

From the MRV limit, we can restore the full α-dependence by using R-symmetry. To accomplish this, we continue to use the Polyakov-Regge blocks in (3.46) and plug in the solution (3.51). However, we now use the R-symmetry polynomials for general α rather than the special value $\alpha = 0$. It is more convenient to write each R-symmetry polynomial (3.47) as an expansion in powers of $(1 - \alpha)$

$$\mathcal{Y}_p(\alpha) = \sum_{i=0}^{\frac{2p-\kappa_t-\kappa_u}{4}} \frac{\Gamma\left[\frac{2p+\kappa_t-\kappa_u+4}{4}\right]\Gamma\left[\frac{4i-2p+\kappa_t+\kappa_u}{4}\right]\Gamma\left[\frac{4i+2p+\kappa_t+\kappa_u+4}{4}\right]}{i!\,p!\,\Gamma\left[\frac{2i+2+\kappa_t}{2}\right]\Gamma\left[\frac{-2p+\kappa_t+\kappa_u}{4}\right]}(1-\alpha)^i \ . \tag{3.53}$$

We find that at each simple pole $s = \epsilon p + 2m$ there is a linear factor of the Mandelstam variable u of the form

$$u + \rho(i, m) \ , \tag{3.54}$$

which breaks the s-channel Bose symmetry since t, the counterpart of u, is missing. To restore the Bose symmetry, we replace m in this factor by its solution from

$$s + t + u = \epsilon(k_1 + k_2 + k_3 + k_4) \ , \quad s = \epsilon p + 2m \ . \tag{3.55}$$

This prescription amounts to choosing a particular contact term in (3.46). The exchange amplitude now becomes

$$\mathcal{S}_p(s, t; \alpha) = \sum_{m=0}^{\infty} \sum_{i=0}^{\frac{2p-\kappa_t-\kappa_u}{4}} \frac{\mathcal{R}^i_{p,m}(t, u)}{s - \epsilon p - 2m}(1-\alpha)^i \ , \tag{3.56}$$

$$\mathcal{R}^i_{p,m}(t, u) = \lambda_{s_p} K^i_p(t, u) B^i_{p,m} E^i_p \tag{3.57}$$

where

$$K_p^i(t,u) = -2i(2i+\kappa_t)u^+ + 2i(\kappa_u-2)t^+ - \frac{1}{4}(t^- - 2i\epsilon)(2p-\kappa_t-\kappa_u)(2p+\kappa_t+\kappa_u-4)\ , \tag{3.58}$$

$$B^i_{p,m} = \frac{(-1)^{1+p-\frac{\kappa_t+\kappa_u}{2}+i}p(p-1)\Gamma[\epsilon(p-1)]\Gamma[\epsilon p]}{p!\,m!\,(m+\epsilon(p-1))!\Gamma[\frac{\epsilon}{2}(k_1+k_2-p)-m]\Gamma[\frac{\epsilon}{2}(k_3+k_4-p)-m]}\ , \tag{3.59}$$

$$\begin{aligned} E^i_p =& \frac{\epsilon\,\Gamma\left[\frac{2p-\kappa_t-\kappa_u+4}{4}\right]\Gamma\left[\frac{2p+\kappa_t-\kappa_u}{4}\right]}{4\,i!\,\Gamma\left[\frac{2i+2+\kappa_t}{2}\right]\Gamma\left[\frac{-4i+2p-\kappa_t-\kappa_u+4}{4}\right]\Gamma\left[\frac{2p\epsilon-(\kappa_t+\kappa_u)\epsilon+4}{4}\right]} \\ &\times \frac{\Gamma\left[\frac{4i+2p+\kappa_t+\kappa_u-4}{4}\right]}{\Gamma\left[\epsilon\,\frac{2p-\kappa_t+\kappa_u}{4}\right]\Gamma\left[\epsilon\,\frac{2p+\kappa_t+\kappa_u}{4}\right]\Gamma\left[\epsilon\,\frac{2p+\kappa_t-\kappa_u}{4}\right]}\ , \end{aligned} \tag{3.60}$$

and

$$u^\pm = u \pm \frac{\epsilon}{2}\kappa_u - \frac{\epsilon}{2}\Sigma\ , \qquad t^\pm = t \pm \frac{\epsilon}{2}\kappa_t - \frac{\epsilon}{2}\Sigma\ . \tag{3.61}$$

The same prescription for restoring Bose symmetry was used in [230, 231] for the maximally superconformal cases, and was shown to have the additional benefit that the full amplitudes can be written only in terms of these exchange amplitudes with no extra contact terms. We will see the same happens for the case with half maximal superconformal symmetry.

3.2.3 Bootstrapping full correlators

We are now ready to assemble the pieces together and write down an ansatz for the full correlator. The ansatz has the general form of (3.38). The s-channel exchange contribution reads

$$\mathcal{M}_s^{I_1I_2I_3I_4} = \mathsf{c}_s \sum_{p\in\mathcal{I}_s} \mathcal{S}_p(s,t;\alpha)\mathcal{Y}_{p-2}(\beta)\ . \tag{3.62}$$

Here c_s is the flavor structure associated with the s-channel exchange defined in (3.39), and $\mathcal{S}_p(s,t;\alpha)$ is the exchange amplitude of the multiplet p given in (3.56). The function $\mathcal{Y}_{p-2}(\beta)$ is the $SU(2)_L$ flavor symmetry polynomial given by (3.47)[15], associated with the spin-$\frac{p-2}{2}$ $SU(2)_L$ representation of multiplet p. Note that each multiplet amplitude $\mathcal{S}_p(s,t;\alpha)$ contains an unknown coefficient λ_{s_p}, see (3.57), which will also be denoted as $\lambda_p^{(k_1k_2|k_3k_4)}$ when we want to emphasize the channel and the external weights. However, we should also note that this coefficient has the interpretation of the

[15] In (3.47) the polynomial was defined for the $SU(2)_R$ symmetry. However, kinematically there is no difference between $SU(2)_R$ and $SU(2)_L$, and therefore they have the same symmetry polynomials.

product of three-point functions of super primary operators

$$\lambda_p^{(k_1k_2|k_3k_4)} = C_{k_1,k_2,p}C_{k_3,k_4,p} \ , \tag{3.63}$$

with our normalizations used for Witten diagrams and $SU(2)_R$, $SU(2)_L$ symmetry polynomials. The summation of p is over a *finite* set $\mathcal{I}_s$

$$\mathcal{I}_s = \{p \mid p - \max\{|k_1 - k_2|, |k_3 - k_4|\} = 2 \, , \, 4 \, , \, \dots \, , \, 2\mathcal{E} - 2\} \ . \tag{3.64}$$

The finite summation range is determined by two constraints: the selection rules of $SU(2)_L$ flavor symmetry and $SU(2)_R$ R-symmetry, and the requirement that extremal couplings must vanish.[16] The t- and u-channels are similar, and can be obtained from the s-channel by Bose symmetry (3.43).

For the contact part $\mathcal{M}_{\text{con}}^{I_1I_2I_3I_4}$, we will allow all possible flavor and R-symmetry structures but only terms that are independent of the Mandelstam variables

$$\mathcal{M}_{\text{con}}^{I_1I_2I_3I_4} = \sum_{\mathbf{R}_a \in \mathbf{adj}_{G_F} \times \mathbf{adj}_{G_F}} \mathrm{P}_a^{I_1I_2|I_3I_4} \sum_{i=0}^{\mathcal{E}} \sum_{j=0}^{\mathcal{E}-2} \delta_{a;i,j} \alpha^i \beta^j \ . \tag{3.65}$$

Here $\delta_{a;i,j}$ are arbitrary coefficients to be fixed. The independence of the Mandelstam variables corresponds to zero-derivative contact interactions. This is required because the Mellin amplitude in the $s, t, u \to \infty$ limit should coincide with the flat space amplitude [73] which has a constant growth behavior. We will make further comments on the flat space behavior of the Mellin amplitudes in section 3.5.

We now impose the superconformal Ward identities on the ansatz $\mathcal{M}^{I_1I_2I_3I_4}$ in Mellin space, as reviewed in section 3.1.3. We find that all contact interaction parameters $\delta_{a;i,j}$ vanish, and all exchange parameters $\lambda_p^{(k_ik_j|k_mk_n)}$ are fixed up to an overall rescaling factor. Furthermore, the ratios of $\lambda_p^{(k_ik_j|k_mk_n)}$ are *independent* of the flavor group. Let us first make a few comments on these general results. The explicit solutions for different theories will be given in the next two sections.

- The independence of the solution on the flavor groups is easy to understand. With all $\delta_{a;i,j}$ vanishing, the four-point amplitudes have only three flavor structures c_s, c_t, c_u. As $\mathsf{c}_{s,t,u}$

[16] This is because the contact Witten diagrams associated with extremal couplings C_{k_1,k_2,k_1+k_2} are divergent. In order for the effective action to remain finite, these couplings must be absent. Note that there is no contradiction with extremal three-point functions being "nonvanishing" on the CFT side, because in CFTs operators in a different basis are usually considered. The natural single-particle operators in the AdS side are linear combinations of the "single-trace" operators and "multi-trace" operators in the CFT side [319, 320]. The mixing is such that the extremal correlators are zero. See, *e.g.*, section 2.1 of [321] and [322, 323] for detailed discussions.

satisfy the Jacobi identity

$$\mathsf{c}_s + \mathsf{c}_t + \mathsf{c}_u = 0 \,, \tag{3.66}$$

only two structures are independent, and can be chosen as c_s and c_t. Therefore, the Mellin superconformal Ward identities are implemented on the combinations

$$M_s^{(k_1,k_2,k_3,k_4)} - M_u^{(k_1,k_2,k_3,k_4)} \,, \quad M_t^{(k_1,k_2,k_3,k_4)} - M_u^{(k_1,k_2,k_3,k_4)} \,, \tag{3.67}$$

which are agnostic about the flavor group G_F.

- The vanishing of the additional contact terms in the ansatz may seem to be merely a computational outcome of applying the superconformal conformal Ward identities, and is far from obvious. However, in section 3.5 we will explain that the absence of these terms is expected by examining the flat space limit of the Mellin amplitudes. Moreover, their absence is also essential for the correlators to exhibit the Parisi-Sourlas supersymmetry as we will discuss in section 3.6.1. A complementary check that there are no contact term solutions to the superconformal Ward identities with less than four derivatives can be found in appendix D of [2].

- Finally, the unfixed overall factors of different correlators are actually not independent. This follows from their interpretation in terms of OPE coefficients (3.63). In fact by considering mixed correlators of the type $\langle kkqq\rangle$, we can extract all three-point function coefficients C_{k_1,k_2,k_3} up to a common factor. This factor is fixed by $C_{2,2,2}$ which is related to the flavor central charge, as the $k = 2$ multiplet contains the flavor current. This allow us to *fully* fix the answer.

3.3 Correlators in theories with eight supercharges

In this section we implement the strategy laid out in section 3.2 to compute all tree-level four-point super gluon correlators for the theories listed in table 3.1. We find that all correlators can be written as the sum of multiplet exchange amplitudes (3.62), with no extra contact terms. Moreover, thanks to (3.63) we can fix all four-point correlators up to an overall coefficient which is determined by the flavor central charge. This computation can be done by examining $\langle kkqq\rangle$ correlators (and their permutations), from which we can also extract all three-point functions for super primary operators.[17]

[17] A famous earlier use of this strategy appeared in the derivation of three-point functions for Virasoro minimal models

Let us expand on this point. We first look at the $\langle 22qq\rangle$ correlators and consider the following ratios

$$\frac{\lambda_2^{(22|qq)}}{\lambda_q^{(2q|2q)}} = \frac{C_{2,2,2}C_{2,q,q}}{C_{2,q,q}C_{2,q,q}} \,, \tag{3.68}$$

which are fully determined since the overall coefficient drops out. This ratio arises from the $p = 2$ multiplet exchange in the s-channel and $p = q$ multiplet exchange in the t-channel, and gives the three-point functions $C_{2,q,q}$ up to the overall factor $C_{2,2,2}$. We then consider the analogous ratio in $\langle kkqq\rangle$

$$\frac{\lambda_2^{(kk|qq)}}{\lambda_r^{(kq|kq)}} = \frac{C_{k,k,2}C_{2,q,q}}{C_{k,q,r}C_{k,q,r}} \,, \tag{3.69}$$

from which we can get $C_{k,q,r}$, again up to the overall factor $C_{2,2,2}$. We can then plug these $C_{k,q,r}$ in (3.62) and verify the superconformal Ward identities for general correlators $\langle pqrs\rangle$. This serves as a consistency check of our method and also confirms the absence of contact terms in the general case as well. Finally, the unfixed coefficient $C_{2,2,2}$, or equivalently $\lambda_2^{(22|22)} = (C_{2,2,2})^2$, is related to the flavor central charge $C_{\mathcal{J}}$ that appears in the two-point function

$$\langle \mathcal{J}_\mu^I(x)\mathcal{J}_\nu^J(0)\rangle = \frac{C_{\mathcal{J}}}{V_{\hat{S}^{d-1}}^2}\frac{\delta^{IJ}\mathcal{I}_{\mu\nu}(x)}{x^{2(d-1)}} \,. \tag{3.70}$$

Here $V_{\hat{S}^{d-1}} = 2\pi^{\frac{d}{2}}/\Gamma[\frac{d}{2}]$ is the volume of the unit $(d-1)$-sphere, and $\mathcal{I}_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x^\mu x^\nu}{x^2}$ is a conformal structure. The central charge $C_{\mathcal{J}}$ was shown in [326] to enter $\lambda_2^{(22|22)}$ as

$$\lambda_2^{(22|22)} = \frac{2(2\epsilon+1)}{\epsilon}\frac{1}{C_{\mathcal{J}}} \,. \tag{3.71}$$

To reiterate, the general super gluon four-point amplitudes in any spacetime dimensions can be assembled as

$$\mathcal{M} = \mathsf{c}_s M_s + \mathsf{c}_t M_t + \mathsf{c}_u M_u \tag{3.72}$$

where $M_{s,t,u}$ are related by the crossing relations (3.43), and in each channel $\mathsf{c}_s M_s$ is given by the finite sum over multiplets (3.62). The λ coefficients for each exchanged multiplet (3.57) are related to the above three-point function coefficients C_{k_1,k_2,k_3} via (3.63). We now give the solution for each theory by writing down the three-point functions.

[324, 325].

3.3.1 4d $\mathcal{N}=2$: D3-branes near F-theory singularities

We start with the 4d $\mathcal{N}=2$ SCFTs arising from D3-branes probing F-theory singularities. We impose the superconformal Ward identities on the ansatz, and find that additional contact terms are absent. As outlined in the above strategy, we focus on the $\langle kkqq\rangle$ correlators. The solution is simply given by

$$\lambda_p^{(kk|qq)} = \lambda_p^{(kq|kq)} = \lambda_2^{(22|22)} \,. \tag{3.73}$$

Hence, from (3.63) the OPE coefficients are simply given by

$$C_{k_1,k_2,k_3} = C_{2,2,2} \,. \tag{3.74}$$

The coefficient $\lambda_2^{(22|22)}$ is given by (3.71), and the central charges were computed in [327]

$$C_{\mathcal{J}} = \frac{12}{2-\nu} N \tag{3.75}$$

where ν characterizes the 7-brane singularity type and was defined in (3.2). By contrast, the stress tensor central charge, which controls the coupling of super gluons to bulk super gravitons, grows as $C_{\mathcal{T}} \sim N^2$. Tree-level super gluon four-point functions exchanging super gravitons are proportional to $C_{\mathcal{T}}^{-1}$, and are therefore subleading to the correlators considered in this chapter where only super gluons are exchanged.

To demonstrate our general result, we give the explicit amplitudes for the $\langle 22kk\rangle$ correlators. The result is

$$\mathcal{M} = \mathsf{c}_s\, M_s + \mathsf{c}_t\, M_t + \mathsf{c}_u\, M_u \,, \tag{3.76}$$

where

$$\begin{aligned} M_s &= \frac{-2\alpha(k+2)+k+\alpha t+\alpha u-u+2}{(k-2)!\,(s-2)} \left(C_{2,2,2}\right)^2 , \\ M_t &= \frac{(1-\alpha)(-2\alpha k-k+\alpha s+u-2)}{(k-2)!\,(t-k)} \left(C_{2,2,2}\right)^2 , \\ M_u &= \frac{-\alpha(2\alpha k-3k-\alpha s+s+t-2)}{(k-2)!\,(u-k)} \left(C_{2,2,2}\right)^2 . \end{aligned} \tag{3.77}$$

3.3.2 5d F_4: Seiberg exceptional theories

We then move onto the Seiberg exceptional theories in 5d. Imposing the Ward identities on $\langle kkqq\rangle$ correlators, we find that there are no further contact terms. The exchange coefficients are given by

$$\begin{aligned}\lambda_p^{(kk|qq)} =& \frac{2^{2(k+q+p)}3^{\frac{1}{2}(7-3k-3q-3p)}\,\Gamma[k]\Gamma[q]\Gamma[p]\,\Gamma[3p/4]^4}{\pi\,\Gamma\left[-\frac{1}{3}+k\right]\Gamma\left[\frac{1}{3}+k\right]\,\Gamma\left[-\frac{1}{3}+q\right]\Gamma\left[\frac{1}{3}+q\right]\,\Gamma\left[-\frac{1}{3}+p\right]\Gamma\left[\frac{1}{3}+p\right]\,\Gamma[p/2]^4}\\ &\frac{\Gamma\left[\frac{1}{12}(-2+6k+3p)\right]\Gamma\left[\frac{1}{12}(2+6k+3p)\right]\Gamma\left[\frac{1}{12}(-2+6q+3p)\right]\Gamma\left[\frac{1}{12}(2+6q+3p)\right]}{\Gamma\left[\frac{1}{4}(2k+p)\right]\Gamma\left[\frac{1}{4}(2q+p)\right]}\\ &\frac{\Gamma\left[\frac{3}{4}(2k-p)\right]\Gamma\left[\frac{3}{4}(2q-p)\right]}{\Gamma\left[k-\frac{p}{2}\right]\Gamma\left[q-\frac{p}{2}\right]}\,\lambda_2^{(kk|qq)}\,,\end{aligned} \tag{3.78}$$

$$\begin{aligned}\lambda_p^{(kq|kq)} =& \frac{2^{2(k+q+p)}3^{\frac{1}{2}(7-3k-3q-3p)}\,\Gamma[k]\Gamma[q]\Gamma[p]}{\pi\,\Gamma\left[-\frac{1}{3}+k\right]\Gamma\left[\frac{1}{3}+k\right]\,\Gamma\left[-\frac{1}{3}+q\right]\Gamma\left[\frac{1}{3}+q\right]\,\Gamma\left[-\frac{1}{3}+p\right]\Gamma\left[\frac{1}{3}+p\right]}\\ &\frac{\Gamma\left[\frac{3}{4}(k+p-q)\right]^2\Gamma\left[\frac{3}{4}(k-p+q)\right]^2\Gamma\left[\frac{3}{4}(-k+p+q)\right]^2}{\Gamma\left[\frac{1}{2}(k+p-q)\right]^2\Gamma\left[\frac{1}{2}(k-p+q)\right]^2\Gamma\left[\frac{1}{2}(-k+p+q)\right]^2}\\ &\frac{\Gamma\left[\frac{1}{12}(-2+3k+3q+3p)\right]^2\Gamma\left[\frac{1}{12}(2+3k+3q+3p)\right]^2}{\Gamma\left[\frac{1}{4}(k+q+p)\right]^2}\,\lambda_2^{(kk|qq)}\,.\end{aligned} \tag{3.79}$$

Using the relation (3.63) between $\lambda_p^{(k_1k_2|k_3k_4)}$ and the super primary three-point functions, the above results allow us to extract the values of the OPE coefficients

$$C_{k_1,k_2,k_3} = \frac{2^{2\Xi}\,3^{\frac{7-6\Xi}{4}}\,\Gamma\left[-\frac{1}{6}+\frac{\Xi}{2}\right]\Gamma\left[\frac{1}{6}+\frac{\Xi}{2}\right]}{\sqrt{\pi}\,\Gamma\left[\frac{\Xi}{2}\right]}\prod_{i=1}^{3}\sqrt{\frac{\Gamma[k_i]}{\Gamma\left[-\frac{1}{3}+k_i\right]\Gamma\left[\frac{1}{3}+k_i\right]}}\,\frac{\Gamma\left[\frac{3\alpha_i}{2}\right]}{\Gamma[\alpha_i]}\,C_{2,2,2}\,. \tag{3.80}$$

Here we have introduced

$$\alpha_1 = \frac{1}{2}(k_2+k_3-k_1)\,,\quad \alpha_2 = \frac{1}{2}(k_3+k_1-k_2)\,,\quad \alpha_3 = \frac{1}{2}(k_1+k_2-k_3)\,, \tag{3.81}$$

and $\Xi = \alpha_1 + \alpha_2 + \alpha_3$. The overall factor $C_{2,2,2}$ is given by (3.71), and the central charges for the flavor group E_{N_f+1} were computed in [328]. We have

$$C_{\mathcal{J}}^{E_{N_f+1}} = \frac{256\sqrt{2}}{3\pi\sqrt{8-N_f}} N^{\frac{3}{2}} + \mathcal{O}(N^{\frac{1}{2}}) \,. \tag{3.82}$$

For comparison, $C_{\mathcal{T}} \sim N^{\frac{5}{2}}$ and super graviton exchanges are again suppressed at large N.

Let us also give the $\langle 22kk \rangle$ correlator as an example. The poles in the Mellin amplitudes can be resummed into hypergeometric functions, and the result reads

$$\mathcal{M} = \mathsf{c}_s \, M_s + \mathsf{c}_t \, M_t + \mathsf{c}_u \, M_u \,, \tag{3.83}$$

where

$$\begin{aligned}
M_s =& \frac{64(2\alpha(-3k+t+u-6)+3k-2u+6)\,{}_3F_2\left(-\frac{1}{2}, \frac{5-3k}{2}, \frac{3-s}{2}; \frac{5}{2}, \frac{5-s}{2}; 1\right)}{3\pi^{5/2}(s-3)\Gamma\left[\frac{3}{2}(k-1)\right]} \left(C_{2,2,2}\right)^2 \,, \\
M_t =& \frac{\Gamma\left[\frac{3k}{2}\right]}{\pi^2\Gamma\left[\frac{1}{2}(3k-1)\right]^2} \frac{24(1-\alpha)(k-1)((6\alpha+3)k-2(\alpha s+u-3))}{3k-2t} \times \\
& {}_3F_2\left(-\frac{1}{2}, -\frac{1}{2}, \frac{3k-2t}{4}; \frac{3k-1}{2}, \frac{3k-2t+4}{4}; 1\right) \left(C_{2,2,2}\right)^2 \,, \\
M_u =& \frac{\Gamma\left[\frac{3k}{2}\right]}{\pi^2\Gamma\left[\frac{1}{2}(3k-1)\right]^2} \frac{24\alpha(k-1)((6\alpha-9)k+2(-\alpha s+s+t-3))}{3k-2u} \times \\
& {}_3F_2\left(-\frac{1}{2}, -\frac{1}{2}, \frac{3k-2u}{4}; \frac{3k-1}{2}, \frac{3k-2u+4}{4}; 1\right) \left(C_{2,2,2}\right)^2 \,.
\end{aligned} \tag{3.84}$$

The $k = 2$ case was first computed in [268].

3.3.3 6d $\mathcal{N} = (1,0)$: E-string theory

Finally, we consider the E-string theory in 6d. Imposing the Ward identities for $\langle kkqq \rangle$ correlators, one finds that it is not possible to introduce extra contact terms, and

$$\begin{aligned}
\lambda_p^{(kk|qq)} =& \frac{6\,\Gamma\left[\frac{p+1}{2}\right]^4 \Gamma\left[\frac{1+2k-p}{2}\right] \Gamma\left[\frac{1+2q-p}{2}\right] \Gamma\left[\frac{-1+2k+p}{2}\right] \Gamma\left[\frac{-1+2q+p}{2}\right]}{\pi\,\Gamma\left[-\frac{1}{2}+k\right] \Gamma\left[\frac{1}{2}+k\right] \Gamma\left[-\frac{1}{2}+q\right] \Gamma\left[\frac{1}{2}+q\right] \Gamma\left[-\frac{1}{2}+p\right] \Gamma\left[\frac{1}{2}+p\right]} \lambda_2^{(kk|qq)} \,, \\
\lambda_p^{(kq|kq)} =& \frac{6\,\Gamma\left[\frac{1-k+q+p}{2}\right]^2 \Gamma\left[\frac{1+k-q+p}{2}\right]^2 \Gamma\left[\frac{1+k+q-p}{2}\right]^2 \Gamma\left[\frac{-1+k+p+q}{2}\right]^2}{\pi\,\Gamma\left[-\frac{1}{2}+k\right] \Gamma\left[\frac{1}{2}+k\right] \Gamma\left[-\frac{1}{2}+q\right] \Gamma\left[\frac{1}{2}+q\right] \Gamma\left[-\frac{1}{2}+p\right] \Gamma\left[\frac{1}{2}+p\right]} \lambda_2^{(kk|qq)} \,.
\end{aligned} \tag{3.85}$$

This allows us to extract the values of the OPE coefficients

$$C_{k_1,k_2,k_3} = \frac{\sqrt{6}\,\Gamma\left[-\frac{1}{2}+\Xi\right]}{\sqrt{\pi}} \prod_{i=1}^{3} \frac{\Gamma\left[\frac{1}{2}+\alpha_i\right]}{\sqrt{\Gamma\left[-\frac{1}{2}+k_i\right]\Gamma\left[\frac{1}{2}+k_i\right]}} C_{2,2,2}\,. \tag{3.86}$$

Finally, the E_8 flavor group central charge of the 6d E-string theory is given by [326]

$$C_{\mathcal{J}} = 60N^2 + 90N\,, \tag{3.87}$$

which determines $C_{2,2,2}$ via (3.71). On the other hand, note that the stress tensor central charge scales as $C_{\mathcal{T}} \sim N^3$ at large N. Hence super graviton exchanges in the four-point functions are again suppressed.

As an explicit example of our results, we give here the Mellin amplitudes for the $\langle 22kk\rangle$ correlators in E-string theory

$$\mathcal{M} = \mathsf{c}_s\, M_s + \mathsf{c}_t M_t + \mathsf{c}_u\, M_u\,, \tag{3.88}$$

where

$$\begin{aligned}
M_s &= -\frac{2(k(s-4)-3)(2(2\alpha-1)k-\alpha(t+u-8)+u-4)}{(2k-3)!\,(s-6)\,(s-4)}\,(C_{2,2,2})^2\,,\\
M_t &= \frac{2(1-\alpha)(k(-2k+t-2)+1)(-2(2\alpha+1)k+\alpha s+u-4)}{(2k-3)!\,(t-2k)\,(t-2k-2)}\,(C_{2,2,2})^2\,,\\
M_u &= -\frac{2\alpha(k(-2k+u-2)+1)(2(2\alpha-3)k-\alpha s+s+t-4)}{(2k-3)!\,(u-2k)\,(u-2k-2)}\,(C_{2,2,2})^2\,.
\end{aligned} \tag{3.89}$$

The $k=2$ case reproduces the result found in [268].

3.4 Correlators from flavor branes

3.4.1 4d SYM with flavors

In our computation we have shown that given the spectrum as an input, all correlators are fixed by superconformal symmetry up to a common overall coefficient $C_{2,2,2}$. The model of $\mathrm{AdS}_5 \times S^5$ with flavor D7-branes wrapping $S^3 \subset S^5$ has the same vector multiplet spectrum and superconformal symmetry as the 4d $\mathcal{N}=2$ SCFTs arising from D3-brane probing F-theory singularities. Therefore the four-point functions of super gluons (or mesons from the dual field theory perspective) are

identical, when expressed in terms of $C_{2,2,2}$, to those given in section 3.3.1. The strength of the interactions among mesons was analyzed in [313] where the expected behavior at large N was found to be

$$C_{2,2,2} \sim \frac{1}{\sqrt{N}}. \tag{3.90}$$

In particular, this interaction becomes small for large N but at the same time is dominant with respect to the exchange of gravitons.

3.4.2 3d ABJM with flavors

We now study the $\text{AdS}_4 \times \mathbb{CP}^3$ model with flavor D6-branes wrapping $\mathbb{RP}^3 \subset \mathbb{CP}^3$, which preserves 3d $\mathcal{N} = 3$ superconformal symmetry. We first clarify the question about what component fields are exchanged in a multiplet, which was alluded to in footnote 9. The Kaluza-Klein spectrum from the D6-branes [308] turns out to be the same as the spectrum of a 7d $\mathcal{N} = 2$ vector multiplet on $\text{AdS}_4 \times S^3$ [329]. The latter shows up when considering M2-branes near orbifold singularities preserving 3d $\mathcal{N} = 4$ superconformal symmetry [330–332], and describes the degrees of freedom localized on the singular locus. These Kaluza-Klein modes are $\frac{1}{3}$-BPS in terms of the 3d $\mathcal{N} = 3$ superconformal algebra, while they are $\frac{1}{2}$-BPS with respect to 3d $\mathcal{N} = 4$. Let us first consider the $\mathcal{N} = 4$ case. The multiplet spectrum is summarized in table 3.4 (see [329]), where we have kept only the bosonic fields, and $SU(2)_a$, $SU(2)_b$ are the two $SU(2)$ R-symmetry groups of 3d $\mathcal{N} = 4$. Notice that the extra

component field	s_p^I	$A_{p,\mu}^I$	r_p^I	t_p^I
Lorentz spin ℓ	0	1	0	0
conformal dimension Δ	$\frac{p}{2}$	$\frac{p}{2}+1$	$\frac{p}{2}+2$	$\frac{p}{2}+1$
$SU(2)_L$ spin j_L	$\frac{p}{2}-1$	$\frac{p}{2}-1$	$\frac{p}{2}-1$	$\frac{p}{2}-1$
$SU(2)_a$ spin j_a	$\frac{p}{2}$	$\frac{p}{2}-1$	$\frac{p}{2}-2$	$\frac{p}{2}-1$
$SU(2)_b$ spin j_b	0	0	0	1

Table 3.4: Bosonic conformal primaries in $3d\ \mathcal{N} = 4$ half-BPS multiplets

scalars t_p^I are charged under $SU(2)_b$ and therefore cannot be exchanged in four-point functions of s_p^I. For $\mathcal{N} = 3$, the spectrum is the same as above but $SU(2)_{a,b}$ combine diagonally to give the $SU(2)_R$ R-symmetry group of 3d $\mathcal{N} = 3$. One might wonder if the t_p^I scalars are now allowed to appear in the exchange. However we will argue that this does not happen. This is because the multiplet exchange amplitude has the same poles and residues as its associated short multiplet superconformal block in Mellin space. The superconformal blocks of this multiplet with external s_p^I satisfy the *same*

superconformal Ward identities in $\mathcal{N}=3$ and $\mathcal{N}=4$. Since the $\mathcal{N}=4$ case admits a unique solution [326, 333, 334], it would require the conformal blocks associated with t_p^I scalars alone to form another solution to the Ward identities, if they were to appear in the exchange. Such solutions do not exist (see appendix B of [2] for details and further comments). To summarize, the 3d $\mathcal{N}=3$ case falls within the situation considered in section 3.2.1, and we can use the techniques developed there to compute four-point functions by setting $\epsilon=\frac{1}{2}$. Moreover, since the exchanged fields are the same as 3d $\mathcal{N}=4$, the correlators we compute can also be interpreted as super gluon four-point functions of 7d $\mathcal{N}=2$ SYM on $\text{AdS}_4\times S^3$ by identifying the $SU(2)_R$ spinors as the $SU(2)_a$ spinors.

Following the same procedure and requiring that the Ward identities are satisfied, we find that there are no additional contact terms. The solution for the $\langle kkqq\rangle$ correlators is given by

$$\begin{aligned}\lambda_p^{(kk|qq)}&=\frac{2^{4-k-p-q}\,\pi\,\Gamma[k]\,\Gamma[q]\,\Gamma[p]}{\Gamma\left[\frac{2+2k-p}{4}\right]\Gamma\left[\frac{2+2q-p}{4}\right]\Gamma\left[\frac{p+2}{4}\right]^4\Gamma\left[\frac{p+2q}{4}\right]\Gamma\left[\frac{p+2k}{4}\right]}\,\lambda_2^{(kk|qq)}\,,\\ \lambda_p^{(kq|kq)}&=\frac{2^{4-k-p-q}\,\pi\,\Gamma[k]\,\Gamma[q]\,\Gamma[p]}{\Gamma\left[\frac{2+k+p-q}{4}\right]^2\Gamma\left[\frac{2+k+q-p}{4}\right]^2\Gamma\left[\frac{2+p+q-k}{4}\right]^2\Gamma\left[\frac{k+p+q}{4}\right]^2}\,\lambda_2^{(kk|qq)}\,.\end{aligned}\tag{3.91}$$

It is straightforward to extract the values of the OPE coefficients which read

$$C_{k_1,k_2,k_3}=\frac{2^{2-\Xi}\sqrt{\pi}}{\Gamma\left[\frac{\Xi}{2}\right]}\prod_{i=1}^{3}\frac{\sqrt{\Gamma[k_i]}}{\Gamma\left[\frac{1+\alpha_i}{2}\right]}\,C_{2,2,2},\tag{3.92}$$

where $C_{2,2,2}$ can be determined via (3.71) once the flavor central charge is inputted.

Let us also write down here the $\langle 22kk\rangle$ correlators as a special case of our general result, which read

$$\mathcal{M}=\mathsf{c}_s\,M_s+\mathsf{c}_t\,M_t+\mathsf{c}_u\,M_u\,,\tag{3.93}$$

where

$$
\begin{aligned}
M_s &= \frac{(2\alpha(-k+t+u-2)+k-2u+2)\,{}_3F_2\left(\frac{1}{2},\frac{3-k}{2},\frac{1-s}{2};\frac{3}{2},\frac{3-s}{2};1\right)}{\pi^{5/2}(s-1)\Gamma\left[\frac{k-1}{2}\right]}\,(C_{2,2,2})^2\,,\\
M_t &= \frac{(1-\alpha)\Gamma\left[\frac{k}{2}\right](2\alpha(s-k)-k+2u-2)\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{k-2t}{4};\frac{k+1}{2},\frac{k-2t+4}{4};1\right)}{\pi^2(2t-k)\Gamma\left[\frac{k-1}{2}\right]\Gamma\left[\frac{k+1}{2}\right]}\,(C_{2,2,2})^2\,,\\
M_u &= \frac{-\alpha\,\Gamma\left[\frac{k}{2}\right](2\alpha k-3k-2(\alpha-1)s+2t-2)\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{k-2u}{4};\frac{k+1}{2},\frac{k-2u+4}{4};1\right)}{\pi^2(2u-k)\Gamma\left[\frac{k-1}{2}\right]\Gamma\left[\frac{k+1}{2}\right]}\,(C_{2,2,2})^2\,.
\end{aligned}
\tag{3.94}
$$

3.5 Flat space limit

In this section we examine the flat space limit of Mellin amplitudes [73], where $s,\,t,\,u\to\infty$ with $s+t+u=0$. From our results in section 3.3 and 3.4 we find the following universal behavior

$$
\mathcal{M}^{I_1I_2I_3I_4}\Big|_{s,t\to\infty}=\mathcal{N}_{\{k_i\}}P_{\{k_i\}}(\sigma,\tau)\mathcal{A}^{I_1I_2I_3I_4}(s,t;\alpha)\,. \tag{3.95}
$$

where $\mathcal{N}_{\{k_i\}}$ is an overall normalization and

$$
P_{\{k_i\}}(\sigma,\tau)=\sum_{\substack{i+j+k=\mathcal{E}-2\\0\le i,j,k\le\mathcal{E}-2}}\frac{\sigma^i\tau^j}{i!\,j!\,k!\,(i+\frac{\kappa_u}{2})!\,(j+\frac{\kappa_t}{2})!\,(k+\frac{\kappa_s}{2})!}\,, \tag{3.96}
$$

where in $P_{\{k_i\}}$ we have introduced

$$
\sigma=\alpha\beta\,,\quad \tau=(1-\alpha)(1-\beta)\,, \tag{3.97}
$$

from the $SU(2)_L$ and $SU(2)_R$ cross ratios α and β. Note that in $P_{\{k_i\}}$, α and β are now on the same footing since σ and τ are invariant under $\alpha\leftrightarrow\beta$. These new cross ratios can be understood as the $SU(2)_R\times SU(2)_L$ spinors $v_i,\,\bar v_i$ regrouping into $SO(4)$ null vectors t_i which satisfy $t_i\cdot t_i=0$. The factor $P_{\{k_i\}}$ is then the Wick contraction of k_i-2 null vectors t_i with $i=1,\dots,4$

$$
P_{\{k_i\}}\propto\prod_{i<j}t_{ij}^{-\gamma^0_{ij}}(t_{12}t_{34})^{2-\mathcal{E}}\,\texttt{Wick}[\underbrace{t_1\dots t_1}_{k_1-2}\dots\underbrace{t_4\dots t_4}_{k_4-2}] \tag{3.98}
$$

where $t_{ij}=t_i\cdot t_j=(v_i\cdot v_j)(\bar v_i\cdot\bar v_j)$ and γ^0_{ij} were defined in (3.13). The cross ratios σ and τ can be

expressed in terms of t_{ij} as

$$\sigma = \frac{t_{13}t_{24}}{t_{12}t_{34}} , \quad \tau = \frac{t_{14}t_{23}}{t_{12}t_{34}} . \tag{3.99}$$

The fact that theories in different dimensions have the same flat space limit is a nontrivial consistency check of our results, as tree-level amplitudes in flat space are blind to the spacetime dimension. Let us now focus on the color-dependent factor $\mathcal{A}^{I_1I_2I_3I_4}$, which reads

$$\mathcal{A}^{I_1I_2I_3I_4} = \mathsf{c}_s \frac{u(1-\alpha)-t\alpha}{s} + \mathsf{c}_t \frac{(\alpha-1)(u+s\alpha)}{t} + \mathsf{c}_u \frac{\alpha(t+s(1-\alpha))}{u} . \tag{3.100}$$

Let us now compare this high energy limit of Mellin amplitudes with the expression of tree-level gluon amplitudes in flat space. In the usual setting of flat space scattering, we consider gluons in the adjoint representation of the $SU(N_c)$ gauge group, such that $A_\mu = A^I_\mu T^I$ with $I = 1, \cdots, N_c^2 - 1$. Tree-level amplitudes are usually organized in terms of color-ordered amplitudes, see for instance [81, 291]

$$A_4^{\text{flat}} = g^2 \sum_{\mathcal{P}(2,3,4)} \text{Tr}\left(T^{I_1}T^{I_2}T^{I_3}T^{I_4}\right) A(1,2,3,4) \tag{3.101}$$

where the partial amplitudes $A(1,2,3,4)$, *etc*, are the color-ordered amplitudes. Color-ordered amplitudes satisfy various relations, including the cyclic and reflection properties

$$A(1,2,3,4) = A(2,3,4,1), \quad A(1,2,3,4) = A(4,3,2,1) . \tag{3.102}$$

For our purposes, however, it becomes more transparent to express the amplitude in the color basis introduced earlier

$$\mathsf{c}_s = f^{I_1I_2J}f^{JI_3I_4}, \quad \mathsf{c}_t = f^{I_1I_4J}f^{JI_2I_3}, \quad \mathsf{c}_u = f^{I_1I_3J}f^{JI_4I_2} , \tag{3.103}$$

which can be written in terms of traces as

$$\mathsf{c}_s = \text{Tr}\left(T^{I_1}T^{I_2}T^{I_3}T^{I_4}\right) - \text{Tr}\left(T^{I_1}T^{I_2}T^{I_4}T^{I_3}\right) - \text{Tr}\left(T^{I_1}T^{I_3}T^{I_4}T^{I_2}\right) + \text{Tr}\left(T^{I_1}T^{I_4}T^{I_3}T^{I_2}\right) , \tag{3.104}$$

and similarly for c_t and c_u. Note that c_s is given by the only combination with the right symmetry properties under the exchange of indices: antisymmetric under the exchange of $I_1 \leftrightarrow I_2$ or $I_3 \leftrightarrow I_4$.

In this basis A_4^{flat} takes the following form (see, *e.g.*, [81])[18]

$$A_4^{\text{flat}} = \frac{\mathsf{c}_s N_s}{s} + \frac{\mathsf{c}_t N_t}{t} + \frac{\mathsf{c}_u N_u}{u} \tag{3.105}$$

which coincides exactly with $\mathcal{A}^{I_1 I_2 I_3 I_4}$ in (3.100) upon the identification

$$N_s = u(1-\alpha) - t\alpha, \quad N_t = (\alpha-1)(u+s\alpha), \quad N_u = \alpha(t+s(1-\alpha)). \tag{3.106}$$

Changing back to the trace basis we can obtain the respective expressions for the color-ordered amplitudes

$$A(1,2,3,4) = -\frac{(s+t-s\alpha)^2}{st}, \tag{3.107}$$

$$A(1,3,4,2) = -\frac{(s+t-s\alpha)^2}{su}, \tag{3.108}$$

$$A(1,4,2,3) = -\frac{(s+t-s\alpha)^2}{tu}. \tag{3.109}$$

These partial amplitudes satisfy both the photon-decoupling identity

$$A(1,2,3,4) + A(1,3,4,2) + A(1,4,2,3) = 0, \tag{3.110}$$

where we have used $s+t+u=0$, as well as the Bern-Carrasco-Johansson (BCJ) relations [291]

$$\begin{gathered} t\,A(1,2,3,4) = u\,A(1,3,4,2)\,, \quad s\,A(1,2,3,4) = u\,A(1,4,2,3)\,, \\ t\,A(1,4,2,3) = s\,A(1,3,4,2). \end{gathered} \tag{3.111}$$

Let us now return to (3.106) and ask whether these expressions can be precisely reproduced from the flat space gluon amplitudes with appropriately restricted kinematic configurations. The functions $N_{s,t,u}$ for four-gluon scattering with momenta $\mathtt{k}_i$ and polarizations ϵ_i, $i=1,2,3,4$, in arbitrary flat space dimensions can be found in [335]. The results are given by[19]

$$\begin{aligned} N_s = \quad & 2\big((\epsilon_1\cdot\epsilon_2)(\mathtt{k}_1-\mathtt{k}_2)^\mu + 2(\epsilon_1\cdot\mathtt{k}_2)\epsilon_2^\mu - 2(\epsilon_2\cdot\mathtt{k}_1)\epsilon_1^\mu\big) \\ & \times\eta_{\mu\nu}\big((\epsilon_3\cdot\epsilon_4)(\mathtt{k}_4-\mathtt{k}_3)^\nu - 2(\epsilon_3\cdot\mathtt{k}_4)\epsilon_4^\nu + 2(\epsilon_4\cdot\mathtt{k}_3)\epsilon_3^\nu\big) \end{aligned}$$

[18] Note that our definition for the Mandelstam variables are $s=-(\mathtt{k}_1+\mathtt{k}_2)^2$, $t=-(\mathtt{k}_1+\mathtt{k}_4)^2$, $u=-(\mathtt{k}_1+\mathtt{k}_3)^2$, where the role of t and u is reversed compared to the convention used in some flat space amplitude literature.

[19] The relevant expressions of [335] are (4.12) to (4.14) where one needs to drop the x^-, y^- dependence and also identify $D_{\mu\nu}$ with the flat space metric.

$$-2\big((\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)-(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)\big)(\mathtt{k}_1\cdot\mathtt{k}_2+\mathtt{k}_3\cdot\mathtt{k}_4)\ , \tag{3.112}$$

with analogous expressions for N_t, N_u fixed by crossing. We first consider the flat space limit of the amplitude for the simplest operator $\mathcal{O}_2^I = \mathcal{O}^{I;\alpha_1,\alpha_2}(x)\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}v^{\beta_1}v^{\beta_2}$. In order to make the comparison, we consider (3.112) in a specially chosen kinematics where the momenta $\mathtt{k}_i$ are restricted to $\mathbb{R}^{d-1,1}\sim\partial\text{AdS}_{d+1}$ while the polarization vectors ϵ_i are restricted to a perpendicular $\mathbb{R}^4$, such that $\mathtt{k}_i\cdot\epsilon_j=0$ for all i,j. This leads to

$$\begin{aligned}N_s =&2(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)((\mathtt{k}_1-\mathtt{k}_2)\cdot(\mathtt{k}_4-\mathtt{k}_3))\\ &-2\left((\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot\epsilon_4)-(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot\epsilon_3)\right)(\mathtt{k}_1\cdot\mathtt{k}_2+\mathtt{k}_3\cdot\mathtt{k}_4)\,.\end{aligned} \tag{3.113}$$

For the polarization vectors we can assemble the doublet of spinors v^{β_1}, v^{β_2} into a null vector ϵ, which leads to $\epsilon_i\cdot\epsilon_j=(v_i\cdot v_j)^2$. This is exactly how we introduced the null vectors earlier in this section, except that we now need to identify $\bar{v}$ with v since $\mathcal{O}_2^I$ contains two v. Extracting the overall factor $(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)$, we obtain

$$N_s=-2(t-u)+2(\alpha^2-(1-\alpha)^2)s=4(\alpha s+u) \tag{3.114}$$

which, upon using $s+t+u=0$, agrees precisely with our expression (3.106) for N_s up to an unimportant overall factor. The flat space limit of general four-point amplitudes can also be matched. For operators with $k_i>2$, one should modify the flat space in- and out-states by allowing them to have a nontrivial wavefunction Ψ_i on the angular directions S^m of the transverse $\mathbb{R}^{m+1}$ ($m=3$ in the above case). This wavefunction is just a scalar spherical harmonic of rank k_i-2, which can be conveniently expressed as[20]

$$\Psi_i(\mathrm{T},t_i)=(\mathrm{T}\cdot t_i)^{k_i-2} \tag{3.115}$$

where t_i is a $(m+1)$-dimensional null vector with $t_i\cdot t_i=0$ and $\mathrm{T}\in\mathbb{R}^{m+1}$, $\mathrm{T}\cdot\mathrm{T}=1$ parametrizing S^m.[21] The flat space gluon amplitudes are now dressed with a factor given by the overlap of the four wavefunctions

$$\int_{S^m}[d\mathrm{T}]\Psi_1(\mathrm{T},t_1)\Psi_2(\mathrm{T},t_2)\Psi_3(\mathrm{T},t_3)\Psi_4(\mathrm{T},t_4) \tag{3.116}$$

[20] To see this, we note that the scalar spherical harmonics can be written as $Y_k^{\mathcal{I}}(\mathrm{T})=C^{\mathcal{I}}_{a_1\ldots a_k}\mathrm{T}^{a_1}\ldots\mathrm{T}^{a_k}$ where $C^{\mathcal{I}}_{a_1\ldots a_k}$ is a symmetric traceless tensor of $SO(m+1)$. To introduce the null vectors, we can contract the spherical harmonics with $C^{\mathcal{I}}_{b_1\ldots b_k}t^{b_1}\ldots t^{b_k}$. The contraction $C^{\mathcal{I}}_{a_1\ldots a_k}C^{\mathcal{I}}_{b_1\ldots b_k}$ is a product of delta functions of the a_i and b_i indices, and gives rise to (3.115).

[21] We need to emphasize here that the sphere polarization t_i and the spacetime polarization ϵ_i are *independent*. From the AdS calculation they turn out to be correlated. However, generally it does not need to be the case.

which is easy to evaluate since the integral over T gives just the Wick contraction [236]. Using this result and the fact that the vectors t_i are null, it is clear that the wavefunction overlap gives the factor $P_{\{k_i\}}(\sigma,\tau)$ in (3.98). The above discussion can be straightforwardly generalized to the case of gravitons (see also [76, 77]). In particular, our derivation explains why the same factor $P_{\{k_i\}}$ also shows up in the flat space limit of Mellin amplitudes in maximally superconformal theories [230, 231]

$$\mathcal{M}^{\text{SUGRA}}\big|_{s,t\to\infty} \propto \frac{(s+t-s\alpha)^2(s+t-s\bar{\alpha})^2}{s\,t\,u} P_{\{k_i\}}(\sigma,\tau)\ . \tag{3.117}$$

Here α, $\bar{\alpha}$ are the two cross ratios of the R-symmetry group $SO(m+1)$ with $m=4,5,7$, and we should identify $\beta=\bar{\alpha}$.

Going back to (3.100), let us also note that using $\mathsf{c}_s+\mathsf{c}_t+\mathsf{c}_u=0$ and $s+t+u=0$ we can write the amplitude as

$$\mathcal{A}^{I_1I_2I_3I_4} = \frac{(t\mathsf{c}_s - s\mathsf{c}_t)(s+t-s\alpha)^2}{s\,t\,u}. \tag{3.118}$$

The factor $(s+t-s\alpha)^2$ exactly agrees with the holomorphic part of the corresponding prefactor present in super graviton amplitudes in flat space. In particular, it implies that the above amplitude is a solution to the flat space limit of the Ward identities. In fact, substituting in (3.105)

$$\mathsf{c}_s \to N_s\ , \quad \mathsf{c}_t \to N_t\ , \quad \mathsf{c}_u \to N_u\ , \tag{3.119}$$

in which $\alpha\to\bar{\alpha}$ is already replaced, the flat space limit of the super gluon amplitudes just become the super graviton amplitudes (3.117). Not surprisingly, this is just the double copy relation in flat space [292].

Let us conclude this section with a remark about how to use the flat space limit to constrain the correlators. Recall that in the last step of our bootstrap strategy we used the superconformal Ward identities to fix the OPE coefficients C_{k_1,k_2,k_3} in terms of $C_{2,2,2}$. Then in this section we checked that the resulting expression for the amplitude has the correct flat space limit. However, we can also turn the logic around. We can leave the OPE coefficients unfixed and require the precise answer (3.95) is reproduced in the flat space limit. It turns out this requirement is powerful enough to fix all coefficients C_{k_1,k_2,k_3} up to an overall factor which we can choose to be $C_{2,2,2}$. Note that in matching (3.95) we do not need to include any contact terms, which are constant in the flat space limit. However, because of this constant growth behavior, contact terms would contradict the limit (3.95) and are not allowed. This provides an intuitive explanation for the vanishing of contact terms

observed earlier as a consequence of the Ward identities. In this new line of attack, however, that the superconformal Ward identities hold now serves as a nontrivial consistency check.

3.6 Hidden structures in holographic correlators

3.6.1 Parisi-Sourlas supersymmetry

The results in section 3.3 and 3.4 exhibit an interesting emergent dimensional Parisi-Sourlas-like reduction structure. A similar structure is present in correlators in maximally superconformal theories [309], as reviewed in chapter 4. However, unlike the maximally superconformal situation where the dimension of the AdS space reduces by four, here we find the AdS space dimension reduces only by two.

To see this structure, we first note that the factor $K_p^i(t,u)$ in the multiplet exchange amplitude residue (3.57) is a polynomial of t, u and i. This allows us to move the factor outside the inverse Mellin integrals and interpret it as a differential operator

$$U\partial_U \leftrightarrow (\tfrac{s}{2}-a_s)\times\ , \quad V\partial_V \leftrightarrow (\tfrac{t}{2}-a_t)\times\ , \quad (\alpha-1)\partial_\alpha \leftrightarrow i\times\ . \tag{3.120}$$

We can now focus on the other two factors $B^i_{p,m}$, E^i_p. The remaining dependence on the Mandelstam variable is contained in a sum over simple poles with residues independent of the Mandelstam variables

$$\sum_{m=0}^{\infty} \frac{1}{m!\,(m+\epsilon(p-1))!\Gamma[\frac{\epsilon(k_1+k_2-p)-2m}{2}]\Gamma[\frac{\epsilon(k_3+k_4-p)-2m}{2}](s-\epsilon p-2m)}\ , \tag{3.121}$$

which turns out to be proportional to the linear combination of two AdS_{d+1} scalar exchange Witten diagrams with conformal dimensions ϵp and $\epsilon p+2$

$$\mathcal{M}^{(s),d}_{\epsilon p,0} - \frac{(k_1-k_2+p)(k_2-k_1+p)(k_3-k_4+p)(k_4-k_3+p)\epsilon^2}{16p(p-1)(1+p\epsilon)(1+(p-1)\epsilon)}\mathcal{M}^{(s),d}_{\epsilon p+2,0}\ . \tag{3.122}$$

Here we have added the superscript d to stress that the Witten diagrams are defined in AdS_{d+1}. On the other hand, we can check by explicit calculations that the above expression is identical to just a

single scalar exchange Witten diagram, however in a lower dimensional space AdS_{d-1}

$$\mathcal{M}^{(s),d-2}_{\epsilon p,0} \,. \tag{3.123}$$

This scalar exchange Witten diagram has exchange dimension ϵp and the same external dimensions ϵk_i. The equivalence of (3.122) and (3.123) can be explained in terms of a holographically realized Parisi-Sourlas supersymmetry [241].[22] However, it is not clear why Parisi-Sourlas supersymmetry should be present in superconformal correlators. Let us turn to the R-symmetry dependence in $B^i_{p,m}$ and E^i_p

$$\sum_i \frac{(-1)^i \Gamma\left[\frac{4i+2p+\kappa_u+\kappa_t-4}{4}\right]}{i!\,\Gamma\left[i+\frac{\kappa_t}{2}+1\right]\Gamma\left[\frac{-4i+2p-\kappa_u-\kappa_t+4}{4}\right]}(1-\alpha)^i \,. \tag{3.124}$$

We can define more generally a polynomial

$$\begin{aligned}\mathbb{Y}_{p,\nu}(\alpha) &= \sum_i \frac{(-1)^i \Gamma\left[\frac{4i+2p+\kappa_u+\kappa_t+4-4\nu}{4}\right]\Gamma[\frac{2p-\kappa_t-\kappa_u+4}{4}]\Gamma[\frac{2p+\kappa_t-\kappa_u+4}{4}]}{i!\,\Gamma\left[\frac{2i+\kappa_t+2}{2}\right]\Gamma\left[\frac{-4i+2p-\kappa_u-\kappa_t+4}{4}\right]\Gamma[1+p-\nu]}(1-\alpha)^i \\ &= \frac{\Gamma[\frac{2p+\kappa_u+\kappa_t+4-4\nu}{4}]\Gamma[\frac{2p+\kappa_t-\kappa_u+4}{4}]}{(\frac{\kappa_t}{2})!\Gamma[1+p-\nu]}{}_2F_1\left(\tfrac{-2p+\kappa_u+\kappa_t}{4},\tfrac{4+2p+\kappa_u+\kappa_t-4\nu}{4},1+\tfrac{\kappa_t}{2};1-\alpha\right) ,\end{aligned} \tag{3.125}$$

which is proportional to (3.124) when $\nu = 2$, and gives the $SO(3)$ R-symmetry polynomials (3.47) when $\nu = 0$

$$\mathcal{Y}_p = \mathbb{Y}_{p,0} \,. \tag{3.126}$$

These functions $\mathbb{Y}_{p,\nu}$ satisfy the differential equation

$$y\,(1-y)\,\frac{d^2}{dy^2}\mathbb{Y}_{p,\nu} + \left(\tfrac{2+\kappa_u-2\nu}{2} - y\left(\tfrac{4+\kappa_t+\kappa_u-2\nu}{2}\right)\right)\frac{d}{dy}\mathbb{Y}_{p,\nu} - \left(\tfrac{\kappa_t+\kappa_u-2p}{4}\right)\left(\tfrac{\kappa_t+\kappa_u+2p+4-4\nu}{4}\right)\mathbb{Y}_{p,\nu} = 0 \,.$$

In [309], the modified R-symmetry dependence was shown to be related to reducing the dimension of the internal sphere also by four. However, it is less clear here how to geometrically interpret $\mathbb{Y}_{p,\nu}$, as the factorization into $SU(2)_R \times SU(2)_L$ is a feature of S^3 only and the $SU(2)_L$ factor is left intact throughout.

Using these observations we can now rewrite the correlators in a more compact form. Restoring

[22] More generally, [241] showed that an AdS_{d-1} exchange Witten diagram with dimension Δ and spin ℓ can be expressed as the linear combination of five exchange Witten diagrams in AdS_{d+1} with shifted dimensions and spins. These Witten diagram relations generalize nontrivially similar dimensional reduction formulae for conformal blocks discovered in [281]. The latter was shown in [281] to be a kinematic consequence of an underlying Parisi-Sourlas supersymmetry [280]. Similar relations also exist for boundary CFTs and CFTs on real projective space [241, 336].

the kinematic factor extracted in (3.10), the s-channel multiplet exchange amplitudes become

$$\begin{aligned}\mathbf{S}_p^{(s)} = \quad & \prod_{i<j}\left(\frac{(v_i\cdot v_j)(\bar{v}_i\cdot\bar{v}_j)}{x_{ij}^{2\epsilon}}\right)^{\gamma_{ij}^0}\frac{(v_1\cdot v_2)^{\mathcal{E}}(v_3\cdot v_4)^{\mathcal{E}}}{(x_{12}^{2\epsilon}x_{34}^{2\epsilon})^{\mathcal{E}}}\left((\bar{v}_1\cdot\bar{v}_2)(\bar{v}_3\cdot\bar{v}_4)\right)^{\mathcal{E}-2} \\ & \times f_{\{k_i,p\}}\,\mathbb{K}_p\circ\int_{-i\infty}^{i\infty}\frac{dsdt}{(4\pi i)^2}U^{\frac{s}{2}-a_s}V^{\frac{t}{2}-a_t}\left(\mathcal{M}_{\epsilon p,0}^{(s),d-2}\mathbb{Y}_{p,2}(\alpha)\mathbb{Y}_{p-2,0}(\beta)\right)\Gamma_{\{k_i\}}^{(\epsilon)}\,.\end{aligned} \tag{3.127}$$

Here $\mathbb{K}_p$ is a differential operator obtained with the replacement (3.120) in the factor $K_p^i(t,u)$, and $f_{\{k_i\},p}$ is an overall factor[23]

$$f_{\{k_i\},p}=\frac{2}{(p-1)\epsilon(2p-\kappa_t-\kappa_u)(2p+\kappa_t-\kappa_u)}\,. \tag{3.128}$$

Expressions with similar structures for the t- and u-channel multiplet exchange amplitudes can be obtained from $\mathbf{S}_p^{(s)}$ by crossing via permuting the external labels. The full correlator can then be written as

$$\begin{aligned}G_{k_1k_2k_3k_4} =& \mathsf{c}_s\sum_{p_s}C_{k_1,k_2,p_s}C_{k_3,k_4,p_s}\mathbf{S}_{p_s}^{(s)}+\mathsf{c}_t\sum_{p_t}C_{k_1,k_4,p_t}C_{k_2,k_3,p_t}\mathbf{S}_{p_t}^{(t)} \\ & +\mathsf{c}_u\sum_{p_u}C_{k_1,k_3,p_u}C_{k_2,k_4,p_u}\mathbf{S}_{p_u}^{(u)}\end{aligned} \tag{3.129}$$

This form of our results suggests that the super gluon correlators in these theories can be obtained from a lower-dimensional scalar "seed theory" in AdS_{d-1}, by dressing the "seed correlators" with differential operators. Finally, we remind the reader that the above Parisi-Sourlas dimensional reduction structure relies crucially on the fact that the four-point functions can be written in terms of only exchange contributions. The existence of additional contact terms would spoil this structure.

3.6.2 Hidden conformal symmetry

In this subsection, we point out another interesting feature, namely, the correlators in 4d $\mathcal{N}=2$ theories exhibit an eight dimensional hidden conformal symmetry. To see it, we start from the solution to superconformal Ward identities (3.19) in 4d, which reads [337]

$$G_{k_1k_2k_3k_4}=G_{0,k_1k_2k_3k_4}+R\,H_{k_1k_2k_3k_4}\,, \tag{3.130}$$

[23] Comparing the original expression (3.56) with the expression here we would find an extra factor $(-1)^{p-\frac{\kappa_t+\kappa_u}{2}}$. However, by selection rules this factor is just 1.

with

$$R = (v_1 \cdot v_2)^2 (v_3 \cdot v_4)^2 x_{13}^2 x_{24}^2 (1 - z\alpha)(1 - \bar{z}\alpha) \ . \tag{3.131}$$

The function $G_{0,k_1k_2k_3k_4}$ is the protected part of the correlator, and gives the meromorphic correlator upon performing the chiral algebra twisting $\alpha = 1/z$ or $\alpha = 1/\bar{z}$ [337, 338] (see appendix C of [2] or chapter 4 for more discussions). Here and below we have left the color indices implicit. We will refer to $H_{k_1k_2k_3k_4}$ as the reduced correlator, which has shifted conformal dimensions $k_i + 1$, and $SU(2)_R \times SU(2)_L$ spins $(\frac{k_i-2}{2}, \frac{k_i-2}{2})$. Recall that the full Mellin amplitude is defined as

$$\begin{aligned} G_{k_1k_2k_3k_4} = & \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} (x_{12}^2)^{\frac{s-k_1-k_2}{2}} (x_{34}^2)^{\frac{s-k_3-k_4}{2}} (x_{14}^2)^{\frac{t-k_1-k_4}{2}} (x_{23}^2)^{\frac{t-k_2-k_3}{2}} \\ & \times (x_{13}^2)^{\frac{u-k_1-k_3}{2}} (x_{24}^2)^{\frac{u-k_2-k_4}{2}} \mathcal{M}_{k_1k_2k_3k_4} \Gamma^{(1)}_{\{k_i\}} \end{aligned} \tag{3.132}$$

where $s+t+u = \sum_{i=1}^4 k_i$, and $\Gamma^{(1)}_{\{k_i\}}$ was defined in (3.27). We define an analogous reduced Mellin amplitude $\widetilde{\mathcal{M}}_{k_1k_2k_3k_4}$ as

$$\begin{aligned} H_{k_1k_2k_3k_4} = & \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} (x_{12}^2)^{\frac{s-k_1-k_2}{2}} (x_{34}^2)^{\frac{s-k_3-k_4}{2}} (x_{14}^2)^{\frac{t-k_1-k_4}{2}} (x_{23}^2)^{\frac{t-k_2-k_3}{2}} \\ & \times (x_{13}^2)^{\frac{\tilde{u}-k_1-k_3}{2}} (x_{24}^2)^{\frac{\tilde{u}-k_2-k_4}{2}} \widetilde{\mathcal{M}}_{k_1k_2k_3k_4} \widetilde{\Gamma}^{(1)}_{\{k_i\}} \end{aligned} \tag{3.133}$$

where $s+t+\tilde{u} = \sum_{i=1}^4 k_i - 2$, and

$$\widetilde{\Gamma}^{(1)}_{\{k_i\}} = \Gamma[\tfrac{k_1+k_2-s}{2}]\Gamma[\tfrac{k_3+k_4-s}{2}]\Gamma[\tfrac{k_1+k_4-t}{2}]\Gamma[\tfrac{k_2+k_3-t}{2}]\Gamma[\tfrac{k_1+k_3-\tilde{u}}{2}]\Gamma[\tfrac{k_2+k_4-\tilde{u}}{2}] \ . \tag{3.134}$$

The shift in $\tilde{u}$ is important because Bose symmetry in the reduced Mellin amplitude acts by permutation of $s, t, \tilde{u}$. Similar to the $\mathcal{N} = 4$ case [227], the protected piece $G_{0,k_1k_2k_3k_4}$ does not contribute to the Mellin amplitude. So it follows from the solution (3.130) that we have the following relation between $\widetilde{\mathcal{M}}_{k_1k_2k_3k_4}$ and $\mathcal{M}_{k_1k_2k_3k_4}$

$$\mathcal{M}_{k_1k_2k_3k_4} = \widehat{R} \circ \widetilde{\mathcal{M}}_{k_1k_2k_3k_4} \tag{3.135}$$

where $\widehat{R}$ is a difference operator with each $U^m V^n$ in R (after extracting the kinematic factor $(v_1 \cdot v_2)^2 (v_3 \cdot v_4)^2 x_{13}^2 x_{24}^2$) interpreted as a difference operator

$$\widehat{U^m V^n}' \circ \widetilde{\mathcal{M}}(s,t) = \frac{\widetilde{\Gamma}^{(1)}_{\{k_i\}}(s-2m, t-2n)}{\Gamma^{(1)}_{\{k_i\}}(s,t)} \widetilde{\mathcal{M}}(s-2m, t-2n) \ . \tag{3.136}$$

Notice these monomials act differently than in (3.37), and their associated difference operators are distinguished by a prime here. This is because the reduced Mellin amplitudes require a different Gamma function factor in their definition. Using this operator, we find that

$$\widetilde{\mathcal{M}}_{2222} = -4(C_{2,2,2})^2 \left(\frac{\mathsf{c}_s}{(s-2)(\tilde{u}-2)} - \frac{\mathsf{c}_t}{(t-2)(\tilde{u}-2)} \right) . \tag{3.137}$$

Here we have used the identity $\mathsf{c}_s + \mathsf{c}_t + \mathsf{c}_u = 0$ to eliminate c_u in order to avoid ambiguities in writing the answer. But we can also rewrite it in the more symmetric form

$$\begin{aligned} \widetilde{\mathcal{M}}_{2222} = & -\frac{4}{3}(C_{2,2,2})^2 \Big(\frac{\mathsf{c}_s}{(s-2)(\tilde{u}-2)} - \frac{\mathsf{c}_s}{(s-2)(t-2)} + \frac{\mathsf{c}_t}{(t-2)(s-2)} \\ & - \frac{\mathsf{c}_t}{(t-2)(\tilde{u}-2)} + \frac{\mathsf{c}_u}{(\tilde{u}-2)(t-2)} - \frac{\mathsf{c}_u}{(\tilde{u}-2)(s-2)} \Big) . \end{aligned} \tag{3.138}$$

This form of summing over pairs of simultaneous poles turns out to be a generic feature of the reduced Mellin amplitudes.

The reduced correlators have a remarkable feature – they are organized by an eight dimensional hidden conformal symmetry. This hidden symmetry allows us to promote H_{2222} into a generating function by replacing AdS distances with higher dimensional distances, and obtain any $H_{k_1k_2k_3k_4}$ by Taylor expansion. More precisely, we find that the $SU(2)_R \times SU(2)_L$ spinors v_i, $\bar{v}_i$ regroup into $SO(4)$ null vectors t_i as they do in the flat space limit (see section 3.5). The reduced correlator $H_{k_1k_2k_3k_4}$ can be viewed as a correlator where each operator transforms in the rank-(k_i-2) symmetric traceless representation of $SO(4)$. The $SO(4)$ indices of the operators are contracted with the null vectors t_i. The null vectors t_i appear in $H_{k_1k_2k_3k_4}$ only as polynomials of $t_{ij} = t_i \cdot t_j$, and under independent scalings $t_i \to \zeta^i t_i$ the reduced correlators scale as $H_{k_1k_2k_3k_4} \to \prod_{i=1}^4 \zeta_i^{k_i-2} H_{k_1k_2k_3k_4}$. For the lowest Kaluza-Klein level $k_i = 2$, H_{2222} is a singlet under $SO(4)$ and therefore depends on x_{ij}^2 only. We can define a generating function from the $k_i = 2$ reduced correlator as

$$\mathbf{H}(x_i, t_i) = H_{2222}(x_{ij}^2 - t_{ij}) . \tag{3.139}$$

To obtain $H_{k_1k_2k_3k_4}$, we only need to Taylor expand $\mathbf{H}$ in powers of t_{ij}, and collect all the monomials of t_{ij} that can appear in this correlator. There are only finitely many such monomials in each $H_{k_1k_2k_3k_4}$ because of the above scalings. The replacement of x_{ij}^2 by $x_{ij}^2 - t_{ij}$ indicates an eight dimensional conformal symmetry. Since the two factors of $\mathrm{AdS}_5 \times S^3$ have the same radius, this background can be conformally mapped to $\mathbb{R}^{7,1}$ where $x_{ij}^2 - t_{ij}$ is the conformally invariant dis-

tance. Furthermore, we can check the dimension by looking at the flat space limit of the reduced Mellin amplitude

$$\widetilde{\mathcal{M}}_{2222} \to -4(C_{2,2,2})^2 \left(\frac{\mathsf{c}_s}{su} - \frac{\mathsf{c}_t}{tu} \right) . \tag{3.140}$$

This amplitude is annihilated by the flat space special conformal transformation generators

$$K_\mu = \sum_{i=1}^{3} \left(\frac{p_{i\mu}}{2} \frac{\partial}{\partial p_i^\nu} \frac{\partial}{\partial p_{i,\nu}} - p_i^\nu \frac{\partial}{\partial p_i^\nu} \frac{\partial}{\partial p_i^\mu} - \frac{d-2}{2} \frac{\partial}{\partial p_i^\mu} \right) , \tag{3.141}$$

only when the spacetime dimension is $d = 8$. This number of dimensions agrees with the fact that we are essentially studying scattering processes constrained inside the eight dimensional world volume of the 7-branes. Using the generating function, it is easy to find that the reduced Mellin amplitudes for general k_1, k_2, k_3, k_4 are

$$\begin{aligned} \widetilde{\mathcal{M}}_{k_1k_2k_3k_4} = & -4(C_{2,2,2})^2 \prod_{a<b} t_{ab}^{\gamma^0_{ab}} (t_{12}t_{34})^{\mathcal{E}-2} \sum_{\substack{i+j+k=\mathcal{E}-2 \\ 0\le i,j,k\le \mathcal{E}-2}} \frac{\sigma^i \tau^j}{i!\, j!\, k!\, (i+\frac{\kappa_u}{2})!\, (j+\frac{\kappa_t}{2})!\, (k+\frac{\kappa_s}{2})!} \\ & \times \left(\frac{\mathsf{c}_s}{(s - s_M + 2k)(\tilde{u} - u_M + 2i)} - \frac{\mathsf{c}_t}{(t - t_M + 2j)(\tilde{u} - u_M + 2i)} \right) \end{aligned} \tag{3.142}$$

where σ, τ were defined in (3.97), and

$$\begin{aligned} s_M &= \min\{k_1 + k_2, k_3 + k_4\} - 2 , \\ t_M &= \min\{k_1 + k_4, k_2 + k_3\} - 2 , \\ u_M &= \min\{k_1 + k_3, k_2 + k_4\} - 2 . \end{aligned} \tag{3.143}$$

Upon acting on $\widetilde{\mathcal{M}}_{k_1k_2k_3k_4}$ with the difference operator $\widehat{R}$ defined in (3.135), we reproduce the results found in section 3.3.1. From the flat space limit of (3.142) and $\widehat{R}$, it is straightforward to see that the flat space limit of $\mathcal{M}_{k_1k_2k_3k_4}$ is given by (3.95).

The above eight dimensional hidden conformal symmetry parallels the ten and six dimensional hidden conformal structures for IIB supergravity on $\text{AdS}_5 \times S^5$ [288] and $\text{AdS}_3 \times S^3 \times K3$ [289] (see also [290]). In the latter two examples, the solutions to the superconformal Ward identity also take the form (3.130), and it is possible to define reduced correlators $H_{k_1k_2k_3k_4}$ which have similar shifted conformal and R-symmetry weights. The lowest-weight reduced correlator H_{2222} is an R-symmetry singlet, and upon the replacement $x_{ij}^2 \to x_{ij}^2 - t_{ij}$ gives the generating function. The flat space limit of the reduced Mellin amplitudes are also conformally invariant in ten and six dimensions

respectively.

However, an important difference is that the 4d $\mathcal{N}=2$ case arises from a supersymmetric *gauge theory* in AdS, while the other two examples are *supergravity* theories. Previously it was not clear whether the existence of such higher dimensional conformal symmetries relies on special properties of IIB supergravity. The 4d $\mathcal{N}=2$ example we presented here proves that is not the case. At the moment, a precise understanding of these structures is still lacking. However, finding the same phenomenon in drastically different theories provides strong indication that such hidden conformal structures are due to the conformal flatness of the background.

3.6.3 Color-kinematic duality

The holographic correlators in sections 3.3 and 3.4 also enjoy another interesting property that can be more easily seen after using the prescription of section 3.2.2 to eliminate the contact terms. This property mimics the celebrated color-kinematics duality [291] for the lowest Kaluza-Klein modes with $k_i=2$, while higher Kaluza-Klein modes satisfy a modification of it. Note the $k_i=2$ multiplet is where the flavor current belongs, dual to a *massless* gauge field in AdS. These four-point amplitudes have the universal factorized structure in each channel

$$\mathcal{M}_{2222}=\mathtt{c}_s\mathtt{n}_s\left(\frac{\mathtt{const}}{s-2\epsilon}+\dots\right)+\mathtt{c}_t\mathtt{n}_t\left(\frac{\mathtt{const}}{t-2\epsilon}+\dots\right)+\mathtt{c}_u\mathtt{n}_u\left(\frac{\mathtt{const}}{u-2\epsilon}+\dots\right) \tag{3.144}$$

where $\mathtt{c}_{s,t,u}$ are the color structures defined in (3.39). The normalization $\mathtt{const}$ depends on the theory but is the same for all three channels. The $\dots$ denote satellite poles at $2\epsilon+2\mathbb{Z}_+$ which complete the series into the sum of two AdS_{d+1} scalar exchange diagrams (3.122), or equivalently, a single scalar exchange diagram in AdS_{d-1} (3.123). The factors $\mathtt{n}_{s,t,u}$ are simple polynomials given by

$$\mathtt{n}_s=u-4\epsilon+\alpha s\ ,\quad \mathtt{n}_t=-(\alpha-1)((\alpha-1)(u-4\epsilon)+\alpha t)\ ,\quad \mathtt{n}_u=-\alpha(\alpha(s-4\epsilon)+u)\ , \tag{3.145}$$

and are related to each other by crossing symmetry. Note that this factorization into polynomial factors $\mathtt{n}_{s,t,u}$ and polar parts corresponding to an AdS_{d-1} scalar exchange diagram is a highly nontrivial feature of our results. The form of (3.144) is very similar to the formula for four-point gluon scattering amplitudes in flat space. The spacetime polarizations in flat space are replaced by R-symmetry polarizations. In analogy with the $\frac{1}{s}$ scalar propagator in flat space, we have the AdS_{d-1} scalar prop-

agator which contains a series of simple poles ($\frac{\mathtt{const}}{s-2\epsilon}+\ldots$). Thanks to the Jacobi identity, we have

$$\mathtt{c}_s+\mathtt{c}_t+\mathtt{c}_u=0\,, \tag{3.146}$$

On the other hand, it is also straightforward to check that the kinematic factors satisfy

$$\mathtt{n}_s+\mathtt{n}_t+\mathtt{n}_u=0\,. \tag{3.147}$$

This relation gives an AdS version of the color-kinematics duality, which is literally the same as that of the flat space gluons! Finding AdS extensions of the flat space color-kinematics duality was also discussed recently in momentum space [339, 340]. However, our realization of this duality in Mellin space appears to be the simplest and bears most resemblance to the flat space relation. One might be tempted by the flat space analogy and replace $\mathtt{c}_{s,t,u}$ by $\mathtt{n}_{s,t,u}$ to obtain new amplitudes. However, we do not recognize the amplitudes following from this naive prescription. It would be very interesting also to find a way to properly "square" the super gluon amplitudes so that super gravitons amplitudes are obtained.

It turns out that a modified version of the color-kinematics duality also holds for correlators with arbitrary weights. For simplicity, we shall discuss here the case of $\langle kkkk\rangle$ correlators, although analogous results are also found in general correlators. We extend the definition of $\mathtt{n}_{s,t,u}$ to each exchanged super multiplet (labelled by p) as the coefficient of the pole due to the super primary ($m=0$), dividing by the corresponding $SU(2)_L$ polynomial in order to get rid of the dependence on β. Explicitly, for each p we have

$$\mathtt{n}_s=\sum_{i=0}^{\frac{p}{2}}\mathcal{R}^i_{p,m=0}(t,u)\,(1-\alpha)^i\,, \tag{3.148}$$

with $\mathtt{n}_t$ and $\mathtt{n}_u$ related to $\mathtt{n}_s$ by crossing as in (3.43) as follows

$$\mathtt{n}_t=(\alpha-1)^k\left(\mathtt{n}_s\big|^{\{t,u\}\to\{u,s\}}_{\alpha\to\frac{1}{1-\alpha}}\right)\,,\qquad \mathtt{n}_u=(-\alpha)^k\left(\mathtt{n}_s\big|^{\{t,u\}\to\{s,t\}}_{\alpha\to\frac{\alpha-1}{\alpha}}\right)\,. \tag{3.149}$$

In this definition we have exploited the fact that the poles of the multiplet exchange amplitude can be completely factored out as an AdS_{d-1} scalar exchange amplitude with the dimension of the super primary (see section 3.6.1). Therefore, stripping away the $SU(2)_L$ polynomials each multiplet exchange amplitude can be expressed as $\mathtt{n}_{s,t,u}$ multiplying a series of simple poles in s, t or u, just as in

(3.144). The simple poles have constant coefficients, and the coefficient of the first pole is normalized to one. These kinematic factors $\mathtt{n}_{s,t,u}$ can be expressed in terms of $SU(2)_R$ R-symmetry polynomials and read

$$\begin{aligned}\mathtt{n}_s =& \frac{\lambda_{s_p}\,(-1)^p\,\Gamma[\epsilon\,p]}{16\,\epsilon\,(p-3)\,(p-1)^2\,\Gamma\left[\frac{p\epsilon}{2}\right]^4\Gamma\left[\frac{\epsilon}{2}(2k-p)\right]^2} \\ &\Big[(p-3)(p-1)\,(-4\epsilon k-\epsilon p+t+u+2\epsilon)\;\mathcal{Y}_p(\alpha) \\ &+8(p-3)(p-1)(t-u)\,\mathcal{Y}_{p-2}(\alpha)+(p-2)^2(-4\epsilon k+\epsilon p+t+u)\,\mathcal{Y}_{p-4}(\alpha)\Big]\,,\end{aligned} \tag{3.150}$$

where for $\langle kkkk\rangle$ we have

$$\mathcal{Y}_p(\alpha)=\frac{1}{p!}\,\Gamma\left[\tfrac{p+2}{2}\right]\;{}_2F_1\left(\frac{p}{2}+1,-\frac{p}{2};1;1-\alpha\right)\,. \tag{3.151}$$

We find that a generalized version of the relation (3.147) holds, which takes the form

$$\mathtt{p}_s(\alpha)\,\mathtt{n}_s+\mathtt{p}_t(\alpha)\,\mathtt{n}_t+\mathtt{p}_u(\alpha)\,\mathtt{n}_u\;=\;0\,. \tag{3.152}$$

Here $\mathtt{p}_{s,t,u}(\alpha)$ are polynomials in α, related by crossing symmetry.[24] Explicitly, they are given by

$$\begin{aligned}\mathtt{p}_s(\alpha)=&\Big[-2(\alpha-1)(p-2)\,{}_2F_1\left(1-\frac{p}{2},\frac{p}{2};1;\frac{\alpha}{\alpha-1}\right)\,{}_2F_1\left(2-\frac{p}{2},\frac{p}{2};1;\frac{1}{\alpha}\right) \\ &+(2(\alpha-2)(p-2)-4k)\,{}_2F_1\left(1-\frac{p}{2},\frac{p}{2};1;\frac{\alpha}{\alpha-1}\right)\,{}_2F_1\left(1-\frac{p}{2},\frac{p}{2};1;\frac{1}{\alpha}\right) \\ &+2(p-2)\,{}_2F_1\left(2-\frac{p}{2},\frac{p}{2};1;\frac{\alpha}{\alpha-1}\right)\,{}_2F_1\left(1-\frac{p}{2},\frac{p}{2};1;\frac{1}{\alpha}\right)\Big]\;((1-\alpha)\alpha)^{\frac{p-2}{2}}\;,\end{aligned} \tag{3.153}$$

with

$$\mathtt{p}_t(\alpha)=(\alpha-1)^{-1-k+3p/2}\,\mathtt{p}_s\left(\frac{1}{1-\alpha}\right)\,,\qquad \mathtt{p}_u(\alpha)=(-\alpha)^{-1-k+3p/2}\,\mathtt{p}_s\left(\frac{\alpha-1}{\alpha}\right)\,. \tag{3.154}$$

Note that when $p=2$ (corresponding to the exchange of massless particles in AdS), the relation (3.152) reduces to

$$\mathtt{n}_s+(\alpha-1)^{k-2}\,\mathtt{n}_t+(-\alpha)^{k-2}\mathtt{n}_u\;=\;0\,, \tag{3.155}$$

[24] One might wonder how nontrivial such relations are. However, as a linear function of s and t, (3.152) gives three equations for two unknowns which are the ratios $\mathtt{p}_{s,t,u}$, and is in general overly-determined. Therefore, the existence of solutions is not guaranteed.

which is a minimal modification of the color-kinematics duality relation (3.147). A remarkable fact is that the $\mathtt{p}_{s,t,u}(\alpha)$ do not depend on ϵ, hence they are the same in any spacetime dimension.

It is also possible to extend the relation (3.152) to general $\langle k_1\, k_2\, k_3\, k_4\rangle$ correlators. We notice that in each channel there are always an equal number of $\mathcal{E}-1$ multiplets being exchanged. The generalized version of (3.152) holds for any triplet of exchanged multiplets which have the same i-th lowest dimension in each channel, where $\mathtt{n}_{s,t,u}$ are defined similarly as in (3.148). The degrees of polynomials $\mathtt{p}_{s,t,u}(\alpha)$ are determined only by the extremality $\mathcal{E}$ and the label i. For example, for next-to-next-to-extremal correlators which have $\mathcal{E}=2$ and include $\langle 2222\rangle$ as a special example, $\mathtt{p}_{s,t,u}(\alpha)$ are just numbers. However, $\mathtt{p}_{s,t,u}(\alpha)$ do depend on ϵ in the general case, and explicit formulae appear to be more cumbersome. A more direct analogue of the standard color-kinematics duality for massive gluons was found in [341] in the AdS_5 case, where using the reduced Mellin amplitude a natural analogue of the double copy in AdS was also found. However, no such results are available in other dimensions.

Chapter 4

Graviton scattering in AdS

The knowledge of all holographic four-point functions of super gravitons at tree level found in of [230, 231] generates a wealth of new theoretical data. The purpose of this chapter is to capitalize on these data and extract useful physical information, focusing two complementary aspects. The first is to find new hidden structures in the correlators. Identifying these hidden structures not only leads to more compact expressions, but also suggests new symmetry properties of the bulk theory. In particular, we will show that the correlators exhibit an emergent dimensional reduction structure, which is closely related to the Parisi-Sourlas supersymmetry [280, 281], completely analogous to that described in section 3.6.1 for super gluon scattering. The second aspect concerns certain protected subsectors of the superconformal field theories. Their information can be cleanly isolated by focusing on special twisted configurations of the supergravity correlators. We also perform independent boundary calculations, which perfectly match the bulk predictions. This provides non-trivial checks for several conjectures in the protected sector. Below we provide a more detailed summary of our main results, embedded in brief reviews of related backgrounds.

Emergent dimensional reduction: It is well known that scattering amplitudes in flat space often contain surprising structures which reveal unexpected symmetries hidden from the Lagrangian description. Viewed as scattering amplitudes in anti de Sitter space, it is not surprising that holographic correlators can similarly exhibit hidden structures which cannot be seen from the diagrammatic expansion. A beautiful example is the hidden ten dimensional conformal symmetry observed in $AdS_5 \times S^5$ [288], which organizes all tree-level four-point functions into a generating function. The generating function is nothing but the stress tensor four-point function after replacing four dimensional distances with ten dimensional distances. A similar six dimensional version was also discovered for $AdS_3 \times S^3 \times K3$ correlators [289, 290]. However, it is generally believed that the hidden conformal symmetry is non-existent for backgrounds which are not conformally flat (such as $AdS_4 \times S^7$ and $AdS_7 \times S^4$).

In this chapter, we will point out a different hidden structure in tree-level four-point functions

that involves dimensional reduction, and is present in *all* maximally superconformal theories. We recall from [230, 231] that four-point functions in $\text{AdS}_{d+1} \times S^{\mathtt{d}-1}$ are sums of exchange amplitudes over finitely many supergravity multiplets, with no additional contact interactions. We will show that the exchange amplitude of a multiplet p can be simplified as a differential operator acting on only three *scalar* exchange amplitudes. Schematically, we have

$$\text{Diff} \circ (\mathcal{M}^{(d)}_{\epsilon p} + \#_1 \mathcal{M}^{(d)}_{\epsilon p+2} + \#_2 \mathcal{M}^{(d)}_{\epsilon p+4}) \tag{4.1}$$

where $\epsilon = (d-2)/2$, and $\mathcal{M}^{(d)}_{\Delta}$ is the scalar exchange amplitude in AdS_{d+1} with internal dimension Δ. Note that in addition to superconformal primary of the multiplet with $\Delta = \epsilon p$, there are also new scalars with shifted conformal dimensions. The spinning field contributions in the multiplet are generated by the differential operator action. Quite magically, the combination of diagrams in (4.1) turns out to be just *a single* scalar exchange diagram in a *lower* dimensional AdS_{d-3} space

$$\mathcal{M}^{(d)}_{\epsilon p} + \#_1 \mathcal{M}^{(d)}_{\epsilon p+2} + \#_2 \mathcal{M}^{(d)}_{\epsilon p+4} = \mathcal{M}^{(d-4)}_{\epsilon p} \ . \tag{4.2}$$

A similar pattern can be found for the R-symmetry part, which leads to a dimensionally reduced internal space $S^{\mathtt{d}-5}$. These observations allow us to repackage the full multiplet amplitude into the following simple form

$$\text{Diff}' \circ (\mathcal{Y}^{\mathtt{d}-4}_{p,0} \mathcal{M}^{(d-4)}_{\epsilon p}) \tag{4.3}$$

where $\mathcal{Y}^{\mathtt{d}-4}_{p,0}$ is the R-symmetry polynomial of the rank-p symmetric traceless representation of the reduced R-symmetry group $SO(\mathtt{d}-4)$ associated with $S^{\mathtt{d}-5}$. Consequently, the correlators in the original theory are now fully captured by the correlators of a simple scalar theory in the lower dimensional spacetime $\text{AdS}_{d-3} \times S^{\mathtt{d}-5}$!

The appearance of dimensionally reduced spacetimes is quite curious, and we do not have a good understanding of its physical origin. Nevertheless, we find strong evidence indicating that this reduction phenomenon is intimately related to the Parisi-Sourlas dimensional reduction, which relates a d dimensional theory with Parisi-Sourlas supersymmetry and a $d-2$ dimensional non-supersymmetric theory. The connection comes from a dimensional reduction formula for exchange Witten diagrams in AdS_{d+1} and AdS_{d-1}, which was shown in [241] as the consequence of the holographically realized Parisi-Sourlas supersymmetry. It turns out that using the dimensional reduction formula twice gives precisely (4.2).

Twisted correlators and protected sectors: Common to the maximally superconformal theories considered in this chapter is the existence of a protected subsector of operators. These operators are formed by restricting a certain class of operators (half-BPS operators are such examples) to a two dimensional plane or a one dimensional line, and "twisting" the operators by giving them special R-symmetry polarizations specified by the locations of the insertions [284, 286, 287, 338]. Crucially, the subsector involves only members of short multiplets, closes on itself under OPE, and does not depend on marginal deformations (if there are any) [338]. Therefore this construction isolates a fully protected subsector. The protected subsector has the form of a unitary chiral algebra in six dimensions [284], a non-unitary chiral algebra in four dimensions [338] and topological quantum mechanics in three dimensions [286, 287]. The presence of the protected subsector immediately imposes strong constraints on the correlation functions, as they must become meromorphic or topological in the twisted configurations. It was also quickly recognized that non-trivial unitarity bounds are encoded in these rigid structures analytically [338, 342–344]. Along the same lines, one can use chiral algebra or topologically protected data to greatly increase the power of the numerical conformal bootstrap [79, 286, 345–353]. When combined with supersymmetric localization, these techniques can also be used to predict new holographic dualities [354–356].

While the existence of a $6d/2d$ or $4d/2d$ correspondence, in the sense described above, is a proven fact, a precise description of the lower dimensional theory is not always readily available. In well studied cases, *e.g.*, theories of Argyres-Douglas type or class $\mathcal{S}$, much of the recent progress in understanding their chiral algebras has been guided by a set of compelling conjectures [284, 338, 357–359]. The $3d/1d$ correspondence is on a somewhat different footing due to the explicit Lagrangians derived in [360–363].[1] Nevertheless, applying these results to ABJM theory requires the use of conjectural dualities [366, 367] and even then the calculations can be very involved.

In this work, we further strengthen the evidence for these conjectures in the case of holographic CFTs with maximal supersymmetry. We perform independent calculations from both the field theory side and the supergravity side, finding perfect agreement. For $6d$ and 4d, where the twisting leads to meromorphic functions, we compute four-point chiral correlators on the field theory side by improving the holomorphic bootstrap method of [368]. The basic idea is that meromorphic functions are determined by singularities which are dictated by the OPE of the chiral algebra. However, the improved algorithm allows us to write down closed form expressions for all chiral four-point functions, in a form that foreshadows the structures anticipated from the supergravity side. We then

[1] See however also [364, 365] for $4d/2d$ examples.

reproduce these results in Mellin space by taking residues, from the MRV limit of the supergravity amplitudes [230, 231]. For the $3d$ case, the topological nature of the twisted correlators makes the situation inevitably more involved and we refer the reader to the original reference [1] for a study of next-to-next-to-extremal correlators of the form $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_k\mathcal{O}_k\rangle$, where the OPE coefficients of certain operators contributing to the protected sector are computed in a $1/c_{\mathcal{T}}$ expansion. To summarize, the calculations outlined above provide non-trivial checks for the conjectures in the protected sector. Given that many aspects of the protected subsectors appear to be within reach of a formal proof, it might also be helpful to turn the logic around and view our results as a tree-level check of AdS/CFT. In performing these checks, we have developed many technical tools for studying holographic four-point functions. We expect these tools will have broader applicability.

The rest of the chapter is organized as follows. We set the stage in section 4.1 by reviewing some basic superconformal kinematics. In section 4.2, we show that the holographic correlators have an emergent dimensional reduction structure, and explain the connection with the Parisi-Sourlas supersymmetry. Discussions of the protected subsectors begin in section 4.3, where we show the agreement between the field theory and supergravity calculations for $6d$. In section 4.4, we present a parallel check for the $4d$ case.

4.1 Superconformal kinematics

To start, we will use this section to fix our notation and review some basic superconformal kinematics of four-point correlation functions. In particular, we will review a powerful kinematic constraint, namely the superconformal Ward identities in diverse dimensions [316, 337], which served as a precursor to the SCFT/chiral algebra correspondence.

4.1.1 Four-point functions in $d > 2$

We focus on the supergravity backgrounds $\text{AdS}_4 \times S^7$, $\text{AdS}_5 \times S^5$ and $\text{AdS}_7 \times S^4$, which have the maximal amount of superconformal symmetry. In these backgrounds all supergravity single-particle states are dual to components of the half-BPS supermultiplets. The super primaries of these half-BPS multiplets are scalar operators of the form $\mathcal{O}_k^{I_1\ldots I_k}(x)$, which transform in rank-k symmetric traceless representations of the R-symmetry group. The scaling dimensions of the half-BPS operators are fixed by the R-symmetry charges

$$\Delta = \epsilon k, \qquad \epsilon \equiv \frac{d-2}{2} \tag{4.4}$$

where $k \in \{2\,,3\,,\ldots\}$. We will saturate the R-symmetry indices of these operators with polarization vectors t_I that are null in order to respect tracelessness

$$\mathcal{O}_k(x,t) = \mathcal{O}_k^{I_1\ldots I_k}(x)\; t_{I_1}\ldots t_{I_k}, \qquad t\cdot t = 0\;. \tag{4.5}$$

The central objects of this chapter are the four-point correlation functions of these operators

$$\langle \mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)\mathcal{O}_3(x_3,t_3)\mathcal{O}_4(x_4,t_4)\rangle\;. \tag{4.6}$$

The extremality for a correlator is defined as

$$\mathcal{E} = \begin{cases} \frac{k_{\bar{1}}+k_{\bar{2}}+k_{\bar{3}}-k_{\bar{4}}}{2}, & k_{\bar{1}}+k_{\bar{4}} \geq k_{\bar{2}}+k_{\bar{3}} \\ k_{\bar{1}}, & k_{\bar{1}}+k_{\bar{4}} < k_{\bar{2}}+k_{\bar{3}} \end{cases} \tag{4.7}$$

where $k_{\bar{i}}$ is the i-th smallest element of $\{k_1,k_2,k_3,k_4\}$. The standard conformal and R-symmetry cross-ratios are

$$U = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\;\; V = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2},\;\; \sigma = \frac{t_{13}t_{24}}{t_{12}t_{34}},\;\; \tau = \frac{t_{14}t_{23}}{t_{12}t_{34}}\;, \tag{4.8}$$

with $x_{ij} \equiv x_i - x_j$ and $t_{ij} \equiv t_i\cdot t_j$. We will exploit conformal symmetry and R-symmetry to extract a kinematic factor such that the correlator becomes a function of the cross-ratios, and depends on σ and τ as a polynomial of total degree $\mathcal{E}$. If the weights are ordered as $k_1 \leq k_2 \leq k_3 \leq k_4$, a convention that accomplishes this is

$$\begin{aligned}\langle \mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)\mathcal{O}_3(x_3,t_3)\mathcal{O}_4(x_4,t_4)\rangle = &\left(\frac{t_{34}}{x_{34}^{2\epsilon}}\right)^{\frac{1}{2}(k_3+k_4-k_1-k_2)}\left(\frac{t_{24}}{x_{24}^{2\epsilon}}\right)^{\frac{1}{2}(k_2+k_4-k_1-k_3)}\\ &\left(\frac{t_{23}}{x_{23}^{2\epsilon}}\right)^{\frac{1}{2}(k_1+k_2+k_3-k_4)-\mathcal{E}}\left(\frac{t_{14}}{x_{14}^{2\epsilon}}\right)^{k_1-\mathcal{E}}\left(\frac{t_{12}t_{34}}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^{\mathcal{E}}\mathcal{G}_{1234}(U,V;\sigma,\tau)\;.\end{aligned} \tag{4.9}$$

However, we should stress that when working with various correlators, we will specify the ordering and often it will be different from $k_1 \leq k_2 \leq k_3 \leq k_4$. The formulas we introduce in this section and section 4.2, which form the basis for later results, will be valid for arbitrary ordering of k_i as long as the σ and τ dependence remains polynomial with degree $\mathcal{E}$. In practice, we implement this by interchanging the four labels in (4.9) according to the permutation which takes us from $k_1 \leq k_2 \leq k_3 \leq k_4$ to whatever the new ordering is.

The dynamical function $\mathcal{G}_{1234}(U,V;\sigma,\tau)$ will have two important pieces in this work – the dis-

connected part and the tree-level part. We are allowed to use these terms because holographic CFTs admit a "large N limit", and disconnected and tree-level correspond to the first two orders in the $1/N$ expansion. In AdS/CFT, N is typically the number of branes that give rise to the near-horizon geometry that has an AdS_{d+1} factor. Since tree-level corrections scale with a different power of N in each dimension, namely $N^{-1-\epsilon}$, it is useful to quote results in terms of the central charge.[2]

$$c_{\mathcal{T}}^{(6d)} = 70(4N^3 - 3N - 1)\,, \tag{4.10}$$

$$c_{\mathcal{T}}^{(4d)} = 30(N^2 - 1)\,, \tag{4.11}$$

$$c_{\mathcal{T}}^{(3d)} \approx 1 - \frac{112}{3\pi^2} - (8N+9)\left(\frac{\pi^2}{2}\right)^{-\frac{2}{3}} \frac{8\text{Ai}'\left[\left(N-\frac{3}{8}\right)\left(\frac{\pi^2}{2}\right)^{\frac{1}{3}}\right]}{3\text{Ai}\left[\left(N-\frac{3}{8}\right)\left(\frac{\pi^2}{2}\right)^{\frac{1}{3}}\right]} \approx \frac{64\sqrt{2}}{3\pi} N^{\frac{3}{2}}\,. \tag{4.12}$$

The $3d$ expression, which includes all perturbative terms, can be computed using the techniques of [45, 46]. Note that we are using conventions such that

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{d}{d-1}\frac{\Gamma\left(\frac{d}{2}\right)^2}{4\pi^d}\left[\frac{1}{2}I_{\mu\sigma}I_{\nu\rho} + \frac{1}{2}I_{\mu\rho}I_{\nu\sigma} - \frac{1}{d}\delta_{\mu\nu}\delta_{\sigma\rho}\right]\frac{c_{\mathcal{T}}}{x^{2d}} \tag{4.13}$$

$$I_{\mu\nu} \equiv \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$$

which means that a single free boson has $c_{\mathcal{T}} = 1$ [37].

4.1.2 Superconformal Ward identities

The four-point functions (4.9), by construction, obey all of the Ward identities from ordinary conformal symmetry and R-symmetry. In addition to this, we must consider superconformal Ward identities associated with the fermionic generators. The action of the fermionic generators generically relate correlation functions of superconformal primaries to those of their super descendants – a complicated bootstrap problem in general (see, *e.g.*, [370]). The situation becomes nicer when the external operators are the super primaries of the half-BPS multiplets. In this case, Ward identities from superconformal symmetry lead to constraints which only involve the original four-point function. More

[2] CFTs in even dimension have an additional notion of central charge because they exhibit a Weyl anomaly when placed on a curved manifold [369]. A universal prefactor, computed in [284, 338], relates the coefficient of the Euler density to the corresponding Virasoro central charge which we will call c_{2d} in later sections.

precisely, this constraint takes the universal form [316]

$$\left[\chi'\frac{\partial}{\partial\chi'}-\epsilon\alpha'\frac{\partial}{\partial\alpha'}\right]\mathcal{G}_{1234}(\chi,\chi';\alpha,\alpha')\Big|_{\alpha'=\frac{1}{\chi'}}=0 \tag{4.14}$$

which is often called the *superconformal Ward identity*. Here we have introduced the change of variables

$$U=\chi\chi',\ \ V=(1-\chi)(1-\chi'),\ \ \sigma=\alpha\alpha',\ \ \tau=(1-\alpha)(1-\alpha'). \tag{4.15}$$

These new variables will be identified with cross-ratios in one dimension (*i.e.*, for $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$), as we will be more specific about in the next subsection.

A direct consequence of (4.14) is that there are certain loci in cross-ratio space, on which the four-point functions are topological along certain directions. We will call these special choices of cross-ratio configurations *twisted configurations*. These configurations exist in $d=3,4,6$, with the prescription being slightly different in each case. We will be interested in seeing how this structure is captured in the results of $\text{AdS}_4\times S^7$, $\text{AdS}_5\times S^5$ and $\text{AdS}_7\times S^4$ supergravity respectively.

Starting with $d=6$, (4.14) implies

$$\epsilon=2:\qquad \frac{\partial}{\partial\chi'}\mathcal{G}_{1234}\left(\chi,\chi';\frac{1}{\chi'},\frac{1}{\chi'}\right)=0. \tag{4.16}$$

This is the statement that the twisted correlator is a holomorphic function. It is now understood that, far from being an arbitrary holomorphic function, the holomorphic function is really a four-point function of an auxiliary chiral algebra in two dimensions [284]. Since this conclusion required us to have two independent R-symmetry cross-ratios, it is easy to see that it only applies to the $6d$ superconformal algebra $\mathfrak{osp}(8^*|4)$.

Chiral symmetry appears again in $d=4$ according to

$$\epsilon=1:\qquad \frac{\partial}{\partial\chi'}\mathcal{G}_{1234}\left(\chi,\chi';\alpha,\frac{1}{\chi'}\right)=0. \tag{4.17}$$

This time, the structure is somewhat richer since we have a holomorphic dependence on α as well. This means there is still a notion of R-symmetry for the twisted correlator. As explained in [338], this is a consequence of the chiral algebra including super-Virasoro generators whenever the $4d$ theory has $\mathfrak{psu}(2,2|4)$ or $\mathfrak{su}(2,2|3)$ symmetry. The chiral algebra is non-supersymmetric for $\mathfrak{su}(2,2|2)$, in agreement with the fact that α is not an independent cross-ratio in these theories.

Finally, for $d = 3$ we have

$$\epsilon = \frac{1}{2}: \qquad \frac{\partial}{\partial \chi'} \mathcal{G}_{1234}\left(\chi', \chi'; \alpha, \frac{1}{\chi'}\right) = 0. \tag{4.18}$$

This indicates that, if we take a diagonal limit first, the remaining spatial cross-ratio drops out after the twist. More precisely, the correlator only depends on space-time positions through the ordering since the superconformal Ward identity is a local statement. This is what gives the one dimensional protected subsector the structure of topological quantum mechanics [286, 287]. To still interpret this object as a CFT in one dimension, it would have to consist solely of dimension zero operators which can be thought of as conserved currents for a higher-spin algebra [355]. This $3d/1d$ correspondence applies to $\mathfrak{osp}(\mathcal{N}|4)$ with $\mathcal{N} \geq 4$ which means that the cross-ratio α is always present.

Although (4.14) is a statement about half-BPS four-point functions, the results of [284, 286, 287, 338] apply to a larger family of operators in short multiplets, and also to higher-point functions. From a holographic point of view, the easiest way to study correlators of these operators would be to perform the OPE on half-BPS correlators. However, this will not be pursued here. It is also worth noting that we will encounter the $d = 2$ superconformal Ward identity in section 4.4 when we construct $\mathfrak{psu}(1,1|2)$ blocks. The implications of this for protected quantities were discussed in [289].

4.1.3 Four-point functions in lower dimensions

Correlators of operators in lower dimensions emerge from correlators in higher dimensional theories after performing superconformal twisting, as we have seen from the discussion on superconformal Ward identities in the last subsection. In this subsection, we give a summary of the kinematics of correlators of local operators on the plane (or restricted to a line).

Let us consider an operator with conformal weight h and $\mathfrak{su}(2)$ spin j. The $\mathfrak{su}(2)$ refers to the possible residual R-symmetry after twisting. We can denote the operator as

$$\mathcal{O}_{h,j}(z, y) = \mathcal{O}_{h,j|a_1 \ldots a_{2j}}(z)\, y^{a_1} \ldots y^{a_{2j}} \tag{4.19}$$

where we have similarly used $\mathfrak{su}(2)$ spinors y^a to keep track of the R-symmetry indices. For $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$, only one independent cross-ratio can be formed from four points. These lower dimensional

cross-ratios are defined in terms of the coordinates as

$$\chi = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad \alpha = \frac{y_{13}y_{24}}{y_{12}y_{34}}\,, \tag{4.20}$$

and can be identified with the symbols in (4.15). Note that $z_{ij} \equiv z_i - z_j$ and $y_{ij} \equiv y_i^a y_j^b \epsilon_{ab}$.

Covariance under $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$ constrains the possible terms in the OPE of the operators (4.19). To discuss these constraints individually, let us pretend for a moment that the y and z dependence can be separated.[3]

$$\mathcal{O}_{h_1}(z_1)\mathcal{O}_{h_2}(z_2) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} \sum_{m=0}^{\infty} \frac{(h_{12}+h)_m}{m!(2h)_m} \frac{\partial^m \mathcal{O}_h(z_2)}{z_{12}^{h_1+h_2-h-m}} \tag{4.21a}$$

$$\mathcal{O}_{j_1}(y_1)\mathcal{O}_{j_2}(y_2) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} \frac{1}{2j!} \sum_{\sigma\in S_{2j}} \mathcal{O}_{j|a_{\sigma(1)}\ldots a_{\sigma(2j)}} y_1^{a_1}\ldots y_1^{a_{j+j_{12}}} y_2^{a_{j+j_{12}+1}}\ldots y_2^{a_{2j}} \tag{4.21b}$$

In both cases, the outer sum runs over (quasi)primaries while the inner sum covers the states other than the highest weight. The chiral OPE (4.21a), first applied to the search for $\mathcal{W}$-algebras in [373], leads to the familiar s-channel expansion of the four-point function.

$$\langle \mathcal{O}_{h_1}(z_1)\mathcal{O}_{h_2}(z_2)\mathcal{O}_{h_3}(z_3)\mathcal{O}_{h_4}(z_4)\rangle = \left(\frac{z_{24}}{z_{14}}\right)^{h_{12}} \left(\frac{z_{14}}{z_{13}}\right)^{h_{34}} \frac{\mathcal{F}_{1234}(\chi)}{z_{12}^{h_1+h_2} z_{34}^{h_3+h_4}} \tag{4.22}$$

$$\mathcal{F}_{1234}(\chi) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g_h^{h_{12},h_{34}}(\chi)$$

where the $\mathfrak{sl}(2)$ blocks are given by

$$g_h^{h_{12},h_{34}}(\chi) = \chi^h\, {}_2F_1(h-h_{12}, h+h_{34}; 2h; \chi)\,. \tag{4.23}$$

In fact, the coefficients in (4.21a) can be fixed by demanding that (4.22) is reproduced. A convenient result, which can be shown with embedding space methods [374, 375], is that the second OPE (4.21b) leads to an entirely analogous expression with $h \mapsto -j$ everywhere.[4]

$$\langle \mathcal{O}_{j_1}(y_1)\mathcal{O}_{j_2}(y_2)\mathcal{O}_{j_3}(y_3)\mathcal{O}_{j_4}(y_4)\rangle = \left(\frac{y_{14}}{y_{24}}\right)^{j_{12}} \left(\frac{y_{13}}{y_{14}}\right)^{j_{34}} y_{12}^{j_1+j_2} y_{34}^{j_3+j_4}\, \mathcal{F}_{1234}(\alpha) \tag{4.24}$$

$$\mathcal{F}_{1234}(\alpha) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g_{-j}^{j_{21},j_{43}}\left(\frac{1}{\alpha}\right)$$

[3] Clearly, a pure $\mathfrak{su}(2)$ representation – one with j as the only non-zero quantum number – cannot satisfy the unitarity bounds. We will also not see operators like this in the non-unitary chiral algebra discussed in section 4.4. Instead, to apply (4.21a) and (4.21b) to a reasonable theory, they should be combined into a single OPE. Note however that non-unitary multiplets, even when absent from the theory, play a role in determining the superconformal blocks [371, 372].

[4] This is far from the only reason to consider the OPE directly in embedding space [376].

$$= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} \frac{(-1)^j (j+j_{34})! P_{j+j_{34}}^{j_{21}-j_{34},j_{12}-j_{34}}(1-2\alpha)}{(-\alpha)^{j_{34}}(j-j_{34}+1)_{j+j_{34}}}$$

We have used a standard hypergeometric function identity in the last line, relating $\mathfrak{sl}(2)$ blocks with half-integer weights and Jacobi polynomials, in order to make contact with another convention in the literature. Note also that the kinematic y_{ij} factors we extracted here are different from the ones in (4.9).

4.2 Emergent Parisi-Sourlas supersymmetry

In this section, we point out interesting hidden structures in tree-level maximally supersymmetric four-point Mellin amplitudes. We start by reviewing the results of [230, 231] in section 4.2.1. We then show in section 4.2.2 that correlators can be expressed as finite linear combinations of scalar exchange Witten diagrams, acted on by differential operators of cross-ratios. The combination coefficients are highly special and exhibit an emergent Parisi-Sourlas symmetry. This allows the sum of Witten diagrams in each multiplet to be written in terms of a single scalar exchange Witten diagram in an AdS space with four dimensions fewer. An analogous structure exists for the R-symmetry dependence, where the dimension of the internal manifold is similarly reduced by four. In section 4.2.3 we use these observations to provide a compact way to rewrite the holographic correlators, which manifests supergraph-like structures. The form of the result also suggests a lower dimensional scalar seed theory which encodes all the essential data. Correlators of the original theory can be obtained by dressing the seed theory correlators with differential operators.

4.2.1 Holographic Mellin amplitudes

The natural language for holographic correlators is the Mellin representation [73, 223, 260] which makes the analytic structure manifest. In the four-point case, the Mellin amplitude involves Mandelstam-like variables satisfying $s+t+u = \sum_{i=1}^{4} \Delta_i$. Tree-level Mellin amplitudes for a CFT with a weakly coupled gravity dual take the form

$$\mathcal{M}(s,t) = \mathcal{M}^{(s)}(s,t) + \mathcal{M}^{(t)}(s,t) + \mathcal{M}^{(u)}(s,t) + \mathcal{M}_{\text{contact}}(s,t) \ . \tag{4.25}$$

Here $\mathcal{M}^{(s)}$, $\mathcal{M}^{(t)}$, $\mathcal{M}^{(u)}$ correspond to exchange contributions, and have simple poles in the Mandelstam variable of the respective channel. $\mathcal{M}_{\text{contact}}$ is regular and accounts for additional con-

tact interactions. Upon transforming back to position space, the poles in the Mellin amplitude – from exchange Witten diagrams – produce conformal blocks for an internal single-particle operator. Poles corresponding to double-particle operators are instead contained in the measure for the inverse Mellin transformation which is ideally suited to theories at large N.[5]

Restoring the dependence on σ and τ, the inverse Mellin transformation is given by the following contour integral[6]

$$\begin{aligned}\mathcal{G}^{\text{conn}}_{1234}(U,V;\sigma,\tau) = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}s\mathrm{d}t}{(4\pi i)^2} U^{\frac{s}{2}-a_s} V^{\frac{t}{2}-a_t} \mathcal{M}_{1234}(s,t;\sigma,\tau) \\ \times \Gamma\left[\frac{\Delta_1+\Delta_2-s}{2}\right]\Gamma\left[\frac{\Delta_1+\Delta_4-t}{2}\right]\Gamma\left[\frac{\Delta_1+\Delta_3-u}{2}\right] \\ \times \Gamma\left[\frac{\Delta_3+\Delta_4-s}{2}\right]\Gamma\left[\frac{\Delta_2+\Delta_3-t}{2}\right]\Gamma\left[\frac{\Delta_2+\Delta_4-u}{2}\right]\end{aligned} \tag{4.26}$$

where

$$a_s = \frac{\epsilon}{2}(k_1+k_2) - \epsilon\mathcal{E}\ , \quad a_t = \epsilon\mathcal{E} - \frac{\epsilon}{2}(k_1-k_4)\ , \tag{4.27}$$

and we recall that $\Delta_i = \epsilon k_i$. To make sense of this, we need to realize that the shape of the contour is different from the schematic one above. The well known prescription is to choose a contour which makes the real part positive in the arguments of all six gamma functions. This is always possible for a G_{conn} which does not require regularization. This restriction on the contour, based on double-particle poles, is not sufficient because the critical strip described above usually contains some of the single-particle poles of the Mellin amplitude itself. To resolve this ambiguity, we must put single-particle and double-particle exchanges on the same footing, *i.e.*, by choosing a contour such that the poles of $\mathcal{M}^{(s)}$ and $\mathcal{M}^{(t)}$ lie to the right while those of $\mathcal{M}^{(u)}$ lie to the left. In other words, whether we are talking about the gamma functions or Mellin amplitudes, we keep semi-infinite sequences of poles that increase the exponents but not semi-infinite sequences of poles that decrease them.[7] Clearly, these towers have integer spacing because an exchanged primary appears together with its

[5] One often hears these types of operators referred to as "single-trace" and "double-trace" respectively. Strictly speaking, this is only correct in the strict $N \to \infty$ limit. Even if one is not concerned with any microscopic Lagrangian, there is a well defined basis of double-particle operators which have correction terms in addition to normal ordered squares of single-particle operators [321, 322].

[6] The $4\pi i$, which might look strange, becomes $2\pi i$ again after mapping s, t and u to the variables which generalize more easily to arbitrary n-point correlators.

[7] In the case of $\text{AdS}_5 \times S^5$ and $\text{AdS}_7 \times S^4$, the spectrum is such that the single-particle sequences are in fact finite [240]. The truncation occurs as a consistency condition of the $1/N$ expansion [267]. The residues of $\mathcal{M}(s,t;\sigma,\tau)$ must conspire to stop the single-particle and double-particle poles from overlapping which would lead to higher order singularities that are unphysical at tree-level. This phenomenon was historically very important in the holographic correlator program before methods were developed in [267] to crack the $\text{AdS}_4 \times S^7$ case as well.

descendants.[8]

The Mellin amplitudes of interest to us were constructively derived for all external half-BPS operators in [230, 231] for all three maximally supersymmetric backgrounds. The derivation exploited a special R-symmetry configuration, dubbed maximally R-symmetry violating (MRV) in [230, 231], where major simplifications occur. The full amplitudes were then obtained from the MRV amplitudes by using symmetries. A remarkable feature in these results is that all the contact terms in the amplitudes vanish, after using a natural prescription to symmetrize the exchange amplitudes recovered from the MRV limit. Therefore the full amplitudes can be written as the sum over only exchange amplitudes in three channels. Specifically, [231] gives a formula for the Mellin amplitudes as a sum over simple poles

$$\mathcal{M}^{(s)}_{1234}(s,t;\sigma,\tau) = \sum_{\substack{p=\max\{|k_{12}|,|k_{34}|\}+2\\ \text{step } 2}}^{\max\{|k_{12}|,|k_{34}|\}+2\mathcal{E}-2} \sum_{m=0}^{\infty} \sum_{0\le i+j\le\mathcal{E}} \sigma^i\tau^j \frac{\mathcal{R}^{p,m;i,j}_{1234}(t,u)}{s-2m-\epsilon p} \tag{4.28}$$

with

$$\begin{aligned} \mathcal{M}^{(t)}_{1234}(s,t;\sigma,\tau) &= \tau^{\mathcal{E}}\mathcal{M}^{(s)}_{3214}\left(t,s;\frac{\sigma}{\tau},\frac{1}{\tau}\right) \\ \mathcal{M}^{(u)}_{1234}(s,t;\sigma,\tau) &= \sigma^{\mathcal{E}}\mathcal{M}^{(s)}_{4231}\left(u,t;\frac{1}{\sigma},\frac{\tau}{\sigma}\right) \end{aligned} \tag{4.29}$$

determining the other channels.[9] The integer p labels the exchanged supergravity multiplets whose super primaries are dual to the half-BPS operators $\mathcal{O}_p$. The R-symmetry selection rule, and the requirement of the effective action being finite, restrict the range of p to the finite set as indicated in the above sum.

For the reader's convenience, we now give explicit expressions for the residues. In [230, 231], they took the form

$$\mathcal{R}^{i,j}_{p,m}(t,u) = K^{i,j}_p(t,u)\, L^{i,j}_{p,m}\, N^{i,j}_p \ , \tag{4.31}$$

where we have dropped the position labels as we will often do when they are clear from the context.

[8] We should note that this logic only defines a correlator when there is some notion of which poles belong together. In the non-perturbative context [377], a Mellin amplitude is only useful if the appropriate contour is specified along with it.

[9] Identities for the s-channel amplitude itself, which might be useful to keep in mind, are

$$\mathcal{M}^{(s)}_{1234}(s,t;\sigma,\tau) = \mathcal{M}^{(s)}_{3412}(s,t;\sigma,\tau) = \mathcal{M}^{(s)}_{2134}(s,u;\tau,\sigma). \tag{4.30}$$

The factor of (4.31) which encodes all the t and u dependence is

$$\begin{aligned} K_p^{i,j} &= 2i(2i+\kappa_u)t^-t^+ + 2j(2j+\kappa_t)u^-u^+ - 2j(\tfrac{2}{\epsilon}-2+\kappa_u)t^+u^- - 2i(\tfrac{2}{\epsilon}-2+\kappa_t)u^+t^- \\ &+ \frac{1}{4}(2p-\kappa_t-\kappa_u)(2p+\tfrac{4}{\epsilon}-4+\kappa_t+\kappa_u)(u^-t^- + 4\epsilon^2 ij) \\ &+ \tfrac{\epsilon}{2}(\kappa_u+\kappa_t-2p)(\kappa_u+\kappa_t+2p+\tfrac{4}{\epsilon}-4)(it^- + ju^-) \\ &+ 4\epsilon ij(t^+(\kappa_u+\tfrac{2}{\epsilon}-2)+u^+(\kappa_t+\tfrac{2}{\epsilon}-2)) - 8ijt^+u^+ \,, \end{aligned} \tag{4.32}$$

where we have defined

$$\kappa_s \equiv |k_3+k_4-k_1-k_2| \,, \quad \kappa_t \equiv |k_1+k_4-k_2-k_3| \,, \quad \kappa_u \equiv |k_2+k_4-k_1-k_3| \,, \tag{4.33}$$

and introduced the shorthand notation

$$u^\pm = u \pm \frac{\epsilon}{2}\kappa_u - \frac{\epsilon}{2}\Sigma \,, \qquad t^\pm = t \pm \frac{\epsilon}{2}\kappa_t - \frac{\epsilon}{2}\Sigma \tag{4.34}$$

with $\Sigma = k_1+k_2+k_3+k_4$.

For the other two factors in (4.31), we will find it convenient to rewrite them in a way which is manifestly proportional to the OPE coefficients of the exchanged super primaries $\mathcal{O}_p$. In the resulting expression

$$\mathcal{R}_{p,m}^{i,j}(t,u) = C_{k_1k_2p}\, C_{k_3k_4p}\, K_p^{i,j}(t,u)\, H_{p,m}^{i,j} \,, \tag{4.35}$$

we have

$$\begin{aligned} H_{p,m}^{i,j} = {} & \frac{2^{-\frac{1}{3}\left(2\epsilon-\frac{2}{\epsilon}+9\right)}\Gamma[\epsilon p]\Gamma[\epsilon(p+\frac{1}{\epsilon}-1)]\Gamma[\frac{p+k_{12}}{2}]\Gamma[\frac{p-k_{12}}{2}]\Gamma[\frac{p+k_{34}}{2}]\Gamma[\frac{p-k_{34}}{2}]}{\Gamma[p]\Gamma[p+\frac{1}{\epsilon}-1]\Gamma[\frac{\epsilon}{2}(p+k_{12})]\Gamma[\frac{\epsilon}{2}(p-k_{12})]\Gamma[\frac{\epsilon}{2}(p+k_{34})]\Gamma[\frac{\epsilon}{2}(p-k_{34})]} \\ & \times \frac{(-1)^{i+j+\frac{2p-\kappa_t-\kappa_u}{4}}\Gamma[\frac{\kappa_u+2+2i}{2}]^{-1}\Gamma[\frac{1}{4}(\frac{4}{\epsilon}-4+2p+\Sigma-\kappa_s-4l)]\Gamma[\frac{\kappa_t+2+2j}{2}]^{-1}}{i!j!m!\Gamma[2-\epsilon+m+\epsilon p]\Gamma[\frac{\epsilon(k_1+k_2-p)}{2}-m]\Gamma[\frac{2(p+2)-\Sigma+\kappa_s+4l}{4}]\Gamma[\frac{\epsilon(k_3+k_4-p)}{2}-m]} \,. \end{aligned} \tag{4.36}$$

and $l = \mathcal{E}-i-j$, which is the analogue of u for the R-symmetry. The OPE coefficients, which come from the cubic couplings in the supergravity effective action, are given by [236, 238, 239]

$$C_{k_1k_2k_3} = \begin{cases} \frac{\pi}{N^{\frac{3}{4}}}\frac{2^{-\beta-\frac{1}{4}}}{\Gamma[\frac{\beta}{2}+1]}\prod_{i=1}^3 \frac{\sqrt{\Gamma[k_i+2]}}{\Gamma[\frac{\alpha_i+1}{2}]}, & d=3 \\ \frac{\sqrt{k_1k_2k_3}}{N}, & d=4 \\ \frac{2^{2\beta-2}}{(\pi N)^{\frac{3}{2}}}\Gamma[\beta]\prod_{i=1}^3 \frac{\Gamma[\alpha_i+\frac{1}{2}]}{\sqrt{\Gamma[2k_i-1]}}, & d=6 \end{cases} \tag{4.37}$$

with

$$\alpha_1 = \frac{1}{2}(k_2 + k_3 - k_1), \quad \alpha_2 = \frac{1}{2}(k_3 + k_1 - k_2), \quad \alpha_3 = \frac{1}{2}(k_1 + k_2 - k_3) \ , \tag{4.38}$$

and $\beta = \alpha_1 + \alpha_2 + \alpha_3$.

4.2.2 Parisi-Sourlas-like dimensional reduction

The above answer gives explicit expressions for holographic correlators of four arbitrary half-BPS operators in all three maximally supersymmetry theories. The fact that they can be cast into the same form as functions of ϵ already shows an unexpected universality of amplitudes in diverse dimensions. In this subsection, we further analyze these amplitudes to uncover more interesting properties. We will show that the maximally supersymmetric Mellin amplitudes exhibit a surprising hidden structure, where a dimensionally reduced spacetime naturally arises. The emergence of a lower dimensional spacetime in these amplitudes is highly reminiscent of the Parisi-Sourlas dimensional reduction.

Let us expose this structure by separating the amplitudes into different parts. We start from the AdS part of the amplitudes, which concerns only the dependence on the Mandelstam variables. First we observe that the Mandelstam variables appear in $K_p^{i,j}(t,u)$ as polynomials. We can bring this factor outside of the Mellin representation (4.26) as differential operators via the following identification

$$U\partial_U \leftrightarrow (\tfrac{s}{2} - a_s)\times \ , \quad V\partial_V \leftrightarrow (\tfrac{t}{2} - a_t) \times \ . \tag{4.39}$$

Therefore, we only need to focus on the sum over poles in s which are labelled by m. Let us isolate the m-dependence from $H_{p,m}^{i,j}$, which reads

$$\mathcal{A}_{p,m} = \frac{1}{m!\Gamma[2-\epsilon+m+\epsilon p]\Gamma[\frac{\epsilon(k_1+k_2-p)}{2}-m]\Gamma[\frac{\epsilon(k_3+k_4-p)}{2}-m]} \ , \tag{4.40}$$

and denote the sum as

$$F_p(s) = \sum_{m=0}^{\infty} \frac{\mathcal{A}_{p,m}}{s-\epsilon p-2m} \ . \tag{4.41}$$

A priori, we do not expect the function $F_p(s)$ to have any special property. Remarkably, however, we find that the sum can always be written as a linear combination of exactly *three* scalar exchange Witten diagrams! More precisely, we have

$$F_p(s) = \mathcal{N}(\mathcal{M}_{\epsilon p}^{(d)}(s) + c_1\mathcal{M}_{\epsilon p+2}^{(d)}(s) + c_2\mathcal{M}_{\epsilon p+4}^{(d)}(s)) \tag{4.42}$$

where

$$\begin{aligned} c_1 &= -\frac{\left(\frac{\epsilon(p+k_{12})}{2}\right)\left(\frac{\epsilon(p-k_{12})}{2}\right)\left(\frac{\epsilon(p+k_{34})}{2}\right)\left(\frac{\epsilon(p-k_{34})}{2}\right)}{2(\epsilon p)_2\left(\frac{\epsilon(p-1)}{2}\right)_2}\,, \\ c_2 &= \frac{\left(\frac{\epsilon(p+k_{12})}{2}\right)_2\left(\frac{\epsilon(p-k_{12})}{2}\right)_2\left(\frac{\epsilon(p+k_{34})}{2}\right)_2\left(\frac{\epsilon(p-k_{34})}{2}\right)_2}{(1+\epsilon(p-1))_3(\epsilon p)_4(2+\epsilon(p-1))}\,, \end{aligned} \tag{4.43}$$

with $k_{ij} = k_i - k_j$. The overall coefficient $\mathcal{N}$ is not important here. $\mathcal{M}_\Delta^{(d)}(s)$ is the Mellin amplitude of a scalar exchange Witten diagram in AdS_{d+1} of dimension Δ, and with external dimensions Δ_i. These scalar exchange Witten diagram amplitudes are given by

$$\begin{aligned} \mathcal{M}_\Delta^{(d)}(s) = \sum_{m=0}^{\infty} &\frac{(-2)\Gamma[\Delta]\Gamma[\Delta-\epsilon]\Gamma[m+\Delta-\epsilon]^{-1}}{m!\Gamma[\frac{\Delta+\Delta_{12}}{2}]\Gamma[\frac{\Delta-\Delta_{12}}{2}]\Gamma[\frac{\Delta+\Delta_{34}}{2}]\Gamma[\frac{\Delta-\Delta_{34}}{2}]} \\ &\times\frac{1}{\Gamma[\frac{\Delta_1+\Delta_2-\Delta-2m}{2}]\Gamma[\frac{\Delta_3+\Delta_4-\Delta-2m}{2}](s-\Delta-2m)} \end{aligned} \tag{4.44}$$

where $\Delta_{ij} = \Delta_i - \Delta_j$.

Meanwhile, we can make another striking observation: $F_p(s)$ is also secretly the scalar exchange Witten diagram $\mathcal{M}_{\epsilon p}^{(d-4)}(s)$ in *four dimensions lower*, with internal dimension ϵp and the same external dimensions! The appearance of AdS_{d-3} in an AdS_{d+1} supergravity amplitude is quite unexpected. However, this equivalence across dimensions can be naturally explained in terms of an emergent Parisi-Sourlas supersymmetry, as we will see below.

We first recall that Parisi-Sourlas supersymmetry is a geometrization of a hidden symmetry of stochastic equations. This supersymmetry is key to the celebrated conjecture of Parisi and Sourlas [280], and links together the IR fixed points of three seemingly unrelated models: a random field model in d dimensions, a supersymmetric field theory without disorder in d dimensions, and a model without disorder in $d-2$ dimensions. In particular, it claims that certain correlation functions in the supersymmetric theory should coincide with the $d-2$ dimensional theory. The Parisi-Sourlas supersymmetry is non-unitary, as the supercharges violate spin-statistics. For this reason, theories with Parisi-Sourlas supersymmetry have very unusual features, from a high energy physicist's point of view. Recently, this supersymmetry was scrutinized in [281] using conformal bootstrap techniques, where a number of kinematical results were derived. In particular, [281] showed that the Parisi-Sourlas supersymmetry leads to a beautiful identity for conformal blocks across different spacetime

dimensions

$$g^{(d-2)}_{\Delta,\ell} = g^{(d)}_{\Delta,\ell} + C_{2,0} g^{(d)}_{\Delta+2,\ell} + C_{1,-1} g^{(d)}_{\Delta+1,\ell-1} + C_{0,-2} g^{(d)}_{\Delta,\ell-2} + C_{2,-2} g^{(d)}_{\Delta+2,\ell-2} \tag{4.45}$$

where the $C_{i,j}$ are pure numbers found in [281]. The combination on the RHS can be interpreted as a single superconformal block under Parisi-Sourlas supersymmetry. Later it was pointed out in [241] that the Parisi-Sourlas supersymmetry can also be realized holographically. This gives rise to similar dimensional reduction identities for exchange Witten diagrams, obtained by replacing conformal blocks with the corresponding exchange Witten diagrams in AdS.[10] Here we will only focus on the relevant case with $\ell = 0$. The relation (4.45) simplifies into

$$g^{(d-2)}_{\Delta,0} = g^{(d)}_{\Delta,0} + C_{2,0} g^{(d)}_{\Delta+2,0} \,, \tag{4.46}$$

with

$$C_{2,0} = -\frac{(\Delta-\Delta_{12})(\Delta+\Delta_{12})(\Delta-\Delta_{34})(\Delta+\Delta_{34})}{4(d-2\Delta-4)(d-2\Delta-2)\Delta(\Delta+1)} \,. \tag{4.47}$$

The corresponding Witten diagram relation is [241]

$$\mathcal{M}^{(d-2)}_{\Delta} = \mathcal{M}^{(d)}_{\Delta} + C_{2,0} \mathcal{M}^{(d)}_{\Delta+2} \,. \tag{4.48}$$

By using the relation twice on the AdS_{d-3} scalar exchange amplitude, we reach AdS_{d+1} and reproduce precisely the combination (4.42).

A similar dimensional reduction structure is also present in the R-symmetry part. Since i, j appear in $K^{i,j}_p(t,u)$ polynomially, we can treat them as differential operators acting on the monomial $\sigma^i \tau^j$ in (4.28)

$$\sigma\partial_\sigma \leftrightarrow i\times \,, \quad \tau\partial_\tau \leftrightarrow j\times \,, \tag{4.49}$$

on the same footing as the Mandelstam variables. The remaining i-, j-dependence is contained in $H^{i,j}_{p,m}$, which we denote as

$$\mathcal{B}^{i,j}_p = \frac{(-1)^{i+j+\frac{2p-\kappa_t-\kappa_u}{4}} \Gamma[\frac{1}{4}(\frac{4}{\epsilon}-4+2p+\Sigma-\kappa_s-4l)]}{i!j!\Gamma[\frac{\kappa_u+2+2i}{2}]\Gamma[\frac{2(p+2)-\Sigma+\kappa_s+4l}{4}]\Gamma[\frac{\kappa_t+2+2j}{2}]} \,. \tag{4.50}$$

We now resum i and j to get $\sum_{i,j} \mathcal{B}^{i,j}_p \sigma^i \tau^j$. It turns out that the sum is nothing but the R-symmetry

[10] Similar generalizations are also found for two-point functions in boundary CFTs [241], and CFTs on real projective space [336].

polynomial associated with the exchange of the rank-p symmetric traceless representation

$$\mathtt{Y}_p = \frac{(-1)^{\frac{p}{2}-\frac{\kappa_t+\kappa_u}{4}}}{\Gamma[\frac{8+2p-\kappa_t-\kappa_u}{4}]\Gamma[\frac{8+2p-\kappa_t+\kappa_u}{4}]\Gamma[\frac{2+\kappa_t}{2}]}\mathcal{Y}_{p,0}\Big|_{\mathtt{d}=\frac{2}{\epsilon}}\,, \tag{4.51}$$

but for a *different* R-symmetry group $SO(\frac{2}{\epsilon})$. Here the polynomial $\mathcal{Y}_{p,0}$ was given in [230, 231] for any $SO(\mathtt{d})$ R-symmetry group

$$\begin{aligned}\mathcal{Y}_{p,0} = \sum_{i,j}(-1)^{\frac{p}{2}-\frac{\kappa_t+\kappa_u}{4}}\sigma^i\tau^j\Bigg[&\frac{\Gamma[\frac{4+2p+\kappa_u-\kappa_t}{4}]}{i!\,j!\,(\frac{\kappa_u}{2})!\Gamma[\frac{\mathtt{d}}{2}+p-1]}\\ \times&\frac{(\frac{\kappa_t+\kappa_u-2p}{4})_{i+j}(\frac{\kappa_t+\kappa_u+2p+2\mathtt{d}-4}{4})_{i+j}\Gamma[\frac{-4+2\mathtt{d}+2p+\kappa_t+\kappa_u}{4}]}{(1+\frac{\kappa_u}{2})_i(1+\frac{\kappa_t}{2})_j}\Bigg]\,,\end{aligned} \tag{4.52}$$

and we have changed its normalization to make $\mathtt{Y}_p$ more symmetric. For the physical spacetime dimensions $\epsilon = \frac{1}{2}, 1, 2$, the new R-symmetry dimension $\mathtt{d} = \frac{2}{\epsilon}$ is the same as shifting the original one by 4

$$\mathtt{d} \to \mathtt{d} - 4\,. \tag{4.53}$$

This implies the dimension of the internal sphere is reduced by four, which perfectly parallels the dimensional reduction structure for the AdS part.

The above dimensional reduction may lead to negative dimensions. However, this does not give rise to problems. We formally view d and $\mathtt{d}$ as continuous parameters, and analytically continue them to negative values. Clearly, Mellin amplitudes and R-symmetry polynomials can be defined for any values of d and $\mathtt{d}$. Moreover, we regard U, V and σ, τ as independent variables, as such is the case for $d > 1$ and $\mathtt{d} > 3$.

As a side comment, the above Parisi-Sourlas-like dimensional reduction structure also explains why the prescription of [230, 231] for recovering the full exchange amplitudes from the MRV limit is unique. Had a different prescription been used, the multiplet exchange amplitudes would differ by additional contact terms, which would ruin the above dimensional reduction structure: as discussed in [241] the only prescription to fix contact terms in exchange diagrams that leads to a clean Parisi-Sourlas dimensional reduction structure is that of using Polyakov-Regge blocks, which naturally emerge from the MRV limit.

4.2.3 Full amplitudes from bosonic seed amplitudes

It is time to take stock and put all the ingredients together. Using observations from the last subsection, we will now give a compact new look to the results of [230, 231]. We first restore the kinematic factor in (4.9) to manifest the Bose symmetry, and define

$$\mathbf{Y}_p^{(s)}(t_i) \quad = \prod_{i<j} t_{ij}^{\gamma_{ij}^0} (t_{12}t_{34})^{\mathcal{E}} \mathtt{Y}_p(\sigma,\tau) \ , \tag{4.54}$$

$$\mathbf{M}_p^{(s)}(x_i) \quad = \prod_{i<j} x_{ij}^{-2\epsilon\gamma_{ij}^0} (x_{12}^{2\epsilon}x_{34}^{2\epsilon})^{-\mathcal{E}} \times \int_{-i\infty}^{i\infty} \frac{dsdt}{(4\pi i)^2} U^{\frac{s}{2}-a_s} V^{\frac{t}{2}-a_t} \mathcal{M}_{\epsilon p}^{(d-4)}(s)\, \Gamma_{\{k_i\}} \ , \tag{4.55}$$

The exponents of $x_{ij}^{-2\epsilon}$ and t_{ij} in the restored kinematic factor are collectively denoted here as γ_{ij}^0. Then the four-point correlators for the $\text{AdS}_{d+1} \times S^{\mathtt{d}-1}$ background can be compactly written as

$$G_{\{k_i\}}(x_i,t_i) = \sum_{p_s} \mathbf{S}_{p_s}^{(s)} + \sum_{p_t} \mathbf{S}_{p_t}^{(t)} + \sum_{p_u} \mathbf{S}_{p_u}^{(u)} \tag{4.56}$$

where the contribution of each multiplet $\mathbf{S}_p^{(s)}$ is obtained from the action of a differential operator $\mathbf{K}_p^{(s)}$ on a scalar exchange diagram in $\text{AdS}_{d-3} \times S^{\mathtt{d}-5}$

$$\mathbf{S}_p^{(s)} = C_{k_1k_2p}C_{k_3k_4p} f_{\{k_i\},p}\, \mathbf{K}_p^{(s)} \circ (\mathbf{Y}_p^{(s)}\mathbf{M}_p^{(s)}) \ . \tag{4.57}$$

Here $f_{\{k_i\},p}$ is a simple universal factor

$$f_{\{k_i\},p} \quad = \quad -2^{-\frac{1}{3}\left(2\epsilon-\frac{2}{\epsilon}+12\right)} \frac{\Gamma[\frac{p+k_{12}}{2}]\Gamma[\frac{p-k_{12}}{2}]\Gamma[\frac{p+k_{34}}{2}]\Gamma[\frac{p-k_{34}}{2}]}{\epsilon\Gamma[p]\Gamma[p+\frac{1}{\epsilon}]} \tag{4.58}$$

describing what is left after (4.40), (4.50) and the normalization from (4.42) are taken out of $H_{p,m}^{i,j}$.[11] The operator $\mathbf{K}_p^{(s)}$ follows from $K_p^{i,j}(t,u)$ in (4.35), and reads

$$\begin{aligned}
\mathbf{K}_p^{(s)} \quad &= \quad 16(D_{13}D_{24}\theta_{14}\theta_{23} + D_{14}D_{23}\theta_{13}\theta_{24} - 2D_{13}D_{23}\theta_{13}\theta_{23}) \\
&- \quad 8D_{23}\theta_{23}(D_{24}-\epsilon\theta_{13})(\tfrac{2}{\epsilon}-2+\kappa_u) - 8D_{13}\theta_{13}(D_{14}-\epsilon\theta_{23})(\tfrac{2}{\epsilon}-2+\kappa_t) \\
&+ \quad (D_{24}-\epsilon\theta_{13})(D_{14}-\epsilon\theta_{23})(2p-\kappa_t-\kappa_u)(\tfrac{4}{\epsilon}-4+2p+\kappa_t+\kappa_u) \ ,
\end{aligned} \tag{4.61}$$

[11] If we wanted to go back and write the original residue in the form (4.57), we would have

$$\mathcal{R}_{p,m}^{i,j}(t,u) = C_{k_1k_2p}C_{k_3k_4p}\mathcal{N}^{-1}\, f_{\{k_i\},p}\, K_p^{i,j}(t,u)\, \mathcal{A}_{p,m}\, \mathcal{B}_p^{i,j} \tag{4.59}$$

with

$$\mathcal{N}^{-1} = -\frac{2\Gamma[\epsilon p]\Gamma[\epsilon p-\epsilon+2]}{\Gamma[\frac{\epsilon}{2}(p+k_{12})]\Gamma[\frac{\epsilon}{2}(p-k_{12})]\Gamma[\frac{\epsilon}{2}(p+k_{34})]\Gamma[\frac{\epsilon}{2}(p-k_{34})]}. \tag{4.60}$$

where we have defined the shorthand notations

$$\theta_{ij} = t_{ij}\frac{\partial}{\partial t_{ij}} , \quad D_{ij} = x_{ij}^2\frac{\partial}{\partial x_{ij}^2} . \tag{4.62}$$

We note that the structure of (4.57) is very reminiscent of the flat space results using supergraphs.

The final answer (4.56) and (4.57) also suggests an effective lower dimensional "seed" theory which encodes all the essential data. The spectrum of the seed theory consists of only scalars, and coincides with the super-primary spectrum of the higher dimensional theory. Moreover, the scalars interact with the same cubic couplings as in the original theory. In the seed theory, correlators can be easily computed by

$$G^{\text{seed}}_{\{k_i\}}(x_i,t_i) = \sum_p C_{k_1k_2p}C_{k_3k_4p}\,\mathbf{Y}_p^{(s)}\mathbf{M}_p^{(s)} + (t) + (u) . \tag{4.63}$$

We can then simply perform the replacement

$$\mathbf{Y}_p^{(s)}\mathbf{M}_p^{(s)} \to f_{\{k_i\},p}\,\mathbf{K}_p^{(s)} \circ (\mathbf{Y}_p^{(s)}\mathbf{M}_p^{(s)}) , \tag{4.64}$$

to obtain the full correlator in the original theory

$$G^{\text{seed}}_{\{k_i\}}(x_i,t_i) \to G_{\{k_i\}}(x_i,t_i) . \tag{4.65}$$

4.3 Chiral symmetry from six dimensions

In this section we focus on the case of $D = 11$ supergravity on $\text{AdS}_7 \times S^4$, which is conjecturally dual to a $6d$ $\mathcal{N} = (2,0)$ SCFT of type $\mathfrak{g} = A_{N-1}$ at large N. As discussed in [284], the $\mathfrak{osp}(8^*|4)$ superconformal algebra contains an $\mathfrak{su}(1,1|2)$ subalgebra, and therefore according to the construction of [338] such a theory admits a protected subsector of operators whose (twisted) correlation functions restricted to a plane form a chiral algebra.[12] In [284] it was furthermore conjectured that the chiral algebra associated to the type $\mathfrak{g} = A_{N-1}$ theory is $\mathcal{W}_\mathfrak{g}$: precisely the algebra appearing on the $2d$ side of the AGT correspondence [378]. In this case, the full higher-spin algebra is known: it is the

[12] The R-symmetry decomposes as $\mathfrak{su}(2) \times \mathfrak{u}(1) \subset \mathfrak{usp}(4)$ and all operators that can contribute to the chiral algebra are neutral under $\mathfrak{u}(1)$. The "twisting-translating" procedure

$$\mathcal{O}(z) = \mathcal{O}^{b_1\ldots b_{2j}}(z,\bar{z},0)\,u_{b_1}(\bar{z})\ldots u_{b_{2j}}(\bar{z}), \qquad u(\bar{z}) \equiv (1,\bar{z}) \tag{4.66}$$

defines their representatives away from the origin.

quantization of the classical $\mathcal{W}_N$ algebra, which in turn is the outcome of the Drinfel'd-Sokolov reduction of $\mathfrak{sl}(N)$. This also appears as the asymptotic symmetry algebra of AdS_3 higher-spin gravity, as discussed in [379, 380].

In [284] two tests of this conjecture were proposed: the first based on the superconformal index, the second on the computation of the three-point functions between half-BPS operators at large N, comparing the latter with certain three-point functions in the chiral algebra $\mathcal{W}_{\mathfrak{g}}$. Here we focus on the second kind of test, providing further evidence for the validity of the conjecture. In the following, we shall focus on the sector of half-BPS operators and compute the four-point functions between the dual operators in the chiral algebra. Then, we shall use the four-point functions computed in [230, 231] at large N, perform the twist described in section 4.1, and derive the same four-point functions directly from Mellin space, finding perfect agreement.

4.3.1 $\mathcal{W}$-algebra correlators at large N

As discussed in [284], the ring of half-BPS operators in the $6d$ $\mathcal{N} = (2,0)$ theory is isomorphic to the ring that is freely generated by the Casimir invariants of $\mathfrak{g}$. This led to the conjecture that the chiral algebra associated with the $6d$ theory is $\mathcal{W}_{\mathfrak{g}}$, generated precisely by the meromorphic currents associated to the half-BPS operators. It was also shown that the central charge of $\mathcal{W}_{\mathfrak{g}}$ is given by

$$c_{2d} = 4\,N^3 - 3\,N - 1\,. \tag{4.67}$$

The chiral algebra generators, which we shall refer to as $W^{(k)}$, with $k \geq 2$, are bosonic primaries of dimension k. The three-point functions of such operators read

$$\langle W^{(k_1)}(z_1)\, W^{(k_1)}(z_2)\, W^{(k_3)}(z_3)\rangle = \frac{C_{k_1 k_2 k_3}}{z_{12}^{k_1+k_2-k_3}\, z_{13}^{k_1+k_3-k_2}\, z_{23}^{k_2+k_3-k_1}}\,, \tag{4.68}$$

where the coefficients $C_{k_1 k_2 k_3}$ are given in the supergravity approximation by (4.37) (in the case $d = 6$). Note that these three-point functions are non-vanishing only when

$$k_3 = |k_{12}| + 2,\ |k_{12}| + 4,\ \dots, k_1 + k_2 - 2\,. \tag{4.69}$$

This means that, to this order, the OPE between generators is

$$W^{(k_1)}(z_1)W^{(k_2)}(z_2) = \frac{\delta_{k_1,k_2}}{z_{12}^{2k_1}} + \sum_{\substack{p=|k_{12}|+2 \\ \text{step } 2}}^{k_1+k_2-2} C_{k_1k_2p} \sum_{m=0}^{k_1+k_2-p-1} \frac{(k_{12}+p)_m}{m!(2p)_m} \frac{\partial^m W^{(p)}(z_2)}{z_{12}^{k_1+k_2-p-m}} + \dots \quad (4.70)$$

where we have specialized (4.21a) to this case and explicitly singled out the contribution of the identity. The terms hidden by $\dots$ in (4.70) come in two types – those suppressed by higher powers of N and/or those that are regular in z_{12}. A careful accounting of regular and singular terms will be important for the calculation of four-point functions which we now discuss. In particular, for operators that lead to a singular contribution to the four-point function (found by taking the OPE twice), we must keep track of whether they belong to singular terms of both OPEs or just one of them.

We are concerned with solving for the dynamical part of a four-point function $\mathcal{F}_{1234}(\chi)$ which should be expanded in the $\mathfrak{sl}(2)$ blocks given by (4.23). Once this is known, the kinematic prefactor can be restored by (4.22). The chiral algebra four-point functions for equal weights ($k_i = k$) have been computed in [267] for $k = 2, 3, 4$, using the holomorphic bootstrap method of [368]. Being a bootstrap approach, this exploits the fact that the dynamical function must satisfy

$$\mathcal{F}_{1234}(\chi) = (-1)^{\Sigma}\, \chi^{k_1+k_2}\, (1-\chi)^{-k_2-k_3}\, \mathcal{F}_{3214}(1-\chi) \quad (4.71a)$$

$$\mathcal{F}_{1234}(\chi) = (-1)^{\Sigma}\, \chi^{k_1+k_4}\, \mathcal{F}_{4231}\left(\frac{1}{\chi}\right) \quad (4.71b)$$

which generate the S_3 group of crossing transformations.[13] Here we compute such correlators using an alternative (but equivalent) method that we shall summarize below. This also allows us to give closed form expressions for the four-point functions with general weights. To illustrate the idea, let us first recall the basic fact that chiral algebra correlators are meromorphic functions of the coordinates. Here, after stripping off the kinematic prefactor, we obtain meromorphic functions of the cross-ratio χ, whose only possible singularities in the complex χ-plane are poles at $\chi = 0, 1, \infty$, corresponding to singular terms in the OPE. This is, at least in principle, enough to fix the correlation functions completely. To exploit this information in a convenient way, we make an ansatz for the four-point function as

$$\mathcal{F}_{1234}(\chi) = \mathcal{F}^{(s)}_{1234}(\chi) + \mathcal{F}^{(t)}_{1234}(\chi)\,, \quad (4.72)$$

[13] The choice to focus on the crossed configuration (4.71a) which switches $(1 \leftrightarrow 3)$, instead of the one that switches $(2 \leftrightarrow 4)$, is such to simplify the argument that we shall present below, which allows us to neglect normal ordered products of operators both in the s- and in the t-channel OPE.

where $\mathcal{F}^{(s)}$ contains the singular terms coming from the s-channel OPE, while $\mathcal{F}^{(t)}$ is its crossing-symmetric completion. By this, we mean that $\mathcal{F}^{(t)}$ is the image of $\mathcal{F}^{(s)}$ under (4.71a). This ensures that (4.72) is crossing symmetric. To fix the correlators, all that is left to do is then to constrain the form of $\mathcal{F}^{(s)}$. As anticipated, this is done by requiring that, in the small-χ expansion, it contains the correct powers of χ with coefficients that correspond to singular terms in the s-channel OPE.

A comment is now in order. We would like to find these correlators purely from the knowledge of the three-point functions (4.68), but this is possible only if, at large N, these are the only contributions to singular terms in the OPE. The OPE between two generators $W^{(k_1)}$ and $W^{(k_2)}$ contains, in general, two kinds of $\mathfrak{sl}(2)$ multiplets: those built on top of another chiral generator, $W^{(p)}$, but also those coming from normal ordered products of generators. At large N, the former have OPE coefficients that scale with $N^{-\frac{3}{2}}$, while the latter generically scale with more negative powers of N starting at N^{-3}. There is, however, an exception: the operator $:W^{(k_1)}\,W^{(k_2)}:$, which appears with an order 1 coefficient. This will contribute to $\mathcal{F}^{(s)}$ if it is a singular term in the four-point function as a whole.

Let us then look at the four-point function $\langle W^{(k_1)}\,W^{(k_2)}\,W^{(k_3)}\,W^{(k_4)}\rangle$. There are two ways one can have a contribution of order N^{-3} from potentially singular terms in the s-channel OPE: either the exchange of a generator $W^{(p)}$, or that of $:W^{(k_3)}\,W^{(k_4)}:$.[14] The latter appears with an OPE coefficient of order N^{-3} in $W^{(k_1)}\times W^{(k_2)}$, and order 1 in $W^{(k_3)}\times W^{(k_4)}$, hence we are allowed to ignore it only if it is a regular term, which happens if $k_1+k_2\leq k_3+k_4$. A similar story applies to the t-channel OPE, which leads to the requirement that $k_2+k_3\leq k_1+k_4$. Hence, we should carry out the computation with these two assumptions, so that we will be able to neglect all the normal ordered products of operators in the OPE. All other orderings can be obtained from this using crossing symmetry.

Under the assumptions above, the chiral algebra generators $W^{(p)}$ and their $\mathfrak{sl}(2)$ descendants are the only singular terms relevant to tree-level in the s- and the t-channel OPEs. Hence, we make an ansatz for $\mathcal{F}^{(s)}$ as

$$\mathcal{F}^{(s)}_{1234}(\chi)=\sum_{n=\max\{|k_{12}|,|k_{34}|\}+2}^{\min\{k_1+k_2,k_3+k_4\}-1}a_n\,\chi^n\,, \tag{4.74}$$

[14] The prefactor chosen for the four-point function is such that

$$\langle W^{(k_1)}(0)W^{(k_2)}(\chi)\,W^{(k_3)}(1)\,W^{(k_4)}(\infty)\rangle=\chi^{-k_1-k_2}\,\mathcal{F}_{1234}(\chi)\,, \tag{4.73}$$

so that $:W^{(k_1)}\,W^{(k_2)}:$ is always a regular term, while $:W^{(k_3)}\,W^{(k_4)}:$ is regular only when $k_1+k_2\leq k_3+k_4$. The crossing relation chosen in (4.71a), instead, is such that $:W^{(k_2)}\,W^{(k_3)}:$ is always a regular term, while $:W^{(k_1)}\,W^{(k_4)}:$ is regular only when $k_2+k_3\leq k_1+k_4$.

for some coefficients a_n to be determined, and we demand that this satisfies the right OPE, namely that

$$\mathcal{F}^{(s)}_{1234}(\chi) = \left[\sum_{\substack{p=\max\{|k_{12}|,|k_{34}|\}+2 \\ \text{step } 2}}^{k_1+k_2-2} C_{k_1k_2p}\, C_{k_3k_4p}\, g_p^{k_{12},k_{34}}(\chi) \right]_{\text{trunc}} , \tag{4.75}$$

where the subscript "trunc" refers to the fact that we should expand the expression enclosed in brackets for small χ, and truncate it at an order corresponding to the highest power of χ that appears in (4.74), namely $k_1 + k_2 - 1$. This allows us to extract the coefficients a_n in a direct way: we simply project the right hand side of (4.75) onto each power of χ appearing in the ansatz (4.74), for instance using the orthogonality relation

$$\oint_{C_0} \frac{d\chi}{2\pi i\, \chi} \chi^{n-m} = \delta_{n,m} \,, \tag{4.76}$$

where C_0 is a contour the encircles $\chi = 0$ in the complex plane. A direct computation then gives the result

$$a_n = \sum_{\substack{p=\max\{|k_{12}|,|k_{34}|\}+2 \\ \text{step } 2}}^{k_1+k_2-2} C_{k_1k_2p} C_{k_3k_4p} \frac{(p-k_{12})_{n-p}\,(p+k_{34})_{n-p}}{(2h)_{n-p}\,(n-p)!} \,. \tag{4.77}$$

As discussed above, this expression is valid only under the assumption that $k_1 + k_2 \leq k_3 + k_4$ and $k_2 + k_3 \leq k_1 + k_4$. Using crossing symmetry one can extend these results to any configuration. The four-point function that follows from (4.74), with these coefficients, can now be recast into the form

$$\left\langle W^{(k_1)}(z_1) W^{(k_2)}(z_2) W^{(k_3)}(z_3) W^{(k_4)}(z_4) \right\rangle = \left(\frac{z_{24}}{z_{14}}\right)^{k_{12}} \left(\frac{z_{14}}{z_{13}}\right)^{k_{34}} \frac{\mathcal{F}_{1234}(\chi)}{z_{12}^{k_1+k_2}\, z_{34}^{k_3+k_4}}$$
$$k_1 + k_2 \leq k_3 + k_4 \quad , \quad k_2 + k_3 \leq k_1 + k_4 \tag{4.78}$$

with

$$\mathcal{F}_{1234}(\chi) = \chi^{k_1+k_2} \left[\sum_{\substack{p=|k_{12}|+2 \\ \text{step } 2}}^{k_1+k_2-2} C_{k_1k_2p} C_{k_3k_4p} \sum_{m=0}^{k_1+k_2-p-1} \frac{(p-k_{12})_m (p+k_{34})_m}{m!(2p)_m \chi^{k_1+k_2-p-m}} \right.$$
$$\left. + \sum_{\substack{p=|k_{23}|+2 \\ \text{step } 2}}^{k_2+k_3-2} C_{k_2k_3p} C_{k_1k_4p} \sum_{m=0}^{k_2+k_3-p-1} \frac{(p+k_{23})_m (p+k_{14})_m}{m!(2p)_m (1-\chi)^{k_2+k_3-p-m}} \right] . \tag{4.79}$$

This will be convenient for the next subsection.

We give a final reminder that, due to the δ_{k_1,k_2} in (4.70), there is one more term that should be added when the weights are pairwise equal. This is $a_0 = 1$, in contrast to the other ones that are proportional to $C_{k_1k_2p}C_{k_3k_4p} = O(N^{-3})$. It is easy to see that this gives the same disconnected correlator that we get after applying the superconformal twist to generalized free theory in six dimensions.

4.3.2 Matching from Mellin space

We now turn our attention to the Mellin amplitudes presented in (4.35) for $\text{AdS}_7 \times S^4$. We wish to derive the same chiral algebra correlators (4.79) by evaluating (4.26) in a twisted configuration.[15] Due to the inequalities in (4.78), we will focus on the ordering $k_2 \leq k_1 \leq k_3 \leq k_4$.[16] Based on this, the Mellin amplitudes quoted in (4.28) are associated with the correlator that comes from switching $(1 \leftrightarrow 2)$ in (4.9).

$$\begin{aligned}\langle \mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)\mathcal{O}_3(x_3,t_3)\mathcal{O}_4(x_4,t_4)\rangle = \left(\frac{t_{34}}{x_{34}^4}\right)^{\frac{1}{2}(k_3+k_4-k_1-k_2)} \left(\frac{t_{14}}{x_{14}^4}\right)^{\frac{1}{2}(k_1+k_4-k_2-k_3)} \\ \left(\frac{t_{13}}{x_{13}^4}\right)^{\frac{1}{2}(k_1+k_2+k_3-k_4)-\mathcal{E}} \left(\frac{t_{24}}{x_{24}^4}\right)^{k_2-\mathcal{E}} \left(\frac{t_{12}t_{34}}{x_{12}^4x_{34}^4}\right)^{\mathcal{E}} \mathcal{G}_{2134}\left(\frac{U}{V},\frac{1}{V};\tau,\sigma\right)\end{aligned} \tag{4.80}$$

On the other hand, we should change to a different prefactor to facilitate direct comparisons to (4.79).

$$\begin{aligned}\langle \mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)\mathcal{O}_3(x_3,t_3)\mathcal{O}_4(x_4,t_4)\rangle = \left(\frac{t_{24}}{x_{24}^4}\right)^{-\frac{1}{2}k_{12}} \left(\frac{t_{14}}{x_{14}^4}\right)^{\frac{1}{2}k_{12}-\frac{1}{2}k_{34}} \left(\frac{t_{13}}{x_{13}^4}\right)^{\frac{1}{2}k_{34}} \\ \left(\frac{t_{12}}{x_{12}^4}\right)^{\frac{1}{2}(k_1+k_2)} \left(\frac{t_{34}}{x_{34}^4}\right)^{\frac{1}{2}(k_3+k_4)} \mathcal{F}_{1234}(U,V;\sigma,\tau)\end{aligned} \tag{4.81}$$

Relating one function to the other and going to the Mellin representation gives

$$\begin{aligned}\mathcal{F}_{1234}(U,V;\sigma,\tau) &= (\sigma U^2)^{\frac{1}{2}(k_1+k_2)-\mathcal{E}}\, \mathcal{G}_{2134}\left(\frac{U}{V},\frac{1}{V};\tau,\sigma\right) \\ &= \int_{-i\infty}^{i\infty} \frac{\mathrm{d}s\mathrm{d}t}{(4\pi i)^2} U^{\frac{s}{2}} V^{-\frac{s}{2}-\frac{t}{2}+k_1+k_4} \sigma^{\frac{1}{2}(k_1+k_2)-\mathcal{E}} \mathcal{M}_{2134}(s,t;\tau,\sigma)\Gamma_{\{k_i\}}.\end{aligned} \tag{4.82}$$

Again, the contour keeps poles such that the exponents on U and V form increasing sequences. In this case, that means the s-channel and the u-channel.

[15] To explain the abbreviated notation, notice that the sums in (4.79) actually do not start until $p = \max\{|k_{12}|,|k_{34}|\}+2$ and $p = \max\{|k_{23}|,|k_{14}|\}+2$. This is because the $C_{k_1k_2k_3}$ are only given by (4.37) when the triangle inequality is satisfied. As stated in (4.69), they are zero otherwise.

[16] We could just as well consider $k_2 \leq k_3 \leq k_1 \leq k_4$ since (4.79) is manifestly symmetric under $(1 \leftrightarrow 3)$.

After we implement the superconformal twist, the integrand of (4.82) will depend on χ'. The key observation we make is that, by meromorphy of the final answer, χ' may be set to any value which makes the evaluation of (4.82) convenient. This does not constitute any additional assumption on the four-point function apart from the statement that the superconformal Ward identity (4.16) holds. We will now take $\chi' \to 0$ which corresponds precisely to the s-channel MRV limit. We should therefore expect the resulting function to decompose into the two pieces of (4.72).

After we do this, the largest pole in s that we need is $s = 2k_1 + 2k_2$. Due to the ordering being considered, this is also the smallest pole in s. It just barely contributes if we set $\tau^{\mathcal{E}-j}\sigma^j \mapsto \chi'^{-2\mathcal{E}}$ in the polynomial and drop all lower degree terms. We now get

$$\begin{aligned}\mathcal{F}_{1234}(\chi) = \frac{1}{2}\Gamma(\kappa_s)\int_{-i\infty}^{i\infty}\frac{\mathrm{d}t}{2\pi i}\chi^{k_1+k_2}(1-\chi)^{k_{42}-\frac{t}{2}}\mathcal{M}_{2134}(2k_1+2k_2,t;\chi'^{-2},\chi'^{-2}) \\ \times\Gamma\left[k_1+k_3-\frac{t}{2}\right]\Gamma\left[k_2+k_4-\frac{t}{2}\right]\Gamma\left[k_{13}+\frac{t}{2}\right]\Gamma\left[k_{24}+\frac{t}{2}\right].\end{aligned} \tag{4.83}$$

Since the function is guaranteed to have the form $\mathcal{F}(\chi) = \mathcal{F}^{(s)}(\chi)+\mathcal{F}^{(t)}(\chi)$, it is enough to compute $\mathcal{F}^{(t)}(\chi)$ by extracting the terms that have a negative power of $1-\chi$. These come from the single-particle poles in u – namely $t = 2k_3+2k_4-2m-2p$. Clearly, $\mathcal{F}^{(s)}(\chi)$ can then be computed using crossing symmetry. Looking at the expression for $\mathcal{R}_{4132}^{p,m;0,j}(2k_3+2k_4-2m-2p,2k_1+2k_2)$,

$$\begin{aligned}\mathcal{F}^{(t)}(\chi) = \chi^{k_1+k_2}\frac{\Gamma(\kappa_s)}{16}\sum_{\substack{p=\max\{|k_{14}|,|k_{23}|\}+2\\ \text{step } 2}}^{k_2+k_3-2} C_{k_1k_4p}C_{k_2k_3p}\sum_{m=0}^{k_2+k_3-p-1}\frac{\pi^{-\frac{1}{2}}2^{2p-2}}{m!(2p)_m(1-\chi)^{k_2+k_3-p-m}} \\ \frac{\Gamma\left[\frac{p+k_{14}}{2}\right]\Gamma\left[\frac{p-k_{14}}{2}\right]\Gamma\left[\frac{p+k_{23}}{2}\right]\Gamma\left[\frac{p-k_{23}}{2}\right]}{\Gamma(p-k_{14})\Gamma(p-k_{23})(p+k_{14})_m^{-1}(p+k_{23})_m^{-1}}\sum_{i=0}^{\mathcal{E}}\mathcal{B}_p^{0,j}K_p^{0,j}\begin{pmatrix}2k_3+2k_4-2m-2p,\\ 2k_1+2k_2\end{pmatrix}\end{aligned} \tag{4.84}$$

where

$$\begin{aligned}&K_p^{0,j}(2k_3+2k_4-2m-2p,2k_1+2k_2) = K'+jK'' \\ &K' \equiv -\frac{1}{2}\kappa_s(\kappa_s-\kappa_u-2m-2p)(2p-\kappa_s-\kappa_u)(2p-2+\kappa_s+\kappa_u) \\ &K'' \equiv 4\kappa_s(\kappa_s-1)(\kappa_s+\kappa_u-2m-2p)+2\kappa_s(2p-\kappa_s-\kappa_u)(2p-2+\kappa_s+\kappa_u).\end{aligned} \tag{4.85}$$

Note that, compared to (4.32), the permutation of the four points causes κ_s to appear in place of κ_u and κ_u to appear in place of κ_t. There are now $O(1)$ and $O(j)$ contributions to the inner sum of

(4.84). For $O(1)$,

$$\begin{aligned}\sum_{j=0}^{\mathcal{E}}\mathcal{B}_p^{0,j}K' &= \frac{\Gamma\left(\frac{3-\kappa_s}{2}\right)\Gamma\left(\frac{p-1-\kappa_u-2\mathcal{E}+k_2+k_3}{2}\right)\Gamma\left(\frac{p-1-2\mathcal{E}+k_2+k_3}{2}\right)}{\Gamma\left(\frac{\kappa_s+2}{2}\right)\Gamma\left(\frac{p+2+\kappa_u+2\mathcal{E}-k_2-k_3}{2}\right)\Gamma\left(\frac{p+2+2\mathcal{E}-k_2-k_3}{2}\right)}\frac{K'}{\pi}\\ &= 2^{\kappa_s-1}\frac{\kappa_s-1}{\Gamma(\kappa_s+1)}\frac{\Gamma\left(\frac{p-1}{2}+\frac{\kappa_s\pm\kappa_u}{4}\right)}{\Gamma\left(\frac{p+2}{2}-\frac{\kappa_s\pm\kappa_u}{4}\right)}\frac{K'}{\sqrt{\pi}}\\ &= 2^{\kappa_s-1}\frac{\kappa_s-1}{\Gamma(\kappa_s+1)}\frac{\Gamma\left(\frac{p-1-k_{14}}{2}\right)\Gamma\left(\frac{p-1-k_{23}}{2}\right)}{\Gamma\left(\frac{p+2+k_{14}}{2}\right)\Gamma\left(\frac{p+2+k_{23}}{2}\right)}\frac{K'}{\sqrt{\pi}}.\end{aligned} \tag{4.86}$$

In the second line, we have used a gamma function identity. We have also used the fact that, even though $\mathcal{E}$ has a piecewise dependence on the weights, it can be traded for κ_u which also does. In the third line, we have used the fact that $\{\kappa_s+\kappa_u,\kappa_s-\kappa_u\}=\{-2k_{14},-2k_{23}\}$ even if we do not know which is which. The analysis for $O(j)$ proceeds similarly with

$$\begin{aligned}\sum_{j=0}^{\mathcal{E}}j\mathcal{B}_p^{0,j}K'' &= \frac{2^{\kappa_s-4}}{\Gamma(\kappa_s+1)}(2p-2+\kappa_s+\kappa_u)(2p-\kappa_s-\kappa_u)\\ &\times\frac{\Gamma\left(\frac{p-1-k_{14}}{2}\right)\Gamma\left(\frac{p-1-k_{23}}{2}\right)}{\Gamma\left(\frac{p+2+k_{14}}{2}\right)\Gamma\left(\frac{p+2+k_{23}}{2}\right)}\frac{K''}{\sqrt{\pi}}.\end{aligned} \tag{4.87}$$

Putting (4.86) and (4.87) back into the original formula,

$$\begin{aligned}\mathcal{F}^{(t)}(\chi) = \chi^{k_1+k_2}\sum_{\substack{p=\max\{|k_{14}|,|k_{23}|\}+2\\ \text{step } 2}}^{k_2+k_3-2}C_{k_1k_4p}C_{k_2k_3p}\sum_{m=0}^{k_2+k_3-p-1}\frac{(p+k_{14})_m(p+k_{23})_m}{m!(2p)_m(1-\chi)^{k_2+k_3-p-m}}\\ \times\frac{\frac{1}{2}(\kappa_s-1)K'+\frac{1}{16}(2p-2+\kappa_s+\kappa_u)(2p-\kappa_s-\kappa_u)K''}{2\kappa_s(p+k_{14})(p+k_{23})(p-k_{14}-1)(p-k_{23}-1)}.\end{aligned} \tag{4.88}$$

Since this is almost the desired expression, we just have to go back to (4.85) to verify that the fraction at the end is indeed 1.

As a check of our result, the inverse Mellin transform can also be arranged so that we compute the dynamical function all at once instead of $\mathcal{F}^{(s)}$ and $\mathcal{F}^{(t)}$ individually. This is explained in appendix A of [1].

4.4 Chiral symmetry from four dimensions

Let us now move to the case of Type IIB supergravity on $AdS_5\times S^5$, which is conjecturally dual to $4d$ $\mathcal{N}=4$ super Yang-Mills (SYM) with gauge algebra $\mathfrak{g}=A_{N-1}$, at large N. The relevant

superconformal algebra in this case is $\mathfrak{psu}(2,2|4)$ which, as discussed in [338], admits a chiral algebra subsector whose currents come from *Schur multiplets* of the $4d$ theory.[17] Furthermore, when $\mathcal{N}>2$ in $d=4$, the chiral algebra contains supercurrents of its own. This leads to a global subalgebra of $\mathfrak{psu}(1,1|2)$ when $\mathcal{N}=4$. The minimal extension of this to a chiral algebra is the "small" $\mathcal{N}=4$ superconformal algebra, which turns out to be the full answer for $N=2$. In more general cases, the proposed chiral algebra for $4d$ $\mathcal{N}=4$ SYM is an $\mathcal{N}=4$ super $\mathcal{W}$-algebra with $\text{rank}(\mathfrak{g})$ generators that are in one-to-one correspondence with the Casimir invariants of $\mathfrak{g}$. As an initial check of this conjecture, the authors of [338] showed that it predicts the correct superconformal index. They also showed that the chiral algebra for a superconformal gauge theory can in principle be constructed from BRST quantization.

More recent results include free-field realizations for $N=2,3,4$ in [359] and a duality between the chiral algebra of $4d$ $\mathcal{N}=4$ SYM at large N and an AdS_3 higher-spin Chern-Simons theory in [354]. The latter echoes the discussion in section 4.3 wherein the $6d$ $\mathcal{N}=(2,0)$ theory is associated with the higher-spin algebra $\mathfrak{hs}[\mu]$. Here we perform a test based on computing the four-point functions between chiral algebra generators at large N, and comparing them with the holographic correlators between half-BPS operators [225, 267] using Mellin space techniques. Again, the symmetric form of the Mellin amplitudes found in [230, 231] will be most convenient for our purposes.

4.4.1 Super $\mathcal{W}$-algebra correlators at large N

As in the case of the $6d$ $\mathcal{N}=(2,0)$ theory, the ring of half-BPS operators is freely generated by the Casimir invariants of the gauge algebra $\mathfrak{g}$. In the case $\mathfrak{g}=A_{N-1}$, the central charge is[18]

$$c_{2d}=-12\,c_{4d}=-3\,(N^2-1)\,. \tag{4.89}$$

The global $\mathfrak{psu}(1,1|2)$ of the full higher-spin algebra has an $\mathfrak{su}(2)$ R-symmetry, and the chiral algebra generators form non-trivial irreducible representations of this algebra. Furthermore, they are superconformal primaries of half-BPS multiplets with respect to the global $\mathfrak{psu}(1,1|2)$. We will denote these currents as $J^{(k)}_{a_1\ldots a_{2j}}$ and contract them with polarization vectors as in (4.19). Their $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$ spins (which must be equal) are given by $h=j=\frac{k}{2}$. Note that for odd k they are fermionic,

[17] Again, the twisted translation (4.66) must be applied to the Schur operators. This time, the maximal subalgebra is $\mathfrak{su}(2)\times\mathfrak{su}(2)\times\mathfrak{u}(1)\subset\mathfrak{su}(4)$ which means that an $\mathfrak{su}(2)$ R-symmetry survives in the chiral algebra. In general there can be Schur operators with non-zero $\mathfrak{u}(1)$ charge but not in the OPE between two scalars [338].

[18] Note that c_{4d} here differs by $c_T^{(4d)}$ given in (1.2) and (4.11) by an overall normalization factor.

having half-integer dimension. In this notation, their three-point functions can be written as

$$\langle J^{(k_1)}(z_1,y_1)\,J^{(k_2)}(z_2,y_2)\,J^{(k_3)}(z_3,y_3)\rangle = \frac{C_{k_1k_2k_3}}{\left(\frac{z_{12}}{y_{12}}\right)^{\frac{k_1+k_2-k_3}{2}}\left(\frac{z_{13}}{y_{13}}\right)^{\frac{k_1+k_3-k_2}{2}}\left(\frac{z_{23}}{y_{23}}\right)^{\frac{k_2+k_3-k_1}{2}}}\,, \tag{4.90}$$

where $C_{k_1k_2k_3}$ to the order we need should be read off from (4.37) (in the $d=4$ case). Once again, the third operator weight satisfies the selection rule (4.69) in terms of the other two, which we restate as

$$k_3 = |k_{12}|+2,\ |k_{12}|+4,\ \dots,k_1+k_2-2\,. \tag{4.91}$$

As anticipated at the end of section 4.1, four-point functions in this chiral algebra must satisfy an additional constraint, namely

$$\left[\alpha^2\frac{\partial}{\partial\alpha}-\frac{\partial}{\partial\chi}\right]\mathcal{F}_{1234}(\chi;\alpha)\Big|_{\alpha=\frac{1}{\chi}}=0 \tag{4.92}$$

in terms of the cross-ratios (4.20). This is the superconformal Ward identity (SCWI) for the global superalgebra $\mathfrak{psu}(1,1|2)$, much like (4.14) for the respective superalgebras. Additionally, it can be seen as arising from a chiral algebra twist as described in appendix A of [289].

Based on the structure of these half-BPS supermultiplets, we now have a way to compute their superconformal blocks – making a suitable ansatz and then fixing any free coefficients by demanding that the SCWI (4.92) is satisfied. Since the $J^{(k)}$ are annihilated by half of the supercharges of a "small" $\mathcal{N}=4$ superconformal algebra, there are 2 directions by which we can descend and therefore 2^2 conformal primaries in their superconformal multiplet. However, when $J^{(k)}$ is Grassman even (odd), it will only be the state obtained by acting with *both* supercharges which is again Grassman even (odd) and therefore admissible in the same OPE. This super descendant, which we will denote by $T^{(k)}_{a_1\dots a_{2j}}$, has the quantum numbers $h=\frac{k+2}{2}$ and $j=\frac{k-2}{2}$. Notice that it is precisely the stress tensor for the case of $k=2$. In the primary and descendant terms of the superconformal block, the cross-ratio χ must appear through the $\mathfrak{sl}(2)$ block (4.23). Due to the "isomorphism" with $\mathfrak{su}(2)$, the cross-ratio α will appear in the same function, continued to negative values of the conformal dimensions. We arrive at

$$\begin{aligned}\mathcal{G}_h^{k_{12},k_{34}}(\chi;\alpha) = \Big[&g_h^{(k_{12}/2,k_{34}/2)}(\chi)\,g_{-h}^{(-k_{12}/2,-k_{34}/2)}(\alpha^{-1})\\ &+\frac{(4h^2-k_{12}^2)\,(4h^2-k_{34}^2)}{64h^2\,(1-4h^2)}\,g_{h+1}^{(k_{12}/2,k_{34}/2)}(\chi)\,g_{1-h}^{(-k_{12}/2,-k_{34}/2)}(\alpha^{-1})\Big]\end{aligned} \tag{4.93}$$

for a $J^{(2h)}$ multiplet exchanged in $\langle J^{(k_1)}J^{(k_2)}J^{(k_3)}J^{(k_4)}\rangle$. Knowing the relative coefficient in (4.93) enables us to write the singular OPE of two generators. Combining (4.21a) for space-time and (4.21b) for R-symmetry, we find

$$\begin{aligned}
J^{(k_1)}(z_1,y_1)J^{(k_2)}(z_2,y_2) = {}& \frac{\delta_{k_1,k_2}}{(z_{12}/y_{12})^{k_1}} + \sum_{\substack{p=|k_{12}|+2\\ \text{step } 2}}^{k_1+k_2-2} C_{k_1k_2p} \sum_{m=0}^{\frac{k_1+k_2-p-2}{2}} \frac{\left(\frac{k_{12}+p}{2}\right)_m}{m!(p)_m} \frac{y_{12}^{\frac{k_1+k_2-p}{2}}}{z_{12}^{\frac{k_1+k_2-p}{2}-m}} \\
& \frac{\partial^m}{k!} \sum_{\sigma\in S_p} J^{(p)}_{a_{\sigma(1)}\dots a_{\sigma(p)}}(z_2)\, y_1^{a_1}\dots y_1^{a_{(p+k_{12})/2}} y_2^{a_{(p+k_{12}+2)/2}}\dots y_2^{a_p} \\
& - \sum_{\substack{p=|k_{12}|+2\\ \text{step } 2}}^{k_1+k_2-2} C_{k_1k_2p} \frac{(p+k_{12})(p-k_{12})}{4p\sqrt{p^2-1}} \sum_{m=0}^{\frac{k_1+k_2-p-4}{2}} \frac{\left(\frac{k_{12}+p+2}{2}\right)_m}{m!(p+2)_m} \frac{y_{12}^{\frac{k_1+k_2-p+2}{2}}}{z_{12}^{\frac{k_1+k_2-p-2}{2}-m}} \\
& \frac{\partial^m}{(p-2)!} \sum_{\sigma\in S_{p-2}} T^{(p)}_{a_{\sigma(1)}\dots a_{\sigma(p-2)}}(z_2)\, y_1^{a_1}\dots y_1^{a_{(p+k_{12}-2)/2}} y_2^{a_{(p+k_{12})/2}}\dots y_2^{a_{p-2}}
\end{aligned} \tag{4.94}$$

plus terms involving normal ordered products that are suppressed by at least $1/N$.

We are now ready to compute the four-point functions in the chiral algebra. Again, it is possible to use (an adaptation of) the holomorphic bootstrap of [368], but instead it turns out to be better to use a similar strategy to the one of section 4.3. That is, we start by making an ansatz

$$\mathcal{F}_{1234}(\chi;\alpha) = \mathcal{F}^{(s)}_{1234}(\chi;\alpha) + \mathcal{F}^{(t)}_{1234}(\chi;\alpha)\,, \tag{4.95}$$

where $\mathcal{F}^{(t)}$ is identified as the completion of $\mathcal{F}^{(s)}$ under crossing symmetry, which is generated by

$$\mathcal{F}_{1234}(\chi;\alpha) = (\alpha\,\chi)^{\frac{k_1+k_2}{2}} \left(\frac{\alpha(1-\chi)}{\alpha-1}\right)^{-\frac{k_2+k_3}{2}} \mathcal{F}_{3214}\left(1-\chi;\frac{\alpha}{\alpha-1}\right) \tag{4.96a}$$

$$\mathcal{F}_{1234}(\chi;\alpha) = (\alpha\,\chi)^{\frac{k_1+k_4}{2}} \mathcal{F}_{4231}\left(\frac{1}{\chi};\frac{1}{\alpha}\right). \tag{4.96b}$$

As in section 4.3, we shall work under the assumption that $k_1+k_2 \le k_3+k_4$ and $k_2+k_3 \le k_1+k_4$, so that we can neglect normal ordered products of operators in the OPE. By using (4.96a) and (4.96b), all other choices for the weights are within reach. If we fix the terms in $\mathcal{F}^{(s)}$ such that each power of χ appears multiple times, once for each power of z_{12} in (4.94), this leads us to a four-point function with the same structure as (4.79):

$$\left\langle J^{(k_1)}(z_1,y_1)J^{(k_2)}(z_2,y_2)J^{(k_3)}(z_3,y_3)J^{(k_4)}(z_4,y_4)\right\rangle = \left(\frac{z_{24}y_{14}}{z_{14}y_{24}}\right)^{\frac{k_{12}}{2}} \left(\frac{z_{14}y_{13}}{z_{13}y_{14}}\right)^{\frac{k_{34}}{2}}$$

$$\left(\frac{y_{12}}{z_{12}}\right)^{\frac{k_1+k_2}{2}}\left(\frac{y_{34}}{z_{34}}\right)^{\frac{k_3+k_4}{2}}\mathcal{F}_{1234}(\chi;\alpha)$$
$$k_1+k_2\leq k_3+k_4\quad,\quad k_2+k_3\leq k_1+k_4 \tag{4.97}$$

where

$$\begin{aligned}\frac{\mathcal{F}_{1234}(\chi;\alpha)}{\chi^{\frac{k_1+k_2}{2}}}=&\sum_{\substack{p=|k_{12}|+2\\ \text{step }2}}^{k_1+k_2-2}C_{k_1k_2p}C_{k_3k_4p}\left[g^{\frac{k_{21}}{2},\frac{k_{43}}{2}}_{-\frac{p}{2}}(\alpha^{-1})\sum_{m=0}^{\frac{k_1+k_2-p-2}{2}}\frac{\left(\frac{k_{21}+p}{2}\right)_m\left(\frac{k_{34}+p}{2}\right)_m}{m!(p)_m\chi^{\frac{k_1+k_2-p}{2}-m}}\right.\\&\left.+\frac{(p^2-k_{12}^2)(p^2-k_{34}^2)}{16p^2(1-p^2)}g^{\frac{k_{21}}{2},\frac{k_{43}}{2}}_{1-\frac{p}{2}}(\alpha^{-1})\sum_{m=0}^{\frac{k_1+k_2-p-4}{2}}\frac{\left(\frac{k_{21}+p+2}{2}\right)_m\left(\frac{k_{34}+p+2}{2}\right)_m}{m!(p+2)_m\chi^{\frac{k_1+k_2-p-2}{2}-m}}\right]\\&+\frac{(\alpha-1)^{\frac{k_2+k_3}{2}}}{\alpha^{\frac{k_{31}}{2}}}\sum_{\substack{p=|k_{23}|+2\\ \text{step }2}}^{k_2+k_3-2}C_{k_1k_4p}C_{k_2k_3p}\left[g^{\frac{k_{23}}{2},\frac{k_{41}}{2}}_{-\frac{p}{2}}(1-\alpha^{-1})\sum_{m=0}^{\frac{k_2+k_3-p-2}{2}}\frac{\left(\frac{k_{23}+p}{2}\right)_m\left(\frac{k_{14}+p}{2}\right)_m}{m!(p)_m(1-\chi)^{\frac{k_2+k_3-p}{2}-m}}\right.\\&\left.+\frac{(p^2-k_{14}^2)(p^2-k_{23}^2)}{16p^2(1-p^2)}g^{\frac{k_{23}}{2},\frac{k_{41}}{2}}_{1-\frac{p}{2}}(1-\alpha^{-1})\sum_{m=0}^{\frac{k_2+k_3-p-4}{2}}\frac{\left(\frac{k_{23}+p+2}{2}\right)_m\left(\frac{k_{14}+p+2}{2}\right)_m}{m!(p+2)_m(1-\chi)^{\frac{k_2+k_3-p-2}{2}-m}}\right].\end{aligned} \tag{4.98}$$

However, computing the coefficient for a given power of χ all at once will lead us to a significant simplification. An ansatz for $\mathcal{F}^{(s)}$ which facilitates this latter approach and takes into account the singular OPE is then

$$\mathcal{F}^{(s)}_{1234}(\chi;\alpha)=\sum_{n=1+\frac{1}{2}\max\{|k_{12}|,|k_{34}|\}}^{\frac{1}{2}(\min\{k_1+k_2,k_3+k_4\}-2)}P_n(\alpha)\,\chi^n\,, \tag{4.99}$$

where $P_n(\alpha)$ are functions of α that we can fix by requiring consistency with the OPE, including only the chiral algebra generators. Namely, we demand that

$$\mathcal{F}^{(s)}_{1234}(\chi;\alpha)=\left[\sum_{\substack{p=\max\{|k_{12}|,|k_{34}|\}+2\\ \text{step }2}}^{k_1+k_2-2}C_{k_1k_2p}\,C_{k_3k_4p}\,\mathcal{G}^{k_{12},k_{34}}_{p/2}(\chi;\alpha)\right]_{\text{trunc}}\,, \tag{4.100}$$

where the subscript "trunc" refers to the fact that we should expand the expression enclosed in brackets for small χ, and truncate it at an order corresponding to the highest power of χ that appears in (4.99), namely $\frac{1}{2}(k_1+k_2-2)$. Projecting (4.100) onto each power of χ, we can extract the

coefficients $P_n(\alpha)$, which turn out to be given by

$$P_n(\alpha) = \sum_{\substack{p=\max\{|k_{12}|,|k_{34}|\}+2 \\ \text{step } 2}}^{k_1+k_2-2} C_{k_1k_2p}\, C_{k_3k_4p}\, \mathcal{I}_{p/2,n}(\alpha)\,, \tag{4.101}$$

where we have introduced

$$\begin{aligned}\mathcal{I}_{h,n}(\alpha) &= \oint_{C_0} \frac{d\chi}{2\pi i\,\chi}\chi^{-n}\,\mathcal{G}_h^{k_{12},k_{34}}(\chi;\alpha) \\ &= \frac{(h-k_{12}/2)_{n-h}\,(h+k_{34}/2)_{n-h}}{(n-h)!(2h)_{n-h}}\Big[g_{-h}^{(-k_{12}/2,-k_{34}/2)}(\alpha^{-1}) \\ &\quad + \frac{(h-n)(2h+k_{12})(2h-k_{34})}{8h(2h-1)(h+n)}\,g_{-h-1}^{(-k_{12}/2,-k_{34}/2)}(\alpha^{-1})\Big]\,.\end{aligned} \tag{4.102}$$

While the sum over $\mathcal{I}_{h,n}(\alpha)$ looks complicated *a priori*, with $P_n(\alpha)$ being in principle arbitrary polynomials of degree n,[19] for the specific values of n that enter the sum (4.101) they turn out to have a particularly simple form, in which the dependence on α is almost completely factorized:

$$P_n(\alpha) = \frac{\sqrt{k_1\,k_2\,k_3\,k_4}}{N^2}\,\alpha^{n-1}\left(\frac{2+k_1-k_2-k_3+k_4}{2} - 2\,n\,(1-\alpha)\right)\,. \tag{4.103}$$

As in section 4.3, using crossing, we can continue the results above to all configurations of the $\{k_i\}$, and therefore this fixes all the four-point functions between chiral generators at tree-level.

4.4.2 Matching from Mellin space

The next step is to show that the same answer in encoded in the Mellin amplitudes (4.35) for $\text{AdS}_5 \times S^5$. Our discussion here will run parallel to that of subsection 4.3.2. As explained there, we will consider weights that satisfy $k_2 \le k_1 \le k_3 \le k_4$. The correlators we derive will initially take the form (4.98) but the simplification apparent in (4.103) will turn out to have an interpretation in Mellin space as well.

Since the convention that will allow the most direct comparison is

$$\begin{aligned}\langle \mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)\mathcal{O}_3(x_3,t_3)\mathcal{O}_4(x_4,t_4)\rangle &= \left(\frac{t_{24}}{x_{24}^2}\right)^{-\frac{1}{2}k_{12}}\left(\frac{t_{14}}{x_{14}^2}\right)^{\frac{1}{2}k_{12}-\frac{1}{2}k_{34}}\left(\frac{t_{13}}{x_{13}^2}\right)^{\frac{1}{2}k_{34}} \\ &\left(\frac{t_{12}}{x_{12}^2}\right)^{\frac{1}{2}(k_1+k_2)}\left(\frac{t_{34}}{x_{34}^2}\right)^{\frac{1}{2}(k_3+k_4)}\mathcal{F}_{1234}(U,V;\sigma,\tau),\end{aligned} \tag{4.104}$$

[19] This can be argued from the relation between the $\mathfrak{sl}(2)$ blocks with negative arguments appearing in (4.102) and the Jacobi polynomials (see (4.24)).

we can solve for the dynamical part as

$$\begin{aligned}\mathcal{F}_{1234}(U,V;\sigma,\tau) &= (\sigma U)^{\frac{1}{2}(k_1+k_2)-\mathcal{E}}\mathcal{G}_{2134}\left(\frac{U}{V},\frac{1}{V};\tau,\sigma\right) \\ &= \int_{-i\infty}^{i\infty}\frac{\mathrm{d}s\mathrm{d}t}{(4\pi i)^2}U^{\frac{s}{2}}V^{-\frac{s}{2}-\frac{t}{2}+\frac{1}{2}(k_1+k_4)}\sigma^{\frac{1}{2}(k_1+k_2)-\mathcal{E}}\mathcal{M}_{2134}(s,t;\tau,\sigma)\Gamma_{\{k_i\}}.\end{aligned} \tag{4.105}$$

Evaluating (4.105) in the twisted configuration will again be based on taking $\chi' \to 0$. Clearly, the freedom to set χ' to a convenient value is a consequence of the superconformal Ward identity. We have been able to prove analytically that it holds for all weights by showing that the Mellin amplitudes in (4.35) can be written in the form

$$\mathcal{M}(s,t;\sigma,\tau) = \hat{R}\circ\widetilde{\mathcal{M}}(s,t;\sigma,\tau) \tag{4.106}$$

where the operator $\hat{R}$ is the Mellin space version of $(1-\chi\alpha)(1-\chi\alpha')(1-\chi'\alpha)(1-\chi'\alpha')$. It should come as no surprise that the auxiliary Mellin amplitude on the right hand side which accomplishes this is the one conjectured in [225, 267] with the normalization from [381].

Based on our chosen ordering, the superconformal twist with $\chi' \to 0$ ensures that $s = k_1 + k_2$ is the only pole that contributes (effectively giving $\tau^{\mathcal{E}-j}\sigma^j \mapsto \chi'^{-\mathcal{E}}\alpha^j(\alpha-1)^{\mathcal{E}-j}$). After this, the u-channel poles of $t = k_3+k_4-2m-p$ that lead to singularities can be extracted to give the second term in $\mathcal{F}(\chi;\alpha) = \mathcal{F}^{(s)}(\chi;\alpha)+\mathcal{F}^{(t)}(\chi;\alpha)$. Continuing, the expression for $\mathcal{R}_{4132}^{p,m;0,j}(k_3+k_4-2m-p,k_1+k_2)$ allows us to write

$$\begin{aligned}\frac{\mathcal{F}^{(t)}(\chi;\alpha)}{(\alpha\chi)^{\frac{k_1+k_2}{2}}} = \frac{\Gamma(\frac{\kappa_s}{2})}{16}\sum_{\substack{p=\max\{|k_{14}|,|k_{23}|\}+2\\ \text{step } 2}}^{k_2+k_3-2} C_{k_1k_4p}C_{k_2k_3p}\sum_{m=0}^{\frac{k_2+k_3-p-2}{2}}\frac{\Gamma[p+m+1]^{-1}}{m!(1-\chi)^{\frac{k_2+k_3-p}{2}-m}} \\ \Gamma\left[\frac{k_{14}+p+2m}{2}\right]\Gamma\left[\frac{k_{23}+p+2m}{2}\right]\sum_{j=0}^{\mathcal{E}}(1-\alpha^{-1})^{\mathcal{E}-j}\mathcal{B}_p^{0,j}K_p^{0,j}\begin{pmatrix}k_3+k_4-2m-p,\\ k_1+k_2\end{pmatrix}\end{aligned} \tag{4.107}$$

in terms of the factors from (4.35). By scrutinizing the gamma functions in (4.50), we can see that the inner sum only goes up to $\mathcal{E}-\frac{k_2+k_3-p}{2}$ instead of $\mathcal{E}$. This allows us to perform the reflection $j \mapsto \mathcal{E}-\frac{k_2+k_3-p}{2}-j$ which yields

$$\sum_{j=0}^{\mathcal{E}}(1-\alpha^{-1})^{\mathcal{E}-j}\mathcal{B}_p^{0,j}K_p^{0,j}\begin{pmatrix}k_3+k_4-2m-p,\\ k_1+k_2\end{pmatrix} = \frac{\Gamma(p)\Gamma\left(\frac{\kappa_s+2}{2}\right)^{-1}}{\Gamma\left(\frac{p+k_{14}+2}{2}\right)\Gamma\left(\frac{p+k_{23}+2}{2}\right)} \tag{4.108}$$

$$\left(\frac{\alpha}{\alpha-1}\right)^{\frac{p-k_2-k_3}{2}}\mathcal{E}^{-\frac{k_2+k_3-p}{2}}\sum_{j=0}^{}(1-\alpha^{-1})^j\frac{\left(\frac{k_{41}-p}{2}\right)_j\left(\frac{k_{32}-p}{2}\right)_j}{j!(1-p)_j}K_p^{0,\mathcal{E}^{-\frac{k_2+k_3-p}{2}-j}}\begin{pmatrix}k_3+k_4-2m-p,\\k_1+k_2\end{pmatrix}.$$

To evaluate (4.108), we need to plug in

$$\begin{aligned}K_p^{0,\mathcal{E}^{-\frac{k_2+k_3-p}{2}-j}}&(k_3+k_4-2m-p,k_1+k_2)\\&=\frac{1}{2}\kappa_s(m+p)[(\kappa_s-2p)^2-\kappa_u^2]+\frac{1}{2}\kappa_s j[\kappa_u^2-\kappa_s^2+4\kappa_s(2m+p)-4p^2]\\&=\kappa_s\left[\frac{2(p-j)(p+m)}{p}(p+k_{14})(p+k_{23})+\frac{2jm}{p}(p-k_{14})(p-k_{23})\right].\end{aligned}\tag{4.109}$$

The fact that it is an even function of κ_u is what allows us to write it in terms of the weights analytically. It is now straightforward to see that our expression for the t-channel singularity becomes

$$\begin{aligned}\frac{\mathcal{F}^{(t)}(\chi;\alpha)}{(\alpha\chi)^{\frac{k_1+k_2}{2}}}=\frac{1}{8}&\sum_{\substack{p=\max\{|k_{14}|,|k_{23}|\}+2\\ \text{step }2}}^{k_2+k_3-2}C_{k_1k_4p}C_{k_2k_3p}\sum_{m=0}^{\frac{k_2+k_3-p-2}{2}}(1-\chi)^{\frac{p+2m-k_2-k_3}{2}}\\&\times\frac{\Gamma(p)\Gamma\left(\frac{k_{14}+2m+p}{2}\right)\Gamma\left(\frac{k_{23}+2m+p}{2}\right)\left(\frac{\alpha}{\alpha-1}\right)^{\frac{p-k_2-k_3}{2}}}{m!\Gamma(p+m+1)\Gamma\left(\frac{k_{14}+2+p}{2}\right)\Gamma\left(\frac{k_{23}+2+p}{2}\right)}\sum_{j=0}^{\infty}\frac{\left(\frac{k_{41}-p}{2}\right)_j\left(\frac{k_{32}-p}{2}\right)_j}{j!(1-p)_j}\left(\frac{\alpha-1}{\alpha}\right)^j\\&\times\left[\frac{2(p-j)(p+m)}{p}(p+k_{14})(p+k_{23})+\frac{2jm}{p}(p-k_{14})(p-k_{23})\right].\end{aligned}\tag{4.110}$$

In the splitting that we have chosen for (4.109), the first term has the property that all factors may be readily absorbed into the gamma functions. This shifts their arguments to exactly the values that we expect for a superconformal primary. From this point of view, it is reassuring that the second term is proportional to m. This allows us to reindex the sum over m so that the $(1-z)^{\frac{p-k_2-k_3}{2}}$ singularity, which cannot come from a super descendant, is manifestly absent. When this is done,

$$\begin{aligned}\frac{\mathcal{F}^{(t)}(\chi;\alpha)}{(\alpha\chi)^{\frac{k_1+k_2}{2}}}=\sum_{\substack{p=|k_{23}|+2\\ \text{step }2}}^{k_2+k_3-2}&C_{k_1k_4p}C_{k_2k_3p}\left[\frac{g_{-\frac{p}{2}}^{\frac{k_{23}}{2},\frac{k_{41}}{2}}(1-\alpha^{-1})}{(1-\alpha^{-1})^{\frac{-k_2-k_3}{2}}}\sum_{m=0}^{\frac{k_2+k_3-p-2}{2}}\frac{\left(\frac{k_{23}+p}{2}\right)_m\left(\frac{k_{14}+p}{2}\right)_m}{m!(p)_m(1-\chi)^{\frac{k_2+k_3-p}{2}-m}}+\right.\\&\left.\frac{(p^2-k_{14}^2)(p^2-k_{23}^2)}{16p^2(1-p^2)}\frac{g_{1-\frac{p}{2}}^{\frac{k_{23}}{2},\frac{k_{41}}{2}}(1-\alpha^{-1})}{(1-\alpha^{-1})^{\frac{-k_2-k_3}{2}}}\sum_{m=0}^{\frac{k_2+k_3-p-4}{2}}\frac{\left(\frac{k_{23}+p+2}{2}\right)_m\left(\frac{k_{14}+p+2}{2}\right)_m}{m!(p+2)_m(1-\chi)^{\frac{k_2+k_3-p-2}{2}-m}}\right]\end{aligned}\tag{4.111}$$

which is nothing but the second part of (4.98).

We have already mentioned in (4.106) that these Mellin amplitudes can be obtained from the action of a difference operator $\hat{R}$ on an auxiliary Mellin amplitude $\widetilde{\mathcal{M}}(s,t;\sigma,\tau)$. Returning to

this point can help us understand the absence of $\alpha^{n-2}, \alpha^{n-3}, \ldots$ terms in (4.103). Consider the poles of the s-channel amplitude delineated by (4.28). Although every value in $(\max\{|k_{12}|, |k_{34}|\} + 2, \ldots, \min\{k_1 + k_2, k_3 + k_4\} - 2)$ appears as a pole of $\mathcal{M}^{(s)}(s,t;\sigma,\tau)$, closer inspection reveals that each one has at most two R-symmetry monomials $\sigma^i\tau^j$ in its residue. This non-trivial fact follows from a key property of the auxiliary amplitude bootstrapped in [225, 267]. Namely that, for a fixed monomial $\sigma^i\tau^j$, it has exactly one pole in each Mandelstam variable. The action of the difference operator is then only able to produce one additional pole in the full result.

We can see how this structure appears by changing the order of the sums in (4.28). Specifically, we should focus on a specific pole $s = s_0$ (taken to be even for simplicity) and then set $m = \frac{s_0 - p}{2}$. Performing the sum over p and keeping u as an independent variable leads us to

$$\begin{aligned}\operatorname*{Res}_{s\to s_0}\mathcal{M}^{i,j}(s,t) = {}& \frac{(-1)^{i+j-\frac{\kappa_t+\kappa_u}{4}}\sqrt{k_1k_2k_3k_4}\,\Gamma\left(\frac{s_0+2}{2}\right)^{-1}}{2i!j!\Gamma\left(\frac{\kappa_u+2+2i}{2}\right)\Gamma\left(\frac{\kappa_t+2+2j}{2}\right)\Gamma\left(\frac{k_1+k_2-s_0}{2}\right)\Gamma\left(\frac{k_3+k_4-s_0}{2}\right)} \\ &\times\left[\frac{K_0^{i,j}(t,u)\Gamma\left(1+i+j-\mathcal{E}+\frac{\Sigma-\kappa_s}{4}\right)\Gamma\left(\frac{s_0}{2}\right)^{-1}}{(4i+4j-4\mathcal{E}-\kappa_s+\Sigma-2s_0)\Gamma\left(1-i-j+\mathcal{E}-\frac{\Sigma-\kappa_s}{4}\right)}\right. \\ &-\frac{32s_0(s_0+2)-16(4i+4j-4\mathcal{E}+\Sigma-\kappa_s)(s_0+4)}{(4i+4j-4\mathcal{E}-\kappa_s+\Sigma-2s_0)^2(4+4i+4j-4\mathcal{E}-\kappa_s+\Sigma-2s_0)^2} \\ &\left.\frac{(t^- - 2j)(u^- - 2i)\Gamma\left(\frac{s_0+2}{2}+i+j-\mathcal{E}+\frac{\Sigma-\kappa_s}{4}\right)\Gamma\left(-\frac{s_0}{2}\right)^{-1}}{\Gamma(s_0)\Gamma\left(\frac{s_0-2}{2}-i-j+\mathcal{E}-\frac{\Sigma-\kappa_s}{4}\right)}\right].\end{aligned} \tag{4.112}$$

We now observe that the first term forces us to have $2s_0 = 4i + 4j - 4\mathcal{E} - \kappa_s + \Sigma$ because $1 - i - j + \mathcal{E} - \frac{\Sigma-\kappa_s}{4}$, which appears in the gamma function, is a non-positive number.[20] By a similar argument, the second term allows us to have either this value or $2s_0 = 4 + 4i + 4j - 4\mathcal{E} - \kappa_s + \Sigma$.

Now that we know $\mathcal{M}^{i,j}(s,t)$ can be written as a sum of only two residues, it is this form of the Mellin amplitude, when subjected to the procedure of this subsection, that produces the polynomial (4.103). Indeed, we can look at all of the single-particle poles in s which can contribute singular terms to $\mathcal{F}^{(s)}$. These have the form $s_0 = p + 2m$ which, as shown above, can only appear with $\tau^j\sigma^{(p+2m+2\mathcal{E}-k_1-k_2)/2-j}$ and $\tau^j\sigma^{(p+2m+2\mathcal{E}-k_1-k_2-2)/2-j}$. If we then perform the superconformal twist with $\chi' \to 1$, we pick out only those monomials with $j = 0$. This causes the integrand of (4.105) to become a linear combination of $\alpha^{\frac{p}{2}+m}$ and $\alpha^{\frac{p}{2}+m-1}$ as required.

[20] The only exception occurs when $i = j = 0$ and all the weights are identical. But then, $K_0^{0,0}(t,u) = 0$ anyway.

Chapter 5

Discussion and outlook

In **part I** of this thesis we have discussed holographic four-point functions between half-BPS operators in a vast array of SCFTs, at first non-trivial order in the large-N (and large λ, in some cases) expansion. In particular, we have discussed how the recently introduced idea of the MRV limit provides a powerful tool to compute all such observables in a unique and systematic way, while also giving insights on their structure. We have also discussed how, in the theories that we examined, half-BPS multiplets are dual to super-gluon and super-graviton multiplets in AdS and CFT correlation functions are naturally interpreted as scattering amplitudes in AdS. Finally, we have examined some interesting hidden features of holographic correlators, either related to the supersymmetry of the associated theories or to properties of gauge theories and gravity that are known to hold in flat space and that we have shown, at least partially, to carry on in AdS. In this chapter we would like to discuss some possible extensions of our work, part of which are currently under investigation by various authors. Recent reviews of the literature on holographic correlators containing a number of interesting open problems can be found in [71, 382, 383].

5.1 New observables

The first type of extension that we would like to discuss consists in introducing new classes of observables in the panorama of holographic correlators. A first interesting extension is the idea of considering higher-point functions of half-BPS operators, dual to n-point scattering amplitudes of gluons and gravitons in AdS. This program, still in its infancy, was started in [266] with the computation of a stress-tensor five-point function at tree level in $AdS_5 \times S^5$ and an analogous result was recently obtained for the flavour current five-point function in $AdS_5 \times S^3$ [384], while higher KK modes and more than five points are currently unexplored territory. There are various reasons for the interest in higher-point functions. First, their more intricate OPE structure allows to access several different channels and novel OPE coefficients between unprotected operators in strongly coupled

theories. Second, as we shall discuss more in detail in the last subsection of this chapter, the hidden properties of gluon and graviton scattering in flat space become particularly interesting at higher points, so that these explorations would allow to perform non-trivial tests of the relations between scattering amplitudes in flat space and in AdS, while also sharpening our understanding of the existing analogies. Finally, the OPE limits of higher-point functions between half-BPS operators allow to access correlators with non-protected operators, a largely unexplored subject that we shall discuss in the next paragraph. A technical obstruction to this program is our currently limited toolkit for the study of higher-point functions in generic CFTs: our knowledge of higher-point conformal blocks is still poor if compared to the case of four points [385–398] and only very recently the adaptation of analytic bootstrap techniques to higher-point correlators has become subject of active investigation [399]. For other discussions of higher-point functions in AdS, without reference to a specific theory, see [400, 401]. See also [402] for an investigation of multi-point correlators in a one-dimensional defect CFT.

Still concerning the possibility of exploring a wider range of holographic observables, the study of correlation functions with external operators preserving less than half of the Poincaré supercharges has only been considered in very few cases but could give several insights on the structure and spectrum of large-N CFTs. Once again, this presents technical difficulties and interesting challenges even just from a kinematical perspective, since superconformal blocks and Ward identities have not been worked out for the vast majority of cases, although some progress has recently been made in, *e.g.*, [403–408]. Despite the difficulties in the study of less-protected observables, these could be instrumental to make progress in the understanding of the spectrum of large-N CFTs, providing fundamental inputs for the solution of mixing problems that could allow to perform high-loop computations. Preliminary explorations in this direction were started in [409], where external quarter-BPS operators in $\mathcal{N} = 4$ SYM were considered. At large N, in their OPE with half-BPS multiplets one finds triple trace operators, which are essential to understand the mixing problem at one loop and to compute higher-loop corrections to graviton four-point functions, an aspect that we will discuss in the next subsection. A similar problem was considered in the context of a one-dimensional supserymmetric defect CFT in [11, 12], where the bootstrap with an external unprotected operator has proven essential to solve an intricate mixing problem, allowing for the computation of a four-point function at three loops.

We would also like to mention that [2] opened the way to the analytic study of holographic SCFTs less than the maximal supersymmetry. On the one hand it would be interesting to consider

even more theories of this kind, possibly to include cases with only $U(1)$ R symmetry where the MRV method, at least in its original incarnation, would not be applicable. On the other hand, even just focusing on the theories analyzed in chapter 3, one has the interesting feature that gravitons and gluons sit in different supermultiplets and therefore one has access to entirely new classes of observables where the interaction between gluons and gravitons is accounted for, or mixed correlators of gluons and gravitons. Related to this point, an interesting question is whether perturbative bootstrap calculations are capable of showing that the gauge theories in AdS that one is studying must *necessarily* be coupled to dynamical gravity (as their holographic description would suggest) or if, instead, one could treat the pure gauge theory results and the interaction with gravity independently from the bootstrap perspective, with their relation only enforced by relating the flavor current and stress tensor central charges (*e.g.* via $c_{\mathcal{T}} \sim c_{\mathcal{J}}^2$ in $d = 4$).

5.2 Higher orders

In this thesis we have focused only on the case of tree-level correlators, which represent the first correction to the (trivial) generalized free theory results and are dual to semi-classical supergravity or gauge theory. Of course a very interesting problem is that of studying subleading corrections in the $1/N$ expansion (and $1/\lambda$ in string theory), which give access to perturbative quantum gravity results in AdS (and in flat space, through a suitable limit [73]) as well as allowing to extract OPE data in strongly interacting CFTs. The study of loop corrections in AdS with bootstrap methods was started in [226] and has been so far particularly successful in $\text{AdS}_5 \times S^5$ [228, 248, 249, 321, 322] even up to two loops [250, 251], with results for the stress-tensor four-point functions in $\text{AdS}_7 \times S^4$ [410] and $\text{AdS}_4 \times S^7$ [80], while more recently loop results for gluon amplitudes [411] and $4d\,\mathcal{N} = 2$ $\mathcal{S}$-fold theories [412] were obtained.

The problem of loops in AdS is also closely related to that of higher-derivative corrections, which in the case of $\text{AdS}_5 \times S^5$ arise as a separate expansion in $1/\lambda$, while for M-theory backgrounds they also come with powers of $1/N$. Higher-derivative terms in quantum effective actions of string and M-theory manifest themselves as contact terms in holographic four-point functions, which tend to appear with free parameters that cannot be fixed by bootstrap considerations alone. This is why recent years have witnessed a lot of progress in the combination of bootstrap methods with results from supersymmetric localization, in the two maximally supersymmetric theories where a Lagrangian description is available: $\mathcal{N} = 4$ SYM [413–423] and the ABJM theory [76–78, 424]. Another useful

aspect is the existence of protected sectors such as those studied in chapter 4, where the computation of perturbative corrections to tree-level results is easier than in the full theory but still provides interesting constraints that can be used on correlation functions.

A particularly interesting application of these results is the study of the quantum effective actions of string and M-theory, where terms protected by supersymmetry have already been computed and a goal for the bootstrap is to fix the coefficient of the first non-protected higher-derivative correction, containing sixteen derivatives. This is particularly interesting for M-theory, where our understanding is particularly poor beyond the $D = 11$ supergravity approximation, and the program of bootstrapping correlators to high order provides one of our few hopes to access quantum corrections in the theory. On the other hand, in the string theory case a microscopic definition is available and a particularly ambitious goal in this case would be that to compute the analogue of the Shapiro-Virasoro amplitude (*i.e.*, the stress-tensor four-point function) in $\text{AdS}_5 \times S^5$, which would provide a remarkable achievement also from the point of solving $\mathcal{N} = 4$ SYM (for progress in this direction, see [425, 426]). For a summary of results and goals in these directions, see the recent review [382].

In terms of what general lessons one can learn about quantum gravity from the study of loops in holographic correlators, this is still unclear and it would be very interesting to have a sharp answer to this question, besides the idea of computing quantum effective actions of string or M-theory. However, we would like to stress one point where holographic correlators can help to answer fundamental questions about the AdS/CFT correspondence and therefore, in a sense, about quantum gravity. As it is well-known, typically long double-trace operators appear in degenerate families in the generalized free theory at infinite N, so that from a single tree-level correlator one cannot extract the exact anomalous dimension of each operator but rather its average $\langle \gamma^{(1)}_{n,\ell} \rangle$, weighted by OPE coefficients, where n is the twist (dimension minus spin) of the double trace operators and ℓ their spin. A careful analysis of this mixing problem allows in some cases to compute $\langle (\gamma^{(1)}_{n,\ell})^2 \rangle$ and, interestingly, it was first observed in [229] that the different behavior of $\langle \gamma^{(1)}_{n,\ell} \rangle^2$ and $\langle (\gamma^{(1)}_{n,\ell})^2 \rangle$ is related (in a precise way) to the number of internal dimensions in the supergravity description. In [427] this observation was turned into a number of comments and conjectures regarding the interesting problem of scale separation in AdS, namely whether there are examples of AdS/CFT where the internal space has a small characteristic length scale compared to the AdS radius, which is a debated question in the literature on swampland conjectures [428].

5.3 Hidden structures

A third natural direction to extend the results of this thesis is that of continuing our exploration of the hidden structures of AdS scattering amplitudes. In addition to what we have already observed, one would certainly like to sharpen the connection with flat-space amplitudes and their properties. We have already stressed how the MRV limit of [230, 231] is inspired by the famous MHV limit of gluon and graviton amplitudes: a very concrete ambition in the study of higher-point functions would be that of generalizing the MRV limit to any number of external insertions to find an AdS analogue of the Parke-Taylor formula [270]. Besides the aesthetic appeal of such a result, it could be instrumental to a derivation of closed-form expressions for n-point functions of gluons and gravitons in AdS, opening the way to a study of their properties. In particular, tree-level amplitudes for certain theories in flat space are known to satisfy recursion relations relating n-point amplitudes to three-point interactions [271, 272] and can be expressed uniformly for any number of external points, in a representation that manifests connections between gravity and gauge theories as well as relations with ambitwistor string theory [429–431]. The generalization of all this to AdS has been already partially explored (at least in the case of recursion relations) [432–435], but no results are available for full-fledged holographic setups like those considered in this thesis. Moreover, it is natural to expect such properties to be apparent in Mellin space and therefore an extension to higher-point functions of the bootstrap methods discussed here seems the most natural framework where to explore these structures in AdS.

Another interesting feature of scattering amplitudes in flat space is that they reveal hidden connections between gravity and gauge theories, as first observed in [436] in the study of string amplitudes and then generalized to field theories starting with [291, 292]. These results, now known as color-kinematics duality and the double copy relations, are instrumental for high-loop computations in gauge theories and gravity, see [83, 84] for recent reviews. In [2] we have observed a color-kinematics duality structure in gluon-four point functions, which was then sharpened and related to graviton four-point functions in $AdS_5 \times S^5$ in [341], where a very precise double-copy structure in AdS was found. The relation is particularly clean in four dimensions (from the CFT perspective), thanks to the simplicity of the associated *reduced correlators* (namely the solution to the superconformal Ward identities), while the realization of the AdS double copy relations in other dimensions is still unclear. Moreover, the study of higher-point functions would be once again particularly interesting in order to sharpen the connection with the usual double-copy relations in flat space. We

would like to stress that understanding the relations between gauge theories and gravity in AdS serves both a technical purpose, in that it would allow to obtain gravity results from simpler ones in gauge theories, as well as providing a non-trivial test of the double copy relations beyond the flat space regime, possibly providing new insights on their origin.

Finally, in the study of holographic correlators some structures have been highlighted that do not have direct analogues in flat space scattering, in particular hidden conformal symmetry and Parisi-Sourlas supersymmetry. It is natural to ask whether these are still present once loops and higher derivative corrections are taken into account, or whether they have analogues for higher-point functions. So far as hidden conformal symmetry is concerned, it has been shown that although it is broken by loops and higher-derivative corrections, it can still be employed to build a particularly simple ansatz at higher orders in $\text{AdS}_5 \times S^5$, which greatly simplifies the bootstrap program with respect to other backgrounds [321, 322, 437]. It would be interesting to explore analogous structures for $\text{AdS}_5 \times S^3$ [438].

Part II

A bestiary of spindles

Chapter 6

RG flow across dimensions

In the second part of this thesis we shall consider a different, in a sense complementary, approach to AdS/CFT. We shall work mostly in supergravity, where following the introduction of the holographic correspondence a lot of activity has been devoted to the study of solutions containing AdS factors. While the expectation is that CFT duals should exist for most, or all, such solutions, it is not always clear what the dual should be. In this approach one can then unveil novel and interesting strongly interacting CFTs by working with classical solutions in supergravity, which is a much more studied and controlled field. Of particular interest in what follows will be solutions interpolating between AdS spaces of different dimensions, dual to the RG flow between CFTs in similarly different dimensions. In particular, the spacetime is taken to interpolate (roughly speaking) between AdS_{d+1} and $\text{AdS}_{d+1-n} \times M_n$, where M_n is a compact manifold of dimension n: such configuration is dual to the compactification of a CFT_d on M_n, which in the infra-red (IR) gives rise to a CFT_{d-n}. This phenomenon, known as the RG flow across dimensions, has the twofold advantage of providing a (geometrical) description of otherwise hardly accessible theories, as well as that of connecting AdS/CFT to the interesting problem of black hole microstates and their counting, since the CFT provides a natural way to account for the quantum states of black holes.

We shall focus in particular on the case where $n = 2$ with magnetic flux across M_2, where typically[1] one takes the compact space to be a constant-curvature Riemann surface $M_2 = \Sigma_{\mathsf{g}}$[2] and supersymmetry is preserved in the compactification with a topological twist [441]. Starting with [3] a new class of supergravity solutions was introduced, where the compact manifold M_2 is a *spindle* $\mathbb{\Sigma} \equiv \mathbb{WCP}^1_{[n_1,n_2]}$, which is a sphere with conical singularities at the two poles (specified by $n_{1,2} \in \mathbb{N}$), and supersymmetry is *not* realized with a topological twist. We will discuss solutions with AdS_2 [4, 6, 7], AdS_3 [3, 7] and AdS_5 [5, 7] factors.

[1] See, *e.g.*, [140, 148, 439] for some examples.

[2] An exception is the case of Riemann surfaces with punctures [119, 440], but spindle solutions are qualitatively different from those, as we shall discuss.

6.1 Consistent truncations

The classification of AdS solutions in $D = 10/11$ supergravities described in section 1.4 is a useful tool to understand the possible geometries of internal spaces and can give invaluable intuition on the dual field theories. However, the canonical form of the metric that arises from such classification results is not always the most straightforward to describe explicit solutions. Moreover, solving the equations of motion directly in $D = 10/11$ can be a daunting task. For these reasons it is often convenient to consider special subsectors of the full $D = 10/11$ supergravities, where the equations of motion can be solved by studying lower-dimensional theories. This is possible when so-called *consistent truncations* are available: given a certain vacuum of the higher-dimensional theory of the form $X \times Y$, with Y a compact manifold, one can perform a KK reduction on Y which naturally gives rise to a finite number of "light" fields and infinite towers of "heavy" ones[3]. In certain cases it is consistent, at the level of equations of motion, to set to zero all the heavy fields, in the sense that such operation does not imply any constraint for the light fields. One then obtains a supergravity theory in lower dimensions where, by construction, X is a vacuum solution. We will be interested in the case where $X = \text{AdS}_{d+1}$, which gives rise to *gauged* [4] supergravities in $d + 1$ dimensions. A general overview of consistent truncations, with specific focus on the case where Y is a sphere, can be found in [442] (see also [443] for the case where Y is a general SE manifold), but many more examples are known and we shall not attempt a review here.

The type of consistent truncations that will be relevant for us is that in which the $X \times Y$ vacuum of interest is one of the three maximally supersymmetric backgrounds discussed in section 1.2. From the point of view of the AdS/CFT correspondence, solutions to consistent truncations on these backgrounds are particularly interesting because they allow us to naturally identify the dual field theories as related to the world-volume theories of M2-, D3- and M5-branes. These consistent truncations all give rise to gauged supergravities in dimensions $d = 4, 5, 7$ with maximal supersymmetry and gauge group related to the isometry group of the sphere Y: a summary can be found in table 6.1. In chapter 7 it will be particularly useful to consider further truncations where one only has abelian gauge groups, obtaining the theories listed in the third column of table 6.1, which were first discussed in [454]. The basic idea is that solutions are most easily found in the case where the gauge

[3] In all known examples of top-down controllable AdS vacua there is no scale separation between the AdS radius and the length scale of the internal space, so the notion of "light" is actually subtle. One could instead say "massless modes" or "zero modes", but then again this would only apply to certain modes in the supermultiplet, much like the supergravity multiplet in $\text{AdS}_5 \times S^5$.

[4] This means that the gravitini are charged under (some of) the gauge fields of the theory. Moreover, the radius of the AdS vacuum solution is specified by the gauge coupling.

Gauged supergravity	$D = 10/11$ origin	Abelian truncation
$D = 4, \mathcal{N} = 8, SO(8)$ [444, 445]	$\text{AdS}_4 \times S^7$ [446, 447]	$D = 4, \mathcal{N} = 2, U(1)^4$
$D = 5, \mathcal{N} = 8, SO(6)$ [448]	$\text{AdS}_5 \times S^5$ [449, 450]	$D = 5, \mathcal{N} = 2, U(1)^3$
$D = 7, \mathcal{N} = 4, SO(5)$ [451, 452]	$\text{AdS}_7 \times S^4$ [453]	$D = 7, \mathcal{N} = 2, U(1)^2$

Table 6.1: Maximally supersymmetric gauged supergravities arising from consistent truncations of type IIB supergravity in $D = 10$ ($\text{AdS}_5 \times S^5$) and $D = 11$ supergravity ($\text{AdS}_4 \times S^7$ and $\text{AdS}_4 \times S^5$). In the first column we have specified the corresponding gauge group, while in the third we have stressed the possibility of further truncating the theories by only retaining the gauge fields belonging to the Cartan generators of the original non-abelian gauge groups.

groups are abelian, so one can restrict to the sector of gauge fields along the Cartan generators of the gauge groups appearing in the first column of table 6.1, suitably restricting the other fields in such a way as to preserve some supersymmetry. While this procedure certainly breaks some supercharges, the resulting theories are still supersymmetric and describe interesting gauged supergravities with abelian gauge groups. Their bosonic sector includes a graviton, a certain number of abelian gauge fields and some scalar fields. While we shall always consider bosonic solutions where all fermions are set to zero, it is important to stress that the gravitini are charged under a specific linear combination of the abelian gauge fields of the theories just described, which is commonly referred to as the R-symmetry gauge field and is dual to the R-symmetry current in the CFT. Further truncations are also possible, for instance to minimal gauged supergravities, where only the gravitational supermultiplet is present: it is interesting to observe that such minimal theories in four and five dimensions can be uplifted to $D = 11$ and type IIB supergravity, respectively, on any SE manifold of the correct dimension [443].

6.2 RG flows in gauged supergravity

Among the various generalizations of the AdS/CFT correspondence which are related to the existence of various solutions containing AdS factors as detailed in the previous section, a particularly interesting one is the possibility of describing holographically RG flows between CFTs defined in spacetimes of different dimensions. We are particularly interested in the case of SCFTs, where one would like to preserve some supersymmetry in the RG flow. From the field theory side, the idea is that given a certain SCFT defined on $\mathbb{R}^d$, it can be placed on a curved background $\mathbb{R}^{d-n} \times M_n$, with M_n an n-dimensional compact and orientable manifold. A systematic way to preserve supersymmetry on a curved background was introduced in [455] and consists in coupling the SCFT to a supergravity background. However, for several years in the study of RG flows across dimensions

the dominant paradigm has been that of Witten's (partial) topological twist [441], based on the idea of turning on a background gauge field for the R symmetry of the SCFT[5], tuned in such a way as to (partially) cancel the spin connection on M_n. This ensures the existence of covariantly constant spinors, that can be used to define supersymmetry generators for the IR SCFT defined on $\mathbb{R}^{d-n}$. The background fields that are turned on to preserve supersymmetry couple to certain local operators in the ultra-violet (UV) SCFT in $\mathbb{R}^d$, thus triggering an RG flow across different dimensions, as at low enough energies one can neglect the size of M_n and the resulting IR SCFT is defined on $\mathbb{R}^{d-n}$.

While charting the space of CFTs is of the uttermost importance to understand more general QFTs, and despite the enhanced symmetries given by conformal invariance and supersymmetry, it is not at all trivial to find and characterize all possible SCFTs. Significant progress was made in six dimensions [456–459] and a lot is now known about five-dimensional SCFTs [460–465], but in four and lower dimensions there is a much wider zoo and a full classification is hard to achieve[6]. It is therefore extremely interesting to find novel ways of constructing interacting SCFTs, and a tool that has proven particularly successful is the compactification of higher-dimensional SCFTs. The most well-studied and understood case is the compactification of six-dimensional SCFTs with $\mathcal{N} = (2, 0)$ supersymmetry to four dimensions [119, 467], defining the so-called theories of "class $\mathcal{S}$", and to three dimensions [468], leading to "class $\mathcal{R}$" theories [468] – see [469, 470] for reviews. Such setups also often lead to interesting dualities between the lower-dimensional CFTs and (topological) theories defined on the compactification space – see [378, 471–473] for some well-known examples.

For these reasons, it is interesting to explore more compactifications of known SCFTs and an interesting way to do so is, when the theories admit a holographic description, to use AdS/CFT and therefore describe the compactification with supergravity. The holographic dual of this phenomenon was first introduced in [113] and consists of branes of string or M-theory wrapping the compact manifold M_n. The first studies of branes wrapping compact manifolds date back to [474, 475], where it was already realized that, to preserve supersymmetry, such branes should wrap calibrated cycles in special holonomy manifolds, see [476] for a detailed review and references. This can be understood in the probe-brane approximation, in which one treats a single brane as a light object moving in a fixed geometry (including the special holonomy manifold), neglecting the gravitational

[5]We will only consider SCFTs with a continuous R symmetry. For SCFTs in one dimension this restricts the focus to theories with at least $\mathcal{N} = 2$ supersymmetry, while for SCFTs in two dimensions we consider theories with at least $\mathcal{N} = (0, 2)$.

[6]In four dimensions it is believed that $\mathcal{N} = 4$ SYM is the only SCFT with maximal superconformal symmetry, see [345, 349] for a bootstrap approach to the problem. For recent progress in the classification of $\mathcal{N} = 2$ SCFTs, see [466] and references therein.

back-reaction. However, if one wants to use supergravity to describe a certain CFT holographically, it is necessary to use a large number of branes for supergravity to provide a reliable approximation to the full string or M-theory. When many branes are present, though, one can no longer trust the probe approximation and is forced to consider the full back-reacted geometry. The idea of [113] was then to focus on the near-horizon region of the brane: when the brane is in flat space one has a rigid AdS geometry, while having the brane wrap a cycle in a special holonomy manifold corresponds to having a metric that is only locally AdS, with the deviation from AdS being dual to the operators that trigger the RG flow in the UV CFT. Moreover, another input from [113] is that of looking for such solutions in consistent truncations of $D = 10/11$ supergravity and only then uplift such solutions on suitable internal spaces.

One should then look for solutions in gauged supergravities describing objects with an asymptotically locally AdS_{d+1} conformal boundary, with a "radial" direction r that interpolates between such boundary and a near-horizon region of the form $\text{AdS}_{d+1-n} \times M_n$. These are nothing but asymptotically AdS black holes (when $n = d - 1$) and black branes (their higher-dimensional generalizations) in gauged supergravity where r, which can be thought of as a generalization of the Schwarzschild radial coordinate, parametrizes the distance from the black brane, while the $\text{AdS}_{d+1-n} \times M_n$ geometry arises near the horizon of the black brane when the latter is extremal. The graviphoton field of the gauged supergravity then provides the background gauge field that couples to the spinors of the field theory, cancelling the spin connection and realizing the topological twist. When the solution is uplifted to ten or eleven dimensions on some compact manifold Y (a sphere, in the simple examples considered in table 6.1), the supergravity solution at the conformal boundary of the black brane is dual to a perturbation of the UV SCFT (corresponding to the presence of a curved background), the coordinate r parametrizes the RG flow and the solution in the near-horizon limit describes the resulting SCFT in the IR. The picture that we are describing is represented schematically in figure 6.1. As discussed in the previous section, the consistent truncations described in table 6.1 do not exhaust all the possibilities and given a certain solution of gauged supergravity, different uplifts may be possible, leading to flows between different SCFTs. Moreover, as usual in AdS/CFT, the behaviour of the supergravity fields at the boundary gives background fields that couple to the dual operators in the CFT. Therefore, when multiple gauge fields are turned on in supergravity, one is not only coupling the CFT to a background R-symmetry gauge field implementing the topological twist, but also to backgrounds for the flavour symmetries of the theory. The R-symmetry gauge field, however, is universally present in gauged supergravities as it sits in the same super-multiplet as the graviton,

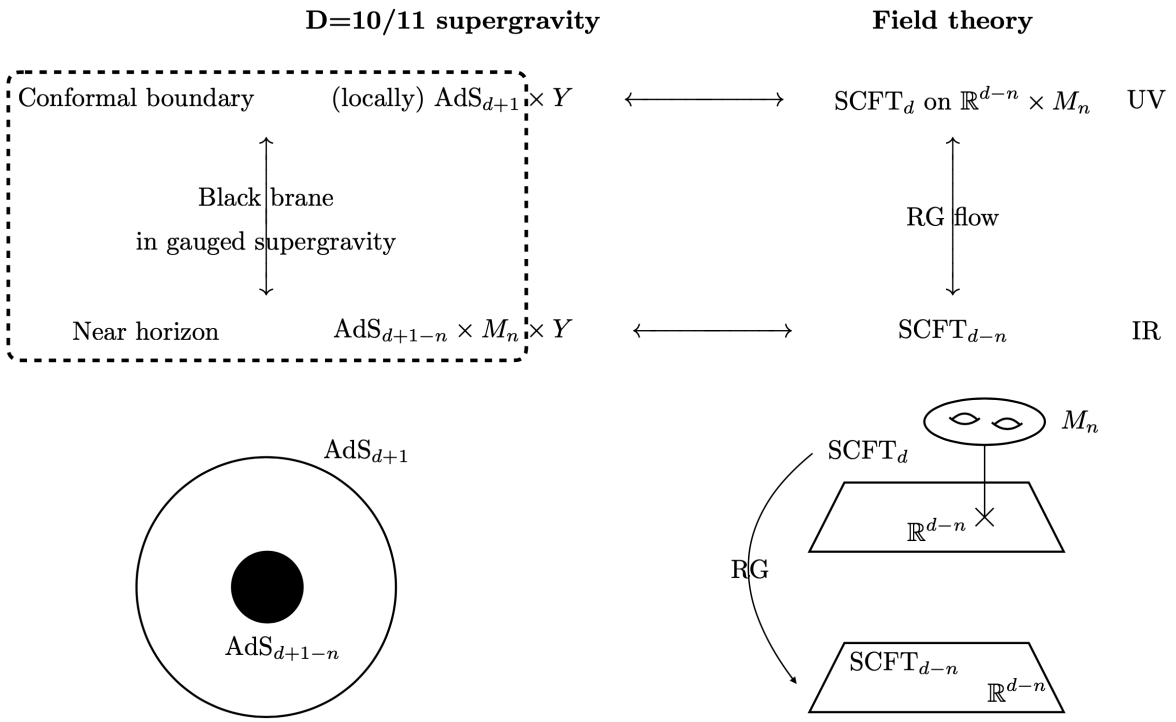


Figure 6.1: The RG flow across dimensions and its holographic interpretation in AdS/CFT. Here Y is a compact manifold used to uplift a solution of gauged supergravity to $D = 10/11$ supergravity.

a situation which is mirrored in the CFT by the fact that the R-symmetry current sits in the same multiplet as the stress tensor. One can therefore limit to consider *minimal* gauged supergravities, where only the R-symmetry background is turned on: these will trigger *universal* RG flows, whose features are common to all SCFTs in a given dimension and with a certain amount of supersymmetry, independently of the details. Such universal RG flows were discussed in [122] for SCFTs in various dimensions, alongside the corresponding supergravity solutions.

It is important to remark that solutions describing the *full flow* from AdS_{d+1} to AdS_{d+1-n} are quite rare (see, *e.g.*, [140, 141, 477] for examples) and in most cases they are only known numerically [148, 439, 478]. On the other hand, it is simpler to obtain solutions that describe the region near the horizon of the black brane, which is what we will consider from now on. We will also limit to consider the case $n = 2$, where the CFTs are compactified on two-dimensional spaces. In this context, if one focuses on the gauged supergravities given in table 6.1, supergravity solutions describing the near-horizon region can be found in [140, 479] for $\mathrm{AdS}_2 \times \Sigma_\mathrm{g}$, in [148] for $\mathrm{AdS}_3 \times \Sigma_\mathrm{g}$ and in [439] for $\mathrm{AdS}_5 \times \Sigma_\mathrm{g}$. A review of these solutions in the case $\mathrm{g} = 0$ (spherical horizon) can be found in appendix A of [7]. Notice that in these references Σ_g is always equipped with a constant curvature metric, in agreement with the idea of *holographic uniformization* proposed in [480]: given any metric

for Σ_g in the UV, the RG flow "washes out" the details of the metric moduli, leading to a rigid metric in the IR.

6.3 Extremization principles and microstates counting

While in certain cases more is known about the SCFTs that arise in the IR after the compactification, it is in general non trivial to compute physical observables in such theories. One therefore often limits to computing the simplest quantities for which a match between the field theory and the supergravity result is available. In most cases, such observables are central charges in even dimensions or suitable analogous thereof in odd dimensions. This also ties in with the problem of determining the R symmetry of an SCFT, as we shall discuss. To illustrate the setup, it is convenient to start from the case of four-dimensional CFTs, whose stress tensor is known to have a Weyl anomaly determined by two central charges, a and c (see, *e.g.*, [481]). It was conjectured in [482] that the a central charge describes, loosely speaking, the number of degrees of freedom of a $4d$ CFT and should therefore be decreasing along RG flows: this was proved in [483] and named "a-theorem", see also [484]. For $4d$ $\mathcal{N}=1$ SCFTs, there is a connection between the central charge and the correct R symmetry: if a theory admits flavour symmetries with abelian factors, it is not a priori clear which of these contributes to giving the exact $U(1)_R$ symmetry of the SCFT. Since the R-symmetry current sits in the same protected supermultiplet as the stress tensor, as discussed in [485, 486], there is a precise relation between the 't Hooft anomalies of the R symmetry and the central charges, which allowed to prove in [98] that, out of all possible mixtures of R symmetry and abelian flavour symmetries, the correct $U(1)_R$ symmetry of a $4d$ $\mathcal{N}=1$ SCFT is the one that maximizes the a central charge. For holographic theories the geometric dual of this statement in supergravity was explored in [99, 487]. The situation is very similar in two dimensions, where one only has one central charge, called c. A "c-theorem" was already found in [488], while in [147, 148] it was found that for $2d$ $\mathcal{N}=(0,2)$ SCFTs the correct $U(1)_R$ symmetry is the one that extremizes c [147, 148]. A geometric dual of this statement for holographic theories was discussed in [489–491].

In odd dimensions there is no Weyl anomaly and therefore no central charges. However, a quantity that behaves in a similar way was introduced in [492, 493], where it was noticed that the partition function of a $3d$ SCFT on the three-sphere, $\mathcal{F}_{S^3}$, is maximized by the exact superconformal R symmetry. This was termed $\mathcal{F}$ maximization, in analogy with the procedures described above, and an $\mathcal{F}$-theorem was proposed in [494, 495]. Some applications in holography can be found in [496, 497]

– see also [498] for a review. Finally, a particularly interesting case from the point of view of supergravity and holography is that of one-dimensional CFTs, which naturally arise as the duals to the near-horizon geometry of extremal black holes in four dimensions. A quantity that counts the number of degrees of freedom and decreases along RG flows is the "g-function" introduced in [499, 500] and recently revisited in [501], but we are going to be interested in something different that is connected to the physics of black holes. In particular, extremal black holes in AdS_4 realize holographically an RG flow across dimensions between a CFT_3 and a CFT_1 and an interesting question is what quantity in the field theory can reproduce the entropy of such black holes. In the supersymmetric case the answer was found in [149, 502, 503], where the *topologically twisted index* $\mathcal{I}$ was introduced. This can be seen as a twisted partition function for the $3d$ CFT on $S^2 \times S^1$, or as a sort of (twisted) Witten index for the $1d$ CFT and extremizing it over certain chemical potential for the flavor symmetries gives the correct R symmetry of the theory as well as reproducing the entropy of the dual black hole at large N with a CFT counting of the associated microstates. This has started a great activity in the attempt of reproducing the entropy of more and more black holes in AdS_4, see [142, 504–507] for some examples and [153] for a review. A geometric dual of $\mathcal{I}$ extremization was discussed in [508, 509]. In dimensions five and six the R symmetry is always non abelian, therefore similar extremization procedures are not present, but for analogues of the a- and c-theorems in other dimensions see, *e.g.*, [510–515]

In the following chapter we will be interested in the case of $1d$ $\mathcal{N} = 2$, $2d$ $\mathcal{N} = (0,2)$ and $4d$ $\mathcal{N} = 1$ SCFTs that arise at the IR fixed point of an RG flow starting from an SCFT living in two dimensions higher in the UV. Actually, the $1d$ and $2d$ theories have no intrinsic or microscopic formulation and the only evidence for their existence arises from the holographic setup described in this section. Computing the topologically twisted index requires a localization computation, for which in principle all the steps are already laid out but which is technically involved. So far as field theory computations are involved we shall then focus on the two- and four-dimensional case, where the procedures of c- and a-extremization can be used to reproduce the R symmetry and central charge of the theory in combination with the suitable use of anomaly polynomials. In particular, we will obtain the anomaly polynomials of lower-dimensional theories by integrating those of their parent theories in the UV over the compactification manifold M_2, in the spirit of [516–519] and, in particular, [520]. On the other hand, these quantities are easily computed in gravity since they are, up to an overall constant, the inverse Newton constant in $d+1$ dimensions for an AdS_{d+1} solution dual to a CFT_d [521, 522].

6.4 Spindles

The remaining chapters of this part will be based on [3–7] , where a special case of RG flow across dimensions has been introduced and investigated for the first time. Specifically, we will consider certain solutions to the gauged supergravities described in the third column of table 6.1 of the form AdS$\times\mathbb{\Sigma}$, where $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_1,n_2]}$ is topologically a weighted projective space, also known as a *spindle*: a two-sphere with conical singularities at the poles, parametrized by the integers $n_{1,2} \in \mathbb{N}$. When uplifted to ten or eleven dimensions, such solutions are conjectured to describe the IR fixed point of M2-, D3- or M5-branes wrapping $\mathbb{\Sigma}$ and should be therefore dual to the compactification of certain SCFTs on $\mathbb{\Sigma}$. Moreover, as we shall discuss, the uplift is such that the conical singularities of the gauged supergravity solution are absent in $D = 10/11$ dimensions or, as it is the case for $\text{AdS}_5 \times \mathbb{\Sigma}$ solutions [5], localized at special points of the internal space. In the general case we only have control on the IR fixed point, which in gauged supergravity is the region near the horizon of an extremal black brane. Solutions describing the full black brane, or "full flows", are only known in the case of $D = 4$ gauged supergravity and they describe accelerating black holes in AdS_4 [4, 6]. We conjecture that in general spindle solutions should be related to accelerating black branes in AdS, but no such solution is known explicitly to date. After they were introduced in [3, 4], spindles were considered by various authors [523–529], whose work we will briefly review in section 7.3.

Spindle solutions have several interesting features that represent technical and conceptual novelties compared to the pre-existing literature on supergravity solutions describing RG flows across dimensions. Manifolds with singularities were famously considered in the context of class $\mathcal{S}$ theories [467], where $6d$ $\mathcal{N} = (2, 0)$ SCFTs compactified on punctured Riemann surfaces, but in that case a description of the metric in supergravity is only given near the punctures [119] and the metric of the Riemann surface is not given explicitly. More solutions with punctured Riemann surfaces were considered in [440], but only constant curvature metrics were allowed. In the case of spindles, on the other hand, one has an explicit metric on a singular space, which is not of constant curvature and describes the orbifold globally. Perhaps the main conceptual novelty introduced by spindle solutions is that, although they are purely magnetically charged[7] and therefore one might expect a typical realization of supersymmetry through a topological twist, this turns out not to be the case. In particular, as we shall discuss in detail in chapter 7, under mild assumptions there are two possibilities

[7] Generalizations to include electric charges and rotation are known in four dimensions [4, 6], but the R-symmetry flux is independent of those charges so that the observations on the type of twist are identical for those cases.

for the R-symmetry magnetic flux across a spindle, which we call the *twist* and the *anti-twist*:

$$\begin{aligned} \text{Twist:} \qquad & \frac{1}{2\pi}\int_\Sigma F = \chi \equiv \frac{n_1+n_2}{n_1 n_2}\,, \\ \text{Anti-twist:} \qquad & \frac{1}{2\pi}\int_\Sigma F = \chi_- \equiv \frac{n_2-n_1}{n_1 n_2}\,, \end{aligned} \tag{6.1}$$

where F is the field strength for the R-symmetry gauge field. Notice in the case of the twist χ is in fact the Euler number of the spindle, so this is a feature that is also shared with the standard topological twist. However, the spin connection on the spindle is not the same as the gauge field (not even up to gauge transformations) so that the conditions for a topological twist are only met globally, but not locally. On the other hand, the anti-twist case is distinguished from the topological twist also at the global level and also less understood.

To conclude, it is also worth mentioning that the recent works on spindles have inspired other interesting supergravity solutions where M_2 is taken to be a punctured disc, or a "half-spindle", often arising from a different type of regularity analysis of the same local metrics that describe spindle solutions. Discs share some of the unusual features of spindles, primarily the lack of a constant curvature metric, the presence of singularities and the absence of a topological twist. However, they generically preserve more supersymmetry (as their existence is related to the vanishing of the magnetic flux for some of the gauge fields) and therefore allow for a higher degree of control on the dual field theories. This is particularly true in the case of [530, 531], where disc solutions were first introduced and the dual $4d$ $\mathcal{N}=2$ SCFTs were identified. Other works on discs include [197, 527, 532–535].

6.5 Outline of part II

The rest of **part II** of this thesis is organized as follows. In chapter 7 we consider solutions of gauged supergravity of the form AdS $\times\, \Sigma$ in various dimensions, which can all be uplifted to either type IIB or $D=11$ supergravity and are conjectured (in some cases known) to describe the near-horizon geometry of supersymmetric accelerating black branes. We start by analyzing the conditions for the existence of spin^c spinors on a spindle and then focus for the most part on the case of five-dimensional gauged supergravity, where generic features of spindle solutions are illustrated most explicitly. We will also comment on similarities and differences with solutions in other dimensions. This chapter is mostly based on [7], as well as on the series of papers [3–6]. In chapter 8 we shall focus on the only case where "full flow" spindle solutions are known, namely four-dimensional gauged

supergravity. We will consider supersymmetric accelerating black holes that interpolate between an asymptotically locally AdS_4 boundary and an $AdS_2 \times \Sigma$ near-horizon geometry, elucidating some aspects of how supersymmetry is preserved when field theories are placed on spindles. This chapter is based on [4, 6]. We will conclude with some discussion and outlook in chapter 9.

Chapter 7

Supersymmetric spindles

A new facet of the AdS/CFT correspondence involves a class of two-dimensional orbifolds known as spindles [3]. Recall that a spindle is topologically a two-sphere but with quantized conical deficit angles at the two poles. More precisely, we define a spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_1,n_2]}$ to be a weighted projective space that is specified by two positive, co-prime integers n_1, n_2. Various AdS$\times\mathbb{\Sigma}$ supersymmetric solutions of gauged supergravity theories in $D = 4, 5$ and 7 dimensions have now been constructed, which are naturally associated with[1] M2, D3 and M5-branes, respectively [3–6, 523, 524, 527]. These can then be uplifted on certain compact manifolds to give supersymmetric solutions of $D = 10$ and $D = 11$ supergravity that are dual to new SCFTs. Remarkably, in the examples involving M2 and D3-branes, one finds uplifted solutions that are completely regular and free from any orbifold singularities. In other examples, involving M5-branes, orbifold singularities remain in the $D = 11$ solutions, but strong supporting evidence has been provided that they are in fact dual to *bona fide* SCFTs.

The regular uplifted solutions in $D = 10$ and $D = 11$ supergravity just mentioned were in fact first constructed some time ago using a different approach [178, 179], but the dual SCFTs remained obscure. The new insight that they can be recast as uplifted solutions of a lower-dimensional gauged supergravity leads to the natural conjecture that the dual SCFTs are the low-energy limit of the field theories arising from branes wrapping a spindle. Evidence supporting this picture has been provided in the examples involving D3- and M5-branes, by calculating the central charge of the proposed SCFT using anomaly polynomials, and finding exact agreement with the gravity result.

The picture that has emerged has a direct parallel with the well-known constructions involving branes wrapping Riemann surfaces $\Sigma_{\mathfrak{g}}$, with genus $\mathfrak{g}$, but there are some crucial differences. In the standard constructions, starting with [113], supersymmetry on the world-volume of the branes is realized with a partial "topological twist", which involves switching on background R-symmetry gauge fields that are determined, locally, by the spin connection of the metric on the Riemann surface.

[1] See also [526] for solutions of massive type IIA that are associated with D4-branes wrapping spindles.

An immediate consequence of the topological twist is that the integrated R-symmetry flux through the Riemann surface, suitably normalized, is equal to the Euler number of the Riemann surface. Another consequence of the topological twist is that the Killing spinors for the AdS$\times\Sigma_g$ solutions are chiral and constant on Σ_g. As is also well-known, this standard set-up is realized by wrapping branes on a Riemann surface that is holomorphically embedded in a CY manifold. By contrast, in the new AdS$\times\mathbb{\Sigma}$ solutions involving spindles the Killing spinors are neither chiral nor constant on $\mathbb{\Sigma}$. In fact two distinct ways of realizing supersymmetry for spindles have now emerged, which we refer to as the *twist* and *anti-twist*, and they can be characterized via the R-symmetry flux through the spindle, as described in (6.1).

All of the known AdS$\times\mathbb{\Sigma}$ solutions have a spindle metric that is invariant under azimuthal rotations about the poles. One of the main results of [7], that we shall review here, is to show (assuming such a spindle metric) that the twist and the anti-twist are the *only* possible ways of realizing supersymmetry on the spindle. The proof follows from analysing the regularity conditions for the two-component spin^c spinors on the spindle $\mathbb{\Sigma}$, which are charged with respect to the R-symmetry gauge field, near each of the two orbifold points. The only other assumption that goes into the proof is that the Killing spinor should be non-vanishing at these orbifold points, which holds on general grounds for supersymmetric AdS solutions, as we shall discuss. Our analysis also clarifies why the regularity arguments that have been used for supergravity solutions with spindles constructed in [3–6, 523, 524, 526–528] do in fact give the correct results.

For the Killing spinors arising in the supersymmetric AdS$\times\mathbb{\Sigma}$ solutions, we show that a vector bilinear which can be constructed from the Killing spinor is the Killing vector which generates the azimuthal rotations on the spindle. This is true for both the twist and the anti-twist cases. This aspect of the way in which the supersymmetry algebra is being realized on the spindle can be contrasted[2] with the usual topological twist for Riemann surfaces, where the corresponding vector bilinear vanishes identically. This further underscores the fact that while the twist case for the spindle and the topological twist share the feature that the R-symmetry flux is the Euler number as in (6.1), they are in fact quite distinct. That being said, if one formally sets $n_1 = n_2 = 1$ in the central charge of the dual SCFTs for various AdS$\times\mathbb{\Sigma}$ solutions, one gets the same result that one gets for some known AdS$\times S^2$ solutions with a topological twist.

In this chapter we present AdS$\times\mathbb{\Sigma}$ solutions of various gauged supergravity theories, which

[2]From the dual field theory point of view, a closely related point is that in considering a SCFT on the spindle, one finds that the R-symmetry of the resulting lower-dimensional SCFT is a mixture of the spindle symmetry and the R-symmetry of the higher-dimensional SCFT.

were analyzed in a systematic way in [7]. We will present the explicit details for $AdS_3 \times \mathbb{\Sigma}$ D3-brane solutions and only schematically describe the solutions in other dimensions. Solutions from D3- and M2-branes are found to realize both the twist[3] and the anti-twist, while for M5-brane solutions the construction in [5] only leads to twist solutions. One can think of the space transverse to the brane, together with the spindle, as the total space of a $\mathbb{C}^n$ fibration over $\mathbb{\Sigma}$ (for a suitable n depending on the type of brane), with $U(1)$ connections given by the supergravity gauge fields, whose fluxes are suitably quantized. In the twist case, this space has vanishing first Chern class, so at least at the topological level it is analogous to the case of the topological twist where Riemann surfaces are holomorphically embedded in CY manifolds. However, it has not yet been clarified how the spindle can be embedded in a CY manifold.On the other hand, it is an interesting open problem to identify a UV geometry for the many anti-twist solutions that have now been constructed, involving wrapped branes, a point we discuss further in chapter 9.

The outline of this chapter is as follows. In section 7.1 we give a general proof that (6.1) are the only possibilities for the R-symmetry flux through a spindle, under some rather general assumptions. Our discussion includes a global description of $U(1)$ orbibundles, the associated Killing spinors and some properties of the uplifted solutions. We will explain how the standard topological twist arises as a special case of the twist, but we emphasize that in actual solutions the Killing spinors have distinct features. In section 7.2 we present a fairly comprehensive analysis of $AdS_3 \times \mathbb{\Sigma}$ D3-brane solutions, in particular constructing new twist solutions. We interpret the new twist solutions as near-horizon limits of D3-branes wrapped on $\mathbb{\Sigma}$ embedded holomorphically inside a Calabi-Yau four-fold, and reproduce the supergravity central charge by computing the anomaly polynomial of $\mathcal{N} = 4$ SYM theory wrapped on $\mathbb{\Sigma}$. Some of the new $D = 5$ gauged supergravity solutions also uplift to new $AdS_3 \times \mathbb{\Sigma} \times \Sigma_g$ twist M5-brane solutions, or more generally the product of $\mathbb{\Sigma}$ with a punctured Riemann surface.

7.1 Twists and anti-twists

As already mentioned, several different classes of supersymmetric supergravity solutions involving spindles have recently been constructed. Typically these are realized as solutions to a lower-dimensional gauged supergravity theory, and then uplifted to $D = 10$ string theory or $D = 11$ M-theory solutions, where the solutions describe D3-brane, M2-brane or M5-brane backgrounds. In

[3] With the caveat that these require multiple magnetic charges, and thus in particular do not exist in minimal gauged supergravity, as shown in [3, 4].

every case the total magnetic flux for the R-symmetry gauge field through the spindle $\mathbb{\Sigma}$ has been found to be one of the two possibilities in (6.1). Here A is the Abelian R-symmetry gauge field, with curvature $F = \mathrm{d}A$, normalized so that the Killing spinor has charge $\frac{1}{2}$. In this section we give a general proof that these are the only two possibilities, under some fairly general assumptions: (i) azimuthal rotations of the spindle are a symmetry of the solution, (ii) the Killing spinor is non-zero at the poles of the spindle. We shall discuss the assumption (ii) further in section 7.1.5, where we argue that the Killing spinor is in fact nowhere zero for supersymmetric AdS solutions in string and M-theory. The precise definition of assumption (i) will be made clear in the next subsections. The proof follows from global regularity conditions on the Killing spinor, independently of precisely which equation it solves.

7.1.1 Setup

We begin in this subsection by summarizing the basic set-up. We focus on a gauged supergravity perspective, as this is the framework where we construct solutions in later sections, but we also discuss the implications for the uplifted supergravity solutions in section 7.1.6.

The precise details of the Killing spinor equation will not be relevant for what follows. In general we can take the Killing spinor to carry a possible "flavour index" and write ζ^i, which for each $i = 1, \ldots, n$ is a two-component spin^c spinor on $\mathbb{\Sigma}$. We assume it satisfies[4] a Killing spinor equation (KSE) of the schematic form

$$(\nabla - \tfrac{\mathrm{i}}{2}A)\zeta^i + \mathcal{M}^i{}_j\zeta^j = 0\,. \tag{7.1}$$

Here $\nabla = \mathrm{d} + \frac{1}{4}\omega^{ab}\gamma_{ab}$ is the Levi-Civita spin connection and, for definiteness, we have taken the spinor to have charge $\frac{1}{2}$ with respect to the R-symmetry gauge field A. The matrix $\mathcal{M}^i{}_j$ could depend on scalar fields, as well as other gauge fields that act on the flavour indices but, as we will see, this matrix will not play any role in the following discussion; indeed we can suppress the flavour index on ζ^i and simply write ζ.

We will be interested in studying such Killing spinors ζ on a spindle. By definition, a spindle $\mathbb{\Sigma}$ is topologically a two-sphere with conical deficit angles $2\pi(1 - 1/n_S)$ at the "south pole" and $2\pi(1 - 1/n_N)$ at the "north pole", where $n_S, n_N \in \mathbb{N}$ are relatively prime with $\mathrm{hcf}(n_S, n_N) = 1$. This may equivalently be described as the weighted projective space $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_S,n_N]}$, which is an orbifold Riemann surface. We will exclude the special case $n_S = n_N = 1$ from the definition of a

[4]The Killing spinor might also satisfy additional algebraic equations, but these play no role in the following arguments.

spindle, but point out that these values give rise to a smooth two-sphere. We assume that the metric on the spindle is invariant under azimuthal rotations around the poles. That is, there is a Killing vector $k \equiv \partial_\varphi$, with coordinates (ρ, φ) in which the metric on the spindle takes the form

$$\mathrm{d}s^2_{\Sigma} \;=\; \mathrm{d}\rho^2 + f^2(\rho)\mathrm{d}\varphi^2\,. \tag{7.2}$$

We normalize φ to have period $\Delta\varphi = 2\pi$, and take $\rho \in [\rho_1, \rho_2]$. The metric (7.2) describes a spindle with the above deficit angles if $f(\rho)$ has the boundary behaviour $f(\rho) = (\rho-\rho_1)/n_N + O((\rho-\rho_1)^2)$ and $f(\rho) = (\rho_2-\rho)/n_S + O((\rho_2-\rho)^2)$ near the north and south poles, respectively, with f strictly positive on the open interval (ρ_1, ρ_2).

The Killing vector $k = \partial_\varphi$ is assumed to generate a symmetry of the full solution. In particular we assume that the matrix $\mathcal{M}$ in the KSE (7.1) invariant, i.e. $\mathcal{L}_k\mathcal{M} = 0$, where $\mathcal{L}_k$ is the Lie derivative. Similarly we assume that $\mathcal{L}_k F = 0$, and will typically work in local gauges which satisfy $\mathcal{L}_k A = 0$, and hence we can write

$$A \;=\; a(\rho)\mathrm{d}\varphi\,. \tag{7.3}$$

Notice this leaves a freedom to make *local* gauge transformations

$$A \;\to\; A + \gamma\,\mathrm{d}\varphi\,, \tag{7.4}$$

with γ a constant, which equivalently shifts $a(\rho) \to a(\rho) + \gamma$. We shall present a careful global discussion of gauge fields on spindles in section 7.1.2 below.

With the above assumptions, the KSE (7.1) is invariant under k. A consequence is that the Killing spinor solution ζ has definite charge Q under $k = \partial_\varphi$. That is, $\mathcal{L}_k\zeta = \mathrm{i}Q\,\zeta$, where $\mathcal{L}_k = \nabla_k + \frac{1}{8}(\mathrm{d}k^\flat)^{ab}\gamma_{ab}$ denotes the spinor Lie derivative along k, where $k^\flat$ is the one-form metrically dual to k. Indeed, since the KSE is invariant under k, its space of solutions will form a representation of the $U(1)$ symmetry generated by k. But the irreducible representations are then one-dimensional, classified by their charge Q, and it follows that without loss of generality we may take a solution ζ with definite charge.[5] Notice that the charge Q of the Killing spinor depends on the residual local

[5]Equivalently, notice that if ζ solves the KSE then so does $\mathcal{L}_k\zeta$, which is then also a Killing spinor.

gauge freedom (7.4), which acts on ζ as

$$\zeta \to \mathrm{e}^{\mathrm{i}\gamma\varphi/2}\,\zeta\,, \tag{7.5}$$

and thus shifts $Q \to Q + \gamma/2$.

7.1.2 Orbibundles on spindles

Since a spindle is a singular space, with orbifold singularities at its poles, we need to be careful about what we mean by a "regular gauge field" and "regular spinor" on a spindle. We begin in this subsection with a global description of an Abelian gauge field on an orbifold Riemann surface, in terms of a connection on a principal $U(1)$-orbibundle, and then move on to spinors in section 7.1.3.

In appendix A of [4], $U(1)$-orbibundles over spindles were described in terms of the quotient of S^3 by a weighted $U(1)$ action on $\mathbb{C}^2 \supset S^3$. This makes use of the fact that a locally free[6] action of a compact Lie group G on a manifold M has orbit space M/G which is naturally an orbifold, with the total space M then being a principal G-orbibundle over M/G. Starting with $M = S^3$ gives a particular $U(1)$-orbibundle over the spindle $\mathbb{\Sigma} = S^3/U(1)$, and different bundles may then be constructed by factoring through the Lens space quotient $S^3/\mathbb{Z}_\lambda$, where $\mathbb{Z}_\lambda \subset U(1)$. Here, we instead present the more usual local description of orbibundles, and associated connections, in terms of covering spaces and gluing, specializing to the case of interest where $G = U(1)$ is fibred over a spindle.

We begin with a local model near to one of the poles of the spindle (or a more general orbifold Riemann surface). Thus let $\mathbb{C} \times S^1$ be a trivial principal $U(1) = S^1$-bundle over $\mathbb{C} \cong \mathbb{R}^2$, and introduce complex coordinates (z, ξ), where $|\xi| = 1$ is a unit norm complex number parametrizing the circle fibre. The copy of $\mathbb{C}$ will be the covering space for $U \equiv \mathbb{C}/\mathbb{Z}_n$, where the generator of $\mathbb{Z}_n$ acts on $z \in \mathbb{C}$ via $z \mapsto \omega z$, with $\omega \equiv \mathrm{e}^{2\pi\mathrm{i}/n}$ being a primitive n^{th} root of unity. In general, to specify a principal G-orbibundle over $\mathbb{R}^k/\Gamma$, one picks a homomorphism h from the local orbifold group Γ into the fibre group G. In the current setting this amounts to picking a "charge" $m \in \mathbb{Z}$:

$$h : \mathbb{Z}_n \to U(1)\,, \quad \text{where} \quad h(\omega) = \omega^m\,. \tag{7.6}$$

More precisely, since $\omega^n = 1$, we have $m \in \mathbb{Z}_n$. A principal G-orbibundle over $\mathbb{R}^k/\Gamma$ is then defined

[6] Meaning all isotropy groups are finite.

as the quotient $B \equiv (\mathbb{R}^k \times G)/\Gamma$, where the action on G is given by the homomorphism h. In our case this is

$$B \equiv (\mathbb{C} \times S^1)/\mathbb{Z}_n\,, \quad \text{where} \quad \omega \cdot (z,\xi) = (\omega z, \omega^m \xi)\,. \tag{7.7}$$

The bundle projection map $\pi : B \to U = \mathbb{C}/\mathbb{Z}_n$ projects onto the first factor. Since the action of $h(\Gamma)$ on the fibre G is free, it follows that the total space of B will be smooth (*i.e.* the lifted action of Γ using h has no fixed points) if h is injective. In our case, this is so precisely[7] when $\mathrm{hcf}(m,n) = 1$.

We may further analyse this local model by writing $\xi = \mathrm{e}^{\mathrm{i}\chi}$, $z = \rho\,\mathrm{e}^{\mathrm{i}\hat{\phi}}$, where $\rho \geq 0$ is a radial coordinate, and on the covering space $\mathbb{C} \times S^1$ we have $\Delta\chi = \Delta\hat{\phi} = 2\pi$. We may geometrize a connection one-form for the $U(1)$ bundle by writing the metric

$$\mathrm{d}s^2_{\mathbb{C}\times S^1} = (\mathrm{d}\chi + A_{(0)})^2 + \mathrm{d}\rho^2 + h(\rho)^2 \mathrm{d}\hat{\phi}^2\,, \tag{7.8}$$

where $h(\rho) = \rho + O(\rho^2)$ near $\rho = 0$ in order for this rotationally-invariant metric on $\mathbb{C}$ to be smooth, and importantly $A_{(0)}$ is an arbitrary global one-form on $\mathbb{C}$ that is invariant under the $\mathbb{Z}_n$ action. For the case of interest in this chapter we impose the stronger condition that

$$\mathcal{L}_{\partial_{\hat{\phi}}} A_{(0)} = 0\,, \tag{7.9}$$

so that rotations of $\mathbb{C}$ generated by $\partial_{\hat{\phi}}$ are a symmetry. The generator of $\mathbb{Z}_n$ acts on the angular coordinates as

$$(\chi, \hat{\phi}) \mapsto \left(\chi + \frac{2\pi m}{n}, \hat{\phi} + \frac{2\pi}{n}\right)\,. \tag{7.10}$$

We may then make the $SL(2,\mathbb{Z})$ transformation of the torus parametrized by $(\chi, \hat{\phi})$ given by

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} \equiv \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi \\ \hat{\phi} \end{pmatrix}\,, \tag{7.11}$$

with $\Delta\psi = \Delta\phi = 2\pi$ on the covering space. Notice that this transformation is only defined for $\rho > 0$, as the azimuthal coordinate $\hat{\phi}$ is not defined at the origin $\rho = 0$ of $\mathbb{C}$. Moreover, although so far $m \in \mathbb{Z}_n$, the transformation (7.11) requires picking a particular choice of integer lift $m \in \mathbb{Z}$. This

[7] If $\mathrm{hcf}(m,n) \neq 1$ we can write $m = bm'$, $n = bn'$ for some $b \in \mathbb{Z}$. Then if we carry out the action of (7.7) n' times, it takes $(z,\xi) \to (\omega^{n/b} z, \xi)$, which defines a $\mathbb{Z}_b$ sub-action that leaves the fibre fixed.

then introduces some redundancy/gauge freedom in the subsequent description, that we comment on further below. The new coordinate $\psi = \chi - m\,\hat{\phi}$ is invariant under (7.10), while the metric (7.8) becomes

$$\mathrm{d}s^2_{\mathbb{C}\times S^1} \;=\; (\mathrm{d}\psi + A)^2 + \mathrm{d}\rho^2 + h(\rho)^2\mathrm{d}\phi^2\,. \tag{7.12}$$

Notice that now $\mathbb{Z}_n$ acts only on ϕ:

$$(\psi,\phi) \;\mapsto\; \left(\psi, \phi + \frac{2\pi}{n}\right)\,, \tag{7.13}$$

while the new connection one-form is

$$A \;=\; A_{(0)} + m\,\mathrm{d}\phi\,. \tag{7.14}$$

The metric on the quotient $B = (\mathbb{C}\times S^1)/\mathbb{Z}_n$ is then similarly given by (7.12), but where ψ has period 2π and ϕ has period $2\pi/n$. Because of the latter period, the connection term $m\,\mathrm{d}\phi$ is a non-trivial flat connection on $(\mathbb{C}\setminus\{0\})/\mathbb{Z}_n$, *i.e.* it cannot be removed by any non-singular gauge transformation. To match with the conventions in (7.2), on the quotient space we write $\phi = \varphi/n$, with φ having period 2π, so that $A = A_{(0)} + \frac{m}{n}\mathrm{d}\varphi$. This gauge field is not defined at the origin in $\mathbb{C}$, precisely because the azimuthal coordinates $\hat{\phi}$, φ are not defined there.

The above construction shows that we may equivalently view $U(1)$-orbibundles over $U = \mathbb{C}/\mathbb{Z}_n$ as specifying a flat connection that is pulled back from the unit circle in $(\mathbb{C}\setminus\{0\})/\mathbb{Z}_n$, with holonomy precisely $\exp\left(\mathrm{i}\int_{S^1_\varphi} A_{\text{flat}}\right) = \omega^m = h(\omega) \in U(1)$. Also notice that if $m = pn$ is an integer multiple p of n, then $A = A_{(0)} - \mathrm{i}g^{-1}\mathrm{d}g$ is gauge equivalent to the global connection form $A_{(0)}$ on[8] $(\mathbb{C}\setminus\{0\})/\mathbb{Z}_n$, where $g = \mathrm{e}^{\mathrm{i}p\varphi}$ is a large gauge transformation on this circle (*i.e.* not homotopic to the identity if $p \neq 1$). This is another manifestation of the fact that $m \in \mathbb{Z}_n$.

With this local model to hand, it is now straightforward to glue two[9] copies to obtain a spindle. Associated with the north and south poles, we take local models $(\mathbb{C}\times S^1)/\mathbb{Z}_{n_N}$ and bundle data specified by $m_N \in \mathbb{Z}_{n_N}$, along with $(\mathbb{C}\times S^1)/\mathbb{Z}_{n_S}$ and bundle data specified by $m_S \in \mathbb{Z}_{n_S}$,

[8] While $A = A_{(0)}$ is globally defined on $\mathbb{C}/\mathbb{Z}_n$, the large gauge transformation $g = \mathrm{e}^{\mathrm{i}p\varphi}$ is singular at the origin of $\mathbb{C}$.

[9] We can similarly glue an arbitrary number of such local models together using $U(1)$ gauge transformations; the spindle is special in that we can consider metrics with azimuthal symmetry as we are interested in here. We also note that the total space of the $U(1)$-orbibundles that we are constructing, when $\mathrm{hcf}(m,n) = 1$ for all local models, give smooth 3-dimensional manifolds known as Seifert manifolds.

respectively. The metrics in each patch on the total space of the bundle are as in (7.12) and given by

$$\begin{aligned} \mathrm{d}s^2_{(\mathbb{C}\times S^1)/\mathbb{Z}_{n_N}} &= \left(\mathrm{d}\psi_N + A^N_{(0)} + \frac{m_N}{n_N}\mathrm{d}\varphi\right)^2 + \mathrm{d}\rho_N^2 + h(\rho_N)^2\frac{1}{n_N^2}\mathrm{d}\varphi^2\,, \\ \mathrm{d}s^2_{(\mathbb{C}\times S^1)/\mathbb{Z}_{n_S}} &= \left(\mathrm{d}\psi_S + A^S_{(0)} + \frac{m_S}{n_S}\mathrm{d}\varphi\right)^2 + \mathrm{d}\rho_S^2 + h(\rho_S)^2\frac{1}{n_S^2}\mathrm{d}\varphi^2\,, \end{aligned} \tag{7.15}$$

where we may use the same name for the coordinate φ in each patch, with $\Delta\varphi = 2\pi$, as this is identified in the gluing. Here ψ_N, ψ_S each have period 2π, and $A^N_{(0)}, A^S_{(0)}$ are global one-forms in each of their respective covering space patches. The total gauge fields in each patch, $A^N = A^N_{(0)} + \frac{m_N}{n_N}\,\mathrm{d}\varphi$ and $A^S = A^S_{(0)} + \frac{m_S}{n_S}\,\mathrm{d}\varphi$, are defined only on an open disc in $\mathbb{C}\setminus\{0\}$, with $\rho_N, \rho_S > 0$, as discussed above, with the flat connection capturing the orbibundle data. We glue on the circles at some fixed values of $\rho_N, \rho_S > 0$, reversing the orientation of the south patch before gluing, where here A^N and A^S may in general differ by a $U(1)$ gauge transformation $g : S^1_{\mathrm{glue}} \to U(1)$, of the form $g = \mathrm{e}^{\mathrm{i}p\varphi}$ with $p \in \mathbb{Z}$. That is, on the overlap

$$A^N \;=\; A^S - \mathrm{i}g^{-1}\mathrm{d}g \;=\; A^S + p\,\mathrm{d}\varphi\,. \tag{7.16}$$

The total gauge field flux $F = \mathrm{d}A$ through the spindle may then be computed using Stokes' theorem, being careful around the poles where the flat gauge fields are singular. This leads to an integral over two small circles around each pole, $S^1_{N,\epsilon}$, $S^1_{S,\epsilon}$, where $A^{(0)}_N$, $A^{(0)}_S$ are non-singular and hence give zero as $\epsilon \to 0$, but the flat gauge fields contribute. We must also be careful where we glue over S^1_{glue} and (7.33) holds. Thus we compute

$$\begin{aligned} \frac{1}{2\pi}\int_{\Sigma} F &= -\frac{m_N}{n_N} + \frac{1}{2\pi}\int_{S^1_{\mathrm{glue}}} A^N - \left[-\frac{m_S}{n_S} + \frac{1}{2\pi}\int_{S^1_{\mathrm{glue}}} A^S\right] \\ &= p - \frac{m_N}{n_N} + \frac{m_S}{n_S} \;\equiv\; \frac{\lambda}{n_N n_S}\,, \end{aligned} \tag{7.17}$$

where we have defined the integer $\lambda = -m_N n_S + m_S n_N + p\, n_N n_S \in \mathbb{Z}$. This shows that $U(1)$-orbibundles over a spindle have gauge field flux given by $\lambda/n_N n_S$ with $\lambda \in \mathbb{Z}$. We refer to the associated complex line orbibundles as $\mathcal{O}(\lambda)$.

In fact $\lambda \in \mathbb{Z}$ specifies uniquely the orbibundle, up to isomorphism, and in particular it determines the local models near the orbifold points. To see this, note that using Bezout's Lemma it is immediate that for given n_N, n_S it is possible to choose the m_N, m_S with $p = 0$ so that $\lambda = -m_N n_S + m_S n_N = 1$. These m_N, m_S are unique up to simultaneously shifting $m_N \mapsto m_N + k n_N$,

$m_S \mapsto m_S + kn_S$, with $k \in \mathbb{Z}$, but this means that the mod reductions $m_N \in \mathbb{Z}_{n_N}$, $m_S \in \mathbb{Z}_{n_S}$ are unique, specifying the local model uniquely at each pole. Notice here that different choices of integer lifts $m_N \in \mathbb{Z}$, $m_S \in \mathbb{Z}$ in the $SL(2,\mathbb{Z})$ transformation (7.11) may also then be absorbed into the choice of $p \in \mathbb{Z}$ in (7.16). Indeed, the gluing (7.16) may also be regarded as an $SL(2,\mathbb{Z})$ transformation (see (7.20) below). Having discussed the case $\lambda = 1$, other integer values for λ may be obtained simply by multiplying m_N, m_S by λ.

Notice from (7.16) that the non-singular gauge fields in each patch are related on the intersection by the gluing

$$A^N_{(0)} = A^S_{(0)} + \frac{\lambda}{n_N n_S}\mathrm{d}\varphi\,. \tag{7.18}$$

While this characterization of the orbibundle on the spindle is in some sense equivalent to what we have just discussed, it is potentially confusing as one is gluing the gauge fields in each patch using fractional gauge transformations. By contrast, we reiterate that the gauge fields A_N and A_S, with their flat connection pieces, are glued using $U(1)$ gauge transformations on the overlap as in (7.16). Nevertheless, in previous papers discussing spindles we note that smooth gauges for the gauge fields have been used at the poles, with the corresponding fractional transition function in (7.18). While (7.18) is perfectly correct for a spindle, the discussion we have presented in this subsection is a precise global description of the construction and, moreover, also extends straightforwardly to more general orbifold Riemann surfaces.

7.1.3 Spinors on spindles

In the previous section we have "geometrized" the R-symmetry gauge field, introducing the $U(1)$ fibre direction for which it is a corresponding connection. In the cases where the spindles arise in gauged supergravity, with an uplift to $D = 10$ or $D = 11$ dimensions, this is in any case part of the construction. We elaborate on this further in section 7.1.6

The total space of the $U(1)$-orbibundle is a Seifert fibred space, which is a smooth three-manifold M_3 when $\mathrm{hcf}(m,n) = 1$ for all the local models. The one-form

$$\eta \equiv \mathrm{d}\psi + A\,, \tag{7.19}$$

is a smooth global one-form on this three-manifold. Focusing on the case of the spindle with two

orbifold points, in the above construction we defined this in patches, and in particular on the overlap we have

$$\psi_N \;=\; \psi_S - p\,\varphi\,, \tag{7.20}$$

which from (7.16) then ensures that $\mathrm{d}\psi_N + A^N = \mathrm{d}\psi_S + A^S$ on the overlap. This is precisely the $U(1)$ gauge transformation $\mathrm{e}^{\mathrm{i}\psi_N} = \mathrm{e}^{\mathrm{i}\psi_S}\cdot\mathrm{e}^{-\mathrm{i}p\varphi}$. Near to the poles one should invert the $SL(2,\mathbb{Z})$ transformation (7.11) to obtain the connection one-forms $\mathrm{d}\chi_N + A^N_{(0)}$, $\mathrm{d}\chi_S + A^S_{(0)}$, which are smooth one-forms on the covering copies of $\mathbb{C}$. Whenever one checks smoothness of local expressions in the orbifold setting, one necessarily has to resort to using patches and covering spaces in this way. We note in passing that the total space M_3 is a Lens space, when the orbifold base $\mathbb{\Sigma}$ is a spindle.

By definition the Killing spinor ζ has charge $\frac{1}{2}$ under A. One can then introduce a complex line bundle L, on which A is a connection, so that the spinor is a section of $S\otimes L^{1/2}$, where S is the spin bundle on $\mathbb{\Sigma}$. The vector field ∂_ψ on M_3 generates the $U(1)$ action, such that $\mathbb{\Sigma} = M_3/U(1)$ is the quotient. Moreover, for any function or more generally tensor field on M_3 we may Fourier expand in eigenmodes of $\mathcal{L}_{\partial_\psi}$. A function of charge r under ∂_ψ, with corresponding phase $\mathrm{e}^{\mathrm{i}r\psi}$, is then, in its dependence on $\mathbb{\Sigma}$, precisely a section of L^r. One thus obtains a spinor on $\mathbb{\Sigma}$ of charge $\frac{1}{2}$ under A by starting with a spinor field on M_3 that has charge $\frac{1}{2}$ under $\mathcal{L}_{\partial_\psi}$. Indeed, notice that the tangent bundle of M_3 naturally splits as $T_{\partial_\psi}\oplus T_{\mathbb{\Sigma}}$, and the spinor bundle on M_3 then likewise splits as a tensor product of the spinor bundle S on $\mathbb{\Sigma}$ with a spinor on the circle fibres of M_3. A charge $\frac{1}{2}$ spinor under $\mathcal{L}_{\partial_\psi}$ then has phase $\mathrm{e}^{\mathrm{i}\psi/2}$, which is the anti-periodic spin structure around the circle fibres. The resulting spinor field on M_3 is then a genuine spinor, rather than a spinor charged under a gauge field. This is precisely how supersymmetry is realized in consistent truncations to gauged supergravity. The comments above apply whether $\mathbb{\Sigma}$ is smooth or an orbifold, as long as one uses the appropriate patches and covering spaces in the orbifold setting when gluing local descriptions together.

With these preliminary global comments to hand, as in the previous subsection we may now introduce local coordinate patches and covering spaces in order to describe the Killing spinor ζ. We thus introduce coordinate patches $\mathcal{U}_N\cong\mathbb{C}/\mathbb{Z}_{n_N}$ and $\mathcal{U}_S\cong\mathbb{C}/\mathbb{Z}_{n_S}$, containing the north and south poles, respectively, and analyse how the spinor ζ behaves near to the poles, as well as how it patches together at the equator, where $\mathcal{U}_N$ and $\mathcal{U}_S$ overlap. We begin by considering $\mathcal{U}_N$, equipped with polar coordinates $(\rho_N\geq 0,\varphi)$ such that $\rho_N = 0$ is the north pole. We may also introduce corresponding

Euclidean coordinates $x_N + \mathrm{i}y_N \equiv \rho_N \, \mathrm{e}^{\mathrm{i}\varphi/n_N}$, identifying $\mathcal{U}_N \cong \mathbb{C}/\mathbb{Z}_{n_N}$. Recalling (7.14) and (7.3) we write the gauge field in this patch as a sum of a regular one-form plus a flat connection:

$$A^N \;=\; a^N_{(0)}(\rho_N)\mathrm{d}\varphi + \frac{m_N}{n_N}\mathrm{d}\varphi \,, \tag{7.21}$$

where $m_N \in \mathbb{Z}$. We demand that $a^N_{(0)}(0) = 0$ to ensure that first term on the right hand side is a well-defined one-form at the origin of polar coordinates (where $\mathrm{d}\varphi$ is a singular one-form) and notice this condition fixes the residual gauge freedom in (7.4).

Since we wish to describe spinors we introduce an orthonormal frame e^1_N, e^2_N, such that $e^1_N(0) = \mathrm{d}x_N, e^2_N(0) = \mathrm{d}y_N$ agrees with the usual Euclidean frame at the north pole. We use gamma matrices $\gamma_1 = \sigma^1, \gamma_2 = \sigma^2$, where σ^a are Pauli matrices, and take the chirality operator to be $\gamma_3 \equiv -\mathrm{i}\gamma_1\gamma_2 = \sigma^3$. We may then write our spinor ζ in this patch and in the given frame in the form

$$\zeta^N \;=\; \begin{pmatrix} \zeta^N_+ \\ \zeta^N_- \end{pmatrix} \,. \tag{7.22}$$

We will discuss the smoothness of $\zeta^N_\pm$ at the north pole momentarily.

The frame e^1_N, e^2_N is not invariant under $k = \partial_\varphi$, so the charge of the spinor (7.22) under k is not immediately manifest in the Euclidean frame. We can make it manifest by changing to a k-invariant frame, defined by

$$\hat{e}^1 \;\equiv\; \mathrm{d}\rho \,, \qquad \hat{e}^2 \;\equiv\; f(\rho)\,\mathrm{d}\varphi \,. \tag{7.23}$$

This is an orthonormal frame for the metric (7.2), where note that near the north pole this is a polar coordinate frame with $\hat{e}^1 = \mathrm{d}\rho_N$, $\hat{e}^2 = \rho_N \,\mathrm{d}\varphi/n_N + O(\rho_N^2)$. The two frames are related by the $U(1) \cong SO(2)$ rotation

$$\begin{pmatrix} \hat{e}^1 \\ \hat{e}^2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\varphi}{n_N} & \sin\frac{\varphi}{n_N} \\ -\sin\frac{\varphi}{n_N} & \cos\frac{\varphi}{n_N} \end{pmatrix} \begin{pmatrix} e^1_N \\ e^2_N \end{pmatrix} \,. \tag{7.24}$$

When vectors are rotated by an orthogonal matrix $L^a{}_b = \exp(s^a{}_b)$, spinors transform in the spin

representation with $\tilde{L} = \exp(\frac{1}{4}s^{ab}\gamma_{ab})$. Hence in the new frame (7.23) the spinor ζ^N reads

$$\hat{\zeta}^N = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\varphi/2n_N}\zeta^N_+ \\ \mathrm{e}^{-\mathrm{i}\varphi/2n_N}\zeta^N_- \end{pmatrix} . \tag{7.25}$$

Since the frame (7.23) is k-invariant, $\mathcal{L}_k\hat{e}^a = 0$, we have $\mathcal{L}_k\hat{\zeta}^N = \partial_\varphi\hat{\zeta}^N$. Recall that in a fixed gauge of the form (7.3) we take the spinor to have definite charge under k. More precisely, in the patch $\mathcal{U}_N$ covering the north pole with gauge field (7.21) we take the charge of the spinor to be Q_N. It then follows that the spinor in the two frames takes the form

$$\zeta^N = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\left(Q_N - \frac{1}{2n_N}\right)\varphi} f^N_+(\rho_N) \\ \mathrm{e}^{\mathrm{i}\left(Q_N + \frac{1}{2n_N}\right)\varphi} f^N_-(\rho_N) \end{pmatrix} , \qquad \hat{\zeta}^N = \mathrm{e}^{\mathrm{i}Q_N\varphi}\begin{pmatrix} f^N_+(\rho_N) \\ f^N_-(\rho_N) \end{pmatrix} . \tag{7.26}$$

We now return to the issue of smoothness at the north pole. If the spinor ζ were *uncharged* under the gauge field A, then by definition it is smooth at the north pole of the spindle precisely if its lift to the covering space $\mathbb{C}$, where φ has period $2\pi n_N$, is smooth at the origin $\rho_N = 0$. The expression for ζ^N in (7.26) is with respect to a smooth frame at the origin, and thus the spinor will be regular precisely if its components are smooth functions at the origin. Notice now that if the positive chirality component ζ^N_+ is *non-zero* at the origin, so $f^N_+(0) \neq 0$, then we must have $Q_N = 1/2n_N$ for ζ^N_+ to be smooth, as φ is singular at the origin. Similarly, if instead ζ^N_- is non-zero at the origin we would instead deduce that $Q_N = -1/2n_N$.

However, the spinor ζ instead has charge $\frac{1}{2}$ under A in the KSE as in (7.1), where in the north pole patch $A = A^N$ takes the form in (7.21). As discussed earlier in this subsection, this charged spinor ζ arises via reduction of a spinor on M_3 that has charge $\frac{1}{2}$ under $\mathcal{L}_{\partial_\psi}$. Moreover, to examine regularity of this spinor at the origin of $\mathbb{C}$ we must invert the $SL(2,\mathbb{Z})$ transformation (7.11), which removes the singular flat connection term $\frac{m_N}{n_N}\mathrm{d}\varphi$ in the gauge field A^N. This leads to coordinates $(\rho_N, \hat{\phi}_N, \chi_N)$, with $\hat{\phi}_N$ and χ_N having period 2π in the covering space, with the twisted $\mathbb{Z}_{n_N}$ action:

$$(\chi_N, \hat{\phi}_N) \mapsto \left(\chi_N + \frac{2\pi m_N}{n_N}, \hat{\phi}_N + \frac{2\pi}{n_N}\right) . \tag{7.27}$$

After uplifting the spinor ζ^N to M_3 we thus obtain

$$\zeta^N \;\to\; \mathrm{e}^{\mathrm{i}\frac{\chi_N}{2}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\left(Q_N-\frac{1}{2n_N}-\frac{m_N}{2n_N}\right)n_N\hat{\phi}_N} f_+^N(\rho_N) \\ \mathrm{e}^{\mathrm{i}\left(Q_N+\frac{1}{2n_N}-\frac{m_N}{2n_N}\right)n_N\hat{\phi}_N} f_-^N(\rho_N) \end{pmatrix} . \tag{7.28}$$

This is then smooth precisely if the components are smooth functions at $\rho_N \to 0$. As in the paragraph above, we then immediately deduce that if the positive chirality (upper) component is non-vanishing at the north pole then we must have $Q_N - \frac{1}{2n_N} - \frac{m_N}{2n_N} = 0$, and similarly if the negative chirality (lower) component is non-vanishing at the north pole then we must have $Q_N + \frac{1}{2n_N} - \frac{m_N}{2n_N} = 0$.

A similar analysis may be carried out in the south pole patch $\mathcal{U}_S$, equipped with polar coordinates $(\rho_S \geq 0, \varphi)$ such that $\rho_S = 0$ is the south pole. The corresponding Euclidean coordinates are $x_S + \mathrm{i}y_S \equiv -\rho_S\,\mathrm{e}^{-\mathrm{i}\varphi/n_S}$, and we take

$$A^S \;=\; a^S_{(0)}(\rho_S)\mathrm{d}\varphi + \frac{m_S}{n_S}\mathrm{d}\varphi\,, \tag{7.29}$$

where $m_S \in \mathbb{Z}$ and we demand that $a^S_{(0)}(0) = 0$ to ensure that first term on the right hand side is smooth one-form at the origin of polar coordinates The only difference now is that the k-invariant frame (7.23) has $\hat{e}^1 = -\mathrm{d}\rho_S$, $\hat{e}^2 = \rho_S\,\mathrm{d}\varphi/n_S + O(\rho_S^2)$. The analogous $U(1) \cong SO(2)$ rotation to (7.24) reads

$$\begin{pmatrix} \hat{e}^1 \\ \hat{e}^2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\varphi}{n_S} & -\sin\frac{\varphi}{n_S} \\ \sin\frac{\varphi}{n_S} & \cos\frac{\varphi}{n_S} \end{pmatrix} \begin{pmatrix} e^1_S \\ e^2_S \end{pmatrix} , \tag{7.30}$$

where at the south pole we take $e^1_S(0) = \mathrm{d}x_S$, $e^2_S(0) = \mathrm{d}y_S$. With respect to the non-singular frame e^a_S, and the k-invariant frame (7.23), a similar analysis to that at the north pole then leads to the respective spinors

$$\zeta^S \;=\; \begin{pmatrix} \mathrm{e}^{\mathrm{i}\left(Q_S+\frac{1}{2n_S}\right)\varphi} f_+^S(\rho_S) \\ \mathrm{e}^{\mathrm{i}\left(Q_S-\frac{1}{2n_S}\right)\varphi} f_-^S(\rho_S) \end{pmatrix} , \qquad \hat{\zeta}^S \;=\; \mathrm{e}^{\mathrm{i}Q_S\varphi} \begin{pmatrix} f_+^S(\rho_S) \\ f_-^S(\rho_S) \end{pmatrix} . \tag{7.31}$$

Also, after uplifting the spinor ζ^S to M_3 we obtain

$$\zeta^S \;\to\; \mathrm{e}^{\mathrm{i}\frac{\chi_S}{2}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\left(Q_S+\frac{1}{2n_S}-\frac{m_S}{2n_S}\right)n_S\hat{\phi}_S} f_+^S(\rho_S) \\ \mathrm{e}^{\mathrm{i}\left(Q_S-\frac{1}{2n_S}-\frac{m_S}{2n_S}\right)n_S\hat{\phi}_S} f_-^S(\rho_S) \end{pmatrix} , \tag{7.32}$$

with the components smooth functions at $\rho_S \to 0$.

Although the hatted spinors $\hat{\zeta}^N$, $\hat{\zeta}^S$ are defined using the same frame (7.23), note that the former uses the gauge field A^N, while the latter uses the gauge field A^S. As in (7.33) these gauge fields are related by a $U(1)$ gauge transformation:

$$A^N \;=\; A^S - \mathrm{i}g^{-1}\mathrm{d}g \;=\; A^S + p\,\mathrm{d}\varphi\,, \tag{7.33}$$

with $p \in \mathbb{Z}$. Since the Killing spinor has charge $\frac{1}{2}$ under A, we correspondingly have the relation

$$\hat{\zeta}^N \;=\; \mathrm{e}^{\mathrm{i}(p/2)\varphi}\,\hat{\zeta}^S\,. \tag{7.34}$$

This is satisfied by the expressions in (7.26), (7.31) provided that $f^N_\pm(\rho_N) = f^S_\pm(\rho_S)$ on the overlap, as well as

$$Q_N \;=\; Q_S + \frac{p}{2}\,. \tag{7.35}$$

With these results in hand, we may now investigate the consequences of our assumption (ii), namely that the spinor is non-zero at the poles. At the north pole at least one of the chiral components must be non-zero. Let us assume for the moment that $\zeta^N_+(0) \neq 0$, i.e. $f^N_+(0) \neq 0$. Then from (7.28), as already discussed regularity of the spinor ζ^N at the north pole implies

$$Q_N \;=\; \frac{1}{2n_N} + \frac{m_N}{2n_N}\,. \tag{7.36}$$

The negative chirality component of the uplifted spinor is then $\mathrm{e}^{\mathrm{i}\chi_N/2}\mathrm{e}^{\mathrm{i}\varphi/n_N} f^N_-(\rho_N)$, and for this to be regular at the north pole we must have $f^N_-(\rho_N) = 0$ so that the spinor is necessarily *chiral* at the pole. In fact we shall see in specific examples that $f^N_-(\rho_N)$ vanishes linearly in ρ_N, near to $\rho_N = 0$, which implies that to leading order near the pole $\mathrm{e}^{-\mathrm{i}\chi_N/2}\zeta^N_- \sim z_N \equiv \rho_N\,\mathrm{e}^{\mathrm{i}\varphi/n_N}$, which is smooth.

Looking next at the south pole, there are two possibilities that follow from regularity of ζ^S:

$$\begin{aligned}
\text{Twist}: \quad & \zeta^S_+(0) \;\neq\; 0 \quad \Rightarrow \quad Q_S \;=\; -\frac{1}{2n_S} + \frac{m_S}{2n_S}\,,\\
\text{Anti-twist}: \quad & \zeta^S_-(0) \;\neq\; 0 \quad \Rightarrow \quad Q_S \;=\; +\frac{1}{2n_S} + \frac{m_S}{2n_S}\,.
\end{aligned} \tag{7.37}$$

Notice that similarly to the north pole, if $\zeta^S_\pm(0) \neq 0$ then correspondingly $\zeta^S_\mp(0) = 0$.

We now recall the formula for the total flux (7.17). Using (7.35) we can rewrite this as $\frac{\lambda}{n_N n_S} = 2(Q_N - Q_S) - \frac{m_N}{n_N} + \frac{m_S}{n_S}$ and hence using (7.36), (7.37) we deduce that

$$\begin{aligned} \text{Twist}: \quad & \lambda = n_S + n_N \,, \\ \text{Anti-twist}: \quad & \lambda = n_S - n_N \,. \end{aligned} \tag{7.38}$$

These are precisely the two possibilities (6.1), and imply that the gauge field A is a connection on $\mathcal{O}(n_S \pm n_N)$, respectively. In the twist case, with the upper sign, this is the tangent bundle of the spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_S,n_N]}$.

In reaching this conclusion we assumed that $\zeta_+^N(0) \neq 0$. If instead $\zeta_-^N(0) \neq 0$, a similar analysis leads to the two cases

$$\lambda = -n_S \mp n_N \,. \tag{7.39}$$

These are the twist and anti-twist cases (7.38), but with the opposite overall sign for the gauge field. Such solutions will be related to twist and anti-twist solutions by charge conjugation, in the following sense. Recall that we took the gamma matrices to be $\gamma_1 = \sigma^1$, $\gamma_2 = \sigma^2$. The charge conjugation matrix[10] is then $B \equiv \sigma^1$, which satisfies $B^{-1}\gamma_a B = \gamma_a^*$, where we define the charge conjugate spinor as

$$\zeta^c \equiv B\zeta^* = \begin{pmatrix} \zeta_-^* \\ \zeta_+^* \end{pmatrix} \,. \tag{7.40}$$

If ζ satisfies (7.1), then it is straightforward to check that ζ^c satisfies the same equation with A replaced by $A^c \equiv -A$ and $\mathcal{M}$ replaced by $\mathcal{M}^c$, where $\mathcal{M}^* = B^{-1}\mathcal{M}^c B$. Since our analysis did not depend on the precise form of $\mathcal{M}$, the solutions with λ given by (7.39) may equivalently be viewed as twist and anti-twist solutions (7.38), but solving the Killing spinor equation with $\mathcal{M}$ replaced by $\mathcal{M}^c$. It might be the case that there is a $\mathbb{Z}_2$ action on the fields such that $\mathcal{M}^c$ is the same function of the transformed fields as $\mathcal{M}$, i.e. charge conjugation leaves the Killing spinor equation (7.1) invariant. However, either way one can effectively reduce to the two cases in (7.38).

The above analysis showed that the spinor is necessarily *chiral* at each of the two poles. The twist case is then compatible with having $\zeta_- \equiv 0$, so that the spinor is positive chirality. This is

[10] We can also run the same argument that follows by replacing B with $\tilde{B} \equiv B\gamma_3$, which satisfies $\tilde{B}^{-1}\gamma_a\tilde{B} = -\gamma_a^*$.

the case for the topological twist, where the spinor is chiral and moreover constant. On the other hand, for the anti-twist the spinor ζ is necessarily non-chiral, with opposite chirality components vanishing at the two poles.

It is illuminating to consider the global structure of spin^c bundles we have constructed above. Recall that on any oriented two-manifold or orbifold, chiral spinors are sections of the bundles

$$S^+ = K^{1/2}\,, \qquad S^- = K^{-1/2} = \Lambda^{0,1}\otimes K^{1/2}\,. \tag{7.41}$$

Here $K=\Lambda^{1,0}$ is the canonical bundle, namely the cotangent bundle, and $\Lambda^{0,1}$ denote $(0,1)$-forms with respect to the canonical complex structure, specified by the orientation. For the spindle $K = \mathcal{O}(-n_S - n_N)$, which notice has a genuine square root only when n_S+n_N is even. On the other hand, our Killing spinor is also charged under A, which is a connection on the line bundle $L\equiv\mathcal{O}(\lambda)$. Since the spinor has charge $\frac{1}{2}$, the chiral components $\zeta_\pm$ of ζ are globally sections of the bundles

$$\begin{aligned}
\zeta_+ :\ S^+\otimes L^{1/2} &= \mathcal{O}(-\tfrac{1}{2}(n_S+n_N))\otimes\mathcal{O}(\tfrac{1}{2}(n_S\pm n_N))\\
&= \begin{cases} \mathcal{O}(0) & \text{Twist}\\ \mathcal{O}(-n_N) & \text{Anti-twist}\end{cases},\\
\zeta_- :\ S^-\otimes L^{1/2} &= \mathcal{O}(\tfrac{1}{2}(n_S+n_N))\otimes\mathcal{O}(\tfrac{1}{2}(n_S\pm n_N))\\
&= \begin{cases} \mathcal{O}(n_S+n_N) & \text{Twist}\\ \mathcal{O}(n_S) & \text{Anti-twist}\end{cases}.
\end{aligned} \tag{7.42}$$

Notice that these are well-defined line bundles over the spindle, even though in general neither $S^\pm$ nor $L^{1/2}$ is. This is a consequence of the fact that $\lambda = n_S\pm n_N\equiv n_S+n_N$ mod 2, meaning that A is a spin^c gauge field. In the twist case notice that ζ_+ is a section of the trivial line bundle $\mathcal{O}(0)$.

Before continuing, we briefly pause to contrast the detailed global analysis for the regularity of the spin^c spinors that we have just carried out with what has appeared in the literature in the context of specific solutions [3–6, 523, 524, 526, 527]. In essence, these papers used patches with regular gauge fields $A^N_{(0)}$, $A^S_{(0)}$ that are related by fractional gauge transformations as in (7.18). This is then combined with demanding smoothness of the unhatted spinors given in (7.26), rather than (7.28). While this approach does in fact give the correct results for a spindle, our new discussion here provides a precise global description.

7.1.4 Topological twist

The "topological twist" is a particular case of the twist solutions. Specifically, one solves the original Killing spinor equation (7.1) by taking $\zeta_- \equiv 0$, so that $\zeta = \zeta_+$ is a chiral spinor. Since then $\gamma_{12}\zeta = \mathrm{i}\sigma^3\zeta = \mathrm{i}\zeta$, equation (7.1) reads[11]

$$\mathrm{d}\zeta + \tfrac{\mathrm{i}}{2}(\omega_{12} - A)\zeta + \mathcal{M}\zeta \;=\; 0\;. \tag{7.43}$$

Assuming that $\mathcal{M}\zeta = 0$, this is solved by taking

$$A \;=\; \omega_{12}\;, \qquad \zeta \;=\; \text{constant}\;. \tag{7.44}$$

Taking A to be the spin connection ω_{12} immediately implies that this is a connection on the tangent bundle of $\mathbb{\Sigma}$, and hence (6.1) holds with the upper sign. Moreover, $\zeta = \zeta_+$ being nowhere zero is then necessarily a section of a trivial line bundle $\mathcal{O}(0)$. This is clearly a special case of what we have called "twist" solutions in the previous subsection.

However, more generally there are twist solutions that are not "topological twists" in the above strict sense, as seen in [5]. We will later see new examples of twist solutions with ζ_- not identically zero (but necessarily zero at the poles of the spindle), and similarly $\mathcal{M}\zeta \neq 0$ and ζ_+ non-constant. We thus find it convenient to distinguish between topological twist and twist, although this distinction is not always made in the literature.

Finally, we note here that for the topological twist case we immediately deduce that the vector bilinears vanish: $\zeta^\dagger\gamma_a\zeta = \zeta^\dagger\gamma_a\gamma_3\zeta = 0$. This is true not just for the two-sphere but also for a topological twist on an arbitrary Riemann surface. By contrast, as we will see later in explicit examples, for the twist case on a spindle there is a vector bilinear constructed from the Killing spinors which gives the Killing vector of the spindle. This demonstrates, at the level of gauged supergravity, that supersymmetry for the topological twist and the twist on the spindle are realized in quite distinct ways.

7.1.5 Nowhere zero spinors

The analysis so far was quite general, and apart from $k = \partial_\varphi$ generating a symmetry of the solution, the only other assumption we made was that the Killing spinor ζ is non-zero at the poles of the

[11] If instead $\zeta = \zeta_-$ has negative chirality, one instead takes $A = -\omega_{12}$, which is the upper sign solution in (7.39) for which A is a connection on the cotangent bundle of $\mathbb{\Sigma}$, rather than tangent bundle.

spindle. In fact in almost all cases of interest ζ is *nowhere zero*, as we now briefly discuss.

Starting with a non-trivial solution to the Killing spinor equation (7.1), we can follow an argument given in [536]. Assume that $\zeta(x_0) = 0$ for some point x_0. Assuming the manifold is path-connected we can connect to any other point by a smooth curve $x(s)$. Along this curve (7.1) has the schematic form $\frac{\mathrm{d}}{\mathrm{d}s}\zeta(x(s)) = \tilde{M}(s)\zeta(x(s))$ for a smooth matrix $\tilde{M}$. Invoking uniqueness of first order linear ODEs, with $\zeta(x_0) = 0$ we deduce that $\zeta(x(s)) = 0$ and hence $\zeta \equiv 0$ identically, which contradicts our assumption of non-triviality. Thus, we deduce that ζ is nowhere vanishing.

This result is immediately applicable to AdS$\times\Sigma$ solutions[12] of gauged supergravity. After uplifting to string theory and M-theory the supergravity metric takes the general form

$$\mathrm{d}s^2 = \mathrm{e}^{2\Delta}\mathrm{d}s^2_{\mathrm{AdS}} + \mathrm{d}s^2_{\mathrm{internal}} \,, \tag{7.45}$$

where the warp factor Δ is a function of the internal space coordinates, so as to preserve the symmetries of AdS. The argument given in the preceding paragraph can in fact be immediately adapted to arbitrary supersymmetric solutions of the form (7.45), utilizing the Killing spinor equations for $D = 10, 11$ supergravity. In particular one can conclude that the internal space Killing spinor η is nowhere vanishing. In particular, for solutions where $\mathrm{d}s^2_{\mathrm{internal}}$ contains a spindle, the restriction of η to the spindle is also nowhere zero.

As somewhat of an aside it is interesting to note that in many cases the internal space Killing spinor η for such supersymmetric solutions satisfies

$$\bar{\eta}\eta = \mathrm{e}^{\Delta} \,, \tag{7.46}$$

which implies that η is nowhere zero. For example, the equation (7.46) holds for general supersymmetric AdS$_3$ D3-brane solutions and M2-brane solutions i.e. $\mathcal{N} = (2,0)$ AdS$_3$ solutions of type IIB with five-form flux and $\mathcal{N} = 2$ AdS$_2$ solutions of $D = 11$ supergravity with electric four-form flux, respectively. The same formula holds for the most general supersymmetric AdS$_5$ solutions of $D = 11$ supergravity [168], and for the most general $\mathcal{N} = 2$ AdS$_4$ solutions of $D = 11$ [175]. A similar conclusion is reached for the $\mathcal{N} = (1,0)$ AdS$_3$ solutions of type IIB studied in [190].

The analysis in the previous subsections can also be applied to rigid supersymmetric backgrounds

[12] The AdS$\times D$ solutions of e.g. [197, 527, 530–532], where D is topologically a disc, are constructed from the same family of local solutions that we are considering in this chapter. In these solutions the Killing spinors vanish as one approaches the boundary of the disc. However, since this is a boundary, in fact a singular boundary (since the warp factor vanishes), our argument does not apply.

that contain a spindle. For supersymmetric backgrounds of so-called new minimal supergravity in $d = 3$ and $d = 4$ dimensions, the Killing spinors are necessarily nowhere zero – see, e.g. [536] and [537], respectively, using the above argument. Localization results for supersymmetric field theories with four supercharges in three and four dimensions have mainly been derived in this setting. However, we note that if one uses conformal supergravity the spinor can have zeroes[13]; it would be interesting to know if supersymmetry on a spindle can be realised in this specific context in a manner that is different from the twist and the anti-twist. Furthermore, it would also be of interest to know if such an alternative does exist, then could it ever arise with the rigid background arising as the conformal boundary of an AdS solution.

7.1.6 Uplifted solutions

Let us return to the general formula (7.17) for the flux of the gauge field strength through the spindle. For $\lambda = n_S \pm n_N$ in the twist and anti-twist cases, respectively, we may solve this by taking $p = 0$ and $m_N = -1$, $m_S = \pm 1$. As discussed after equation (7.17) these choices are not unique, but $m_N \in \mathbb{Z}_{n_N}$, $m_S \in \mathbb{Z}_{n_S}$ *are* unique, and this is the invariant data for the local models at each pole. Moreover, notice that $m = \mp 1$ precisely give the tangent and cotangent bundle actions at a pole: if the generator ω of $\mathbb{Z}_n$ acts as $z \mapsto \omega z$ on $\mathbb{C}$, then the one-form $\mathrm{d}z \mapsto \omega\, \mathrm{d}z$, while the tangent vector $\partial_z \mapsto \omega^{-1}\partial_z$. With the above choices we then have that

$$\tilde{A} \equiv A^N = A^S \tag{7.47}$$

is a *global* one-form on the spindle minus both of its poles. We may also then write this near each pole as

$$\tilde{A} = A^N_{(0)} - \frac{1}{n_N}\mathrm{d}\varphi = A^S_{(0)} \pm \frac{1}{n_S}\mathrm{d}\varphi\,, \tag{7.48}$$

where recall that $A^N_{(0)}$ and $A^S_{(0)}$ are smooth one-forms in their respective patches, and in particular smooth at each pole, respectively. It then follows that $\tilde{A}$ is singular at *both* poles, where from (7.48)

[13] To see why the above argument can fail, consider the conformal Killing spinor equation $(\delta^\nu_\mu - \frac{1}{d}\gamma_\mu\gamma^\nu)\nabla_\nu\zeta = \ldots$, where the right hand side could contain other fields, and notice that $(\delta^\nu_\mu - \frac{1}{d}\gamma_\mu\gamma^\nu)$ is a projection.

we deduce

$$\tilde{A}\Big|_N = -\frac{1}{n_N}\mathrm{d}\varphi\,, \qquad \tilde{A}\Big|_S = \begin{cases} +\dfrac{1}{n_S}\mathrm{d}\varphi & \text{Twist} \\ -\dfrac{1}{n_S}\mathrm{d}\varphi & \text{Anti-twist} \end{cases}\,. \tag{7.49}$$

We shall make use of this gauge for the field theory analysis later in the chapter. In particular, from (7.36) and (7.37) we see that $Q_N = Q_S = 0$, and the spinor ζ is *uncharged* in this gauge.

Recall that in sections 7.1.2 and 7.1.3 we "geometrized" the R-symmetry gauge field by introducing a corresponding circle direction ψ, where after appropriate gluing one verifies that $\eta = \mathrm{d}\psi + A$ is a global one-form on the total space M_3. This precisely happens when the Killing spinor equation (7.1) arises by dimensionally reducing a $D = 10$ or $D = 11$ Killing spinor equation, associated with a solution of the form (7.45). On uplifting, the Abelian R-symmetry gauge field A becomes a connection in the internal space metric, with the gauge transformations we have discussed becoming coordinate transformations. The Killing vector ∂_ψ then generates an isometry of the internal space. For example, to be concrete we may consider the original D3-brane spindle solutions in [3]. These are $\mathrm{AdS}_3 \times Y_7$ solutions of type IIB supergravity, where Y_7 is the total space of a regular Sasaki-Einstein 5-manifold SE_5 fibred over a spindle $\mathbb{\Sigma}$. Here SE_5 by definition itself fibres over a Kähler-Einstein 4-manifold KE_4, and fixing a point in the later space inside Y_7 is precisely our (Lens space) M_3, with ψ being fibred both over the KE_4 and also over the spindle. The fibration over the KE_4 doesn't play any role in the regularity analysis we have been discussing thus far.

On the other hand, in the lifted geometry one can phrase things more globally and invariantly. The $D = 10$ or $D = 11$ Killing spinor will form a representation of the isometry group, and if there is a single complex spinor then this statement means it will have definite charge Q_i under the ith $U(1)$ isometry ∂_{φ_i}. Generically, in the examples discussed in this chapter, there are at least two such isometries. However, we can always choose a basis in which the spinor has charge $\frac{1}{2}$ under $\partial_{\varphi_1} \equiv \partial_\psi$, and is uncharged under ∂_{φ_2}. The latter is by definition then a flavour isometry. The gauge $\tilde{A}$ for A in (7.48) is then precisely a choice of gauge for which $\partial_\varphi = \partial_{\varphi_2}$ is a flavour symmetry.

In the known examples of the anti-twist and twist on the spindle, as highlighted at the end of section 7.1.4, there is a vector bilinear for the Killing spinors of the gauged supergravity solution that involves the Killing direction on the spindle. After uplifting, one finds this implies that there is a vector bilinear for the higher-dimensional Killing spinors which is a linear combination of this Killing vector with the Killing vector ∂_ψ. The dual interpretation of this is that the lower-dimensional

SCFT has an R-symmetry that is a linear combination of the rotational symmetry on the spindle with the R-symmetry of the higher-dimensional SCFT upon which the SCFT is wrapped.

7.2 AdS_3 spindles in five dimensions

We now re-examine the case of D3-branes wrapping spindles that was considered in [3, 523, 524]. For the cases associated with solutions of minimal $D = 5$ gauged supergravity we recover exactly the same results as [3] where it was shown, in particular, that only the anti-twist case is allowed. For the cases associated with the $D = 5$ STU gauged supergravity we will recover the results of [523, 524] for the anti-twist case, but we will show here, as a new result, that the twist case is also possible. In contrast to the anti-twist solutions, these new twist solutions are naturally interpreted as the near-horizon limits of D3-branes wrapping a spindle embedded as a holomorphic curve inside a Calabi-Yau four-fold. Moreover, taking a (formal) limit of the central charge for these twist solutions, in which the spindle becomes a two-sphere, we reproduce the central charge of the *topologically* twisted solutions of [148]. We will also discuss how a special sub-class of solutions are associated with M5-branes wrapped on the product of a spindle with a Riemann surface. We also make some comparisons with field theory.

7.2.1 The $D = 5$ gauged supergravity model

The $D = 5$ STU, $U(1)^3$ gauged supergravity model that we shall consider has a Lagrangian given by (see *e.g.* [538])

$$\mathcal{L} = \sqrt{-g}\Big[R - \mathcal{V} - \frac{1}{2}\sum_{I=1}^{3}\left(X^{(I)}\right)^{-2}\left(\partial X^{(I)}\right)^2 - \frac{1}{4}\sum_{I=1}^{3}\left(X^{(I)}\right)^{-2}(F^{(I)})^2\Big] - F^{(1)}\wedge F^{(2)}\wedge A^{(3)}\,, \tag{7.50}$$

where $A^{(I)}$ are three $U(1)$ gauge fields, $I = 1, 2, 3$, with field strengths $F^{(I)} = \mathrm{d}A^{(I)}$. The three scalar fields $X^{(I)} > 0$ satisfy the constraint $X^{(1)}\,X^{(2)}\,X^{(3)} = 1$ and can be written in terms of two canonically normalized scalars φ_1, φ_2 via

$$X^{(1)} = \mathrm{e}^{-\frac{\varphi_1}{\sqrt{6}}-\frac{\varphi_2}{\sqrt{2}}}\,, \qquad X^{(2)} = \mathrm{e}^{-\frac{\varphi_1}{\sqrt{6}}+\frac{\varphi_2}{\sqrt{2}}}\,, \qquad X^{(3)} = \mathrm{e}^{\frac{2\varphi_1}{\sqrt{6}}}\,. \tag{7.51}$$

The potential is given by

$$\mathcal{V} = -4\sum_{I=1}^{3}\left(X^{(I)}\right)^{-1} . \tag{7.52}$$

The model is a consistent truncation of $D = 5$, $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiplets. For a bosonic solution to preserve supersymmetry, we require that the variation of the gravitino and the two gauginos both vanish

$$\begin{aligned} 0 &= \left[\nabla_\mu - \frac{\mathrm{i}}{2}\sum_{I=1}^{3} A_\mu^{(I)} + \frac{1}{6}\sum_{I=1}^{3} X^{(I)}\,\Gamma_\mu + \frac{\mathrm{i}}{24}\sum_{I=1}^{3}\left(X^{(I)}\right)^{-1}\left(\Gamma_\mu{}^{\nu\rho} - 4\,\delta_\mu^\nu\,\Gamma^\rho\right) F_{\nu\rho}^{(I)}\right]\epsilon\,, \\ 0 &= \left[\not{\partial}\varphi_i - 2\sum_{I=1}^{3}\partial_{\varphi_i}X^{(I)} + \frac{\mathrm{i}}{2}\sum_{I=1}^{3}\partial_{\varphi_i}\left(X^{(I)}\right)^{-1} F_{\mu\nu}^{(I)}\Gamma^{\mu\nu}\right]\epsilon\,, \end{aligned} \tag{7.53}$$

where Γ^μ are $D = 5$ Gamma matrices, $\nabla = \mathrm{d} + \frac{1}{4}\omega^{ab}\Gamma_{ab}$, with ω^{ab} the Levi-Civita connection, and ϵ is a Dirac spinor. We can define the R-symmetry gauge field via

$$A^R \equiv \sum_{I=1}^{3} A^{(I)}\,, \tag{7.54}$$

which is then normalized as in section 7.1.

Any supersymmetric solution of this model can be uplifted on S^5 to give a supersymmetric solution of type IIB supergravity using the results of [454]. In particular, the vacuum AdS_5 solution, which has unit radius, uplifts to the $\mathrm{AdS}_5 \times S^5$ solution, dual to $d = 4$ $\mathcal{N} = 4$ SYM theory. In the special case that two gauge fields are equal, and also reducing to one scalar field, for example $A^{(1)} = A^{(2)}$ and $X^{(1)} = X^{(2)}$ (*i.e.* $\varphi_2 = 0$), we can then uplift on a class of M_6 geometries that arise in the most general $\mathrm{AdS}_5 \times M_6$ solutions of $D = 11$ supergravity dual to $\mathcal{N} = 2$ SCFTs [169], to give supersymmetric solutions of $D = 11$ supergravity [539]. In particular, this includes uplifting on M_6 which are an S^4 fibration over a Riemann surface Σ_g with genus $\mathsf{g} > 1$, that arise in the Maldacena-Núñez solution describing M5-branes wrapped on a Riemann surface embedded in a Calabi-Yau two-fold [113], or more generally a punctured Riemann surface as in [119]. Setting three gauge fields equal and the scalars to zero leads to minimal gauged supergravity, and the solutions can be uplifted to type IIB on a Sasaki-Einstein manifold [540]; this case is discussed in [3].

7.2.2 Supergravity solution and global analysis

We consider the supersymmetric solutions of $D = 5$ STU gauged supergravity that were presented in [541], and then recently analysed in [523, 524]. Following[14] [524], the local form of the solutions is given by

$$\begin{aligned} \mathrm{d}s_5^2 &= H^{1/3}\left[\mathrm{d}s^2_{\mathrm{AdS}_3} + \frac{1}{4P}\mathrm{d}y^2 + \frac{P}{H}\mathrm{d}z^2\right] = H^{1/3}\left[\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{\mathbb{\Sigma}}\right], \\ A^{(I)} &= \frac{y}{h_I}\mathrm{d}z\,, \qquad X^{(I)} = \frac{H^{1/3}}{h_I}\,, \end{aligned} \tag{7.55}$$

where $\mathrm{d}s^2_{\mathrm{AdS}_3}$ is a unit radius metric and h_I, H and P are all functions of y given by

$$h_I = y + q_I\,, \qquad H = \prod_{I=1}^{3} h_I\,, \qquad P = H - y^2\,, \tag{7.56}$$

and q_I, $I = 1, 2, 3$ are constants. Note that the field strength can be written as

$$\left(X^{(I)}\right)^{-2} F^{(I)} = \frac{2q_I}{H^{1/2}}\widetilde{\mathrm{vol}}_{\mathbb{\Sigma}}\,, \tag{7.57}$$

where $\widetilde{\mathrm{vol}}(\mathbb{\Sigma})$ is the volume form associated to the conformally rescaled spindle metric $\widetilde{\mathrm{d}s^2}_{\mathbb{\Sigma}} = H^{1/3}\mathrm{d}s^2_{\mathbb{\Sigma}}$.

We want to choose the parameters so that y, z parametrize a compact spindle and with a positive warp factor, $H > 0$. Since $X^{(I)} > 0$, we must have $h_I > 0$. Also note that $P > 0$ implies that $H > 0$. We assume that P has three real roots, which we take to be $y_1 < y_2 < y_3$, and we take $y \in [y_1, y_2]$ with $P \geq 0$. Expanding near the boundaries of this interval, after setting $\varrho_i = 2|y - y_i|^{1/2}$ for $i = 1, 2$, we find

$$\mathrm{d}s^2_{\mathbb{\Sigma}} \simeq \frac{1}{4\,|P'(y_i)|}\left(\mathrm{d}\varrho_i^2 + \kappa_i^2\,\varrho_i^2\,\mathrm{d}z^2\right), \quad \text{where} \quad \kappa_i = \left|\frac{P'(y_i)}{y_i}\right|, \quad i = 1, 2\,. \tag{7.58}$$

We would now like to remove the absolute value in the definition of κ_i. From the expression for P it is clear that $P'(y_1) > 0$ and $P'(y_2) < 0$ and so, with $y_1 < y_2$, we need to distinguish three cases corresponding to the three possible signs for y_i ($i = 1, 2$). This is conveniently achieved by writing

$$\eta_1\, y_1 < 0\,, \qquad \eta_2\, y_2 > 0\,, \tag{7.59}$$

[14] We have slightly changed the parametrization of the solutions as compared to [524] in order to be more uniform with later sections.

with the three cases determined by the signs

$$\begin{aligned} \text{Case A}: \quad & (\eta_1, \eta_2) \;=\; (-1, +1)\,, \\ \text{Case B}: \quad & (\eta_1, \eta_2) \;=\; (+1, +1)\,, \\ \text{Case C}: \quad & (\eta_1, \eta_2) \;=\; (+1, -1)\,. \end{aligned} \tag{7.60}$$

Then, for all three cases we can write

$$\kappa_i \;=\; -\frac{\eta_i P'(y_i)}{y_i}\,, \tag{7.61}$$

and $\mathbb{\Sigma}$ will be a spindle provided we impose

$$\Delta z \;=\; \frac{2\pi}{\kappa_1 n_1} \;=\; \frac{2\pi}{\kappa_2 n_2}\,, \tag{7.62}$$

where $n_1, n_2 \in \mathbb{N}$ with $\text{hcf}(n_1, n_2) = 1$. As we will see, only cases A and B will in fact be realized.

We next demand that the three gauge field fluxes on the spindle are suitably quantized. Specifically, as in [524] with the $D = 5$ spinors carrying charge $\frac{1}{2}$ with respect to each of the gauge fields, we demand

$$\begin{aligned} Q_I \;&\equiv\; \frac{1}{2\pi}\int_{\mathbb{\Sigma}} F^{(I)} \;=\; \left(\frac{y_2}{h_I(y_2)} - \frac{y_1}{h_I(y_1)}\right)\frac{\Delta z}{2\pi}\,, \\ &\equiv\; \frac{p_I}{n_1\, n_2}\,, \qquad p_I \in \mathbb{Z}\,. \end{aligned} \tag{7.63}$$

After writing

$$P \;=\; (y - y_1)\,(y - y_2)\,(y - y_3)\,, \tag{7.64}$$

and comparing with the expression for P in (7.56), one can now solve (7.62) and (7.63) for $y_{1,2}$, $q_{1,2,3}$ and Δz in terms of the spindle data n_1, n_2 and the quantized fluxes $p_{1,2,3}$. We will give the explicit solution below.

However, without knowing the explicit solution one can compute the Euler number of the spindle, $\chi(\mathbb{\Sigma})$. Using the metric

$$\mathrm{d}s^2_{\mathbb{\Sigma}} \;=\; \frac{1}{4P}\mathrm{d}y^2 + \frac{P}{H}\mathrm{d}z^2\,, \tag{7.65}$$

one finds $\sqrt{g_{\mathbb{\Sigma}}}\, R_{\mathbb{\Sigma}} = 2\,[H^{-3/2}(PH' - HP')]'$ from which we calculate

$$\chi(\mathbb{\Sigma}) \;=\; \frac{1}{4\pi}\int_{\mathbb{\Sigma}} \sqrt{g_{\mathbb{\Sigma}}}\, R_{\mathbb{\Sigma}}\, \mathrm{d}y\, \mathrm{d}z \;=\; \frac{\Delta z}{2\pi}\left(\frac{P'(y_1)}{|y_1|} - \frac{P'(y_2)}{|y_2|}\right) \;=\; \frac{n_1+n_2}{n_1\, n_2}\,, \tag{7.66}$$

where we have used $H(y_i) = y_i^2$, (7.58)–(7.62), and the result is independent of which case is considered. Equivalently, using $\mathrm{e}^1 = \mathrm{d}y/(2P^{1/2})$, $\mathrm{e}^2 = (P^{1/2}/H^{1/2})\mathrm{d}z$ as an orthonormal frame, the spin connection for $\mathrm{d}s^2_{\mathbb{\Sigma}}$ is given by $\omega^{12}_{\mathbb{\Sigma}} = [H^{-3/2}(PH' - HP')]\mathrm{d}z$. When evaluated at the roots we have

$$\omega^{12}_{\mathbb{\Sigma}}(y_1) = -\frac{\mathrm{d}\varphi}{n_1}, \qquad \omega^{12}_{\mathbb{\Sigma}}(y_2) = +\frac{\mathrm{d}\varphi}{n_2}, \tag{7.67}$$

and hence $\chi(\mathbb{\Sigma}) = (2\pi)^{-1}\int_{\mathbb{\Sigma}} \mathrm{d}\omega^{12}_{\mathbb{\Sigma}}$ gives the same result. Now $\omega^{12}_{\mathbb{\Sigma}}$ is an $SO(2) \cong U(1)$ connection on the tangent bundle, and it is interesting to highlight that the behaviour at the poles given in (7.67) is in alignment with the comments in section 7.1.6.

In addition, we can also compute the total R-symmetry flux $Q^R \equiv (Q_1 + Q_2 + Q_3)$. To that end, one can use the identities

$$\begin{aligned} P' \;&=\; h_1\, h_2 + h_2\, h_3 + h_1\, h_3 - 2y\,, \\ y_i^2 \;&=\; h_1(y_i)\, h_2(y_i)\, h_3(y_i)\,, \qquad (i = 1, 2)\,, \end{aligned} \tag{7.68}$$

in (7.61), which allows us to rewrite (7.62) as

$$\frac{\Delta z}{2\pi}\left[2 - y_i \sum_{I=1}^{3} \frac{1}{h_I(y_i)}\right] \;=\; \frac{\eta_i}{n_i}\,, \quad \text{(no sum on } i)\,. \tag{7.69}$$

One can now immediately prove that the R-symmetry flux is given by

$$Q^R \;=\; \frac{\eta_1\, n_2 - \eta_2\, n_1}{n_1 n_2} \qquad \Leftrightarrow \qquad p_1 + p_2 + p_3 \;=\; \eta_1\, n_2 - \eta_2\, n_1\,. \tag{7.70}$$

Case B is the anti-twist that was considered in [523, 524]. We now see that the same local solutions can also potentially realize the twist in cases A and C, although we will ultimately find that case C is not possible, while case A is.

The central charge for the $(0,2)$ SCFT dual to the $\mathrm{AdS}_3\times\mathbb{\Sigma}$ solution can be expressed in the form

$$c = \frac{3\,L}{2\,G_{(3)}} = \frac{3}{2\pi}\,\Delta z\,(y_2-y_1)\,N^2\,, \tag{7.71}$$

where L is the AdS radius and $G_{(3)}$ the three-dimensional Newton constant, which can be computed from the ten-dimensional one by a suitable integral over the internal space geometry, as in [524]. To express this in terms of spindle data we now need to solve for $y_{1,2}$ and Δz in terms of n_i and p_1, p_2 (recall that p_3 is given by the constraint (7.70)), as outlined below (7.64). The explicit solution given in [524] is valid for case B. With a little effort one can rewrite the solution of [524] by reintroducing p_3 and then using the signs $\eta_{1,2}$ introduced in (7.60) to deduce expressions that are valid for all three cases. Specifically, for the constants q_I we find

$$\begin{aligned} q_1 &= \frac{8}{s^3}\,p_2\,p_3\,(n_1+\eta_2\,p_2)\,(n_1+\eta_2\,p_3)\,(n_2-\eta_1\,p_2)\,(n_2-\eta_1\,p_3)\,,\\ q_2 &= q_1\big|_{p_1\leftrightarrow p_2}\,,\\ q_3 &= q_1\big|_{p_1\leftrightarrow p_3}\,, \end{aligned} \tag{7.72}$$

where we have introduced

$$s \equiv n_1^2+n_2^2-(p_1^2+p_2^2+p_3^2)\,. \tag{7.73}$$

The roots $y_{1,2}$ are given by

$$\begin{aligned} y_1 &= -\frac{8\eta_2}{s^3}p_1p_2p_3(n_1+\eta_2\,p_1)(n_1+\eta_2\,p_2)(n_1+\eta_2\,p_3)\,,\\ y_2 &= y_1\big|_{n_1\leftrightarrow n_2,\;\eta_2\to-\eta_1}\,, \end{aligned} \tag{7.74}$$

in terms of which y_3 is fixed using $\sum_{I=1}^3(y_I+q_I)=1$, and explicitly we have

$$y_3 = \frac{8\eta_1\eta_2}{s^3}\prod_{I=1}^{3}(n_1+\eta_2p_I)(n_2-\eta_1p_I)\,. \tag{7.75}$$

Finally, we have

$$\frac{\Delta z}{2\pi} = \frac{s}{2n_1n_2(\eta_2\,n_1+\eta_1\,n_2)}\,. \tag{7.76}$$

With these results we can now express the central charge, for all three cases, in the simple form

$$c = \frac{6\, p_1 p_2 p_3}{n_1 n_2\, s} N^2 \,. \tag{7.77}$$

We now examine whether it is possible to choose the spindle data $n_{1,2}$ and quantized fluxes p_I, satisfying (7.70), such that cases A, B and C are in fact realized as *bona fide* solutions. Recall we have assumed from the start that $n_{1,2} > 0$. We also need to check that

$$h_I > 0 \,, \qquad y_1 < y_2 < y_3 \,, \qquad \eta_1\, y_1 < 0 \,, \qquad \eta_2\, y_2 > 0 \,, \tag{7.78}$$

where the first condition comes from the fact that the scalar fields $X^{(I)} > 0$, and the last two come from (7.59). To analyse these constraints one can utilize the expressions given in (7.72), (7.74). Doing so, we find that case C is not possible, leaving just cases A and B.

For case B, which realizes the anti-twist with $\eta_1 = \eta_2 = 1$, we find that the above conditions (7.78) are all satisfied provided that

$$\text{Anti-twist}: \qquad p_1, p_2 > 0 \,, \qquad p_3 = n_2 - n_1 - p_1 - p_2 > 0 \,. \tag{7.79}$$

We can then show that, in particular, the central charge given in (7.77) is positive, $c > 0$. This is the same result that was obtained in [524].

We also find that case A, which realizes the twist with $\eta_1 = -1$, $\eta_2 = 1$, is possible provided that

$$\text{Twist}: \qquad n_2 > n_1 \quad \text{and two } p_i \text{ are positive} \,. \tag{7.80}$$

Notice here that since $p_1 + p_2 + p_3 = -n_1 - n_2$, given that two of the p_i are positive, the third must then be negative. Moreover, these conditions imply that the central charge is positive, $c > 0$. We also note that after *formally* setting $n_1 = n_2$ in the expression for the central charge in case A with $\eta_1 = -1$, $\eta_2 = 1$, we recover the expression for the central charge obtained in [148] for D3-branes wrapping a two-sphere with a topological twist. However, we emphasize that this is a formal statement, since *e.g.* from (7.74) we see that $y_1 = y_2$ in this limit, giving a double root for the polynomial P, which is incompatible with having a compact spindle. Instead the solutions of [148] may be recovered from our local solutions by utilizing a scaling limit, as shown in appendix A of [7].

It is interesting to ask whether case A or B can be realized in minimal gauged supergravity. We

will obtain solutions of minimal gauged supergravity after setting $p_1 = p_2 = p_3 = 1/3(\eta_1 n_2 - \eta_2 n_1)$, and we immediately find that only the anti-twist, case B, is allowed, confirming what was already shown in [3]. It is also worth noting that if we formally set one of the fluxes to zero, say $p_1 = 0$, we will obtain a local solution that preserves twice as much supersymmetry. However, we can immediately see from the expression for the central charge, which vanishes in this limit, that this cannot make sense. Indeed, from (7.74) we see that we obtain a double root for the polynomial P in this limit, and hence we will not obtain a compact spindle.

Another interesting limit to consider is when two charges are equal, for example $p_1 = p_2 \neq 0$, with $p_3 = -p_1 - p_2 + (\eta_1\, n_2 - \eta_2\, n_1)$. For case B, the anti-twist, we just need $p_1 > 0$ and $n_2 - n_1 > 2p_1$, as shown in [524]. We also find that case A, the twist, is possible when we have $n_2 > n_1$ and $p_1 > 0$. As mentioned earlier, these solutions can be uplifted on M_6 to give solutions of $D = 11$ supergravity, as we discuss further in section 7.2.6.

7.2.3 Uplift to type IIB supergravity

In this section we briefly consider the regularity of the uplift of the solutions (7.55). Any solution of $d = 5, U(1)^3$ gauged supergravity can be uplifted on S^5 to a solution of type IIB supergravity. From [454] we read that the corresponding metric in ten dimensions is

$$L^{-2}\,\mathrm{d}s_{10}^2 = W^{1/2}\,\mathrm{d}s_5^2 + W^{-1/2}\sum_{I=1}^{3}(X^{(I)})^{-1} + [\mathrm{d}\mu_I + \mu_I\,(\mathrm{d}\phi_I + A^{(I)})^2]\,, \tag{7.81}$$

with self-dual five-form flux[15]

$$\begin{aligned} L^{-4}\,F_5 = (1+\star_{10})\{2\sum_{I=1}^{3}[(X^{(I)})^2\,\mu_I^2 - W\,X^{(I)}]\,\mathrm{vol}_5 + \frac{1}{2}\sum_{I=1}^{3}(X^{(I)})^{-1}\,\star_5\,\mathrm{d}X^{(I)}\wedge\mathrm{d}(\mu_I^2) \\ + \frac{1}{2}\sum_{I=1}^{3}(X^{(I)})^{-2}\,\mathrm{d}(\mu_I^2)\wedge(\mathrm{d}\phi_I + A^{(I)})\wedge\star_5 F^{(I)}\}\,, \end{aligned} \tag{7.82}$$

where $L > 0$ is an arbitrary length scale, fixed by flux quantization, while the warp factor W is given by

$$W = \sum_{I=1}^{3}\mu_I^2\,X^{(I)} > 0\,. \tag{7.83}$$

[15] As pointed out in [178] there is a typo in the five-form flux in [454], which we correct below following [524].

Finally, the three pairs of coordinates $\{\mu_I,\, \phi_I\}$ parametrize $S^5 \subset \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$, with $\sum_{I=1}^3 \mu_I^2 = 1$ and the angular coordinates ϕ_I have period 2π.

The ten-dimensional metric can be re-written as

$$\mathrm{d}s_{10}^2 = L^2\, W^{1/2}\, H^{1/3}\, (\mathrm{d}s_{\mathrm{AdS}_3}^2 + \mathrm{d}s_{Y_7}^2)\,, \tag{7.84}$$

where the seven-dimensional manifold Y_7 is an S^5 bundle over the spindle $\mathbb{\Sigma}$:

$$S^5 \hookrightarrow Y_7 \to \mathbb{\Sigma}\,, \tag{7.85}$$

which is completely smooth thanks to the quantization condition (7.63), which as described in section 7.1 is sufficient to guarantee regularity of a $U(1)$ fibration over $\mathbb{\Sigma}$. The five-form flux should also be suitably quantized, integrating F_5 through a copy of S^5 at any point in the five-dimensional spacetime. We find

$$N \equiv \frac{1}{(2\pi\ell_s)^4\, g_s} \int_{S^5} F_5 = \frac{L^4}{4\pi\ell_s^4 g_s}\,, \tag{7.86}$$

which, together with the standard formula [542]

$$c = \frac{3L}{2G_{(3)}}\,, \tag{7.87}$$

allows to derive

$$c = \frac{3L}{2}\frac{L^7}{G_{(10)}} \int_{Y_7} W^2\, H^{4/3}\, \mathrm{vol}_7 = 3(y_2 - y_1)\, \frac{\Delta z}{2\pi}\, N^2\,, \tag{7.88}$$

where we have used

$$G_{(10)} = \frac{(2\pi)^7 g_s^2 \ell_s^8}{16\pi}\,. \tag{7.89}$$

This gives precisely (7.71).

7.2.4 Killing spinors

In this subsection we explicitly present the solutions to the Killing spinor equations (7.53). Some details of the derivation are presented in appendix C of [7]. We begin by choosing the following

orthonormal frame for the metric (7.55):

$$\mathrm{e}^a \;=\; H^{1/6}\,\bar{\mathrm{e}}^a, \quad a=0,1,2\,, \qquad \mathrm{e}^3 \;=\; \frac{H^{1/6}}{2P^{1/2}}\,\mathrm{d}y\,, \qquad \mathrm{e}^4 \;=\; \frac{P^{1/2}}{H^{1/3}}\,\mathrm{d}z\,, \tag{7.90}$$

where $\bar{\mathrm{e}}^a$ is an orthonormal frame for the unit radius metric on AdS_3. We use five-dimensional gamma matrices

$$\Gamma_a \;=\; \beta_a\otimes\gamma_3\,, \qquad \Gamma_3 \;=\; 1\otimes\gamma_1\,, \qquad \Gamma_4 \;=\; 1\otimes\gamma_2\,, \tag{7.91}$$

where $\beta_0 = \mathrm{i}\sigma^2$, $\beta_1 = \sigma^1$, $\beta_2 = \sigma^3$ are three-dimensional gamma matrices for AdS_3 directions, while $\gamma_1 = \sigma^1$, $\gamma_2 = \sigma^2$ are two-dimensional gamma matrices for the spindle directions, with chirality matrix $\gamma_3 = -\mathrm{i}\,\gamma_1\gamma_2 = \sigma^3$.

We next introduce two-component AdS_3 Killing spinors ϑ satisfying

$$\nabla_a\vartheta \;=\; \frac{1}{2}\,\beta_a\vartheta\,, \tag{7.92}$$

whose two solutions $\vartheta_+^{(A)}$, $A=1,2$, are described explicitly in appendix B of [7]. We then find that the two $D=5$ Killing spinors $\epsilon^{(A)}$, $A=1,2$, solving (7.53) are

$$\epsilon^{(A)} \;=\; \vartheta_+^{(A)}\otimes\zeta\,, \tag{7.93}$$

where the two-component spinor ζ on the spindle is given by

$$\zeta \;\equiv\; \begin{pmatrix}\zeta_+\\ \zeta_-\end{pmatrix} \;=\; \frac{\mathrm{e}^{\mathrm{i}z}}{\sqrt{2}H^{1/6}}\begin{pmatrix}[\sqrt{H}-y]^{1/2}\\ -[\sqrt{H}+y]^{1/2}\end{pmatrix} \;=\; H^{1/12}\,\mathrm{e}^{\mathrm{i}z}\begin{pmatrix}\sin\frac{\alpha}{2}\\ -\cos\frac{\alpha}{2}\end{pmatrix}\,, \tag{7.94}$$

and the function $\alpha=\alpha(y)$ satisfies

$$\sin\alpha \;=\; \frac{P^{1/2}}{H^{1/2}}\,, \qquad \cos\alpha \;=\; \frac{y}{H^{1/2}}\,. \tag{7.95}$$

In particular, the Killing spinors given by (7.93) satisfy the single projection condition

$$M\epsilon^{(A)} \;=\; -\mathrm{i}\,\epsilon^{(A)}\,, \quad \text{where} \quad M \;\equiv\; \cos\alpha\,\Gamma_{34}+\mathrm{i}\sin\alpha\,\Gamma_3\,, \tag{7.96}$$

and one can verify that $M^2=-1$. We thus preserve half of the supersymmetries, corresponding to

$d = 2$ $\mathcal{N} = (0,2)$ supersymmetry. One also sees from (7.94) that the squared norm of the Killing spinor is

$$\zeta^\dagger \zeta \;=\; H^{1/6}\,, \tag{7.97}$$

which is nowhere vanishing, and proportional to the square root of the warp factor, as expected from the general discussion in section 7.1.5. Moreover, on the spindle we have the Killing vector bilinear

$$\frac{\mathrm{i}}{\zeta^\dagger\zeta}(\zeta^\dagger\gamma^m\gamma_3\zeta)\partial_m \;=\; \partial_z\,, \tag{7.98}$$

where here $x^m = (y,z)$ are coordinates on the spindle. On the $D = 5$ spacetime we then deduce

$$(\bar{\epsilon}^{(A)}\Gamma^\mu\epsilon^{(B)})\partial_\mu \;=\; (\bar{\vartheta}^{(A)}_+\beta^a\vartheta^{(B)}_+)\partial_a + \mathrm{i}(\bar{\vartheta}^{(A)}_+\vartheta^{(B)}_+)\partial_z\,, \tag{7.99}$$

which reveals the way in which supersymmetry is being realized for the AdS$_3 \times \mathbb{\Sigma}$ solutions. In particular, recalling the explicit Killing spinors on AdS$_3$ (see appendix B of [7]) we conclude that, schematically, the last term in (7.99) implies that the azimuthal rotations on the spindle are acting as an R-symmetry for the $d = 2, \mathcal{N} = (0,2)$ supersymmetry algebra. By contrast this term is absent in the supersymmetry algebra for AdS$_3 \times \mathbb{\Sigma}_g$ solutions associated with a topological twist on the Riemann surface $\mathbb{\Sigma}_g$. One might wonder how it is possible to realize $\mathcal{N} = (0,2)$ supersymmetry given that (in contrast with the standard topological twist) the spinor ζ on the internal space is non chiral. The answer to that puzzle is that the chirality is inherited by the fact that both Killing spinors are built in (7.93) using the AdS spinors $\vartheta^{(A)}_+$, whose vector bilinears realize only a chiral $SL(2,\mathbb{R}) \subset SL(2,\mathbb{C})$ subalgebra of the AdS$_3$ isometries.

Using the expression (7.94) for the Killing spinors, together with (7.59), one can immediately deduce

- Case A, Twist: $\zeta_+(y_1) = 0$ and $\zeta_+(y_2) = 0$, with $\zeta_- \neq 0$ everywhere.
- Case B, Anti-Twist: $\zeta_-(y_1) = 0$ but ζ_- is non-vanishing everywhere else, while $\zeta_+(y_2) = 0$ but ζ_+ is non-vanishing everywhere else.

The corresponding R-symmetry flux in the two cases is

$$Q^R \equiv \frac{1}{2\pi}\int_{\Sigma} F^R = \frac{p_1+p_2+p_3}{n_1 n_2} = \begin{cases} -\dfrac{n_2+n_1}{n_1 n_2} & \text{Case A, Twist}\,, \\ \dfrac{n_2-n_1}{n_1 n_2} & \text{Case B, Anti-Twist}\,. \end{cases} \tag{7.100}$$

Comparing to the general analysis in section 7.1.3, recall that to obtain the R-symmetry flux in (7.38) we assumed $\zeta_+^N(0)$, $\zeta_+^S(0) \neq 0$, while if instead $\zeta_-^N(0)$, $\zeta_-^S(0) \neq 0$ one obtains the upper sign in (7.39). The latter then agrees with the R-symmetry flux in (7.100), where n_1 may be identified with either of n_S or n_N (and then respectively n_2 with either n_N or n_S). On the other hand, for case B since ζ_+ is non-vanishing at y_1, we identify $y_1 = y_N$, $y_2 = y_S$, and the R-symmetry flux in (7.100) agrees with the anti-twist in equation (7.38).

Note that in the gauge we are using for $A^{(I)}$, the R-symmetry gauge field is not regular at either pole located at $y = y_1$ or $y = y_2$. However, using (7.69) we may obtain a smooth gauge at $y = y_1$ via the gauge transformation $A^R \to A^R_{(0)} = A^R - 2\,\mathrm{d}z + \frac{\eta_1}{n_1}\mathrm{d}\varphi$. Here we have introduced $\varphi = \frac{2\pi}{\Delta z} z$, with $\Delta\varphi = 2\pi$, and the subscript on $A^R_{(0)}$ indicates that this gauge field is smooth at the pole, as in the notation and discussion in section 7.1. In particular the shift by $\frac{\eta_1}{n_1}\mathrm{d}\varphi$ is equivalent to an $SL(2,\mathbb{Z})$ transformation in the covering space. This gauge transformation will act on the spinor, which is only charged under the R-symmetry, and replace the $\mathrm{e}^{\mathrm{i}z}$ phase in (7.94) with $\mathrm{e}^{\frac{\mathrm{i}\eta_1\varphi}{2n_1}}$. Noting that the frame we are using is the analogue of the hatted frame in section 2 (which is the frame that is not regular at $y = y_1$), we see that this behaviour is in agreement with (7.26). Recall here that for the twist case A, $\zeta_+(y_1) = 0$ and $\eta_1 = -1$, while for the anti-twist case B, $\zeta_-(y_1) = 0$ and $\eta_1 = +1$. It then follows from (7.94) that in the locally Euclidean frame and in the new gauge, for twist case A we have $\zeta_+ \sim \mathrm{e}^{-\frac{\mathrm{i}\varphi}{n_1}}\rho_1$, where ρ_1 is the radial coordinate at $y = y_1$, while for the anti-twist case B we instead have $\zeta_- \sim \mathrm{e}^{+\frac{\mathrm{i}\varphi}{n_1}}\rho_1$. In both cases we have a simple holomorphic zero of the spinor to leading order at the pole $\rho_1 = 0$. Similar comments apply, *mutatis mutandis*, to the behaviour of the spinors at the other pole located at $y = y_2$.

Finally, as already noted, the spinor (7.94) is charged under azimuthal rotations of the spindle, due to the phase $\mathrm{e}^{\mathrm{i}z}$. We consider the following two-parameter family of gauge transformations on the gauge fields

$$\tilde{A}^{(I)} \equiv A^{(I)} + \mathfrak{s}^I \mathrm{d}z = \rho_I(y)\,\mathrm{d}\varphi\,, \tag{7.101}$$

where the $\mathfrak{s}^I$ are constants satisfying the constraint

$$\mathfrak{s}^1 + \mathfrak{s}^2 + \mathfrak{s}^3 = -2\,. \tag{7.102}$$

The latter ensures that the R-symmetry gauge field transforms to $\tilde{A}^R = A^R - 2\,\mathrm{d}z$, which cancels the phase $\mathrm{e}^{\mathrm{i}z}$ in the spinor as the spinor has charge $\frac{1}{2}$ under A^R. In (7.101) we have introduced the functions

$$\rho_I(y) \equiv \left(\frac{y}{h_I} + \mathfrak{s}^I\right)\frac{\Delta z}{2\pi}\,, \tag{7.103}$$

which will play a role in the field theory analysis in the next subsection (where we will also see that the gauge choice of all $\mathfrak{s}^I$ equal, namely $\mathfrak{s}^I = -2/3$, has a notable property). From (7.101) one can check that

$$\tilde{A}^R\Big|_{y=y_1} = -\frac{\eta_1}{n_1}\mathrm{d}\varphi\,, \qquad \tilde{A}^R\Big|_{y=y_2} = -\frac{\eta_2}{n_2}\mathrm{d}\varphi\,. \tag{7.104}$$

This agrees with the general conditions for a local gauge in which the spinor is uncharged with respect to azimuthal rotations that we derived earlier (7.49), where in particular recall that our twist case A is a charge conjugate of the twist considered in section 7.1.

7.2.5 Comparison with field theory

As in [524], there is a natural field theory interpretation[16] of the solutions we have constructed. One begins with $d = 4$, $\mathcal{N} = 4$ SYM theory, which is dual to the $\mathrm{AdS}_5 \times S^5$ vacuum of the $D = 5$, STU $U(1)^3$ gauged supergravity theory. One then compactifies this field theory on the spindle $\mathbb{\Sigma}$, together with magnetic fluxes Q_I (7.63) for the three Abelian gauge fields. The existence of the AdS_3 supergravity solutions suggests that this compactified theory flows to a $d = 2$, $(0,2)$ SCFT in the IR. There is an important distinction here between the twist and anti-twist solutions, that we discuss further in chapter 9.

Following [3, 524], we consider the anomaly polynomial of $\mathcal{N} = 4$ SYM, with background gauge fields fluxes $F^{(I)}$ for the $U(1)^3 \subset SO(6)$ Abelian global symmetry group. In the large N limit for

[16] See [3] for a more general interpretation for the solutions of minimal gauged supergravity and also section 7.2.6 for an M5-brane interpretation of a sub-class.

$SU(N)$ gauge group this reads

$$\mathcal{A}_{4\mathrm{d}} \;=\; c_1(F^{(1)})\, c_1(F^{(2)})\, c_1(F^{(3)})\, \frac{N^2}{2}\,. \tag{7.105}$$

Here $c_1(F^{(I)})$ denote the first Chern classes of the $U(1)$ bundles with gauge field curvatures $F^{(I)}$. Recalling the expression (7.101) for the gauge field, we next introduce connection one-forms

$$\mathscr{A}^{(I)} \;=\; \rho_I(y)\,(\mathrm{d}\varphi + \mathcal{A}_{\mathcal{J}})\;, \tag{7.106}$$

with curvature

$$\mathscr{F}^{(I)} \;=\; \mathrm{d}\mathscr{A}^{(I)} \;=\; \rho_I'(y)\,\mathrm{d}y\wedge(\mathrm{d}\varphi + \mathcal{A}_{\mathcal{J}}) + \rho_I(y)\,\mathcal{F}_{\mathcal{J}}\,, \tag{7.107}$$

where $\mathcal{F}_{\mathcal{J}} = \mathrm{d}\mathcal{A}_{\mathcal{J}}$. Here $\mathcal{A}_{\mathcal{J}}$ gauges the azimuthal symmetry of the spindle. In particular, this means that setting $\mathcal{A}_{\mathcal{J}} = 0$ the $\mathscr{A}^{(I)}$ reduce to the gauge fields $\tilde{A}^{(I)}$ in (7.101) *i.e.* in a gauge where the Killing spinor is uncharged under azimuthal rotations of the spindle generated by ∂_φ. This implies the flux conditions

$$\frac{1}{2\pi}\int_{\Sigma}\mathscr{F}^{(I)} \;=\; \rho_I(y_2) - \rho_I(y_1) \;=\; \frac{p_I}{n_2 n_1} \qquad (I=1,2,3)\,, \tag{7.108}$$

together with the constraints (7.104) at the poles:

$$\sum_{I=1}^{3}\rho_I(y_1) \;=\; -\frac{\eta_1}{n_1}\,, \qquad \sum_{I=1}^{3}\rho_I(y_2) \;=\; -\frac{\eta_2}{n_2}\,. \tag{7.109}$$

The general solution to the constraints (7.108), (7.109) for $\rho_I(y_i)$, with p_3 determined by (7.70), is

$$\begin{aligned}
&\rho_1(y_1) \;=\; \alpha_1\,, \quad \rho_2(y_1) \;=\; \alpha_2\,, \quad \rho_3(y_1) \;=\; -\frac{\eta_1}{n_1} - \alpha_1 - \alpha_2\,,\\
&\rho_1(y_2) \;=\; \frac{p_1}{n_1\, n_2} + \alpha_1\,, \quad \rho_2(y_2) \;=\; \frac{p_2}{n_1\, n_2} + \alpha_2\,,\\
&\rho_3(y_2) \;=\; -\frac{\eta_2}{n_2} - \frac{p_1}{n_1\, n_2} - \frac{p_2}{n_2 n_1} - \alpha_1 - \alpha_2\,,
\end{aligned} \tag{7.110}$$

where α_1, α_2 are *a priori* arbitrary. Notice that the AdS_3 supergravity solution gave explicit functions (7.103) for $\rho_I(y)$, although these are determined up to a choice of $\mathfrak{s}^I$ satisfying the constraint (7.102). This is equivalent to the freedom of choosing α_1, α_2 above. However, we shall see momentarily that

the central charge of the theory is independent of α_1, α_2.

The curvature form $\mathscr{F}^{(I)}$ defines a corresponding first Chern class $c_1(\mathcal{L}_I) = [\mathscr{F}^{(I)}/2\pi]$, and similarly we define $c_1(\mathcal{J}) = [\mathcal{F}_{\mathcal{J}}/2\pi]$. We then write

$$c_1(F^{(I)}) = \Delta_I\, c_1(R_{2\mathrm{d}}) + c_1(\mathcal{L}_I)\,, \tag{7.111}$$

with the trial R-charges Δ_I constrained to satisfy $\Delta_1 + \Delta_2 + \Delta_3 = 2$, so that the superpotential of $\mathcal{N} = 4$ SYM has R-charge 2, in $\mathcal{N} = 1$ language. The $d = 2$ anomaly polynomial is then obtained by integrating $\mathcal{A}_{4\mathrm{d}}$ over $\mathbb{\Sigma}$:

$$\begin{aligned}\mathcal{A}_{2\mathrm{d}} = \int_{\mathbb{\Sigma}} \mathcal{A}_{4\mathrm{d}} = \Big[&(\Delta_1\Delta_2 I_3 + \Delta_2\Delta_3 I_1 + \Delta_3\Delta_1 I_2)\, c_1(R_{2\mathrm{d}})^2 \\ &+ (\Delta_1 I_4 + \Delta_2 I_5 + \Delta_3 I_6)\, c_1(R_{2\mathrm{d}}) c_1(\mathcal{J}) + I_7 c_1(\mathcal{J})^2\Big]\, \frac{N^2}{2}\,.\end{aligned} \tag{7.112}$$

Here

$$\begin{aligned} &I_I = \rho_I(y_2) - \rho_I(y_1) \equiv [\rho_I]_{y_1}^{y_2}\,, \quad I_4 = [\rho_2\rho_3]_{y_1}^{y_2}\,, \quad I_5 = [\rho_3\rho_1]_{y_1}^{y_2}\,, \\ &I_6 = [\rho_1\rho_2]_{y_1}^{y_2}\,, \quad I_7 = [\rho_1\rho_2\rho_3]_{y_1}^{y_2}\,. \end{aligned} \tag{7.113}$$

The $d = 2$ superconformal R-symmetry extremizes the trial function

$$c_{\mathrm{trial}} = 3\,\mathrm{tr}\,\gamma_3 R_{\mathrm{trial}}^2\,, \tag{7.114}$$

where

$$R_{\mathrm{trial}} = R_{2\mathrm{d}} + \varepsilon\,\mathcal{J}\,. \tag{7.115}$$

From (7.112) we find

$$\begin{aligned} c_{\mathrm{trial}} = 3\Big[&(\Delta_1\Delta_2 I_3 + \Delta_2\Delta_3 I_1 + \Delta_3\Delta_1 I_2) + (\Delta_1 I_4 + \Delta_2 I_5 + \Delta_3 I_6)\,\varepsilon \\ &+ I_7\varepsilon^2\Big] N^2\,. \end{aligned} \tag{7.116}$$

Extremizing over Δ_1, Δ_2 and ε, we obtain the extremal values

$$\begin{aligned} \Delta_1^\star &= -2\frac{p_1}{s}(\eta_2 n_1 + p_1) - \varepsilon^* \alpha_1 \,, & \Delta_2^\star &= -2\frac{p_2}{s}(\eta_2 n_1 + p_2) - \varepsilon^* \alpha_2 \,, \\ \varepsilon^\star &= 2\frac{n_2 n_1}{s}(\eta_1 n_2 + \eta_2 n_1) \,, & c^\star &= \frac{6\, p_1 p_2 p_3}{n_2 n_1\, s} N^2 \,, \end{aligned} \tag{7.117}$$

where note that $c^\star$ agrees with the general supergravity result (7.77). Notice that neither of the critical values $\varepsilon^\star$ nor $c^\star$ depend on the parameters α_1, α_2, but the critical R-charges $\Delta_I^\star$ do. This is a gauge freedom, and in fact the parameters α_1, α_2 may be removed from c_{trial} via the change of variable $\Delta_I \to \Delta_I - \varepsilon\,\alpha_I$, $I = 1, 2$ (where $\Delta_3 = 2 - \Delta_1 - \Delta_2$). We also note that by choosing α_1, α_2 with $\mathfrak{s}^I = -2/3$ in (7.103) we find the extremal values $\Delta_1^\star = \Delta_2^\star = \Delta_3^\star = 2/3$, the notable property we alluded to earlier (just below (7.103)).

In the limit $n_1 = n_2 = 1$, when the spindle reduces to a two-sphere, we should be able to *formally* compare our results with (3.12) of reference [148], where their $\eta_{\mathbb{\Sigma}} = 2$, $d_G = N^2$, $\mathfrak{g} = 0$ and $\kappa = 1$. We find

$$c_{BB} = \frac{24\, a_1 a_2 (1 + a_1 + a_2)}{1 + 4(a_1^2 + a_2^2 + a_1 a_2 + a_1 + a_2)} N^2 \,. \tag{7.118}$$

This indeed agrees with the $n_1 = n_2 = 1$ limit of $c^\star$ in the twist case A, identifying $a_i = p_i/2$. Notice the configurations of [148] are a topological twist for the two-sphere, and arise here as a limit of our more general twist case A spindle configurations. Note, however, that the supergravity solutions of [148] do not arise as a limit of the solutions for the spindle, but instead can be obtained from the local solutions (7.55) after taking a certain scaling limit, as discussed in appendix A of [7].

7.2.6 M5-branes

As noted above there is an alternative field theory interpretation of the solutions of $D = 5$ gauged supergravity that we have constructed in the special case that two of the magnetic fluxes are equal. Start with a $d = 4$, $\mathcal{N} = 2$ field theory that is dual to an $\text{AdS}_5 \times M_6$ solution of $D = 11$ supergravity of the type discussed in [119, 169]. One can then compactify on the spindle $\mathbb{\Sigma}$, together with magnetic fluxes Q_I (7.63) for the three Abelian gauge fields, with two equal. The existence of our corresponding AdS_3 supergravity solutions, uplifted to $D = 11$ using the results of [539], indicates that this compactified theory flows to a $d = 2$, $\mathcal{N} = (0, 2)$ SCFT in the IR.

Focusing, for simplicity, on the explicit case of the $\text{AdS}_5 \times M_6$ solution given in [113], associated

with N M5-branes wrapping a Riemann surface of genus g, and setting $p_1 = p_2$ for definiteness, it was shown in [524] for the anti-twist solutions that the central charge of the $d = 2$ SCFT can be written in the form

$$c = \frac{8\,p_1^2 p_3}{n_1 n_2\, s}\,(\mathsf{g}-1)N^3\,, \tag{7.119}$$

where

$$\text{Anti-Twist}: \quad p_1 = p_2 > 0\,, \qquad p_3 = n_2 - n_1 - 2p_1 > 0\,. \tag{7.120}$$

In fact the analysis in [524] can immediately be adapted to the twist case and we find a central charge as in (7.119), but now with $p_1 = p_2 > 0$, $n_2 > n_1$ and

$$\text{Twist}: \quad p_1 = p_2 > 0, \quad n_2 > n_1 \quad \Rightarrow \quad p_3 = -n_1 - n_2 - 2p_1 < 0\,. \tag{7.121}$$

Furthermore, the field theory analysis of [524] using the large N limit of the M5-brane anomaly polynomial which was used to recover the gravity central charge for the anti-twist case, can also be adapted for the case of the twist and again we find exact agreement.

All of these configurations preserve $\mathcal{N} = (0,2)$ supersymmetry. Configurations with $\mathcal{N} = (0,4)$ or $\mathcal{N} = (2,2)$ supersymmetry are linked with setting $p_1 = p_2 = 0$ or $p_3 = 0$, associated, in the twist class, with M5-branes wrapping a product of $\mathbb{\Sigma} \times \Sigma_{\mathsf{g}}$ in a Calabi-Yau three fold or a product of two Calabi-Yau two-folds, respectively. However, we do not have supergravity solutions with compact spindles for either of these values as one can conclude from the vanishing of the central charge.

It is also interesting to point out that if we *formally* set $n_1 = n_2 = 1$ in the twist class we find

$$c = \frac{8\,p_1^2}{1+3p_1}(\mathsf{g}-1)N^3\,, \tag{7.122}$$

and this should be compared with the central charge of the $S^2 \times \Sigma_{\mathsf{g}}$ $D = 11$ supergravity solutions, first found in section 3.4 of [539] and subsequently discussed in [148]. In the notation of the latter reference one should set $\kappa_1 = 1$, $\kappa_2 = -1$, $\eta_1 = 2$, $\eta_2 = 2(\mathsf{g}-1)$, $z_1 = 2a_1 + 1$ and $z_2 = 2a_2 - 1$, and then set $a_2 = 0$ (*i.e.* $z_2 = -1$). One then finds the central charge

$$c_{BB} = \frac{8a_1^2}{1+3a_1}(\mathsf{g}-1)N^3\,. \tag{7.123}$$

This agrees with (7.122), identifying $a_1 = p_1$.

7.3 An overview of spindles in other dimensions

In the previous section we have discussed in detail some aspects of $AdS_3 \times \mathbb{\Sigma}$ solutions to five-dimensional STU gauged supergravity. In this section we will present a schematic overview of the other spindle solutions that have appeared in the literature so far, also connecting to the work of other authors, in dimensions $4 \leq D \leq 7$. The general features are very similar in all dimensions: one has a warped product of a certain AdS space and a spindle $\mathbb{\Sigma}$ with a $U(1)$-invariant metric, a set of gauge fields with magnetic fluxes across $\mathbb{\Sigma}$ and some non-constant scalar fields, much like in (7.55). The key features highlighted in section 7.2 are present in all dimensions: the absence of a topological twist, which is replaced by the two options of twist or anti-twist, and the lack of a constant curvature metric for the spindle (which is topologically obstructed). In table 7.1 we give an overview of the solutions analyzed in the literature so far, which we then describe more in detail in the next subsections.

Supergravity	Uplift	Twist	Minimal	FT	Full flow	References
$4d\ U(1)^4$	Regular	Both	Anti twist	No	Yes	[4, 6, 7, 527, 528]
$5d\ U(1)^3$	Regular	Both	Anti twist	Yes	No	[3, 7, 523, 524]
$6d\ F(4)$	Singular	Twist	No sol.	No	No	[526, 529]
$7d\ U(1)^2$	Singular	Twist	No sol.	Yes	No	[5, 7]

Table 7.1: Summary of the purely magnetic spindle solutions in the consistent truncations discussed in table 6.1, as well as the $6d\ F(4)$ supergravity model which uplifts to massive type IIA supergravity. We emphasize if the AdS $\times\ \mathbb{\Sigma}$ solution is regular when uplifted to $D = 10/11$, which type of twist is possible in the most general solution and which in the corresponding minimal gauged supergravity. We also stress that a field theory (FT) computation that reproduces the supergravity central charge is possible only for field theories in even dimensions and that solutions describing the full RG flow across dimensions are present only in $D = 4$.

7.3.1 Four dimensions

In four dimensions, we consider solutions of the so-called STU gauged supergravity [543, 544], which can be thought of as truncation of the more general $\mathcal{N} = 8\ SO(8)$ gauged supergravity to its abelian $U(1)^4$ sector, or alternatively as a Fayet-Iliopoulos gauged $\mathcal{N} = 2$ supergravity coupled to three vector multiplets, with prepotential $F = -\mathrm{i}\sqrt{X^0\,X^1\,X^2\,X^3}$ [545]. Its bosonic fields are a metric tensor, four abelian gauge fields and six scalars, which can be divided in three dilatons and three axions (pseudo-scalars). While various truncations of this model are possible (see [546] and references

therein for a summary and review), we are particularly interested in two of them. The first is the case in which the four gauge fields are set to be pairwise equal and two of the axio-dilatons set to zero: in $\mathcal{N}=2$ language this gives a gauged supergravity coupled to one vector multiplet, with pre-potential $F=-\mathrm{i}\,X^0\,X^1$, where the remaining bosonic fields are the metric, two $U(1)^2$ gauge fields and one axio-dilaton pair of scalars. Finally, one can consider a truncation where the four gauge fields of the original theory are all identical and all scalar fields set to zero: this gives a minimal gauged supergravity, whose bosonic action is the same as that of Einstein-Maxwell theory with a cosmological constant. While solutions to all these models can be uplifted to eleven dimensions of S^7, it is interesting to observe that solutions to four-dimensional minimal gauged supergravity can be uplifted on any Sasaki-Einstein seven-manifold [443].

The first spindle solutions in four dimensions appeared in [4], in the context of minimal gauged supergravity. An interesting aspect of the four-dimensional case is that, as opposed to the other dimensions, it was possible in certain cases to obtain "full flow" solutions, interpolating between an asymptotically locally AdS_4 boundary and an $\mathrm{AdS}_2\times\mathbb{\Sigma}$ near-horizon geometry, with $\mathbb{\Sigma}$ a spindle: it was already realized in [4] that these are accelerating black holes in four dimensions, where the conical deficits of the spindle provide the source of the (constant) acceleration, allowing for a stationary solution. A more detailed discussion on accelerating black holes can be found in chapter 8. Another interesting feature of four-dimensional solutions is that, on top of the magnetic charges that are present for all solutions found in the literature so far, generalizations to include electric charge(s) and rotation are known: they were are presented in [4] for minimal gauged supergravity and in [6] for the $U(1)^2$ model discussed previously. Full-flow solutions were also found in [6] for $U(1)^2$ and $U(1)^4$ gauged supergravity, although including only magnetic charges and acceleration but no electric charges or rotation.

Focusing now on the near-horizon region, it is interesting to observe that the local form of the spindle solutions with purely magnetic charges can be thought of as arising from a double Wick rotation of the electrically charged AdS_4 black holes presented in [454], which uplift to rotating M2-branes in eleven-dimensions, with the electric charges related to the angular momenta of the M2-branes. This connection is also common to solutions in other dimensions, as we shall discuss. Moreover, when uplifted to $D=11$ on S^7, the $\mathrm{AdS}_2\times\mathbb{\Sigma}$ solutions with only magnetic charge are diffeomorphic to the $\mathrm{AdS}_2\times Y_9$ solutions presented in [178], where the regularity in $D=11$ was already considered and Y_9 is a manifold with GK geometry [215]. It is interesting to highlight two alternative approaches to the regularity analysis in the minimal supergravity case. In analogy with

[179] one can think of Y_9 as a fibration of a lens space $L(p;q)$ over a Kähler-Einstein six-manifold KE_6

$$L(p;q) \hookrightarrow Y_9 \to \mathrm{KE}_6\,, \tag{7.124}$$

with the result that $\mathrm{AdS}_2 \times Y_9$ gives a regular solution of $D = 11$ supergravity but where the four-dimensional origin is unclear. One of the main observations of [4] from this perspective is that one can alternatively think Y_9 as the fibration of a SE_7 over a spindle $\mathbb{\Sigma}$

$$\mathrm{SE}_7 \hookrightarrow Y_9 \to \mathbb{\Sigma}\,, \tag{7.125}$$

where the SE_7 is a $U(1)$ fibration over the KE_6, while the lens space $L(p;q)$ is a $U(1)$ fibration over $\mathbb{\Sigma}$, using the same $U(1)$ direction. While both points of view are valid, since in moving between the two one is just re-writing the same metric in two ways which make one or the other fibration structure manifest, this shift in perspective is crucial as it allows to identify $\mathbb{\Sigma}$ as building, together with the AdS_2 part of the metric, a solution to $4d$ minimal gauged supergravity.

In terms of how these solutions preserve supersymmetry, it is worth remarking that only anti-twist solutions are present in minimal gauged supergravity [4] and in the $U(1)^2$ model [6], while both twist and anti-twist solutions are possible in the $U(1)^4$ model [7, 528], with an explicit solution to the regularity conditions in the case with four independent magnetic charges given in [527]. A common feature to all four-dimensional spindle solutions discussed so far in the literature is that, when uplifted, they give rise to completely regular solutions of $D = 11$ supergravity, with no conical singularities. In the general case of uplift on S^7 this works in a completely analogous way to the discussion of section 7.1.6, while in the case of minimal gauged supergravity a more complicated analysis is required in order to show that such solutions uplift to regular solutions of $D = 11$ supergravity on any *regular* SE_7. More details on this second case can be found in [4]. Once uplifted to eleven dimensions, the four-dimensional spindle solutions are conjectured to be dual to models of $1d$ superconformal quantum mechanics (SCQM) that arise compactifying $d = 3, \mathcal{N} = 2$ SCFTs on a spindle, so that the $\mathrm{AdS}_2 \times \mathbb{\Sigma}$ supergravity solutions, once uplifted to eleven dimensions, are thought of as describing M2-branes wrapped on a spindle. In this context, it would be interesting to reproduce the two-dimensional inverse Newton constant of the AdS_2 solutions (corresponding to the entropy of an accelerating black hole) from a computation of the topologically twisted index [502, 503] of

the dual field theory using supersymmetric localization [74]. Unfortunately, such computation has not been attempted to date and the evidence for the holographic interpretation of four-dimensional spindle solutions mostly comes from the analogy with other cases where field theory results are available.

To conclude, we would like to stress a feature that is shared by all spindle solutions, namely that the existence of a solution to the regularity conditions (the analogue of (7.62) and (7.63)) relies on the fact that the four magnetic charges are all non-vanishing, which implies that the dual $1d$ superconformal quantum mechanics has $\mathcal{N}=2$ supersymmetry (in particular, $\mathfrak{su}(1,1|1)$ invariance), which is the minimum amount still allowing for a continuous R symmetry in one dimension.

7.3.2 Five dimensions

In five dimensions, we consider solutions of the so-called STU gauged supergravity, which can be thought of as truncation of the more general $\mathcal{N}=8$ $SO(6)$ gauged supergravity to its abelian $U(1)^3$ sector [454], or alternatively as a gauged $\mathcal{N}=2$ supergravity coupled to two vector multiplets [538]. Its bosonic fields are a metric tensor, three abelian gauge fields and two real scalars. Solutions to this model generically uplift to type IIB supergravity on S^5, but there are two special cases that are of interest for us. One is when two of the gauge fields are equal, in which case solutions can be uplifted to $D=11$ supergravity [539] and one is the case of minimal gauged supergravity, which is again Einstein-Maxwell theory with a cosmological constant, as well as a Chern-Simons term. Solutions of minimal gauged supergravity in five dimensions can be uplifted to type IIB supergravity on any SE_5.

The first spindle solutions in five dimensions were discussed in [3], where the case of minimal gauged supergravity is discussed, as well as the regularity conditions for the uplift on any SE_5. Solutions with three distinct magnetic charges were found soon after [523, 524] and as in the four-dimensional case they arise from local solutions that had already appeared in the literature. In particular, the local AdS_3 solutions can again be seen as a double Wick rotation of the charged black holes presented in [454], and were also later rediscovered in [541]. After the uplift on S^5, they were shown to give rise to regular $\mathrm{AdS}_3\times Y_7$ solutions of type IIB supergravity in [178], with Y_7 a GK manifold. Once again, therefore, the local form of the solutions was known and the main point of [3] is the study of the regularity conditions in gauged supergravity, as well as the interpretation of the uplifted solution as describing D3-branes wrapped on a spindle. The kind of shift in the paradigm in the interpretation of the ten-dimensional solution from [178] to [3] is completely analogous to that

described in the four-dimensional case.

For these five-dimensional spindles no full-flow solution is known, although in analogy with four dimension one might expect them to arise as the near-horizon limit of accelerating black strings in five dimensions. Unfortunately, accelerating black string solutions have not been considered so far in the literature, a point that is worth of further consideration in the future. On the other hand, replacing the AdS_3 factor with a suitable BTZ black hole, a generalization to include electric charges and rotation (in addition to the magnetic charges) was presented in [524]. Regarding how supersymmetry is preserved, it is interesting to observe that local solutions with three distinct magnetic charges can lead to both twist and anti-twist solutions, as discussed in the previous section, and similarly when two charges are identical. However, when all three charges are the same (corresponding to minimal gauged supergravity), only anti-twist solutions are possible. At the level of field theory computations, we showed in section 7.2.5 how the central charge of the dual $2d$ SCFT can be reproduced in the case of the uplift to type IIB supergravity – a completely analogous computation leads to similar results for the case of the uplift to $D = 11$ supergravity considered in section 7.2.6 – see [524] for more details.

To conclude, let us stress that in analogy with the case of four-dimensional supergravity and as discussed in the previous section, for spindle solutions in five dimensions one requires that all magnetic charges are distinct. Therefore, once uplifted to type IIB supergravity, such solutions are dual to $d = 2$ SCFTs with $\mathcal{N} = (0, 2)$ supersymmetry, which is the minimum amount for which a continuous R symmetry is present.

7.3.3 Six dimensions

Although we have not included, nor we will discuss thoroughly, the case of six-dimensional gauged supergravity in table 6.1, we briefly summarize this case here for completeness. In $D = 6$ there is an $\mathcal{N} = 2$ gauged supergravity consisting of an extensions of Romans $F(4)$ theory [547] by vector multiplets [548], which arises as a consistent truncation of massive type IIA supergravity on a warped S^4 [549] and is related to a system of D4- and D8-branes. A further truncation [550] leads to a theory with a metric tensor, a $U(1)^2$ pair of Abelian gauge fields and two scalars. It is in this theory that spindle solutions have been constructed in [526, 529], conjecturally describing the IR fixed point of D4-branes wrapped on a spindle (in the presence of D8-branes), or, according to the holographic dictionary, the compactification of $5d$ SCFTs on a spindle, giving rise to $3d$ $\mathcal{N} = 2$ SCFTs. Once again, this is related to the fact that none of the two magnetic fluxes through the spindle can be

set to zero, and the resulting superalgebra in three dimensions is the smallest one allowing for a continuous R symmetry.

We would like to emphasize mainly two points regarding spindles in six dimensions. The first is that for the first time we find that only twist solutions are present, and no anti-twist ones, and moreover the existence of a solution to the regularity conditions for the spindle requires the two magnetic fluxes to be distinct, so that there is no spindle solution in minimal gauged supergravity. The second is related to the uplift to massive type IIA, which is singular even just for the AdS_6 vacuum: the six-dimensional solutions are uplifted on a hemi-sphere HS^4 and the geometry is singular at the equator of the sphere. Besides this point, which is not related to spindles, we would like to remark that although at a generic point in the internal space the HS^4 is fibered over the spindle $\mathbb{\Sigma}$ in such a way as to make the total space regular (the analogue of (7.63)), at the north pole of HS^4 one has a copy of the spindle with no fibration terms and therefore one still sees the conical singularities in the uplifted metric at this special point.

7.3.4 Seven dimensions

In seven dimensions we consider solutions to (a truncation of) the $U(1)^2$ supergravity considered in table 6.1, whose only interesting further truncation from our perspective is minimal gauged supergravity, where the only gauge field is the graviphoton. Interestingly, spindles in seven dimensions share many features with the six-dimensional solutions. In particular, they only admit twist solutions and it is not possible to solve the regularity conditions in the case of identical magnetic charges, so that no solution is present in minimal gauged supergravity. Moreover, since once again the theory is uplifted on S^4 (but this time on the full sphere, rather than just one hemi-sphere), the conical singularities of the spindle are still visible at the poles of the sphere, where the metric coefficients of the fibration terms that "smoothen" the singularities vanish. As in all other cases, it is not possible to set any of the magnetic charges to zero, so that the amount of supersymmetry preserved by the dual $4d$ SCFTs is the minimal one, $\mathcal{N} = 1$. A detailed analysis of $\text{AdS}_5 \times \mathbb{\Sigma}$ solutions is presented in [5, 7], where the central charge of the dual $4d$ SCFTs is reproduced using the techniques of the anomaly polynomial and a-maximization, in analogy with section 7.2.5

Chapter 8

Accelerating black holes

In this chapter we will discuss aspects of spindle solutions that aim at a better understanding of the UV regime, where we conjecture to have $(d-1)$ branes wrapping spindles from the supergravity perspective, holographically dual to SCFTs in d dimensions placed on a background $\mathbb{R}^{1,d-3} \times \mathbb{\Sigma}$. The solutions described in the previous chapter, on the other hand, should correspond to the IR fixed point of the RG flow triggered by placing the SCFT on a curved space (with background fields coupling to certain operators). One of the main puzzles related to the IR solutions is the way in which they preserve supersymmetry, which as discussed several times is distinct from the usual topological twist. In section 7.1 we have shown that there are only two ways for spindles to preserve supersymmetry: the twist and the anti-twist. However, this does not clarify the precise deformations that one should do to the SCFT in the UV and to that end it is useful to look for "full flow" solutions in gauged supergravity, that interpolate between AdS_{d+1} and $\text{AdS}_{d-1} \times \mathbb{\Sigma}$. Unfortunately, these are hard to find and only a few cases are known in the literature on spindles, all related to the case $d = 3$: these are accelerating black holes in four-dimensional gauged supergravity and they will be the main subject of this chapter.

For simplicity, we shall mostly focus on the case of minimal gauged supergravity considered in [4], where the relevant black holes were originally found by Plebański and Demiański (PD) [551]. Accelerating black holes with multiple magnetic charges are discussed in [6], to which we will also refer at times. The rest of the chapter in two sections: in section 8.1 we will present certain aspects of PD black holes, including a discussion of supersymmetry, extremality and their connection to spindles; in section 8.2 we will then discuss the conformal boundary of accelerating black holes in four dimensions, which will help us to shed light on certain aspects of the UV origin of spindle solutions.

8.1 Accelerating black holes

The C-metrics of $D = 4$ Einstein-Maxwell theory [552] describe two charged black holes undergoing uniform acceleration. The force for the acceleration is provided by a conical angle excess (a strut) between the two black holes, a conical angle deficit (a string) attached to the black holes and extending out to infinity, or a combination of the two. By extending the model to include additional matter fields that allow for cosmic strings, the conical singularity can be smoothed out by having two cosmic strings pull the black holes apart [553, 554]. Another approach for removing the conical singularities, in the case of electrically or magnetically charged black holes, is provided by the Ernst metric in which the black holes are being accelerated by a background electric or magnetic field, respectively [555].

Generalizations of the Ernst metric were found for Einstein-Maxwell-dilaton gravity in [556]. For the particular case associated with $D = 5$ Kaluza-Klein theory it was shown in [557] that the accelerating $D = 4$ extremal, magnetically charged black hole solutions can be obtained by a dimensional reduction of a double Wick rotation of the $D = 5$ generalization of the Kerr solution [558]. Recall the extremal magnetically charged black holes in Kaluza-Klein theory are Kaluza-Klein monopoles [559, 560], which are in fact naked singularities from the four-dimensional point of view. A key ingredient in the construction of [557] is that the Kaluza-Klein circle action associated with the reduction has two fixed points which precisely correspond to a Kaluza-Klein monopole, anti-monopole pair.

Here we study generalizations of the C-metric to include, in addition to mass and acceleration, also electric and magnetic charge, rotation and a cosmological constant, known as PD black holes [551][1]. These were extensively studied by various authors (see, *e.g.*, [561–563]) and here we will only be interested in the case of small acceleration for which there is just a single black hole. Focusing on the case where the conical deficits are quantized in such a way that the black hole horizon is a spindle, it was shown in [4] that the uplift of such on *regular* Sasaki-Einstein 7-manifolds using the formulas of [443] gives rise to smooth solutions of $D = 11$ supergravity, providing an alternative way to regularizing the conical deficits. Here we will focus on a discussion of the black hole metric, the supersymmetry conditions and the near-horizon geometry.

[1] In [551] the possibility of including a NUT (Newman-Unti-Tamburino) parameter was also considered, but here we set it to zero since we want to avoid closed time-like curves.

8.1.1 Minimal gauged supergravity

We will be interested in solutions of $D = 4$ Einstein-Maxwell theory with action given by

$$S = \frac{1}{16\pi G_{(4)}} \int \mathrm{d}^4x\sqrt{-g}(R + 6 - F^2)\,, \tag{8.1}$$

with $F = \mathrm{d}A$ and where we have set the cosmological constant $\Lambda = -3$. This is the action for the bosonic fields of $D = 4$, $\mathcal{N} = 2$ minimal gauged supergravity. A solution to the equations of motion will preserve supersymmetry provided that it admits appropriate Killing spinors ϵ, which are solutions of

$$\left(\nabla_a - \mathrm{i}A_a + \frac{1}{2}\gamma_a + \frac{\mathrm{i}}{4}F_{bc}\gamma^{bc}\gamma_a\right)\epsilon = 0\,, \tag{8.2}$$

with $\nabla = \mathrm{d} + \frac{1}{4}\omega^{ab}\gamma_{ab}$ the covariant derivative and γ_a four-dimensional gamma matrices. All (supersymmetric) solutions to this theory uplift to (supersymmetric) solutions of $D = 11$ supergravity on any Sasaki-Einstein 7-manifold SE_7 [443], using

$$\begin{aligned} \mathrm{d}s_{11}^2 &= L^2\left[\tfrac{1}{4}\mathrm{d}s_4^2 + \left(\eta + \tfrac{1}{2}A\right)^2 + \mathrm{d}s_T^2\right]\,, \\ G &= L^3\left[\tfrac{3}{8}\mathrm{vol}_4 - \tfrac{1}{2}*_4 F\wedge J\right]\,, \end{aligned} \tag{8.3}$$

where $L > 0$ is a constant which is eventually fixed by flux quantization. Here η is the contact one-form of the Sasaki-Einstein manifold, with transverse Kähler two-form J and associated transverse metric $\mathrm{d}s_T^2$, with $\mathrm{d}\eta = 2J$. The vacuum AdS_4 solution, with $\mathrm{d}s_4^2$ a unit radius AdS_4 and $A = 0$, uplifts to the supersymmetric $\mathrm{AdS}_4\times\mathrm{SE}_7$ solution, dual to an $\mathcal{N} = 2$ SCFT in $d = 3$.

8.1.2 The PD solution

We start with a sub-class of the class of Plebański–Demiański (PD) solutions [551] of Einstein-Maxwell theory, as presented in [564], where we set the NUT parameter to zero. The metric is given by

$$\begin{aligned} \mathrm{d}s^2 = \frac{1}{H^2}\Big\{ &-\frac{Q}{\Sigma}\left[\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right]^2 + \frac{\Sigma}{Q}\,\mathrm{d}r^2 + \frac{\Sigma}{P}\mathrm{d}\theta^2 \\ &+ \frac{P}{\Sigma}\sin^2\theta\left[a\mathrm{d}t - (r^2 + a^2)\mathrm{d}\phi\right]^2\Big\}, \end{aligned} \tag{8.4}$$

where

$$
\begin{aligned}
H &= 1 - \alpha\, r \cos\theta\,, \\
\Sigma &= r^2 + a^2 \cos^2\theta\,, \\
P &= 1 - 2\alpha m \cos\theta + \Big(\alpha^2(a^2+e^2+g^2) - a^2\Big)\cos^2\theta\,, \\
Q &= \Big(r^2 - 2mr + a^2 + e^2 + g^2\Big)(1-\alpha^2 r^2) + (a^2+r^2)r^2\,,
\end{aligned}
\tag{8.5}
$$

and we note that $P = P(\theta)$ and $Q = Q(r)$ while H and Σ depend on both r and θ. The gauge field is given by

$$
A = -e\frac{r}{\Sigma}(\mathrm{d}t - a\sin^2\theta \mathrm{d}\phi) + g\frac{\cos\theta}{\Sigma}(a\mathrm{d}t - (r^2+a^2)\mathrm{d}\phi)\,. \tag{8.6}
$$

The solution depends on five free parameters m, e, g, a and α, which we can loosely associate with mass, electric charge, magnetic charge, rotation and acceleration, respectively. We have fixed the cosmological constant to be $\Lambda = -3$, as in (8.1), and finally we note that we have changed the sign of g compared with [564].

We will assume $m > 0$. Note that there are various coordinate changes $\theta \to \pi - \theta$, $t \to -t$, $\phi \to -\phi$ as well as $A \to -A$, which change the signs of the pairs (α, g), (a, e), (a, g) and (e, g), respectively. If $\alpha = 0$ we can thus choose $e, g, a \geq 0$ without loss of generality. If $\alpha \neq 0$ we can choose $m, \alpha > 0$ and $e, g \geq 0$ with either $a \geq 0$ or $a \leq 0$. However, in order to get supersymmetric, extremal black holes one should take $a \geq 0$. Thus, in the sequel we focus on

$$
m > 0\,, \qquad \alpha, e, g, a \geq 0\,. \tag{8.7}
$$

The range of the θ coordinate is taken to be $0 \leq \theta \leq \pi$. We will assume that $P(\theta) > 0$ in this range.[2] It will be convenient to define

$$
\theta_1 = \pi\,, \qquad \text{and} \qquad \theta_2 = 0\,. \tag{8.8}
$$

On slices of constant t, r, we can examine the behaviour of the metric as we approach $\theta = \theta_{1,2}$ and

[2] As in [563], this can be achieved if $m\alpha < \frac{1}{2}\Xi$ for $\Xi \in (0, 2]$ and $m\alpha < \frac{1}{2}(\Xi - 1)^{1/2}$ for $\Xi > 2$, where $\Xi \equiv 1 + \alpha^2(a^2 + e^2 + g^2) - a^2$.

we find

$$\mathrm{d}s^2_{\theta,\phi} \approx \left[\frac{\Sigma}{PH^2}\right]_{\theta=\theta_{1,2}} \left[\mathrm{d}\theta^2 + P^2_{1,2}(\theta-\theta_{1,2})^2\mathrm{d}\phi^2\right] , \tag{8.9}$$

where

$$P_{1,2} \equiv P(\theta_{1,2}) = 1 \pm 2\alpha m + \alpha^2(a^2+e^2+g^2) - a^2 , \tag{8.10}$$

are constants, with the positive sign associated with θ_1 and the negative with θ_2. With $\alpha m \neq 0$ it is not possible to choose a period for ϕ so that we obtain a smooth metric on a round two sphere as there will always be conical deficits at one or both of the poles. Thus, as is well known, the mismatch of these conical deficits at the two poles is directly connected with the non-vanishing of the acceleration parameter.

A simple observation, which will turn out to be important in obtaining regular solutions in $D = 11$, is that we can suitably constrain the parameters in the metric, and choose a corresponding period for ϕ, so that the conical defects give rise to spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_1,n_2]}$. We can demand that the ratio P_1/P_2 is a rational number, which we write as

$$\frac{P_1}{P_2} = \frac{n_2}{n_1} , \tag{8.11}$$

with $n_\pm \in \mathbb{N}$, and then choose the period of ϕ to be

$$\Delta\phi = \frac{2\pi}{n_1 P_1} = \frac{2\pi}{n_2 P_2} , \tag{8.12}$$

and hence from (8.10) we can also write $\Delta\phi = \frac{\pi}{2m\alpha}\frac{n_2-n_1}{n_2 n_1}$. The two-dimensional space parametrized by θ and ϕ is then a spindle. We will see that we are always able to desingularize the singularities of this spindle after uplifting to $D = 11$. As we will see, our procedure will require that we impose the condition $m\alpha = g$; we will also find that $m\alpha = g$ is implied by supersymmetry. It will also be useful in the sequel to introduce a coordinate φ on the spindle that has period 2π:

$$\varphi \equiv \left(\frac{2\pi}{\Delta\phi}\right)\phi . \tag{8.13}$$

In order to have a spacetime with a black hole horizon, located at $r = r_+$, we demand that

$Q(r_+) = 0$ and the region exterior to the black hole has $r \geq r_+$. We also require $H > 0$ and hence the radial coordinate is constrained via

$$\alpha r \cos\theta < 1 \,, \tag{8.14}$$

and so, in particular, $\alpha r_+ < 1$. The conformal boundary is approached when $\alpha r \cos\theta \to 1$ and one finds that the conical defects are still present when $\alpha \neq 0$. There can also be an acceleration horizon. We discuss this in section 8.2, where we study the global structure of the non-rotating black holes in more detail. In particular, although Q has no other roots for $r > r_+$, one can continue the radial coordinate r past $r = \infty$, and there can effectively be another root beyond this, corresponding to an acceleration horizon. A detailed analysis of the various cases and their Penrose diagrams may be found in [562]. Some further discussion of these black holes may also be found in [563] which calculates various thermodynamic quantities (when $g = 0$). Here we record that the entropy of the black holes is given by

$$S_{BH} = \frac{\text{Area}}{4G_{(4)}} = \frac{(r_+^2 + a^2)\Delta\phi}{2G_{(4)}(1 - \alpha^2 r_+^2)} \,. \tag{8.15}$$

We also record that the magnetic flux through the horizon, Q_m, is defined by

$$G_{(4)} Q_m \equiv \frac{1}{4\pi} \int_{r=r_+} F = g \, \frac{\Delta\phi}{2\pi} \tag{8.16}$$

and the electric flux, Q_e, is defined by

$$G_{(4)} Q_e \equiv \frac{1}{4\pi} \int_{r=r_+} *F = e \, \frac{\Delta\phi}{2\pi} \,. \tag{8.17}$$

Finally, we can introduce the angular momentum of the black hole. Rather than a simple Komar integral, as often used, we define the angular momentum as a conserved quantity that can be equally evaluated either at the conformal boundary or at the black hole horizon (see, *e.g.* ,[565]). Associated with the Killing vector $k = \partial_\varphi$, where $\varphi \in [0, 2\pi)$ was introduced in (8.13), we introduce the two-form

$$Z = \mathrm{d}k + 4(A \cdot k)F \,, \tag{8.18}$$

and then define the angular momentum via

$$J(A) = \frac{1}{16\pi G_{(4)}} \int_{r=r_+} *Z \,. \tag{8.19}$$

The angular momentum is then a kind of Page charge that depends on the gauge. Using the gauge as in (8.6) we obtain

$$G_{(4)} J = m\, a \left(\frac{\Delta\phi}{2\pi}\right)^2 , \tag{8.20}$$

which agrees with formulas in the literature for the non-accelerating limit. We refer to appendix E of [4] for more details on the definition (8.19) of angular momentum and its dependence on the gauge choice.

8.1.3 Supersymmetry and extremality

In this section we analyse the additional conditions for supersymmetry, as well as the conditions required to have an extremal black hole horizon with vanishing surface gravity. In general, the extremality condition is not implied by supersymmetry. We also derive an expression for the black hole entropy in the supersymmetric and extremal case.

By definition the supersymmetric (BPS) limit occurs when the solutions admit solutions to the $D = 4$ Killing spinor equation of minimal gauged supergravity given by (8.2). The conditions for the PD solutions to admit Killing spinors were determined in [566] and we are primarily interested in the case $\alpha \neq 0$. By examining the integrability conditions for (8.2), as in [566, 567] we find that supersymmetry implies that the five parameters are constrained by the following two conditions

$$\begin{aligned}
0 &= 2aeg\alpha + g^4\alpha^4 + g^2(-1+\alpha^2+(a^2+2e^2)\alpha^4) + e^2\alpha^2(1+e^2\alpha^2+a^2(-1+\alpha^2))\,,\\
0 &= 2a^2g^2\alpha + 2e^4\alpha^3 + 2g^4\alpha^3 - 2aeg(-1+a^2\alpha^2) - 2e^2\alpha(a^2-2g^2\alpha^2)\\
&\quad - \alpha(m+a^2m\alpha^2)^2 - (e^2+g^2)\alpha(1+a^2\alpha^2)(-1+e^2\alpha^2+g^2\alpha^2+a^2(-1+\alpha^2))\,.
\end{aligned} \tag{8.21}$$

With $\alpha \neq 0$, and $m > 0$ we must have $g > 0$. With the signs of the parameters as in (8.7), we can solve these conditions to obtain

$$m = \frac{g}{\alpha}\,,$$

$$a = \frac{-eg\alpha + (e^2+g^2)\,\alpha^2\sqrt{1+(-1+e^2)\alpha^2-(e^2+g^2)\,\alpha^4}}{g^2\alpha^4+e^2\alpha^2(-1+\alpha^2)}\,. \tag{8.22}$$

Interestingly, the first condition also plays a role in the regularity analysis of the uplifted $D = 11$ solutions, see section 4 of [4] for more details.

We now consider the additional conditions imposed by extremality when $r = r_+$ becomes a double root of $Q(r)$. Assuming that both equations in (8.21) are satisfied we find

$$a^3\alpha^2 eg^3 + a^2\alpha g^2(e-g)(e+g) - aeg^3 + \alpha^3 e^2\left(e^2+g^2\right)^2 = 0\,. \tag{8.23}$$

In principle, one could solve these three constraints in terms of two independent parameters, and then solve (8.11) to find the relation between these and $n_{1,2}$. However, this is a little cumbersome to do in practice, and it is clearer to keep all the parameters in our expressions, where it is then understood that they must solve the constraints given by (8.21) and (8.23). For example, a simple expression for the horizon radius of the supersymmetric and extremal black hole is given by

$$r_+ = \sqrt{\frac{a\,(m-a\,e)}{e+a\,m\,\alpha^2}}\,, \tag{8.24}$$

which would not be so simple if one were to use the explicit solutions of the constraints above. However, for the special case when we set the rotation parameter $a = 0$, we will find simple and explicit expressions as we discuss in section 8.2.1. For supersymmetric and extremal black holes with $a = 0$ we also have $e = 0$, and the expression (8.24) doesn't apply directly. Notice that substituting (8.24) into the general black hole entropy formula (8.15) immediately gives

$$S_{BH} = \frac{am}{2G_{(4)}e}\Delta\phi\,, \tag{8.25}$$

as long as the parameter $e \neq 0$.

8.1.4 The near-horizon limit

We now elucidate the near horizon limit of these supersymmetric extremal black holes. The result is a new class of supersymmetric $AdS_2 \times \mathbb{WCP}^1_{[n_1,n_2]}$ solutions of $D = 4$ gauged supergravity. These uplift to regular $AdS_2 \times Y_9$ solutions, which generalize those of [178] by an extra rotation parameter.

A convenient way to find the near horizon solution is to implement the following coordinate

transformation,

$$r \to r_+ + \lambda\, s\, \rho\,, \quad t \to \lambda^{-1}\, s\, \tau\,, \quad \phi \to \phi' + \lambda^{-1}\, s\, W \frac{\Delta\phi}{2\pi}\, \tau\,, \tag{8.26}$$

where s is a constant, and then take the $\lambda \to 0$ limit. Here W is given by

$$W = \frac{a}{r_+^2 + a^2} \frac{2\pi}{\Delta\phi}\,, \tag{8.27}$$

with $\partial_t + W \frac{\Delta\phi}{2\pi} \partial_\phi = \partial_t + W \partial_\varphi$ a null generator of the horizon.

Some details of the limiting procedure are given in appendix C of [4]. After carrying out various coordinate and gauge transformations, as well as redefining the parameters, we eventually end up with the following class of $AdS_2 \times \mathbb{WCP}^1_{[n_1,n_2]}$ solutions:

$$\begin{aligned} \mathrm{d}s^2 &= \tfrac{1}{4}(y^2 + \mathtt{j}^2) \left(-\rho^2\, d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + \frac{y^2 + \mathtt{j}^2}{q(y)} dy^2 + \frac{q(y)}{4(y^2 + \mathtt{j}^2)} (dz + \mathtt{j}\, \rho\, d\tau)^2\,, \\ A &= h(y)\, (dz + \mathtt{j}\, \rho\, d\tau)\,, \end{aligned} \tag{8.28}$$

where we have defined

$$\begin{aligned} q(y) &= \left(y^2 + \mathtt{j}^2\right)^2 - 4\left(1 - \mathtt{j}^2\right) y^2 + 4\mathtt{a}\sqrt{1 - \mathtt{j}^2}\, y - \mathtt{a}^2\,, \\ h(y) &= \frac{\sqrt{1 - \mathtt{j}^2}}{2} - \frac{1}{2(y^2 + \mathtt{j}^2)} (\mathtt{a} y + 2\mathtt{j}^2 \sqrt{1 - \mathtt{j}^2})\,. \end{aligned} \tag{8.29}$$

The solutions depend on two free parameters $\mathtt{j}$, $\mathtt{a}$, which are functions of the original m, α, e, g, a, given implicitly in appendix C of [4]. Indeed, it is remarkable how simple the solution is in the above parametrization. The parameter $\mathtt{j}$ is a rotation parameter and $\mathtt{j} = 0$ is the non-rotating limit. In fact if we set $\mathtt{j} = 0$ then we precisely recover the $AdS_2 \times Y_9$ solutions of [178], after dimensional reduction on a SE_7, as shown in appendix D of [4].

Continuing[3] with $\mathtt{j} \geq 0$, we need $\mathtt{j} \in [0, 1]$ in order to get a real solution. We next analyse the roots of $q(y)$, which are given by $y = \sqrt{1 - \mathtt{j}^2} \pm \sqrt{1 - \mathtt{a} - 2\mathtt{j}^2}$ and $y = -\sqrt{1 - \mathtt{j}^2} \pm \sqrt{1 + \mathtt{a} - 2\mathtt{j}^2}$. Given that $q(y)$ is a quartic in y with positive coefficient of y^4, in order for $q(y) \geq 0$ we need to choose y to lie in between the middle two roots of the quartic, with all roots real, which fixes $y \in [y_1, y_2]$

[3] Note that we can change the sign of $\mathtt{j}$ by changing the sign of τ, z and also the gauge field A.

with

$$y_1 = -\sqrt{1-\mathtt{j}^2} + \sqrt{1+\mathtt{a}-2\mathtt{j}^2}\,, \qquad y_2 = \sqrt{1-\mathtt{j}^2} - \sqrt{1-\mathtt{a}-2\mathtt{j}^2}\,. \tag{8.30}$$

For these to be real we need to take

$$\mathtt{j} \in [0, \tfrac{1}{\sqrt{2}})\,. \tag{8.31}$$

By analysing how the metric behaves at the roots, we demand that the (y,z) part of the metric becomes $\mathbb{WCP}^1_{[n_1,n_2]}$, which fixes

$$\sqrt{1+\mathtt{a}-2\mathtt{j}^2}\cdot\Delta z = \frac{2\pi}{n_1}\,, \qquad \sqrt{1-\mathtt{a}-2\mathtt{j}^2}\cdot\Delta z = \frac{2\pi}{n_2}\,, \tag{8.32}$$

where we have identified $y_{1,2}$ with $\theta_{1,2}$ of previous sections, respectively, which has solution

$$\mathtt{a} = \frac{(1-2\mathtt{j}^2)(n_2^2-n_1^2)}{n_1^2+n_2^2}\,, \qquad \Delta z = \frac{\sqrt{2}\sqrt{n_1^2+n_2^2}}{n_1 n_2\sqrt{1-2\mathtt{j}^2}}\pi\,. \tag{8.33}$$

We may now compute the magnetic flux (8.16) of the gauge field and find

$$G_{(4)}Q_m = \frac{1}{4\pi}\int_\Sigma \mathrm{d}A = \frac{n_2-n_1}{4n_1n_2}\,, \tag{8.34}$$

where $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_1,n_2]}$ is the spindle horizon, precisely as we had for the general PD black holes. In particular notice this result is independent of the continuous rotation parameter $\mathtt{j}$. We can also derive a very useful expression for the electric charge Q_e (8.17) by calculating it directly in the near horizon solution. Doing so we find

$$G_{(4)}Q_e = \frac{1}{4\pi}\int_{\mathbb{\Sigma}} *F = \mathtt{j}\frac{\Delta z}{4\pi}\,. \tag{8.35}$$

Given the expression (8.33) for Δz, we may now solve for $\mathtt{j}$ in terms of the physical black hole parameter Q_e:

$$\mathtt{j} = \frac{2\sqrt{2}n_1n_2(G_{(4)}Q_e)}{\sqrt{16n_1^2n_2^2(G_{(4)}Q_e)^2+n_1^2+n_2^2}}\,. \tag{8.36}$$

The area of the horizon is then: Area $= \frac{1}{2}(y_3-y_2)\Delta z$. Substituting for $\mathtt{j}$ in terms of Q_e using (8.36),

we find the entropy is

$$S_{BH} = \frac{\text{Area}}{4G_{(4)}} = \frac{\sqrt{2}\sqrt{8n_1^2n_2^2(G_{(4)}Q_e)^2 + n_1^2 + n_2^2} - (n_1+n_2)}{n_2n_1}\frac{\pi}{4G_{(4)}} \,. \tag{8.37}$$

Notice that setting $\mathtt{j} = 0$ is equivalent to $Q_e = 0$, which gives the non-rotating limit. The expression (8.37) with $Q_e = 0$ gives the entropy of the non-rotating but accelerating extremal supersymmetric black holes, studied in more detail in section 8.2. We also note that the angular momentum J and electric charge Q_e are related by

$$J = Q_e \frac{\sqrt{2}\sqrt{8n_1^2n_2^2(G_{(4)}Q_e)^2 + n_1^2 + n_2^2} - (n_1+n_2)}{4n_2n_1} \,, \tag{8.38}$$

for these extremal solutions. Formally setting $n_2 = n_1 = 1$ into the relation (8.38) gives the corresponding relation for the supersymmetric extremal Kerr-Newman-AdS black holes [568] (see appendix B of [4]).

We note that we can also express the black hole entropy as

$$S_{BH} = \left(\frac{J_{AdS_2}}{Q_e} - \frac{1}{4}\chi(\mathbb{\Sigma})\right)\frac{\pi}{G_{(4)}} \,. \tag{8.39}$$

Here $\chi(\mathbb{\Sigma})$ is the Euler number of the spindle horizon $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_1,n_2]}$ given by

$$\chi(\mathbb{\Sigma}) = \frac{1}{4\pi}\int_{\mathbb{\Sigma}} R_2 \,\text{vol}_2 = \frac{n_1+n_2}{n_1n_2} \,, \tag{8.40}$$

where R_2 denotes the Ricci scalar of the spindle, while J_{AdS_2} is the angular momentum of the solution computed with the prescription (8.19), but where the gauge (8.28) is used, which is such that the gauge field is invariant under the AdS_2 isometries.More details on the gauge dependence of the angular momentum can be found in appendix F of [4], while here we limit to observe that in terms of the angular momentum J introduced in (8.20) we have

$$J_{AdS_2} - J = \frac{Q_e}{4}\chi(\mathbb{\Sigma}) \,. \tag{8.41}$$

This formula as well as (8.39) are also valid for non-accelerating Kerr-Newman-AdS black holes upon setting $n_2 = n_1 = 1$.

We can also express the entropy in another form[4], by replacing the orbifold parameters $n_\pm$ with the magnetic flux Q_m, given in (8.34), and the Euler number (8.40):

$$S_{BH} = \frac{\pi}{G_{(4)}}\frac{J}{Q_e} = \frac{\pi}{4G_{(4)}}\left(\sqrt{\chi^2 + 16\left[(G_{(4)}Q_e)^2 + (G_{(4)}Q_m)^2\right]} - \chi\right) . \tag{8.42}$$

Correspondingly, this implies that the near horizon angular momentum can be written in the form

$$J_{AdS_2} = \frac{Q_e}{4}\sqrt{\chi^2 + 16\left[(G_{(4)}Q_e)^2 + (G_{(4)}Q_m)^2\right]} . \tag{8.43}$$

A detailed discussion of the Killing spinors associated to this solution can be found in section 5 of [4], where through an analysis of the Killing spinor bilinears it is also shown that the R symmetry of the $1d$ SCQM dual to this solution is related to that of the parent $3d$ $\mathcal{N}=2$ SCFT via

$$R = R_{3d} + 2\sqrt{1-\mathtt{j}^2}\partial_z = R_{3d} + 2\sqrt{2}\frac{n_1 n_2\sqrt{8n_1^2n_2^2(G_{(4)}Q_e)^2 + n_1^2 + n_2^2}}{\sqrt{16n_1^2n_2^2(G_{(4)}Q_e)^2 + n_1^2 + n_2^2}}\partial_\varphi . \tag{8.44}$$

An interesting feature of this formula is that the R symmetry of the $1d$ SCQM is a mixture of the R symmetry of the parent $3d$ theory with the $U(1)$ isometry of the spindle, generated by ∂_z (or equivalently ∂_φ), with a mixing parameter that is non vanishing even in the non rotating case ($\mathtt{j} = 0$).This is a general feature of spindle solutions and is related to the presence of acceleration.

8.2 The conformal boundary for non-rotating solutions: $a = 0$

An ultimate goal for holography would be to reproduce the black hole entropy (8.42) for the extremal supersymmetric solutions via a dual field theory computation. The dual theory lives on the conformal boundary three-manifold of the full black hole solution. In this section we therefore turn to looking at this conformal boundary and, for simplicity, we will now set the rotation parameter $a = 0$. We will see that for the supersymmetric extremal black holes we must have $e = 0$, hence $J = Q_e = 0$, and the near horizon solutions can be obtained by setting $\mathtt{j} = 0$ in the near horizon metric of section 8.1.4.

We shall see that the global black hole geometry of the solutions with $a = 0$ have some interesting features, including an acceleration horizon beyond $r = \infty$. For the supersymmetric and extremal

[4]It is interesting to wonder if this formula can also be used for supersymmetric black holes with no acceleration, electric charge or rotation. The answer is no. However, we note that for the so called universal twist black holes, with horizon consisting of Riemann surface with genus $\mathtt{g} > 1$, one obtains the correct entropy formula (as in, *e.g.*, [506]), after setting $Q_e = 0$ and formally taking $G_{(4)}Q_m \to \mathrm{i}\chi/4$ with $\chi = 2(1-\mathtt{g})$.

solution this acceleration horizon intersects the conformal boundary, effectively dividing the latter in half. Moreover, we shall find that the Killing spinor on this conformal boundary is given by a topological twist, so that the spinor is constant, but it is a *different* constant spinor on each half of the space.

These exotic features will make a dual field theory calculation more challenging, but in the remaining subsections we show that the features arise as a limit of more well-behaved solutions, still with $a = 0$. In particular in section 8.2.2 we relax the extremality condition, while preserving supersymmetry. The boundary three-manifold is now a smooth product of the time direction with a spindle, with a single component, and has a smooth boundary Killing spinor. It is therefore natural to perform any field theory localization calculation in this setting. One could then take the extremal limit. However, in the bulk of this solution there is a naked curvature singularity. In section 8.2.3 we discuss a simple way of regulating this feature by also relaxing the requirement of supersymmetry, to obtain completely regular accelerating black holes with a smooth conformal boundary.

8.2.1 Supersymmetric and extremal solutions

In this section we focus on the solutions in (8.4) with $a = 0$, which depend on four free parameters m, e, g and α. Explicitly, we then have

$$\mathrm{d}s^2 = \frac{1}{H^2}\left[-\frac{Q}{r^2}\mathrm{d}t^2 + \frac{r^2}{Q}\,\mathrm{d}r^2 + \frac{r^2}{P}\mathrm{d}\theta^2 + Pr^2\sin^2\theta\mathrm{d}\phi^2\right], \tag{8.45}$$

with

$$\begin{aligned} H &= 1 - \alpha\, r\cos\theta\,, \\ P &= 1 - 2\alpha m\cos\theta + \alpha^2(e^2+g^2)\cos^2\theta\,, \\ Q &= (r^2 - 2mr + e^2 + g^2)(1-\alpha^2 r^2) + r^4\,. \end{aligned} \tag{8.46}$$

The gauge field is given by

$$A = -\frac{e}{r}\mathrm{d}t - g\cos\theta\mathrm{d}\phi\,. \tag{8.47}$$

By directly examining the integrability conditions for the Killing spinor equations when $a = 0$,

we find that supersymmetry requires

$$\begin{aligned} 0 &= m^2 - (e^2+g^2)(1+(e^2+g^2)\alpha^2)\,, \\ 0 &= m^4 - m^2(e^2+g^2) - g^2(e^2+g^2)^2\,. \end{aligned} \tag{8.48}$$

In fact we get the same system of equations from (8.21) after setting $a = 0$ and assuming that $\alpha \neq 0$ as well as one of e, g to be non-vanishing. These can be solved to give

$$\begin{aligned} m^2 &= \frac{1-\alpha^2-2\alpha^4e^2+\sqrt{(1-\alpha^2)^2-4\alpha^4e^2}}{2\alpha^6}\,, \\ g^2 &= \frac{1-\alpha^2-2\alpha^4e^2+\sqrt{(1-\alpha^2)^2-4\alpha^4e^2}}{2\alpha^4}\,, \end{aligned} \tag{8.49}$$

where we have taken the positive square roots in order to continuously connect with the extremal solution below. Note that (8.49) implies that $m = g/\alpha$.

The extremal limit is given by setting $e = 0$. Everything may then be expressed in terms of one parameter, for example α, via

$$e = 0\,, \quad g = \frac{\sqrt{1-\alpha^2}}{\alpha^2}\,, \quad m = \frac{\sqrt{1-\alpha^2}}{\alpha^3}\,. \tag{8.50}$$

The black hole horizon radius is given by the largest double root $r_+ > 0$ of $Q(r)$, where

$$r_\pm = \frac{-1 \pm \sqrt{5-4\alpha^2}}{2\alpha\sqrt{1-\alpha^2}}\,. \tag{8.51}$$

We require the function $P \geq 0$ and so we should restrict the range of α to be

$$\frac{\sqrt{3}}{2} < \alpha < 1\,, \tag{8.52}$$

and we also observe that in this range r_+ is positive and r_- is negative. The regularity analysis fixes the parameter α in terms of the integers n_1, n_2, so that there are no remaining free parameters.In particular, we note that in terms of the integers $n_2 > n_1$ specifying the orbifold singularities at the poles $\theta = 0, \theta = \pi$, we have

$$\alpha = \frac{\sqrt{(3n_1+n_2)(3n_1+n_2)}}{2(n_1+n_2)}\,. \tag{8.53}$$

Thus the lower limit $\sqrt{3}/2$ for α in (8.52) is the limit $n_2 \to \infty$, holding n_1 fixed, while the upper limit of 1 corresponds[5] to $n_2 - n_1 \to 0$.

Next we look in more detail at the global structure of this extremal solution. We take $r \geq r_+$, where recall that the conformal boundary is at $\alpha r \cos\theta = 1$. Notice immediately that for $\theta > \pi/2$ this requires r *negative*. Globally r is not a good coordinate, and we instead put $y = 1/r$. The black hole metric (8.45) now reads

$$\mathrm{d}s^2 = \frac{1}{(y - \alpha\cos\theta)^2}\Big[-Y\mathrm{d}t^2 + \frac{\mathrm{d}y^2}{Y} + \frac{1}{P}\mathrm{d}\theta^2 + P\sin^2\theta\mathrm{d}\phi^2\Big]\,, \tag{8.54}$$

where for the extremal supersymmetric solution we have introduced

$$Y = Y(y) = (1-\alpha^2)(1-r_-y)^2(1-r_+y)^2\,. \tag{8.55}$$

For the original r coordinate we have $r > r_+ > 0$, which implies

$$y \leq \frac{1}{r_+} \equiv y_+\,, \tag{8.56}$$

and we may then continue y past zero to negative values (effectively extending beyond $r = \infty$). The y coordinate decreases as one moves away from the horizon at $y = y_+$, eventually hitting the conformal boundary at $y = \alpha\cos\theta$, with $y - \alpha\cos\theta > 0$ in the interior of the spacetime. However, although $1 - r_+y > 0$ for $y < y_+$, for negative y one can reach the double root of the metric function Y at $y = 1/r_-$, where we recall that $r_- < 0$ given (8.52). One can show that this is an *acceleration horizon*. To emphasize this we write $r_- \equiv r_A$, and then the acceleration horizon is located at

$$y = \frac{1}{r_A} \equiv y_A\,, \qquad r_A \equiv r_-\,. \tag{8.57}$$

One can ask when the acceleration horizon at $y = y_A$ intersects the conformal boundary. This is determined by the equation

$$y_A = \frac{1}{r_A} = \alpha\cos\theta\,, \tag{8.58}$$

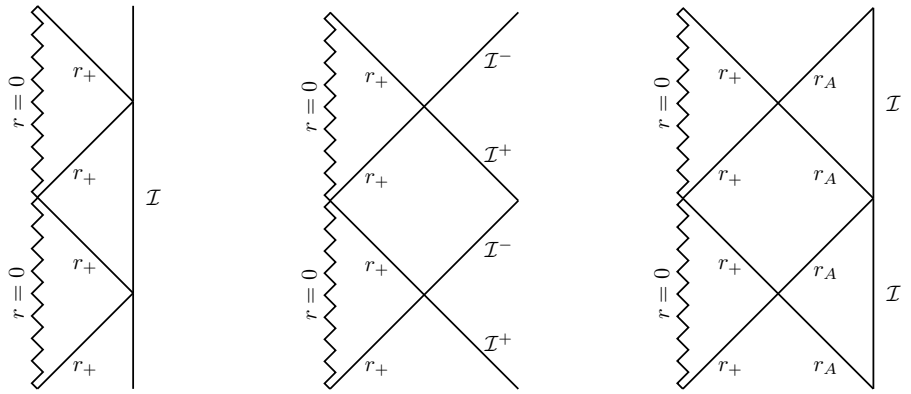
[5]Note that in this limit we have $\alpha = 1$, $g = m = 0$ and hence, in particular, vanishing gauge field, and the metric is locally that of AdS_4.

which can be solved to give

$$\theta = \theta_0 \equiv \arccos\left(\frac{1-\sqrt{5-4\alpha^2}}{2\sqrt{1-\alpha^2}}\right) = \arccos\frac{n_1+n_2-\sqrt{2n_2^2+2n_1^2}}{n_2-n_1}\,. \tag{8.59}$$

On the other hand, for given α and fixed $\theta > \theta_0$, note $y_A > \alpha\cos\theta$, which means that as one approaches from the black hole one hits the acceleration horizon before the conformal boundary. On the conformal boundary itself the lower half of the spindle, with $\theta \in (\theta_0, \pi]$, then effectively lies behind the acceleration horizon. Interestingly, the acceleration horizon is also extremal, and there is an asymptotic AdS_2 region as one approaches $y \to y_A$ from above.

An extensive analysis of the causal structure of the AdS C-metrics is presented in [562], for general values of the parameters. The Penrose diagram for the extremal supersymmetric black hole solution is shown in figure 8.1. In particular we note that the lower half of the spindle boundary, with $\theta > \theta_0$, lies behind the acceleration horizon $r = r_A$.



(a) $\theta < \theta_0$ (b) $\theta = \theta_0$ (c) $\theta > \theta_0$

Figure 8.1: Penrose diagram for the non-rotating supersymmetric extremal black hole, for different slices of constant θ. The black hole horizon is denoted by r_+. The left panel is associated with a conformal boundary, consisting of the product of the time direction with half a spindle with $\theta < \theta_0$. In the right panel one reaches the other half of the spindle with $\theta > \theta_0$ at the conformal boundary after passing through the extremal acceleration horizon, denoted by r_A. For $\theta = \theta_0$, in the middle panel, a null infinity appears.

Next let us look more closely at the conformal boundary itself. Starting from the general non-rotating black hole metric (8.45), the conformal boundary is located at $H = 0$. Choosing the conformal factor so that the timelike Killing vector ∂_t has unit norm on the boundary, we find that the general form of the conformal boundary metric is

$$\mathrm{d}s^2_{3d} = -\mathrm{d}t^2 + \mathrm{d}s^2_{\Sigma} = -\mathrm{d}t^2 + \frac{\mathrm{d}\theta^2}{P\,(1-\alpha^2\,\sin^2\theta\,P)^2} + \frac{P\,\sin^2\theta}{1-\alpha^2\,\sin^2\theta\,P}\,\mathrm{d}\phi^2\,. \tag{8.60}$$

That is, the conformal boundary is a static product metric, where the induced metric on a constant time slice is $\mathrm{d}s^2_{\mathbb{\Sigma}}$.

We have seen that for the extremal supersymmetric black hole the acceleration horizon intersects the conformal boundary at $\theta = \theta_0$, and in fact the boundary actually splits in half along this slice. Indeed, although $P = P(\theta) > 0$ for all $\theta \in [0, \pi]$, we find that for the extremal solution $1 - \alpha^2 \sin^2\theta P(\theta) \geq 0$, with equality if and only if $\theta = \theta_0$. The metric on $\mathbb{\Sigma}$ in (8.60) is then singular for $\theta = \theta_0$. Introducing the new coordinate

$$\rho = \frac{\alpha\sqrt{1+\sqrt{5-4\alpha^2}}}{\sqrt{2}\,(5-4\alpha^2)}\frac{1}{\theta-\theta_0}\,, \tag{8.61}$$

we find that near to $\theta = \theta_0$ (which is $\rho = \infty$), the metric on $\mathbb{\Sigma}$ takes the form

$$\mathrm{d}s^2_{\mathbb{\Sigma}} \simeq \mathrm{d}\rho^2 + \frac{5-4\alpha^2}{\alpha^4}\rho^2\mathrm{d}\phi^2\,. \tag{8.62}$$

Thus each side of $\theta = \theta_0$ opens out into a non-compact asymptotically locally Euclidean end, with each of the poles $\theta = 0$ and $\theta = \pi$ being an infinite distance from $\theta = \theta_0$. The conformal boundary thus effectively has two halves, with the lower half $\theta \in (\theta_0, \pi]$ lying behind the acceleration horizon.

We can now discuss the behaviour of the Killing spinor on the conformal boundary. The bulk Killing spinor equation for minimal $d = 4$, $\mathcal{N} = 2$ gauged supergravity induces the following conformal Killing spinor equation (CKSE) on the conformal boundary [569]:

$$\nabla_a\zeta = \frac{1}{3}\gamma_a\not{\nabla}\zeta\,, \tag{8.63}$$

where we have introduced the covariant derivative $\nabla_a = \partial_a + \frac{1}{4}\omega_a^{bc}\gamma_{bc} - \mathrm{i}A_a$. Here $a = 0, 1, 2$ is a tangent space index, and we may take the gamma matrices to be $\gamma_0 = \mathrm{i}\sigma^1$, $\gamma_1 = \sigma^2$, $\gamma_2 = \sigma^3$, in terms of Pauli matrices. To solve this equation we begin by introducing the obvious orthonormal frame

$$e^0 = \mathrm{d}t\,, \qquad e^1 = \frac{\mathrm{d}\theta}{\sqrt{P}(1-\alpha^2\sin^2\theta P)}\,, \qquad e^2 = \sqrt{\frac{P}{1-\alpha^2\sin^2\theta P}}\sin\theta\mathrm{d}\phi\,. \tag{8.64}$$

For the extremal solution with $e = 0$ the gauge field is

$$A = -g\cos\theta\mathrm{d}\phi = -\frac{\sqrt{1-\alpha^2}}{\alpha^2}\cos\theta\mathrm{d}\phi\,. \tag{8.65}$$

It is convenient to define the gauge-equivalent gauge field

$$A_0 \equiv A + \frac{1}{2\alpha^2}\mathrm{d}\phi\,. \tag{8.66}$$

Then remarkably we find that this gauge field is equal to plus or minus $\frac{1}{2}\omega^{12}$, where ω^{12} is the non-zero spin connection component for $\mathbb{\Sigma}$, with the sign depending on which half of $\mathbb{\Sigma}$ the expressions are compared in:

$$\tfrac{1}{2}\omega^{12} = \begin{cases} -A_0 & \theta \in [0,\theta_0)\,, \\ +A_0 & \theta \in (\theta_0,\pi]\,. \end{cases} \tag{8.67}$$

We find that the solution to (8.63) is in fact covariantly constant, so $\nabla_a\zeta = 0$, and moreover due to (8.67) in the gauge (8.66) the solution for ζ is in fact constant:

$$\zeta = \begin{cases} \begin{pmatrix} -\mathrm{i} \\ \mathrm{i} \end{pmatrix} & \theta \in [0,\theta_0)\,, \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \theta \in (\theta_0,\pi]\,. \end{cases} \tag{8.68}$$

There is thus a topological twist on each half of the conformal boundary, with the gauge field effectively cancelling the spin connection and leading to a constant spinor, but with a discontinuity in the spinor as one moves across the slice $\theta = \theta_0$ that intersects the acceleration horizon of the bulk black hole. Of course we may multiply each spinor in (8.68) by any constant, and this will still be a solution. The reason for normalizing the spinors in the way that we have, in particular taking a purely imaginary spinor in $\theta \in [0,\theta_0)$, will be come apparent in the next subsection.

Finally in this subsection we note that it is possible to derive an explicit expression for the entropy of the non-rotating extremal supersymmetric solution directly from (8.15). In terms of $n_\pm$ we find

$$g = \frac{2(n_2^2 - n_1^2)}{(3n_1+n_2)(n_2+3n_1)}\,, \qquad \Delta\phi = \frac{(3n_1+n_2)(n_2+3n_1)\pi}{4n_1n_2(n_1+n_2)}\,. \tag{8.69}$$

The horizon radii $r = r_\pm$ are given by

$$r_\pm = \frac{2\,(n_1+n_2)\left(\pm\sqrt{2}\sqrt{n_1^2+n_2^2} - (n_1+n_2)\right)}{(n_2-n_1)\,\sqrt{(3n_1+n_2)\,(n_2+3n_1)}}\,. \tag{8.70}$$

Using (8.15) we then find the entropy is given by

$$S_{BH} = \frac{\pi}{4G_{(4)}} \frac{\sqrt{2}\sqrt{n_1^2+n_2^2}-(n_1+n_2)}{n_2 n_1} \tag{8.71}$$

The first expression agrees with (8.37) after setting $Q_e = 0$, where recall that (8.37) was instead computed using the near horizon metric for the general supersymmetric extremal solution.

8.2.2 Supersymmetric and non-extremal solutions

The non-rotating, extremal supersymmetric black hole has some slightly exotic features, especially the behaviour of the conformal boundary and its Killing spinor. In this section we relax the extremality condition $e = 0$, instead imposing the BPS relations (8.49) on the conformal boundary geometry with $e \neq 0$. We shall find that the conformal boundary has the same form as (8.60), but now with a completely regular metric on the spindle $\mathbb{\Sigma}$, apart from the usual orbifold singularities at $\theta = 0$ and $\theta = \pi$, so that $\mathbb{\Sigma} \cong \mathbb{WCP}^1_{[n_1,n_2]}$ has the same topology as the black hole horizon in the extremal limit. The circumference of $\mathbb{\Sigma}$ near to $\theta = \theta_0$ grows as $e \to 0$, as does the distance between the poles and $\theta = \theta_0$, with the spindle effectively completely splitting in half in the extremal limit $e = 0$. There is correspondingly a smooth solution ζ to the $d = 3$ Killing spinor equation for $e \neq 0$, that approaches the piecewise constant solution (8.68) in the extremal limit. As we discuss, the non-rotating BPS and non-extremal solutions no longer have a smooth black horizon but a naked singularity.

We first note that with $a = 0$, the BPS conditions (8.49) and the regularity conditions (8.11) imply that

$$\begin{aligned} m &= \frac{g}{\alpha}, \qquad \alpha = \frac{\sqrt{(3n_1+n_2)(3n_1+n_2)}}{2(n_1+n_2)\sqrt{1+\frac{e^2(3n_1+n_2)^2(3n_1+n_2)^2}{4(n_2^2-n_1^2)^2}}}, \\ g &= \frac{2(n_2^2-n_1^2)}{(3n_1+n_2)(n_2+3n_1)}, \qquad \Delta\phi = \frac{(3n_1+n_2)(n_2+3n_1)}{4n_1n_2(n_1+n_2)}\pi . \end{aligned} \tag{8.72}$$

These expressions can be obtained by solving the regularity condition (8.11) for e and then substiuting this into the first line of (8.21) to derive the expression for g. Then substituting this expression for g into (8.11) we get the expression for α. We also note that if we set $e = 0$ then we recover the same conditions for the BPS and extremal solutions that we considered in the previous subsection.

To analyse the conformal boundary and the Killing spinors both on the boundary and in the bulk,

it is convenient to change to PD-type coordinates via

$$t = \alpha\tau\,, \qquad \cos\theta = p\,, \qquad \phi = \alpha^2\sigma\,, \qquad r = -1/(\alpha q)\,, \tag{8.73}$$

where $p \in [-1, 1]$. We also change parameters by introducing

$$g = \frac{\mathsf{P}}{\alpha^2}\,, \qquad e = \frac{\mathsf{Q}}{\alpha^2}\,, \qquad \mathsf{C} = \mathsf{P}^2 + \mathsf{Q}^2\,. \tag{8.74}$$

The BPS conditions (8.49) imply $m = g/\alpha$, as usual, and $\alpha = (\mathsf{P}^2/\mathsf{C} - \mathsf{C})^{1/2}$. Using these relations, we can then write the full non-rotating solution (8.45) in these coordinates:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{(p+q)^2}\left(-\mathcal{Q}(q)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}q^2}{\mathcal{Q}(q)} + \frac{\mathrm{d}p^2}{\mathcal{P}(p)} + \mathcal{P}(p)\,\mathrm{d}\sigma^2\right)\,, \\ A &= \mathsf{Q}\,q\,d\tau - \mathsf{P}\,p\,\mathrm{d}\sigma\,, \end{aligned} \tag{8.75}$$

where the metric functions are

$$\mathcal{P}(p) = \mathsf{C}^{-1}\,\mathcal{P}_1(p)\,\mathcal{P}_2(p)\,, \quad \mathcal{Q}(q) = \mathsf{C}^{-1}\,\mathcal{Q}_1(q)\,\mathcal{Q}_2(q)\,, \tag{8.76}$$

and we have introduced

$$\begin{aligned} \mathcal{P}_1(p) &= -(1-p)(\mathsf{C}p + \mathsf{C} - \mathsf{P})\,, & \mathcal{Q}_1(q) &= \mathsf{C}q^2 - \mathsf{C} + \mathsf{P}q + \mathrm{i}\mathsf{Q}\,, \\ \mathcal{P}_2(p) &= -(1+p)(\mathsf{C}p - \mathsf{C} - \mathsf{P})\,, & \mathcal{Q}_2(q) &= \mathsf{C}q^2 - \mathsf{C} + \mathsf{P}q - \mathrm{i}\mathsf{Q}\,. \end{aligned} \tag{8.77}$$

The regularity conditions on the metric imply that

$$\mathsf{P} = \frac{n_2 - n_1 + \sqrt{(n_2-n_1)^2 - 16(n_1+n_2)^2\mathsf{Q}^2}}{4(n_1+n_2)}\,, \tag{8.78}$$

with $\mathsf{P} < \frac{1}{2}$, while the period of σ can be expressed as

$$\Delta\sigma = 2\pi\frac{n_2^2 - n_1^2}{n_1 n_2(n_2 - n_1 + \sqrt{(n_1-n_2)^2 - 16(n_1+n_2)^2\mathsf{Q}^2})}\,. \tag{8.79}$$

Notice that we can parametrize this class of solutions in terms of $n_{1,2}$ and Q, with the extremal limit

obtained when $\mathsf{Q} \to 0$. The reality of P in (8.78) requires that we impose

$$\mathsf{Q} \leq \frac{n_2 - n_1}{4(n_1 + n_2)}\,, \tag{8.80}$$

The conformal boundary metric (8.60), obtained at $p = -q$, is then (after a rescaling by the constant α^2) given by

$$\mathrm{d}s^2_{3d} = -\mathrm{d}\tau^2 + \mathrm{d}s^2_{\mathbb{\Sigma}} = -\mathrm{d}\tau^2 + \frac{\mathrm{d}p^2}{\mathcal{P}(p)(1-\mathcal{P}(p))^2} + \frac{\mathcal{P}(p)}{1-\mathcal{P}(p)}\mathrm{d}\sigma^2\,. \tag{8.81}$$

Notice that $\mathcal{P}(p) > 0$ for $p \in [-1,1]$ is implied by (8.80). The circumference $\mathcal{C}$ of the spindle, at fixed $p \in [-1,1]$, is given by the function

$$\mathcal{C} = \mathcal{C}(p) = \sqrt{\frac{\mathcal{P}(p)}{1-\mathcal{P}(p)}}\Delta\sigma\,. \tag{8.82}$$

We have plotted this in figure 8.2 for the spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[3,1]}$, with progressively smaller values of Q, tending to the extremal solution with $\mathsf{Q} = 0$. The circumference at $p = p_0$ is infinite for $\mathsf{Q} = 0$, where $p_0 = \cos\theta_0$, with θ_0 given by (8.59):

$$p_0 = \frac{n_1 + n_2 - \sqrt{2n_2^2 + 2n_1^2}}{n_2 - n_1}\,. \tag{8.83}$$

In the $\mathsf{Q} = 0$ limit the spindle has then effectively split in half.

At this point one might picture the geometry as breaking up into two "pancakes" as $\mathsf{Q} \to 0$. However, this is not correct and is clarified by calculating the proper distance from $p = \pm 1$ to $p = p_0$, both of which diverge as $\mathsf{Q} \to 0$. It is illuminating to present the geometry of the spindle as an embedding in three-dimensional Euclidean space, as in figure 8.3. As $\mathsf{Q} \to 0$, while the circumference at p_0 is diverging, so too is the height of the figures.

Introducing the orthonormal frame[6] for the conformal boundary metric (8.81)

$$e^0 = \mathrm{d}\tau\,, \qquad e^1 = -\frac{\mathrm{d}p}{\sqrt{\mathcal{P}(p)}(1-\mathcal{P}(p))}\,, \qquad e^2 = \sqrt{\frac{\mathcal{P}(p)}{1-\mathcal{P}(p)}}\mathrm{d}\sigma\,, \tag{8.84}$$

we may solve the conformal Killing spinor equation (8.63) using the same basis of gamma matrices

[6]Notice the overall minus sign in e^1 is so as to match the orientation in the corresponding frame (8.64), where $dp = -\sin\theta d\theta$.

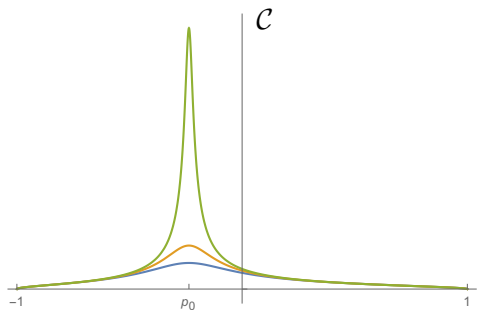


Figure 8.2: Circumference $\mathcal{C}$ of the metric on the spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[3,1]}$ on the conformal boundary as a function of $p \in [-1, 1]$, for the supersymmetric non-extremal black holes. The blue, orange, and green curves have progressively smaller values of Q, namely $\mathsf{Q} = 0.04$, $\mathsf{Q} = 0.01$ and $\mathsf{Q} = 0.0002$, respectively, tending to the extremal solution with $\mathsf{Q} = 0$. The same values of Q are plotted also in figures 8.4 and 8.5.

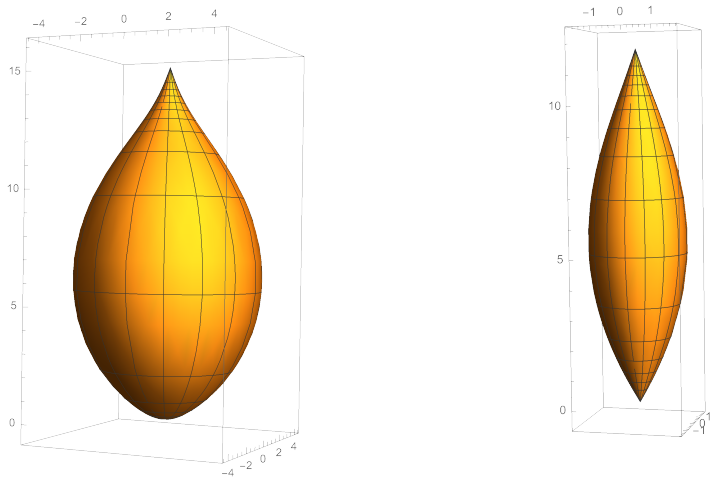


Figure 8.3: Embedding of two spindles on the conformal boundary for the supersymmetric non-extremal black holes in three dimensional Euclidean space. The left plot is for $\mathbb{\Sigma} = \mathbb{WCP}^1_{[3,1]}$ and $\mathsf{Q} = 0.04$ and the right plot is for $\mathbb{\Sigma} = \mathbb{WCP}^1_{[3,2]}$ and $\mathsf{Q} = 0.025$.

as the previous subsection (see below (8.63)). We find that

$$\zeta = \mathrm{e}^{-\mathrm{i}(\kappa_1\tau+\kappa_2\sigma)} \begin{pmatrix} \zeta_1(p) \\ \zeta_2(p) \end{pmatrix} , \tag{8.85}$$

where we have introduced the constants

$$\kappa_1 = \frac{\mathsf{PQ}}{2(\mathsf{P}^2+\mathsf{Q}^2)} , \qquad \kappa_2 = \frac{\mathsf{P}^2}{2(\mathsf{P}^2+\mathsf{Q}^2)} . \tag{8.86}$$

Notice that in the extremal limit $\mathsf{Q} = 0$ the phase in (8.85) is $\mathrm{e}^{-\mathrm{i}\sigma/2} = \mathrm{e}^{-\mathrm{i}\phi/2\alpha^2}$, which was compensated for in the previous subsection by making the gauge transformation (8.66). The components

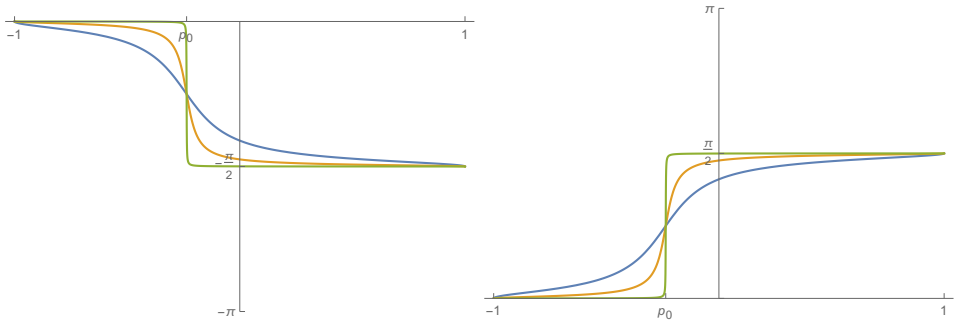
$\zeta_1(p)$, $\zeta_2(p)$ satisfy the equations

$$\begin{aligned}\zeta_1' &= \frac{\mathrm{i}(p\mathsf{Q}-\kappa_1)}{\sqrt{\mathcal{P}(p)}(1-\mathcal{P}(p))}\zeta_1\,,\\ \zeta_2 &= \frac{4\mathrm{i}(p\mathsf{Q}-\kappa_1)\mathcal{P}(p)-\sqrt{\mathcal{P}(p)}\mathcal{P}'(p)}{4(p\mathsf{P}-\kappa_2)\sqrt{\mathcal{P}(p)(1-\mathcal{P}(p))}}\zeta_1\,.\end{aligned} \tag{8.87}$$

After some effort, one finds the solution

$$\begin{aligned}\zeta_1(p) &= \sqrt{\frac{\mathcal{P}_1(p)-\mathcal{P}_2(p)-2\mathrm{i}\mathsf{Q}\sqrt{\mathcal{P}(p)}}{2\mathsf{P}\sqrt{1-\mathcal{P}(p)}}}\,,\\ \zeta_2(p) &= \sqrt{\frac{\mathcal{P}_1(p)-\mathcal{P}_2(p)+2\mathrm{i}\mathsf{Q}\sqrt{\mathcal{P}(p)}}{2\mathsf{P}\sqrt{1-\mathcal{P}(p)}}}\,.\end{aligned} \tag{8.88}$$

Notice that $\zeta_1^* = \zeta_2$ and $|\zeta_1| = |\zeta_2|$. In fact we have chosen the overall normalization constant of the spinor so that the components lie on the unit circle in the complex plane, namely $|\zeta_1| = |\zeta_2| = 1$.



(a) $\arg\zeta_1(p)$ (b) $\arg\zeta_2(p)$

Figure 8.4: Arguments of $\zeta_1(p)$, $\zeta_2(p)$, appearing in the conformal Killing spinors on the conformal boundary for the supersymmetric non-extremal black holes, as a function of $p\in[-1,1]$ and plotted for the spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[3,1]}$ The blue, orange, and green curves have progressively smaller values of Q, as given in figure 8.2, tending to the extremal solution with $\mathsf{Q}=0$. For $\mathsf{Q}\neq 0$ we have a smooth conformal Killing spinor on the spindle that approaches two different constant values on each half of the spindle in the extremal limit $\mathsf{Q}\to 0$.

In figure 8.4 we have plotted the arguments $\arg\zeta_1(p)$, $\arg\zeta_2(p)$, for the spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[3,1]}$ with progressively smaller values of Q, tending to the extremal solution with $\mathsf{Q}=0$. In the latter case notice that in the $\mathsf{Q}\to 0$ limit we have

$$\text{for } \mathsf{Q}=0\text{:}\qquad \begin{cases}\arg\zeta_2(p) = -\arg\zeta_1(p) = \frac{\pi}{2}\,, & p\in(p_0,1]\,,\\ \arg\zeta_2(p) = \arg\zeta_1(p) = 0\,, & p\in[-1,p_0)\,.\end{cases} \tag{8.89}$$

This precisely corresponds to the extremal solution (8.68).

We can also display the way in which the two different topological twists for the extremal case arise in the limit that $\mathsf{Q} \to 0$. In figure 8.5 we have plotted the spin connection and gauge field for the conformal boundary geometries corresponding to figure 8.4. More precisely, the non-trivial spin connection component is ω^{12}, and we define its holonomy around a circle in the spindle $\mathbb{\Sigma}$ at constant p, parametrized by σ, via

$$\omega(p) = \frac{1}{2\pi}\int_{S^1} \omega^{12}\,. \tag{8.90}$$

The solid curves in figure 8.5 are then $\frac{1}{2}\omega(p)$, while the dashed lines are $\pm$ the corresponding holonomy of the gauge field $A + \kappa_2 d\sigma$, *cf.* equation (8.67).

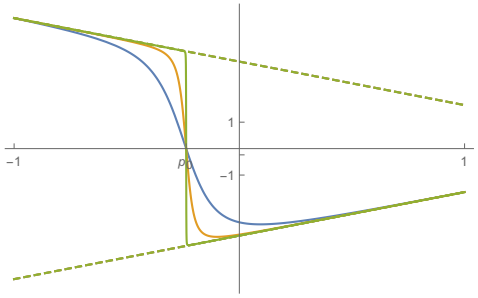


Figure 8.5: Spin connection and gauge field holonomies around a circle in the spindle $\mathbb{\Sigma}$ for the supersymmetric non-extremal black holes, as a function of $p \in [-1, 1]$ and plotted for the spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[3,1]}$. The dashed lines are $\pm$ the holonomy of the gauge field, while the solid blue, orange, and green lines are the corresponding holonomy of the spin connection for progressively smaller values of Q, as given in figure 8.2, tending to the extremal solution with $\mathsf{Q} = 0$. For $\mathsf{Q} \neq 0$ we do not have a topological twist, but in the extremal limit $\mathsf{Q} \to 0$ we get a different topological twist on each half of the spindle.

Although the conformal boundary of this non-extremal supersymmetric solution is perfectly regular, in the bulk the black hole horizon has disappeared and there is a naked curvature singularity. To see this, we return to the black hole metric given in (8.45) in PD-type coordinates and observe that the function $\mathcal{Q}(q)$ has no real roots for $\mathsf{Q} \neq 0$. There is then a naked curvature singularity at $q = \infty$, with Penrose diagram given by figure 8.6.

8.2.3 A one-parameter family of non-supersymmetric and non-extremal solutions

In the previous two subsections we have discussed special cases that lie inside the more general class of non-rotating PD black holes. While interesting because they preserve supersymmetry, as discussed they both have some pathologies: either an acceleration horizon that cuts the conformal boundary, or a naked singularity. These pathologies arise because of the specific restrictions we have

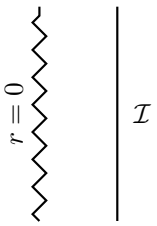


Figure 8.6: Penrose diagram for the non-extremal supersymmetric black holes, with $e \neq 0$. The black hole horizon has disappeared leaving a naked singularity. There is now a smooth conformal boundary which consists of the product of the time direction with a spindle.

imposed on the parameters in those cases. According to the number and value of the roots of the functions Q and P there are many other possibilities. A detailed analysis of the causal structure in various cases can be found in [562]. In this subsection we will consider another special case that, while allowing some degree of analytic control over the roots of the metric functions, gives a black hole with completely regular conformal boundary, and two ordinary horizons. This configuration is also smoothly connected with the extremal and BPS black hole, thus providing a kind of "regulator" of the latter solution, while staying within[7] the non-rotating family of solutions.

We start again with the general metric (8.4) with $a = 0$, and we further restrict the parameters to satisfy

$$m = \frac{g}{\alpha}\,, \quad e = 0\,, \tag{8.91}$$

and hence, in particular, we have $J = Q_e = 0$. The first of the conditions in (8.91) is satisfied when *both* BPS conditions (8.48) are met, so it amounts to imposing only one of the two conditions. In addition, we recall that this condition is also necessary to construct regular uplifted solutions, as discussed in section 4 of [4]. The second is the extremality condition in the BPS case. It follows that the black hole that we obtain with these restrictions is neither BPS nor extremal, but it is continuously connected with the case discussed in section 8.2.1 by taking the limit

$$g \to g_{BPS} = \frac{\sqrt{1-\alpha^2}}{\alpha^2}\,. \tag{8.92}$$

[7] We can also regulate the solutions in sections 8.2.1 and 8.2.2 by turning on the rotation parameter a as considered in section 8.1. In particular, the supersymmetric and extremal rotating black holes have no acceleration horizons.

Since we are taking $a = e = 0$ in (8.6), we note that the gauge field is simply

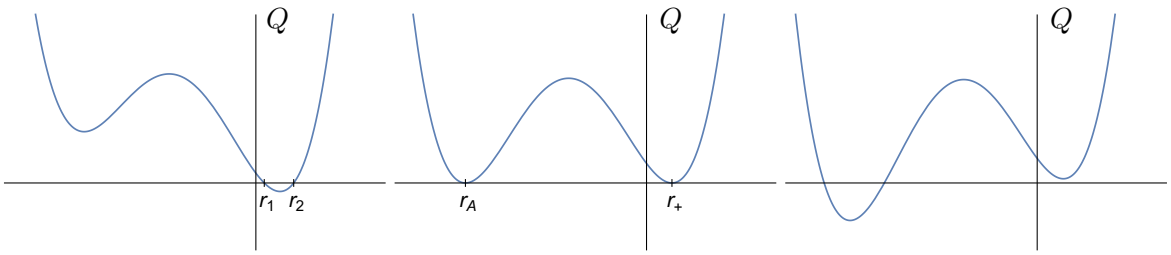
$$A = -g \cos\theta \, \mathrm{d}\phi \,. \tag{8.93}$$

We continue to take $\sqrt{3}/2 < \alpha < 1$ as in section 8.2.1, or equivalently $0 < \mathsf{P} < 1/2$ as in section 8.2.2. For fixed α, the roots of $Q(r)$ depend on g in a very simple way, as illustrated in figure 8.7. In particular, we find:

- When $g > g_{BPS}$, $Q(r)$ has two real roots for *negative* r which correspond to acceleration horizons, and no black hole horizon. As in section 8.2.1, the former intersect the conformal boundary, which causes pathologies: this can be seen from the fact that in this case the combination $1 - \alpha^2 \sin^2\theta \, P(\theta)$ appearing in the boundary metric (8.60) has two real roots, and is negative between the two.

- When $g < g_{BPS}$, $Q(r)$ has two real roots for *positive* r, which give two black hole horizons (an inner and an outer horizon) with no acceleration horizons. For this case $1 - \alpha^2 \sin^2\theta \, P(\theta)$ has no roots for $0 \leq \theta \leq \pi$, which means the conformal boundary is a smooth spindle as in section 8.2.2.

- Finally, when $g = g_{BPS}$ then $Q(r)$ has two pairs of coincident real roots, which is the case discussed in section 8.2.1.



(a) $g < g_{BPS}$ (b) $g = g_{BPS}$ (c) $g > g_{BPS}$

Figure 8.7: The function $Q(r)$ for three different cases in the class of non-supersymmetric and non-extremal black holes with $m = g/\alpha$ and $a = e = 0$. We focus on the cases $g < g_{BPS}$ and $g = g_{BPS}$. When $g < g_{BPS}$ the two positive roots, r_1, r_2, correspond to an inner and outer black hole horizon. When $g = g_{BPS}$ these two horizons coalesce to give an extremal horizon, r_+, and, in addition, an acceleration horizon, r_A, also appears. For $g > g_{BPS}$ there are no positive roots of Q, and hence no black hole horizon, but instead two negative roots which correspond to two acceleration horizons.

Since the main point of this section is showing that we can have an ordinary black hole with no

acceleration horizons, we shall focus on the case

$$g < g_{BPS} \,. \tag{8.94}$$

Furthermore, note that while so far we have focused on the roots of $Q(r)$, the restrictions we have put on α and g are also such as to guarantee $P(\theta) > 0$ for $0 \leq \theta \leq \pi$. Hence this case indeed corresponds to a completely regular black hole, whose Penrose diagram is given in figure 8.8, where we have denoted by $r_{1,2}$ the two positive roots of $Q(r)$, with $r_1 < r_2$. In the supersymmetric and extremal limit $g = g_{BPS}$ we have $r_1 = r_2 = r_+$.

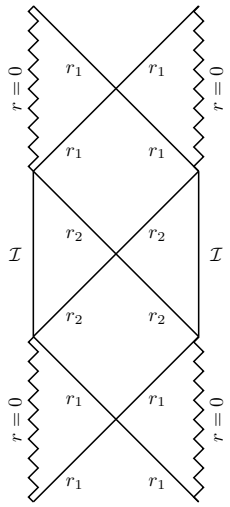


Figure 8.8: Penrose diagram for the non-supersymmetric and non-extremal black holes with $m = g/\alpha$ and $a = e = 0$ and $g < g_{BPS}$. In addition to the black hole event horizon, denoted by r_2, there is also an inner horizon, denoted by r_1. There is a smooth conformal boundary which consists of the product of the time direction with a spindle.

We can then require that the topology in the θ, ϕ directions is that of a spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_1,n_2]}$, by appropriately quantizing the conical deficits at $\theta = 0, \pi$. This gives

$$\alpha = \sqrt{\frac{4\,n_2}{g(n_2 - n_1)} - \frac{2g+1}{g^2}} \,, \tag{8.95}$$

while the periodicity of ϕ is given by

$$\Delta\phi = \frac{\pi}{2\,g}\frac{n_2 - n_1}{n_2\,n_1} \,. \tag{8.96}$$

As discussed above, in this case the black hole is completely smooth, and the spindle topology at fixed t and r persists to the conformal boundary, where the metric is again given by (8.60) and has

the topology $\mathbb{R}\times\mathbb{WCP}^1_{[n_1,n_2]}$. This is regular for any $0<g<g_{BPS}$, but degenerates as described in section 8.2.1 when g approaches the BPS value. The behaviour of the circumference of the spindle at the boundary and of the spin connection of the boundary metric are very similar to those given in figures 8.2 and 8.5, respectively. Namely, when $g\to g_{BPS}$, the spindle splits in half at

$$\theta=\theta_0\equiv\arccos\left(\frac{1-\sqrt{5-4\alpha^2}}{2\sqrt{1-\alpha^2}}\right)\,, \tag{8.97}$$

while the spin connection approaches $\pm$ the gauge field, up to the pure gauge term discussed in section 8.2.1.

8.2.4 Comments on non-minimal supergravity

To conclude this chapter, let us make some comments on accelerating black holes beyond the case of minimal gauged supergravity. Solutions in the $4d$ gauged STU model including only mass, acceleration and four independent magnetic charges were introduced in [6], where the conditions for supersymmetry and extremality are also investigated. The global analysis that should relate the parameters of the solution to the conical deficits of the spindle is however particularly involved in the general case: for this reason in [6] we focused on a truncation with pairwise equal charges. This only allows for anti-twist solutions, as in minimal gauged supergravity, and the considerations on the conformal boundary are completely analogous to those discussed in this section for PD black holes. It would be interesting to solve the regularity conditions in the case of four independent charges since, as discussed in chapter 7, this is the case that allows for twist solutions as well. In the case of twist solutions one has spinors of the same chirality at both poles of the spindle and the expectation is that on the conformal boundary one should have a topological twist (rather than two topological twists with opposite signs). An interesting question is then whether the spindle still splits into two halves also in the twist case, or rather it is smooth (except for the conical singularities) also on the boundary.

We would also like to comment on the inclusion of other parameters, such as electric charges and rotation. While the black holes considered in [6] are only magnetically charged, a family of $AdS_2\times\Sigma$ solutions was constructed that also have two electric charges and a rotation parameter. Such solutions are conjectured to describe the near-horizon limit of black holes with an analogous set of parameters and it would be interesting to find them explicitly. Similar results should be possible in the full STU model, with four independent dyonic charges, mass, acceleration and angular momentum.

Chapter 9

Discussion and outlook

In **part II** of this thesis we have discussed a novel realization of the RG flow across dimensions in gauged supergravity, where a certain SCFT is compactified on a spindle $\mathbb{\Sigma} = \mathbb{WCP}^1_{[n_1,n_2]}$, which is a two-dimensional orbifold. In most cases one only has control over the IR fixed point of the RG flow, described in gauged supergravity by certain AdS $\times$ $\mathbb{\Sigma}$ solutions which have a number of novel and interesting features compared to analogous solutions previously discussed in the literature, where $\mathbb{\Sigma}$ is usually taken to be a genus g Riemann surface Σ_g. One of the most interesting aspects of spindles is the novel way in which they preserve supersymmetry, which is distinct from the usual topological twist, as well as the fact that they sometimes uplift to smooth solutions of $D = 10/11$ supergravity despite the presence of orbifold singularities in gauged supergravity. Finally, we have discussed the relation between spindles and accelerating black holes in four-dimensional gauged supergravity, which we also conjecture to exist in other dimensions with black holes replaced by appropriate accelerating black branes. This is interesting because the study of the conformal boundary of such solutions can provide essential inputs to understand the way supersymmetry is preserved in the dual CFTs and provides the background for field theory computations. In this chapter we would like to discuss a number of possible extensions of our work.

9.1 Supersymmetry and the twist

The most subtle aspect of spindle solutions, which has raised several questions while also being one of the main reasons for the interest in this subject, is related to how supersymmetry is preserved in the RG flow. The analysis of [7], revisited in chapter 7, clarifies that only two types of twist are possible on spindles but does not give a clear understanding of how this is realized in field theory. On the other hand, the study of accelerating black holes in four dimensions [4, 6], also discussed in chapter 8, gives some additional intuition about what happens in the UV but so far the scope of this type of investigations is only limited to anti-twist solutions and only in some cases. Further

investigations of these aspects are therefore a priority in this subject and a more field-theory oriented analysis could probably shed light on the most obscure aspects. Here we would like to collect a few observations related to these points that have so far emerged in the discussion of spindle solutions and that could constitute a starting point for future endeavors.

A first important point that we would like to clarify is the current understanding of the wrapped brane geometry that spindle solutions should originate from. In all cases the twist solutions have a natural physical interpretation as the IR limit of branes wrapped on $\mathbb{\Sigma}$ holomorphically embedded in a CY $(n+1)$-fold, where $n = 2, 3$ or 4, in the cases of M5-branes, D3-branes or M2-branes, respectively. The CY $(n+1)$-fold is the total space of a $\mathbb{C}^n$ bundle over $\mathbb{\Sigma}$, given by

$$\bigoplus_{i=1}^{n} \mathcal{O}(p_i) \to \mathbb{\Sigma}\,, \tag{9.1}$$

where $p_i \in \mathbb{Z}$ give the quantized magnetic charges of the n $U(1)$ gauge fields in the gauged supergravity constructions. The first Chern class of the space (9.1) is zero if and only if

$$\sum_{i=1}^{n} \frac{p_i}{n_1 n_2} + \chi = 0\,, \tag{9.2}$$

where $\chi = \chi(\mathbb{\Sigma})$ is the Euler number of the spindle, which is precisely the twist R-symmetry flux condition in (6.1). One starts with a Minkowski$\times$CY$_n$ solution to string theory or M-theory, appropriately, and wraps N D3-branes, M2-branes or M5-branes over the holomorphically embedded $\mathbb{\Sigma} \subset$ CY$_n$. The resulting low-energy theory on the Minkowski space part of the world-volume is supersymmetric, utilizing a topological twist[1]. The AdS twist solutions we have constructed in all cases are then naturally interpreted as the near-horizon limits of these wrapped brane configurations, which correspondingly flow to superconformal field theories in the IR. For the D3-brane solutions we have presented dual field theory calculations that corroborate this picture and analogous ones are possible for M5-branes. More puzzling is the corresponding interpretation of the anti-twist solutions. One can formally still write down a complex manifold (9.1) in this case, but it is *not* CY, as the first Chern class is not zero. However, deleting the zero section gives a complex cone that does have vanishing first Chern class; indeed this is a necessary condition for all GK geometries [215], which are relevant for the D3-brane AdS$_3 \times Y_7$ and M2-brane AdS$_2 \times Y_9$ solutions. Thus there is a CY cone

[1]The picture here should be somewhat analogous to the case of accelerating black holes, where the conformal boundary is topologically (anti-)twisted, while the IR supergravity solution presents a more complicated twist. One can imagine that in the UV supersymmetry is realized with a simple topological twist but somehow moving to the IR the twist becomes less trivial. This is of course speculative and it would be interesting to verify this picture in an explicit solution.

geometry, but there is no "blown-up" CY geometry with $\mathbb{\Sigma}$ a holomorphic curve for the anti-twist. The AdS anti-twist solutions strongly suggest an interpretation in terms of N D3-branes or N M2-branes wrapping $\mathbb{\Sigma}$, and the field theory calculation in section 7.2.5 assumes this in the D3-brane case and then correctly reproduces the supergravity central charge. But there is not a UV interpretation as N branes wrapping a holomorphic curve in a CY, as there is for the twist solutions. On the other hand, the anti-twist solutions have Killing spinors which are sections of non-trivial bundles and preserve supersymmetry differently than standard wrapped branes. It remains an interesting open problem to identify a UV geometry for this case.

Other interesting observations come from the study of Killing spinors and their vector bilinears, which we discuss in section 7.2.4 and can elucidate the associated supersymmetry algebra. At the level of the gauged supergravity, the rotational symmetry of the spindle enters the supersymmetry algebra as an R symmetry and this is in contrast to the case of the topological twist where it does not. After uplifting to $D = 10/11$ supergravity the internal space of the AdS solutions has an R-symmetry Killing vector which appears in the SCFT algebra. In the case of the topological twist, this is just an R-symmetry Killing vector of the space one uplifts upon, and dual to an R-symmetry of the brane world-volume theory. For the spindle solutions, it is instead a linear combination of an R-symmetry Killing vector of the space one uplifts upon with the Killing vector of the spindle, and from a dual point of view corresponds to the fact that in this case there is a mixing of the R symmetry in the brane world-volume theory. These features are typical of solutions describing rotating black holes with no R symmetry magnetic flux, where there is a different realization of supersymmetry that is reflected on the superalgebra satisfied by Killing spinors [570, 571] and the rotation of the black hole corresponds to turning on an Omega background [572] in the dual CFT[2]. It would be interesting to make this relationship more transparent. Finally, so far only the mixing of the spindle isometry with R- and mesonic symmetries has been considered, but in cases where the gauged supergravity solutions are uplifted on non-trivial Sasaki-Einstein manifolds one could also consider the mixing with $U(1)$ baryonic symmetries, as it was first pointed out in [573] and later studied more in detail in [491] for the case of branes wrapped on Riemann surfaces.

Finally, we would like to comment on the conformal boundary of the accelerating black holes discussed in chapter 8. This should provide the three-dimensional background where a $3d$ $\mathcal{N} = 2$ SCFT is placed, which then flows to a SCQM in the IR after the parent theory is compactified on spindle. A systematic way to place field theories on curved backgrounds preserving supersymmetry

[2]For a review of these developments, see [153].

was introduced in [455] and consists in coupling the theory to a suitable supergravity. The case of three-dimensional backgrounds was considered in [536], where a classification of the backgrounds allowed by supersymmetry was presented. As we discussed, for the non-rotating black holes the boundary is split into two halves with a topological twist on one and an anti-topological twist on the other[3], so in each half of the conformal boundary one has a standard situation. In the rotating case, on the other hand, one has a more standard configuration where, as noticed in [525], the boundary Killing vector obtained as a bilinear in the conformal Killing spinors is complex in Euclidean signature. This should be contrasted with the analysis of [536] (see in particular section 5.1), where such Killing vector is assumed to be real: the conformal boundary of rotating and accelerating black holes is therefore not automatically included in that classification.

9.2 Accelerating black holes and branes

Another interesting aspect of spindles is their relation to accelerating black holes in four dimensions. So far, this connection has been our only way to access the UV regime of the RG flow of SCFTs placed on spindles through the study of their conformal boundary, which we reviewed in chapter 8. Moreover, we conjecture that even beyond the four-dimensional case spindle solutions should arise as the near-horizon region of extremal black branes in AdS undergoing a uniform acceleration due to the presence of conical deficits in the geometry. However, no such solution is known besides the black hole ones and this certainly constitutes a challenge for future investigations in gauged supergravity. A better understanding of general accelerating black brane solutions would certainly help to clarify some aspects of the field theories dual to spindle solutions, starting from the possible twists. For example, it remains unclear at a fundamental level why one can have both twist and anti-twist solutions in four and five dimensions, while only the former are allowed in six and seven. Is there a relation with the geometry of the associated black branes?

Restricting to the case of accelerating black holes, which is the only one that we control so far, there are still several different puzzles that ought to be clarified. First of all, even in this case, explicit solutions are scarce. In the abelian truncation of $4d$ $\mathcal{N}=8$ $SO(8)$ gauged supergravity that we have considered in this thesis, the only known solutions are the PD black holes, whose relation to spindles was pointed out in [4] and the magnetically charged accelerating black holes with four independent charges discussed in [6]. However, in the latter case a regularity analysis was only

[3] This is reminiscent of the so-called topological-antitopological fusion [574] and it would be interesting to investigate potential connections. We thank Nikita Nekrasov for pointing out this analogy.

given in the truncation with pairwise equal charges, which leads to anti-twist solutions, so a first interesting step would be that of analyzing the conformal boundary of black holes realizing a twist on the associated spindles. Is the boundary singular as in the anti-twist case, and are the spinors smooth on the boundary? Moreover, one expects a generalization of PD black holes to exist in the abelian truncation of the $\mathcal{N}=8$ theory, to include mass, acceleration, rotation as well as four electric and four magnetic charges, generalizing the solutions of [575] with the addition of rotation and acceleration. Evidence for the existence of such more general accelerating solutions was presented in [6] for the $U(1)^2$ truncation of the theory, where their near-horizon geometry was found together with a guess for their on-shell action.

The thermodynamics of accelerating black holes has only recently attracted some attention and, interestingly, certain points are yet to be fully clarified. In particular, the definition of mass for such black holes has been debated for a few years, which have witnessed the co-existence of conflicting results [576–580]. A more systematic approach was considered in [563, 581] and extended in [525] where a detailed discussion of supersymmetry is also presented. Despite the recent progress, some aspects are still unclear: in particular, the notion of mass is associated to the existence of a time-like Killing vector whose normalization is, *a priori*, arbitrary. While it can be fixed by trial and error requiring the first law of thermodynamics and the Smarr relation to work, it would be certainly useful to develop a completely systematic understanding of how to compute the mass of accelerating black holes.

Another interesting aspect of accelerating black holes is the possibility or reproducing their entropy by studying the dual M2-branes field theories, using localization to compute a topologically twisted index and the technique of $\mathcal{I}$-extremization [142, 149, 150], which was recently generalized to electrically charged and rotating black holes [146, 582, 583]. For PD black holes, the renormalized on-shell action was computed in [525], from which one can extract an *entropy function* and, extremizing the latter, the black hole entropy. A guess for the entropy function of dyonic, rotating and accelerating black holes in the $U(1)^2$ theory was also proposed in [6]. From the field theory side, one should then focus on the Euclidean version of the conformal boundary geometry of the accelerating black holes and compute a certain twisted topological index associated with the $3d$ $\mathcal{N}=2$ SCFT living on the M2-branes wrapping the spindle $\mathbb{\Sigma}$. In this computation, some care would be required in the study of the boundary conditions for the fields at the conical deficits of the spindle and this might also help to clarify their role in field theory.

Finally, we would like to mention an approach that might help to shed light on some aspects

of accelerating black holes: primarily their mass and dual field theories. The idea is to consider theories of three-dimensional gauged supergravity that uplift to $D = 10/11$ supergravities [584, 585] and explore generalizations of the famous Bañados-Teitelboim-Zanelli (BTZ) black holes [586–588] to include acceleration, extending the results obtained in [589–591] for pure Einstein gravity with a cosmological constant. The simpler kinematics in three dimensions should make computations more straightforward and provide a fertile ground for the study of mass for such black holes. Even just considering pure Einstein gravity, one could try and understand the role of acceleration in field theory in light of the recent works [592–594] where the role of conical defects in AdS was understood as a way to cure certain negativities in the spectrum of $2d$ CFTs.

9.3 More general orbifolds

The study of black branes with spindle horizons naturally leads to the idea of considering more general solutions of gauged supergravity of the form AdS $\times M_n$, where M_n is allowed to have certain types of singularities and the dual field theory is not necessarily topologically twisted. In [440] the case of M_n an orbifold Riemann surface was considered, but only in the case of a constant curvature metric (which is not possible for a spindle) with a topological twist. A first question is whether more general twists are possible on this type of surfaces. Moreover, generalizing spindles, a class of solutions where M_n is a punctured disc (or, a "half-spindle") was introduced in [530, 531]. In this case the supergravity solution has a curvature singularity which persists when uplifted to $D = 11$ supergravity and is interpreted as a smeared M5-branes source and the increased supersymmetry compared to spindles allows for the identification of the dual $4d$ $\mathcal{N} = 2$ SCFTs. In that case, one interprets the orbifold point of the disc as a "regular puncture" and the boundary as an "irregular puncture", in the language of [595]. It would be interesting to have analogous field theory interpretations for the conical deficits of spindles. Moreover, reversing the logic one could picture spindles as obtained from "gluing" two punctured discs on their singular boundaries, which raises the question of whether the field theories dual to spindle solutions can be obtained from the process of "gluing" two field theories dual two discs along the irregular punctures, a procedure that, however, seems to only have been explored in the literature for the case of regular punctures.

More in general, it would be interesting to understand what type of singularities are allowed for M_n in gauged supergravity in such a way that the uplifted solutions to $D = 10/11$ supergravity are regular, or more in general still admit interesting dual CFTs. One could consider for M_n more general

orbifolds, not only in two but also in higher dimensions, possibly still representing the horizon of certain black holes or branes. A possibility that was explored in [596] for the first time is that of taking M_n to be the fibration of a spindle over a Riemann surface ($M_4 = \mathbb{\Sigma}_1 \ltimes \Sigma_{\mathsf{g}}$) or of a spindle over another spindle ($M_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$). Interestingly, solutions of this kind can be obtained from consistent truncations over spindles, which is another topic worth of future investigations. Such explorations could open a window on the study of entirely new classes of solutions in supergravity, as well as possibly shed some light on the meaning of specific types of singularities in field theory. Finally, it is known since [178] that the same local solutions that describe spindles in four- and five-dimensional gauged supergravity are related, by a different regularity analysis, to defects in CFTs, a fact that was recently revisited for seven-dimensional gauged supergravity in [597], where their relation to disc solutions has been briefly discussed. It would be interesting to make this connection more precise.

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