

# Lectures on special Lagrangian geometry

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## 1 Introduction

*Calabi–Yau  $m$ -folds*  $(M, J, \omega, \Omega)$  are compact complex manifolds  $(M, J)$  of complex dimension  $m$ , equipped with a Ricci-flat Kähler metric  $g$  with Kähler form  $\omega$ , and a holomorphic  $(m, 0)$ -form  $\Omega$  of constant length  $|\Omega|^2 = 2^m$ . Using Algebraic Geometry and Yau’s solution of the Calabi Conjecture, one can construct them in huge numbers. String Theorists (a species of theoretical physicist) are very interested in Calabi–Yau 3-folds, and have made some extraordinary conjectures about them, in the subject known as *Mirror Symmetry*.

*Special Lagrangian submanifolds*, or *SL  $m$ -folds*, are a distinguished class of real  $m$ -dimensional minimal submanifolds that may be defined in  $\mathbb{C}^m$ , or in Calabi–Yau  $m$ -folds, or more generally in *almost Calabi–Yau  $m$ -folds*. They are calibrated with respect to the  $m$ -form  $\text{Re}\Omega$ . They are fairly rigid and well-behaved, so that compact SL  $m$ -folds  $N$  occur in smooth moduli spaces of dimension  $b^1(N)$ , for instance. They are important in String Theory, and are expected to play a rôle in the eventual explanation of Mirror Symmetry.

This article is intended as an introduction to special Lagrangian geometry, and a survey of the author’s research on the singularities of SL  $m$ -folds, of directions in which the subject might develop in the next few years, and of possible applications of it to Mirror Symmetry and the SYZ Conjecture.

Sections 2 and 3 discuss general properties of special Lagrangian submanifolds of  $\mathbb{C}^m$ , and ways to construct examples. Then Section 4 defines Calabi–Yau and almost Calabi–Yau manifolds, and their special Lagrangian submanifolds. Section 5 discusses the deformation and obstruction theory of *compact* SL  $m$ -folds, and properties of their moduli spaces.

In Section 6 we describe a theory of *isolated conical singularities* in compact SL  $m$ -folds. Finally, Section 7 briefly introduces String Theory, Mirror Symmetry and the *SYZ Conjecture*, a conjectural explanation of Mirror Symmetry of Calabi–Yau 3-folds, and discusses mathematical progress towards clarifying and proving the conjecture.

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## 2 Special Lagrangian submanifolds in $\mathbb{C}^m$

We begin by defining *calibrations* and *calibrated submanifolds*, following Harvey and Lawson [11].

**Definition 2.1** Let  $(M, g)$  be a Riemannian manifold. An *oriented tangent  $k$ -plane*  $V$  on  $M$  is a vector subspace  $V$  of some tangent space  $T_x M$  to  $M$  with  $\dim V = k$ , equipped with an orientation. If  $V$  is an oriented tangent  $k$ -plane on  $M$  then  $g|_V$  is a Euclidean metric on  $V$ , so combining  $g|_V$  with the orientation on  $V$  gives a natural *volume form*  $\text{vol}_V$  on  $V$ , which is a  $k$ -form on  $V$ .

Now let  $\varphi$  be a closed  $k$ -form on  $M$ . We say that  $\varphi$  is a *calibration* on  $M$  if for every oriented  $k$ -plane  $V$  on  $M$  we have  $\varphi|_V \leq \text{vol}_V$ . Here  $\varphi|_V = \alpha \cdot \text{vol}_V$  for some  $\alpha \in \mathbb{R}$ , and  $\varphi|_V \leq \text{vol}_V$  if  $\alpha \leq 1$ . Let  $N$  be an oriented submanifold of  $M$  with dimension  $k$ . Then each tangent space  $T_x N$  for  $x \in N$  is an oriented tangent  $k$ -plane. We say that  $N$  is a *calibrated submanifold* if  $\varphi|_{T_x N} = \text{vol}_{T_x N}$  for all  $x \in N$ .

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [11, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in  $\mathbb{C}^m$ , taken from [11, §III].

**Definition 2.2** Let  $\mathbb{C}^m \cong \mathbb{R}^{2m}$  have complex coordinates  $(z_1, \dots, z_m)$  and complex structure  $I$ , and define a metric  $g'$ , Kähler form  $\omega'$  and complex volume form  $\Omega'$  on  $\mathbb{C}^m$  by

$$g' = |dz_1|^2 + \dots + |dz_m|^2, \quad \omega' = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \dots + dz_m \wedge d\bar{z}_m), \quad (1)$$

and  $\Omega' = dz_1 \wedge \dots \wedge dz_m$ .

Then  $\text{Re } \Omega'$  and  $\text{Im } \Omega'$  are real  $m$ -forms on  $\mathbb{C}^m$ . Let  $L$  be an oriented real submanifold of  $\mathbb{C}^m$  of real dimension  $m$ . We call  $L$  a *special Lagrangian submanifold* in  $\mathbb{C}^m$ , or *SL  $m$ -fold* for short, if  $L$  is calibrated with respect to  $\text{Re } \Omega'$ , in the sense of Definition 2.1.

In fact there is a more general definition involving a *phase*  $e^{i\theta}$ : if  $\theta \in [0, 2\pi)$ , we say that  $L$  is *special Lagrangian with phase*  $e^{i\theta}$  if it is calibrated with respect to  $\cos \theta \text{Re } \Omega' + \sin \theta \text{Im } \Omega'$ . But we will not use this.

We shall identify the family  $\mathcal{F}$  of tangent  $m$ -planes in  $\mathbb{C}^m$  calibrated with respect to  $\text{Re } \Omega'$ . The subgroup of  $\text{GL}(2m, \mathbb{R})$  preserving  $g', \omega'$  and  $\Omega'$  is the Lie

group  $SU(m)$  of complex unitary matrices with determinant 1. Define a real vector subspace  $U$  in  $\mathbb{C}^m$  to be

$$U = \{(x_1, \dots, x_m) : x_j \in \mathbb{R}\} \subset \mathbb{C}^m, \quad (2)$$

and let  $U$  have the usual orientation. Then  $U$  is calibrated with respect to  $\operatorname{Re} \Omega'$ .

Furthermore, any oriented real vector subspace  $V$  in  $\mathbb{C}^m$  calibrated with respect to  $\operatorname{Re} \Omega'$  is of the form  $V = \gamma \cdot U$  for some  $\gamma \in SU(m)$ . Therefore  $SU(m)$  acts transitively on  $\mathcal{F}$ . The stabilizer subgroup of  $U$  in  $SU(m)$  is the subset of matrices in  $SU(m)$  with real entries, which is  $SO(m)$ . Thus  $\mathcal{F} \cong SU(m)/SO(m)$ , and we prove:

**Proposition 2.3** *The family  $\mathcal{F}$  of oriented real  $m$ -dimensional vector subspaces  $V$  in  $\mathbb{C}^m$  with  $\operatorname{Re} \Omega'|_V = \operatorname{vol}_V$  is isomorphic to  $SU(m)/SO(m)$ , and has dimension  $\frac{1}{2}(m^2 + m - 2)$ .*

The dimension follows because  $\dim SU(m) = m^2 - 1$  and  $\dim SO(m) = \frac{1}{2}m(m - 1)$ . It is easy to see that  $\omega'|_U = \operatorname{Im} \Omega'|_U = 0$ . As  $SU(m)$  preserves  $\omega'$  and  $\operatorname{Im} \Omega'$  and acts transitively on  $\mathcal{F}$ , it follows that  $\omega'|_V = \operatorname{Im} \Omega'|_V = 0$  for any  $V \in \mathcal{F}$ . Conversely, if  $V$  is a real  $m$ -dimensional vector subspace of  $\mathbb{C}^m$  and  $\omega'|_V = \operatorname{Im} \Omega'|_V = 0$ , then  $V$  lies in  $\mathcal{F}$ , with some orientation. This implies an alternative characterization of special Lagrangian submanifolds, [11, Cor. III.1.11]:

**Proposition 2.4** *Let  $L$  be a real  $m$ -dimensional submanifold of  $\mathbb{C}^m$ . Then  $L$  admits an orientation making it into a special Lagrangian submanifold of  $\mathbb{C}^m$  if and only if  $\omega'|_L \equiv 0$  and  $\operatorname{Im} \Omega'|_L \equiv 0$ .*

Note that an  $m$ -dimensional submanifold  $L$  in  $\mathbb{C}^m$  is called *Lagrangian* if  $\omega'|_L \equiv 0$ . (This is a term from symplectic geometry, and  $\omega'$  is a symplectic structure.) Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that  $\operatorname{Im} \Omega'|_L \equiv 0$ , which is how they get their name.

## 2.1 Special Lagrangian 2-folds in $\mathbb{C}^2$ and the quaternions

The smallest interesting dimension,  $m = 2$ , is a special case. Let  $\mathbb{C}^2$  have complex coordinates  $(z_1, z_2)$ , complex structure  $I$ , and metric  $g'$ , Kähler form  $\omega'$  and holomorphic 2-form  $\Omega'$  defined in (1). Define real coordinates  $(x_0, x_1, x_2, x_3)$  on  $\mathbb{C}^2 \cong \mathbb{R}^4$  by  $z_0 = x_0 + ix_1$ ,  $z_1 = x_2 + ix_3$ . Then

$$\begin{aligned} g' &= dx_0^2 + \dots + dx_3^2, & \omega' &= dx_0 \wedge dx_1 + dx_2 \wedge dx_3, \\ \operatorname{Re} \Omega' &= dx_0 \wedge dx_2 - dx_1 \wedge dx_3 & \text{and} & \quad \operatorname{Im} \Omega' = dx_0 \wedge dx_3 + dx_1 \wedge dx_2. \end{aligned}$$

Now define a *different* set of complex coordinates  $(w_1, w_2)$  on  $\mathbb{C}^2 = \mathbb{R}^4$  by  $w_1 = x_0 + ix_2$  and  $w_2 = x_1 - ix_3$ . Then  $\omega' - i \operatorname{Im} \Omega' = dw_1 \wedge dw_2$ .

But by Proposition 2.4, a real 2-submanifold  $L \subset \mathbb{R}^4$  is special Lagrangian if and only if  $\omega'|_L \equiv \operatorname{Im} \Omega'|_L \equiv 0$ . Thus,  $L$  is special Lagrangian if and only if

$(dw_1 \wedge dw_2)|_L \equiv 0$ . But this holds if and only if  $L$  is a *holomorphic curve* with respect to the complex coordinates  $(w_1, w_2)$ .

Here is another way to say this. There are *two different* complex structures  $I$  and  $J$  involved in this problem, associated to the two different complex coordinate systems  $(z_1, z_2)$  and  $(w_1, w_2)$  on  $\mathbb{R}^4$ . In the coordinates  $(x_0, \dots, x_3)$ ,  $I$  and  $J$  are given by

$$\begin{aligned} I\left(\frac{\partial}{\partial x_0}\right) &= \frac{\partial}{\partial x_1}, & I\left(\frac{\partial}{\partial x_1}\right) &= -\frac{\partial}{\partial x_0}, & I\left(\frac{\partial}{\partial x_2}\right) &= \frac{\partial}{\partial x_3}, & I\left(\frac{\partial}{\partial x_3}\right) &= -\frac{\partial}{\partial x_2}, \\ J\left(\frac{\partial}{\partial x_0}\right) &= \frac{\partial}{\partial x_2}, & J\left(\frac{\partial}{\partial x_1}\right) &= -\frac{\partial}{\partial x_3}, & J\left(\frac{\partial}{\partial x_2}\right) &= -\frac{\partial}{\partial x_0}, & J\left(\frac{\partial}{\partial x_3}\right) &= \frac{\partial}{\partial x_1}. \end{aligned}$$

The usual complex structure on  $\mathbb{C}^2$  is  $I$ , but a 2-fold  $L$  in  $\mathbb{C}^2$  is special Lagrangian if and only if it is holomorphic with respect to the alternative complex structure  $J$ . This means that special Lagrangian 2-folds are already very well understood, so we generally focus our attention on dimensions  $m \geq 3$ .

We can express all this in terms of the *quaternions*  $\mathbb{H}$ . The complex structures  $I, J$  anticommute, so that  $IJ = -JI$ , and  $K = IJ$  is also a complex structure on  $\mathbb{R}^4$ , and  $\langle 1, I, J, K \rangle$  is an algebra of automorphisms of  $\mathbb{R}^4$  isomorphic to  $\mathbb{H}$ .

## 2.2 Special Lagrangian submanifolds in $\mathbb{C}^m$ as graphs

In symplectic geometry, there is a well-known way of manufacturing *Lagrangian* submanifolds of  $\mathbb{R}^{2m} \cong \mathbb{C}^m$ , which works as follows. Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  be a smooth function, and define

$$\Gamma_f = \left\{ \left( x_1 + i \frac{\partial f}{\partial x_1}(x_1, \dots, x_m), \dots, x_m + i \frac{\partial f}{\partial x_m}(x_1, \dots, x_m) \right) : x_1, \dots, x_m \in \mathbb{R} \right\}.$$

Then  $\Gamma_f$  is a smooth real  $m$ -dimensional submanifold of  $\mathbb{C}^m$ , with  $\omega'|_{\Gamma_f} \equiv 0$ . Identifying  $\mathbb{C}^m \cong \mathbb{R}^{2m} \cong \mathbb{R}^m \times (\mathbb{R}^m)^*$ , we may regard  $\Gamma_f$  as the graph of the 1-form  $df$  on  $\mathbb{R}^m$ , so that  $\Gamma_f$  is the *graph of a closed 1-form*. Locally, but not globally, every Lagrangian submanifold arises from this construction.

Now by Proposition 2.4, a special Lagrangian  $m$ -fold in  $\mathbb{C}^m$  is a Lagrangian  $m$ -fold  $L$  satisfying the additional condition that  $\text{Im } \Omega'|_L \equiv 0$ . We shall find the condition for  $\Gamma_f$  to be a special Lagrangian  $m$ -fold. Define the *Hessian*  $\text{Hess } f$  of  $f$  to be the  $m \times m$  matrix  $\left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^m$  of real functions on  $\mathbb{R}^m$ . Then it is easy to show that  $\text{Im } \Omega'|_{\Gamma_f} \equiv 0$  if and only if

$$\text{Im } \det_c(I + i \text{Hess } f) \equiv 0 \quad \text{on } \mathbb{C}^m. \quad (3)$$

This is a *nonlinear second-order elliptic partial differential equation* upon the function  $f : \mathbb{R}^m \rightarrow \mathbb{R}$ .

## 2.3 Local discussion of special Lagrangian deformations

Suppose  $L_0$  is a special Lagrangian submanifold in  $\mathbb{C}^m$  (or, more generally, in some (almost) Calabi–Yau  $m$ -fold). What can we say about the family of *special*

*Lagrangian deformations* of  $L_0$ , that is, the set of special Lagrangian  $m$ -folds  $L$  that are “close to  $L_0$ ” in a suitable sense? Essentially, deformation theory is one way of thinking about the question “how many special Lagrangian submanifolds are there in  $\mathbb{C}^m$ ”?

Locally (that is, in small enough open sets), every special Lagrangian  $m$ -fold looks quite like  $\mathbb{R}^m$  in  $\mathbb{C}^m$ . Therefore deformations of special Lagrangian  $m$ -folds should look like special Lagrangian deformations of  $\mathbb{R}^m$  in  $\mathbb{C}^m$ . So, we would like to know what special Lagrangian  $m$ -folds  $L$  in  $\mathbb{C}^m$  close to  $\mathbb{R}^m$  look like.

Now  $\mathbb{R}^m$  is the graph  $\Gamma_f$  associated to the function  $f \equiv 0$ . Thus, a graph  $\Gamma_f$  will be close to  $\mathbb{R}^m$  if the function  $f$  and its derivatives are small. But then  $\text{Hess } f$  is small, so we can approximate equation (3) by its *linearization*. For

$$\text{Im det}_c(I + i \text{Hess } f) = \text{Tr Hess } f + \text{higher order terms.}$$

Thus, when the second derivatives of  $f$  are small, equation (3) reduces approximately to  $\text{Tr Hess } f \equiv 0$ . But

$$\text{Tr Hess } f = \frac{\partial^2 f}{(\partial x_1)^2} + \cdots + \frac{\partial^2 f}{(\partial x_m)^2} = -\Delta f,$$

where  $\Delta$  is the *Laplacian* on  $\mathbb{R}^m$ .

Hence, the small special Lagrangian deformations of  $\mathbb{R}^m$  in  $\mathbb{C}^m$  are approximately parametrized by small *harmonic functions* on  $\mathbb{R}^m$ . Actually, because adding a constant to  $f$  has no effect on  $\Gamma_f$ , this parametrization is degenerate. We can get round this by parametrizing instead by  $df$ , which is a closed and coclosed 1-form. This justifies the following:

**Principle.** *Small special Lagrangian deformations of a special Lagrangian  $m$ -fold  $L$  are approximately parametrized by closed and coclosed 1-forms  $\alpha$  on  $L$ .*

This is the idea behind McLean’s Theorem, Theorem 5.1 below.

We have seen using (3) that the deformation problem for special Lagrangian  $m$ -folds can be written as an *elliptic equation*. In particular, there are the same number of equations as functions, so the problem is neither overdetermined nor underdetermined. Therefore we do not expect special Lagrangian  $m$ -folds to be very few and very rigid (as would be the case if (3) were overdetermined), nor to be very abundant and very flabby (as would be the case if (3) were underdetermined).

If we think about Proposition 2.3 for a while, this may seem surprising. For the set  $\mathcal{F}$  of special Lagrangian  $m$ -planes in  $\mathbb{C}^m$  has dimension  $\frac{1}{2}(m^2 + m - 2)$ , but the set of all real  $m$ -planes in  $\mathbb{C}^m$  has dimension  $m^2$ . So the special Lagrangian  $m$ -planes have codimension  $\frac{1}{2}(m^2 - m + 2)$  in the set of all  $m$ -planes.

This means that the condition for a real  $m$ -submanifold  $L$  in  $\mathbb{C}^m$  to be special Lagrangian is  $\frac{1}{2}(m^2 - m + 2)$  real equations on each tangent space of  $L$ . However, the freedom to vary  $L$  is the sections of its normal bundle in  $\mathbb{C}^m$ , which is  $m$

real functions. When  $m \geq 3$ , there are more equations than functions, so we would expect the deformation problem to be *overdetermined*.

The explanation is that because  $\omega'$  is a *closed* 2-form, submanifolds  $L$  with  $\omega'|_L \equiv 0$  are much more abundant than would otherwise be the case. So the closure of  $\omega'$  is a kind of integrability condition necessary for the existence of many special Lagrangian submanifolds, just as the integrability of an almost complex structure is a necessary condition for the existence of many complex submanifolds of dimension greater than 1 in a complex manifold.

### 3 Constructions of SL $m$ -folds in $\mathbb{C}^m$

We now describe five methods of constructing special Lagrangian  $m$ -folds in  $\mathbb{C}^m$ , drawn from papers by the author [15, 16, 18, 19, 20, 21, 22, 23], Bryant [1], Castro and Urbano [2], Goldstein [4, 5], Harvey [10, p. 139–143], Harvey and Lawson [11, §III], Haskins [12], Lawlor [32], Ma and Ma [33], McIntosh [35] and Sharipov [40]. These yield many examples of singular SL  $m$ -folds, and so hopefully will help in understanding general singularities of SL  $m$ -folds in Calabi–Yau  $m$ -folds.

#### 3.1 SL $m$ -folds with large symmetry groups

Here is a method used in [18] (and also by Harvey and Lawson [11, §III.3], Haskins [12] and Goldstein [4, 5]) to construct examples of SL  $m$ -folds in  $\mathbb{C}^m$ . The group  $SU(m) \times \mathbb{C}^m$  acts on  $\mathbb{C}^m$  preserving all the structure  $g', \omega', \Omega'$ , so that it takes SL  $m$ -folds to SL  $m$ -folds in  $\mathbb{C}^m$ . Let  $G$  be a Lie subgroup of  $SU(m) \times \mathbb{C}^m$  with Lie algebra  $\mathfrak{g}$ , and  $N$  a connected  $G$ -invariant SL  $m$ -fold in  $\mathbb{C}^m$ .

Since  $G$  preserves the symplectic form  $\omega'$  on  $\mathbb{C}^m$ , one can show that it has a *moment map*  $\mu : \mathbb{C}^m \rightarrow \mathfrak{g}^*$ . As  $N$  is Lagrangian, one can show that  $\mu$  is constant on  $N$ , that is,  $\mu \equiv c$  on  $N$  for some  $c \in Z(\mathfrak{g}^*)$ , the *center* of  $\mathfrak{g}^*$ .

If the orbits of  $G$  in  $N$  are of codimension 1 (that is, dimension  $m - 1$ ), then  $N$  is a 1-parameter family of  $G$ -orbits  $\mathcal{O}_t$  for  $t \in \mathbb{R}$ . After reparametrizing the variable  $t$ , it can be shown that the special Lagrangian condition is equivalent to an ODE in  $t$  upon the orbits  $\mathcal{O}_t$ .

Thus, we can construct examples of cohomogeneity one SL  $m$ -folds in  $\mathbb{C}^m$  by solving an ODE in the family of  $(m - 1)$ -dimensional  $G$ -orbits  $\mathcal{O}$  in  $\mathbb{C}^m$  with  $\mu|_{\mathcal{O}} \equiv c$ , for fixed  $c \in Z(\mathfrak{g}^*)$ . This ODE usually turns out to be *integrable*.

Now suppose  $N$  is a *special Lagrangian cone* in  $\mathbb{C}^m$ , invariant under a subgroup  $G \subset SU(m)$  which has orbits of dimension  $m - 2$  in  $N$ . In effect the symmetry group of  $N$  is  $G \times \mathbb{R}_+$ , where  $\mathbb{R}_+$  acts by *dilations*, as  $N$  is a cone. Thus, in this situation too the symmetry group of  $N$  acts with cohomogeneity one, and we again expect the problem to reduce to an ODE.

One can show that  $N \cap \mathcal{S}^{2m-1}$  is a 1-parameter family of  $G$ -orbits  $\mathcal{O}_t$  in  $\mathcal{S}^{2m-1} \cap \mu^{-1}(0)$  satisfying an ODE. By solving this ODE we construct SL cones in  $\mathbb{C}^m$ . When  $G = U(1)^{m-2}$ , the ODE has many *periodic solutions* which give

large families of distinct SL cones on  $T^{m-1}$ . In particular, we can find many examples of SL  $T^2$ -cones in  $\mathbb{C}^3$ .

### 3.2 Evolution equations for SL $m$ -folds

The following method was used in [15] and [16] to construct many examples of SL  $m$ -folds in  $\mathbb{C}^m$ . A related but less general method was used by Lawlor [32], and completed by Harvey [10, p. 139–143].

Let  $P$  be a real analytic  $(m-1)$ -dimensional manifold, and  $\chi$  a nonvanishing real analytic section of  $\Lambda^{m-1}TP$ . Let  $\{\phi_t : t \in \mathbb{R}\}$  be a 1-parameter family of real analytic maps  $\phi_t : P \rightarrow \mathbb{C}^m$ . Consider the ODE

$$\left(\frac{d\phi_t}{dt}\right)^b = (\phi_t)_*(\chi)^{a_1 \dots a_{m-1}} (\operatorname{Re} \Omega')_{a_1 \dots a_{m-1} a_m} g'^{a_m b}, \quad (4)$$

using the index notation for (real) tensors on  $\mathbb{C}^m$ , where  $g'^{ab}$  is the inverse of the Euclidean metric  $g'_{ab}$  on  $\mathbb{C}^m$ .

It is shown in [15, §3] that if the  $\phi_t$  satisfy (4) and  $\phi_0^*(\omega') \equiv 0$ , then  $\phi_t^*(\omega') \equiv 0$  for all  $t$ , and  $N = \{\phi_t(p) : p \in P, t \in \mathbb{R}\}$  is an SL  $m$ -fold in  $\mathbb{C}^m$  wherever it is nonsingular. We think of (4) as an *evolution equation*, and  $N$  as the result of evolving a 1-parameter family of  $(m-1)$ -submanifolds  $\phi_t(P)$  in  $\mathbb{C}^m$ .

Here is one way to understand this result. Suppose we are given  $\phi_t : P \rightarrow \mathbb{C}^m$  for some  $t$ , and we want to find an SL  $m$ -fold  $N$  in  $\mathbb{C}^m$  containing the  $(m-1)$ -submanifold  $\phi_t(P)$ . As  $N$  is Lagrangian, a necessary condition for this is that  $\omega'|_{\phi_t(P)} \equiv 0$ , and hence  $\phi_t^*(\omega') \equiv 0$  on  $P$ .

The effect of equation (4) is to flow  $\phi_t(P)$  in the direction in which  $\operatorname{Re} \Omega'$  is “largest”. The result is that  $\operatorname{Re} \Omega'$  is “maximized” on  $N$ , given the initial conditions. But  $\operatorname{Re} \Omega'$  is maximal on  $N$  exactly when  $N$  is calibrated with respect to  $\operatorname{Re} \Omega'$ , that is, when  $N$  is special Lagrangian. The same technique also works for other calibrations, such as the associative and coassociative calibrations on  $\mathbb{R}^7$ , and the Cayley calibration on  $\mathbb{R}^8$ .

Now (4) evolves amongst the infinite-dimensional family of real analytic maps  $\phi : P \rightarrow \mathbb{C}^m$  with  $\phi^*(\omega') \equiv 0$ , so it is an *infinite-dimensional* problem, and thus difficult to solve explicitly. However, there are *finite-dimensional* families  $\mathcal{C}$  of maps  $\phi : P \rightarrow \mathbb{C}^m$  such that evolution stays in  $\mathcal{C}$ . This gives a *finite-dimensional* ODE, which can hopefully be solved fairly explicitly. For example, if we take  $G$  to be a Lie subgroup of  $\operatorname{SU}(m) \times \mathbb{C}^m$ ,  $P$  to be an  $(m-1)$ -dimensional homogeneous space  $G/H$ , and  $\phi : P \rightarrow \mathbb{C}^m$  to be  $G$ -equivariant, we recover the construction of Section 3.1.

But there are also other possibilities for  $\mathcal{C}$  which do not involve a symmetry assumption. Suppose  $P$  is a submanifold of  $\mathbb{R}^n$ , and  $\chi$  the restriction to  $P$  of a linear or affine map  $\mathbb{R}^n \rightarrow \Lambda^{m-1}\mathbb{R}^n$ . (This is a strong condition on  $P$  and  $\chi$ .) Then we can take  $\mathcal{C}$  to be the set of restrictions to  $P$  of linear or affine maps  $\mathbb{R}^n \rightarrow \mathbb{C}^m$ .

For instance, set  $m = n$  and let  $P$  be a quadric in  $\mathbb{R}^m$ . Then one can construct SL  $m$ -folds in  $\mathbb{C}^m$  with few symmetries by evolving quadrics in La-

grangian planes  $\mathbb{R}^m$  in  $\mathbb{C}^m$ . When  $P$  is a quadric cone in  $\mathbb{R}^m$  this gives many SL cones on products of spheres  $\mathcal{S}^a \times \mathcal{S}^b \times \mathcal{S}^1$ .

### 3.3 Ruled special Lagrangian 3-folds

A 3-submanifold  $N$  in  $\mathbb{C}^3$  is called *ruled* if it is fibered by a 2-dimensional family  $\mathcal{F}$  of real lines in  $\mathbb{C}^3$ . A cone  $N_0$  in  $\mathbb{C}^3$  is called *two-sided* if  $N_0 = -N_0$ . Two-sided cones are automatically ruled. If  $N$  is a ruled 3-fold in  $\mathbb{C}^3$ , we define the *asymptotic cone*  $N_0$  of  $N$  to be the two-sided cone fibered by the lines passing through 0 and parallel to those in  $\mathcal{F}$ .

Ruled SL 3-folds are studied in [19], and also by Harvey and Lawson [11, §III.3.C, §III.4.B] and Bryant [1, §3]. Each (oriented) real line in  $\mathbb{C}^3$  is determined by its *direction* in  $\mathcal{S}^5$  together with an orthogonal *translation* from the origin. Thus a ruled 3-fold  $N$  is determined by a 2-dimensional family of directions and translations.

The condition for  $N$  to be special Lagrangian turns out [19, §5] to reduce to two equations, the first involving only the direction components, and the second *linear* in the translation components. Hence, if a ruled 3-fold  $N$  in  $\mathbb{C}^3$  is special Lagrangian, then so is its asymptotic cone  $N_0$ . Conversely, the ruled SL 3-folds  $N$  asymptotic to a given two-sided SL cone  $N_0$  come from solutions of a linear equation, and so form a *vector space*.

Let  $N_0$  be a two-sided SL cone, and let  $\Sigma = N_0 \cap \mathcal{S}^5$ . Then  $\Sigma$  is a *Riemann surface*. Holomorphic vector fields on  $\Sigma$  give solutions to the linear equation (though not all solutions) [19, §6], and so yield new ruled SL 3-folds. In particular, each SL  $T^2$ -cone gives a 2-dimensional family of ruled SL 3-folds, which are generically diffeomorphic to  $T^2 \times \mathbb{R}$  as immersed 3-submanifolds.

### 3.4 Integrable systems

Let  $N_0$  be a special Lagrangian cone in  $\mathbb{C}^3$ , and set  $\Sigma = N_0 \cap \mathcal{S}^5$ . As  $N_0$  is calibrated, it is minimal in  $\mathbb{C}^3$ , and so  $\Sigma$  is minimal in  $\mathcal{S}^5$ . That is,  $\Sigma$  is a *minimal Legendrian surface* in  $\mathcal{S}^5$ . Let  $\pi : \mathcal{S}^5 \rightarrow \mathbb{C}\mathbb{P}^2$  be the Hopf projection. One can also show that  $\pi(\Sigma)$  is a *minimal Lagrangian surface* in  $\mathbb{C}\mathbb{P}^2$ .

Regard  $\Sigma$  as a *Riemann surface*. Then the inclusions  $\iota : \Sigma \rightarrow \mathcal{S}^5$  and  $\pi \circ \iota : \Sigma \rightarrow \mathbb{C}\mathbb{P}^2$  are *conformal harmonic maps*. Now harmonic maps from Riemann surfaces into  $\mathcal{S}^n$  and  $\mathbb{C}\mathbb{P}^m$  are an *integrable system*. There is a complicated theory for classifying them in terms of algebro-geometric “spectral data”, and finding “explicit” solutions. In principle, this gives all harmonic maps from  $T^2$  into  $\mathcal{S}^n$  and  $\mathbb{C}\mathbb{P}^m$ . So, the field of integrable systems offers the hope of a *classification* of all SL  $T^2$ -cones in  $\mathbb{C}^3$ .

For a good general introduction to this field, see Fordy and Wood [3]. Sharipov [40] and Ma and Ma [33] apply this integrable systems machinery to describe minimal Legendrian tori in  $\mathcal{S}^5$ , and minimal Lagrangian tori in  $\mathbb{C}\mathbb{P}^2$ , respectively, giving explicit formulae in terms of Prym theta functions. McIntosh [35] provides a more recent, readable, and complete discussion of special Lagrangian cones in  $\mathbb{C}^3$  from the integrable systems perspective.

The families of SL  $T^2$ -cones constructed by U(1)-invariance in Section 3.1, and by evolving quadrics in Section 3.2, turn out to come from a more general, very explicit, “integrable systems” family of conformal harmonic maps  $\mathbb{R}^2 \rightarrow \mathcal{S}^5$  with Legendrian image, involving two commuting, integrable ODEs, described in [20]. So, we can fit some of our examples into the integrable systems framework.

However, we know a good number of other constructions of SL  $m$ -folds in  $\mathbb{C}^m$  which have the classic hallmarks of integrable systems — elliptic functions, commuting ODEs, and so on — but which are not yet understood from the point of view of integrable systems.

### 3.5 Analysis and U(1)-invariant SL 3-folds in $\mathbb{C}^3$

Next we summarize the author’s three papers [21, 22, 23], which study SL 3-folds  $N$  in  $\mathbb{C}^3$  invariant under the U(1)-action

$$e^{i\theta} : (z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2, z_3) \quad \text{for } e^{i\theta} \in \text{U}(1). \quad (5)$$

These three papers are surveyed in [24]. Locally we can write  $N$  in the form

$$N = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v(x, y) + iy, \quad z_3 = x + iu(x, y), \right. \\ \left. |z_1|^2 - |z_2|^2 = 2a, \quad (x, y) \in S \right\}, \quad (6)$$

where  $S$  is a domain in  $\mathbb{R}^2$ ,  $a \in \mathbb{R}$  and  $u, v : S \rightarrow \mathbb{R}$  are continuous.

Here we may take  $|z_1|^2 - |z_2|^2 = 2a$  to be one of the equations defining  $N$  as  $|z_1|^2 - |z_2|^2$  is the *moment map* of the U(1)-action (5), and so  $|z_1|^2 - |z_2|^2$  is constant on any U(1)-invariant Lagrangian 3-fold in  $\mathbb{C}^3$ . Effectively (6) just means that we are choosing  $x = \text{Re}(z_3)$  and  $y = \text{Im}(z_1 z_2)$  as local coordinates on the 2-manifold  $N/\text{U}(1)$ . Then we find [21, Prop. 4.1]:

**Proposition 3.1** *Let  $S, a, u, v$  and  $N$  be as above. Then*

- (a) *If  $a = 0$ , then  $N$  is a (possibly singular) SL 3-fold in  $\mathbb{C}^3$  if  $u, v$  are differentiable and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2(v^2 + y^2)^{1/2} \frac{\partial u}{\partial y}, \quad (7)$$

*except at points  $(x, 0)$  in  $S$  with  $v(x, 0) = 0$ , where  $u, v$  need not be differentiable. The singular points of  $N$  are those of the form  $(0, 0, z_3)$ , where  $z_3 = x + iu(x, 0)$  for  $(x, 0) \in S$  with  $v(x, 0) = 0$ .*

- (b) *If  $a \neq 0$ , then  $N$  is a nonsingular SL 3-fold in  $\mathbb{C}^3$  if and only if  $u, v$  are differentiable in  $S$  and satisfy*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2(v^2 + y^2 + a^2)^{1/2} \frac{\partial u}{\partial y}. \quad (8)$$

Now (7) and (8) are *nonlinear Cauchy–Riemann equations*. Thus, we may treat  $u + iv$  as like a holomorphic function of  $x + iy$ . Many of the results in [21, 22, 23] are analogues of well-known results in elementary complex analysis.

In [21, Prop. 7.1] we show that solutions  $u, v \in C^1(S)$  of (8) come from a potential  $f \in C^2(S)$  satisfying a second-order quasilinear elliptic equation.

**Proposition 3.2** *Let  $S$  be a domain in  $\mathbb{R}^2$  and  $u, v \in C^1(S)$  satisfy (8) for  $a \neq 0$ . Then there exists  $f \in C^2(S)$  with  $\frac{\partial f}{\partial y} = u$ ,  $\frac{\partial f}{\partial x} = v$  and*

$$P(f) = \left( \left( \frac{\partial f}{\partial x} \right)^2 + y^2 + a^2 \right)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial y^2} = 0. \quad (9)$$

*This  $f$  is unique up to addition of a constant,  $f \mapsto f + c$ . Conversely, all solutions of (9) yield solutions of (8).*

In the following result, a condensation of [21, Th. 7.6] and [22, Th.s 9.20 & 9.21], we prove existence and uniqueness for the *Dirichlet problem* for (9).

**Theorem 3.3** *Suppose  $S$  is a strictly convex domain in  $\mathbb{R}^2$  invariant under  $(x, y) \mapsto (x, -y)$ , and  $\alpha \in (0, 1)$ . Let  $a \in \mathbb{R}$  and  $\phi \in C^{3,\alpha}(\partial S)$ . Then if  $a \neq 0$  there exists a unique solution  $f$  of (9) in  $C^{3,\alpha}(S)$  with  $f|_{\partial S} = \phi$ . If  $a = 0$  there exists a unique  $f \in C^1(S)$  with  $f|_{\partial S} = \phi$ , which is twice weakly differentiable and satisfies (9) with weak derivatives. Furthermore, the map  $C^{3,\alpha}(\partial S) \times \mathbb{R} \rightarrow C^1(S)$  taking  $(\phi, a) \mapsto f$  is continuous.*

Here a domain  $S$  in  $\mathbb{R}^2$  is *strictly convex* if it is convex and the curvature of  $\partial S$  is nonzero at each point. Also domains are by definition compact, with smooth boundary, and  $C^{3,\alpha}(\partial S)$  and  $C^{3,\alpha}(S)$  are *Hölder spaces* of functions on  $\partial S$  and  $S$ . For more details see [21, 22].

Combining Propositions 3.1 and 3.2 and Theorem 3.3 gives existence and uniqueness for a large class of  $U(1)$ -invariant  $SL$  3-folds in  $\mathbb{C}^3$ , with boundary conditions, and including *singular*  $SL$  3-folds. It is interesting that this existence and uniqueness is *entirely unaffected* by singularities appearing in  $S^\circ$ .

Here are some other areas covered in [21, 22, 23]. Examples of solutions  $u, v$  of (7) and (8) are given in [21, §5]. In [22] we give more precise statements on the regularity of singular solutions of (7) and (9). In [21, §6] and [23, §7] we consider the zeroes of  $(u_1, v_1) - (u_2, v_2)$ , where  $(u_j, v_j)$  are (possibly singular) solutions of (7) and (8).

We show that if  $(u_1, v_1) \neq (u_2, v_2)$  then the zeroes of  $(u_1, v_1) - (u_2, v_2)$  in  $S^\circ$  are *isolated*, with a positive integer *multiplicity*, and that the zeroes of  $(u_1, v_1) - (u_2, v_2)$  in  $S^\circ$  can be counted with multiplicity in terms of boundary data on  $\partial S$ . In particular, under some boundary conditions we can show  $(u_1, v_1) - (u_2, v_2)$  has no zeroes in  $S^\circ$ , so that the corresponding  $SL$  3-folds do not intersect. This will be important in constructing  $U(1)$ -invariant  $SL$  fibrations in Section 7.5.

In [23, §9–§10] we study singularities of solutions  $u, v$  of (7). We show that either  $u(x, -y) \equiv u(x, y)$  and  $v(x, -y) \equiv -v(x, y)$ , so that  $u, v$  are singular all along the  $x$ -axis, or else the singular points of  $u, v$  in  $S^\circ$  are all *isolated*, with

a positive integer *multiplicity*, and one of two *types*. We also show that singularities exist with every multiplicity and type, and multiplicity  $n$  singularities occur in codimension  $n$  in the family of all  $U(1)$ -invariant SL 3-folds.

### 3.6 Examples of singular special Lagrangian 3-folds in $\mathbb{C}^3$

We shall now describe four families of SL 3-folds in  $\mathbb{C}^3$ , as examples of the material of Sections 3.1–3.4. They have been chosen to illustrate different kinds of singular behavior of SL 3-folds, and also to show how nonsingular SL 3-folds can converge to a singular SL 3-fold, to serve as a preparation for our discussion of singularities of SL  $m$ -folds in Section 6.

Our first example derives from Harvey and Lawson [11, §III.3.A], and is discussed in detail in [17, §3].

**Example 3.4** Define a subset  $L_0$  in  $\mathbb{C}^3$  by

$$L_0 = \{(re^{i\theta_1}, re^{i\theta_2}, re^{i\theta_3}) : r \geq 0, \theta_1, \theta_2, \theta_3 \in \mathbb{R}, \theta_1 + \theta_2 + \theta_3 = 0\}.$$

Then  $L_0$  is a *special Lagrangian cone* on  $T^2$ . An alternative definition is

$$L_0 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| = |z_2| = |z_3|, \operatorname{Im}(z_1 z_2 z_3) = 0, \operatorname{Re}(z_1 z_2 z_3) \geq 0\}.$$

Let  $t > 0$ , write  $\mathcal{S}^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ , and define a map  $\phi_t : \mathcal{S}^1 \times \mathbb{C} \rightarrow \mathbb{C}^3$  by

$$\phi_t : (e^{i\theta}, z) \mapsto ((|z|^2 + t^2)^{1/2} e^{i\theta}, z, e^{-i\theta} \bar{z}).$$

Then  $\phi_t$  is an *embedding*. Define  $L_t = \operatorname{Image} \phi_t$ . Then  $L_t$  is a nonsingular special Lagrangian 3-fold in  $\mathbb{C}^3$  diffeomorphic to  $\mathcal{S}^1 \times \mathbb{R}^2$ . An equivalent definition is

$$L_t = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - t^2 = |z_2|^2 = |z_3|^2, \\ \operatorname{Im}(z_1 z_2 z_3) = 0, \operatorname{Re}(z_1 z_2 z_3) \geq 0\}.$$

As  $t \rightarrow 0_+$ , the nonsingular SL 3-fold  $L_t$  converges to the singular SL cone  $L_0$ . Note that  $L_t$  is *asymptotic* to  $L_0$  at infinity, and that  $L_t = t L_1$  for  $t > 0$ , so that the  $L_t$  for  $t > 0$  are all homothetic to each other. Also, each  $L_t$  for  $t \geq 0$  is invariant under the  $T^2$  subgroup of  $SU(3)$  acting by

$$(z_1, z_2, z_3) \mapsto (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{i\theta_3} z_3) \text{ for } \theta_1, \theta_2, \theta_3 \in \mathbb{R} \text{ with } \theta_1 + \theta_2 + \theta_3 = 0,$$

and so fits into the framework of Section 3.1. By [21, Th. 5.1] the  $L_a$  may also be written in the form (6) for continuous  $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ , as in Section 3.5.

Our second example is adapted from Harvey and Lawson [11, §III.3.B].

**Example 3.5** For each  $t > 0$ , define

$$L_t = \{(e^{i\theta} x_1, e^{i\theta} x_2, e^{i\theta} x_3) : x_j \in \mathbb{R}, \theta \in (0, \pi/3), x_1^2 + x_2^2 + x_3^2 = t^2 (\sin 3\theta)^{-2/3}\}.$$

Then  $L_t$  is a nonsingular embedded SL 3-fold in  $\mathbb{C}^3$  diffeomorphic to  $\mathcal{S}^2 \times \mathbb{R}$ . As  $t \rightarrow 0_+$  it converges to the singular union  $L_0$  of the two SL 3-planes

$$\Pi_1 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R}\} \text{ and } \Pi_2 = \{(e^{i\pi/3} x_1, e^{i\pi/3} x_2, e^{i\pi/3} x_3) : x_j \in \mathbb{R}\},$$

which intersect at 0. Note that  $L_t$  is invariant under the action of the Lie subgroup  $\mathrm{SO}(3)$  of  $\mathrm{SU}(3)$ , acting on  $\mathbb{C}^3$  in the obvious way, so again this comes from the method of Section 3.1. Also  $L_t$  is asymptotic to  $L_0$  at infinity.

Our third example is taken from [18, Ex. 9.4 & Ex. 9.5].

**Example 3.6** Let  $a_1, a_2$  be positive, coprime integers, and set  $a_3 = -a_1 - a_2$ . Let  $c \in \mathbb{R}$ , and define

$$L_c^{a_1, a_2} = \{(e^{ia_1\theta}x_1, e^{ia_2\theta}x_2, ie^{ia_3\theta}x_3) : \theta \in \mathbb{R}, x_j \in \mathbb{R}, a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = c\}.$$

Then  $L_c^{a_1, a_2}$  is an SL 3-fold, which comes from the ‘‘evolving quadrics’’ construction of Section 3.2. It is also symmetric under the  $\mathrm{U}(1)$ -action

$$(z_1, z_2, z_3) \mapsto (e^{ia_1\theta}z_1, e^{ia_2\theta}z_2, ie^{ia_3\theta}z_3) \quad \text{for } \theta \in \mathbb{R},$$

but this is not a necessary feature of the construction; these are just the easiest examples to write down.

When  $c = 0$  and  $a_3$  is odd,  $L_0^{a_1, a_2}$  is an embedded special Lagrangian cone on  $T^2$ , with one singular point at 0. When  $c = 0$  and  $a_3$  is even,  $L_0^{a_1, a_2}$  is two opposite embedded SL  $T^2$ -cones with one singular point at 0.

When  $c > 0$  and  $a_3$  is odd,  $L_c^{a_1, a_2}$  is an embedded 3-fold diffeomorphic to a nontrivial real line bundle over the Klein bottle. When  $c > 0$  and  $a_3$  is even,  $L_c^{a_1, a_2}$  is an embedded 3-fold diffeomorphic to  $T^2 \times \mathbb{R}$ . In both cases,  $L_c^{a_1, a_2}$  is a *ruled* SL 3-fold, as in Section 3.3, since it is fibered by hyperboloids of one sheet in  $\mathbb{R}^3$ , which are ruled in two different ways.

When  $c < 0$  and  $a_3$  is odd,  $L_c^{a_1, a_2}$  an immersed copy of  $\mathcal{S}^1 \times \mathbb{R}^2$ . When  $c < 0$  and  $a_3$  is even,  $L_c^{a_1, a_2}$  two immersed copies of  $\mathcal{S}^1 \times \mathbb{R}^2$ .

All the singular SL 3-folds we have seen so far have been *cones* in  $\mathbb{C}^3$ . Our final example, taken from [16], has more complicated singularities which are not cones. They are difficult to describe in a simple way, so we will not say much about them. For more details, see [16].

**Example 3.7** In [16, §5] the author constructed a family of maps  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{C}^3$  with special Lagrangian image  $N = \text{Image } \Phi$ . It is shown in [16, §6] that generic  $\Phi$  in this family are immersions, so that  $N$  is nonsingular as an immersed SL 3-fold, but in codimension 1 in the family they develop isolated singularities.

Here is a rough description of these singularities, taken from [16, §6]. Taking the singular point to be at  $\Phi(0, 0, 0) = 0$ , one can write  $\Phi$  as

$$\begin{aligned} \Phi(x, y, t) = & (x + \frac{1}{4}g'(\mathbf{u}, \mathbf{v})t^2) \mathbf{u} + (y^2 - \frac{1}{4}|\mathbf{u}|^2t^2) \mathbf{v} \\ & + 2yt \mathbf{u} \times \mathbf{v} + O(x^2 + |xy| + |xt| + |y|^3 + |t|^3), \end{aligned} \quad (10)$$

where  $\mathbf{u}, \mathbf{v}$  are linearly independent vectors in  $\mathbb{C}^3$  with  $\omega'(\mathbf{u}, \mathbf{v}) = 0$ , and  $\times : \mathbb{C}^3 \times \mathbb{C}^3 \rightarrow \mathbb{C}^3$  is defined by

$$(r_1, r_2, r_3) \times (s_1, s_2, s_3) = \frac{1}{2}(\bar{r}_2\bar{s}_3 - \bar{r}_3\bar{s}_2, \bar{r}_3\bar{s}_1 - \bar{r}_1\bar{s}_3, \bar{r}_1\bar{s}_2 - \bar{r}_2\bar{s}_1).$$

The next few terms in the expansion (10) can also be given very explicitly, but we will not write them down as they are rather complex, and involve further choices of vectors  $\mathbf{w}, \mathbf{x}, \dots$ .

What is going on here is that the lowest order terms in  $\Phi$  are a *double cover* of the special Lagrangian plane  $\langle \mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle_{\mathbb{R}}$  in  $\mathbb{C}^3$ , *branched* along the real line  $\langle \mathbf{u} \rangle_{\mathbb{R}}$ . The branching occurs when  $y = t = 0$ . Higher order terms deviate from the 3-plane  $\langle \mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v} \rangle_{\mathbb{R}}$ , and make the singularity isolated.

## 4 Almost Calabi–Yau geometry

*Calabi–Yau  $m$ -folds*  $(M, J, \omega, \Omega)$  are compact complex  $m$ -folds  $(M, J)$  equipped with a Ricci-flat Kähler metric  $g$  with Kähler form  $\omega$ , and a holomorphic  $(m, 0)$ -form  $\Omega$  of constant length  $|\Omega|^2 = 2^m$ . Then  $\operatorname{Re} \Omega$  is a *calibration* on  $(M, g)$ , and the corresponding calibrated submanifolds are called *special Lagrangian  $m$ -folds*. They are a natural generalization of the idea of special Lagrangian submanifolds in  $\mathbb{C}^m$ .

However, we will actually define and study special Lagrangian submanifolds in the much larger class of *almost Calabi–Yau manifolds*  $(M, J, \omega, \Omega)$ , in which  $g$  is not required to be Ricci-flat, and  $\Omega$  not required to have constant length. Apart from greater generality, this has the advantage that by restricting to a *generic* almost Calabi–Yau manifold one can (the author believes) much simplify the singular behavior of the special Lagrangian submanifolds within it.

The idea of extending special Lagrangian geometry to almost Calabi–Yau manifolds appears in the work of Goldstein [4, §3.1], Bryant [1, §1], who uses the term “special Kähler” instead of “almost Calabi–Yau”, and the author [25].

### 4.1 Calabi–Yau and almost Calabi–Yau manifolds

Here is our definition of Calabi–Yau and almost Calabi–Yau manifolds.

**Definition 4.1** Let  $m \geq 2$ . An *almost Calabi–Yau  $m$ -fold*, or *ACY  $m$ -fold* for short, is a quadruple  $(M, J, \omega, \Omega)$  such that  $(M, J)$  is a compact  $m$ -dimensional complex manifold,  $\omega$  is the Kähler form of a Kähler metric  $g$  on  $M$ , and  $\Omega$  is a non-vanishing holomorphic  $(m, 0)$ -form on  $M$ .

We call  $(M, J, \omega, \Omega)$  a *Calabi–Yau  $m$ -fold*, or *CY  $m$ -fold* for short, if in addition  $\omega$  and  $\Omega$  satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega}. \quad (11)$$

Furthermore,  $g$  is Ricci-flat and its holonomy group is a subgroup of  $\operatorname{SU}(m)$ .

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it. Using Yau’s proof of the Calabi Conjecture [42], [14, §5] one can prove:

**Theorem 4.2** *Let  $(M, J)$  be a compact complex manifold with trivial canonical bundle  $K_M$ , admitting Kähler metrics. Then in each Kähler class on  $M$  there*

is a unique Ricci-flat Kähler metric  $g$ , with Kähler form  $\omega$ . Given such  $g$  and  $\omega$ , there exists a holomorphic section  $\Omega$  of  $K_M$ , unique up to change of phase  $\Omega \mapsto e^{i\theta}\Omega$ , such that  $(M, J, \omega, \Omega)$  is a Calabi–Yau manifold.

Thus, to find examples of Calabi–Yau manifolds all one needs is complex manifolds  $(M, J)$  satisfying certain essentially topological conditions. Using algebraic geometry one can construct very large numbers of such complex manifolds, particularly in complex dimension 3, and thus Calabi–Yau manifolds are very abundant. For a review of such constructions, and other general properties of Calabi–Yau manifolds, see [14, §6].

## 4.2 SL $m$ -folds in almost Calabi–Yau $m$ -folds

Next, we define *special Lagrangian  $m$ -folds* in almost Calabi–Yau  $m$ -folds.

**Definition 4.3** Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold, and  $N$  a real  $m$ -dimensional submanifold of  $M$ . We call  $N$  a *special Lagrangian submanifold*, or *SL  $m$ -fold* for short, if  $\omega|_N \equiv \text{Im } \Omega|_N \equiv 0$ . It easily follows that  $\text{Re } \Omega|_N$  is a nonvanishing  $m$ -form on  $N$ . Thus  $N$  is orientable, with a unique orientation in which  $\text{Re } \Omega|_N$  is positive.

Let  $(M, J, \omega, \Omega)$  be a Calabi–Yau  $m$ -fold, with metric  $g$ . Then equation (11) ensures that for each  $x \in M$  there exists an isomorphism  $T_x M \cong \mathbb{C}^m$  that identifies  $g_x, \omega_x$  and  $\Omega_x$  with the flat versions  $g', \omega', \Omega'$  on  $\mathbb{C}^m$  in (1). From Proposition 2.4 we then deduce:

**Proposition 4.4** *Let  $(M, J, \omega, \Omega)$  be a Calabi–Yau  $m$ -fold, with metric  $g$ , and  $N$  a real  $m$ -submanifold of  $M$ . Then  $N$  is special Lagrangian, with the natural orientation, if and only if it is calibrated with respect to  $\text{Re } \Omega$ .*

Thus, in the Calabi–Yau case Definition 4.3 is equivalent to the conventional definition of special Lagrangian  $m$ -folds in Calabi–Yau  $m$ -folds, which is that they should be calibrated with respect to  $\text{Re } \Omega$ , as in Definition 2.2. In the almost Calabi–Yau case, we can still interpret SL  $m$ -folds as calibrated submanifolds, but with respect to a conformally rescaled metric  $\tilde{g}$ . We explain how in the next proposition, which is easily proved using Proposition 2.4.

**Proposition 4.5** *Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold with metric  $g$ , define  $f : M \rightarrow (0, \infty)$  by  $f^{2m}\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m\Omega \wedge \bar{\Omega}$ , and let  $\tilde{g}$  be the conformally equivalent metric  $f^2g$  on  $M$ . Then  $\text{Re } \Omega$  is a calibration on the Riemannian manifold  $(M, \tilde{g})$ .*

*A real  $m$ -submanifold  $N$  in  $M$  is special Lagrangian in  $(M, J, \omega, \Omega)$  if and only if it admits an orientation for which it is calibrated with respect to  $\text{Re } \Omega$  in  $(M, \tilde{g})$ . In particular, special Lagrangian  $m$ -folds in  $M$  are minimal in  $(M, \tilde{g})$ .*

Thus, we can give an equivalent definition of SL  $m$ -folds in terms of calibrated geometry. Nonetheless, in the author’s view the definition of SL  $m$ -folds in terms of the vanishing of closed forms is more fundamental than the definition in terms

of calibrated geometry, at least in the almost Calabi–Yau case, and so should be taken as the primary definition.

One important reason for considering SL  $m$ -folds in almost Calabi–Yau rather than Calabi–Yau  $m$ -folds is that they have much stronger *genericness properties*. There are many situations in geometry in which one uses a genericity assumption to control singular behavior.

For instance, pseudo-holomorphic curves in an arbitrary almost complex manifold may have bad singularities, but the possible singularities in a generic almost complex manifold are much simpler. In the same way, it is reasonable to hope that in a *generic* Calabi–Yau  $m$ -fold, compact SL  $m$ -folds may have better singular behavior than in an arbitrary Calabi–Yau  $m$ -fold.

But because Calabi–Yau manifolds come in only finite-dimensional families, choosing a generic Calabi–Yau structure is a fairly weak assumption, and probably will not help very much. However, almost Calabi–Yau manifolds come in *infinite-dimensional* families, so choosing a generic almost Calabi–Yau structure is a much more powerful thing to do, and will probably simplify the singular behavior of compact SL  $m$ -folds considerably. We will return to this idea in Section 6.

## 5 Compact SL $m$ -folds in ACY $m$ -folds

In this section we shall discuss *compact* special Lagrangian submanifolds in almost Calabi–Yau manifolds. Here are three important questions which motivate work in this area.

1. Let  $N$  be a compact special Lagrangian  $m$ -fold in a fixed almost Calabi–Yau  $m$ -fold  $(M, J, \omega, \Omega)$ . Let  $\mathcal{M}_N$  be the moduli space of *special Lagrangian deformations* of  $N$ , that is, the connected component of the set of special Lagrangian  $m$ -folds containing  $N$ . What can we say about  $\mathcal{M}_N$ ? For instance, is it a smooth manifold, and of what dimension?
2. Let  $\{(M, J_t, \omega_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$  be a smooth 1-parameter family of almost Calabi–Yau  $m$ -folds. Suppose  $N_0$  is an SL  $m$ -fold in  $(M, J_0, \omega_0, \Omega_0)$ . Under what conditions can we extend  $N_0$  to a smooth family of special Lagrangian  $m$ -folds  $N_t$  in  $(M, J_t, \omega_t, \Omega_t)$  for  $t \in (-\epsilon, \epsilon)$ ?
3. In general the moduli space  $\mathcal{M}_N$  in Question 1 will be noncompact. Can we enlarge  $\mathcal{M}_N$  to a compact space  $\overline{\mathcal{M}}_N$  by adding a “boundary” consisting of *singular* special Lagrangian  $m$ -folds? If so, what is the nature of the singularities that develop?

Briefly, these questions concern the *deformations* of special Lagrangian  $m$ -folds, *obstructions* to their existence, and their *singularities* respectively. The local answers to Questions 1 and 2 are well understood, and we shall discuss them in this section. Question 3 is the subject of Sections 6–7.

## 5.1 Deformations of compact special Lagrangian $m$ -folds

The deformation theory of compact SL  $m$ -folds  $N$  was studied by McLean [36], who proved the following result in the Calabi–Yau case. Because McLean’s proof only relies on the fact that  $\omega|_N \equiv \text{Im } \Omega|_N \equiv 0$ , it also applies equally well to SL  $m$ -folds in almost Calabi–Yau  $m$ -folds.

**Theorem 5.1** *Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold, and  $N$  a compact special Lagrangian  $m$ -fold in  $M$ . Then the moduli space  $\mathcal{M}_N$  of special Lagrangian deformations of  $N$  is a smooth manifold of dimension  $b^1(N)$ , the first Betti number of  $N$ .*

*Proof of Sketch of proof.* Suppose for simplicity that  $N$  is an embedded submanifold. There is a natural orthogonal decomposition  $TM|_N = TN \oplus \nu$ , where  $\nu \rightarrow N$  is the *normal bundle* of  $N$  in  $M$ . As  $N$  is Lagrangian, the complex structure  $J : TM \rightarrow TM$  gives an isomorphism  $J : \nu \rightarrow TN$ . But the metric  $g$  gives an isomorphism  $TN \cong T^*N$ . Composing these two gives an isomorphism  $\nu \cong T^*N$ .

Let  $T$  be a small *tubular neighborhood* of  $N$  in  $M$ . Then we can identify  $T$  with a neighborhood of the zero section in  $\nu$ . Using the isomorphism  $\nu \cong T^*N$ , we have an identification between  $T$  and a neighborhood of the zero section in  $T^*N$ . This can be chosen to identify the Kähler form  $\omega$  on  $T$  with the natural symplectic structure on  $T^*N$ . Let  $\pi : T \rightarrow N$  be the obvious projection.

Under this identification, submanifolds  $N'$  in  $T \subset M$  which are  $C^1$  close to  $N$  are identified with the graphs of small smooth sections  $\alpha$  of  $T^*N$ . That is, submanifolds  $N'$  of  $M$  close to  $N$  are identified with 1-forms  $\alpha$  on  $N$ . We need to know: which 1-forms  $\alpha$  are identified with *special Lagrangian* submanifolds  $N'$ ?

Well,  $N'$  is special Lagrangian if  $\omega|_{N'} \equiv \text{Im } \Omega|_{N'} \equiv 0$ . Now  $\pi|_{N'} : N' \rightarrow N$  is a diffeomorphism, so we can push  $\omega|_{N'}$  and  $\text{Im } \Omega|_{N'}$  down to  $N$ , and regard them as functions of  $\alpha$ . Calculation shows that

$$\pi_*(\omega|_{N'}) = d\alpha \quad \text{and} \quad \pi_*(\text{Im } \Omega|_{N'}) = F(\alpha, \nabla\alpha),$$

where  $F$  is a nonlinear function of its arguments. Thus, the moduli space  $\mathcal{M}_N$  is locally isomorphic to the set of small 1-forms  $\alpha$  on  $N$  such that  $d\alpha \equiv 0$  and  $F(\alpha, \nabla\alpha) \equiv 0$ .

Now it turns out that  $F$  satisfies  $F(\alpha, \nabla\alpha) \approx d(*\alpha)$  when  $\alpha$  is small. Therefore  $\mathcal{M}_N$  is locally approximately isomorphic to the vector space of 1-forms  $\alpha$  with  $d\alpha = d(*\alpha) = 0$ . But by Hodge theory, this is isomorphic to the de Rham cohomology group  $H^1(N, \mathbb{R})$ , and is a manifold with dimension  $b^1(N)$ .

To carry out this last step rigorously requires some technical machinery: one must work with certain *Banach spaces* of sections of  $T^*N$ ,  $\Lambda^2 T^*N$  and  $\Lambda^m T^*N$ , use *elliptic regularity results* to prove that the map  $\alpha \mapsto (d\alpha, F(\alpha, \nabla\alpha))$  has *closed image* in these Banach spaces, and then use the *Implicit Function Theorem for Banach spaces* to show that the kernel of the map is what we expect.  $\square$

## 5.2 Obstructions to the existence of compact SL $m$ -folds

Next we address Question 2 above. First, observe that if  $(M, J, \omega, \Omega)$  is an almost Calabi–Yau  $m$ -fold and  $N$  a compact SL  $m$ -fold in  $M$  then  $\omega|_N \equiv \text{Im } \Omega|_N \equiv 0$ , and thus  $[\omega|_N]$  and  $[\text{Im } \Omega|_N]$  are zero in  $H^2(N, \mathbb{R})$  and  $H^m(N, \mathbb{R})$ . But  $[\omega|_N]$  and  $[\text{Im } \Omega|_N]$  are unchanged under continuous variations of  $N$  in  $M$ . So we deduce:

**Lemma 5.2** *Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold, and  $N$  a compact real  $m$ -submanifold in  $M$ . Then a necessary condition for  $N$  to be isotopic to a special Lagrangian submanifold  $N'$  in  $M$  is that  $[\omega|_N] = 0$  in  $H^2(N, \mathbb{R})$  and  $[\text{Im } \Omega|_N] = 0$  in  $H^m(N, \mathbb{R})$ .*

This gives a simple, necessary topological condition for an isotopy class of  $m$ -submanifolds in an almost Calabi–Yau  $m$ -fold to contain an SL  $m$ -fold. Our next result, following from Marshall [34, Th. 3.2.9], shows that locally, this is also a *sufficient* condition for an SL  $m$ -fold to persist under deformations of the almost Calabi–Yau structure.

**Theorem 5.3** *Let  $\{(M, J_t, \omega_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$  be a smooth family of almost Calabi–Yau  $m$ -folds. Let  $N_0$  be a compact SL  $m$ -fold in  $(M, J_0, \omega_0, \Omega_0)$ , and suppose that  $[\omega_t|_{N_0}] = 0$  in  $H^2(N_0, \mathbb{R})$  and  $[\text{Im } \Omega_t|_{N_0}] = 0$  in  $H^m(N_0, \mathbb{R})$  for all  $t \in (-\epsilon, \epsilon)$ . Then  $N_0$  extends to a smooth family  $\{N_t : t \in (-\delta, \delta)\}$ , where  $0 < \delta \leq \epsilon$  and  $N_t$  is a compact SL  $m$ -fold in  $(M, J_t, \omega_t, \Omega_t)$ .*

This is proved using similar techniques to Theorem 5.1, though McLean did not prove it. Note that the condition  $[\text{Im } \Omega_t|_{N_0}] = 0$  for all  $t$  can be satisfied by choosing the phases of the  $\Omega_t$  appropriately, and if the image of  $H_2(N, \mathbb{Z})$  in  $H_2(M, \mathbb{R})$  is zero, then the condition  $[\omega|_N] = 0$  holds automatically.

Thus, the obstructions  $[\omega_t|_{N_0}] = [\text{Im } \Omega_t|_{N_0}] = 0$  in Theorem 5.3 are actually fairly mild restrictions, and special Lagrangian  $m$ -folds should be thought of as pretty stable under small deformations of the almost Calabi–Yau structure.

**Remark.** The deformation and obstruction theory of compact special Lagrangian  $m$ -folds are *extremely well-behaved* compared to many other moduli space problems in differential geometry. In other geometric problems (such as the deformations of complex structures on a complex manifold, or pseudo-holomorphic curves in an almost complex manifold, or instantons on a Riemannian 4-manifold, and so on), the deformation theory often has the following general structure.

There are vector bundles  $E, F$  over a compact manifold  $M$ , and an elliptic operator  $P : C^\infty(E) \rightarrow C^\infty(F)$ , usually first-order. The kernel  $\text{Ker } P$  is the set of *infinitesimal deformations*, and the cokernel  $\text{Coker } P$  the set of *obstructions*. The actual moduli space  $\mathcal{M}$  is locally the zeros of a nonlinear map  $\Psi : \text{Ker } P \rightarrow \text{Coker } P$ .

In a *generic* case,  $\text{Coker } P = 0$ , and then the moduli space  $\mathcal{M}$  is locally isomorphic to  $\text{Ker } P$ , and so is locally a manifold with dimension  $\text{ind}(P)$ . However,

in nongeneric situations Coker  $P$  may be nonzero, and then the moduli space  $\mathcal{M}$  may be nonsingular, or have an unexpected dimension.

However, SL  $m$ -folds do not follow this pattern. Instead, the obstructions are *topologically determined*, and the moduli space is *always* smooth, with dimension given by a topological formula. This should be regarded as a minor mathematical miracle.

### 5.3 Natural coordinates on the moduli space $\mathcal{M}_N$

Let  $N$  be a compact SL  $m$ -fold in an almost Calabi–Yau  $m$ -fold  $(M, J, \omega, \Omega)$ . Theorem 5.1 shows that the moduli space  $\mathcal{M}_N$  has dimension  $b^1(N)$ . By Poincaré duality  $b^1(N) = b^{m-1}(N)$ . Thus  $\mathcal{M}_N$  has the same dimension as the de Rham cohomology groups  $H^1(M, \mathbb{R})$  and  $H^{m-1}(M, \mathbb{R})$ .

We shall construct natural local diffeomorphisms  $\Phi$  from  $\mathcal{M}_N$  to  $H^1(N, \mathbb{R})$ , and  $\Psi$  from  $\mathcal{M}_N$  to  $H^{m-1}(N, \mathbb{R})$ . These induce two natural *affine structures* on  $\mathcal{M}_N$ , and can be thought of as two *natural coordinate systems* on  $\mathcal{M}_N$ . The material of this section can be found in Hitchin [13, §4].

Here is how to define  $\Phi$  and  $\Psi$ . Let  $U$  be a connected and simply-connected open neighborhood of  $N$  in  $\mathcal{M}_N$ . We will construct smooth maps  $\Phi : U \rightarrow H^1(N, \mathbb{R})$  and  $\Psi : U \rightarrow H^{m-1}(N, \mathbb{R})$  with  $\Phi(N) = \Psi(N) = 0$ , which are local diffeomorphisms.

Let  $N' \in U$ . Then as  $U$  is connected, there exists a smooth path  $\gamma : [0, 1] \rightarrow U$  with  $\gamma(0) = N$  and  $\gamma(1) = N'$ , and as  $U$  is simply-connected,  $\gamma$  is unique up to isotopy. Now  $\gamma$  parametrizes a family of submanifolds of  $M$  diffeomorphic to  $N$ , which we can lift to a smooth map  $\Gamma : N \times [0, 1] \rightarrow M$  with  $\Gamma(N \times \{t\}) = \gamma(t)$ .

Consider the 2-form  $\Gamma^*(\omega)$  on  $N \times [0, 1]$ . As each fiber  $\gamma(t)$  is Lagrangian, we have  $\Gamma^*(\omega)|_{N \times \{t\}} \equiv 0$  for each  $t \in [0, 1]$ . Therefore we may write  $\Gamma^*(\omega) = \alpha_t \wedge dt$ , where  $\alpha_t$  is a closed 1-form on  $N$  for  $t \in [0, 1]$ . Define

$$\Phi(N') = \left[ \int_0^1 \alpha_t dt \right] \in H^1(N, \mathbb{R}).$$

That is, we integrate the 1-forms  $\alpha_t$  with respect to  $t$  to get a closed 1-form  $\int_0^1 \alpha_t dt$ , and then take its cohomology class.

Similarly, write  $\Gamma^*(\text{Im } \Omega) = \beta_t \wedge dt$ , where  $\beta_t$  is a closed  $(m-1)$ -form on  $N$  for  $t \in [0, 1]$ , and define  $\Psi(N') = \left[ \int_0^1 \beta_t dt \right] \in H^{m-1}(N, \mathbb{R})$ . Then  $\Phi$  and  $\Psi$  are independent of choices made in the construction (exercise). We need to restrict to a simply-connected subset  $U$  of  $\mathcal{M}_N$  so that  $\gamma$  is unique up to isotopy. Alternatively, one can define  $\Phi$  and  $\Psi$  on the universal cover  $\tilde{\mathcal{M}}_N$  of  $\mathcal{M}_N$ .

## 6 Singularities of special Lagrangian $m$ -folds

Now we move on to Question 3 of Section 5, and discuss the *singularities* of special Lagrangian  $m$ -folds. We can divide it into two sub-questions:

- 3(a)** What kinds of singularities are possible in singular special Lagrangian  $m$ -folds, and what do they look like?

**3(b)** How can singular SL  $m$ -folds arise as limits of nonsingular SL  $m$ -folds, and what does the limiting behavior look like near the singularities?

These questions are addressed in the author's series of papers [26, 27, 28, 29, 30] for *isolated conical singularities* of SL  $m$ -folds, that is, singularities locally modelled on an SL cone  $C$  in  $\mathbb{C}^m$  with an isolated singularity at 0. We now explain the principal results. Readers of the series are advised to begin with the final paper [30], which surveys the others.

## 6.1 Special Lagrangian cones

We define *SL cones*, and some notation.

**Definition 6.1** A (singular) SL  $m$ -fold  $C$  in  $\mathbb{C}^m$  is called a *cone* if  $C = tC$  for all  $t > 0$ , where  $tC = \{t\mathbf{x} : \mathbf{x} \in C\}$ . Let  $C$  be a closed SL cone in  $\mathbb{C}^m$  with an isolated singularity at 0. Then  $\Sigma = C \cap \mathcal{S}^{2m-1}$  is a compact, nonsingular  $(m-1)$ -submanifold of  $\mathcal{S}^{2m-1}$ , not necessarily connected. Let  $g_\Sigma$  be the restriction of  $g'$  to  $\Sigma$ , where  $g'$  is as in (1).

Set  $C' = C \setminus \{0\}$ . Define  $\iota : \Sigma \times (0, \infty) \rightarrow \mathbb{C}^m$  by  $\iota(\sigma, r) = r\sigma$ . Then  $\iota$  has image  $C'$ . By an abuse of notation, *identify*  $C'$  with  $\Sigma \times (0, \infty)$  using  $\iota$ . The *cone metric* on  $C' \cong \Sigma \times (0, \infty)$  is  $g' = \iota^*(g') = dr^2 + r^2g_\Sigma$ . For  $\alpha \in \mathbb{R}$ , we say that a function  $u : C' \rightarrow \mathbb{R}$  is *homogeneous of order  $\alpha$*  if  $u \circ t \equiv t^\alpha u$  for all  $t > 0$ . Equivalently,  $u$  is homogeneous of order  $\alpha$  if  $u(\sigma, r) \equiv r^\alpha v(\sigma)$  for some function  $v : \Sigma \rightarrow \mathbb{R}$ .

In [26, Lem. 2.3] we study *homogeneous harmonic functions* on  $C'$ .

**Lemma 6.2** *In the situation of Definition 6.1, let  $u(\sigma, r) \equiv r^\alpha v(\sigma)$  be a homogeneous function of order  $\alpha$  on  $C' = \Sigma \times (0, \infty)$ , for  $v \in C^2(\Sigma)$ . Then*

$$\Delta u(\sigma, r) = r^{\alpha-2}(\Delta_\Sigma v - \alpha(\alpha + m - 2)v),$$

where  $\Delta$ ,  $\Delta_\Sigma$  are the Laplacians on  $(C', g')$  and  $(\Sigma, g_\Sigma)$ . Hence,  $u$  is harmonic on  $C'$  if and only if  $v$  is an eigenfunction of  $\Delta_\Sigma$  with eigenvalue  $\alpha(\alpha + m - 2)$ .

Following [26, Def. 2.5], we define:

**Definition 6.3** In the situation of Definition 6.1, suppose  $m > 2$  and define

$$\mathcal{D}_\Sigma = \{\alpha \in \mathbb{R} : \alpha(\alpha + m - 2) \text{ is an eigenvalue of } \Delta_\Sigma\}. \quad (12)$$

Then  $\mathcal{D}_\Sigma$  is a countable, discrete subset of  $\mathbb{R}$ . By Lemma 6.2, an equivalent definition is that  $\mathcal{D}_\Sigma$  is the set of  $\alpha \in \mathbb{R}$  for which there exists a nonzero homogeneous harmonic function  $u$  of order  $\alpha$  on  $C'$ .

Define  $m_\Sigma : \mathcal{D}_\Sigma \rightarrow \mathbb{N}$  by taking  $m_\Sigma(\alpha)$  to be the multiplicity of the eigenvalue  $\alpha(\alpha + m - 2)$  of  $\Delta_\Sigma$ , or equivalently the dimension of the vector space of homogeneous harmonic functions  $u$  of order  $\alpha$  on  $C'$ . Define  $N_\Sigma : \mathbb{R} \rightarrow \mathbb{Z}$  by

$$N_\Sigma(\delta) = - \sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta, 0)} m_\Sigma(\alpha) \text{ if } \delta < 0, \text{ and } N_\Sigma(\delta) = \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0, \delta]} m_\Sigma(\alpha) \text{ if } \delta \geq 0. \quad (13)$$

Then  $N_\Sigma$  is monotone increasing and upper semicontinuous, and is discontinuous exactly on  $\mathcal{D}_\Sigma$ , increasing by  $m_\Sigma(\alpha)$  at each  $\alpha \in \mathcal{D}_\Sigma$ . As the eigenvalues of  $\Delta_\Sigma$  are nonnegative, we see that  $\mathcal{D}_\Sigma \cap (2 - m, 0) = \emptyset$  and  $N_\Sigma \equiv 0$  on  $(2 - m, 0)$ .

We define the *stability index* of  $C$ , and *stable cones* [27, Def. 3.6].

**Definition 6.4** Let  $C$  be an SL cone in  $\mathbb{C}^m$  for  $m > 2$  with an isolated singularity at 0, let  $G$  be the Lie subgroup of  $SU(m)$  preserving  $C$ , and use the notation of Definitions 6.1 and 6.3. Then [27, eq. (8)] shows that

$$m_\Sigma(0) = b^0(\Sigma), \quad m_\Sigma(1) \geq 2m \quad \text{and} \quad m_\Sigma(2) \geq m^2 - 1 - \dim G. \quad (14)$$

Define the *stability index*  $s\text{-ind}(C)$  to be

$$s\text{-ind}(C) = N_\Sigma(2) - b^0(\Sigma) - m^2 - 2m + 1 + \dim G. \quad (15)$$

Then  $s\text{-ind}(C) \geq 0$  by (14), as  $N_\Sigma(2) \geq m_\Sigma(0) + m_\Sigma(1) + m_\Sigma(2)$  by (13). We call  $C$  *stable* if  $s\text{-ind}(C) = 0$ .

Here is the point of these definitions. By the Principle in Section 2.3, homogeneous harmonic functions  $v$  of order  $\alpha$  on  $C'$  correspond to infinitesimal deformations  $dv$  of  $C'$  as an SL  $m$ -fold in  $\mathbb{C}^m$ , which grow like  $O(r^{\alpha-1})$ . Hence,  $N_\Sigma(\lambda)$  is effectively the dimension of a space of *infinitesimal deformations* of  $C'$  as an SL  $m$ -fold, which grow like  $O(r^{\alpha-1})$  for  $\alpha \in [0, \lambda]$ , when  $\lambda \geq 0$ .

For  $\lambda = 2$  this space of harmonic functions, or infinitesimal deformations of  $C'$ , contains some from obvious geometrical sources: locally constant functions on  $C'$ , and infinitesimal deformations of  $C'$  from translations in  $\mathbb{C}^m$ , or  $\mathfrak{su}(m)$  rotations. We get  $s\text{-ind}(C)$  by subtracting off these obvious geometrical deformations from  $N_\Sigma(2)$ . Hence,  $s\text{-ind}(C)$  is the dimension of a space of *excess infinitesimal deformations* of  $C'$  as an SL  $m$ -fold, with growth between  $O(r^{-1})$  and  $O(r)$ , which do not arise from infinitesimal automorphisms of  $\mathbb{C}^m$ .

We shall see in Section 6.2 that  $s\text{-ind}(C)$  is the dimension of an *obstruction space* to deforming an SL  $m$ -fold  $X$  with a conical singularity with cone  $C$ , and that if  $C$  is *stable* then the deformation theory of  $X$  simplifies.

## 6.2 Special Lagrangian $m$ -folds with conical singularities

Now we can define *conical singularities* of SL  $m$ -folds, following [26, Def. 3.6].

**Definition 6.5** Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold for  $m > 2$ . Suppose  $X$  is a compact singular SL  $m$ -fold in  $M$  with singularities at distinct points  $x_1, \dots, x_n \in X$ , and no other singularities.

Fix isomorphisms  $v_i : \mathbb{C}^m \rightarrow T_{x_i}M$  for  $i = 1, \dots, n$  such that  $v_i^*(\omega) = \omega'$  and  $v_i^*(\Omega) = a_i\Omega'$ , where  $\omega', \Omega'$  are as in (1) and  $a_1, \dots, a_n > 0$ . Let  $C_1, \dots, C_n$  be SL cones in  $\mathbb{C}^m$  with isolated singularities at 0. For  $i = 1, \dots, n$  let  $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$ , and let  $\mu_i \in (2, 3)$  with

$$(2, \mu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset, \quad \text{where } \mathcal{D}_{\Sigma_i} \text{ is defined in (12)}. \quad (16)$$

Then we say that  $X$  has a *conical singularity* or *conical singular point* at  $x_i$ , with rate  $\mu_i$  and cone  $C_i$  for  $i = 1, \dots, n$ , if the following holds.

By Darboux Theorem there exist embeddings  $\Upsilon_i : B_R \rightarrow M$  for  $i = 1, \dots, n$  satisfying  $\Upsilon_i(0) = x_i$ ,  $d\Upsilon_i|_0 = v_i$  and  $\Upsilon_i^*(\omega) = \omega'$ , where  $B_R$  is the open ball of radius  $R$  about 0 in  $\mathbb{C}^m$  for some small  $R > 0$ . Define  $\iota_i : \Sigma_i \times (0, R) \rightarrow B_R$  by  $\iota_i(\sigma, r) = r\sigma$  for  $i = 1, \dots, n$ .

Define  $X' = X \setminus \{x_1, \dots, x_n\}$ . Then there should exist a compact subset  $K \subset X'$  such that  $X' \setminus K$  is a union of open sets  $S_1, \dots, S_n$  with  $S_i \subset \Upsilon_i(B_R)$ , whose closures  $\bar{S}_1, \dots, \bar{S}_n$  are disjoint in  $X$ . For  $i = 1, \dots, n$  and some  $R' \in (0, R]$  there should exist a smooth  $\phi_i : \Sigma_i \times (0, R') \rightarrow B_R$  such that  $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \rightarrow M$  is a diffeomorphism  $\Sigma_i \times (0, R') \rightarrow S_i$ , and

$$|\nabla^k(\phi_i - \iota_i)| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \rightarrow 0 \text{ for } k = 0, 1. \quad (17)$$

Here  $\nabla$  is the Levi-Civita connection of the cone metric  $\iota_i^*(g')$  on  $\Sigma_i \times (0, R')$ ,  $|\cdot|$  is computed using  $\iota_i^*(g')$ . If the cones  $C_1, \dots, C_n$  are *stable* in the sense of Definition 6.4, then we say that  $X$  has *stable conical singularities*.

We show in [26, Th.s 4.4 & 5.5] that if (17) holds for  $k = 0, 1$  and some  $\mu_i$  satisfying (16), then we can choose a natural  $\phi_i$  for which (17) holds for *all*  $k \geq 0$ , and for *all* rates  $\mu_i$  satisfying (16). Thus the number of derivatives required in (17) and the choice of  $\mu_i$  both make little difference. We choose  $k = 0, 1$  in (17), and some  $\mu_i$  in (16), to make the definition as weak as possible.

Suppose we did not require (16), and that  $\alpha \in (2, \mu_i) \cap \mathcal{D}_{\Sigma_i}$ . Then there exists a homogeneous harmonic function  $v$  on  $C'_i$  of order  $\alpha$ . By the Principle in Section 2.3,  $dv$  yields an infinitesimal deformation of  $C'_i$  as an SL  $m$ -fold in  $\mathbb{C}^m$ , growing like  $O(r^{\alpha-1})$ . Locally this gives a way to deform  $X$  into an SL  $m$ -fold which would *not* satisfy (17), as  $\alpha < \mu_i$ . Effectively,  $v$  acts as an *obstruction* to deforming  $X$  through SL  $m$ -folds satisfying Definition 6.5. So the point of (16) is to reduce to a minimum the obstructions to existence and deformation of SL  $m$ -folds with isolated conical singularities.

In [27] we study the *deformation theory* of compact SL  $m$ -folds with conical singularities, generalizing Theorem 5.1 in the nonsingular case. Following [27, Def. 5.4], we define the space  $\mathcal{M}_X$  of compact SL  $m$ -folds  $\hat{X}$  in  $M$  with conical singularities deforming a fixed SL  $m$ -fold  $X$  with conical singularities.

**Definition 6.6** Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold and  $X$  a compact SL  $m$ -fold in  $M$  with conical singularities at  $x_1, \dots, x_n$  with identifications  $v_i : \mathbb{C}^m \rightarrow T_{x_i}M$  and cones  $C_1, \dots, C_n$ . Define the *moduli space*  $\mathcal{M}_X$  of *deformations of  $X$*  to be the set of  $\hat{X}$  such that

- (i)  $\hat{X}$  is a compact SL  $m$ -fold in  $M$  with conical singularities at  $\hat{x}_1, \dots, \hat{x}_n$  with cones  $C_1, \dots, C_n$ , for some  $\hat{x}_i$  and identifications  $\hat{v}_i : \mathbb{C}^m \rightarrow T_{\hat{x}_i}M$ .
- (ii) There exists a homeomorphism  $\hat{\iota} : X \rightarrow \hat{X}$  with  $\hat{\iota}(x_i) = \hat{x}_i$  for  $i = 1, \dots, n$  such that  $\hat{\iota}|_{X'} : X' \rightarrow \hat{X}'$  is a diffeomorphism and  $\hat{\iota}$  and  $\iota$  are isotopic as continuous maps  $X \rightarrow M$ , where  $\iota : X \rightarrow M$  is the inclusion.

In [27, Def. 5.6] we define a *topology* on  $\mathcal{M}_X$ , and explain why it is the natural choice. We will not repeat the complicated definition here. In [27, Th. 6.10] we describe  $\mathcal{M}_X$  near  $X$ , in terms of a smooth map  $\Phi$  between the *infinitesimal deformation space*  $\mathcal{I}_{X'}$  and the *obstruction space*  $\mathcal{O}_{X'}$ .

**Theorem 6.7** *Suppose  $(M, J, \omega, \Omega)$  is an almost Calabi–Yau  $m$ -fold and  $X$  a compact  $SL$   $m$ -fold in  $M$  with conical singularities at  $x_1, \dots, x_n$  and cones  $C_1, \dots, C_n$ . Let  $\mathcal{M}_X$  be the moduli space of deformations of  $X$  as an  $SL$   $m$ -fold with conical singularities in  $M$ , as in Definition 6.6. Set  $X' = X \setminus \{x_1, \dots, x_n\}$ .*

*Then there exist natural finite-dimensional vector spaces  $\mathcal{I}_{X'}$ ,  $\mathcal{O}_{X'}$  such that  $\mathcal{I}_{X'}$  is the image of  $H_{\text{CS}}^1(X', \mathbb{R})$  in  $H^1(X', \mathbb{R})$  and  $\dim \mathcal{O}_{X'} = \sum_{i=1}^n \text{s-ind}(C_i)$ , where  $\text{s-ind}(C_i)$  is the stability index of Definition 6.4. There exists an open neighbourhood  $U$  of 0 in  $\mathcal{I}_{X'}$ , a smooth map  $\Phi : U \rightarrow \mathcal{O}_{X'}$  with  $\Phi(0) = 0$ , and a map  $\Xi : \{u \in U : \Phi(u) = 0\} \rightarrow \mathcal{M}_X$  with  $\Xi(0) = X$  which is a homeomorphism with an open neighbourhood of  $X$  in  $\mathcal{M}_X$ .*

If the  $C_i$  are stable then  $\mathcal{O}_{X'} = \{0\}$  and we deduce [27, Cor. 6.11]:

**Corollary 6.8** *Suppose  $(M, J, \omega, \Omega)$  is an almost Calabi–Yau  $m$ -fold and  $X$  a compact  $SL$   $m$ -fold in  $M$  with stable conical singularities, and let  $\mathcal{M}_X$  and  $\mathcal{I}_{X'}$  be as in Theorem 6.7. Then  $\mathcal{M}_X$  is a smooth manifold of dimension  $\dim \mathcal{I}_{X'}$ .*

This has clear similarities with Theorem 5.1. Here is another simple condition for  $\mathcal{M}_X$  to be a manifold near  $X$ , [27, Def. 6.12].

**Definition 6.9** Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold and  $X$  a compact  $SL$   $m$ -fold in  $M$  with conical singularities, and let  $\mathcal{I}_{X'}$ ,  $\mathcal{O}_{X'}$ ,  $U$  and  $\Phi$  be as in Theorem 6.7. We call  $X$  *transverse* if the linear map  $d\Phi|_0 : \mathcal{I}_{X'} \rightarrow \mathcal{O}_{X'}$  is surjective.

If  $X$  is transverse then  $\{u \in U : \Phi(u) = 0\}$  is a manifold near 0, so Theorem 6.7 yields [27, Cor. 6.13]:

**Corollary 6.10** *Suppose  $(M, J, \omega, \Omega)$  is an almost Calabi–Yau  $m$ -fold and  $X$  a transverse compact  $SL$   $m$ -fold in  $M$  with conical singularities, and let  $\mathcal{M}_X$ ,  $\mathcal{I}_{X'}$  and  $\mathcal{O}_{X'}$  be as in Theorem 6.7. Then  $\mathcal{M}_X$  is near  $X$  a smooth manifold of dimension  $\dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$ .*

We would like to conclude that by choosing a sufficiently generic perturbation  $\omega^s$  we can make  $\mathcal{M}_X^s$  smooth everywhere. This is the idea of the following conjecture, [27, Conj. 9.5]:

**Conjecture 6.11** *Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold,  $X$  a compact  $SL$   $m$ -fold in  $M$  with conical singularities, and  $\mathcal{I}_{X'}$ ,  $\mathcal{O}_{X'}$  be as in Theorem 6.7. Then for a second category subset of Kähler forms  $\hat{\omega}$  in the Kähler class of  $\omega$ , the moduli space  $\hat{\mathcal{M}}_X$  of compact  $SL$   $m$ -folds  $\hat{X}$  with conical singularities in  $(M, J, \hat{\omega}, \Omega)$  isotopic to  $X$  consists of transverse  $\hat{X}$ , and so is a smooth manifold of dimension  $\dim \mathcal{I}_{X'} - \dim \mathcal{O}_{X'}$ .*

For a partial proof of this, see [27, Th.s 9.1 & 9.3]. Basically, we can prove the conjecture for  $\hat{\omega}$  close to  $\omega$  and  $\hat{X} \in \hat{\mathcal{M}}_X$  close to  $X$ , or more generally, close to a fixed compact subset of the moduli space  $\mathcal{M}_X$  in  $(M, J, \omega, \Omega)$ .

### 6.3 Asymptotically Conical SL $m$ -folds

The local models for how to *desingularize* compact SL  $m$ -folds with isolated conical singularities are *Asymptotically Conical* SL  $m$ -folds  $L$  in  $\mathbb{C}^m$ , so we discuss these briefly. Here is the definition, [26, Def. 7.1].

**Definition 6.12** Let  $C$  be a closed SL cone in  $\mathbb{C}^m$  with isolated singularity at 0 for  $m > 2$ , and let  $\Sigma = C \cap \mathcal{S}^{2m-1}$ , so that  $\Sigma$  is a compact, nonsingular  $(m-1)$ -manifold, not necessarily connected. Let  $g_\Sigma$  be the metric on  $\Sigma$  induced by the metric  $g'$  on  $\mathbb{C}^m$  in (1), and  $r$  the radius function on  $\mathbb{C}^m$ . Define  $\iota : \Sigma \times (0, \infty) \rightarrow \mathbb{C}^m$  by  $\iota(\sigma, r) = r\sigma$ . Then the image of  $\iota$  is  $C \setminus \{0\}$ , and  $\iota^*(g') = r^2 g_\Sigma + dr^2$  is the cone metric on  $C \setminus \{0\}$ .

Let  $L$  be a closed, nonsingular SL  $m$ -fold in  $\mathbb{C}^m$ . We call  $L$  *Asymptotically Conical (AC)* with *rate*  $\lambda < 2$  and *cone*  $C$  if there exists a compact subset  $K \subset L$  and a diffeomorphism  $\varphi : \Sigma \times (T, \infty) \rightarrow L \setminus K$  for some  $T > 0$ , such that

$$|\nabla^k(\varphi - \iota)| = O(r^{\lambda-1-k}) \quad \text{as } r \rightarrow \infty \text{ for } k = 0, 1.$$

Here  $\nabla, |\cdot|$  are computed using the cone metric  $\iota^*(g')$ .

The deformation theory of Asymptotically Conical SL  $m$ -folds in  $\mathbb{C}^m$  has been studied independently by Pacini [37] and Marshall [34]. Pacini's results are earlier, but Marshall's are more complete.

**Definition 6.13** Suppose  $L$  is an Asymptotically Conical SL  $m$ -fold in  $\mathbb{C}^m$  with cone  $C$  and rate  $\lambda < 2$ , as in Definition 6.12. Define the *moduli space*  $\mathcal{M}_L^\lambda$  of deformations of  $L$  with rate  $\lambda$  to be the set of AC SL  $m$ -folds  $\hat{L}$  in  $\mathbb{C}^m$  with cone  $C$  and rate  $\lambda$ , such that  $\hat{L}$  is diffeomorphic to  $L$  and isotopic to  $L$  as an Asymptotically Conical submanifold of  $\mathbb{C}^m$ . One can define a natural *topology* on  $\mathcal{M}_L^\lambda$ .

The following result can be deduced from Marshall [34, Th. 6.2.15] and [34, Table 5.1]. (See also Pacini [37, Th. 2 & Th. 3].)

**Theorem 6.14** *Let  $L$  be an Asymptotically Conical SL  $m$ -fold in  $\mathbb{C}^m$  with cone  $C$  and rate  $\lambda < 2$ , and let  $\mathcal{M}_L^\lambda$  be as in Definition 6.13. Set  $\Sigma = C \cap \mathcal{S}^{2m-1}$ , and let  $\mathcal{D}_\Sigma, N_\Sigma$  be as in Section 6.1 and  $b^k(L), b_{\text{cs}}^k(L)$  be the Betti numbers in ordinary and compactly-supported de Rham cohomology  $H^k(L, \mathbb{R}), H_{\text{cs}}^k(L, \mathbb{R})$ . Then*

- (a) *If  $\lambda \in (0, 2) \setminus \mathcal{D}_\Sigma$  then  $\mathcal{M}_L^\lambda$  is a manifold with*

$$\dim \mathcal{M}_L^\lambda = b^1(L) - b^0(L) + N_\Sigma(\lambda). \quad (18)$$

*Note that if  $0 < \lambda < \min(\mathcal{D}_\Sigma \cap (0, \infty))$  then  $N_\Sigma(\lambda) = b^0(\Sigma)$ .*

- (b) *If  $\lambda \in (2 - m, 0)$  then  $\mathcal{M}_L^\lambda$  is a manifold of dimension  $b_{\text{cs}}^1(L) = b^{m-1}(L)$ .*

This is the analogue of Theorems 5.1 and 6.7 for AC SL  $m$ -folds. If  $\lambda \in (2 - m, 2) \setminus \mathcal{D}_\Sigma$  then the deformation theory for  $L$  with rate  $\lambda$  is *unobstructed* and  $\mathcal{M}_L^\lambda$  is a *smooth manifold* with a given dimension.

## 6.4 Desingularizing singular SL $m$ -folds

Suppose  $(M, J, \omega, \Omega)$  is an almost Calabi–Yau  $m$ -fold, and  $X$  a compact SL  $m$ -fold in  $M$  with conical singularities at  $x_1, \dots, x_n$  and cones  $C_1, \dots, C_n$ . In [28, 29] we study *desingularizations* of  $X$ , realizing  $X$  as a limit of a family of compact, nonsingular SL  $m$ -folds  $\tilde{N}^t$  in  $M$  for small  $t > 0$ .

Here is the basic method. Let  $L_1, \dots, L_n$  be *Asymptotically Conical* SL  $m$ -folds in  $\mathbb{C}^m$ , as in Section 6.3, with  $L_i$  asymptotic to the cone  $C_i$  at infinity. We shrink  $L_i$  by a small factor  $t > 0$ , and glue  $tL_i$  into  $X$  at  $x_i$  for  $i = 1 \dots, n$  to get a 1-parameter family of compact, nonsingular *Lagrangian*  $m$ -folds  $N^t$  in  $(M, \omega)$  for small  $t > 0$ .

Then we show using analysis that when  $t$  is sufficiently small we can deform  $N^t$  to a compact, nonsingular *special* Lagrangian  $m$ -fold  $\tilde{N}^t$  via a small Hamiltonian deformation. This  $\tilde{N}^t$  depends smoothly on  $t$ , and as  $t \rightarrow 0$  it converges to the singular SL  $m$ -fold  $X$ , in the sense of currents.

Our simplest desingularization result is [28, Th. 6.13].

**Theorem 6.15** *Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold and  $X$  a compact SL  $m$ -fold in  $M$  with conical singularities at  $x_1, \dots, x_n$  and cones  $C_1, \dots, C_n$ . Let  $L_1, \dots, L_n$  be Asymptotically Conical SL  $m$ -folds in  $\mathbb{C}^m$  with cones  $C_1, \dots, C_n$  and rates  $\lambda_1, \dots, \lambda_n$ . Suppose  $\lambda_i < 0$  for  $i = 1, \dots, n$ , and  $X' = X \setminus \{x_1, \dots, x_n\}$  is connected.*

*Then there exists  $\epsilon > 0$  and a smooth family  $\{\tilde{N}^t : t \in (0, \epsilon]\}$  of compact, nonsingular SL  $m$ -folds in  $(M, J, \omega, \Omega)$ , such that  $\tilde{N}^t$  is constructed by gluing  $tL_i$  into  $X$  at  $x_i$  for  $i = 1, \dots, n$ . In the sense of currents,  $\tilde{N}^t \rightarrow X$  as  $t \rightarrow 0$ .*

The theorem contains two *simplifying assumptions*: that  $\lambda_i < 0$  for all  $i$ , and that  $X'$  is connected. These avoid two kinds of *obstructions* to desingularizing  $X$  using the  $L_i$ . For the first, the  $L_i$  have *cohomological invariants*  $Y(L_i)$  in  $H^1(\Sigma_i, \mathbb{R})$  derived from the relative cohomology class of  $\omega'$ . If  $\lambda_i < 0$  then  $Y(L_i) = 0$ . But if  $\lambda_i \geq 0$  and  $Y(L_i) \neq 0$  then there are obstructions to the existence of  $N^t$  as a *Lagrangian*  $m$ -fold. That is, we can only define  $N^t$  if the  $Y(L_i)$  satisfy an equation.

For the second, if  $X'$  is not connected then there is an analytic obstruction to deforming  $N^t$  to  $\tilde{N}^t$ , because the Laplacian  $\Delta$  on functions on  $N^t$  has *small eigenvalues*. Again, the  $L_i$  have cohomological invariants  $Z(L_i)$  in  $H^{m-1}(\Sigma_i, \mathbb{R})$  derived from the relative cohomology class of  $\text{Im } \Omega'$ , and we can only deform  $N^t$  to  $\tilde{N}^t$  if the  $Z(L_i)$  satisfy an equation.

In the obstructed cases we prove generalizations of Theorem 6.15 showing that SL desingularizations  $\tilde{N}^t$  exist when  $Y(L_i), Z(L_i)$  satisfy equations, and also generalize the results to *families* of almost Calabi–Yau  $m$ -folds. As the details are complicated we will not give them, but we refer the reader to [28, 29] and [30, §7].

## 6.5 The index of singularities of SL $m$ -folds

We now consider the *boundary*  $\partial\mathcal{M}_N$  of a moduli space  $\mathcal{M}_N$  of SL  $m$ -folds.

**Definition 6.16** Let  $(M, J, \omega, \Omega)$  be an almost Calabi–Yau  $m$ -fold,  $N$  a compact, nonsingular SL  $m$ -fold in  $M$ , and  $\mathcal{M}_N$  the moduli space of deformations of  $N$  in  $M$ . Then  $\mathcal{M}_N$  is a smooth manifold of dimension  $b^1(N)$ , in general noncompact. We can construct a natural *compactification*  $\overline{\mathcal{M}}_N$  as follows.

Regard  $\mathcal{M}_N$  as a moduli space of special Lagrangian *integral currents* in the sense of Geometric Measure Theory, as discussed in [26, §6]. Let  $\overline{\mathcal{M}}_N$  be the closure of  $\mathcal{M}_N$  in the space of integral currents. As elements of  $\mathcal{M}_N$  have uniformly bounded volume,  $\overline{\mathcal{M}}_N$  is *compact*. Define the *boundary*  $\partial\mathcal{M}_N$  to be  $\overline{\mathcal{M}}_N \setminus \mathcal{M}_N$ . Then elements of  $\partial\mathcal{M}_N$  are *singular SL integral currents*.

In good cases, say if  $(M, J, \omega, \Omega)$  is suitably generic, it seems reasonable that  $\partial\mathcal{M}_N$  should be divided into a number of *strata*, each of which is a moduli space of singular SL  $m$ -folds with singularities of a particular type, and is itself a manifold with singularities. In particular, some or all of these strata could be moduli spaces  $\mathcal{M}_X$  of SL  $m$ -folds with isolated conical singularities, as in Section 6.2.

Let  $\mathcal{M}_N$  be a moduli space of compact, nonsingular SL  $m$ -folds  $N$  in  $M$ , and  $\mathcal{M}_X$  a moduli space of singular SL  $m$ -folds in  $\partial\mathcal{M}_N$  with singularities of a particular type, and  $X \in \mathcal{M}_X$ . Following [30, §8.3], we (loosely) define the *index* of the singularities of  $X$  to be  $\text{ind}(X) = \dim \mathcal{M}_N - \dim \mathcal{M}_X$ , provided  $\mathcal{M}_X$  is smooth near  $X$ . Note that  $\text{ind}(X)$  depends on  $N$  as well as  $X$ .

In [30, Th. 8.10] we use the results of [27, 28, 29] to compute  $\text{ind}(X)$  when  $X$  is *transverse* with conical singularities, in the sense of Definition 6.9. Here is a simplified version of the result, where we assume that  $H_{\text{cs}}^1(L_i, \mathbb{R}) \rightarrow H^1(L_i, \mathbb{R})$  is surjective to avoid a complicated correction term to  $\text{ind}(X)$  related to the obstructions to defining  $N^t$  as a Lagrangian  $m$ -fold.

**Theorem 6.17** *Let  $X$  be a compact, transverse SL  $m$ -fold in  $(M, J, \omega, \Omega)$  with conical singularities at  $x_1, \dots, x_n$  and cones  $C_1, \dots, C_n$ . Let  $L_1, \dots, L_n$  be AC SL  $m$ -folds in  $\mathbb{C}^m$  with cones  $C_1, \dots, C_n$ , such that the natural projection  $H_{\text{cs}}^1(L_i, \mathbb{R}) \rightarrow H^1(L_i, \mathbb{R})$  is surjective. Construct desingularizations  $N$  of  $X$  by gluing AC SL  $m$ -folds  $L_1, \dots, L_n$  in at  $x_1, \dots, x_n$ , as in Section 6.4. Then*

$$\text{ind}(X) = 1 - b^0(X') + \sum_{i=1}^n b_{\text{cs}}^1(L_i) + \sum_{i=1}^n \text{s-ind}(C_i). \quad (19)$$

If the cones  $C_i$  are not *rigid*, for instance if  $C_i \setminus \{0\}$  is not connected, then (19) should be corrected, as in [30, §8.3]. If Conjecture 6.11 is true then for a *generic* Kähler form  $\omega$ , *all* compact SL  $m$ -folds  $X$  with conical singularities are transverse, and so Theorem 6.17 and [30, Th. 8.10] allow us to calculate  $\text{ind}(X)$ .

Now singularities with *small index* are the most commonly occurring, and so arguably the most interesting kinds of singularity. Also, as  $\text{ind}(X) \leq \dim \mathcal{M}_N$ , for various problems it will only be necessary to know about singularities with index up to a certain value.

For example, in [17] the author proposed to define an invariant of almost Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres (which occur in 0-dimensional moduli spaces) in a given homology class, with a certain

topological weight. This invariant will only be interesting if it is essentially conserved under deformations of the underlying almost Calabi–Yau 3-fold. During such a deformation, nonsingular SL 3-folds can develop singularities and disappear, or new ones appear, which might change the invariant.

To prove the invariant is conserved, we need to show that it is unchanged along generic 1-parameter families of almost Calabi–Yau 3-folds. The only kinds of singularities of SL homology 3-spheres that arise in such families will have index 1. Thus, to resolve the conjectures in [17], we only have to know about index 1 singularities of SL 3-folds in almost Calabi–Yau 3-folds.

Another problem in which the index of singularities will be important is the *SYZ Conjecture*, to be discussed in Section 7. This has to do with dual 3-dimensional families  $\mathcal{F}, \hat{\mathcal{F}}$  of SL 3-tori in (almost) Calabi–Yau 3-folds  $M, \hat{M}$ . If  $M, \hat{M}$  are generic then the only kinds of singularities that can occur at the boundaries of  $\mathcal{F}, \hat{\mathcal{F}}$  are of index 1, 2 or 3. So, to study the SYZ Conjecture in the generic case, we only have to know about singularities of SL 3-folds with index 1, 2 and 3.

## 7 The SYZ Conjecture and SL fibrations

*Mirror Symmetry* is a mysterious relationship between pairs of Calabi–Yau 3-folds  $M, \hat{M}$ , arising from a branch of physics known as *String Theory*, and leading to some very strange and exciting conjectures about Calabi–Yau 3-folds, many of which have been proved in special cases.

The *SYZ Conjecture* is an attempt to explain Mirror Symmetry in terms of dual “fibrations”  $f : M \rightarrow B$  and  $\hat{f} : \hat{M} \rightarrow B$  of  $M, \hat{M}$  by special Lagrangian 3-folds, including singular fibers. We give brief introductions to String Theory, Mirror Symmetry, and the SYZ Conjecture, and then a short survey of the state of mathematical research into the SYZ Conjecture, biased in favor of the author’s own interests.

### 7.1 String Theory and Mirror Symmetry

String Theory is a branch of high-energy theoretical physics in which particles are modeled not as points but as 1-dimensional objects – “strings” – propagating in some background space-time  $S$ . String theorists aim to construct a *quantum theory* of the string’s motion. The process of quantization is extremely complicated, and fraught with mathematical difficulties that are as yet still poorly understood.

The most popular version of String Theory requires the universe to be 10-dimensional for this quantization process to work. Therefore, String Theorists suppose that the space we live in looks locally like  $S = \mathbb{R}^4 \times M$ , where  $\mathbb{R}^4$  is Minkowski space, and  $M$  is a compact Riemannian 6-manifold with radius of order  $10^{-33}$ cm, the Planck length. Since the Planck length is so small, space then appears to macroscopic observers to be 4-dimensional.

Because of supersymmetry,  $M$  has to be a *Calabi–Yau 3-fold*. Therefore String Theorists are very interested in Calabi–Yau 3-folds. They believe that each Calabi–Yau 3-fold  $M$  has a quantization, which is a *Super Conformal Field Theory* (SCFT), a complicated mathematical object. Invariants of  $M$  such as the Dolbeault groups  $H^{p,q}(M)$  and the number of holomorphic curves in  $M$  translate to properties of the SCFT.

However, two entirely different Calabi–Yau 3-folds  $M$  and  $\hat{M}$  may have the *same* SCFT. In this case, there are powerful relationships between the invariants of  $M$  and of  $\hat{M}$  that translate to properties of the SCFT. This is the idea behind *Mirror Symmetry* of Calabi–Yau 3-folds.

It turns out that there is a very simple automorphism of the structure of a SCFT — changing the sign of a  $U(1)$ -action — which does *not* correspond to a classical automorphism of Calabi–Yau 3-folds. We say that  $M$  and  $\hat{M}$  are *mirror* Calabi–Yau 3-folds if their SCFT’s are related by this automorphism. Then one can argue using String Theory that

$$H^{1,1}(M) \cong H^{2,1}(\hat{M}) \quad \text{and} \quad H^{2,1}(M) \cong H^{1,1}(\hat{M}).$$

Effectively, the mirror transform exchanges even- and odd-dimensional cohomology. This is a very surprising result!

More involved String Theory arguments show that, in effect, the Mirror Transform exchanges things related to the complex structure of  $M$  with things related to the symplectic structure of  $\hat{M}$ , and vice versa. Also, a generating function for the number of holomorphic rational curves in  $M$  is exchanged with a simple invariant to do with variation of complex structure on  $\hat{M}$ , and so on.

Because the quantization process is poorly understood and not at all rigorous — it involves non-convergent path-integrals over horrible infinite-dimensional spaces — String Theory generates only conjectures about Mirror Symmetry, not proofs. However, many of these conjectures have been verified in particular cases.

## 7.2 Mathematical interpretations of Mirror Symmetry

In the beginning (the 1980’s), Mirror Symmetry seemed mathematically completely mysterious. But there are now two complementary conjectural theories, due to Kontsevich and Strominger–Yau–Zaslow, which explain Mirror Symmetry in a fairly mathematical way. Probably both are true, at some level.

The first proposal was due to Kontsevich [31] in 1994. This says that for mirror Calabi–Yau 3-folds  $M$  and  $\hat{M}$ , the derived category  $D^b(M)$  of coherent sheaves on  $M$  is equivalent to the derived category  $D^b(\text{Fuk}(\hat{M}))$  of the Fukaya category of  $\hat{M}$ , and vice versa. Basically,  $D^b(M)$  has to do with  $M$  as a complex manifold, and  $D^b(\text{Fuk}(\hat{M}))$  with  $\hat{M}$  as a symplectic manifold, and its Lagrangian submanifolds. We shall not discuss this here.

The second proposal, due to Strominger, Yau and Zaslow [41] in 1996, is known as the *SYZ Conjecture*. Here is an attempt to state it.

**The SYZ Conjecture** *Suppose  $M$  and  $\hat{M}$  are mirror Calabi–Yau 3-folds. Then (under some additional conditions) there should exist a compact topological*

3-manifold  $B$  and surjective, continuous maps  $f : M \rightarrow B$  and  $\hat{f} : \hat{M} \rightarrow B$ , such that

- (i) There exists a dense open set  $B_0 \subset B$ , such that for each  $b \in B_0$ , the fibers  $f^{-1}(b)$  and  $\hat{f}^{-1}(b)$  are nonsingular special Lagrangian 3-tori  $T^3$  in  $M$  and  $\hat{M}$ . Furthermore,  $f^{-1}(b)$  and  $\hat{f}^{-1}(b)$  are in some sense dual to one another.
- (ii) For each  $b \in \Delta = B \setminus B_0$ , the fibers  $f^{-1}(b)$  and  $\hat{f}^{-1}(b)$  are expected to be singular special Lagrangian 3-folds in  $M$  and  $\hat{M}$ .

We call  $f$  and  $\hat{f}$  *special Lagrangian fibrations*, and the set of singular fibers  $\Delta$  is called the *discriminant*. In part (i), the nonsingular fibers of  $f$  and  $\hat{f}$  are supposed to be *dual tori*. What does this mean?

On the topological level, we can define duality between two tori  $T, \hat{T}$  to be a choice of isomorphism  $H^1(T, \mathbb{Z}) \cong H_1(\hat{T}, \mathbb{Z})$ . We can also define duality between tori equipped with flat Riemannian metrics. Write  $T = V/\Lambda$ , where  $V$  is a Euclidean vector space and  $\Lambda$  a *lattice* in  $V$ . Then the dual torus  $\hat{T}$  is defined to be  $V^*/\Lambda^*$ , where  $V^*$  is the dual vector space and  $\Lambda^*$  the dual lattice. However, there is no notion of duality between non-flat metrics on dual tori.

Strominger, Yau and Zaslow argue only that their conjecture holds when  $M, \hat{M}$  are close to the “large complex structure limit”. In this case, the diameters of the fibers  $f^{-1}(b), \hat{f}^{-1}(b)$  are expected to be small compared to the diameter of the base space  $B$ , and away from singularities of  $f, \hat{f}$ , the metrics on the nonsingular fibers are expected to be approximately flat.

So, part (i) of the SYZ Conjecture says that for  $b \in B \setminus B_0$ ,  $f^{-1}(b)$  is approximately a flat Riemannian 3-torus, and  $\hat{f}^{-1}(b)$  is approximately the dual flat Riemannian torus. Really, the SYZ Conjecture makes most sense as a statement about the limiting behavior of *families* of mirror Calabi–Yau 3-folds  $M_t, \hat{M}_t$  which approach the “large complex structure limit” as  $t \rightarrow 0$ .

### 7.3 The symplectic topological approach to SYZ

The most successful approach to the SYZ Conjecture so far could be described as *symplectic topological*. In this approach, we mostly forget about complex structures, and treat  $M, \hat{M}$  just as *symplectic manifolds*. We mostly forget about the ‘special’ condition, and treat  $f, \hat{f}$  just as *Lagrangian fibrations*. We also impose the condition that  $B$  is a *smooth* 3-manifold and  $f : M \rightarrow B$  and  $\hat{f} : \hat{M} \rightarrow B$  are *smooth maps*. (It is not clear that  $f, \hat{f}$  can in fact be smooth at every point, though).

Under these simplifying assumptions, Gross [6, 7, 8, 9], Ruan [38, 39], and others have built up a beautiful, detailed picture of how dual SYZ fibrations work at the global topological level, in particular for examples such as the quintic and its mirror, and for Calabi–Yau 3-folds constructed as hypersurfaces in toric 4-folds, using combinatorial data.

## 7.4 Local geometric approach, and SL singularities

There is also another approach to the SYZ Conjecture, begun by the author in [23, 25], and making use of the ideas and philosophy set out in Section 6. We could describe it as a *local geometric* approach.

In it we try to take the special Lagrangian condition seriously from the outset, and our focus is on the local behavior of special Lagrangian submanifolds, and especially their singularities, rather than on global topological questions. Also, we are interested in what fibrations of *generic* (almost) Calabi–Yau 3-folds might look like.

One of the first-fruits of this approach has been the understanding that for *generic* (almost) Calabi–Yau 3-folds  $M$ , special Lagrangian fibrations  $f : M \rightarrow B$  will not be smooth maps, but only piecewise smooth. Furthermore, their behavior at the singular set is rather different to the smooth Lagrangian fibrations discussed in Section 7.3.

For smooth special Lagrangian fibrations  $f : M \rightarrow B$ , the discriminant  $\Delta$  is of codimension 2 in  $B$ , and the typical singular fiber is singular along an  $\mathcal{S}^1$ . But in a generic special Lagrangian fibration  $f : M \rightarrow B$  the discriminant  $\Delta$  is of codimension 1 in  $B$ , and the typical singular fiber is singular at finitely many points.

One can also show that if  $M, \hat{M}$  are a mirror pair of generic (almost) Calabi–Yau 3-folds and  $f : M \rightarrow B$  and  $\hat{f} : \hat{M} \rightarrow B$  are dual special Lagrangian fibrations, then in general the discriminants  $\Delta$  of  $f$  and  $\hat{\Delta}$  of  $\hat{f}$  cannot coincide in  $B$ , because they have different topological properties in the neighborhood of a certain kind of codimension 3 singular fiber.

This contradicts part (ii) of the SYZ Conjecture, as we have stated it in Section 7.2. In the author’s view, these calculations support the idea that the SYZ Conjecture in its present form should be viewed primarily as a limiting statement, about what happens at the “large complex structure limit”, rather than as simply being about pairs of Calabi–Yau 3-folds. A similar conclusion is reached by Mark Gross in [9, §5].

## 7.5 U(1)-invariant SL fibrations in $\mathbb{C}^3$

We finish by describing work of the author in [23, §8] and [25], which aims to describe what the singularities of SL fibrations of *generic* (almost) Calabi–Yau 3-folds look like, providing they exist.

This proceeds by first studying SL fibrations of subsets of  $\mathbb{C}^3$  invariant under the U(1)-action (5), using the ideas of Section 3.5. For a brief survey of the main results, see [24]. Then we argue that the kinds of singularities we see in codimension 1 and 2 in generic U(1)-invariant SL fibrations in  $\mathbb{C}^3$ , also occur in codimension 1 and 2 in SL fibrations of generic (almost) Calabi–Yau 3-folds.

Following [23, Def. 8.1], we use the results of Section 3.5 to construct a family of SL 3-folds  $N_\alpha$  in  $\mathbb{C}^3$ , depending on boundary data  $\Phi(\alpha)$ .

**Definition 7.1** Let  $S$  be a strictly convex domain in  $\mathbb{R}^2$  invariant under  $(x, y) \mapsto (x, -y)$ , let  $U$  be an open set in  $\mathbb{R}^3$ , and  $\alpha \in (0, 1)$ . Suppose  $\Phi :$

$U \rightarrow C^{3,\alpha}(\partial S)$  is a continuous map such that if  $(a, b, c) \neq (a', b', c')$  in  $U$  then  $\Phi(a, b, c) - \Phi(a', b', c')$  has exactly one local maximum and one local minimum in  $\partial S$ .

For  $\alpha = (a, b, c) \in U$ , let  $f_\alpha \in C^{3,\alpha}(S)$  or  $C^1(S)$  be the unique (weak) solution of (9) with  $f_\alpha|_{\partial S} = \Phi(\alpha)$ , which exists by Theorem 3.3. Define

$$u_\alpha = \frac{\partial f_\alpha}{\partial y} \quad \text{and} \quad v_\alpha = \frac{\partial f_\alpha}{\partial x}.$$

Then  $(u_\alpha, v_\alpha)$  is a solution of (8) in  $C^{2,\alpha}(S)$  if  $a \neq 0$ , and a weak solution of (7) in  $C^0(S)$  if  $a = 0$ . Also  $u_\alpha, v_\alpha$  depend continuously on  $\alpha \in U$  in  $C^0(S)$ , by Theorem 3.3.

For each  $\alpha = (a, b, c)$  in  $U$ , define  $N_\alpha$  in  $\mathbb{C}^3$  by

$$N_\alpha = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 = v_\alpha(x, y) + iy, \quad z_3 = x + iu_\alpha(x, y), \right. \\ \left. |z_1|^2 - |z_2|^2 = 2a, \quad (x, y) \in S^\circ \right\}. \quad (20)$$

Then  $N_\alpha$  is a noncompact SL 3-fold without boundary in  $\mathbb{C}^3$ , which is nonsingular if  $a \neq 0$ , by Proposition 3.1.

In [23, Th. 8.2] we show that the  $N_\alpha$  are the fibers of an *SL fibration*.

**Theorem 7.2** *In the situation of Definition 7.1, if  $\alpha \neq \alpha'$  in  $U$  then  $N_\alpha \cap N_{\alpha'} = \emptyset$ . There exists an open set  $V \subset \mathbb{C}^3$  and a continuous, surjective map  $F : V \rightarrow U$  such that  $F^{-1}(\alpha) = N_\alpha$  for all  $\alpha \in U$ . Thus,  $F$  is a special Lagrangian fibration of  $V \subset \mathbb{C}^3$ , which may include singular fibers.*

It is easy to produce families  $\Phi$  satisfying Definition 7.1. For example [23, Ex. 8.3], given any  $\phi \in C^{3,\alpha}(\partial S)$  we may define  $U = \mathbb{R}^3$  and  $\Phi : \mathbb{R}^3 \rightarrow C^{3,\alpha}(\partial S)$  by  $\Phi(a, b, c) = \phi + bx + cy$ . So this construction produces very large families of  $U(1)$ -invariant SL fibrations, including singular fibers, which can have any multiplicity and type.

Here is a simple, explicit example. Define  $F : \mathbb{C}^3 \rightarrow \mathbb{R} \times \mathbb{C}$  by

$$F(z_1, z_2, z_3) = (a, b), \quad \text{where} \quad 2a = |z_1|^2 - |z_2|^2 \\ \text{and} \quad b = \begin{cases} z_3, & a = z_1 = z_2 = 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_1|, & a \geq 0, z_1 \neq 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_2|, & a < 0. \end{cases} \quad (21)$$

This is a piecewise-smooth SL fibration of  $\mathbb{C}^3$ . It is not smooth on  $|z_1| = |z_2|$ .

The fibers  $F^{-1}(a, b)$  are  $T^2$ -cones singular at  $(0, 0, b)$  when  $a = 0$ , and nonsingular  $\mathcal{S}^1 \times \mathbb{R}^2$  when  $a \neq 0$ . They are isomorphic to the SL 3-folds of Example 3.4 under transformations of  $\mathbb{C}^3$ , but they are assembled to make a fibration in a novel way.

As  $a$  goes from positive to negative the fibers undergo a surgery, a Dehn twist on  $\mathcal{S}^1$ . The reason why the fibration is only piecewise-smooth, rather than smooth, is really this topological transition, rather than the singularities

themselves. The fibration is not differentiable at every point of a singular fiber, rather than just at singular points, and this is because we are jumping from one moduli space of SL 3-folds to another at the singular fibers.

I claim that  $F$  is a local model for codimension one singularities of SL fibrations of generic almost Calabi–Yau 3-folds. The reason for this is that these  $T^2$ -cone singularities are *stable*, as in Definition 6.4, so SL 3-folds  $X$  with these singularities form *smooth moduli spaces*  $\mathcal{M}_X$  by Corollary 6.8.

The singularities are automatically *transverse*, as in Definition 6.9, so we can apply [30, Th. 8.10] to compute the *index*  $\text{ind}(X)$  of the singularities, as in Section 6.5. This is done in detail in [30, §10]. If the topology of  $X$  is suitably chosen then  $\text{ind}(X) = 1$ , so  $\mathcal{M}_X$  has codimension one in  $\mathcal{M}_N$ . The singular behavior is stable under small exact perturbations of the underlying almost Calabi–Yau structure.

I also have a  $U(1)$ -invariant model for codimension two singularities, described in [25], in which two of the codimension one  $T^2$ -cones come together and cancel out. I conjecture that it too is a typical codimension two singular behavior in SL fibrations of generic almost Calabi–Yau 3-folds. I do not expect codimension three singularities in generic SL fibrations to be locally  $U(1)$ -invariant, and so this approach will not help.

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