

A study on the random vortex methods and their related equations



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Abstract

This thesis is mainly divided into three parts. In Chapter 3, we propose a simple yet powerful random vortex method to approximate the fluid flow dynamics. The idea is that we sample the initial vortices of the fluid flow over some Brownian fluid particles and then track their vortex dynamics. The weak convergence of the approximation from our scheme is shown. We also demonstrate its effectiveness through several numerical experiments. This chapter is based on a published paper [93].

Chapter 4 establishes a closed stochastic system which is equivalent to the Navier-Stokes equations for incompressible flows. This system consists of a forward stochastic differential equation and a ordinary functional differential equation, which reveals the vortex dynamics for a viscous fluid flow. We investigate two tools to derive the system: the duality of pinned diffusion measure and a forward type Feynman-Kac formula for nonlinear parabolic equations. We also conduct numerical simulations based on the system. This chapter is based on a published paper [94].

In Chapter 5, the random vortex system proposed in Chapter 4 is generalized to a class of stochastic differential equations of the random vortex type (RVSDEs). We investigate various problems on RVSDEs, such as existence and uniqueness theories, large deviation principles, and numerical methods, under certain nice assumptions.

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Notations

- $C(\mathbb{R}^d; \mathbb{R}^k)$ is the space of continuous functions from $\mathbb{R}^d$ to $\mathbb{R}^k$.
- $C^n(\mathbb{R}^d; \mathbb{R}^k)$ is the space of n -times differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^k$.
- $C_0^n(\mathbb{R}^d; \mathbb{R}^k)$ is the space of n -times differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^k$ with compact support.
- $C^\infty(\mathbb{R}^d; \mathbb{R}^k)$ is the space of smooth functions from $\mathbb{R}^d$ to $\mathbb{R}^k$.
- $C_0^\infty(\mathbb{R}^d; \mathbb{R}^k)$ is the space of smooth functions from $\mathbb{R}^d$ to $\mathbb{R}^k$ with compact support.
- $L^p(\mathbb{R}^d)$ ($1 \leq p < \infty$) is the space of measurable functions on $\mathbb{R}^d$ with finite L^p -norms under the Lebesgue measure:

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^d} |f|^p dx \right)^{1/p} < \infty$$

$L_{loc}^p(\mathbb{R}^d)$ ($1 \leq p < \infty$) is the space of locally L^p -integrable functions on $\mathbb{R}^d$, i.e. functions with finite L^p -norms on every compact set in $\mathbb{R}^d$.

$L^\infty(\mathbb{R}^d)$ is the space of measurable functions on $\mathbb{R}^d$ with finite L^∞ -norms:

$$\|f\|_{L^\infty} := \inf \{C \geq 0 : |f(x)| \leq C \text{ with almost everywhere } x\}$$

- δ_j^i :

$$\delta_j^i = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}$$

with $i, j = 1, 2, \dots, d$.

- $\delta(x)$ is the Dirac- δ function.
- I_d is a $d \times d$ identity matrix.
- $C(K_1, K_2, \dots, K_n)$ denotes a constant depending only on $K_1, K_2, \dots, K_n$.
- Let x be a vector in $\mathbb{R}^d$, then x^i is the i -th component of x , i.e.

$$x = (x^1, x^2, \dots, x^d).$$

Similarly, suppose that X is a vector field in $\mathbb{R}^d$, then X^i is the i -th component of X :

$$X = (X^1, X^2, \dots, X^d).$$

- Let f be a function on $\mathbb{R}^d$. $\partial_{x^i} f$ denotes the distributional derivative of f with respect to x^i ($i = 1, 2, \dots, d$). If f is differentiable, $\partial_{x^i} f$ coincides with the classical derivative of $f : \frac{\partial f}{\partial x^i}$.
- $\times$ denotes the cross product on $\mathbb{R}^3$. ϵ^{ijk} ($i, j, k \in \{1, 2, 3\}$) is a constant such that for all $x, y \in \mathbb{R}^3$:

$$(x \times y)^i = \sum_{j=1}^3 \sum_{k=1}^3 \epsilon^{ijk} x^j y^k$$

- $\otimes$ is the outer product between two vectors.

Chapter 1

Introduction

1.1 Random vortex methods

1.1.1 The Navier-Stokes equations and random vortex methods

1.1.1.1 The Navier-Stokes equations

The Navier-Stokes equations for an incompressible flow are non-linear partial differential equations (PDEs) of second order

$$\begin{aligned}\frac{\partial}{\partial t}u + u \cdot \nabla u &= \nu \Delta u + \nabla p, \\ \nabla \cdot u &= 0,\end{aligned}\tag{1.1.1}$$

where $\nu \geq 0$ is the kinematic constant which represents the viscosity coefficient of the fluid flow, which can be regarded as the reciprocal of the Reynolds number Re . (Although a more precise definition of Re is $Re = \frac{UL}{\nu}$, where U and L are the characteristic speed and length of the flow respectively, we adopt the convention used in [73] in this thesis.) When $\nu > 0$, the flow is *viscous*. When $\nu = 0$, the flow is *inviscid*, (Helium is the only fluid known to exhibit zero viscosity.) and (1.1.1) is thus referred as *Euler equations*. In (1.1.1), $u(x, t)$ is the velocity field of the flow,

and $p(x, t)$ is a scalar function which represents the pressure. The first equation of (1.1.1) follows from the conservation of momentum for a continuum, and the second equation follows from the incompressibility of the fluid.

The Navier-Stokes equations are widely regarded as one of the most important types of PDEs. The Navier-Stokes existence and smoothness problem is one of the Millennium Prize Problems by the Clay Mathematics Institute. In the two-dimensional (2D) case, the global existence and uniqueness of both strong solutions and weak solutions hold for the solutions [75]. In the three-dimensional (3D) case, while weak solutions exist globally, as demonstrated by Leray’s pioneering work [64], the existence of strong solutions is usually only guaranteed on a certain time interval, and uniqueness is ensured on the same interval. It is uncertain whether strong solutions exist for all time given some generic initial conditions. For an overview of more recent advancements in the research of the existence and regularity problem of Navier Stokes equations, one may refer to [43, 99].

In addition to their significant role in pure mathematics, the Navier-Stokes equations are also of great importance in various physical and engineering applications. For example, they can be utilised to model ocean currents [76, 108] and the air flow around wings [111]. Therefore, developing algorithms to solve the Navier-Stokes equations is crucial. Due to the inherent difficulty in finding analytical solutions to the Navier-Stokes equations with generic initial data, numerical schemes must be developed. In our thesis, we focus primarily on approximating the flows that occupy the entire space ($\mathbb{R}^2$ or $\mathbb{R}^3$), although fluid flows of interest in real-world applications are often confined to bounded regions. This is due to the consideration of mathematical simplicity — for fluid flows in bounded domains, (1.1.1) is often supplied with some boundary conditions, which makes it even more complicated and challenging to solve, and the approach to find the solutions may vary a lot depending on the specific region. Nevertheless, we would like to point out that some tools developed in this thesis remain useful for some bounded flows [92], and can be further

extended to heat-conducting flows [66].

1.1.1.2 Numerical methods for solving the Navier-Stokes equations

The numerical approximations to the Navier-Stokes equations have been extensively studied for many years. Direct numerical simulation (DNS) is a standard way in simulating the flows. Various classical numerical methods have been utilized for DNS, including finite differences [20, 21, 23, 50], finite elements [44], finite volumes [65] and pseudo-spectral methods [15, 86], but the need for very fine meshes to capture rapid energy transfers in turbulent flows makes these methods impractical.

To avoid this problem, researchers have developed alternative methods that calculate average quantities of fluid solutions instead of directly approximating the solution to the Navier-Stokes equations. As a matter of fact, the average quantities are of more interest in many engineering applications. One such approach is the Reynolds-averaged Navier-Stokes (RANS) equations, which applies an averaging operation to the Navier-Stokes equations. The RANS approach was introduced by Osborne Reynolds [52, 97, 98] and has been a subject of extensive research [3, 19, 70]. However, the intrinsic nature of RANS approach makes it difficult to predict the transient behavior of the flow.

To overcome this limitation, an alternative approach, large-eddy simulation (LES), was proposed for achieving a balance between accuracy and computational cost. LES, initially proposed by [103], models the large scale motions directly while filtering out the small scale motions, thereby reducing the computational cost compared with classical methods while capturing the key information about the flow dynamics. In recent years, LES has become a popular tool for simulating flows in different practical scenarios, such as atmosphere modeling [104], combustors [42]. A comprehensive overview of LES can be found in [102], while a more recent review can be found in [114]. Despite the success of LES, the underlying sub-grid scale (SGS) models can impede its performance, calling for the development of more accurate SGS models.

Moreover, as demonstrated in [18], vortex methods, which are the primary numerical methods we will focus in this thesis, have more potential dealing with flows with high Reynolds number compared to LES.

More recently, machine learning techniques have been employed to model fluid flows [110]. For instance, DNS for Navier-Stokes equations have been accelerated by neural networks [54, 58]. Machine learning also helps improve the performance of RANS turbulence modeling [68, 112, 113] and LES [11, 62]. Although the applications of machine learning techniques look promising in modeling fluid dynamics, they require a large amount of data and are expensive to train.

1.1.1.3 Vortex methods

This thesis focuses on the vortex methods as an attractive numerical scheme for simulating fluid flows based on the vorticity transport equation. As a promising approach, vortex methods have received significant attention from the scientific community. For a comprehensive overview of the vortex methods, one may refer to the relevant monographs including [27, 73]. Additionally, a more recent review can be found in [79].

Unlike Eulerian numerical methods, vortex methods approximate the solution based on the dynamics of a set of particles which move with respect to the flow velocity, rather than on a fixed lattice. To understand the key idea of vortex methods, we can consider the following vorticity transport equation in $\mathbb{R}^3 \times [0, T]$:

$$\frac{\partial}{\partial t}\omega + u \cdot \nabla \omega = \nu \Delta \omega + \omega \cdot \nabla u. \quad (1.1.2)$$

Here, $\omega = \text{curl} u$ is the vorticity field. The vorticity transport equations (1.1.2) are obtained by taking the curl of the Navier-Stokes equations (1.1.1). The LHS of (1.1.2) represents the material derivative of the vorticity ω , so (1.1.2) provides a description of how the vorticity of a particle changes within a flow.

In essence, vortex methods aim to approximate the locations and local vorticity

of a set of particles based on the vorticity transport equation (1.1.2). The vorticity-stream formulation of incompressible flows (1.1.2) also highlights a crucial difference between 2D and 3D flows. For 2D fluid flows, the vortex stretching term $\omega \cdot \nabla u$ vanishes, so that (1.1.2) reduces to

$$\frac{\partial}{\partial t}\omega + u \cdot \nabla \omega = \nu \Delta \omega. \quad (1.1.3)$$

As expected, the investigation of vortex methods for 2D flows is a much more straight-forward task.

The origins of vortex methods can be traced back to the work of Rosenhead in 1932 [101]. However, a more modern formulation was introduced by Chorin in [22], where they proposed a numerical method to solve the Navier-Stokes equations in 2D spaces. They discretized the vorticity into a sum of finite number of vortex blobs and derived the motion dynamics for the vortex blobs. Theoretical studies of vortex methods are more complete for inviscid flows due to the absence of the diffusion term in (1.1.2). Several convergence results have been established for Chorin's numerical scheme and some other similar schemes for inviscid 2D flows, as shown in [46, 47, 48]. In contrast to 2D flows, where the vorticity remains constant as the particles move, the vorticity in 3D flows undergoes changes. To address the issue of vortex stretching, several methods have been proposed. For example, convergence with Lagrangian stretching was proved in [8]. A different approach, which updates the vorticity through the velocity field, was proposed, and the corresponding convergence result was shown in [6]. Furthermore, in [4], the convergence analysis of the vortex methods with respect to the temporal discretisation was conducted.

For viscous fluid flows, generally speaking, there are random vortex methods and several deterministic vortex schemes. This thesis focuses on the random vortex methods, which were also proposed by Chorin in [22] for approximating 2D flows. Essentially, the motion of the vortex blobs is now replaced by a random dynamical system in which a Brownian motion term is added to the equation of motion of

the fluid particles, and the fluid particles become Brownian fluid particles. The convergence results for 2D random vortex methods have been successively proved by [45, 69, 74]. However, for 3D viscous fluid flows, to the best of our knowledge, there is no satisfactory solution yet so far.

For reference, we give an overview of some representative deterministic vortex schemes here. The particle strength exchange method (PSE) [28, 30] approximates the Laplace operator in the vorticity equation with an integral operator, and incorporates the diffusion of vorticity directly into the dynamics of point vortices. In the core-spreading method (CSM) [60, 63], the volume of the vortex blobs expands in time at a rate determined by viscosity. The diffusion velocity method (DVM) was investigated in several studies [10, 39, 83]. In DVM, particles are no longer solely driven by the advection velocity, i.e., the underlying flow velocity. Rather, an artificial velocity field, known as the diffusion velocity, is incorporated into the particle dynamics.

Some hybrid vortex methods were also proposed, which combine the Eulerian schemes focusing on the grids and the vortex methods focusing on the particles. In [26], Cottet introduced a particle-grid superposition method, which employs a domain decomposition strategy to utilize both grid methods and vortex methods in different parts of the computation domains. More recently, the vortex particle-mesh method (VPM) has gained popularity, as seen in studies such as [16, 78, 88]. In VPM, when updating the locations and local vorticity of the particles, the advection of the particles is based on the point vortex methods, while the diffusion of vorticity is based on grid calculations. Another hybrid method, proposed by Kornev [59], combines LES with vortex methods, where the underlying SGS models are based on vortex simulations.

While these deterministic methods are useful for numerically solving the 3D Navier-Stokes equations under specific circumstances, they do not reveal the underlying vortex dynamics. Our goal is extend Chorin's random vortex method [22] to 3D flows.

To obtain an exact stochastic representation of the 3D vortex dynamics, various attempts have been tried. One example is in [25], where a stochastic Lagrangian representation of the solution to the Navier-Stokes equations is obtained using stochastic flows and Leray's projection. Other studies, such as [84, 115, 116], represent solutions as stochastic flows and investigate existence and uniqueness results based on these representations. Different approaches were also investigated. For example, in [2, 14], the solution was represented by a McKean-Vlasov type stochastic differential equation, while a time-reversal in the coefficients was involved. More complex representations have been proposed, such as expressing solutions as functionals of a Markov process indexed by the nodes of a binary tree [87]. Although these are all possible stochastic representations of the viscous fluid flows, these representations are too complicated, and it is difficult to derive an efficient numerical algorithm from these representations.

Therefore, we are going to focus on two important problems regarding 3D viscous flows:

1. How to efficiently simulate 3D viscous fluid flows based on random vortex methods?
2. How to better formulate the 3D random vortex dynamics system?

1.1.2 Main contribution

Our work is based on two published papers [93, 94], which is presented in Chapter 3 and Chapter 4 of this thesis. In Chapter 3, we propose a simple random vortex method, which could be used to approximate the dynamics of an incompressible 3D viscous fluid flow. In Chapter 4, an exact 3D random vortex dynamics system is derived and can be utilized for Monte-Carlo simulations. We would like to point out that while we have empirically verified that our methods lead to good approximations to the Navier-Stokes equations through numerical experiments, a rigorous theoretical proof of the approximation is beyond the scope of this thesis.

The idea of the random vortex method proposed in Chapter 3 can be summarized as follows. First, we sample the distribution of the initial vortices of the fluid flow using a smooth wavelet function, denoted by φ , as a smooth approximation of the Dirac- δ function:

$$W(x, 0) \approx \sum_{n=1}^N A_n \varphi(x - x_n),$$

where N is the total number of vortices, and (x_n, A_n) are their initial positions and strengths.

Next, we consider stochastic processes $(X_n(t), A_n(t))_{n=1}^N$ which satisfy the initial condition $X_n(0) = x_n, A_n(0) = A_n$. Here, $X_n(t)$ represents the trajectory of the particle starting at x_n , and $A_n(t)$ is the vorticity that the particle $X_n(t)$ carries at time t . The distribution of vortices at time $t > 0$ is then be updated as

$$W(x, t) = \sum_n A_n(t) \varphi(x - X_n(t)). \quad (1.1.4)$$

Our observation is as follows. By applying the Biot-Savart law, the velocity field can be determined by (1.1.4), which allows us to derive the dynamics of $(X_n(t), A_n(t))$. The main contribution of Chapter 3 is to demonstrate that this method gives rise to good approximations by showing the convergence in some weak sense and presenting simulation results.

Although this numerical scheme seems to be powerful, we have yet to establish an exact relationship between our numerical system and the vorticity transport equation. Our ultimate goal is to propose a probabilistic formulation of the Navier-Stokes equation which can aid in developing the Monte-Carlo simulations for fluid flows.

To achieve this goal, in Chapter 4, a novel, closed 3D random vortex dynamics system which is equivalent to the Navier-Stokes equations for incompressible viscous fluid flows will be presented. This work is based on a published paper [94]. The system consists of a forward stochastic differential equation (SDE) coupled with a backward functional ordinary differential equation (ODE):

Theorem 1.1.1. *Suppose that u and ω decay sufficiently fast as $|x| \rightarrow \infty$ and are sufficiently regular, and they satisfy the vorticity transport equation (1.1.2) on $[0, T]$. Let $X(\xi, t), G(\xi, t, s)$ be the solution to the following system:*

$$\begin{aligned} dX(\xi, t) &= u(X(\xi, t), t)dt + \sqrt{2\nu}dB(t), X(\xi, 0) = \xi \in \mathbb{R}^3 \\ \frac{d}{ds}G(\xi, t, s) &= -G(\xi, t, s)S(X(\xi, s), s), G(\xi, t, t) = I_3 \in \mathbb{R}^{3 \times 3} \end{aligned} \quad (1.1.5)$$

where $S = \frac{1}{2}(\nabla u + \nabla u^T), \xi \in \mathbb{R}^3, X(\xi, t) \in \mathbb{R}^3, G(\xi, t, s) \in \mathbb{R}^{3 \times 3}, 0 \leq s \leq t \leq T$.

Then

$$u^i(x, t) = \int_{\mathbb{R}^3} \mathbb{E} [\epsilon^{ilk} K^l(x - X(\xi, t)) G_j^k(\xi, t, 0) \omega_0^j(\xi)] d\xi, \quad (1.1.6)$$

and

$$S_j^i(x, t) = \text{PV} \int_{\mathbb{R}^3} \mathbb{E} [H_{j,k}^i(x - X(\xi, t)) G_l^k(\xi, t, 0) \omega_0^l(\xi)] d\xi, \quad (1.1.7)$$

where

$$K(x) = -\frac{1}{4\pi} \frac{x}{|x|^3}, \quad H_{j,k}^i(z) = \frac{3}{2} \frac{z^l}{4\pi|z|^5} (\epsilon^{ilk} z^j + \epsilon^{jlk} z^i).$$

and PV is the Cauchy principal value defined in (4.5.7).

It is worth noticing that given the initial vorticity field ω_0 , (1.1.5), (1.1.6), (1.1.7) form a closed system of (X, G) , as the coefficients u and S in (1.1.5) depend on the distribution of the solution (X, G) . This implies that we can calculate the solution to (1.1.5) and thereby obtain the solution to the vorticity transport equation (1.1.2). Consequently, Monte-Carlo methods can be employed to approximate the Navier-Stokes equations.

In order to derive the system (1.1.5), we mainly investigate two tools, which help us overcome the difficulty in dealing with the time-reversal problem as we aim to run a forward Taylor diffusion to represent the solution to a forward PDE. The first one is the duality of the pinned diffusion measures, and the second one is a forward-type Feynman–Kac formula. These tools both leverage the fact that the velocity field of an incompressible flow is divergence-free, so that we are able to represent the distribution of diffusion driven by a time-reversed velocity field by a diffusion that

does not involve time-reversal. These tools will be discussed in detail in Chapter 4, and it is our belief that they are quite general and could potentially be applied to some other non-linear PDEs.

We give a brief explanation of these tools here. Let us introduce the canonical paths space Ω , which consists of all continuous paths on $\mathbb{R}^d$ in $[0, T]$. The pinned diffusion measure is the probability measure on Ω induced by a diffusion conditional on the initial and terminal conditions. For a time-varying vector field $b(x, t)$ on $\mathbb{R}^d$, $\mathbb{P}_b^{\xi, 0 \rightarrow \eta, T}$ on Ω is the distribution of the L_b -diffusion process X conditional on $X_0 = \xi, X_T = \eta$. The following theorem states the duality of pinned diffusion measures.

Theorem 1.1.2. (Duality of pinned diffusion measures) *Let $b(x, t)$ be a bounded, Borel measurable and divergence-free vector field in $\mathbb{R}^d \times [0, T]$ and $b^T(x, t) = b(x, T-t)$, then the associated pinned measures satisfy the following duality relation:*

$$\mathbb{P}_b^{\xi, 0 \rightarrow \eta, T} = \mathbb{P}_{-b^T}^{\eta, 0 \rightarrow \xi, T} \circ \tau_T, \quad (1.1.8)$$

where τ_T is the time-reversal operator at T , that is, $\tau_T : \Omega \rightarrow \Omega$ is the mapping which sends $w \in \Omega$ to $\tau_T w \in \Omega$, where $\tau_T w(t) = w(T-t)$ for $t \in [0, T]$.

Based on the duality theorem, we are able to establish the following Feynman–Kac formula, which gives a probabilistic representation of the solution to a forward parabolic PDE.

Theorem 1.1.3. (A forward Feynman–Kac’s formula) *Let $T > 0$. Suppose the vector field $f(x, t)$ is a strong solution to the following forward parabolic equation:*

$$\left(\frac{\partial}{\partial t} - L_{-b(x, t)} \right) f^i(x, t) = S_j^i(x, t) f^j(x, t) \quad (1.1.9)$$

where $L_{-b(x, t)} = -b(x, t) \cdot \nabla + \nu \Delta$, $(x, t) \in \mathbb{R}^d \times (0, T]$ and $i = 1, 2, \dots, d$. $f(x, t)$ is subject to the initial condition $f(x, 0) = f_0$, and S_j^i are globally Lipschitz continuous

for $i, j = 1, \dots, d$. Assume that $b(x, t)$ is a time dependent divergence-free vector field on $\mathbb{R}^d$, then

$$f(x, T) = \int_{\mathbb{R}^d} f_0(\xi) p_b(0, \xi, T, x) \mathbb{E}[Q(0)|X(0) = \xi, X(T) = x] d\xi \quad (1.1.10)$$

where

$$dX^i(t) = b^i(X(t), t) dt + \sqrt{2\nu} dB^i(t)$$

and

$$\frac{d}{dt} Q_j^i(t) = -Q_k^i(t) S_j^k(X(t), t), \quad Q_j^i(T) = \delta_j^i$$

for $i, j = 1, \dots, d$, where B is a standard d -dimensional Brownian motion on a probability space, p_b is the transition probability density function of X .

The property that b is a divergence-free field is essential for deriving the forward Feynman–Kac’s formula (1.1.10). If we apply the standard Feynman–Kac’s formula, where the forward parabolic PDE (1.1.9) must be solved via a time reversal of the PDE, the solution will be represented by a diffusion driven by $b^T(x, t) := b(x, T - t)$. Instead, (1.1.10) enables us to represent the solution via a L_b diffusion directly. This is especially important in modelling fluid flows — the dynamics of the fluid flows in the past is the only information available to us for the prediction of the motion of the flows in the future.

An implementable Monte-Carlo numerical scheme to approximate the 3D viscous fluid flows is developed based on the space discretisation of the system (1.1.10). In this scheme, we consider the dynamics of X and G starting from a lattice in $\mathbb{R}^3$, and the velocity field (1.1.6) is approximated by the Riemann sum of these discrete particles. Although a theoretical proof of the convergence of our numerical scheme is beyond the scope of the thesis, we will justify our scheme by comparing the simulation results with some analytical solutions to the Navier-Stokes equations and results from some other numerical schemes.

1.2 Random vortex type stochastic differential equations

1.2.1 Introducing random vortex type stochastic differential equations

The exact random vortex system (1.1.5) consists of a forward SDE and a backward functional ODE, and the coefficients in this system depend on the distribution of the solution. Indeed, by applying the same approach, we could derive stochastic representations of other non-linear PDEs in a similar form. We would like to study the properties of such stochastic representations. To this end, we generalize the random vortex system (1.1.5) to a class of *random vortex type stochastic differential equations* (RVSDs) and some problems on RVSDs will be investigated.

The RVSDs are the system of equations of the following type:

$$\begin{aligned} dX_t &= b(X_t, t, \mathcal{L}(X_t, G_{t,0}))dt + \sigma(X_t, t, \mathcal{L}(X_t, G_{t,0}))dB_t, \quad X_0 = \xi \in \mathbb{R}^d, \\ dG_{t,s} &= -S(G_{t,s}, X_s, s, \mathcal{L}(X_s, G_{s,0}))ds, \quad G_{t,t} = \zeta_t \in \mathbb{R}^l, \end{aligned} \quad (1.2.1)$$

where $X_t \in \mathbb{R}^d, G_{t,s} \in \mathbb{R}^l, 0 \leq s \leq t \leq T$, B_t is an r -dimensional $(\mathcal{F}_t)_{t \geq 0}$ Brownian motion, b, σ and S are Borel-measurable functions of proper dimensions.

The system (1.2.1) could be regarded as a generalization of the random vortex system (1.1.5) if the initial vorticity field is distributed at some discrete points $x_1, x_2, \dots, x_n$: $\omega_0(x) = \sum_k \omega_k \delta(x - x_k)$, so that the random vortex system (1.1.5) becomes the following finite-dimensional system:

$$\begin{aligned} dX_t(k) &= b_{(X,G)}(X_t(k), t)dt + \sqrt{2\nu}dB_t, \quad X_0(k) = x_k, \\ \frac{d}{ds}G_{t,s}(k) &= -G_{t,s}(k)S_{(X,G)}(X_s(k), s), \quad G(k)_{t,t} = I_3 \end{aligned} \quad (1.2.2)$$

where

$$b_{(X,G)}(x, t) = \sum_k \mathbb{E} \left[K(x - X_t(k)) \times [G(k)_{t,0} \omega_k] \right],$$

and

$$S_{(X,G)}^h(x, t) = \sum_k \mathbb{E} \left[H(x - X_t(k)) [G(k)_{t,0} \omega_k] \right].$$

We may concatenate $X_t(k), G_{t,s}(k)$ over all k together into two high dimensional vectors $X_t, G_{t,s}$. The vectors $X_t, G_{t,s}$ now belong to some finite-dimensional spaces $\mathbb{R}^d = \mathbb{R}^{3n}$ and $\mathbb{R}^l = \mathbb{R}^{3 \times 3 \times n}$ respectively. Therefore, the equations (1.2.2) can be regarded as a special case of (1.2.1).

If the initial vorticity is not distributed at some discrete points, the random vortex system (1.1.5) is an infinite-dimensional system, where $X_t, G_{t,s}$ are defined on every $\xi \in \mathbb{R}^3$. The space discretisation (see Section 4.6) reduces (1.1.5) to the following finite-dimensional system:

$$\begin{aligned} dX_t(k) &= b_{(X,G)}^h(X_t(k), t)dt + \sqrt{2\nu}dW_t, \quad X_0(k) = x_k, \\ \frac{d}{ds}G_{t,s}(k) &= -G_{t,s}(k)S_{(X,G)}^h(X_s(k), s), \quad G(k)_{t,t} = I_3, \end{aligned} \tag{1.2.3}$$

where

$$b_{(X,G)}^h(x, t) = \sum_k \mathbb{E} \left[K(x - X(k)_t) \times [G(k)_{t,0} \omega_0(k)] \right] h^3,$$

and

$$S_{(X,G)}^h(x, t) = \sum_k \mathbb{E} \left[H(x - X(k)_t) [G(k)_{t,0} \omega_0(k)] \right] h^3.$$

$h > 0$ is the mesh size of the discretisation and k runs over some finite index set. Similar to (1.2.2), the system (1.2.3) also has the same structure as the RVSDE (1.2.1).

1.2.2 Main contribution

In Chapter 5 of our thesis, we focus on investigating various aspects of RVSDEs (1.2.1), including the existence and uniqueness of solutions, propagation of chaos,

large deviation principles, and numerical approximations. Additionally, we will study the infinite-dimensional RVSEs (see (1.2.5)) and demonstrate that they can be approximated by the finite-dimensional RVSEs system.

Recall that a standard McKean-Vlasov SDE (MVSDE) reads as

$$dX_t = b(X_t, t, \mathcal{L}(X_t))dt + \sigma(X_t, t, \mathcal{L}(X_t))dB_t, \quad X_0 = \xi \in \mathbb{R}^d \quad (1.2.4)$$

In RVSEs, similar to MVSDEs, the functions b , σ , and S depend on the joint distribution of the solution $(X_t, G_{t,0})$. However, since a SDE is coupled with another functional ordinary differential equation, the RVSEs are more complicated than MVSDEs and require a more complex analysis. Despite this complexity, most results in RVSEs are similar to those in MVSDEs.

In our thesis, **we shall always assume that the global Lipschitz and linear growth conditions hold for b, σ and S in (1.2.1)**. The study on the solution with a weaker assumption is beyond the scope of the thesis. In this overview, we summarize our results and provide related literature for reference. For a brief description of the techniques we employ to obtain our findings and accurate statements of our theorems, we refer readers to the introduction in Chapter 5. Our study is divided into the following parts.

Existence, uniqueness, stability and propagation of chaos of the solutions to RVSEs

The solution to the RVSE will be properly defined and the existence and uniqueness results of such solutions will be presented. Besides, we demonstrate the stability of the solutions with respect to the coefficients. The result is similar to that in MVSDEs (1.2.4). It has been established in the literature that when $b(x, t, \mu)$ and $\sigma(x, t, \mu)$ are Lipschitz and of linear growth with respect to x and the $\mathbb{R}^d$ probability measure μ , the MVSDE admits a unique strong solution. Some existence and uniqueness results for the McKean-Vlasov SDE under weaker assumptions on b and σ are investigated

in [41, 61, 80], while it is not easy to generalize the result to RVSDEs.

Furthermore, we investigate the propagation of chaos for RVSDEs — the solutions to the RVSDEs could be regarded as a limit of interacting particles, which justifies the procedure that we run independent copies to conduct the direct numerical scheme in simulating the fluid flows (see Section 4.6). Indeed, for the MVSDEs (1.2.4), the propagation of chaos has been well studied in previous literature [41, 61, 77, 106]. In this thesis, a similar result will be established for RVSDEs.

Large deviation principle for RVSDEs

The large deviation principle (LDP) will be established for RVSDEs. Since the RVSDEs arise from the vorticity equation, we believe that it may have implications for the limiting behavior of the Navier-Stokes equation as the viscosity constant $\nu \rightarrow 0$, as well as other non-linear PDEs in which the coefficients of Laplacian converge to 0. For MVSDEs (1.2.4), several LDP results for the MVSDEs have been attained. For example, in [29, 91], the authors established LDP results by exploring the limit of N -particle systems. A different approach was presented in [96], where arguments concerning the convergence of empirical measures are avoided. We will generalize the method in [96] to establish the LDP for RVSDEs.

Euler-Maruyama type numerical scheme of RVSDEs

In order to numerically approximate the solution, an Euler-Maruyama (EM) type numerical scheme will be proposed and its strong convergence will be proved. For MVSDEs, a standard way to approximate the solution is to first introduce a reference SDE via interacting particle systems and then apply the EM algorithm to this SDE. We mention a few of the results of the EM algorithms on MVSDEs for reference. In [13], an Euler scheme was established to approximate a specific MVSDE arising from the biological interactions. The convergence of EM algorithm for MVSDEs under weaker assumptions of the coefficients was investigated in [67, 95]. In [33], without

utilizing the empirical measure of interacting particles, they proposed an EM scheme to approximate the MVSDE under non-Lipschitz conditions directly. With the aid of the propagation of chaos results, the strong convergence of the EM scheme for RVSEs will be obtained in this thesis.

Infinite-dimensional RVSEs and the approximations by finite-dimensional RVSEs

We study the following infinite-dimensional RVSEs:

$$\begin{aligned} dX_t(\xi) &= b(X_t(\xi), t)dt + \sigma(X_t(\xi), t)dB_t, \\ dG_{t,s}(\xi) &= -S(G_{t,s}(\xi), X_s(\xi), s)ds, \end{aligned} \tag{1.2.5}$$

where $\xi \in \Omega$, and Ω is a compact set in some Euclidean space $\mathbb{R}^\rho$. $X_t(\xi) \in \mathbb{R}^d$, $G_{t,s}(\xi) \in \mathbb{R}^l$, $0 \leq s \leq t \leq T$, B is a standard r -dimensional Brownian motion, and b, σ, S depend on the joint distribution of (X, G) by the following equations:

$$\begin{aligned} b(x, t) &= \int_{\Omega} \mathbb{E} [b'(x, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi, \\ \sigma(x, t) &= \int_{\Omega} \mathbb{E} [\sigma'(x, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi, \\ S(g, x, t) &= \int_{\Omega} \mathbb{E} [S'(x, g, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi, \end{aligned} \tag{1.2.6}$$

where b', σ', S' are Borel-measurable functions of proper dimensions. The initial data is given by $X_0(\xi)$ and $G_{t,t}(\xi)$ for $t \in [0, T]$, $\xi \in \Omega$.

Under the assumption of the global Lipschitz and linear growth on b', σ', S' , the existence and uniqueness of the solutions to the infinite-dimensional RVSEs (1.2.5) will be shown. We will also show how to numerically approximate (1.2.5) by finite-dimensional RVSEs via space discretisation.

The infinite-dimensional RVSEs system (1.2.5) is a direct generalization of the original random vortex dynamics. Comparably, the finite-dimensional RVSEs (1.2.1) are a generalization of the discretisation of the exact random vortex dynamics

system. The space discretisation reduces an original infinite-dimensional random vortex system into a finite-dimensional system. In the original form of random vortex dynamics, compared with (1.2.1), X_t and $G_{t,s}$ are actually defined on every $\xi \in \mathbb{R}^3$, or more specifically, on a compact set in $\mathbb{R}^3$ if the vorticity is compactly supported, which motivates us to study (1.2.5).

Recall that when approximating the 3D viscous fluid flows, we first discretise the random vortex system and then consider the particles starting from a lattice in $\mathbb{R}^3$, so that the approximated dynamics of the fluid flow is determined by these particles. In this section, we aim to demonstrate that this discretisation provides a reliable approximation. We acknowledge that proving convergence from the discretisation for the exact random vortex dynamics is difficult due to the high-order singularity in the system. However, for general infinite-dimensional RVSDEs with nice assumptions of the coefficients, we are able to prove convergence.

Chapter 2

Preliminaries

2.1 The vorticity-stream formulation of the Navier-Stokes equations

In this section, we derive the vorticity-stream formulation for the Navier-Stokes equations. First, we present some results on singular integrals and distributional derivatives in Section 2.1.1. These results will serve as a useful tool when we introduce the vorticity-stream formulation in Section 2.1.2.

2.1.1 Singular integrals and distributional derivatives

Let K be a function in $L^1_{\text{loc}}(\mathbb{R}^n)$ which could be singular at 0 and smooth outside. The distributional derivative ∂_{x^j} of K is defined as the linear functional $\partial_{x^j} K$ which satisfies

$$(\partial_{x^j} K, \varphi) = -(K, \partial_{x^j} \varphi), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n) \quad (2.1.1)$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R}^n)$.

Let us compute the distributional derivative of K . By Green's formula,

$$\begin{aligned} -(K, \partial_{x^j} \varphi) &= -\lim_{\epsilon \downarrow 0} \int_{|x| \geq \epsilon} K \partial_{x^j} \varphi dx \\ &= \lim_{\epsilon \downarrow 0} \left\{ \int_{|x| \geq \epsilon} \partial_{x^j} K \varphi dx - \int_{|x|=\epsilon} K \varphi \frac{x^j}{|x|} ds \right\}. \end{aligned} \quad (2.1.2)$$

In some cases, $\partial_{x^j} K \varphi$ is not integrable, but $\lim_{\epsilon \downarrow 0} \int_{|x| \geq \epsilon} \partial_{x^j} K \varphi dx$ exists, then we call the limit the *Cauchy principal-value integral* of $\partial_{x^j} K \varphi$. Generally speaking, the Cauchy principal-value integral of a function f with singularity at 0 is defined as:

$$\text{PV} \int_{\mathbb{R}^n} f dx = \lim_{\epsilon \downarrow 0} \int_{|x| \geq \epsilon} f dx. \quad (2.1.3)$$

Remark 2.1.1. *For the functions with singularity at $x \neq 0$, by integrating outside $B(x, \epsilon)$, the definition of the distributional derivative and Cauchy principal-value can be easily extended.*

Next, we give a few examples of the distributional derivatives of some functions, which will be useful later.

Example 2.1.2. *If $K(x) = \frac{1}{|x|}$ with $x \in \mathbb{R}^3$, then*

$$\partial_{x^j} K = -\frac{x^j}{|x|^3}. \quad (2.1.4)$$

Proof. For every $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \int_{|x|=\epsilon} K \varphi \frac{x^j}{|x|} ds \\ &= \lim_{\epsilon \downarrow 0} \int_{|x|=\epsilon} \varphi \frac{x^j}{\epsilon^2} ds \\ &= 0. \end{aligned} \quad (2.1.5)$$

When $x \neq 0$,

$$\frac{\partial K(x)}{\partial x^j} = -\frac{x^j}{|x|^3}.$$

□

Example 2.1.3. If $K(x) = \frac{x}{|x|^3}$ with $x \in \mathbb{R}^3$, then

$$\partial_{x^j} K^i(x) = -\frac{4\pi}{3} \delta_j^i \delta(x) - \frac{3x^i x^j}{|x|^5} + \frac{1}{|x|^3} \delta_j^i. \quad (2.1.6)$$

Proof. For every $\varphi \in C_0^\infty(\mathbb{R}^3)$, by a change of variables $x \rightarrow \epsilon x$,

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} \int_{|x|=\epsilon} K^i \varphi \frac{x^j}{|x|} ds \\ &= \lim_{\epsilon \downarrow 0} \epsilon^2 \int_{|x|=1} K^i(\epsilon x) \varphi(\epsilon x) \frac{x^j}{|x|} ds \\ &= \lim_{\epsilon \downarrow 0} \int_{|x|=1} \frac{x^i}{|x|^3} \varphi(\epsilon x) \frac{x^j}{|x|} ds \\ &= \varphi(0) \int_{|x|=1} x^i x^j ds \\ &= \frac{4\pi}{3} \varphi(0) \delta_j^i. \end{aligned} \quad (2.1.7)$$

When $x \neq 0$, by direct computation,

$$\nabla K(x) = -\frac{3x \otimes x}{|x|^5} + \frac{1}{|x|^3} I_3, \quad (2.1.8)$$

where $\otimes$ is the outer product between two vectors. □

Example 2.1.4. If $K(x) = \ln |x|$ with $x \in \mathbb{R}^2$, then

$$\partial_{x^j} K = \frac{x^j}{|x|^2}. \quad (2.1.9)$$

Proof. For every $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned} & \lim_{\epsilon \downarrow 0} \int_{|x|=\epsilon} K \varphi \frac{x^j}{|x|} ds \\ &= \lim_{\epsilon \downarrow 0} \int_{|x|=\epsilon} \varphi \ln \epsilon \frac{x^j}{\epsilon} ds \\ &= 0. \end{aligned} \quad (2.1.10)$$

When $x \neq 0$,

$$\nabla K(x) = \left(\frac{x^1}{|x|^2}, \frac{x^2}{|x|^2} \right). \quad (2.1.11)$$

□

2.1.2 Vorticity-stream formulation of the Navier-Stokes equations

Recall that the Navier-Stokes equations for an incompressible fluid in $\mathbb{R}^3$ are

$$\begin{aligned} \frac{\partial}{\partial t} u + u \cdot \nabla u &= \nu \Delta u + \nabla p, \\ \nabla \cdot u &= 0, \\ u|_{t=0} &= u_0, \end{aligned} \quad (2.1.12)$$

where $u(x, t) = (u^1, u^2, u^3)$ is the fluid velocity with $(x, t) \in \mathbb{R}^3 \times [0, \infty)$, and $p(x, t)$ is the scalar pressure.

By taking the curl of the Navier-Stokes equation, we obtain the following equation for the vorticity $\omega = \operatorname{curl} u$:

$$\frac{\partial}{\partial t} \omega + u \cdot \nabla \omega = \omega \cdot \nabla u + \nu \Delta \omega. \quad (2.1.13)$$

Since $\omega \cdot (\nabla u - \nabla u^T) = 0$, we also have the vorticity equation in the symmetric form:

$$\frac{\partial}{\partial t} \omega + u \cdot \nabla \omega = \omega \cdot S + \nu \Delta \omega, \quad (2.1.14)$$

where $S = \frac{1}{2}(\nabla u + \nabla u^T)$.

In fact, given the vorticity ω , the velocity u can be obtained by solving the following PDEs:

$$\begin{aligned} \operatorname{curl} u &= \omega, \\ \operatorname{div} u &= 0. \end{aligned} \quad (2.1.15)$$

The solution to (2.1.15) can be obtained using the following lemma.

Lemma 2.1.5. (3D Biot-Savart law) *Let $u, \omega \in L^2$ be smooth vector fields in $\mathbb{R}^3$, which both vanish sufficiently rapidly when $|x| \rightarrow \infty$. Suppose u, ω satisfy the PDEs (2.1.15), then u has the following explicit formula:*

$$u(x) = \int_{\mathbb{R}^3} K(x-y)\omega(y)dy, \quad (2.1.16)$$

where K is a 3×3 matrix such that $K(x)h = -\frac{1}{4\pi} \frac{x}{|x|^3} \times h$ for every $h \in \mathbb{R}^3$.

Proof. Consider the following Poisson equation:

$$\Delta\psi = \omega. \quad (2.1.17)$$

Let $\Phi(x) = \frac{1}{4\pi|x|}$ be the fundamental solution to the Poisson equation (see [35]), then

$$\psi(x) = - \int_{\mathbb{R}^3} \Phi(x-y)\omega(y)dy$$

is a solution to (2.1.17).

Now, since ω is a curl of some vector field, $\operatorname{div}\omega = 0$. Hence,

$$\operatorname{div}\psi(x) = \int_{\mathbb{R}^3} \Phi(y)\operatorname{div}\omega(x-y)dy = 0.$$

Therefore, by the identity $\operatorname{curl}(-\operatorname{curl}\psi) = \Delta\psi - \nabla\operatorname{div}\psi$, we have

$$\operatorname{curl}(-\operatorname{curl}\psi) = \omega.$$

Let $\phi = u + \operatorname{curl}\psi$, we have $\operatorname{curl}\phi = 0$ and $\operatorname{div}\phi = 0$. Since ϕ vanishes sufficiently rapidly as $|x| \rightarrow \infty$, $\phi \equiv 0$, which means $u = -\operatorname{curl}\psi$.

By taking the curl of ψ , from (2.1.4),

$$\begin{aligned} u^i(x) &= -\epsilon^{ijk} \int_{\mathbb{R}^3} \partial_{x^j} \Phi(x-y) \omega^k(y) dy \\ &= -\epsilon^{ijk} \int_{\mathbb{R}^3} \frac{1}{4\pi} \frac{x^j - y^j}{|x-y|^3} \omega^k(y) dy, \end{aligned} \quad (2.1.18)$$

which concludes the proof. $\square$

Furthermore, by taking the distributional derivative of $\frac{x}{|x|^3}$ (see (2.1.6)), we are able to represent ∇u and S by ω :

$$\begin{aligned} \nabla u(x) &= -\frac{1}{4\pi} \left\{ \text{PV} \int_{\mathbb{R}^3} \left(-\frac{3[(x-y) \times \omega(y)] \otimes (x-y)}{|x-y|^5} - \frac{1}{|x-y|^3} \omega(y)^\times \right) dy \right\} \\ &\quad - \frac{1}{3} \omega(x)^\times, \end{aligned} \quad (2.1.19)$$

where $\alpha^\times = (\alpha^1, \alpha^2, \alpha^3)^\times$ is the 3×3 matrix such that $\alpha^\times h = \alpha \times h$ for all $h \in \mathbb{R}^3$:

$$\alpha^\times = \begin{pmatrix} 0 & -\alpha^3 & \alpha^2 \\ \alpha^3 & 0 & -\alpha^1 \\ -\alpha^2 & \alpha^1 & 0 \end{pmatrix},$$

and ∇u is arranged as:

$$\nabla u = \begin{pmatrix} \partial_{x^1} u^1 & \partial_{x^2} u^1 & \partial_{x^3} u^1 \\ \partial_{x^1} u^2 & \partial_{x^2} u^2 & \partial_{x^3} u^2 \\ \partial_{x^1} u^3 & \partial_{x^2} u^3 & \partial_{x^3} u^3 \end{pmatrix}.$$

It is easy to obtain S from ∇u by taking the symmetric form. $\omega(x)^\times$ will vanish since it is antisymmetric, which yields

$$S = \frac{3}{8\pi} \text{PV} \int_{\mathbb{R}^3} \left(\frac{[(x-y) \times \omega(y)] \otimes (x-y)}{|x-y|^5} + \frac{(x-y) \otimes [(x-y) \times \omega(y)]}{|x-y|^5} \right) dy. \quad (2.1.20)$$

From (2.1.19), we are able to obtain the vorticity-stream formulation for the

Navier-Stokes equations.

Proposition 2.1.6. *Suppose that u is a smooth velocity field which vanish sufficiently rapidly when $|x| \rightarrow \infty$ and satisfies the Navier-Stokes equations (2.1.12). Then the vorticity ω is a solution to the vorticity transport equation:*

$$\begin{aligned} \frac{\partial}{\partial t} \omega + u \cdot \nabla \omega &= \omega \cdot \nabla u + \nu \Delta \omega, \\ \omega|_{t=0} &= \omega_0 = \operatorname{curl} u_0. \end{aligned} \tag{2.1.21}$$

where u is determined by ω according to the Biot-Savart law (2.1.16):

$$u(x, t) = \int_{\mathbb{R}^3} K(x - y) \omega(y, t) dy,$$

with $K(x)h = -\frac{1}{4\pi} \frac{x}{|x|^3} \times h$.

Besides, ∇u is determined by ω according to

$$\begin{aligned} \nabla u(x, t) &= -\frac{1}{4\pi} \left\{ \operatorname{PV} \int_{\mathbb{R}^3} \left(-\frac{3[(x - y) \times \omega(y, t)] \otimes (x - y)}{|x - y|^5} - \frac{1}{|x - y|^3} \omega(y, t)^\times \right) dy \right\} \\ &\quad - \frac{1}{3} \omega(x, t)^\times. \end{aligned}$$

Similarly, from (2.1.20), the vorticity-stream formulation in a symmetric form can be obtained.

Proposition 2.1.7. *Suppose that u is a smooth velocity field which vanish sufficiently rapidly when $|x| \rightarrow \infty$ and satisfies the Navier-Stokes equations (2.1.12). The the vorticity ω is a solution to the following equation*

$$\begin{aligned} \frac{\partial}{\partial t} \omega + u \cdot \nabla \omega &= S\omega + \nu \Delta \omega \\ \omega|_{t=0} &= \omega_0 = \operatorname{curl} u_0 \end{aligned} \tag{2.1.22}$$

where u is determined by ω according to the Biot-Savart law (2.1.16):

$$u(x, t) = \int_{\mathbb{R}^3} K(x - y)\omega(y, t)dy$$

with $K(x)h = -\frac{1}{4\pi} \frac{x}{|x|^3} \times h$.

Besides, S could be determined by

$$S(x, t) = \int_{\mathbb{R}^3} H(x - y)\omega(y, t)dy$$

with

$$H(x)h = \frac{3}{8\pi} [(x \times h) \otimes x + x \otimes (x \times h)] / |x|^5$$

The velocity field for a 2D flow is represented as $u = (u^1, u^2, 0)$, which is zero in the third component. The vorticity, denoted by $\omega = (0, 0, \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1})$, can be treated as a scalar and is perpendicular to the velocity field. Therefore, the vortex stretching term $\omega \cdot \nabla u$ in (2.1.13) vanishes. so that the vorticity transport equations (2.1.13) for 2D flows degenerate to the following evolution equations for a scalar ω :

$$\frac{\partial}{\partial t}\omega + u \cdot \nabla \omega = \nu \Delta \omega \quad (2.1.23)$$

In fact, for the 2D flows, u can be also determined by ω from the following lemma:

Lemma 2.1.8. (2D Biot-Savart law) *Let $u \in L^2$ be a smooth vector field in $\mathbb{R}^2$, which vanishes sufficiently rapidly when $|x| \rightarrow \infty$. Suppose u is divergence free and $\omega = \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1}$, then u can be determined by ω by the following equation:*

$$u(x) = \int_{\mathbb{R}^2} K(x - y)\omega(y)dy \quad (2.1.24)$$

where $K(x) = -\frac{1}{2\pi} \frac{(x^2, -x^1)}{|x|^2}$.

Proof. Since $\operatorname{div} u = 0$, from Green's formula, it is easy to verify that $(-u^2, u^1)$ is a

conservative field. Therefore, let ψ be the potential such that

$$\nabla\psi = (-u^2, u^1).$$

By taking the divergence on both sides, $\Delta\psi = -\omega$, which means

$$\psi(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-y| \omega(y) dy.$$

By differentiating under the integral using (2.1.9),

$$\nabla\psi(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x^1, x^2)}{|x|^2} \omega(y) dy.$$

Therefore,

$$u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x^2 - y^2, -(x^1 - y^1))}{|x - y|^2} \omega(y) dy.$$

□

From the 2D Biot-Savart law, we are able to derive the vorticity-stream formulation for 2D flows:

Proposition 2.1.9. *For the smooth 2D flows with the velocity u vanishing sufficiently rapidly as $|x| \rightarrow \infty$, the scalar vorticity ω satisfies the following vorticity transport equation:*

$$\begin{aligned} \frac{\partial}{\partial t} \omega + u \cdot \nabla \omega &= \nu \Delta \omega, \\ \omega|_{t=0} &= \omega_0, \end{aligned} \tag{2.1.25}$$

where the velocity u is determined from the vorticity ω by the 2D Biot-Savart law

$$u(x, t) = \int_{\mathbb{R}^2} K(x - y) \omega(y, t) dy,$$

where $K(x) = -\frac{1}{2\pi} \frac{(x^2, -x^1)}{|x|^2}$.

2.2 Fundamental solutions to the parabolic equations

In this section, we will provide an overview of PDE theories related to fundamental solutions.

Consider the following PDE:

$$\left(L - \frac{\partial}{\partial t}\right) u = \sum_{i,j=1}^n a^{ij}(x, t) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i(x, t) \frac{\partial u}{\partial x^i} + c(x, t)u - \frac{\partial u}{\partial t} = 0, \quad (2.2.1)$$

where $(x, t) \in \mathbb{R}^n \times [T_0, T_1]$ and $a^{ij} = a^{ji}$. We say that L is parabolic if the matrix $(a^{ij}(x, t))$ is positive definite for all $(x, t) \in \mathbb{R}^n \times [T_0, T_1]$. If there exist positive $\overline{\lambda}_0, \overline{\lambda}_1$ such that for all vectors $\xi \in \mathbb{R}^n$,

$$\overline{\lambda}_0 |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x, t) \xi^i \xi^j \leq \overline{\lambda}_1 |\xi|^2$$

for all $(x, t) \in \mathbb{R}^n \times [T_0, T_1]$, then we say L is uniformly parabolic. *From now on, we always assume that L is uniformly parabolic and the coefficients of L are bounded continuous functions in $\mathbb{R}^n \times [T_0, T_1]$.* We also let λ_0 be some positive constant such that $x'(a(y, t))^{-1}x \geq \lambda_0 |x|^2$ for all $(y, t) \in \mathbb{R}^n \times [T_0, T_1], x \in \mathbb{R}^n$.

Definition 2.2.1. *A fundamental solution of $(L - \frac{\partial}{\partial t})u = 0$ is a function $\Gamma(x, t; \xi, \tau)$ defined for all $(x, t), (\xi, \tau) \in \mathbb{R}^n \times [T_0, T_1]$ and $t > \tau$, which satisfies the following conditions:*

1. *For fixed (ξ, τ) , $\Gamma(x, t; \xi, \tau)$ satisfies the equation $(L - \frac{\partial}{\partial t})u = 0$ as a function of (x, t) .*
2. *For every continuous function $f(x)$ which satisfies*

$$|f(x)| \leq C \exp(h|x|^2)$$

for some constants $C > 0, 0 < h < \frac{\lambda_0}{4T}$ on $\mathbb{R}^n$, the following equation holds for every $x \in \mathbb{R}^n$:

$$\lim_{t \downarrow \tau} \int_{\mathbb{R}^n} \Gamma(x, t; \xi, \tau) f(\xi) d\xi = f(x).$$

Theorem 2.2.2. (Existence of fundamental solutions) ([38] Theorem 10)

Suppose that a, b, c are all continuous functions in $\mathbb{R}^n \times [T_0, T_1]$, and for some constant $A > 0, 0 < \alpha \leq 1$, we have

$$\begin{aligned} |a(x_1, t_1) - a(x_2, t_2)| &\leq A(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}), \\ |b(x_1, t_1) - b(x_2, t_1)| &\leq A|x_1 - x_2|^\alpha, \\ |c(x_1, t_1) - c(x_2, t_1)| &\leq A|x_1 - x_2|^\alpha, \end{aligned} \tag{A1}$$

for all $(x_1, t_1), (x_2, t_2) \in \mathbb{R}^n \times [T_0, T_1]$, then there exists a fundamental solution $\Gamma(x, t; \xi, \tau)$ to (2.2.1).

The fundamental solutions are useful to construct the solution to the Cauchy problems. We are interested in the following parabolic equation:

$$\left(L - \frac{\partial}{\partial t}\right) u = f(x, t), u(x, 0) = \varphi(x), \tag{2.2.2}$$

where $(x, t) \in \mathbb{R}^n \times (0, T]$. Suppose that

$$\begin{aligned} |f(x, t)| &\leq C \exp(h|x|^2), \\ |\varphi(x)| &\leq C \exp(h|x|^2) \end{aligned}$$

for some constants $C > 0, 0 < h < \frac{\lambda_0}{4T}$. Let $\Gamma(x, t; \xi, \tau)$ be a fundamental solution to $(L - \frac{\partial}{\partial t}) u = 0$, then

$$u(x, t) = \int_{\mathbb{R}^n} \Gamma(x, t; \xi, 0) \varphi(\xi) d\xi - \int_0^t \int_{\mathbb{R}^n} \Gamma(x, t; \xi, \tau) f(\xi, \tau) d\xi d\tau \tag{2.2.3}$$

is a solution to the Cauchy problem (2.2.2). (see [38] Theorem 12).

Next, we present the results on adjoint parabolic equations. Let L^* be adjoint operator of L , i.e.

$$L^*v = \sum_{i,j=1}^n a^{ij}(x,t) \frac{\partial^2 v}{\partial x^i \partial x^j} + \sum_{i=1}^n b^{i*}(x,t) \frac{\partial v}{\partial x^i} + c^*(x,t)v, \quad (2.2.4)$$

where

$$b^{i*} = -b^i + 2 \sum_{j=1}^n \frac{\partial a^{ij}}{\partial x^j},$$

$$c^* = c - \sum_{i=1}^n \frac{\partial b^i}{\partial x^i} + \sum_{i,j=1}^n \frac{\partial^2 a^{ij}}{\partial x^i \partial x^j}.$$

The adjoint equation of (2.2.1) is $(L^* + \frac{\partial}{\partial t})v = 0$.

Definition 2.2.3. A fundamental solution $(L^* + \frac{\partial}{\partial t})v = 0$ is a function $\Gamma^*(x, t; \xi, \tau)$ where $(x, t), (\xi, \tau) \in \mathbb{R}^n \times [T_0, T_1]$, $t < \tau$, which satisfies the following conditions:

1. For fixed (ξ, τ) , $\Gamma^*(x, t; \xi, \tau)$ satisfies the equation $L^*v = 0$ as a function of (x, t) .
2. For every continuous function $f(x)$ which satisfies

$$|f(x)| \leq C \exp(h|x|^2)$$

for some constants $C > 0, 0 < h < \frac{\lambda_0}{4T}$ on $\mathbb{R}^n$, the following equation holds for every $x \in \mathbb{R}^n$:

$$\lim_{t \uparrow \tau} \int_{\mathbb{R}^n} \Gamma^*(x, t; \xi, \tau) f(\xi) d\xi = f(x).$$

The duality of the fundamental solutions to the adjoint equations is demonstrated in the following theorem.

Theorem 2.2.4. (Duality of the fundamental solutions) ([38] Theorem 15)

Suppose that (A1) holds. Furthermore, suppose that $a^{ij}, \frac{\partial}{\partial x^l} a^{ij}, \frac{\partial}{\partial x^l \partial x^k} a^{ij}, b^i, \frac{\partial}{\partial x^k} b^i, c$ are

all bounded continuous functions in $\mathbb{R}^n \times [T_0, T_1]$ for all $i, j, k, l = 1, 2, \dots, n$, and they satisfy

$$|f(x, t) - f(y, t)| \leq A|x - y|^\alpha$$

for some constant $0 < \alpha \leq 1, A > 0$ for all $t \in [T_0, T_1], x, y \in \mathbb{R}^n$, then a fundamental solution $\Gamma^*(\xi, \tau; x, t)$ to $(L^* + \frac{\partial}{\partial t})v = 0$ exists, and the following duality holds:

$$\Gamma(x, t; \xi, \tau) = \Gamma^*(\xi, \tau; x, t). \quad (t > \tau) \quad (2.2.5)$$

2.3 Sobolev spaces

In this section, we will give a review of some concepts in Sobolev spaces.

Definition 2.3.1. (Sobolev spaces) ([1] 3.1) Suppose Ω is a domain in $\mathbb{R}^n$. Let m be a positive integer and $1 \leq p \leq \infty$.

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\},$$

where $D^\alpha u$ is the weak partial derivative of u . We also define $W_0^{m,p}(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Theorem 2.3.2. ([1] Theorem 3.3) $W^{m,p}(\Omega)$ is a Banach space.

Next we study $(W^{m,p}(\Omega))'$, the dual space of $W^{m,p}(\Omega)$. Let p' be the conjugate exponent of p .

$$p' = \begin{cases} \infty, & \text{if } p = 1 \\ p/(p-1), & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty \end{cases} \quad (2.3.1)$$

From the Riesz Representation Theorem, $L^{p'}(\Omega)$ is isomorphic to $(L^p(\Omega))'$ when $1 \leq p < \infty$. Hence, we have the following theorem:

Theorem 2.3.3. ([1] Theorem 3.9) Let $1 \leq p < \infty$. For every $L \in (W^{m,p}(\Omega))'$, there exists $v_\alpha \in L^{p'}(\Omega)$, $(0 \leq |\alpha| \leq m)$ such that for all $u \in W^{m,p}(\Omega)$

$$L(u) = \sum_{0 \leq |\alpha| \leq m} \langle D^\alpha u, v_\alpha \rangle. \quad (2.3.2)$$

Moreover, $\|L\| = \inf \sum_{0 \leq |\alpha| \leq m} \|v_\alpha\|_{L^{p'}(\Omega)}$ where infimum is taken over for all v such that (2.3.2) hold.

Since every $L \in (W^{m,p}(\Omega))'$ is an extension of a distribution $T \in \mathcal{D}'(\Omega)$ to $W^{m,p}(\Omega)$. Suppose that L is given by (2.3.2), then L is an extension of

$$T = \sum_{0 \leq |\alpha| \leq m} (-1)^{|\alpha|} D^\alpha T_{v_\alpha}, \quad (2.3.3)$$

where $T_{v_\alpha} \in \mathcal{D}'(\Omega)$ such that $T_{v_\alpha}(\phi) = \langle \phi, v_\alpha \rangle = \int_\Omega \phi(x) v_\alpha(x) dx$ for all test functions $\phi \in \mathcal{D}(\Omega)$. Conversely, if we have a distribution $T \in \mathcal{D}'(\Omega)$ which has the form (2.3.3), T could be continuously extended to $W^{m,p}(\Omega)$. There might exist non-unique extensions, while the extension to $W_0^{m,p}(\Omega)$ is unique. In fact, $(W_0^{m,p}(\Omega))'$ could be identified with the space of distributions $T \in \mathcal{D}'(\Omega)$ with the form (2.3.3), which could be summarized by the following theorem.

Theorem 2.3.4. ([1] Theorem 3.12) Let $1 \leq p < \infty$. Let $W^{-m,p'}(\Omega)$ be the Banach space consisting of all the distributions $T \in \mathcal{D}'(\Omega)$ which have the form (2.3.3), and the norm is given by

$$\|T\|_{W^{-m,p'}(\Omega)} = \inf \left\{ \sum_{0 \leq |\alpha| \leq m} \|v_\alpha\|_{L^{p'}(\Omega)} \right\},$$

where infimum is taken over for all v such that (2.3.3) hold. Then $W^{-m,p'}$ is isomorphic to $(W_0^{m,p}(\Omega))'$.

2.4 Stochastic differential equations

We will introduce the necessary notation and recall some standard results from SDE theory in this section. We work with a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$. Let $b^i(x, t)$ and $\sigma_j^i(x, t)$, $1 \leq i \leq d$, $1 \leq j \leq r$, be Borel-measurable functions defined on $\mathbb{R}^d \times [0, \infty)$. Given an r -dimensional Brownian motion $B_t = (B_t^1, B_t^2, \dots, B_t^r)$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, we say that a continuous stochastic process $X = \{X_t : 0 \leq t \leq \infty\}$ is a strong solution to the following stochastic differential equation

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t, \quad (2.4.1)$$

or, equivalently,

$$dX_t^i = b^i(X_t, t)dt + \sum_{j=1}^r \sigma_j^i(X_t, t)dB_t^j \quad (2.4.2)$$

with initial datum ξ , if the following properties hold:

- (i) X is adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$;
- (ii) $\mathbb{P}(X_0 = \xi) = 1$;
- (iii) $\mathbb{P}\left(\int_0^t [|b^i(X_s, s)| + \sigma_j^{i^2}(X_s, s)]ds < \infty\right) = 1$ for every $1 \leq i \leq d$, $1 \leq j \leq r$ and $0 \leq t < \infty$;
- (iv) The following equality holds:

$$X_t = X_0 + \int_0^t b(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s, \quad (2.4.3)$$

or equivalently,

$$X_t^i - X_0^i = \int_0^t b^i(X_s, s)ds + \int_0^t \sum_{j=1}^r \sigma_j^i(X_s, s)dB_s^j \quad (2.4.4)$$

holds for all $t \geq 0$ and $1 \leq i \leq d$.

Theorem 2.4.1. (Existence and uniqueness of strong solutions to SDEs)

(cf. [57] Theorem 5.2.9) Suppose that the coefficients $b(x, t)$ and $\sigma(x, t)$ satisfy the global Lipschitz and linear growth conditions

$$|b(x, t) - b(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq K|x - y|,$$

$$|b(x, t)|^2 + |\sigma(x, t)|^2 \leq K^2(1 + |x|^2)$$

for every $0 \leq t < \infty$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, where K is a positive constant. Let B_t be an r -dimensional Brownian motion defined on $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$, and $\xi \in \mathcal{F}_0$, with finite second moment: $\mathbb{E}|\xi|^2 < \infty$. Then, there exists a strong solution to the stochastic differential equation (2.4.1), and the solution satisfies the following bound: for each $T \geq 0$ there is a constant C , depending only on T and K , such that

$$\mathbb{E}|X_t|^2 \leq C(K, T)(1 + \mathbb{E}|\xi|^2) e^{C(K, T)t}$$

for all $t \in [0, T]$. Moreover, the strong solution is unique, that is, if X_t and Y_t are both strong solutions to this stochastic differential equation, then X_t and Y_t are indistinguishable in the sense that $\mathbb{P}(X_t = Y_t; 0 \leq t < \infty) = 1$.

In particular, we have the following moment estimate.

Theorem 2.4.2. *The following estimate holds for the solution X_t :*

$$\mathbb{E} \sup_{0 \leq t \leq T} |X_t|^2 \leq C(K, T)(1 + \mathbb{E}|\xi|^2), \quad (2.4.5)$$

where $C(K, T)$ is a constant depending only on K and T .

2.5 Wasserstein metrics and McKean-Vlasov SDEs

In this section, we will discuss McKean-Vlasov SDEs, which are closely related to the RVSEs in this thesis.

2.5.1 Wasserstein metrics

Given a complete metric space (E, d) together with its Borel σ -algebra $\mathcal{E}$, we denote the set of all probability measures on $(E, \mathcal{E})$ by $\mathcal{P}(E)$. Let $\mathcal{P}_2(E)$ be the subset of $\mathcal{P}(E)$, consisting of all probability measures μ such that $\int_E d(x, x_0)^2 d\mu(x) < \infty$, where x_0 is any point in E ; obviously this definition does not depend on the specific choice of x_0 . The distance W_2 on $\mathcal{P}_2(E)$ is defined as follows. For $\mu, \nu \in \mathcal{P}_2(E)$,

$$W_2(\mu, \nu) = \inf \left\{ \left[\int d(x, y)^2 \pi(dx, dy) \right]^{1/2} : \pi \in \mathcal{P}(E \times E) \right. \\ \left. \text{with marginal distributions } \mu \text{ and } \nu \right\}.$$

Remark 2.5.1. ([109] Theorem 6.18) $(\mathcal{P}_2(E), W_2(E))$ is a Polish space as long as (E, d) is a Polish space.

For a specific example of $\mathcal{P}_2(E)$, consider the Banach space $C([0, T]; \mathbb{R}^d)$, which is the space of all continuous functions from $[0, T]$ to $\mathbb{R}^d$ equipped with the norm

$$\|f\| = \sup_{0 \leq t \leq T} |f(t)|$$

for $f \in C([0, T]; \mathbb{R}^d)$. Then for two probability measures $\mu, \nu \in \mathcal{P}_2(C([0, T]; \mathbb{R}^d))$,

$$W_2(\mu, \nu) = \inf_{m \in \Lambda} \left[\int \sup_{0 \leq t \leq T} |\omega_1(t) - \omega_2(t)|^2 dm(\omega) \right]^{1/2}, \quad (2.5.1)$$

where $\omega = (\omega_1, \omega_2) \in C([0, T]; \mathbb{R}^d) \times C([0, T]; \mathbb{R}^d)$, and Λ is the set of all probability measures with marginal distributions μ and ν .

From (2.5.1), it is easy to deduce that, given two continuous stochastic processes $(X_t)_{0 \leq t \leq T}$ and $(Y_t)_{0 \leq t \leq T}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking their values in $\mathbb{R}^d$, whose distributions are probability measures on $C([0, T]; \mathbb{R}^d)$, denoted by μ_X

and μ_Y respectively, then

$$W_2(\mu_X, \mu_Y) \leq \left[\mathbb{E} \sup_{0 \leq t \leq T} |X_t - Y_t|^2 \right]^{1/2}. \quad (2.5.2)$$

2.5.2 McKean-Vlasov SDEs

After introducing the Wasserstein metrics, we will present some classical results on MVSDEs in this section. A standard MVSDE is of the following form

$$dX_t = b(X_t, t, \mathcal{L}(X_t))dt + \sigma(X_t, t, \mathcal{L}(X_t))dB_t, X_0 = \xi \quad (2.5.3)$$

where $\mathcal{L}(X_t)$ is distribution of X_t . We work with the same dimension as in Section 2.4, so that b, σ are Borel-measurable functions from $\mathbb{R}^d \times [0, T] \times \mathcal{P}(\mathbb{R}^d)$ to $\mathbb{R}^d$ and $\mathbb{R}^{d \times r}$ respectively.

In fact, when b and σ satisfy global Lipschitz and linear growth conditions, the strong existence and uniqueness holds for the solution to (2.5.3).

Theorem 2.5.2. (*[17] Theorem 1.7*) *Suppose that the coefficients b and σ satisfies the global Lipschitz and linear growth conditions:*

$$\begin{aligned} |b(x, t, \mu) - b(y, t, \nu)| + |\sigma(x, t, \mu) - \sigma(y, t, \nu)| &\leq K (|x - y| + W_2(\mu, \nu)), \\ |b(x, t, \mu)| + |\sigma(x, t, \mu)| &\leq K (1 + |x| + W_2(\mu, \mu_0)) \end{aligned}$$

for every $0 \leq t \leq T, x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, where μ_0 is some fixed distribution in $\mathcal{P}_2(\mathbb{R}^d)$. Then there exists a unique strong solution to (2.5.3).

For MVSDEs, the solution could be approximated by running independent particles, which is justified by the law of large numbers. Let X_t be the solution to the MVSDE (2.5.3) with $X_0 = \xi \sim \mu \in \mathcal{P}_2(\mathbb{R}^d)$. Let $X_t^{(i), N}$ be the solution to the

following system of interacting particles:

$$dX_t^{(i),N} = b(X_s, t, \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{(i),N}})dt + \sigma(X_t, t, \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{(i),N}})dB_t^{(i)}, X_0 = \xi^{(i)}$$

where $i = 1, 2, \dots, N$, $(\xi^{(i)})_{1 \leq i \leq N}$ are i.i.d random variables with common distribution μ and $(B^{(i)})_{1 \leq i \leq N}$ are independent Brownian motions. When $N \rightarrow \infty$, by the law of large numbers, the empirical process $\frac{1}{N} \sum_{i=1}^N \delta_{X_t^{(i),N}}$ will become deterministic, which minimizes the coupling impact among $X_t^{(i),N}$ for $i = 1, 2, \dots, N$ from $\frac{1}{N} \sum_{i=1}^N \delta_{X_t^{(i),N}}$ in the coefficients b, σ . $\frac{1}{N} \sum_{i=1}^N \delta_{X_t^{(i),N}}$ will converge to the $\mathcal{L}(X_t)$. Therefore, $X_t^{(i),N}$ will converge to the following independent processes which all have the same distribution with X_t :

$$dX_t^{(i)} = b(X_s^{(i)}, t, \mathcal{L}(X_t^{(i)}))dt + \sigma(X_t^{(i)}, t, \mathcal{L}(X_t^{(i)}))dB_t^{(i)}, X_0^{(i)} = \xi^{(i)}$$

This phenomenon is called the propagation of chaos [41, 77, 106].

Theorem 2.5.3. (Propogation of chaos for MVSDEs) ([17] *Theorem 1.10.*)

Under the above assumptions and the the global Lipschitz and linear growth conditions of b, σ , we have

$$\lim_{N \rightarrow \infty} \sup_{1 \leq i \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |X_t^{(i),N} - X_t^{(i)}|^2 = 0. \quad (2.5.4)$$

2.6 Large deviation principles

Let (E, d) be a polish space, and $\mathcal{B}(E)$ be its Borel σ -algebra. A function $I : E \rightarrow \mathbb{R} \cup \{\infty\}$ is lower semi-continuous if for every $c \in \mathbb{R} \cup \{\infty\}$, $I^{-1}((-\infty, c])$ is closed. We call a lower semi-continuous function $I : E \rightarrow [0, \infty]$ *rate function*. Moreover, if $I^{-1}([0, c])$ is compact for every $c \geq 0$, then I is a *good rate function*.

Let us begin with the definition of the large deviation principle.

Definition 2.6.1. *A family $\{P_\epsilon : \epsilon > 0\}$ of probability measures on $(E, \mathcal{B}(E))$ is said to satisfy the large deviation principle with rate function I if for every closed*

sets F in E

$$\limsup_{\epsilon \downarrow 0} \epsilon \log P_\epsilon(F) \leq - \inf_{s \in F} I(s),$$

and every open set G in E ,

$$\limsup_{\epsilon \downarrow 0} \epsilon \log P_\epsilon(E) \geq - \inf_{s \in F} I(s),$$

We can also say a family of E -valued random variables $\{X_\epsilon : \epsilon > 0\}$ satisfies the LDP with the rate function I if P_ϵ , the law of X_ϵ , satisfies the LDP with I .

For the finite-dimensional case, let $\{x^i\}_{i \in \mathbb{N}}$ be a sequence of $\mathbb{R}^d$ -valued i.i.d. random variables with distribution μ , and let μ_n be the law of $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n x^i$. The following Cramér's theorem establishes the LDP for μ_n .

Theorem 2.6.2. (Cramér's theorem) ([32] Theorem 2.2.30) *Suppose for any $\lambda \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} e^{\langle \lambda, z \rangle} \mu(dz) < \infty$, then $\{\mu_n\}_{n \geq 1}$ satisfies the LDP with the good rate function*

$$I(x) = \sup_{\lambda \in \mathbb{R}^d} \left\{ \langle \lambda, x \rangle - \log \int_{\mathbb{R}^d} e^{\langle \lambda, z \rangle} \mu(dz) \right\}.$$

If E is an infinite-dimensional space, e.g. $C([0, T]; \mathbb{R}^d)$, the LDP is usually established by projecting E onto some finite-dimensional space and then the limit of a sequence of these projections is considered.

The following contraction principle establishes the LDP between two spaces under a continuous map.

Theorem 2.6.3. (Contraction principle) ([32] Theorem 4.2.1) *Let E, E' be two Polish spaces, and f be a continuous map from E to E' . Suppose that $\{Z_\epsilon : \epsilon > 0\}$ satisfies the LDP in E with a good rate function I , then $X^\epsilon = f(Z^\epsilon)$ satisfies the LDP with the good rate function I' :*

$$I'(s') = \inf \{I(s) : s \in E \text{ and } f(s) = s'\}.$$

If the map is only measurable but can be approximated by continuous maps, the

contraction principle could be extended. Before presenting the generalized contraction principle, let us introduce the concept of exponential approximations first.

Definition 2.6.4. *Let (E, d) be a polish space and let $\{X_\epsilon\}_{\epsilon>0}$ and $\{Y_\epsilon\}_{\epsilon>0}$ be two families of E -valued random variables. We say that $\{X_\epsilon\}_{\epsilon>0}$ and $\{Y_\epsilon\}_{\epsilon>0}$ are exponentially equivalent if for every $\delta > 0$*

$$\limsup_{\epsilon \downarrow 0} \epsilon \log P(d(X_\epsilon, Y_\epsilon) > \delta) = -\infty.$$

The exponential equivalent probability measures are equivalent in the LDP sense.

Theorem 2.6.5. *([32] Theorem 4.2.13) Suppose that $\{X_\epsilon\}_{\epsilon>0}$ satisfies an LDP with a good rate function I . If $\{Y_\epsilon\}_{\epsilon>0}$ are exponentially equivalent to $\{X_\epsilon\}_{\epsilon>0}$, then the same LDP holds for $\{Y_\epsilon\}_{\epsilon>0}$.*

Now the exponential approximations are defined as follows.

Definition 2.6.6. *Let $\{X_\epsilon^{(n)}\}_{\epsilon>0}$ be a sequence of family of E -valued random variables, $n = 1, 2, \dots$ and $\{X_\epsilon\}_{\epsilon>0}$ be a family of E -valued random variables. We say that $\{X_\epsilon^{(n)}\}_{\epsilon>0}$ are exponentially good approximations of X if*

$$\lim_{n \rightarrow \infty} \limsup_{\epsilon \downarrow 0} \epsilon \log P(d(X_\epsilon^{(n)}, X_\epsilon) > \delta) = -\infty.$$

We are in the place to present the contraction principle for the measurable map.

Theorem 2.6.7. *([32] Theorem 4.2.23) Let E, E' be two Polish spaces, and f be a measurable map from E to E' . Suppose that $\{X_\epsilon\}$ satisfies the LDP with a good rate function I . Let f_n be a continuous map from E to E' for $n = 1, 2, \dots$. Assume that for every $\alpha < \infty$,*

$$\limsup_{n \rightarrow \infty} \sup_{\{x: I(x) \leq \alpha\}} d(f_n(x), f(x)) = 0.$$

Then any family measures $\{\mu_\epsilon\}$ for which $\{f_n(X_\epsilon)\}$ are exponentially good approxi-

mations satisfies the LDP in E' with the good rate function

$$I'(s') = \inf \{I(s) : s \in E \text{ and } f(s) = s'\}.$$

This generalized contraction principle allows us to establish the LDP for the Brownian motion and the diffusion process, which are illustrated in the following two theorems.

Theorem 2.6.8. (Schilder's Theorem) ([32] Theorem 5.2.3) *Let $B(t)$ be a standard Brownian motion in $\mathbb{R}^d$ on $[0, 1]$, then $B^\epsilon(t) = \sqrt{\epsilon}B(t)$ satisfies the LDP in $C_0([0, 1]; \mathbb{R}^d)$, the space of all continuous functions $\phi : [0, 1] \rightarrow \mathbb{R}^d$ with $\phi(0) = 0$, with the good rate function*

$$I(\phi) = \begin{cases} \frac{1}{2} \int_0^1 |\phi'(s)|^2 ds, & \phi \in H_1 \\ \infty & \text{otherwise} \end{cases} \quad (2.6.1)$$

where H_1 is the subspace of absolutely continuous functions in $C_0([0, 1]; \mathbb{R}^d)$ with square integrable derivatives.

Theorem 2.6.9. (Freidlin–Wentzell's Theorem) ([32] Theorem 5.6.7) *Let $\{x_t^\epsilon\}$ be the diffusion process that is the unique solution of the SDE*

$$dx_t^\epsilon = b(x_t^\epsilon)dt + \sqrt{\epsilon}\sigma(x_t^\epsilon)dB_t, 0 \leq t \leq 1, x_0^\epsilon = x$$

where $x \in \mathbb{R}^d$, b is a uniformly continuous function from $\mathbb{R}^d$ to $\mathbb{R}^d$, $\sigma(\cdot)$ is bounded and uniformly Lipschitz continuous, B is a standard Brownian motion in $\mathbb{R}^d$. Then $\{x_t^\epsilon\}$ satisfies the LDP in $C([0, 1]; \mathbb{R}^d)$ with the good rate function

$$I(f) = \inf_{g \in \Lambda_f} \frac{1}{2} \int_0^1 |g'(s)|^2 ds,$$

where Λ_f is the set

$$\Lambda_f = \left\{ g \in H_1 : f(t) = x + \int_0^t b(f(s))ds + \int_0^t \sigma(f(s))g'(s)ds \right\}.$$

2.7 Euler-Maruyama methods

For a given SDE on $[0, 1]$,

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t, X_0 = x_0 \in \mathbb{R}^d \quad (2.7.1)$$

The Euler-Maruyama scheme is a discrete approximation to the solution. We fix the grid length $h = \frac{1}{N}$ first, then we let $X_0^h = x_0$ and

$$X_n^h = X_{n-1}^h + b(X_{n-1}^h, t_{n-1}^h)h + \sigma(X_{n-1}^h, t_{n-1}^h)\Delta_n^h B$$

for $n = 1, 2, \dots, N$, where $t_n^h = \frac{n}{N}$ and $\Delta B_n^h = B_{t_{n+1}^h} - B_{t_n^h}$, then we have the following strong convergence result.

Theorem 2.7.1. (Strong convergence of the Euler scheme) (*[5] Theorem 3.1*) Suppose that b, σ satisfy global Lipschitz and of linear growth conditions, and for some constant K

$$|b(x, t) - b(x, s)| + |\sigma(x, t) - \sigma(x, s)| \leq K(1 + |x|)\sqrt{t - s}$$

for $0 \leq s \leq t \leq T, x \in \mathbb{R}^d$, then

$$\mathbb{E}|X(1) - X^h(1)| = \mathcal{O}(h^{\frac{1}{2}}).$$

Chapter 3

Tracking the vortex motion by using Brownian fluid particles

3.1 Introduction

This chapter aims to present a simple vortex dynamics system which appears to approximate the motions of vortex dynamics. Let $u = (u^1, u^2, u^3)$ be the velocity of an incompressible fluid flow, and $\omega = (\omega^1, \omega^2, \omega^3)$ be the vorticity of u .

Recall that the vorticity equation is

$$\frac{\partial}{\partial t}\omega^i + u^j \frac{\partial}{\partial x^j}\omega^i = \nu \Delta \omega^i + \omega^j \frac{\partial}{\partial x^j}u^i \quad (3.1.1)$$

for $i = 1, 2, 3$. The study of this equation can be formulated as a Cauchy's initial value problem to the vorticity equation (3.1.1) together with the equation that $\omega = \text{curl} u$, subject to the initial vorticity $\omega_0 = \omega(\cdot, 0)$. We aim to provide numerical schemes to the initial value problem.

The main idea of our scheme is as follows. From now on, we use W and V instead of ω and u when referring to the approximated vector fields. Suppose the

initial vorticity of the fluid flow is given by

$$W^i(x, 0) = \sum_n A_n^i(0) \varphi(x - x_n) \quad (3.1.2)$$

where $A_n(0)$, vectors with components $A_n^i(0)$ (for $i = 1, 2, 3$) are the initial vortices, and φ is a smooth approximation of the Dirac delta function. The initial vorticity $W(x, 0)$ will be transported along the Brownian fluid particles. More precisely, the initial vorticity $A_n(0)$ is transported to a new location $X_n(t)$, and due to the stretching of vorticity, $A_n(0)$ becomes a new vorticity $A_n(t) = (A_n^1(t), A_n^2(t), A_n^3(t))$, so the vorticity field at $t > 0$ is given by

$$W^i(x, t) = \sum_n A_n^i(t) \varphi(x - X_n(t)) \quad (3.1.3)$$

for $i = 1, 2, 3$. Now the key step would be to determine the dynamics of $A_n(t)$ and $X_n(t)$. From the Biot-Savart law (2.1.16),

$$V^i(x, t) = \int_{\mathbb{R}^3} \epsilon^{ijk} G^j(x - y) W^k(y, t) dy \quad (3.1.4)$$

where $G = (G^1, G^2, G^3)$ and

$$G(z) = -\frac{1}{4\pi} \frac{z}{|z|^3}$$

The dynamics of X_n should be a diffusion with V being its drift, so that

$$dX_n^i(t) = V^i(X_n(t), t)dt + \sqrt{2\nu}dB^i(t), X_n(0) = x_n \quad (3.1.5)$$

for $i = 1, 2, 3$ and $B = (B^1, B^2, B^3)$ is a standard 3D Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We should regard X_n as some *imaginary* particles in the flow which carries the vorticity. In the absence of viscosity, vorticity is transported along the actual particle trajectories. However, in the presence of viscosity, the vorticity diffuses during convection. We introduce the Brownian motion here to capture this diffusion

phenomenon. This idea draws inspiration from [22], where it was utilized in simulating 2D viscous flows.

To determine the dynamics of $A_n = (A_n^1, A_n^2, A_n^3)$, from the vorticity equation (3.1.1), the derivative of the vorticity with respect to the particle trajectory is driven by the diffusion and the vortex stretching. The diffusion has been accounted for by incorporating Brownian motion into the particle dynamics. Now we propose the following ordinary differential equations to deal with the influence for the vortex stretching:

$$dA_n^j(t) = A_n^k(t) \frac{\partial V^j}{\partial x^k}(X_n(t), t) dt, A_n(0) = A_n(0) \quad (3.1.6)$$

for $i = 1, 2, 3$, which are responsible for the vortex stretching of the fluid flow. We will show that the system constructed by (3.1.3), (3.1.4), (3.1.5) and (3.1.6) is a good approximation to the motion of vortex dynamics by presenting several theoretical results and simulations.

The chapter is based on a submitted paper (cf. [93]), and is organized as the following. In Section 3.2, we give a more detailed description of the SDE system (3.1.3), (3.1.4), (3.1.5) and (3.1.6). This system of SDEs allows us to employ the Monte Carlo simulation to the study of incompressible fluid flows, which will be demonstrated in Section 3.7. In Section 3.3, a approximation vorticity equation is derived based on our expressions of the vorticity and velocity fields, which takes a form of a simple stochastic partial differential equation (SPDE). In Sections 3.4 and 3.5, we discuss the weak convergence results with respect to certain Sobolev norms, which justify our simple vortex method. In Section 3.6, we discuss the random vortex methods for 2D incompressible flows, which is unsurprisingly simpler and the weak convergence holds for all time.

3.2 Description of the random vortex method

In this section we give a detailed description of our random vortex system. The random vortex dynamics system proposed here would be in a slightly general setting compared with the approach outlined in the Introduction.

The main change will be made for the relation between the vorticity field W and the velocity V which may be not given by the vector identity $W = \text{curl} V$, instead it will be given in terms of a vector integral kernel $K = (K^1, K^2, K^3)$, where K^i are locally integrable functions on $\mathbb{R}^3$. The reason to work with a slightly general kernel K than the Biot-Savart kernel G is the following. For solving the initial value problem to the vorticity equations (3.1.1) where $\omega = \text{curl} u$ and $\text{div} u = 0$, which are equivalent to the system (3.1.1) and (3.1.4), mathematical difficulty arises since the Biot-Savart kernel G is singular, so it is natural to replace G by its smooth approximations. Moreover, we will see that our numerical schemes can be naturally generalized to other similar PDEs as well. Therefore, we employ a general kernel, rather than the specific Biot-Savart kernel, to illustrate a broader framework. A simple way to create an approximation is to mollify the Biot-Savart kernel G . That is, choosing a smooth function $\psi \geq 0$ with a compact support in $(-\frac{1}{2}, \frac{1}{2})^3$ and $\int_{\mathbb{R}^3} \psi(x) dx = 1$. For each $\delta > 0$, set $\psi_\delta(x) = \delta^{-3} \psi(\delta^{-1} x)$ and $G_\delta(x) = G \star \psi_\delta(x)$, where $\star$ denotes the convolution, i.e.

$$G_\delta(x) = \int_{\mathbb{R}^3} G(y) \psi_\delta(x - y) dy. \quad (3.2.1)$$

Then G_δ is smooth and

$$\|D^k G_\delta\|_\infty \leq C_k \frac{1}{\delta^{2+k}} \quad (3.2.2)$$

for some constant $C_k > 0$, for $k = 1, 2, \dots$, depending only on the regularization ψ . Moreover $G_\delta \rightarrow G$ as $\delta \downarrow 0$ in distribution sense. Therefore the solution $\omega(x, t)$ to the following initial value problem

$$\frac{\partial}{\partial t} \omega^i + u^j \frac{\partial}{\partial x^j} \omega^i = \nu \Delta \omega^i + \omega^j \frac{\partial}{\partial x^j} u^i \quad (3.2.3)$$

with the initial data $\omega(x, 0) = \omega_0(x)$, and

$$u^i(x, t) = \int_{\mathbb{R}^3} \varepsilon^{ijk} G_\delta^j(x - y) \omega^k(y, t) dy \quad (3.2.4)$$

for $\delta > 0$ approximates the solution to the vorticity equation. We denote the solution by ω^δ and u^δ . In fact as long as the initial data ω_0 is smooth with a compact support, then $\omega^\delta \rightarrow \omega$ as $\delta \downarrow 0$ with respect to some Sobolev norm at least for small time. Therefore, from the computational view-point, we only need to develop numerical schemes for the approximation solutions ω^δ and u^δ , thus it is important to work with kernels K such as G_δ . We would like to point out that in implementing the random vortex method, it seems that we can use a singular kernel G directly instead of using a smooth approximation G_δ , but a smooth kernel will ease our efforts in proving the convergence of our numerical schemes.

Let us now define the random vortex system. In defining the distribution of the initial vortices (3.1.2) the wavelet type function φ is assumed to be smooth with a compact support about the origin 0. Let us assume that the support of $\varphi \geq 0$ lies inside the ball at 0 with radius $r_\varphi > 0$ which will be chosen to be small, and the total mass $\int_{\mathbb{R}^3} \varphi(x) dx = 1$. These are the structure data for our random vortex scheme, which are fixed if we are not caring about the convergence issue.

Let $W(x, 0)$ and $W(x, t)$ (for $t > 0$) be given by (3.1.2) and (3.1.3) respectively, where x_n and $A_n(0)$ are sampled initially, so they are the fixed data too. We then define the vector field $V(x, t)$ by the convolution (together with the cross product)

$$V^i(x, t) = \int_{\mathbb{R}^3} \varepsilon^{ijk} K^j(x - y) W^k(y, t) dy \quad (3.2.5)$$

which coincides with (3.1.4) if K is the Biot-Savart kernel G . The dynamics of (X_n, A_n) are still defined by (3.1.5) and (3.1.6). Together with (3.1.3) we deduce that

$$V^i(x, t) = - \sum_n \varepsilon^{ikj} A_n^k(t) K_\varphi^j(x - X_n(t)) \quad (3.2.6)$$

for $i = 1, 2, 3$, $t \geq 0$ and $x \in \mathbb{R}^3$, where

$$K_\varphi^j(x) = \int_{\mathbb{R}^3} K^j(x - y) \varphi(y) dy$$

(where $j = 1, 2, 3$) turns out to be the mollification of K^j by φ . This is a nice feature in this scheme. By utilizing equation (3.2.6) and substituting it into (3.1.5), the dynamic system for the Brownian fluid particles X_n can be reformulated as the following SDEs

$$\begin{aligned} dX_n^i(t) &= - \sum_m \varepsilon^{ikj} A_m^k(t) K_\varphi^j(X_n(t) - X_m(t)) dt + \sqrt{2\nu} dB^i(t), \\ X_n(0) &= x_n, \end{aligned} \tag{3.2.7}$$

Similarly, the dynamics system (3.1.6) for A_n may be written as the following ODEs

$$\begin{aligned} dA_n^i(t) &= A_n^l(t) \sum_m \varepsilon^{ijk} \frac{\partial K_\varphi^j}{\partial x^l} (X_n(t) - X_m(t)) A_m^k(t) dt, \\ A_n(0) &= A_n(0), \end{aligned} \tag{3.2.8}$$

where $i = 1, 2, 3$, m and n run through the finite range of the initial locations x_n .

The system of SDEs (3.2.7), (3.2.8) is closed and depends only on the structure data K and φ . We observe that K_φ is smooth if K is locally integrable under our assumption on φ . Therefore, we have the following theorem regarding the existence of the solution of our dynamical system (3.2.7), (3.2.8).

Theorem 3.2.1. *Given a finite collection of loci $x_n \in \mathbb{R}^3$ and a family of initial vortices $A_n(0)$, there is a unique maximal strong solution (X_n, A_n) to SDEs (3.2.7, 3.2.8) up to the explosion time $\tau > 0$. The vector fields defined by*

$$W^i(x, t) = \sum_n A_n^i(t) \varphi(x - X_n(t)) \tag{3.2.9}$$

and

$$V^i(x, t) = - \sum_n \varepsilon^{ikj} A_n^k(t) K_\varphi^j(x - X_n(t)) \quad (3.2.10)$$

are smooth in x for $t < \tau$, where $i = 1, 2, 3$. Moreover the explosion time $\tau \geq \tau_{A(0), \varphi}$, where

$$\tau_{A(0), \varphi} = \frac{1}{\|A(0)\| \|DK_\varphi\|_\infty}, \quad (3.2.11)$$

$\|A\| = \sum_n |A_n|$ and $\|f\|_\infty$ denotes the L^∞ -norm of a function on $\mathbb{R}^3$. We note that $\tau_{A(0), \varphi} > 0$ is deterministic.

Proof. The proof follows from the standard result in Itô's theory of stochastic differential equations (cf. Theorem 2.4.1. The coefficients in defining SDEs (3.2.7) and (3.2.8) are in general not globally Lipschitz, but nevertheless locally Lipschitz continuous. Therefore the explosion time τ may be finite, and the maximal strong solution (X_n, A_n) is unique for a given 3D Brownian motion $W = (W^1, W^2, W^3)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. To derive the estimate (3.2.11), we utilize a truncation technique. For each $L > 0$, let $\theta_L(s)$ be the cutting-off function which equals s if $|s| \leq L$ and equals L if $|s| > L$. The unique strong solution (X^L, A^L) to the truncated SDEs

$$dX_n^i(t) = - \sum_m \varepsilon^{ikj} \theta_L(A_m^k(t)) K_\varphi^j(X_n(t) - X_m(t)) dt + \sqrt{2\nu} dB^i(t)$$

and

$$dA_n^i(t) = \theta_L(A_n^l(t)) \sum_m \varepsilon^{ijk} \frac{\partial K_\varphi^j}{\partial x^l}(X_n(t) - X_m(t)) \theta_L(A_m^k(t)) dt$$

subject to the same initial data exist for $t \in [0, \infty)$ since the coefficients are globally Lipschitz and of finite linear growth (cf. Theorem 2.4.1). It follows from the second equation that

$$|A_n^L(t)| \leq |A_n^L(0)| + \int_0^t |A_n^L(s)| \sum_m \|DK_\varphi\|_\infty |A_m^L(s)| ds. \quad (3.2.12)$$

Then by adding the inequalities where n runs through the index set, we obtain that

$$\|A^L(t)\| \leq \|A(0)\| + \|DK_\varphi\|_\infty \int_0^t \|A^L(s)\|^2 ds \quad (3.2.13)$$

for $t > 0$. By Gronwall's inequality,

$$\|A^L(t)\| \leq \frac{\|A(0)\|}{1 - \|A(0)\| \|DK_\varphi\|_\infty t} \quad (3.2.14)$$

for all $t < \tau_{A(0),\varphi}$. The estimate (3.2.11) now follows immediately by sending $L \uparrow \infty$. $\square$

The following remark will be useful later.

Remark 3.2.2. *From (3.2.14), it is clear that*

$$\|A(t)\| \leq 2\|A(0)\| \quad (3.2.15)$$

for $t < \frac{1}{2}\tau_{A(0),\varphi}$.

By Gronwall's inequality, it follows from (3.2.12) that

$$|A_n(t)| \leq |A_n(0)| e^{\|DK_\varphi\|_\infty \int_0^t \|A(s)\| ds} \quad (3.2.16)$$

for $t < \tau_{A(0),\varphi}$, and therefore, by (3.2.15)

$$|A_n(t)| \leq |A_n(0)| e^{2\|DK_\varphi\|_\infty \|A(0)\| t} \quad (3.2.17)$$

for $t < \frac{1}{2}\tau_{A(0),\varphi}$.

3.3 Approximating the vorticity equation

In this section, we demonstrate that (W, V) defined in Theorem 3.2.1 is an approximation solution to the vorticity equation in a certain sense. We shall assume that

same conditions hold for the structure data K, φ as in the previous section, i.e. K is locally integrable, and φ is a smooth non-negative function of total mass 1 with a compact support around 0, and we let (X_n, A_n) be the unique maximal solution pair to the SDEs (3.2.7) and (3.2.8), and $V(x, t)$ and $W(x, t)$ are defined in Theorem 3.2.1.

The relationship between $W(x, t)$ and $V(x, t)$ will be established by a SPDE as shown in the following theorem.

Theorem 3.3.1. *For each $x \in \mathbb{R}^3$, $W(x, t)$ and $V(x, t)$ are continuous semi-martingales for $t \in [0, \tau_{A(0), \varphi})$, and*

$$\begin{aligned} dW^i &= \left(W^j \frac{\partial V^i}{\partial x^j} - V^j \frac{\partial W^i}{\partial x^j} + \nu \Delta W^i \right) dt \\ &\quad - \sqrt{2\nu} \frac{\partial W^i}{\partial x^j} dB^j(t) + (F^i + G^i) dt \end{aligned} \quad (3.3.1)$$

for $t \in [0, \tau_{A(0), \varphi}]$, where the error terms

$$F^i(x, t) = \sum_n A_n^i(t) (V^j(x, t) - V^j(X_n(t), t)) \frac{\partial \varphi}{\partial x^j}(x - X_n(t)) \quad (3.3.2)$$

and

$$G^i(x, t) = \sum_n A_n^j(t) \left(\frac{\partial V^i}{\partial x^j}(X_n(t), t) - \frac{\partial V^i}{\partial x^j}(x, t) \right) \varphi(x - X_n(t)). \quad (3.3.3)$$

Moreover, according to (3.2.10) and (3.2.9), $V = K \star W$ where the convolution is made with the wedge product of two vectors.

Proof. Since $A_n(x, t)$ are continuous semi-martingales with finite variations (in t), by applying Itô's formula to W^i define in (3.2.9), we have

$$\begin{aligned} &dW^i(x, t) \\ &= \sum_n \varphi(x - X_n(t)) dA_n^i(t) - \sum_n A_n^i(t) \frac{\partial \varphi}{\partial x^j}(x - X_n(t)) dX_n^j(t) \\ &\quad + \nu \sum_n A_n^i(t) \Delta \varphi(x - X_n(t)) dt. \end{aligned}$$

Next we substitute dX_n and dA_n by using the SDEs (3.2.7) and (3.2.8) to obtain that

$$\begin{aligned}
& dW^i(x, t) \\
&= \sum_n \sum_m \varepsilon^{iqk} A_n^l(t) A_m^k(t) \varphi(x - X_n(t)) \frac{\partial K_\varphi^q}{\partial x^l}(X_n(t) - X_m(t)) dt \\
&+ \sum_n \sum_m \varepsilon^{jkq} A_m^k(t) A_n^i(t) K_\varphi^q(X_n(t) - X_m(t)) \frac{\partial \varphi}{\partial x^j}(x - X_n(t)) dt \\
&+ \nu \sum_n A_n^i(t) \Delta \varphi(x - X_n(t)) dt \\
&- \sqrt{2\nu} \sum_n A_n^i(t) \frac{\partial \varphi}{\partial x^j}(x - X_n(t)) dB^j(t). \tag{3.3.4}
\end{aligned}$$

On the other hand, according to the construction (3.2.9), (3.2.10) we have

$$\begin{aligned}
& V^j \frac{\partial W^i}{\partial x^j}(x, t) \\
&= - \sum_m \sum_n \varepsilon^{jkq} A_m^k(t) A_n^i(t) K_\varphi^q(x - X_m(t)) \frac{\partial \varphi}{\partial x^j}(x - X_n(t)) \\
&= - \sum_m \sum_n \varepsilon^{jkq} A_m^k(t) A_n^i(t) K_\varphi^q(X_n(t) - X_m(t)) \frac{\partial \varphi}{\partial x^j}(x - X_n(t)) \\
&+ \sum_n A_n^i(t) (V^j(x, t) - V^j(X_n(t), t)) \frac{\partial \varphi}{\partial x^j}(x - X_n(t)),
\end{aligned}$$

$$\begin{aligned}
& W^j \frac{\partial V^i}{\partial x^j}(x, t) \\
&= - \sum_n \sum_m \varepsilon^{ikq} A_m^j(t) A_n^k(t) \varphi(x - X_m(t)) \frac{\partial K_\varphi^q}{\partial x^j}(x - X_n(t)) \\
&= \sum_n \sum_m \varepsilon^{iqk} A_n^l(t) A_m^k(t) \varphi(x - X_n(t)) \frac{\partial K_\varphi^q}{\partial x^l}(x - X_m(t)) \\
&= \sum_n \sum_m \varepsilon^{iqk} A_n^l(t) A_m^k(t) \varphi(x - X_n(t)) \frac{\partial K_\varphi^q}{\partial x^l}(X_n(t) - X_m(t)) \\
&- \sum_n A_n^l(t) \left(\frac{\partial V^i}{\partial x^l}(X_n(t), t) - \frac{\partial V^i}{\partial x^l}(x, t) \right) \varphi(x - X_n(t))
\end{aligned}$$

and

$$\Delta W^i(x, t) = \sum_n A_n^i(t) \Delta \varphi(x - X_n(t)).$$

(3.3.1) follows from substituting these equations into (3.3.4). $\square$

We next show that the SPDE (3.3.1) is a good approximation to the vorticity transport equation (3.1.1) with the initial vorticity $W(x, 0)$. More precisely, in the following lemma, we will show that the error terms F and G which appeared in (3.3.1) can be controlled by r_φ .

Lemma 3.3.2. *Suppose the support of φ is contained in the ball centered at 0 with radius $r_\varphi > 0$.*

1) *It holds that*

$$|F(x, t)| \leq 2 \|A(t)\|^2 \|DK_\varphi\|_\infty \|D\varphi\|_\infty r_\varphi$$

and

$$|G(x, t)| \leq 2 \|A(t)\|^2 \|D^2 K_\varphi\|_\infty \|\varphi\|_\infty r_\varphi$$

for $t < \tau_{A(0), \varphi}$ and $x \in \mathbb{R}^3$.

2) *There is a positive constant C depending only on $\|A(0)\|$, $\|DK_\varphi\|_\infty$, $\|D^2 K_\varphi\|_\infty$, $\|D\varphi\|_\infty$ and $\|\varphi\|_\infty$ such that*

$$|F(x, t)| \leq Cr_\varphi \quad \text{and} \quad |G(x, t)| \leq Cr_\varphi$$

for all $t \in [0, \frac{1}{2}\tau_{A(0), \varphi}]$ and $x \in \mathbb{R}^3$.

Proof. By definition (3.2.10),

$$V^i(x, t) = - \sum_n \varepsilon^{ikj} A_n^k(t) K_\varphi^j(x - X_n(t)). \quad (3.3.5)$$

Hence, we have

$$|V(x, t) - V(y, t)| \leq 2 \|A(t)\| \|DK_\varphi\|_\infty |x - y| \quad (3.3.6)$$

and

$$|DV(x, t) - DV(y, t)| \leq 2 \|A(t)\| \|D^2K_\varphi\|_\infty |x - y|. \quad (3.3.7)$$

Therefore, 1) follows immediately from (3.3.2) and (3.3.3) and 2) follows from 1). $\square$

Corollary 3.3.3. *Suppose the support of the wavelet function φ is contained in the ball centered at 0 with radius $r_\varphi > 0$. Let*

$$\begin{aligned} & U^i(x, t) \\ &= W^i(x, t) - W^i(x, 0) \\ & - \int_0^t \left(W^j \frac{\partial V^i}{\partial x^j} + V^j \frac{\partial W^i}{\partial x^j} + \nu \Delta W^i \right) (x, s) ds \end{aligned} \quad (3.3.8)$$

for $i = 1, 2, 3$ and $t < \tau_{A(0), \varphi}$ and $x \in \mathbb{R}^3$. Then

$$|\mathbb{E}[U^i(x, t)]| \leq C_1 \tau_{A(0), \varphi} r_\varphi \quad (3.3.9)$$

for all $t \in [0, \frac{1}{2} \tau_{A(0), \varphi}]$.

Proof. By (3.3.1) we have

$$U^i(x, t) = -\sqrt{2\nu} \int_0^t \frac{\partial W^i}{\partial x^j} dB^j(s) + \int_0^t (F^i + G^i) ds.$$

Under the assumptions on φ , $\int_0^t \frac{\partial W^i}{\partial x^j} dB^j(s)$ are martingales for $t \in [0, \frac{1}{2} \tau_{A(0), \varphi}]$, which means

$$\mathbb{E}[U^i(x, t)] = \mathbb{E} \int_0^t (F^i + G^i) ds$$

for $i = 1, 2, 3$ and (3.3.9) follows from Lemma 3.3.2, 2). $\square$

Remark 3.3.4. *If $K = G$ is the Biot-Savart kernel, then $\operatorname{div} V = 0$ according to the*

formula (3.2.10) and

$$V^i(x, t) = \int_{\mathbb{R}^3} \varepsilon^{ijk} G^j(x - y) W^k(y, t) dy \quad (3.3.10)$$

which implies that $\Delta V = -\text{curl} W$. Therefore $W = \text{curl} V + \nabla f$ for some scalar function f , and V and W satisfy the vorticity equation approximately in the sense stated in Corollary 3.3.3. The equation that $W = \text{curl} V$ may fail in general due to the fact that $W(x, 0)$ may be not divergence-free. The reason why the approximation V and W still do the job (see the simulations below) nicely is that in practice φ is close to an indicator function of a small ball, and therefore $W(x, t)$ is nearly divergence-free, and the relation $W \simeq \text{curl} V$ may be restored approximately. In fact, outside the support of φ , W is always 0 so that the divergence of W vanishes in the almost whole space. We admit that it is challenging to give a satisfactory bound for the $\text{div} W$ near the particles $X_n(t)$. However, during practical simulations, our primary focus lies on the approximated velocity fields. Therefore, the divergence issue of W is not a matter of significant concern.

3.4 Weak convergence of the approximation

A simple procedure of sampling the initial vortices, for the sake of theoretical study, may be described as the following. Suppose ω_0 is the initial vorticity, which is smooth with a compact support S , to the initial value problem to the vorticity equations (3.1.1). Let ψ be described at the beginning of the previous section which defines G_δ for every $\delta > 0$.

Lemma 3.4.1. *Let $p \geq 1$, and the error terms in the random vortex scheme, $F(x, t)$ and $G(x, t)$ be defined by (3.3.2) and (3.3.3).*

1) *The following two estimates hold:*

$$\|F(\cdot, t)\|_{W^{-1,p}} \leq 6 \|A(t)\|^2 (\|DK_\varphi\|_\infty \|\varphi\|_{L^p} r_\varphi + \|\text{curl} K_\varphi\|_\infty \|\varphi\|_{L^p}) \quad (3.4.1)$$

and

$$\|G(x, t)\|_{L^p} \leq 2 \|A(t)\|^2 \|D^2 K_\varphi\|_\infty \|\varphi\|_{L^p} r_\varphi \quad (3.4.2)$$

for all $t \in [0, \frac{1}{2}\tau_{A(0), \varphi}]$, where $\|\cdot\|_{W^{s,p}}$ denote the Sobolev norms, see Theorem 2.3.4.

2) If the singular kernel $K = G_\delta$, where $\delta > 0$, given in (3.2.1), then there are universal constant C_1, C_2 such that

$$\|F(\cdot, t)\|_{W^{-1,p}} \leq C_2 \frac{r_\varphi}{\delta^3} e^{C_1 \frac{2}{\delta^3} \|A(0)\|} \|A(0)\|^2 \|\varphi\|_{L^p}$$

and

$$\|G(x, t)\|_{L^p} \leq C_2 \frac{r_\varphi}{\delta^4} e^{C_1 \frac{2}{\delta^3} \|A(0)\|} \|A(0)\|^2 \|\varphi\|_{L^p}$$

for $t \in [0, \frac{1}{2}\tau_{A(0), \varphi} \wedge 1]$.

Proof. It is clear that from definition and the property of mollification,

$$\|D^k K_\varphi\|_\infty \leq \|D^k K\|_\infty.$$

In particular, if $K = G_\delta$, then

$$\|D^k K_\varphi\|_\infty \leq C_k \frac{1}{\delta^{2+k}}$$

for some universal constants $C_1, C_2, \dots$. Consider the linear functional

$$f_i(h) = \int_{\mathbb{R}^3} F^i(x, t) h(x) dx,$$

where h defined on $\mathbb{R}^3$ is smooth with a compact support, i.e. a test function. Since

$$F^i(x, t) = \sum_n A_n^i(t) (V^j(x, t) - V^j(X_n(t), t)) \frac{\partial \varphi}{\partial x^j}(x - X_n(t)), \quad (3.4.3)$$

we have

$$\begin{aligned}
& f_i(h) \\
&= - \sum_n A_n^i(t) \int_{\mathbb{R}^3} (V^j(x, t) - V^j(X_n(t), t)) \varphi(x - X_n(t)) \frac{\partial h(x)}{\partial x^j} dx \\
& - \int_{\mathbb{R}^3} \sum_n \operatorname{div} V(x, t) h(x) \varphi(x - X_n(t)) A_n^i(t) dx.
\end{aligned}$$

By the definition for V (see (3.2.10)) one has the elementary estimate (see the proof of Lemma 3.3.2)

$$|V^j(x, t) - V^j(y, t)| \leq 2 \|A(t)\| \|DK_\varphi\|_\infty |x - y|.$$

Since

$$\begin{aligned}
\operatorname{div} V(x, t) &= - \sum_n \varepsilon^{ikj} A_n^k(t) \frac{\partial}{\partial x^i} K_\varphi^j(x - X_n(t)) \\
&= \sum_n A_n(t) \cdot \operatorname{curl} K_\varphi(x - X_n(t)),
\end{aligned}$$

we obtain the following bound for $\operatorname{div} V(x, t)$:

$$|\operatorname{div} V(x, t)| \leq \|A(t)\| \|\operatorname{curl} K_\varphi\|_\infty.$$

By using these estimates and the assumption that the support of φ lies in the ball centered at 0 with radius r_φ , we therefore deduce that

$$\begin{aligned}
|f_i(h)| &\leq 2 \|A(t)\| \|DK_\varphi\|_\infty r_\varphi \sum_n |A_n^i(t)| \int_{\mathbb{R}^3} |\varphi(x - X_n(t))| |Dh(x)| dx \\
&+ \|A(t)\| \|\operatorname{curl} K_\varphi\|_\infty \sum_n |A_n^i(t)| \int_{\mathbb{R}^3} |\varphi(x - X_n(t))| h(x) dx \\
&\leq 2 \|A(t)\|^2 \|DK_\varphi\|_\infty r_\varphi \max_n \int_{\mathbb{R}^3} |\varphi(x - X_n(t))| |Dh(x)| dx \\
&+ \|A(t)\|^2 \|\operatorname{curl} K_\varphi\|_\infty r_\varphi \max_n \int_{\mathbb{R}^3} |\varphi(x - X_n(t))| h(x) dx
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \|A(t)\|^2 \|DK_\varphi\|_\infty \|\varphi\|_{L^p} r_\varphi \|Dh\|_{L^q} \\
&+ \|A(t)\|^2 \|\operatorname{curl} K_\varphi\|_\infty \|\varphi\|_{L^p} \|h\|_{L^q},
\end{aligned}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, for every test function h and $i = 1, 2, 3$. Therefore

$$\|F(\cdot, t)\|_{W^{-1,p}} \leq 6 \|A(t)\|^2 (\|DK_\varphi\|_\infty \|\varphi\|_{L^p} r_\varphi + \|\operatorname{curl} K_\varphi\|_\infty \|\varphi\|_{L^p}).$$

The estimate for the error term G^i is trivial. In fact by (3.3.3) and (3.3.7) to obtain

$$\begin{aligned}
&\|G^i(x, t)\|_{L^p} \\
&\leq \left\| \sum_n A_n^j(t) \left(\frac{\partial V^i}{\partial x^j}(X_n(t), t) - \frac{\partial V^i}{\partial x^j}(x, t) \right) \varphi(x - X_n(t)) \right\|_{L^p} \\
&\leq 2 \|A(t)\|^2 \|D^2 K_\varphi\|_\infty \|\varphi\|_{L^p} r_\varphi
\end{aligned}$$

In particular, if $K = G_\delta$, then $\operatorname{curl} K_\varphi = 0$, so that

$$\begin{aligned}
\|F(\cdot, t)\|_{W^{-1,p}} &\leq 6 \|A(t)\|^2 \|DK_\varphi\|_\infty \|\varphi\|_{L^p} r_\varphi \\
&\leq 6C_1 \frac{r_\varphi}{\delta^3} \|A(t)\|^2 \|\varphi\|_{L^p},
\end{aligned}$$

and

$$\|G(x, t)\|_{L^p} \leq 6C_2 \frac{r_\varphi}{\delta^4} \|A(t)\|^2 \|\varphi\|_{L^p}.$$

The other conclusions follow from the following estimate: if $K = G_\delta$, then $\|DK_\varphi\| \leq C_1 \frac{1}{\delta^3}$, thus by estimate (3.2.17) we obtain that

$$\begin{aligned}
\|A(t)\| &\leq \|A(0)\| e^{2\|DK_\varphi\|_\infty \|A(0)\| t} \\
&\leq \|A(0)\| e^{2C_1 \frac{1}{\delta^3} \|A(0)\| t} \\
&\leq \|A(0)\| e^{C_1 \frac{1}{\delta^3} \|A(0)\|}
\end{aligned}$$

for all $t \in [0, \frac{1}{2}\tau_{A(0),\varphi} \wedge 1]$. □

We are now in a position to state a weak convergence theorem. Choose $\delta > 0$ so the solutions ω^δ and u^δ solving (3.2.3) and (3.2.4) are good approximations to the vorticity equations (3.1.1). Let $h > 0$. Let x_n where $n = (n^1, n^2, n^3) \in \mathbb{Z}^3$ the center of the lattice (open) box B_n with size h whose lower-left corner is hn . Then ω_0 may be approximated by $\sum_n \omega_0(x_n) 1_{B_n}$ in L^p space for $p \geq 1$. Let $W^h(x, 0) = \sum_n A_{n,h}(0) \phi_h(x - x_n)$ with $A_{n,h}(0) = \omega_0(x_n) h^3$, where n runs over all n such that $B_n \cap S \neq \emptyset$ (S is the support of ω_0), and $\phi_h(x) = \frac{8}{h^3} \phi\left(\frac{2}{h}x\right)$, where $\phi \geq 0$ is chosen so that ϕ is an approximation of 1_B in some Sobolev space, ϕ is smooth with a compact support in B , where $B = (-1/2, 1/2)^3$. Then

$$\|A_{\cdot,h}(0)\| = \sum_n |\omega_0(x_n)| h^3 \leq h|S| \|D\omega_0\|_\infty + \|\omega_0\|_{L^1} \leq |S| \|D\omega_0\|_\infty + \|\omega_0\|_{L^1}$$

which is independent of $h \in (0, 1)$ and the lattice size. With this $W^h(x, 0)$ as the initial sampling distribution, according to Corollary 3.3.3, $V^{\delta,h}, W^{\delta,h}$ defined by (3.2.10) and (3.2.9) with $K = G_\delta$ and $W^{\delta,h}(x, 0) = W^h(x, 0)$, are approximations in mean of the vorticity equations (3.1.1) with initial vorticity ω_0 . Notice that

$$\begin{aligned} K_{\phi_h}^j(x) &= \frac{8}{h^3} \int_{\mathbb{R}^3} K^j(y) \phi_h\left(\frac{2}{h}(x - y)\right) dy \\ &= \frac{8}{h^3} \int_{\mathbb{R}^3} K^j(x - y) \phi_h\left(\frac{2}{h}y\right) dy \\ &= \int_{|y|_\infty < \frac{1}{2}} K^j\left(x - \frac{h}{2}y\right) \phi(y) dy \end{aligned}$$

which yields that

$$\|D^k K_{\phi_h}\|_\infty \leq \|D^k K\|_\infty.$$

Suppose $K = G_\delta$, then

$$\|D^k K_{\phi_h}\|_\infty \leq C_k \frac{1}{\delta^{2+k}}.$$

Theorem 3.4.2. *Let $\delta > 0$ and $h > 0$. Let ψ , which is used to define $K = G_\delta$ for every $\delta > 0$, and ϕ , which is used to define the initial $W^h(x, 0)$ for every $h > 0$, be*

two non-negative smooth functions with supports lying inside the box $(-\frac{1}{2}, \frac{1}{2})^3$ with $\int_{\mathbb{R}^3} \psi(x)dx = \int_{\mathbb{R}^3} \phi(x)dx = 1$ as above. Let

$$\begin{aligned} & U^{\delta,h}(x, t) \\ &= W^{\delta,h}(x, t) - W^h(x, 0) \\ & - \int_0^t (W^{\delta,h} \cdot \nabla V^{\delta,h} - V^{\delta,h} \cdot \nabla W^{\delta,h} + \nu \Delta W^{\delta,h})(x, s) ds. \end{aligned} \quad (3.4.4)$$

Then there is T depending only on $|S| \|D\omega_0\|_\infty + \|\omega_0\|_{L^1}$ such that

$$\|\mathbb{E} [U^{\delta,h}(\cdot, t)]\|_{W^{-1,p}} \rightarrow 0$$

for all $t \in [0, T]$, as $h \downarrow 0$ for any fixed $\delta > 0$ as long as $p < 4/3$.

Proof. This follows from the above estimates and Lemma 3.4.1 item 2). $\square$

While from the proof we can see that the error term tends to zero as $\delta \downarrow 0$ and $h \downarrow 0$ such that $h \ll \delta^4 e^{-C/\delta^3}$ (with some positive constant C). This means that we need to choose the lattice size h much smaller than the regularization δ in order to ensure the convergence result. We should point out that this estimate is very crude. Deriving the convergence rate with respect to both h and δ presents challenges due to the absence of prior assessments for solutions to the vorticity transport equations. The current understanding lacks a satisfactory evaluation of how solutions to the vorticity transport equations depend on initial data and kernel mollification. Consequently, the analysis of the convergence rate becomes inherently intricate.

We should also notice that the convergence is limited to a small time interval, which is in contrast to the convergence for the 2D flows which holds uniformly over the whole time interval (see Section 3.6). This distinction arises from the fact that the vorticity may explode after a short time interval in 3D flows. The existence of the numerical system (X_n, A_n) can only be guaranteed over a small time, a concern not encountered when dealing with 2D flows. The choice of p here is only technical

as we want F and G in Lemma 3.4.1 to converge to 0 in a suitable Banach space.

3.5 Modified random vortex dynamics

In the previous section, we have shown our random vortex system (3.2.7, 3.2.8) converges to the vorticity equations in mean, while if the viscosity $\nu > 0$ is not small, then the random perturbation term, i.e. the noise part involving Brownian motion, appearing in (3.3.1) may be not small although its mean always stays zero. To reduce the error caused by the noise, we may split each particle into N copies of the same particle and apply independent copies of Brownian motion for these particles. By incorporating independent Brownian motions for individual particles, the noise term in (3.3.1) is now driven by a sum of independent Brownian motions instead of identical Brownian motions. This modification leads to a reduction in the variance of the noise. To amplify this effect, the process of splitting particles is employed. More independent Brownian motions are incorporated in the system, while retaining the initial data and the same structure of (3.3.1) (see (3.5.3), which further minimizes the variance. Hence we propose the following SDEs

$$\begin{aligned} dX_{n,a}^i(t) &= - \sum_{m,b} \varepsilon^{ikj} A_{m,a}^k(t) K_\varphi^j(X_{n,a}(t) - X_{m,b}(t)) dt \\ &\quad + \sqrt{2\nu} dB_{n,a}^i(t), \\ X_{n,a}(0) &= x_n, \end{aligned} \tag{3.5.1}$$

and

$$\begin{aligned} dA_{n,a}^i(t) &= A_{n,a}^l(t) \sum_{m,b} \varepsilon^{ijk} \frac{\partial K_\varphi^j}{\partial x^l} (X_{n,a}(t) - X_{m,b}(t)) A_{m,b}^k(t) dt, \\ A_{n,a}(0) &= \frac{1}{N} A_n(0), \end{aligned} \tag{3.5.2}$$

where a, b runs from 1 up to N , where N is a fixed natural number, and $W_{n,a}$ are independent copies of 3D Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus we have split the distribution of the initial vortices as

$$\begin{aligned} W(x, 0) &= \sum_n A_n(0) \varphi(x - x_n) \\ &= \sum_{n,a} A_{n,a}(0) \varphi(x - x_n) \end{aligned}$$

and we have used independent Brownian motions for different locations to reduce the eventual noise in the approximation vorticity equation (3.3.1). $W(x, t)$ and $V(x, t)$ are still defined in terms of (3.2.10) and (3.2.9), where the only modifications one has to make are the sums over n, a rather than n . The approximation vorticity equations (3.3.1) has the same form but with different noise term:

$$dW^i = \left(W^j \frac{\partial V^i}{\partial x^j} - V^j \frac{\partial W^i}{\partial x^j} + \nu \Delta W^i \right) dt + H^i + (F^i + G^i) dt \quad (3.5.3)$$

where $i = 1, 2, 3$ and

$$H(x, t) = -\sqrt{2\nu} \sum_{n,a} A_{n,a}(t) \cdot \nabla \varphi(x - X_{n,a}(t)) dB_{n,a}(t)$$

which can be verified by using Itô's formula too. Hence

$$\begin{aligned} &\mathbb{E} |H(x, t)|^2 \\ &\leq 2\nu \sum_{n,a} \mathbb{E} \int_0^t |A_{n,a}(s)|^2 |\nabla \varphi(x - X_{n,a}(s))|^2 ds \\ &\leq \frac{4e\nu}{N} \|\nabla \varphi\|_\infty^2 \|A(0)\| \max_n |A_n(0)| t \end{aligned}$$

for $t \leq \frac{1}{2} \tau_{A(0), \varphi}$, which goes to zero uniformly for $x \in \mathbb{R}^3$ as $N \rightarrow \infty$. Meanwhile, the error term estimates for F and G remain the same which are independent of N . In fact from the previous estimate we may deduce that an estimate of the splitting

number N we should use in implementing the proposed method: $N \propto \nu$.

3.6 2D case

We retain the same notations as used in the previous sections, but we work with the two dimensional vorticity equation instead. 2D case is special as we have indicated – there is no non-linear stretching term in the vorticity transport equation. The vorticity $\omega(x, t)$ of an incompressible fluid flow with velocity $u = (u^1, u^2)$ can be identified with the scalar function $\frac{\partial}{\partial x^1}u^2 - \frac{\partial}{\partial x^2}u^1$, and the vorticity equation

$$\frac{\partial}{\partial t}\omega + u^j \frac{\partial}{\partial x^j}\omega = \nu \Delta \omega \quad (3.6.1)$$

appears as a “linear” parabolic equation. Hence the initial vortices are transported along the motion of Brownian fluid particles. Since $\operatorname{div} u = 0$, so that

$$\Delta u^1 = -\frac{\partial}{\partial x^2}\omega, \quad \Delta u^2 = \frac{\partial}{\partial x^1}\omega \quad (3.6.2)$$

and

$$u(x, t) = \int_{\mathbb{R}^2} G(x - y)\omega(y, t)dy$$

where in 2D case, the Biot-Savart kernel $G(x) = \frac{1}{2\pi}(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2})$ which is singular at the origin 0. We therefore need to treat it as for the 3D case to introduce $G_\delta(x) = G * \psi_\delta(x)$, where $\psi_\delta(x) = \delta^{-2}\psi(\delta^{-1}x)$, and $\psi(x)$ is a smooth function with total integral 1 and a compact support lying inside $(-1/2, 1/2)^2$. Then we have $\|DG_\delta(x)\| \leq C\frac{1}{\delta^2}$, where C depends only on the regularization ψ . We then implement the vortex method to the approximation system (3.6.1) together with the integral relation

$$u(x, t) = \int_{\mathbb{R}^2} G_\delta(x - y)\omega(y, t)dy \quad (3.6.3)$$

for each fixed $\delta > 0$. Since $\omega(x, t)$ is scalar, so that there is a simplification when we sample the distribution of initial vortices. As for the 3D case, (x_n) is a finite collection of points in $\mathbb{R}^2$ and the initial vortices $A_n(0)$ are scalars. The dynamic equation for $A_n(t)$ is no longer needed as $A_n(t) = A_n(0) = A_n$ are independent of t , and the dynamic equation for $X_n(t)$ is reduced to the following SDE

$$\begin{aligned} dX_n^i(t) &= \sum_m A_m K_\varphi^i(X_n(t) - X_m(t)) dt + \sqrt{2\nu} dB^i(t), \\ X_n(0) &= x_n, \end{aligned} \tag{3.6.4}$$

whose coefficients are globally Lipschitz, so that it has a unique strong solution $X_n(t)$ defined for all t . Let

$$W(x, t) = \sum_n A_n \varphi(x - X_n(t)) \tag{3.6.5}$$

and

$$V^i(x, t) = \sum_n A_n K_\varphi^i(x - X_n(t)) \tag{3.6.6}$$

By using Itô's formula one may deduce that

$$\begin{aligned} dW(x, t) &= -V(x, t) \cdot \nabla W(x, t) dt + \nu \Delta W(x, t) dt \\ &\quad - \sum_n A_n \nabla \varphi(x - X_n(t)) \cdot \sqrt{2\nu} dB(t) \\ &\quad - \sum_n A_n \nabla \varphi(x - X_n(t)) \cdot (V(X_n(t), t) - V(x, t)) dt. \end{aligned}$$

The non-martingale error term

$$F(x, t) = \sum_n A_n \nabla \varphi(x - X_n(t)) \cdot (V(X_n(t), t) - V(x, t)).$$

Following the same procedure as in the proof of Lemma 3.4.1, we may deduce the following global estimates for 2D case.

Lemma 3.6.1. 1) For any $p \geq 1$

$$\|F\|_{W^{-1,p}} \leq \|A\|^2 \left(\|DK_\varphi\|_\infty \|\varphi\|_{L^p} r_\varphi + \|\operatorname{div} K_\varphi\|_\infty \|\varphi\|_{L^p} \right).$$

2) If the singular kernel $K = G_\delta$, where $\delta > 0$, then

$$\|F(\cdot, t)\|_{W^{-1,p}} \leq C_5 \frac{1}{\delta^2} \|A\|^2 \|\varphi\|_{L^p} r_\varphi$$

for all $t > 0$ where $\|A\| = \sum_n |A_n|$.

Now we state the weak convergence theorem. Let $h > 0$, and let x_n where $n = (n_1, n_2) \in \mathbb{Z}^2$ the center of the lattice box B_n with size h whose lower left corner is hn . Then the initial vorticity ω_0 may be approximated by $\sum_n \omega_0(x_n) 1_{B_n}$ for example in an L^p -space. Let $W^h(x, 0) = \sum_n A_{n,h} \varphi_h(x - x_n)$ with $A_{n,h} = \omega_0(x_n) h^2$, where n runs over all n such that $B_n \cap S \neq \emptyset$, and $\varphi_h(x) = \frac{1}{h^2} \phi(\frac{x}{h})$, where ϕ is an approximation of 1_B where B is the unit square $(-1/2, 1/2)^2$ in L^p space for $p > 1$. Then

$$\|A_{\cdot,h}\| = \sum_n |\omega_0(x_n)| h^2 \leq h |S| \|D\omega_0\|_\infty + \|\omega_0\|_{L^1} \leq |S| \|D\omega_0\|_\infty + \|\omega_0\|_{L^1},$$

which is independent of $h \in (0, 1)$ and the lattice size. With this $W^h(x, 0)$ as the initial sampling distribution, let $V^{\delta,h}, W^{\delta,h}$ be defined as in equations (3.6.6) and (3.6.5) with $K = G_\delta$. Therefore, by Lemma 3.6.1 2), we have the following theorem:

Theorem 3.6.2. Let

$$\begin{aligned} U^{\delta,h}(x, t) &= W^{\delta,h}(x, t) - W^h(x, 0) \\ &\quad - \int_0^t (-V^{\delta,h} \cdot \nabla W^{\delta,h} + \nu \Delta W^{\delta,h})(x, s) ds. \end{aligned} \quad (3.6.7)$$

Then

$$\|\mathbb{E} [U^{\delta,h}(\cdot, t)]\|_{W^{-1,p}} \rightarrow 0$$

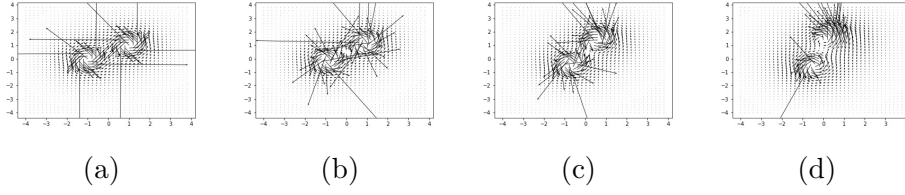


Figure 3.7.1: An inviscid flow: (a) $t = 0$, (b) $t = 1$, (c) $t = 3$, (d) $t = 5$

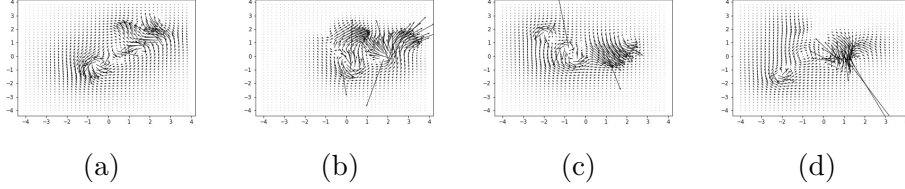


Figure 3.7.2: Split each particle into 5 parts. (a)(b)(c)(d) correspond to different randomisation

for all $t > 0$ as $h \downarrow 0$ for any fixed $\delta > 0$ as long as $p < 3/2$. This convergence is uniform for all $t \times (0, T]$.

From Lemma 3.6.1 2), we can see that that the error term tends to zero as $\delta \downarrow 0$ and $h \downarrow 0$ such that $h \ll \delta^2$.

3.7 Simulation results

In this section, we will present some simulation results based on our simple random vortex method.

First we show some simulation results of 3D flows. In Figure. 3.7.1, we consider an inviscid ($\nu = 0$) flow which starts with two points at $(-1, 0, 0)$ and $(1, 1, 0)$ with vorticity $(0, 0, 10)$ and $(10, 10, 10)$ respectively. i.e $x_1 = (-1, 0, 0)$, $x_2 = (1, 1, 0)$, $A_1(0) = (0, 0, 10)$, $A_2(0) = (10, 10, 10)$. Throughout the simulation in 3D flows, we choose the Biot-Savart kernel G , and we let φ be $c(|x|^2 - 1)^2$, where c is a constant such that the total integral of $\varphi = 1$. h is set at 0.5. Figure. 3.7.1 shows the the velocity field projected at the plane $z = 0$ by our approximation scheme for this flow.

When the flow is viscous ($\nu \neq 0$), we need to split the particles as suggested by Section 3.4, Figure. 3.7.2 and Figure. 3.7.3 demonstrate that more splitting helps

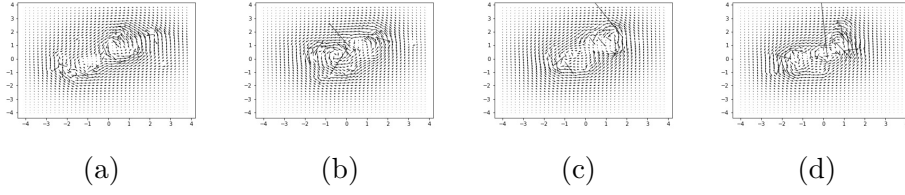


Figure 3.7.3: Split each particle into 50 parts.(a)(b)(c)(d) correspond to different randomisation

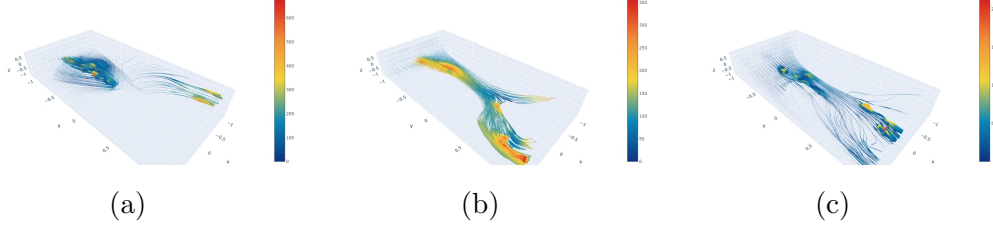


Figure 3.7.4: The streamline plots of an inviscid flow:(a) $t = 0$,(b) $t = 0.1$,(c) $t = 0.2$

minimise the stochastic error. We keep the choice of the kernel, φ , h , the initial condition as in the inviscid flow. The only difference here is that the viscosity $\nu = 1$ in this case, which means we have to solve a system of SDEs with the appropriate noise. The projections of velocity fields when $t = 0.5$ are shown in Figure. 3.7.2 and Figure. 3.7.3. When we split each particle into 50 parts, the result from different randomisation is more consistent.

For a more complicated 3D flow simulation, we start with 10 points $(x_1, x_2, \dots, x_n)$ uniformly distributed in $[-1, 1]^3$ with vorticity $(a_1, a_2, \dots, a_n)$ uniformly distributed in $[-10, 10]^3$. We set viscosity $\nu = 0$, and the streamlines plots for the velocity fields is shown in Figure. 3.7.4. The color bar represents the magnitude for the velocity.

For 2D flows simulation, we use the 2D Biot-Savart kernel G , and $\varphi = c(|x|^2 - 1)^2$, $h = 0.5$. we start the flow with randomized (x_n, A_n) $n = 1, 2, \dots, 10$. x_n are sampled in $[-1, 1] \times [-1, 1]$ by uniform distribution independently, and A_n are sampled in $[-10, 10]$ by uniform distribution. The particles inside the flow are denoted by different colors, and the movement of each particle is presented in Figure. 3.7.5.

Figure. 3.7.6 is the vorticity picture for a non-inviscid flow with viscosity $\nu = 1$. We use the same kernel, φ , h as in the previous simulation. We let (x_1, x_2, x_3, x_4) be

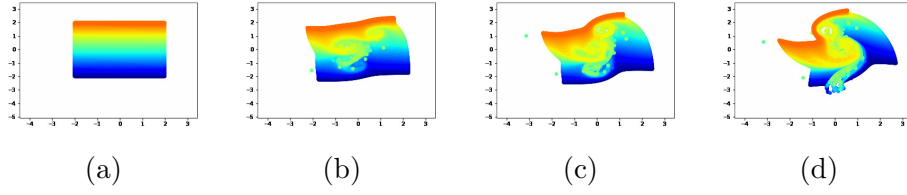


Figure 3.7.5: The dynamics of an inviscid flow:(a) $t = 0$,(b) $t = 0.3$,(c) $t = 0.6$,(d) $t = 1$

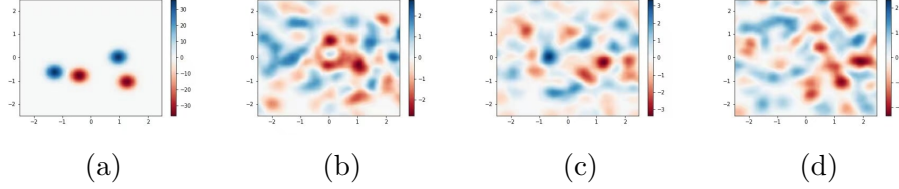


Figure 3.7.6: Vorticity of a non-inviscid flow(a) $t = 0$,(b) $t = 1$,(c) $t = 2$,(d) $t = 3$

sampled in $\mathbb{R}^2$ by $N(0, I)$, with vorticity $(10, 10, -10, -10)$ respectively. We split each particle into 50 parts to minimise the stochastic error. The vorticity value at each point is represented by its color, and the relations between the color and vorticity are shown in the bar on the right of each figure. The blue color represents positive vorticity, and the red color represents negative vorticity. From the figure, it is clear that the vorticity is being diffused into the whole flow as the time goes.

Figure. 3.7.7 illustrates the impacts of the kernel regularization and the number of splitting in approximations. We consider the solution of the vorticity equation associated with 2D Navier-Stokes equation [40]:

$$\frac{\partial}{\partial t}\omega + u^j \frac{\partial}{\partial x^j}\omega = \Delta\omega$$

with initial vorticity condition $W(x, 0) = \delta(x - 0)$. The explicit solution for this is given by

$$\omega(x, t) = \frac{1}{t}G\left(\frac{x}{\sqrt{t}}\right), \quad (3.7.1)$$

$$u(x, t) = \frac{1}{\sqrt{t}}v^G\left(\frac{x}{\sqrt{t}}\right), \quad (3.7.2)$$

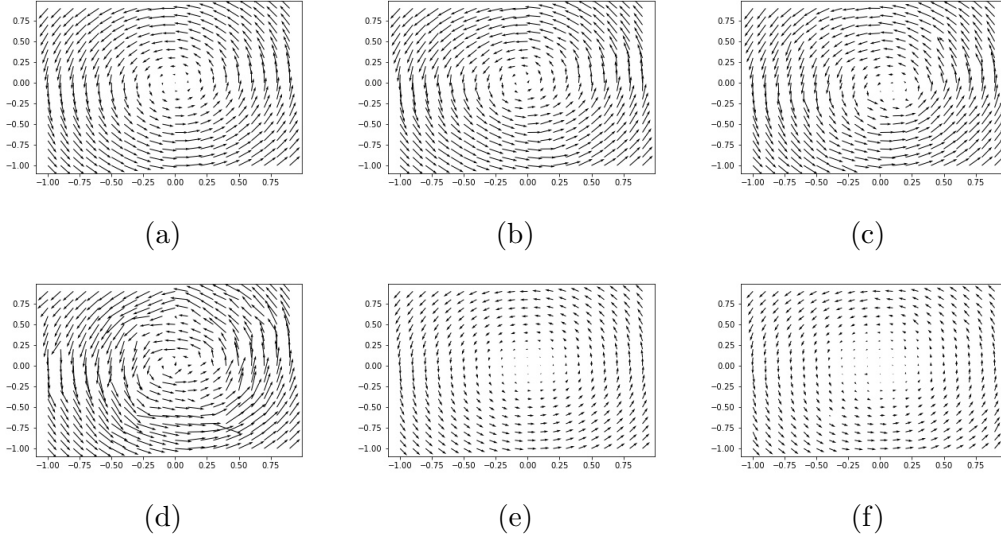


Figure 3.7.7: Velocity field computed by different approximations (a) Original solution. (b) $G, N = 1000, h = 0.1$. (c) $G, N = 500, h = 0.1$. (d) $G, N = 100, h = 0.1$. (e) $G_\delta, N = 1000, h = 0.1, \delta = 1$. (f) $G, N = 1000, h = 2$

where

$$G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \quad (3.7.3)$$

$$v^G(\xi) = \frac{1}{2\pi} \frac{(-\xi^2, \xi^1)}{|\xi|^2} (1 - e^{-|\xi|^2/4}). \quad (3.7.4)$$

The figure shows the exact solution and numerical simulations via splitting method for the velocity field at instance $t = 0.2$. The same scaling of arrows in different figures. That is, if the arrows in different figures have the same length, then the corresponding velocities have the same magnitude. $G_\delta(x) = \frac{1 - e^{-(|x|/\delta)^2}}{2\pi} \left(-\frac{x^2}{|x|^2}, \frac{x^1}{|x|^2} \right)$ as in (6.47)[73]. The numerical simulations show that the splitting works well and the noise perturbation is invisible already if N is chosen to be about 100, and becomes quite close to the exact solution if N is chosen to be around 1000.

Chapter 4

Random vortex dynamics via functional stochastic differential equations

4.1 Introduction

In this chapter, we will propose a novel random vortex system, which is equivalent to the Navier-Stokes equation for the incompressible viscous fluid flows. To numerically approximate the 3D viscous fluid flows has always been an interesting and challenging problem. As we have mentioned in the introduction part of the thesis, different stochastic representations of the Navier-Stokes equations were proposed, including [2, 14, 25, 84, 87, 115, 116]. However, the underlying vortex dynamics were not revealed in a clear way so that they are not useful for deriving a feasible numerical algorithm.

Unlike the representations mentioned above, a novel random vortex system will be shown in this chapter, which includes a forward SDE coupled with a backward functional ordinary differential equation, and the coefficients involve the law of the solution to the whole system. This system is an exact closure of Taylor's diffusion

equation, which is an equation of motion for the "imaginary" Brownian fluid particles. A Monte Carlo numerical algorithm to simulate the viscous fluid flows could be easily developed based on our representation.

To derive our representation, we utilized two powerful tools. The first is the duality of the distribution of the pinned diffusion process, which enables us to overcome the time reversal problem that has been encountered in some previous types of representations [2, 14]. The second tool is a forward-type Feynman–Kac formula, which allows us to derive a stochastic representation of a forward nonlinear parabolic equation. These tools are not only applicable to the specific problem we are considering but are also quite general and can be used for other nonlinear PDEs as well.

This chapter is organized as follows. We begin in Section 4.2 by providing a brief review of 2D random vortex dynamics and describing the main challenges and ideas in our approach. In Section 4.3 and Section 4.4, we will introduce two essential tools that we subsequently use in our methods. The first one is the pinned diffusion measure, which is the conditional of an irreversible diffusion given a terminal location, and a duality theorem of the pinned diffusion under time reversal. The second one is a forward-type Feynman–Kac formula, which gives a probabilistic representation of the solutions to the vorticity transport equation. Next, in Section 4.5, we utilize the tools developed in the previous sections to obtain a closed and exact system of equations for random vortex dynamics. In Section 4.6, we discretise this system to develop a numerical scheme for approximating the solution to the 3D Navier-Stokes equations. Finally, we show the simulation results for the 3D viscous fluid flows in Section 4.7.

4.2 Several basic facts and the main ideas

Let $u = (u^1, u^2, u^3)$ be the velocity field of an incompressible fluid flow with viscosity $\nu > 0$, and let $\omega := \operatorname{curl} u$ be its vorticity, then they satisfy the following vorticity transport equation (2.1.13):

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega = \nu \Delta \omega + (\omega \cdot \nabla) u, \quad (4.2.1)$$

where

$$\omega = \operatorname{curl} u. \quad (4.2.2)$$

Actually, from Lemma 2.1.5, (4.2.1) could be regarded as a closed PDE of the vorticity ω . The velocity u is determined by the second (4.2.2), together with the incompressibility condition $\nabla \cdot u = 0$. In fact, for 3D viscous fluid flows, under the assumption that both u and ω decay to zero at infinity sufficiently fast, we have the following Biot-Savart law:

$$u(x, t) = \int_{\mathbb{R}^3} K(x - y) \times \omega(y, t) \, dy, \quad (4.2.3)$$

where $K = (K^1, K^2, K^3)$ is the singular vector kernel with components

$$K^i(z) := -\frac{z^i}{4\pi|z|^3} \quad (4.2.4)$$

for $i = 1, 2, 3$.

For 2D flows (i.e., flows for which the third velocity component $u^3 = 0$, and the first two velocity components, u^1 and u^2 , depend on the first two coordinates, x^1 and x^2 , only), the vorticity transport equation (4.2.1) degenerates into a much simpler form (2.1.23), where the vorticity $\omega = (0, 0, \omega^3)$ is identified with the scalar function

ω^3 , which evolves according to the following parabolic equation:

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega = \nu \Delta \omega, \quad \omega(\cdot, 0) = \omega_0. \quad (4.2.5)$$

and similar to the 3D viscous fluid flows, u could be retrieved by ω by using the 2D Biot-Savart law Lemma 2.1.8:

$$u(x, t) = \int_{\mathbb{R}^2} K(x - y) \omega(y, t) \, dy, \quad (4.2.6)$$

where $K(z) := -\frac{1}{2\pi} \frac{(z^2, -z^1)}{|z|^2}$ is the 2D Biot-Savart kernel. This crucial fact allows one to close the stochastic differential equations of Taylor's Brownian motion X (cf. [7, 8, 9, 107] and [69]):

$$dX(t) = u(X(t), t) \, dt + \sqrt{2\nu} \, dB(t), \quad X(0) = x,$$

issued from an initial state x , where B is a two-dimensional Brownian motion, to provide powerful random vortex methods for computing 2D fluid flows; cf. [27, 69] and [73] for details. We would like to emphasize that this random vortex dynamics heavily relies on the special feature of 2D fluid flows, which is not easy to generalize to 3D fluid flows with nonlinear stretching of vortex motions.

To describe our approach and the contributions in the present work, we shall outline the key steps in the derivation of the 2D random vortex method in more detail and establish the notation that we will use throughout the chapter. Let us begin by recalling several important mathematical structures associated with the velocity vector field $u(x, t)$. We shall discuss these constructions for a general vector field $b(x, t)$ for reasons that will become clear in what follows.

Let $b(x, t)$ be a time-dependent, bounded and Borel measurable vector field. (where $x \in \mathbb{R}^d$, $t \geq 0$, and for the physically most relevant case of interest to us here,

the spatial dimension $d = 3$), and consider

$$L_{b(x,t)} := \nu \Delta + b(x,t) \cdot \nabla \quad (4.2.7)$$

which is a one-parameter family of second-order elliptic differential operators on $\mathbb{R}^d$. For simplicity, if no confusion is involved, we denote $L_{b(x,t)}$ by L_b . Unless otherwise stated, we shall assume that the drift vector field $b(x,t)$ is *solenoidal* (i.e., *divergence-free*), that is, for every t , the divergence $\nabla \cdot b(\cdot, t)$ vanishes identically. Since b is divergence-free, for the formal adjoint operator L_b^* , the following relation holds (cf. (2.2.4)):

$$L_b^* = L_{-b}$$

Hence, L_b^* is an elliptic operator of the same type as L_b . Let $\Gamma_b(x, t; \xi, \tau)$ (for $0 \leq \tau < t$ and $\xi, x \in \mathbb{R}^d$) denote the fundamental solution to the forward heat operator $L_b - \frac{\partial}{\partial t}$, and $\Gamma_b^*(x, t; \xi, \tau)$ (for $0 \leq t < \tau$, $x, \xi \in \mathbb{R}^d$) the fundamental solution to the backward heat operator $L_b^* + \frac{\partial}{\partial t}$, respectively (cf. Definition 2.2.1 and Definition 2.2.3 for their definitions and basic properties). Since b is bounded, both $\Gamma_b(x, t; \xi, \tau)$ and $\Gamma_b^*(x, t; \xi, \tau)$ are positive and continuous in x, ξ and $t > \tau \geq 0$ (cf. [38] and [82], for example), and $\Gamma_b(x, t; \xi, \tau) = \Gamma_b^*(\xi, \tau; x, t)$ for $t > \tau \geq 0$ and $x, \xi \in \mathbb{R}^d$ (cf. Theorem 2.2.4).

There is a relationship between the transition probability density function of L_b diffusion and the fundamental solutions. Let $p_b(\tau, \xi, t, x)$ denote the transition probability density function of the diffusion process with associated infinitesimal generator L_b (cf. [51] and [105]). To define $p_b(\tau, \xi, t, x)$ more rigorously, suppose that $X(\xi, \tau, t)$ (where $\xi \in \mathbb{R}^d$ is a (weak) solution to the SDE

$$dX(\xi, \tau, t) = b(X(\xi, \tau, t), t) dt + \sqrt{2\nu} dB(t) \quad \text{for } t \geq \tau \quad (4.2.8)$$

and

$$X(\xi, \tau, s) = \xi \quad \text{for } s \leq \tau, \quad (4.2.9)$$

where B is a d -dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then,

$$p_b(\tau, \xi, t, x) dx = \mathbb{P}[X(\xi, \tau, t) \in dx] \quad (4.2.10)$$

for any $t > \tau \geq 0$. When $\tau = 0$, we will simply write $X(\xi, t)$ for $X(\xi, 0, t)$ unless otherwise indicated. Since $b(x, t)$ is divergence-free, $L_b^* = L_{-b}$. Therefore, the following relation between p_b and the fundamental solutions holds:

$$p_b(\tau, \xi, t, x) = \Gamma_{-b}^*(\xi, \tau; x, t) = \Gamma_{-b}(x, t; \xi, \tau) \quad (4.2.11)$$

for $t > \tau \geq 0$ and $\xi, x \in \mathbb{R}^d$. We will see that (4.2.11) is the key to derive the random vortex dynamics equations in two dimensions.

For a 2D viscous flow with velocity $u(x, t)$, the vorticity $\omega = \text{curl} u$ solves the forward “linear” parabolic equation (4.2.5), which may be written as

$$\left(\frac{\partial}{\partial t} - L_{-u} \right) \omega = 0, \quad \omega(x, 0) = \omega_0(x), \quad (4.2.12)$$

subject to appropriate boundary conditions at infinity. Hence,

$$\omega(x, t) = \int_{\mathbb{R}^2} \Gamma_{-u}(x, t; \xi, 0) \omega_0(\xi) d\xi \quad (4.2.13)$$

for $(x, t) \in \mathbb{R}^2 \times [0, \infty)$, provided that $\omega_0(x)$ tends to zero sufficiently fast as $|x| \rightarrow \infty$.

From (4.2.11),

$$\omega(x, t) = \int_{\mathbb{R}^2} p_u(0, \xi, t, x) \omega_0(\xi) d\xi, \quad (4.2.14)$$

where $p_u(0, \xi, t, x)$ is the probability density function of the Taylor diffusion $X(\xi, t)$ with velocity u , which started at $\xi \in \mathbb{R}^2$ at $t = 0$. We know that $X(\xi, t)$ satisfies the

following SDE

$$dX(\xi, t) = u(X(\xi, t), t) dt + \sqrt{2\nu} dB(t), \quad X(\xi, 0) = \xi \quad (4.2.15)$$

for $t \geq 0$. The idea in random vortex methods is to rewrite the SDE (4.2.15) as a closed stochastic differential equation depending only on the initial vorticity ω_0 and the distribution of Taylor's diffusion X . To achieve this goal, we use the 2D Biot-Savart law (4.2.6), together with (4.2.14), one may rewrite

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}^2} \left[K(x - y) \int_{\mathbb{R}^2} p_u(0, \eta, t, y) \omega_0(\eta) d\eta \right] dy \\ &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} K(x - y) p_u(0, \eta, t, y) dy \right] \omega_0(\eta) d\eta \\ &= \int_{\mathbb{R}^2} \mathbb{E}[K(x - X(\eta, t))] \omega_0(\eta) d\eta. \end{aligned}$$

Thanks to this representation for the velocity $u(x, t)$ one is therefore able to close the stochastic differential equation (4.2.15):

$$dX(\xi, t) = \left(\int_{\mathbb{R}^2} \mathbb{E}[K(x - X(\eta, t))] \omega_0(\eta) d\eta \Big|_{x=X(\xi, t)} \right) dt + \sqrt{2\nu} dB(t) \quad (4.2.16)$$

and

$$X(\xi, 0) = \xi \quad (4.2.17)$$

for $(\xi, t) \in \mathbb{R}^2 \times [0, \infty)$. The system of (4.2.16), (4.2.17) is a stochastic differential equation dependent on the law of its solution, i.e., it is a system of distributional stochastic differential equations.

An efficient numerical method arises naturally from the stochastic dynamical system (4.2.16), (4.2.17). By employing a discretisation of the finite-dimensional integral appearing in (4.2.16) and by appealing to the strong law of large numbers for the expectation, the closed system of stochastic differential equations (4.2.16), (4.2.17) will be simplified to a finite-dimensional stochastic differential equation, which leads

to random vortex numerical schemes based on Monte Carlo simulation techniques for 2D incompressible fluid flows.

It would be ideal if we can apply the same idea to the 3D viscous fluid flows. However, due to the nonlinear stretching term appearing in the 3D vorticity equation, the same approach may fail to give us a nice stochastic representation. Let us explain the key difficulty we have to overcome.

We begin with the vorticity equations (4.2.1), (4.2.2) for incompressible fluid flow with viscosity $\nu > 0$ and velocity $u(x, t)$, which are rewritten as follows (cf. (2.1.14)):

$$\left(\frac{\partial}{\partial t} - L_{-u}\right)\omega = S\omega, \quad \omega(x, 0) = \omega_0(x), \quad (4.2.18)$$

where $S = (S_j^i)$ is the symmetric velocity gradient tensor, with entries $S_j^i := \frac{1}{2} \left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i} \right)$, and $\omega = \text{curl} u$ is now, unlike in the 2D setting, a three-component vector field. Because of the nonlinear stretching term, there is now no explicit representation for the vorticity ω , which is considered as a solution to (4.2.18), in terms of the initial vorticity ω_0 and the fundamental solution of L_{-u} . We have the following implicit representation:

$$\omega(x, t) = \int_{\mathbb{R}^3} p_u(0, \eta, t, x) \omega_0(\eta) d\eta + \int_0^t \int_{\mathbb{R}^3} p_u(s, \xi, t, x) S(\xi, s) \omega(\xi, s) d\xi ds. \quad (4.2.19)$$

This expression, however, is not useful for the purpose of closing the system of stochastic differential equations (4.2.15) for Taylor diffusion, because it involves a family of diffusion starting at different time. New ideas are therefore needed in order to formulate an exact system of random vortex dynamics.

A natural approach is to consider the symmetric velocity gradient tensor $S = (S_j^i)$ as a multiplicative factor and apply the Feynman–Kac formula to the initial-value problem (4.2.18) by constructing a gauge functional from S (cf. [36, 55, 56] for the linear case and [89, 90] for semilinear parabolic equations). Since the elliptic operator $L_{-u(x,t)}$ is temporally nonconstant and irreversible (which is canonical for

flows observed in nature), the Feynman–Kac formula for ω thus obtained involves the distribution of a backward Taylor diffusion. Therefore there are at least two substantial difficulties one needs to overcome. First, this backward-type Feynman–Kac formula cannot be used to close the system of stochastic differential equations (4.2.15), as we have to run a time-reversed Taylor diffusion in order to formulate the Feynman–Kac formula. Second, if our system is based on the backward-type Feynman–Kac formulation, there are currently no satisfactory numerical methods to approximate the solutions.

We aim to somehow reverse the backward Taylor diffusion so that we have a forward SDE with the initial conditions known to us. To achieve this goal, two powerful new tools are devised – the duality between pinned diffusion and a forward-type Feynman–Kac formula, which provide the means to close the system of stochastic differential equations (4.2.15) for Taylor diffusion with velocity $u(x, t)$. Recall that in two space dimensions, in order to close the system of stochastic differential equations (4.2.15) by eliminating the velocity u , two key steps are involved. The first is to devise a representation of the vorticity ω by ‘solving’ the vorticity equation, and the second step is to apply the Biot–Savart law to obtain a representation of the velocity u in terms of ω , then perform integration by parts (in two space dimensions one has to do this at any fixed time t) relying on the fact that u is divergence-free, whereby $L_{-u} = L_u^*$). In our approach, in the three-dimensional case, we devise a new representation formula, called a forward-type Feynman–Kac formula, stated in (4.4.2) below, which helps us represent the solution ω to the vorticity equation by using a Taylor diffusion with velocity $u(x, t)$ instead of a time-reversed velocity field. The forward Feynman–Kac formulas we developed relies on the fact that $u(x, t)$ is divergence-free, which implies that there is a convenient duality between the L_u -diffusion and the L_{-u} -diffusion conditional on their initial and final states (cf. Theorem 4.3.1 below). We then close the system of stochastic differential equations (4.2.15) by performing integration by parts at the path space level, – a technique

which also appears to be new.

4.3 Pinned diffusion processes

In this section we introduce the first main technical tool in our study – pinned diffusion measures and a duality theorem of time reversal distribution.

Let $\nu > 0$ be a constant, $b(x, t)$ be a Borel measurable and bounded vector field depending on a parameter $t \geq 0$, and let $L_b := \nu \Delta + b \cdot \nabla$. Let $\Omega := \{w | w \in C([0, \infty), \mathbb{R}^d)\}$ be the path space of all continuous paths in $\mathbb{R}^d$. For $0 \leq t < \infty$, $w(t)$ is the canonical mapping of the path w at t . Let $\mathcal{F}_t^0 := \sigma\{w(s) : s \leq t\}$ be the smallest σ -algebra in Ω such that the coordinate mappings $w \rightarrow w(s)$ are measurable for all $s \leq t$, and $\mathcal{F}^0 := \sigma\{w(s) : s \geq 0\}$. We have thus turned the path space $C([0, \infty), \mathbb{R}^d)$ into a measurable space $(\Omega, \mathcal{F}^0)$. By the diffusion process $X = (X_t)_{t \geq 0}$, with associated infinitesimal generator L_b , we mean the Markov family of probability measures

$$\{\mathbb{P}_b^{\xi, \tau} : \xi \in \mathbb{R}^d \text{ and } \tau \geq 0\},$$

where $\mathbb{P}_b^{\xi, \tau}$ is the unique probability measure on the space $(\Omega, \mathcal{F}^0)$ such that the finite-dimensional joint distribution

$$\mathbb{P}_b^{\xi, \tau} [w \in \Omega : w(s) = \xi \text{ for all } s \leq \tau, w(t_1) \in dx_1, \dots, w(t_k) \in dx_k]$$

is given by

$$p_b(\tau, \xi, t_1, x_1) p_b(t_1, x_1, t_2, x_2) \cdots p_b(t_{k-1}, x_{k-1}, t_k, x_k) dx_1 \cdots dx_k$$

for all $k = 0, 1, 2, \dots$, and $\tau < t_1 < \cdots < t_k$.

Next we define the pinned diffusion process. Let $T > \tau \geq 0$, and $\xi, \eta \in \mathbb{R}^d$, we define $\mathbb{P}_b^{\xi, \tau \rightarrow \eta, T}$ to be the distribution of the L_b -diffusion process $X = (X_t)_{t \geq 0}$ conditional on the event that $X_s = \xi$ for $s \leq \tau$ and $X_s = \eta$ for $s \geq T$. $\mathbb{P}_b^{\xi, \tau \rightarrow \eta, T}$ is

called the pinned diffusion measure, which is a distribution on the continuous path space Ω characterising a diffusion starting from ξ at instance τ and ending at state η at the terminal time T . It is known that the laws of a Markov process conditional on the terminal values at some fixed time $T > 0$ are also Markovian (cf. Section 14 in [31] and [24]). According to eq.(14.1) in [31, pp. 161], we define

$$q_b(s, x, t, y) := \frac{p_b(s, x, t, y) p_b(t, y, T, \eta)}{p_b(s, x, T, \eta)} \quad (4.3.1)$$

for $\tau < s < t < T$ and $x, y \in \mathbb{R}^d$. It is then easy to verify that

$$\int_{\mathbb{R}^d} q_b(s, x, t, y) dy = 1$$

and, for $\tau < s < t < r < T$, one has that

$$\int_{\mathbb{R}^d} q_b(s, x, t, y) q_b(t, y, r, z) dy = q_b(s, x, r, z).$$

Hence, $\{q_b(s, x, t, y) : \tau \leq s < t < T\}$ defines a temporal inhomogeneous Markov family. The L_b -pinned diffusion measure or the L_b -diffusion bridge measure $\mathbb{P}_b^{\xi, \tau \rightarrow \eta, T}$ is by definition the conditional law of $\mathbb{P}_b^{\xi, \tau}[\cdot | w(T) = \eta]$. It turns out that $\mathbb{P}_b^{\xi, \tau \rightarrow \eta, T}$ is the unique probability measure which is determined by the transition probability q_b on $(\Omega, \mathcal{F}^0)$. Indeed, for any

$$\tau = t_0 < t_1 < \dots < t_n < t_{n+1} = T,$$

consider the finite-dimensional marginal law

$$\mathbb{P}_b^{\xi, \tau \rightarrow \eta, T} [w \in \Omega : w(t_0) \in dx_0, \dots, w(t_k) \in dx_k, \dots, w(t_{n+1}) \in dx_{n+1}] \quad (4.3.2)$$

we have

$$\begin{aligned}
& \mathbb{P}_b^{\xi, \tau \rightarrow \eta, T} [w \in \Omega : w(t_0) \in dx_0, \dots, w(t_k) \in dx_k, \dots, w(t_{n+1}) \in dx_{n+1}] \\
&= \mathbb{P}_b^{\xi, \tau} [w \in \Omega : w(t_0) \in dx_0, \dots, w(t_n) \in dx_n] \frac{p_b(t_n, x_n, T, \eta)}{p_b(\tau, \xi, T, \eta)} \delta_\eta(x_{n+1}) \\
&= \frac{p_b(t_n, x_n, T, \eta)}{p_b(\tau, \xi, T, \eta)} p_b(\tau, \xi, t_1, x_1) \prod_{k=1}^{n-1} p_b(t_k, x_k, t_{k+1}, x_{k+1}) \delta_\xi(x_0) dx_1 \cdots dx_n \delta_\eta(x_{n+1}) \\
&= q_b(\tau, \xi, t_1, x_1) \prod_{k=1}^{n-1} q_b(t_k, x_k, t_{k+1}, x_{k+1}) \delta_\xi(x_0) dx_1 \cdots dx_n \delta_\eta(x_{n+1})
\end{aligned}$$

which means that $\mathbb{P}_b^{\xi, \tau \rightarrow \eta, T}$ is a Markov probability measure with transition probability q_b and

$$\mathbb{P}_b^{\xi, \tau \rightarrow \eta, T} [w \in \Omega : w(s) = \xi \text{ and } w(t) = \eta \text{ for } s \leq \tau \text{ and } t \geq T] = 1. \quad (4.3.3)$$

If $\tau = 0$ and $T > 0$, then, via the canonical embedding, $\mathbb{P}_b^{\xi, \tau \rightarrow \eta, T}$ can be considered (by an abuse of notation) as the probability measure on Ω of all continuous paths in $\mathbb{R}^d$ with running time from 0 to T .

Now, we are able to state the duality theorem, which reveals the relationship between the laws of the L_b -diffusions and the laws of the L_{-b} -diffusions. This theorem plays an important role in the proposed new 3D random vortex method.

Theorem 4.3.1. *Let $b^T(x, t)$ be a bounded, Borel measurable and divergence-free vector field such that $b^T(x, t) = b(x, T - t)$ for all x and $t \leq T$. Then, the associated pinned measures satisfy the following duality relation:*

$$\mathbb{P}_b^{\xi, 0 \rightarrow \eta, T} = \mathbb{P}_{-b^T}^{\eta, 0 \rightarrow \xi, T} \circ \tau_T, \quad (4.3.4)$$

where τ_T is the time-reversal operator at T , that is, $\tau_T : \Omega \rightarrow \Omega$ is the mapping, which sends $w \in \Omega$ to $\tau_T w \in \Omega$, where $\tau_T w(t) = w(T - t)$ for $t \in [0, T]$.

The theorem may be reformulated as follows: suppose that (X_t) is an L_b -diffusion

and (Y_t) is an L_{-b^T} -diffusion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$; then,

$$\mathbb{E}[F(X_\cdot)|X_0 = \xi, X_T = \eta] = \mathbb{E}[F(Y_{T-\cdot})|Y_0 = \eta, Y_T = \xi] \quad (4.3.5)$$

for any bounded or positive Borel measurable function F on Ω .

To prove this theorem, we need the following simple fact concerning fundamental solutions.

Lemma 4.3.2. *The following equality holds:*

$$p_{b^T}(T - t, x, T - \tau, \xi) = \Gamma_b(x, t; \xi, \tau), \quad (4.3.6)$$

and therefore

$$p_b(s, x, t, y) = p_{-b^T}(T - t, y, T - s, x) \quad (4.3.7)$$

for $0 \leq s < t \leq T$ and all $x, y \in \mathbb{R}^d$.

Proof. Recall that $p_{b^T}(t, x, \tau, \xi)$, for fixed τ and ξ , as a function of (x, t) is the unique solution to the backward equation

$$\left(L_{b^T} + \frac{\partial}{\partial t} \right) f = 0, \quad (4.3.8)$$

for $0 \leq t < \tau$ by Feynman–Kac formula, such that $f(x, t) \rightarrow \delta_\xi$ as $t \uparrow \tau$. By performing the change of variables $t \rightarrow T - t$ and $\tau \rightarrow T - \tau$, we have that

$$\left(L_b - \frac{\partial}{\partial t} \right) p_{b^T}(T - t, x, T - \tau, \xi) = 0 \quad (4.3.9)$$

for $0 \leq \tau < t \leq T$. Moreover, $p_{b^T}(T - t, x, T - \tau, \xi) \rightarrow \delta_\xi$ as $t \downarrow \tau$. Thus, by the definition and the uniqueness of $\Gamma_b(x, t; \xi, \tau)$, the equality (4.3.6) directly follows. $\square$

Proof of Theorem 4.3.1. Since $\mathbb{P}_b^{\eta, 0 \rightarrow \zeta, T}$ is a Markov probability measure with

transition probability q_b , for every $0 < t_1 < t_2 < \dots < t_k < T$,

$$\mathbb{P}_b^{\eta, 0 \rightarrow \zeta, T} [w(t_1) \in dx_1, \dots, w(t_k) \in dx_k]$$

is equal to the measure

$$q_b(0, \eta, t_1, x_1) \cdots q_b(t_{i-1}, x_{i-1}, t_i, x_i) \cdots q_b(t_{k-1}, x_{k-1}, t_k, x_k) dx_1 \cdots dx_k,$$

which may be written as the following by (4.3.1):

$$\frac{p_b(0, \eta, t_1, x_1) \cdots p_b(t_{i-1}, x_{i-1}, t_i, x_i) \cdots p_b(t_k, x_k, T, \zeta)}{p_b(0, \eta, T, \zeta)} dx_1 \cdots dx_k.$$

By Lemma 4.3.2, the above finite-dimensional distribution can be written as

$$\frac{p_{-b^T}(0, \zeta, T-t_k, x_k) \cdots p_{-b^T}(T-t_i, x_i, T-t_{i-1}, x_{i-1}) \cdots p_{-b^T}(T-t_1, x_1, T, \eta)}{p_{-b^T}(0, \zeta, T, \eta)} dx_1 \cdots dx_k,$$

which is the same as

$$\mathbb{P}_{-b^T}^{\zeta, 0 \rightarrow \eta, T} [w(T-t_k) \in dx_k, \dots, w(T-t_1) \in dx_1].$$

Now the conclusion follows, as a distribution is determined uniquely by its family of finite-dimensional marginal distributions. This completes the proof. $\square$

Remark 4.3.3. *The well-known setting of generalised integration by parts is the concept of dual (or adjoint) operators in functional analysis, and the concept of a pair of dual Markov processes [12, 34]. To the best of our knowledge, the class of functionals considered in the dual Markov processes is quite limited [53]. Conversely, within the framework of the duality of pinned diffusion measures, a much broader class of functionals on the paths space can be applied. However, it is important*

to acknowledge that the divergence-free condition that we require in the duality of pinned diffusion measures could be overly restrictive for a general diffusion process. The path space version (with respect to stationary measures of diffusion processes) was first formulated in [72] via forward and backward martingale decompositions (Lyons–Zheng’s decomposition), and their approach was subsequently generalised to nonsymmetric diffusion processes; cf. [71] and the literature therein.

4.4 Forward Feynman–Kac’s formula

In this section we develop our second key tool, a forward Feynman–Kac formula, which is based on the duality theorem in the previous section and is the key in deriving the 3D random vortex dynamics equations. We shall continue to assume that $b(x, t)$ is a bounded, Borel measurable and *divergence-free* vector field, with $x \in \mathbb{R}^d$ and $t \geq 0$.

Theorem 4.4.1. *Let $T > 0$. Suppose that the vector field $f(x, t)$ is a strong solution to the parabolic equation:*

$$\left(\frac{\partial}{\partial t} - L_{-b(x,t)} \right) f^i(x, t) = S_j^i(x, t) f^j(x, t) \quad \text{in } \mathbb{R}^d \times (0, T], \quad (4.4.1)$$

where $i = 1, \dots, d$, subject to the initial condition $f(x, 0) = f_0$, and S_j^i are globally Lipschitz continuous for $i, j = 1, \dots, d$. Then,

$$f(x, T) = \int_{\mathbb{R}^d} f_0(\xi) p_b(0, \eta, T, x) \mathbb{E}[Q(0) | X(0) = \eta, X(T) = x] d\xi, \quad (4.4.2)$$

where

$$dX^i(t) = b^i(X(t), t) dt + \sqrt{2\nu} dB^i(t) \quad (4.4.3)$$

and

$$\frac{d}{dt} Q_j^i(t) = -Q_k^i(t) S_j^k(X(t), t), \quad Q_j^i(T) = \delta_j^i \quad (4.4.4)$$

for $i, j = 1, \dots, d$, B is a standard d -dimensional Brownian motion on a probability space.

Proof. First, we perform a change of variables $t \rightarrow T - t$ of (4.4.1), which gives us

$$\left(\frac{\partial}{\partial t} + L_{-b^T} \right) f^i(x, T - t) = -S_j^i(x, T - t) f^j(x, T - t) \quad \text{in } \mathbb{R}^d \times (0, T] \quad (4.4.5)$$

where $b^T(x, t) = b(x, T - t)$.

Now we can apply the Feynman–Kac formula to $f(x, T - t)$. Let Y be the weak solution to the following SDE:

$$dY^i(t) = -b^i(Y(t), T - t) dt + \sqrt{2\nu} dB^i(t), \quad Y(0) = \eta, \quad (4.4.6)$$

where $t \in [0, T]$ $i = 1, \dots, d$. That is to say, Y is the diffusion with infinitesimal generator L_{-b^T} , Define a gauge functional $Z = (Z_j^i)$ by solving the system of ordinary differential equations

$$\frac{d}{dt} Z_j^i(t) = Z_k^i(t) S_j^k(Y(t), T - t), \quad Z_j^i(0) = \delta_j^i, \quad (4.4.7)$$

where $i, j = 1, \dots, d$, whose solution depends on T and η . Then, by using Itô's formula,

$$dM^i(t) = \sqrt{2\nu} Z_k^i(t) \nabla f^k(Y(t), T - t) \cdot dB(t).$$

Hence, $M^i(t) = Z_k^i(t) f^k(Y(t), T - t)$ is a martingale for $t \in [0, T]$.

Therefore, we obtain the following Feynman–Kac's formula:

$$f(\eta, T) = \mathbb{E}[Z(T) f_0(Y(T))]. \quad (4.4.8)$$

Next, we aim to transform the time-reversed vector field b^T back to its original form.

Disintegrating the right-hand side by conditioning on $Y(T) = \zeta$, we have that

$$\begin{aligned} f(\eta, T) &= \int_{\mathbb{R}^d} \mathbb{E}[Z(T)f_0(Y(T))|Y(T) = \zeta] \mathbb{P}[Y(T) \in d\zeta] \\ &= \int_{\mathbb{R}^d} \mathbb{E}[Z(T)|Y(0) = \eta, Y(T) = \zeta] f_0(\zeta) p_{-b^T}(0, \eta, T, \zeta) d\zeta \\ &= \int_{\mathbb{R}^d} \mathbb{E}[Z(T)|Y(0) = \eta, Y(T) = \zeta] f_0(\zeta) p_b(0, \zeta, T, \eta) d\zeta. \end{aligned}$$

Let us compute the expectation $\mathbb{E}[Z(T)|Y(0) = \eta, Y(T) = \zeta]$. To this end we employ the canonical setting, that is, Ω is the path space of all continuous paths in $\mathbb{R}^d$ with time duration $[0, T]$, and $\tau_T : \Omega \rightarrow \Omega$ is the time reversal map for a path, which maps $w(t)$ into $w(T - t)$. Let $\mathbb{P}_{-b^T}^{\eta, 0}$ be the law of the diffusion started at η , with associated infinitesimal generator L_{-b^T} . Z_t (written as $Z(w, t)$) could be regarded as a functional on Ω . For each $w \in \Omega$, Z_t is defined as the unique solution to the system of ordinary differential equations

$$\frac{d}{dt} Z_j^i = Z_k^i S_j^k(w(t), T - t), \quad Z_j^i(w, 0) = \delta_j^i,$$

By definition,

$$Z_j^i(w, t) = \delta_j^i + \int_0^t Z_k^i(w, s) S_j^k(w(s), T - s) ds,$$

which means

$$Z_j^i(\tau_T w, t) = \delta_j^i + \int_0^t Z_k^i(\tau_T w, s) S_j^k(\tau_T w(s), T - s) ds.$$

By a change of variable $t \rightarrow T - t$,

$$Z_j^i(\tau_T w, T - t) = \delta_j^i - \int_T^t Z_k^i(\tau_T w, T - s) S_j^k(w(s), s) ds.$$

Let $Q(\cdot, t) = Z(\cdot, T - t) \circ \tau_T$. Then Q solves the following system of ODEs:

$$\frac{d}{dt} Q_j^i(w, t) = -Q_k^i(w, t) S_j^k(w(t), t), \quad Q_j^i(w, T) = \delta_j^i$$

for every continuous path $w \in \Omega$. By Theorem 4.3.1,

$$\begin{aligned}\mathbb{E}_{-b^T}^{\eta,0} [Z(\cdot, T-t)|w(T) = \zeta] &= \mathbb{E}_b^{\zeta,0} [Z(\cdot, T-t) \circ \tau_T | w(T) = \eta] \\ &= \mathbb{E}_b^{\zeta,0} [Q(\cdot, t) | w(T) = \eta],\end{aligned}$$

which yields the desired conclusion. $\square$

4.5 The random vortex dynamics equations

In this section we derive the 3D random vortex dynamics SDE. To this end, let $u(x, t)$ be the velocity field of a viscous incompressible fluid flow (with viscosity $\nu > 0$) without an external force applied, which moves freely in $\mathbb{R}^3$, and let $\omega := \text{curl} u$. We assume that the velocity $u(x, t)$ and the vorticity $\omega(x, t)$ decay sufficiently fast as $|x| \rightarrow \infty$ and are sufficiently regular. Recall the vorticity equation

$$\left(\frac{\partial}{\partial t} - L_{-u(x,t)} \right) \omega^i(x, t) = S_j^i(x, t) \omega^j(x, t), \quad (4.5.1)$$

where S is the symmetric velocity gradient tensor associated with u . Since u is divergence-free, it follows that $L_{-u} = L_u^*$.

Our goal is to rewrite the stochastic equations of Taylor diffusion with velocity $u(x, t)$ in closed form such that u depends only on distribution of the solution:

$$dX(\xi, t) = u(X(\xi, t), t) dt + \sqrt{2\nu} dB(t), \quad X(\xi, 0) = \xi, \quad (4.5.2)$$

for all $\xi \in \mathbb{R}^3$, where B is a three-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In three space dimensions we need to couple it with the following system of ordinary differential equations to define the gauge functional $G(\xi, t, s) =$

$$G(s) = (G_j^i(s)):$$

$$\frac{d}{ds} G_j^i(\xi, t, s) = -G_k^i(\xi, t, s) S_j^k(X(\xi, s), s), \quad G_j^i(\xi, t, t) = \delta_j^i, \quad (4.5.3)$$

for $0 \leq s \leq t$, where $S_j^k := \frac{1}{2}(\frac{\partial u^k}{\partial x^j} + \frac{\partial u^j}{\partial x^k})$. We want to devise a closed SDE system equivalent to (4.5.2) and (4.5.3), whose solutions give rise to the velocity field $u(x, t)$.

By applying the forward Feynman–Kac formula (4.4.2) to (4.5.1), we obtain the following representation:

$$\omega^k(y, t) = \int_{\mathbb{R}^3} p_u(0, \xi, t, y) \omega_0^j(\xi) \mathbb{E}[G_j^k(\xi, t, 0) | X(\xi, t) = y] d\xi. \quad (4.5.4)$$

On the other hand, according to the Biot–Savart law

$$u^i(x, t) = - \int_{\mathbb{R}^3} \varepsilon^{ilk} \frac{x^l - y^l}{4\pi|x - y|^3} \omega^k(y, t) dy.$$

Substituting ω^k given by (4.5.4) into the Biot–Savart law, we then have that

$$\begin{aligned} u^i(x, t) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varepsilon^{ilk} \frac{y^l - x^l}{4\pi|y - x|^3} \mathbb{E}[G_j^k(\xi, t, 0) | X(\xi, t) = y] \omega_0^j(\xi) p_u(0, \xi, t, y) d\xi dy \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{E} \left[\varepsilon^{ilk} \frac{y^l - x^l}{4\pi|y - x|^3} G_j^k(\xi, t, 0) | X(\xi, t) = y \right] \omega_0^j(\xi) p_u(0, \xi, t, y) d\xi dy \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{E} \left[\varepsilon^{ilk} \frac{X(\xi, t)^l - x^l}{4\pi|X(\xi, t) - x|^3} G_j^k(\xi, t, 0) | X(\xi, t) = y \right] \omega_0^j(\xi) p_u(0, \xi, t, y) d\xi dy \\ &= \int_{\mathbb{R}^3} \mathbb{E} \left[\varepsilon^{ilk} \frac{X(\xi, t)^l - x^l}{4\pi|X(\xi, t) - x|^3} G_j^k(\xi, t, 0) \right] \omega_0^j(\xi) d\xi, \end{aligned}$$

which expresses the velocity $u(x, t)$ in terms of the law of (X, G) . To close the SDE system, we need to perform a similar computation for the symmetric tensor $S = (S_j^i)$.

From (2.1.20), we obtain

$$S_j^k(x, t) = \text{PV} \int_{\mathbb{R}^3} H_{j,i}^k(x - y) \omega^i(y, t) dy, \quad (4.5.5)$$

where the singular integral kernel $H_{j,i}^k$ is defined by

$$H_{j,i}^k(z) := \frac{3}{2} \frac{z^l}{4\pi|z|^5} (\varepsilon^{kli} z^j + \varepsilon^{jli} z^k). \quad (4.5.6)$$

Hence,

$$\begin{aligned} & S_j^k(x, t) \\ &= \lim_{\epsilon \downarrow 0} \int_{|y-x|>\epsilon} \int_{\xi \in \mathbb{R}^3} H_{j,i}^k(x-y) \mathbb{E}[G_l^i(\xi, t, 0) | X(\xi, t) = y] p_u(0, \xi, t, y) \omega_0^l(\xi) d\xi dy \\ &= \lim_{\epsilon \downarrow 0} \int_{|y-x|>\epsilon} \int_{\xi \in \mathbb{R}^3} \mathbb{E}[H_{j,i}^k(x-y) G_l^i(\xi, t, 0) | X(\xi, t) = y] \omega_0^l(\xi) p_u(0, \xi, t, y) dy d\xi \\ &= \lim_{\epsilon \downarrow 0} \int_{|y-x|>\epsilon} \int_{\xi \in \mathbb{R}^3} \mathbb{E}[H_{j,i}^k(x-X(\xi, t)) G_l^i(\xi, t, 0) | X(\xi, t) = y] \omega_0^l(\xi) p_u(0, \xi, t, y) dy d\xi \\ &= \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^3} \mathbb{E}[\mathbb{1}_{\{|x-X(\xi, t)|>\epsilon\}} H_{j,i}^k(x-X(\xi, t)) G_l^i(\xi, t, 0)] \omega_0^l(\xi) d\xi \\ &:= \text{PV} \int_{\mathbb{R}^3} \mathbb{E}[H_{j,i}^k(x-X(\xi, t)) G_l^i(\xi, t, 0)] \omega_0^l(\xi) d\xi, \end{aligned} \quad (4.5.7)$$

Here we use the Cauchy principal-value in (2.1.3) in a more complicated integral (by an abuse of notation).

Now we are able to formulate a closed system of SDEs which is equivalent to vorticity transport equation.

Assume that $X(\xi, t) = (X^i(\xi, t))$ is a family of $\mathbb{R}^d$ -valued continuous semimartingales, and suppose that $G(\xi, s, t) = (G_j^i(\xi, s, t))$ (for each ξ and $t > 0$) is a family of $d \times d$ symmetric matrix-valued continuous processes, differentiable in s , adapted to the filtration generated by X ; we then define

$$b_{(X,G)}^i(x, t) := \int_{\mathbb{R}^3} \mathbb{E} \left[\varepsilon^{ilk} \frac{X(\xi, t)^l - x^l}{4\pi|X(\xi, t) - x|^3} G_\alpha^k(\xi, t, 0) \right] \omega_0^\alpha(\xi) d\xi$$

and

$$S_{(X,G),j}^k(x, t) := \text{PV} \int_{\mathbb{R}^3} \mathbb{E}[H_{j,\alpha}^k(x-X(\xi, t)) G_l^\alpha(\xi, t, 0)] \omega_0^l(\xi) d\xi.$$

What is important here is the fact that $b_{(X,G)}^i$ and $S_{(X,G),j}^k$ are determined by the

joint distribution of (X, G) .

The random vortex system may now be formulated as follows:

$$dX^k(\xi, t) = b_{(X, G)}^k(X(\xi, t), t) dt + \sqrt{2\nu} dB^k(t), \quad X(\xi, 0) = \xi \quad (4.5.8)$$

and

$$\frac{d}{ds} G_j^i(\xi, t, s) = -G_l^i(\xi, t, s) S_{(X, G), j}^l(X(\xi, s), s), \quad G_j^i(x, t, t) = \delta_j^i \quad (4.5.9)$$

for $0 \leq s \leq t$ and for all $t \geq 0$, where $i, j, k = 1, 2, 3$.

Equations (4.5.8) and (4.5.9) together represent a closed system of distributional stochastic differential equations. The first equation (4.5.8) is a forward-type distributional SDE, and the second equation is a system of ordinary differential equations. The system (4.5.8), (4.5.9) is not Markovian but a system of ordinary functional differential equations (cf. [49]). Although (4.5.9) is a backward type equation, since it is an ordinary differential equation, we are always able to deal with it.

This result is of course a direct consequence of our definition.

4.6 Direct numerical scheme

Based on the random vortex dynamical system (4.5.8), (4.5.9) developed in the previous section, we are now able to devise a numerical scheme to approximate the solutions to the solutions of the Navier–Stokes equations by performing Monte Carlo simulations.

There are two kinds of terms that we need to numerically approximate. The first is the integral over the whole $\mathbb{R}^3$ space, which could be approximated by discrete Riemann sums. The second is the expected value within the integral, which could be approximated by algebraic average which is justified by law of large numbers. We will call this scheme a direct numerical scheme based on the exact random vortex

dynamics. Because this will be beneficial in future studies, we break up the scheme into several steps.

We begin by recalling the singular kernels

$$K_k^i(z) := -\varepsilon^{ilk} \frac{z^l}{4\pi|z|^3}, \quad H_{j,i}^k(z) := \frac{3}{2} \frac{z^l}{4\pi|z|^5} (\varepsilon^{kli} z^j + \varepsilon^{jli} z^k).$$

Besides, we let $K(0) = 0, H(0) = 0$ for simplicity.

Let $h > 0$ be the spatial lattice size in our numerical scheme, and let $x_{\mathbf{k}}$ be the lattice points defined by $x_{\mathbf{k}} := h\mathbf{k}$ with $\mathbf{k} := (k^1, k^2, k^3) \in \mathbb{Z}^3$. Let $X_t^{\mathbf{k}} := X(x_{\mathbf{k}}, t)$, $G_j^{\mathbf{k},l}(t, s) := G_j^l(x_{\mathbf{k}}, t, s)$ (where $t \geq s \geq 0$) and $\omega_{\mathbf{k}}^j := \omega_0^j(x_{\mathbf{k}})$. The first step is to replace the integrals on $\mathbb{R}^3$ with the corresponding Riemann sums, so that

$$b_{(X,G)}^i(x, t) = \int_{\mathbb{R}^3} \mathbb{E}[K_k^i(x - X(\xi, t)) G_j^k(\xi, t, 0)] \omega_0^j(\xi) d\xi$$

is approximated by the following discrete sum:

$$h^3 \sum_{\mathbf{k} \in \mathbb{Z}^3} \omega_{\mathbf{k}}^j \mathbb{E}[K_l^i(x - X_t^{\mathbf{k}}) G_j^{\mathbf{k},l}(t, 0)],$$

and

$$S_{(X,G),j}^l(x, t) = \text{PV} \int_{\mathbb{R}^3} \mathbb{E}[H_{j,\alpha}^l(x - X(\xi, t)) G_{\beta}^{\alpha}(\xi, t, 0)] \omega_0^{\beta}(\xi) d\xi$$

is approximated by the following discrete sum if we fix some small $\epsilon > 0$ in PV first:

$$h^3 \sum_{\mathbf{k} \in \mathbb{Z}^3, |x - X_t^{\mathbf{k}}| > \epsilon} \omega_{\mathbf{k}}^{\beta} \mathbb{E}[H_{j,\alpha}^l(x - X_t^{\mathbf{k}}) G_{\beta}^{\mathbf{k},\alpha}(t, 0)]. \quad (4.6.1)$$

Therefore, on a spatial mesh of spacing $h > 0$, we may run the following approximate random vortex system:

$$dX_t^{i,i} = h^3 \sum_{\mathbf{k} \in \mathbb{Z}^3} \omega_{\mathbf{k}}^j \mathbb{E}[K_l^i(x - X_t^{\mathbf{k}}) G_j^{\mathbf{k},l}(t, 0)] \Big|_{x=X_t^i} dt + \sqrt{2\nu} dB_t \quad (4.6.2)$$

with the initial condition $X_0^i = x_i$, where $t \geq 0$, and

$$\frac{d}{ds} G_j^{l,i}(t, s) = -G_l^{l,i}(t, s) h^3 \sum_{\mathbf{k} \in \mathbb{Z}^3} \omega_{\mathbf{k}}^\beta \mathbb{E} \left[H_{j,\alpha}^l(x - X_s^{\mathbf{k}}) G_\beta^{\mathbf{k},\alpha}(s, 0) \right] \Big|_{x=X_s^l} \quad (4.6.3)$$

with the terminal value $G_j^{l,i}(t, t) = \delta_j^i$, where $0 \leq s \leq t$ for $t \geq 0$. Here we may ignore ϵ as appeared in (4.6.1), because when ϵ is small enough, the distance among different discrete particles should always be greater than ϵ .

In the next step we substitute the expectations by running N independent copies via independent 3D Brownian motions B^ρ , to define the following stochastic differential equations

$$dX_t^{(i,\rho),i} = \frac{h^3}{N} \sum_{\mathbf{k} \neq \mathbf{i}} \omega_{\mathbf{k}}^\beta \sum_{\sigma=1}^N \left[K_\alpha^i(X_t^{(\mathbf{i},\rho)} - X_t^{(\mathbf{k},\sigma)}) G_\beta^{(\mathbf{k},\sigma),\alpha}(t, 0) \right] dt + \sqrt{2\nu} dB_t^\rho, \quad (4.6.4)$$

$$X_0^{(i,\rho)} = x_i, \quad \rho = 1, \dots, N, \quad (4.6.5)$$

coupled with

$$\frac{d}{ds} G_j^{(l,\lambda),i}(t, s) = -\frac{h^3}{N} G_l^{(l,\lambda),i}(t, s) \sum_{\mathbf{k} \neq \mathbf{l}} \omega_{\mathbf{k}}^\beta \sum_{\sigma=1}^N \left[H_{j,\alpha}^l(X_s^{(l,\lambda)} - X_s^{(\mathbf{k},\sigma)}) G_\beta^{(\mathbf{k},\sigma),\alpha}(s, 0) \right], \quad (4.6.6)$$

$$G_j^{(l,\lambda),i}(t, t) = \delta_j^i \quad (4.6.7)$$

for $0 \leq s \leq t$. The approximate velocity is therefore given by

$$\tilde{u}(x, t) = \frac{h^3}{N} \sum_{\mathbf{k} \neq \mathbf{i}} \omega_{\mathbf{k}}^\beta \sum_{\sigma=1}^N \left[K_\alpha^i(x - X_t^{(\mathbf{k},\sigma)}) G_\beta^{(\mathbf{k},\sigma),\alpha}(t, 0) \right].$$

The rate of convergence of the sequence of solutions of the discretisation scheme (4.6.4), (4.6.5) towards the solution of the random vortex dynamics system (4.5.8), (4.5.9) is an interesting question to consider. Due to the complexity of our random dynamics system, which consists of a forward distributional SDE and a family of backward ordinary differential equations and the singular kernel appearing in the

system, the convergence rate is difficult to analyze. What we can certainly say at this stage regardless is that the convergence rate will necessarily depend on the regularity of the solution to the Navier–Stokes equations. The convergence analysis of the scheme is both interesting and relevant, but is beyond the scope of the present study.

4.7 Simulation results

According to equations (4.6.4)–(4.6.7), the numerical algorithm is implemented as follows.

1. We select initial data (x_i, ω_i) , $i = 1, 2, \dots, n$. We scale ω_i by the factor h^3 here, so there is no need to include the constant h^3 in our numerical scheme.
2. An iterative algorithm is used to solve the equations. First, the time interval $[0, T]$ is partitioned into $0 = t_0 \leq t_1 \leq \dots \leq t_m = T$, where the partitions are of the same length $\frac{T}{m}$, and then we sample the *i.i.d.* white noise $Z_{t_r}^\rho \sim N(0, \frac{T}{m} I_3)$ at $t_0, t_1, \dots, t_{m-1}$, and for each $1 \leq \rho \leq N$, where N is the number of independent copies.
3. We aim to compute $(X_{t_r}^{(\mathbf{i}, \rho)}, G_{t_r, t_{r'}}^{(\mathbf{i}, \rho)})$, where $0 \leq r' \leq r \leq m, 1 \leq \mathbf{i} \leq n, 1 \leq \rho \leq N$, such that it approximates the solution of (4.6.4), (4.6.5). That is,

$$\begin{aligned}
X_{t_{r+1}}^{(\mathbf{i}, \rho), i} - X_{t_r}^{(\mathbf{i}, \rho), i} &= \sum_{\mathbf{k} \neq \mathbf{i}} \frac{\omega_{\mathbf{k}}^\beta}{N} \sum_{\sigma=1}^N \left[K_\alpha^i (X_{t_r}^{(\mathbf{i}, \rho)} - X_{t_r}^{(\mathbf{k}, \sigma)}) G_\beta^{(\mathbf{k}, \sigma), \alpha}(t_r, 0) \right] \Delta t + \sqrt{2\nu} Z_{t_r}^\rho, \\
X_0^{(\mathbf{i}, \rho)} &= x_{\mathbf{i}},
\end{aligned} \tag{4.7.1}$$

$$\begin{aligned}
&G_j^{(\mathbf{l}, \lambda), i}(t_r, t_{r'}) - G_j^{(\mathbf{l}, \lambda), i}(t_r, t_{r'-1}) \\
&= -G_l^{(\mathbf{l}, \lambda), i}(t_r, t_{r'}) \sum_{\mathbf{k} \neq \mathbf{l}} \frac{\omega_{\mathbf{k}}^\beta}{N} \sum_{\sigma=1}^N \left[H_{j, \alpha}^l (X_{t_{r'-1}}^{(\mathbf{l}, \lambda)} - X_{t_{r'-1}}^{(\mathbf{k}, \sigma)}) G_\beta^{(\mathbf{k}, \sigma), \alpha}(t_{r'-1}, 0) \right] \Delta t
\end{aligned} \tag{4.7.2}$$

and $G_j^{(\mathbf{l}, \lambda), i}(t_r, t_r) = \delta_j^i$ for $0 \leq r \leq m$.

4. To compute (4.7.1) and (4.7.2), suppose that we have computed $(X_{t_r}^{(\mathbf{i}, \rho)}, G_{t_r, t_{r'}}^{(\mathbf{i}, \rho)})$, where $0 \leq r' \leq r \leq m', 1 \leq \mathbf{i} \leq n, 1 \leq \rho \leq N$. To compute $(X_{t_{m'+1}}^{(\mathbf{i}, \rho)}, G_{t_{m'+1}, t_{r'}}^{(\mathbf{i}, \rho)})$, where $0 \leq r' \leq m' + 1$. We observe that $X_{t_{m'+1}}^{(\mathbf{i}, \rho)}$ can be obtained from (4.7.1), and $(G_{t_{m'+1}, t_{r'}}^{(\mathbf{i}, \rho)})$ can be obtained by utilizing (4.7.2) iteratively from $r' = m' + 1$ to $r' = 1$:

$$\begin{aligned} & G_j^{(\mathbf{l}, \lambda), i}(t_{m'+1}, t_{r'-1}) \\ &= G_j^{(\mathbf{l}, \lambda), i}(t_{m'+1}, t_{r'}) \\ &+ G_l^{(\mathbf{l}, \lambda), i}(t_r, t_{r'}) \sum_{\mathbf{k} \neq \mathbf{l}} \frac{\omega_{\mathbf{k}}^\beta}{N} \sum_{\sigma=1}^N \left[H_{j, \alpha}^l(X_{t_{r'-1}}^{(\mathbf{l}, \lambda)} - X_{t_{r'-1}}^{(\mathbf{k}, \sigma)}) G_\beta^{(\mathbf{k}, \sigma), \alpha}(t_{r'-1}, 0) \right] \Delta t \end{aligned} \quad (4.7.3)$$

with $G_j^{(\mathbf{l}, \lambda), i}(t_{m'+1}, t_{m'+1}) = \delta_j^i$.

The overall computational complexity is $\mathcal{O}(m(Nn)^2 + m^2Nn)$. At time point t'_m , on the right-hand sides of (4.7.1), we essentially compute Nn matrices and then sum them up. We need to execute this computation for all $1 \leq \mathbf{i} \leq n, 1 \leq \rho \leq N$, which gives complexity of $\mathcal{O}(N^2n^2)$ to update X at $t_{m'+1}$. To update G , we need to solve a backward ODE by using (4.7.3) for m' times and execute this computation for $1 \leq \mathbf{l} \leq n, 1 \leq \lambda \leq N$, which gives complexity of $\mathcal{O}(m'Nn)$. We can compute and store the term $\sum_{\mathbf{k} \neq \mathbf{l}} \frac{\omega_{\mathbf{k}}^\beta}{N} \sum_{\sigma=1}^N \left[H_{j, \alpha}^l(X_{t_{r'}}^{(\mathbf{l}, \lambda)} - X_{t_{r'}}^{(\mathbf{k}, \sigma)}) G_\beta^{(\mathbf{k}, \sigma), \alpha}(t_{r'}, 0) \right]$ as appeared in (4.7.3) for $1 \leq \mathbf{l} \leq n, 1 \leq \lambda \leq N$ every time we update X and G , (This gives us $\mathcal{O}(N^2n^2)$ complexity at each time step.) so that we do not need to compute this again when we solve the backward ODE (4.7.3). Therefore, the overall complexity of this algorithm is $\mathcal{O}(mN^2n^2 + m^2Nn)$.

In our actual experiments, because of the time discretisation, there is a possibility that $X_{t_r}^{(\mathbf{i}, \rho)}$ might be close to $X_{t_r}^{(\mathbf{k}, \sigma)}$ for some $i \neq k$, which could lead to an arithmetic overflow, because of the presence of singularities in the kernels H and K at zero. Therefore, mollifications of the singular kernels H and K are used in the experiments

reported below in order to avoid the appearance of overflow, and we sum through all k instead of $k \neq l$ in (4.7.2) or $k \neq i$ in (4.7.1), as there are no singularities in the mollified kernels. We ran the experiments on an AMD EPYC 7713 64-Core Processor, and the computations in (4.7.2) for $1 \leq l \leq n$, $1 \leq \lambda \leq N$ and in (4.7.1) for $1 \leq i \leq n$, $1 \leq \rho \leq N$ were parallelised.

In the following simulations the time interval was in all cases (unless otherwise stated) partitioned into subintervals of length 0.02 and the viscosity ν was set to $\frac{1}{2}$. The aim of the first simulation is to present a simple 3D visualisation of the computed streamlines of the flow emanating from a chosen initial condition. In our second and third experiment we compared our method with an exact solution and with another numerical scheme, respectively, to validate the proposed algorithm. For the sake of clarity, in the second and third experiment we chose to depict in the corresponding figures snapshots of the evolving velocity field, projected on the horizontal plane $z = 0$.

In the first simulation (cf. Figure. 4.7.2) the initial data are chosen as

$$(X_0^1, \omega_0^1) = ((0, 0, 0), (0, 0, 0.1)), \quad (X_0^2, \omega_0^2) = ((0, 1, 0), (0, 0, -0.1))$$

and

$$(X_0^3, \omega_0^3) = ((0, 0, 1), (0, 0.1, 0)),$$

each with 100 independent copies. Because of the diffusion, additional swirling eddies appear in the flow, and the magnitude of the velocity is diffused into the whole flow.

In the second simulation, we assess our numerical scheme by considering the known evolution of the Lamb–Oseen vortex, a line vortex that decays because of the presence of viscosity. Although the Lamb–Oseen vortex is typically regarded as a 2D solution with initial vorticity $\delta(x)\delta(y)$ for $(x, y) \in \mathbb{R}^2$, we consider it here as a 3D vortex solution with initial vorticity $(0, 0, \delta(x)\delta(y))$ for $(x, y, z) \in \mathbb{R}^3$. Then, the exact solution of the resulting 3D Lamb–Oseen vortex evolution problem, with $\nu = \frac{1}{2}$,

is given by

$$u(x, y, z, t) = \frac{1}{2\pi} \frac{(-y, x, 0)}{x^2 + y^2} (1 - e^{-\frac{x^2 + y^2}{2t}}), \quad (x, y, z) \in \mathbb{R}^3, \quad t > 0. \quad (4.7.4)$$

In our numerical simulations the evolution of the computed solution is initialized at $x_k = (0, 0, k/2)$ with $\omega_k = (0, 0, 1/2)$ for $-20 \leq k \leq 20$, and we consider the velocity field projected on the plane $z = 0$ when $t = 0.1$ (cf. Figure. 4.7.3). It is clear from the figures that the resemblance between the exact velocity field depicted in Figure. 4.7.3 (a) and the computed velocity field with $N = 1, 20$ and 100 independent copies, shown in Figure. 4.7.3 (b), (c), (d), respectively, improves as N increases. To compare the computed velocity fields with the exact velocity field more precisely, a lattice consisting of 25 points in the horizontal plane, $z = 0$, was chosen; the values at these points of the exact and approximate velocity fields projected on the plane $z = 0$ are shown in Table 4.1. In addition, we have calculated the error between the exact solution and the approximate solution in the discrete counterpart of the $L^1_{[0,1]^3}$ norm. More precisely, we sum $|u_{\text{approx}} - u_{\text{exact}}|$ over the lattice $(\frac{i}{10}, \frac{j}{10}, \frac{k}{10})_{-10 \leq i, j, k \leq 9}$ and then multiply the resulting sum by 0.1^3 , where u_{approx} is the velocity field calculated using our scheme, while u_{exact} is the exact velocity field. In Table 7.1, only the first two coordinates of the 25 lattice points and the first two components of the three-component vectors of the exact and approximate velocity fields at the 25 lattice points are shown. The third component of the exact velocity field is identically 0; for the values of the approximate velocity fields at the lattice points, listed in the third, fourth and fifth columns of the table, the third component is not exactly 0, but was in all cases less than 10^{-4} .

We apply the numerical scheme, slightly modified, instead of (4.6.4) and (4.6.6). More precisely we use the following scheme:

$$dX_t^{(i,\rho),i} = \frac{h^3}{N} \sum_{\mathbf{k} \neq \mathbf{i}} \omega_{\mathbf{k}}^\beta \sum_{\sigma=1}^N \left[K_\alpha^i(X_t^{(i,\rho)} - X_t^{(\mathbf{k},\sigma)}) G_\beta^{(\mathbf{k},\sigma),\alpha}(t, 0) \right] dt + \sqrt{2\nu} dB_t^{i,\rho}, \quad (4.7.5)$$

$$\frac{d}{ds} G_j^{(l,\lambda),i}(t,s) = -\frac{h^3}{N} G_l^{(l,\lambda),i}(t,s) \sum_{\mathbf{k} \neq \mathbf{l}} \omega_{\mathbf{k}}^\beta \sum_{\sigma=1}^N \left[H_{j,\alpha}^l(X_s^{(l,\lambda)} - X_s^{(\mathbf{k},\sigma)}) G_{\beta}^{(\mathbf{k},\sigma),\alpha}(s,0) \right]. \quad (4.7.6)$$

There are $N\rho$ independent Brownian motions instead of ρ independent Brownian motions for the whole system and $X^{i,\rho}$ are driven by independent Brownian motions. A numerical algorithm is developed based on this system by time discretization. The numerical experiment with this scheme helps avoid using a large number of copies but still gives a stable result. This scheme however reduces the computational costs.

The scheme is tested for the Taylor–Green vortex with the Reynolds number set to 1600. The initial velocity has components $\cos(x) \sin(y) \sin(z)$, $-\sin(x) \cos(y) \sin(z)$ and 0. The simulation is initialized with the lattice points $(\frac{\pi}{16}i, \frac{\pi}{16}j, \frac{\pi}{16}k)_{-16 \leq i,j,k \leq 48}$, and $N = 1$. We compare the evolution of the velocity field projected on the plane $z = 0.1$. We see that the simulation results resemble the real solution quite well (cf. Figure. 4.7.4). In order to explore the improvement resulting from reduction of the step size, we conducted another simulation with the same initialization but with step size $\Delta t = 0.01$ instead of 0.02 (cf. Figure. 4.7.4). In Table 4.2, a lattice consisting of 25 points on this plane was chosen, and the values at these points of the exact and approximate velocity fields projected on the plane $z = 0.1$ are shown. In addition, we have calculated the $L^1_{[\frac{\pi}{2}, \frac{3\pi}{2}]^2}$ norm error between the exact solution and the approximate solutions. It is clear that the accuracy of the approximation is improved by reducing the step size Δt . Furthermore, in order to analyze the convergence rate with respect to the spatial lattice size, we have conducted simulations with the lattice points inside $[-\pi, 3\pi]^3$ with lattice size $h = \frac{\pi}{k}, k = 2, 3, \dots, 13$, so that the number of lattice points is $(4k + 1)^3$. The step size is set at $\Delta t = 0.02$. The Reynolds numbers are set to 1600 and 160000. We have computed the $L^1_{[\frac{\pi}{2}, \frac{3\pi}{2}]^3}$ norm error between the exact solution and the approximate solutions. The result is demonstrated in Figure. 4.7.1, which shows a roughly linear relationship between the lattice size h and the L^1 error.

Next, the scheme is tested in the case of artificial homogeneous isotropic turbulence.

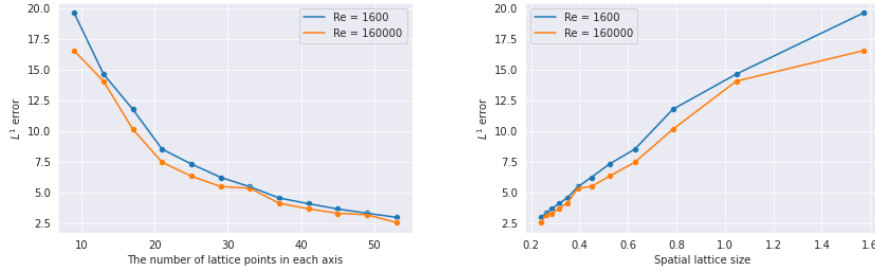


Figure 4.7.1: The L^1 error of the Taylor-Green vortex simulations with different lattice sizes. The left image shows the relations between $L^1_{[\frac{\pi}{2}, \frac{3\pi}{2}]^3}$ error and the number of lattice points in each axis. The right image shows the relations between $L^1_{[\frac{\pi}{2}, \frac{3\pi}{2}]^3}$ error and the lattice size. The blue curve and the red curve show the results for the flows with Reynolds number = 1600 and 160000 respectively.

The turbulence is initiated by a solenoidal velocity field

$$\begin{aligned}
 u(x, y, z) = & (\cos(x) \sin(y) \sin(z) + \cos(x) \sin(y) \sin(2z), \\
 & -\sin(x) \cos(y) \sin(z) + \sin(x) \cos(y) \sin(2z), \\
 & -\sin(x) \sin(y) \cos(2z)).
 \end{aligned} \tag{4.7.7}$$

We compare our numerical results with the results obtained by DNS. The DNS is based on a pseudo-spectral Fourier–Galerkin method [15]. The code for the implementation of DNS is based on the algorithm suggested in [81, 85]. The Reynolds number is 1600 as in the numerical verification in [81], so that the viscosity ν is $\frac{1}{1600}$, accordingly. The simulation is initialized using the lattice points $(\frac{\pi}{16}i, \frac{\pi}{16}j, \frac{\pi}{16}k)_{-16 \leq i, j, k \leq 48}$ and $N = 1$. We compare the evolution of the velocity field projected onto the plane $z = 0.1$. The simulation and the DNS solution produce very similar results (cf. Figure. 4.7.5). We would like to point out that, in the context of homogeneous isotropic turbulence, the pseudo-spectral methods may outperform the vortex methods. This advantage arises from that the pseudo-spectral methods benefit enormously from the periodic boundary conditions. The primary objective of the numerical experiment conducted here is to verify the validity of our vortex methods. An optimisation of our methods requires further exploration.

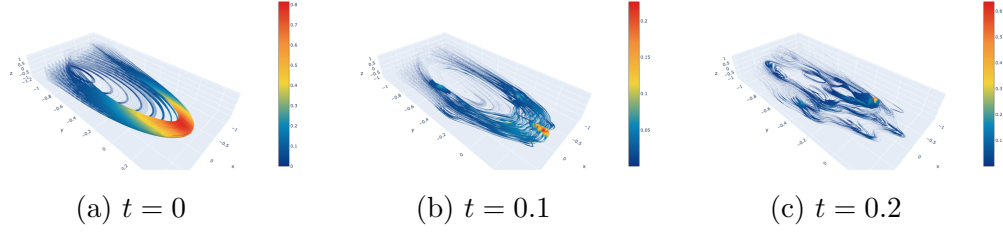


Figure 4.7.2: Computed streamlines of the flow at $t = 0, 0.1, 0.2$ in the first experiment.

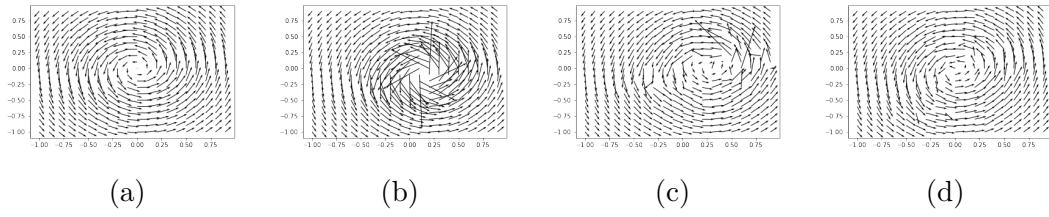


Figure 4.7.3: The velocity field of the Lamb–Oseen vortex at $t = 0.1$: (a) the exact solution; and the numerical solution computed with, respectively, (b) $N = 1$, (c) $N = 20$, (d) $N = 100$ independent copies.

Point	Exact value	$N = 1$	$N = 20$	$N = 100$
(-1.0, -1.0)	(0.08, -0.08)	(0.06, -0.08)	(0.06, -0.08)	(0.08, -0.08)
(-0.5, -1.0)	(0.13, -0.06)	(0.11, -0.08)	(0.10, -0.07)	(0.13, -0.07)
(0.0, -1.0)	(0.16, 0.00)	(0.17, -0.03)	(0.14, -0.03)	(0.15, 0.01)
(0.50, -1.0)	(0.13, 0.06)	(0.15, 0.06)	(0.14, 0.04)	(0.12, 0.07)
(1.00, -1.0)	(0.08, 0.08)	(0.09, 0.08)	(0.10, 0.07)	(0.08, 0.08)
(-1.0, -0.5)	(0.06, -0.13)	(0.04, -0.12)	(0.05, -0.11)	(0.05, -0.13)
(-0.5, -0.5)	(0.15, -0.15)	(0.11, -0.17)	(0.11, -0.13)	(0.11, -0.14)
(0.0, -0.5)	(0.23, 0.00)	(0.30, -0.16)	(0.16, -0.09)	(0.17, 0.01)
(0.5, -0.5)	(0.15, 0.15)	(0.21, 0.18)	(0.16, 0.10)	(0.13, 0.14)
(1.0, -0.5)	(0.06, 0.13)	(0.07, 0.15)	(0.10, 0.13)	(0.06, 0.12)
(-1.0, 0.0)	(-0.00, -0.16)	(-0.01, -0.13)	(-0.01, -0.13)	(-0.02, -0.14)
(-0.5, 0.0)	(-0.00, -0.23)	(-0.04, -0.24)	(-0.04, -0.18)	(-0.04, -0.22)
(0.0, 0.0)	(-0.00, 0.00)	(-0.26, -0.37)	(0.06, -0.12)	(-0.09, 0.04)
(0.5, 0.0)	(-0.00, 0.23)	(-0.18, 0.32)	(0.02, 0.06)	(0.02, 0.21)
(1.0, 0.0)	(-0.00, 0.16)	(-0.02, 0.18)	(-0.03, 0.24)	(0.00, 0.15)
(-1.0, 0.5)	(-0.06, -0.13)	(-0.06, -0.10)	(-0.05, -0.10)	(-0.06, -0.11)
(-0.5, 0.5)	(-0.15, -0.15)	(-0.12, -0.13)	(-0.10, -0.12)	(-0.13, -0.13)
(0.0, 0.5)	(-0.23, 0.00)	(-0.24, -0.07)	(-0.21, -0.11)	(-0.20, 0.00)
(0.5, 0.5)	(-0.15, 0.15)	(-0.19, 0.11)	(-0.06, 0.20)	(-0.13, 0.12)
(1.0, 0.5)	(-0.06, 0.13)	(-0.09, 0.12)	(-0.10, 0.13)	(-0.06, 0.13)
(-1.0, 1.0)	(-0.08, -0.08)	(-0.07, -0.07)	(-0.06, -0.07)	(-0.07, -0.07)
(-0.5, 1.0)	(-0.13, -0.06)	(-0.10, -0.06)	(-0.10, -0.07)	(-0.11, -0.06)
(0.0, 1.0)	(-0.16, 0.00)	(-0.14, -0.02)	(-0.14, -0.04)	(-0.14, -0.01)
(0.5, 1.0)	(-0.13, 0.06)	(-0.13, 0.04)	(-0.14, 0.04)	(-0.13, 0.05)
(1.0, 1.0)	(-0.08, 0.08)	(-0.09, 0.07)	(-0.10, 0.07)	(-0.08, 0.08)
$L^1_{[-1,1]^3}$ norm error		Error = 0.91	Error = 0.66	Error = 0.19

Table 4.1: The exact values of the velocity field in the Lamb–Oseen vortex problem at $t = 0.1$ (second column) at 25 lattice points (first column), and the values of the computed velocity field at those points, with $N = 1$ (third column), $N = 20$ (fourth column), and $N = 100$ (fifth column) independent copies, corresponding to the images depicted in Figure. 4.7.3. The discrete $L^1_{[-1,1]^3}$ norm error based on the 20^3 lattice points $(\frac{i}{10}, \frac{j}{10}, \frac{k}{10})_{-10 \leq i,j,k \leq 9}$ is shown in the last row of the table.

Point	Exact value	$\Delta t = 0.02$	$\Delta t = 0.01$
(1.57 , 1.57)	(-0.00 , -0.00)	(0.00 , -0.00)	(0.00 , -0.01)
(2.20 , 1.57)	(0.05 , -0.00)	(0.04 , -0.00)	(0.03 , -0.01)
(2.83 , 1.57)	(-0.04 , -0.00)	(-0.08 , -0.01)	(-0.07 , -0.01)
(3.46 , 1.57)	(-0.17 , -0.00)	(-0.15 , -0.00)	(-0.18 , -0.01)
(4.08 , 1.57)	(-0.17 , -0.00)	(-0.15 , -0.00)	(-0.17 , -0.00)
(1.57 , 2.20)	(-0.00 , 0.17)	(-0.00 , 0.15)	(0.00 , 0.17)
(2.20 , 2.20)	(0.06 , 0.15)	(0.06 , 0.13)	(0.04 , 0.15)
(2.83 , 2.20)	(-0.02 , 0.12)	(-0.13 , 0.14)	(-0.04 , 0.11)
(3.46 , 2.20)	(-0.15 , 0.09)	(-0.15 , 0.08)	(-0.16 , 0.08)
(4.08 , 2.20)	(-0.15 , 0.06)	(-0.14 , 0.04)	(-0.15 , 0.04)
(1.57 , 2.83)	(-0.00 , 0.17)	(-0.00 , 0.11)	(0.00 , 0.18)
(2.20 , 2.83)	(0.09 , 0.15)	(0.08 , 0.13)	(0.08 , 0.15)
(2.83 , 2.83)	(0.04 , 0.09)	(0.04 , 0.08)	(0.02 , 0.10)
(3.46 , 2.83)	(-0.09 , 0.04)	(-0.09 , 0.03)	(-0.10 , 0.03)
(4.08 , 2.83)	(-0.12 , -0.02)	(-0.11 , -0.03)	(-0.12 , -0.04)
(1.57 , 3.46)	(-0.00 , 0.04)	(-0.00 , 0.03)	(0.00 , 0.07)
(2.20 , 3.46)	(0.12 , 0.02)	(0.12 , -0.01)	(0.12 , 0.04)
(2.83 , 3.46)	(0.09 , -0.04)	(0.11 , -0.03)	(0.09 , -0.02)
(3.46 , 3.46)	(-0.04 , -0.09)	(-0.03 , -0.08)	(-0.03 , -0.09)
(4.08 , 3.46)	(-0.09 , -0.15)	(-0.07 , -0.14)	(-0.08 , -0.15)
(1.57 , 4.08)	(-0.00 , -0.05)	(-0.00 , -0.02)	(0.00 , -0.03)
(2.20 , 4.08)	(0.15 , -0.06)	(0.14 , -0.05)	(0.15 , -0.04)
(2.83 , 4.08)	(0.15 , -0.09)	(0.14 , -0.10)	(0.15 , -0.08)
(3.46 , 4.08)	(0.02 , -0.12)	(0.05 , -0.11)	(0.04 , -0.12)
(4.08 , 4.08)	(-0.06 , -0.15)	(-0.06 , -0.13)	(-0.04 , -0.15)
$L^1_{[\frac{\pi}{2}, \frac{3\pi}{2}]^2}$ norm error		Error = 0.26	Error = 0.20

Table 4.2: The exact values of the velocity field of the Taylor–Green vortex at $t = 1$ (second column) at 25 lattice points (first column), and the values of the computed velocity field at those points, with $\Delta t = 0.02$ (third column), $\Delta t = 0.01$ (fourth column) corresponding to the images depicted in Figure. 4.7.4.

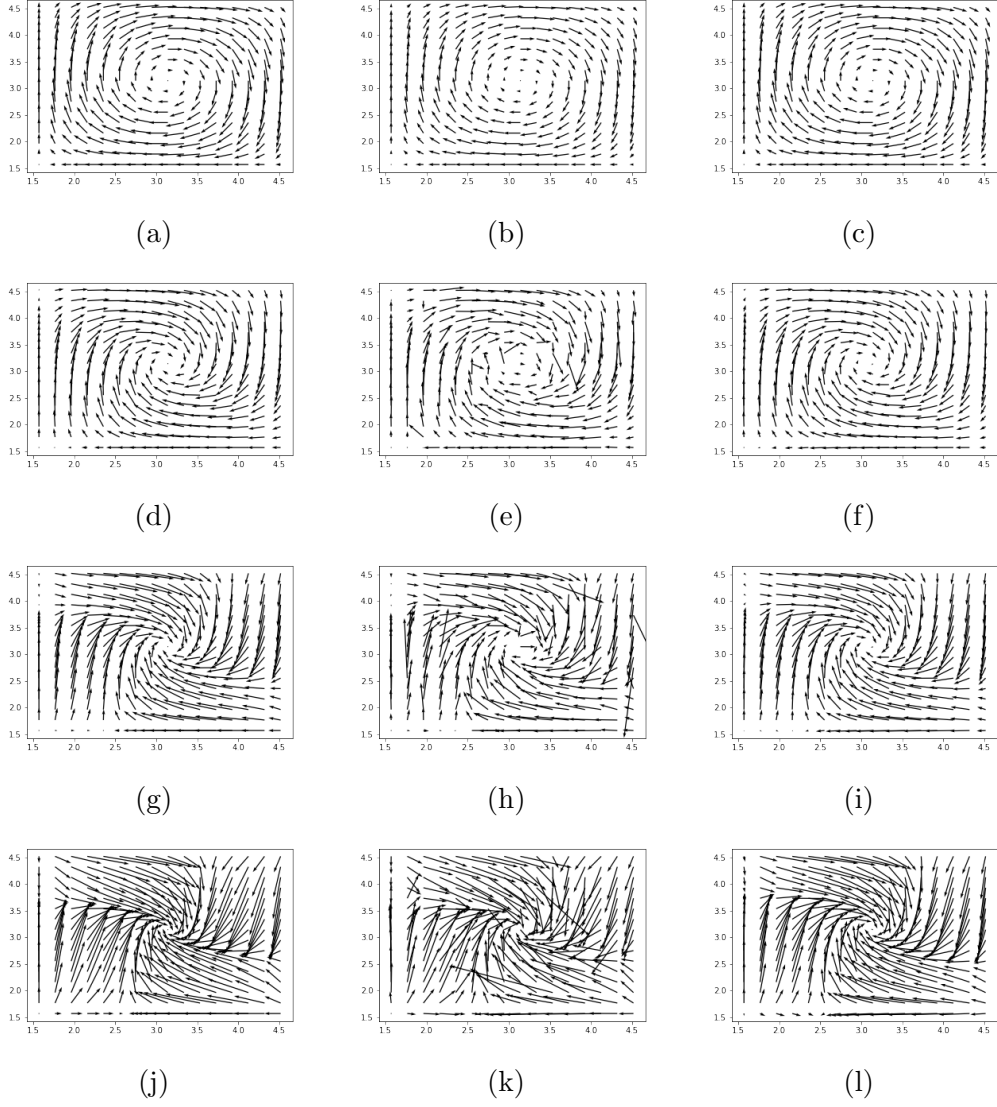


Figure 4.7.4: The images in the left column represent the Taylor–Green vortex solution. Those in the right column have been obtained using the proposed scheme. From the top to the bottom, $t = 0, 0.3, 0.6, 1$, respectively. The images in the left column represent the Taylor–Green vortex solution. Those in the middle column and right column have been obtained using the proposed scheme with step size $\Delta t = 0.02$ and 0.01 , respectively; from the top to the bottom, $t = 0, 0.3, 0.6, 1$.

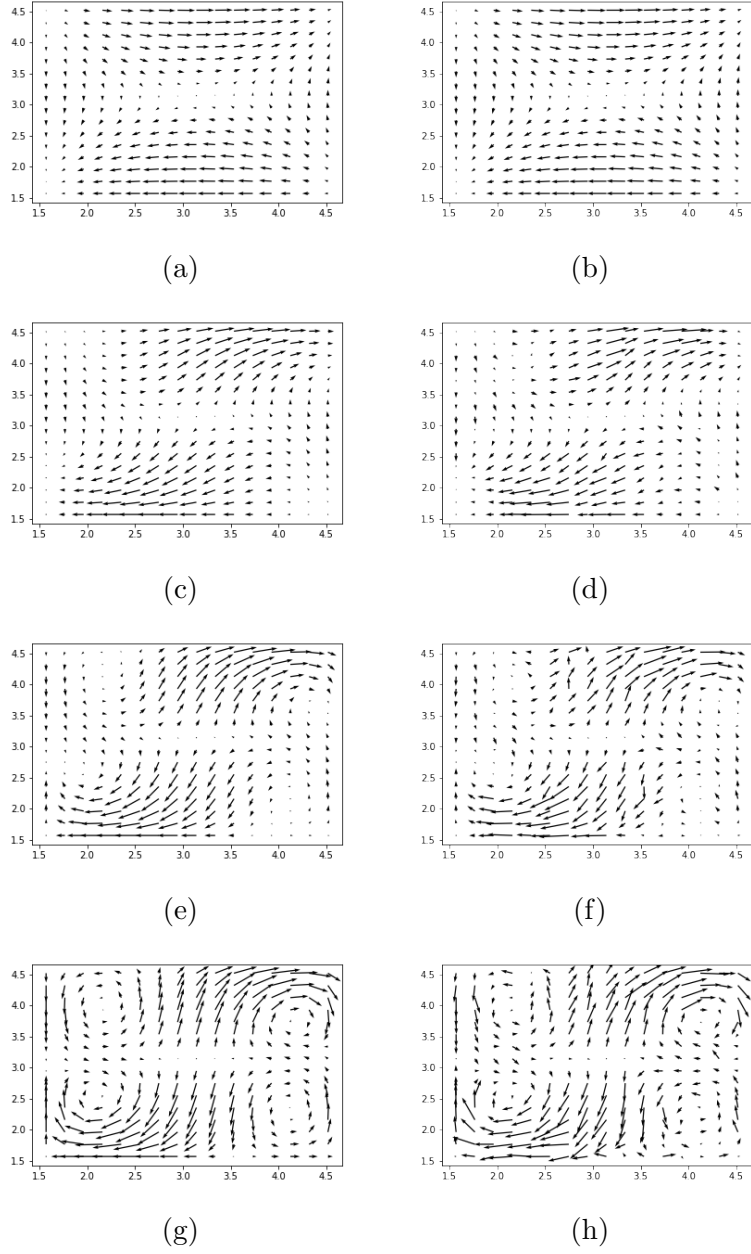


Figure 4.7.5: The images in the left column represent the DNS solution. Those in the right column have been obtained using the proposed scheme; from the top to the bottom, $t = 0, 0.3, 0.6, 1$, respectively.

Chapter 5

A general theory of random vortex type SDEs

5.1 Introduction

In this chapter, we will study some fundamental questions on the following RVSDEs:

$$\begin{aligned} dX_t &= b(X_t, t, \mathcal{L}(X_t, G_{t,0}))dt + \sigma(X_t, t, \mathcal{L}(X_t, G_{t,0}))dB_t, \quad X_0 = \xi \in \mathbb{R}^d, \\ dG_{t,s} &= -S(G_{t,s}, X_s, s, \mathcal{L}(X_s, G_{s,0}))ds, \quad G_{t,t} = \zeta_t \in \mathbb{R}^l \end{aligned} \tag{5.1.1}$$

The following topics will be investigated:

Existence, uniqueness, stability and propagation of chaos of the solutions to RVSDEs

We will establish the existence and uniqueness of solutions to the system of RVSDEs (5.1.1). Since our paper is mainly an introductory work of RVSDEs, we always assume the global Lipschitz and linear growth condition. The technique of proof is similar to the fixed point argument used for McKean–Vlasov SDEs. However, a more careful analysis of G must be conducted in order to apply the fixed point argument. Another slight technical complication is the continuity of $G_{t,s}$. Although the function $G_{t,s}$

appearing in (5.1.1) is differentiable in s for a fixed value of t , the continuity of $G_{t,s}$ with respect to t is unclear. We will show that $G_{t,s}$ is in fact continuous in the whole simplex $0 \leq s \leq t \leq T$, which may help us avoid some technical issues involving the continuity of $\mathcal{L}(X_t, G_{t,0})$. Besides, since G is continuous in the simplex, we can make a better sense of what should be a proper space that the solution (X, G) lies in, which will help us formulate the large deviation principle for the RVSDs more accurately. A more general theory of existence and uniqueness for RVSDs is beyond the scope of this paper. The stability of the solutions with respect to the coefficients will be shown, which serves as a tool in deriving some other results on RVSDs. In summary, we have the following theorem:

Theorem 5.1.1. (Existence, uniqueness and stability of solutions to RVSDs) *Suppose that b, σ and S satisfy the global Lipschitz and linear growth condition, then (5.1.1) admits a unique solution. Moreover, the solution to the RVSD (5.1.1) is continuous with respect to the initial data ξ, ζ_t , and the coefficients b, σ and S .*

Besides, we will formulate and prove the propagation of chaos for the RVSDs, where we run independent copies of Brownian motions and use interacting particles to approximate the solutions, so that the law of the solutions in RVSDs is replaced by the empirical measure of the particles.

Theorem 5.1.2. (Propagation of chaos for RVSDs) *Let $(X^{(i),N}, G^{(i),N})$ be interacting particles satisfying the following equations:*

$$\begin{aligned} dX_t^{(i),N} &= b(X_t^{(i),N}, t, \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{(i),N}, G_{t,0}^{(i),N}}) dt + \sigma(X_t^{(i),N}, t, \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{(i),N}, G_{t,0}^{(i),N}}) dB_t^{(i)} \\ dG_{t,s}^{(i),N} &= -S(G_{t,s}^{(i),N}, X_s, s, \frac{1}{N} \sum_{i=1}^N \delta_{X_s^{(i),N}, G_{s,0}^{(i),N}}) ds \end{aligned}$$

where $i = 1, 2, \dots, N$, and $B_t^{(i)}$ are independent Brownian motions. Then

$$\lim_{N \rightarrow \infty} \max_{1 \leq i \leq N} \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{(i)} - X_s^{(i),N}|^2 + |G_{s,0}^{(i)} - G_{s,0}^{(i),N}|^2 = 0,$$

where $(X^{(i)}, G^{(i)})$ satisfies the following decoupled RVSDDE:

$$\begin{aligned} dX_t^{(i)} &= b(X_t^{(i)}, t, \mathcal{L}(X_t^{(i)}, G_{t,0}^{(i)}))dt + \sigma(X_t^{(i)}, t, \mathcal{L}(X_t^{(i)}, G_{t,0}^{(i)}))dB_t^{(i)} \\ dG_{t,s}^{(i)} &= -S(G_{t,s}^{(i)}, X_s^{(i)}, s, \mathcal{L}(X_t^{(i)}, G_{t,0}^{(i)}))ds \end{aligned}$$

Large deviation principles for RVSDDEs

An interesting question to consider is the LDP for RVSDDEs. The LDP describes the probability of a rare event. General LDP theories may be found in literature [32, 37]. Our approach is motivated by [96], where they established the LDP for MVSDDEs directly without making use of propagation of chaos. The result is based on the following observation. When $\epsilon \rightarrow 0$, the limiting distribution of X^ϵ satisfying

$$dX_t^\epsilon = b(X_t^\epsilon, t, \mathcal{L}(X_t^\epsilon))dt + \sqrt{\epsilon}\sigma(X_t^\epsilon, t, \mathcal{L}(X_t^\epsilon))dB_t, \quad X_0^\epsilon = \xi$$

will be the delta distribution at ϕ , where ϕ solves the following ODE:

$$d\phi_t = b(\phi_t, t, \delta_{\phi_t})dt, \quad \phi_0 = \xi.$$

The rate function for X^ϵ is given by

$$I(f) = \inf_{h \in F} \frac{1}{2} \int_0^T |\dot{h}(s)|^2 ds,$$

where

$$F = \left\{ h \in H^1 \left| f(t) = \xi + \int_0^t b(f(s), s, \delta_{\phi_s})ds + \int_0^t \sigma(f(s), s, \delta_{\phi_s})\dot{h}(s)ds \right. \right\}.$$

In RVSDDEs, when $\epsilon \rightarrow 0$, the limit distribution of $(X, G)^\epsilon$ is the delta distribution

at (ϕ, ψ) , where ϕ and ψ solve the following system of equations:

$$\begin{aligned} d\phi_t &= b(\phi_t, t, \delta_{(\phi_t, \psi_{t,0})})dt, \quad \phi_0 = \xi, \\ d\psi_{t,s} &= -S(\psi_{t,s}, \phi_s, s, \delta_{(\phi_s, \psi_{s,0})})ds, \quad \psi_{t,t} = \zeta_t. \end{aligned} \tag{5.1.2}$$

We will show the following LDP for the RVSEs:

Theorem 5.1.3. (Large deviation principles for RVSEs) *Let (X^ϵ, G^ϵ) be the solution to the following RVSEs:*

$$\begin{aligned} dX_t^\epsilon &= b(X_t^\epsilon, t, \mathcal{L}(X_t^\epsilon, G_{t,0}^\epsilon))dt + \sqrt{\epsilon}\sigma(X_t^\epsilon, t, \mathcal{L}(X_t^\epsilon, G_{t,0}^\epsilon))dB_t, \quad X_0 = \xi \in \mathbb{R}^d, \\ dG_{t,s}^\epsilon &= -S(G_{t,s}^\epsilon, X_s^\epsilon, s, \mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon))ds, \quad G_{t,t} = \zeta_t \in \mathbb{R}^l \end{aligned} \tag{5.1.3}$$

Then (X^ϵ, G^ϵ) satisfies the LDP with the rate function

$$I(f, g) = \begin{cases} \inf \frac{1}{2} \int_0^T |\dot{h}|^2 dt & \text{if } g_{t,t} - g_{t,s} = \int_s^t -S(g_{t,u}, f_u, u, \delta_{\phi_u, \psi_{u,0}})du, \quad g_{t,t} = \zeta_t, \\ \infty & \text{otherwise} \end{cases} \tag{5.1.4}$$

where the infimum is taken over the set

$$\left\{ h \in H_1 : f(t) = \xi + \int_0^t b(f(s), s, \delta_{\phi_s, \psi_{s,0}})ds + \int_0^t \sigma(f(s), s, \delta_{\phi_s, \psi_{s,0}})\dot{h}(s)ds \right\}$$

and ϕ, ψ are defined in (5.1.2).

We would like to admit that the general idea to deal with the LDP for RVSEs (5.1.1) is similar to that in [96]. However, when dealing with RVSEs, the limiting behavior of $G_{t,s}$ needs to be investigated, which requires a more thorough analysis. The proof of LDP relies on the stability of the solutions to RVSEs, which we will show along with the existence and uniqueness result.

Euler-Maruyama type numerical scheme of RVSDs

To approximate the solution to the RVSDs, we propose an Euler-Maruyama type numerical scheme that extends and justifies the time discretisation step of the numerical program to simulate 3D fluid flows in Section 4.7. EM algorithms could be thought of as a natural extension of the classical Euler scheme for the ODE to the SDE settings. To approximate the solution to RVSDs, we first utilize our propagation of chaos results to replace the law by empirical measures, then we discretize the time interval to solve the RVSD step by step. The strong convergence of our scheme will be demonstrated in a similar way to that of the classical SDEs. The difference is that at each time step, we need to compute the whole path of G in a backward way, which leads to a more complicated implementation. We will prove the strong convergence of the Euler-Maruyama type numerical scheme, which has exactly the same convergence rate as the Euler-Maruyama method for a classical forward SDE, as expected.

Theorem 5.1.4. (Strong convergence of Euler scheme of RVSDs) *Let $h > 0$ and X^h, G^h defined by the following system*

$$\begin{aligned} X_k^h &= X_{k-1}^h + b(X_{k-1}^h, G_{k-1,0}^h, t_{k-1}^h)h + \sigma(X_{k-1}^h, G_{k-1,0}^h, t_{k-1}^h)\Delta B_{t_{k-1}^h} \text{ for } 1 \leq k \leq n \\ G_{t_i^h, t_j^h}^h - G_{t_i^h, t_{j-1}^h}^h &= -S(G_{t_i^h, t_j^h}^h, X_{t_{j-1}^h}^h, G_{t_{j-1},0}^h, t_j^h)h \text{ for } 1 \leq j \leq i \leq n \end{aligned} \quad (5.1.5)$$

where $0 = t_0^h \leq t_1^h \leq t_2^h \leq \dots \leq t_n^h = T$, $t_i^h - t_{i-1}^h = h$ for $i = 1, 2, \dots, n$. X^h is defined on $0 = t_0^h \leq t_1^h \leq t_2^h \leq \dots \leq t_n^h = T$ and G^h is defined on (t_i^h, t_j^h) where $0 \leq j \leq i \leq n$. $X_0^h = x_0$, $G_{t_i^h, t_i^h}^h = g_{t_i^h}^h$. We extend $X_t^h, G_{t,0}^h$ to the whole interval $[0, T]$ by letting $X_t^h = X_{t_i}^h, G_t^h = G_{t_i,0}^h$ when $t \in [t_i, t_{i+1})$. Then

$$\sup_{0 \leq t \leq T} \mathbb{E}|G_{t,0}^h - G_{t,0}|^2 + |X_t^h - X_t|^2 \leq C(K, T)(1 + |x_0|^2 + \sup_{0 \leq t \leq T} |g_t|^2)h$$

Infinite-dimensional RVSDEs and the approximations by finite-dimensional RVSDEs

We are going to consider the infinite-dimensional RVSDE, which is a direct generalization of the original random vortex system. The following system will be studied:

$$\begin{aligned} dX_t(\xi) &= b(X_t(\xi), t)dt + \sigma(X_t(\xi), t)dB_t \\ dG_{t,s}(\xi) &= -S(G_{t,s}(\xi), X_s(\xi), s)ds \end{aligned} \tag{5.1.6}$$

where $X_t, G_{t,s}$ is defined on every $\xi \in \Omega$ and Ω is some region in an Euclidean space $\mathbb{R}^p$. b, σ and S depend on the distribution of $(X_t(\xi), G_{t,0}(\xi))_{\xi \in \Omega}$ by the following equations:

$$\begin{aligned} b(x, t) &= \int_{\Omega} \mathbb{E} [b'(x, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi \\ \sigma(x, t) &= \int_{\Omega} \mathbb{E} [\sigma'(x, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi \\ S(g, x, t) &= \int_{\Omega} \mathbb{E} [S'(x, g, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi \end{aligned} \tag{5.1.7}$$

We are going to prove the existence and uniqueness of the solutions to (5.1.6) under nice assumptions. Besides, we are going to show how to approximate the solutions to (5.1.6) by space discretisation. The space discretisation reduces (5.1.6) to finite-dimensional RVSDEs, which can be numerically computed via EM algorithms. To prove the convergence of the space discretisation, we borrow the idea from the random vortex methods [8, 45, 69], where they proved that the dynamics of 2D fluid flows could be approximated by discrete particles as the mesh size goes to 0. Since our system is nicely formulated with the assumption of Lipschitz continuity, the main difficulty of proving the convergence of the random vortex methods that the system is of high singularity is avoided. Therefore, the analysis here is relatively simpler with just a more careful consideration when dealing with $G_{t,s}$.

Theorem 5.1.5. (Approximation to infinite-dimensional RVSDEs via space discretisation (For a precise statement, please refer to Theorem 5.7.3))

Let (X, G) be the solution to (5.1.6) with b, σ and S determined by the solution via (5.1.7). Discretise (5.1.6) with some mesh size $h > 0$. Let ih be the lattice points in Ω , $i \in \mathbb{Z}^\rho$. Since Ω is a compact set, there are finitely many i , so we relabel them by $i = 1, 2, \dots, N$, and denote ih by $\xi^{(i)}$. Let $(\tilde{X}, \tilde{G})_{i=1}^N$ be the solution to the following closed system of finite-dimensional RVSDs,

$$\begin{aligned} d\tilde{X}_t^{(i)} &= \tilde{b}(\tilde{X}_t^{(i)}, t)dt + \tilde{\sigma}(\tilde{X}_t^{(i)}, t)dB_t, \tilde{X}_0^i = X_0(\xi^{(i)}) \\ d\tilde{G}_{t,s}^{(i)} &= -\tilde{S}(\tilde{G}_{t,s}^{(i)}, \tilde{X}_s^{(i)}, s)ds, \tilde{G}_{t,t}^i = G_{t,t}(\xi^{(i)}) \end{aligned} \quad (5.1.8)$$

where

$$\begin{aligned} \tilde{b}(x, t) &= \sum_{i=1}^N \mathbb{E} \left[b'(x, t, \tilde{X}_t^{(i)}, \tilde{G}_{t,0}^{(i)}, \xi^{(i)}) \right] h^\rho \\ \tilde{\sigma}(x, t) &= \sum_{i=1}^N \mathbb{E} \left[\sigma'(x, t, \tilde{X}_t^{(i)}, \tilde{G}_{t,0}^{(i)}, \xi^{(i)}) \right] h^\rho \\ \tilde{S}(g, x, t) &= \sum_{i=1}^N \mathbb{E} \left[S'(x, g, t, \tilde{X}_t^{(i)}, \tilde{G}_{t,0}^{(i)}, \xi^{(i)}) \right] h^\rho \end{aligned}$$

Suppose that b', σ' and S' are L ($L \geq \rho + 1$) times bounded differentiable. Then

$$\begin{aligned} \sup_{x \in \mathbb{R}^d, t \in [0, T]} |\tilde{b}(x, t) - b(x, t)| &\leq C(K, T, \Omega, L)h^L \\ \sup_{x \in \mathbb{R}^d, t \in [0, T]} |\tilde{\sigma}(x, t) - \sigma(x, t)| &\leq C(K, T, \Omega, L)h^L \\ \sup_{g \in \mathbb{R}^l, x \in \mathbb{R}^d, t \in [0, T]} |\tilde{S}(g, x, t) - S(g, x, t)| &\leq C(K, T, \Omega, L)h^L \end{aligned}$$

This chapter is organized as follows. In Sections 5.2 and 5.3, we will introduce the RVSDs and study the basic problems of the solutions to RVSDs, including existence, uniqueness, stability and propagation of chaos. In Section 5.4, the large deviation principle of RVSDs is established. In Section 5.5, EM algorithm is proposed and its strong convergence is shown. In Sections 5.6 and 5.7, we will study the infinite-dimensional RVSDs and show how to approximate their solutions via

space discretisation.

5.2 A general random vortex dynamics

The Cauchy problem for the vorticity equation for an incompressible viscous flow is

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \Delta \omega + S\omega, \quad \omega(x, 0) = \omega_0(x). \quad (5.2.1)$$

From the forward Feynman–Kac formula (see Theorem 4.4.1), the solution ω to (5.2.1) can be represented in terms of the following functional integral:

$$\omega(x, T) = \int_{\mathbb{R}^3} \omega_0(\xi) p_u(0, \xi; T, x) \mathbb{E}[Q_0 | X_0 = \xi, X_T = x] d\xi,$$

where X_t is the diffusion process given by the SDE

$$dX_t = u(X_t, t)dt + \sqrt{2\nu}dB_t,$$

whose transition probability density function is denoted by p_u , and

$$\frac{d}{dt}Q_t = -Q_t S(X_t, t), \quad Q_T = I_3.$$

The approach in Chapter 4 can be applied to PDEs of the following form:

$$\frac{\partial \omega}{\partial t} + f(u, \nabla u, x, t) \cdot \nabla \omega = \nu \Delta \omega + \omega \cdot g(u, \nabla u, x, t), \quad \omega(x, 0) = \omega_0(x), \quad (5.2.2)$$

where $\omega, u, f \in \mathbb{R}^m$ and $g \in \mathbb{R}^{m \times m}$; in component-wise form this may be rewritten as

$$\frac{\partial \omega^i}{\partial t} + f^j(u, \nabla u, x, t) \cdot \frac{\partial \omega^i}{\partial x^j} = \nu \Delta \omega^i + \omega^j \cdot g_j^i(u, \nabla u, x, t), \quad \omega^i(x, 0) = \omega_0^i(x),$$

for $i = 1, 2, \dots, m$. Such PDEs appear in fluid flow models, but also in other physical models.

From the forward Feynman–Kac formula, if $f(u, \nabla u, x, t)$ is divergence-free, for example, when $f(u, \nabla u, x, t)$ is a linear combination of u and $\frac{\partial u}{\partial x^i}$, $i = 1, 2, \dots, m$, and u is the velocity field of an incompressible fluid, then the solution of the above system of PDEs has the following functional integral representation:

$$\omega(x, T) = \int_{\mathbb{R}^m} \omega_0(\xi) p_b(0, \xi; T, x) \mathbb{E}[Q_0 | X_0 = \xi, X_T = x] d\xi,$$

where X_t is the diffusion process defined by solving the following system of SDEs:

$$dX_t = f(u(X_t, t), \nabla u(X_t, t), X_t, t) dt + \sqrt{2\nu} dB_t, \quad (5.2.3)$$

with p_b being its transition probability density function, and

$$\frac{d}{dt} Q_t = -Q_t g(u(X_t, t), \nabla u(X_t, t), X_t, t), \quad Q_T = I_m. \quad (5.2.4)$$

Under the assumption that ω_0 has a compact support the vorticity transport equation (5.2.1) is a closed system of PDEs depending only on ω by using the relation $u(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y, t) dy$. Therefore, if one can close the system (5.2.2) by means of the nonlocal singular integrals

$$u^i = \int_{\mathbb{R}^m} K_j^i(x-y) \omega^j(y, t) dy$$

and

$$\nabla u_j^i = \int_{\mathbb{R}^m} H_{j,k}^i(x-y) \omega^k(y, t) dy$$

where K and H_j , $j = 1, 2, \dots, m$, are all $m \times m$ matrices, then we may employ the same idea as in the formulation of the random vortex method developed in Chapter 4 to close the previous system of SDEs (5.2.3), (5.2.4). In fact, we may rewrite (5.2.3) and (5.2.4) so that they form a closed system of equations involving only X and Q .

Let

$$dX_t(\xi) = f(u(X_t(\xi), t), \nabla u(X_t(\xi), t), X_t(\xi), t)dt + \sqrt{2\nu}dB_t, \quad X_0(\xi) = \xi \quad (5.2.5)$$

and

$$\frac{d}{ds}G_{t,s}(\xi) = -G_{t,s}(\xi) g(u(X_s(\xi), s), \nabla u(X_s(\xi), s), X_s(\xi), s), \quad G_{t,t}(\xi) = I_m. \quad (5.2.6)$$

Then,

$$\begin{aligned} u^i(x, t) &= \int_{\mathbb{R}^m} K_j^i(x - y) \omega^j(y, t) dy \\ &= \int_{\mathbb{R}^m} K_j^i(x - y) \int_{\mathbb{R}^m} \omega_0^l(\xi) p_b(0, \xi; t, y) \mathbb{E}[Q_l^j(t) | X_0 = \xi, X_t = y] d\xi dy \\ &= \int_{\mathbb{R}^m} \omega_0^l(\xi) \mathbb{E}[K_j^i(x - X_t) Q_l^j(t) | X_0 = \xi] d\xi \\ &= \int_{\mathbb{R}^m} \omega_0^l(\xi) \mathbb{E}[K_j^i(x - X_t(\xi)) G_l^j(t, 0; \xi)] d\xi. \end{aligned}$$

(We shall sometimes write $G(t, s)$ instead of $G_{t,s}$ if we need to write down a particular component of G .) Similarly,

$$\nabla u_j^i(x, t) = \int_{\mathbb{R}^m} \omega_0^l(\xi) \mathbb{E}[H_{j,k}^i(x - X_t(\xi)) G_l^k(t, 0; \xi)] d\xi.$$

By substituting these expressions into (5.2.5) and (5.2.6), we have a closed system for the pair (X, G) ; X_t and $G_{t,0}$ are both infinite-dimensional stochastic processes since they are defined at each $\xi \in \mathbb{R}^m$. To avoid the difficulty of analysis in infinite-dimensional spaces, we replace u and ∇u with the corresponding Riemann sums $\tilde{u}$ and $\widetilde{\nabla} u$ with mesh size $h > 0$ as follows.

$$\tilde{u}^i(x, t) = \sum_{k=(k_1, k_2, \dots, k_m) \in \mathbb{Z}^m \cap D} \omega_0^l(kh) \mathbb{E}[K_j^i(x - \tilde{X}_t(kh)) \tilde{G}_l^j(t, 0; kh)] h^m$$

and

$$\widetilde{\nabla} u_j^i(x, t) = \sum_{k=(k_1, k_2, \dots, k_m) \in \mathbb{Z}^m \cap D} \omega_0^l(kh) \mathbb{E} \left[H_{j,k}^i(x - \tilde{X}_t(kh)) \tilde{G}_l^k(t, 0; kh) \right] h^m$$

where D is some bounded set in $\mathbb{Z}^m$, and $\{\tilde{X}(kh), \tilde{G}(kh)\}_{k=(k_1, k_2, \dots, k_m) \in \mathbb{Z}^m \cap D}$ is then determined by (5.2.5) and (5.2.6) by replacing u and ∇u with $\tilde{u}$ and $\widetilde{\nabla} u$.

We denote the finite collection of $\{\tilde{X}(kh), \tilde{G}(kh)\}_{k=(k_1, k_2, \dots, k_m) \in \mathbb{Z}^m \cap D}$ by $(\tilde{X}, \tilde{G})$, so that $f(\tilde{u}(x, t), \widetilde{\nabla} u(x, t), x, t)$ and $g(\tilde{u}(x, t), \widetilde{\nabla} u(x, t), x, t)$ are both of the form $\varphi(x, t, \mathcal{L}(\tilde{X}_t, \tilde{G}_{t,0}))$. That is, they depend on (x, t) and the joint distribution of $(\tilde{X}_t, \tilde{G}_{t,0})$. Hence, the equations that $(\tilde{X}, \tilde{G})$ satisfy can be written as

$$\begin{aligned} d\tilde{X}_t(kh) &= \varphi_1(\tilde{X}_t(kh), t, \mathcal{L}(\tilde{X}_t, \tilde{G}_{t,0}))dt + \sqrt{2\nu}dB_t, \quad \tilde{X}_0(kh) = kh \\ \frac{d}{ds}\tilde{G}_{t,s}(kh) &= -\tilde{G}_{t,s}(kh)\varphi_2(\tilde{X}_s(kh), s, \mathcal{L}(\tilde{X}_s, \tilde{G}_{s,0})), \quad \tilde{G}_{t,t}(\xi) = I_m. \end{aligned} \tag{5.2.7}$$

for some φ_1 and φ_2 . By concatenating this system of equations for all $k \in \mathbb{Z}^m \cap D$, it can be rewritten in the following form

$$\begin{aligned} d\tilde{X}_t &= b(\tilde{X}_t, t, \mathcal{L}(\tilde{X}_t, \tilde{G}_{t,0}))dt + \sqrt{2\nu}dB_t, \\ \frac{d}{ds}\tilde{G}_{t,s} &= -S(\tilde{G}_{t,s}, \tilde{X}_s, s, \mathcal{L}(\tilde{X}_s, \tilde{G}_{s,0})) \end{aligned}$$

with given $\tilde{X}_0$ and $\tilde{G}_{t,t}$ for some b and S , and S is linear with respect to $\tilde{G}_{t,s}$. Here, we shall study a slightly more general nonlinear system by allowing nonlinear dependence of S on $\tilde{G}_{t,s}$ and a nontrivial diffusion coefficient. That is, in this chapter we study the following system of SDEs:

$$\begin{aligned} dX_t &= b(X_t, t, \mathcal{L}(X_t, G_{t,0}))dt + \sigma(X_t, t, \mathcal{L}(X_t, G_{t,0}))dB_t \\ dG_{t,s} &= -S(G_{t,s}, X_s, s, \mathcal{L}(X_s, G_{s,0}))ds \end{aligned}$$

A more detailed definition of this system will be given in the next section.

5.3 Equations arising from random vortex dynamics

We begin with the following definition.

Definition 5.3.1. (RVSDEs) *On a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, by a RVSDE system we mean a system of SDEs coupled with a system of functional differential equations depending on the distribution of its solution:*

$$\begin{aligned} dX_t &= b(X_t, t, \mathcal{L}(X_t, G_{t,0}))dt + \sigma(X_t, t, \mathcal{L}(X_t, G_{t,0}))dB_t, \\ dG_{t,s} &= -S(G_{t,s}, X_s, s, \mathcal{L}(X_s, G_{s,0}))ds \end{aligned} \tag{5.3.1}$$

for $0 \leq s \leq t \leq T$, where $X_t \in \mathbb{R}^d, G_{t,s} \in \mathbb{R}^l$. $b(x, t, m) = \{b^i(x, t, m)\}, 1 \leq i \leq d$, is a Borel-measurable function on $\mathbb{R}^d \times [0, T] \times \mathcal{P}_2(\mathbb{R}^{d+l})$, and $\sigma = \{\sigma_j^i(x, t, m)\}, 1 \leq i \leq d, 1 \leq j \leq r$, is a Borel-measurable function on $\mathbb{R}^d \times [0, T] \times \mathcal{P}_2(\mathbb{R}^{d+l})$ and $S = \{S^i(g, x, t, m)\}, 1 \leq i \leq l$, is Borel-measurable function on $\mathbb{R}^l \times \mathbb{R}^d \times [0, T] \times \mathcal{P}_2(\mathbb{R}^{d+l})$; B_t is an r -dimensional $(\mathcal{F}_t)_{t \geq 0}$ Brownian motion.

$G_{t,s}$ is defined on the simplex $0 \leq s \leq t \leq T$. We call a family of random variables defined on the simplex a *stochastic simplex-indexed process* for convenience.

Definition 5.3.2. (Stochastic simplex-indexed process) *Given a real number $T \in (0, \infty]$ and a measurable space $(\Omega, \mathcal{F})$, a stochastic simplex-indexed process is a collection of random variables $G = \{G_{t,s}; 0 \leq s \leq t \leq T\}$ defined on $(\Omega, \mathcal{F})$, which take values in a topological space S with its Borel σ -algebra. We say that a stochastic simplex-indexed process G is continuous if, for almost surely ω , $G_{t,s}(\omega)$ is a continuous mapping from $0 \leq s \leq t \leq T$ to S .*

We say that a continuous d -dimensional stochastic processes $X = \{X_t; 0 \leq t \leq T\}$ together with an l -dimensional continuous stochastic simplex-indexed process $G = \{G_{t,s}; 0 \leq s \leq t \leq T\}$ is a solution to (5.3.1) if the following properties hold:

- (i) $\mathcal{L}(X_t, G_{t,0})$ is the joint distribution of $(X_t, G_{t,0})$ on $\mathbb{R}^{d+l}$;
- (ii) $X_t, G_{t,0} \in \mathcal{F}_t$ for all $0 \leq t \leq T$;
- (iii)

$$\mathbb{P}\left[\int_0^t |b^i(X_s, s, \mathcal{L}(X_s, G_{s,0}))| + |S^j(G_{t,s}, X_s, s, \mathcal{L}(X_s, G_{s,0}))| + \sigma_n^{m2} |(X_s, s, \mathcal{L}(X_s, G_{s,0}))| ds < \infty\right] = 1$$

for every $1 \leq i, m \leq d, 1 \leq j \leq l, 1 \leq n \leq r$ and $0 \leq t \leq T$ if $T < \infty$ or $0 \leq t < \infty$ if $T = \infty$;

- (iv) $X_t, G_{t,s}$ satisfy the integral equations

$$X_t - X_0 = \int_0^t b(X_s, s, \mathcal{L}(X_s, G_{s,0})) ds + \int_0^t \sigma(X_s, s, \mathcal{L}(X_s, G_{s,0})) dB_s, \quad (5.3.2)$$

$$G_{t,t} - G_{t,s} = \int_s^t -S(G_{t,u}, X_u, u, \mathcal{L}(X_u, G_{u,0})) du, \quad (5.3.3)$$

or equivalently,

$$X_t^i - X_0^i = \int_0^t b^i(X_s, s, \mathcal{L}(X_s, G_{s,0})) ds + \sum_{j=1}^r \int_0^t \sigma_j^i(X_s, s, \mathcal{L}(X_s, G_{s,0})) dB_s^j,$$

$$G_{t,t}^k - G_{t,s}^k = \int_s^t -S^k(G_{t,u}, X_u, u, \mathcal{L}(X_u, G_{u,0})) du$$

for all $1 \leq i \leq d, 1 \leq k \leq l$.

The equations (5.3.2) and (5.3.3) form a coupled system of equations. The distribution $\mathcal{L}(X_t, G_{t,0})$ in b and σ depends on both X_t and $G_{t,0}$, which determines the dynamics of X . It is clear from (5.3.3) that $G_{t,0}$ is dependent on the whole history path of X up to t .

Such a pair (X, G) is referred as a *weak solution*. We say that a weak solution (X, G) is a *strong solution* if $(X_t, G_{t,0})_{0 \leq t \leq T}$ is adapted to the filtration generated by

the Brownian motion B .

5.3.1 Existence and uniqueness of strong solutions

The next theorem identifies suitable conditions on b, σ and S that ensure the existence and uniqueness of strong solutions to RVSDEs.

Theorem 5.3.3. (Existence and uniqueness under Lipschitz conditions)

Suppose that the coefficients b, σ and S satisfy the following global Lipschitz and linear growth conditions for some constant $K > 0$, in the sense that

$$\begin{aligned} & |b(x, t, m) - b(x', t, m')| + |\sigma(x, t, m) - \sigma(x', t, m')| + |S(g, x, t, m) - S(g', x', t, m')| \\ & \leq K(|x - x'| + |g - g'| + W_2(m, m')), \end{aligned} \quad (\text{RVSDE-A1})$$

$$\begin{aligned} & |b(x, t, m)|^2 + |\sigma(x, t, m)|^2 \leq K^2(1 + |x|^2 + W_2(m, \pi_0)^2), \\ & |S(g, x, t, m)| \leq K(1 + |g| + |x| + W_2(m, \pi_0)) \end{aligned} \quad (\text{RVSDE-A2})$$

for every $0 \leq t \leq T$, $x, x' \in \mathbb{R}^d$, $g, g' \in \mathbb{R}^l$, $m, m' \in \mathcal{P}_2(\mathbb{R}^{d+l})$. Here, π_0 the Dirac- δ distribution in $\mathcal{P}_2(\mathbb{R}^{d+l})$.i.e.

$$\pi_0(\{0\}) = 1$$

Assume that $\xi \in \mathcal{F}_0, \zeta_t \in \mathcal{F}_t$, $\mathbb{E}|\xi|^2 + \sup_{0 \leq t \leq T} |\zeta_t|^2 < \infty$. Then there exists a unique solution (X, G) to the RVSDE (5.3.1) on $[0, T]$ with $X_0 = \xi$ and $G_{t,t} = \zeta_t$.

Proof. Given a distribution $m \in \mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$, for $t \in [0, T]$ let m_t be the push-forward measure of m on $\mathbb{R}^{d+l}$ induced by the projection from $C([0, T]; \mathbb{R}^{d+l})$ to $\mathbb{R}^{d+l}$ at t . For $s, t \in [0, T]$,

$$W_2(m_s, m_t)^2 \leq \int_{C([0, T]; \mathbb{R}^{d+l})} |\omega_t - \omega_s|^2 m(d\omega).$$

Hence, by Lebesgue's dominated convergence theorem, $W_2(m_s, m_t) \rightarrow 0$ when $s \rightarrow t$.

Therefore, $t \mapsto m_t$ is a continuous map from $[0, T]$ to $\mathcal{P}_2(\mathbb{R}^{d+l})$.

Now from the existence and uniqueness of strong solutions to SDEs stated in Theorem 2.4.1, there exists a unique strong solution $X = \{X_t; 0 \leq t \leq T\}$ to

$$X_t - X_0 = X_t - \xi = \int_0^t b(X_s, s, m_s) ds + \int_0^t \sigma(X_s, s, m_s) dB_s, \quad (5.3.4)$$

which in turn gives a solution $G = \{G_{t,s}; 0 \leq s \leq t \leq T\}$ to

$$G_{t,t} - G_{t,s} = \zeta_t - G_{t,s} = \int_s^t -S(G_{t,u}, X_u, u, m_u) du \quad (5.3.5)$$

pathwise. Clearly, $G_{t,0} \in \mathcal{F}_t$ since $G_{t,0}$ can be regarded as a functional from the path $X_{[0,t]}$, and actually $G_{t,s} \in \mathcal{F}_t$ for all $0 \leq s \leq t$ for the same reason. We shall divide the rest of our proof into two parts.

Part 1: G is continuous and the law of $(X_t, G_{t,0})_{0 \leq t \leq T} \in \mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$, which naturally gives us a map Φ from $\mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$ to itself as follows:

$$m \xrightarrow{(5.3.4), (5.3.5)} (X, G) \xrightarrow{\mathcal{L}(X_t, G_{t,0})} \Phi(m)$$

Part 2: There exists a unique fixed point of the map Φ .

Proof of Part 1: By Lemma 5.3.5, G is continuous.

To show that $\mathcal{L}(X, G_{\cdot,0}) \in \mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$, from (5.3.5), we have

$$|G_{t,s}| \leq |\zeta_t| + K \int_s^t (1 + |G_{t,u}| + |X_u| + W_2(m_u, \pi_0)) du.$$

By Gronwall's inequality,

$$|G_{t,0}| \leq C(K, T) \left(1 + |\zeta_t| + \int_0^t (|X_u| + W_2(m_u, \pi_0)) du \right),$$

which yields that

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq t \leq T} |G_{t,0}|^2 \\ & \leq C(K, T) \left(1 + \mathbb{E} \sup_{0 \leq t \leq T} |\zeta_t|^2 + \int_0^T \mathbb{E} |X_u|^2 du + \int_0^T W_2(m_u, \pi_0)^2 du \right). \end{aligned} \quad (5.3.6)$$

Moreover, given m , from (2.4.5) we have that $\mathbb{E}(\sup_{0 \leq t \leq T} |X_t|^2) < \infty$, which implies that $\mathbb{E}(\sup_{0 \leq t \leq T} |G_{t,0}|^2) < \infty$. Hence,

$$\mathcal{L}(X_t, G_{t,0})_{0 \leq t \leq T} \in \mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l})).$$

Proof of Part 2: Let $m, m' \in \mathcal{P}_2(\mathbb{R}^{d+l})$. For every $t \in [0, T]$, by using the Lipschitz continuity of the coefficients and Doob's inequality in L^2 , we have that

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq t} |X_s - X'_s|^2 \right] \\ & \leq 2\mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \int_0^s b(X_u, u, m_u) - b(X'_u, u, m'_u) du \right|^2 \right] \\ & \quad + 2\mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \int_0^s \sigma(X_u, u, m_u) - \sigma(X'_u, u, m'_u) dB_u \right|^2 \right] \\ & \leq 2\mathbb{E} \left[\left(\int_0^t |b(X_s, s, m_s) - b(X'_s, s, m'_s)| ds \right)^2 \right] \\ & \quad + C\mathbb{E} \int_0^t |\sigma(X_u, u, m_u) - \sigma(X'_u, u, m'_u)|^2 du \\ & \leq C(T)\mathbb{E} \left[\int_0^t |b(X_s, s, m_s) - b(X'_s, s, m'_s)|^2 + |\sigma(X_s, s, m_s) - \sigma(X'_s, s, m'_s)|^2 ds \right] \\ & \leq C(K, T) \int_0^t \mathbb{E} \left[\sup_{0 \leq u \leq s} |X_u - X'_u|^2 \right] ds + C(K, T) \int_0^t W_2(m_s, m'_s)^2 ds. \end{aligned}$$

It then follows by Gronwall's inequality that

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |X_s - X'_s|^2 \right] \leq C(K, T) \int_0^t W_2(m_s, m'_s)^2 ds. \quad (5.3.7)$$

On the other hand, for all $0 \leq s \leq t \leq T$,

$$G_{t,s} - G'_{t,s} = \int_s^t (S(G_{t,u}, X_u, u, m_u) - S(G'_{t,u}, X'_u, u, m'_u)) \, du,$$

which leads to

$$\begin{aligned} & |G_{t,s} - G'_{t,s}|^2 \\ & \leq \left(\int_s^t (K|G_{t,u} - G'_{t,u}| + K|X_u - X'_u| + KW_2(m_u, m'_u)) \, du \right)^2 \\ & \leq C(K, T) \left[\int_s^t |G_{t,u} - G'_{t,u}|^2 \, du + \int_s^t (|X_u - X'_u|^2 + W_2(m_u, m'_u)^2) \, du \right]. \end{aligned} \quad (5.3.8)$$

By using Gronwall's inequality with respect to s we have that

$$|G_{t,0} - G'_{t,0}|^2 \leq C(K, T) \int_0^t (|X_u - X'_u|^2 + W_2(m_u, m'_u)^2) \, du. \quad (5.3.9)$$

By combining (5.3.7) and (5.3.9) we deduce that

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq s \leq t} (|X_s - X'_s|^2 + |G_{s,0} - G'_{s,0}|^2) \right] \\ & \leq C(K, T) \left(\int_0^t W_2(m_u, m'_u)^2 \, du + \mathbb{E} \int_0^t |X_u - X'_u|^2 \, du \right) \\ & \leq C(K, T) \int_0^t W_2(m_u, m'_u)^2 \, du. \end{aligned} \quad (5.3.10)$$

For every $\mu, \nu \in \mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$, and $0 \leq t \leq T$, let

$$W_{2,[0,t]}(\mu, \nu) := \inf_{m \in \Lambda} \left[\int \sup_{0 \leq s \leq t} |X_1(s) - X_2(s)|^2(\omega) \, dm(\omega) \right]^{1/2},$$

where $\omega = (\omega_1, \omega_2) \in C([0, T]; \mathbb{R}^{d+l}) \times C([0, T]; \mathbb{R}^{d+l})$, and $(X_1(t), X_2(t))(\omega) = (\omega_1(t), \omega_2(t))$. Λ is the set of probability measures with marginal distributions μ and ν . Clearly, $W_{2,[0,T]}(\mu, \nu) = W_2(\mu, \nu)$. From (5.3.10), we have that

$$W_{2,[0,t]}(\Phi(m), \Phi(m'))^2 \leq \mathbb{E} \left[\sup_{0 \leq s \leq t} (|X_s - X'_s|^2 + |G_{s,0} - G'_{s,0}|^2) \right]$$

$$\leq C(K, T) \int_0^t W_2(m_s, m'_s)^2 ds.$$

As $W_2(m_s, m'_s) \leq W_{2,[0,s]}(m, m')$, it follows that

$$W_{2,[0,t]}(\Phi(m), \Phi(m'))^2 \leq C(K, T) \int_0^t W_{2,[0,s]}(m, m')^2 ds \quad (5.3.11)$$

for all $t \leq T$. Therefore to show the existence of some fixed point, consider a sequence $\{m, \Phi(m), \Phi^2(m), \dots\} \in \mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$. From (5.3.11) we have that, for all $k \geq 1$,

$$W_{2,[0,t]}(\Phi^{k+1}(m), \Phi^k(m))^2 \leq C(K, T) \int_0^t W_{2,[0,s]}(\Phi^k(m), \Phi^{k-1}(m))^2 ds. \quad (5.3.12)$$

By recursively using this inequality, we deduce that

$$W_{2,[0,t]}(\Phi^{k+1}(m), \Phi^k(m))^2 \leq \frac{[C(K, T)t]^k M}{k!}, \quad (5.3.13)$$

where $M = W_2(\Phi^1(m), \Phi^0(m))^2$. It is now easy to see that $\{\Phi^k(m); k \geq 0\}$ is a Cauchy sequence in $\mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$, which, by completeness on this space when equipped with the Wasserstein metric, converges to some $m' \in \mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$. By (5.3.11), Φ is also a Lipschitz map with constant $\sqrt{C(K, T)T}$, which means that $\Phi(m') = m'$.

To prove the uniqueness, we can apply the same argument as in proving (5.3.13). For every $m, m' \in \mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$, we have that

$$W_{2,[0,T]}(\Phi^k(m), \Phi^k(m'))^2 \leq \frac{[C(K, T)T]^k}{k!} W_2(m, m')^2,$$

which means that Φ^k is a contraction on $\mathcal{P}_2(C([0, T]; \mathbb{R}^{d+l}))$ when k is large enough, and this then implies the uniqueness of the fixed point of Φ . $\square$

The following bound of the solutions will be useful in the future.

Corollary 5.3.4. *Let the solution to the RVSDE (5.3.1) be (X, G) , then we have the following bound for the solution:*

$$\mathbb{E} \sup_{0 \leq t \leq T} (|X_t|^2 + |G_{t,0}|^2) \leq C(K, T) \left(1 + \mathbb{E} \sup_{0 \leq t \leq T} |\zeta_t|^2 + \mathbb{E} |\xi|^2 \right).$$

Proof. Denote $\mathcal{L}(X_t, G_{t,0})$ by m_t . Since

$$X_t = \xi + \int_0^t b(X_s, s, m_s) ds + \int_0^t \sigma(X_s, s, m_s) dB_s,$$

we have

$$\mathbb{E} \sup_{0 \leq s \leq t} |X_s|^2 \leq C(K, T) \left[1 + \mathbb{E} |\xi|^2 + \int_0^t \mathbb{E} |X_u|^2 du + \int_0^t W_2(m_u, \pi_0)^2 du \right]$$

for every $t \in [0, T]$. On the other hand, according to (5.3.6),

$$\mathbb{E} \sup_{0 \leq s \leq t} |G_{s,0}|^2 \leq C(K, T) \left[1 + \mathbb{E} \sup_{0 \leq s \leq t} |\zeta_s|^2 + \int_0^t \mathbb{E} |X_u|^2 du + \int_0^t W_2(m_u, \pi_0)^2 du \right].$$

Combining these two inequalities, we have

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq t} |X_s|^2 + \mathbb{E} \sup_{0 \leq s \leq t} |G_{s,0}|^2 \\ & \leq C(K, T) \left[1 + \mathbb{E} |\xi|^2 + \mathbb{E} \sup_{0 \leq s \leq t} |\zeta_s|^2 + \int_0^t \mathbb{E} |X_u|^2 du + \int_0^t W_2(m_u, \pi_0)^2 du \right] \\ & \leq C(K, T) \left[1 + \mathbb{E} |\xi|^2 + \mathbb{E} \sup_{0 \leq s \leq t} |\zeta_s|^2 + \int_0^t \mathbb{E} |X_u|^2 du + \int_0^t (\mathbb{E} |X_u|^2 + \mathbb{E} |G_{u,0}|^2) du \right] \end{aligned}$$

The result now follows from the Gronwall's inequality. $\square$

Lemma 5.3.5. *Let $S(g, t)$ be a continuous function from $\mathbb{R}^l \times [0, T]$ to $\mathbb{R}^l$ which satisfies the global Lipschitz and linear growth conditions for constant K , i.e.*

$$S(g, t) \leq K(1 + |g|)$$

$$|S(g, t) - S(g', t)| \leq K|g - g'|$$

for every $g, g' \in \mathbb{R}^l, t \in [0, T]$. Let $\{G_{t,s}, 0 \leq s \leq t \leq T\} \in \mathbb{R}^l$ be the unique solution to

$$dG_{t,s} = -S(G_{t,s}, s)ds \quad (5.3.14)$$

with $G_{t,t} = \zeta_t$. Suppose ζ_t is continuous, then $G_{t,s}$ is continuous in the simplex $0 \leq s \leq t \leq T$, and we have the following bound for $G_{t,s}$:

$$\sup_{0 \leq s \leq t \leq T} |G_{t,s}| \leq \left(\sup_{0 \leq t \leq T} |\zeta_t| + KT \right) e^{KT} \quad (5.3.15)$$

Proof. Denote the simplex $0 \leq s \leq t \leq T$ by Δ . First, we show that $G_{t,s}$ is bounded in Δ . In fact, given any $(t, s) \in \Delta$, by (5.3.14)

$$|G_{t,s}| \leq |\zeta_t| + K \int_s^t (1 + |G_{t,u}|) du.$$

By Gronwall's inequality, $|G_{t,s}| \leq (\sup_{0 \leq t \leq T} |\zeta_t| + KT)e^{KT}$, which proves (5.3.15).

To show continuity, let $(t_1, s_1), (t_2, s_2)$ be any two points in Δ , and let $s_0 = \min(s_1, s_2)$, then

$$\begin{aligned} & |G_{t_2, s_2} - G_{t_1, s_1}| \\ & \leq |G_{t_2, s_2} - G_{t_2, s_0}| + |G_{t_2, s_0} - G_{t_1, s_0}| + |G_{t_1, s_0} - G_{t_1, s_1}| \end{aligned}$$

Since $dG_{t,s} = -S(G_{t,s}, s)ds$,

$$|G_{t_2, s_2} - G_{t_2, s_0}| + |G_{t_1, s_0} - G_{t_1, s_1}| \leq K(|s_2 - s_0| + |s_1 - s_0|) \left(1 + \sup_{0 \leq s \leq t \leq T} |G_{t,s}|\right). \quad (5.3.16)$$

To bound $|G_{t_2, s_0} - G_{t_1, s_0}|$, w.l.o.g, assume $t_1 \leq t_2$. Consider $G_{t_2, u} - G_{t_1, u}$ for $0 \leq u \leq t_1$. From (5.3.14) and the Lipschitz continuity of S , we have

$$|G_{t_2, u} - G_{t_1, u}| \leq |G_{t_2, t_1} - G_{t_1, t_1}| + K \int_u^{t_1} |G_{t_2, u} - G_{t_1, u}| du.$$

By Gronwall's inequality, we are able to obtain that

$$\begin{aligned}
& |G_{t_2, s_0} - G_{t_1, s_0}| \\
& \leq C(K, T) |G_{t_2, t_1} - G_{t_1, t_1}| \\
& \leq C(K, T) \left[|\zeta_{t_2} - \zeta_{t_1}| + |t_2 - t_1| \left(1 + \sup_{0 \leq s \leq t \leq T} |G_{t, s}| \right) \right]
\end{aligned} \tag{5.3.17}$$

Therefore, by combining (5.3.16) and (5.3.17), $G_{t, s}$ is continuous in Δ .

□

5.3.2 Stability

Next, we show the stability of solutions.

Theorem 5.3.6. (Stability of RVSDs) *Suppose that b, σ, S satisfy (RVSD-A1) and (RVSD-A2) with constant K , let (X, G) be the unique solution to the RVSD (5.3.1), and let (X', G') be a solution to (5.3.1) with $(b', S', \sigma', \xi', \zeta')$. Then,*

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} \left(|X_s - X'_s|^2 + |G_{s, 0} - G'_{s, 0}|^2 \right) \\
& \leq C(K, T) \left[\|\sigma - \sigma'\|_{L^\infty}^2 + \|b - b'\|_{L^\infty}^2 + \|S - S'\|_{L^\infty}^2 \right] t \\
& \quad + \mathbb{E} \sup_{0 \leq s \leq t} |\zeta_s - \zeta'_s|^2 + \mathbb{E} |\xi - \xi'|^2.
\end{aligned} \tag{5.3.18}$$

Proof. For every $0 \leq t \leq T$,

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} |X_s - X'_s|^2 \\
& \leq C(T) \left\{ \mathbb{E} \int_0^t |b(X_s, s, \mathcal{L}(X_s, G_{s, 0})) - b'(X'_s, s, \mathcal{L}(X'_s, G'_{s, 0}))|^2 ds \right. \\
& \quad \left. + \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \int_0^s (\sigma(X_s, s, \mathcal{L}(X_s, G_{s, 0})) - \sigma'(X'_s, s, \mathcal{L}(X'_s, G'_{s, 0}))) dB_s \right|^2 + |\xi - \xi'|^2 \right] \right\} \\
& \leq C(T) \left\{ \mathbb{E} \int_0^t (|b(X_s, s, \mathcal{L}(X_s, G_{s, 0})) - b(X'_s, s, \mathcal{L}(X'_s, G'_{s, 0}))|^2 + \|b - b'\|_{L^\infty}^2) ds \right. \\
& \quad \left. + \mathbb{E} \left[\sup_{0 \leq s \leq t} \left| \int_0^s (\sigma(X_s, s, \mathcal{L}(X_s, G_{s, 0})) - \sigma(X'_s, s, \mathcal{L}(X'_s, G'_{s, 0}))) dB_s \right|^2 \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sup_{0 \leq s \leq t} \left| \int_0^s (\sigma(X'_s, s, \mathcal{L}(X'_s, G'_{s,0})) - \sigma'(X'_s, s, \mathcal{L}(X'_s, G'_{s,0}))) dB_s \right|^2 + |\xi - \xi'|^2 \Big] \Big\} \\
& \leq C(K, T) \left\{ \mathbb{E} \left[\int_0^t |X_s - X'_s|^2 + W_2(\mathcal{L}(X_s, G_{s,0}), \mathcal{L}(X'_s, G'_{s,0}))^2 ds \right. \right. \\
& \quad \left. \left. + (\|\sigma - \sigma'\|_{L^\infty}^2 + \|b - b'\|_{L^\infty}^2)t + |\xi - \xi'|^2 \right] \right\} \\
& \leq C(K, T) \left\{ \mathbb{E} \left[\int_0^t (|X_s - X'_s|^2 + |G_{s,0} - G'_{s,0}|^2) ds \right. \right. \\
& \quad \left. \left. + (\|\sigma - \sigma'\|_{L^\infty}^2 + \|b - b'\|_{L^\infty}^2)t + |\xi - \xi'|^2 \right] \right\}. \tag{5.3.19}
\end{aligned}$$

On the other hand, by the same argument, for every $0 \leq s \leq t \leq T$,

$$\begin{aligned}
& |G_{t,s} - G'_{t,s}|^2 \\
& \leq C(K, T) \left\{ \int_s^t (|G_{t,u} - G'_{t,u}|^2 + |X_u - X'_u|^2 + W_2(\mathcal{L}(X_u, G_{u,0}), \mathcal{L}(X'_u, G'_{u,0}))^2) du \right. \\
& \quad \left. + |\zeta_t - \zeta'_t|^2 + \|S - S'\|_{L^\infty}^2(t - s) \right\},
\end{aligned}$$

so, by Gronwall's inequality, one may deduce that

$$\begin{aligned}
|G_{t,0} - G'_{t,0}|^2 & \leq C(K, T) \left\{ \int_0^t (|X_u - X'_u|^2 + W_2(\mathcal{L}(X_u, G_{u,0}), \mathcal{L}(X'_u, G'_{u,0}))^2) du \right. \\
& \quad \left. + |\zeta_t - \zeta'_t|^2 + \|S - S'\|_{L^\infty}^2 t \right\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |G_{s,0} - G'_{s,0}|^2 & \leq C(K, T) \left\{ \mathbb{E} \left[\int_0^t (|X_s - X'_s|^2 + |G_{s,0} - G'_{s,0}|^2) ds \right] \right. \\
& \quad \left. + \mathbb{E} \sup_{0 \leq s \leq t} |\zeta_s - \zeta'_s|^2 + \|S - S'\|_{L^\infty}^2 t \right\}.
\end{aligned}$$

Together with (5.3.19) we have that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} (|X_s - X'_s|^2 + |G_{s,0} - G'_{s,0}|^2) \\
& \leq C(K, T) \left\{ \int_0^t \mathbb{E} \sup_{0 \leq s \leq u} (|X_s - X'_s|^2 + |G_{s,0} - G'_{s,0}|^2) du \right.
\end{aligned}$$

$$\begin{aligned}
& + \mathbb{E} \sup_{0 \leq s \leq t} |\zeta_s - \zeta'_s|^2 + \mathbb{E} |\xi - \xi'|^2 \\
& + (\|\sigma - \sigma'\|_{L^\infty}^2 + \|b - b'\|_{L^\infty}^2 + \|S - S'\|_{L^\infty}^2)t \Big\}.
\end{aligned}$$

By Gronwall's inequality again we therefore deduce that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} (|X_s - X'_s|^2 + |G_{s,0} - G'_{s,0}|^2) \\
& \leq C(K, T) \left[(\|\sigma - \sigma'\|_{L^\infty}^2 + \|b - b'\|_{L^\infty}^2 + \|S - S'\|_{L^\infty}^2)t \right. \\
& \quad \left. + \mathbb{E} \sup_{0 \leq s \leq t} |\zeta_s - \zeta'_s|^2 + \mathbb{E} |\xi - \xi'|^2 \right],
\end{aligned}$$

which completes the proof. $\square$

5.3.3 Propagation of chaos

For MVSEs, the solution could be approximated by running interacting particles (see Theorem 2.5.3), which is referred as the propagation of chaos. In this section, we are going to formulate and prove the propagation of chaos for RVSEs. Consider the following equations:

$$\begin{aligned}
X_t^{(i),N} - X_0^{(i),N} &= X_t^{(i),N} - \xi^{(i)} = \int_0^t b(X_s^{(i),N}, s, \frac{1}{N} \sum_{i=1}^N \delta_{X_s^{(i),N}, G_{s,0}^{(i),N}}) ds \\
&+ \int_0^t \sigma(X_s^{(i),N}, s, \frac{1}{N} \sum_{i=1}^N \delta_{X_s^{(i),N}, G_{s,0}^{(i),N}}) dB_s^{(i)}
\end{aligned} \tag{5.3.20}$$

$$G_{t,t}^{(i),N} - G_{t,s}^{(i),N} = \zeta_t^{(i)} - G_{t,s}^{(i),N} = \int_s^t -S(G_{t,u}^{(i),N}, X_u, u, \frac{1}{N} \sum_{i=1}^N \delta_{X_u^{(i),N}, G_{u,0}^{(i),N}}) du \tag{5.3.21}$$

in $0 \leq s \leq t \leq T$, where $(W^{(i)}, \xi^{(i)}, \zeta^{(i)})_{1 \leq i \leq N}$ are independent and have the same distribution as (W, ξ, ζ) . Denote the solutions by $(X, G)^{(i),N}$. We also consider the solution $(X, G)^{(i)}$ which satisfies the RVSEs (5.3.2) and (5.3.3) with corresponding

$(W^{(i)}, \xi^{(i)}, \zeta^{(i)})$. i.e.

$$X_t^{(i)} - X_0^{(i)} = X_t^{(i)} - \xi^{(i)} = \int_0^t b(X_s^{(i)}, s, \mathcal{L}(X_s^{(i)}, G_{s,0}^{(i)})) ds \quad (5.3.22)$$

$$+ \int_0^t \sigma(X_s^{(i)}, s, \mathcal{L}(X_s^{(i)}, G_{s,0}^{(i)})) dB_s^{(i)}$$

$$G_{t,t}^{(i)} - G_{t,s}^{(i)} = \zeta_t^{(i)} - G_{t,s}^{(i)} = \int_s^t -S(G_{t,u}^{(i)}, X_u^{(i)}, u, \mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)})) du \quad (5.3.23)$$

The result of this section will establish the connection between $(X, G)^{(i),N}$ and $(X, G)^{(i)}$. We recall the following two standard lemmas in Warssenstein spaces, which will be useful in our argument.

The first lemma is self-evident from the definition of Warssenstein spaces.

Lemma 5.3.7. *Let $(x_1, x_2, \dots, x_N), (y_1, y_2, \dots, y_N) \in \mathbb{R}^d$, and δ_x be the Dirac measure at x . Then*

$$W_2 \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \frac{1}{N} \sum_{i=1}^N \delta_{y_i} \right) \leq \left(\frac{1}{N} \sum_{i=1}^N |x_i - y_i|^2 \right)^{1/2}$$

The second lemma states that the empirical measure will converge to the underlying probability measure in Warssenstein spaces.

Lemma 5.3.8. *(see [17] Lemma 1.9) Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Let $\{X_i\}_{i \geq 1}$ be i.i.d. random variables with the law $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, Let μ_n denote the empirical measure $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$. Then*

$$\mathbb{E}[W_2(\mu_n, \mu)^2] \leq C \int_{\mathbb{R}^d} |x|^2 \mu(dx) \text{ for all } n$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E}[W_2(\mu_n, \mu)^2] = 0.$$

Now the propagation of chaos is stated as follows.

Theorem 5.3.9. *Assume that (RVSDE-A1) and (RVSDE-A2) hold, and $(X, G)^{(i),N}$*

and $(X, G)^{(i)}$ are defined by (5.3.20), (5.3.21), (5.3.22), (5.3.23), then

$$\lim_{N \rightarrow \infty} \max_{1 \leq i \leq N} \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{(i)} - X_s^{(i),N}|^2 + |G_{s,0}^{(i)} - G_{s,0}^{(i),N}|^2 = 0,$$

and as a result,

$$\lim_{N \rightarrow \infty} W_2 \left(\mathcal{L} \left(X^{(1)}, G_{\cdot,0}^{(1)} \right), \mathcal{L} \left(X^{(1),N}, G_{\cdot,0}^{(1),N} \right) \right) = 0.$$

Proof. The idea of this proof is borrowed from [17]. For $t \in [0, T]$, and $i = 1, 2, \dots, N$,

$$\begin{aligned} & \mathbb{E} \sup_{0 \leq s \leq t} |X_s^{(i)} - X_s^{(i),N}|^2 \\ & \leq C \left[\mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \left(b(X_u^{(i)}, u, \mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)})) - b(X_u^{(i),N}, u, \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j),N}, G_{u,0}^{(j),N}}) \right) du \right|^2 \right. \\ & \quad \left. + \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s \left(\sigma(X_u^{(i)}, u, \mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)})) - \sigma(X_u^{(i),N}, u, \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j),N}, G_{u,0}^{(j),N}}) \right) dB_u \right|^2 \right] \\ & \leq C(K, T) \mathbb{E} \int_0^t \left(|X_u^{(i)} - X_u^{(i),N}|^2 + W_2(\mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)}), \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j),N}, G_{u,0}^{(j),N}})^2 \right) du \\ & \leq C(K, T) \left\{ \mathbb{E} \int_0^t |X_u^{(i)} - X_u^{(i),N}|^2 du + \mathbb{E} \int_0^t W_2(\mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)}), \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j)}, G_{u,0}^{(j)}})^2 du \right. \\ & \quad \left. + \mathbb{E} \int_0^t W_2(\frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j)}, G_{u,0}^{(j)}}, \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j),N}, G_{u,0}^{(j),N}})^2 du \right\} \\ & \leq C(K, T) \left\{ \mathbb{E} \int_0^t |X_u^{(i)} - X_u^{(i),N}|^2 du + \mathbb{E} \int_0^t W_2(\mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)}), \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j)}, G_{u,0}^{(j)}})^2 du \right. \\ & \quad \left. + \mathbb{E} \int_0^t \frac{1}{N} \sum_{j=1}^N \left(|X_u^{(j)} - X_u^{(j),N}|^2 + |G_{u,0}^{(j)} - G_{u,0}^{(j),N}|^2 \right) du \right\} \\ & = C(K, T) \left\{ \mathbb{E} \int_0^t |X_u^{(i)} - X_u^{(i),N}|^2 du + \mathbb{E} \int_0^t W_2(\mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)}), \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j)}, G_{u,0}^{(j)}})^2 du \right. \\ & \quad \left. + \mathbb{E} \int_0^t \left(|X_u^{(i)} - X_u^{(i),N}|^2 + |G_{u,0}^{(i)} - G_{u,0}^{(i),N}|^2 \right) du \right\}. \tag{5.3.24} \end{aligned}$$

The last equation follows from the fact that $(X^{(j)}, G^{(j)}, X^{(j),N}, G^{(j),N})$ have the same distribution for each $j = 1, 2, \dots, N$. On the other hand, for every $0 \leq s \leq t \leq T$, by

the same argument,

$$\begin{aligned}
|G_{t,s}^{(i)} - G_{t,s}^{(i),N}|^2 &\leq C(K, T) \int_s^t \left[|G_{t,u}^{(i)} - G_{t,u}^{(i),N}|^2 + |X_u^{(i)} - X_u^{(i),N}|^2 \right. \\
&\quad \left. + W_2(\mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)}), \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j)}, G_{u,0}^{(j)}})^2 \right. \\
&\quad \left. + \frac{1}{N} \sum_{j=1}^N (|X_u^{(j)} - X_u^{(j),N}|^2 + |G_{u,0}^{(j)} - G_{u,0}^{(j),N}|^2) \right] du
\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}
|G_{t,0}^{(i)} - G_{t,0}^{(i),N}|^2 &\leq C(K, T) \int_0^t \left(|X_u^{(i)} - X_u^{(i),N}|^2 + W_2(\mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)}), \frac{1}{N} \sum_{j=1}^N \delta_{X_u^{(j)}, G_{u,0}^{(j)}})^2 \right. \\
&\quad \left. + \frac{1}{N} \sum_{j=1}^N (|X_u^{(j)} - X_u^{(j),N}|^2 + |G_{u,0}^{(j)} - G_{u,0}^{(j),N}|^2) \right) du.
\end{aligned}$$

Then by taking the expectation and using the symmetry among each $j = 1, 2, \dots, n$,

$$\begin{aligned}
\mathbb{E} \sup_{0 \leq s \leq t} |G_{s,0}^{(i)} - G_{s,0}^{(i),N}|^2 &\leq C(K, T) \mathbb{E} \int_0^t \left(|X_u^{(i)} - X_u^{(i),N}|^2 + |G_{u,0}^{(i)} - G_{u,0}^{(i),N}|^2 \right. \\
&\quad \left. + W_2(\mathcal{L}(X_u^{(i)}, G_{u,0}^{(i)}), \frac{1}{N} \sum_{i=1}^N \delta_{X_u^{(i)}, G_{u,0}^{(i)}})^2 \right) du,
\end{aligned}$$

which yields

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq s \leq t} |X_s^{(i)} - X_s^{(i),N}|^2 + |G_{s,0}^{(i)} - G_{s,0}^{(i),N}|^2 \\
&\leq C(K, T) \int_0^t \mathbb{E} \left[\sup_{0 \leq u \leq s} |X_u^{(i)} - X_u^{(i),N}|^2 + |G_{u,0}^{(i)} - G_{u,0}^{(i),N}|^2 \right. \\
&\quad \left. + W_2(\mathcal{L}(X_s^{(i)}, G_{s,0}^{(i)}), \frac{1}{N} \sum_{i=1}^N \delta_{X_s^{(i)}, G_{s,0}^{(i)}})^2 \right] ds.
\end{aligned}$$

together with (5.3.24).

Again by Gronwall's inequality,

$$\mathbb{E} \sup_{0 \leq s \leq t} |X_s^{(i)} - X_s^{(i),N}|^2 + |G_{s,0}^{(i)} - G_{s,0}^{(i),N}|^2$$

$$\leq C(K, T) \int_0^t \mathbb{E} W_2(\mathcal{L}(X_s^{(i)}, G_{s,0}^{(i)}), \frac{1}{N} \sum_{i=1}^N \delta_{X_s^{(i)}, G_{s,0}^{(i)}})^2 ds.$$

By Lemma 5.3.8,

$$\mathbb{E} W_2(\mathcal{L}(X_s^{(i)}, G_{s,0}^{(i)}), \frac{1}{N} \sum_{i=1}^N \delta_{X_s^{(i)}, G_{s,0}^{(i)}})^2 \leq C (\mathbb{E}|X_s|^2 + \mathbb{E}|G_{s,0}|^2).$$

Then by the dominated convergence theorem and Lemma 5.3.8 ,

$$\lim_{N \rightarrow \infty} \max_{1 \leq i \leq N} \mathbb{E} \sup_{0 \leq s \leq T} |X_s^{(i)} - X_s^{(i),N}|^2 + |G_{s,0}^{(i)} - G_{s,0}^{(i),N}|^2 = 0.$$

□

5.4 The large deviation principle for RVSDs

The Freidlin–Wentzell theory (cf. Theorem 2.6.9) is concerned with the LDP for a family of diffusion processes $\{X^\epsilon\}_{0 \leq \epsilon < 1}$ defined by the following system of SDEs:

$$dX_t^\epsilon = b(X_t^\epsilon)dt + \sqrt{\epsilon}\sigma(X_t^\epsilon)dB_t, \quad 0 \leq t \leq 1, \quad X_0^\epsilon = x_0.$$

where $x \in \mathbb{R}^d$ and W is an r -dimensional Brownian motion. Suppose that $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times r}$ are both Lipschitz continuous and bounded functions; then, $\{X^\epsilon\}_{0 \leq \epsilon < 1}$ satisfies the LDP in $C([0, 1]; \mathbb{R}^d)$ with good rate function given by

$$I_x(f) := \inf_{g \in H_1 : f(t) = x + \int_0^t b(f(s))ds + \int_0^t \sigma(f(s))\dot{g}(s)ds} \frac{1}{2} \int_0^1 |\dot{g}(t)|^2 dt$$

In this section we will establish a LDP for the RVSD (5.3.1), which is closely related to the viscosity limit problem to the related nonlinear fluid dynamic equations. It is worth exploring this aspect in the future.

Consider the following RVSDE, dependent on a small parameter $\epsilon > 0$:

$$\begin{aligned} dX_t^\epsilon &= b(X_t^\epsilon, t, \mathcal{L}(X_t^\epsilon, G_{t,0}^\epsilon))dt + \sqrt{\epsilon}\sigma(X_t^\epsilon, t, \mathcal{L}(X_t^\epsilon, G_{t,0}^\epsilon))dB_t, \\ dG_{t,s}^\epsilon &= -S(G_{t,s}^\epsilon, X_s^\epsilon, s, \mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon))ds, \end{aligned} \quad (5.4.1)$$

subject to the initial conditions $X_0^\epsilon = \xi, G_{t,t}^\epsilon = \zeta_t$. The aim of this section is to derive a large deviation principle for the distribution of (X^ϵ, G^ϵ) . We adopt the approach initiated in [96] to our situation, but it necessarily becomes much more complicated since the solution couple (X, G) is no longer Markovian. We assume that ζ_t is a deterministic process with $\xi = x_0$ for the rest of this section. Let ϕ, ψ be the unique solution couple of the following system of functional differential equations:

$$\begin{aligned} d\phi_t &= b(\phi_t, t, \delta_{\phi_t, \psi_{t,0}})dt, \quad \phi_0 = x_0, \\ d\psi_{t,s} &= -S(\psi_{t,s}, \phi_s, s, \delta_{\phi_s, \psi_{s,0}})ds, \quad \psi_{t,t} = \zeta_t. \end{aligned}$$

The existence and uniqueness of ϕ, ψ is guaranteed by a special case of Theorem 5.3.3 corresponding to $\sigma = 0$. Now we are in a position to state the following LDP for (5.4.1).

Theorem 5.4.1. *Suppose that b, σ, S satisfy global Lipschitz and linear growth conditions (RVSDE-A1) and (RVSDE-A2). We also assume that, for some $\beta \in (0, 1]$,*

$$|b(x, t, m) - b(x, t', m)| + |\sigma(x, t, m) - \sigma(x, t', m)| \leq K|t - t'|^\beta \quad (\text{RVSDE-A3})$$

for all $x \in \mathbb{R}^d, m \in \mathcal{P}_2(\mathbb{R}^{d+l}), 0 \leq t, t' \leq T$. and σ is bounded. Then the solution to (5.4.1) (X^ϵ, G^ϵ) satisfies the LDP with the good rate function

$$I(f, g) = \begin{cases} \inf \frac{1}{2} \int_0^T |\dot{h}|^2 dt & \text{if } g_{t,t} - g_{t,s} = \int_s^t -S(g_{t,u}, f_u, u, \delta_{\phi_u, \psi_{u,0}})du, g_{t,t} = \zeta_t \\ \infty & \text{otherwise,} \end{cases} \quad (5.4.2)$$

where the infinitum is taken over the set

$$\left\{ h \in H_1 : f(t) = x_0 + \int_0^t b(f(s), s, \delta_{\phi_s, \psi_{s,0}}) ds + \int_0^t \sigma(f(s), s, \delta_{\phi_s, \psi_{s,0}}) \dot{h}(s) ds \right\}.$$

The topology with respect to the (X^ϵ, G^ϵ) here is induced by the uniform norm of the space $C([0, T], \mathbb{R}^d) \times C(0 \leq s \leq t \leq T, \mathbb{R}^l)$.

The proof of the LDP is lengthy, and is divided into two steps.

Step 1: Let $\{Y^\epsilon, H^\epsilon\}_{0 \leq \epsilon < 1}$ be the solution to the following functional stochastic differential equations:

$$\begin{aligned} dY_t^\epsilon &= b(Y_t^\epsilon, t, \delta_{\phi_t, \psi_{t,0}}) dt + \sqrt{\epsilon} \sigma(Y_t^\epsilon, t, \delta_{\phi_t, \psi_{t,0}}) dB_t, \quad Y_0^\epsilon = x_0, \\ dH_{t,s}^\epsilon &= -S(H_{t,s}^\epsilon, Y_s^\epsilon, s, \delta_{\phi_s, \psi_{s,0}}) ds, \quad H_{t,t}^\epsilon = \zeta_t. \end{aligned}$$

We will show that $\{Y^\epsilon, H^\epsilon\}_{0 \leq \epsilon < 1}$ satisfies the LDP with the good rate function (5.4.2) $I(f, g)$ (see Proposition 5.4.2 below).

Step 2: We next to show that $\{Y^\epsilon, H^\epsilon\}_{0 \leq \epsilon < 1}$ and $\{X^\epsilon, G^\epsilon\}_{0 \leq \epsilon < 1}$ are exponentially equivalent, which will be done in Proposition 5.4.4. That is, for every $\delta > 0$,

$$\limsup_{\epsilon \downarrow 0} \epsilon \log \mathbb{P}[d((Y^\epsilon, H^\epsilon), (X^\epsilon, G^\epsilon)) > \delta] = -\infty. \quad (5.4.3)$$

Then by Theorem 2.6.5, $\{X^\epsilon, G^\epsilon\}_{0 \leq \epsilon < 1}$ satisfies the LDP with the good rate function $I(f, g)$.

Proposition 5.4.2. $\{Y^\epsilon, H^\epsilon\}_{0 \leq \epsilon < 1}$ satisfies the LDP with the good rate function $I(f, g)$.

Proof. By Freidlin–Wentzell theory, Y^ϵ satisfies the LDP with the good rate function

$$I(f) = \inf \frac{1}{2} \int_0^t |\dot{h}|^2 dt$$

where the infimum is taken over the set

$$\left\{ h \in H_1 : f(t) = x_0 + \int_0^t b(f(s), s, \delta_{\phi_s, \psi_{s,0}}) ds + \int_0^t \sigma(f(s), s, \delta_{\phi_s, \psi_{s,0}}) \dot{h}(s) ds \right\}$$

Consider a mapping Φ which maps $C([0, T], \mathbb{R}^d)$ to $C([0, T], \mathbb{R}^d) \times C(0 \leq s \leq t \leq T, \mathbb{R}^l)$ defined as follows:

$$\Phi(f) = (f, g)$$

where g is the unique solution to the following equation:

$$dg_{t,s} = -S(g_{t,s}, f_s, s, \delta_{\phi_s, \psi_{s,0}}) ds, \quad g_{t,t} = \zeta_t.$$

We aim to show that Φ is a continuous mapping, then by the contraction principle Theorem 2.6.3, $I(f, g)$ is a good rate function and $\{Y^\epsilon, H^\epsilon\}_{0 \leq \epsilon < 1}$ satisfies the LDP with $I(f, g)$.

In fact, given every $f, f' \in C([0, T], \mathbb{R}^d)$, let $g, g' \in C(0 \leq s \leq t \leq T, \mathbb{R}^l)$ be the image of f, f' under Φ . Then for any $0 \leq s \leq t \leq T$,

$$\begin{aligned} |g_{t,s} - g'_{t,s}| &\leq \int_s^t |S(g_{t,u}, f_u, u, \delta_{\phi_u, \psi_{u,0}}) - S(g'_{t,u}, f'_u, u, \delta_{\phi_u, \psi_{u,0}})| du \\ &\leq K \int_s^t (|g_{t,u} - g'_{t,u}| + |f_u - f'_u|) du. \end{aligned}$$

By Gronwall's inequality,

$$|g_{t,s} - g'_{t,s}| \leq C(K, T) \int_s^t |f_u - f'_u| du \leq C(K, T) \sup_{0 \leq t \leq T} |f_t - f'_t|.$$

Since s, t are arbitrary,

$$\sup_{0 \leq s \leq t \leq T} |g_{t,s} - g'_{t,s}| \leq C(K, T) \sup_{0 \leq t \leq T} |f_t - f'_t|,$$

which implies Φ is continuous. □

To show the exponential closeness, we need the following technical fact.

Lemma 5.4.3. (see [32] Lemma 5.6.18) *Let b_t, σ_t be progressively measurable processes and let $dz_t = b_t dt + \sqrt{\epsilon} \sigma_t dB_t$ where z_0 is deterministic. Let $\tau \in [0, T]$ be a stopping time. Suppose that σ_t is uniformly bounded, and for some constants M, B, ρ and any $t \in [0, \tau]$,*

$$|\sigma_t| \leq M(\rho^2 + |z_t|^2)^{1/2}$$

and

$$|b_t| \leq B(\rho^2 + |z_t|^2)^{1/2},$$

then for any $\delta > 0$ and any $\epsilon \leq 1$,

$$\epsilon \log \mathbb{P} \left[\sup_{0 \leq t \leq \tau} |z_t| \geq \delta \right] \leq K + \log \frac{\rho^2 + |z_0|^2}{\rho^2 + \delta^2},$$

where $K = 2B + M^2(2 + d)$.

Proposition 5.4.4. $\{Y^\epsilon, H^\epsilon\}_{0 \leq \epsilon < 1}$ and $\{X^\epsilon, G^\epsilon\}_{0 \leq \epsilon < 1}$ are exponentially equivalent.

Proof. To prove (5.4.3), it suffices to show that for every $\delta > 0$,

$$\limsup_{\epsilon \downarrow 0} \epsilon \log \mathbb{P} \left[\sup_{0 \leq t \leq T} |Y_t^\epsilon - X_t^\epsilon| > \delta \right] = -\infty \quad (5.4.4)$$

and

$$\limsup_{\epsilon \downarrow 0} \epsilon \log \mathbb{P} \left[\sup_{0 \leq s \leq t \leq T} |G_{t,s}^\epsilon - H_{t,s}^\epsilon| > \delta \right] = -\infty \quad (5.4.5)$$

To prove (5.4.4), let $Z^\epsilon = X^\epsilon - Y^\epsilon$, then

$$\begin{aligned} Z_t^\epsilon &= \int_0^t [b(X_s^\epsilon, s, \mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon)) - b(Y_s^\epsilon, s, \delta_{\phi_s, \psi_{s,0}})] ds \\ &\quad + \sqrt{\epsilon} \int_0^t [\sigma(X_s^\epsilon, s, \mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon)) - \sigma(Y_s^\epsilon, s, \delta_{\phi_s, \psi_{s,0}})] dB_s. \end{aligned}$$

From the Lipschitz continuity of b and σ ,

$$|b(X_s^\epsilon, s, \mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon)) - b(Y_s^\epsilon, s, \delta_{\phi_s, \psi_{s,0}})| \leq C(K) [|Z_s^\epsilon|^2 + W_2^2(\mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon), \delta_{\phi_s, \psi_{s,0}})]^{1/2}$$

and

$$|\sigma(X_s^\epsilon, s, \mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon)) - \sigma(Y_s^\epsilon, s, \delta_{\phi_s, \psi_{s,0}})| \leq C(K) [|Z_s^\epsilon|^2 + W_2^2(\mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon), \delta_{\phi_s, \psi_{s,0}})]^{1/2},$$

so by Lemma 5.4.3,

$$\epsilon \log \mathbb{P} \left(\sup_{0 \leq t \leq T} |Z_t^\epsilon| \geq \delta \right) \leq C(K) + \log \left(\frac{\sup_{0 \leq s \leq T} W_2^2(\mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon), \delta_{\phi_s, \psi_{s,0}})}{\sup_{0 \leq s \leq T} W_2^2(\mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon), \delta_{\phi_s, \psi_{s,0}}) + \delta^2} \right).$$

On the other hand, by Theorem 5.3.6, by substituting σ' with 0, we have

$$\mathbb{E} \sup_{0 \leq s \leq T} \left(|X_s^\epsilon - \phi_s|^2 + |G_{s,0}^\epsilon - \psi_{s,0}|^2 \right) \leq C(K, T) \|\sqrt{\epsilon} \sigma\|_{L^\infty}^2,$$

which implies

$$\sup_{0 \leq s \leq T} W_2^2(\mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon), \delta_{\phi_s, \psi_{s,0}}) \leq C(K, T) \|\sqrt{\epsilon} \sigma\|_{L^\infty}^2$$

Therefore

$$\lim_{\epsilon \downarrow 0} \sup_{0 \leq s \leq T} W_2^2(\mathcal{L}(X_s^\epsilon, G_{s,0}^\epsilon), \delta_{\phi_s, \psi_{s,0}}) = 0, \quad (5.4.6)$$

and this means that

$$\limsup_{\epsilon \downarrow 0} \epsilon \log \mathbb{P} \left(\sup_{0 \leq t \leq T} |Z_t^\epsilon| \geq \delta \right) = -\infty. \quad (5.4.7)$$

To prove (5.4.5), since

$$\zeta_t - H_{t,s}^\epsilon = \int_s^t -S(H_{t,u}^\epsilon, u, Y_u^\epsilon, \delta_{\phi_u, \psi_{u,0}}) du$$

$$\zeta_t - G_{t,s}^\epsilon = \int_s^t -S(G_{t,u}^\epsilon, u, X_u^\epsilon, \mathcal{L}(X_u^\epsilon, G_{u,0}^\epsilon)) du,$$

for every $0 \leq s \leq t \leq T$,

$$|H_{t,s}^\epsilon - G_{t,s}^\epsilon| \leq K \int_s^t (|H_{t,u}^\epsilon - G_{t,u}^\epsilon| + |Y_u^\epsilon - X_u^\epsilon| + W_2(\delta_{\phi_u, \psi_{u,0}}, \mathcal{L}(X_u^\epsilon, G_{u,0}^\epsilon))) du,$$

so, by Gronwall's inequality,

$$|H_{t,s}^\epsilon - G_{t,s}^\epsilon| \leq C(K, T) \int_s^t \left(|Y_u^\epsilon - X_u^\epsilon| + W_2(\delta_{\phi_u, \psi_{u,0}}, \mathcal{L}(X_u^\epsilon, G_{u,0}^\epsilon)) \right) du$$

Hence

$$\sup_{0 \leq s \leq t \leq T} |H_{t,s}^\epsilon - G_{t,s}^\epsilon| \leq C(K, T) \left(\sup_{0 \leq t \leq T} |Y_t^\epsilon - X_t^\epsilon| + \sup_{0 \leq t \leq T} W_2(\delta_{\phi_t, \psi_{t,0}}, \mathcal{L}(X_t^\epsilon, G_{t,0}^\epsilon)) \right),$$

and therefore,

$$\begin{aligned} & \mathbb{P} \left[\sup_{0 \leq s \leq t \leq T} |H_{t,s}^\epsilon - G_{t,s}^\epsilon| > \delta \right] \\ & \leq \mathbb{P} \left[\sup_{0 \leq t \leq T} |Y_t^\epsilon - X_t^\epsilon| > \frac{\delta}{2C(K, T)} \right] \\ & + \mathbb{P} \left[\sup_{0 \leq t \leq T} W_2(\delta_{\phi_t, \psi_{t,0}}, \mathcal{L}(X_t^\epsilon, G_{t,0}^\epsilon)) > \frac{\delta}{2C(K, T)} \right]. \end{aligned}$$

By (5.4.6), when ϵ is sufficiently small,

$$\mathbb{P} \left[\sup_{0 \leq t \leq T} W_2(\delta_{\phi_t, \psi_{t,0}}, \mathcal{L}(X_t^\epsilon, G_{t,0}^\epsilon)) > \frac{\delta}{2C(K, T)} \right] = 0.$$

Therefore, by (5.4.7),

$$\limsup_{\epsilon \downarrow 0} \epsilon \log \mathbb{P} \left[\sup_{0 \leq s \leq t \leq T} |G_{t,s}^\epsilon - H_{t,s}^\epsilon| > \delta \right] = -\infty.$$

This completes the proof. □

Example 1. Consider the following PDE as a particular case of (5.2.2).

$$\frac{\partial \omega}{\partial t} + u^\delta \cdot \nabla \omega = \frac{1}{2} \Delta \omega + C\omega, \quad (5.4.8)$$

where $\omega(x, t)$, $u^\delta(x, t) \in \mathbb{R}^3$ for $(x, t) \in \mathbb{R}^3 \times [0, T]$. C is a constant matrix in $\mathbb{R}^{3 \times 3}$.

The PDE is a closed system of W by the relation

$$u^\delta(x, t) = \int_{\mathbb{R}^3} K^\delta(x - y) \omega(y, t) dy, \quad \omega(x, 0) = \omega_0(x)$$

where $K^\delta \in BC^1(\mathbb{R}^3; \mathbb{R}^{3 \times 3})$, and $K^\delta(0) = 0$. Assume that the solution u^δ is divergence free, and the initial data $\omega_0(x) = \delta(x)\omega$ for some $\omega \in \mathbb{R}^3$.

Since ω_0 vanish outside 0, by (5.2.7), the discretised system is as follows:

$$dX_t = b(X_t, t)dt + dB_t, \quad X_0 = 0,$$

$$dG_{t,s} = -G_{t,s}Cds, \quad G_{t,t} = I_3,$$

where

$$b^i(x, t) = \mathbb{E} \left[\omega^l K_j^{\delta,i}(x - X_t) G_l^j(t, 0) \right].$$

We aim to compute the rate function for $(X^\epsilon, G^\epsilon)_{0 < \epsilon \leq 1}$ where

$$dX_t^\epsilon = b(X_t^\epsilon, t)dt + \sqrt{\epsilon} dB_t, \quad X_0 = 0,$$

$$dG_{t,s}^\epsilon = -G_{t,s}^\epsilon C ds, \quad G_{t,t}^\epsilon = I_3,$$

which corresponds to the PDE

$$\frac{\partial \omega}{\partial t} + u^\delta \cdot \nabla \omega = \frac{1}{2} \epsilon \Delta \omega + C\omega, \quad \omega(x, 0) = \delta(x)\omega.$$

Let $\phi_t, \psi_{t,s}$ be the solution of

$$\begin{aligned} d\phi_t^i &= [\omega^l K_j^{\delta,i}(\phi_t - \phi_t)\psi_l^j(t, 0)]dt, \quad \phi_0^i = 0, \\ d\psi_{t,s} &= -\psi_{t,s}Cds, \quad \psi_{t,t} = I_3. \end{aligned}$$

Since $K^\delta(0) = 0$, so that

$$\phi_t = 0, \psi_{t,s} = \exp(C(t - s))$$

Therefore, by (5.4.2), the rate function for $(X^\epsilon, G^\epsilon)_{0 < \epsilon \leq 1}$ is given by

$$I(f, g) = \frac{1}{2} \sum_{i=1}^3 \int_0^T |\dot{f}(t)^i - \omega^l K_j^{\delta,i}(f(t)) \exp(Ct)_l^j|^2 dt \quad (5.4.9)$$

if $g_{t,s} = \exp(C(t - s))$, $f(0) = 0$, and ∞ otherwise.

Example 2. For a general initial data $\omega_0(x)$ to (5.4.8), by (5.2.7), the discretised system is as follows (we replace the index kh with k here):

$$\begin{aligned} dX_t(k) &= b(X_t(k), t)dt + dB_t, \quad X_0(k) = kh, \\ \frac{d}{ds} G_{t,s}(k) &= -G_{t,s}(k)C, \quad G_{t,t}(k) = I_3, \end{aligned}$$

where

$$b^i(x, t) = \sum_{k \in \Lambda} \omega_0^l(kh) \mathbb{E} \left[K_j^{\delta,i}(x - X_t(k)) G_l^j(t, 0; k) \right] h^3$$

and Λ is a finite set in $\mathbb{Z}^3$. We are going to give the rate function for $(X^\epsilon, G^\epsilon)_{0 < \epsilon \leq 1}$, where

$$\begin{aligned} dX_t^\epsilon(k) &= b(X_t^\epsilon(k), t)dt + \sqrt{\epsilon}dB_t, \quad \tilde{X}_0(k) = kh, \\ \frac{d}{ds} G_{t,s}^\epsilon(k) &= -G_{t,s}^\epsilon(k)C, \quad G_{t,t}^\epsilon(k) = I_3. \end{aligned}$$

Let $\phi_t, \psi_{t,s}$ be the solution of

$$\begin{aligned} d\phi_t^i(k) &= \sum_{k' \in \Lambda} \omega_0^l(kh) [K_j^{\delta,i}(\phi_t(k) - \phi_t(k')) G_l^j(t, 0; k')] h^3 dt, \quad \phi_0(k) = kh, \\ d\psi_{t,s}^i(k) &= -\psi_{t,s}(k) C ds, \quad \psi_{t,t} = I_3, \end{aligned}$$

which is equivalent to

$$\begin{aligned} d\phi_t^i(k) &= \sum_{k' \in \Lambda} \omega_0^l(k'h) [K_j^{\delta,i}(\phi_t(k) - \phi_t(k')) \exp(Ct)_l^j] h^3 dt, \quad \phi_0(k) = kh, \\ \psi_{t,s}(k) &= \exp(C(t-s)), \quad \psi_{t,t} = I_3, \end{aligned}$$

which may be written in a matrix form

$$\omega_0^l(kh) \left[K_j^{\delta,i}(\phi_t(k) - \phi_t(k')) G_l^j(t, 0; k') \right] = \left[K^\delta(\phi_t(k) - \phi_t(k')) G(t, 0; k') \omega_0(kh) \right]^i$$

Therefore, when

$$\begin{aligned} \dot{f}_t(k_1) - \sum_{k' \in \Lambda} K^\delta(f_t(k_1) - \phi_t(k')) \exp(Ct) \omega_0(k'h) h^3 \\ = \dot{f}_t(k_2) - \sum_{k' \in \Lambda} K^\delta(f_t(k_2) - \phi_t(k')) \exp(Ct) \omega_0(k'h) h^3 \end{aligned}$$

for all $k_1, k_2 \in \Lambda$, and $f_0(k) = hk, g_{t,s}(k) = \exp(C(t-s))$ for all $k \in \lambda$, the rate function is

$$\frac{1}{2} \int_0^T \left| \dot{f}_t(k_1) - \sum_{k' \in \Lambda} K^\delta(f_t(k_1) - \phi_t(k')) \exp(Ct) \omega_0(k'h) h^3 \right|^2 dt,$$

otherwise it equals ∞ .

5.5 Euler-Maruyama method

In this section, we are going to propose an efficient numerical program to approximate a RVSD (5.3.1) solution. First, we use a system of finite interacting particles to approximate the law of solutions in RVSDs (see Section 5.3.3), so that the RVSDs reduce to the equations of the following form:

$$\begin{aligned} dX_t &= b(X_t, G_{t,0}, t)dt + \sigma(X_t, G_{t,0}, t)dB_t \\ dG_{t,s} &= -S(G_{t,s}, X_s, G_{s,0}, s)ds \end{aligned} \quad (5.5.1)$$

with the initial state $X_0 = \xi, G_{t,t} = \zeta_t$. Now our goal is to numerically approximate the solution to (5.5.1).

Our proposed method is inspired from the EM method for SDEs (cf. Section 2.7). The EM algorithm could be adjusted to approximate the solution to (5.5.1). Throughout this section, we shall assume that ξ and ζ_t are both deterministic for simplicity, and we denote them by x_0 and g_t respectively, though the argument still works without this assumption. The EM algorithm for RVSDs is the approximation of (5.5.1) by time discretisation. The approximation is denoted by (X^h, G^h) given by the following system:

$$\begin{aligned} X_{t_k^h}^h &= X_{t_{k-1}^h}^h + b(X_{t_{k-1}^h}^h, G_{t_{k-1}^h,0}^h, t_{k-1}^h)h + \sigma(X_{t_{k-1}^h}^h, G_{t_{k-1}^h,0}^h, t_{k-1}^h)\Delta B_{t_{k-1}^h} \text{ for } 1 \leq k \leq n \\ G_{t_i^h, t_j^h}^h - G_{t_i^h, t_{j-1}^h}^h &= -S(G_{t_i^h, t_j^h}^h, X_{t_{j-1}^h}^h, G_{t_{j-1}^h,0}^h, t_j^h)h \text{ for } 1 \leq j \leq i \leq n \end{aligned} \quad (5.5.2)$$

where $0 = t_0^h \leq t_1^h \leq t_2^h \leq \dots \leq t_n^h = T$, $t_i^h - t_{i-1}^h = h$ for $i = 1, 2, \dots, n$, $\Delta B_{t_{k-1}^h} = B_{t_k^h} - B_{t_{k-1}^h}$. X^h is defined on $0 = t_0^h \leq t_1^h \leq t_2^h \leq \dots \leq t_n^h = T$ and G^h is defined on (t_i^h, t_j^h) where $0 \leq j \leq i \leq n$. $X_0^h = x_0$, $G_{t_i^h, t_i^h}^h = g_{t_i^h}$. We extend $X_t^h, G_{t,0}^h$ to the whole interval $[0, T]$ by setting $X_t^h = X_{t_i^h}^h, G_t^h = G_{t_i^h,0}^h$ when $t \in [t_i, t_{i+1})$. For simplicity, we denote t_k^h by t_k , $X_{t_k^h}^h$ by $X_{t_k}^h$ and $G_{t_i^h, t_j^h}^h$ by G_{t_i, t_j}^h if there is no confusion.

Theorem 5.5.1. (Convergence of the EM scheme for RVSDs) *Suppose that*

$b(x, g, t), \sigma(x, g, t), S(g, x, g', t)$ satisfy global Lipschitz and linear growth conditions with constant K in x, g, g' , and are globally $\frac{1}{2}$ -Hölder continuous in t :

$$\begin{aligned}
|b(x_1, g_1, t_1) - b(x_2, g_2, t_1)| + |\sigma(x_1, g_1, t_1) - \sigma(x_2, g_2, t_1)| &\leq K(|x_1 - x_2| + |g_1 - g_2|) \\
|S(g_1, x_1, g'_1, t_1) - S(g_2, x_2, g'_2, t_1)| &\leq K(|x_1 - x_2| + |g_1 - g_2| + |g'_1 - g'_2|) \\
|b(x_1, g_1, t_1)| + |\sigma(x_1, g_1, t_1)| + |S(g_1, x_1, g'_1, t_1)| &\leq K(1 + |x_1| + |g_1| + |g'_1|) \\
|b(x_1, g_1, t_1) - b(x_1, g_1, t_2)| + |\sigma(x_1, g_1, t_1) - \sigma(x_1, g_1, t_2)| &\leq K|t_1 - t_2|^{1/2} \\
|S(g_1, x_1, g'_1, t_1) - S(g_1, x_1, g'_1, t_2)| &\leq K|t_1 - t_2|^{1/2}
\end{aligned}$$

for all $x_1, x_2 \in \mathbb{R}^d, g_1, g_2, g'_1, g'_2 \in \mathbb{R}^l, t_1, t_2 \in [0, T]$, then the strong convergence of the EM algorithm holds:

$$\sup_{0 \leq t \leq T} \mathbb{E}|G_{t,0}^h - G_{t,0}|^2 + |X_t^h - X_t|^2 \leq C(K, T)(1 + |x_0|^2 + \sup_{0 \leq t \leq T} |g_t|^2)h$$

Proof. For $t \in [t_i, t_{i+1})$,

$$\begin{aligned}
X_t^h &= x_0 + \sum_{0 \leq k \leq i-1} (b(X_{t_k}^h, G_{t_k,0}^h, t_k)h + \sigma(X_{t_k}^h, G_{t_k,0}^h, t_k)\Delta B_{t_k}) \\
&= x_0 + \sum_{0 \leq k \leq i-1} \left(\int_{t_k}^{t_{k+1}} b(X_t^h, G_t^h, t_k)dt + \int_{t_k}^{t_{k+1}} \sigma(X_t^h, G_t^h, t_k)dB_t \right),
\end{aligned}$$

and

$$\begin{aligned}
X_t &= x_0 + \sum_{0 \leq k \leq i-1} \left(\int_{t_k}^{t_{k+1}} b(X_t, G_{t,0}, t)dt + \int_{t_k}^{t_{k+1}} \sigma(X_t, G_{t,0}, t)dB_t \right) \\
&\quad + \int_{t_i}^t b(X_s, G_{s,0}, s)ds + \int_{t_i}^t \sigma(X_s, G_{s,0}, s)dB_s,
\end{aligned}$$

which yields

$$\begin{aligned} X_t^h - X_t &= \sum_{0 \leq k \leq i-1} \left(\int_{t_k}^{t_{k+1}} [b(X_t^h, G_{t,0}^h, t_k) - b(X_t, G_{t,0}, t)] dt \right. \\ &\quad \left. + \int_{t_k}^{t_{k+1}} [\sigma(X_t^h, G_{t,0}^h, t_k) - \sigma(X_t, G_{t,0}, t)] dB_t \right) \\ &\quad - \left(\int_{t_i}^t b(X_s, G_{s,0}, s) ds + \int_{t_i}^t \sigma(X_s, G_{s,0}, s) dB_s \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}|X_t^h - X_t|^2 &\leq C \left\{ \mathbb{E} \left| \sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} (b(X_t^h, G_{t,0}^h, t_k) - b(X_t, G_{t,0}, t)) dt \right|^2 \right. \\ &\quad \left. + \mathbb{E} \left| \sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} (\sigma(X_t^h, G_{t,0}^h, t_k) - \sigma(X_t, G_{t,0}, t)) dB_t \right|^2 \right. \\ &\quad \left. + \mathbb{E} \left| \int_{t_i}^t b(X_s, G_{s,0}, s) ds + \int_{t_i}^t \sigma(X_s, G_{s,0}, s) dB_s \right|^2 \right\}. \end{aligned}$$

Now we estimate these three terms individually,

$$\begin{aligned} &\mathbb{E} \left| \sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} (b(X_t^h, G_{t,0}^h, t_k) - b(X_t, G_{t,0}, t)) dt \right|^2 \\ &\leq \mathbb{E} \left| \sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} (|b(X_t^h, G_{t,0}^h, t_k) - b(X_t, G_{t,0}, t_k)| \right. \\ &\quad \left. + |b(X_t, G_{t,0}, t_k) - b(X_t, G_{t,0}, t)|) dt \right|^2 \\ &\leq C(K) \left\{ \mathbb{E} \left[\int_0^t (|X_s^h - X_s| + |G_{s,0}^h - G_{s,0}|) ds \right]^2 \right. \\ &\quad \left. + \mathbb{E} \left[\sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} \sqrt{|t_k - t|} (1 + |X_t| + |G_{t,0}|) dt \right]^2 \right\} \\ &\leq C(K, T) \left\{ \mathbb{E} \int_0^t (|X_s^h - X_s|^2 + |G_{s,0}^h - G_{s,0}|^2) ds \right. \\ &\quad \left. + h \mathbb{E} \left[1 + \sup_{0 \leq t \leq T} |X_t|^2 + \sup_{0 \leq t \leq T} |G_{t,0}|^2 \right] \right\}; \end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left| \sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} (\sigma(X_t^h, G_{t,0}^h, t_k) - \sigma(X_t, G_{t,0}, t)) \, dB_t \right|^2 \\
& \leq \mathbb{E} \sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} |\sigma(X_t^h, G_{t,0}^h, t_k) - \sigma(X_t, G_{t,0}, t)|^2 \, dt \\
& \leq C \mathbb{E} \sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} (|\sigma(X_t^h, G_{t,0}^h, t_k) - \sigma(X_t, G_{t,0}, t_k)|^2 \\
& \quad + |\sigma(X_t, G_{t,0}, t_k) - \sigma(X_t, G_{t,0}, t)|^2) \, dt \\
& \leq C(K) \mathbb{E} \left\{ \int_0^t (|X_s^h - X_s|^2 + |G_{s,0}^h - G_{s,0}|^2) \, ds \right. \\
& \quad \left. + \sum_{0 \leq k \leq i-1} \int_{t_k}^{t_{k+1}} |t - t_k| (1 + |X_t|^2 + |G_{t,0}|^2) \, dt \right\} \\
& \leq C(K, T) \left\{ \mathbb{E} \int_0^t (|X_s^h - X_s|^2 + |G_{s,0}^h - G_{s,0}|^2) \, ds \right. \\
& \quad \left. + h \mathbb{E} \left[1 + \sup_{0 \leq t \leq T} |X_t|^2 + \sup_{0 \leq t \leq T} |G_{t,0}|^2 \right] \right\};
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E} \left| \int_{t_i}^t b(X_s, G_{s,0}, s) \, ds + \int_{t_i}^t \sigma(X_s, G_{s,0}, s) \, dB_s \right|^2 \\
& \leq C(T) \mathbb{E} \int_{t_i}^t (|b(X_s, G_{s,0}, s)|^2 + |\sigma(X_s, G_{s,0}, s)|^2) \, ds \\
& \leq C(K, T) \mathbb{E} \left[1 + \sup_{0 \leq t \leq T} |X_t|^2 + \sup_{0 \leq t \leq T} |G_{t,0}|^2 \right] h.
\end{aligned}$$

Combining these three inequalities together, we have

$$\begin{aligned}
\mathbb{E} |X_t^h - X_t|^2 & \leq C(K, T) \left\{ \mathbb{E} \int_0^t |X_s^h - X_s|^2 + |G_{s,0}^h - G_{s,0}|^2 \, ds \right. \\
& \quad \left. + h \mathbb{E} \left[1 + \sup_{0 \leq t \leq T} |X_t|^2 + \sup_{0 \leq t \leq T} |G_{t,0}|^2 \right] \right\}. \tag{5.5.3}
\end{aligned}$$

Now we estimate $|G - G^h|$. Given t_i , $1 \leq i \leq n$, let $G_{t_i,s}^h = G_{t_i,t_j}^h$ when $s \in (t_{j-1}, t_j]$ and $0 \leq j \leq i$. Then for $s \in (t_{j-1}, t_j]$, notice that

$$G_{t_i,t_i}^h - G_{t_i,s}^h = \sum_{j+1 \leq k \leq i} -S(G_{t_i,t_k}^h, X_{t_{k-1}}^h, G_{t_{k-1},0}^h, t_k) h$$

$$= \sum_{j+1 \leq k \leq i} \int_{t_{k-1}}^{t_k} -S(G_{t_i,u}^h, X_u^h, G_{u,0}^h, t_k) du,$$

and

$$G_{t_i,t_i} - G_{t_i,s} = \sum_{j+1 \leq k \leq i} \int_{t_{k-1}}^{t_k} -S(G_{t_i,u}, X_u, G_{u,0}, u) du + \int_s^{t_j} -S(G_{t_i,u}, X_u, G_{u,0}, u) du.$$

which yields

$$\begin{aligned} G_{t_i,s}^h - G_{t_i,s} &= \sum_{j+1 \leq k \leq i} \int_{t_{k-1}}^{t_k} (S(G_{t_i,u}^h, X_u^h, G_{u,0}^h, t_k) - S(G_{t_i,u}, X_u, G_{u,0}, u)) du \\ &\quad - \int_s^{t_j} S(G_{t_i,u}, X_u, G_{u,0}, u) du. \end{aligned}$$

Therefore,

$$\begin{aligned} &|G_{t_i,s}^h - G_{t_i,s}| \\ &\leq \sum_{j+1 \leq k \leq i} \int_{t_{k-1}}^{t_k} [|S(G_{t_i,u}^h, X_u^h, G_{u,0}^h, t_k) - S(G_{t_i,u}, X_u, G_{u,0}, t_k)| \\ &\quad + |S(G_{t_i,u}, X_u, G_{u,0}, t_k) - S(G_{t_i,u}, X_u, G_{u,0}, u)|] du + \int_s^{t_j} |S(G_{t_i,u}, X_u, G_{u,0}, u)| du \\ &\leq C(K) \left\{ \int_s^{t_i} (|G_{t_i,u}^h - G_{t_i,u}| + |X_u^h - X_u| + |G_{u,0}^h - G_{u,0}|) du \right. \\ &\quad + \sum_{j+1 \leq k \leq i} \int_{t_{k-1}}^{t_k} \sqrt{|u - t_k|} (1 + |X_u| + |G_{u,0}| + |G_{t_i,u}|) du \\ &\quad \left. + h(1 + \sup_{0 \leq s \leq t \leq T} |G_{t,s}| + \sup_{0 \leq t \leq T} |X_t|) \right\} \\ &\leq C(K, T) \left\{ \int_s^{t_i} (|G_{t_i,u}^h - G_{t_i,u}| + |X_u^h - X_u| + |G_{u,0}^h - G_{u,0}|) du \right. \\ &\quad \left. + (h^{1/2} + h)(1 + \sup_{0 \leq s \leq t \leq T} |G_{t,s}| + \sup_{0 \leq t \leq T} |X_t|) \right\}. \end{aligned}$$

By Gronwall's inequality, and we assume $h < 1$,

$$|G_{t_i,0}^h - G_{t_i,0}| \leq C(K, T) \left\{ \int_0^{t_i} (|X_u^h - X_u| + |G_{u,0}^h - G_{u,0}|) du \right.$$

$$+ h^{1/2} \left(1 + \sup_{0 \leq s \leq t \leq T} |G_{t,s}| + \sup_{0 \leq t \leq T} |X_t| \right) \Big\},$$

Hence, for $t \in [t_i, t_{i+1})$,

$$\begin{aligned} |G_{t,0}^h - G_{t,0}| &\leq |G_{t_i,0}^h - G_{t_i,0}| + |G_{t_i,0} - G_{t,0}| \\ &\leq C(K, T) \left\{ \int_0^t (|X_u^h - X_u| + |G_{u,0}^h - G_{u,0}|) du \right. \\ &\quad \left. + h^{1/2} \left(1 + \sup_{0 \leq s \leq t \leq T} |G_{t,s}| + \sup_{0 \leq t \leq T} |X_t| \right) \right\}, \end{aligned}$$

which yields for every $t \in [0, T]$,

$$\begin{aligned} \mathbb{E}|G_{t,0}^h - G_{t,0}|^2 &\leq C(K, T) \left\{ \int_0^t (\mathbb{E}|X_u^h - X_u|^2 + \mathbb{E}|G_{u,0}^h - G_{u,0}|^2) du \right. \\ &\quad \left. + h \mathbb{E} \left(1 + \sup_{0 \leq s \leq t \leq T} |G_{t,s}|^2 + \sup_{0 \leq t \leq T} |X_t|^2 \right) \right\}. \end{aligned}$$

Then, by combining this with (5.5.3),

$$\begin{aligned} \mathbb{E} [|G_{t,0}^h - G_{t,0}|^2 + |X_t^h - X_t|^2] &\leq C(K, T) \left\{ \int_0^t (\mathbb{E}|X_u^h - X_u|^2 + \mathbb{E}|G_{u,0}^h - G_{u,0}|^2) du \right. \\ &\quad \left. + h \mathbb{E} \left(1 + \sup_{0 \leq s \leq t \leq T} |G_{t,s}|^2 + \sup_{0 \leq t \leq T} |X_t|^2 \right) \right\}, \end{aligned}$$

so that we have

$$\sup_{0 \leq t \leq T} \mathbb{E} [|G_{t,0}^h - G_{t,0}|^2 + |X_t^h - X_t|^2] \leq C(K, T) \mathbb{E} \left(1 + \sup_{0 \leq s \leq t \leq T} |G_{t,s}|^2 + \sup_{0 \leq t \leq T} |X_t|^2 \right) h$$

by Gronwall's inequality. Since the L^2 norm of X and G is bounded by the initial data (see Corollary 5.3.4), we deduce that

$$\sup_{0 \leq t \leq T} \mathbb{E} [|G_{t,0}^h - G_{t,0}|^2 + |X_t^h - X_t|^2] \leq C(K, T) (1 + |x_0|^2 + \sup_{0 \leq t \leq T} |g_t|^2) h.$$

□

To implement the EM scheme (5.5.2) for the RVSDs, we first initialize $X_0^h = x_0$ and $G_{0,0}^h = g_0$. Suppose that we have computed $X_{t_i^h}^h, G_{t_i^h,0}^h$ for $i = 0, 1, 2, \dots, k$. To calculate $X_{t_{k+1}^h}^h, G_{t_{k+1}^h,0}^h$, $X_{t_{k+1}^h}^h$ could be computed by

$$X_{t_{k+1}^h}^h = X_{t_k^h}^h + b(X_{t_k^h}^h, G_{t_k^h,0}^h, t_{k-1}^h)h + \sigma(X_{t_k^h}^h, G_{t_k^h,0}^h, t_k^h)\Delta B_{t_k^h},$$

and $G_{t_{k+1}^h,0}^h$ could be computed by recursively using

$$G_{t_{k+1}^h, t_{j-1}^h}^h = G_{t_{k+1}^h, t_j^h}^h + S(G_{t_{k+1}^h, t_j^h}^h, X_{t_{j-1}^h}^h, G_{t_{j-1}^h,0}^h, t_j^h)h,$$

with the initial condition $G_{t_{k+1}^h, t_{k+1}^h}^h = g_{t_{k+1}^h}^h$.

5.6 Infinite-dimensional RVSDs

In this section, we study the infinite-dimensional RVSDs, which is a generalization of the random vortex dynamics system without discretisation. Suppose that Ω is some compact set in $\mathbb{R}^\rho$. We are interested in the following system of equations:

$$dX_t(\xi) = b(X_t(\xi), t)dt + \sigma(X_t(\xi), t)dB_t, \quad (5.6.1)$$

$$dG_{t,s}(\xi) = -S(G_{t,s}(\xi), X_s(\xi), s)ds, \quad (5.6.2)$$

where $\xi \in \Omega$, $X_t(\xi) \in \mathbb{R}^d$, $G_{t,s}(\xi) \in \mathbb{R}^l$, $0 \leq s \leq t \leq T$, B is a standard r -dimensional Brownian motion, and b, σ, S depend on the joint distribution of (X, G) by the following equations:

$$\begin{aligned} b(x, t) &= \int_{\Omega} \mathbb{E}[b'(x, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi, \\ \sigma(x, t) &= \int_{\Omega} \mathbb{E}[\sigma'(x, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi, \\ S(g, x, t) &= \int_{\Omega} \mathbb{E}[S'(x, g, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi, \end{aligned} \quad (5.6.3)$$

where $b' = \{b^{i'}(x, t, x', g', \xi)\}, 1 \leq i \leq d$. $b^{i'}$ are Borel-measurable functions from $\mathbb{R}^d \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^\rho$. $\sigma' = \{\sigma_j^{i'}(x, t, x', g', \xi)\}, 1 \leq i \leq d, 1 \leq j \leq r$, $\sigma_j^{i'}$ are Borel-measurable functions from $\mathbb{R}^d \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^\rho$. $S' = \{S^{i'}(x, g, t, x', g', \xi)\}, 1 \leq i \leq l$, $S^{i'}$ are Borel-measurable functions from $\mathbb{R}^d \times \mathbb{R}^l \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^\rho$. We assume that b', σ', S' vanish outside $\mathbb{R}^d \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^l \times \Omega$. The initial data is given by $X_0(\xi)$ and $G_{t,t}(\xi)$ for $t \in [0, T], \xi \in \Omega$.

We say that the family of continuous stochastic processes of $X_t(\xi), \xi \in \Omega, t \in [0, T]$, and continuous stochastic simplex-indexed processes $G_{t,s}(\xi), \xi \in \Omega, 0 \leq s \leq t \leq T$ constitute a solution to (5.6.1), (5.6.2) if the following properties hold:

- (i) $(X_t(\xi), G_{t,0}(\xi)) \in \mathcal{F}_t$ for all $\xi \in \Omega$.
- (ii) Let b, σ, S defined as in (5.6.3). For every $\xi \in \Omega$, $X_t(\xi)$ is a solution to (5.6.1), and $G_{t,s}(\xi), X_s(\xi)$ is a solution to (5.6.2) for $0 \leq s \leq t \leq T$ with the given initial data $X_0(\xi)$ and $G_{t,t}(\xi)$.

Theorem 5.6.1. *Suppose that the coefficients b', σ' and S' satisfy the global Lipschitz and linear growth conditions, i.e.*

$$\begin{aligned}
& |b'(x_1, t, x'_1, g'_1, \xi) - b'(x_2, t, x'_2, g'_2, \xi)| + |\sigma'(x_1, t, x'_1, g'_1, \xi) - \sigma'(x_2, t, x'_2, g'_2, \xi)| \\
& + |S'(x_1, g_1, t, x'_1, g'_1, \xi) - S'(x_2, g_2, t, x'_2, g'_2, \xi)| \\
& \leq K(|x_1 - x_2| + |x'_1 - x'_2| + |g_1 - g_2| + |g'_1 - g'_2|), \tag{RVSDE-B1}
\end{aligned}$$

$$\begin{aligned}
& |b'(x_1, t, x'_1, g'_1, \xi)| + |\sigma'(x_1, t, x'_1, g'_1, \xi)| \leq K(1 + |x_1| + |x'_1| + |g'_1|), \\
& |S'(x_1, g_1, t, x'_1, g'_1, \xi)| \leq K(1 + |x_1| + |g_1| + |x'_1| + |g'_1|), \tag{RVSDE-B2}
\end{aligned}$$

for every $x_1, x_2, x'_1, x'_2 \in \mathbb{R}^d, g_1, g_2, g'_1, g'_2 \in \mathbb{R}^l, t \in [0, T], \xi \in \Omega$. Besides, suppose that $\sup_{\xi \in \Omega} \mathbb{E} |X_0(\xi)|^2 < \infty, \sup_{\xi \in \Omega} \mathbb{E} \sup_{t \in [0, T]} |G_{t,t}(\xi)|^2 < \infty$. Then there exists a unique solution to (5.6.1), (5.6.2).

Proof. Let $L^2(\mathcal{X}, \mathcal{G})$ be the Banach space consisting of a family of all continuous stochastic processes $\{X_t(\xi)\}_{0 \leq t \leq T}$ and $\{G_{t,0}(\xi)\}_{0 \leq t \leq T}$, indexed by $\xi \in \Omega$, with norm given by $\sup_{\xi \in \Omega} (\mathbb{E}[\sup_{0 \leq t \leq T} |X_t(\xi)|^2 + \sup_{0 \leq t \leq T} |G_{t,0}(\xi)|^2])^{1/2}$.

Given $(X_t, G_{t,0}) \in L^2(\mathcal{X}, \mathcal{G})$, let b, σ, S given by (5.6.3). It could be verified that b, σ and S satisfy global Lipschitz and linear growth conditions. Indeed, for every $x_1, x_2 \in \mathbb{R}^d, t \in [0, T]$,

$$\begin{aligned} & |b(x_1, t) - b(x_2, t)| \\ & \leq \left| \int_{\Omega} \mathbb{E} [b'(x_1, t, X_t(\xi), G_{t,0}(\xi), \xi) - b'(x_2, t, X_t(\xi), G_{t,0}(\xi), \xi)] d\xi \right| \\ & \leq \int_{\Omega} K |x_1 - x_2| d\xi \\ & = C(K, \Omega) |x_1 - x_2|. \end{aligned} \tag{5.6.4}$$

Similarly, for every $x_1, x_2 \in \mathbb{R}^d, g_1, g_2 \in \mathbb{R}^l, t \in [0, T]$,

$$\begin{aligned} & |\sigma(x_1, t) - \sigma(x_2, t)| \leq C(K, \Omega) |x_1 - x_2|, \\ & |S(g_1, x_1, t) - S(g_2, x_2, t)| \leq C(K, \Omega) (|g_1 - g_2| + |x_1 - x_2|). \end{aligned} \tag{5.6.5}$$

On the other hand, for every $x \in \mathbb{R}^d, t \in [0, T]$,

$$\begin{aligned} & |b(x, t)| \\ & \leq \int_{\Omega} \mathbb{E} |b'(x, t, X_t(\xi), G_{t,0}(\xi), \xi)| d\xi \\ & \leq \int_{\Omega} K \mathbb{E} (1 + |x| + |X_t(\xi)| + |G_{t,0}(\xi)|) d\xi \\ & \leq C \left(K, \Omega, \sup_{\xi \in \Omega} \mathbb{E} [\sup_{0 \leq t \leq T} |X_t(\xi)|^2 + \sup_{0 \leq t \leq T} |G_{t,0}(\xi)|^2] \right) (1 + |x|). \end{aligned}$$

Similarly,

$$\begin{aligned} & |\sigma(x, t)| \leq C \left(K, \Omega, \sup_{\xi \in \Omega} \mathbb{E} [\sup_{0 \leq t \leq T} |X_t(\xi)|^2 + \sup_{0 \leq t \leq T} |G_{t,0}(\xi)|^2] \right) (1 + |x|), \\ & |S(g, x, t)| \leq C \left(K, \Omega, \sup_{\xi \in \Omega} \mathbb{E} [\sup_{0 \leq t \leq T} |X_t(\xi)|^2 + \sup_{0 \leq t \leq T} |G_{t,0}(\xi)|^2] \right) (1 + |g| + |x|). \end{aligned}$$

Therefore, with b, σ, S computed above, denote the solution to (5.6.1) by X' , and the solution to (5.6.2)

$$dG'_{t,s}(\xi) = -S(G'_{t,s}(\xi), X'_s(\xi), s)ds$$

by G' . By Lemma 5.3.5, $G'_{t,s}(\xi)$ is continuous in $0 \leq s \leq t \leq T$ for all $\xi \in \Omega$. Together with (2.4.5), we have

$$\sup_{\xi \in \Omega} (\mathbb{E} [\sup_{0 \leq t \leq T} |X'_t(\xi)|^2 + \sup_{0 \leq t \leq T} |G'_{t,0}(\xi)|^2]) < \infty,$$

This yields a map Φ from $L^2(\mathcal{X}, \mathcal{G})$ to itself :

$$X'_t(\xi), G'_{t,0}(\xi) = \Phi(X_t(\xi), G_{t,0}(\xi)).$$

Now it remains to show that there exists a unique fixed point of Φ .

Given any $(X, G)^{(1)}, (X, G)^{(2)} \in L^2(\mathcal{X}, \mathcal{G})$, let $(b, \sigma, S)^{(1)}, (b, \sigma, S)^{(2)}$ be computed by (5.6.3). By (5.6.4) and (5.6.5), they are Lipschitz continuous with a constant only dependent on K, Ω . For every $t \in [0, T], x \in \mathbb{R}^d$,

$$\begin{aligned} & |b^{(1)}(x, t) - b^{(2)}(x, t)| \\ & \leq \int_{\Omega} \mathbb{E} \left| b'(x, t, X_t^{(1)}(\xi), G_{t,0}^{(1)}(\xi), \xi) - b'(x, t, X_t^{(2)}(\xi), G_{t,0}^{(2)}(\xi), \xi) \right| d\xi \\ & \leq \int_{\Omega} K \mathbb{E} \left(\left| X_t^{(1)}(\xi) - X_t^{(2)}(\xi) \right| + \left| G_{t,0}^{(1)}(\xi) - G_{t,0}^{(2)}(\xi) \right| \right) d\xi \\ & \leq C(K, \Omega) \sup_{\xi \in \Omega} \mathbb{E} \left[\left| X_t^{(1)}(\xi) - X_t^{(2)}(\xi) \right| + \left| G_{t,0}^{(1)}(\xi) - G_{t,0}^{(2)}(\xi) \right| \right], \end{aligned}$$

so that

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} |b^{(1)}(x, t) - b^{(2)}(x, t)|^2 \\ & \leq C(K, \Omega) \sup_{\xi \in \Omega} \mathbb{E} \left[\left| X_t^{(1)}(\xi) - X_t^{(2)}(\xi) \right|^2 + \left| G_{t,0}^{(1)}(\xi) - G_{t,0}^{(2)}(\xi) \right|^2 \right]. \end{aligned} \tag{5.6.6}$$

Similarly,

$$\begin{aligned}
& \sup_{x \in \mathbb{R}^d} |\sigma^{(1)}(x, t) - \sigma^{(2)}(x, t)|^2 \\
& \leq C(K, \Omega) \sup_{\xi \in \Omega} \mathbb{E} \left[\left| X_t^{(1)}(\xi) - X_t^{(2)}(\xi) \right|^2 + \left| G_{t,0}^{(1)}(\xi) - G_{t,0}^{(2)}(\xi) \right|^2 \right], \\
& \sup_{g \in \mathbb{R}^l, x \in \mathbb{R}^d} |S^{(1)}(g, x, t) - S^{(2)}(g, x, t)|^2 \\
& \leq C(K, \Omega) \sup_{\xi \in \Omega} \mathbb{E} \left[\left| X_t^{(1)}(\xi) - X_t^{(2)}(\xi) \right|^2 + \left| G_{t,0}^{(1)}(\xi) - G_{t,0}^{(2)}(\xi) \right|^2 \right].
\end{aligned} \tag{5.6.7}$$

Let $(X', G')^{(i)} = \Phi((X, G)^{(i)})$, $i = 1, 2$. For every $t \in [0, T]$, $\xi \in \Omega$,

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} |X_s'^{(1)}(\xi) - X_s'^{(2)}(\xi)|^2 \\
& \leq C \left\{ \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s (b^{(1)}(X_u'^{(1)}(\xi), u) - b^{(2)}(X_u'^{(2)}(\xi), u)) du \right|^2 \right. \\
& \quad \left. + \mathbb{E} \sup_{0 \leq s \leq t} \left| \int_0^s (\sigma^{(1)}(X_u'^{(1)}(\xi), u) - \sigma^{(2)}(X_u'^{(2)}(\xi), u)) dB_u \right|^2 \right\} \\
& \leq C(T) \mathbb{E} \int_0^t \left(|b^{(1)}(X_u'^{(1)}(\xi), u) - b^{(2)}(X_u'^{(2)}(\xi), u)|^2 + \right. \\
& \quad \left. |\sigma^{(1)}(X_u'^{(1)}(\xi), u) - \sigma^{(2)}(X_u'^{(2)}(\xi), u)|^2 \right) du \\
& \leq C(K, T, \Omega) \int_0^t \left[\sup_{x \in \mathbb{R}^d} |b^{(1)}(x, u) - b^{(2)}(x, u)|^2 + \sup_{x \in \mathbb{R}^d} |\sigma^{(1)}(x, u) - \sigma^{(2)}(x, u)|^2 \right. \\
& \quad \left. + \mathbb{E} |X_u'^{(1)}(\xi) - X_u'^{(2)}(\xi)|^2 \right] du.
\end{aligned}$$

By Gronwall's inequality, we deduce that

$$\begin{aligned}
& \mathbb{E} \sup_{0 \leq s \leq t} |X_s'^{(1)}(\xi) - X_s'^{(2)}(\xi)|^2 \\
& \leq C(K, T, \Omega) \int_0^t \left(\sup_{x \in \mathbb{R}^d} |b^{(1)}(x, u) - b^{(2)}(x, u)|^2 + \sup_{x \in \mathbb{R}^d} |\sigma^{(1)}(x, u) - \sigma^{(2)}(x, u)|^2 \right) du.
\end{aligned} \tag{5.6.8}$$

On the other hand,

$$\left| G_{t,s}'^{(1)}(\xi) - G_{t,s}'^{(2)}(\xi) \right|$$

$$\begin{aligned}
&\leq \int_s^t \left| S^{(1)}(G'_{t,u}(1)(\xi), X'_u(1)(\xi), u) - S^{(2)}(G'_{t,u}(2)(\xi), X'_u(2)(\xi), u) \right| du \\
&\leq \int_s^t \left[\sup_{g \in \mathbb{R}^l, x \in \mathbb{R}^d} |S^{(1)}(g, x, u) - S^{(2)}(g, x, u)| \right. \\
&\quad \left. + C(K, \Omega) \left(|G'_{t,u}(1)(\xi) - G'_{t,u}(2)(\xi)| + |X'_u(1)(\xi) - X'_u(2)(\xi)| \right) \right] du.
\end{aligned}$$

By applying Gronwall's inequality again,

$$\begin{aligned}
&|G'_{t,0}(1)(\xi) - G'_{t,0}(2)(\xi)| \\
&\leq C(K, T, \Omega) \int_0^t \left(\sup_{g \in \mathbb{R}^l, x \in \mathbb{R}^d} |S^{(1)}(g, x, u) - S^{(2)}(g, x, u)| + |X'_u(1)(\xi) - X'_u(2)(\xi)| \right) du.
\end{aligned}$$

Together with (5.6.8), we have

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq s \leq t} \left[|G'_{s,0}(1)(\xi) - G'_{s,0}(2)(\xi)|^2 + |X'_s(1)(\xi) - X'_s(2)(\xi)|^2 \right] \\
&\leq C(K, T, \Omega) \int_0^t \left(\sup_{g \in \mathbb{R}^l, x \in \mathbb{R}^d} |S^{(1)}(g, x, u) - S^{(2)}(g, x, u)|^2 \right. \\
&\quad \left. + \sup_{x \in \mathbb{R}^d} |b^{(1)}(x, u) - b^{(2)}(x, u)|^2 + \sup_{x \in \mathbb{R}^d} |\sigma^{(1)}(x, u) - \sigma^{(2)}(x, u)|^2 \right) du
\end{aligned}$$

Therefore, by (5.6.6) and (5.6.7),

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leq s \leq t} \left[|X'_s(1)(\xi) - X'_s(2)(\xi)|^2 + |G'_{s,0}(1)(\xi) - G'_{s,0}(2)(\xi)|^2 \right] \\
&\leq C(K, T, \Omega) \int_0^t \sup_{\xi \in \Omega} \mathbb{E} \left[|X'_u(1)(\xi) - X'_u(2)(\xi)|^2 + |G'_{u,0}(1)(\xi) - G'_{u,0}(2)(\xi)|^2 \right] du,
\end{aligned}$$

which implies

$$\begin{aligned}
&\sup_{\xi \in \Omega} \mathbb{E} \sup_{0 \leq s \leq t} \left[|X'_s(1)(\xi) - X'_s(2)(\xi)|^2 + |G'_{s,0}(1)(\xi) - G'_{s,0}(2)(\xi)|^2 \right] \\
&\leq C(K, T, \Omega) \int_0^t \sup_{\xi \in \Omega} \mathbb{E} \sup_{0 \leq s \leq u} \left[|X'_s(1)(\xi) - X'_s(2)(\xi)|^2 + |G'_{s,0}(1)(\xi) - G'_{s,0}(2)(\xi)|^2 \right] du
\end{aligned}$$

Then our result follows from the same fixed-point arguments in the proof of Theorem 5.3.3 after (5.3.11). $\square$

5.7 Approximating the infinite-dimensional RVS-DEs by discretisation

In this section, we discuss how to approximate the solution of an infinite-dimensional RVSDE by space discretisation, which reduces the system into a finite-dimensional RVSDE. Recall that in approximating the 3D viscous fluid flows, we consider a lattice in $\mathbb{R}^3$ and the flow is then approximated by the particles starting from the lattice. However, for the Navier-Stokes equations, the convergence of the numerical scheme is very difficult to verify due to the high singularity. In this section, we do not tackle the convergence problem of the Navier-Stokes equations. Instead, we show the convergence by space discretisation with the assumption that the conditions in infinite-dimensional RVSDEs are nice.

Let us consider an infinite-dimensional RVSDE defined on Ω by (5.6.1),(5.6.2). We apply the same idea as we used in the random vortex method. Define a mesh size $h > 0$, and we consider the lattice points ih which lie in Ω , $i \in \mathbb{Z}^p$. Since Ω is a compact set, there are finitely many lattice points in Ω , so we relabel them by $i = 1, 2, \dots, N$, and denote ih by $\xi^{(i)}$. The discretized approximation is as follows:

$$\begin{aligned} d\tilde{X}_t^{(i)} &= \tilde{b}(\tilde{X}_t^{(i)}, t)dt + \tilde{\sigma}(\tilde{X}_t^{(i)}, t)dB_t, \tilde{X}_0^{(i)} = X_0(\xi^{(i)}) \\ d\tilde{G}_{t,s}^{(i)} &= -\tilde{S}(\tilde{G}_{t,s}^{(i)}, \tilde{X}_s^{(i)}, s)ds, \tilde{G}_{t,t}^{(i)} = G_{t,t}(\xi^{(i)}) \end{aligned} \tag{5.7.1}$$

where $\tilde{b}, \tilde{\sigma}, \tilde{S}$ only depend on $(\tilde{X}, \tilde{G})$:

$$\begin{aligned} \tilde{b}(x, t) &= \sum_{i=1}^N \mathbb{E} \left[b'(x, t, \tilde{X}_t^{(i)}, \tilde{G}_{t,0}^{(i)}, \xi^{(i)}) \right] h^p \\ \tilde{\sigma}(x, t) &= \sum_{i=1}^N \mathbb{E} \left[\sigma'(x, t, \tilde{X}_t^{(i)}, \tilde{G}_{t,0}^{(i)}, \xi^{(i)}) \right] h^p \\ \tilde{S}(g, x, t) &= \sum_{i=1}^N \mathbb{E} \left[S'(x, g, t, \tilde{X}_t^{(i)}, \tilde{G}_{t,0}^{(i)}, \xi^{(i)}) \right] h^p \end{aligned}$$

We are going to show that $\tilde{b}, \tilde{\sigma}, \tilde{S}$ converges to b, σ, S when $h \rightarrow 0$. First, let us

consider a reference system of $\{X^{(i)}, G^{(i)}\}_{i=1,2,\dots,N}$, where $\{X^{(i)}, G^{(i)}\}$ is the solution to the infinite-dimensional RVSD (5.6.1),(5.6.2) starting from $\xi^{(i)}$:

$$\begin{aligned} dX_t^{(i)} &= b(X_t^{(i)}, t)dt + \sigma(X_t^{(i)}, t)dB_t, X_0^{(i)} = X_0(\xi^{(i)}) \\ dG_{t,s}^{(i)} &= -S(G_{t,s}^{(i)}, X_s^{(i)}, s)ds, G_{t,t}^{(i)} = G_{t,t}(\xi^{(i)}) \end{aligned}$$

and we let

$$\begin{aligned} b^h(x, t) &= \sum_{i=1}^N \mathbb{E} \left[b'(x, t, X_t^{(i)}, G_{t,0}^{(i)}, \xi^{(i)}) \right] h^\rho, \\ \sigma^h(x, t) &= \sum_{i=1}^N \mathbb{E} \left[\sigma'(x, t, X_t^{(i)}, G_{t,0}^{(i)}, \xi^{(i)}) \right] h^\rho, \\ S^h(g, x, t) &= \sum_{i=1}^N \mathbb{E} \left[S'(x, g, t, X_t^{(i)}, G_{t,0}^{(i)}, \xi^{(i)}) \right] h^\rho. \end{aligned}$$

b^h, σ^h, S^h are the direct approximations of b, σ, S by Riemann sum, and we will show that their convergence orders depend on the smoothness of b', σ', S' . We introduce the following quadrature lemma for our proof.

Lemma 5.7.1. (see [4] Lemma 2.2) *Suppose $F \in C_0^L(\mathbb{R}^d)$ with $L \geq d + 1$, then the following estimate holds*

$$\left| \sum_{i \in \mathbb{Z}^d} F(h \cdot i) h^d - \int_{\mathbb{R}^d} F(\alpha) d\alpha \right| \leq Ch^L \max(\|\partial_{x_1}^L F\|_{L^1}, \dots, \|\partial_{x_d}^L F\|_{L^1})$$

where C is a constant only depends on L .

Now we are able to give the estimation.

Lemma 5.7.2. *Suppose that b', σ' and S' are $L(L \geq \rho + 1)$ times differentiable and*

bounded, and they vanish outside $\mathbb{R}^d \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^l \times \Omega$, i.e.

$$|D^l b'(x, t, x', g', \xi)| + |D^l \sigma'(x, t, x', g', \xi)| + |D^l S'(x, g, t, x', g', \xi)| \leq K^1 \quad (5.7.2)$$

for all $l = 1, 2, \dots, L, (x, g, x', t, g', \xi) \in \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^l \times \mathbb{R}^\rho$. Besides,

$$b'(x, t, x', g', \xi), \sigma'(x, t, x', g', \xi), S'(x, g, t, x', g', \xi) \equiv 0$$

when $(x, g, x', t, g', \xi) \in \mathbb{R}^d \times \mathbb{R}^l \times \mathbb{R}^d \times [0, T] \times \mathbb{R}^l \times (\mathbb{R}^\rho \setminus \Omega)$. We also assume that for all $l = 1, 2, \dots, L$, the initial data $X_0(\xi), G_{t,t}(\xi)$ satisfy

$$\sup_{\xi \in \Omega} \mathbb{E}|D^l X_0(\xi)|^2 \leq K, \quad \sup_{\xi \in \Omega, t \in [0, T]} \mathbb{E}|D^l G_{t,t}(\xi)|^2 \leq K$$

then

$$\begin{aligned} \sup_{x \in \mathbb{R}^d, t \in \mathbb{R}^d} |b(x, t) - b^h(x, t)| + |\sigma(x, t) - \sigma^h(x, t)| &\leq C(K, T, \Omega, L)h^L \\ \sup_{x \in \mathbb{R}^d, g \in \mathbb{R}^l, t \in \mathbb{R}^d} |S(g, x, t) - S^h(g, x, t)| &\leq C(K, T, \Omega, L)h^L. \end{aligned} \quad (5.7.3)$$

Proof. We prove that $\sup_{x \in \mathbb{R}^d, t \in \mathbb{R}^d} |b(x, t) - b^h(x, t)| \leq C(K, T, \Omega, L)h^L$, and the rest follows from the same argument.

Given any $x \in \mathbb{R}^d, t \in [0, T]$, let

$$f(\xi) = \begin{cases} \mathbb{E}[b'(x, t, X_t(\xi), G_{t,0}(\xi), \xi)], & \xi \in \Omega \\ 0, & \xi \notin \Omega \end{cases} \quad (5.7.4)$$

By Lemma 5.7.1, it suffices to show that $\|D^l f(\xi)\|_{L^1}$ is bounded by $C(K, T, \Omega, l)$ for $l = 1, 2, \dots, L$. By differentiating $f(\xi)$ for l times, since b' is L times differentiable

¹For a function $f(t, \xi)$, where t is a time variable in $[0, T]$, and ξ is a space variable, $D^l f(t, \xi)$ denotes the vector consisting of all possible l -order partial derivatives of f at (t, ξ) , excluding those involving the derivative w.r.t t . For example, $D^l b'(x, t, x', g', \xi)$ denotes the vector of all possible l -order partial derivatives of b' as long as they involve the derivatives w.r.t x, x', g', ξ only.

and bounded, we only need to show that

$$\sup_{\xi \in \Omega} \mathbb{E}|D^l X_t(\xi)| + |D^l G_{t,0}(\xi)| < C(K, T, \Omega, l) \quad (5.7.5)$$

for $l = 1, 2, \dots, L$. Since $dX_t(\xi) = b(X_t(\xi), t)dt + \sigma(X_t(\xi), t)dB_t$, let $Y_t(\xi) = D^1 X_t(\xi)$, we have (see [100] Theorem 13.12)

$$dY_t(\xi) = \nabla b(X_t(\xi), t)Y_t(\xi)dt + \nabla \sigma(X_t(\xi), t)Y_t(\xi)dB_t.$$

By repeated differentiating ξ on both sides,

$$dD^l X_t(\xi) = \nabla b(X_t(\xi), t)D^l X_t(\xi)dt + \nabla \sigma(X_t(\xi), t)D^l X_t(\xi)dB_t + A_t dt + A'_t dB_t, \quad (5.7.6)$$

where A_t, A'_t only involve $D^j X_t(\xi)$ for $1 \leq j \leq l-1$ and derivatives of b and σ up to order l .

From our assumption (5.7.2) and the definition of b and σ (5.6.3), the derivatives of b and σ up to order L can be bounded by a constant depending only on K and Ω . It follows from (5.7.6) that

$$\begin{aligned} & \mathbb{E}|D^l X_t(\xi)|^2 \\ & \leq C(T, K, \Omega) \left\{ \int_0^t \mathbb{E}|D^l(X_s(\xi))|^2 ds + \int_0^t (\mathbb{E}|A_s|^2 + \mathbb{E}|A'_s|^2) ds + \mathbb{E}|D^l X_0(\xi)|^2 \right\} \end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}|D^l(X_t(\xi))|^2 \leq C(K, T, \Omega) \left\{ \mathbb{E} \int_0^t (|A_s|^2 + |A'_s|^2) ds + \mathbb{E}|D^l X_0(\xi)|^2 \right\}$$

Therefore, by induction we have

$$\sup_{\xi \in \Omega} \mathbb{E}|D^l(X_t(\xi))|^2 < C(K, T, \Omega, l)$$

for $l = 1, 2, \dots, L$. On the other hand, by differentiating ξ on both sides of

$$dG_{t,s}(\xi) = -S(G_{t,s}(\xi), X_s(\xi), s)ds,$$

it follows that

$$\begin{aligned} dD^1 G_{t,s}(\xi) &= -(\nabla_g S(G_{t,s}(\xi), X_s(\xi), s)D^1 G_{t,s}(\xi) \\ &\quad + \nabla_x S(G_{t,s}(\xi), X_s(\xi), s)D^1 X_s(\xi))ds. \end{aligned}$$

By repeated differentiating ξ for l times, we deduce that

$$dD^l G_{t,s}(\xi) = -(\nabla_g S(G_{t,s}(\xi), X_s(\xi), s)D^l G_{t,s}(\xi) + A_s) ds,$$

where A_s only involves the $D^j G_{t,s}(\xi)$ for $1 \leq j \leq l-1$, $D^j X_s(\xi)$ for $1 \leq j \leq l$, and derivatives of S up to order l .

Hence, by applying Gronwall's inequality,

$$|D^l G_{t,0}(\xi)| \leq C(K, T, \Omega) \left(\int_0^t |A_s| ds + |D^l G_{t,t}(\xi)| \right).$$

By induction and the previous estimate on X , we have

$$\sup_{\xi \in \Omega} \mathbb{E} |D^l G_{t,0}(\xi)| < C(K, T, \Omega, l)$$

for $l = 1, 2, \dots, L$. Therefore, (5.7.5) follows. □

Now we are able to prove our main theorem.

Theorem 5.7.3. *With the same assumption as in Lemma 5.7.2,*

$$\begin{aligned} \sup_{x \in \mathbb{R}^d, t \in [0, T]} |\tilde{b}(x, t) - b(x, t)| &\leq C(K, T, \Omega, L)h^L, \\ \sup_{x \in \mathbb{R}^d, t \in [0, T]} |\tilde{\sigma}(x, t) - \sigma(x, t)| &\leq C(K, T, \Omega, L)h^L, \end{aligned}$$

$$\sup_{g \in \mathbb{R}^l, x \in \mathbb{R}^d, t \in [0, T]} |\tilde{S}(g, x, t) - S(g, x, t)| \leq C(K, T, \Omega, L)h^L.$$

Proof. From the definition of $\tilde{b}, b^h, \tilde{\sigma}, \sigma^h, \tilde{S}, S^h$, it is clear that for every $x \in \mathbb{R}^d, g \in \mathbb{R}^l, t \in [0, T]$,

$$\begin{aligned} & |\tilde{b}(x, t) - b^h(x, t)| + |\tilde{\sigma}(x, t) - \sigma^h(x, t)| + |\tilde{S}(g, x, t) - S^h(g, x, t)| \\ & \leq C(K)h^\rho \sum_{i=1}^N \mathbb{E} \left[|X_t^{(i)} - \tilde{X}_t^{(i)}| + |G_{t,0}^{(i)} - \tilde{G}_{t,0}^{(i)}| \right], \end{aligned}$$

which gives us

$$\begin{aligned} & |\tilde{b}(x, t) - b(x, t)| + |\tilde{\sigma}(x, t) - \sigma(x, t)| + |\tilde{S}(g, x, t) - S(g, x, t)| \\ & \leq C(K)h^\rho \sum_{i=1}^N \mathbb{E} \left[|X_t^{(i)} - \tilde{X}_t^{(i)}| + |G_{t,0}^{(i)} - \tilde{G}_{t,0}^{(i)}| \right] + C(K, T, \Omega, L)h^L. \end{aligned} \tag{5.7.7}$$

from our previous lemma.

For $i = 1, 2, \dots, N$, by (5.6.1) and (5.7.1), we have

$$\tilde{X}_t^{(i)} - X_t^{(i)} = \int_0^t \left(\tilde{b}(\tilde{X}_s^{(i)}, s) - b(X_s^{(i)}, s) \right) ds + \int_0^t \left(\tilde{\sigma}(\tilde{X}_s^{(i)}, s) - \sigma(X_s^{(i)}, s) \right) dB_s, \tag{5.7.8}$$

which yields

$$\begin{aligned} & \mathbb{E} |\tilde{X}_t^{(i)} - X_t^{(i)}|^2 \\ & \leq C(K, T) \left\{ \int_0^t \mathbb{E} |\tilde{X}_s^{(i)} - X_s^{(i)}|^2 ds \right. \\ & \quad \left. + \int_0^t \left(\sup_{x \in \mathbb{R}^d} |\tilde{b}(x, s) - b(x, s)|^2 + \sup_{x \in \mathbb{R}^d} |\tilde{\sigma}(x, s) - \sigma(x, s)|^2 \right) ds \right\} \\ & \leq C(K, T, \Omega, L) \left\{ \int_0^t \mathbb{E} |\tilde{X}_s^{(i)} - X_s^{(i)}|^2 ds \right. \\ & \quad \left. + h^{2\rho} N \sum_{i=1}^N \left(\int_0^t \mathbb{E} |X_s^{(i)} - \tilde{X}_s^{(i)}|^2 + |G_{s,0}^{(i)} - \tilde{G}_{s,0}^{(i)}|^2 ds \right) + h^{2L} \right\}. \end{aligned}$$

By summing $i = 1, 2, \dots, N$, since $h^\rho N$ is bounded by some constant only depends

on Ω ,

$$\begin{aligned} & \sum_{i=1}^N \mathbb{E} |\tilde{X}_t^{(i)} - X_t^{(i)}|^2 \\ & \leq C(K, T, \Omega, L) \left\{ \int_0^t \sum_{i=1}^N \left(\mathbb{E} |\tilde{X}_s^{(i)} - X_s^{(i)}|^2 + |\tilde{G}_{s,0}^{(i)} - G_{s,0}^{(i)}|^2 \right) ds + h^{2L} N \right\}. \end{aligned} \quad (5.7.9)$$

Similarly, for any $0 \leq s \leq t \leq T$,

$$\begin{aligned} & |\tilde{G}_{t,s}^{(i)} - G_{t,s}^{(i)}| \\ & \leq C(K, T, \Omega, L) \int_s^t \left(h^\rho \sum_{i=1}^N \mathbb{E} \left[|X_u^{(i)} - \tilde{X}_u^{(i)}| + |G_{u,0}^{(i)} - \tilde{G}_{u,0}^{(i)}| \right] + h^L \right. \\ & \quad \left. + |X_u^{(i)} - \tilde{X}_u^{(i)}| + |G_{t,u}^{(i)} - \tilde{G}_{t,u}^{(i)}| \right) du \end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned} & |\tilde{G}_{t,0}^{(i)} - G_{t,0}^{(i)}| \\ & \leq C(K, T, \Omega, L) \int_0^t \left(h^\rho \sum_{i=1}^N \mathbb{E} \left[|X_u^{(i)} - \tilde{X}_u^{(i)}| + |G_{u,0}^{(i)} - \tilde{G}_{u,0}^{(i)}| \right] + h^L \right. \\ & \quad \left. + |X_u^{(i)} - \tilde{X}_u^{(i)}| \right) du \end{aligned}$$

which yields

$$\begin{aligned} & |\tilde{G}_{t,0}^{(i)} - G_{t,0}^{(i)}|^2 \\ & \leq C(K, T, \Omega, L) \int_0^t \left(h^{2\rho} N \sum_{i=1}^N \mathbb{E} \left[|X_u^{(i)} - \tilde{X}_u^{(i)}|^2 + |G_{u,0}^{(i)} - \tilde{G}_{u,0}^{(i)}|^2 \right] + h^{2L} \right. \\ & \quad \left. + |X_u^{(i)} - \tilde{X}_u^{(i)}|^2 \right) du. \end{aligned}$$

Then we sum i over 1 to N and take the expectation, so that

$$\mathbb{E} \sum_{i=1}^N |\tilde{G}_{t,0}^{(i)} - G_{t,0}^{(i)}|^2$$

$$\leq C(K, T, \Omega, L) \int_0^t \left(\sum_{i=1}^N \mathbb{E} \left[|X_u^{(i)} - \tilde{X}_u^{(i)}|^2 + |G_{u,0}^{(i)} - \tilde{G}_{u,0}^{(i)}|^2 \right] + h^{2L} N \right) du.$$

By summing the above inequality with (5.7.9) and applying Gronwall's inequality, we have

$$\mathbb{E} \sum_{i=1}^N \left[|\tilde{X}_t^{(i)} - X_t^{(i)}|^2 + |\tilde{G}_{t,0}^{(i)} - G_{t,0}^{(i)}|^2 \right] \leq C(K, T, \Omega, L) h^{2L} N \leq C(K, T, \Omega, L) h^{2L-\rho}. \quad (5.7.10)$$

Therefore, by (5.7.7),

$$\begin{aligned} & |\tilde{b}(x, t) - b(x, t)| \\ & \leq C(K, T, \Omega, L) h^\rho \sqrt{N} \sqrt{\sum_{i=1}^N \mathbb{E} \left[|X_t^{(i)} - \tilde{X}_t^{(i)}|^2 + |G_{t,0}^{(i)} - \tilde{G}_{t,0}^{(i)}|^2 \right]} + C(K, T, \Omega, L) h^L \\ & \leq C(K, T, \Omega, L) h^L. \end{aligned}$$

Similar bounds also hold for $\tilde{S} - S$ and $\tilde{\sigma} - \sigma$. □

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