

Lattice Points in the Sphere

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*For Professor Andrzej Schinzel
On the Occasion of his Sixtieth Birthday*

1 Introduction

Our goal in this paper is to give a new estimate for the number of integer lattice points lying in a sphere of radius R centred at the origin. Thus we define

$$S(R) = \#\{\mathbf{x} \in \mathbb{Z}^3 : \|\mathbf{x}\| \leq R\},$$

and seek an asymptotic formula

$$S(R) = \frac{4}{3}\pi R^3 + O(R^\theta)$$

with as small an exponent θ as possible. It is known that the error term is $\Omega(R(\log R)^{1/2})$, and it is conjectured that it should be $O_\varepsilon(R^{1+\varepsilon})$ for any $\varepsilon > 0$. However the best result established to date is

$$S(R) = \frac{4}{3}\pi R^3 + O_\varepsilon(R^{29/22+\varepsilon}), \quad (1)$$

due to Chamizo and Iwaniec [1]. This sharpened a long standing result, due independently to Chen [3] and Vinogradov [5], in which the exponent was $4/3$.

Here we shall build on the method of Chamizo and Iwaniec to obtain the following bound.

Theorem *For any $\varepsilon > 0$ we have*

$$S(R) = \frac{4}{3}\pi R^3 + O_\varepsilon(R^{21/16+\varepsilon}).$$

The extraneous part in the exponent has been reduced from $29/22 - 1 = 7/22$ to $21/16 - 1 = 5/16$. Since

$$\frac{\frac{7}{22} - \frac{5}{16}}{\frac{7}{22}} = \frac{1}{56}$$

the new result may be regarded as being roughly a 2% improvement on (1).

Luckily we are able to use much of Chamizo and Iwaniec's work without major changes. They divide the problem into two parts, one involving the estimation of a three-dimensional exponential sum, and the other involving bounds for an average of real character sums. The key to their advance was the discovery that character sums could be brought into play. Indeed they treat the exponential sum in essentially the same way as Chen and Vinogradov. Following their lead we shall concentrate on the character sums, and provide better bounds, using a mean-value estimate (Lemma 4) obtained recently by the author [4].

We remark that it seems likely that the methods of this paper might be applied to Gauss's problem of the average of the class number, in much the same way as has been done by Chamizo and Iwaniec [2]. One would then hope to show, in the usual notation, that

$$\sum_{n \leq N} h(-n) = \frac{\pi}{18\zeta(3)} N^{3/2} - \frac{3}{2\pi^2} N + O_\varepsilon(N^{21/32+\varepsilon}),$$

for any $\varepsilon > 0$.

2 The Method of Chamizo and Iwaniec

Following the work of Chamizo and Iwaniec, we may view the estimates of Chen and Vinogradov as arising from a Tauberian argument, in which one approximates $S(R)$ by a sum of the form

$$S^{(H)}(R) = \sum_{\mathbf{x} \in \mathbb{Z}^3} f_H(\|\mathbf{x}\|),$$

with $f_H(t) = 1$ or 0 for $t \leq R$ and $t \geq R + H$ respectively, and

$$f_H(t) = \frac{R(R+H-t)}{tH}, \text{ for } R \leq t \leq R+H.$$

Here the parameter H will be taken to lie in the range $R^{-1} \leq H < 1$. The number of solutions of $\|\mathbf{x}\|^2 = n$ with $\mathbf{x} \in \mathbb{Z}^3$ is $r_3(n)$, where $r_3(n)$ counts representations as a sum of 3 squares. Since $r_3(n) \ll_\varepsilon n^{1/2+\varepsilon}$, it follows that

$$S(R) = S^{(H)}(R) + O_\varepsilon(HR^{2+\varepsilon}), \tag{2}$$

on re-defining ε .

One may now apply the Poisson summation formula to evaluate $S^{(H)}(R)$ via an infinite series involving terms of the form $r_3(n) \exp(2\pi i R \sqrt{n})$. The effect of the weight function $f_H(t)$ is to damp the above terms so that the main

contribution is from those with $n \ll H^{-2}$. One then has to consider a three-dimensional exponential sum of the form

$$\sum_{a,b,c} \exp(2\pi i R \sqrt{a^2 + b^2 + c^2}).$$

A suitable bound for this leads, in Chamizo and Iwaniec's version (see [1; (5.3)]), to the estimate

$$S^{(H)}(R) = \frac{4\pi}{3}R^3 + 2\pi HR^2 + O_\varepsilon(H^{-\varepsilon}\{R^{21/16} + RH^{-1/2} + R^{9/8}H^{-1/8}\}), \quad (3)$$

providing that $H \leq R^{-1/2}$.

If one now chooses $H = R^{-2/3}$ it follows from (2) and (3) that

$$S(R) = \frac{4}{3}\pi R^3 + O_\varepsilon(R^{4/3+\varepsilon}),$$

which is the result of Chen and Vinogradov.

The new idea introduced by Chamizo and Iwaniec, was to examine the difference $S^{(H)}(R) - S(R)$, which was previously estimated in (2) by a crude upper bound. Since we may rewrite $S(R)$ as

$$1 + \sum_{n \leq R^2} r_3(n),$$

this leads one to consider sums of the form

$$\sum_{N < n \leq N+K} r_3(n),$$

where, roughly speaking, one has $N = R^2$ and $K = RH$.

One can now use Gauss's formula for the number of primitive representations as a sum of three squares, in terms of class numbers. One then obtains an expression for $S^{(H)}(R) - S(R)$ in terms of averages of L -functions $L(1, \chi_n)$, where

$$\chi_n(m) = \left(\frac{-4n}{m}\right),$$

by applying Dirichlet's class number formula. To be specific, the analysis of Chamizo and Iwaniec [1; pp. 418 & 426], may be used to establish the following relationship.

Lemma 1 *Suppose that $E(N, K)$ is a non-negative function, increasing with respect to K . Suppose further that the estimate*

$$\sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} L(1, \chi_n) = \frac{3\zeta(2)}{28\zeta(3)}K + O(E(N, K))$$

holds for $0 < K \leq N^{1/2}$ and for each of $\nu = 1, 2, 3, 5$ and 6 . Then

$$S^{(H)}(R) - S(R) - 2\pi HR^2 \ll_{\varepsilon} R^{1+\varepsilon} + \sum_{d \leq 2\sqrt{R}} \frac{R}{d} E\left(\left(\frac{R}{d}\right)^2, \frac{3RH}{d^2}\right), \quad (4)$$

for $0 < H \leq 1/3$ and $R \geq 1$.

Chamizo and Iwaniec show that

$$E(N, K) = K^{7/8} N^{\varepsilon} + K^{2/3} N^{1/32+\varepsilon}$$

is admissible, for any $\varepsilon > 0$. The estimate (4) then produces

$$S^{(H)}(R) - S(R) - 2\pi HR^2 \ll_{\varepsilon} R^{1+\varepsilon} + H^{7/8} R^{15/8+\varepsilon} + H^{2/3} R^{83/48+\varepsilon}.$$

When this is combined with (3) the choice $H = R^{-7/11}$ leads to the bound (1).

The remainder of this paper will be devoted to the proof of the following result.

Lemma 2 *One may take*

$$E(N, K) = N^{\varepsilon} (K^{5/6} + N^{2/15} + N^{1/6} \min\{1, K^{-1/4}\})$$

in Lemma 1, for any $\varepsilon > 0$.

As above this produces

$$\begin{aligned} S^{(H)}(R) - S(R) - 2\pi HR^2 \\ \ll_{\varepsilon} R^{1+\varepsilon} + R^{11/6+2\varepsilon} H^{5/6} + R^{19/15+2\varepsilon} + R^{7/6+2\varepsilon} H^{-1/6}. \end{aligned} \quad (5)$$

If we combine this with (3) we now obtain

$$\begin{aligned} S(R) &= \frac{4\pi}{3} R^3 + O_{\varepsilon}(H^{-\varepsilon} \{R^{21/16} + RH^{-1/2} + R^{9/8} H^{-1/8}\}) \\ &\quad + O_{\varepsilon}(R^{2\varepsilon} \{R + R^{11/6} H^{5/6} + R^{19/15} + R^{7/6} H^{-1/6}\}). \end{aligned}$$

The choice $H = R^{-5/8}$ then leads to the estimate claimed in our theorem. We should observe at this point that the error terms in (3) can be improved a little, although the term $RH^{-1/2}$ remains. It is therefore the terms $RH^{-1/2}$ in (3) and $R^{11/6} H^{5/6}$ in (5) that produce the exponent $21/16$ in our theorem, rather than the term $R^{21/16}$ in (3).

Before turning to the proof of Lemma 2 we remark that the Generalized Riemann Hypothesis (or indeed the Generalized Lindelöf Hypothesis) would permit one to take $E(N, K) = N^{\varepsilon} K^{1/2}$. As remarked above, improvements on (3) are possible, but it is difficult to see how the term $RH^{-1/2}$ can be avoided. It therefore appears that an error term $_{\varepsilon}(R^{5/4+\varepsilon})$ in our theorem is the very best that one can hope for by these methods.

3 Averages of Real Character Sums: I

By the Pólya-Vinogradov inequality we have

$$L(1, \chi_n) = \sum_{m \leq N} m^{-1} \left(\frac{-4n}{m} \right) + O_\varepsilon(N^{-1/2+\varepsilon}),$$

for $N < n \leq N + K$, whence

$$\sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} L(1, \chi_n) = \sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} \sum_{m \leq N} m^{-1} \left(\frac{-4n}{m} \right) + O_\varepsilon(N^\varepsilon).$$

We begin by examining the contribution of square values of m . Following the argument of Chamizo and Iwaniec [1; page 425], we see that this is

$$\sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} \sum_{\substack{k \leq \sqrt{N}, (k, 4n)=1}} k^{-2} = \frac{3\zeta(2)}{28\zeta(3)} K + O(1).$$

We proceed to investigate sums of the form

$$S(K, M) = \sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} \sum_{M < m \leq 2M}^* \left(\frac{-4n}{m} \right),$$

where Σ^* indicates that only non-square values of m are to be included. We observe at the outset that Lemma 2 is trivial unless $K \geq 1$, as we henceforth assume. We shall show that, for any positive integer r and any $\varepsilon > 0$ we have

$$S(K, M) \ll_{r, \varepsilon} K^{1-1/2r} (M + M^{1/2} N^{1/2r}) (MN)^\varepsilon, \quad (6)$$

and

$$S(K, M) \ll_\varepsilon (M^{7/5} + M^{3/2} K^{-1/4} + M^{5/4} K^{1/4}) (MK)^\varepsilon. \quad (7)$$

We shall use (6) with $r = 3$ for $M \geq N^{1/3}$, and (7) for $M < N^{1/3}$. Thus (7) yields

$$S(K, M) \ll_\varepsilon M(N^{2/15} + N^{1/6} K^{-1/4} + N^{1/12} K^{1/4}) (NK)^\varepsilon,$$

whence

$$\begin{aligned} S(K, M) &\ll_\varepsilon M(K^{5/6} + N^{2/15} + N^{1/6} K^{-1/4} + N^{1/12} K^{1/4}) N^{2\varepsilon} \\ &\ll_\varepsilon M(K^{5/6} + N^{2/15} + N^{1/6} K^{-1/4}) N^{2\varepsilon}, \end{aligned}$$

in either case. This clearly suffices for Lemma 2.

For the proof of (6) we apply Hölder's inequality to obtain

$$S(K, M)^{2r} \ll_r K^{2r-1} \sum_{N < n \leq N+K} |S_n|^2,$$

where

$$S_n = \left\{ \sum_{M < m \leq 2M}^* \chi_n(m) \right\}^r = \sum_{v \leq (2M)^r} a_v \chi_n(v),$$

say, so that $a_v \ll_{r,\varepsilon} M^\varepsilon$. We now observe that no two of the characters χ_n can be equivalent. For if $\alpha\beta^2 \neq \alpha\gamma^2$ both lie in $(N, N+K]$, with $\beta < \gamma$, say, then

$$K > \alpha\gamma^2 - \alpha\beta^2 \geq \alpha(\beta+1)^2 - \alpha\beta^2 > 2\alpha\beta \geq 2\alpha^{1/2}\beta \geq 2\sqrt{N},$$

which contradicts our initial assumption.

We may therefore call on the following result.

Lemma 3 *Let χ run over a set of inequivalent real characters of modulus at most Q . Then*

$$\sum_{\chi} \left| \sum_{v \leq V} a_v \chi(v) \right|^2 \ll_{\varepsilon} Q^\varepsilon V^{1+\varepsilon} (Q+V) \max_{v \leq V} |a_v|^2,$$

for any $\varepsilon > 0$.

This shows that

$$S(K, M)^{2r} \ll_{r,\varepsilon} K^{2r-1} N^\varepsilon M^{r+r\varepsilon} (N+M^r) M^{2\varepsilon},$$

whence

$$S(K, M) \ll_{r,\varepsilon} K^{1-1/2r} (M + M^{1/2} N^{1/2r}) (MN)^\varepsilon,$$

on re-defining ε if necessary. This proves (6).

Lemma 3 is a simple extension of the following bound of the author [4; Corollary 3].

Lemma 4 *Let χ run over the set of primitive real characters of modulus at most Q . Then*

$$\sum_{\chi} \left| \sum_{v \leq V} a_v \chi(v) \right|^2 \ll_{\varepsilon} Q^\varepsilon V^{1+\varepsilon} (Q+V) \max_{v \leq V} |a_v|^2,$$

for any $\varepsilon > 0$.

For each character χ in Lemma 3 we write χ_0 for the corresponding primitive character, and we let the moduli of χ and χ_0 be qj and q respectively. Then

$$\sum_{\chi} \left| \sum_{v \leq V} a_v \chi(v) \right|^2 = \sum_{\chi_0} \left| \sum_{v \leq V, (v,j)=1} a_v \chi_0(v) \right|^2.$$

However

$$\begin{aligned} \left| \sum_{v \leq V, (v,j)=1} a_v \chi_0(v) \right|^2 &= \left| \sum_{d|j} \mu(d) \sum_{w \leq V/d} a_{dw} \chi_0(d) \chi_0(w) \right|^2 \\ &\leq \left(\sum_{d|j} 1 \right) \left(\sum_{d|j} \left| \sum_{w \leq V/d} a_{dw} \chi_0(d) \chi_0(w) \right|^2 \right) \end{aligned}$$

Since $j \leq Q$ we deduce that

$$\sum_{\chi} \left| \sum_{v \leq V} a_v \chi(v) \right|^2 \ll_{\varepsilon} Q^{\varepsilon} \sum_{d \leq Q} \sum_{\chi_0} \left| \sum_{w \leq V/d} a_{dw} \chi_0(d) \chi_0(w) \right|^2.$$

We may now apply Lemma 4 for each value of d to obtain

$$\begin{aligned} \sum_{\chi} \left| \sum_{v \leq V} a_v \chi(v) \right|^2 &\ll_{\varepsilon} Q^{\varepsilon} \sum_{d \leq Q} Q^{\varepsilon} \left(\frac{V}{d} \right)^{1+\varepsilon} \left(Q + \frac{V}{d} \right) \max_{v \leq V} |a_v|^2 \\ &\ll_{\varepsilon} Q^{2\varepsilon} V^{1+\varepsilon} (Q + V) \max_{v \leq V} |a_v|^2. \end{aligned}$$

Lemma 3 now follows, on re-defining ε .

4 Averages of Real Character Sums: II

Our second bound for $S(K, M)$ comes from the following result.

Lemma 5 *For any $\varepsilon > 0$ we have*

$$\sum_{M < m \leq 2M}^{**} \left| \sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} \left(\frac{-4n}{m} \right) \right| \ll_{\varepsilon} (MK)^{\varepsilon} \{ M^{7/5} + M^{3/2} K^{-1/4} + M^{5/4} K^{1/4} \},$$

for $M \geq 1$, uniformly in N , where Σ^{**} indicates that the integers m are restricted to be square-free.

Since we may remove a factor $(-4/m)$ from the inner sum it suffices to consider

$$\sum_{M < m \leq 2M}^{**} \left| \sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} \left(\frac{n}{m} \right) \right|,$$

where m must be odd as well as square-free. We may regard ν as fixed, and we take $N' = \lfloor (N - \nu)/8 \rfloor$ and $N' + K' = \lfloor (N + K - \nu)/8 \rfloor$, so that the condition on n becomes $n = 8n' + \nu$ with $N' < n' \leq N' + K'$.

We can now transform the inner sum as

$$\sum_{a(\bmod m)} \left(\frac{a}{m}\right) \sum_n \frac{1}{m} \sum_{b(\bmod m)} e\left(\frac{b(n-a)}{m}\right),$$

where $e(x) = \exp(2\pi i x)$ as usual. We proceed to perform the summations over a and n explicitly. On writing τ_m for the Gauss sum, the above then becomes

$$\frac{\tau_m}{2im} \left(\frac{-1}{m}\right) \sum_{b(\bmod m)} \left(\frac{b}{m}\right) \{e(Ab/m) - e(Bb/m)\} \operatorname{cosec}(8\pi b/m),$$

where

$$A = 8N' + 8K' + 4 + \nu, \quad \text{and} \quad B = 8N' - 4 + \nu.$$

Since m cannot be not equal to 1, the term corresponding to $b = 0$ vanishes. We may therefore think of the sum over b as being for $1 \leq |b| \leq m/2$. We shall confine our attention to the interval $1 \leq b \leq m/2$, since the alternative range can be handled analogously. When $b \leq M/K$ we remove a factor

$$\{e(8K'b/m) - 1\} \operatorname{cosec}(8\pi b/m)$$

by partial summation, to show that

$$\begin{aligned} \sum_{b \leq M/K} \left(\frac{b}{m}\right) \{e(Ab/m) - e(Bb/m)\} \operatorname{cosec}(8\pi b/m) \\ \ll K |S_m(M/K)| + K^2 M^{-1} \int_0^{M/K} |S_m(t)| dt, \end{aligned}$$

with

$$S_m(t) = \sum_{b \leq t} \left(\frac{b}{m}\right) e(Bb/m).$$

For the remaining range $M/K < b \leq m/2$ we consider only terms involving $e(Bb/m)$, since the treatment of those containing $e(Ab/m)$ is identical. Here we remove the factor $\operatorname{cosec}(8\pi b/m)$ by partial summation, producing

$$\begin{aligned} \sum_{M/K < b \leq m/2} \left(\frac{b}{m}\right) e(Bb/m) \operatorname{cosec}(8\pi b/m) \\ \ll K |S_m(M/K)| + |S_m(m/2)| + M \int_{M/K}^m t^{-2} |S_m(t)| dt, \end{aligned}$$

with $S_m(t)$ as before.

The standard procedure for estimating incomplete Gauss sums shows that $S_m(t) \ll_\varepsilon m^{1/2+\varepsilon}$. We use this bound both for $S_m(m/2)$ and for that part of

the integral in which $t \geq M^{3/5}$. In view of the above results we then obtain the bound

$$\begin{aligned} \sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} \binom{n}{m} &\ll_{\varepsilon} M^{2/5+\varepsilon} + \frac{K}{\sqrt{M}} |S_m(\frac{M}{K})| + \frac{K^2}{M^{3/2}} \int_0^{M/K} |S_m(t)| dt \\ &+ M^{1/2} \int_{M/K \leq t \leq M^{3/5}} t^{-2} |S_m(t)| dt. \end{aligned} \quad (8)$$

We now proceed to show that

$$\sum_{M < m \leq 2M}^{**} |S_m(t)| \ll_{\varepsilon} M^{\varepsilon} (Mt^{3/4} + M^{3/4}t^{5/4}) \quad (9)$$

for $1 \leq t \leq M$. Taken in conjunction with (8), one readily checks that this suffices to establish Lemma 5.

In order to establish (9) we begin by applying Hölder's inequality, whence

$$\sum_{M < m \leq 2M}^{**} |S_m(t)| \ll M^{3/4} \left\{ \sum_{M < m \leq 2M}^{**} |S_m(t)|^4 \right\}^{1/4}.$$

Now comes the key idea. For each value of m we set $B_m = B - m[B/m]$, so that

$$S_m(t) = \sum_{b \leq t} \left(\frac{b}{m}\right) e(B_m b/m).$$

We then define intervals

$$I_{m,h} = \left[\frac{(h-1)m}{t}, \frac{hm}{t} \right), \quad h = 1, \dots, [t+1],$$

and divide the possible m into subsets $J(h)$, according to the value of h for which $B_m \in I_{m,h}$. It follows that there is some h for which

$$\sum_{M < m \leq 2M}^{**} |S_m(t)| \ll M^{3/4} t^{1/4} \left\{ \sum_{m \in J(h)}^{**} \left| \sum_{b \leq t} \left(\frac{b}{m}\right) e(B_m b/m) \right|^4 \right\}^{1/4}.$$

We can now estimate the inner sum by partial summation, knowing that $B_m = hm/t - m\eta_m/t$, where η_m lies in $[0, 1)$ and is independent of b . Thus, if we set

$$S_m^{(0)}(v) = \sum_{b \leq v} \left(\frac{b}{m}\right) e(hb/t),$$

we have

$$\sum_{b \leq t} \left(\frac{b}{m}\right) e(B_m b/m) \ll |S_m^{(0)}(t)| + t^{-1} \int_0^t |S_m^{(0)}(v)| dv,$$

whence

$$\sum_{m \in J(h)}^{**} \left| \sum_{b \leq t} \left(\frac{b}{m}\right) e(B_m b/m) \right|^4 \ll \sum_{m \in J(h)}^{**} \{ |S_m^{(0)}(t)|^4 + t^{-1} \int_0^t |S_m^{(0)}(v)|^4 dv \},$$

by a further application of Hölder's inequality. We therefore conclude that it suffices, for the proof of (9), to show that

$$\sum_{M < m \leq 2M}^{**} |S_m^{(0)}(v)|^4 \ll_{\varepsilon} \{Mt^2 + t^4\} M^{4\varepsilon}, \quad (10)$$

uniformly for $1 \leq v \leq t \leq M$. The important point here is that $S_m(t)$ contains an exponential factor which varies with m , while $S_m^{(0)}(v)$ does not. This has been achieved at a cost of a factor t however.

In order to prove (10) we write

$$S_m^{(0)}(v)^2 = \sum_{w \leq v^2} b_w \left(\frac{w}{m}\right),$$

where the coefficients b_w are independent of m and satisfy $b_w \ll_{\varepsilon} v^{\varepsilon}$. Lemma 4 then shows that

$$\sum_{M < m \leq 2M}^{**} \left| \sum_{w \leq v^2} b_w \left(\frac{w}{m}\right) \right|^2 \ll_{\varepsilon} M^{\varepsilon} v^{2+4\varepsilon} (M + v^2),$$

and (10) follows, on re-defining ε .

To complete the proof of our theorem it remains to demonstrate how (7) may be derived from Lemma 5. We have therefore to show how the assumption that m should be square-free may be replaced by the condition that m is not a square. To this end we observe that

$$S(K, M) \ll \sum_{M < m \leq 2M}^* \left| \sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} \left(\frac{-4n}{m}\right) \right|.$$

We write $m = qs^2$, where $q > 1$ is square-free. Thus

$$\begin{aligned} S(K, M) &= \sum_{\substack{s \leq \sqrt{M} \\ 2 \nmid s}} \sum_{Ms^{-2} < q \leq 2Ms^{-2}}^{**} \left| \sum_{\substack{N < n \leq N+K, (n,s)=1 \\ n \equiv \nu \pmod{8}}} \left(\frac{n}{q}\right) \right| \\ &= \sum_{\substack{s \leq \sqrt{M} \\ 2 \nmid s}} \sum_{Ms^{-2} < q \leq 2Ms^{-2}}^{**} \left| \sum_{\substack{N < n \leq N+K \\ n \equiv \nu \pmod{8}}} \sum_{d|s, d|n} \mu(d) \left(\frac{n}{q}\right) \right| \\ &\leq \sum_{\substack{s \leq \sqrt{M} \\ 2 \nmid s}} \sum_{d|s} \sum_{Ms^{-2} < q \leq 2Ms^{-2}}^{**} \left| \sum_{\substack{N/d < n' \leq (N+K)/d \\ n' \equiv \nu' \pmod{8}}} \left(\frac{n'}{q}\right) \right|, \end{aligned}$$

where $n = dn'$ and $d\nu' \equiv \nu \pmod{8}$. We may apply Lemma 5 to the sums over q and n' to obtain

$$\begin{aligned} S(K, M) & \ll_{\varepsilon} \sum_{s \leq \sqrt{M}} \sum_{\substack{d|s \\ 2 \nmid s}} (MK)^{\varepsilon} \left\{ \left(\frac{M}{s^2}\right)^{7/5} + \left(\frac{M}{s^2}\right)^{3/2} \left(\frac{K}{d}\right)^{-1/4} + \left(\frac{M}{s^2}\right)^{5/4} \left(\frac{K}{d}\right)^{1/4} \right\} \\ & \ll_{\varepsilon} (MK)^{\varepsilon} \{M^{7/5} + M^{3/2}K^{-1/4} + M^{5/4}K^{1/4}\}, \end{aligned}$$

as required.

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