On the Formation of Singularities for the Compressible Euler Equations



Tianrui Bayles-Rea

Keble College

University of Oxford

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Abstract

The purpose of this thesis is to study the phenomenon of singularity formation in large data problems for C^1 solutions to the Cauchy problem of the compressible Euler equations. The classical theory established by Lax [35] shows that, for 2×2 hyperbolic systems, the break-down of C^1 solutions occurs in finite time if initial data contain any compression in some truly nonlinear characteristic field under some additional conditions, which include genuine nonlinearity and the strict positivity of the difference between two corresponding eigenvalues. These strong structural assumptions mean that it is highly non-trivial to apply this theory to archetypal systems of conservation laws, such as the (1+1)-dimensional relativistic Euler equations.

Actually, in the (1+1)–dimensional spacetime setting, if the mass-energy density ρ does not vanish initially at any finite point, the essential difficulty in considering the possible break-down is coming up with a way to obtain sharp enough control on the lower bound for ρ.

In Chapter 2 and Chapter 3, we consider the (1+1)–dimensional isentropic Euler equations in both classical and relativistic settings. We will establish a sufficiently sharp lower bound of ρ, and show Lax's theory indeed holds for these systems. Furthermore, we will give a better lower bound of ρ, at the expense of restricting the class of initial data further.

In Chapter 4 and Chapter 5, we consider the (1+1)–dimensional non-isentropic Euler equations in both classical and relativistic settings. As the non-isentropic Euler equations are 3×3 systems, Lax's theory no longer directly applies. Instead, we will establish a sufficiently sharp lower bound of ρ, and give a sufficient condition for the solution to break down in finite time.

In Chapter 6 and Chapter 7, we consider the multidimensional isentropic Euler equations with spherical symmetry in both classical and relativistic settings. With spherical symmetry, the system will have a singularity at the origin. To bypass this, we consider a specific set of initial data such that the local existence and uniqueness result can be established (see [31]). Same as in the non-isentropic case, we will give a sufficient condition for the solution to break down in finite time.

Contents

1 Introduction **5**

1.1 The mathematical models 5

1.2 An overview of Lax's theory for 2×2 hyperbolic systems 6

1.3 Historical development 11

2 Isentropic Classical Compressible Euler Equations **16**

2.1 Introduction 16

2.2 The main results 17

2.3 Singularity formation in the 2×2 system with far-field vacuum 19

2.3.1 Preliminaries 19

2.3.2 Time-dependent density lower bound for the adiabatic exponent $\gamma \in (1, 3)$ 21

2.3.3 Verification of Lax's theory in 2×2 system: Proof of Theorem 2.2.1 . . . 22

2.4 Time-dependent density lower bound 23

2.4.1 Coupled ODEs 23

2.4.2 Upper bound for gradient quantities $\widetilde{\alpha}$ and $\widetilde{\beta}$ 25

2.4.3 Time-dependent lower bound on the density: Proof of Theorem 2.2.2 . . . 26

3 Isentropic Relativistic Compressible Euler Equations **27**

3.1 Introduction 27

3.2 The main results 29

3.3 Singularity formation in the 2×2 system with far-field vacuum 31

3.3.1 Preliminaries 31

3.3.2 Riccati-type ODE inequality and time-dependent mass-energy density lower bound for the adiabatic exponent $\gamma \in (1,3)$ . . . 32
3.3.3 Verification of Lax's theory in 2×2 system: Proof of Theorem 3.2.1 . . . 35
3.4 Time-dependent mass-energy density lower bound . . . 36
3.4.1 Uniform bound for the velocity and the local sound speed . . . 36
3.4.2 Coupled ODEs . . . 37
3.4.3 Upper bound for gradient quantities $\widetilde{\alpha}$ and $\widetilde{\beta}$ . . . 39
3.4.4 Time-dependent lower bound on the mass-energy density: Proof of Theorem 3.2.2 . . . 40

4 Non-isentropic Classical Euler Equations 42
4.1 Introduction . . . 42
4.2 The main result . . . 44
4.3 PDEs satisfied by the Riemann variables . . . 46
4.4 Uniform bounds on the Riemann variables . . . 48
4.5 Non-homogeneous ODEs of the Riemann variables . . . 55
4.6 Decoupled ODE inequalities . . . 64
4.7 Density lower bound for the adiabatic exponent $\gamma \in (1,3)$ . . . 65
4.8 Singularity formation with strong compression: Proof of Theorem 4.2.1 . . . 69

5 Non-isentropic Relativistic Compressible Euler Equations 71
5.1 A note on the pressure term . . . 71
5.2 Introduction of some important quantities . . . 73
5.3 The main result . . . 75
5.4 PDEs satisfied by the Riemann variables . . . 79
5.5 Uniform bounds on the Riemann variables . . . 80
5.6 Non-homogeneous ODEs of the Riemann variables . . . 87
5.7 Decoupled ODE inequalities . . . 95
5.8 Mass-energy density lower bound for the adiabatic exponent $\gamma \in (1,3)$ . . . 110
5.9 Singularity formation with strong compression: Proof of Theorem 5.3.1 . . . 113

6 Isentropic Classical Compressible Euler Equations with Spherical Symmetry 115

6.1 Introduction . . . 115

6.2 Main results . . . 117

6.3 Density lower bound . . . 120

6.3.1 Coupled ODEs . . . 121

6.3.2 Upper bound for gradient quantities $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widehat{\alpha}$ and $\widehat{\beta}$ . . . 124

6.3.3 Time-dependent lower bound on the density: Proof of Theorem 6.3.1 . . . 130

6.4 Decoupled ODEs . . . 131

6.5 Decoupled ODE inequalities . . . 133

6.6 Singularity formation with strong compression: Proof of Theorem 6.2.2 . . . 134

7 Isentropic Relativistic Compressible Euler Equations with Spherical Symmetry 137

7.1 Introduction . . . 137

7.2 Main results . . . 141

7.3 Local existence and uniform bounds on the Riemann invariants . . . 144

7.4 Density lower bound . . . 153

7.4.1 Coupled ODEs . . . 154

7.4.2 Upper bound for gradient quantities $\widehat{\alpha}$, $\widehat{\beta}$, $\overline{\alpha}$ and $\overline{\beta}$ . . . 157

7.4.3 Time-dependent lower bound on the density: Proof of Theorem 7.4.1 . . . 163

7.5 Decoupled ODEs . . . 164

7.6 ODE comparison . . . 166

7.7 Singularity formation with strong compression: Proof of Theorem 7.2.2 . . . 173

Appendices 176

A On the equation of state for the polytropic gas 177

B A property of continuous functions 180

C The explicit derivation of weight functions: h_1, h_2, h and g 182

Chapter 1

Introduction

1.1 The mathematical models

This thesis is concerned with classical Euler equations and relativistic Euler equations with large data. The compressible Euler equations consist of the conservation laws of mass, momentum, and energy. For a C^1 solution in the Eulerian spatial coordinate, the conservation of energy is equivalent to the entropy equation. They are given by

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho, S) = 0, \\ \partial_t S + \mathbf{u} \nabla S = 0, \end{cases} \tag{1.1.1}$$

where $\rho \geq 0$ denotes the density, $\mathbf{u}$ denotes the velocity, P denotes the pressure and S denotes the entropy. These equations describe inviscid flow in fluid dynamics, they are also applicable as a good approximation to flows with small viscosity. The compressible Euler equations have first appeared in the published article [36] written by Euler, they are one of the first partial differential equations discovered by mathematicians. Till today, the compressible Euler equations are widely used in science and engineering.

Similarly, the relativistic Euler equations are a system of great physical significance, these equations constitute a generalisation of the classical compressible Euler equations which manages

to capture and account for the effects of special relativity. For a C^1 solution in the Eulerian system of coordinates, the Euler equations are given by

$$\begin{cases} \partial_t \left(\dfrac{c^4\rho + P|\mathbf{u}|^2}{c^2(c^2 - |\mathbf{u}|^2)} \right) + \operatorname{div} \left(\dfrac{(c^2\rho + P)\mathbf{u}}{c^2 - |\mathbf{u}|^2} \right) = 0, \\ \partial_t \left(\dfrac{(c^2\rho + P)\mathbf{u}}{c^2 - |\mathbf{u}|^2} \right) + \operatorname{div} \left(\dfrac{(c^2\rho + P)}{c^2 - |\mathbf{u}|^2} \mathbf{u} \otimes \mathbf{u} \right) + \nabla P = 0, \\ \partial_t S + \mathbf{u} \nabla S = 0, \end{cases} \tag{1.1.2}$$

where $\rho \geq 0$ denotes the mass-energy density, $\mathbf{u}$ denotes the (relativistic) velocity, c is a large constant representing the speed of light in appropriate units. Moreover, P denotes the fluid pressure, S denotes the entropy function and $x \in \mathbb{R}, t \geq 0$ denote the Eulerian spatial coordinate and the time coordinate respectively. Be it noted that the formal limit of (1.1.2) as $c \to \infty$ reduces to the classical compressible Euler equations, which is a reflection of the principle that the theory of special relativity, in a regime of low velocities, should produce results consistent and coherent with classical Newtonian theory.

1.2 An overview of Lax's theory for 2×2 hyperbolic systems

We will give an overview of Lax's theory for 2×2 hyperbolic systems [35], and how we can adapt this theory to obtain gradient blow-up results for the compressible Euler equations. We start with the following equations

$$\begin{cases} \partial_t w + \lambda_1 \partial_x w = 0, \\ \partial_t z + \lambda_2 \partial_x z = 0, \end{cases} \tag{1.2.1}$$

where $\lambda_1(w, z)$ and $\lambda_2(w, z)$ are C^1 functions of $w(t, x)$ and $z(t, x)$. We will consider the initial data $(w_0(x), z_0(x)) = (w(0, x), z(0, x)) \in C^1(\mathbb{R})$ with

$$\|(w_0(x), z_0(x))\|_{C^1(\mathbb{R})} < M, \tag{1.2.2}$$

where $M > 0$ is an arbitrary constant. For the simplicity of notations, denote

$$\partial_1 = \partial_t + \lambda_1 \partial_x, \quad \partial_2 = \partial_t + \lambda_2 \partial_x.$$

Given the initial data satisfying (1.2.2), it is well-known that the unique classical solution to the Cauchy problem of (1.2.1) exists in a short time (see [26], [34], [39] etc). Define the characteristics by

$$\frac{dx^i}{dt} = \lambda_i(t, x^i(t)), \quad x^i(0) = x_0^i, \quad i = 1, 2. \tag{1.2.3}$$

By integrating (1.2.1) with respect to the characteristics, one sees that the L^∞ norm of $w(t,x)$ and $z(t,x)$ are bounded by the initial data.

We aim to derive some conditions on the initial data such that the local classical solution breaks down in finite time. To do so, one needs to convert the PDEs (1.2.1) into Riccati-type ODEs. Define

$$\alpha = \partial_x w, \quad \beta = \partial_x z.$$

Differentiating $(1.2.1)_1$ with respect to x, one gets

$$\begin{aligned} &\partial_t \alpha + \partial_x \lambda_1 \alpha + \lambda_1 \alpha^2 = 0 \\ \Longleftrightarrow\ &\partial_1 \alpha + \partial_z \lambda_1 \alpha \beta + \partial_w \lambda_1 \alpha^2 = 0. \end{aligned} \tag{1.2.4}$$

From $(1.2.1)_2$, one can obtain

$$\beta = \frac{\partial_1 z}{\lambda_2 - \lambda_1}.$$

Hence, (1.2.4) becomes

$$\partial_1 \alpha + \partial_z \lambda_1 \alpha \frac{\partial_1 z}{\lambda_2 - \lambda_1} + \partial_w \lambda_1 \alpha^2 = 0. \tag{1.2.5}$$

Define a function h_1 by $\partial_z h_1 = \frac{\partial_z \lambda_1}{\lambda_2 - \lambda_1}$. By (1.2.1), one has

$$\partial_1 h_1 = \partial_w h_1 \partial_1 w + \partial_z h_1 \partial_1 z = \partial_z h_1 \partial_1 z. \tag{1.2.6}$$

By (1.2.6), equation (1.2.5) becomes

$$\partial_1\alpha + \partial_1 h_1\alpha + \partial_w\lambda_1\alpha^2 = 0 \iff \partial_1\left(e^{h_1}\alpha\right) + e^{h_1}\partial_w\lambda_1\alpha^2 = 0. \tag{1.2.7}$$

Define quantity $y = e^{h_1}\alpha$, (1.2.7) becomes

$$\partial_1 y = -e^{h_1}\partial_w\lambda_1 y^2. \tag{1.2.8}$$

By defining functions h_2 and q by $\partial_w h_2 = \frac{\partial_w\lambda_2}{\lambda_1-\lambda_2}$ and $q = e^{h_2}\beta$ respectively, and by doing the same analysis as above expect for equation $(1.2.1)_2$, one can derive

$$\partial_2 q = -e^{h_2}\partial_z\lambda_2 q^2. \tag{1.2.9}$$

Integrating (1.2.8) along the characteristic (1.2.3), one obtains

$$\frac{1}{y(t,x^1(t))} = \frac{1}{y(0,x_0^1)} + \int_0^t e^{-h_1}\partial_w\lambda_1(\sigma,x^1(\sigma))d\sigma. \tag{1.2.10}$$

Assuming that

$$\int_0^t e^{-h_1}\partial_w\lambda_1(\sigma,x^1(\sigma))d\sigma \to \infty \quad \text{as} \quad t\to\infty, \tag{1.2.11}$$

one has the quantity y blows up if and only if there exists some $x_0^1 \in \mathbb{R}$ such that $y(0,x_0^1) < 0$ (this is called compression in the initial data). One can obtain a similar result for q. Note y blows up if and only if $\alpha = \partial_x w$ blows up. This concludes Lax's theory for 2×2 hyperbolic systems.

In practice, one encounters several difficulties when applying Lax's theory to the compressible Euler equations. We will focus on the (1+1)-dimensional isentropic compressible Euler equations first, which is explored in Chapter 2 and Chapter 3. Our aim is to prove the gradient blow-up results stated in Theorem 2.2.1 and Theorem 3.2.1 hold. The PDEs satisfied by the Riemann invariants for both the classical case (2.1.4) and the relativistic case (3.1.6) have the same form as (1.2.1). Hence, by integrating along the characteristics, the Riemann invariants are uniformly bounded by the initial data.

The first difficulty one encounters is to write λ_1 and λ_2 in terms of the Riemann invariants

w and z, which is easy for the classical case but difficult for the relativistic case (see Lemma 3.3.2).

The second difficulty is to give an explicit expression to the functions h_1 and h_2. This is, once again, easy for the classical case, but difficult for the relativistic case (see Lemma 3.3.2).

The third and the last difficulty is to show that assumption (1.2.11) holds. For the isentropic Euler equations, we consider pressure laws satisfying $P(\rho) = k\rho^\gamma$, where $k > 0$ and $\gamma > 1$ are constants. For such pressure laws, one can deduce that

$$e^{-h_1}\partial_w\lambda_1 \approx e^{-h_2}\partial_z\lambda_w \approx \rho^{\frac{3-\gamma}{4}},$$

where the notation $\approx$ means the above quantities differ by a multiplication of a positive constant. Now to show assumption (1.2.11) holds, we have to consider two cases:

Case 1: $\gamma \geq 3$. As the Riemann invariants are uniformly bounded, one can deduce that ρ has a uniform upper bound. Hence (1.2.11) is satisfied for $\gamma \geq 3$.

Case 2: $1 < \gamma < 3$. To show (1.2.11) is satisfied, one has to derive a sharp enough lower bound for the density. This is done by Lemma 2.3.3 for the classical case, and by Lemma 3.3.6 for the relativistic case.

In fact, Theorem 2.2.2 and Theorem 3.2.2 give a better time-dependent density lower bound in comparison to Lemma 2.3.3 and Lemma 3.3.6 respectively. However, Theorem 3.2.2 requires the initial data satisfying some additional assumptions, such that the absolute value of velocity $|u|$ and the local sound speed $\sqrt{P'}$ are bounded from above by ϵc, where ϵ is a small positive constant and c is the speed of light. Whereas Lemma 3.3.6 only requires $|u|$ and $\sqrt{P'}$ to be bounded from above by c.

We will now focus on the (1+1)-dimensional non-isentropic compressible Euler equations, which is explored in Chapter 4 and Chapter 5. Our aim is to prove the gradient blow-up results stated in Theorem 4.2.1 and Theorem 5.3.1 hold. Note the non-isentropic compressible Euler equations are 3×3 systems, so the PDEs satisfied by the Riemann invariants, (4.3.5) for the classical case and (5.4.1) for the relativistic case, no longer take the same form as (1.2.1). Due to this, we shall call them Riemann variables instead.

The first difficulty is to find uniform bounds for the Riemann variables, as simply integrating

along the characteristics will no longer yield the desired result. In fact, one has to use some complex ODE arguments with Gronwall's inequality. This is done by Lemma 4.4.1 for the classical case and Lemma 5.5.1 for the relativistic case.

The second difficulty is to find an explicit expression for the pressure term of a polytropic ideal gas. For the classical case, one can take $P(\rho, S) = k\rho^{\gamma}e^{\mu S}$, where $k > 0$, $\gamma > 1$ and $\mu > 0$. However, for the relativistic case, the pressure term can no longer be explicitly expressed as a function of ρ and S (see section 5.1).

The third difficulty is that the decoupled Riccati type ODEs for the non-isentropic compressible Euler equations (see Theorem 4.5.1 and Theorem 5.6.1) are no longer taking the same form as equation (1.2.8) and (1.2.9), so to prove the gradient blow-up result, one first need to bound the ratio of certain coefficients in the Riccati type ODEs. Furthermore, one can no longer directly integrate them as done in (1.2.10). As a consequence, an if and only if condition for the gradient blow-up can no longer be obtained, instead, one finds a sufficient condition as stated in Theorem 4.2.1 and Theorem 5.3.1.

In addition to the above three difficulties, all the difficulties from the isentropic case carry over as well.

Lastly, we will focus on the multi-dimensional isentropic compressible Euler equations with spherical symmetry, which is explored in Chapter 6 and Chapter 7. Our aim is to prove the gradient blow-up result stated in Theorem 6.2.2 and Theorem 7.2.2 hold.

There are two main difficulties. The first difficulty is the PDEs satisfied by the Riemann variables (6.1.6) and (7.1.9) have a singularity at the origin $r = 0$, where r denotes the radius. Due to this, the classical local existence and uniqueness theory for the hyperbolic system of conservation laws no longer applies. To bypass this difficulty, one way is to carefully choose the initial data, such that the singularity at the origin vanishes. For the classical Euler equations, this has been done in a paper by Hang-Chen-Wang [31], where they established the local existence and uniqueness of C^1 solutions for a class of initial data when $1 < \gamma \leq 3$, they also obtained a uniform bound on the Riemann variables. For the relativistic Euler equations, based on the ideas from [31], we present a similar local existence result.

The second difficulty is to obtain a sharp enough lower bound on the density. The method used to obtain such a lower bound in the (1+1)-dimensional case no longer works, due to the

appearance of singularity at the origin $r = 0$. In order to obtain the desired lower bound, one has to consider some finer structural properties of some ODEs associated with the system, this has been done in section 6.3 and section 7.4.

1.3 Historical development

We will firstly consider the one spatial dimension case. (1.1.1) and (1.1.2) are physical models for hyperbolic conservation laws

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0\,, \tag{1.3.1}$$

where $\mathbf{u} = \mathbf{u}(t,x) \in \mathbb{R}^n$ is the unknown vector and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear flux. It is well-known that if initial data are smooth enough, then the unique classical solution to the Cauchy problem of (1.3.1) exists in a short time (see [26], [34], [39] etc.). A typical phenomenon observed in such systems is the development of singularities, i.e. shock wave, no matter how small and smooth the data are. This has been justified in a series of seminal works by Lax [35] in 1964 for general 2×2 systems, and by John [33], Liu [44] and Li-Zhou-Kong [40, 41] for general $n \times n$ systems. These results confirm that, for the Cauchy problem of general strictly hyperbolic systems, the initial compression[1] in any truly nonlinear (not weakly linearly degenerate [40]) characteristic field develops singularity in finite time.

Particularly, in the last section, one finds that, for the Cauchy problem of 2×2 genuinely nonlinear, strictly hyperbolic systems with large data, if we additionally assume that (1.2.11) holds, then

$$\begin{aligned} &\textit{the existence of compression in the initial data} \\ \Longleftrightarrow\ &\textit{formation of singularities in finite time.} \end{aligned} \tag{1.3.2}$$

As mentioned before, for the 1-D isentropic compressible Euler equations with the pressure law $P(\rho) = \rho^\gamma$ ($\gamma > 1$ is a fixed constant), to apply Lax's theory, one crucial point is to obtain sharp lower bound estimates on the density. In fact, the research on this issue has made some solid progress. For purely rarefactive piecewise Lipschitz continuous solutions, Lin [43] proved

[1] An initial tendency of the characteristic curves propagating the solution to cross each other, defined rigorously through the negativity of certain Riemann invariants' spatial derivatives.

that the density has a $O(1+t)^{-1}$ order lower bound via a polygonal scheme. For general smooth solutions including compressions, in Chen-Pan-Zhu [9], via considering the system in Lagrangian coordinates and making use of the simple structure of the P-system, crucially the continuity equation

$$(1/\rho)_t - u_x = 0,$$

they found a lower bound $O(1+t)^{\frac{4}{\gamma-3}}$ for the density when $1 < \gamma < 3$, which has been further extended in [9] to non-isentropic flow. Recently in Chen-Pan-Zhu [10], they showed that the lower bound $O(1+t)^{-1}$ obtained in [43] actually holds for any Lipschitz continuous solution. Based on these lower bound estimates on the density, in [9] and Chen-Chen-Zhu [8], the theory obtained by Lax in 1964 has been verified in this important physical model: for any C^1 solution with large data to the Cauchy problem of the 1-D isentropic compressible Euler equations, (1.3.2) holds. In [9], $\rho(0,x)$ is assumed to be strictly positive, and in [8], $\rho(0,x)$ is still required to be positive but is allowed to vanish in the far field.

For the 1-D non-isentropic compressible Euler equations. Chen-Frid-Li [14] established uniqueness and L^1 stability for the Riemann problem under arbitrarily large $L^1 \cap L^\infty \cap BV_{loc}$ perturbation of the Riemann initial data. For singularity formation with arbitrarily large C^1 initial data, as the equations written in Lagrangian coordinates possess a linearly degenerate characteristic field, the equivalent relationship (1.3.2) is no longer obtained, instead, authors in [8, 9] insisted that the compression along a genuinely nonlinear characteristic field should be strong enough.

For the (1+1)-D relativistic Euler equations, some progress on the singularity formation and global well-posedness of solutions with large data have also been obtained. Under the assumption that $P = \sigma^2\rho$ for some positive constant $\sigma < c$, Smoller-Temple established in their seminal work [53] the existence of a global BV weak solution to the Cauchy problem, by crucially noticing that the shock curves for the (1+1)-D system satisfy very strong geometric properties. Frid-Perepelitsa [27] proved the global existence of a spatially periodic entropy solution, for any given periodic initial data of locally bounded total variation and strictly positive initial density, and later, the global L^1 stability of such solutions have been established by Calvo-Colombo-Frid in [6]. Furthermore, Chen-Schrecker [16] proved the global existence of entropy solutions, as

well as their convergence to entropy solutions to the classical Euler equations as the speed of light tends to infinity. For a general pressure law $P(\rho)$ satisfying $P'(\rho) > 0$ and $P''(\rho) > 0$, Chen [17] solved the corresponding Riemann problem. Later, by considering specific pressure law associated with neutron stars, Hsu-Lin-Markino [32] established the existence of global L^∞ weak solutions with initial data containing the vacuum state, by using the compactness framework developed in a series of work by Ding-Chen-Lou [12, 24, 25]. Ruan-Zhu [51] firstly proved the global well-posedness of C^1 solutions with large data to the Cauchy problem of the isentropic flow, if the initial data do not contain compression. For the multi-dimensional case, some results on singularity formation along the spirit of Sideris in 1985 have been obtained by Guo-Tahvildar-Zadeh [30], Pan–Smoller [47], Rendall [49] and so on. In a very recent paper Athanasiou-Zhu [1], via introducing an elaborate argument on a particular ODE inequality and some crucial artificial quantity, they can obtain a time-dependent lower bound $O(1+t)^{\frac{4}{\gamma-3}}$ of the mass-energy density for the (1+1)-D isentropic relativistic Euler equations. Ultimately, for C^1 solutions with large data and uniformly positive initial mass-energy density, in [1] one finds that (1.3.2) still holds. Some other interesting results can also be found in Oliynyk [45, 46].

For spatial dimension higher than one, in 1985, Sideris published a remarkable paper on the development of singularities in three-dimensional compressible Euler equations [52], proving that the classical solution has to blow-up in finite time (see also Rammaha [48] for two-dimensional case) under some initial assumptions. His proof was based on studying certain averaged quantities formed out of the solution, showing that they satisfy some differential inequalities whose solutions have finite life span. The idea is that by using the averaged quantities, one is able to avoid local analysis of solutions, which is essentially different from the characteristic arguments used in [8, 9, 11, 33, 35, 44]. For two or three-dimensional isentropic compressible Euler equations with spherical symmetry and the pressure law satisfying $P(\rho) = \rho^\gamma$ ($\gamma > 1$ is a fixed constant), in 1997, Chen [13] established the global existence of entropy solutions in $\mathbb{R}^2$. Later in 2003, Chen-Li [15] established the global existence of entropy solutions for the Euler and the Euler-Poisson equations. In both papers they considered large initial data in L^∞. Via characteristic arguments, Hang-Chen-Wang [31] established the local existence of C^1 solutions for a class of initial data, and obtained a sufficient condition for singularity formation in finite time.

Once one obtains finite time gradient blow-up type results, it is natural to wonder if a

shock wave will propagate from the blow-up point. Some concrete progress has been made in this direction. For the p-system with one constant Riemann invariant, by carefully analysing the differential equation associated to the Rankine–Hugoniot condition, Lebaud [37] successfully constructed a shock curve from the blow-up point in 1994. Later in 2001, Chen-Dong [19] extended this result to the p-system with non-constant Riemann invariants. For spherically symmetric Euler equations, when the initial data is small, Yin not only demonstrated that smooth solutions can blow-up in finite time [55], he was also able to construct a shock curve from the blow-up point [56]. More analysis on the blow-up mechanism can also be found in Christodoulou [21], Christodoulou-Miao [22], Chen-Xin-Yin [20], Chen-Dong [19] and Buckmaster-Shkoller-Vicol [3–5].

In this thesis, we will show that the singularity formation results and the density lower bound obtained for the p-system in the paper by Chen-Chen-Zhu [8] and the paper by Hang-Chen-Wang [31], hold also for the compressible Euler equations in the Eulerian coordinates. Furthermore, we will address the following four issues for the relativistic Euler equations, which are original works by the author of this thesis:

(i). ***Verification of*** (1.3.2) ***for the (1+1)-D isentropic relativistic Euler equations with far-filed vacuum.*** The theory obtained by Lax in 1964 will be verified in this important physical model: for any C^1 solution to the Cauchy problem of the (1+1)-D isentropic relativistic Euler equations with large data and possible far field vacuum, (1.3.2) holds. Compared with the initial assumption in [1], now one can allow the initial density profiles $\rho_0 \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $\lim_{x\to\infty} \rho_0(x) = 0$ and $\lim_{x\to-\infty} \rho_0(x) = 0$. Generally, vacuum will appear in the far field under some physical requirements such as finite total mass and finite total energy in $\mathbb{R}$.

(ii). $O(1+t)^{-1+\epsilon}$ ***mass-energy density lower bound for the (1+1)-D isentropic relativistic Euler equations.*** For any C^1 solutions to the Cauchy problem of the (1+1)-D isentropic relativistic Euler equations, under certain additional assumptions on the initial data ensuring that the relativistic velocity u and the local sound speed $\sqrt{P'}$ are small in comparison to the light speed c, one can find some $\epsilon > 0$ characterising this "smallness" and obtain $O(1+t)^{-1+\epsilon}$ lower bound for the mass-energy density. Formally, if $\epsilon = O(\frac{1}{c})$ and let $c \to \infty$,

one retrieves the $O(1+t)^{-1}$ density lower bound for the classical Euler equations.

(iii). ***The development of singularities for the (1+1)-D non-isentropic relativistic Euler equations with strong compression.*** For any C^1 solutions to the Cauchy problem of the (1+1)-D non-isentropic relativistic Euler equations with large data and strictly positive initial mass–energy density, we exhibit a numerical value N, thought of as representing the strength of an initial compression, above which all initial data lead to a finite-time singularity formation. Compared to the isentropic case, a further difficulty is encountered: one needs to establish the upper bounds of the Riemann variables in the slab of existence. It is no longer as straightforward as that of the isentropic case, since the constructed Riemann variables satisfy some inhomogeneous ODEs along characteristics. To rectify this, an elaborate ODE argument was introduced in our paper, which, together with the Gronwall inequality, yields the the desired boundedness.

This is a joint result with Dr. Athanasiou and Dr. Zhu. The resulting paper has been accepted for publication by the Transactions of the American Mathematical Society [2].

(iv). ***The development of singularities for the multi-dimensional isentropic relativistic Euler equations with strong compression under spherical symmetry.*** By considering a specific set of initial data, we establish local existence and uniqueness of C^1 solutions to the Cauchy problem for the two and three dimensional isentropic relativistic Euler equations with spherical symmetry. We exhibit a blow-up time T, and a numerical value G_T dependent on T, thought of as representing the strength of an initial compression, above which all initial data lead to a singularity formation before the blow-up time T.

Chapter 2

Isentropic Classical Compressible Euler Equations

2.1 Introduction

The (1+1)-dimensional compressible Euler equations in the Eulerian coordinate system are given by

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0)(x) \quad \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{2.1.1}$$

Here, we consider the pressure law

$$P = k^2 \rho^\gamma, \quad k > 0, \quad \gamma > 1,$$

for some constants $k > 0$ and $\gamma > 1$.

We proceed with a rudimentary analysis of the Cauchy problem (2.1.1). First, the two eigenvalues λ_1 and λ_2 of equations $(2.1.1)_1$-$(2.1.1)_2$ can be given by

$$\lambda_1 = u - \sqrt{P'}, \quad \lambda_2 = u + \sqrt{P'}. \tag{2.1.2}$$

We denote the directional derivatives as

$$\partial_1 = \partial_t + \lambda_1 \partial_x, \quad \partial_2 = \partial_t + \lambda_2 \partial_x,$$

along two characteristic directions $x^1(t, x_0^1)$ and $x^2(t, x_0^2)$, defined by

$$\frac{dx^1}{dt} = \lambda_1, \quad \frac{dx^2}{dt} = \lambda_2,$$

where $x^1(0, x_0^1) = x_0^1$ and $x^2(0, x_0^2) = x_0^2$. For simplicity we will abbreviate $x^1(t, x_0^1)$ and $x^2(t, x_0^2)$ as $x^1(t)$ and $x^2(t)$. Introduce the corresponding Riemann invariants

$$z = u + \frac{2\sqrt{P'}}{\gamma - 1}, \quad w = u - \frac{2\sqrt{P'}}{\gamma - 1}. \tag{2.1.3}$$

By explicit calculations, one can verify that

$$\partial_2 z = 0 \quad \text{and} \quad \partial_1 w = 0. \tag{2.1.4}$$

Let $\epsilon > 0$ be a small constant, denote

$$\alpha \coloneqq \partial_x w, \quad \beta \coloneqq \partial_x z, \quad \widetilde{\alpha} = \frac{\alpha}{\rho^{1-\epsilon}}, \quad \widetilde{\beta} = \frac{\beta}{\rho^{1-\epsilon}}. \tag{2.1.5}$$

Lastly, we define two important quantities ξ and ζ, which play important roles in our analysis on singularity formation.

$$\xi = \rho^{\frac{\gamma-3}{4}} \alpha, \quad \zeta = \rho^{\frac{\gamma-3}{4}} \beta. \tag{2.1.6}$$

2.2 The main results

First, define the C^1 solutions as follows:

Definition 2.2.1. *Let $T > 0$ be some time. The pair $(\rho(t,x), u(t,x))$ is called a C^1 solution to the equations $(2.1.1)_1$-$(2.1.1)_2$ on $(0,T) \times \mathbb{R}$ if*

$$\rho > 0, \quad \rho \in C^1([0,T) \times \mathbb{R}), \quad u \in C^1([0,T) \times \mathbb{R}),$$

and the equations $(2.1.1)_1$-$(2.1.1)_2$ *are satisfied in the pointwise sense on* $(0,T)\times\mathbb{R}$. *It is called a* C^1 *solution to the Cauchy problem* (2.1.1) *if it is a* C^1 *solution to the equations* $(2.1.1)_1$-$(2.1.1)_2$ *on* $(0,T)\times\mathbb{R}$ *and admits the initial data* $(2.1.1)_3$ *continuously.*

Assumption 2.2.1. *Denote* $(w_0,z_0)(x)=(w(0,x),z(0,x))$. *Assume* $(w_0,\ z_0)\in C^1(\mathbb{R})$, *and*

$$z_0-w_0>0,\quad \|(z_0,w_0)\|_{C^1(\mathbb{R})}\le M_0,$$

for some constant $M_0>0$.

Assumption 2.2.2. *For* (w_0,z_0) *satisfying Assumption* 2.2.1, *we assume additionally that*

$$\max\left\{\sup_x\widetilde{\alpha}_0(x),\ \sup_x\widetilde{\beta}_0(x)\right\}<K,\quad \inf_x(z_0-w_0)>0,$$

for some constant $K>0$.

Remark 2.2.1. *It is well-known that under Assumption* 2.2.1, *the local-in-time existence of the unique* C^1 *solution* (ρ,u) *to the Cauchy problem* (2.1.1) *can be established by the classical characteristic method; c.f. Theorem 4.1 on p. 66 of* [39], *or see also* [23, 26, 38].

Since now, assume Assumption 2.2.1 holds, and let T denote the life span of the C^1 solution to the Cauchy problem (2.1.1). All the conclusions are proved under the assumption that $t<T$.

The first result of this chapter gives an if and only if condition for singularities to form in finite time. This result was proved for the p-system in [9].

Theorem 2.2.1. *If the initial data* (ρ_0,u_0) *satisfy Assumption* 2.2.1, *then the Cauchy problem* (2.1.1) *has a unique global-in-time* C^1 *solution if and only if*

$$min\left\{\inf_{x\in\mathbb{R}}\xi(0,x),\ \inf_{x\in\mathbb{R}}\zeta(0,x)\right\}\ge 0.$$

The second result of this chapter gives a time-dependent lower bound on the mass-energy density. This result was proved for the p-system in [7].

Theorem 2.2.2. *Let the initial data* (ρ_0,u_0) *satisfy Assumption* 2.2.1 *and Assumption* 2.2.2.

For any $0 < \epsilon < \frac{1}{2}$*, one has*

$$\rho(t,x)^{1-\epsilon} \geq \frac{D}{1+t}, \tag{2.2.1}$$

where $D > 0$ *is a constant depending only on the initial data,* γ*,* k *and* K*.*

Remark 2.2.2. *By explicit calculation, one has*

$$z_0 - w_0 > 0 \iff \rho_0 > 0.$$

Assumption 2.2.1 *allows far-field vacuum, i.e., one can have that*

$$\rho_0(x) \to 0, \quad as \quad |x| \to \infty.$$

Remark 2.2.3. *Theorem* 2.2.2 *holds for* $\epsilon = 0$*, however, in the next chapter, when we consider the relativistic system,* $\epsilon > 0$ *is necessary. Here we leave* $\epsilon > 0$*, so one can compare the proof of Theorem* 2.2.2 *with the proof of Theorem* 3.2.2*.*

2.3 Singularity formation in the 2×2 system with far-field vacuum

This section will be devoted to proving Theorem 2.2.1. All results are proven under the assumption that the initial data (ρ_0, u_0) satisfy Assumption 2.2.1. For simplicity, in this section, we denote by $C > 0$ a generic constant depending only on (z_0, w_0), k, γ and independent of the time.

2.3.1 Preliminaries

Proposition 2.3.1. *One has*

$$\rho < C, \quad |u| < C.$$

Proof. By (2.1.4), one has

$$w(t, x^1(t)) = w_0(x_0^1), \quad z(t, x^2(t)) = z_0(x_0^2).$$

Furthermore, by explicit calculations, one has

$$\sqrt{P'} = \frac{\gamma - 1}{4}(z - w), \quad u = \frac{z + w}{2}.$$

This complete the proof. □

Lemma 2.3.1. *The following Riccati ODEs hold:*

$$\partial_1 \xi = -\frac{\gamma + 1}{4}\rho^{\frac{3-\gamma}{4}}\xi^2, \ \ \partial_2 \zeta = -\frac{\gamma + 1}{4}\rho^{\frac{3-\gamma}{4}}\zeta^2. \tag{2.3.1}$$

Moreover, one has

$$\begin{aligned} \frac{1}{\xi(t, x^1(t, x_0^1))} =& \frac{1}{\xi(0, x_0^1)} + \frac{\gamma + 1}{4}\int_0^t \rho^{\frac{3-\gamma}{4}}(\sigma, x^1(\sigma, x_0^1))\, d\sigma, \\ \frac{1}{\zeta(t, x^2(t, x_0^2))} =& \frac{1}{\zeta(0, x_0^2)} + \frac{\gamma + 1}{4}\int_0^t \rho^{\frac{3-\gamma}{4}}(\sigma, x^2(\sigma, x_0^2))\, d\sigma. \end{aligned} \tag{2.3.2}$$

Proof. We will show the Lemma holds for ξ, as the proof for ζ will be the same. Differentiating $\partial_1 w = 0$ with respect to x yields

$$\begin{aligned} &\partial_{tx} w + \lambda_1 \partial_{xx} w + \partial_z \lambda_1 \partial_x w \partial_x z + \partial_w \lambda_1 (\partial_x w)^2 = 0, \\ \iff\ &\partial_1 \alpha + \partial_z \lambda_1 \alpha\beta + \partial_w \lambda_1 \alpha^2 = 0. \end{aligned} \tag{2.3.3}$$

By (2.1.4), one has

$$\begin{aligned} \partial_t z + \lambda_2 \partial_x z = 0 \implies& \partial_t z + \lambda_1 \partial_x z + (\lambda_2 - \lambda_1)\partial_x z = 0 \\ \implies& \beta = \frac{\partial_1 z}{\lambda_1 - \lambda_2}. \end{aligned}$$

Hence (2.3.3) becomes

$$\partial_1 \alpha + \alpha \frac{\partial_z \lambda_1 \partial_1 z}{\lambda_1 - \lambda_2} + \partial_w \lambda_1 \alpha^2 = 0. \tag{2.3.4}$$

By explicit computation, one can check that

$$\lambda_1 = \frac{z + w}{2} - \frac{\gamma - 1}{4}(z - w), \quad \partial_z \lambda_1 = \frac{3 - \gamma}{4}, \quad \partial_w \lambda_1 = \frac{\gamma + 1}{4}.$$

Furthermore, define a function h by

$$h \coloneqq \frac{\gamma-3}{2(\gamma-1)}\ln(z-w).$$

By (2.1.4), one has

$$\partial_1 h = \frac{\partial_z \lambda_1 \partial_1 z}{\lambda_1 - \lambda_2}.$$

Hence (2.3.4) becomes

$$\begin{aligned} &\partial_1 \alpha + \alpha \partial_1 h + \frac{\gamma+1}{4}\alpha^2 = 0, \\ \Longrightarrow\ &\partial_1(e^h \alpha) + \frac{\gamma+1}{4} e^h \alpha^2 = 0. \end{aligned}$$

By recalling the expression of ξ given by (2.1.6), one arrives at (2.3.1). By integrating (2.3.1) along the $x^1(t)$ characteristics, one gets (2.3.2).

□

2.3.2 Time-dependent density lower bound for the adiabatic exponent $\gamma \in (1,3)$

Lemma 2.3.2.

$$\xi(t,x) \leq Q_1, \quad \zeta(t,x) \leq Q_2, \tag{2.3.5}$$

where

$$Q_1 \coloneqq max\left\{0, \sup_x \xi(0,x)\right\}, \quad Q_2 \coloneqq max\left\{0, \sup_x \zeta(0,x)\right\}. \tag{2.3.6}$$

Proof. This lemma follows directly from (2.3.2) and the fact that $\rho > 0$. □

Lemma 2.3.3. *Given $1 < \gamma < 3$, one has*

$$\rho^{\frac{3-\gamma}{4}}(t, x^1(t)) \geq \frac{1}{\rho_0^{\frac{\gamma-3}{4}}(x_0^1) + \frac{4}{3-\gamma}Q_1 t}, \quad \rho^{\frac{3-\gamma}{4}}(t, x^2(t)) \geq \frac{1}{\rho_0^{\frac{\gamma-3}{4}}(x_0^2) + \frac{4}{3-\gamma}Q_2 t}. \tag{2.3.7}$$

Proof. We will only prove the result for $(2.3.7)_1$. By (2.1.4) and (2.1.6), one has

$$\partial_1(z-w) = \partial_1 z = \partial_2 z + (\lambda_1 - \lambda_2)\partial_x z = -2\sqrt{P'}\rho^{\frac{3-\gamma}{4}}\xi.$$

By viewing $z-w$ as a function of ρ, one has

$$\partial_1(z-w) = 2k\rho^{\frac{\gamma-3}{2}}\partial_1\rho.$$

Hence by Lemma 2.3.2 and $1<\gamma<3$, one has

$$\begin{aligned} 2k\rho^{\frac{\gamma-3}{2}}\partial_1\rho = -2\sqrt{P'}\rho^{\frac{3-\gamma}{4}}\xi, &\implies \rho^{\frac{\gamma-7}{4}}\partial_1\rho = -\xi \\ &\implies \frac{4}{\gamma-3}\partial_1\rho^{\frac{\gamma-3}{4}} \geq -Q_1 \\ &\implies \rho^{\frac{\gamma-3}{4}} \leq \rho_0^{\frac{\gamma-3}{4}}(x_0^1) + \frac{4}{3-\gamma}Q_1 t. \end{aligned}$$

This is the desired result. □

2.3.3 Verification of Lax's theory in 2×2 system: Proof of Theorem 2.2.1

Without loss of generality, assume that there exists a point x_* such that $\xi(0,x_*)<0$. We shall prove that a singularity forms in finite time by contradiction. Indeed, assume otherwise, $T^*=\infty$. It follows from Lemma 2.3.2 that

$$\xi(t,x^1(t,x_*)) = \frac{\xi(0,x_*)}{1+\xi(0,x_*)\frac{\gamma+1}{4}\int_0^t \rho^{\frac{3-\gamma}{4}}(\sigma,x^1(\sigma,x_*))\,\mathrm{d}\sigma}. \tag{2.3.8}$$

There are two cases to consider: the case $\gamma\geq 3$ and the case $1<\gamma<3$.

Case 1: $\gamma\geq 3$. By Proposition 2.3.1, one obtains

$$\int_0^t \rho^{\frac{3-\gamma}{4}}(\sigma,x^1(\sigma,x_*))\,\mathrm{d}\sigma \geq C^{\frac{3-\gamma}{4}}t.$$

Case 2: $1<\gamma<3$. By Lemma 2.3.3, one obtains

$$\int_0^t \rho^{\frac{3-\gamma}{4}}(\sigma, x^1(\sigma, x_*))\,\mathrm{d}\sigma \geq \frac{3-\gamma}{4Q_1}\log\left(\rho_0^{\frac{\gamma-3}{4}}(x_0^1) + \frac{4}{3-\gamma}Q_1 t\right).$$

In both cases, the contradiction now follows from the fact that $\xi(0, x_*) < 0$ and the fact

$$\int_0^t \rho^{\frac{3-\gamma}{4}}(\sigma, x^1(\sigma, x_*))\,\mathrm{d}\sigma \to \infty \quad \text{as} \quad t \to \infty.$$

Second, for the other direction, assume that there exists no compression in the initial data. In that case, the boundedness of the Riemann invariants follows as usual through their constancy along characteristics, and moreover, equation (2.3.8) implies that the L^∞ norm of $\partial_x w$ remains finite for all time. Hence, at every finite time T, one can invoke the local existence result, as the C^1 norm remains bounded. The solution, therefore, exists for all time and the result follows.

2.4 Time-dependent density lower bound

This section will be devoted to proving Theorem 2.2.2. All results are proven under the assumption that the initial data (ρ_0, u_0) satisfy Assumption 2.2.1 and Assumption 2.2.2.

2.4.1 Coupled ODEs

Lemma 2.4.1. *α and β satisfy*

$$\begin{cases} \partial_1 \alpha = -\frac{1}{2}\alpha(\alpha+\beta) - \frac{\gamma-1}{4}\alpha(\alpha-\beta), \\ \\ \partial_2 \beta = -\frac{1}{2}\beta(\alpha+\beta) - \frac{\gamma-1}{4}\beta(\beta-\alpha). \end{cases}$$

Proof. We will only prove the lemma for $\partial_1\alpha$, as the proof for $\partial_2\beta$ will be similar. Differentiating the equation $\partial_1 w = 0$ with respect to x, one gets

$$\begin{aligned} \partial_{tx} w + \partial_x \lambda_1 \partial_x w + \lambda_1 \partial_{xx} w = 0 &\iff \partial_t \alpha + \lambda_1 \partial_x \alpha + \partial_x \lambda_1 \alpha = 0 \\ &\iff \partial_1 \alpha + \partial_x \lambda_1 \alpha = 0 \\ &\iff \partial_1 \alpha + \partial_z \lambda_1 \partial_x z \alpha + \partial_w \lambda_1 \partial_x w \alpha = 0 \\ &\iff \partial_1 \alpha + \partial_z \lambda_1 \beta \alpha + \partial_w \lambda_1 \alpha^2 = 0. \end{aligned} \tag{2.4.1}$$

By explicit calculations, one has

$$\partial_w \lambda_1 = \frac{1}{2} + \frac{\gamma - 1}{4}, \quad \partial_z \lambda_1 = \frac{1}{2} - \frac{\gamma - 1}{4}. \tag{2.4.2}$$

Now substitute (2.4.2) into (2.4.1) and simplify, one will obtain the desired result. □

Lemma 2.4.2. *One has*

$$\begin{cases} \partial_1 \left(\dfrac{1}{\rho^{1-\epsilon}} \right) = \dfrac{(1-\epsilon)}{\rho^{1-\epsilon}} \beta, \\ \partial_2 \left(\dfrac{1}{\rho^{1-\epsilon}} \right) = \dfrac{(1-\epsilon)}{\rho^{1-\epsilon}} \alpha. \end{cases}$$

Proof. One has

$$\partial_1 \left(\frac{1}{\rho^{1-\epsilon}} \right) = -\frac{(1-\epsilon)\partial_1 \rho}{\rho^{2-\epsilon}}. \tag{2.4.3}$$

By (2.1.4) and (2.1.5), one has

$$\partial_1 \rho = \partial_{\frac{z-w}{2}} \rho \partial_1 \left(\frac{z-w}{2} \right) = -\frac{\rho}{\sqrt{P'}} \frac{\partial_1 z}{2} = -\frac{\rho}{\sqrt{P'}} \frac{(\lambda_2 - \lambda_1)\beta}{2} = -\rho\beta.$$

Substituting this into (2.4.3), one obtains the desired result. □

Theorem 2.4.1. *Define the following quantities*

$$\begin{cases} a_0 \coloneqq \dfrac{1}{2} + \dfrac{\gamma - 1}{4}, \\ a_1 \coloneqq \dfrac{1}{2} + \dfrac{\gamma - 1}{4} - \epsilon. \end{cases}$$

Then $\widetilde{\alpha}$ and $\widetilde{\beta}$ satisfy

$$\begin{cases} \partial_1 \widetilde{\alpha} = -\rho^{1-\epsilon} \widetilde{\alpha}(a_0 \widetilde{\alpha} - a_1 \widetilde{\beta}), \\ \partial_2 \widetilde{\beta} = -\rho^{1-\epsilon} \widetilde{\beta}(a_0 \widetilde{\beta} - a_1 \widetilde{\alpha}). \end{cases} \tag{2.4.4}$$

Furthermore, for any $0 < \epsilon < \frac{1}{2}$, one has $a_0 > a_1 > 0$.

Proof. Combining the above two Lemmas and by recalling the definition of $\widetilde{\alpha}$ and $\widetilde{\beta}$ given by (2.1.5), one immediately obtains the desired result.

□

2.4.2 Upper bound for gradient quantities $\widetilde{\alpha}$ and $\widetilde{\beta}$

Lemma 2.4.3.

$$\max\{\widetilde{\alpha}(t,x), \widetilde{\beta}(t,x)\} < K,$$

where $K > 0$ is the constant in Assumption 2.2.2.

Proof. We will prove the lemma by contradiction. Take the first time t such that there exists some $x \in \mathbb{R}$ where

$$\max\{\widetilde{\alpha}(t,x), \widetilde{\beta}(t,x)\} = K.$$

Without any loss of generality, assume that $\widetilde{\alpha}(t,x) = K$ and $\widetilde{\beta}(t,x) \leq K$. By Theorem 2.4.1, one has

$$(a_0\widetilde{\alpha} - a_1\widetilde{\beta})(t,x) > 0.$$

Now consider the x^1 characteristic passing through the point (t,x). By continuity, one can find a t_1 where $t_1 < t$, and for $l \in (t_1, t]$, one has

$$\widetilde{\alpha}(l, x^1(l)) > 0 \quad \text{and} \quad (a_0\widetilde{\alpha} - a_1\widetilde{\beta})(l, x^1(l)) > 0.$$

Hence, for $l \in (t_1, t]$, by Theorem 2.4.1, one has

$$\partial_1\widetilde{\alpha} = -\rho^{1-\epsilon}\widetilde{\alpha}(a_0\widetilde{\alpha} - a_1\widetilde{\beta}) < 0.$$

Integrating the above from t_1 to t, one gets

$$K = \widetilde{\alpha}(t,x) < \widetilde{\alpha}(t_1, x^1(t_1)) < K, \tag{2.4.5}$$

a contradiction. □

2.4.3 Time-dependent lower bound on the density: Proof of Theorem 2.2.2

For this subsection, fix any $(t,x) \in (0,T) \times \mathbb{R}$, we will consider the x^1 characteristic passing through the point (t,x). By (2.1.4), one has

$$\partial_1 \left(\frac{z-w}{2} \right) = \frac{\partial_1 z}{2} = \frac{\lambda_1 - \lambda_2}{2} \beta = -\sqrt{P'}\beta.$$

By viewing $z-w$ as a function of ρ, one has

$$\partial_1 \left(\frac{z-w}{2} \right) = \partial_\rho \left(\frac{z-w}{2} \right) \partial_1 \rho = \frac{\sqrt{P'}}{\rho} \partial_1 \rho.$$

Hence by (2.1.5) and Lemma 2.4.3, one has

$$\partial_1 \rho = -\widetilde{\beta} \rho^{2-\epsilon} \geq -K \rho^{2-\epsilon}.$$

Integrating the above equation, one gets

$$\frac{1}{\rho^{1-\epsilon}(t,x)} < \frac{1}{\rho_0^{1-\epsilon}(x_0^1)} + (1-\epsilon)Kt < \frac{1}{\inf_x \rho_0^{1-\epsilon}} + (1-\epsilon)Kt. \tag{2.4.6}$$

This is the desired result.

Chapter 3

Isentropic Relativistic Compressible Euler Equations

3.1 Introduction

We consider the Cauchy problem of the (1+1)-D isentropic relativistic Euler equations, which, in Eulerian coordinates, is given by

$$\begin{cases} \partial_t \left(\dfrac{c^4\rho + Pu^2}{c^2(c^2-u^2)} \right) + \partial_x \left(\dfrac{(c^2\rho + P)u}{c^2-u^2} \right) = 0, \\ \partial_t \left(\dfrac{(c^2\rho + P)u}{c^2-u^2} \right) + \partial_x \left(\dfrac{(c^2\rho + P)u^2}{c^2-u^2} \right) + \partial_x P = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0)(x) \quad \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{3.1.1}$$

Here, we consider the pressure law

$$P(\rho) = k^2 \rho^\gamma, \tag{3.1.2}$$

for some fixed constants $k > 0$ and $\gamma > 1$.

We proceed with a rudimentary analysis of the Cauchy problem (3.1.1). First, the two

eigenvalues λ_1 and λ_2 of equations $(3.1.1)_1$-$(3.1.1)_2$ can be given by

$$\lambda_1 = \frac{u - \sqrt{P'}}{1 - \frac{u\sqrt{P'}}{c^2}}, \quad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \frac{u\sqrt{P'}}{c^2}}. \tag{3.1.3}$$

We denote the directional derivatives as

$$\partial_1 = \partial_t + \lambda_1 \partial_x, \quad \partial_2 = \partial_t + \lambda_2 \partial_x,$$

along two characteristic directions $x^1(t, x_0^1)$ and $x^2(t, x_0^2)$, defined by

$$\frac{dx^1}{dt} = \lambda_1, \quad \frac{dx^2}{dt} = \lambda_2,$$

where $x^1(0, x_0^1) = x_0^1$ and $x^2(0, x_0^2) = x_0^2$. For simplicity we will abbreviate $x^1(t, x_0^1)$ and $x^2(t, x_0^2)$ as $x^1(t)$ and $x^2(t)$. Introduce the corresponding Riemann invariants

$$z = \frac{c}{2} \ln\Big(\frac{c+u}{c-u}\Big) + \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}} \mathrm{d}\sigma, \quad w = \frac{c}{2} \ln\Big(\frac{c+u}{c-u}\Big) - \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}} \mathrm{d}\sigma. \tag{3.1.4}$$

By explicit calculations, one can verify that

$$\sqrt{P'(\rho)} = c\sqrt{\gamma}\ \tan\Big(\frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}\Big). \tag{3.1.5}$$

Furthermore, one can verify that w and z satisfy

$$\partial_2 z = 0 \quad \text{and} \quad \partial_1 w = 0. \tag{3.1.6}$$

We will now introduce some functions. Let h_1 and h_2 be functions satisfying

$$\partial_z h_1 = \frac{\partial_z \lambda_1}{\lambda_1 - \lambda_2}, \quad \partial_w h_2 = \frac{\partial_w \lambda_2}{\lambda_2 - \lambda_1}. \tag{3.1.7}$$

Let $\epsilon > 0$ be a small positive constant, denote

$$\alpha \coloneqq \partial_x w, \quad \beta \coloneqq \partial_x z, \quad \widetilde{\alpha} = \frac{\alpha}{\rho^{1-\epsilon}}, \quad \widetilde{\beta} = \frac{\beta}{\rho^{1-\epsilon}}. \tag{3.1.8}$$

We define two important quantities ξ and ζ, which play important roles in our analysis on singularity formation.

$$\xi = \mathrm{e}^{h_1}\alpha, \quad \zeta = \mathrm{e}^{h_2}\beta. \tag{3.1.9}$$

3.2 The main results

First, define the C^1 solutions as follows:

Definition 3.2.1. *Let $T > 0$ be some time. The pair $(\rho(t,x), u(t,x))$ is called a C^1 solution to the equations $(3.1.1)_1$-$(3.1.1)_2$ on $(0,T)\times\mathbb{R}$ if*

$$\rho > 0, \quad \rho \in C^1([0,T)\times\mathbb{R}), \quad u \in C^1([0,T)\times\mathbb{R}),$$

and the equations $(3.1.1)_1$-$(3.1.1)_2$ are satisfied in the pointwise sense on $(0,T)\times\mathbb{R}$. It is called a C^1 solution to the Cauchy problem (3.1.1) if it is a C^1 solution to the equations $(3.1.1)_1$-$(3.1.1)_2$ on $(0,T)\times\mathbb{R}$ and admits the initial data $(3.1.1)_3$ continuously.

Assumption 3.2.1. *Denote $(w_0, z_0)(x) = (w(0,x), z(0,x))$. Assume $(w_0,\ z_0) \in C^1(\mathbb{R})$, and*

$$z_0 - w_0 > 0, \quad \|(z_0, w_0)\|_{C^1(\mathbb{R})} \le M_0,$$

for some constant $M_0 > 0$.

Assumption 3.2.2. *For (w_0, z_0) satisfying Assumption 3.2.1, we assume additionally that*

$$\sup_{x\in\mathbb{R}} z_0 - \inf_{x\in\mathbb{R}} w_0 < \frac{4c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\big(\frac{1}{\sqrt{\gamma}}\big).$$

Assumption 3.2.3. *For (w_0, z_0) satisfying Assumption 3.2.1, we assume additionally that*

$$\left|\exp\left(\frac{\sup_x w_0 + \sup_x z_0}{c}\right) - 1\right| < \epsilon, \quad \left|\exp\left(\frac{\inf_x w_0 + \inf_x z_0}{c}\right) - 1\right| < \epsilon,$$
$$\sup_x z_0 - \inf_x w_0 < \frac{4c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\big(\frac{\epsilon}{\sqrt{\gamma}}\big), \quad \max\{\widetilde{\alpha}_0(x), \widetilde{\beta}_0(x)\} < K, \quad \inf_x(z_0 - w_0) > 0,$$

where $\epsilon > 0$ is the constant in (3.1.8), and $K > 0$ is a constant.

Remark 3.2.1. *It is well-known that under Assumption* 3.2.1*, the local-in-time existence of the unique* C^1 *solution* (ρ, u) *to the Cauchy problem* (3.1.1) *can be established by the classical characteristic method; c.f. Theorem 4.1 on p. 66 of* [39]*, or see also* [23, 26, 38].

Remark 3.2.2. *Assumption* 3.2.3 *is stronger than Assumption* 3.2.2*, we need to use Assumption* 3.2.3 *to prove Theorem* 3.2.2*, but to prove Theorem 3.2.1, Assumption* 3.2.2 *will be sufficient.*

Since now, assume Assumption 3.2.1 holds, and let T denote the life span of the C^1 solution to the Cauchy problem (3.1.1). All the conclusions are proved under the assumption that $t < T$.

The first result of this chapter gives an if and only if condition for singularities to form in finite time. For initial data that allows far-field vacuum, the same result has been proved in [1], here we extend their result to initial data with far-field vacuum.

Theorem 3.2.1. *If the initial data* (ρ_0, u_0) *satisfy Assumption* 3.2.1 *and Assumption* 3.2.2*, then the Cauchy problem* (3.1.1) *has a unique global-in-time* C^1 *solution if and only if*

$$min\left\{\inf_{x\in\mathbb{R}}\xi(0,x),\ \inf_{x\in\mathbb{R}}\zeta(0,x)\right\} \geq 0.$$

The second result of this chapter gives a time-dependent lower bound on the mass-energy density. This result is an original work by the author of this thesis.

Theorem 3.2.2. *Assume* $\epsilon > 0$ *is small, and the initial data* (ρ_0, u_0) *satisfy Assumption* 3.2.1 *and Assumption* 3.2.3*, then*

$$\rho(t,x)^{1-\epsilon} \geq \frac{D}{1+t}, \tag{3.2.1}$$

where $D > 0$ *is a constant depending only on the initial data,* γ*,* k *and* K*.*

Remark 3.2.3. *The initial assumption* $\sup_{x\in\mathbb{R}} w_0 - \inf_{x\in\mathbb{R}} z_0 < \frac{4c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\left(\frac{1}{\sqrt{\gamma}}\right)$ *is used to make sure that* $|\sqrt{P'(\rho)}| < c$ *holds for* C^1 *solutions of the problem* (3.1.1) *(see* (3.3.4)*).*
The first three conditions in 3.2.3 *are used to make sure that* $|\sqrt{P'(\rho)}| < \epsilon c$ *and* $|u| < \epsilon c$ *hold for* C^1 *solutions of the problem* (3.1.1)*.*

Remark 3.2.4. *By* (3.1.5)*, one has that*

$$z_0 - w_0 > 0 \iff \rho_0 > 0.$$

Assumption 3.2.1 *allows far-field vacuum, i.e., one can have that*

$$\rho_0(x) \to 0, \quad as \quad |x| \to \infty.$$

3.3 Singularity formation in the 2×2 system with far-field vacuum

This section will be devoted to proving Theorem 3.2.1. All results are proven under the assumption that the initial data (ρ_0, u_0) satisfy Assumption 3.2.2. For simplicity, in this section, we denote $C_g > 0$ a generic constant depending only on (z_0, w_0), κ, γ and c, and independent of the time.

3.3.1 Preliminaries

Lemma 3.3.1.

$$|u(t,x)| < c \quad and \quad \sqrt{P'(\rho)}(t,x) < c \quad uniformly, \quad \rho(t,x) > 0 \quad pointwisely, \tag{3.3.1}$$

and the following Riccati ODEs hold:

$$\partial_1 \xi = -\big(e^{-h_1}\partial_w \lambda_1\big)\xi^2, \;\; \partial_2 \zeta = -\big(e^{-h_2}\partial_z \lambda_2\big)\zeta^2. \tag{3.3.2}$$

Moreover, one has

$$\begin{aligned} \frac{1}{\xi(t, x^1(t, x_0^1))} =& \frac{1}{\xi(0, x_0^1)} + \int_0^t \big(\mathrm{e}^{-h_1}\partial_w \lambda_1\big)(\sigma, x^1(\sigma, x_0^1))\, d\sigma, \\ \frac{1}{\zeta(t, x^2(t, x_0^2))} =& \frac{1}{\zeta(0, x_0^2)} + \int_0^t \big(\mathrm{e}^{-h_2}\partial_z \lambda_2\big)(\sigma, x^2(\sigma, x_0^2))\, d\sigma. \end{aligned} \tag{3.3.3}$$

Proof. First, the facts $|u(t,x)| < c$ and $\sqrt{P'(\rho)}(t,x) < c$ directly follow from

$$\begin{aligned} \big|\ln\Big(\frac{c+u}{c-u}\Big)\big| =& \big|\frac{w+z}{c}\big| \le \frac{2M_0}{c}, \\ \sqrt{P'(\rho)} = k\sqrt{\gamma}\rho^{(\gamma-1)/2} =& c\sqrt{\gamma}\, \tan\Big(\frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}\Big), \end{aligned} \tag{3.3.4}$$

where one has used (3.1.4)-(3.1.6) and Assumption 3.2.1.

Second, the proof for the fact $\rho(t,x) > 0$ can be found on p. 997-998 of [51].

At last, the proof for (3.3.2)-(3.3.3) can be found in [1].

□

Lemma 3.3.2. *Define the following functions*

$$\begin{aligned} H_1(w,z) :=& \frac{w+z}{c} + \ln\Big(\frac{1-\sqrt{\gamma}\ \tan\big(\frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}\big)}{1+\sqrt{\gamma}\ \tan\big(\frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}\big)}\Big), \\ H_2(w,z) :=& \frac{w+z}{c} + \ln\Big(\frac{1+\sqrt{\gamma}\ \tan\big(\frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}\big)}{1-\sqrt{\gamma}\ \tan\big(\frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}\big)}\Big), \\ Y(w,z) :=& \tan^{-1}\big(\frac{k\rho^{(\gamma-1)/2}}{c}\big) = \frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}. \end{aligned} \tag{3.3.5}$$

Then one has

$$\begin{aligned} \lambda_1 =& c\left(1-\frac{2}{e^{H_1}+1}\right), \quad \lambda_2 = c\left(1-\frac{2}{e^{H_2}+1}\right), \\ h_1 =& \frac{3\gamma-1}{2\gamma-2}\ln(\cos(Y)) + \frac{\gamma-3}{2\gamma-2}\ln(\sin(Y)) + \frac{z-w}{2c} \\ & - \ln\Big(\big(1+e^{\frac{w+z}{c}}\big)\cos(Y) - \big(-1+e^{\frac{w+z}{c}}\big)\sqrt{\gamma}\sin(Y)\Big), \\ h_2 =& \frac{3\gamma-1}{2\gamma-2}\ln(\cos(Y)) + \frac{\gamma-3}{2\gamma-2}\ln(\sin(Y)) + \frac{z-w}{2c} \\ & - \ln\Big(\big(e^{\frac{2w}{c}}+e^{\frac{z-w}{c}}\big)\cos(Y) + \big(e^{\frac{2w}{c}}-e^{\frac{z-w}{c}}\big)\sqrt{\gamma}\sin(Y)\Big). \end{aligned} \tag{3.3.6}$$

λ_1 and λ_2 follow from direct calculations. The verification of formulas for h_1 and h_2 follows from very complex calculations, which can be found in Appendix C

3.3.2 Riccati-type ODE inequality and time-dependent mass-energy density lower bound for the adiabatic exponent $\gamma \in (1,3)$

In the following proof, we will focus only on ξ, and ζ can be dealt with similarly. Obviously, by (3.1.6) and (3.3.3), the key point in the proof of Theorem 2.1 is to give one good enough lower bound estimate on $\mathrm{e}^{-h_1}\partial_w\lambda_1$, which will be achieved by some elaborate ODE argument. For this purpose, we first observed that

Lemma 3.3.3. *Denote* $y \coloneqq \frac{k\rho^{(\gamma-1)/2}}{c}$. *One has*

$$C_g^{-1}\mathcal{Y} \le e^{-h_1}\partial_w\lambda_1 \le C_g\mathcal{Y}, \tag{3.3.7}$$

where

$$\mathcal{Y} \coloneqq \left(\frac{k\rho^{(\gamma-1)/2}/c}{\sqrt{1+\frac{k^2\rho^{\gamma-1}}{c^2}}}\right)^{\frac{3-\gamma}{2\gamma-2}} \big(1+\frac{k^2\rho^{\gamma-1}}{c^2}\big)^{\frac{\gamma+1}{4\gamma-4}} = \big(\frac{y}{\sqrt{y^2+1}}\big)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}}. \tag{3.3.8}$$

Proof. First, it follows from Lemma 3.3.2 and direct calculations that

$$\mathrm{e}^{-h_1}\partial_w\lambda_1 = \frac{c\,\mathrm{e}^{-\frac{2\sqrt{\gamma}\tan^{-1}(y)}{\gamma-1}}(c+u)(\gamma+1)\big(\frac{y}{\sqrt{y^2+1}}\big)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}}(1-y^2)}{2c^2-2u\sqrt{P'(\rho)}}. \tag{3.3.9}$$

Second, according to Lemma 3.3.1 and its proof, one can obtain that

$$c\,\mathrm{e}^{-\frac{\pi\sqrt{\gamma}}{\gamma-1}} \le c\,\mathrm{e}^{-\frac{2\sqrt{\gamma}\tan^{-1}(y)}{\gamma-1}} \le c, \quad 1-\gamma^{-1} \le 1-y^2 \le 1, \quad C_g^{-1} \le \frac{(c+u)(\gamma+1)}{2c^2-2u\sqrt{P'(\rho)}} \le C_g,$$

which, along with (3.3.9), yields the desired estimates.

□

Until now, from (3.3.7), it is clear that for proving Theorem 3.2.1, one needs to give a proper time-dependent lower bound on $\mathcal{Y}$. For $\gamma \ge 3$, we observe that $\mathcal{Y} > 1$. Next we consider the case $1 < \gamma < 3$. We first consider the upper bounds of ξ and ζ.

Lemma 3.3.4.

$$\xi(x,t) \le Q_1, \;\; \zeta(x,t) \le Q_2, \tag{3.3.10}$$

where

$$Q_1 \coloneqq max\left\{0, \sup_x \xi(x,0)\right\}, \;\; Q_2 \coloneqq max\left\{0, \sup_x \zeta(x,0)\right\}. \tag{3.3.11}$$

Proof. We only consider ξ. First, it follows from Lemma 3.3.2 and direct calculations that

$$\partial_w\lambda_1 = \frac{\mathrm{e}^{\frac{w+z}{c}}(1+\gamma)\cos(2Y)\,\mathrm{Sec}(Y)^2}{\big(1+\mathrm{e}^{\frac{w+z}{c}}-(\mathrm{e}^{\frac{w+z}{c}}-1)\sqrt{\gamma}\tan(Y)\big)^2}. \tag{3.3.12}$$

Second, we show that $\cos(2Y) > 0$, which, can be obtained from Lemma 3.3.1,

$$Y = \frac{(z-w)(\gamma-1)}{4\,c\sqrt{\gamma}} = \tan^{-1}\Big(\frac{k\rho^{(\gamma-1)/2}}{c}\Big) \quad \text{and} \quad \cos(2\tan^{-1}(x)) = \frac{1-x^2}{1+x^2}.$$

Thus, the desired conclusion follows from the Riccati ODEs established in Lemma 3.3.1. □

Now we are ready to develop an ODE inequality for y.

Lemma 3.3.5. *Assume $1 < \gamma < 3$. One has*

$$\partial_1 y \geq -C_g\, y^{\frac{\gamma+1}{2\gamma-2}}\,(1+y^2)^{3/2}. \tag{3.3.13}$$

Proof. First, we rewrite (3.1.6) in the form of the difference $(w-z)$ as:

$$\partial_1(z-w) = \partial_t(z-w) + \lambda_1\partial_x(z-w) = (\lambda_1-\lambda_2)\partial_x z = \frac{\lambda_1-\lambda_2}{\mathrm{e}^{h_2}}\zeta. \tag{3.3.14}$$

Second, it follows from (3.1.2)-(3.1.3) and direct computations that

$$\frac{\lambda_2-\lambda_1}{\mathrm{e}^{h_2}} = \frac{4c\,\mathrm{e}^{\frac{2\sqrt{\gamma}\,\tan^{-1}(y)}{\gamma-1}}\,(c+u)\,c\sqrt{\gamma}\,y\,\big(\frac{y}{\sqrt{y^2+1}}\big)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}}}{c^2-u\sqrt{P'(\rho)}}. \tag{3.3.15}$$

Notice that, according to Lemma 3.3.1 and its proof, one has

$$C_g^{-1} \leq \mathrm{e}^{\frac{2\sqrt{\gamma}\,\tan^{-1}(y)}{\gamma-1}} \leq C_g, \quad \text{and} \quad C_g^{-1} \leq \frac{c+u}{c^2-u\sqrt{P'(\rho)}} \leq C_g, \tag{3.3.16}$$

which, along with (3.3.14)-(3.3.15), yields that

$$\partial_1(z-w) \geq -C_g\,y\,\big(\frac{y}{\sqrt{y^2+1}}\big)^{\frac{3-\gamma}{2\gamma-2}}(1+y^2)^{\frac{\gamma+1}{4\gamma-4}} = -C_g\,y^{\frac{\gamma+1}{2\gamma-2}}\,(1+y^2)^{\frac{1}{2}}. \tag{3.3.17}$$

Due to

$$z-w = \frac{4\,c\sqrt{\gamma}}{\gamma-1}\tan^{-1}(y), \quad \text{and} \quad \partial_1(z-w) = C_g\frac{\partial_- y}{y^2+1},$$

then one can rewrite (3.3.17) as (3.3.13). □

Finally, it turns out that, one can obtain the following Riccati-type inequality for $\mathcal{Y}$:

Lemma 3.3.6. *Assume* $1 < \gamma < 3$. *One has*

$$\partial_1 \mathcal{Y} \geq -C_g \mathcal{Y}^2. \tag{3.3.18}$$

In particular, along the characteristic $x^1(t, x_*)$ *starting from* $(0, x_*)$, *one has*

$$\mathcal{Y}(t, x^1(t, x_*)) \geq \frac{\mathcal{Y}(0, x_*)}{1 + C_g \mathcal{Y}(0, x_*) t}.$$

Proof. First, it follows from (3.3.8), (3.3.13) and direct calculations that

$$\partial_1 \mathcal{Y} \geq -C_g \, y^{\frac{6-2\gamma}{2\gamma-2}} \, (1 + y^2) \big((\gamma + 1) y^2 + (3 - \gamma) \big), \tag{3.3.19}$$

which can equivalently be rewritten as

$$\partial_1 \mathcal{Y} \geq -C_g \, \mathcal{Y}^2 \big((\gamma + 1) y^2 + (3 - \gamma) \big). \tag{3.3.20}$$

Second, it follows from the fact $\sqrt{P'(\rho)} < c$ that $y < \gamma^{-\frac{1}{2}}$. Therefore

$$(\gamma + 1) y^2 + (3 - \gamma) \leq \frac{\gamma + 1}{\gamma} + (3 - \gamma),$$

which, along with (3.3.20), implies (3.3.18).

At last, by integrating along the characteristic $x^1(t, x_*)$, one obtains the desired result.

□

3.3.3 Verification of Lax's theory in 2×2 system: Proof of Theorem 3.2.1

Without loss of generality, assume that there exists a point x_* such that $\xi(0, x_*) < 0$. We shall prove that a singularity forms in finite time by contradiction. Indeed, assume otherwise, $T^* = \infty$. It follows from Lemma 3.3.1 that

$$\xi(t, x^1(t, x_*)) = \frac{\xi(0, x_*)}{1 + \xi(0, x_*) \int_0^t \left(\mathrm{e}^{-h_1} \partial_w \lambda_1 \right) (\sigma, x^1(\sigma, x_*)) \, \mathrm{d}\sigma}. \tag{3.3.21}$$

There are two cases to consider: the case $\gamma \geq 3$ and the case $1 < \gamma < 3$.

Case 1: $\gamma \geq 3$. By (3.3.7) and the fact that $\mathcal{Y} > 1$, one obtains

$$\int_0^t \left(\mathrm{e}^{-h_1}\partial_w \lambda_1\right)(\sigma, x^1(\sigma, x_*))\,\mathrm{d}\sigma \geq C_g^{-1} t.$$

Case 2: $1 < \gamma < 3$. By Lemma 3.3.6 and (3.3.7), one obtains

$$\left(\mathrm{e}^{-h_1}\partial_w \lambda_1\right)(\sigma, x^1(\sigma, x_*)) \geq C_g^{-1} \frac{\mathcal{Y}(0, x_*)}{1 + C_g \mathcal{Y}(0, x_*)\sigma},$$

which yields that

$$\int_0^t \left(\mathrm{e}^{-h_1}\partial_w \lambda_1\right)(\sigma, x^1(\sigma, x_*))\,\mathrm{d}\sigma \geq C \log(1 + C_g \mathcal{Y}(0, x_*)t),$$

for a constant $C > 0$ depending only on the C^1 norm of the initial data and $\mathcal{Y}(0, x_*)$.

In both cases, the contradiction now follows from the fact that $\xi(0, x_*) < 0$ and the fact

$$\int_0^t \left(\mathrm{e}^{-h_1}\partial_w \lambda_1\right)(\sigma, x^1(\sigma, x_*))\,\mathrm{d}\sigma \to \infty \quad \text{as} \quad t \to \infty.$$

Second, for the other direction, assume that there exists no compression in the initial data. In that case, the boundedness of the Riemann invariants follows as usual through their constancy along characteristics, and moreover, equation (3.3.21) implies that the L^∞ norm of $\partial_x z$ remains finite for all time. Hence, at every finite time T, one can invoke the local existence result, as the C^1 norm remains bounded. The solution, therefore, exists for all time and the result follows.

3.4 Time-dependent mass-energy density lower bound

This section will be devoted to proving Theorem 3.2.2. All results are proven under the assumption that the initial data (ρ_0, u_0) satisfy Assumption 3.2.3.

3.4.1 Uniform bound for the velocity and the local sound speed

Lemma 3.4.1.

$$|u(t, x)| < \epsilon c, \quad \sqrt{P'(\rho)}(t, x) < \epsilon c, \tag{3.4.1}$$

where $\epsilon > 0$ is the constant in (3.1.8).

Proof. By explicit calculation, one has

$$|u| = c\left|\frac{\exp(\frac{z+w}{c})-1}{\exp(\frac{z+w}{c})+1}\right| \le c\left|\exp(\frac{z+w}{c})-1\right|,$$
$$\sqrt{P'(\rho)} = k\sqrt{\gamma}\rho^{(\gamma-1)/2} = c\sqrt{\gamma}\ \tan\Big(\frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}\Big).$$

By (3.1.6), one has $w(t, x^1(t)) = w_0(x_0^1)$, $z(t, x^2(t)) = z_0(x_0^2)$. Combining this with Assumption 3.2.3, one obtains the desired result. □

3.4.2 Coupled ODEs

Lemma 3.4.2. *α and β satisfy*

$$\begin{cases} \partial_1\alpha = -\dfrac{(c^2-u^2)(c^2-P')}{2(c^2-u\sqrt{P'})^2}\alpha(\alpha+\beta) - \dfrac{P''(c^2-u^2)(c^2\rho+P)}{4P'(c^2-u\sqrt{P'})^2}\alpha(\alpha-\beta), \\ \partial_2\beta = -\dfrac{(c^2-u^2)(c^2-P')}{2(c^2+u\sqrt{P'})^2}\beta(\alpha+\beta) - \dfrac{P''(c^2-u^2)(c^2\rho+P)}{4P'(c^2+u\sqrt{P'})^2}\beta(\beta-\alpha). \end{cases}$$

Proof. We will only prove the lemma for $\partial_1\alpha$, as the proof for $\partial_2\beta$ will be similar. Differentiating the equation $\partial_1 w = 0$ with respect to x, one gets

$$\begin{aligned} \partial_{tx}w + \partial_x\lambda_1\partial_x w + \lambda_1\partial_{xx}w = 0 &\iff \partial_t\alpha + \lambda_1\partial_x\alpha + \partial_x\lambda_1\alpha = 0 \\ &\iff \partial_1\alpha + \partial_x\lambda_1\alpha = 0 \\ &\iff \partial_1\alpha + \partial_z\lambda_1\partial_x z\alpha + \partial_w\lambda_1\partial_x w\alpha = 0 \\ &\iff \partial_1\alpha + \partial_z\lambda_1\beta\alpha + \partial_w\lambda_1\alpha^2 = 0. \end{aligned} \tag{3.4.2}$$

By Lemma 3.3.2 and explicit calculations, one has

$$\partial_w\lambda_1 = \frac{2ce^{H_1}\partial_w H_1}{(e^{H_1}+1)^2}, \quad \partial_z\lambda_1 = \frac{2ce^{H_1}\partial_z H_1}{(e^{H_1}+1)^2}, \tag{3.4.3}$$

where

$$\partial_w H_1 = \frac{1}{c} + \frac{P''(c^2\rho + P)}{2cP'(c^2 - P')}, \quad \partial_z H_1 = \frac{1}{c} - \frac{P''(c^2\rho + P)}{2cP'(c^2 - P')},$$
$$\frac{e^{H_1}}{(e^{H_1}+1)^2} = \frac{(c^2-u^2)(c^2-P')}{4(c^2 - u\sqrt{P'})^2}.$$

Substituting (3.4.3) into (3.4.2) followed with simplification, one will obtain the desired result.

□

Lemma 3.4.3.

$$\begin{cases} \partial_1\left(\frac{1}{\rho^{1-\epsilon}}\right) = \frac{(1-\epsilon)(c^2\rho+P)(c^2-u^2)}{\rho^{2-\epsilon}(c^2-u\sqrt{P'})(c^2+u\sqrt{P'})}\beta, \\ \partial_2\left(\frac{1}{\rho^{1-\epsilon}}\right) = \frac{(1-\epsilon)(c^2\rho+P)(c^2-u^2)}{\rho^{2-\epsilon}(c^2-u\sqrt{P'})(c^2+u\sqrt{P'})}\alpha. \end{cases}$$

Proof. One has

$$\partial_1\left(\frac{1}{\rho^{1-\epsilon}}\right) = -\frac{(1-\epsilon)\partial_1\rho}{\rho^{2-\epsilon}}. \tag{3.4.4}$$

By (3.1.6) and (3.1.8), one has

$$\partial_1\rho = \partial_{\frac{z-w}{2}}\rho\partial_1\left(\frac{z-w}{2}\right) = -\frac{c^2\rho+P}{c^2\sqrt{P'}}\frac{\partial_1 z}{2} = -\frac{c^2\rho+P}{c^2\sqrt{P'}}\frac{(\lambda_2-\lambda_1)\beta}{2},$$

where

$$\lambda_2 - \lambda_1 = \frac{2c^2\sqrt{P'}(c^2-u^2)}{(c^2-u\sqrt{P'})(c^2+u\sqrt{P'})}.$$

Substituting this to (3.4.4), one obtains the desired result. □

Theorem 3.4.1. *Define the following quantities*

$$\begin{cases} a_0 \coloneqq \frac{(c^2-u^2)}{(c^2-u\sqrt{P'})^2}\left(\frac{(c^2-P')}{2} + \frac{P''(c^2\rho+P)}{4P'}\right), \\ a_1 \coloneqq \frac{(c^2-u^2)}{(c^2-u\sqrt{P'})^2}\left(-\frac{(c^2-P')}{2} + \frac{P''(c^2\rho+P)}{4P'} + \frac{(1-\epsilon)(c^2+\frac{P}{\rho})(c^2-u\sqrt{P'})}{(c^2+u\sqrt{P'})}\right), \\ b_0 \coloneqq \frac{(c^2-u^2)}{(c^2+u\sqrt{P'})^2}\left(\frac{(c^2-P')}{2} + \frac{P''(c^2\rho+P)}{4P'}\right), \\ b_1 \coloneqq \frac{(c^2-u^2)}{(c^2+u\sqrt{P'})^2}\left(-\frac{(c^2-P')}{2} + \frac{P''(c^2\rho+P)}{4P'} + \frac{(1-\epsilon)(c^2+\frac{P}{\rho})(c^2+u\sqrt{P'})}{(c^2-u\sqrt{P'})}\right). \end{cases}$$

Then $\widetilde{\alpha}$ *and* $\widetilde{\beta}$ *satisfy*

$$\begin{cases} \partial_1 \widetilde{\alpha} = -\rho^{1-\epsilon}\widetilde{\alpha}(a_0\widetilde{\alpha} - a_1\widetilde{\beta}), \\ \partial_2 \widetilde{\beta} = -\rho^{1-\epsilon}\widetilde{\beta}(b_0\widetilde{\beta} - b_1\widetilde{\alpha}). \end{cases} \tag{3.4.5}$$

Furthermore, given $\epsilon > 0$ *is small, one has*

$$a_0 > a_1 > 0, \quad b_0 > b_1 > 0. \tag{3.4.6}$$

Proof. The ODEs for $\partial_1\widetilde{\alpha}$ and $\partial_2\widetilde{\beta}$ can be obtained by direct calculations using the previous two lemmas. We will show $a_0 > a_1 > 0$ given ϵ is small.

By explicit computation, Lemma 3.4.1 and the positivity of ρ, one has

$$\begin{aligned} a_0 > a_1 &\iff \frac{(c^2 - P')}{(c^2 - u\sqrt{P'})} > \frac{(1-\epsilon)(c^2 + \frac{P}{\rho})}{(c^2 + u\sqrt{P'})} \\ &\iff \epsilon c^4 > (1-\epsilon)c^2\frac{P}{\rho} - (2-\epsilon)c^2 u\sqrt{P'} - (1-\epsilon)u\sqrt{P'}\frac{P}{\rho} + u\sqrt{P'}P' - c^2P'. \end{aligned}$$

By Lemma 3.4.1, the above becomes

$$a_0 > a_1 \iff \epsilon > \mathcal{O}(\epsilon^2), \tag{3.4.7}$$

which holds if $\epsilon > 0$ is small. Similarly, by Lemma 3.4.1, one has

$$a_1 > 0 \iff \frac{\gamma - 1}{4} + \frac{1 - 2\epsilon}{2} + \mathcal{O}(\epsilon^2) > 0, \tag{3.4.8}$$

which holds if $\epsilon > 0$ is small, as $\gamma > 1$. Lastly, by the same method, one can obtain $b_0 > b_1 > 0$ for $\epsilon > 0$ small.

□

3.4.3 Upper bound for gradient quantities $\widetilde{\alpha}$ and $\widetilde{\beta}$

Theorem 3.4.2.

$$\max\{\widetilde{\alpha}(t,x), \widetilde{\beta}(t,x)\} < K,$$

where $K > 0$ *is the constant in Assumption* 3.2.3.

Proof. We will prove the theorem by contradiction. Take the first time t such that there exists some $x \in \mathbb{R}$ where

$$\max\{\widetilde{\alpha}(t,x), \widetilde{\beta}(t,x)\} = K.$$

Without any loss of generality, assume that $\widetilde{\alpha}(t,x) = K$ and $\widetilde{\beta}(t,x) \leq K$. By Theorem 3.4.1, one has

$$(a_0\widetilde{\alpha} - a_1\widetilde{\beta})(t,x) > 0.$$

Now consider the x^1 characteristic passing through the point (t,x), by continuity, one can find a t_1 where $t_1 < t$, and for $l \in (t_1, t]$, one has

$$\widetilde{\alpha}(l, x^1(l)) > 0 \quad \text{and} \quad (a_0\widetilde{\alpha} - a_1\widetilde{\beta})(l, x^1(l)) > 0.$$

Hence, for $l \in (t_1, t]$, by Theorem 3.4.1, one has

$$\partial_1\widetilde{\alpha} = -\rho^{1-\epsilon}\widetilde{\alpha}(a_0\widetilde{\alpha} - a_1\widetilde{\beta}) < 0.$$

Integrate the above from t_1 to t one gets

$$K = \widetilde{\alpha}(t,x) < \widetilde{\alpha}(t_1, x^1(t_1)) < K, \tag{3.4.9}$$

a contradiction. □

3.4.4 Time-dependent lower bound on the mass-energy density: Proof of Theorem 3.2.2

For this subsection, fix any $(t,x) \in (0,T) \times \mathbb{R}$, let D denote a constant depending only on k, γ, the initial data and K. We will consider the x^1 characteristic passing through the point (t,x). By (3.1.6), one has

$$\partial_1\left(\frac{z-w}{2}\right) = \frac{\partial_1 z}{2} = \frac{\lambda_1 - \lambda_2}{2}\beta = -\frac{c^2\sqrt{P'}(c^2-u^2)}{(c^2+u\sqrt{P'})(c^2-u\sqrt{P'})}\beta.$$

By viewing $z - w$ as a function of ρ, one has

$$\partial_1 \left(\frac{z-w}{2} \right) = \partial_\rho \left(\frac{z-w}{2} \right) \partial_1 \rho = \frac{c^2 \sqrt{P'}}{c^2 \rho + P} \partial_1 \rho.$$

Hence one gets

$$\partial_1 \rho = -\frac{(c^2 + \frac{P}{\rho})(c^2 - u^2)}{(c^2 + u\sqrt{P'})(c^2 - u\sqrt{P'})} \widetilde{\beta} \rho^{2-\epsilon}.$$

By Lemma 3.4.1 and Theorem 3.4.2, one gets

$$\partial_1 \rho > -D\rho^{2-\epsilon}.$$

Integrating the above equation, one gets

$$\frac{1}{\rho^{1-\epsilon}(t,x)} < \frac{1}{\rho_0^{1-\epsilon}(x_0^1)} + (1-\epsilon)Dt < \frac{1}{\inf_x \rho_0^{1-\epsilon}} + (1-\epsilon)Dt. \tag{3.4.10}$$

This is the desired result.

Chapter 4

Non-isentropic Classical Euler Equations

4.1 Introduction

For $t \geq 0$ and $x \in \mathbb{R}$, the non-isentropic compressible Euler equations for C^1 solutions are given by:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho, S) = 0, \\ \partial_t S + u\partial_x S = 0, \end{cases} \tag{4.1.1}$$

where the pressure $P(\rho, S)$ takes the form:

$$P(\rho, S) \coloneqq Ae^{\mu S}\rho^\gamma, \quad A > 0, \quad \mu > 0, \quad \gamma > 1. \tag{4.1.2}$$

The initial data are given by

$$(\rho, u, S)|_{t=0} = (\rho_0, u_0, S_0). \tag{4.1.3}$$

We can write (4.1.1) in the following form:

$$\begin{bmatrix} \rho \\ \rho u \\ S \end{bmatrix}_t + \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + \partial_\rho P & 2u & \partial_S P \\ 0 & 0 & u \end{bmatrix} \begin{bmatrix} \rho \\ \rho u \\ S \end{bmatrix}_x = 0. \tag{4.1.4}$$

The matrix has eigenvalues:

$$\lambda_1 = u, \quad \lambda_2 = u + \sqrt{\partial_\rho P}, \quad \lambda_3 = u - \sqrt{\partial_\rho P}. \tag{4.1.5}$$

The directional derivatives will be defined by

$$\partial_1 \coloneqq \partial_t + \lambda_1 \partial_x, \quad \partial_2 \coloneqq \partial_t + \lambda_2 \partial_x, \quad \partial_3 \coloneqq \partial_t + \lambda_3 \partial_x, \tag{4.1.6}$$

and the characteristics $x^i(t, x_0^i)$ will be defined by

$$\frac{dx^1}{dt} = \lambda_1, \quad \frac{dx^2}{dt} = \lambda_2, \quad \frac{dx^3}{dt} = \lambda_3, \tag{4.1.7}$$

with $x_0^i = x^i(0, x_0^i)$, $i = 1, 2, 3$. For simplicity, $x^i(t, x_0^i)$ will be abbreviated as $x^i(t)$.

We introduce the Riemann variables:

$$\begin{cases} w = u - \displaystyle\int_0^\rho \frac{\sqrt{\partial_\sigma P(\sigma, S)}}{\sigma} \mathrm{d}\sigma = u - \frac{2\sqrt{\gamma A}}{\gamma - 1} \exp\left(\frac{\mu S}{2}\right) \rho^{\frac{\gamma-1}{2}}, \\ z = u + \displaystyle\int_0^\rho \frac{\sqrt{\partial_\sigma P(\sigma, S)}}{\sigma} \mathrm{d}\sigma = u + \frac{2\sqrt{\gamma A}}{\gamma - 1} \exp\left(\frac{\mu S}{2}\right) \rho^{\frac{\gamma-1}{2}}. \end{cases} \tag{4.1.8}$$

Denote the following quantities

$$
\begin{aligned}
&a \coloneqq -\frac{\mu}{\sqrt{\gamma}(\gamma-1)}\sqrt{Ae^{\mu S}}\rho^{\frac{\gamma-1}{2}}, \quad b \coloneqq \frac{1}{\rho},\\
&\alpha \coloneqq \partial_x w = \partial_x u - \frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x\rho - \frac{\mu}{2}\sqrt{\partial_\rho P}\partial_x S,\\
&\beta \coloneqq \partial_x z = \partial_x u + \frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x\rho + \frac{\mu}{2}\sqrt{\partial_\rho P}\partial_x S,\\
&\eta = \partial_x S, \quad I(S) \coloneqq \exp\left(\frac{3\mu(\gamma-3)}{4(3\gamma-1)}S\right),\\
&\widetilde{\alpha} \coloneqq \alpha - a\eta = \partial_x u - \frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x\rho - \frac{\mu}{\gamma}\sqrt{\partial_\rho P}\eta,\\
&\widetilde{\beta} \coloneqq \beta + a\eta = \partial_x u + \frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x\rho + \frac{\mu}{\gamma}\sqrt{\partial_\rho P}\eta,\\
&a_0 \coloneqq \frac{\gamma+1}{4}\rho^{-\frac{\gamma-3}{4}}\frac{1}{I(S)},\\
&a_1 \coloneqq \frac{\mu I(S)}{\gamma(3\gamma-1)}\partial_\rho P\rho^{\frac{\gamma-3}{4}}\left[-\frac{\mu}{3\gamma-1}\eta^2 + \left(\partial_{xx}S - \frac{\partial_x S}{\rho}\partial_x\rho\right)\right].
\end{aligned}
\tag{4.1.9}
$$

Now we can define two important quantities $\widetilde{y}$ and $\widetilde{q}$, which play important roles in our analysis on singularity formation:

$$
\begin{cases}
\widetilde{y} \coloneqq I(S)\left(\rho^{\frac{\gamma-3}{4}}\widetilde{\alpha} - \frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\eta\rho^{\frac{\gamma-3}{4}}\right),\\
\widetilde{q} \coloneqq I(S)\left(\rho^{\frac{\gamma-3}{4}}\widetilde{\beta} + \frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\eta\rho^{\frac{\gamma-3}{4}}\right),
\end{cases}
\tag{4.1.10}
$$

4.2 The main result

We begin by defining the notion of a C^1 solution.

Definition 4.2.1. *Let $T > 0$ be some time. The triple $(\rho(t,x), u(t,x), S(t,x))$ is called a C^1 solution to* (4.1.1) *on $(0,T)\times\mathbb{R}$ if it satisfies the equations* (4.1.1) *pointwise in $(0,T)\times\mathbb{R}$ and*

$$
\begin{aligned}
&\rho > 0, \quad \rho \in C^1([0,T)\times\mathbb{R}), \quad u \in C^1([0,T)\times\mathbb{R}),\\
&S \in C^1([0,T)\times\mathbb{R}), \quad \partial_{xx}S \in C([0,T)\times\mathbb{R}).
\end{aligned}
$$

It is said to be a C^1 solution to the Cauchy problem (4.1.1)–(4.1.3) *if moreover* (4.1.3) *holds in the pointwise sense at $t = 0$.*

Next, we will assume the initial data satisfy the following conditions

Assumption 4.2.1. *We assume the initial data satisfy the following conditions:*

$$\|(w_0, z_0)\|_{C^1(\mathbb{R})} \leq M, \quad \|S_0\|_{C^2(\mathbb{R})} \leq M, \quad \inf_x (z_0 - w_0) > 0, \quad S_0(x) \in BV(x), \tag{4.2.1}$$

where $M > 0$ is a fixed positive constant.

Remark 4.2.1. *Using the expression of the Riemann invariants, we have*

$$\rho_0^{\frac{\gamma-1}{2}} = \frac{4\sqrt{\gamma A}}{\gamma - 1} \exp\left(-\frac{\mu S}{2}\right)(z_0 - w_0).$$

Combining this with the assumptions $\|S_0\|_{C^1(\mathbb{R})} \leq M$ and $\gamma > 1$, we have

$$\inf_x (z_0 - w_0) > 0 \iff \inf_x \rho_0 > 0.$$

Remark 4.2.2. *It is well-known that under Assumption* 4.2.1*, the local-in-time existence of the unique C^1 solution (ρ, u, S) to the Cauchy problem* (4.1.1) *with* (4.1.3) *can be established by the classical characteristic method; c.f. Theorem* 4.1 *on p. 66 of* [39]*, or see also* [23, 26, 38].

Define the following quantities:

$$\begin{aligned}
K &\coloneqq \frac{\mu}{4\gamma}, \\
V &\coloneqq \exp\left(K\mathrm{BV}(S_0(x))\right), \\
U_1 &\coloneqq \sup_x |w_0| V + KV^2 \sup_x |z_0| \mathrm{BV}(S_0(x)), \\
U_2 &\coloneqq \sup_x |z_0| V + KV^2 \sup_x |w_0| \mathrm{BV}(S_0(x)), \\
\max|w| &\coloneqq U_1 + K^2V^2\mathrm{BV}(S_0(x))^2 \exp\left(K^2V^2\mathrm{BV}(S_0(x))^2\right) U_1, \\
\max|z| &\coloneqq U_2 + K^2V^2\mathrm{BV}(S_0(x))^2 \exp\left(K^2V^2\mathrm{BV}(S_0(x))^2\right) U_2.
\end{aligned} \tag{4.2.2}$$

Furthermore, define the constant $N > 0$ by

$$N \coloneqq \sup_{\mathcal{N}} \sqrt{\left|\frac{a_1}{a_0}\right|}, \tag{4.2.3}$$

where a_0, a_1 are defined in (4.1.9), and $\mathcal{N}$ is defined by

$$\mathcal{N} \coloneqq \{(z, w, S) \mid |z| \leq \max|z|,\ |w| \leq \max|w|,\ |S| \leq M\}.$$

Finally, we can state the main theorem of this chapter. This result was proved for the p-system in [9].

Theorem 4.2.1. *If we assume that there exists a point x^* such that*

$$\widetilde{y}_0(x^*) < -N \quad or \quad \widetilde{q}_0(x^*) < -N. \tag{4.2.4}$$

Then the smooth solution to (4.1.1) *blows up in finite time, more precisely, either $\partial_x \rho$ or $\partial_x u$ will blow-up in finite time.*

Since now, let T denote the life span of the C^1 solution to the Cauchy problem (4.1.1) with (4.1.3). All the conclusions are proved under the assumption that $t < T$.

4.3 PDEs satisfied by the Riemann variables

The matrix in (4.1.4) has right eigenvectors:

$$r_1 = \begin{bmatrix} 1 \\ u \\ -\frac{\partial_\rho P}{\partial_S P} \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ u + \sqrt{\partial_\rho P} \\ 0 \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 \\ u - \sqrt{\partial_\rho P} \\ 0 \end{bmatrix}, \tag{4.3.1}$$

and left eigenvectors:

$$l_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} -(u - \sqrt{\partial_\rho P}) \\ 1 \\ \frac{\partial_S P}{\sqrt{\partial_\rho P}} \end{bmatrix}, \quad l_3 = \begin{bmatrix} -(u + \sqrt{\partial_\rho P}) \\ 1 \\ -\frac{\partial_S P}{\sqrt{\partial_\rho P}} \end{bmatrix}. \tag{4.3.2}$$

Note that

$$\nabla w \cdot r_2 = 0 \implies \nabla w = a l_1 + b l_3, \tag{4.3.3}$$

where a and b are defined in (4.1.9) and

$$\nabla w \coloneqq \begin{bmatrix} \partial_\rho w \\ \partial_{\rho u} w \\ \partial_S w \end{bmatrix}.$$

Equation (4.3.3) holds because

$$r_i \cdot l_j \begin{cases} = 0 & \text{if} \quad i \neq j, \\ \neq 0 & \text{if} \quad i = j, \end{cases}$$

and $\{l_1, l_2, l_3\}$ spans $\mathbb{R}^3$. So given $\nabla w r_2 = 0$, ∇w can be expressed as a linear combination of l_1 and l_2. Similarly, one can obtain that

$$\nabla z \cdot r_3 = 0 \implies \nabla z = -a l_1 + b l_2.$$

Define

$$v \coloneqq \begin{bmatrix} \rho \\ \rho u \\ S \end{bmatrix}.$$

An explicit calculation will yield

$$\begin{cases} \partial_t w + \lambda_3 \partial_x w = a(\lambda_3 - \lambda_1) l_1 \cdot v_x, \\ \partial_t w + \lambda_1 \partial_x w = b(\lambda_1 - \lambda_3) l_3 \cdot v_x, \\ \partial_t z + \lambda_2 \partial_x z = a(\lambda_1 - \lambda_2) l_1 \cdot v_x, \\ \partial_t z + \lambda_1 \partial_x z = b(\lambda_1 - \lambda_2) l_2 \cdot v_x. \end{cases} \tag{4.3.4}$$

By $(4.1.1)_3$ and explicit calculations, one has $l_1 \cdot \partial_x v = \partial_x S$, so

$$\begin{aligned} a(\lambda_3 - \lambda_1) l_1 \cdot v_x &= a(\partial_t S + \lambda_3 \partial_x S), \\ a(\lambda_1 - \lambda_2) l_1 \cdot v_x &= -a(\partial_t S + \lambda_2 \partial_x S). \end{aligned}$$

Hence the first and the third equation in (4.3.4) become

$$\begin{cases} \partial_t w + \lambda_3 \partial_x w = a(\partial_t S + \lambda_3 \partial_x S), \\ \partial_t z + \lambda_2 \partial_x z = -a(\partial_t S + \lambda_2 \partial_x S). \end{cases} \tag{4.3.5}$$

Remark 4.3.1. *We can write a in terms of the Riemann variables, one can verify that*

$$a = -\frac{\mu}{4\gamma}(z - w) = -K(z - w).$$

4.4 Uniform bounds on the Riemann variables

Proposition 4.4.1.

$$|S(t, x)| \leq M \quad \forall (t, x) \in [0, T) \times \mathbb{R}, \tag{4.4.1}$$

where $M > 0$ is the constant in Assumption 4.2.1.

Proof. Integrate $(4.1.1)_3$ with respect to $x^3(t)$ characteristics, one has that

$$S(t, x^3(t)) = S_0(x_0^3).$$

Now by Assumption 4.2.1, the proposition is proved. □

Let $\epsilon > 0$ be a fixed positive constant independent of anything. Define the time T^* by

$$T^* := \sup\{t \in [0, T) \mid \rho > 0 \ \text{ pointwisely, } \ |w| < \max|w| + \epsilon \ \text{ and } \ |z| < \max|z| + \epsilon \ \text{ uniformly}\},$$

where $\max|z|$ and $\max|w|$ are the two constants defined in (4.2.2). It follows from Assumption 4.2.1 and the continuity of the solution that $T^* > 0$. Next, we want to prove that (w, z) are uniformly bounded up to the time T. We will first show that such uniform bounds indeed exist for $t \in [0, T^*)$, and then prove $T^* = T$.

Proposition 4.4.2. *When $t \in [0, T^*)$, one has*

$$\lambda_3(t, x) < \lambda_1(t, x) < \lambda_2(t, x) \quad \textit{for} \quad x \in \mathbb{R}, \tag{4.4.2}$$

and any two characteristics $x^i(t)$ and $x^j(t)$ $(i \neq j)$ intersect each other at most once.

Proof. First, for $t < T^*$, (4.4.2) follows directly from the definition of λ_i by (4.1.5).

Second, provided $i \neq j$, define the function $f(t)$ by

$$f(t) \coloneqq x^i(t) - x^j(t).$$

One observes that if for some $t_* \in [0, T^*)$, $f(t_*) = 0$, then $f'(t_*) \neq 0$. Actually, note that $f(t_*) = 0$ implies that

$$x^i(t_*) = x^j(t_*) \coloneqq x_*.$$

Then if $f'(t_*) = 0$, one can get

$$0 = f'(t_*) = \lambda_i(t_*, x_*) - \lambda_j(t_*, x_*),$$

which contradicts (4.4.2).

Now we assume that $x^i(t)$ and $x^j(t)$ intersect each other more than once in $[0, T^*)$. Then $f(t) = 0$ holds at least for two different points: $\bar{a}_1$ and $\bar{a}_2$. One can choose a time $\widetilde{t} \in (\bar{a}_1, \bar{a}_2)$ such that $f(\widetilde{t}) \neq 0$. Without loss of generality, assume $f(\widetilde{t}) > 0$. It is clear that such kind of points indeed exist, or else $f(t) \equiv 0$ for $t \in (\bar{a}_1, \bar{a}_2)$, which is impossible.

By the continuity of $f(t)$, there exists two points $\bar{a}_3 \in [\bar{a}_1, \bar{a}_2)$ and $\bar{a}_4 \in (\bar{a}_1, \bar{a}_2]$ such that $\widetilde{t} \in (\bar{a}_3, \bar{a}_4)$, $f(t) > 0$ for all $t \in (\bar{a}_3, \bar{a}_4)$, and $f(\bar{a}_3) = f(\bar{a}_4) = 0$, which implies that

$$x^i(\bar{a}_3) = x^j(\bar{a}_3) \coloneqq \bar{A} \quad \text{and} \quad x^i(\bar{a}_4) = x^j(\bar{a}_4) \coloneqq \bar{B}.$$

As $f'(\bar{a}_3) \neq 0$ and $f'(\bar{a}_4) \neq 0$, one has $f'(\bar{a}_3) > 0$ and $f'(\bar{a}_4) < 0$. Then

$$\lambda_i(\bar{a}_3, \bar{A}) > \lambda_j(\bar{a}_3, \bar{A}), \quad \lambda_i(\bar{a}_4, \bar{B}) < \lambda_j(\bar{a}_4, \bar{B}),$$

which contradicts (5.5.1). The proof of this lemma is complete. □

Lemma 4.4.1.

$$\sup_{(t,x)\in[0,T^*)\times\mathbb{R}}|w(t,x)|\leq\max|w|,\qquad\sup_{(t,x)\in[0,T^*)\times\mathbb{R}}|z(t,x)|\leq\max|z|,\tag{4.4.3}$$

where $\max|w|$ *and* $\max|z|$ *are the two constants defined in* (4.2.2).

Proof. We will focus only on w, and z can be dealt with similarly. For simplicity, let $\oint$ denote the integral along the characteristic $x^3(t)$, and $\not\oint$ the integral along the characteristic $x^2(t)$, i.e., for any function $f(t,x)$,

$$\oint_0^t f(\sigma)\mathrm{d}\sigma\equiv\int_0^t f(\sigma,x^3(\sigma))\mathrm{d}\sigma,\quad\not\oint_0^t f(\sigma)\mathrm{d}\sigma\equiv\int_0^t f(\sigma,x^2(\sigma))\mathrm{d}\sigma.$$

The proof is divided into two steps: integrating along the characteristics and using the Gronwall Inequality to show the boundedness of (w,z).

Step 1: integrating along the characteristics. By $(4.3.5)_1$ and Remark 4.3.1, one has that

$$\partial_t w(t,x^3(t))=-K(z-w)(t,x^3(t))\partial_t S(t,x^3(t)).$$

Hence

$$-K(z-w)|\partial_t S(t,x^3(t))|\leq\partial_t w(t,x^3(t))\leq K(z-w)|\partial_t S(t,x^3(t))|.\tag{4.4.4}$$

Define the function $I(t)$ by

$$I(t)\coloneqq\exp\left(K\int_0^t|\partial_\sigma S(\sigma,x^3(\sigma))|\mathrm{d}\sigma\right).\tag{4.4.5}$$

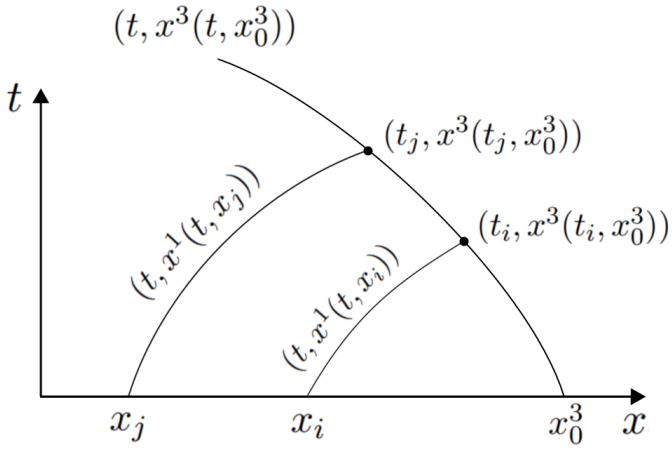
Multiplying (4.4.4) by $I(t)$ and integrating over $[0,t]$ along $x^3(t)$, one has

$$w_0(x_0^3)I(t)-KI(t)\oint_0^t z\frac{1}{I}|\partial_\sigma S|\mathrm{d}\sigma\leq w(t,x^3(t))\leq w_0(x_0^3)\frac{1}{I(t)}+K\frac{1}{I(t)}\oint_0^t zI|\partial_\sigma S|\mathrm{d}\sigma.\tag{4.4.6}$$

According to the continuity of $\partial_t S(t,x^3(t))$ and Proposition B.1 in Appendix B, one can split $[0,t]$ into countably many disjoint intervals $\bigcup_{n=0}^{\infty}[t_n,t_{n+1}]$ with $t_0=0$ and $t_\infty=t$, such that $\partial_t S(t,x^3(t))$ has a constant sign on each $[t_n,t_{n+1}]$.

Now we consider the characteristic $x^3(t,x_0^3)$ that starts from the point $(0,x_0^3)$. For any two

different time endpoints $t_i, t_j \in (0, T^*)$ obtained above, one can introduce two characteristics: $x^1(t, x_i)$ that intersects $x^3(t, x_0^3)$ at $(t_i, x^3(t_i, x_0^3))$, and intersects x-axis at $(0, x_i)$; and $x^1(t, x_j)$ that intersects $x^3(t, x_0^3)$ at $(t_j, x^3(t_j, x_0^3))$, and intersects x-axis at $(0, x_j)$. According to Proposition 4.4.2, each of the characteristics $x^1(t)$ intersects $x^3(t)$ at most once, so one finds that x_i is different from x_j provided $i \neq j$. See the figure below.



Then since $\partial_t S(t, x^1(t)) = 0$, one can obtain that

$$\begin{aligned} &\int_0^t |\partial_\sigma S(\sigma, x^3(\sigma))| \mathrm{d}\sigma = \sum_{i=1}^{\infty} \int_{t_{i-1}}^{t_i} |\partial_\sigma S(\sigma, x^3(\sigma))| \mathrm{d}\sigma \\ =& \sum_{i=1}^{\infty} |S(t_{i-1}, x^3(t_{i-1})) - S(t_i, x^3(t_i))| = \sum_{i=1}^{\infty} |S_0(x_{i-1}) - S_0(x_i)| \leq \|S_0\|_{BV}, \end{aligned} \tag{4.4.7}$$

which, along with (4.4.5), yields that

$$\frac{1}{I(t)} \leq 1 \leq I(t) \leq \exp(K\|S_0\|_{BV}) = V. \tag{4.4.8}$$

Then it follows from (4.4.6), (4.4.8), and the fact that the integrals in (4.4.6) depends on x_0^3, that

$$|w(t,x)| \leq \sup_{x \in \mathbb{R}} |w_0(x)| V + KV \sup_{x_0^3 \in \mathbb{R}} \int_0^t |z \partial_\sigma S| \mathrm{d}\sigma. \tag{4.4.9}$$

Similarly, for z satisfying $(4.3.5)_2$, one can also obtain

$$|z(t,x)| \leq \sup_{x\in\mathbb{R}} |z_0(x)|V + KV \sup_{x_0^2\in\mathbb{R}} \fint_0^t |w\partial_\sigma S|\mathrm{d}\sigma. \tag{4.4.10}$$

Substituting (4.4.10) into (4.4.9), one can obtain that

$$\begin{aligned} |w(t,x)| \leq & \sup_{x\in\mathbb{R}} |w_0(x)|V + KV^2 \sup_{x\in\mathbb{R}} |z_0(x)|\|S_0\|_{BV} \\ & + K^2V^2 \sup_{x_0^3\in\mathbb{R}} \fint_0^t |\partial_\sigma S|\Big(\sup_{x_0^2\in\mathbb{R}} \fint_0^\sigma |\partial_\phi S||w(\phi)|\mathrm{d}\phi\Big)\mathrm{d}\sigma, \end{aligned} \tag{4.4.11}$$

where one has used (4.4.7).

Step 2: using the Gronwall Inequality to show the boundedness of $|w|$. Replacing $w(t,x)$ by $w(\tau, x^2(\tau))$ in (4.4.11), multiplying by $|\partial_\tau S|$ and integrating over $[0,t]$ for τ along $x^2(t)$, one gets

$$\begin{aligned} \fint_0^t |w(\tau)||\partial_\tau S|\mathrm{d}\tau \leq & \sup_{x\in\mathbb{R}} |w_0(x)|V\|S_0\|_{BV} + KV^2 \sup_{x\in\mathbb{R}} |z_0(x)|\|S_0\|_{BV}^2 \\ & + K^2V^2 \fint_0^t |\partial_\tau S|\Big(\sup_{x_0^3\in\mathbb{R}} \fint_0^\tau |\partial_\sigma S| \sup_{x_0^2\in\mathbb{R}} \Big(\fint_0^\sigma |\partial_\phi S||w(\phi)|\mathrm{d}\phi\Big)\mathrm{d}\sigma\Big)\mathrm{d}\tau, \end{aligned} \tag{4.4.12}$$

where one has used (4.4.7).

Denote $W(t)$ by

$$W(t) \coloneqq \sup_{x_0^2\in\mathbb{R}} \fint_0^t |w(\tau)||\partial_\tau S|\mathrm{d}\tau.$$

Then one has

$$\begin{aligned} W(t) \leq & V \sup_{x\in\mathbb{R}} |w_0(x)|\|S_0\|_{BV} + KV^2 \sup_{x\in\mathbb{R}} |z_0(x)|\|S_0\|_{BV}^2 \\ & + K^2V^2 \sup_{x_0^2\in\mathbb{R}} \fint_0^t |\partial_\tau S|\Big(\sup_{x_0^3\in\mathbb{R}} \fint_0^\tau |\partial_\sigma S|W(\sigma)\mathrm{d}\sigma\Big)\mathrm{d}\tau, \end{aligned}$$

where

$$\fint_0^\tau |\partial_\sigma S|W(\sigma)\mathrm{d}\sigma \equiv \int_0^\tau |\partial_\sigma S(\sigma, x^3(\sigma))|W(\sigma)\mathrm{d}\sigma.$$

Noting that $W(\widetilde{\sigma}) \leq W(\tau)$ for all $\widetilde{\sigma} \in [0, \tau]$, one has

$$\begin{aligned} W(t) \leq & V \sup_{x\in\mathbb{R}} |w_0(x)| \|S_0\|_{BV} + KV^2 \sup_{x\in\mathbb{R}} |z_0(x)| \|S_0\|_{BV}^2 \\ & + K^2V^2 \sup_{x_0^2\in\mathbb{R}} \fint_0^t W(\tau)|\partial_\tau S| \left(\sup_{x_0^3\in\mathbb{R}} \fint_0^\tau |\partial_\sigma S| \mathrm{d}\sigma \right) \mathrm{d}\tau \\ \leq & V \sup_{x\in\mathbb{R}} |w_0(x)| \|S_0\|_{BV} + KV^2 \sup_{x\in\mathbb{R}} |z_0(x)| \|S_0\|_{BV}^2 \\ & + \|S_0\|_{BV} K^2V^2 \sup_{x_0^2\in\mathbb{R}} \fint_0^t W(\tau)|\partial_\tau S| \mathrm{d}\tau. \end{aligned}$$

According to the Gronwall inequality and (4.4.7), one has

$$W(t) \leq \|S_0\|_{BV} U_1 \exp\left(K^2V^2 \|S_0\|_{BV}^2\right),$$

which, along with (4.4.11) and (4.4.7), implies the desired bound for w.

□

Corollary 4.4.1. *There exists a constant $C > 0$ that depends on nothing but the initial data, A, γ and μ, such that*

$$\sup_{(t,x)\in[0,T^*)\times\mathbb{R}} |u| < C, \qquad \sup_{(t,x)\in[0,T^*)\times\mathbb{R}} \rho < C.$$

Proof. From the definition of Riemann invariants, one has

$$u = \frac{w+z}{2}, \quad \rho^{\frac{\gamma-1}{2}} = \frac{4\sqrt{\gamma A}}{\gamma-1} \exp\left(-\frac{\mu S}{2}\right)(z-w).$$

Now using the uniform bounds on the Riemann invariants, the uniform bound on the entropy S given by Proposition 4.4.1 and the fact that $\gamma > 1$, one obtains the result. □

Theorem 4.4.1. $T^* = T$.

Proof. Assume for contradiction that $T^* < T$. Via the continuity of our solution, in order to prove this theorem, we just need to show that the facts $|w| < \max|w| + \epsilon$ uniformly and $|z| < \max|z| + \epsilon$ uniformly both hold at $t = T^*$, and $\rho(T^*, x) > 0$ for $(t, x) \in [0, T^*] \times \mathbb{R}$.

First, according to Lemma 4.4.1, one has $|w| < \max|w| + \epsilon$ uniformly and $|z| < \max|z| + \epsilon$ uniformly both hold at $t = T^*$.

Second, we consider the positivity of ρ. By (4.3.5), one has

$$\partial_2(z-w) = (\lambda_3 - \lambda_2)\partial_x w - a(\partial_2 S + \partial_3 S), \tag{4.4.13}$$

where

$$\lambda_2 - \lambda_3 = 2\partial_\rho P > 0. \tag{4.4.14}$$

It follows from (4.1.8) and Remark 4.3.1 that

$$\sqrt{\partial_\rho P} = \widetilde{K}\,(z-w)\,, \quad |a| = \widetilde{K}(z-w),$$

which, together with (4.4.13)-(4.4.14), yields that

$$\begin{aligned}\partial_2(z-w) &\geq -\frac{2c^2\sqrt{\partial_\rho P}(c^2-u^2)}{c^4-u^2\partial_\rho P}\,|\partial_x w| - |a||\partial_2 S + \partial_3 S| \\ &\geq -\widetilde{K}\left(\frac{2c^2(c^2-u^2)}{c^4-u^2\partial_\rho P}|\partial_x w| + |\partial_2 S + \partial_3 S|\right)(z-w).\end{aligned}$$

Then it follows from the Gronwall inequality and Assumption 4.2.1 that

$$z - w \geq (z_0 - w_0)\exp\left(\int_0^t -\widetilde{K}\left(\frac{2c^2(c^2-u^2)}{c^4-u^2\partial_\rho P}|\partial_x w| + |\partial_2 S + \partial_3 S|\right)(\tau, x^2(\tau))\mathrm{d}\tau\right) > 0.$$

At last, one has

$$\rho^{\frac{\gamma-1}{2}} = \frac{\gamma-1}{2\sqrt{\gamma A}}\exp\left(-\frac{\mu S}{2}\right)(z-w) > 0 \quad \text{for} \quad (t,x) \in [0,T^*]\times\mathbb{R}.$$

The proof of this theorem is complete. □

4.5 Non-homogeneous ODEs of the Riemann variables

Lemma 4.5.1.

$$\begin{cases} \partial_3\widetilde{\alpha} = -\left[\dfrac{(\gamma+1)\widetilde{\alpha}+(3-\gamma)\widetilde{\beta}}{4}\right]\widetilde{\alpha} + \dfrac{\mu\eta\sqrt{\partial_\rho P}}{2\gamma}\left(\dfrac{\widetilde{\alpha}-\widetilde{\beta}}{2}\right), \\ \partial_2\widetilde{\beta} = -\left[\dfrac{(3-\gamma)\widetilde{\alpha}+(\gamma+1)\widetilde{\beta}}{4}\right]\widetilde{\beta} + \dfrac{\mu\eta\sqrt{\partial_\rho P}}{2\gamma}\left(\dfrac{\widetilde{\alpha}-\widetilde{\beta}}{2}\right), \\ \partial_1\eta = -\partial_x u\eta, \end{cases} \tag{4.5.1}$$

where $\widetilde{\alpha}$, $\widetilde{\beta}$ and η are defined in (4.1.9).

Proof. By differentiating $(4.1.1)_3$ with respect to x, one has η satisfies the equation

$$\partial_t\eta + u\partial_x\eta = -\partial_x u\eta. \tag{4.5.2}$$

We will only prove $(4.5.1)_1$, as $(4.5.1)_1$ follows from similar considerations. We begin by differentiating $(4.3.5)_1$ with respect to x. This gives

$$\partial_t\alpha + \lambda_3\partial_x\alpha + \partial_x\lambda_3\alpha = \partial_x a(\lambda_3 - u)\eta + a\left(\partial_t\eta + \lambda_3\partial_x\eta + \partial_x\lambda_3\eta\right). \tag{4.5.3}$$

Hence we get

$$\begin{aligned} \partial_3\alpha &= -\partial_x\lambda_3\alpha + \partial_x a(\lambda_3 - u)\eta + a\partial_3\eta + a\eta\partial_x\lambda_3 \\ &= -\partial_x\lambda_3(\alpha - a\eta) + \partial_x a(\lambda_3 - u)\eta + a\partial_3\eta. \end{aligned} \tag{4.5.4}$$

Recalling the defined quantity $\widetilde{\alpha} = \alpha - a\eta$, we arrive at

$$\begin{aligned} \partial_3\widetilde{\alpha} &= \partial_3\alpha - \partial_3(a\eta) \\ &= -\partial_x\lambda_3(\alpha - a\eta) + \partial_x a(\lambda_3 - u)\eta + a\partial_3\eta - a\partial_3\eta - \eta\partial_3 a \\ &= -\partial_x\lambda_3(\alpha - a\eta) + \eta\left((\lambda_3 - u)\partial_x a - \partial_3 a\right) \\ &= -\partial_x\lambda_3\widetilde{\alpha} - \eta(\partial_t a + u\partial_x a). \end{aligned} \tag{4.5.5}$$

A similar calculation yields

$$\partial_2\widetilde{\beta} = -\partial_x\lambda_2\widetilde{\beta} + \eta(\partial_t a + u\partial_x a)..\tag{4.5.6}$$

To prove the coupled ODEs, one therefore needs expressions for $\partial_x\lambda_2, \partial_x\lambda_3, \partial_t a$ and $\partial_x a$. Given the polytropic constitutive assumption (4.1.2), one can calculate that

$$\partial_x\lambda_2 = \partial_x u + \frac{\gamma-1}{2}\frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x\rho + \frac{\mu}{2}\sqrt{\partial_\rho P}\partial_x S.\tag{4.5.7}$$

$$\partial_x\lambda_3 = \partial_x u - \frac{\gamma-1}{2}\frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x\rho - \frac{\mu}{2}\sqrt{\partial_\rho P}\partial_x S.\tag{4.5.8}$$

To compute $\partial_t a + u\partial_x a$, notice that (4.1.2) gives

$$a = -\frac{\mu}{\gamma(\gamma-1)}\sqrt{\partial_\rho P}.$$

By using (4.1.2) and the chain rule, we can compute

$$\partial_t a + u\partial_x a = -\frac{\mu}{2\gamma}\frac{1}{\rho}\sqrt{\partial_\rho P}(\partial_t\rho + u\partial_x\rho) = \frac{\mu}{2\gamma}\sqrt{\partial_\rho P}\partial_x u.\tag{4.5.9}$$

In the last equality we have used the conservation of mass equation. Plugging the above calculations to (4.5.5) and (4.5.6) we get

$$\partial_3\widetilde{\alpha} = \left(-\partial_x u + \frac{\gamma-1}{2\rho}\sqrt{\partial_\rho P}\partial_x\rho + \frac{\mu}{2}\sqrt{\partial_\rho P}\right)\widetilde{\alpha} - \frac{\mu}{2\gamma}\sqrt{\partial_\rho P}\partial_x u\eta,\tag{4.5.10}$$

$$\partial_2\widetilde{\beta} = \left(-\partial_x u - \frac{\gamma-1}{2\rho}\sqrt{\partial_\rho P}\partial_x\rho - \frac{\mu}{2}\sqrt{\partial_\rho P}\right)\widetilde{\beta} + \frac{\mu}{2\gamma}\sqrt{\partial_\rho P}\partial_x u\eta.\tag{4.5.11}$$

A simple calculation also yields

$$\widetilde{\alpha} = \partial_x u - \frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x \rho - \frac{\mu}{\gamma}\sqrt{\partial_\rho P}\eta, \tag{4.5.12}$$

$$\widetilde{\beta} = \partial_x u + \frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x \rho + \frac{\mu}{\gamma}\sqrt{\partial_\rho P}\eta, \tag{4.5.13}$$

whence we get

$$\partial_x u = \frac{\widetilde{\alpha} + \widetilde{\beta}}{2}, \tag{4.5.14}$$

$$\frac{1}{\rho}\sqrt{\partial_\rho P}\partial_x \rho = \frac{\widetilde{\beta} - \widetilde{\alpha}}{2} - \frac{\mu}{\gamma}\sqrt{\partial_\rho P}\eta. \tag{4.5.15}$$

We are now in a position to rewrite $\partial_3\widetilde{\alpha}, \partial_2\widetilde{\beta}$ in terms of $\widetilde{\alpha}, \widetilde{\beta}$ and η. We have

$$\partial_3\widetilde{\alpha} = \left(-\frac{\widetilde{\alpha} + \widetilde{\beta}}{2} + \frac{\gamma - 1}{2}\left(\frac{\widetilde{\beta} - \widetilde{\alpha}}{2} - \frac{\mu}{\gamma}\sqrt{\partial_\rho P}\eta\right) + \frac{\mu}{2}\sqrt{\partial_\rho P}\eta\right)\widetilde{\alpha} - \frac{\mu}{2\gamma}\sqrt{\partial_\rho P}\left(\frac{\widetilde{\alpha} + \widetilde{\beta}}{2}\right)\eta. \tag{4.5.16}$$

Rearranging this equation yields the desired result. Finally, the same procedure works for the second equation.

□

Lemma 4.5.2.

$$\frac{\partial_x S(t, x^1(t))}{\rho(t, x^1(t))} = \frac{\partial_x S_0(x_0)}{\rho_0(x_0)}, \tag{4.5.17}$$

and

$$\partial_{xx} S(t, x^1(t)) - \frac{\partial_x S(t, x^1(t))}{\rho(t, x^1(t))}\partial_x \rho(t, x^1(t)) = \rho(t, x^1(t))^2 \left(\partial_{xx} S_0 \frac{1}{\rho_0^2} - \partial_x S_0 \frac{\partial_x \rho_0}{\rho_0^3}\right)(x_0), \tag{4.5.18}$$

where $x_0 = x^1(0)$.

Proof. Let us show (4.5.17) holds first. By Lemma 4.5.1, one has $\partial_1 \eta = -\partial_x u \eta$. Solving this equation alone the $x^1(t)$ characteristic, one has

$$\eta(t, x^1(t)) = \eta_0(x_0) \exp\left(-\int_0^t \partial_x u(s, x(s)) \mathrm{d}s\right). \tag{4.5.19}$$

Notice that by $(4.1.1)_1$, one has

$$\partial_t \rho + u\partial_x \rho = -\partial_x u\rho \iff \partial_1 \rho = -\partial_x u\rho. \tag{4.5.20}$$

Integrating along the x^1 characteristic, one has

$$\rho(t, x^1(t)) = \rho_0(x_0) \exp\left(-\int_0^t \partial_x u(s, x(s))\mathrm{d}s\right). \tag{4.5.21}$$

Hence (4.5.17) is proved. It remains to prove (4.5.18). By differentiating $(4.1.1)_1$ with respect to x, we have

$$\begin{aligned}\partial_{tx}\rho + \partial_{xx}(\rho u) = 0 &\iff \partial_{tx}\rho + u\partial_{xx}\rho + \partial_{xx}u\rho + 2\partial_x u\partial_x \rho = 0\\ &\iff \partial_1(\partial_x \rho) = -\partial_{xx}u\rho - 2\partial_x u\partial_x \rho.\end{aligned}$$

Define an integrating factor $I(t)$ by

$$I(t) \coloneqq \exp\left(2\int_0^t \partial_x u(s, x(s))\mathrm{d}s\right). \tag{4.5.22}$$

Multiplying the integrating factor with the equation above, we get

$$\partial_1(\partial_x \rho I) = -\partial_{xx}u\rho I. \tag{4.5.23}$$

By integrating along the $x^1(t)$ characteristic, one obtains

$$\partial_x \rho(t, x^1(t))I(t) = \partial_x \rho_0(x_0) - \int_0^t \partial_{xx}u(s, x(s))\rho(s, x(s))I(s)\mathrm{d}s. \tag{4.5.24}$$

By differentiating the equation $\partial_1 \eta = -\partial_x u\eta$ obtained in Lemma 4.5.1, one has

$$\partial_1(\partial_x \eta) = -\partial_{xx}u\eta - 2\partial_x u\partial_x \eta. \tag{4.5.25}$$

Multiplying the above equation by the integrating factor $I(t)$ and integrating along the $x^1(t)$

characteristic, one gets

$$\partial_x \eta(t, x^1(t)) I(t) = \partial_x \eta_0(x_0) - \int_0^t \partial_{xx} u(s, x(s)) \eta(s, x(s)) I(s) \mathrm{d}s. \tag{4.5.26}$$

Recall by (4.5.17) we have that

$$\eta(t, x^1(t)) = \frac{\eta_0(x_0)}{\rho_0(x_0)} \rho(t, x^1(t)). \tag{4.5.27}$$

Using this we get that

$$\partial_x \eta(t, x^1(t)) I(t) = \partial_x \eta_0(x_0) - \frac{\eta_0(x_0)}{\rho_0(x_0)} \int_0^t \partial_{xx} u(s, x(s)) \rho(s, x(s)) I(s) \mathrm{d}s. \tag{4.5.28}$$

Combining this with (4.5.24), the above equation becomes

$$\partial_x \eta(t, x^1(t)) I(t) = \partial_x \eta_0(x_0) + \frac{\eta_0(x_0)}{\rho_0(x_0)} \left(\partial_x \rho(t, x^1(t)) I(t) - \partial_x \rho_0(x_0) \right). \tag{4.5.29}$$

By rearranging the above equation we get

$$\partial_x \eta(t, x^1(t)) - \frac{\eta_0(x_0)}{\rho_0(x_0)} \partial_x \rho(t, x^1(t)) = \frac{1}{I(t)} \left(\partial_x \eta_0(x_0) - \frac{\eta_0(x_0)}{\rho_0(x_0)} \partial_x \rho_0(x_0) \right). \tag{4.5.30}$$

Now using (4.5.17) once again, as well as that

$$\exp\left(- \int_0^t \partial_x u(s, x(s)) \mathrm{d}s \right) = \frac{\rho(t, x^1(t))}{\rho_0(x_0)} \implies \frac{1}{I(t)} = \frac{\rho(t, x^1(t))^2}{\rho_0(x_0)^2}, \tag{4.5.31}$$

we will get

$$\partial_x \eta(t, x^1(t)) - \frac{\eta(t, x^1(t))}{\rho(t, x^1(t))} \partial_x \rho(t, x^1(t)) = \frac{\rho(t, x^1(t))^2}{\rho_0(x_0)^2} \left(\partial_x \eta_0 - \frac{\eta_0}{\rho_0} \partial_x \rho_0 \right)(x_0). \tag{4.5.32}$$

This is precisely (4.5.18) that we wish to prove. □

Remark 4.5.1. *From the uniform bounds on the Riemann Invariants, we have that ρ is bounded above uniformly, with the uniform bound depending only on the initial data. If we assume our*

initial data is bounded, then from the above lemma we can conclude that

$$\partial_x S(t,x) < C\rho, \tag{4.5.33}$$

and

$$\partial_{xx} S(t,x) - \frac{\partial_x S}{\rho}\partial_x \rho(t,x) < C\rho^2. \tag{4.5.34}$$

Where C is a positive constant depending only on the initial data.

Corollary 4.5.1.

$$\partial_1 \eta = \frac{\eta}{\rho}\partial_1 \rho. \tag{4.5.35}$$

$$\partial_3 \eta - \frac{\eta}{\rho}\partial_3 \rho = -\sqrt{\partial_\rho P}\left(\partial_{xx} S - \frac{\partial_x S}{\rho}\partial_x \rho\right). \tag{4.5.36}$$

$$\partial_2 \eta - \frac{\eta}{\rho}\partial_2 \rho = \sqrt{\partial_\rho P}\left(\partial_{xx} S - \frac{\partial_x S}{\rho}\partial_x \rho\right). \tag{4.5.37}$$

Proof. First of all, let us prove (4.5.35). By differentiating (4.5.17) with respect to t, one has

$$\partial_t\left(\frac{\eta(t,x^1(t))}{\rho(t,x^1(t))}\right) = \partial_1\left(\frac{\eta(t,x)}{\rho(t,x)}\right) = 0. \tag{4.5.38}$$

By the chain rule, one has

$$\frac{\partial_1 \eta}{\rho} - \frac{\eta}{\rho^2}\partial_1 \rho = 0. \tag{4.5.39}$$

Rearranging the above equation, (4.5.35) is proved.

To prove (4.5.36), one notes

$$\begin{aligned}
\partial_3 \eta - \frac{\eta}{\rho}\partial_3 \rho &= \partial_1 \eta - \sqrt{\partial_\rho P}\partial_{xx} S - \frac{\eta}{\rho}\partial_1 \rho + \sqrt{\partial_\rho P}\frac{\eta}{\rho}\partial_x \rho \\
&= \left(\partial_1 \eta - \frac{\eta}{\rho}\partial_1 \rho\right) - \sqrt{\partial_\rho P}\left(\partial_{xx} S - \frac{\eta}{\rho}\partial_x \rho\right) \\
&= -\sqrt{\partial_\rho P}\left(\partial_{xx} S - \frac{\eta}{\rho}\partial_x \rho\right).
\end{aligned}$$

Lastly, to prove (4.5.37). One notes

$$\begin{aligned}\partial_2\eta - \frac{\eta}{\rho}\partial_2\rho &= \partial_1\eta + \sqrt{\partial_\rho P}\partial_{xx}S - \frac{\eta}{\rho}\partial_1\rho - \sqrt{\partial_\rho P}\frac{\eta}{\rho}\partial_x\rho \\ &= \left(\partial_1\eta - \frac{\eta}{\rho}\partial_1\rho\right) + \sqrt{\partial_\rho P}\left(\partial_{xx}S - \frac{\eta}{\rho}\partial_x\rho\right) \\ &= \sqrt{\partial_\rho P}\left(\partial_{xx}S - \frac{\eta}{\rho}\partial_x\rho\right).\end{aligned}$$

□

Theorem 4.5.1. *The quantities $\widetilde{y}$ and $\widetilde{q}$ defined in* (4.1.10) *satisfy the following Riccati-type ODEs:*

$$\begin{cases}\partial_3\widetilde{y} = -a_0\widetilde{y}^2 + a_1 \\ \partial_2\widetilde{q} = -a_0\widetilde{q}^2 + a_1\end{cases} \tag{4.5.40}$$

where a_0 and a_1 are defined in (4.1.9).

Proof. We will only prove $\partial_3\widetilde{y} = -a_0\widetilde{y}^2 + a_1$, as the proof for $\widetilde{q}$ is similar. In the first instance, we are interested in decoupling the ODEs obtained in Lemma 4.5.1. Thus, we wish to express $\widetilde{\beta}$ in term of other quantities and substituting it in $(4.5.1)_1$ given by Lemma 4.5.1. By the definition of $\widetilde{\alpha}$ and $\widetilde{\beta}$ from (4.1.9), we see that

$$\widetilde{\beta} = \widetilde{\alpha} + \frac{2}{\rho}\sqrt{\partial_\rho P}\partial_x\rho + \frac{2\mu}{\gamma}\sqrt{\partial_\rho P}\eta. \tag{4.5.41}$$

By the conservation of mass $(4.1.1)_1$, one has

$$\begin{aligned}\partial_3\rho = \partial_t\rho + (u - \sqrt{\partial_\rho P})\partial_x\rho &= -\partial_x(\rho u) + (u - \sqrt{\partial_\rho P})\partial_x\rho \\ &= -\rho\partial_x u - \sqrt{\partial_\rho P}\partial_x\rho \\ &= -\rho\left(\frac{\widetilde{\alpha} + \widetilde{\beta}}{2}\right) - \sqrt{\partial_\rho P}\partial_x\rho.\end{aligned}$$

So

$$\frac{2}{\rho}\sqrt{\partial_\rho P}\partial_x\rho = -\frac{2}{\rho}\partial_3\rho - (\widetilde{\alpha} + \widetilde{\beta}). \tag{4.5.42}$$

Substituting this expression into (4.5.41), we get

$$\widetilde{\beta} = \widetilde{\alpha} - \frac{2}{\rho}\partial_3\rho - (\widetilde{\alpha} + \widetilde{\beta}) + \frac{2\mu}{\gamma}\sqrt{\partial_\rho P}\eta \iff \widetilde{\beta} = -\frac{1}{\rho}\partial_3\rho + \frac{\mu}{\gamma}\sqrt{\partial_\rho P}\eta. \tag{4.5.43}$$

By substituting $\widetilde{\beta}$ into $(4.5.1)_1$ and simplify, one has

$$\partial_3\widetilde{\alpha} = -\frac{(\gamma+1)}{4}\widetilde{\alpha}^2 - \frac{(\gamma-3)}{4\rho}\widetilde{\alpha}\partial_3\rho + \frac{\mu(\gamma-2)}{4\gamma}\sqrt{\partial_\rho P}\eta\widetilde{\alpha} + \frac{\mu}{4\gamma\rho}\sqrt{\partial_\rho P}\eta\partial_3\rho - \frac{\mu^2}{4\gamma^2}\partial_\rho P\eta^2.$$

Hence

$$\partial_3\widetilde{\alpha} + \frac{(\gamma-3)}{4\rho}\widetilde{\alpha}\partial_3\rho - \frac{\mu}{4\gamma\rho}\sqrt{\partial_\rho P}\eta\partial_3\rho = -\frac{(\gamma+1)}{4}\widetilde{\alpha}^2 + \frac{\mu(\gamma-2)}{4\gamma}\sqrt{\partial_\rho P}\eta\widetilde{\alpha} - \frac{\mu^2}{4\gamma^2}\partial_\rho P\eta^2.$$

Multiplying the above equation by $\rho^{\frac{\gamma-3}{4}}$, one gets

$$\begin{aligned}\rho^{\frac{\gamma-3}{4}}\partial_3\widetilde{\alpha} + \widetilde{\alpha}\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right) - \frac{\mu}{\gamma(\gamma-3)}\eta\sqrt{\partial_\rho P}\,\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right) &= -\frac{(\gamma+1)}{4}\rho^{\frac{\gamma-3}{4}}\widetilde{\alpha}^2 \\ &+ \frac{\mu(\gamma-2)\eta\sqrt{\partial_\rho P}}{4\gamma}\widetilde{\alpha}\rho^{\frac{\gamma-3}{4}} - \frac{\mu^2}{4\gamma^2}\eta^2\rho^{\frac{\gamma-3}{4}}\partial_\rho P.\end{aligned} \tag{4.5.44}$$

This expression can be further simplified to

$$\begin{aligned}\partial_3\left(\widetilde{\alpha}\rho^{\frac{\gamma-3}{4}}\right) - \frac{\mu}{\gamma(\gamma-3)}\sqrt{\partial_\rho P}\eta\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right) &= -\frac{(\gamma+1)}{4}\rho^{\frac{\gamma-3}{4}}\widetilde{\alpha}^2 \\ &+ \frac{\mu(\gamma-2)}{4\gamma}\sqrt{\partial_\rho P}\eta\widetilde{\alpha}\rho^{\frac{\gamma-3}{4}} - \frac{\mu^2}{4\gamma^2}(\partial_\rho P)\eta^2\rho^{\frac{\gamma-3}{4}}.\end{aligned} \tag{4.5.45}$$

We wish to express the left hand side of the above equation as a total derivative of ∂_3, so we wish to find the expression of $\partial_3\left(\sqrt{\partial_\rho P}\eta\rho^{\frac{\gamma-3}{4}}\right)$. First of all, because $P(\rho, S) \coloneqq A\rho^\gamma e^{\mu S}$, we have

$$\begin{aligned}\partial_3(\sqrt{\partial_\rho P}) &= \partial_\rho(\sqrt{\partial_\rho P})\partial_3\rho + \partial_S(\sqrt{\partial_\rho P})\partial_3 S \\ &= \frac{\gamma-1}{2}\frac{1}{\rho}\sqrt{\partial_\rho P}\partial_3\rho + \frac{\mu}{2}\sqrt{\partial_\rho P}\partial_3 S \\ &= \frac{\gamma-1}{2}\frac{1}{\rho}\sqrt{\partial_\rho P}\partial_3\rho - \frac{\mu}{2}\sqrt{\partial_\rho P}\eta.\end{aligned}$$

Using this we have

$$\begin{aligned}
&\partial_3\left(\sqrt{\partial_\rho P}\eta\rho^{\frac{\gamma-3}{4}}\right)\\
=&\partial_3\left(\sqrt{\partial_\rho P}\right)\eta\rho^{\frac{\gamma-3}{4}}+\sqrt{\partial_\rho P}\,(\partial_3\eta)\,\rho^{\frac{\gamma-3}{4}}+\sqrt{\partial_\rho P}\eta\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right)\\
=&\frac{\gamma-1}{2}\frac{1}{\rho}\sqrt{\partial_\rho P}(\partial_3\rho)\eta\rho^{\frac{\gamma-3}{4}}-\frac{\mu}{2}\sqrt{\partial_\rho P}\eta^2\rho^{\frac{\gamma-3}{4}}+\sqrt{\partial_\rho P}(\partial_3\eta)\rho^{\frac{\gamma-3}{4}}+\sqrt{\partial_\rho P}\eta\,\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right)\\
=&\frac{3\gamma-5}{\gamma-3}\sqrt{\partial_\rho P}\eta\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right)-\frac{\mu}{2}\sqrt{\partial_\rho P}\eta^2\rho^{\frac{\gamma-3}{4}}+\sqrt{\partial_\rho P}(\partial_3\eta)\rho^{\frac{\gamma-3}{4}}.
\end{aligned}$$

At this point, we can use (4.5.36) from Corollary 4.5.1 to get that

$$\begin{aligned}
&\partial_3\left(\sqrt{\partial_\rho P}\,\eta\,\rho^{\frac{\gamma-3}{4}}\right)\\
=&\frac{3\gamma-5}{\gamma-3}\sqrt{\partial_\rho P}\,\eta\,\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right)-\frac{\mu}{2}\sqrt{\partial_\rho P}\eta^2\rho^{\frac{\gamma-3}{4}}+\sqrt{\partial_\rho P}\,(\partial_3\eta)\,\rho^{\frac{\gamma-3}{4}}\\
=&\frac{3\gamma-5}{\gamma-3}\sqrt{\partial_\rho P}\,\eta\,\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right)-\frac{\mu}{2}\sqrt{\partial_\rho P}\eta^2\rho^{\frac{\gamma-3}{4}}+\sqrt{\partial_\rho P}\rho^{\frac{\gamma-3}{4}}\left(\partial_3\eta-\frac{\eta}{\rho}\partial_3\rho\right)+\sqrt{\partial_\rho P}\rho^{\frac{\gamma-3}{4}}\frac{\eta}{\rho}\partial_3\rho\\
=&\frac{3\gamma-1}{\gamma-3}\sqrt{\partial_\rho P}\,\eta\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right)-\frac{\mu}{2}\sqrt{\partial_\rho P}\,\eta^2\,\rho^{\frac{\gamma-3}{4}}-(\partial_\rho P)\,\rho^{\frac{\gamma-3}{4}}\left(\partial_{xx}S-\frac{\eta}{\rho}\partial_x\rho\right).
\end{aligned}$$

From this we can conclude that

$$\begin{aligned}
-\frac{\mu\sqrt{\partial_\rho P}\eta}{\gamma(\gamma-3)}\partial_3\left(\rho^{\frac{\gamma-3}{4}}\right)=&-\frac{\mu^2}{2\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\,\eta^2\,\rho^{\frac{\gamma-3}{4}}\\
&-\frac{\mu}{\gamma(3\gamma-1)}\,\partial_\rho P\,\rho^{\frac{\gamma-3}{4}}\left(\partial_{xx}S-\frac{\eta}{\rho}\partial_x\rho\right)-\frac{\mu}{\gamma(3\gamma-1)}\,\partial_3\left(\sqrt{\partial_\rho P}\,\eta\,\rho^{\frac{\gamma-3}{4}}\right).
\end{aligned}\tag{4.5.46}$$

Substituting this expression into (4.5.45) and simplifying, one gets

$$\begin{aligned}
\partial_3\left(\widetilde{\alpha}\rho^{\frac{\gamma-3}{4}}-\frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\,\eta\,\rho^{\frac{\gamma-3}{4}}\right)=&-\frac{(\gamma+1)}{4}\rho^{\frac{\gamma-3}{4}}\widetilde{\alpha}^2+\frac{\mu(\gamma-2)}{4\gamma}\sqrt{\partial_\rho P}\,\eta\widetilde{\alpha}\,\rho^{\frac{\gamma-3}{4}}\\
&-\frac{\mu^2(\gamma-1)}{4\gamma^2(3\gamma-1)}(\partial_\rho P)\,\eta^2\,\rho^{\frac{\gamma-3}{4}}+\frac{\mu}{\gamma(3\gamma-1)}(\partial_\rho P)\,\rho^{\frac{\gamma-3}{4}}\left(\partial_{xx}S-\frac{\eta}{\rho}\,\partial_x\rho\right).
\end{aligned}\tag{4.5.47}$$

Now let us define y by

$$y\coloneqq\widetilde{\alpha}\rho^{\frac{\gamma-3}{4}}-\frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\,\eta\rho^{\frac{\gamma-3}{4}}.\tag{4.5.48}$$

Then we have

$$\widetilde{\alpha}=\rho^{-\frac{\gamma-3}{4}}\,y+\frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\,\eta\tag{4.5.49}$$

and

$$\widetilde{\alpha}^2 = \rho^{-\frac{\gamma-3}{2}}\, y^2 + \frac{2\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\,\eta\,\rho^{-\frac{\gamma-3}{4}}\, y + \frac{\mu^2}{\gamma^2(3\gamma-1)^2}\,(\partial_\rho P)\,\eta^2. \tag{4.5.50}$$

Substituting these expressions into (4.5.47) and simplifying, we have an ODE for y, namely

$$\begin{aligned}\partial_3 y = &-\frac{(\gamma+1)}{4}\rho^{\frac{3-\gamma}{4}}\, y^2 + \frac{3\mu(\gamma-3)}{4(3\gamma-1)}\sqrt{\partial_\rho P}\,\eta\, y \\ &- \frac{\mu^2}{\gamma(3\gamma-1)^2}(\partial_\rho P)\,\eta^2\,\rho^{\frac{\gamma-3}{4}} + \frac{\mu}{\gamma(3\gamma-1)}\,(\partial_\rho P)\,\rho^{\frac{\gamma-3}{4}}\left(\partial_{xx}S - \frac{\eta}{\rho}\partial_x\rho\right).\end{aligned} \tag{4.5.51}$$

Now we note that

$$\partial_3 S = \partial_t S + u\,\partial_x S - \sqrt{\partial_\rho P}\,\partial_x S = -\sqrt{\partial_\rho P}\,\eta. \tag{4.5.52}$$

Hence, if we define an integrating factor $I(S)$ by

$$I(S) \coloneqq \exp\left(\frac{3\mu(\gamma-3)}{4(3\gamma-1)}\,S\right), \tag{4.5.53}$$

and define $\widetilde{y}$ by

$$\widetilde{y} = y\,I(S), \tag{4.5.54}$$

then by multiplying (4.5.51) with $I(S)$, we get that

$$\begin{aligned}\partial_3\widetilde{y} = &-\frac{(\gamma+1)}{4}\,\rho^{\frac{3-\gamma}{4}}\,\widetilde{y}^2\frac{1}{I(S)} - \frac{\mu^2}{\gamma(3\gamma-1)^2}(\partial_\rho P)\,\eta^2\,\rho^{\frac{\gamma-3}{4}}\,I(S) \\ &+ \frac{\mu}{\gamma(3\gamma-1)}(\partial_\rho P)\,\rho^{\frac{\gamma-3}{4}}\,I(S)\left(\partial_{xx}S - \frac{\eta}{\rho}\partial_x\rho\right).\end{aligned} \tag{4.5.55}$$

This is exactly what we wish to prove. □

4.6 Decoupled ODE inequalities

Lemma 4.6.1.

$$\begin{cases}\partial_3\widetilde{y} \le -a_0\left(\widetilde{y}^2 - N^2\right), \\ \partial_2\widetilde{q} \le -a_0\left(\widetilde{q}^2 - N^2\right),\end{cases} \tag{4.6.1}$$

where a_0 is defined in (4.1.9)*, N is defined in* (4.2.3)*.*

Proof. In order to prove the lemma, we just need to show

$$\left|\frac{a_1}{a_0}\right| \leq N^2. \tag{4.6.2}$$

By using the explicate expressions for a_0 and a_1 given in Theorem 4.5.1, as well as using Lemma 4.5.2, we can get

$$\begin{aligned}\frac{a_1}{a_0} &= \frac{4\mu I^2}{\gamma(\gamma+1)(3\gamma-1)}\partial_\rho P \rho^{\frac{\gamma-3}{2}}\left[-\frac{\mu}{3\gamma-1}\eta^2 + \left(\partial_{xx}S - \frac{\partial_x S}{\rho}\partial_x\rho\right)\right] \\ &= \frac{4\mu I^2}{\gamma(\gamma+1)(3\gamma-1)}\partial_\rho P \rho^{\frac{\gamma+1}{2}}\left[-\frac{\mu}{3\gamma-1}\left(\frac{\eta_0}{\rho_0}\right)^2 + \frac{1}{\rho_0^2}\left(\partial_{xx}S_0 - \frac{\partial_x S_0}{\rho_0}\partial_x\rho_0\right)\right].\end{aligned}$$

By the uniform bound on entropy S given by Proposition 4.4.1, the uniform bound on the Riemann invariants, the above quantity is bounded by N, hence our lemma is proved. □

4.7 Density lower bound for the adiabatic exponent $\gamma \in (1, 3)$

Lemma 4.7.1. *We have the following bounds on $\widetilde{y}$ and $\widetilde{q}$.*

$$\begin{cases}\widetilde{y} \leq \max\{N, \sup_x \widetilde{y}_0(x)\} \coloneqq \overline{Y}, \\ \widetilde{q} \leq \max\{N, \sup_x \widetilde{q}_0(x)\} \coloneqq \overline{Q}.\end{cases} \tag{4.7.1}$$

Proof. We will focus only on $\widetilde{q}$, and $\widetilde{y}$ can be dealt with similarly. We consider 3 cases: $\widetilde{q}_0(x_0^2) < N$, $\widetilde{q}_0(x_0^2) > N$, and $\widetilde{q}_0(x_0^2) = N$.

First of all, consider the case $\widetilde{q}_0(x_0^2) < N$. By the continuity of $\widetilde{q}$, in a small open interval around $t = 0$, $\widetilde{q}(t, x^2(t)) < N$. Define time $\widetilde{T}$ by

$$\widetilde{T} \coloneqq \inf_t\{t : \widetilde{q}(t, x^2(t)) > N\}. \tag{4.7.2}$$

Then $\widetilde{T} > 0$. Note for $t \in [0, \widetilde{T}]$, one has $\widetilde{q}(t, x^2(t)) \leq N$.

It remains to show that $T = \widetilde{T}$. Suppose for contradiction that $\widetilde{T} < T$. Via the continuity of

$\widetilde{q}(t, x^2(t))$, there exists some small $\upsilon > 0$, such that if $t \in (\widetilde{T}, \widetilde{T} + \upsilon)$, then $\widetilde{q}(t, x^2(t)) > N$. By $(4.6.1)_2$, one has

$$\partial_2 \widetilde{q} \leq 0 \quad \text{for} \quad t \in (\widetilde{T}, \widetilde{T} + \upsilon).$$

Integrating this inequality along $x^2(t)$ over $(\widetilde{T}, \widetilde{T} + \delta)$ for any δ satisfying $0 < \delta < \upsilon$, one gets

$$\widetilde{q}(\widetilde{T} + \delta, x^2(\widetilde{T} + \delta)) \leq \widetilde{q}(\widetilde{T}, x^2(\widetilde{T})) \leq N.$$

This is a contradiction to $\widetilde{q}(t, x^2(t)) > N$ if $t \in (\widetilde{T}, \widetilde{T} + \upsilon)$, so one must have $\widetilde{T} = T$.

Second, consider the case $\widetilde{q}_0(x_0^2) > N$. Once again by the continuity of $\widetilde{q}$, in a small open interval around $t = 0$, one has $\widetilde{q}(t, x^2(t)) > N$. Define time $\widetilde{T}$ by

$$\widetilde{T} \coloneqq \inf_t \{t : \widetilde{q}(t, x^2(t)) < N\}.$$

Then $\widetilde{T} > 0$. Note for $t \in [0, \widetilde{T}]$, $\widetilde{q}(t, x^2(t)) \geq N$. By $(4.6.1)_2$, one has

$$\partial_2 \widetilde{q} \leq 0 \quad \text{for} \quad t \in [0, \widetilde{T}].$$

Integrating this inequality along $x^2(t)$ over $[0, t]$ for any $t \in (0, \widetilde{T}]$, one has

$$\widetilde{q}(t, x^2(t)) \leq \widetilde{q}_0(x_0^2) \leq \sup_x \widetilde{q}_0(x) \quad \text{for} \quad t \in [0, \widetilde{T}],$$

so $\widetilde{q}$ has the upper bound $\overline{Q}$ in this interval. Now by the definition of $\widetilde{T}$ and the continuity of $\widetilde{q}$, there exists some $\upsilon > 0$ such that $\widetilde{q}(t, x^2(t)) < N$ on $t \in (\widetilde{T}, \widetilde{T} + \upsilon)$. In particular, fixing a δ such that $0 < \delta < \upsilon$, one has $\widetilde{q}(\widetilde{T} + \delta, x^2(\widetilde{T} + \delta)) < N$. Then the rest of the proof is reduced to the case $\widetilde{q}_0(x_0^2) < N$. Hence $\widetilde{q}(t, x^2(t)) \leq \overline{Q}$.

Finally, the case $\widetilde{q}_0(x_0^2) = N$ can be dealt with similarly. □

Lemma 4.7.2. *Given $1 < \gamma < 3$, there exist a positive constant $D > 0$ depending only on the initial data, A, μ and γ, such that*

$$\rho^{\frac{3-\gamma}{4}}(t, x^2(t)) \geq \frac{D}{t+1}, \quad \rho^{\frac{3-\gamma}{4}}(t, x^3(t)) \geq \frac{D}{t+1}. \tag{4.7.3}$$

Proof. We will only do the proof for $\rho^{\frac{3-\gamma}{4}}(t, x^3(t))$, as the proof for the other case is similar. From (4.3.5), we can get that

$$\begin{aligned}\partial_3(z-w) &= \partial_3 z - a\partial_3 S \\ &= \partial_2 z + (\lambda_3 - \lambda_2)\partial_x z - a\partial_3 S \\ &= -a\partial_2 S - a\partial_3 S + (\lambda_3 - \lambda_2)(\widetilde{\beta} - a\eta) \\ &= 2a\eta\sqrt{\partial_\rho P} - 2\sqrt{\partial_\rho P}\widetilde{\beta}.\end{aligned}$$

Recall that

$$\widetilde{\beta} = \frac{1}{I}\widetilde{q}\rho^{\frac{3-\gamma}{4}} - \frac{\mu}{(3\gamma-1)\gamma}\sqrt{\partial_\rho P}\eta. \tag{4.7.4}$$

Hence upon simplification we get that

$$\partial_3(z-w) = 2\sqrt{\partial_\rho P}\eta\left(a + \frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\right) - 2\sqrt{\partial_\rho P}\frac{1}{I}\widetilde{q}\rho^{\frac{3-\gamma}{4}}\widetilde{q}. \tag{4.7.5}$$

Now using Lemma 4.5.2, the above equation becomes

$$\partial_3(z-w) = 2\sqrt{\partial_\rho P}\rho\frac{\eta_0}{\rho_0}\left(a + \frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\right) - 2\sqrt{\partial_\rho P}\frac{1}{I}\rho^{\frac{3-\gamma}{4}}\widetilde{q}. \tag{4.7.6}$$

Dividing both sides by $\sqrt{\partial_\rho P}\rho^{\frac{3-\gamma}{4}}$ and using Lemma 4.7.1, we get

$$\begin{aligned}\frac{1}{\sqrt{\partial_\rho P}}\rho^{\frac{3-\gamma}{4}}\partial_3(z-w) &= 2\rho^{\frac{\gamma+1}{4}}\frac{\eta_0}{\rho_0}\left(a + \frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\right) - 2\frac{1}{I}\widetilde{q} \\ &\geq 2\rho^{\frac{\gamma+1}{4}}\inf_x\frac{\eta_0}{\rho_0}\left(a + \frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\right) - 2\frac{1}{I}\overline{Q}.\end{aligned}$$

Now using the definition of I and the uniform bound on S, we can find a positive constant $D_1 > 0$ such that

$$-2\frac{1}{I}\overline{Q} \geq -\frac{D_1}{2}. \tag{4.7.7}$$

We can also assume the constant D_1 is such that

$$2\rho^{\frac{\gamma+1}{4}}\inf_x\frac{\eta_0}{\rho_0}\left(a + \frac{\mu}{\gamma(3\gamma-1)}\sqrt{\partial_\rho P}\right) \geq -\frac{D_1}{2}. \tag{4.7.8}$$

Hence we get that

$$\frac{1}{\sqrt{\partial_\rho P}}\rho^{\frac{3-\gamma}{4}}\partial_3(z-w) \geq -D_1. \tag{4.7.9}$$

By direct calculation one can verify that

$$\frac{1}{\sqrt{\partial_\rho P}} = \frac{4}{(\gamma-1)(z-w)}, \tag{4.7.10}$$

and

$$\rho^{\frac{\gamma-3}{4}} = \left(\frac{\gamma-1}{4\sqrt{\gamma A}}\exp\left(-\frac{\mu}{2}S\right)\right)^{\frac{\gamma-3}{2(\gamma-1)}}(z-w)^{\frac{\gamma-3}{2(\gamma-1)}}. \tag{4.7.11}$$

Substituting these into (4.7.9), we get that

$$\frac{4}{(\gamma-1)(z-w)}\left(\frac{\gamma-1}{4\sqrt{\gamma A}}\exp\left(-\frac{\mu}{2}S\right)\right)^{\frac{\gamma-3}{2(\gamma-1)}}(z-w)^{\frac{\gamma-3}{2(\gamma-1)}}\partial_3(z-w) \geq -D_1. \tag{4.7.12}$$

Upon simplification and by using the uniform bounds on S, we can find some constant $D_2 > 0$ such that

$$\frac{2(\gamma-1)}{\gamma-3}\partial_3\left[(z-w)^{\frac{\gamma-3}{2(\gamma-1)}}\right] \geq -D_2. \tag{4.7.13}$$

Now as we assumed that $1 < \gamma < 3$, we have $\gamma - 3 < 0$, hence

$$\partial_3\left[(z-w)^{\frac{\gamma-3}{2(\gamma-1)}}\right] \leq \frac{3-\gamma}{2(\gamma-1)}D_2 = D_3. \tag{4.7.14}$$

Now integrating along the x^3 characteristic, we get

$$(z-w)^{\frac{\gamma-3}{2(\gamma-1)}}(t, x_3(t)) \leq \sup_x (z_0 - w_0)^{\frac{\gamma-3}{2(\gamma-1)}} + D_3 t = D_4 + D_3 t. \tag{4.7.15}$$

By (4.7.11), we get

$$\rho^{\frac{\gamma-3}{4}}\left(\frac{\gamma-1}{4\sqrt{\gamma A}}\exp\left(-\frac{\mu}{2}S\right)\right)^{\frac{3-\gamma}{2(\gamma-1)}} \leq D_4 + D_3 t. \tag{4.7.16}$$

Once again using the uniform bounds on S, we can find some positive constants D_5 and D_6 such that

$$\rho^{\frac{\gamma-3}{4}} \leq D_5 + D_6 t. \tag{4.7.17}$$

This is the desired result. □

4.8 Singularity formation with strong compression: Proof of Theorem 4.2.1

In order to prove the theorem, we just need either $\widetilde{y}$ or $\widetilde{q}$ blows up. We will show $\widetilde{y}$ blows up under the assumption (4.2.4). From our assumption that $\widetilde{y}_0(x^*) < -N$, we can conclude that $\exists \epsilon > 0$ such that $\widetilde{y}_0(x^*) \leq -N\sqrt{1+\epsilon}$. By the continuity of $\widetilde{y}$, we can find some time $\widetilde{T} > 0$ defined by

$$\inf_t \{t : \widetilde{y}(t, x^3(t)) > -N\}. \tag{4.8.1}$$

So we have $\widetilde{y}(t, x^3(t)) \leq -N$ for all $t \in [0, \widetilde{T}]$, where $x^3(t)$ is the characteristic defined by λ_3 starting from x^*. Now we claim that $\widetilde{T} = T$, where T is the maximum time for the smooth solution to exist. Indeed, for contradiction assume that $\widetilde{T} < T$. Then, by the continuity of $\widetilde{y}$, we must have some $\delta > 0$ such that $\widetilde{y}(t, x^3(t)) > -N$ for all $t \in (\widetilde{T}, \widetilde{T} + \delta)$. However, let us consider the interval $t \in [0, \widetilde{T}]$, as $\widetilde{y} < -N$ in this interval, by Lemma 4.6.1, we get that

$$\partial_3 \widetilde{y} \leq 0 \implies \widetilde{y}(\widetilde{T}, x^3(\widetilde{T})) \leq y_0(x^*) \leq -N\sqrt{1+\epsilon}. \tag{4.8.2}$$

As $\widetilde{T} < T$, so we are still in the region of smooth solution. Then use once again the continuity of $\widetilde{y}$, we can conclude $\exists \delta_1 > 0$ such that $\widetilde{y}(t, x^3(t)) < -N$ for all $t \in (\widetilde{T}, \widetilde{T} + \delta_1)$. This is a contradiction. Hence we must have that $\widetilde{T} = T$.

Now that we have established that $\widetilde{T} = T$, using Lemma 4.6.1 we can get that

$$\partial_3 \widetilde{y} \leq 0 \implies \widetilde{y}(t, x^3(t)) \leq y_0(x^*) \leq -N\sqrt{1+\epsilon} \quad \forall t \in [0, T). \tag{4.8.3}$$

From this we can conclude that

$$\widetilde{y}^2 \geq N^2(1+\epsilon) \iff N^2 \leq \frac{\widetilde{y}^2}{1+\epsilon}. \tag{4.8.4}$$

Using this, by Lemma 4.6.1 we get

$$\begin{aligned}\partial_3\widetilde{y} &\leq -\frac{\gamma+1}{4}\rho^{-\frac{\gamma-3}{4}}\frac{1}{I}\widetilde{y^2} + \frac{\gamma+1}{4}\rho^{-\frac{\gamma-3}{4}}\frac{1}{I}N^2,\\ &\leq -\frac{\gamma+1}{4}\rho^{-\frac{\gamma-3}{4}}\frac{1}{I}\widetilde{y^2} + \frac{\gamma+1}{4}\rho^{-\frac{\gamma-3}{4}}\frac{1}{I}\frac{\widetilde{y}^2}{1+\epsilon}\\ &= -\frac{\epsilon}{1+\epsilon}\frac{\gamma+1}{4}\rho^{-\frac{\gamma-3}{4}}\frac{1}{I}\widetilde{y}^2.\end{aligned}$$

Dividing by $\widetilde{y}^2$, one gets

$$-\partial_2\left(\frac{1}{\widetilde{y}}\right) \leq -\frac{\epsilon}{1+\epsilon}\frac{\gamma+1}{4}\rho^{-\frac{\gamma-3}{4}}\frac{1}{I}. \tag{4.8.5}$$

By integrating, one has

$$\frac{1}{\widetilde{y}(t,x^3(t))} \geq \frac{1}{\widetilde{y}_0(x^*)} + \frac{\epsilon}{1+\epsilon}\frac{\gamma+1}{4}\int_0^t \rho^{-\frac{\gamma-3}{4}}\frac{1}{I}\mathrm{d}t. \tag{4.8.6}$$

We see now in order to prove the theorem, we need the right hand side of the above inequality to be greater than zero, so we just need that

$$\int_0^t \rho^{-\frac{\gamma-3}{4}}\frac{1}{I}\mathrm{d}t \quad \to \infty \quad \text{as} \quad t \to \infty. \tag{4.8.7}$$

By the uniform bound on S, we can conclude there exists positive constants C_1 and C_2 such that $C_1 < I < C_2$. Hence we only need to consider $\rho^{\frac{3-\gamma}{4}}$. We see that if $\gamma \geq 3$, we have $\rho^{\frac{3-\gamma}{4}} \geq \rho_{\max}^{\frac{3-\gamma}{4}}$, so (4.8.7) will be satisfied. Otherwise if $1 < \gamma < 3$, then by using Lemma 4.7.2, (4.8.7) is also satisfied. Hence the theorem is proved.

Chapter 5

Non-isentropic Relativistic Compressible Euler Equations

For $t \geq 0$ and $x \in \mathbb{R}$, the non-isentropic relativistic compressible Euler equations for C^1 solutions are given by:

$$\begin{cases} \partial_t \left(\dfrac{c^4\rho + Pu^2}{c^2(c^2-u^2)} \right) + \partial_x \left(\dfrac{(c^2\rho+P)u}{c^2-u^2} \right) = 0, \\ \partial_t \left(\dfrac{(c^2\rho+P)u}{c^2-u^2} \right) + \partial_x \left(\dfrac{(c^2\rho+P)u^2}{c^2-u^2} \right) + \partial_x P = 0, \\ \partial_t S + u\partial_x S = 0, \end{cases} \tag{5.0.1}$$

with the initial data

$$(\rho, u, S)|_{t=0} = (\rho_0, u_0, S_0)(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{5.0.2}$$

5.1 A note on the pressure term

We first consider the constitutive relation. Denote e the specific internal energy, n the rest-mass density, and θ is the absolute temperature. Usually, one has

$$\rho = n\left(1 + \frac{e}{c^2}\right), \tag{5.1.1}$$

which can be seen in [18, 28, 29, 42, 54]. For ideal gas, it holds that

$$P = R\theta n, \tag{5.1.2}$$

where $R > 0$ is the gas constant. For the polytropic gas, one still has

$$e = C_v\theta,$$

for the specific heat at constant volume $C_v = \frac{R}{\gamma-1}$ with $\gamma > 1$, which, along with (5.1.2), yields that

$$P = R/C_v en = (\gamma - 1)en. \tag{5.1.3}$$

Based on the above relations and the first law of thermodynamics:

$$\theta \mathrm{d}S = \mathrm{d}e + P\mathrm{d}n^{-1},$$

one can obtain that

$$S - S_* = R\log n^{-1} + \frac{R}{\gamma - 1}\log(e) = \log\left(\frac{e^{C_v}}{n^{C_v(\gamma-1)}}\right), \tag{5.1.4}$$

where S^* is some constant. In this manuscript, without loss of generality, we assume that $S^* = 0$, which, along with (5.1.4) and (5.1.3), yields that

$$P = (\gamma - 1)n^\gamma \exp\left(\frac{S}{C_v}\right). \tag{5.1.5}$$

Moreover, it follows from (5.1.1) and (5.1.3) that

$$P = c^2(\gamma - 1)(\rho - n), \tag{5.1.6}$$

which, along with (5.1.5), implies that

$$P - (\gamma - 1)\left(\rho - \frac{P}{(\gamma - 1)c^2}\right)^\gamma \exp\left(\frac{S}{C_v}\right) = 0. \tag{5.1.7}$$

5.2 Introduction of some important quantities

One can rewrite (5.0.1) in the following form:

$$\partial_t v + A(\rho, u, S)\partial_x v = 0,$$

where $v = (\rho, u, S)^\top$, and $A(\rho, u, S)$ is a 3×3 matrix defined by

$$A(\rho, u, S) \coloneqq \begin{bmatrix} \frac{c^2 u(c^2 - \partial_\rho P)}{c^4 - u^2\partial_\rho P} & \frac{c^2(P + \rho c^2)}{c^4 - u^2\partial_\rho P} & -\frac{\partial_S P u(c^2 - u^2)}{c^4 - u^2\partial_\rho P} \\ \frac{c^2\partial_\rho P(c^2 - u^2)^2}{(c^4 - u^2\partial_\rho P)(P + \rho c^2)} & \frac{c^2 u(c^2 - \partial_\rho P)}{c^4 - u^2\partial_\rho P} & \frac{c^2\partial_S P(c^2 - u^2)^2}{(c^4 - u^2\partial_\rho P)(P + \rho c^2)} \\ 0 & 0 & u \end{bmatrix}. \tag{5.2.1}$$

It follows from direct calculations that the above system has eigenvalues

$$\lambda_1 = u, \quad \lambda_2 = \frac{c^2(u + \sqrt{\partial_\rho P})}{c^2 + u\sqrt{\partial_\rho P}}, \quad \lambda_3 = \frac{c^2(u - \sqrt{\partial_\rho P})}{c^2 - u\sqrt{\partial_\rho P}}. \tag{5.2.2}$$

The characteristics $x^i(t, x_0^i)$ can be defined by

$$\frac{dx^1}{dt} = \lambda_1, \quad \frac{dx^2}{dt} = \lambda_2, \quad \frac{dx^3}{dt} = \lambda_3, \tag{5.2.3}$$

with $x_0^i = x^i(0, x_0^i)$ $(i = 1, 2, 3)$. For simplicity, $x^i(t, x_0^i)$ will be abbreviated as $x^i(t)$.

Comparing with the isentropic system, one of significant differences for the non-isentropic system is the disappearance of Riemann invariances, in fact, the Riemann variables are:

$$w = \frac{c}{2}\ln\left(\frac{c+u}{c-u}\right) - \int_0^\rho \frac{\sqrt{\partial_\sigma P(\sigma, S)}}{\sigma + \frac{P(\sigma,S)}{c^2}}\mathrm{d}\sigma, \quad z = \frac{c}{2}\ln\left(\frac{c+u}{c-u}\right) + \int_0^\rho \frac{\sqrt{\partial_\sigma P(\sigma, S)}}{\sigma + \frac{P(\sigma,S)}{c^2}}\mathrm{d}\sigma,$$

which vary along characteristics as shown in (5.4.1). The functions below are all expressed in the coordinate $(w(\rho, u, S), z(\rho, u, S), \hat{S}(\rho, u, S))$, with $\hat{S}(\rho, u, S) = S$. Denote

$$\Lambda\left(\frac{z-w}{2}, \hat{S}\right) \coloneqq \partial_\rho P\left(\mathcal{J}^{-1}\left(\frac{z-w}{2}, \hat{S}\right)\right), \tag{5.2.4}$$

where $\mathcal{J}^{-1}$ is the inverse function of $\mathcal{J}(\rho,S)=\left(\frac{z-w}{2}(\rho,S),\hat{S}\right)^{\top}$, and

$$\begin{cases} H(w,z,\hat{S}) \coloneqq \dfrac{w+z}{c}+\ln\left(\dfrac{c-\sqrt{\Lambda}}{c+\sqrt{\Lambda}}\right)=\ln\left(\dfrac{c+\lambda_3}{c-\lambda_3}\right), \\ G(w,z,\hat{S}) \coloneqq \dfrac{w+z}{c}+\ln\left(\dfrac{c+\sqrt{\Lambda}}{c-\sqrt{\Lambda}}\right)=\ln\left(\dfrac{c+\lambda_2}{c-\lambda_2}\right). \end{cases} \tag{5.2.5}$$

Then one has that

$$\lambda_2=c\left(1-\frac{2}{e^G+1}\right),\quad \lambda_3=c\left(1-\frac{2}{e^H+1}\right). \tag{5.2.6}$$

Next, denote

$$\begin{aligned} &\alpha \coloneqq \partial_x w,\quad \beta \coloneqq \partial_x z,\quad \widetilde{\alpha}=\alpha-a\eta,\quad \widetilde{\beta}=\beta+a\eta,\quad \eta \coloneqq \partial_x S,\\ &\widetilde{n} \coloneqq \frac{cn}{\sqrt{c^2-u^2}},\quad a=\partial_S\left(\frac{w-z}{2}\right)+\frac{c^2}{(P+\rho c^2)}\frac{\partial_S P}{\sqrt{\partial_\rho P}},\\ &h \coloneqq -\ln\left(e^H+1\right)+\frac{1}{2}\ln\left(e^{G+H}-e^{2H}\right)-\int_{\epsilon}^{\frac{z-w}{2}}\frac{(c+\sqrt{\Lambda})^2}{2c^2\sqrt{\Lambda}}(\Xi,\hat{S})\,\mathrm{d}\Xi,\\ &g \coloneqq -\ln\left(e^G+1\right)+\frac{1}{2}\ln\left(e^{2G}-e^{G+H}\right)-\int_{\epsilon}^{\frac{z-w}{2}}\frac{(c-\sqrt{\Lambda})^2}{2c^2\sqrt{\Lambda}}(\Xi,\hat{S})\,\mathrm{d}\Xi,\\ &L(z,w,\hat{S}) \coloneqq -\frac{1}{\widetilde{n}^2}\int_w^z e^h\widetilde{n}\partial_\vartheta a\left(\frac{\lambda_1-\lambda_2}{\lambda_3-\lambda_2}\right)(w,\vartheta,\hat{S})\,\mathrm{d}\vartheta,\\ &M(z,w,\hat{S}) \coloneqq -\frac{1}{\widetilde{n}^2}\int_z^w e^g\widetilde{n}\partial_\varsigma a\left(\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\right)(\varsigma,z,\hat{S})\,\mathrm{d}\varsigma, \end{aligned} \tag{5.2.7}$$

where $\epsilon=\frac{1}{2}\inf_x(z_0-w_0)$ and $z_0=z(0,x)$, $w_0=w(0,x)$. Now we can introduce two key quantities, which play important roles in our analysis on singularity formation:

$$r \coloneqq e^h\widetilde{\alpha}-L\eta\widetilde{n},\quad q \coloneqq e^g\widetilde{\beta}-M\eta\widetilde{n}. \tag{5.2.8}$$

Lastly, define

$$\begin{cases} a_0 \coloneqq e^h \left[\partial_3 S(a\partial_w h + \partial_{\hat{S}} h) - \partial_z \lambda_3 \dfrac{a\partial_2 S}{\lambda_3 - \lambda_2} - \eta(a\partial_w \lambda_3 - \partial_{\hat{S}} \lambda_3) - \eta \partial_w a(\lambda_1 - \lambda_3) \right], \\ a_1 \coloneqq e^h \eta \partial_z a \left(\dfrac{\lambda_1 - \lambda_2}{\lambda_3 - \lambda_2} \right) \left[\dfrac{\partial_w \widetilde{n}}{\partial_z \widetilde{n}} \partial_3 w + \dfrac{\partial_{\hat{S}} \widetilde{n}}{\partial_z \widetilde{n}} \partial_3 S - a\partial_2 S - \eta a(\lambda_3 - \lambda_2) \right], \\ a_2 \coloneqq \widetilde{n}\eta \left[\dfrac{\partial_w \widetilde{n}}{\partial_z \widetilde{n}} \partial_3 w \partial_z L + \dfrac{\partial_{\hat{S}} \widetilde{n}}{\partial_z \widetilde{n}} \partial_3 S \partial_z L - \partial_3 w \partial_w L - \partial_{\hat{S}} L \partial_3 S \right] - L\widetilde{n} \left(\partial_3 \eta - \dfrac{\eta}{\widetilde{n}} \partial_3 \widetilde{n} \right), \\ a_3 \coloneqq e^{-h}(a_0 - 2\partial_w \lambda_3 L \widetilde{n} \eta), \\ a_4 \coloneqq e^{-h} L \eta \widetilde{n} a_0 + a_1 + a_2 - \partial_w \lambda_3 e^{-h} L^2 \eta^2 \widetilde{n}^2, \end{cases} \tag{5.2.9}$$

and

$$\begin{cases} b_0 \coloneqq e^g \left[\partial_2 S(-a\partial_z g + \partial_{\hat{S}} g) - \partial_w \lambda_2 \dfrac{a\partial_3 S}{\lambda_3 - \lambda_2} + \eta(a\partial_z \lambda_2 - \partial_{\hat{S}} \lambda_2) + \eta \partial_z a(\lambda_1 - \lambda_2) \right], \\ b_1 \coloneqq e^g \eta \partial_w a \left(\dfrac{\lambda_1 - \lambda_3}{\lambda_3 - \lambda_2} \right) \left[\dfrac{\partial_z \widetilde{n}}{\partial_w \widetilde{n}} \partial_2 z + \dfrac{\partial_{\hat{S}} \widetilde{n}}{\partial_w \widetilde{n}} \partial_2 S + a\partial_3 S - \eta a(\lambda_3 - \lambda_2) \right], \\ b_2 \coloneqq \widetilde{n}\eta \left[\dfrac{\partial_z \widetilde{n}}{\partial_w \widetilde{n}} \partial_2 z \partial_w M + \dfrac{\partial_{\hat{S}} \widetilde{n}}{\partial_w \widetilde{n}} \partial_2 S \partial_w M - \partial_2 z \partial_z M - \partial_{\hat{S}} M \partial_2 S \right] - M\widetilde{n} \left(\partial_2 \eta - \dfrac{\eta}{\widetilde{n}} \partial_2 \widetilde{n} \right), \\ b_3 \coloneqq e^{-g}(b_0 - 2\partial_z \lambda_2 M \widetilde{n} \eta), \\ b_4 \coloneqq e^{-g} M \eta \widetilde{n} b_0 + b_1 + b_2 - \partial_z \lambda_2 e^{-g} M^2 \eta^2 \widetilde{n}^2, \end{cases} \tag{5.2.10}$$

where

$$\partial_1 = \partial_t - \lambda_1 \partial_x, \quad \partial_2 = \partial_t - \lambda_2 \partial_x, \quad \partial_3 = \partial_t - \lambda_3 \partial_x. \tag{5.2.11}$$

5.3 The main result

First of all, we define the C^1 solutions as follows:

Definition 5.3.1. *Let $T > 0$ be some time. The triple $(\rho(t,x), u(t,x), S(t,x))$ is called a C^1 solution to the non-isentropic relativistic Euler equations on $(0,T) \times \mathbb{R}$ if*

$$\rho > 0, \quad \rho \in C^1([0,T) \times \mathbb{R}), \quad u \in C^1([0,T) \times \mathbb{R}),$$
$$S \in C^1([0,T) \times \mathbb{R}), \quad \partial_{xx} S \in C([0,T) \times \mathbb{R}),$$

and the equations (5.0.1) *are satisfied in the pointwise sense on* $(0,T)\times\mathbb{R}$*. It is called a* C^1 *solution to the Cauchy problem* (5.0.1) *with* (5.0.2)*, if it is a* C^1 *solution to the equations* (5.0.1) *on* $(0,T)\times\mathbb{R}$*, and it admits the initial data* (5.0.2) *continuously.*

For simplicity, we give the following initial assumption.

Assumption 5.3.1. *Assume that* $(w_0,z_0)\in C^1(\mathbb{R})$*,* $S_0\in C^2(\mathbb{R})\cap BV$*, and*

$$\inf_{x\in\mathbb{R}}(z_0-w_0)>0,\quad \sup_{x\in\mathbb{R}}|u_0(x)|<c,\quad \|(z_0,w_0)\|_{C^1(\mathbb{R})}\le M_1,\quad \|S_0\|_{C^2(\mathbb{R})}\le B,$$

where M_1 *and* B *are both positive constants, and* BV *denotes the space of functions with bounded total variation.*

Second, for stating our main result precisely, we give the following several propositions.

Proposition 5.3.1. *The following function*

$$\psi(\rho)\coloneqq\int_0^{\rho}\inf_{S\in[-B,B]}\frac{c^2\sqrt{\partial_\sigma P(\sigma,S)}}{c^2\sigma+P(\sigma,S)}d\sigma$$

is strictly monotonically increasing, continuous,

$$0\le\psi(\rho)\le\frac{z-w}{2},\quad \textit{and}\quad \psi(\rho)=0\iff\rho=0.$$

Proof. It follows from Proposition A.4 in Appendix A that

$$\frac{c^2\sqrt{\partial_\rho P(\rho,S)}}{c^2\rho+P(\rho,S)}\sim\mathcal{O}(\rho^{\frac{\gamma-3}{2}}),\quad \text{as}\quad \rho\to0,$$

which quickly implies the desired conclusions. □

Proposition 5.3.2. *Let* $\kappa>0$ *be a constant. Then there exists an increasing function:*

$$\Psi(\kappa)\coloneqq\max_{(\rho,S)\in[0,\kappa]\times[-B,B]}\frac{2\sqrt{\partial_\rho P(\rho,S)}}{(z-w)(\rho,S)}.\tag{5.3.1}$$

Proof. It follows from Proposition A.4 in Appendix A that

$$\sqrt{\partial_\rho P} \sim \mathcal{O}(\rho^{\frac{\gamma-1}{2}}) \quad \text{and} \quad \frac{(z-w)}{2}(\rho, S) \sim \mathcal{O}(\rho^{\frac{\gamma-1}{2}}) \quad \text{as} \quad \rho \to 0,$$

which quickly yields the desired conclusion. □

Denote $E(c, B) \coloneqq \sup\left\{\rho \mid \sqrt{\partial_\rho P} < c,\ |S| \leq B\right\}$. It follows from Proposition A.2 in Appendix A and the relation between n and ρ that, such $E(c, B)$ exists and is finite.

Proposition 5.3.3. *There exists a constant $K(c, B)$ depending only on (c, B) such that*

$$K(c, B) = \sup_{\mathcal{M}} \frac{|a(\rho, S)|}{(z-w)(\rho, S)} \quad \textit{with} \quad \mathcal{M} = \{(\rho, S) \in [0, E(c, B)) \times [-B, B]\}. \tag{5.3.2}$$

Proof. By explicit computation, one can check that

$$a = -\int_0^\rho \frac{c^2 \partial_{\sigma S} P}{2\sqrt{\partial_\sigma P}(c^2\sigma + P)} \mathrm{d}\sigma + \int_0^\rho \frac{c^2\sqrt{\partial_\sigma P}\partial_S P}{(c^2\sigma + P)^2} \mathrm{d}\sigma + \frac{c^2}{(P + \rho c^2)} \frac{\partial_S P}{\sqrt{\partial_\rho P}}.$$

It follows from Proposition A.4 in Appendix A that

$$a(\rho, S) \sim \mathcal{O}\left(\rho^{\frac{\gamma-1}{2}}\right) \quad \text{and} \quad \frac{z-w}{2}(\rho, S) \sim \mathcal{O}\left(\rho^{\frac{\gamma-1}{2}}\right) \quad \text{as} \quad \rho \to 0,$$

which quickly yields the desired conclusion.

□

Then one can denote $V \coloneqq \exp\left(K(c, B)\|S_0\|_{BV}\right)$, and the following quantities:

$$\begin{aligned}
U_1 \coloneqq& \sup_x |w_0| V + K(c, B) V^2 \sup_x |z_0| \|S_0\|_{BV}, \\
U_2 \coloneqq& \sup_x |z_0| V + K(c, B) V^2 \sup_x |w_0| \|S_0\|_{BV}, \\
\max|w| \coloneqq& U_1 + K(c, B)^2 V^2 \|S_0\|_{BV}^2 \exp\left(K(c, B)^2 V^2 \|S_0\|_{BV}^2\right) U_1, \\
\max|z| \coloneqq& U_2 + K(c, B)^2 V^2 \|S_0\|_{BV}^2 \exp\left(K(c, B)^2 V^2 \|S_0\|_{BV}^2\right) U_2, \\
M_2 \coloneqq& \psi^{-1}\left(\frac{\max|z| + \max|w|}{2}\right).
\end{aligned} \tag{5.3.3}$$

Assumption 5.3.2.

$$\Psi(M_2)\left(\frac{\max|z|+\max|w|}{2}\right)<c.$$

Finally, denote two constants N_1 and N_2 depending only on (z_0,w_0,S_0), R, c and γ, by

$$N_1\coloneqq\sup_{\mathcal{N}}\sqrt{\frac{2}{\partial_w\lambda_3e^{-h}}\left(\frac{|a_3|^2}{2\partial_w\lambda_3e^{-h}}+|a_4|\right)},\quad N_2\coloneqq\sup_{\mathcal{N}}\sqrt{\frac{2}{\partial_z\lambda_2e^{-g}}\left(\frac{|b_3|^2}{2\partial_z\lambda_2e^{-g}}+|b_4|\right)},$$

where

$$\mathcal{N}\coloneqq\{(z,w,S)\mid|z|\le\max|z|,\ |w|\le\max|w|,\ |S|\le B\}.$$

Based on the above preparations, now we can state the main result of this chapter. This is a joint result with Dr. Athanasiou and Dr. Zhu [2].

Theorem 5.3.1. *Let Assumptions* 5.3.1-5.3.2 *hold. If there exists a point* $x^*\in\mathbb{R}$ *such that*

$$r(0,x^*)<-N_1\quad or\quad q(0,x^*)<-N_2,\tag{5.3.4}$$

then for the C^1 *solution* (ρ,u,S) *to the Cauchy problem* (5.0.1) *with* (5.0.2)*,* $\partial_x\rho$ *and/or* ∂_xu *must blow-up in a finite time.*

Remark 5.3.1. *First, Assumption* 5.3.2 *is used to make sure that* $|\sqrt{\partial_\rho P}|<c$ *holds for* C^1 *solutions of the Cauchy problem* (5.0.1) *with* (5.0.2) *(see Theorem* 5.5.1*).*

Second, owing to the complex structure of (5.0.1)*, it is not easy to give a clear definition of R/C characters as the isentropic case. Here we formally say that* (ρ_0,u_0,S_0) *contain compression, if there exists a point* $x^*\in\mathbb{R}$ *such that* $r(0,x^*)<0$ *or* $q(0,x^*)<0$*. That's why we state that* N_i *(i=1,2) represents the strength of the initial compression.*

Remark 5.3.2. *It is well-known that under Assumptions* 5.3.1-5.3.2*, the local-in-time existence of the unique* C^1 *solution* (ρ,u,S) *to the Cauchy problem* (5.0.1) *with* (5.0.2) *can be established by the classical characteristic method; c.f. Theorem 4.1 on p. 66 of* [39]*, or see also* [23, 26, 38].

Since now, let T denote the life span of the C^1 solution to the Cauchy problem (5.0.1) with (5.0.2). For the rest of the chapter, all the conclusions are proved under the assumption that $t<T$.

5.4 PDEs satisfied by the Riemann variables

We will derive the equations satisfied by the Riemann variables (z, w) from the original system (5.0.1). It follows from direct calculations that the matrix $A(\rho, u, S)$ given by (5.2.1) has right eigenvectors

$$r_1 = \begin{bmatrix} -\partial_S P \\ 0 \\ \partial_\rho P \end{bmatrix}, \quad r_2 = \begin{bmatrix} P + \rho c^2 \\ \sqrt{\partial_\rho P}(c^2 - u^2) \\ 0 \end{bmatrix}, \quad r_3 = \begin{bmatrix} -(P + \rho c^2) \\ \sqrt{\partial_\rho P}(c^2 - u^2) \\ 0 \end{bmatrix},$$

and left eigenvectors

$$l_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} \sqrt{\partial_\rho P}(c^2 - u^2) \\ P + \rho c^2 \\ (c^2 - u^2)\frac{\partial_S P}{\sqrt{\partial_\rho P}} \end{bmatrix}, \quad l_3 = \begin{bmatrix} -\sqrt{\partial_\rho P}(c^2 - u^2) \\ P + \rho c^2 \\ -(c^2 - u^2)\frac{\partial_S P}{\sqrt{\partial_\rho P}} \end{bmatrix}.$$

Moreover, the corresponding Riemann variables satisfy the following properties:

$$\nabla w \cdot r_2 = 0 \implies \nabla w = a l_1 + b l_3;$$
$$\nabla z \cdot r_3 = 0 \implies \nabla z = -a l_1 + b l_2,$$

where $\nabla w = (\partial_\rho w, \partial_u w, \partial_S w)^\top$, $\nabla z = (\partial_\rho z, \partial_u z, \partial_S z)^\top$, a is defined in (5.2.7) and

$$b = \frac{c^2}{(c^2 - u^2)(P + \rho c^2)}.$$

Then via direct calculations, one can obtain that

$$\partial_t w + \lambda_3 \partial_x w = a(\lambda_3 - \lambda_1) l_1 \cdot \partial_x v,$$

$$\partial_t w + \lambda_1 \partial_x w = b(\lambda_1 - \lambda_3) l_3 \cdot \partial_x v,$$

$$\partial_t z + \lambda_2 \partial_x z = a(\lambda_1 - \lambda_2) l_1 \cdot \partial_x v,$$

$$\partial_t z + \lambda_1 \partial_x z = b(\lambda_1 - \lambda_2) l_2 \cdot \partial_x v,$$

which, along with $l_1 \cdot \partial_x v = \partial_x S$ and $(5.0.1)_3$, yields that

$$\begin{cases} \partial_t w + \lambda_3 \partial_x w = a(\partial_t S + \lambda_3 \partial_x S), \\ \partial_t z + \lambda_2 \partial_x z = -a(\partial_t S + \lambda_2 \partial_x S). \end{cases} \tag{5.4.1}$$

For simplicity, denote

$$F(\rho, S) = \frac{z - w}{2} = \int_0^\rho \frac{c^2 \sqrt{\partial_\sigma P(\sigma, S)}}{c^2 \sigma + P(\sigma, S)} \mathrm{d}\sigma, \tag{5.4.2}$$

then $a(\rho, S)$ can be expressed in terms of $F(\rho, S)$ and $P(\rho, S)$ as

$$a(\rho, S) = -\partial_S F + \frac{\partial_S P}{\partial_\rho P} \partial_\rho F. \tag{5.4.3}$$

5.5 Uniform bounds on the Riemann variables

First, one has

Proposition 5.5.1.

$$|S(t, x)| \leq \sup_{x \in \mathbb{R}} |S_0(x)| \quad \textit{for} \quad (t, x) \in [0, T) \times \mathbb{R}.$$

The proof is obvious, and then we omit it here.

Second, define the time T^* by

$$T^* \coloneqq \sup\{t \in [0, T) \mid \rho > 0 \ \text{ pointwisely}, \ \ |u| < c \ \text{ and } \ \sqrt{\partial_\rho P} < c \ \text{ uniformly}\}.$$

Here, for the function $f(t, x)$, at the time t_0, $|f(t_0, x)| < c$ uniformly means: there exists one constant $C_0 > 0$ independent of $(t, x) \in [0, t_0] \times \mathbb{R}$ such that

$$c - |f(t, x)| > C_0 \quad \text{for} \quad (t, x) \in [0, t_0] \times \mathbb{R}.$$

It follows from Assumptions 5.3.1-5.3.2 and the continuity of the solution that $T^* > 0$. Next we

want to prove that (w, z) are uniformly bounded up to the time T. We will first show that such uniform bounds indeed exist for $t \in [0, T^*)$, and then prove $T^* = T$.

Proposition 5.5.2. *When $t \in [0, T^*)$, one has*

$$\lambda_3(t, x) < \lambda_1(t, x) < \lambda_2(t, x) \quad \textit{for} \quad x \in \mathbb{R}, \tag{5.5.1}$$

and any two characteristics $x^i(t)$ and $x^j(t)$ $(i \neq j)$ intersect each other at most once.

Proof. First, for $t < T^*$, (5.5.1) follows directly from (5.2.2).

Second, provided $i \neq j$, define the function $f(t)$ by

$$f(t) \coloneqq x^i(t) - x^j(t).$$

One observes that if for some $t_* \in [0, T^*)$, $f(t_*) = 0$, then $f'(t_*) \neq 0$. Actually, note that $f(t_*) = 0$ implies that

$$x^i(t_*) = x^j(t_*) \coloneqq x_*.$$

Then if $f'(t_*) = 0$, one can get

$$0 = f'(t_*) = \lambda_i(t_*, x_*) - \lambda_j(t_*, x_*),$$

which contradicts (5.5.1).

Now we assume that $x^i(t)$ and $x^j(t)$ intersect each other more than once in $[0, T^*)$. Then $f(t) = 0$ holds at least for two different points: $\bar{a}_1$ and $\bar{a}_2$. One can choose a time $\widetilde{t} \in (\bar{a}_1, \bar{a}_2)$ such that $f(\widetilde{t}) \neq 0$. Without loss of generality, assume $f(\widetilde{t}) > 0$. It is clear that such kind of points indeed exist, or else $f(t) \equiv 0$ for $t \in (\bar{a}_1, \bar{a}_2)$, which is impossible.

By the continuity of $f(t)$, there exists two points $\bar{a}_3 \in [\bar{a}_1, \bar{a}_2)$ and $\bar{a}_4 \in (\bar{a}_1, \bar{a}_2]$ such that $\widetilde{t} \in (\bar{a}_3, \bar{a}_4)$, $f(t) > 0$ for all $t \in (\bar{a}_3, \bar{a}_4)$, and $f(\bar{a}_3) = f(\bar{a}_4) = 0$, which implies that

$$x^i(\bar{a}_3) = x^j(\bar{a}_3) \coloneqq \bar{A} \quad \text{and} \quad x^i(\bar{a}_4) = x^j(\bar{a}_4) \coloneqq \bar{B}.$$

As $f'(\bar{a}_3) \neq 0$ and $f'(\bar{a}_4) \neq 0$, one has $f'(\bar{a}_3) > 0$ and $f'(\bar{a}_4) < 0$. Then

$$\lambda_i(\bar{a}_3, \bar{A}) > \lambda_j(\bar{a}_3, \bar{A}), \quad \lambda_i(\bar{a}_4, \bar{B}) < \lambda_j(\bar{a}_4, \bar{B}),$$

which contradicts (5.5.1). The proof of this lemma is complete. □

Next we show that (w, z) are uniformly bounded up to the time T^*.

Lemma 5.5.1.

$$\sup_{(t,x)\in[0,T^*)\times\mathbb{R}} |w(t,x)| \leq \max|w|, \qquad \sup_{(t,x)\in[0,T^*)\times\mathbb{R}} |z(t,x)| \leq \max|z|, \tag{5.5.2}$$

where $\max|w|$ *and* $\max|z|$ *are the constants defined in* (5.3.3)*, and are independent of* T^*.

Proof. We will focus only on w, and z can be dealt with similarly. For simplicity, let $\oint$ denote the integral along the characteristic $x^3(t)$, and $\not\oint$ the integral along the characteristic $x^2(t)$, i.e., for any function $f(t,x)$,

$$\oint_0^t f(\sigma)\mathrm{d}\sigma \equiv \int_0^t f(\sigma, x^3(\sigma))\mathrm{d}\sigma, \quad \not\oint_0^t f(\sigma)\mathrm{d}\sigma \equiv \int_0^t f(\sigma, x^2(\sigma))\mathrm{d}\sigma.$$

Moreover, $K(c, B)$ will be abbreviated as K. The proof is divided into two steps: integrating along the characteristics and using the Gronwall Inequality to show the boundedness of (w, z).

Step 1: integrating along the characteristics. There are two cases that need to be dealt with separately: $w(t, x^3(t)) \geq 0$ and $w(t, x^3(t)) < 0$. First of all, when $w(t, x^3(t)) \geq 0$, it follows from Proposition 5.3.3 and $(5.4.1)_1$ that

$$\partial_t w(t, x^3(t)) \leq K(z - w)|\partial_t S(t, x^3(t))|. \tag{5.5.3}$$

Define the function $I(t)$ by

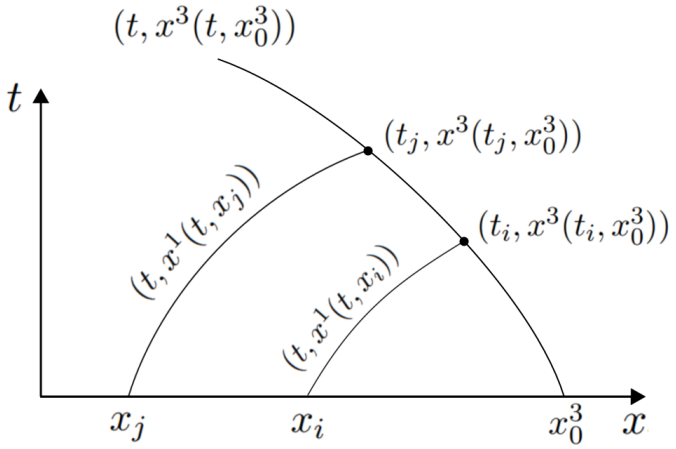
$$I(t) \coloneqq \exp\left(K \int_0^t |\partial_\sigma S(\sigma, x^3(\sigma))|\mathrm{d}\sigma\right). \tag{5.5.4}$$

Multiplying (5.5.3) by $I(t)$ and integrating over $[0,t]$ along $x^3(t)$, one has

$$w(t,x^3(t)) \le w_0(x_0^3)\frac{1}{I(t)} + K\frac{1}{I(t)}\int_0^t zI|\partial_\sigma S|\mathrm{d}\sigma. \tag{5.5.5}$$

According to the continuity of $\partial_t S(t,x^3(t))$ and Proposition B.1 in Appendix B, one can split $[0,t]$ into countably many disjoint intervals $\bigcup_{n=0}^{\infty}[t_n,t_{n+1}]$ with $t_0=0$ and $t_\infty = t$, such that $\partial_t S(t,x^3(t))$ has a constant sign on each $[t_n,t_{n+1}]$.

Now we consider the characteristic $x^3(t,x_0^3)$ that starts from the point $(0,x_0^3)$. For any two different time endpoints $t_i,t_j \in (0,T^*)$ obtained above, one can introduce two characteristics: $x^1(t,x_i)$ that intersects $x^3(t,x_0^3)$ at $(t_i,x^3(t_i,x_0^3))$, and intersects x-axis at $(0,x_i)$; and $x^1(t,x_j)$ that intersects $x^3(t,x_0^3)$ at $(t_j,x^3(t_j,x_0^3))$, and intersects x-axis at $(0,x_j)$. According to Proposition 5.5.2, each of the characteristics $x^1(t)$ intersects $x^3(t)$ at most once, so one finds that x_i is different from x_j provided $i\neq j$. See the figure below.



Then since $\partial_t S(t,x^1(t))=0$, one can obtain that

$$\begin{aligned}&\int_0^t |\partial_\sigma S(\sigma,x^3(\sigma))|\mathrm{d}\sigma = \sum_{i=1}^{\infty}\int_{t_{i-1}}^{t_i}|\partial_\sigma S(\sigma,x^3(\sigma))|\mathrm{d}\sigma\\ =&\sum_{i=1}^{\infty}|S(t_{i-1},x^3(t_{i-1}))-S(t_i,x^3(t_i))| = \sum_{i=1}^{\infty}|S_0(x_{i-1})-S_0(x_i)| \le \|S_0\|_{BV},\end{aligned} \tag{5.5.6}$$

which, along with (5.5.4), yields that

$$\frac{1}{I(t)} \le 1 \le I(t) \le \exp(K\|S_0\|_{BV}) = V. \tag{5.5.7}$$

Then it follows from (5.5.5) and (5.5.7) that

$$|w(t, x^3(t))| \le |w_0(x_0^3)|V + KV \fint_0^t |z\partial_\sigma S| \mathrm{d}\sigma. \tag{5.5.8}$$

Second, when $w(t, x^3(t)) < 0$, it follows from Proposition 5.3.3 and $(5.4.1)_1$ that

$$\partial_t w(t, x^3(t)) \ge -K(z - w)|\partial_t S(t, x_3(t))|.$$

Hence

$$0 > w(t, x^3(t)) \ge w_0(x_0^3)I(t) - KI(t) \fint_0^t z\frac{1}{I}|\partial_\sigma S| \mathrm{d}\sigma, \tag{5.5.9}$$

which, along with (5.5.7), implies that (5.5.8) still holds when $w(t, x^3(t)) < 0$.

Note the integral in (5.5.8) depends on x_0^3, hence

$$|w(t, x)| \le \sup_{x\in\mathbb{R}} |w_0(x)|V + KV \sup_{x_0^3\in\mathbb{R}} \fint_0^t |z\partial_\sigma S| \mathrm{d}\sigma. \tag{5.5.10}$$

Similarly, for z satisfying $(5.4.1)_2$, one can also obtain

$$|z(t, x)| \le \sup_{x\in\mathbb{R}} |z_0(x)|V + KV \sup_{x_0^2\in\mathbb{R}} \fint_0^t |w\partial_\sigma S| \mathrm{d}\sigma. \tag{5.5.11}$$

Substituting (5.5.11) into (5.5.10), one can obtain that

$$\begin{aligned} |w(t, x)| \le & \sup_{x\in\mathbb{R}} |w_0(x)|V + KV^2 \sup_{x\in\mathbb{R}} |z_0(x)|\|S_0\|_{BV} \\ & + K^2V^2 \sup_{x_0^3\in\mathbb{R}} \fint_0^t |\partial_\sigma S| \Big(\sup_{x_0^2\in\mathbb{R}} \fint_0^\sigma |\partial_\phi S||w(\phi)| \mathrm{d}\phi \Big) \mathrm{d}\sigma, \end{aligned} \tag{5.5.12}$$

where one has used (5.5.6).

Step 2: using the Gronwall Inequality to show the boundedness of $|w|$. Replacing $w(t, x)$ by

$w(\tau, x^2(\tau))$ in (5.5.12), multiplying by $|\partial_\tau S|$ and integrating over $[0,t]$ for τ along $x^2(t)$, one gets

$$\begin{aligned}\fint_0^t |w(\tau)||\partial_\tau S|\mathrm{d}\tau \leq & \sup_{x\in\mathbb{R}}|w_0(x)|V\|S_0\|_{BV} + KV^2\sup_{x\in\mathbb{R}}|z_0(x)|\|S_0\|_{BV}^2 \\ & + K^2V^2\fint_0^t |\partial_\tau S|\Big(\sup_{x_0^3\in\mathbb{R}}\fint_0^\tau |\partial_\sigma S|\sup_{x_0^2\in\mathbb{R}}\Big(\fint_0^\sigma |\partial_\phi S||w(\phi)|\mathrm{d}\phi\Big)\mathrm{d}\sigma\Big)\mathrm{d}\tau,\end{aligned} \tag{5.5.13}$$

where one has used (5.5.6).

Denote $W(t)$ by

$$W(t) \coloneqq \sup_{x_0^2\in\mathbb{R}}\fint_0^t |w(\tau)||\partial_\tau S|\mathrm{d}\tau.$$

then one has

$$\begin{aligned}W(t) \leq & V\sup_{x\in\mathbb{R}}|w_0(x)|\|S_0\|_{BV} + KV^2\sup_{x\in\mathbb{R}}|z_0(x)|\|S_0\|_{BV}^2 \\ & + K^2V^2\sup_{x_0^2\in\mathbb{R}}\fint_0^t |\partial_\tau S|\Big(\sup_{x_0^3\in\mathbb{R}}\fint_0^\tau |\partial_\sigma S|W(\sigma)\mathrm{d}\sigma\Big)\mathrm{d}\tau,\end{aligned}$$

where

$$\fint_0^\tau |\partial_\sigma S|W(\sigma)\mathrm{d}\sigma \equiv \int_0^\tau |\partial_\sigma S(\sigma, x^3(\sigma))|W(\sigma)\mathrm{d}\sigma.$$

Note that $W(\widetilde{\sigma}) \leq W(\tau)$ for all $\widetilde{\sigma}\in[0,\tau]$, one has

$$\begin{aligned}W(t) \leq & V\sup_{x\in\mathbb{R}}|w_0(x)|\|S_0\|_{BV} + KV^2\sup_{x\in\mathbb{R}}|z_0(x)|\|S_0\|_{BV}^2 \\ & + K^2V^2\sup_{x_0^2\in\mathbb{R}}\fint_0^t W(\tau)|\partial_\tau S|\left(\sup_{x_0^3\in\mathbb{R}}\fint_0^\tau |\partial_\sigma S|\mathrm{d}\sigma\right)\mathrm{d}\tau \\ \leq & V\sup_{x\in\mathbb{R}}|w_0(x)|\|S_0\|_{BV} + KV^2\sup_{x\in\mathbb{R}}|z_0(x)|\|S_0\|_{BV}^2 \\ & + \|S_0\|_{BV}K^2V^2\sup_{x_0^2\in\mathbb{R}}\fint_0^t W(\tau)|\partial_\tau S|\mathrm{d}\tau.\end{aligned}$$

According to the Gronwall inequality and (5.5.6), one has

$$W(t) \leq \|S_0\|_{BV}U_1\exp\left(K^2V^2\|S_0\|_{BV}^2\right),$$

which, along with (5.5.12) and (5.5.6), implies the desired bound for w. □

Theorem 5.5.1. $T^* = T$.

Proof. Assume for contradiction that $T^* < T$. Via the continuity of our solution, in order to prove this theorem, we just need to show that the facts $|u| < c$ uniformly and $\sqrt{\partial_\rho P} < c$ uniformly both hold at $t = T^*$, and $\rho(T^*, x) > 0$ for $(t, x) \in [0, T^*] \times \mathbb{R}$.

First, according to Lemma 5.5.1, one has

$$\left|\ln\left(\frac{c+u}{c-u}\right)\right| = \frac{1}{c}|w+z| \leq \frac{max|z| + max|w|}{c} \quad \text{for} \quad (t,x) \in [0,T^*] \times \mathbb{R},$$

which means that the fact that $|u| < c$ uniformly still holds at $t = T^*$.

Second, according to Proposition 5.3.1 and Lemma 5.5.1, one gets

$$\psi(\rho) \leq \frac{z-w}{2} \leq \frac{\max|z| + \max|w|}{2} \quad \text{for} \quad (t,x) \in [0,T^*] \times \mathbb{R}.$$

By the properties of $\psi(\rho)$, one can take its inverse and get

$$\rho \leq \psi^{-1}\left(\frac{\max|z| + \max|w|}{2}\right), \tag{5.5.14}$$

which, along with Assumption 5.3.2, yields that the fact that $\sqrt{\partial_\rho P} < c$ uniformly still holds at $t = T^*$.

Next we consider the positivity of ρ. By (5.4.1) and (5.2.11), one has

$$\partial_2(z-w) = (\lambda_3 - \lambda_2)\partial_x w - a(\partial_2 S + \partial_3 S). \tag{5.5.15}$$

By (5.2.2) and Proposition 5.5.2, one has

$$\lambda_2 - \lambda_3 = \frac{2c^2\sqrt{\partial_\rho P}(c^2 - u^2)}{c^4 - u^2\partial_\rho P} > 0. \tag{5.5.16}$$

It follows from (5.5.14) and Proposition 5.3.2-5.3.3 that, there exists a fixed constant $\widetilde{K} > 0$ independent of T^*, such that

$$\sqrt{\partial_\rho P} \leq \widetilde{K}(z-w), \quad |a| \leq \widetilde{K}(z-w),$$

which, together with (5.5.15)-(5.5.16), yields that

$$\begin{aligned}\partial_2(z-w) &\geq -\frac{2c^2\sqrt{\partial_\rho P}(c^2-u^2)}{c^4-u^2\partial_\rho P}\,|\partial_x w| - |a||\partial_2 S+\partial_3 S| \\ &\geq -\widetilde{K}\left(\frac{2c^2(c^2-u^2)}{c^4-u^2\partial_\rho P}|\partial_x w| + |\partial_2 S+\partial_3 S|\right)(z-w).\end{aligned}$$

Then it follows from the Gronwall inequality and Assumption 5.3.1 that

$$z-w \geq (z_0-w_0)\exp\left(\int_0^t -\widetilde{K}\left(\frac{2c^2(c^2-u^2)}{c^4-u^2\partial_\rho P}|\partial_x w| + |\partial_2 S+\partial_3 S|\right)(\tau,x^2(\tau))\mathrm{d}\tau\right) > 0.$$

At last, according to Proposition 5.3.1, one has

$$\rho = \psi^{-1}(z-w) > 0 \quad \text{for} \quad (t,x)\in[0,T^*]\times\mathbb{R}.$$

The proof of this theorem is complete. □

5.6 Non-homogeneous ODEs of the Riemann variables

The aim of this section is to converts the PDEs (5.4.1) into two decoupled ODEs of (r,q) defined in (5.2.8). For this purpose, one needs to differentiate (5.4.1) with respect to x, and carefully analysis each of the terms via the quantities and lemmas shown below. For C^1 solutions to the Cauchy problem (5.0.1) with (5.0.2), the quantity $\widetilde{n} \coloneqq \frac{cn}{\sqrt{c^2-u^2}}$ introduced in (5.2.7) satisfies the following equation (see Chapter 5 in [42]),

$$\partial_t\widetilde{n} + \partial_x(u\widetilde{n}) = 0. \tag{5.6.1}$$

The first lemma is of crucial importance, which gives some of the quantities that can be expressed in terms of the initial data, i.e. have an uniform bound.

Lemma 5.6.1. *Let $x^1(t)$ be the characteristic defined by $(5.2.3)_1$, then*

$$\begin{aligned}
&\frac{\partial_x S(t,x^1(t))}{\widetilde{n}(t,x^1(t))} = \frac{\partial_x S_0(x_0^1)}{\widetilde{n}_0(x_0^1)},\\
&\partial_{xx} S(t,x^1(t)) - \frac{\partial_x S(t,x^1(t))}{\widetilde{n}(t,x^1(t))}\partial_x \widetilde{n}(t,x^1(t))\\
=&\widetilde{n}(t,x^1(t))^2\left(\partial_{xx}S_0\frac{1}{\widetilde{n}_0^2} - \partial_x S_0\frac{\partial_x \widetilde{n}_0}{\widetilde{n}_0^3}\right)(x_0^1).
\end{aligned} \tag{5.6.2}$$

Proof. We divided the proof into two steps: the proof for $(5.6.2)_1$ and the proof for $(5.6.2)_2$.

Step 1: the proof for $(5.6.2)_1$. Differentiating $(5.0.1)_3$ with respect to x, then along the characteristic $x^1(t)$, one can obtain that

$$\eta(t,x^1(t)) = \eta_0(x_0^1)\exp\left(-\int_0^t \partial_x u(s,x^1(s))\mathrm{d}s\right), \tag{5.6.3}$$

where $\eta = \partial_x S$.

It follows from (5.6.1) that

$$\widetilde{n}(t,x^1(t)) = \widetilde{n}_0(x_0^1)\exp\left(-\int_0^t \partial_x u(s,x^1(s))\mathrm{d}s\right),$$

which, together with (5.6.3), implies $(5.6.2)_1$.

Step 2: the proof for $(5.6.2)_2$. First, differentiating (5.6.1) with respect to x, one can get

$$\partial_1(\partial_x \widetilde{n}) = -\partial_{xx}u\widetilde{n} - 2\partial_x u\partial_x \widetilde{n}.$$

Define an integration factor $\mathcal{I}(t)$ by

$$\mathcal{I}(t) \coloneqq \exp\left(2\int_0^t \partial_x u(s,x^1(s))\mathrm{d}s\right).$$

Multiplying the equation above by $\mathcal{I}(t)$ on both sides, one has that

$$\partial_1(\partial_x \widetilde{n}\mathcal{I}) = -\partial_{xx}u\widetilde{n}\mathcal{I}.$$

Integrating the equation along the characteristic $x^1(t)$, one has that

$$\partial_x\widetilde{n}(t,x^1(t))\mathcal{I}(t)=\partial_x\widetilde{n}_0(x_0^1)-\int_0^t\partial_{xx}u(s,x^1(s))\widetilde{n}(s,x^1(s))\mathcal{I}(s)\mathrm{d}s. \tag{5.6.4}$$

Second, differentiating the equation $(5.0.1)_3$ with respect to x twice, one can obtain

$$\partial_1(\partial_x\eta)=-\partial_{xx}u\eta-2\partial_x u\partial_x\eta.$$

Multiplying the above equation by $\mathcal{I}(t)$, and integrating over $[0,t]$ along the characteristic $x^1(t)$, one has

$$\partial_x\eta(t,x^1(t))\mathcal{I}(t)=\partial_x\eta_0(x_0^1)-\int_0^t\partial_{xx}u(s,x^1(s))\eta(s,x^1(s))\mathcal{I}(s)\mathrm{d}s,$$

which, along with $(5.6.2)_1$, yields that

$$\partial_x\eta(t,x^1(t))\mathcal{I}(t)=\partial_x\eta_0(x_0^1)-\frac{\eta_0(x_0^1)}{\widetilde{n}_0(x_0^1)}\int_0^t\partial_{xx}u(s,x^1(s))\widetilde{n}(s,x^1(s))\mathcal{I}(s)\mathrm{d}s. \tag{5.6.5}$$

Then according to (5.6.4) and (5.6.5), one has

$$\partial_x\eta(t,x^1(t))-\frac{\eta_0(x_0^1)}{\widetilde{n}_0(x_0^1)}\partial_x\widetilde{n}(t,x^1(t))=\frac{1}{\mathcal{I}(t)}\left(\partial_x\eta_0(x_0^1)-\frac{\eta_0(x_0^1)}{\widetilde{n}_0(x_0^1)}\partial_x\widetilde{n}_0(x_0^1)\right). \tag{5.6.6}$$

At last, according to $(5.6.2)_1$ and the following fact

$$\exp\left(-\int_0^t\partial_x u(s,x^1(s))\mathrm{d}s\right)=\frac{\widetilde{n}(t,x^1(t))}{\widetilde{n}_0(x_0^1)}\implies\frac{1}{\mathcal{I}(t)}=\frac{\widetilde{n}(t,x^1(t))^2}{\widetilde{n}_0(x_0^1)^2},$$

one can show that $(5.6.2)_2$ holds. □

According to Lemma 4.2, one can get the following several relations.

Corollary 5.6.1.

$$\partial_1\eta=\frac{\eta}{\widetilde{n}}\partial_1\widetilde{n}. \tag{5.6.7}$$

$$\partial_3\eta-\frac{\eta}{\widetilde{n}}\partial_3\widetilde{n}=(\lambda_3-\lambda_1)\left(\partial_{xx}S-\frac{\partial_x S}{\widetilde{n}}\partial_x\widetilde{n}\right). \tag{5.6.8}$$

$$\partial_2\eta-\frac{\eta}{\widetilde{n}}\partial_2\widetilde{n}=(\lambda_2-\lambda_1)\left(\partial_{xx}S-\frac{\partial_x S}{\widetilde{n}}\partial_x\widetilde{n}\right). \tag{5.6.9}$$

Proof. First, differentiating $(5.6.2)_1$ in Lemma 5.6.1 with respect to t, one gets that

$$\partial_t\left(\frac{\eta(t,x^1(t))}{\widetilde{n}(t,x^1(t))}\right)=\partial_1\left(\frac{\eta(t,x)}{\widetilde{n}(t,x)}\right)=0,$$

which, implies (5.6.7).

Second, for (5.6.8), actually one has

$$\begin{aligned}\partial_3\eta-\frac{\eta}{\widetilde{n}}\partial_3\widetilde{n}&=\partial_1\eta+(\lambda_3-\lambda_1)\partial_{xx}S-\frac{\eta}{\widetilde{n}}\partial_1\widetilde{n}-(\lambda_3-\lambda_1)\frac{\eta}{\widetilde{n}}\partial_x\widetilde{n}\\&=\left(\partial_1\eta-\frac{\eta}{\widetilde{n}}\partial_1\widetilde{n}\right)+(\lambda_3-\lambda_1)\left(\partial_{xx}S-\frac{\eta}{\widetilde{n}}\partial_x\widetilde{n}\right)\\&=(\lambda_3-\lambda_1)\left(\partial_{xx}S-\frac{\eta}{\widetilde{n}}\partial_x\widetilde{n}\right).\end{aligned}$$

Finally, (5.6.9) can be obtained similarly. □

Next we introduce one lemma to transfer the coordinate system from (ρ,u,S) to $(w,z,\hat{S})$.

Lemma 5.6.2. *The vector function*

$$\Phi(\rho,u,S)\coloneqq\Big(\frac{z-w}{2}(\rho,u,S),\ \frac{z+w}{c}(\rho,u,S),\ \hat{S}(\rho,u,S)\Big)^{\top}\tag{5.6.10}$$

is a diffeomorphism, where $\hat{S}=S$. Moreover, denote $\nabla=(\partial_\rho,\partial_u,\partial_S)^{\top}$, one has

$$\nabla\Phi(\rho,u,S)=\begin{bmatrix}\partial_\rho F & 0 & \partial_S F\\ 0 & \frac{2c}{c^2-u^2} & 0\\ 0 & 0 & 1\end{bmatrix},\tag{5.6.11}$$

and

$$(\nabla\Phi)^{-1}(\rho,u,S)=\begin{bmatrix}\frac{1}{\partial_\rho F} & 0 & -\frac{\partial_S F}{\partial_\rho F}\\ 0 & \frac{c^2-u^2}{2c} & 0\\ 0 & 0 & 1\end{bmatrix}.\tag{5.6.12}$$

The proof for the above lemma follows from direct calculations, and here we omit it.

Lemma 5.6.3. *Let g and h be the quantities defined in* (5.2.7)*. One has*

$$\partial_z h = \frac{\partial_z \lambda_3}{\lambda_3 - \lambda_2}, \quad \partial_w g = \frac{\partial_w \lambda_2}{\lambda_2 - \lambda_3}. \tag{5.6.13}$$

The verification of formulas shown in (5.6.13) follows from very complex calculations, which can be found in Appendix C.

At last, one can obtain the following decoupled ODEs of (r, q).

Theorem 5.6.1. *Let r and q be defined in* (5.2.8)*. Then they satisfy the following ODEs*

$$\begin{cases} \partial_3 r = -\partial_w \lambda_3 e^{-h} r^2 + a_3 r + a_4, \\ \partial_2 q = -\partial_z \lambda_2 e^{-g} q^2 + b_3 q + b_4, \end{cases} \tag{5.6.14}$$

where the coefficients a_i and b_i are defined in (5.2.9) *and* (5.2.10) *respectively.*

Proof. We will focus only on q, and r can be dealt with similarly.

First, differentiating $(5.4.1)_2$ with respect to x, along with $(5.0.1)_3$, one can get

$$\begin{aligned} \partial_t \beta + \partial_x \lambda_2 \beta + \lambda_2 \partial_x \beta &= -\partial_x a(\lambda_2 - u)\eta - a(\partial_t \eta + \lambda_2 \partial_x \eta + \partial_x \lambda_2 \eta) \\ &= -\partial_x a(\lambda_2 - u)\eta - a\partial_2 \eta - a\partial_x \lambda_2 \eta, \end{aligned}$$

which yields that

$$\partial_2 \beta = -\partial_x \lambda_2 (\beta + a\eta) - \partial_x a(\lambda_2 - u)\eta - a\partial_2 \eta.$$

Then it follows from direct calculations and the definition of $\widetilde{\beta}$ that

$$\begin{aligned} \partial_2 \widetilde{\beta} &= \partial_2 \beta + \partial_2 (a\eta) \\ &= -\partial_x \lambda_2 (\beta + a\eta) - \partial_x a(\lambda_2 - u)\eta - a\partial_2 \eta + \partial_2 a\eta + a\partial_2 \eta \\ &= -\partial_x \lambda_2 \widetilde{\beta} + \eta \partial_1 a. \end{aligned} \tag{5.6.15}$$

Considering the coefficients of the ODE $(5.6.14)_2$ for q, namely b_3 and b_4, one observes that they do not contain terms: derivatives of some quantities with respect to x or t, except for the conserved quantities listed in Corollary 5.6.1 and $\partial_2 z$. Note by (5.4.1) and $(5.0.1)_3$, $\partial_2 z$ can be

expressed in terms of the conserved quantities as well. Then in order to derive $(5.6.14)_2$, one has to express $\partial_x\lambda_2$ and $\partial_1 a$ in terms of total derivatives with respect to ∂_2, as well as the terms in b_3 and b_4. For this purpose, the rest of this proof is divided into 3 steps: the reformulations of $\partial_x\lambda_2$, $\partial_1 a$ and $\partial_2\widetilde{n}$.

Step 1: the reformulation of $\partial_x\lambda_2$. According to (5.2.7), one has

$$\begin{aligned}\partial_x\lambda_2 &= \partial_w\lambda_2\partial_x w + \partial_z\lambda_2\partial_x z + \partial_{\hat{S}}\lambda_2\partial_x\hat{S}\\ &= \partial_w\lambda_2(\widetilde{\alpha}+a\eta) + \partial_z\lambda_2(\widetilde{\beta}-a\eta) + \partial_{\hat{S}}\lambda_2\partial_x\hat{S}\\ &= \partial_w\lambda_2\widetilde{\alpha} + \partial_z\lambda_2\widetilde{\beta} + \eta(\partial_w\lambda_2 a - \partial_z\lambda_2 a + \partial_{\hat{S}}\lambda_2).\end{aligned} \tag{5.6.16}$$

Now one has to rewrite $\widetilde{\alpha}$. According to $(5.4.1)_1$, one can get that

$$\partial_t w + \lambda_2\partial_x w + (\lambda_3-\lambda_2)\partial_x w = a\partial_3 S,$$

which, implies that

$$\partial_x w = \frac{-\partial_2 w + a\partial_3 S}{\lambda_3-\lambda_2}. \tag{5.6.17}$$

Then according to (5.2.7) and (5.6.17), one has

$$\widetilde{\alpha} = -\frac{\partial_2 w}{\lambda_3-\lambda_2} + \frac{a\partial_3 S}{\lambda_3-\lambda_2} - a\eta. \tag{5.6.18}$$

Substituting (5.6.18) into (5.6.16), one gets that

$$\begin{aligned}\partial_x\lambda_2 =& -\partial_w\lambda_2\frac{\partial_2 w}{\lambda_3-\lambda_2} + \partial_w\lambda_2\frac{a\partial_3 S}{\lambda_3-\lambda_2} - \partial_w\lambda_2 a\eta\\ &+ \partial_z\lambda_2\widetilde{\beta} + \eta(\partial_w\lambda_2 a - \partial_z\lambda_2 a + \partial_{\hat{S}}\lambda_2).\end{aligned} \tag{5.6.19}$$

According to (5.6.13) and $(5.4.1)_2$, one has

$$\begin{aligned}\partial_2 g &= \frac{\partial_w\lambda_2}{\lambda_2-\lambda_3}\partial_2 w + \partial_z g\partial_2 z + \partial_{\hat{S}} g\partial_2\hat{S}\\ &= \frac{\partial_w\lambda_2}{\lambda_2-\lambda_3}\partial_2 w - a\partial_z g\partial_2 S + \partial_{\hat{S}} g\partial_2 S,\end{aligned} \tag{5.6.20}$$

which, together with (5.6.19), implies that

$$\begin{aligned}\partial_x\lambda_2 =&\partial_2 g + a\partial_z g\partial_2 S - \partial_{\hat{S}} g\partial_2 S + \partial_w\lambda_2\frac{a\partial_3 S}{\lambda_3-\lambda_2}\\ &- \partial_w\lambda_2 a\eta + \partial_z\lambda_2\widetilde{\beta} + \eta(\partial_w\lambda_2 a - \partial_z\lambda_2 a + \partial_{\hat{S}}\lambda_2).\end{aligned} \tag{5.6.21}$$

Substituting (5.6.21) into (5.6.15), one gets that

$$\begin{aligned}\partial_2\widetilde{\beta} =& -\partial_2 g\widetilde{\beta} - (a\partial_z g - \partial_{\hat{S}} g)\partial_2 S\widetilde{\beta}\\ &- \partial_w\lambda_2\frac{a\partial_3 S}{\lambda_3-\lambda_2}\widetilde{\beta} - \partial_z\lambda_2\widetilde{\beta}^2 + \widetilde{\beta}\eta(\partial_z\lambda_2 a - \partial_{\hat{S}}\lambda_2) + \eta\partial_1 a.\end{aligned}$$

Multiplying the above equation on both sides by e^g, one has

$$\partial_2\left(\widetilde{\beta}e^g\right) = -\partial_z\lambda_2 e^g\widetilde{\beta}^2 + \left[b_0 - e^g\eta\partial_z a(\lambda_1-\lambda_2)\right]\widetilde{\beta} + e^g\eta\partial_1 a. \tag{5.6.22}$$

Step 2: the reformulation of $\partial_1 a$. According to (5.4.1), one has

$$\begin{aligned}\partial_1 a &= \partial_w a\partial_1 w + \partial_z a\partial_1 z + \partial_{\hat{S}} a\partial_1\hat{S}\\ &= \partial_w a[\partial_3 w + (\lambda_1-\lambda_3)\partial_x w] + \partial_z a[\partial_2 z + (\lambda_1-\lambda_2)\partial_x z]\\ &= \partial_w a[a\partial_3 S + (\lambda_1-\lambda_3)\partial_x w] + \partial_z a[-a\partial_2 S + (\lambda_1-\lambda_2)\partial_x z]\\ &= \partial_w a[a(\lambda_3-\lambda_1)\eta + (\lambda_1-\lambda_3)\partial_x w] + \partial_z a[-a(\lambda_2-\lambda_1)\eta + (\lambda_1-\lambda_2)\partial_x z]\\ &= (\lambda_1-\lambda_3)\partial_w a\widetilde{\alpha} + (\lambda_1-\lambda_2)\partial_z a\widetilde{\beta}.\end{aligned} \tag{5.6.23}$$

It follows from direct calculations that

$$\partial_2\widetilde{n} = \partial_z\widetilde{n}\partial_2 z + \partial_w\widetilde{n}\partial_2 w + \partial_{\hat{S}}\widetilde{n}\partial_2\hat{S},$$

which implies that

$$\partial_2 w = \frac{\partial_2\widetilde{n}}{\partial_w\widetilde{n}} - \frac{\partial_z\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 z - \frac{\partial_{\hat{S}}\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 S. \tag{5.6.24}$$

Substituting (5.6.24) into (5.6.18), one gets that

$$\widetilde{\alpha} = -\frac{1}{\lambda_3-\lambda_2}\left[\frac{\partial_2\widetilde{n}}{\partial_w\widetilde{n}} - \frac{\partial_z\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 z - \frac{\partial_{\hat{S}}\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 S\right] + \frac{a\partial_3 S}{\lambda_3-\lambda_2} - a\eta, \tag{5.6.25}$$

which, along with (5.6.23), yields that

$$e^g\eta\partial_1 a = -\frac{(\lambda_1-\lambda_3)\partial_w a}{(\lambda_3-\lambda_2)\partial_w\widetilde{n}}e^g\eta\partial_2\widetilde{n} + b_1 + e^g\eta(\lambda_1-\lambda_2)\partial_z a\widetilde{\beta}.$$

Then according to the above relation and (5.6.22), one has

$$\partial_2\left(\widetilde{\beta}e^g\right) = -\partial_z\lambda_2 e^g\widetilde{\beta}^2 + b_0\widetilde{\beta} + b_1 - \frac{(\lambda_1-\lambda_3)\partial_w a}{(\lambda_3-\lambda_2)\partial_w\widetilde{n}}e^g\eta\partial_2\widetilde{n}. \tag{5.6.26}$$

Step 3: the reformulation of $\partial_2\widetilde{n}$. Considering the function $M(w,z,\hat{S})$ defined in (5.2.7), it follows from direct calculations and (5.6.24) that

$$\begin{aligned}\partial_2(M\widetilde{n}\eta) &= \partial_w M\partial_2 w\widetilde{n}\eta + \partial_z M\partial_2 z\widetilde{n}\eta + \partial_{\hat{S}}M\partial_2\hat{S}\widetilde{n}\eta + M\partial_2\widetilde{n}\eta + M\widetilde{n}\partial_2\eta\\ &= \partial_w M\widetilde{n}\eta\frac{\partial_2\widetilde{n}}{\partial_w\widetilde{n}} - \partial_w M\widetilde{n}\eta\frac{\partial_z\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 z - \partial_w M\widetilde{n}\eta\frac{\partial_{\hat{S}}\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 S + \partial_z M\partial_2 z\widetilde{n}\eta\\ &\quad + \partial_{\hat{S}}M\partial_2\hat{S}\widetilde{n}\eta + M\widetilde{n}\left(\partial_2\eta - \frac{\eta}{\widetilde{n}}\partial_2\widetilde{n}\right) + 2M\eta\partial_2\widetilde{n}.\end{aligned} \tag{5.6.27}$$

One can check that, actually the function M satisfies:

$$\partial_w M\widetilde{n} + 2M\partial_w\widetilde{n} = -\frac{(\lambda_1-\lambda_3)\partial_w a}{(\lambda_3-\lambda_2)}e^g.$$

Multiplying the above equation by $\widetilde{n}$, one has that

$$\partial_w\left(M\widetilde{n}^2\right) = -\frac{(\lambda_1-\lambda_3)\partial_w a}{(\lambda_3-\lambda_2)}e^g\widetilde{n},$$

which, along with (5.6.27), implies that

$$\begin{aligned}-\frac{(\lambda_1-\lambda_3)\partial_w a}{(\lambda_3-\lambda_2)\partial_w\widetilde{n}}e^g\eta\partial_2\widetilde{n} =& \partial_2(M\widetilde{n}\eta) + \partial_w M\widetilde{n}\eta\frac{\partial_z\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 z + \partial_w M\widetilde{n}\eta\frac{\partial_{\hat{S}}\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 S\\ &- \partial_z M\partial_2 z\widetilde{n}\eta - \partial_{\hat{S}}M\partial_2\hat{S}\widetilde{n}\eta - M\widetilde{n}\left(\partial_2\eta - \frac{\eta}{\widetilde{n}}\partial_2\widetilde{n}\right).\end{aligned}$$

Substituting the above relation into (5.6.26), one has that

$$\partial_2\left(\widetilde{\beta}e^g - M\widetilde{n}\eta\right) = -\partial_z\lambda_2 e^g\widetilde{\beta}^2 + b_0\widetilde{\beta} + b_1 + b_2. \tag{5.6.28}$$

Notice that

$$\widetilde{\beta} = e^{-g}q + e^{-g}M\widetilde{n}\eta \quad \text{and} \quad \widetilde{\beta}^2 = e^{-2g}q^2 + 2e^{-2g}qM\widetilde{n}\eta + e^{-2g}M^2\widetilde{n}^2\eta^2.$$

which, along with (5.6.28), yields the desired ODE for q.

□

5.7 Decoupled ODE inequalities

The main purpose of this section is to derive some decoupled ODE inequalities on (r, q) with good structures from (5.6.14), which play a key role on the estimate of the upper bound for (r, q).

Theorem 5.7.1. *Let N_1 and N_2 be the two constants introduced in Theorem 5.3.1. The following ODE inequalities hold:*

$$\begin{cases} \partial_3 r \leq -\frac{1}{2}\partial_w \lambda_3 e^{-h}(r^2 - N_1^2), \\ \partial_2 q \leq -\frac{1}{2}\partial_z \lambda_2 e^{-g}(q^2 - N_2^2). \end{cases} \tag{5.7.1}$$

For simplicity, in the rest of this section, we denote $D \geq 1$ a generic constant depending only on (z_0, w_0, S_0), R, γ and c, and independent of the time. Next, a definition and some additional lemmas will be given as follows.

Definition 5.7.1. *Let f and g be two functions. We say $f \approx g$ if $D^{-1}f \leq g \leq Df$; and we say $f \lesssim g$ if $f \leq Dg$.*

Lemma 5.7.1.

$$\widetilde{n} \approx n \approx \rho. \tag{5.7.2}$$

Furthermore, there exists a fixed positive constant $n_{\max}$ such that $0 < n \leq n_{\max}$.

Proof. It follows from $n > 0$, (5.2.7), Theorem 5.5.1 and its proof that $n \approx \widetilde{n}$. It remains to show $n \approx \rho$. Actually, according to Proposition A.1, one has

$$n = \frac{c^2\rho}{n^{\gamma-1}\exp\left(\frac{S}{C_v}\right) + c^2}, \tag{5.7.3}$$

where one regards n as a continuous function of ρ and S.

According to Theorem 5.5.1, there is a positive constant $n_{\max}$ such that $0 < n \leq n_{\max}$, which, along with (5.7.3), yields $n \approx \rho$. □

According to Lemma 4.5, one can obtain that

Corollary 5.7.1.

$$P(\rho, S) \approx \rho^{\gamma}, \quad \partial_{\rho} P(\rho, S) \approx \rho^{\gamma-1}, \quad \partial_{S} P(\rho, S) \approx \rho^{\gamma}, \tag{5.7.4}$$

$$\partial_{\rho S} P(\rho, S) \approx \rho^{\gamma-1}, \quad \partial_{\rho\rho} P(\rho, S) \approx \rho^{\gamma-2}, \quad F(\rho, S) \approx \rho^{\frac{\gamma-1}{2}}, \tag{5.7.5}$$

$$\partial_{\rho} F(\rho, S) \approx \rho^{\frac{\gamma-3}{2}}, \quad |\partial_{S} F(\rho, S)| \lesssim \rho^{\frac{\gamma-1}{2}} + \rho^{\frac{3\gamma-3}{2}}, \quad |\partial_{\rho S} F(\rho, S)| \lesssim \rho^{\frac{\gamma-3}{2}} + \rho^{\frac{3\gamma-5}{2}}, \tag{5.7.6}$$

$$|a(\rho, S)| \lesssim \rho^{\frac{\gamma-1}{2}}, \quad |\partial_{\rho} a(\rho, S)| \lesssim \rho^{\frac{\gamma-3}{2}} + \rho^{\frac{3\gamma-5}{2}} + \rho^{\frac{\gamma-1}{2}}. \tag{5.7.7}$$

Proof. First, the estimates on P, $\partial_{\rho} P$, $\partial_{S} P$, $\partial_{\rho S} P$ and $\partial_{\rho\rho} P$ follow from (5.1.5), Lemma 5.7.1, Theorem 5.5.1, Propositions 5.5.1 and A.2.

Second, according to the definition of $F(\rho, S)$ in (5.4.2), one has

$$\partial_{\rho} F = \frac{c^2 \sqrt{\partial_{\rho} P}}{P + \rho c^2}, \quad \partial_{S} F = \int_0^{\rho} \frac{c^2 \partial_{\sigma S} P}{2\sqrt{\partial_{\sigma} P}(P + \sigma c^2)} - \frac{c^2 \partial_{S} P \sqrt{\partial_{\sigma} P}}{(P + \sigma c^2)^2} \mathrm{d}\sigma, \tag{5.7.8}$$

which, along with the estimates on P obtained above, implies the desired estimates on F, $\partial_{\rho} F$ and $\partial_{S} F$. The term $\partial_{\rho S} F$ can be dealt with similarly.

Finally, it follows from (5.4.3) that

$$\partial_{\rho} a = -\partial_{\rho S} F - \frac{\partial_{S} P c^2 (\partial_{\rho} P + c^2)}{(P + \rho c^2)^2 \sqrt{\partial_{\rho} P}} + \frac{c^2}{P + \rho c^2} \left(\frac{2 \partial_{\rho} P \partial_{\rho S} P - \partial_{\rho S} P \partial_{S} P}{2 \sqrt{\partial_{\rho} P} \partial_{\rho} P} \right).$$

Then one can obtain the desired estimates on a and $\partial_{\rho} a$ based on the estimates associated with P and F obtained above. □

Next we consider the relation between (g, h) and ρ.

Lemma 5.7.2.

$$e^{-g} \approx e^{-h} \approx \frac{\sqrt{\rho}}{(\partial_{\rho} P)^{\frac{1}{4}}}. \tag{5.7.9}$$

Proof. We will focus only on e^{-g}, and e^{-h} can be dealt with similarly. According to (5.2.4), (5.2.5) and (5.2.7), one has

$$e^{-g} = \frac{e^G + 1}{\sqrt{e^{2G} - e^{G+H}}} e^J \quad \text{with} \quad J \coloneqq \int_\epsilon^{\frac{z-w}{2}} \frac{(c - \sqrt{\Lambda})^2}{2c^2\sqrt{\Lambda}} (\Xi, \hat{S}) \mathrm{d}\Xi.$$

The rest of the proof is divided into 2 steps: estimates on e^J and $\frac{e^G+1}{\sqrt{e^{2G}-e^{G+H}}}$.

Step 1: the estimate on e^J. First, it follows from (5.2.4) and Lemma 4.3 that

$$\begin{aligned} \frac{1}{S - S_0} \int_{S_0}^{S} J\left(\frac{z-w}{2}, \widetilde{S}\right) \mathrm{d}\widetilde{S} &= \frac{1}{S - S_0} \int_{S_0}^{S} \int_\epsilon^{\frac{z-w}{2}} \frac{(c - \sqrt{\Lambda})^2}{2c^2\sqrt{\Lambda}} (\Xi, \widetilde{S}) \mathrm{d}\Xi \mathrm{d}\widetilde{S} \\ &= \frac{1}{S - S_0} \int_{S_0}^{S} \int_{\widetilde{\epsilon}(\widetilde{S})}^{\rho} \frac{(c - \sqrt{\partial_\sigma P})^2}{2c^2\sqrt{\partial_\sigma P}} \partial_\sigma F(\sigma, \widetilde{S}) \mathrm{d}\sigma \mathrm{d}\widetilde{S}. \end{aligned} \tag{5.7.10}$$

In the above integral, after the change of variable from Ξ to σ, the integration area has changed from $(\widetilde{S}, \Xi) \in [S_0, S] \times [\epsilon, \frac{z-w}{2}]$ to $(\widetilde{S}, \sigma) \in [S_0, S] \times [\widetilde{\epsilon}(\widetilde{S}), \rho]$. By (5.2.4) and (5.4.2), one has

$$\Xi(\sigma, \widetilde{S}) \equiv \frac{z-w}{2}(\sigma, \widetilde{S}) \equiv F(\sigma, \widetilde{S}).$$

Next, we consider the mapping

$$\Upsilon : (\rho, S) \mapsto \left(\int_0^\rho G(\sigma, S) \, \mathrm{d}\sigma, S \right) = \left(\frac{z-w}{2}, S \right), \tag{5.7.11}$$

where

$$G(\sigma, S) := \frac{\sqrt{\partial_\sigma P(\sigma, S)}}{\sigma + \frac{P(\sigma, S)}{c^2}}.$$

Denote $\rho_{\max} = \sup_{(t,x) \in [0,T] \times \mathbb{R}} \rho(t, x)$, then it follows from (4.18) and Theorem 4.1 that

$$\rho_{\max} \leq \psi^{-1} \left(\frac{\max |z| + \max |w|}{2} \right) < \infty.$$

Then, according to the properties of P and $\partial_\rho P$ in Proposition A.3, one has that, for a fixed $S \in [-B, B]$,

- the mapping $\rho \mapsto \int_0^\rho G(\sigma, S) \, \mathrm{d}\sigma$ is well-defined for all $\rho \in [0, \rho_{\max}]$;
- $\int_{\rho_1}^{\rho_2} G(\sigma, S) \, \mathrm{d}\sigma = 0$ if and only if $\rho_1 = \rho_2$.

Therefore, for a fixed $S \in [-B, B]$, the mapping $\rho \mapsto \int_0^\rho G(\sigma, S)\,\mathrm{d}\sigma$ is strictly increasing and continuous for $\rho \in [0, \rho_{\max}]$, thus a bijection between $[0, \rho_{\max}]$ and $\left[0, \int_0^{\rho_{\max}} G(\sigma, S)\,\mathrm{d}\sigma\right]$. Then one gets that the mapping Υ is a bijection between $(\rho, S) \in [0, \rho_{max}] \times [-B, B]$ and its image.

Taking the inverse of Υ, one has

$$(\rho, S)^\top = \Upsilon^{-1}\left(\frac{z-w}{2}, S\right) = (\Upsilon_1^{-1}, \Upsilon_2^{-1})^\top.$$

In particular, when $\Xi(\sigma, \widetilde{S}) = \epsilon$, one has that $(\sigma, \widetilde{S})^\top = \Upsilon^{-1}(\epsilon, \widetilde{S})$, which implies that

$$\sigma = \Upsilon_1^{-1}(\epsilon, \widetilde{S}) := \widetilde{\epsilon}(\widetilde{S}) > 0."$$

Now taking $S_0 \to S$, by the Lebesgue Differentiation Theorem (see [26]), one has

$$\begin{aligned}
&\lim_{S_0 \to S} \frac{1}{S - S_0} \int_{S_0}^{S} J\left(\frac{z-w}{2}, \widetilde{S}\right) \mathrm{d}\widetilde{S} = J\left(\frac{z-w}{2}, S\right), \\
&\lim_{S_0 \to S} \frac{1}{S - S_0} \int_{S_0}^{S} \int_{\widetilde{\epsilon}(\widetilde{S})}^{\rho} \frac{(c - \sqrt{\partial_\sigma P})^2}{2c^2\sqrt{\partial_\sigma P}} \partial_\sigma F(\sigma, \widetilde{S}) \mathrm{d}\sigma \mathrm{d}\widetilde{S} \\
=& \int_{\widetilde{\epsilon}(S)}^{\rho} \frac{(c - \sqrt{\partial_\sigma P})^2}{2c^2\sqrt{\partial_\sigma P}} \partial_\sigma F(\sigma, S) \mathrm{d}\sigma,
\end{aligned}$$

which, along with (5.7.10) and (5.7.8), implies that

$$J\left(\frac{z-w}{2}, S\right) = \int_{\widetilde{\epsilon}(S)}^{\rho} \frac{(c - \sqrt{\partial_\sigma P})^2}{2c^2\sqrt{\partial_\sigma P}} \partial_\sigma F(\sigma, S) \mathrm{d}\sigma = \int_{\widetilde{\epsilon}(S)}^{\rho} \frac{(c - \sqrt{\partial_\sigma P})^2}{2(c^2\sigma + P)} \mathrm{d}\sigma.$$

On the one hand, for the upper bound estimate, according to Theorem 5.5.1, one has

$$\frac{(c - \sqrt{\partial_\rho P})^2}{2(c^2\rho + P)} \leq \frac{c^2}{2(c^2\rho + P)} \leq \frac{c^2}{2c^2\rho} = \frac{1}{2\rho}. \tag{5.7.12}$$

On the other hand, for the lower bound estimate, one claims that there exists a constant $A > 0$, such that for all $\rho > 0$, one has

$$\frac{(c - \sqrt{\partial_\rho P})^2}{2(c^2\rho + P)} \geq -A\frac{\sqrt{\partial_\rho P}}{\rho} + \frac{1}{2\rho}. \tag{5.7.13}$$

The above inequality holds if and only if

$$\sqrt{\partial_\rho P} + 2Ac^2 + 2A\frac{P}{\rho} \geq 2c + \frac{P}{2\rho\sqrt{\partial_\rho P}}. \tag{5.7.14}$$

Actually, according to Corollary 5.7.1 and Theorem 5.5.1, one has

$$\frac{P}{\rho\sqrt{\partial_\rho P}} \approx \rho^{\frac{\gamma-1}{2}} \approx \sqrt{\partial_\rho P} < c, \tag{5.7.15}$$

so (5.7.13)-(5.7.14) are indeed true if A is large enough and depends only on the initial data and c.

Then it follows from (5.7.12)-(5.7.13) that

$$-A\int_{\widetilde{\epsilon}(S)}^{\rho} \frac{\sqrt{\partial_\sigma P}}{\sigma}\mathrm{d}\sigma + \frac{1}{2}\log\left(\frac{\rho}{\widetilde{\epsilon}(S)}\right) \leq J \leq \frac{1}{2}\log\left(\frac{\rho}{\widetilde{\epsilon}(S)}\right). \tag{5.7.16}$$

Now by Corollary 5.7.1, one has $\frac{\sqrt{\partial_\rho P}}{\rho} \approx \rho^{\frac{\gamma-3}{2}}$, hence

$$\int_{\widetilde{\epsilon}(S)}^{\rho} \frac{\sqrt{\partial_\sigma P}}{\sigma}\mathrm{d}\sigma \approx \rho^{\frac{\gamma-1}{2}} - \widetilde{\epsilon}(S)^{\frac{\gamma-1}{2}},$$

which, along with (5.7.15)-(5.7.16), Theorem 5.5.1 and its proof , yields that

$$e^J \approx \sqrt{\rho}. \tag{5.7.17}$$

Step 2: the estimate on $\frac{e^G+1}{\sqrt{e^{2G}-e^{G+H}}}$. First of all, by (5.2.4) and (5.2.5), one has

$$e^G = e^{\frac{w+z}{2}}\left(\frac{c+\sqrt{\partial_\rho P}}{c-\sqrt{\partial_\rho P}}\right). \tag{5.7.18}$$

According to Lemma 5.5.1 and Theorem 5.5.1, $e^G > 0$ has uniform upper and lower bounds.

Second, by (5.2.4), (5.2.5) and Theorem 5.5.1, one has

$$e^G - e^H = \left(\frac{c+u}{c-u}\right)\left(\frac{4c\sqrt{\partial_\rho P}}{c^2-\partial_\rho P}\right) \approx \sqrt{\partial_\rho P}, \tag{5.7.19}$$

which implies that

$$\frac{e^G+1}{\sqrt{e^{2G}-e^{G+H}}} \approx \frac{1}{(\partial_\rho P)^{\frac{1}{4}}}. \tag{5.7.20}$$

Then the desired estimate on e^{-g} follows from (5.7.17) and (5.7.20). The proof of this lemma is complete. □

Next, in order to prove the positivity of $e^{-g}\partial_z\lambda_2$ and $e^{-h}\partial_w\lambda_3$, one needs the following relations:

Lemma 5.7.3.

$$\begin{aligned}
\partial_w\Lambda =& -\frac{\partial_{\rho\rho}P}{2\partial_\rho F}, \quad \partial_z\Lambda = \frac{\partial_{\rho\rho}P}{2\partial_\rho F}, \quad \partial_{\hat{S}}\Lambda = -\frac{\partial_{\rho\rho}P\partial_S F}{\partial_\rho F} + \partial_{\rho S}P,\\
\partial_w H =& \frac{1}{c} + \frac{c\partial_{\rho\rho}P}{2(c^2-\partial_\rho P)\sqrt{\partial_\rho P}\partial_\rho F}, \quad \partial_z H = \frac{1}{c} - \frac{c\partial_{\rho\rho}P}{2(c^2-\partial_\rho P)\sqrt{\partial_\rho P}\partial_\rho F},\\
\partial_w G =& \frac{1}{c} - \frac{c\partial_{\rho\rho}P}{2(c^2-\partial_\rho P)\sqrt{\partial_\rho P}\partial_\rho F}, \quad \partial_z G = \frac{1}{c} + \frac{c\partial_{\rho\rho}P}{2(c^2-\partial_\rho P)\sqrt{\partial_\rho P}\partial_\rho F},\\
\partial_{\hat{S}} H =& -\frac{c}{(c^2-\partial_\rho P)\sqrt{\partial_\rho P}}\left[\partial_{\rho S}P - \partial_{\rho\rho}P\frac{\partial_S F}{\partial_\rho F}\right] = -\partial_{\hat{S}}G,\\
\partial_w a =& -\frac{1}{2\partial_\rho F}\partial_\rho a = -\partial_z a, \quad \partial_{\hat{S}}\widetilde{n} = -\frac{c\partial_\rho n\partial_S F}{\partial_\rho F\sqrt{c^2-u^2}} + \frac{c\partial_S n}{\sqrt{c^2-u^2}},\\
\partial_w\widetilde{n} =& -\frac{c\partial_\rho n}{2\partial_\rho F\sqrt{c^2-u^2}} + \frac{un}{2c\sqrt{c^2-u^2}}, \quad \partial_z\widetilde{n} = \frac{c\partial_\rho n}{2\partial_\rho F\sqrt{c^2-u^2}} + \frac{un}{2c\sqrt{c^2-u^2}}.
\end{aligned}$$

Proof. The above formulae can be directly obtained by the chain rule. We only prove the one on $\partial_w a$ as an example. First, via the chain rule, one has

$$\partial_w a = -\frac{1}{2}\partial_{\frac{z-w}{2}}a + \frac{1}{c}\partial_{\frac{z+w}{c}}a. \tag{5.7.21}$$

By Lemma 5.6.2, and note that $\partial_u a = 0$ as a is independent of u, one can obtain that

$$\begin{aligned}
\partial_{\frac{z-w}{2}}a &= \partial_\rho a\partial_{\frac{z-w}{2}}\rho + \partial_u a\partial_{\frac{z-w}{2}}u + \partial_S a\partial_{\frac{z-w}{2}}S = \frac{1}{\partial_\rho F}\partial_\rho a,\\
\partial_{\frac{z+w}{c}}a &= \partial_\rho a\partial_{\frac{z+w}{c}}\rho + \partial_u a\partial_{\frac{z+w}{c}}u + \partial_S a\partial_{\frac{z+w}{c}}S = 0,
\end{aligned}$$

which, along with (5.7.21), yields the desired formula on $\partial_w a$. □

Now based on Lemma 5.7.3, one has

Corollary 5.7.2.

$$e^{-g}\partial_z\lambda_2 > 0 \quad \textit{and} \quad e^{-h}\partial_w\lambda_3 > 0.$$

Proof. It follows from (5.2.6), Lemma 5.7.3, Corollary 5.7.1 and Theorem 5.5.1 that

$$\partial_z\lambda_2 = \frac{2\partial_z G}{(e^G+1)^2} = \frac{2}{(e^G+1)^2}\left(\frac{1}{c} + \frac{c\partial_{\rho\rho}P}{2(c^2-\partial_\rho P)\sqrt{\partial_\rho P}\partial_\rho F}\right) > 0. \tag{5.7.22}$$

Hence $e^{-g}\partial_z\lambda_2 > 0$. Similarly, one can prove that $e^{-h}\partial_w\lambda_3 > 0$. □

Now, based on the above lemmas, one can prove the boundedness of the coefficients a_i $(i = 3, 4)$ and b_i $(i = 3, 4)$ in the decoupled ODEs (5.6.14).

Lemma 5.7.4.

$$\frac{|b_3|}{\partial_z\lambda_2 e^{-g}} \le D.$$

Proof. According to (5.7.18), Lemma 5.5.1 and Theorem 5.5.1, one gets

$$e^G \approx 1, \tag{5.7.23}$$

which, along with Corollary 5.7.1, (5.7.22) and Theorem 5.5.1, implies

$$\partial_z\lambda_2 \approx 1. \tag{5.7.24}$$

Hence by the definition of b_3 given by (5.2.10), one can get that

$$\frac{|b_3|}{\partial_z\lambda_2 e^{-g}} \lesssim |M\widetilde{n}\eta| + |b_0|.$$

The rest of the proof can be divided into two steps: estimates on $|M\widetilde{n}\eta|$ and $|b_0|$.

Step 1: the estimate on $|M\widetilde{n}\eta|$. By the definition of M given by (5.2.7), one has

$$|M\widetilde{n}\eta| \lesssim \left|\frac{\eta}{\widetilde{n}}\right|\left|\int_z^w e^g\widetilde{n}\partial_\varsigma a\left(\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\right)(\varsigma, z, \hat{S})\mathrm{d}\varsigma\right|. \tag{5.7.25}$$

First, according to Lemma 5.7.3, Corollary 5.7.1, $\gamma > 1$, Theorem 5.5.1 and its proof, one has

$$\partial_w a \lesssim 1 + \rho^{\gamma-1} + \rho \lesssim 1. \tag{5.7.26}$$

Second, it follows from (5.2.2) and direct calculations that

$$\lambda_2 - \lambda_3 = \frac{2c^2(c^2-u^2)}{c^4 - u^2\partial_\rho P}\sqrt{\partial_\rho P}, \quad \lambda_1 - \lambda_3 = \frac{c^2-u^2}{c^2 - u\sqrt{\partial_\rho P}}\sqrt{\partial_\rho P},$$

which, along with Theorem 5.5.1, implies that

$$\left|\frac{\lambda_1 - \lambda_3}{\lambda_3 - \lambda_2}\right| \approx 1. \tag{5.7.27}$$

At last, according to Lemmas 5.7.1-5.7.2, one has

$$|e^g \widetilde{n}| \lesssim \left|(\partial_\rho P)^{\frac{1}{4}}\sqrt{\rho}\right|,$$

which, along with the above estimates, Lemma 5.6.1, Lemma 5.7.1 and Corollary 5.7.1, implies that

$$\begin{aligned} |M\widetilde{n}\eta| &\lesssim \left|\int_0^{\frac{z-w}{2}} (\partial_\rho P)^{\frac{1}{4}}\sqrt{\rho}(\Xi, \hat{S})\mathrm{d}\Xi\right| \\ &\lesssim \left|\int_0^{\rho} (\partial_\sigma P)^{\frac{1}{4}}\sqrt{\sigma}\partial_\sigma F \mathrm{d}\sigma\right| \lesssim \rho^{\frac{3\gamma-1}{4}}. \end{aligned}$$

As $\gamma > 1$ and ρ has an uniform upper bound by Theorem 5.5.1 and its proof, $|M\widetilde{n}\eta|$ is bounded.

Step 2: the estimate on $|b_0|$. First, one has $|b_0| \leq \sum_{i=1}^6 |b_0^i|$, where b_0^i is the i-th term of b_0. First, by $(5.0.1)_3$, Lemma 5.7.2 and Corollary 5.7.1, one has

$$|b_0^1| = |e^g \partial_2 S a \partial_z g| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}}(\lambda_2 - \lambda_1)\left|\frac{\eta}{\widetilde{n}}\right| \widetilde{n}\rho^{\frac{\gamma-1}{2}}|\partial_z g|. \tag{5.7.28}$$

It follows from (5.2.2), (5.2.7), (5.7.19), Lemma 5.7.3, Corollary 5.7.1, Theorem 5.5.1 and direct

calculations that

$$\begin{aligned}\lambda_2-\lambda_1=&\frac{c^2-u^2}{c^2+u\sqrt{\partial_\rho P}}\sqrt{\partial_\rho P}\lesssim\sqrt{\partial_\rho P},\\ |\partial_z g|=&\left|-\frac{e^G\partial_z G}{e^G+1}+\frac{2e^G\partial_z G-e^H(\partial_z G+\partial_z H)}{2(e^G-e^H)}-\frac{(c-\sqrt{\partial_\rho P})^2}{4c^2\sqrt{\partial_\rho P}}\right|\lesssim 1+\frac{1}{\sqrt{\partial_\rho P}},\end{aligned}\tag{5.7.29}$$

which, along with (5.7.28), implies that

$$|b_0^1|\lesssim(\partial_\rho P)^{\frac{5}{4}}\sqrt{\rho}\left|\frac{\eta}{\widetilde{n}}\right|\left(1+\frac{1}{\sqrt{\partial_\rho P}}\right).$$

Similarly, by $(5.0.1)_3$, (5.7.29) and Lemma 5.7.2, one has

$$|b_0^2|=|e^g\partial_2 S\partial_{\hat{S}}g|\lesssim\frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}}\left|\frac{\eta}{\widetilde{n}}\right|\widetilde{n}\sqrt{\partial_\rho P}|\partial_{\hat{S}}g|.\tag{5.7.30}$$

According to (5.2.7), one has

$$\begin{aligned}\partial_{\hat{S}}g=&-\frac{e^G\partial_{\hat{S}}G}{e^G+1}+\frac{2e^G\partial_{\hat{S}}G-e^H(\partial_{\hat{S}}G+\partial_{\hat{S}}H)}{2(e^G-e^H)}-\partial_{\hat{S}}J,\\ \partial_{\hat{S}}J=&-\int_\epsilon^{\frac{z-w}{2}}\frac{c^2-\partial_\rho P}{4c^2(\partial_\rho P)^{\frac{3}{2}}}\left[\partial_{\rho S}P-\frac{\partial_{\rho\rho}P\partial_S F}{2\partial_\rho F}\right](\Xi,\hat{S})\mathrm{d}\Xi.\end{aligned}\tag{5.7.31}$$

According to (5.2.5), Lemma 5.7.3, Corollary 5.7.1 and Theorem 5.5.1, one can obtain

$$|\partial_{\hat{S}}G|=|\partial_{\hat{S}}H|\lesssim\sqrt{\partial_\rho P}.\tag{5.7.32}$$

By Corollary 5.7.1 and a change of variable as in the proof of Lemma 5.7.2, one can obtain

$$\begin{aligned}|\partial_{\hat{S}}J|=&\left|\int_{\widetilde{\epsilon}(S)}^{\rho}\frac{c^2-\partial_\sigma P}{4c^2(\partial_\sigma P)^{\frac{3}{2}}}\left[\partial_{\sigma S}P-\frac{\partial_{\sigma\sigma}P\partial_S F}{2\partial_\sigma F}\right]\partial_\sigma F\mathrm{d}\sigma\right|,\\ \lesssim&\left|\int_{\widetilde{\epsilon}(S)}^{\rho}\frac{1}{\sigma}\mathrm{d}\sigma\right|\lesssim 1+|\log(\rho)|,\end{aligned}$$

which, along with (5.7.19), (5.7.30)-(5.7.32) and direct calculations, yields that

$$|b_0^2|\lesssim\frac{(\partial_\rho P)^{\frac{3}{4}}}{\sqrt{\rho}}\left|\frac{\eta}{\widetilde{n}}\right|\widetilde{n}(1+\sqrt{\partial_\rho P}+|\log(\rho)|).$$

Next, by $(5.0.1)_3$, (5.7.23), (5.7.24), (5.7.27), (5.7.32), Lemma 5.7.2-5.7.3, Corollary 5.7.1 and Theorem 5.5.1, one has $|\partial_w \lambda_2| \approx 1$ and

$$\begin{aligned}
|b_0^3| =& \left| e^g \partial_w \lambda_2 \frac{a \partial_3 S}{\lambda_3 - \lambda_2} \right| = \left| e^g \partial_w \lambda_2 \frac{a(\lambda_3 - \lambda_1)\eta}{\lambda_3 - \lambda_2} \right| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}} \rho^{\frac{\gamma-1}{2}} \widetilde{n} \left| \frac{\eta}{\widetilde{n}} \right|, \\
|b_0^4| =& |e^g \eta a \partial_z \lambda_2| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}} \widetilde{n} \left| \frac{\eta}{\widetilde{n}} \right| \rho^{\frac{\gamma-1}{2}}, \\
|b_0^5| =& |e^g \eta \partial_{\hat{S}} \lambda_2| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}} \widetilde{n} \left| \frac{\eta}{\widetilde{n}} \right| |\partial_{\hat{S}} \lambda_2| \\
=& \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}} \widetilde{n} \left| \frac{\eta}{\widetilde{n}} \right| \left| \frac{2e^G \partial_{\hat{S}} G}{(e^G + 1)^2} \right| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}} \widetilde{n} \left| \frac{\eta}{\widetilde{n}} \right| \sqrt{\partial_\rho P}.
\end{aligned}$$

Finally, by (5.7.29), Lemma 5.7.2, Corollary 5.7.1 and Theorem 5.5.1, one has

$$|b_0^6| = |e^g \eta \partial_z a (\lambda_1 - \lambda_2)| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}} \widetilde{n} \left| \frac{\eta}{\widetilde{n}} \right| |\partial_z a| \sqrt{\partial_\rho P}.$$

According to Lemma 5.7.3 and Corollary 5.7.1, one gets

$$|\partial_z a| \lesssim 1 + \rho^{\gamma-1} + \rho.$$

Hence

$$|b_0^6| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}} \widetilde{n} \left| \frac{\eta}{\widetilde{n}} \right| \sqrt{\partial_\rho P} (1 + \rho^{\gamma-1} + \rho).$$

Thus, the desired boundedness follows from the above estimates, Lemma 5.6.1, Lemma 5.7.1, Corollary 5.7.1, Theorem 5.5.1 and its proof. □

Lemma 5.7.5.

$$\frac{|b_1|}{\partial_z \lambda_2 e^{-g}} \leq D.$$

Proof. The proof will be divided into two steps: the estimate on a key term $\frac{1}{|\partial_w \widetilde{n}|}$ and then the boundedness of $\frac{|b_1|}{\partial_z \lambda_2 e^{-g}}$.

Step 1: the estimate on $\frac{1}{|\partial_w \widetilde{n}|}$. According to Lemma 5.7.3, if $u \leq 0$, then

$$|\partial_w \widetilde{n}| = \left| \frac{c \partial_\rho n}{2 \partial_\rho F \sqrt{c^2 - u^2}} - \frac{un}{2c\sqrt{c^2 - u^2}} \right| \geq \frac{c \partial_\rho n}{2 \partial_\rho F \sqrt{c^2 - u^2}}. \tag{5.7.33}$$

If $u > 0$, by Theorem 5.5.1, one gets that

$$\begin{aligned} |\partial_w \widetilde{n}| &\geq \frac{c\partial_\rho n}{2\partial_\rho F\sqrt{c^2-u^2}} - \frac{n}{2\sqrt{c^2-u^2}} \\ &= \frac{n}{2\sqrt{\partial_\rho P}\sqrt{c^2-u^2}}\left(\frac{c\sqrt{\partial_\rho P}\partial_\rho n}{n\partial_\rho F} - \sqrt{\partial_\rho P}\right). \end{aligned} \tag{5.7.34}$$

We claim that there exists some constant $D^* > 0$ depending only on (z_0, w_0, S_0), R, γ and c, and independent of $(t, x) \in [0, T) \times \mathbb{R}$, such that

$$\frac{c\sqrt{\partial_\rho P}\partial_\rho n}{n\partial_\rho F} - \sqrt{\partial_\rho P} > D^*. \tag{5.7.35}$$

According to Proposition A.1 and (A.0.4), one has $\rho\partial_\rho n = n$. Then

$$\begin{aligned} &\frac{c\sqrt{\partial_\rho P}\partial_\rho n}{n\partial_\rho F} - \sqrt{\partial_\rho P} = \frac{\sqrt{\partial_\rho P}}{n\partial_\rho F}(c\partial_\rho n - n\partial_\rho F) \\ =&\frac{\sqrt{\partial_\rho P}}{\rho n\partial_\rho F}(cn - \rho n\partial_\rho F) = \frac{\sqrt{\partial_\rho P}}{\rho\partial_\rho F}(c - \rho\partial_\rho F). \end{aligned} \tag{5.7.36}$$

By (5.7.8), Theorem 5.5.1 and the fact that $\frac{P}{\rho} > 0$, one has

$$\rho\partial_\rho F = \frac{c^2\sqrt{\partial_\rho P}}{c^2 + \frac{P}{\rho}} < \sqrt{\partial_\rho P} < c - D^{**},$$

for some constant $D^{**} > 0$ depending only on (z_0, w_0, S_0), R, γ and c, and independent of $(t, x) \in [0, T) \times \mathbb{R}$. Then one has

$$c - \rho\partial_\rho F > D^{**}. \tag{5.7.37}$$

Also note that, according to (5.7.8) and Corollary 5.7.1, one has

$$\frac{\sqrt{\partial_\rho P}}{\rho\partial_\rho F} = \frac{c^2 + \frac{P}{\rho}}{c^2} \approx 1,$$

which, along with (5.7.36)-(5.7.37), implies that (5.7.35) is indeed true.

Then according to (A.0.4), (5.7.33)-(5.7.35), Proposition 5.5.1, Lemma 5.7.1, Corollary 5.7.1 and Theorem 5.5.1, one has

$$\frac{1}{|\partial_w \widetilde{n}|} \lesssim \frac{\sqrt{\partial_\rho P}}{\rho}. \tag{5.7.38}$$

Step 2: the boundedness of $|b_1|e^g$. According to (5.7.24), one has

$$\frac{|b_1|}{\partial_z\lambda_2 e^{-g}} \approx |b_1|e^g,$$

where $|b_1|e^g \leq \sum_{i=1}^4 |b_1^i|e^g$, and b_1^i is the i-th term in b_1.

First, by $(5.4.1)_2$, (5.7.26), (5.7.27), (5.7.38) and Lemma 5.7.2, one has

$$\begin{aligned}|b_1^1|e^g =& \left|e^{2g}\eta\partial_w a\left(\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\right)\frac{\partial_z\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 z\right| \\ \lesssim& \frac{\sqrt{\partial_\rho P}}{\rho}\left|\frac{\eta}{\widetilde{n}}\right|\widetilde{n}\frac{|\partial_z\widetilde{n}|\sqrt{\partial_\rho P}}{\rho}|a||\partial_2 S|.\end{aligned}$$

By $(5.0.1)_3$, (5.7.29), (A.0.4), Proposition 5.5.1, Lemma 5.7.1, Lemma 5.7.3, Corollary 5.7.1 and Theorem 5.5.1, one can get that

$$|\partial_z\widetilde{n}| \lesssim \frac{\rho}{\sqrt{\partial_\rho P}}, \quad |\partial_2 S| \lesssim \sqrt{\partial_\rho P}\left|\frac{\eta}{\widetilde{n}}\right|\widetilde{n}, \quad |a| \approx \sqrt{\partial_\rho P}. \tag{5.7.39}$$

Hence one has

$$|b_1^1|e^g \leq \frac{(\partial_\rho P)^{\frac{3}{2}}}{\rho^2}\left|\frac{\eta}{\widetilde{n}}\right|^2\widetilde{n}^2.$$

Next, by (5.7.26), (5.7.27), (5.7.38), (5.7.39) and Lemma 5.7.2, one has

$$|b_1^2|e^g = \left|e^{2g}\eta\partial_w a\left(\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\right)\frac{\partial_{\hat{S}}\widetilde{n}}{\partial_w\widetilde{n}}\partial_2 S\right| \lesssim \left|\frac{\eta}{\widetilde{n}}\right|^2\frac{(\partial_\rho P)^{\frac{3}{2}}}{\rho^2}\widetilde{n}^2|\partial_{\hat{S}}\widetilde{n}|. \tag{5.7.40}$$

From Theorem 5.5.1, Lemma 5.7.3 and Corollary 5.7.1, one can get that

$$|\partial_{\hat{S}}\widetilde{n}| \lesssim \rho|\partial_\rho n| + |\partial_S n|.$$

According to (A.0.4), Proposition 5.5.1 and Lemma 5.7.1, one can conclude

$$|\partial_S n| \lesssim \rho^\gamma, \quad \rho|\partial_\rho n| = \rho,$$

so

$$|\partial_{\hat{S}}\widetilde{n}| \lesssim \rho, \tag{5.7.41}$$

which, along with (5.7.40), implies

$$|b_1^2|e^g \leq \left|\frac{\eta}{\widetilde{n}}\right|^2 \frac{(\partial_\rho P)^{\frac{3}{2}}}{\rho^2}\widetilde{n}^2\rho.$$

At last, by $(5.0.1)_3$, (5.2.2), (5.7.26), (5.7.27), Lemma 5.7.1-5.7.2, Corollary 5.7.1 and Theorem 5.5.1, $|b_1^3|e^g$ and $|b_1^4|e^g$ can be easily dealt with as

$$\begin{aligned}
|b_1^3|e^g =& \left|e^{2g}\eta\partial_w a\left(\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\right)a\partial_3 S\right| \lesssim \left|\frac{\eta}{\widetilde{n}}\right|^2(\partial_\rho P)^{\frac{3}{2}}\rho,\\
|b_1^4|e^g =& \left|e^{2g}\eta^2\partial_w a\,(\lambda_1-\lambda_3)\,a\right| \lesssim \left|\frac{\eta}{\widetilde{n}}\right|^2(\partial_\rho P)^{\frac{3}{2}}\rho.
\end{aligned}$$

Thus, the desired boundedness follows from the above estimates, Lemma 5.6.1, Lemma 5.7.1, Corollary 5.7.1, Theorem 5.5.1 and its proof. □

Lemma 5.7.6.

$$\frac{|b_2|}{\partial_z\lambda_2 e^{-g}} \leq D.$$

Proof. According to (5.7.24), one needs to show that that

$$e^g|b_2| \leq D,$$

where $|b_2| \leq \sum_{i=1}^5 |b_2^i|$, and b_2^i is the i-th term of b_2.

First, by the definition of M given in (5.2.7), one can obtain

$$\begin{aligned}
\partial_w M =& -2\frac{\partial_w\widetilde{n}}{\widetilde{n}^3}\int_z^w e^g\widetilde{n}\partial_\varsigma a\left(\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\right)(\varsigma,z,\hat{S})\mathrm{d}\varsigma - \frac{1}{\widetilde{n}}e^g\partial_w a\left(\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\right)\\
=& -2\frac{\partial_w\widetilde{n}}{\widetilde{n}}M - \frac{1}{\widetilde{n}}e^g\partial_w a\left(\frac{\lambda_1-\lambda_3}{\lambda_3-\lambda_2}\right).
\end{aligned}$$

By the same method as how $|M\widetilde{n}\eta| \lesssim \rho^{\frac{3\gamma-1}{4}}$ was obtained in the proof of Lemma 5.7.4, one has that $|M| \lesssim \rho^{\frac{3\gamma-9}{4}}$, which, along with (A.0.4), (5.7.26), (5.7.27), Lemma 5.7.1-5.7.3, Corollary 5.7.1 and Theorem 5.5.1, yields

$$|\partial_w M| \lesssim \frac{\rho^{\frac{3\gamma-9}{4}}}{\sqrt{\partial_\rho P}} + \frac{(\partial_\rho P)^{\frac{1}{4}}}{\rho\sqrt{\rho}}. \tag{5.7.42}$$

Then, by $(5.4.1)_2$, (5.7.38), (5.7.39), (5.7.41), (5.7.42), Lemma 5.7.1-5.7.2 and Corollary 5.7.1, one has

$$\begin{aligned}
|b_2^1|e^g \leq & \left| e^g \eta \widetilde{n} \frac{\partial_z \widetilde{n}}{\partial_w \widetilde{n}} \partial_2 z \partial_w M \right| \\
\lesssim & (\partial_\rho P)^{\frac{1}{4}} \sqrt{\rho} \left| \frac{\eta}{\widetilde{n}} \right|^2 \left(\sqrt{\partial_\rho P} \rho^{\frac{3\gamma-1}{4}} + \sqrt{\rho} (\partial_\rho P)^{\frac{5}{4}} \right), \\
|b_2^2|e^g = & \left| e^g \eta \widetilde{n} \frac{\partial_{\hat{S}} \widetilde{n}}{\partial_w \widetilde{n}} \partial_2 S \partial_w M \right| \\
\lesssim & (\partial_\rho P)^{\frac{1}{4}} \sqrt{\rho} \rho^2 \left| \frac{\eta}{\widetilde{n}} \right|^2 \partial_\rho P |\partial_w M| \\
\lesssim & (\partial_\rho P)^{\frac{1}{4}} \sqrt{\rho} \left| \frac{\eta}{\widetilde{n}} \right|^2 \left(\sqrt{\partial_\rho P} \rho^{\frac{3\gamma-1}{4}} + \sqrt{\rho} (\partial_\rho P)^{\frac{5}{4}} \right).
\end{aligned}$$

Second, by $(5.4.1)_2$ and Lemma 5.7.1-5.7.2, one has

$$|b_2^3|e^g = |e^g \eta \widetilde{n} \partial_2 z \partial_z M| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}} \left| \frac{\eta}{\widetilde{n}} \right|^2 \rho^3 \partial_\rho P |\partial_z M|.$$

Define the function

$$\Omega(w, z, \hat{S}) \coloneqq e^g \widetilde{n} \partial_w a \frac{\lambda_1 - \lambda_3}{\lambda_3 - \lambda_2} = -e^g \widetilde{n} \partial_w a \frac{(c^2 + u\sqrt{\partial_\rho P})}{2c^2}.$$

According to (5.2.7) and the fact that: $z = w$ if and only if $\rho = 0$, one has $\Omega(z, z, \hat{S}) = 0$ and

$$\begin{aligned}
\partial_z M = & -2 \frac{\partial_z \widetilde{n}}{\widetilde{n}^3} \int_z^w \Omega(\varsigma, z, \hat{S}) \mathrm{d}\varsigma - \frac{1}{\widetilde{n}^2} \Omega(z, z, \hat{S}) - \frac{1}{\widetilde{n}^2} \int_z^w \partial_z \Omega(\varsigma, z, \hat{S}) \mathrm{d}\varsigma \\
= & -2 \frac{\partial_z \widetilde{n}}{\widetilde{n}} M - \frac{1}{\widetilde{n}^2} \int_z^w \partial_z \Omega(\varsigma, z, \hat{S}) \mathrm{d}\varsigma.
\end{aligned}$$

Now by (5.7.39), $|M| \lesssim \rho^{\frac{3\gamma-9}{4}}$ and Lemma 5.7.1, one has

$$|b_2^3|e^g \lesssim (\partial_\rho P)^{\frac{1}{4}} \sqrt{\rho} \left| \frac{\eta}{\widetilde{n}} \right|^2 \left(\rho^{\frac{3\gamma-1}{4}} + \partial_\rho P \left| \int_z^w \partial_z \Omega(\varsigma, z, \hat{S}) \, \mathrm{d}\varsigma \right| \right).$$

All the terms associated with the integral involving $\partial_z \Omega$ can be easily bounded by the previous procedures, except for the term

$$\int_z^w e^g \widetilde{n} \partial_{z\varsigma} a \frac{(c^2 + u\sqrt{\partial_\rho P})}{2c^2} (\varsigma, z, \hat{S}) \mathrm{d}\varsigma.$$

As $a(\rho, S)$ is a function independent of u, by Lemma 5.6.2 and explicit computations, one can obtain that

$$\partial_{wz} a = -\frac{1}{4\partial_\rho F}\partial_\rho\left(\frac{\partial_\rho a}{\partial_\rho F}\right),$$

which, along with the Mean Value Theorem, yields that

$$\begin{aligned}
&\int_z^w e^g \widetilde{n}\partial_{z\varsigma} a \frac{(c^2 + u\sqrt{\partial_\rho P})}{2c^2}(\varsigma, z, \hat{S})\mathrm{d}\varsigma \\
&= -\frac{1}{4} e^g(z, w^*, \hat{S})\widetilde{n}(z, w^*, \hat{S})\frac{(c^2 + u\sqrt{\partial_\rho P})}{2c^2}(z, w^*, \hat{S})\int_z^w \frac{1}{\partial_\rho F}\partial_\rho\left(\frac{\partial_\rho a}{\partial_\rho F}\right)(\varsigma, z, \hat{S})\mathrm{d}\varsigma,
\end{aligned}$$

for some $w^* \in [z, w]$.

Here, via the change of variable formula, Corollary 5.7.1, Theorem 5.5.1 and its proof, one has

$$\left|\int_z^w \frac{1}{\partial_\rho F}\partial_\rho\left(\frac{\partial_\rho a}{\partial_\rho F}\right)(\varsigma, z, \hat{S})\mathrm{d}\varsigma\right| = \left|\int_0^\rho \partial_\sigma\left(\frac{\partial_\sigma a}{\partial_\sigma F}\right)\mathrm{d}\sigma\right| = \left|\frac{\partial_\rho a}{\partial_\rho F}\right| \lesssim 1.$$

Hence, by Lemma 5.7.1-5.7.2, one has

$$|b_2^3| e^g \lesssim (\partial_\rho P)^{\frac{1}{4}}\sqrt{\rho}\left|\frac{\eta}{\widetilde{n}}\right|^2\left(\rho^{\frac{3\gamma-1}{4}} + \partial_\rho P\right).$$

Similarly, $|\partial_{\hat{S}} M| \lesssim 1$ can be proved by the same argument. Hence, by (5.7.39), Lemma 5.7.1-5.7.2, one has

$$\begin{aligned}
|b_2^4| e^g &= \left|e^g \eta\widetilde{n}\partial_2 S\partial_{\hat{S}} M\right| \\
&\lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}}\left|\frac{\eta}{\widetilde{n}}\right|^2 \rho^3\sqrt{\partial_\rho P}|\partial_{\hat{S}} M| \lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}}\left|\frac{\eta}{\widetilde{n}}\right|^2 \rho^3\sqrt{\partial_\rho P}, \\
|b_2^5| e^g &= \left|e^g \eta\widetilde{n}^2 M\left(\partial_2\eta - \frac{\eta}{n}\partial_2\widetilde{n}\right)\right| \\
&\lesssim \frac{(\partial_\rho P)^{\frac{1}{4}}}{\sqrt{\rho}}\left|\frac{\eta}{\widetilde{n}}\right| \rho^3 |M|\left|\partial_2\eta - \frac{\eta}{n}\partial_2\widetilde{n}\right|.
\end{aligned}$$

Thus, the desired boundedness follows from the above estimates, Lemmas 5.6.1 and 5.7.1, Corollaries 5.6.1-5.7.1, Theorem 5.5.1 and its proof. □

Lemma 5.7.7.

$$\frac{|b_4|}{\partial_z\lambda_2 e^{-g}} \le D.$$

Proof. The desired conclusion follows from the proof of Lemmas 5.7.5-5.7.6 and the fact that $|M\widetilde{n}\eta|$ is bounded from the proof of Lemma 5.7.4. □

Now we can give the proof of Theorem 5.7.1

We will focus only on q, and r can be dealt with similarly. First, according to Young's inequality, one has

$$b_3 q \leq |b_3 q| \leq \frac{\partial_z \lambda_2 e^{-g}}{2} q^2 + \frac{|b_3|^2}{2\partial_z \lambda_2 e^{-g}},$$

which, along with Theorem 5.6.1, implies that

$$\partial_2 q \leq -\frac{\partial_z \lambda_2 e^{-g}}{2} q^2 + \frac{|b_3|^2}{2\partial_z \lambda_2 e^{-g}} + |b_4|. \tag{5.7.43}$$

Then the inequality $(5.7.1)_2$ follows from (5.7.43) and Lemmas 5.7.4-5.7.7.

5.8 Mass-energy density lower bound for the adiabatic exponent $\gamma \in (1, 3)$

This section will be devoted to giving a sharp enough lower bound estimate on ρ for $1 < \gamma < 3$. For this purpose, we first give the upper bounds of r and q.

Lemma 5.8.1. *One has the following bounds on r and q*

$$\begin{cases} r \leq \max\{N_1, \sup_x r_0(x)\} \coloneqq \overline{R}, \\ q \leq \max\{N_2, \sup_x q_0(x)\} \coloneqq \overline{Q}, \end{cases}$$

where $r_0(x) = r(0, x)$ and $q_0(x) = q(0, x)$.

Proof. We will focus only on q, and r can be dealt with similarly. We consider 3 cases: $q_0(x_0^2) < N_2$, $q_0(x_0^2) > N_2$, and $q_0(x_0^2) = N_2$.

First of all, consider the case $q_0(x_0^2) < N_2$. By the continuity of q, in a small open interval around $t = 0$, $q(t, x^2(t)) < N_2$. Define time $\widetilde{T}$ by

$$\widetilde{T} \coloneqq \inf_t \{t : q(t, x^2(t)) > N_2\}. \tag{5.8.1}$$

Then $\widetilde{T} > 0$. Note for $t \in [0, \widetilde{T}]$, one has $q(t, x^2(t)) \leq N_2$.

It remains to show that $T = \widetilde{T}$. Suppose for contradiction that $\widetilde{T} < T$. Via the continuity of $q(t, x^2(t))$, there exists some small $\upsilon > 0$, such that if $t \in (\widetilde{T}, \widetilde{T} + \upsilon)$, then $q(t, x^2(t)) > N_2$. By $(5.7.1)_2$, one has

$$\partial_2 q \leq 0 \quad \text{for} \quad t \in (\widetilde{T}, \widetilde{T} + \upsilon).$$

Integrating this inequality along $x^2(t)$ over $(\widetilde{T}, \widetilde{T} + \delta)$ for any δ satisfying $0 < \delta < \upsilon$, one gets

$$q(\widetilde{T} + \delta, x^2(\widetilde{T} + \delta)) \leq q(\widetilde{T}, x^2(\widetilde{T})) \leq N_2.$$

This is a contradiction to $q(t, x^2(t)) > N_2$ if $t \in (\widetilde{T}, \widetilde{T} + \upsilon)$, so one must have $\widetilde{T} = T$.

Second, consider the case $q_0(x_0^2) > N_2$. Once again by the continuity of q, in a small open interval around $t = 0$, one has $q(t, x^2(t)) > N_2$. Define time $\widetilde{T}$ by

$$\widetilde{T} \coloneqq \inf_t \{t : q(t, x^2(t)) < N_2\}.$$

Then $\widetilde{T} > 0$. Note for $t \in [0, \widetilde{T}]$, $q(t, x^2(t)) \geq N_2$. By $(5.7.1)_2$, one has

$$\partial_2 q \leq 0 \quad \text{for} \quad t \in [0, \widetilde{T}].$$

Integrating this inequality along $x^2(t)$ over $[0, t]$ for any $t \in (0, \widetilde{T}]$, one has

$$q(t, x^2(t)) \leq q_0(x_0^2) \leq \sup_x q_0(x) \quad \text{for} \quad t \in [0, \widetilde{T}],$$

so q has the upper bound $\overline{Q}$ in this interval. Now by the definition of $\widetilde{T}$ and the continuity of q, there exists some $\upsilon > 0$ such that $q(t, x^2(t)) < N_2$ on $t \in (\widetilde{T}, \widetilde{T} + \upsilon)$. In particular, fix a δ such that $0 < \delta < \upsilon$, one has $q(\widetilde{T} + \delta, x^2(\widetilde{T} + \delta)) < N_2$. Then the rest of the proof is reduced to the case $q_0(x_0^2) < N_2$. Hence $q(t, x^2(t)) \leq \overline{Q}$.

Finally, the case $q_0(x_0^2) = N_2$ can be dealt with similarly. □

The lower bound on the ρ now can be given along the characteristics $x^2(t)$ and $x^3(t)$.

Lemma 5.8.2. *Assume* $1 < \gamma < 3$. *One has*

$$\rho^{\frac{3-\gamma}{4}}(t, x^2(t)) \geq \frac{1}{D(t+1)}, \quad \rho^{\frac{3-\gamma}{4}}(t, x^3(t)) \geq \frac{1}{D(t+1)}.$$

Proof. We will focus only on $\rho^{\frac{3-\gamma}{4}}(t, x^2(t))$, and $\rho^{\frac{3-\gamma}{4}}(t, x^3(t))$ can be dealt with similarly. First, according to $(5.0.1)_3$, (5.2.7), (5.2.8) and (5.4.1), one has

$$\begin{aligned}
\partial_2(z-w) &= -a\partial_2 S - (\partial_t w + \lambda_2 \partial_x w) \\
&= -a\partial_2 S - (\partial_3 w + (\lambda_2 - \lambda_3)\partial_x w) \\
&= -a\partial_2 S - a\partial_3 S - (\lambda_2 - \lambda_3)\partial_x w \\
&= -2a\eta(\lambda_2 - \lambda_1) - (\lambda_2 - \lambda_3)\widetilde{\alpha} \\
&= -2a\eta(\lambda_2 - \lambda_1) - (\lambda_2 - \lambda_3) r e^{-h} - (\lambda_2 - \lambda_3) L\eta\widetilde{n} e^{-h},
\end{aligned} \tag{5.8.2}$$

which, along with Proposition 5.5.2, Lemma 5.8.1 and the fact

$$\begin{aligned}
-2a\eta(\lambda_2 - \lambda_1) \geq & -2|a||\eta|(\lambda_2 - \lambda_1), \\
-L\eta\widetilde{n}(\lambda_2 - \lambda_3)e^{-h} \geq & -|L||\eta|\widetilde{n}(\lambda_2 - \lambda_3)e^{-h},
\end{aligned}$$

yields that

$$\partial_2(z-w) \geq -2|a||\eta|(\lambda_2 - \lambda_1) - |L||\eta|\widetilde{n}(\lambda_2 - \lambda_3)e^{-h} - (\lambda_2 - \lambda_3)\overline{R}e^{-h}. \tag{5.8.3}$$

Notice that, according to (5.4.2), one has

$$\partial_2(z-w) = 2\partial_\rho F \partial_2 \rho + 2\partial_S F \partial_2 S,$$

which, along with (5.8.3), yields

$$2\partial_\rho F \partial_2 \rho \geq -2|a||\eta|(\lambda_2 - \lambda_1) - |L||\eta|\widetilde{n}(\lambda_2 - \lambda_3)e^{-h} - (\lambda_2 - \lambda_3)\overline{R}e^{-h} - 2|\partial_S F||\partial_2 S|.$$

Multiplying the above inequality on both sides by $\frac{e^h}{(\partial_\rho P)^{\frac{1}{4}}} > 0$, one has that

$$\begin{aligned} 2\frac{e^h}{(\partial_\rho P)^{\frac{1}{4}}}\partial_\rho F\partial_2\rho \geq & -2\frac{e^h}{(\partial_\rho P)^{\frac{1}{4}}}|a||\eta|(\lambda_2-\lambda_1)-\frac{1}{(\partial_\rho P)^{\frac{1}{4}}}|L||\eta|\widetilde{n}(\lambda_2-\lambda_3) \\ & -\frac{1}{(\partial_\rho P)^{\frac{1}{4}}}(\lambda_2-\lambda_3)\overline{R}-2\frac{e^h}{(\partial_\rho P)^{\frac{1}{4}}}|\partial_S F||\partial_2 S|. \end{aligned} \tag{5.8.4}$$

Via the similar argument used in the last section, one can check that

$$\begin{aligned} \frac{e^h}{(\partial_\rho P)^{\frac{1}{4}}}|a||\eta|(\lambda_2-\lambda_1)\leq & D, \quad \frac{1}{(\partial_\rho P)^{\frac{1}{4}}}|L||\eta|\widetilde{n}(\lambda_2-\lambda_3)\leq D, \\ \frac{1}{(\partial_\rho P)^{\frac{1}{4}}}(\lambda_2-\lambda_3)\overline{R}\leq & D, \quad \frac{e^h}{(\partial_\rho P)^{\frac{1}{4}}}|\partial_S F||\partial_2 S|\leq D, \end{aligned}$$

which, along with (5.8.4), implies that

$$2\frac{e^h}{(\partial_\rho P)^{\frac{1}{4}}}\partial_\rho F\partial_2\rho \geq -D. \tag{5.8.5}$$

According to Lemma 5.7.2 and Corollary 5.7.1, one has $\frac{e^h}{(\partial_\rho P)^{\frac{1}{4}}}\partial_\rho F \approx \rho^{\frac{\gamma-5}{2}}$, which, along with (5.8.5), yields

$$\partial_2\left(\rho^{\frac{\gamma-3}{2}}\right)\leq D.$$

Integrating the above inequality along the characteristic $x^2(t)$, one finally gets

$$\rho^{\frac{\gamma-3}{2}}(t,x^2(t))\leq D(t+1).$$

□

5.9 Singularity formation with strong compression: Proof of Theorem 5.3.1

We will focus only on q, and r can be dealt with similarly. Due to $q_0(x^*) < -N_2$, one has: $\exists\ \upsilon > 0$ such that

$$q_0(x^*)\leq -N_2\sqrt{1+\upsilon},$$

which, along with Theorem 5.7.1, implies that

$$q(t, x^2(t)) \leq q_0(x^*) \leq -N_2\sqrt{1+\upsilon} \quad \forall\ t \in [0, T]. \tag{5.9.1}$$

According to Theorem 5.7.1 and (5.9.1), one can obtain

$$\begin{aligned}
\partial_2 q \leq & -\frac{1}{2}\partial_z\lambda_2 e^{-g}q^2 + \frac{1}{2}\partial_z\lambda_2 e^{-g}N_2^2 \\
\leq & -\frac{1}{2}\partial_z\lambda_2 e^{-g}q^2 + \frac{1}{2}\partial_z\lambda_2 e^{-g}\frac{q^2}{1+\upsilon} \\
= & -\frac{1}{2}\partial_z\lambda_2 e^{-g}q^2\frac{\upsilon}{1+\upsilon},
\end{aligned}$$

which yields that

$$-\partial_2\left(\frac{1}{q}\right) \leq -\frac{1}{2}\partial_z\lambda_2 e^{-g}\frac{\upsilon}{1+\upsilon}.$$

Integrating the above inequality along the characteristic $x^2(t)$, one gets

$$\frac{1}{q(t, x^2(t))} \geq \frac{1}{q_0(x^*)} + \frac{\upsilon}{2(1+\upsilon)}\int_0^t \partial_z\lambda_2 e^{-g}(s, x^2(s))\mathrm{d}s.$$

As $q_0(x^*) < 0$, for proving this theorem, it is sufficient to show

$$\int_0^t \partial_z\lambda_2 e^{-g}(s, x^2(s))\mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty. \tag{5.9.2}$$

According to (5.7.24), Corollary 5.7.1-5.7.2 and Lemma 5.7.2, one has $\partial_z\lambda_2 e^{-g} > 0$ and $\partial_z\lambda_2 e^{-g} \approx \rho^{\frac{3-\gamma}{4}}$. From this, for the case $\gamma \geq 3$, then

$$\rho^{\frac{3-\gamma}{4}} \geq (\sup_{(t,x)\in[0,T)\times\mathbb{R}} \rho)^{\frac{3-\gamma}{4}},$$

so (5.9.2) will be satisfied. Otherwise, for the case $1 < \gamma < 3$, by Lemma 5.8.2, (5.9.2) is also satisfied. Hence Theorem 5.3.1 is proved.

Chapter 6

Isentropic Classical Compressible Euler Equations with Spherical Symmetry

6.1 Introduction

The multi-dimensional compressible Euler equations in the Eulerian coordinate system with spherical symmetry are given by

$$\begin{cases} \partial_t(A\rho) + \partial_r(A\rho u) = 0, \\ \partial_t(A\rho u) + \partial_r(A\rho u^2) + A\partial_r P = 0, \qquad \text{with} \quad A(r) \coloneqq r^{d-1}, \quad d = 2, 3, \\ (\rho, u)|_{t=0} = (\rho_0, u_0)(r) \quad \text{for} \quad r \in \mathbb{R}^+. \end{cases} \tag{6.1.1}$$

We consider the initial value problem of (6.1.1) for $(t, r) \in \mathbb{R}^+ \times \mathbb{R}^+$, where the pressure term takes the form

$$P = k\rho^\gamma, \quad k > 0, \quad \gamma > 1.$$

Define the quantity

$$v \coloneqq A\rho, \tag{6.1.2}$$

we now proceed with a rudimentary analysis of the Cauchy problem (6.1.1). First, the two eigenvalues λ_1 and λ_2 of equations $(6.1.1)_2$ can be given by

$$\begin{aligned} \lambda_1 &= u - \sqrt{P'} = u - \sqrt{k\gamma} A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-1}{2}}, \\ \lambda_2 &= u + \sqrt{P'} = u + \sqrt{k\gamma} A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-1}{2}}. \end{aligned} \tag{6.1.3}$$

We denote the directional derivatives as

$$\partial_1 = \partial_t + \lambda_1 \partial_r, \quad \partial_2 = \partial_t + \lambda_2 \partial_r,$$

along two characteristic directions $x^1(t, x_0^1)$ and $x^2(t, x_0^2)$, defined by

$$\frac{dx^1}{dt} = \lambda_1, \quad \frac{dx^2}{dt} = \lambda_2, \tag{6.1.4}$$

where $x^1(0, x_0^1) = x_0^1$ and $x^2(0, x_0^2) = x_0^2$. For simplicity we will abbreviate $x^1(t, x_0^1)$ and $x^2(t, x_0^2)$ as $x^1(t)$ and $x^2(t)$. Introducing the corresponding Riemann variables

$$\begin{aligned} z &= u + \frac{2\sqrt{P'}}{\gamma - 1} = u + \frac{2\sqrt{k\gamma}}{\gamma - 1} A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-1}{2}}, \\ w &= u - \frac{2\sqrt{P'}}{\gamma - 1} = u - \frac{2\sqrt{k\gamma}}{\gamma - 1} A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-1}{2}}. \end{aligned} \tag{6.1.5}$$

By explicit calculations, one can verify that

$$\begin{cases} \partial_1 w = \frac{A'}{A} u \sqrt{P'} = \sqrt{k\gamma} u A^{-\frac{\gamma+1}{2}} v^{\frac{\gamma-1}{2}} A', \\ \partial_2 z = -\frac{A'}{A} u \sqrt{P'} = -\sqrt{k\gamma} u A^{-\frac{\gamma+1}{2}} v^{\frac{\gamma-1}{2}} A'. \end{cases} \tag{6.1.6}$$

Define the following quantities:

$$\begin{aligned}
\alpha = \partial_r u - \sqrt{k\gamma} A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-3}{2}} \partial_r v, \quad \beta = \partial_r u + \sqrt{k\gamma} A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-3}{2}} \partial_r v, \\
\widetilde{\alpha} = \frac{\alpha}{v}, \quad \widetilde{\beta} = \frac{\beta}{v}, \quad \widehat{\alpha} = \frac{\alpha}{v^{\frac{\gamma-1}{2}}}, \quad \widehat{\beta} = \frac{\beta}{v^{\frac{\gamma-1}{2}}}, \\
d_2 = -\frac{\gamma+1}{4} v^{\frac{3-\gamma}{4}}, \quad d_1 = L_1 \frac{1}{r} u + L_2 r^{-\frac{(\gamma-1)(d-1)+2}{2}} v^{\frac{\gamma-1}{2}}, \\
d_0 = L_3 r^{-\frac{(d-1)(\gamma-1)+4}{2}} u v^{\frac{3\gamma-5}{4}} + L_4 \frac{1}{r^2} u^2 v^{\frac{\gamma-3}{4}} + L_5 r^{-(\gamma-1)(d-1)-2} v^{\frac{5\gamma-7}{4}}, \\
\overline{d}_2 = -\frac{\gamma+1}{4} v^{\frac{3-\gamma}{4}}, \quad \overline{d}_1 = \overline{L}_1 \frac{1}{r} u + \overline{L}_2 r^{-\frac{(\gamma-1)(d-1)+2}{2}} v^{\frac{\gamma-1}{2}}, \\
\overline{d}_0 = \overline{L}_3 r^{-\frac{(d-1)(\gamma-1)+4}{2}} u v^{\frac{3\gamma-5}{4}} + \overline{L}_4 \frac{1}{r^2} u^2 v^{\frac{\gamma-3}{4}} + \overline{L}_5 r^{-(\gamma-1)(d-1)-2} v^{\frac{5\gamma-7}{4}},
\end{aligned} \tag{6.1.7}$$

where L_i and $\overline{L}_i$ are constants depending only on k, γ and d. Lastly, define two quantities Y and Q, which play important roles in our analysis on singularity formation:

$$\begin{aligned}
Y = \alpha v^{\frac{\gamma-3}{4}} + \frac{4\sqrt{k\gamma}}{(\gamma-3)(3\gamma-5)} v^{\frac{3\gamma-5}{4}} A' A^{-\frac{\gamma+1}{2}} + \frac{\gamma+1}{\gamma-3} \frac{A'}{A} 2uv^{\frac{\gamma-3}{4}}, \\
Q = \beta v^{\frac{\gamma-3}{4}} - \frac{4\sqrt{k\gamma}}{(\gamma-3)(3\gamma-5)} v^{\frac{3\gamma-5}{4}} A' A^{-\frac{\gamma+1}{2}} + \frac{\gamma+1}{\gamma-3} \frac{A'}{A} 2uv^{\frac{\gamma-3}{4}}.
\end{aligned} \tag{6.1.8}$$

6.2 Main results

Define the C^1 solutions as follows:

Definition 6.2.1. *Let $T > 0$ be some time. The pair $(\rho(t,r), u(t,r))$ is called a C^1 solution to the equations $(6.1.1)_1$-$(6.1.1)_2$ on $(0,T) \times \mathbb{R}$ if*

$$\rho > 0, \quad \rho \in C^1([0,T) \times \mathbb{R}^+), \quad u \in C^1([0,T) \times \mathbb{R}^+),$$

and the equations $(6.1.1)_1$-$(6.1.1)_2$ are satisfied in the pointwise sense on $(0,T) \times \mathbb{R}$. It is called a C^1 solution to the Cauchy problem (6.1.1) if it is a C^1 solution to the equations $(6.1.1)_1$-$(6.1.1)_2$ on $(0,T) \times \mathbb{R}$ and admits the initial data $(6.1.1)_3$ continuously.

We assume the initial data satisfy the following conditions:

Assumption 6.2.1. *Denote $(w_0, z_0)(r) = (w(0,r), z(0,r))$. Assume $(w_0, z_0) \in C^1(\mathbb{R}^+)$,*

$(\frac{w_0}{r},\ \frac{z_0}{r}) \in C^1(\mathbb{R}^+)$ *and there exists a constant* C_0 *such that*

$$0 < w_0 \le z_0 \le C_0, \quad \forall r \in (0, \infty),$$

and

$$w_0(0) = z_0(0) = 0.$$

Remark 6.2.1. *By the definition of the Riemann variables* (6.1.5)*, Assumption* 6.2.1 *holds if and only if* $(\rho_0,\ u_0) \in C^1(\mathbb{R}^+)$, $(\frac{\sqrt{P'(\rho_0)}}{r},\ \frac{u_0}{r}) \in C^1(\mathbb{R}^+)$ *and*

$$0 < \frac{2\sqrt{P'(\rho_0)}}{\gamma - 1} \le u_0 \le C_0 \quad \forall r \in (0, \infty), \quad \textit{and} \quad \rho_0(0) = u_0(0) = 0.$$

We will now state the result on local existence of C^1 solutions found in [31].

Theorem 6.2.1. *Let* $1 < \gamma \le 3$ *and let the initial data* (w_0, z_0) *satisfy Assumption* 6.2.1*. There exists* $\delta > 0$ *and a unique* C^1 *solution* $(\rho, u)(t, r)$ *on* $(t, r) \in [0, \delta) \times \mathbb{R}^+$*, with*

$$0 < w \le z \le C_0, \quad \forall (t, r) \in [0, \delta) \times (0, \infty),$$

and $\rho(t, 0) = u(t, 0) = 0$*. Furthermore, for any point* $(\widetilde{t}, \widetilde{r}) \in [0, \delta) \times (0, \infty)$*, the characteristic* $(t, x^2(t))$ *passing through the point* $(\widetilde{t}, \widetilde{r})$ *will not touch the line* $r = 0$ *if* $t < \widetilde{t}$*.*

We will now introduce another assumption, which will help us to obtain the gradient blow-up result.

Assumption 6.2.2. *For initial data* (w_0, z_0) *satisfying Assumption* 6.2.1*, assume furthermore that*

$$v_{\min}(\underline{r}) \coloneqq \min_{r \in [\underline{r}, \infty)} v_0(r) > 0, \quad \forall \underline{r} > 0,$$

and

$$\max_{r \in [\underline{r}, \infty)} \{\widetilde{\alpha}_0(r), \widetilde{\beta}_0(r), \widehat{\alpha}_0(r), \widehat{\beta}_0(r)\} < K(\underline{r}), \quad \forall \underline{r} > 0,$$

where $K(\underline{r}) > 0$ *is some positive constant depending on* $\underline{r}$*.*

Since now, assume Assumption 6.2.1 and Assumption 6.2.2 hold. All the conclusions are

proved under the assumption that $t < \delta$. We will now introduce some important definitions, which will be used throughout the rest of the chapter.

Definition 6.2.2. *Define the domain of dependence* $\Omega(\underline{r})$ *as*

$$\Omega(\underline{r}) \coloneqq \{(t,r) | r \geq x^2(t,\underline{r}), \quad t \in \mathbb{R}^+\},$$

for some $\underline{r} > 0$.

Definition 6.2.3. *Let* $\underline{r} > 0$, *take any point* $(t,r) \in \Omega(\underline{r})$, *define the constant* $N(\underline{r})$ *by*

$$N(\underline{r}) \coloneqq \max\left\{K(\underline{r}), E(k,d,\gamma)\underline{r}^{-\frac{(d-1)(\gamma-1)+2}{2}}\right\},$$

where $E(k,d,\gamma)$ *is defined by*

$$E(k,d,\gamma) \coloneqq \max\left\{\frac{12(\gamma-1)(d-1)\sqrt{k\gamma}}{3-\gamma}, \frac{6(\gamma-1)(d-1)\sqrt{k\gamma}}{3\gamma-5}, \frac{6(\gamma-1)(d-1)\sqrt{k\gamma}}{\gamma+1}\right\}.$$

Definition 6.2.4. *Define a constant* $\widetilde{L}$ *by*

$$\widetilde{L} \coloneqq \max\left\{\sup_{0<w\leq z\leq C_0} rv^{\frac{3-\gamma}{4}}\sqrt{16\left|\frac{d_1}{d_2}\right|^2 + \left|\frac{d_0}{d_2}\right|}, \sup_{0<w\leq z\leq C_0} rv^{\frac{3-\gamma}{4}}\sqrt{16\left|\frac{\overline{d}_1}{\overline{d}_2}\right|^2 + \left|\frac{\overline{d}_0}{\overline{d}_2}\right|}\right\}.$$

Let $\underline{r} > 0$, *define the blow-up time* $T > 0$ *by*

$$T \coloneqq \frac{8}{\gamma+1}\frac{\underline{r}}{\widetilde{L}}.$$

Define the initial compression strength $G_T(\underline{r})$ *by*

$$G_T(\underline{r}) \coloneqq \begin{cases} \dfrac{\widetilde{L}}{\underline{r}}\left(\dfrac{1}{v_{\min}(\underline{r})} + K(\underline{r})T\right)^{\frac{3-\gamma}{4}}, & \text{if} \quad 1 < \gamma < \dfrac{5}{3}, \\ \dfrac{\widetilde{L}}{\underline{r}}\left(\dfrac{1}{v_{\min}(\underline{r})^{\frac{\gamma-1}{2}}} + \dfrac{\gamma-1}{2}N(\underline{r})T\right)^{\frac{3-\gamma}{2(\gamma-1)}}, & \text{if} \quad \dfrac{5}{3} < \gamma < 3. \end{cases}$$

Finally, we can state the gradient blow-up result. This result was proved for the p-system in [31].

Theorem 6.2.2. *Let $1 < \gamma < 3$, $\gamma \neq \frac{5}{3}$ and $\underline{r} > 0$. Let T and $G_T(\underline{r})$ be the constants in Definition* 6.2.4. *For initial data (w_0, z_0) satisfy Assumption* 6.2.1 *and Assumption* 6.2.2, *if*

$$Q(0, \underline{r}) < -G_T(\underline{r}), \tag{6.2.1}$$

then the C^1 solution will break down before $t = T$, more precisely, one has $|Q| \to \infty$ before $t = T$. Moreover, if there exists some $r_0 > \underline{r}$ with $r_0 - \underline{r} \geq 2C_0\gamma T$, such that

$$Y(0, r_0) < -G_T(\underline{r}), \tag{6.2.2}$$

then the C^1 solution will break down before $t = T$, more precisely, one has $|Y| \to \infty$ before $t = T$.

6.3 Density lower bound

In this section, we aim to prove the following theorem concerning the time-dependent lower bound on the quantity v given by (6.1.2) :

Theorem 6.3.1. *Let $(t, r) \in \Omega(\underline{r})$ for some $\underline{r} > 0$, where $\Omega(\underline{r})$ is defined in Definition* 6.2.2.

For $1 < \gamma \leq \frac{5}{3}$, one has

$$\frac{1}{v(t, r)} < \frac{1}{v_{\min}(\underline{r})} + K(\underline{r})t, \tag{6.3.1}$$

where $v_{\min}(\underline{r})$ and $K(\underline{r})$ are given by Assumption 6.2.2.

For $\frac{5}{3} < \gamma < 3$, one has

$$\frac{1}{v(t, r)^{\frac{\gamma-1}{2}}} < \frac{1}{v_{\min}(\underline{r})^{\frac{\gamma-1}{2}}} + \frac{\gamma - 1}{2} N(\underline{r})t, \tag{6.3.2}$$

where $N(\underline{r})$ is given by Definition 6.2.3.

6.3.1 Coupled ODEs

Lemma 6.3.1. *Define the following quantities:*

$$\begin{aligned} k_0 &= \frac{(\gamma-1)^2(z-w)^2}{16}\frac{d}{dr}\left(\frac{A'}{A}\right) - \frac{(\gamma-1)^3(z-w)^2}{16}\left(\frac{A'}{A}\right)^2, \\ k_1^- &= -\frac{\gamma-1}{4}\frac{A'}{A}\left(\frac{3(\gamma-1)(z-w)}{4} + \frac{z+w}{2}\right), \\ k_1^+ &= -\frac{\gamma-1}{4}\frac{A'}{A}\left(\frac{3(\gamma-1)(z-w)}{4} - \frac{z+w}{2}\right). \end{aligned} \tag{6.3.3}$$

Then α and β satisfy

$$\partial_1\alpha = -\frac{1}{2}\alpha(\alpha+\beta) - \frac{\gamma-1}{4}\alpha(\alpha-\beta) + k_1^-(\alpha-\beta) + k_0,$$

$$\partial_2\beta = -\frac{1}{2}\beta(\alpha+\beta) - \frac{\gamma-1}{4}\beta(\beta-\alpha) + k_1^+(\beta-\alpha) + k_0.$$

Proof. We will only prove the lemma for $\partial_2\beta$, as the proof for $\partial_1\alpha$ will be similar. Noticing that in terms of w and z, (6.1.6) can be written as

$$\begin{cases} \partial_1 w = \frac{A'}{A}\frac{\gamma-1}{8}(w+z)(w-z), \\ \partial_2 z = -\frac{A'}{A}\frac{\gamma-1}{8}(w+z)(w-z). \end{cases} \tag{6.3.4}$$

Furthermore, by (6.1.7) and (6.1.5), one has

$$\begin{cases} \partial_r w = \alpha + \sqrt{k\gamma}A^{-\frac{\gamma+1}{2}}v^{\frac{\gamma-1}{2}}A' = \alpha + \frac{(z-w)(\gamma-1)}{4}\frac{A'}{A}, \\ \partial_r z = \beta - \sqrt{k\gamma}A^{-\frac{\gamma+1}{2}}v^{\frac{\gamma-1}{2}}A' = \beta - \frac{(z-w)(\gamma-1)}{4}\frac{A'}{A}. \end{cases} \tag{6.3.5}$$

Differentiating (6.3.4) with respect to r and using (6.3.5), one has

$$\begin{aligned}\partial_2\left(\beta-\frac{(z-w)(\gamma-1)}{4}\frac{A'}{A}\right)+\partial_w\lambda_2\left(\alpha+\frac{(z-w)(\gamma-1)}{4}\frac{A'}{A}\right)\left(\beta-\frac{(z-w)(\gamma-1)}{4}\frac{A'}{A}\right)\\ +\partial_z\lambda_2\left(\beta-\frac{(z-w)(\gamma-1)}{4}\frac{A'}{A}\right)^2=-\frac{(\gamma-1)(z-w)(\alpha+\beta)}{8}\frac{A'}{A}\\ -\frac{(\gamma-1)(w+z)}{4}\left(\beta-\alpha-\frac{(\gamma-1)(z-w)}{2}\frac{A'}{A}\right)\frac{A'}{A}-\frac{(\gamma-1)(z-w)(z+w)}{8}\frac{d}{dr}\left(\frac{A'}{A}\right).\end{aligned} \tag{6.3.6}$$

By explicit calculations, one can check that

$$\lambda_2=\frac{w+z}{2}+\frac{(\gamma-1)(z-w)}{4}\implies\begin{cases}\partial_w\lambda_2=\frac{1}{2}-\frac{\gamma-1}{4},\\ \partial_z\lambda_2=\frac{1}{2}+\frac{\gamma-1}{4}.\end{cases} \tag{6.3.7}$$

Substituting (6.3.7) into (6.3.6) and simplify, one gets

$$\begin{aligned}\partial_2\beta-\frac{A'}{A}\frac{\gamma-1}{4}\partial_2(z-w)+\frac{1}{2}\beta(\alpha+\beta)\\ -\frac{\gamma-1}{4}\left(\beta-\frac{(\gamma-1)(z-w)}{4}\frac{A'}{A}+\frac{w+z}{2}\frac{A'}{A}\right)\left(\alpha-\beta+\frac{(\gamma-1)(z-w)}{2}\frac{A'}{A}\right)\\ =\frac{(\gamma-1)^2(z-w)^2}{16}\frac{d}{dr}\left(\frac{A'}{A}\right).\end{aligned} \tag{6.3.8}$$

Now, by (6.3.4) and (6.3.5), one can check that

$$\begin{aligned}\partial_2(z-w)&=\partial_2 z-\partial_1 w-(\lambda_2-\lambda_1)\partial_r w\\ &=-\frac{(\gamma-1)(z-w)(z+w)}{4}\frac{A'}{A}-\frac{(\gamma-1)(z-w)}{2}\left(\alpha+\frac{(z-w)(\gamma-1)}{4}\frac{A'}{A}\right).\end{aligned} \tag{6.3.9}$$

Substituting (6.3.9) into (6.3.8) and simplify, one gets the desired result. □

Lemma 6.3.2.

$$\begin{cases}\partial_1 v=-v\left(\frac{\alpha+\beta}{2}\right)-v\left(\frac{\beta-\alpha}{2}\right),\\ \partial_2 v=-v\left(\frac{\alpha+\beta}{2}\right)+v\left(\frac{\beta-\alpha}{2}\right).\end{cases}$$

Proof. We will only prove the lemma for $\partial_2 v$. Notice that, $(6.1.1)_1$ can be written in terms of v

as

$$\partial_t v + \partial_r(vu) = 0. \tag{6.3.10}$$

By (6.3.10), one has

$$\begin{aligned}\partial_2 v = \partial_t v + \lambda_2 \partial_r v &= -\partial_r(vu) + \left(u + \sqrt{k\gamma} A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-1}{2}}\right)\partial_r v \\ &= -v\partial_r u + \sqrt{k\gamma} A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-1}{2}} \partial_r v \\ &= -v\partial_r u + v A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-3}{2}} \partial_r v.\end{aligned}$$

Now by (6.1.7) and explicit calculations, one can check that

$$\partial_r u = \frac{\alpha+\beta}{2}, \quad A^{-\frac{\gamma-1}{2}} v^{\frac{\gamma-3}{2}} \partial_r v = \frac{\beta-\alpha}{2}.$$

This completes the proof. □

Lemma 6.3.3.

$$\begin{aligned}\partial_1 \widetilde{\alpha} &= -\frac{\gamma+1}{4} v\widetilde{\alpha}(\widetilde{\alpha} - \widetilde{\beta}) + k_1^-(\widetilde{\alpha} - \widetilde{\beta}) + \frac{1}{v} k_0, \\ \partial_2 \widetilde{\beta} &= -\frac{\gamma+1}{4} v\widetilde{\beta}(\widetilde{\beta} - \widetilde{\alpha}) + k_1^+(\widetilde{\beta} - \widetilde{\alpha}) + \frac{1}{v} k_0,\end{aligned}$$

where k_0, $k_1^{\pm}$ are defined in Lemma 6.3.1.

Proof. Combining Lemma 6.3.1 and Lemma 6.3.2, one can obtain the result by explicit calculations. □

Lemma 6.3.4.

$$\begin{aligned}\partial_1 \widehat{\alpha} =& -\frac{3\gamma-5}{4} v^{\frac{\gamma-1}{2}} \widehat{\alpha}(\widehat{\alpha} - \widehat{\beta}) - v^{\frac{\gamma-1}{2}} \left(\frac{3-\gamma}{2}\widehat{\alpha}^2 + \frac{3(\gamma-1)}{4}\frac{A'}{A}\sqrt{k\gamma} A^{-\frac{\gamma-1}{2}}(\widehat{\alpha} - \widehat{\beta})\right) \\ &- \frac{\gamma-1}{2}\frac{A'}{A} u(\widehat{\alpha} - \widehat{\beta}) + v^{\frac{\gamma-1}{2}} k_0, \\ \partial_2 \widehat{\beta} =& -\frac{3\gamma-5}{4} v^{\frac{\gamma-1}{2}} \widehat{\beta}(\widehat{\beta} - \widehat{\alpha}) - v^{\frac{\gamma-1}{2}} \left(\frac{3-\gamma}{2}\widehat{\beta}^2 - \frac{3(\gamma-1)}{4}\frac{A'}{A}\sqrt{k\gamma} A^{-\frac{\gamma-1}{2}}(\widehat{\beta} - \widehat{\alpha})\right) \\ &- \frac{\gamma-1}{2}\frac{A'}{A} u(\widehat{\beta} - \widehat{\alpha}) + v^{\frac{\gamma-1}{2}} k_0.\end{aligned}$$

where k_0 is defined in Lemma 6.3.1.

Proof. Combining Lemma 6.3.1 and Lemma 6.3.2, one can obtain the result by explicit calculations.

□

6.3.2 Upper bound for gradient quantities $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widehat{\alpha}$ and $\widehat{\beta}$

Proposition 6.3.1. *For $1 < \gamma \leq 3$ and $r > 0$, one has*

$$0 < \lambda_1(t, r) < \lambda_2(t, r), \tag{6.3.11}$$

and the characteristics $x^1(t)$ and $x^2(t)$ can intersect each other at most once.

Proof. First, $\lambda_1 < \lambda_2$ follows directly from the definition of λ_i by (6.1.3) and the fact that $\rho > 0$ if $r > 0$.

Second, by Theorem 6.2.1, if $r > 0$, one has

$$w > 0 \implies u > \frac{2}{\gamma - 1}\sqrt{P'} > 0. \tag{6.3.12}$$

For $1 < \gamma \leq 3$, (6.3.12) implies $\lambda_1 > 0$.

It remains to show $x^1(t)$ and $x^2(t)$ intersect each other at most once. Define the function $f(t)$ by

$$f(t) \coloneqq x^1(t) - x^2(t).$$

One observes that if for some t_*, $f(t_*) = 0$, then $f'(t_*) \neq 0$. Actually, note that $f(t_*) = 0$ implies that

$$x^1(t_*) = x^2(t_*) \coloneqq x_*.$$

Then if $f'(t_*) = 0$, one can get

$$0 = f'(t_*) = \lambda_1(t_*, x_*) - \lambda_2(t_*, x_*),$$

which contradicts (6.3.11).

Now we assume that $x^1(t)$ and $x^2(t)$ intersect each other more than once. Then $f(t) = 0$ holds at least for two different points: $\bar{a}_1$ and $\bar{a}_2$. One can choose a time $\widetilde{t} \in (\bar{a}_1, \bar{a}_2)$ such that $f(\widetilde{t}) \neq 0$. Without loss of generality, assume $f(\widetilde{t}) > 0$. It is clear that such kind of points indeed exist, or else $f(t) \equiv 0$ for $t \in (\bar{a}_1, \bar{a}_2)$, which is impossible.

By the continuity of $f(t)$, there exists two points $\bar{a}_3 \in [\bar{a}_1, \bar{a}_2)$ and $\bar{a}_4 \in (\bar{a}_1, \bar{a}_2]$ such that $\widetilde{t} \in (\bar{a}_3, \bar{a}_4)$, $f(t) > 0$ for all $t \in (\bar{a}_3, \bar{a}_4)$, and $f(\bar{a}_3) = f(\bar{a}_4) = 0$, which implies that

$$x^1(\bar{a}_3) = x^1(\bar{a}_3) \coloneqq \bar{A} \quad \text{and} \quad x^1(\bar{a}_4) = x^2(\bar{a}_4) \coloneqq \bar{B}.$$

As $f'(\bar{a}_3) \neq 0$ and $f'(\bar{a}_4) \neq 0$, one has $f'(\bar{a}_3) > 0$ and $f'(\bar{a}_4) < 0$. Then

$$\lambda_1(\bar{a}_3, \bar{A}) > \lambda_2(\bar{a}_3, \bar{A}), \quad \lambda_1(\bar{a}_4, \bar{B}) < \lambda_2(\bar{a}_4, \bar{B}),$$

which contradicts (6.3.11). The proof of this lemma is complete. □

Proposition 6.3.2. *For $1 < \gamma \leq 3$, one has*

$$k_0 < 0, \quad k_1^- < 0,$$

furthermore,

$$k_1^+ \leq 0 \quad \textit{if} \quad 1 < \gamma \leq \frac{5}{3}.$$

Lastly, for any $(t, r) \in \Omega(\underline{r})$ for some $\underline{r} > 0$, where $\Omega(\underline{r})$ is defined in Definition 6.2.2. There exists some constant $D(\underline{r}) > 0$ depending only on the initial data, γ, d and $\underline{r}$, such that

$$\max\left\{|k_0(t,r)|, |k_1^-(t,r)|, |k_1^+(t,r)|\right\} \leq D(\underline{r}).$$

Proof. By (6.3.12) and $1 < \gamma \leq 3$, the fact that $k_0 < 0$ and $k_1^- < 0$ come from direct calculation. Next, we show $k_1^+ < 0$ for $1 < \gamma < \frac{5}{3}$. By direct calculations, one has

$$k_1^+ \leq 0 \iff -3(\gamma-1)(z-w) + 4(z+w) \geq 0 \iff 6u + (-6\gamma+4)\frac{2}{\gamma+1}\sqrt{P'} \geq 0.$$

Now by (6.3.12), one has

$$6u + (-6\gamma + 4)\frac{2}{\gamma + 1}\sqrt{P'} > (-6\gamma + 10)u \geq 0 \iff \gamma \leq \frac{5}{3}.$$

It remains to show $|k_0(t, r)|$ and $|k_1^{\pm}(t, r)|$ have the required upper bound. We will only do this k_1^-, as the other ones are similar. By Theorem 6.2.1, one has

$$|k_1^-(t, r)| \leq \frac{(3\gamma + 1)(\gamma - 1)}{16} C_0 \frac{A'}{A} = \frac{(3\gamma + 1)(\gamma - 1)}{16} C_0 \frac{d - 1}{r}. \tag{6.3.13}$$

Now by Definition 6.2.2, one has $r \geq x^2(t, \underline{r})$. By Proposition 6.3.1, one has

$$\frac{dx^2}{dt} = \lambda_2 > 0,$$

which implies $r \geq x^2(t, \underline{r}) > \underline{r}$. Combining this with (6.3.13), one gets

$$|k_1^-(t, r)| < \frac{(3\gamma + 1)(\gamma - 1)}{16} C_0 \frac{d - 1}{\underline{r}}.$$

This is the desired result. □

Lemma 6.3.5. *Let $1 < \gamma \leq \frac{5}{3}$ and $(t, r) \in \Omega(\underline{r})$ for some $\underline{r} > 0$, where $\Omega(\underline{r})$ was defined in Definition* 6.2.2. *Then*

$$\max\{\widetilde{\alpha}(t, r), \widetilde{\beta}(t, r)\} < K(\underline{r}),$$

where $K(\underline{r}) > 0$ is the constant in Assumption 6.2.2.

Proof. We will prove the lemma by contradiction. Suppose there exists a point $(t, r) \in \Omega(\underline{r})$ such that

$$\max\{\widetilde{\alpha}(t, r), \widetilde{\beta}(t, r)\} = K(\underline{r}).$$

Let us consider the two characteristics $x^1(t)$ and $x^2(t)$ passing through the point (t, r). Let Ω_0 be the characteristic triangle with left and right boundaries being the $x^1(t)$, $x^2(t)$ characteristics, bottom boundary being the line $t = 0$, and crest being the point (t, r). Note by Proposition 6.3.1 and the fact that $|\lambda_i|$ is uniformly bounded, such characteristic triangle exists, moreover,

$\Omega_0 \subset \Omega(\underline{r})$. Let $(t_1, r_1) \in \Omega_0$ be the first time that $\widetilde{\alpha} = K(\underline{r})$ or $\widetilde{\beta} = K(\underline{r})$, so

$$\max_{(t',r')\in\Omega_0, t'\in[0,t_1)} \{\widetilde{\alpha}(t',r'), \widetilde{\beta}(t',r')\} < K(\underline{r}).$$

Without any loss of generality, assume that $\widetilde{\alpha}(t_1, r_1) = K(\underline{r})$ and $\widetilde{\beta}(t_1, r_1) \leq K(\underline{r})$.

Now consider the x^1 characteristic passing through the point (t_1, r_1). By continuity, one can find a t_2 where $t_2 < t_1$, and for $l \in (t_2, t_1)$, one has

$$0 < \widetilde{\alpha}(l, x^1(l)) < K(\underline{r}).$$

Hence, for $l \in (t_2, t_1)$, by Theorem 6.3.3 and Proposition 6.3.2, one has

$$\partial_1 \widetilde{\alpha} < \left(\frac{\gamma+1}{4} v\widetilde{\alpha} + |k_1^-|\right)(\widetilde{\beta} - \widetilde{\alpha}) < L(K(\underline{r}) - \widetilde{\alpha}),$$

where $L > 0$ is some constant depending on the initial data, γ, $K(\underline{r})$, $D(\underline{r})$ and t_1. Integrating the above equation, one has

$$\ln\left(\frac{1}{K(\underline{r}) - \widetilde{\alpha}(l, x^1(l))}\right) < \ln\left(\frac{1}{K(\underline{r}) - \widetilde{\alpha}(t_2, x^1(t_2))}\right) + L(l - t_2).$$

As $l \to t_1$, the left hand side tends to infinity, while the right hand side tends to a constant. This is a contradiction. □

Lemma 6.3.6. *Let $\frac{5}{3} < \gamma < 3$ and $(t, r) \in \Omega(\underline{r})$ for some $\underline{r} > 0$, where $\Omega(\underline{r})$ was defined in Definition* 6.2.2. *Then*

$$\max\{\widehat{\alpha}(t,r), \widehat{\beta}(t,r)\} < N(\underline{r}),$$

where $N(\underline{r}) > 0$ is the constant in Definition 6.2.3.

Proof. We will prove the lemma by contradiction. We will denote $N(\underline{r})$ as N for the proof. Suppose there exists a point $(t, r) \in \Omega(\underline{r})$ such that

$$\max\{\widehat{\alpha}(t,r), \widehat{\beta}(t,r)\} = N.$$

Let $\Omega_0 \subset \Omega(\underline{r})$ be the characteristic triangle with crest (t, r), defined the same as in Lemma

6.3.5. Let $(t_1, r_1) \in \Omega_0$ be the first time that $\widehat{\alpha} = N$ or $\widehat{\beta} = N$, so

$$\max_{(t',r')\in\Omega_0, t'\in[0,t_1)} \{\widehat{\alpha}(t',r'), \widehat{\beta}(t',r')\} < N.$$

Without any loss of generality, assume that $\widehat{\beta}(t_1, r_1) = N$ and $\widehat{\alpha}(t_1, r_1) \leq N$. We will now consider two cases: the case $-\frac{N}{2} < \widehat{\alpha}(t_1, r_1) < N$ and the case $\widehat{\alpha}(t_1, r_1) \leq -\frac{N}{2}$.

Case 1: $-\frac{N}{2} < \widehat{\alpha}(t_1, r_1) < N$. Let us consider the x^2 characteristic passing the point (t_1, r_1). By continuity, one can find t_2 with $t_2 < t_1$, such that for $l \in (t_2, t_1)$, one has

$$|\widehat{\alpha}(l, x^2(l))| < N, \quad \frac{N}{2} < \widehat{\beta}(l, x^2(l)) < N. \tag{6.3.14}$$

For $\widehat{\alpha}$ and $\widehat{\beta}$ satisfying (6.3.14), by explicit calculations, one has

$$\left(\frac{3-\gamma}{2}\widehat{\beta}^2 - \frac{3(\gamma-1)}{4}\frac{A'}{A}\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}(\widehat{\beta}-\widehat{\alpha})\right)(l, x^2(l)) \geq 0 \tag{6.3.15}$$

if

$$\begin{aligned} &\frac{3-\gamma}{2}\left(\frac{N}{2}\right)^2 - \frac{3(\gamma-1)}{4}\frac{A'}{A}\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}2N \geq 0 \\ \iff &N \geq \frac{12(\gamma-1)(d-1)\sqrt{k\gamma}}{3-\gamma}x^2(l)^{-\frac{(d-1)(\gamma-1)+2}{2}}. \end{aligned} \tag{6.3.16}$$

By the fact that $(t_1, r_1) \in \Omega_1$ and the fact that $\lambda_2 > 0$ from Proposition 6.3.1, one has $\underline{r} > x^2(l)$. Hence by the definition of N from Definition 6.2.3, (6.3.16) holds, hence (6.3.15) also holds. Now, by Lemma 6.3.4, Proposition 6.3.2, (6.3.12) and (6.3.15), one has

$$\partial_2\widehat{\beta} \leq \left(\frac{3\gamma-5}{4}v^{\frac{\gamma-1}{2}}\widehat{\beta} + \frac{\gamma-1}{2}\frac{A'}{A}u\right)(\widehat{\alpha}-\widehat{\beta}) < L(N-\widehat{\beta}),$$

where $L > 0$ is a constant depending only on the initial data, γ, k, N, $\underline{r}$ and t_1. Integrating the above equation, one has

$$\ln\left(\frac{1}{N-\widehat{\beta}(l, x^2(l))}\right) < \ln\left(\frac{1}{N-\widehat{\beta}(t_2, x^2(t_2))}\right) + L(l-t_2).$$

As $l \to t_1$, the left hand side tends to infinity, while the right hand side tends to a constant. This is a contradiction.

Case 2: $\widehat{\alpha}(t_1, r_1) \leq -\frac{N}{2}$. Let us consider the x^2 characteristic passing the point (t_1, r_1). By continuity, one can find t_2 with $t_2 < t_1$, such that for $l \in (t_2, t_1)$, one has

$$\widehat{\alpha}(l, x^2(l)) < -\frac{N}{4}, \quad \frac{N}{2} < \widehat{\beta}(l, x^2(l)) < N. \tag{6.3.17}$$

For $\widehat{\alpha}$ and $\widehat{\beta}$ satisfying (6.3.17), by explicit calculations, one has

$$\left(\frac{3\gamma-5}{4}\widehat{\alpha}\widehat{\beta} - \frac{3(\gamma-1)}{4}\frac{A'}{A}\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}\widehat{\alpha}\right)(l, x^2(l)) \leq 0 \tag{6.3.18}$$

if

$$N \geq \frac{6(\gamma-1)(d-1)\sqrt{k\gamma}}{3\gamma-5}x^2(l)^{-\frac{(d-1)(\gamma-1)+2}{2}}. \tag{6.3.19}$$

Moreover, one has

$$\left(-\frac{3\gamma-5}{4}\widehat{\beta}^2 - \frac{3-\gamma}{4}\widehat{\beta}^2 - \frac{3(\gamma-1)}{4}\frac{A'}{A}\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}\widehat{\beta}\right)(l, x^2(l)) \leq 0 \tag{6.3.20}$$

if

$$N \geq \frac{6(\gamma-1)(d-1)\sqrt{k\gamma}}{\gamma+1}x^2(l)^{-\frac{(d-1)(\gamma-1)+2}{2}}. \tag{6.3.21}$$

By the fact that $\underline{r} < x^2(l)$ and the definition of N from Definition 6.2.3, (6.3.21) and (6.3.19) hold, so (6.3.18) and (6.3.20) also hold. Now, by Lemma 6.3.4, Proposition 6.3.2, (6.3.12) , (6.3.18) and (6.3.20), one has

$$\partial_2\widehat{\beta} \leq \left(\frac{\gamma-1}{2}\frac{A'}{A}u\right)(\widehat{\alpha} - \widehat{\beta}) < L(N - \widehat{\beta}),$$

for some $L > 0$. Integrating the above equation one can obtain a contradiction same as in case 1. □

6.3.3 Time-dependent lower bound on the density: Proof of Theorem 6.3.1

For this subsection, fix any $\underline{r} > 0$ and any $(t, r) \in \Omega(\underline{r})$, we will consider the $x^2(t, x_0^2)$ characteristic passing through the point (t, x). Note that $\underline{r} < x_0^2$. By (6.1.6) and (6.3.5), one has

$$\begin{aligned}\partial_2\left(\frac{z-w}{2}\right) &= \frac{\partial_2 z - \partial_1 w}{2} - \frac{(\lambda_2 - \lambda_1)\partial_r w}{2} \\ &= -\frac{A'}{A}u\sqrt{P'} - \sqrt{P'}\left(\alpha + \frac{A'}{A}\sqrt{P'}\right) \\ &= -\sqrt{P'}\alpha - \frac{A'}{A}\sqrt{P'}\lambda_2.\end{aligned}$$

By viewing $z - w$ as a function of ρ, one has

$$\partial_2\left(\frac{z-w}{2}\right) = \partial_\rho\left(\frac{z-w}{2}\right)\partial_2\rho = \frac{\sqrt{P'}}{\rho}\partial_2\rho = \frac{\sqrt{P'}}{v}\partial_2 v - \frac{\sqrt{P'}}{A}\lambda_2 A'.$$

Combining the above two equations, one has

$$\frac{1}{v}\partial_2 v = -\alpha. \tag{6.3.22}$$

For $1 < \gamma \leq \frac{5}{3}$, by Lemma 6.3.5, one has

$$\begin{aligned}\frac{1}{v^2}\partial_2 v = -\widetilde{\alpha} > -K(\underline{r}) &\implies \frac{1}{v(t,r)} < \frac{1}{v_0(x_0^2)} + K(\underline{r})t \\ &\implies \frac{1}{v(t,r)} < \frac{1}{v_{\min}(\underline{r})} + K(\underline{r})t.\end{aligned}$$

For $\frac{5}{3} < \gamma < 3$, by Lemma 6.3.6, one has

$$\begin{aligned}\frac{1}{v^{\frac{\gamma+1}{2}}}\partial_2 v = -\widehat{\alpha} > -N(\underline{r}) &\implies \frac{1}{v(t,r)^{\frac{\gamma-1}{2}}} < \frac{1}{v_0(x_0^2)^{\frac{\gamma-1}{2}}} + \frac{\gamma-1}{2}N(\underline{r})t \\ &\implies \frac{1}{v(t,r)^{\frac{\gamma-1}{2}}} < \frac{1}{v_{\min}(\underline{r})^{\frac{\gamma-1}{2}}} + \frac{\gamma-1}{2}N(\underline{r})t.\end{aligned}$$

This is the desired result.

6.4 Decoupled ODEs

Theorem 6.4.1. *Let $1 < \gamma < 3$ and $\gamma \neq \frac{5}{3}$. The quantities Y and Q defined in* (6.1.8) *satisfy the following decoupled ODEs:*

$$\begin{cases} \partial_1 Y = d_0 + d_1 Y + d_2 Y^2, \\ \partial_2 Q = \overline{d}_0 + \overline{d}_1 Q + \overline{d}_2 Q^2, \end{cases} \tag{6.4.1}$$

where the coefficients d_i and $\overline{d}_i$ are given in (6.1.7).

Proof. We will only prove this theorem for Q. By Lemma 6.3.2, one has

$$\alpha = -\frac{1}{v}\partial_2 v. \tag{6.4.2}$$

Substituting (6.4.2) into the expression for $\partial_2\beta$ given by Lemma 6.3.1, by explicit calculations one can obtain

$$\partial_2\beta = -\frac{\gamma+1}{4}\beta^2 + \frac{3-\gamma}{4}\frac{1}{v}\partial_2 v\beta - \frac{\gamma-1}{4}\left(-3\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}v^{\frac{\gamma-1}{2}} + 2u\right)\left(\frac{1}{v}\partial_2 v + \beta\right) + k_0.$$

Multiplying the above equation by $v^{\frac{\gamma-3}{4}}$ and simplify, one gets

$$\begin{aligned} \partial_2\left(v^{\frac{\gamma-3}{4}}\beta\right) =& -\frac{\gamma+1}{4}\beta^2 v^{\frac{\gamma-3}{4}} - \frac{\gamma-1}{4}\left(-3\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}v^{\frac{\gamma-1}{2}} + 2u\right)\beta v^{\frac{\gamma-3}{4}} \\ &+ \frac{\gamma-1}{\gamma-3}\frac{A'}{A}\left(\frac{12\sqrt{k\gamma}}{3\gamma-5}A^{-\frac{\gamma-1}{2}}\partial_2\left(v^{\frac{3\gamma-5}{4}}\right) - 2u\partial_2\left(v^{\frac{\gamma-3}{4}}\right)\right) + k_0 v^{\frac{\gamma-3}{4}}. \end{aligned} \tag{6.4.3}$$

Now by (6.1.5), (6.1.6), (6.3.5) and (6.4.2), one has

$$\begin{aligned} \partial_2(2u) = \partial_2 z + \partial_2 w &= \partial_2 z + \partial_1 w + (\lambda_2 - \lambda_1)\partial_r w \\ &= 2\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}v^{\frac{\gamma-1}{2}}\alpha + 2k\gamma A^{-\gamma}A'v^{\frac{\gamma-1}{2}} \\ &= -2\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}v^{\frac{\gamma-3}{2}}\partial_2 v + 2k\gamma A^{-\gamma}A'v^{\frac{\gamma-1}{2}}, \end{aligned}$$

hence one has

$$\begin{aligned} 2u\partial_2\left(v^{\frac{\gamma-3}{4}}\right) &= \partial_2\left(2uv^{\frac{\gamma-3}{4}}\right) - v^{\frac{\gamma-3}{4}}\partial_2(2u) \\ &= \partial_2\left(2uv^{\frac{\gamma-3}{4}}\right) + \frac{8\sqrt{k\gamma}}{3\gamma-5}A^{-\frac{\gamma-1}{2}}\partial_2\left(v^{\frac{3\gamma-5}{4}}\right) - 2k\gamma A^{-\gamma}A'v^{\frac{\gamma-1}{2}}. \end{aligned} \tag{6.4.4}$$

Substituting (6.4.4) into (6.4.3), one gets

$$\begin{aligned} \partial_2\left(v^{\frac{\gamma-3}{4}}\beta\right) =& -\frac{\gamma+1}{4}\beta^2 v^{\frac{\gamma-3}{4}} - \frac{\gamma-1}{4}\left(-3\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}v^{\frac{\gamma-1}{2}} + 2u\right)\beta v^{\frac{\gamma-3}{4}} + k_0 v^{\frac{\gamma-3}{4}} \\ &+ \frac{\gamma-1}{\gamma-3}\frac{A'}{A}\left(\frac{4\sqrt{k\gamma}}{3\gamma-5}A^{-\frac{\gamma-1}{2}}\partial_2\left(v^{\frac{3\gamma-5}{4}}\right) - \partial_2\left(2uv^{\frac{\gamma-3}{4}}\right) + 2k\gamma A^{-\gamma}A'v^{\frac{\gamma-1}{2}}\right). \end{aligned} \tag{6.4.5}$$

By explicit calculations, one has

$$\begin{aligned} \frac{A'}{A}A^{-\frac{\gamma-1}{2}}\partial_2\left(v^{\frac{3\gamma-5}{4}}\right) &= \partial_2\left(v^{\frac{3\gamma-5}{4}}A'A^{-\frac{\gamma+1}{2}}\right) - v^{\frac{3\gamma-5}{4}}\lambda_2\frac{d}{dr}\left(A'A^{-\frac{\gamma+1}{2}}\right), \\ \frac{A'}{A}\partial_2\left(2uv^{\frac{\gamma-3}{4}}\right) &= \partial_2\left(\frac{A'}{A}2uv^{\frac{\gamma-3}{4}}\right) - 2uv^{\frac{\gamma-3}{4}}\lambda_2\frac{d}{dr}\left(\frac{A'}{A}\right). \end{aligned}$$

Hence upon simplifications, (6.4.5) becomes

$$\begin{aligned} &\partial_2\left(\beta v^{\frac{\gamma-3}{4}} - \frac{4\sqrt{k\gamma}}{(\gamma-3)(3\gamma-5)}v^{\frac{3\gamma-5}{4}}A'A^{-\frac{\gamma+1}{2}} + \frac{\gamma+1}{\gamma-3}\frac{A'}{A}2uv^{\frac{\gamma-3}{4}}\right) \\ =& -\frac{\gamma+1}{4}\beta^2 v^{\frac{\gamma-3}{4}} - \frac{\gamma-1}{4}\left(-3\sqrt{k\gamma}A^{-\frac{\gamma-1}{2}}v^{\frac{\gamma-1}{2}} + 2u\right)\beta v^{\frac{\gamma-3}{4}} + k_0 v^{\frac{\gamma-3}{4}} \\ &- \frac{4\sqrt{k\gamma}(\gamma-1)}{(\gamma-3)(3\gamma-5)}v^{\frac{3\gamma-5}{4}}\lambda_2\frac{d}{dr}\left(A'A^{-\frac{\gamma+1}{2}}\right) + \frac{\gamma-1}{\gamma-3}2uv^{\frac{\gamma-3}{4}}\lambda_2\frac{d}{dr}\left(\frac{A'}{A}\right) + \frac{\gamma-1}{\gamma-3}2k\gamma A^{-\gamma}A'v^{\frac{\gamma-1}{2}}. \end{aligned}$$

Now by recalling the definition of Q given by (6.1.8), one can simplify the above expression and obtain the desired result. □

6.5 Decoupled ODE inequalities

Theorem 6.5.1. *Let $1 < \gamma < 3$ and $\gamma \neq \frac{5}{3}$. Let $\underline{r} > 0$ and $(t, r) \in \Omega(\underline{r})$ with $t < T$, the quantities Q and Y given by* (6.1.8) *satisfy the following ODE inequality:*

$$\begin{cases} \partial_1 Y \leq \frac{1}{2} d_2 Y^2 + \frac{1}{4} d_2 \left(Y^2 - G_T(\underline{r})^2 \right), \\ \partial_2 Q \leq \frac{1}{2} \overline{d}_2 Q^2 + \frac{1}{4} \overline{d}_2 \left(Q^2 - G_T(\underline{r})^2 \right), \end{cases} \tag{6.5.1}$$

where the coefficients $\overline{d}_2$ and d_2 are given in (6.1.7)*, and $G_T(\underline{r})$ is given by Definition* 6.2.4.

Proof. By Theorem 6.4.1, Young's inequality and $\overline{d}_2 < 0$, one has

$$\begin{aligned} \partial_2 Q = \frac{1}{2}\overline{d}_2 Q^2 + \frac{1}{2}\overline{d}_2 Q^2 + \overline{d}_1 Q + \overline{d}_0 &\leq \frac{1}{2}\overline{d}_2 Q^2 + \frac{1}{2}\overline{d}_2 Q^2 + \frac{1}{4}|\overline{d}_2| Q^2 + 4\frac{\overline{d}_1^2}{|\overline{d}_2|} + \overline{d}_0 \\ &= \frac{1}{2}\overline{d}_2 Q^2 + \frac{1}{4}\overline{d}_2 \left(Q^2 + 16\frac{\overline{d}_1^2}{\overline{d}_2 |\overline{d}_2|} + \frac{\overline{d}_0}{\overline{d}_2} \right) \\ &\leq \frac{1}{2}\overline{d}_2 Q^2 + \frac{1}{4}\overline{d}_2 \left(Q^2 - 16\frac{\overline{d}_1^2}{\overline{d}_2^2} - \left| \frac{\overline{d}_0}{\overline{d}_2} \right| \right) \end{aligned}$$

Hence to prove this theorem, one just needs

$$\sqrt{16\frac{\overline{d}_1^2}{\overline{d}_2^2} + \left| \frac{\overline{d}_0}{\overline{d}_2} \right|} \leq G_T(\underline{r}). \tag{6.5.2}$$

By explicit calculations using the expression of $\overline{d}_i$ from (6.1.7), one has

$$\begin{aligned} \frac{\overline{d}_1}{\overline{d}_2} &= -r^{-1} v^{\frac{\gamma-3}{4}} \frac{\gamma+1}{4} \left(\overline{L}_1 u + \overline{L}_2 \rho^{\frac{\gamma-1}{2}} \right), \\ \frac{\overline{d}_0}{\overline{d}_2} &= -r^{-2} v^{\frac{\gamma-3}{2}} \frac{\gamma+1}{4} \left(\overline{L}_3 u \rho^{\frac{\gamma-1}{2}} + \overline{L}_4 u^2 + \overline{L}_5 \rho^{\gamma-1} \right). \end{aligned}$$

Hence, by the uniform upper bound on ρ and u from Theorem 6.2.1 and the definition of $\widetilde{L}$ given by Definition 6.2.4, one gets

$$\sqrt{16\frac{\overline{d}_1^2}{\overline{d}_2^2} + \left| \frac{\overline{d}_0}{\overline{d}_2} \right|} \leq \widetilde{L} r^{-1} v^{\frac{\gamma-3}{4}}.$$

Now notice that $r \geq \underline{r}$ as $(t, r) \in \Omega(\underline{r})$. Combining this with the time-dependent lower bound on v given by Theorem 6.3.1, one sees that (6.5.2) indeed hold, hence the theorem is proved. $\square$

6.6 Singularity formation with strong compression: Proof of Theorem 6.2.2

First, we will show if (6.2.1) holds, then Q blows up before time $t = T$. We will consider the characteristic $x^2(t, \underline{r})$. The proof is divided into two steps.

Step 1: We will show if (6.2.1) holds and $t \leq T$, then alone the $x^2(t, \underline{r})$ characteristic, one has

$$\partial_2 Q < \frac{1}{2}\overline{d}_2 Q^2. \tag{6.6.1}$$

By the ODE inequality associated to Q given by Theorem 6.5.1, (6.6.1) holds if

$$Q(t, x^2(t, \underline{r})) + G_T(\underline{r}) < 0. \tag{6.6.2}$$

By (6.2.1) and continuity, we know (6.6.2) holds up to some $t > 0$. Assume for contradiction that there exists some $t^* < T$, being the first time that

$$Q(t^*, x^2(t^*, \underline{r})) + G_T(\underline{r}) = 0 \iff Q(t^*, x^2(t^*, \underline{r})) = -G_T(\underline{r}).$$

Then by by Theorem 6.5.1, $\overline{d}_2 < 0$ and (6.2.1), for $t \in [0, t^*]$, one has

$$\partial_2 Q \leq \frac{1}{2}\overline{d}_2 Q^2 < 0 \implies -G_T(\underline{r}) = Q(t^*, x^2(t^*, \underline{r})) < Q(0, \underline{r}) < -G_T(\underline{r}).$$

This is a contradiction, hence (6.6.2) holds, so (6.6.1) holds as well.

Step 2: We will show Q blows up before time reaches $t = T$. By integrating (6.6.1) alone the $x^2(t, \underline{r})$ characteristic from $t \in [0, T]$, and recalling the expression for $\overline{d}_2$ given by (6.1.7), one has

$$\begin{aligned} \frac{1}{Q(T, x^2(T, \underline{r}))} &> \frac{1}{Q(0, \underline{r})} + \frac{\gamma + 1}{8}\int_0^T v^{\frac{3-\gamma}{4}}(t, x^2(t, \underline{r}))dt \\ &> -\frac{1}{G_T(\underline{r})} + \frac{\gamma + 1}{8}\int_0^T v^{\frac{3-\gamma}{4}}(t, x^2(t, \underline{r}))dt. \end{aligned} \tag{6.6.3}$$

Now, as $Q(0,\underline{r}) < -G_T(\underline{r}) < 0$, for Q to blow-up before time reaches T, one just needs the right hand side of the above quantity to be non-negative.

For $1 < \gamma < \frac{3}{5}$, by Theorem 6.3.1, (6.6.3) becomes

$$\frac{1}{Q(T, x^2(T,\underline{r}))} > -\frac{1}{G_T(\underline{r})} + \frac{\gamma+1}{8} T \left(\frac{1}{v_{\min}(\underline{r})} + K(\underline{r})T \right)^{\frac{\gamma-3}{4}}. \tag{6.6.4}$$

For $\frac{3}{5} < \gamma < 1$, by Theorem 6.3.1, (6.6.3) becomes

$$\frac{1}{Q(T, x^2(T,\underline{r}))} > -\frac{1}{G_T(\underline{r})} + \frac{\gamma+1}{8} T \left(\frac{1}{v_{\min}(\underline{r})^{\frac{\gamma-1}{2}}} + \frac{\gamma-1}{2} N(\underline{r})T \right)^{\frac{\gamma-3}{2(\gamma-1)}}. \tag{6.6.5}$$

By the explicit expression of T and $G_T(\underline{r})$ given by Definition 6.2.4, one can check that the right hand side of (6.6.4) and (6.6.5) equal to 0. Hence the case for Q is proved.

It remains to prove the theorem for Y. We will consider the characteristic $x^1(t, r_0)$. One firstly has to show that, $(t, x^1(t, r_0)) \in \Omega(\underline{r})$ for $t \in [0, T]$, i.e., we need to show that

$$x^1(t, r_0) - x^2(t, \underline{r}) \geq 0 \quad \forall t \in [0, T]. \tag{6.6.6}$$

By the definition of characteristics (6.1.4), one has

$$\begin{aligned} x^1(t, r_0) - x^2(t, \underline{r}) &= r_0 - \underline{r} + \int_0^t (u - \sqrt{P'})(\sigma, x^1(\sigma, \underline{r}))d\sigma - \int_0^t (u + \sqrt{P'})(\sigma, x^2(\sigma, r_0))d\sigma \\ &\geq r_0 - \underline{r} - \int_0^t \sqrt{P'}(\sigma, x^1(\sigma, \underline{r}))d\sigma - \int_0^t (u + \sqrt{P'})(\sigma, x^2(\sigma, r_0))d\sigma. \end{aligned}$$

Now by Theorem 6.2.1, one has

$$\begin{aligned} &\int_0^t \sqrt{P'}(\sigma, x^1(\sigma, \underline{r}))d\sigma + \int_0^t (u + \sqrt{P'})(\sigma, x^2(\sigma, r_0))d\sigma \\ =& \int_0^t \left(\frac{\gamma-1}{4}(z-w) \right)(\sigma, x^1(\sigma, \underline{r}))d\sigma + \int_0^t \left(\frac{w+z}{2} + \frac{\gamma-1}{4}(z-w) \right)(\sigma, x^2(\sigma, r_0))d\sigma \\ \leq& 2\gamma C_0 T. \end{aligned}$$

By the assumption that $r_0 - \underline{r} \geq 2\gamma C_0 T$, (6.6.6) indeed holds, i.e. $(t, x^1(t, r_0)) \in \Omega(\underline{r})$ for $t \in [0, T]$. Now, by the same procedure as how we proved Q blows up, we have Y blows up

before time reaches T if (6.2.2) holds.

Chapter 7

Isentropic Relativistic Compressible Euler Equations with Spherical Symmetry

7.1 Introduction

We consider the Cauchy problem of the multi-dimensional isentropic relativistic Euler equations, which, in the Eulerian coordinates, are given by

$$\begin{cases} \partial_t \left(\dfrac{c^4\rho + Pu^2}{c^2(c^2-u^2)} \right) + \partial_r \left(A \dfrac{(c^2\rho+P)u}{c^2-u^2} \right) = 0, \\ \partial_t \left(\dfrac{(c^2\rho+P)u}{c^2-u^2} \right) + \partial_r \left(\dfrac{(c^2\rho+P)u^2}{c^2-u^2} \right) + A\partial_r P = 0, \quad \text{with} \quad A(r) = r^{d-1}, \quad d = 2, 3, \\ (\rho, u)|_{t=0} = (\rho_0, u_0)(r) \quad \text{for} \quad r \in \mathbb{R}^+. \end{cases} \tag{7.1.1}$$

Here, we consider the pressure law

$$P(\rho) = k\rho^\gamma, \tag{7.1.2}$$

for some fixed constants $k > 0$ and $\gamma > 1$.

We proceed with a rudimentary analysis of the Cauchy problem (7.1.1). First, the two eigenvalues λ_1 and λ_2 of equations $(7.1.1)_1$-$(7.1.1)_2$ can be given by

$$\lambda_1 = \frac{c^2(u - \sqrt{P'})}{c^2 - u\sqrt{P'}}, \quad \lambda_2 = \frac{c^2(u + \sqrt{P'})}{c^2 + u\sqrt{P'}}. \tag{7.1.3}$$

Define the following functions

$$\begin{aligned} H_1(w,z) :=& \frac{w+z}{c} + \ln\left(\frac{c - \sqrt{P'}}{c + \sqrt{P'}}\right), \\ H_2(w,z) :=& \frac{w+z}{c} + \ln\left(\frac{c + \sqrt{P'}}{c - \sqrt{P'}}\right). \end{aligned} \tag{7.1.4}$$

By explicit calculation, one can check that

$$\lambda_1 = c\left(1 - \frac{2}{\mathrm{e}^{H_1} + 1}\right), \quad \lambda_2 = c\left(1 - \frac{2}{\mathrm{e}^{H_2} + 1}\right). \tag{7.1.5}$$

We denote the directional derivatives as

$$\partial_1 = \partial_t + \lambda_1 \partial_r, \quad \partial_2 = \partial_t + \lambda_2 \partial_r,$$

along two characteristic directions $x^1(t, x_0^1)$ and $x^2(t, x_0^2)$, defined by

$$\frac{dx^1}{dt} = \lambda_1, \quad \frac{dx^2}{dt} = \lambda_2, \tag{7.1.6}$$

where $x^1(0, x_0^1) = x_0^1$ and $x^2(0, x_0^2) = x_0^2$. For simplicity we will abbreviate $x^1(t, x_0^1)$ and $x^2(t, x_0^2)$ as $x^1(t)$ and $x^2(t)$. Introducing the corresponding Riemann invariants

$$z = \frac{c}{2}\ln\Big(\frac{c+u}{c-u}\Big) + \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}}\mathrm{d}\sigma, \quad w = \frac{c}{2}\ln\Big(\frac{c+u}{c-u}\Big) - \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}}\mathrm{d}\sigma. \tag{7.1.7}$$

By explicit calculations, one can verify that

$$\sqrt{P'(\rho)} = c\sqrt{\gamma}\ \tan\Big(\frac{(z-w)(\gamma-1)}{4c\sqrt{\gamma}}\Big). \tag{7.1.8}$$

Furthermore, one can verify that w and z satisfy

$$\begin{cases} \partial_1 w = \dfrac{A'}{A}\dfrac{c^2 u\sqrt{P'}}{c^2 - u\sqrt{P'}} = \dfrac{A'}{A}\left(\dfrac{c^4}{c^2 - u\sqrt{P'}} - c^2\right), \\ \partial_2 z = -\dfrac{A'}{A}\dfrac{c^2 u\sqrt{P'}}{c^2 + u\sqrt{P'}} = \dfrac{A'}{A}\left(\dfrac{c^4}{c^2 + u\sqrt{P'}} - c^2\right). \end{cases} \tag{7.1.9}$$

Let $\epsilon > 0$ be a small positive constant, denote

$$\begin{aligned} &\alpha \coloneqq \partial_r w, \quad \beta \coloneqq \partial_r z, \quad \widetilde{\alpha} = \alpha - \frac{A'}{A}\frac{c^2\sqrt{P'}}{c^2 - P'}, \quad \widetilde{\beta} = \beta + \frac{A'}{A}\frac{c^2\sqrt{P'}}{c^2 - P'}, \\ &M \coloneqq \exp\left(\int_1^\rho \frac{c^2 - P'(\sigma)}{c^2\sigma + P(\sigma)} d\sigma\right), \quad \widehat{\alpha} \coloneqq \frac{\widetilde{\alpha}}{(AM)^{1-\epsilon}}, \quad \widehat{\beta} \coloneqq \frac{\widetilde{\beta}}{(AM)^{1-\epsilon}}, \\ &\overline{\alpha} \coloneqq \frac{\widetilde{\alpha}}{(AM)^{\frac{\gamma-1}{2}-\epsilon}}, \quad \overline{\beta} \coloneqq \frac{\widetilde{\beta}}{(AM)^{\frac{\gamma-1}{2}-\epsilon}}, \quad k_0 \coloneqq \frac{(c^2 - P')(c^2 - u^2)}{(c^2 - u\sqrt{P'})(c^2 + u\sqrt{P'})}, \\ &F \coloneqq \frac{c^2 u\sqrt{P'}}{c^2 - u\sqrt{P'}}, \quad G \coloneqq \frac{c^2 u\sqrt{P'}}{c^2 + u\sqrt{P'}}. \end{aligned} \tag{7.1.10}$$

Furthermore, let $\widetilde{\epsilon}$ satisfy

$$2\widetilde{\epsilon} = (z-w)(1) = 2\int_0^1 \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}} \mathrm{d}\sigma \iff \left(\frac{z-w}{2}\right)^{-1}(\widetilde{\epsilon}) = 1, \tag{7.1.11}$$

denote

$$\begin{aligned} h_1 &\coloneqq -\ln(e^{H_1} + 1) + \frac{1}{2}\ln(e^{H_1+H_2} - e^{2H_2}) - \int_{\widetilde{\epsilon}}^{\frac{z-w}{2}} \frac{(c^2 + \sqrt{P'})^2}{2c^2\sqrt{P'}} d\left(\frac{z-w}{2}\right), \\ h_2 &\coloneqq -\ln(e^{H_2} + 1) + \frac{1}{2}\ln(e^{2H_2} - e^{H_1+H_2}) - \int_{\widetilde{\epsilon}}^{\frac{z-w}{2}} \frac{(c^2 - \sqrt{P'})^2}{2c^2\sqrt{P'}} d\left(\frac{z-w}{2}\right), \\ \Phi &\coloneqq -\int_{w+2\widetilde{\epsilon}}^{z} \frac{\partial_z F e^{h_1}}{\lambda_2 - \lambda_1} dz, \quad \Psi \coloneqq \int_{z-2\widetilde{\epsilon}}^{w} \frac{\partial_w G e^{h_2}}{\lambda_2 - \lambda_1} dw, \\ Y &\coloneqq e^{h_1}\alpha - \frac{A'}{A}\Phi, \quad Q \coloneqq e^{h_2}\beta - \frac{A'}{A}\Psi. \end{aligned} \tag{7.1.12}$$

Now we can introduce two key quantities, which play important roles in our analysis on singu-

larity formation:

$$\widetilde{Y} \coloneqq A^{\frac{\gamma-3}{4}} Y, \quad \widetilde{Q} \coloneqq A^{\frac{\gamma-3}{4}} Q. \tag{7.1.13}$$

We also introduce the coefficients for the coupled ODEs given in section 7.4.1:

$$\begin{aligned}
a_0 &= -\frac{2e^{H_1}}{(e^{H_1}+1)^2}, \quad a_1 = -\frac{P''(c^2\rho+P)(c^2-u^2)}{4P'(c^2-u\sqrt{P'})^2}, \\
a_2 &= -\frac{A'}{A}\left[\frac{c^2P''(c^2\rho+P)}{4P'(c^2-u\sqrt{P'})^2}\left(\frac{c^2-u^2}{c^2-P'}\sqrt{P'}+u\right) + \frac{c^2P''(c^2\rho+P)(c^2+P')(c^2-u^2)}{2\sqrt{P'}(c^2-P')^2(c^2+u\sqrt{P'})(c^2-u\sqrt{P'})}\right], \\
a_3 &= -\frac{A'}{A}\frac{c^4P''(c^2\rho+P)(u-\sqrt{P'})(c^2-u^2)}{2(c^2-P')^2(c^2+u\sqrt{P'})(c^2-u\sqrt{P'})^2}, \\
a_4 &= \frac{d}{dr}\left(\frac{A'}{A}\right)\frac{c^2P'}{c^2-P'} - \left(\frac{A'}{A}\right)^2\frac{c^6uP''(c^2\rho+P)(u-\sqrt{P'})}{2(c^2-P')^2(c^2+u\sqrt{P'})(c^2-u\sqrt{P'})^2} \\
&\quad - \left(\frac{A'}{A}\right)^2\frac{c^4P''(c^2\rho+P)(c^2-u^2)}{2(c^2-P')^2(c^2-u\sqrt{P'})^2}\left[\frac{(c^2+P')(c^2-u\sqrt{P'})}{(c^2+u\sqrt{P'})(c^2-P')}+1\right],
\end{aligned}$$

and

$$\begin{aligned}
b_0 &= -\frac{2e^{H_2}}{(e^{H_2}+1)^2}, \quad b_1 = -\frac{P''(c^2\rho+P)(c^2-u^2)}{4P'(c^2+u\sqrt{P'})^2}, \\
b_2 &= -\frac{A'}{A}\left[\frac{c^2P''(c^2\rho+P)}{4P'(c^2+u\sqrt{P'})^2}\left(u-\frac{c^2-u^2}{c^2-P'}\sqrt{P'}\right) - \frac{c^2P''(c^2\rho+P)(c^2+P')(c^2-u^2)}{2\sqrt{P'}(c^2-P')^2(c^2+u\sqrt{P'})(c^2-u\sqrt{P'})}\right], \\
b_3 &= -\frac{A'}{A}\frac{c^4P''(c^2\rho+P)(u+\sqrt{P'})(c^2-u^2)}{2(c^2-P')^2(c^2+u\sqrt{P'})^2(c^2-u\sqrt{P'})}, \\
b_4 &= \frac{d}{dr}\left(\frac{A'}{A}\right)\frac{c^2P'}{c^2-P'} - \left(\frac{A'}{A}\right)^2\frac{c^6uP''(c^2\rho+P)(u+\sqrt{P'})}{2(c^2-P')^2(c^2+u\sqrt{P'})^2(c^2-u\sqrt{P'})} \\
&\quad - \left(\frac{A'}{A}\right)^2\frac{c^4P''(c^2\rho+P)(c^2-u^2)}{2(c^2-P')^2(c^2+u\sqrt{P'})^2}\left[\frac{(c^2+P')(c^2+u\sqrt{P'})}{(c^2-u\sqrt{P'})(c^2-P')}+1\right].
\end{aligned}$$

Lastly, we introduce coefficients for the decoupled ODEs given in section 7.5:

$$\begin{aligned}
d_2 &= -\left(\frac{A'}{A}\right)^2\left[\partial_w\lambda_1 e^{-h_1}\Phi^2 + \left(\partial_w F - \frac{\partial_z\lambda_1 G}{\lambda_2-\lambda_1} - \partial_w h_1 F\right)\Phi + \frac{\partial_z FG}{\lambda_2-\lambda_1}e^{h_1} + \partial_w\Phi F\right]A^{\frac{\gamma-3}{4}} \\
&\quad + \frac{d}{dr}\left(\frac{A'}{A}\right)(Fe^{h_1}-\lambda_1\Phi)A^{\frac{\gamma-3}{4}}, \\
d_1 &= -\frac{A'}{A}\left(2\Phi\partial_w\lambda_1 e^{-h_1} + \partial_w F - \frac{\partial_z\lambda_1 G}{\lambda_2-\lambda_1} - \partial_w h_1 F - \frac{\gamma-3}{4}\lambda_1\right), \\
d_0 &= -\partial_w\lambda_1 e^{-h_1}A^{\frac{3-\gamma}{4}},
\end{aligned}$$

and

$$\begin{aligned}
\overline{d}_2 =& -\left(\frac{A'}{A}\right)^2\left[\partial_z\lambda_2 e^{-h_2}\Psi^2+\left(\partial_z G-\frac{\partial_w\lambda_2 F}{\lambda_1-\lambda_2}-\partial_z h_2 G\right)\Psi+\frac{\partial_w GF}{\lambda_2-\lambda_1}e^{h_2}+\partial_z\Psi G\right]A^{\frac{\gamma-3}{4}}\\
&-\frac{d}{dr}\left(\frac{A'}{A}\right)(Fe^{h_1}-\lambda_1\Phi)A^{\frac{\gamma-3}{4}},\\
\overline{d}_1 =& -\frac{A'}{A}\left(2\Psi\partial_z\lambda_2 e^{-h_2}+\partial_z G-\frac{\partial_w\lambda_2 F}{\lambda_1-\lambda_2}-\partial_z h_2 G-\frac{\gamma-3}{4}\lambda_2\right),\\
\overline{d}_0 =& -\partial_z\lambda_2 e^{-h_2}A^{\frac{3-\gamma}{4}}.
\end{aligned}$$

7.2 Main results

Define the C^1 solutions as follows:

Definition 7.2.1. *Let $T>0$ be some time. The pair $(\rho(t,r),u(t,r))$ is called a C^1 solution to the equations $(7.1.1)_1$-$(7.1.1)_2$ on $(0,T)\times\mathbb{R}^+$ if*

$$\rho>0,\quad \rho\in C^1([0,T)\times\mathbb{R}^+),\quad u\in C^1([0,T)\times\mathbb{R}^+),$$

and the equations $(7.1.1)_1$-$(7.1.1)_2$ are satisfied in the pointwise sense on $(0,T)\times\mathbb{R}^+$. It is called a C^1 solution to the Cauchy problem (7.1.1) if it is a C^1 solution to the equations $(7.1.1)_1$-$(7.1.1)_2$ on $(0,T)\times\mathbb{R}^+$ and admits the initial data $(7.1.1)_3$ continuously.

Assumption 7.2.1. *Denote $(w_0,z_0)(r)=(w(0,r),z(0,r))$. We will assume $(w_0,z_0)\in C^1(\mathbb{R}^+)$, $(\frac{w_0}{r},\ \frac{z_0}{r})\in C^1(\mathbb{R}^+)$. Moreover, $\forall r\in[0,\infty)$,*

$$0\le w_0(r)\le z_0(r)\le\sup_{r\in\mathbb{R}^+}z_0(r)<\max\left\{\frac{8c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\left(\frac{\epsilon}{\sqrt{\gamma}}\right),\quad\frac{c}{2}\log(1+2\epsilon)\right\},$$

and

$$w_0(r)=z_0(r)=0\iff r=0,$$

where $\epsilon>0$ is the small constant in (7.1.10).

We now give a theorem that tells us local in time C^1 solutions indeed exist. This result is an original work by the author of this thesis. The proof of this result (section 7.3) takes inspiration from [31].

Theorem 7.2.1. *Assume $1 < \gamma < 3 - \epsilon$, where $\epsilon > 0$ is the small constant in* (7.1.10). *Assume the initial data (w_0, z_0) satisfy Assumption* 7.2.1. *There exists $\delta > 0$ and a unique C^1 solution $(w, z)(t, r)$ on $(t, r) \in [0, \delta) \times \mathbb{R}^+$, where $\forall (t, r) \in [0, \delta) \times [0, \infty)$, one has*

$$0 \leq w(t, r) \leq z(t, r) \leq \sup_{r \in \mathbb{R}^+} z_0(r) < \max \left\{ \frac{8c\sqrt{\gamma}}{\gamma - 1} \tan^{-1} \left(\frac{\epsilon}{\sqrt{\gamma}} \right), \quad \frac{c}{2} \log(1 + 2\epsilon) \right\},$$

and

$$w(t, r) = z(t, r) = 0 \iff r = 0.$$

In particular, one has $\forall (t, r) \in [0, \delta) \times [0, \infty)$,

$$0 \leq u(t, r) < \epsilon c, \quad 0 \leq \sqrt{P'}(t, r) < \epsilon c,$$

and

$$u(t, r) = \sqrt{P'}(t, r) = 0 \iff r = 0.$$

Furthermore, for any point $(\widetilde{t}, \widetilde{r}) \in [0, \delta) \times (0, \infty)$, the characteristic $(t, x^2(t))$ passing through the point $(\widetilde{t}, \widetilde{r})$ will not touch the line $r = 0$ if $t < \widetilde{t}$.

Assumption 7.2.2. *For initial data (w_0, z_0) satisfy Assumption* 7.2.1*, assume furthermore that*

$$AM_{\min}(\underline{r}) \coloneqq \min_{r \in [\underline{r}, \infty)} A(r) M_0(r) > 0, \quad \forall \underline{r} > 0,$$

and

$$\max_{r \in [\underline{r}, \infty)} \{ \overline{\alpha}_0(r), \overline{\beta}_0(r), \widehat{\alpha}_0(r), \widehat{\beta}_0(r) \} < K(\underline{r}), \quad \forall \underline{r} > 0,$$

where $K(\underline{r}) > 0$ is some positive constant depending on $\underline{r}$.

Since now, we will assume Assumption 7.2.1 holds. All the conclusions are proved under the assumption that $t < \delta$, where δ is the maximal time for the C^1 solution to exist.

Definition 7.2.2. *Define the domain of dependence $\Omega(\underline{r})$ as*

$$\Omega(\underline{r}) \coloneqq \{ (t, r) | r \geq x^2(t, \underline{r}), \quad t \in \mathbb{R}^+ \},$$

for some $\underline{r} > 0$.

Definition 7.2.3. *Define the constant* $N(\underline{r})$ *depending only on* $\underline{r}$*, initial data,* k*,* γ *and* d *by*

$$\begin{aligned} N(\underline{r}) \coloneqq \underline{r}^{\frac{(d-1)(\gamma-1)}{2}-\epsilon(d-1)+1} \max\Big\{ & \sup \frac{r^{\frac{(d-1)(\gamma-1)}{2}-\epsilon(d-1)+1}|b_2|}{k_0(AM)^{\frac{\gamma-1}{2}-\epsilon}}, \\ & \sup \frac{2r^{\frac{(d-1)(\gamma-1)}{2}-\epsilon(d-1)+1}|b_2|}{\left(-b_1+b_0+(\frac{\gamma-1}{2}-\epsilon)k_0\right)(AM)^{\frac{\gamma-1}{2}-\epsilon}}, \\ & \sup \frac{2r^{\frac{(d-1)(\gamma-1)}{2}-\epsilon(d-1)+1}|b_2|}{(-b_1+b_0)(AM)^{\frac{\gamma-1}{2}-\epsilon}}\Big\}. \end{aligned}$$

Remark 7.2.1. *In the proof of Lemma 7.4.9, it is shown that* $N(\underline{r})$ *indeed only depends on* $\underline{r}$*, initial data,* k*,* γ *and* d.

Definition 7.2.4. *Define a constant* $\widetilde{L}$ *by*

$$\widetilde{L} \coloneqq \max\left\{ \sup_{0<w\le z\le \sup z_0} r(AM)^{\frac{3-\gamma}{4}} \sqrt{16\left|\frac{d_1}{d_0}\right|^2+\left|\frac{d_2}{d_0}\right|}, \sup_{0<w\le z\le \sup z_0} r(AM)^{\frac{3-\gamma}{4}} \sqrt{16\left|\frac{\bar{d}_1}{\bar{d}_0}\right|^2+\left|\frac{\bar{d}_2}{\bar{d}_0}\right|} \right\}.$$

Define a constant $\widetilde{M}$ *by*

$$\widetilde{M} = \min\left\{ \inf_{0<w\le z\le \sup z_0} |d_0|(AM)^{\frac{3-\gamma}{4}}, \inf_{0<w\le z\le \sup z_0} |\bar{d}_0|(AM)^{\frac{3-\gamma}{4}} \right\}.$$

Let $\underline{r} > 0$*, define the blow-up time* $T > 0$ *by*

$$T \coloneqq \frac{2\underline{r}}{\widetilde{M}\widetilde{L}}.$$

Define the initial compression strength $G_T(\underline{r})$ *by*

$$G_T(\underline{r}) \coloneqq \begin{cases} \dfrac{\widetilde{L}}{\underline{r}}\left(\dfrac{1}{AM_{\min}(\underline{r})^{1-\epsilon}}+K(\underline{r})T(1-\epsilon)\sup\limits_{0<w\le z\le \sup z_0}|k_0|\right)^{\frac{3-\gamma}{4(1-\epsilon)}}, & \text{if} \quad 1<\gamma<\dfrac{5}{3}-\dfrac{7}{6}\epsilon, \\ \dfrac{\widetilde{L}}{\underline{r}}\left(\dfrac{1}{AM_{\min}(\underline{r})^{\frac{\gamma-1}{2}}-\epsilon}+N(\underline{r})T\left(\dfrac{\gamma-1}{2}-\epsilon\right)\sup\limits_{0<w\le z\le \sup z_0}|k_0|\right)^{\frac{3-\gamma}{4\left(\frac{\gamma-1}{2}-\epsilon\right)}}, & \\ \text{if} \quad \frac{5}{3}+\frac{8}{3}\epsilon<\gamma<3-6\epsilon. & \end{cases}$$

Finally, we can state the gradient blow-up result. This result is an original work by the

author of this thesis.

Theorem 7.2.2. *Let $\epsilon > 0$ be small and let $\underline{r} > 0$. Assume $1 < \gamma < \frac{5}{3} - \frac{7}{6}\epsilon$ or $\frac{5}{3} + \frac{8}{3}\epsilon < \gamma < 3 - 6\epsilon$. Let T and $G_T(\underline{r})$ be the constants in Definition 7.2.4. For initial data (w_0, z_0) satisfy Assumption 7.2.1, if*

$$\widetilde{Q}(0, \underline{r}) < -G_T(\underline{r}), \tag{7.2.1}$$

then the C^1 solution will break down before $t = T$, more precisely, one has $|Q| \to \infty$ before $t = T$. Moreover, if there exists some $r_0 > \underline{r}$ satisfying

$$r_0 - \underline{r} \geq \frac{3\epsilon c T}{(1-\epsilon)^2}, \tag{7.2.2}$$

such that

$$\widetilde{Y}(0, r_0) < -G_T(\underline{r}), \tag{7.2.3}$$

then the C^1 solution will break down before $t = T$, more precisely, one has $|Y| \to \infty$ before $t = T$.

7.3 Local existence and uniform bounds on the Riemann invariants

This section is devoted to proving Theorem 7.2.1. Before starting to prove the result, we introduce two useful lemmas.

Lemma 7.3.1. *Let $T > 0$. Assume $\rho(t, r) > 0$ for all $(t, r) \in [0, T) \times (0, \infty)$, and $(w, z) \in C^1([0, T) \times \mathbb{R}^+)$, $(\frac{w}{r}, \frac{z}{r}) \in C^1([0, T) \times \mathbb{R}^+)$ with*

$$0 \leq w \leq z \leq \sup_{(t,r)} z < \frac{8c\sqrt{\gamma}}{\gamma - 1} \tan^{-1}\big(\frac{1}{\sqrt{\gamma}}\big), \quad \forall (t, r) \in [0, T) \times [0, \infty), \tag{7.3.1}$$

and

$$w(t, r) = z(t, r) = 0 \iff r = 0. \tag{7.3.2}$$

Then one has $(u,\rho)\in C^1([0,T)\times\mathbb{R}^+)$, $(\frac{u}{r},\frac{\sqrt{P'}}{r})\in C^1([0,T)\times\mathbb{R}^+)$ *with*

$$0\le u<c,\quad 0\le\sqrt{P'}<c,\quad \forall(t,r)\in[0,T)\times[0,\infty), \tag{7.3.3}$$

and

$$u(t,r)=\sqrt{P'}(t,r)=0\iff r=0. \tag{7.3.4}$$

Proof. First of all, by the explicit expression of w and z given by (7.1.7), one immediately has $(u,\rho)\in C^1([0,T)\times\mathbb{R}^+)$. Next, we check (7.3.3) holds. By (7.3.1), (7.1.8), (7.1.7) and the assumption $\rho>0$, for $(t,r)\in[0,T)\times(0,\infty)$ one has $\sqrt{P'}>0$,

$$\begin{aligned}
&\sqrt{P'}\le c\sqrt{\gamma}\,\tan\Big(\frac{z(\gamma-1)}{4c\sqrt{\gamma}}\Big)<c,\\
w>0\iff\frac{c}{2}\ln\Big(\frac{c+u}{c-u}\Big)&>\frac{4c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\left(\frac{\sqrt{P'}}{c\sqrt{\gamma}}\right)>0\implies u>0,\\
w+z<\frac{16c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\left(\frac{1}{\sqrt{\gamma}}\right)\iff\frac{c}{2}\ln\Big(\frac{c+u}{c-u}\Big)&<\frac{16c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\left(\frac{1}{\sqrt{\gamma}}\right)\implies u<c.
\end{aligned}$$

Hence (7.3.3) indeed holds. Lastly, we will show $(\frac{u}{r},\frac{\sqrt{P'}}{r})\in C^1([0,T)\times\mathbb{R}^+)$ and (7.3.4) holds. By (7.3.3) and the Mean Value Theorem, there exists some $0\le\widetilde{u}<c$ and $0\le\sqrt{P'(\widetilde{\rho})}<c$, such that

$$\frac{u}{r}=\frac{2}{c^2-\widetilde{u}^2}\frac{w+z}{r},\quad \frac{\sqrt{P'}}{r}=\frac{c^2\gamma+\sqrt{P'(\widetilde{\rho})}}{c^2\gamma}\frac{\gamma-1}{2}\frac{z-w}{r},$$

which immediately implies the desired result. □

Lemma 7.3.2. *Let the assumptions in Lemma* 7.3.1 *hold, assume additionally that*

$$\sup_{(t,r)}z(t,r)<\max\left\{\frac{8c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\big(\frac{\epsilon}{\sqrt{\gamma}}\big),\quad\frac{c}{2}\log(1+2\epsilon)\right\}, \tag{7.3.5}$$

where $\epsilon>0$ *the small constant in* (7.1.10)*. Then one has*

$$u<\epsilon c,\quad \sqrt{P'}<\epsilon c\quad\forall(t,r)\in[0,T)\times[0,\infty). \tag{7.3.6}$$

Proof. Using (7.1.8) and the fact that

$$u = c\frac{\exp(\frac{z+w}{c}) - 1}{\exp(\frac{z+w}{c}) + 1} \leq \frac{c}{2}\exp\left(\frac{z+w}{c}\right) - \frac{1}{2} \leq \frac{c}{2}\exp\left(\frac{2\sup_{(t,r)} z}{c}\right) - \frac{1}{2},$$

(7.3.5) immediately yields the desired result. □

Now we will show a local C^1 solution exists for $r \geq 0$. For the simplicity of notations, denote

$$h \coloneqq \sqrt{P'}, \quad s(\rho) \coloneqq \frac{2c\sqrt{\gamma}}{\gamma - 1}\tan^{-1}\left(\frac{h}{c\sqrt{\gamma}}\right),$$

and introduce the following functions:

$$\widetilde{w} = \frac{w}{r}, \quad \widetilde{z} = \frac{z}{r}, \quad \widetilde{u} = \frac{u}{r}, \quad \widetilde{h} = \frac{h}{r}, \quad \widetilde{s}(\rho) = \frac{s(\rho)}{r}, \quad \widetilde{\lambda}_1 = \frac{\lambda_1}{r}, \quad \widetilde{\lambda}_2 = \frac{\lambda_2}{r}.$$

We will first consider the initial value problem on $r \in \mathbb{R}$ by making the following reflection on the initial data when $r < 0$:

$$w_0(r) = -z_0(-r), \quad z_0(r) = -w_0(-r). \tag{7.3.7}$$

Using Assumption 7.2.1, one can check that w_0, z_0, $\widetilde{w}_0$ and $\widetilde{z}_0$ are Lipschitz continuous after the reflection, and they are C^1 away from the point $r = 0$. Moreover, when $r < 0$, one has

$$u_0(r) = -u_0(-r), \quad \rho_0(r) = \rho_0(-r), \quad \lambda_1(0,r) = -\lambda_2(0,r), \quad \lambda_2(0,r) = -\lambda_1(0,-r). \tag{7.3.8}$$

Notice that Lemma 7.3.1 holds for Lipschitz continuous functions, so u_0, h_0, $\widetilde{u}_0$ and $\widetilde{h}_0$ are Lipschitz continuous. Moreover, by (7.3.8), one can deduce that

$$|u_0| < c, \quad h_0 < c. \tag{7.3.9}$$

Formally, (7.1.9) can be reformulated as:

$$\begin{cases} \partial_1 \widetilde{w} = (d-1)\dfrac{c^2\widetilde{u}\widetilde{h}}{c^2-uh} - \widetilde{\lambda}_1\widetilde{w}, \\ \partial_2 \widetilde{z} = -(d-1)\dfrac{c^2\widetilde{u}\widetilde{h}}{c^2+uh} - \widetilde{\lambda}_2\widetilde{z}. \end{cases} \tag{7.3.10}$$

By the properties of the initial data, it is well-known that the local-in-time existence of the unique Lipschitz continuous solution $(\widetilde{w}, \widetilde{z})$ to the Cauchy problem (7.3.10) can be established by the classical characteristic method; c.f. Theorem 4.1 on p. 66 of [39], or see also [23], [26], [38]. Henceforth, we let $\delta > 0$ denote the maximal time for the Lipschitz continuous solution to exist.

Define the time δ^* by

$$\delta^* \coloneqq \sup\{t \in [0,\delta) \mid \ |u| < \epsilon c \ \text{ and } \ h < \epsilon c \ \text{ uniformly}\}.$$

Here, for the function $f(t,r)$, at the time t_0, $|f(t_0,r)| < \epsilon c$ uniformly means: there exists one constant $C_0 > 0$ independent of $(t,r) \in [0,t_0]\times\mathbb{R}$ such that

$$\epsilon c - |f(t,r)| > C_0 \quad \text{for} \quad (t,r) \in [0,t_0]\times\mathbb{R}.$$

It follows from Assumption 7.2.1 and the continuity of the solution that $\delta^* > 0$. To prove Theorem 7.2.1, we first show the solution has certain properties for $t \in [0,\delta^*)$, and then prove that $\delta^* = \delta$. Furthermore, we will focus only on the region $r \geq 0$.

Lemma 7.3.3. *For $t \in [0,\delta^*)$, $\rho(t,r) > 0$ if $r \neq 0$.*

Proof. Subtracting the two equations in (7.3.10) and simplify, one has

$$\partial_t \widetilde{s}(\rho) + r(\widetilde{\lambda}_1 + \widetilde{\lambda}_2)\partial_r \widetilde{s}(\rho) = (d-1)\frac{c^2\widetilde{u}\widetilde{h}}{c^2-uh} - (d-1)\frac{c^2\widetilde{u}\widetilde{h}}{c^2+uh} + \frac{c^2\widetilde{h}(c^2-u^2)}{(c^2+uh)(c^2-uh)} r\partial_r(\widetilde{w}+\widetilde{z}),$$

this can be further rewritten as

$$\begin{aligned}\partial_t\widetilde{h}+r(\widetilde{\lambda}_1+\widetilde{\lambda}_2)\partial_r\widetilde{h}=&\frac{(\gamma-1)(c^2\gamma+h^2)(d-1)\widetilde{u}\widetilde{h}}{2\gamma(c^2-uh)}-\frac{(\gamma-1)(c^2\gamma+h^2)(d-1)\widetilde{u}\widetilde{h}}{2\gamma(c^2+uh)}\\&+\frac{(\gamma-1)(c^2\gamma+h^2)\widetilde{h}(c^2-u^2)r\partial_r(\widetilde{w}+\widetilde{z})}{2\gamma(c^2+uh)(c^2-uh)}-\widetilde{h}(\widetilde{\lambda}_1+\widetilde{\lambda}_2)\\&+\frac{(\gamma-1)(c^2\gamma+h^2)(\widetilde{\lambda}_1+\widetilde{\lambda}_2)\widetilde{h}}{2c^2\gamma}\frac{\widetilde{s}(\rho)}{\widetilde{h}}.\end{aligned}\tag{7.3.11}$$

As $0<h<c$, by explicit calculations, one can find two constants $0<D_1<D_2$ depending only on k, γ and c, such that

$$D_1<\frac{\widetilde{s}(\rho)}{\widetilde{h}}=\frac{2c\sqrt{\gamma}}{(\gamma-1)h}\tan^{-1}\left(\frac{h}{c\sqrt{\gamma}}\right)<D_2.$$

Integrating (7.3.11) along the characteristic defined by

$$\frac{dx}{dt}=x(\widetilde{\lambda}_1+\widetilde{\lambda}_2),$$

then, along with our assumption on $\widetilde{h}$, one has

$$\widetilde{h}(t,x(t))\geq 0\quad\text{with}\quad\widetilde{h}(t,x(t))=0\iff\widetilde{h}(0,x(0))=0\iff x(0)=0,$$

which corresponds to the characteristics $x(t)\equiv 0$. Hence one has

$$\widetilde{h}(t,r)\neq 0\quad\text{if}\quad r\neq 0.$$

By the explicit expression of $\widetilde{h}$, the lemma is proved. □

Lemma 7.3.4. *Let $\epsilon>0$ be the small constant defined in* (7.1.10)*. Define the function*

$$f(x)\coloneqq\exp\left(\frac{4\sqrt{\gamma}}{\gamma-1}\tan^{-1}(x)\right)-1-\frac{4\sqrt{\gamma}}{\gamma-1}\left(1-\frac{2\epsilon}{3}\right)x.\tag{7.3.12}$$

Then for $\epsilon>0$ small, one has

$$f(x)\geq 0,\quad\forall x\in\left[0,\frac{\epsilon}{\sqrt{\gamma}}\right],$$

with $f(x) = 0 \iff x = 0$. *Moreover, if* $t \in [0, \delta^*)$ *and* $1 < \gamma < 3 - 6\epsilon$, *one can deduce that*

$$u(t, r) \geq \sqrt{P'}(t, r),$$

with $u(t, r) = \sqrt{P'}(t, r) \iff r = 0$.

Proof. We first show $f(x) \geq 0$ with $f(x) = 0 \iff x = 0$. Noticing that $f(0) = 0$, so to show this, it is sufficient to show $f'(x) > 0$. By explicit calculations, one has

$$f'(x) = \frac{4\sqrt{\gamma}}{\gamma - 1} \frac{1}{1 + x^2} \exp\left(\frac{4\sqrt{\gamma}}{\gamma - 1} \tan^{-1}(x)\right) - \frac{4\sqrt{\gamma}}{\gamma - 1}\left(1 - \frac{2\epsilon}{3}\right), \tag{7.3.13}$$

and

$$f''(x) = \frac{16\gamma}{(\gamma - 1)^2} \frac{1}{(1 + x^2)^2} \exp\left(\frac{4\sqrt{\gamma}}{\gamma - 1} \tan^{-1}(x)\right) - \frac{8\sqrt{\gamma}}{\gamma - 1} \frac{x}{(1 + x^2)^2} \exp\left(\frac{4\sqrt{\gamma}}{\gamma - 1} \tan^{-1}(x)\right).$$

One has

$$f''(x) > 0 \quad \forall x \in \left[0, \frac{2\sqrt{\gamma}}{\gamma - 1}\right) \implies f''(x) > 0 \quad \forall x \in \left[0, \frac{\epsilon}{\sqrt{\gamma}}\right],$$

given $\epsilon > 0$ is small. Hence as $f'(0) > 0$, one has

$$f'(x) > 0, \quad \forall x \in \left[0, \frac{\epsilon}{\sqrt{\gamma}}\right],$$

which implies $f(x) \geq 0$ with $f(x) = 0 \iff x = 0$ for $x \in \left[0, \frac{\epsilon}{\sqrt{\gamma}}\right]$.

It remains to show $u(t, r) \geq \sqrt{P'}(t, r)$ with $u(t, r) = \sqrt{P'}(t, r) \iff r = 0$. As $\widetilde{w}(t, r) = \frac{w(t,r)}{r}$ is a continuous function, one has $w(t, r) \geq 0$ with $w(t, r) = 0$ if $r = 0$. By (7.1.7) and (7.1.8), one has

$$w \geq 0 \iff u \geq \frac{c \exp\left(\frac{4\sqrt{\gamma}}{\gamma - 1} \tan^{-1}\left(\frac{\sqrt{P'}}{c\sqrt{\gamma}}\right)\right) - c}{1 + \exp\left(\frac{4\sqrt{\gamma}}{\gamma - 1} \tan^{-1}\left(\frac{\sqrt{P'}}{c\sqrt{\gamma}}\right)\right)}. \tag{7.3.14}$$

As $t \in [0, \delta^*)$, using Taylor expansion, one has

$$1 + \exp\left(\frac{4\sqrt{\gamma}}{\gamma - 1} \tan^{-1}\left(\frac{\sqrt{P'}}{c\sqrt{\gamma}}\right)\right) \leq 2 + \frac{4\epsilon}{\gamma - 1} + \frac{\mathcal{O}(\epsilon^2)}{\gamma - 1}.$$

Combining this with the fact that $f(x) \geq 0$, (7.3.14) becomes

$$u \geq \frac{c}{2+\frac{4\epsilon}{\gamma-1}+\frac{\mathcal{O}(\epsilon^2)}{\gamma-1}} \frac{4\sqrt{\gamma}}{\gamma-1}\left(1-\frac{3}{2}\epsilon\right)\frac{\sqrt{P'}}{c\sqrt{\gamma}} = \sqrt{P'}\frac{4\left(1-\frac{3}{2}\epsilon\right)}{2(\gamma-1)+4\epsilon+\mathcal{O}(\epsilon^2)}.$$

Now as $1 < \gamma < 3 - 6\epsilon$, for $\epsilon > 0$ small, one can check that

$$\frac{4\left(1-\frac{3}{2}\epsilon\right)}{2(\gamma-1)+4\epsilon+\mathcal{O}(\epsilon^2)} > 1.$$

Hence this lemma is proved. □

Proposition 7.3.1. *For $t \in [0, \delta^*)$, $1 < \gamma < 3 - 6\epsilon$ and $r > 0$, one has*

$$0 < \lambda_1(t, r) < \lambda_2(t, r), \tag{7.3.15}$$

and the characteristics $x^1(t)$ and $x^2(t)$ intersect each other at most once.

Proof. First, $0 < \lambda_1 < \lambda_2$ follows directly from (7.1.3) and Lemma 7.3.4. It remains to show $x^1(t)$ and $x^2(t)$ intersect each other at most once. Define the function $g(t)$ by

$$g(t) \coloneqq x^1(t) - x^2(t).$$

One observes that if for some t_*, $g(t_*) = 0$, then $g'(t_*) \neq 0$. Actually, note that $g(t_*) = 0$ implies that

$$x^1(t_*) = x^2(t_*) \coloneqq x_*.$$

Then if $g'(t_*) = 0$, one can get

$$0 = g'(t_*) = \lambda_1(t_*, x_*) - \lambda_2(t_*, x_*),$$

which contradicts (7.3.15).

Now we assume that $x^1(t)$ and $x^2(t)$ intersect each other more than once. Then $g(t) = 0$ holds at least for two different points: $\bar{a}_1$ and $\bar{a}_2$. One can choose a time $\widetilde{t} \in (\bar{a}_1, \bar{a}_2)$ such that $g(\widetilde{t}) \neq 0$. Without loss of generality, assume $g(\widetilde{t}) > 0$. It is clear that such kind of points indeed

exist, or else $g(t) \equiv 0$ for $t \in (\bar{a}_1, \bar{a}_2)$, which is impossible.

By the continuity of $g(t)$, there exists two points $\bar{a}_3 \in [\bar{a}_1, \bar{a}_2)$ and $\bar{a}_4 \in (\bar{a}_1, \bar{a}_2]$ such that $\widetilde{t} \in (\bar{a}_3, \bar{a}_4)$, $g(t) > 0$ for all $t \in (\bar{a}_3, \bar{a}_4)$, and $g(\bar{a}_3) = g(\bar{a}_4) = 0$, which implies that

$$x^1(\bar{a}_3) = x^1(\bar{a}_3) \coloneqq \bar{A} \quad \text{and} \quad x^1(\bar{a}_4) = x^2(\bar{a}_4) \coloneqq \bar{B}.$$

As $g'(\bar{a}_3) \neq 0$ and $g'(\bar{a}_4) \neq 0$, one has $g'(\bar{a}_3) > 0$ and $g'(\bar{a}_4) < 0$. Then

$$\lambda_1(\bar{a}_3, \bar{A}) > \lambda_2(\bar{a}_3, \bar{A}), \quad \lambda_1(\bar{a}_4, \bar{B}) < \lambda_2(\bar{a}_4, \bar{B}),$$

which contradicts (7.3.15). The proof of this lemma is complete. □

Lemma 7.3.5. *For any $(t, r) \in [0, \delta^*) \times \mathbb{R}$ with $r > 0$, the two characteristics passing through the point (t, r) will not cross the line $r = 0$.*

Proof. Let us consider a point $(t', r') \in [0, \delta^*) \times \mathbb{R}$ with $r' > 0$, and two characteristics $(t, x^1(t))$ and $(t, x^2(t))$ passing through the point (t', r'). By Lemma 7.3.1, one has $\lambda_1 < \lambda_2$, so to prove the lemma for this case, one just needs to show $x^2(t) > 0$ for $t \in [0, \delta^*)$. For contradiction, suppose $t^* \in [0, t')$ is the first time that $x^2(t^*) = 0$, so $x^2(t) > 0$ for $t \in (t^*, t')$. As $\widetilde{h}(t, r)$ and $\widetilde{u}(t, r)$ are Lipschitz continuous, one has $h(t, 0) = u(t, 0) = 0$, hence

$$-Lr < \lambda_2(r, t) = \lambda_2(r, t) - \lambda_2(0, t) < Lr,$$

where $L > 0$ is a constant depending on the Lipschitz constant of h and u. Hence

$$\frac{dx^2(t)}{dt} < Lx^2(t). \tag{7.3.16}$$

Integrating (7.3.16), one gets

$$x^2(t') < e^{L(t'-t^*)} x^2(t^*) = 0,$$

a contradiction. □

Lemma 7.3.6. *Let $t \in [0, \delta^*)$. Let Assumption 7.2.1 holds. If $r > 0$, then*

$$0 < w \leq z \leq \sup_{(t,r)\in[0,\delta^*)\times\mathbb{R}^+} z < \max\left\{\frac{8c\sqrt{\gamma}}{\gamma-1}\tan^{-1}\left(\frac{\epsilon}{\sqrt{\gamma}}\right), \quad \frac{c}{2}\log(1+2\epsilon)\right\}.$$

If $r = 0$, then

$$w(t,0) = z(t,0) = 0.$$

Proof. The fact $w(t,0) = z(t,0) = 0$ follows directly from $\widetilde{w}$ and $\widetilde{z}$ are Lipschitz continuous. It remains to prove for the case $r > 0$.

First, one proves $w \geq 0$ by contradiction. Assuming there exists some $(t', r') \in [0, \delta^*) \times \mathbb{R}$ such that $w(t', r') < 0$. Let us consider the $x^1(t)$ characteristic passing through the point r'. By Lemma 7.3.5, $x^1(0) = x_0^1 > 0$. So by the continuity of w, one can find some t_1 and t_1' such that $0 < t_1 < t_1' \leq t'$, with

$$w(t, x^1(t)) \begin{cases} > 0 & \forall t \in [0, t_1), \\ = 0 & \text{if} \quad t = t_1, \\ < 0 & \forall t \in (t_1, t_1']. \end{cases}$$

Then by Lemma 7.3.3, for $t \in [0, t_1]$, one has

$$w(t, x^1(t)) \geq 0 \iff \frac{c}{2}\ln\Big(\frac{c+u}{c-u}\Big) \geq \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}}\mathrm{d}\sigma > 0 \implies u > 0.$$

Hence by integrating $(7.1.9)_1$ along the $x^1(t)$ characteristic, one gets

$$\partial_1 w = \frac{A'}{A}\frac{c^2 u\sqrt{P'}}{c^2 - u\sqrt{P'}} > 0 \implies 0 = w(t_1, x^1(t_1)) > w_0(x_0^1) > 0,$$

a contradiction.

As $w \leq z$ follows directly from (7.1.7), it remains to show the upper bound for z. By the same analysis as what we did above, one sees that for $t > 0$ and $r > 0$, one has $u(t,r) > 0$. Use this fact, by integrating $(7.1.9)_2$ along the $x^2(t)$ characteristics, one has

$$z(t, x^2(t)) \leq z_0(x_0^2).$$

By Assumption 7.2.1, the case $r > 0$ is proved.

□

Theorem 7.3.1. *One has $\delta^* = \delta$.*

Proof. By Lemma 7.3.6 and Lemma 7.3.2, one has for $t \in [0, \delta^*)$,

$$|u| < \epsilon c, \quad \text{and} \quad \sqrt{P'} < \epsilon c \quad \text{uniformly.} \tag{7.3.17}$$

Now for contradiction, assume $\delta^* < \delta$. But since (7.3.17) holds up to time $t = \delta^*$, one can find some small $\sigma > 0$, such that (7.3.17) holds up to $t = \delta^* + \sigma$, this is a contradiction. □

Remark 7.3.1. *For $r > 0$, the Lipschitz solutions (z, w) are in fact C^1. Take any point $(t', r') \in (0, \delta) \times \mathbb{R}^+$. Consider the characteristics $x^1(t)$ and $x^2(t)$ passing through the point (t', r'). Next, consider the characteristic triangle Ω with vertices (t', r'), $(0, x_0^1)$ and $(0, x_0^2)$, where $x_0^1 = x^1(0)$, $x_0^2 = x^2(0)$. By Lemma 7.3.5, one has $0 < x_0^2 < x_0^1$. Then as $(w_0(r), z_0(r)) \in C^1([x_0^2, x_0^1])$, by the classical local existence and uniqueness result from* [39]*, one has a unique solution $(z(t, r), w(t, r)) \in C^1(\Omega)$.*

7.4 Density lower bound

In this section, we will show the following lower bound on the function AM. Note that AM depends on the density ρ, so the theorem below serves as a lower bound on the density.

Theorem 7.4.1. *Let $(t, r) \in \Omega(\underline{r})$ for some $\underline{r} > 0$, where $\Omega(\underline{r})$ is defined in Definition* 7.2.2*. Let $\epsilon > 0$ be the small constant in* (7.1.10)*.*

For $1 < \gamma < \frac{5}{3} - \frac{7}{6}\epsilon$, one has

$$\frac{1}{AM(t, r)^{1-\epsilon}} < \frac{1}{AM_{\min}(\underline{r})^{1-\epsilon}} + K(\underline{r})t(1-\epsilon) \sup_{0<w\le z\le \sup z_0} |k_0|, \tag{7.4.1}$$

where $AM_{\min}(\underline{r})$ and $K(\underline{r})$ are given by Assumption 7.2.2*.*

For $\frac{5}{3} + \frac{8}{3}\epsilon < \gamma < 3 - 6\epsilon$, one has

$$\frac{1}{AM(t, r)^{\frac{\gamma-1}{2}-\epsilon}} < \frac{1}{AM_{\min}(\underline{r})^{\frac{\gamma-1}{2}-\epsilon}} + N(\underline{r})t\left(\frac{\gamma-1}{2} - \epsilon\right) \sup_{0<w\le z\le \sup z_0} |k_0|, \tag{7.4.2}$$

where $N(\underline{r})$ is given by Definition 7.2.3.

7.4.1 Coupled ODEs

Lemma 7.4.1. *α and β satisfy*

$$\begin{cases} \partial_1\alpha + \dfrac{2e^{H_1}}{(e^{H_1}+1)^2}\alpha(\alpha+\beta) + \dfrac{e^{H_1}}{(e^{H_1}+1)^2}\dfrac{P''(c^2\rho+P)}{P'(c^2-P')}\alpha(\alpha-\beta) \\ = \dfrac{d}{dr}\left(\dfrac{A'}{A}\right)\dfrac{c^2u\sqrt{P'}}{c^2-u\sqrt{P'}} + \dfrac{A'}{A}\dfrac{c^2(c^2-u^2)\sqrt{P'}}{2(c^2-u\sqrt{P'})^2}(\alpha+\beta) - \dfrac{A'}{A}\dfrac{uP''c^2(c^2\rho+P)}{4P'(c^2-u\sqrt{P'})^2}(\alpha-\beta), \\ \partial_2\beta + \dfrac{2e^{H_2}}{(e^{H_2}+1)^2}\beta(\alpha+\beta) + \dfrac{e^{H_2}}{(e^{H_2}+1)^2}\dfrac{P''(c^2\rho+P)}{P'(c^2-P')}\beta(\beta-\alpha) \\ = -\dfrac{d}{dr}\left(\dfrac{A'}{A}\right)\dfrac{c^2u\sqrt{P'}}{c^2+u\sqrt{P'}} - \dfrac{A'}{A}\dfrac{c^2(c^2-u^2)\sqrt{P'}}{2(c^2+u\sqrt{P'})^2}(\alpha+\beta) - \dfrac{A'}{A}\dfrac{uP''c^2(c^2\rho+P)}{4P'(c^2+u\sqrt{P'})^2}(\alpha-\beta). \end{cases}$$

Proof. We will only prove the lemma for $\partial_2\beta$, as the proof for $\partial_1\alpha$ will be similar. Differentiating the equation $(7.1.9)_2$ with respect to r, one gets

$$\begin{aligned} &\partial_{tr}z + \partial_r\lambda_2\partial_r z + \lambda_2\partial_{rr}z = -\frac{d}{dr}\left(\frac{A'}{A}\right)\frac{c^2u\sqrt{P'}}{c^2+u\sqrt{P'}} + \frac{A'}{A}\partial_r\left(\frac{c^2u\sqrt{P'}}{c^2+u\sqrt{P'}}\right) \\ \iff &\partial_2\beta + \partial_w\lambda_2\alpha\beta + \partial_z\beta^2 = -\frac{d}{dr}\left(\frac{A'}{A}\right)\frac{c^2u\sqrt{P'}}{c^2+u\sqrt{P'}} - \frac{A'}{A}\frac{c^4\partial_r(u\sqrt{P'})}{(c^2+u\sqrt{P'})^2}. \end{aligned} \tag{7.4.3}$$

By (7.1.4), (7.1.5) and explicit calculations, one has

$$\partial_w\lambda_2 = \frac{2ce^{H_2}\partial_w H_2}{(e^{H_1}+1)^2}, \quad \partial_z\lambda_2 = \frac{2ce^{H_2}\partial_z H_2}{(e^{H_2}+1)^2}, \tag{7.4.4}$$

where

$$\partial_w H_2 = \frac{1}{c} - \frac{P''(c^2\rho+P)}{2cP'(c^2-P')}, \quad \partial_z H_1 = \frac{1}{c} + \frac{P''(c^2\rho+P)}{2cP'(c^2-P')}.$$

Furthermore, by (7.1.7) and (7.1.8), one can check that

$$\frac{c^4\partial_r(u\sqrt{P'})}{(c^2+u\sqrt{P'})^2} = \frac{c^2(c^2-u^2)\sqrt{P'}}{2(c^2+u\sqrt{P'})^2}(\alpha+\beta) + \frac{uP''c^2(c^2\rho+P)}{4P'(c^2-u\sqrt{P'})^2}(\alpha-\beta). \tag{7.4.5}$$

Substituting (7.4.4) and (7.4.5) into (7.4.3), one can obtain the desired result upon simplification.

□

Lemma 7.4.2. *$\widetilde{\alpha}$ and $\widetilde{\beta}$ satisfy*

$$\begin{cases} \partial_1\widetilde{\alpha} = a_0\widetilde{\alpha}(\widetilde{\alpha}+\widetilde{\beta}) + a_1\widetilde{\alpha}(\widetilde{\alpha}-\widetilde{\beta}) + a_2(\widetilde{\alpha}-\widetilde{\beta}) + a_3\widetilde{\alpha} + a_4, \\ \partial_2\widetilde{\beta} = b_0\widetilde{\beta}(\widetilde{\beta}+\widetilde{\alpha}) + b_1\widetilde{\beta}(\widetilde{\beta}-\widetilde{\alpha}) + b_2(\widetilde{\beta}-\widetilde{\alpha}) + b_3\widetilde{\beta} + b_4, \end{cases}$$

where a_i and b_i are defined in section 7.1.

Proof. We will only prove the lemma for $\partial_2\widetilde{\beta}$, as the proof for $\partial_1\widetilde{\alpha}$ will be similar. By Lemma 7.1.10, (7.1.10), and the fact that

$$\frac{e^{H_2}}{(e^{H_2}+1)^2} = \frac{(c^2-u^2)(c^2-P')}{4(c^2+u\sqrt{P'})^2},$$

one has

$$\begin{aligned} \partial_2\widetilde{\beta} - \partial_2\left(\frac{A'}{A}\frac{c^2\sqrt{P'}}{c^2-P'}\right) - b_0\widetilde{\beta}(\widetilde{\beta}+\widetilde{\alpha}) - b_1\widetilde{\beta}(\widetilde{\beta}-\widetilde{\alpha}) + b_1\frac{A'}{A}\frac{c^2\sqrt{P'}}{c^2-P'}(\widetilde{\beta}-\widetilde{\alpha}) + b_1\frac{A'}{A}\frac{2c^2\sqrt{P'}}{c^2-P'}\widetilde{\beta} \\ = -\frac{d}{dr}\left(\frac{A'}{A}\right)\frac{c^2u\sqrt{P'}}{c^2+u\sqrt{P'}} - \frac{A'}{A}\frac{uP''c^2(c^2\rho+P)}{4P'(c^2+u\sqrt{P'})^2}(\widetilde{\beta}-\widetilde{\alpha}) \\ + \left(\frac{A'}{A}\right)^2\frac{c^4uP''(c^2\rho+P)}{2\sqrt{P'}(c^2-P')(c^2+u\sqrt{P'})^2} - \left(\frac{A'}{A}\right)^2\frac{c^4P''(c^2\rho+P)(c^2-u^2)}{2\sqrt{P'}(c^2-P')^2(c^2+u\sqrt{P'})^2}. \end{aligned} \tag{7.4.6}$$

Now, one has

$$\begin{aligned} \partial_2\left(\frac{A'}{A}\frac{c^2\sqrt{P'}}{c^2-P'}\right) &= \lambda_2\frac{d}{dr}\left(\frac{A'}{A}\right)\frac{c^2\sqrt{P'}}{c^2-P'} + \frac{A'}{A}\partial_2\left(\frac{c^2\sqrt{P'}}{c^2-P'}\right) \\ &= \lambda_2\frac{d}{dr}\left(\frac{A'}{A}\right)\frac{c^2\sqrt{P'}}{c^2-P'} + \frac{A'}{A}\partial_\rho\left(\frac{c^2\sqrt{P'}}{c^2-P'}\right)\partial_{\frac{z-w}{2}}\rho\partial_2\left(\frac{z-w}{2}\right). \end{aligned} \tag{7.4.7}$$

By (7.1.7) and explicit calculation, one has

$$\partial_\rho\left(\frac{c^2\sqrt{P'}}{c^2-P'}\right) = \frac{c^2P''(c^2+P')}{2\sqrt{P'}(c^2-P')^2}, \quad \partial_{\frac{z-w}{2}}\rho = \frac{1}{\partial_\rho\left(\frac{z-w}{2}\right)} = \frac{c^2\rho+P}{c^2\sqrt{P'}}. \tag{7.4.8}$$

Also, by (7.1.9) and explicit calculations, one has

$$\begin{aligned}
&\partial_2(z-w)\\
=&\partial_2 z-\partial_1 w-(\lambda_2-\lambda_1)\partial_r w\\
=&\partial_2 z-\partial_1 w-(\lambda_2-\lambda_1)\left(\widetilde{\alpha}+\frac{A'}{A}\frac{c^2\sqrt{P'}}{c^2-P'}\right)\\
=&-\frac{A'}{A}\frac{2c^4\sqrt{P'}}{(c^2+u\sqrt{P'})(c^2-u\sqrt{P'})}\left[u+\frac{c^2-u^2}{c^2-P'}\sqrt{P'}\right]-\frac{2c^2\sqrt{P'}(c^2-u^2)}{(c^2+u\sqrt{P'})(c^2-u\sqrt{P'})}\widetilde{\alpha}.
\end{aligned}\tag{7.4.9}$$

Then by (7.4.7), (7.4.8) and (7.4.9), (7.4.6) can be simplified to the desired expression. □

Lemma 7.4.3.

$$\begin{cases}\partial_1(AM)=-AMk_0\widetilde{\beta},\\[2ex] \partial_2(AM)=-AMk_0\widetilde{\alpha},\end{cases}$$

where k_0 and M are defined in (7.1.10).

Proof. We will only prove this lemma for $\partial_1(AM)$. By (7.1.9) and (7.1.10), one has

$$\begin{aligned}
\partial_1(AM)&=A'\lambda_1 M+A(\partial_z M\partial_1 z+\partial_w M\partial_1 w)\\
&=A'\lambda_1 M+A(\partial_z M\partial_2 z+\partial_z M(\lambda_1-\lambda_2)\partial_r z+\partial_w M\partial_1 w)\\
&=A'\lambda_1 M-A'\left(\partial_z M\frac{c^2u\sqrt{P'}}{c^2+u\sqrt{P'}}-\partial_w M\frac{c^2u\sqrt{P'}}{c^2-u\sqrt{P'}}\right)\\
&\quad-\partial_z M(\lambda_1-\lambda_2)A'\frac{c^2\sqrt{P'}}{c^2-P'}+A\partial_z M(\lambda_1-\lambda_2)\widetilde{\beta}.
\end{aligned}\tag{7.4.10}$$

Now, by the explicit expression of M given by (7.1.10), one has

$$\partial_z M=\frac{1}{2}M\frac{c^2-P'}{c^2\sqrt{P'}},\quad \partial_w M=-\frac{1}{2}M\frac{c^2-P'}{c^2\sqrt{P'}}.$$

Hence, by (7.1.3) and explicit calculations, one can check that

$$\lambda_1 M-\left(\partial_z M\frac{c^2u\sqrt{P'}}{c^2+u\sqrt{P'}}-\partial_w M\frac{c^2u\sqrt{P'}}{c^2-u\sqrt{P'}}\right)-\partial_z M(\lambda_1-\lambda_2)\frac{c^2\sqrt{P'}}{c^2-P'}=0.$$

By the above and (7.1.3), (7.4.10) simplifies to the desired result. □

Lemma 7.4.4. *$\widehat{\alpha}$ and $\widehat{\beta}$ satisfy*

$$\begin{cases} \partial_2\widehat{\alpha} = -(AM)^{1-\epsilon}\widehat{\alpha}\left[(-a_1-a_0)\widehat{\alpha} - (-a_1+a_0+(1-\epsilon)k_0)\,\widehat{\beta}\right] \\ \qquad\qquad +a_2(\widehat{\alpha}-\widehat{\beta}) + a_3\widehat{\alpha} + \dfrac{a_4}{(AM)^{1-\epsilon}}, \\ \partial_2\widehat{\beta} = -(AM)^{1-\epsilon}\widehat{\beta}\left[(-b_1-b_0)\widehat{\beta} - (-b_1+b_0+(1-\epsilon)k_0)\,\widehat{\alpha}\right] \\ \qquad\qquad +b_2(\widehat{\beta}-\widehat{\alpha}) + b_3\widehat{\beta} + \dfrac{b_4}{(AM)^{1-\epsilon}}. \end{cases} \tag{7.4.11}$$

Proof. The result follows directly from Lemma 7.4.2 and Lemma 7.4.3. □

Lemma 7.4.5. *$\overline{\alpha}$ and $\overline{\beta}$ satisfy*

$$\begin{cases} \partial_2\overline{\alpha} = -(AM)^{\frac{\gamma-1}{2}-\epsilon}\overline{\alpha}\left[\left(-a_1-a_0+\dfrac{\gamma-3}{2}k_0\right)\overline{\alpha} - \left(-a_1+a_0+\left(\dfrac{\gamma-1}{2}-\epsilon\right)k_0\right)\overline{\beta}\right] \\ \qquad\qquad +\dfrac{\gamma-3}{2}k_0(AM)^{\frac{\gamma-1}{2}-\epsilon}\overline{\alpha}^2 + a_2(\overline{\alpha}-\overline{\beta}) + a_3\overline{\alpha} + \dfrac{a_4}{(AM)^{\frac{\gamma-1}{2}-\epsilon}}, \\ \partial_2\overline{\beta} = -(AM)^{\frac{\gamma-1}{2}-\epsilon}\overline{\beta}\left[\left(-b_1-b_0+\dfrac{\gamma-3}{2}k_0\right)\overline{\beta} - \left(-b_1+b_0+\left(\dfrac{\gamma-1}{2}-\epsilon\right)k_0\right)\overline{\alpha}\right] \\ \qquad\qquad +\dfrac{\gamma-3}{2}k_0(AM)^{\frac{\gamma-1}{2}-\epsilon}\overline{\beta}^2 + b_2(\overline{\beta}-\overline{\alpha}) + b_3\overline{\beta} + \dfrac{b_4}{(AM)^{\frac{\gamma-1}{2}-\epsilon}}. \end{cases} \tag{7.4.12}$$

Proof. The result follows directly from Lemma 7.4.2 and Lemma 7.4.3. □

7.4.2 Upper bound for gradient quantities $\widehat{\alpha}$, $\widehat{\beta}$, $\overline{\alpha}$ and $\overline{\beta}$

Lemma 7.4.6. *For $1 < \gamma < 3 - 6\epsilon$, one has*

$$-a_1 - a_0 > -a_1 + a_0 + (1-\epsilon)k_0 > 0,$$

$$-b_1 - b_0 > -b_1 + b_0 + (1-\epsilon)k_0 > 0,$$

$$a_2 < 0, \quad a_3 < 0, \quad a_4 < 0, \quad b_3 < 0, \quad b_4 < 0.$$

Furthermore, for $1 < \gamma < \frac{5}{3} - \frac{7}{6}\epsilon$, *one has* $b_2 < 0$. *For* $\frac{5}{3} + \frac{8}{3}\epsilon < \gamma < 3 - 6\epsilon$, *one has*

$$-a_1 - a_0 + \frac{\gamma - 3}{2} k_0 > -a_1 + a_0 + \left(\frac{\gamma - 1}{2} - \epsilon \right) k_0 > 0,$$
$$-b_1 - b_0 + \frac{\gamma - 3}{2} k_0 > -b_1 + b_0 + \left(\frac{\gamma - 1}{2} - \epsilon \right) k_0 > 0.$$

Proof. By Theorem 7.2.1 and Lemma 7.3.4, one immediately has for $1 < \gamma < 3 - 6\epsilon$, $a_2 < 0$, $a_3 < 0$, $a_4 < 0$, $b_3 < 0$ and $b_4 < 0$. Next, we will show $-a_1 - a_0 > -a_1 + a_0 + (1 - \epsilon)k_0 > 0$. By (7.1.4) and (7.1.10), one has

$$\begin{aligned} -a_1 - a_0 > -a_1 + a_0 + (1 - \epsilon)\, k_0 &\iff -2a_0 > (1 - \epsilon)k_0 \\ &\iff (c^2 + u\sqrt{P'}) > (1 - \epsilon)(c^2 - u\sqrt{P'}), \end{aligned}$$

which holds true for any $0 < \epsilon < 1$. And

$$\begin{aligned} & - a_1 + a_0 + (1 - \epsilon)\, k_0 > 0 \\ \iff & \frac{\gamma - 1}{4}(c^2 + u\sqrt{P'})(c^2 + \frac{\sqrt{P'}}{\rho}) - \frac{1}{2}(c^2 - P')(c^2 + u\sqrt{P'}) + (1 - \epsilon)(c^2 - P')(c^2 - u\sqrt{P'}) > 0 \\ \iff & \frac{\gamma - 1}{4} + \frac{1}{2} + \mathcal{O}(\epsilon) > 0, \end{aligned}$$

which holds true for $\epsilon > 0$ small. The proof for $-b_1 - b_0 > -b_1 + b_0 + (1 - \epsilon)k_0 > 0$ is similar, here we ignore it.

Next, we show $b_2 < 0$ if $1 < \gamma < \frac{5}{3} - \frac{7}{6}\epsilon$. By explicit calculations, one has

$$b_2 < 0 \iff \sqrt{P'} \left(\frac{2(c^2 + P')(c^2 + u\sqrt{P'})}{(c^2 - P')(c^2 - u\sqrt{P'})} + 1 \right) < \frac{c^2 - P'}{c + u} \frac{u}{c - u}. \tag{7.4.13}$$

From Theorem 7.2.1, one has $w \geq 0$, from this one can conclude that

$$\frac{2u}{c - u} \geq \exp\left(\frac{4\sqrt{\gamma}}{\gamma - 1} \tan^{-1}\left(\frac{\sqrt{P'}}{c\sqrt{\gamma}} \right) - 1 \right).$$

By this, (7.4.13), Theorem 7.2.1 and Lemma 7.3.4, one has $b_2 < 0$ if one can show that

$$\begin{aligned}&\frac{2(c^2+P')(c^2+u\sqrt{P'})}{(c^2-P')(c^2-u\sqrt{P'})}+1<\frac{c^2-P'}{c(c+u)(\gamma-1)}(2-3\epsilon)\\ \iff &6\left(1+\frac{u}{c}\right)-\frac{2}{\gamma-1}(2-3\epsilon)+\mathcal{O}(\epsilon^2)<0.\end{aligned}$$

Now from Theorem 7.2.1, one has $0 \le u < \epsilon c$, hence the above inequality holds if

$$6-\frac{2}{\gamma-1}(2-3\epsilon)+\mathcal{O}(\epsilon^2)<0 \iff 6\gamma-10+6\epsilon+\mathcal{O}(\epsilon^2)<0.$$

Given $\epsilon > 0$ is small, an explicit calculation yields that the above inequality holds if $1 < \gamma < \frac{5}{3} - \frac{7}{6}\epsilon$. Hence we showed $b_2 < 0$.

Next we wish to show $-a_1 - a_0 + \frac{\gamma-3}{2}k_0 > -a_1 + a_0 + \left(\frac{\gamma-1}{2} - \epsilon\right)k_0 > 0$ if $\frac{5}{3} + \frac{8}{3}\epsilon < \gamma < 3 - 6\epsilon$. By Theorem 7.2.1 and direct calculations, one has

$$-a_1-a_0+\frac{\gamma-3}{2}k_0>-a_1+a_0+\left(\frac{\gamma-1}{2}-\epsilon\right)k_0 \iff \epsilon+\mathcal{O}(\epsilon^2)>0,$$

which holds if $\epsilon > 0$ is small. Furthermore, one has

$$-a_1+a_0+\left(\frac{\gamma-1}{2}-\epsilon\right)k_0>0 \iff \frac{3\gamma-5}{4}-\epsilon+\mathcal{O}(\epsilon^2)>0,$$

which holds if $\epsilon > 0$ is small and $\frac{5}{3} + \frac{8}{3}\epsilon < \gamma < 3 - 6\epsilon$. The proof for $-b_1 - b_0 + \frac{\gamma-3}{2}k_0 > -b_1 + b_0 + \left(\frac{\gamma-1}{2} - \epsilon\right)k_0 > 0$ is similar, here we ignore it. □

For simplicity, for the rest of the chapter, we denote $D \ge 1$ a generic constant depending only on (z_0, w_0), k, γ and c, and independent of the time and r.

Definition 7.4.1. *Let f and g be two functions. We say $f \approx g$ if $D^{-1}f \le g \le Df$; and we say $f \lesssim g$ if $f \le Dg$.*

Lemma 7.4.7.

$$M(\rho) \approx \rho.$$

Proof. One has

$$\begin{aligned}\int_1^\rho \frac{c^2 - P'(\sigma)}{c^2\sigma + P(\sigma)} d\sigma &= \int_1^\rho \frac{1}{\sigma} d\sigma + \int_1^\rho \frac{-\frac{P}{\sigma} - P'}{c^2\sigma + P} d\sigma \\ &= \ln(\rho) - \frac{\gamma - 1}{\gamma} \int_1^\rho \frac{P'}{c^2\sigma + P} d\sigma,\end{aligned}$$

where

$$\left| \int_1^\rho \frac{P'}{c^2\sigma + P} d\sigma \right| \approx 1.$$

By the definition of $M(\rho)$ given by (7.1.10), the lemma is proved. □

Lemma 7.4.8. *Let $1 < \gamma < \frac{5}{3} - \frac{7}{6}\epsilon$ and $(t, r) \in \Omega(\underline{r})$ for some $\underline{r} > 0$, where $\Omega(\underline{r})$ is defined in Definition 7.2.2. Then*

$$\max\{\widehat{\alpha}(t, r), \widehat{\beta}(t, r)\} < K(\underline{r}),$$

where $K(\underline{r}) > 0$ is the constant in Assumption 7.2.2.

Proof. We will prove the lemma by contradiction. Fix some $\underline{r} > 0$. Let t_1 be the first time such that there exists a r_1 where $(t_1, r_1) \in \Omega(\underline{r})$ and

$$\max\{\widehat{\alpha}(t_1, r_1), \widehat{\beta}(t_1, r_1)\} = K(\underline{r}).$$

Without any loss of generality assume $\widehat{\beta}(t_1, r_1) = K(\underline{r})$ and $\widehat{\alpha}(t_1, r_1) \leq K(\underline{r})$. Let us consider the characteristic $(t, x^2(t))$ passing through the point (t_1, r_1). By Lemma 7.4.4 and Lemma 7.4.6, for $t \in [0, t_1]$ and along the x^2 characteristic, one has

$$\partial_2 \widehat{\beta} < 0.$$

Integrating the above inequality for $t \in [0, t_1]$, one has

$$K(\underline{r}) = \widehat{\beta}(t_1, x^2(t_1)) < \widehat{\beta}(0, x^2(0)).$$

Now by Proposition 7.3.1, one has $\underline{r} \leq x^2(0)$. Hence by Assumption 7.2.2, the above equation becomes

$$K(\underline{r}) = \widehat{\beta}(t_1, x^2(t_1)) < \widehat{\beta}(0, x^2(0)) < K(\underline{r}).$$

This is a contradiction. □

Lemma 7.4.9. *Let* $\frac{5}{3}+\frac{8}{3}\epsilon<\gamma<3-6\epsilon$ *and* $(t,r)\in\Omega(\underline{r})$ *for some* $\underline{r}>0$*, where* $\Omega(\underline{r})$ *is defined in Definition* 7.2.2*. Then*

$$\max\{\overline{\alpha}(t,r),\overline{\beta}(t,r)\}<N(\underline{r}),$$

where $N(\underline{r})>0$ *is the constant in Definition* 7.2.3*.*

Proof. We will prove the lemma by contradiction. We will denote $N(\underline{r})$ as N for the proof. Fix some $\underline{r}>0$. Let t_1 be the first time such that there exists a r_1 where $(t_1,r_1)\in\Omega(\underline{r})$ and

$$\max\{\overline{\alpha}(t_1,r_1),\overline{\beta}(t_1,r_1)\}=N.$$

Without any loss of generality assume $\overline{\beta}(t_1,r_1)=N$ and $\overline{\alpha}(t_1,r_1)\leq N$. We will now consider two cases: the case $-\frac{N}{2}<\overline{\alpha}(t_1,r_1)\leq N$ and the case $\overline{\alpha}(t_1,r_1)\leq-\frac{N}{2}$.

Case 1: $-\frac{N}{2}<\overline{\alpha}(t_1,r_1)<N$. Consider the x^2 characteristic passing the point (t_1,r_1). By continuity, one can find t_2 with $t_2<t_1$, such that for $l\in(t_2,t_1)$, one has

$$|\overline{\alpha}(l,x^2(l))|<N,\quad \frac{N}{2}<\overline{\beta}(l,x^2(l))<N. \tag{7.4.14}$$

For $\overline{\alpha}$ and $\overline{\beta}$ satisfying (7.4.14), by explicit calculations, one has

$$\left(\frac{\gamma-3}{2}k_0(AM)^{\frac{\gamma-1}{2}-\epsilon}\overline{\beta}^2+b_2(\overline{\beta}-\overline{\alpha})\right)(l,x^2(l))<0 \tag{7.4.15}$$

if

$$\left(\frac{N}{2}\right)^2-\frac{2}{3-\gamma}\frac{|b_2|}{k_0(AM)^{\frac{\gamma-1}{2}-\epsilon}}\frac{3N}{2}>0\iff N>\frac{12}{3-\gamma}\frac{|b_2|}{k_0(AM)^{\frac{\gamma-1}{2}-\epsilon}}. \tag{7.4.16}$$

By Lemma 7.3.4, Theorem 7.2.1, Lemma 7.4.7 and explicit calculations, one has

$$|b_2|\approx\frac{\sqrt{P'}}{r},\quad k_0\approx1,\quad (AM)^{\frac{\gamma-1}{2}-\epsilon}\approx r^{\frac{(d-1)(r-1)}{2}-\epsilon(d-1)}\sqrt{P'}\rho^{-\epsilon}. \tag{7.4.17}$$

Furthermore, by Proposition 7.3.1, one has $\underline{r}>x^2(l)$. Hence by the definition of N from Definition 7.2.3, (7.4.16) holds, hence (7.4.15) also holds. Now, by Lemma 7.4.5, Lemma 7.4.6

and (7.4.15), one has

$$\partial_2 \overline{\beta} < 0.$$

Integrating the above over $l \in [t_2, t_1]$, one gets

$$N = \overline{\beta}(t_1, r_1) < \overline{\beta}(t_2, r_2) < N,$$

this is a contradiction.

Case 2: $\widehat{\alpha}(t_1, r_1) \leq -\frac{N}{2}$. Let us consider the x^2 characteristic passing the point (t_1, r_1). By continuity, one can find t_2 with $t_2 < t_1$, such that for $l \in (t_2, t_1)$, one has

$$\widehat{\alpha}(l, x^2(l)) < -\frac{N}{4}, \quad \frac{N}{2} < \widehat{\beta}(l, x^2(l)) < N. \tag{7.4.18}$$

For $\widehat{\alpha}$ and $\widehat{\beta}$ satisfying (7.4.18), by explicit calculation, one has

$$(AM)^{\frac{\gamma-1}{2}-\epsilon}\overline{\beta}\overline{\alpha}\left(-b_1 + b_0 + \left(\frac{\gamma-1}{2} - \epsilon\right)k_0\right) - b_2\overline{\alpha} < 0 \tag{7.4.19}$$

if

$$\begin{aligned} &-\frac{N^2}{8}(AM)^{\frac{\gamma-1}{2}-\epsilon}\left(-b_1 + b_0 + \left(\frac{\gamma-1}{2} - \epsilon\right)k_0\right) + \frac{|b_2|}{4}N < 0 \\ \iff & N > \frac{2}{(AM)^{\frac{\gamma-1}{2}-\epsilon}} \frac{|b_2|}{-b_1 + b_0 + \left(\frac{\gamma-1}{2} - \epsilon\right)k_0}. \end{aligned} \tag{7.4.20}$$

Moreover, one has

$$\begin{aligned} &-(AM)^{\frac{\gamma-1}{2}-\epsilon}\overline{\beta}^2\left(-b_1 + b_0 + \frac{\gamma-3}{2}k_0\right) - \frac{3-\gamma}{2}k_0(AM)^{\frac{\gamma-1}{2}-\epsilon}\overline{\beta}^2 + b_2\overline{\beta} < 0 \\ \iff & -(AM)^{\frac{\gamma-1}{2}-\epsilon}(-b_1 + b_0)\overline{\beta}^2 + b_2\overline{\beta} < 0, \end{aligned} \tag{7.4.21}$$

if

$$-(AM)^{\frac{\gamma-1}{2}-\epsilon}(-b_1 + b_0)N^2 + |b_2|N < 0 \iff N > \frac{N}{(AM)^{\frac{\gamma-1}{2}-\epsilon}(-b_1 + b_0)}. \tag{7.4.22}$$

By Lemma 7.3.4, Theorem 7.2.1, Lemma 7.4.7 and explicit calculations, one has

$$-b_1 + b_0 + \left(\frac{\gamma-1}{2} - \epsilon\right)k_0 \approx 1, \quad -b_1 + b_0 \approx 1,$$

which combined with (7.4.17) and the fact that $\underline{r} > x^2(l)$, one sees that (7.4.20) and (7.4.22) hold, hence (7.4.19) and (7.4.21) hold. Now, by Lemma 7.4.5, Lemma 7.4.6, (7.4.19) and (7.4.21), one has

$$\partial_2 \overline{\beta} < 0.$$

Integrating the above over $l \in [t_2, t_1]$, one gets

$$N = \overline{\beta}(t_1, r_1) < \overline{\beta}(t_2, r_2) < N,$$

this is a contradiction. Hence the lemma is proved. □

7.4.3 Time-dependent lower bound on the density: Proof of Theorem 7.4.1

Take any $\underline{r} > 0$ and any $(t, r) \in \Omega(\underline{r})$. We will consider the x^2 characteristic passing through the point (t, r). Note by Proposition 7.3.1, one has $\underline{r} < x^2(0)$. By Lemma 7.4.3, one has

$$\partial_2(AM) = -AMk_0\widetilde{\alpha}.$$

For $1 < \gamma < \frac{5}{3} - \frac{7}{6}\epsilon$, one has

$$\frac{1}{(AM)^{2-\epsilon}}\partial_2(AM) = -k_0\widehat{\alpha} \iff \partial_2\left(\frac{1}{(AM)^{1-\epsilon}}\right) = k_0\widehat{\alpha}(1-\epsilon).$$

Note that $k_0 > 0$ by Theorem 7.2.1. Integrating the above and by Lemma 7.4.8, one has

$$\begin{aligned}\frac{1}{(AM)^{1-\epsilon}(t,r)} &< \frac{1}{(AM)^{1-\epsilon}(0, x^2(0))} + K(\underline{r})\sup k_0(1-\epsilon) \\ &< \frac{1}{AM_{\min}(\underline{r})^{1-\epsilon}} + K(\underline{r})\sup k_0(1-\epsilon).\end{aligned}$$

For $\frac{5}{3} + \frac{8}{3}\epsilon < \gamma < 3 - 6\epsilon$, one has

$$\frac{1}{(AM)^{\frac{\gamma-1}{2}+1-\epsilon}}\partial_2(AM) = -k_0\overline{\alpha} \iff \partial_2\left(\frac{1}{(AM)^{\frac{\gamma-1}{2}-\epsilon}}\right) = k_0\overline{\alpha}\left(\frac{\gamma-1}{2} - \epsilon\right).$$

Integrating the above and by Lemma 7.4.9, one has

$$\begin{aligned}\frac{1}{(AM)^{\frac{\gamma-1}{2}-\epsilon}(t,r)} &< \frac{1}{(AM)^{\frac{\gamma-1}{2}-\epsilon}(0,x^2(0))} + N(\underline{r})\sup k_0\left(\frac{\gamma-1}{2}-\epsilon\right)\\ &< \frac{1}{AM_{\min}(\underline{r})^{\frac{\gamma-1}{2}-\epsilon}} + N(\underline{r})\sup k_0\left(\frac{\gamma-1}{2}-\epsilon\right).\end{aligned}$$

Hence the theorem is proved.

7.5 Decoupled ODEs

Theorem 7.5.1. *$\widetilde{Y}$ and $\widetilde{Q}$ satisfy*

$$\begin{cases}\partial_1\widetilde{Y} = d_0\widetilde{Y}^2 + d_1\widetilde{Y} + d_2,\\ \partial_2\widetilde{Q} = \overline{d}_0\widetilde{Q}^2 + \overline{d}_1\widetilde{Q} + \overline{d}_2,\end{cases}$$

where d_i and $\overline{d}_i$ are defined in section 7.1.

Proof. We will only prove the theorem for $\widetilde{Y}$, as the proof for $\widetilde{Q}$ will be the same. Differentiating $(7.1.9)_1$ with respect to r, and recall the definition of α and F given by (7.1.10), one gets

$$\partial_1\alpha + \partial_w\lambda_1\alpha^2 + \partial_z\lambda_1\alpha\beta = \frac{d}{dr}\left(\frac{A'}{A}\right)F + \frac{A'}{A}\left(\partial_z F\beta + \partial_w F\alpha\right). \tag{7.5.1}$$

Now from $(7.1.9)_2$, one can deduce

$$\beta = -\frac{A'}{A}\frac{G}{\lambda_2-\lambda_1} - \frac{\partial_1 z}{\lambda_2-\lambda_1},$$

hence (7.5.1) becomes

$$\begin{aligned}&\partial_1\alpha + \partial_w\lambda_1\alpha^2 - \frac{A'}{A}\frac{\partial_z\lambda_1 G}{\lambda_2-\lambda_1} - \frac{\partial_z\lambda_1\alpha}{\lambda_2-\lambda_1}\partial_1 z\\ =&\frac{d}{dr}\left(\frac{A'}{A}\right)F - \left(\frac{A'}{A}\right)^2\frac{\partial_z FG}{\lambda_2-\lambda_1} - \frac{A'}{A}\frac{\partial_z F\partial_1 z}{\lambda_2-\lambda_1} - \frac{A'}{A}\partial_z F\alpha.\end{aligned} \tag{7.5.2}$$

By explicit calculations, one can verify that the function h_1 defined in (7.1.12) satisfies

$$\partial_z h_1 = \frac{\partial_z \lambda_1}{\lambda_1 - \lambda_2},$$

hence by (7.1.9), one has

$$\partial_1 h_1 = \partial_z h_1 \partial_1 z + \partial_w h_1 \partial_1 w = \frac{\partial_z \lambda_1}{\lambda_1 - \lambda_2} \partial_1 z + \partial_w h_1 \frac{A'}{A} F.$$

Substituting the above to (7.5.2), one obtains

$$\begin{aligned} &\partial_1 \alpha + \partial_w \lambda_1 \alpha^2 + \alpha \partial_1 h_1 + \frac{A'}{A} \left(\partial_w F - \frac{\partial_z \lambda_1 G}{\lambda_2 - \lambda_1} - \partial_w h_1 F \right) \alpha \\ =&\frac{d}{dr} \left(\frac{A'}{A} \right) F - \left(\frac{A'}{A} \right)^2 \frac{\partial_z F G}{\lambda_2 - \lambda_1} - \frac{A'}{A} \frac{\partial_z F \partial_1 z}{\lambda_2 - \lambda_1}. \end{aligned}$$

Multiplying the above by e^{h_1} and simplify, one gets

$$\begin{aligned} &\partial_1 (e^{h_1} \alpha) + \partial_w \lambda_1 e^{h_1} \alpha^2 + \frac{A'}{A} \left(\partial_w F - \frac{\partial_z \lambda_1 G}{\lambda_2 - \lambda_1} - \partial_w h_1 F \right) e^{h_1} \alpha \\ =&\frac{d}{dr} \left(\frac{A'}{A} \right) F e^{h_1} - \left(\frac{A'}{A} \right)^2 \frac{\partial_z F G}{\lambda_2 - \lambda_1} e^{h_1} - \frac{A'}{A} \frac{\partial_z F \partial_1 z}{\lambda_2 - \lambda_1} e^{h_1}. \end{aligned} \tag{7.5.3}$$

By explicit calculations, one can verify that the function Φ defined in (7.1.12) satisfies

$$\partial_z \Phi = -\frac{\partial_z F e^{h_1}}{\lambda_2 - \lambda_1},$$

hence one can obtain

$$\partial_1 \Phi = -\frac{\partial_z F e^{h_1}}{\lambda_2 - \lambda_1} \partial_1 z + \partial_w \Phi \frac{A'}{A} F \implies -\frac{\partial_z F e^{h_1}}{\lambda_2 - \lambda_1} \partial_1 z = \partial_1 \Phi - \partial_w \Phi \frac{A'}{A} F.$$

Substituting the above into (7.5.3) and simplify, one obtains

$$\begin{aligned} \partial_1 \left(e^{h_1} \alpha - \frac{A'}{A} \Phi \right) =& - \partial_w \lambda_1 e^{h_1} \alpha^2 - \frac{A'}{A} \left(\partial_w F - \frac{\partial_z \lambda_1 G}{\lambda_2 - \lambda_1} - \partial_w h_1 F \right) e^{h_1} \alpha \\ &+ \frac{d}{dr} \left(\frac{A'}{A} \right) \left(F e^{h_1} - \lambda_1 \Phi \right) - \left(\frac{A'}{A} \right)^2 \left(\frac{\partial_z F G}{\lambda_2 - \lambda_1} e^{h_1} + \partial_w \Phi F \right). \end{aligned}$$

Recalling the definition of Y given in (7.1.12), one can further simplify the above equation into

$$\begin{aligned}\partial_1 Y =& -\partial_w\lambda_1 e^{-h_1}Y^2 - \frac{A'}{A}\left(\partial_w F - \frac{\partial_z\lambda_1 G}{\lambda_2-\lambda_1} - \partial_w h_1 F + 2\Phi\partial_w\lambda_1 e^{-h_1}\right)Y \\ &- \left(\frac{A'}{A}\right)^2\left[\partial_w\lambda_1 e^{-h_1}\Phi^2 + \left(\partial_w F - \frac{\partial_z\lambda_1 G}{\lambda_2-\lambda_1} - \partial_w h_1 F\right)\Phi + \frac{\partial_z FG}{\lambda_2-\lambda_1}e^{h_1} + \partial_w\Phi F\right] \\ &+ \frac{d}{dr}\left(\frac{A'}{A}\right)\left(Fe^{h_1} - \lambda_1\Phi\right).\end{aligned} \tag{7.5.4}$$

Lastly, by recalling the definition of $\widetilde{Y}$ given by (7.1.13), one has

$$\partial_1\widetilde{Y} = A^{\frac{\gamma-3}{4}}\partial_1 Y + \frac{\gamma-3}{4}A^{\frac{\gamma-7}{4}}A'\lambda_1 Y.$$

By using (7.5.4), the above equation can be simplified to the desired result. □

7.6 ODE comparison

Theorem 7.6.1. *Let $1<\gamma<\frac{5}{3}-\frac{7}{6}\epsilon$ or $\frac{5}{3}+\frac{8}{3}\epsilon<\gamma<3-6\epsilon$. Let $\underline{r}>0$ and $(t,r)\in\Omega(\underline{r})$, where $\Omega(\underline{r})$ is defined in Definition 7.2.2. Then the quantities $\widetilde{Y}$ and $\widetilde{Q}$ given by (7.1.13) satisfy the following ODE inequality:*

$$\begin{cases}\partial_1\widetilde{Y} \le \frac{1}{2}d_2 Y^2 + \frac{1}{4}d_2\left(Y^2 - G_T(\underline{r})^2\right), \\ \partial_2\widetilde{Q} \le \frac{1}{2}\overline{d}_2 Q^2 + \frac{1}{4}\overline{d}_2\left(Q^2 - G_T(\underline{r})^2\right),\end{cases} \tag{7.6.1}$$

where the coefficients $\overline{d}_2$ and d_2 are given in section 7.1, and $G_T(\underline{r})$ is given by Definition 7.2.4.

In order to prove this theorem, we will firstly introduce some lemmas, and we will use the notations $\approx$ and $\lesssim$ defined in Definition 7.4.1.

Also as a reminder, we denote by $D\ge 1$ a generic constant depending only on (z_0,w_0), k, γ and c, and independent of the time and r.

Lemma 7.6.1.

$$e^{-h_1}\approx e^{-h_2}\approx\frac{\sqrt{\rho}}{(P')^{\frac{1}{4}}}. \tag{7.6.2}$$

Proof. We will focus only on e^{-h_1}, and e^{-h_2} can be dealt with similarly. According to (7.1.12),

one has

$$e^{-h_1} = \frac{e^{H_1}+1}{\sqrt{e^{H_1+H_2}-e^{2H_2}}}e^J \quad \text{with} \quad J \coloneqq \int_{\widetilde{\epsilon}}^{\frac{z-w}{2}} \frac{(c+\sqrt{P'})^2}{2c^2\sqrt{P'}}\mathrm{d}\left(\frac{z-w}{2}\right).$$

The rest of the proof is divided into 2 steps: estimates on e^J and $\frac{e^{H_1}+1}{\sqrt{e^{H_1+H_2}-e^{2H_2}}}$.

Step 1: the estimate on e^J. By (7.1.11) and (7.1.7), under a change of integration variable, one has

$$J = \int_1^{\rho} \frac{(c+\sqrt{P'})^2}{2(c^2\rho+P)}d\rho. \tag{7.6.3}$$

On the one hand, for the upper bound estimate, according to Theorem 7.2.1, one has

$$\frac{(c+\sqrt{P'})^2}{2(c^2\rho+P)} \leq \frac{1}{2\rho} + D(\rho^{\gamma-2}+\rho^{\frac{\gamma-3}{2}}). \tag{7.6.4}$$

On the other hand, for the lower bound estimate, one claims that there exists a constant $A>0$, such that for all $\rho>0$, one has

$$\frac{(c+\sqrt{\partial_\rho P})^2}{2(c^2\rho+P)} \geq \frac{1}{2\rho}. \tag{7.6.5}$$

Then it follows from (7.6.4)-(7.6.5) that

$$\frac{1}{2}\log(\rho) \leq J \leq \frac{1}{2}\log(\rho) + D(\rho^{\gamma-1}+\rho^{\frac{\gamma-1}{2}}+1),$$

which yields that

$$e^J \approx \sqrt{\rho}. \tag{7.6.6}$$

Step 2: the estimate on $\frac{e^{H_1}+1}{\sqrt{e^{H_1+H_2}-e^{2H_2}}}$. First of all, by (7.1.4) and (7.1.12), one has

$$e^{H_1} = e^{\frac{w+z}{2}}\left(\frac{c-\sqrt{P'}}{c+\sqrt{P'}}\right). \tag{7.6.7}$$

According to Theorem 7.2.1, $e^{H_1}>0$ has uniform upper and lower bounds.

Second, by (7.1.4) and (7.1.12) and Theorem 7.2.1, one has

$$e^{H_1} - e^{H_2} = \left(\frac{c+u}{c-u}\right)\left(\frac{4c\sqrt{P'}}{c^2+P'}\right) \approx \sqrt{P'}, \tag{7.6.8}$$

which implies that

$$\frac{e^{H_1}+1}{\sqrt{e^{H_1+H_2} - e^{2H_2}}}. \tag{7.6.9}$$

Hence the proof of this lemma is complete. □

Lemma 7.6.2.

$$d_0 = -A^{\frac{3-\gamma}{4}} e^{-h_1}\partial_w\lambda_1 < 0 \quad \textit{and} \quad \overline{d}_0 = -A^{\frac{3-\gamma}{4}} e^{-h_2}\partial_z\lambda_2 < 0.$$

Proof. By explicit calculation, one has

$$\begin{aligned}
\partial_w\lambda_1 &= \frac{2}{\left(e^{H_1}+1\right)^2}\left(\frac{1}{c} + \frac{cP''(c^2\rho+P)}{2(c^2-\partial_\rho P)P'}\right) > 0,\\
\partial_z\lambda_2 &= \frac{2}{\left(e^{H_2}+1\right)^2}\left(\frac{1}{c} + \frac{cP''(c^2\rho+P)}{2(c^2-\partial_\rho P)P'}\right) > 0.
\end{aligned}$$

Combining this with the fact that e^{-h_1}, e^{-h_2} and $A^{\frac{3-\gamma}{4}} > 0$, the lemma is proved. □

Lemma 7.6.3.

$$\left|\frac{d_1}{d_0}\right| \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\rho^{\frac{\gamma-3}{4}} \quad \textit{and} \quad \left|\frac{\overline{d}_1}{\overline{d}_0}\right| \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\rho^{\frac{\gamma-3}{4}}.$$

Proof. We will only prove the lemma for $\left|\frac{d_1}{d_0}\right|$, as the proof for $\left|\frac{\overline{d}_1}{\overline{d}_0}\right|$ is similar. One has

$$\left|\frac{d_1}{d_0}\right| \le \sum_{i=1}^{5} |d^i|,$$

where d^i is the i-th term in $\left|\frac{d_1}{d_0}\right|$. To prove the lemma, one needs to bound each of the $|d^i|$.

First of all, one has

$$|d^1| = \left|\frac{A'}{A}A^{\frac{\gamma-3}{4}}\Phi\right| \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\left|\int_{w+2\tilde{\epsilon}}^{z}\frac{|\partial_z F|e^{h_1}}{|\lambda_2-\lambda_1|}dz\right|.$$

Now by explicit calculations and Theorem 7.2.1, one has

$$|\partial_z F| = \frac{c^2}{2(c^2 - u\sqrt{P'})^2}\left|\sqrt{P'}(c^2-u^2) + \frac{uP''(c^2\rho+P)}{2P'}\right| \lesssim 1. \tag{7.6.10}$$

Moreover, one can check that

$$|\lambda_2 - \lambda_1| \approx \sqrt{P'}.$$

Combining with Lemma 7.6.2 and (7.1.11), one gets

$$\begin{aligned}|d^1| \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\left|\int_{w+2\widetilde{\epsilon}}^{z}\frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}}dz\right| &\approx \frac{1}{r}A^{\frac{\gamma-3}{4}}\left|\int_{\widetilde{\epsilon}}^{\frac{z-w}{2}}\frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}}d\left(\frac{z-w}{2}\right)\right| \\ &\lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\left|\int_1^{\rho}\rho^{\frac{\gamma-3}{4}}d\rho\right| \\ &\lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}.\end{aligned}$$

Now by Theorem 7.2.1 and $1<\gamma<3$, one has

$$|d^1| \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\rho^{\frac{\gamma-3}{4}}\rho^{\frac{3-\gamma}{4}} \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\rho^{\frac{\gamma-3}{4}}.$$

Next, by Lemma 7.6.1, one has

$$|d^2| = \left|\frac{A'}{A}A^{\frac{\gamma-3}{4}}\frac{\partial_w F}{\partial_w\lambda_1 e^{-h_1}}\right| \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\rho^{\frac{\gamma-3}{4}}\left|\frac{\partial_w F}{\partial_w\lambda_1}\right|.$$

By explicit calculations and Theorem 7.2.1, one has

$$|\partial_w F| = \frac{c^2}{2(c^2 - u\sqrt{P'})^2}\left|\sqrt{P'}(c^2-u^2) - \frac{uP''(c^2\rho+P)}{2P'}\right| \lesssim 1, \tag{7.6.11}$$

also one can check that $\partial_w\lambda_1 \approx 1$. Hence one obtains

$$|d^2| \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\rho^{\frac{\gamma-3}{4}}.$$

Next, by (7.1.10) and Theorem 7.2.1, one can obtain

$$|d^3| = \left|\frac{A'}{A}A^{\frac{\gamma-3}{4}}\frac{1}{\partial_w\lambda_1 e^{-h_1}}\frac{\partial_z\lambda_1 G}{\lambda_2-\lambda_1}\right| \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\rho^{\frac{\gamma-3}{4}}u \lesssim \frac{1}{r}A^{\frac{\gamma-3}{4}}\rho^{\frac{\gamma-3}{4}}.$$

Next, by Theorem 7.2.1 and Lemma 7.6.1, one has

$$|d^4| = \left| \frac{A'}{A} A^{\frac{\gamma-3}{4}} \frac{1}{\partial_w \lambda_1 e^{-h_1}} \frac{\partial_z \lambda_1 G}{\lambda_2 - \lambda_1} \right| \lesssim \frac{1}{r} A^{\frac{\gamma-3}{4}} \rho^{\frac{\gamma-3}{4}} u \sqrt{P'} |\partial_w h_1| \lesssim \frac{1}{r} A^{\frac{\gamma-3}{4}} \rho^{\frac{\gamma-3}{4}} \sqrt{P'} |\partial_w h_1|.$$

By explicit calculations, one can check that

$$|\partial_w h_1| \lesssim 1 + \frac{1}{\sqrt{P'}}, \tag{7.6.12}$$

hence one has

$$|d^4| \lesssim \frac{1}{r} A^{\frac{\gamma-3}{4}} \rho^{\frac{\gamma-3}{4}}.$$

Lastly, by Theorem 7.2.1 and Lemma 7.6.1, one has

$$|d^5| = \left| \frac{A'}{A} A^{\frac{\gamma-3}{4}} \frac{1}{\partial_w \lambda_1 e^{-h_1}} \frac{\gamma-3}{4} \lambda_1 \right| \lesssim \frac{1}{r} A^{\frac{\gamma-3}{4}} \rho^{\frac{\gamma-3}{4}}.$$

Hence the lemma is proved. □

Lemma 7.6.4.

$$\left| \frac{d_2}{d_0} \right| \lesssim \frac{1}{r^2} A^{\frac{\gamma-3}{2}} \rho^{\frac{\gamma-3}{2}} \quad \textit{and} \quad \left| \frac{\bar{d}_2}{\bar{d}_0} \right| \lesssim \frac{1}{r^2} A^{\frac{\gamma-3}{2}} \rho^{\frac{\gamma-3}{2}}.$$

Proof. We will only prove the lemma for $\left|\frac{d_2}{d_0}\right|$, as the proof for $\left|\frac{\bar{d}_2}{\bar{d}_0}\right|$ is similar. One has

$$\left| \frac{d_2}{d_0} \right| \leq \sum_{i=1}^{8} |d^i|,$$

where d^i is the i-th term in $\left|\frac{d_2}{d_0}\right|$. To prove the lemma, one needs to bound each of the $|d^i|$.

In the proof of Lemma 7.6.3, we showed that

$$|\Phi| \lesssim \rho^{\frac{\gamma-3}{4}}.$$

Combining this with (7.6.11), (7.6.10), (7.6.12), Lemma 7.6.1 and Theorem 7.2.1, one has

$$|d^1| = \left|\left(\frac{A'}{A}\right)^2 \Phi^2 A^{\frac{\gamma-3}{2}}\right| \lesssim \frac{1}{r^2}\rho^{\frac{\gamma-3}{2}} A^{\frac{\gamma-3}{2}},$$
$$|d^2| = \left|\left(\frac{A'}{A}\right)^2 \Phi A^{\frac{\gamma-3}{2}} \frac{\partial_w F}{\partial_w \lambda_1 e^{-h_1}}\right| \lesssim \frac{1}{r^2}\rho^{\frac{\gamma-3}{2}} A^{\frac{\gamma-3}{2}},$$

$$|d^3| = \left|\left(\frac{A'}{A}\right)^2 \Phi A^{\frac{\gamma-3}{2}} \frac{\partial_z \lambda_1 G}{\partial_w \lambda_1 e^{-h_1}(\lambda_2-\lambda_1)}\right| \lesssim \frac{1}{r^2}\rho^{\frac{\gamma-3}{2}} A^{\frac{\gamma-3}{2}},$$
$$|d^4| = \left|\left(\frac{A'}{A}\right)^2 \Phi A^{\frac{\gamma-3}{2}} \frac{\partial_w h_1 F}{\partial_w \lambda_1 e^{-h_1}}\right| \lesssim \frac{1}{r^2}\rho^{\frac{\gamma-3}{2}} A^{\frac{\gamma-3}{2}},$$
$$|d^5| = \left|\left(\frac{A'}{A}\right)^2 A^{\frac{\gamma-3}{2}} \frac{\partial_z F G e^{h_1}}{\partial_w \lambda_1 e^{-h_1}(\lambda_2-\lambda_1)}\right| \lesssim \frac{1}{r^2}\rho^{\frac{\gamma-3}{2}} A^{\frac{\gamma-3}{2}},$$
$$|d^7| = \left|\frac{d}{dr}\left(\frac{A'}{A}\right) A^{\frac{\gamma-3}{2}} \frac{F e^{h_1}}{\partial_w \lambda_1 e^{-h_1}}\right| \lesssim \frac{1}{r^2}\rho^{\frac{\gamma-3}{2}} A^{\frac{\gamma-3}{2}},$$
$$|d^8| = \left|\frac{d}{dr}\left(\frac{A'}{A}\right) A^{\frac{\gamma-3}{2}} \frac{\lambda_1 \Phi}{\partial_w \lambda_1 e^{-h_1}}\right| \lesssim \frac{1}{r^2}\rho^{\frac{\gamma-3}{2}} A^{\frac{\gamma-3}{2}}.$$

It remains to bound $|d^6|$, where

$$|d^6| = \left|\left(\frac{A'}{A}\right)^2 A^{\frac{\gamma-3}{2}} \frac{F\partial_w \Phi}{\partial_w \lambda_1 e^{-h_1}}\right| \lesssim \frac{1}{r^2}\rho^{\frac{\gamma-3}{4}} A^{\frac{\gamma-3}{2}} \sqrt{P'}|\partial_w \Phi|. \tag{7.6.13}$$

By explicit calculation, one has

$$\partial_w \Phi = \frac{\partial_z F e^{h_1}}{\lambda_2-\lambda_1} - \int_{w+2\widetilde{\epsilon}}^z \frac{\partial_{zw} F e^{h_1}}{\lambda_2-\lambda_1} dz - \int_{w+2\widetilde{\epsilon}}^z \frac{\partial_z F e^{h_1} \partial_z h_1}{\lambda_2-\lambda_1} dz + \int_{w+2\widetilde{\epsilon}}^z \frac{\partial_z F e^{h_1} \partial_z(\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)^2} dz.$$

Hence

$$\begin{aligned} |\partial_w \Phi| &\lesssim \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} + \int_{w+2\widetilde{\epsilon}}^z \frac{|\partial_{zw} F|\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} dz + \int_{w+2\widetilde{\epsilon}}^z \frac{|\partial_z h_1|\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} dz + \int_{w+2\widetilde{\epsilon}}^z \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} dz \\ &= \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} + I_1 + I_2 + I_3. \end{aligned} \tag{7.6.14}$$

By the definition of $\widetilde{\epsilon}$ given by (7.1.11) and a change of integration variables, one has

$$I_3 \lesssim \int_1^\rho \frac{\rho^{\frac{\gamma-3}{4}}}{P'} \frac{c^2\sqrt{P'}}{c^2\rho + P} d\rho \lesssim \int_1^\rho \frac{\rho^{\frac{\gamma-7}{4}}}{\sqrt{P'}} d\rho \lesssim \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} + 1.$$

Next, by explicate calculations, one can get

$$|\partial_z h_1| \lesssim 1 + \frac{1}{\sqrt{P'}},$$

so by a change of integration variables, one has

$$\begin{aligned} I_2 &\lesssim \int_1^\rho \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} \frac{c^2\sqrt{P'}}{c^2\rho + P} d\rho + \int_1^\rho \frac{\rho^{\frac{\gamma-3}{4}}}{P'} \frac{c^2\sqrt{P'}}{c^2\rho + P} d\rho \\ &\lesssim \int_1^\rho \rho^{\frac{\gamma-7}{4}} d\rho + \int_1^\rho \frac{\rho^{\frac{\gamma-7}{4}}}{\sqrt{P'}} d\rho \\ &\lesssim \rho^{\frac{\gamma-3}{4}} + \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} + 1. \end{aligned}$$

Next, by explicit calculations, one has

$$\begin{aligned} \partial_{zw} F =& \frac{1}{2(c^2 - u\sqrt{P'})^3} \left[\sqrt{P'}(c^2 - u^2) + u\frac{\gamma - 1}{2}\left(c^2 + \frac{P}{\rho}\right) \right]^2 \\ &+ \frac{c^2}{2(c^2 - u\sqrt{P'})^2} \left[2\sqrt{P'} u \frac{c^2 - u^2}{2c^2} - \frac{P''(c^2\rho + P)(c^2 - u^2)}{4c^2 P'} \right] \\ &- \frac{c^2}{2(c^2 - u\sqrt{P'})^2} \left[\frac{(\gamma - 1)(c^2 + \frac{P}{\rho})(c^2 - u^2)}{4c^2} + uP'' \frac{\gamma - 1}{4\gamma} \frac{c^2\rho + P}{c^2\sqrt{P'}} \right]. \end{aligned}$$

By Theorem 7.2.1, one has $\partial_{zw} F \lesssim 1$, so

$$I_1 \lesssim \int_{w+2\widetilde{\epsilon}}^z \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} dz \lesssim \int_1^\rho \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} \frac{c^2\sqrt{P'}}{c^2\rho + P} d\rho \lesssim \int_1^\rho \rho^{\frac{\gamma-7}{4}} d\rho \lesssim \rho^{\frac{\gamma-3}{4}} + 1.$$

Substituting these into (7.6.14), by Theorem 7.2.1, one has

$$|\partial_w \Phi| \lesssim \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} + \rho^{\frac{\gamma-3}{4}} + 1 \approx \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}} \left(1 + \sqrt{P'} + \sqrt{P'}\rho^{\frac{3-\gamma}{4}}\right) \lesssim \frac{\rho^{\frac{\gamma-3}{4}}}{\sqrt{P'}}.$$

Substituting the above into (7.6.14), one has

$$|d^6| \lesssim \frac{1}{r^2} \rho^{\frac{\gamma-3}{2}} A^{\frac{\gamma-3}{2}}.$$

Hence the lemma is proved. □

Proof of Theorem 7.6.1:

We will only prove the theorem for $\widetilde{Q}$, as the proof for $\widetilde{Y}$ is similar. By Theorem 7.5, Young's inequality and Lemma 7.6.2, one has

$$\begin{aligned}\partial_2\widetilde{Q} = \frac{1}{2}\overline{d}_0Q^2 + \frac{1}{2}\overline{d}_0\widetilde{Q}^2 + \overline{d}_1\widetilde{Q} + \overline{d}_2 &\leq \frac{1}{2}\overline{d}_0\widetilde{Q}^2 + \frac{1}{2}\overline{d}_0\widetilde{Q}^2 + \frac{1}{4}|\overline{d}_0|Q^2 + 4\frac{\overline{d}_1^2}{|\overline{d}_0|} + \overline{d}_2 \\ &= \frac{1}{2}\overline{d}_0\widetilde{Q}^2 + \frac{1}{4}\overline{d}_0\left(\widetilde{Q}^2 + 16\frac{\overline{d}_1^2}{\overline{d}_0|\overline{d}_0|} + \frac{\overline{d}_2}{\overline{d}_0}\right) \\ &\leq \frac{1}{2}\overline{d}_0\widetilde{Q}^2 + \frac{1}{4}\overline{d}_0\left(\widetilde{Q}^2 - 16\frac{\overline{d}_1^2}{\overline{d}_0^2} - \left|\frac{\overline{d}_2}{\overline{d}_0}\right|\right).\end{aligned}$$

Hence to prove this theorem, one just needs

$$\sqrt{16\frac{\overline{d}_1^2}{\overline{d}_0^2} + \left|\frac{\overline{d}_2}{\overline{d}_0}\right|} \leq G_T(\underline{r}). \tag{7.6.15}$$

First of all, as $(t,r) \in \Omega(\underline{r})$, if we consider the $(t, x^2(t))$ characteristic passing through (t,r), then by definition of $\Omega(\underline{r})$ one has $\underline{r} < x^2(0)$. By proposition 7.3.1, one obtains

$$\underline{r} \leq x^2(0) < r.$$

Combining this with Lemma 7.4.7, Lemma 7.6.4, Lemma 7.6.3, Theorem 7.4.1 and the definition of $G_T(\underline{r})$ given by Definition 7.2.4, (7.6.15) indeed holds, hence the theorem is proved. □

7.7 Singularity formation with strong compression: Proof of Theorem 7.2.2

First, we will show if (7.2.1) is satisfied, then $\widetilde{Q}$ blows up before time $t = T$. We will consider the characteristic $x^2(t, \underline{r})$. The proof is divided into two steps.

Step 1: We will show if (7.2.1) holds and $t \leq T$, then along the $x^2(t,\underline{r})$ characteristic, one has

$$\partial_2\widetilde{Q} < \frac{1}{2}\overline{d}_0\widetilde{Q}^2. \tag{7.7.1}$$

By the ODE inequality associated to $\widetilde{Q}$ given by Theorem 6.5.1, (6.6.1) holds if

$$\widetilde{Q}(t, x^2(t, \underline{r})) + G_T(\underline{r}) < 0. \tag{7.7.2}$$

By (7.2.1) and continuity, we know (7.7.2) holds up to some $t > 0$. Assume for contradiction that there exists some $t^* < T$, being the first time that

$$\widetilde{Q}(t^*, x^2(t^*, \underline{r})) + G_T(\underline{r}) = 0 \iff \widetilde{Q}(t^*, x^2(t^*, \underline{r})) = -G_T(\underline{r}).$$

Then by by Theorem 7.6.1, Lemma 7.6.2 and (7.2.1), for $t \in [0, t^*]$, one has

$$\partial_2 \widetilde{Q} \leq \frac{1}{2}\overline{d}_0 \widetilde{Q}^2 < 0 \implies -G_T(\underline{r}) = \widetilde{Q}(t^*, x^2(t^*, \underline{r})) < \widetilde{Q}(0, \underline{r}) < -G_T(\underline{r}).$$

This is a contradiction, hence (7.7.2) holds, so (7.7.1) holds as well.

Step 2: We will show $\widetilde{Q}$ blows up before time reaches $t = T$. By integrating (7.7.1) along the $x^2(t, \underline{r})$ characteristic from $t \in [0, T]$, Lemma 7.6.1 and the explicit expression of $\widetilde{M}$ from Definition 7.2.4, one has

$$\begin{aligned} \frac{1}{\widetilde{Q}(T, x^2(T, \underline{r}))} &> \frac{1}{\widetilde{Q}(0, \underline{r})} + \frac{1}{2}\int_0^T d_0(t, x^2(t, \underline{r}))dt \\ &= -\frac{1}{G_T(\underline{r})} + \frac{1}{2}\int_0^T d_0 (AM)^{\frac{3-\gamma}{4}} (AM)^{\frac{\gamma-3}{4}}(t, x^2(t, \underline{r}))dt \\ &> -\frac{1}{G_T(\underline{r})} + \frac{\widetilde{M}}{2}\int_0^T (AM)^{\frac{3-\gamma}{4}}(t, x^2(t, \underline{r}))dt. \end{aligned} \tag{7.7.3}$$

Now, as $\widetilde{Q}(0, \underline{r}) < -G_T(\underline{r}) < 0$, for $\widetilde{Q}$ to blow-up before time reaches T, one just needs the right hand side of the above quantity to be non-negative.

For $1 < \gamma < \frac{3}{5} - \frac{7}{6}\epsilon$, by Theorem 7.4.1, (7.7.3) becomes

$$\frac{1}{\widetilde{Q}(T, x^2(T, \underline{r}))} > -\frac{1}{G_T(\underline{r})} + \frac{\widetilde{M}}{2} T \left(\frac{1}{AM_{\min}(\underline{r})^{1-\epsilon}} + K(\underline{r})T(1-\epsilon) \sup_{0<w\leq z\leq \sup z_0} |k_0| \right)^{\frac{\gamma-3}{4(1-\epsilon)}}. \tag{7.7.4}$$

For $\frac{3}{5} < \gamma < 1$, by Theorem 7.4.1, (7.7.3) becomes

$$\frac{1}{\widetilde{Q}(T, x^2(T, \underline{r}))} > -\frac{1}{G_T(\underline{r})} + \frac{\widetilde{M}}{2} T \left(\frac{1}{A M_{\min}(\underline{r})^{\frac{\gamma-1}{2}} - \epsilon} + N(\underline{r}) T \left(\frac{\gamma-1}{2} - \epsilon \right) \sup_{0<w\leq z\leq \sup z_0} |k_0| \right)^{\frac{\gamma-3}{4\left(\frac{\gamma-1}{2}-\epsilon\right)}}. \tag{7.7.5}$$

By the explicit expression of T and $G_T(\underline{r})$ given by Definition 7.2.4, one can check that the right hand side of (7.7.4) and (7.7.5) equal to 0. Hence the case for $\widetilde{Q}$ is proved.

It remains to prove the theorem for $\widetilde{Y}$. Consider the characteristic $x^1(t, r_0)$, one firstly has to show that, $(t, x^1(t, r_0)) \in \Omega(\underline{r})$ for $t \in [0, T]$, i.e., we need to show that

$$x^1(t, r_0) - x^2(t, \underline{r}) \geq 0 \quad \forall t \in [0, T]. \tag{7.7.6}$$

By the definition of characteristics (7.1.6), one has

$$\begin{aligned} x^1(t, r_0) - x^2(t, \underline{r}) &= r_0 - \underline{r} + \int_0^t \lambda_1(\sigma, x^1(\sigma, \underline{r}))d\sigma - \int_0^t \lambda_2(\sigma, x^2(\sigma, r_0))d\sigma \\ &\geq r_0 - \underline{r} - \int_0^t \frac{c^2\sqrt{P'}}{c^2 - u\sqrt{P'}}(\sigma, x^1(\sigma, \underline{r}))d\sigma - \int_0^t \frac{c^2(u+\sqrt{P'})}{c^2 - u\sqrt{P'}}(\sigma, x^2(\sigma, r_0))d\sigma. \end{aligned}$$

Now by Theorem 7.2.1, one has

$$\begin{aligned} &\int_0^t \frac{c^2\sqrt{P'}}{c^2 - u\sqrt{P'}}(\sigma, x^1(\sigma, \underline{r}))d\sigma + \int_0^t \frac{c^2(u+\sqrt{P'})}{c^2 - u\sqrt{P'}}(\sigma, x^2(\sigma, r_0))d\sigma \\ \leq & T\frac{\epsilon c^3}{c^2(1-\epsilon^2)} + T\frac{2\epsilon c^3}{c^2(1-\epsilon^2)} \\ = & T\frac{3\epsilon c}{(1-\epsilon^2)}. \end{aligned}$$

By assumption (7.2.2), (7.7.6) indeed holds, i.e. $(t, x^1(t, r_0)) \in \Omega(\underline{r})$ for $t \in [0, T]$. Now, by the same procedure as how we proved $\widetilde{Q}$ blows up, we have $\widetilde{Y}$ blows up before time reaches T if (7.2.3) holds.

Appendices

Appendix A

On the equation of state for the polytropic gas

This section will be devoted to showing some properties on the equation of state for the polytropic gas.

Proposition A.1. *ρ, n and S satisfy the relation*

$$n^\gamma \exp\left(\frac{S}{C_v}\right) + c^2(n-\rho) = 0. \tag{A.0.1}$$

Moreover, if $|S| \leq C$ for some positive constant C, one has

$$n(\rho, S) \sim \mathcal{O}(\rho) \quad as \quad \rho \to 0. \tag{A.0.2}$$

Proof. First, (A.0.1) can be obtained directly from (5.1.5) and (5.1.6).

Second, denote

$$G(n, \rho, S) \coloneqq n^\gamma \exp\left(\frac{S}{C_v}\right) + c^2(n-\rho).$$

Due to $n \geq 0$, a simple calculation will yield that $\frac{\partial G}{\partial n} > 0$. Hence by the implicit function theorem, one can express n as a function of ρ and S.

Since $n \geq 0$ and $\rho \geq 0$, it follows from (5.1.1) that $\rho = 0 \Longleftrightarrow n = 0$. We show ρ and n tends

to 0 at the same decay rate. When $n > 0$, we can divide (A.0.1) by n and obtain

$$\frac{\rho}{n} = 1 + \frac{1}{c^2} n^{\gamma-1} \exp\left(\frac{S}{C_v}\right).$$

Since $n(\rho, S) \to 0$ as $\rho \to 0$, one has

$$\lim_{\rho \to 0} \frac{\rho}{n(\rho, S)} = 1,$$

which implies (A.0.2). □

Proposition A.2. *If the pressure P is given in terms of (ρ, S), i.e., $P = P(\rho, S)$, then*

$$\begin{aligned}
\partial_\rho P =& \frac{c^2\gamma(\gamma-1)n^{\gamma-1}\exp\left(\frac{S}{C_v}\right)}{\gamma n^{\gamma-1}\exp\left(\frac{S}{C_v}\right) + c^2}, \\
\partial_S P =& \frac{1}{C_v}\frac{c^2(\gamma-1)n^{\gamma}\exp\left(\frac{S}{C_v}\right)}{\gamma n^{\gamma-1}\exp\left(\frac{S}{C_v}\right) + c^2}, \\
\partial_{\rho S} P =& \frac{c^4\gamma(\gamma-1)n^{\gamma-1}\exp\left(\frac{S}{C_v}\right)\left(c^2 + n^{\gamma-1}\exp\left(\frac{S}{C_v}\right)\right)}{C_v\left(\gamma n^{\gamma-1}\exp\left(\frac{S}{C_v}\right) + c^2\right)^3}, \\
\partial_{\rho\rho} P =& \frac{c^6\gamma(\gamma-1)^2 n^{\gamma-2}\exp\left(\frac{S}{C_v}\right)}{\left(\gamma n^{\gamma-1}\exp\left(\frac{S}{C_v}\right) + c^2\right)^3}.
\end{aligned} \tag{A.0.3}$$

Proof. We consider the relation $n = n(\rho, S)$. It follows from $G(n, \rho, S) = 0$ that

$$\partial_\rho n = \frac{c^2}{\gamma n^{\gamma-1}\exp\left(\frac{S}{C_v}\right) + c^2}, \quad \partial_S n = \frac{-n^\gamma \exp\left(\frac{S}{C_v}\right)}{C_v\left(\gamma n^{\gamma-1}\exp\left(\frac{S}{C_v}\right) + c^2\right)}. \tag{A.0.4}$$

Denote $P(\rho, S) = \widetilde{P}\left(n(\rho, S), S\right)$. By the chain rule, one has

$$\partial_\rho P = \partial_n \widetilde{P} \partial_\rho n, \quad \partial_S P = \partial_n \widetilde{P} \partial_S n + \partial_S \widetilde{P}, \tag{A.0.5}$$

which, along with (5.1.5) and (A.0.4), yields $(\text{A.0.3})_1 - (\text{A.0.3})_2$.

For the second derivatives, by the chain rule, one has

$$\frac{\partial^2 P}{\partial\rho\partial S}=\frac{\partial^2\widetilde{P}}{\partial n\partial S}\frac{\partial n}{\partial\rho}+\frac{\partial^2 n}{\partial\rho\partial S}\frac{\partial\widetilde{P}}{\partial n}+\frac{\partial^2\widetilde{P}}{\partial n^2}\frac{\partial n}{\partial\rho}\frac{\partial n}{\partial S}. \tag{A.0.6}$$

It follows from (5.1.5) that

$$\begin{aligned}
\frac{\partial^2\widetilde{P}}{\partial n^2}=&\gamma(\gamma-1)^2 n^{\gamma-2}\exp\left(\frac{S}{C_v}\right),\\
\frac{\partial^2\widetilde{P}}{\partial n\partial S}=&\gamma(\gamma-1)\frac{1}{C_v}n^{\gamma-1}\exp\left(\frac{S}{C_v}\right),\\
\frac{\partial^2 n}{\partial\rho\partial S}=&-\frac{c^2\gamma n^{\gamma-1}\exp\left(\frac{S}{C_v}\right)\left(n^{\gamma-1}\exp\left(\frac{S}{C_v}\right)+c^2\right)}{C_v\left(\gamma n^{\gamma-1}\exp\left(\frac{S}{C_v}\right)+c^2\right)^3},
\end{aligned}$$

which, along with (A.0.6), yields $(A.0.3)_3$. Similarly, one can obtain $(A.0.3)_4$. □

Proposition A.3.

$$\begin{aligned}
P(\rho,S)\geq&0,\quad P(\rho,S)=0\iff\rho=0,\\
\partial_\rho P(\rho,S)\geq&0,\quad \partial_\rho P(\rho,S)=0\iff\rho=0.
\end{aligned} \tag{A.0.7}$$

Proof. $(A.0.7)_1$ follows from (5.1.5) and the fact that $n=0$ if and only if $\rho=0$. $(A.0.7)_2$ follows from (A.0.3) and the fact that $n=0$ if and only if $\rho=0$. □

It follows from Propositions A.1 and A.2 that

Proposition A.4.

$$\begin{aligned}
P(\rho,S)\sim\mathcal{O}(\rho^\gamma)\quad as\quad\rho\to0;\quad\partial_\rho P(\rho,S)\sim\mathcal{O}(\rho^{\gamma-1})\quad as\quad\rho\to0;\\
\partial_S P(\rho,S)\sim\mathcal{O}(\rho^\gamma)\quad as\quad\rho\to0;\quad\partial_{S\rho} P(\rho,S)\sim\mathcal{O}(\rho^{\gamma-1})\quad as\quad\rho\to0.
\end{aligned} \tag{A.0.8}$$

Appendix B

A property of continuous functions

We will give one property of continuous functions in this section that will be used in our proof.

Proposition B.1. *Let $f(x)$ be a continuous function on $x \in [a, b]$, where a and b are two fixed numbers. Then we can split the interval $[a, b]$ into countably many sub-intervals*

$$[a, b] = \bigcup_{i=0}^{\infty} [x_i, x_{i+1}], \tag{B.0.1}$$

where $x_0 = a$ and $x_\infty = b$, (x_i, x_{i+1}) and (x_j, x_{j+1}) are disjoint if $i \neq j$, such that on each of the (x_i, x_{i+1}) we have either $f(x) > 0$, $f(x) < 0$ or $f(x) = 0$.

Proof. It is obvious that we only need to consider the case that the sign of $f(x)$ will change in the whole interval $[a, b]$.

First, for any point $x_0 \in (a, b)$ such that $f(x_0) \neq 0$,

1. if $x_0 = a$, there is an interval $[a, x_b]$ $(a < x_b < b)$ such that $f(x) \neq 0$ for $x \in [a, x_b)$, and $f(x_b) = 0$;

2. if $x_0 = b$, there is an interval $[x_a, b]$ $(a < x_a < b)$ such that $f(x) \neq 0$ for $x \in (x_a, b]$, and

$f(x_a) = 0$;

3. if $x_0 \in [a, b]$, one has three cases:

   (a) there is an interval $x_0 \in [a, x_b]$ $(a < x_b < b)$ such that $f(x) \neq 0$ for $x \in (a, x_b)$;

   (b) there is an interval $x_0 \in [x_a, b]$ $(a < x_a < b)$ such that $f(x) \neq 0$ for $x \in (x_a, b)$;

   (c) there is an interval $x_0 \in [x_a, x_b]$ $(a < x_a < x_b < b)$ such that $f(x) \neq 0$ for $x \in (x_a, x_b)$, and $f(x_a) = f(x_b) = 0$,

where one has used the continuity of the function f.

Second, consider the set of all rational numbers $\mathbf{Q} \cap [a, b] = \{q_i\}_{i=0}^{\infty}$. We denote (y_i, y_{i+1}) the interval satisfying:

1. there at least exists one real number $q_i \in (y_i, y_{i+1})$ for some $i \in \mathbb{N}$;

2. $f(x) \neq 0$ for $x \in (y_i, y_{i+1})$.

Because $\mathbf{Q} \cap [a, b]$ is dense in $[a, b]$, such kind of intervals indeed exists. Denote

$$O \coloneqq (a, b) \setminus \Big(\bigcup_{i=0}^{\infty} [y_i, y_{i+1}] \Big).$$

Note O is an open set as it is a open set minus a closed set, and $f \equiv 0$ on O by construction. As O is open, one can express it as countably many disjoint open intervals (see [50], section 1.4, Proposition 9), hence one has

$$O = \bigcup_{i=0}^{\infty} (z_i, z_{i+1})$$

Now note

$$[a, b] = \Big(\bigcup_{i=0}^{\infty} [y_i, y_{i+1}] \Big) \bigcup \overline{O} = \Big(\bigcup_{i=0}^{\infty} [y_i, y_{i+1}] \Big) \bigcup \Big(\bigcup_{j=0}^{\infty} [z_j, z_{j+1}] \Big)$$

After some relabelling we will get (B.0.1), and all the properties stated in the proposition are satisfied by construction. □

Appendix C

The explicit derivation of weight functions: h_1, h_2, h and g

This section is divided into four steps, where we will show that the functions (h_1, h_2) given in (3.3.6) satisfy (3.1.7), and (h, g) given in (5.2.7) satisfy (5.6.13).

Step 1: h_1 satisfies (3.1.7). First, according to the formulas of (λ_1, λ_2) in (3.1.7), one has

$$\partial_z \lambda_1 = \frac{2c\partial_z(e^{H_1})}{(e^{H_1}+1)^2}, \quad \lambda_1 - \lambda_2 = -2c\left(\frac{e^{H_2}-e^{H_1}}{(e^{H_2}+1)(e^{H_1}+1)}\right),$$

which yields that

$$\begin{aligned}\frac{\partial_z \lambda_1}{\lambda_1-\lambda_2} &= -\frac{\partial_z(e^{H_1})(e^{H_2}+1)}{(e^{H_1}+1)(e^{H_2}-e^{H_1})} = -\left(\frac{1}{e^{H_1}+1}+\frac{1}{e^{H_2}-e^{H_1}}\right)\partial_z(e^{H_1}) \\ &= -\partial_z \ln(e^{H_1}+1) - \frac{\partial_z(e^{H_1})}{e^{H_2}-e^{H_1}}.\end{aligned} \tag{C.0.1}$$

Second, it follows from (3.3.5), that $H_2 + H_1 = \frac{2(w+z)}{c}$. Then

$$\begin{aligned}\partial_z \ln(e^{H_2+H_1}-e^{2H_1}) &= \frac{1}{e^{H_2+H_1}-e^{2H_1}}\left(\frac{2}{c}e^{\frac{2(w+z)}{c}} - 2e^{H_1}\partial_z(e^{H_1})\right) \\ &= \frac{1}{e^{H_2}-e^{H_1}}\left(\frac{2}{c}e^{H_2} - 2\partial_z(e^{H_1})\right),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{\partial_z(e^{H_1})}{e^{H_2}-e^{H_1}} &= \frac{e^{H_2}}{c(e^{H_2}-e^{H_1})} - \frac{1}{2}\partial_z \ln(e^{H_2+H_1}-e^{2H_1}) \\ &= \frac{1}{c(1-e^{H_1-H_2})} - \frac{1}{2}\partial_z \ln(e^{H_2+H_1}-e^{2H_1}).\end{aligned} \tag{C.0.2}$$

According to (3.3.5), one can obtain that

$$1-e^{H_1-H_2} = 1-\left(\frac{1-\sqrt{\gamma}\ \tan(Y)}{1+\sqrt{\gamma}\ \tan(Y)}\right)^2 = \frac{4\sqrt{\gamma}\ \tan(Y)}{(1+\sqrt{\gamma}\ \tan(Y))^2},$$

which, along with (C.0.2), provides that

$$\frac{\partial_z(e^{H_1})}{e^{H_2}-e^{H_1}} = \frac{(1+\sqrt{\gamma}\ \tan(Y))^2}{4c\sqrt{\gamma}\ \tan(Y)} - \frac{1}{2}\partial_z \ln(e^{H_2+H_1}-e^{2H_1}). \tag{C.0.3}$$

Then it follows from (C.0.1) and (C.0.3) that

$$\frac{\partial_z\lambda_1}{\lambda_1-\lambda_2} = -\partial_z \ln(e^{H_1}+1) + \frac{1}{2}\partial_z \ln(e^{H_2+H_1}-e^{2H_1}) - \frac{(1+\sqrt{\gamma}\ \tan(Y))^2}{4c\sqrt{\gamma}\ \tan(Y)}. \tag{C.0.4}$$

Moreover, according to (3.3.5) and the fact $\frac{dz}{dY} = \frac{4c\sqrt{\gamma}}{\gamma-1}$, one has

$$\begin{aligned}\partial_z \ln(e^{H_1}+1) &= \partial_z \ln\left(e^{\frac{w+z}{c}}\left(\frac{1-\sqrt{\gamma}\ \tan(Y)}{1+\sqrt{\gamma}\ \tan(Y)}\right)+1\right) \\ &= \partial_z \ln\left(\frac{(1+e^{\frac{w+z}{c}})-\sqrt{\gamma}(-1+e^{\frac{w+z}{c}})\tan(Y)}{1+\sqrt{\gamma}\ \tan(Y)}\right) \\ &= \partial_z \ln\Big((1+e^{\frac{w+z}{c}})\cos(Y)-\sqrt{\gamma}(-1+e^{\frac{w+z}{c}})\sin(Y)\Big) \\ &\quad - \partial_z \ln(\cos(Y)) - \partial_z \ln(1+\sqrt{\gamma}\ \tan(Y)),\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\partial_z \ln(e^{H_1+H_2}-e^{2H_1}) &= \frac{1}{2}\partial_z \ln\left(e^{2(\frac{w+z}{c})}-e^{2(\frac{w+z}{c})}\left(\frac{1-\sqrt{\gamma}\ \tan(Y)}{1+\sqrt{\gamma}\ \tan(Y)}\right)^2\right)\\
&= \frac{1}{2}\partial_z\left(\ln(e^{2(\frac{w+z}{c})})+\ln\left(\frac{4\sqrt{\gamma}\ \tan(Y)}{(1+\sqrt{\gamma}\ \tan(Y))^2}\right)\right)\\
&= \frac{1}{c}+\frac{1}{2}\partial_z \ln(4\sqrt{\gamma}\ \tan(Y))-\partial_z \ln(1+\sqrt{\gamma}\ \tan(Y))\\
&= \frac{1}{c}+\frac{1}{2}\partial_z \ln(\tan(Y))-\partial_z \ln(1+\sqrt{\gamma}\ \tan(Y))\\
&= \frac{1}{c}+\frac{1}{2}\partial_z \ln(\sin(Y))-\frac{1}{2}\partial_z \ln(\cos(Y))-\partial_z \ln(1+\sqrt{\gamma}\ \tan(Y)),
\end{aligned}$$

$$\begin{aligned}
\frac{(1+\sqrt{\gamma}\ \tan(Y))^2}{4c\sqrt{\gamma}\ \tan(Y)} &= \frac{1}{4c\sqrt{\gamma}\ \tan(Y)}+\frac{1}{2c}+\frac{\sqrt{\gamma}\ \tan(Y)}{4c}\\
&= \frac{1}{4c\sqrt{\gamma}}\partial_Y \ln(\sin(Y))+\frac{1}{2c}-\frac{\sqrt{\gamma}}{4c}\partial_Y \ln(\cos(Y))\\
&= \frac{1}{\gamma-1}\partial_z \ln(\sin(Y))+\frac{1}{2c}-\frac{\gamma}{\gamma-1}\partial_z \ln(\cos(Y)),
\end{aligned}$$

which, along with (C.0.4), yields that

$$\begin{aligned}
\frac{\partial_z\lambda_1}{\lambda_1-\lambda_2} =& -\partial_z \ln\left((1+e^{\frac{w+z}{c}})\cos(Y)-\sqrt{\gamma}(-1+e^{\frac{w+z}{c}})\sin(Y)\right)\\
&+\frac{\gamma-3}{2(\gamma-1)}\partial_z \ln(\sin(Y))+\frac{3\gamma-1}{2(\gamma-1)}\partial_z \ln(\cos(Y))+\frac{1}{2c}.
\end{aligned}$$

From this one can easily see that the h_1 given by (3.3.6) indeed satisfies (3.1.7).

Step 2: h_2 satisfies (3.1.7). First, via the similar argument used in Step 1, one can verify that

$$\frac{\partial_w\lambda_2}{\lambda_1-\lambda_2} = -\partial_w \ln(e^{H_2}+1)+\frac{1}{2}\partial_w \ln(e^{2H_2}-e^{H_1+H_2})+\frac{(1-\sqrt{\gamma}\ \tan(Y))^2}{4c\sqrt{\gamma}\ \tan(Y)}. \tag{C.0.5}$$

Second, it follows from (3.3.5), the fact $\frac{dw}{dY}=\frac{4c\sqrt{\gamma}}{1-\gamma}$ and direct calculations that the function h_2 given in (3.3.6) satisfies (3.1.7).

Step 3: h satisfies (5.6.13). According to (5.2.5) and (5.2.6), one can verify that

$$\frac{\partial_z\lambda_3}{\lambda_3-\lambda_2} = -\partial_z \ln(e^{H}+1)+\frac{1}{2}\partial_z \ln(e^{H+G}-e^{2H})-\frac{(c+\sqrt{\Lambda})}{4c^2\sqrt{\Lambda}}. \tag{7.0.6}$$

It is easy to see that the h defined in (5.2.7) indeed satisfies (5.6.13).

Step 4: g satisfies (5.6.13). According to (5.2.5) and (5.2.6), one can verify that

$$\frac{\partial_w \lambda_2}{\lambda_3 - \lambda_2} = -\partial_w \ln(e^G + 1) + \frac{1}{2}\partial_w \ln(e^{2G} - e^{G+H}) + \frac{(c - \sqrt{\Lambda})}{4c^2\sqrt{\Lambda}}. \tag{7.0.7}$$

It is easy to see that the g defined in (5.2.7) indeed satisfies (5.6.13).

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