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The Central Hull and Central Kernel in JBW*-Triples[1]

C. Martin Edwards

The Queen's College, Oxford, United Kingdom
E-mail: martin.edwards@queens.ox.ac.uk

and

Gottfried T. Rüttimann†

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The complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in a JBW*-triple A has as its centre the complete Boolean algebra $\mathcal{Z}\mathcal{I}(A)$ of weak*-closed ideals in A. The annihilator $L^{\perp}$ of the subset L of A consists of elements b of A for which $\{L\, b\, A\}$ is equal to zero, and the kernel Ker(L) of L consists of those elements b in A for which $\{L\, b\, L\}$ is equal to zero. For each element J of $\mathcal{I}(A)$, $J^{\perp}$ also lies in $\mathcal{I}(A)$, and A enjoys the generalized Peirce decomposition

$$A = J \oplus_M J^{\perp} \oplus J_1,$$

where J_1 is the intersection of the kernels of J and $J^{\perp}$. To investigate the properties of the weak*-closed subspace J_1 of A, which is not, in general, a subtriple, the notions of the central hull $c(L)$ and central kernel $k(L)$ of a subspace L are introduced. These are, respectively, the smallest element of $\mathcal{Z}\mathcal{I}(A)$ containing L and the largest element of $\mathcal{Z}\mathcal{I}(A)$ contained in L. For any element J of $\mathcal{I}(A)$, the relationships that exist between the central hull and central kernel of J and $J^{\perp}$ are examined and it is shown that $(J_1)^{\perp} \cap J$ is the weak*-closed ideal $k(J)$, that $(J_1)^{\perp} \cap J^{\perp}$ is the weak*-closed ideal $k(J^{\perp})$, and, when J is a Peirce inner ideal, that $(J_1)^{\perp}$ is the weak*-closed ideal $(k(J) \oplus_M k(J^{\perp}))$. © 2002 Elsevier Science (USA)

† Deceased.

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1. INTRODUCTION

This paper presents a further investigation into the structure of JBW*-triples. The work of Kaup and Upmeier [34–36] and Vigué [45–48] shows how the holomorphic structure of the open unit ball in a complex Banach space A leads to the existence of a closed subspace A_s of A and a triple product $\{\dots\}$ from $A \times A_s \times A$ to A. The purely algebraic properties of the triple product, namely the linearity and symmetry in the first and third variables, the conjugate linearity in the second variable and, most important, the existence of a Jordan triple identity, relate any complex Banach space to the Jordan triple systems studied by Koecher [37], Loos [38], and Meyberg [40]. When A_s exhausts A or, equivalently, when the open unit ball in A is a bounded symmetric domain, the complex Banach space A is said to be a JB*-triple. A JB*-triple that is the dual of a complex Banach space is said to be a JBW*-triple. Because of the very intimate nature of the relationship between their geometric and algebraic structures, considerable attention has been given to the properties of JBW*-triples in recent years. See, for example, [3–5, 16, 17, 22, 23, 28–30, 32].

The structure of the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in a JBW*-triple A has been the subject of an intensive investigation by the authors in the past [16, 18–22, 26, 27]. After a precise description of $\mathcal{I}(A)$ was given in many cases, it was shown that, in general, every weak*-closed inner ideal in A is the range of a unique structural projection on A and, as a consequence, that the set $\mathcal{S}(A)$ of structural projections on A forms a complete lattice order isomorphic to $\mathcal{I}(A)$. Furthermore, the algebraically defined structural projections are automatically contractive and weak*-continuous.

More recently in [15, 25], the authors studied the structure of the complete lattices $\mathcal{I}(A)$ and $\mathcal{S}(A)$ from a rather different point of view. For each element J in $\mathcal{I}(A)$, the kernel $\mathrm{Ker}(J)$ of J is defined to be the set of elements a in A for which the triple product $\{J\,a\,J\}$ is equal to zero, and the annihilator $J^\perp$ of J is defined to be the set of elements a in A for which $\{J\,a\,A\}$ is equal to zero. For each element J in $\mathcal{I}(A)$, the annihilator $J^\perp$ also lies in $\mathcal{I}(A)$, and A enjoys the generalized Peirce decomposition

$$A = J_0 \oplus J_1 \oplus J_2,$$

where

$$J_0 = J^\perp,\ J_2 = J,\ J_1 = \mathrm{Ker}(J) \cap \mathrm{Ker}(J^\perp).$$

The structural projections onto J and $J^\perp$ are denoted by $P_2(J)$ and $P_0(J)$, respectively, and the projection $\mathrm{id}_A - P_2(J) - P_0(J)$ onto J_1 is denoted by $P_1(J)$. Furthermore,

$$\{A\,J_0\,J_2\} = \{0\},\ \{A\,J_2\,J_0\} = \{0\} \tag{1.1}$$

and, for j, k, and l equal to 0, 1, or 2, the Peirce arithmetical relations,

$$\{J_j\, J_k\, J_l\} \subseteq J_{j+l-k}, \tag{1.2}$$

when $j+l-k$ is equal to 0, 1, or 2, and

$$\{J_j\, J_k\, J_l\} = \{0\}, \tag{1.3}$$

otherwise, hold, except in the cases where (j, k, l) is equal to $(0, 1, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(2, 1, 1)$, $(1, 1, 2)$, $(1, 2, 1)$, or $(1, 1, 1)$. A weak*-closed inner ideal J for which the Peirce relations hold in all cases is said to be a Peirce inner ideal. An element $P_2(I)$ in $\mathcal{S}(A)$ is said to be central if $P_2(I)$ commutes with every element of $\mathcal{S}(A)$. The results of [15] show that $P_2(I)$ is central if and only if I is an ideal in A or, equivalently, is an M-summand in A. The sets $\mathcal{Z}\mathcal{I}(A)$ of weak*-closed ideals in A and $\mathcal{Z}\mathcal{S}(A)$ of central elements of $\mathcal{S}(A)$ form Boolean subcomplete lattices of $\mathcal{I}(A)$ and $\mathcal{S}(A)$, respectively.

For any weak*-closed inner ideal J in A, the Peirce-two and Peirce-zero spaces J_2 and J_0 are weak*-closed inner ideals. However, the Peirce-one space J_1, which is a weak*-closed subspace of A, may not, in general, even be a subtriple of A. Consequently, the algebraic properties of J_1 are not so transparent. To make an investigation of the annihilator $(J_1)^\perp$ of J_1, which is, of course, a weak*-closed inner ideal, the notions of the central hull $c(L)$ and central kernel $k(L)$ of a subspace L of A are introduced. These are, respectively, the smallest element of $\mathcal{Z}\mathcal{I}(A)$ containing L and the largest element of $\mathcal{Z}\mathcal{I}(A)$ contained in L. A study of the central hull and central kernel of J and $J^\perp$ reveals that the intersection $(J_1)^\perp \cap J$, which is clearly a weak*-closed inner ideal, is, in fact, an ideal, the central kernel $k(J)$ of J. Similarly, it is shown that $(J_1)^\perp \cap J^\perp$ is also an ideal, the central kernel $k(J^\perp)$ of $J^\perp$. When J is a Peirce inner ideal, where J_1 is a subtriple, rather more can be said about its annihilator. In this case $(J_1)^\perp$ is itself an ideal, the orthogonal sum $k(J) \oplus_M k(J^\perp)$ of the ideals $k(J)$ and $k(J^\perp)$.

The paper is organized as follows. In Section 2 definitions are given, notation is established, and certain preliminary results are described. In Section 3 the definitions of the central hull and central kernel are given, their properties are investigated, and the main results of the paper are proved. In Section 4 the results are applied to various examples, including the case where A is a W*-algebra, for which the results do not appear to have been known previously. We are grateful to the referee for the helpful remarks on an earlier version of the paper.

2. PRELIMINARIES

A complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a\, b\, c\}$ from $A \times A \times A$ to A, which is symmetric and linear in the first and third variables and conjugate linear in the second variable and, for elements a, b, c, and d in A, satisfies the identity

$$[D(a,b), D(c,d)] = D(\{a\,b\,c\}, d) - D(c, \{d\,a\,b\}), \tag{2.1}$$

where [,] denotes the commutator and D is the mapping from $A \times A$ to the algebra of linear operators on A defined by

$$D(a,b)c = \{a\,b\,c\},$$

is said to be a *Jordan*-triple*. A Jordan*-triple A for which the vanishing of $\{a\,a\,a\}$ implies that a itself vanishes is said to be *anisotropic*. For each element a in A, the conjugate linear mapping $Q(a)$ from A to itself is defined, for each element b in A, by

$$Q(a)b = \{a\,b\,a\}.$$

For details about the properties of Jordan*-triples the reader is referred to [38].

A Jordan*-triple A, which is also a Banach space such that D is continuous from $A \times A$ to the Banach algebra $B(A)$ of bounded linear operators on A, and, for each element a in A, $D(a,a)$ is hermitian in the sense of [7, Definition 5.1], with nonnegative spectrum, and satisfies

$$\|D(a,a)\| = \|a\|^2,$$

is said to be a *JB*-triple*. A subspace B of a JB*-triple A is said to be a *subtriple* if $\{B\,B\,B\}$ is contained in B. A subspace B is clearly a subtriple if and only if, for each element a in B, the element $\{a\,a\,a\}$ lies in B. Observe that every subtriple of a JB*-triple is an anisotropic Jordan*-triple. A subspace J of a JB*-triple A is said to be an *inner ideal* if $\{J\,A\,J\}$ is contained in J and is said to be an *ideal* if $\{A\,A\,J\}$ and $\{A\,J\,A\}$ are contained in J. Every norm-closed subtriple of a JB*-triple A is a JB*-triple [34], and a norm-closed subspace J of A is an ideal if and only if $\{J\,J\,A\}$ is contained in J [8]. A JB*-triple A, which is the dual of a Banach space A_*, is said to be a *JBW*-triple*. In this case the *predual* A_* of A is unique and, for each element a in A, the operators $D(a,b)$ and $Q(a)$ are weak*-continuous. It follows that a weak*-closed subtriple B of a JBW*-triple A is a JBW*-triple. Examples of JB*-triples are JB*-algebras and examples of JBW*-triples are JBW*-algebras. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [3, 4, 11, 12, 28, 33–35, 43, 44].

A pair a and b of elements in a JBW*-triple A is said to be *orthogonal*, denoted by $a \perp b$, when $D(a,b)$ is equal to zero. By [23, Lemma 3.1], it follows that orthogonality is a symmetric relation. Also, for an element a in A, $a \perp a$ if and only if a is equal to zero. For a subset L of A, denote by

$L^\perp$ the subset which consists of all elements in A which are orthogonal to all elements in L. The subset $L^\perp$ is said to be the *annihilator* of L. By [23, Lemma 3.2], $L^\perp$ is a weak*-closed inner ideal in A. Moreover, for subsets L, M of A, $L^\perp \cap L \subseteq \{0\}$, $L \subseteq L^{\perp\perp}$, $L \subseteq M$ implies that $M^\perp \subseteq L^\perp$, and $L^\perp$ and $L^{\perp\perp\perp}$ coincide.

For each nonempty subset B of the JBW*-triple A, the *kernel* $\mathrm{Ker}(B)$ of B is the weak*-closed subspace of elements a in A for which $\{B\,a\,B\}$ is equal to $\{0\}$. It follows that the annihilator $B^\perp$ of B is contained in $\mathrm{Ker}(B)$ and that $B \cap \mathrm{Ker}(B)$ is contained in $\{0\}$. A subtriple B of A is said to be *complemented* [22] if A coincides with $B \oplus \mathrm{Ker}(B)$. It can easily be seen that every complemented subtriple is a weak*-closed inner ideal. A linear projection R on the JBW*-triple A is said to be a *structural projection* [39] if, for each element a in A,

$$RQ(a)R = Q(Ra). \tag{2.2}$$

The main results of [16, 21, 22] show that the range RA of a structural projection R is a complemented subtriple, that the kernel $\ker R$ of the map R coincides with $\mathrm{Ker}(RA)$, that every structural projection is contractive and weak*-continuous, and, most significantly, that every weak*-closed inner ideal is complemented.

Let $\mathcal{I}(A)$ denote the complete lattice of weak*-closed inner ideals in the JBW*-triple A and let $\mathcal{S}(A)$ denote the set of structural projections on A. The results of [16] can be used to show that the set $\mathcal{S}(A)$ of structural projections on A is a complete lattice with respect to the ordering defined, for elements R_1 and R_2, by $R_1 \leq R_2$ if R_2R_1 is equal to R_1 and the mapping $R \mapsto RA$ is an order isomorphism from $\mathcal{S}(A)$ onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A.

For each element J of $\mathcal{I}(A)$, the annihilator $J^\perp$ also lies in $\mathcal{I}(A)$ and A enjoys the *generalized Peirce decomposition*

$$A = J_2 \oplus J_1 \oplus J_0, \tag{2.3}$$

relative to J, where

$$J_2 = J,\ \ J_0 = J^\perp,\ \ J_1 = \mathrm{Ker}(J) \cap \mathrm{Ker}(J^\perp). \tag{2.4}$$

The structural projections the ranges of which are J_2 and J_0 are denoted by $P_2(J)$ and $P_0(J)$, respectively, and the projection

$$P_1(J) = \mathrm{id}_A - P_2(J) - P_0(J) \tag{2.5}$$

denotes the projection onto J_1. Then $P_0(J)$, $P_1(J)$, and $P_2(J)$ are mutually orthogonal weak*-continuous linear projections on A with sum id_A. For j

equal to 0, 1, or 2, the range J_j of $P_j(J)$ is the set of eigenvectors of the weak*-continuous linear operator $D(J)$ defined by

$$D(J) = \frac{1}{2}(\mathrm{id}_A + P_2(J) - P_0(J))$$

with eigenvalue $\frac{1}{2}j$. Furthermore, the Peirce relations given in (1.1), (1.2), and (1.3) hold, except in the cases where (j,k,l) is equal to $(0,1,1)$, $(1,1,0)$, $(1,0,1)$, $(2,1,1)$, $(1,1,2)$, $(1,2,1)$, or $(1,1,1)$. When the relations hold in all cases, then J is said to be a *Peirce inner ideal*.

A pair J and K of elements of $\mathcal{I}(A)$ is said to be *compatible* if, for j and k equal to 0, 1, or 2,

$$P_j(J)P_k(K) = P_k(K)P_j(J). \tag{2.6}$$

Let A be a complex Banach space. A linear projection S on A is said to be an *M-projection* if, for each element a in A,

$$\|a\| = \max\{\|Sa\|, \|a - Sa\|\}.$$

A closed subspace, which is the range of an M-projection, is said to be an *M-summand* of A, and A is said to be equal to the *M-sum*

$$A = SA \oplus_M (\mathrm{id}_A - S)A$$

of the M-summands SA and $(\mathrm{id}_A - S)A$. For details the reader is referred to [1, 2, 9, 10].

The results of [3, 33] show that the set of M-summands of a JBW*-triple A coincides with the set of its weak*-closed ideals. Recall that, the *centroid* $\mathcal{Z}^b(A)$ of A is the set of bounded linear operators T on A such that, for all elements a in A,

$$TD(a,a) = D(a,a)T. \tag{2.7}$$

For each element T in $\mathcal{Z}^b(A)$ there exists a unique element $T^\dagger$ in $\mathcal{Z}^b(A)$ such that, for all elements a and b in A,

$$T\{a\,b\,a\} = \{Ta\,b\,a\} = \{a\,T^\dagger b\,a\}. \tag{2.8}$$

The following result is an immediate consequence of those of [1, 2, 6, 13, 15].

LEMMA 2.1. *Let A be a JBW*-triple, with centroid $\mathcal{Z}^b(A)$. Then*:

(i) *with respect to the operator norm and product, and the involution $T \mapsto T^\dagger$, defined by* (2.8), *$\mathcal{Z}^b(A)$ forms a commutative W*-algebra*;

(ii) *the set of M-projections on A, when ordered by the set inclusion of the corresponding M-summands, with complementation $P \mapsto \mathrm{id}_A - P$ is identical to the complete Boolean lattice $\mathcal{P}(\mathcal{Z}^b(A))$ of self-adjoint idempotents in the commutative W*-algebra $\mathcal{Z}^b(A)$.*

A structural projection $P_2(I)$ on the JBW*-triple A, which commutes with every structural projection on A, is said to be *central*. The proof of the following result can be found in [15].

LEMMA 2.2. *Let A be a JBW*-triple, let I be an element of the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A, and let*

$$A = I_2 \oplus I_1 \oplus I_0, \ \mathrm{id}_A = P_2(I) + P_1(I) + P_0(I)$$

be the generalized Peirce decomposition of A relative to I. Then, the following are equivalent:

(i) *the operator $P_2(I)$ is a central structural projection*;

(ii) *the operator $P_2(I)$ lies in the centroid $\mathcal{Z}^b(A)$ of A*;

(iii) *the operator $P_2(I)$ is an M-projection on A*;

(iv) *the weak*-closed inner ideal I is an ideal in A*;

(v) *the weak*-closed inner ideal I is compatible with every weak*-closed inner ideal J in A*;

(vi) *for all elements a, b in A,*

$$\{P_2(I)a\, b\, P_2(I)a\} = \{a\, P_2(I)b\, a\};$$

(vii) *the Peirce one-space I_1 coincides with $\{0\}$.*

3. THE CENTRAL HULL AND CENTRAL KERNEL

This section is concerned with the notions of the central hull $c(J)$ and the central kernel $k(J)$ of a weak*-closed inner ideal J in a JBW*-triple A. Recall that the set $\mathcal{Z}\mathcal{I}(A)$ of weak*-closed ideals in A is a Boolean subcomplete lattice of the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A, the lattice operations being defined for a family $\{I_j : j \in \Lambda\}$ of elements of $\mathcal{Z}\mathcal{I}(A)$ by

$$\bigwedge_{j\in\Lambda} I_j = \bigcap_{j\in\Lambda} I_j, \ \bigvee_{j\in\Lambda} I_j = \overline{\mathrm{lin}\Big(\bigcup_{j\in\Lambda} I_j\Big)}^{w^*},$$

and the lattice complement being the annihilator.

It is now possible to give the first of the main definitions in the paper. The *central hull* $c(L)$ of a subspace L of the JBW*-triple A is defined by

$$c(L) = \bigwedge\{I \in \mathcal{Z}\mathcal{I}(A) : L \subseteq I\},$$

the smallest weak*-closed ideal in A that contains L. Observe that, from elementary lattice-theoretic properties it follows that, for a family

$\{J_j\colon j \in \Lambda\}$ of elements of the complete lattice $\mathcal{J}(A)$ of weak*-closed inner ideals in A,

$$c\Big(\bigvee_{j\in\Lambda} J_j\Big) = \bigvee_{j\in\Lambda} c(J_j). \tag{3.1}$$

Before the study of the properties of the central hull is continued, some elementary results about inner ideals and ideals are needed.

LEMMA 3.1. *Let A be a JBW*-triple, let J be a weak*-closed inner ideal in A, and let I be a weak*-closed ideal in A. Then, $I + J$ is a weak*-closed inner ideal in A such that*

$$I + J = I \oplus_M (I^\perp \cap J), \;\; \mathrm{Ker}(I + J) = I^\perp \cap \mathrm{Ker}(J).$$

Proof. Simple calculations show that $I + J$ is an inner ideal in A and, by [15, Lemma 4.7],

$$\begin{aligned} I + J &= I + (I \cap J) + (I^\perp \cap J) \\ &= I \oplus (I^\perp \cap J) = I \oplus_M (I^\perp \cap J), \end{aligned}$$

using the fact that $I^\perp \cap J$ is contained in the annihilator $I^\perp$ of the weak*-closed ideal I. Since both I and $I^\perp \cap J$ are weak*-closed, it can be seen that their M-sum $I + J$ is also weak*-closed. Finally, since $I^\perp$ and $\mathrm{Ker}(I)$ coincide, it follows that

$$\mathrm{Ker}(I + J) = \mathrm{Ker}(I) \cap \mathrm{Ker}(J) = I^\perp \cap \mathrm{Ker}(J),$$

as required. ∎

LEMMA 3.2. *Let A be an anisotropic Jordan *-triple, let I be a complemented ideal in A, and let B and C be subtriples of A such that B is orthogonal to C. Then,*

$$I \cap (B + C) = (I \cap B) + (I \cap C).$$

Proof. It is clear that $(I \cap B) + (I \cap C)$ is contained in $I \cap (B + C)$. Let a be an element of $I \cap (B + C)$, and let b and c be elements of B and C, respectively, such that

$$a = b + c.$$

Then,

$$\{b\,b\,b\} = \{b\,b\,b\} + \{b\,c\,b\} = \{b\,a\,b\},$$

which lies in $\{A\, I\, A\}$, which itself lies in I. Since, by [15, Proposition 4.1], A coincides with $I \oplus I^{\perp}$, there exist unique elements b_0 in $I^{\perp}$ and b_2 in I, such that

$$b = b_0 + b_2.$$

Furthermore, by orthogonality,

$$\{b\, b\, b\} = \{b_0\, b_0\, b_0\} + \{b_2\, b_2\, b_2\}.$$

Since both $\{b\, b\, b\}$ and $\{b_2\, b_2\, b_2\}$ lie in I, $\{b_0\, b_0\, b_0\}$ lies in $I \cap I^{\perp}$ and is, therefore, equal to zero. Since A is anisotropic, it follows that b_0 is equal to zero, and b, which coincides with b_2, lies in $I \cap B$. Similarly, c lies in $I \cap C$, and the proof is complete. ∎

It is now possible to present a further property of the central hull.

THEOREM 3.3. *Let A be a JBW*-triple, and let J and K be elements of the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A. Then*

$$c(c(J) \cap K) = c(J) \cap c(K).$$

Proof. Let I be a weak*-closed ideal in A. Then, by [15, Lemma 4.7],

$$K = (I \cap K) \oplus (I^{\perp} \cap K) = (I \wedge K) \vee (I^{\perp} \wedge K).$$

Since $c(I \cap K)$ and $c(I^{\perp} \cap K)$ are weak*-closed ideals in A, it follows from (3.1) and Lemma 3.1 that

$$c(K) = c(I \wedge K) \vee c(I^{\perp} \wedge K) = c(I \cap K) + c(I^{\perp} \cap K).$$

Since $c(I \cap K)$ is contained in I and $c(I^{\perp} \cap K)$ is contained in $I^{\perp}$, by Lemma 3.2,

$$I \cap c(K) = I \cap c(I \cap K) + I \cap c(I^{\perp} \cap K) = c(I \cap K) + \{0\} = c(I \cap K).$$

Replacing I by $c(J)$ gives the required result. ∎

For each subspace L of the JBW*-triple A, the *central kernel* $k(L)$ of L is defined by

$$k(L) = \bigvee \{I \in \mathcal{Z}\mathcal{I}(A) : I \subseteq L\},$$

the greatest weak*-closed ideal in A that is contained in L. Observe that, for a family $\{J_j \colon j \in \Lambda\}$ of elements of the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A,

$$k\Big(\bigwedge_{j \in \Lambda} J_j\Big) = \bigwedge_{j \in \Lambda} k(J_j). \tag{3.2}$$

To study the properties of the central kernel, the following preliminary results are required.

LEMMA 3.4. *Let A be a JBW*-triple and let J be a weak*-closed inner ideal in A with kernel* $\mathrm{Ker}(J)$ *and central kernel* $k(J)$. *Then*

$$\{J\, k(J)\, \mathrm{Ker}(J)\} = \{0\}.$$

Proof. Since $k(J)$ is an ideal,

$$\{J\, k(J)\, \mathrm{Ker}(J)\} \subseteq k(J) \subseteq J,$$

and, by [16, Lemma 3.1],

$$\{J\, k(J)\, \mathrm{Ker}(J)\} \subseteq \{A\, J\, \mathrm{Ker}(J)\} \subseteq \mathrm{Ker}(J).$$

Hence

$$\{J\, k(J)\, \mathrm{Ker}(J)\} \subseteq J \cap \mathrm{Ker}(J) = \{0\},$$

as required. ∎

LEMMA 3.5. *Let A be a JBW*-triple and let J be a weak*-closed inner ideal in A with annihilator $J^\perp$. Then the central kernel $k(J^\perp)$ of $J^\perp$ coincides with the annihilator $c(J)^\perp$ of the central hull $c(J)$ of J.*

Proof. Since J is contained in $c(J)$, it follows that $c(J)^\perp$ is contained in $J^\perp$. Therefore,

$$c(J)^\perp = k(c(J)^\perp) \subseteq k(J^\perp).$$

On the other hand $k(J^\perp)$ is contained in $J^\perp$ and, therefore,

$$J \subseteq J^{\perp\perp} \subseteq k(J^\perp)^\perp.$$

Therefore,

$$c(J) \subseteq c(k(J^\perp)^\perp) = k(J^\perp)^\perp,$$

and, taking annihilators,

$$k(J^\perp) = k(J^\perp)^{\perp\perp} \subseteq c(J)^\perp.$$

This completes the proof of the lemma. ∎

This result has the following corollary, which shows that the result of [24, Lemma 4.1] holds in much greater generality.

COROLLARY 3.6. *Let A be a JBW*-triple and let J be a weak*-closed inner ideal in A with bi-annihilator $J^{\perp\perp}$. Then, the central hulls $c(J)$ and $c(J^{\perp\perp})$ of J and $J^{\perp\perp}$, respectively, coincide.*

Proof. By Lemma 3.5,

$$c(J^{\perp\perp}) = k(J^{\perp\perp\perp})^{\perp} = k(J^{\perp})^{\perp} = c(J),$$

as required. ∎

The next result describes a striking property of the central kernel of a weak*-closed inner ideal.

THEOREM 3.7. *Let A be a JBW*-triple and let J be a weak*-closed inner ideal in A with kernel* $\mathrm{Ker}(J)$. *Then, the central kernel $k(J)$ of J consists of the set of elements a in J for which*

$$\{a\, J\, \mathrm{Ker}(J)\} = \{0\}.$$

Proof. Let K be the set of elements a in J for which $\{a\, J\, \mathrm{Ker}(J)\}$ is equal to $\{0\}$. It is clear that K is a subspace of A, and, by the separate weak*-continuity of the triple product, that K is weak*-closed. Let a be an element of K, let b, c, and d be elements of J, and let e be an element of $\mathrm{Ker}(J)$. Then, by (2.1), using the definitions of $\mathrm{Ker}(J)$ and K, and [16], Lemma 3.1,

$$\begin{aligned}\{\{a\, b\, c\}\, d\, e\} &= \{\{e\, d\, a\}\, b\, c\} - \{a\, \{d\, e\, b\}\, c\} + \{a\, b\, \{e\, d\, c\}\}\\ &\subseteq \{\{a\, J\, \mathrm{Ker}(J)\}\, J\, J\} + \{a\, \{J\, \mathrm{Ker}(J)\, J\}\, c\}\\ &\quad + \{a\, J\, \{J\, J\, \mathrm{Ker}(J)\}\}\\ &\subseteq \{\{0\}\, J\, J\} + \{a\, \{0\}\, c\} + \{a\, J\, \mathrm{Ker}(J)\} = \{0\}.\end{aligned}$$

It follows that the set $\{a\, J\, J\}$ is contained in K and, hence, that $\{K\, J\, J\}$ is contained in K.

Now let a be an element of A. Then, by [16, Theorem 5.4], there exist elements b in J and c in $\mathrm{Ker}(J)$ such that

$$a = b + c.$$

Then, from above,

$$\begin{aligned}\{K\, K\, a\} &= \{K\, K\, b\} + \{K\, K\, c\}\\ &\subseteq \{K\, J\, J\} + \{K\, J\, \mathrm{Ker}(J)\}\\ &\subseteq K + \{0\} = K.\end{aligned}$$

It now follows from [8, Proposition 1.3], that K is a weak*-closed ideal in A. Since K is contained in J, it follows that K is contained in $k(J)$. Furthermore, since $k(J)$ is an ideal, and, again using [16, Lemma 3.1],

$$\{k(J)\, J\, \mathrm{Ker}(J)\} \subseteq k(J) \cap \{A\, J\, \mathrm{Ker}(J)\} \subseteq J \cap \mathrm{Ker}(J) = \{0\}.$$

Therefore, $k(J)$ is contained in K, and the proof is complete. ∎

This theorem has the following significant corollary.

COROLLARY 3.8. *Let A be a JBW*-triple, let J be a weak*-closed inner ideal in A, with annihilator $J^\perp$ and Peirce one-space J_1. Then*:

(i) $k(J) = \{a \in J : \{a\,J\,J_1\} = \{0\}\}$;

(ii) $k(J^\perp) = \{a \in J^\perp : \{a\,J^\perp J_1\} = \{0\}\}$.

Proof. Let $P_0(J)$, $P_1(J)$, and $P_2(J)$ be the Peirce projections corresponding to J, defined in (2.3)–(2.5). By [16, Lemma 3.3 and Theorem 3.4],

$$\mathrm{Ker}(J) = \ker(P_2(J)) = (P_0(J) + P_1(J))(A) = J^\perp \oplus J_1,$$

and

$$\mathrm{Ker}(J^\perp) = \ker(P_0(J)) = (P_2(J) + P_1(J))(A) = J \oplus J_1.$$

By Theorem 3.7, an element a in J lies in $k(J)$ if and only if

$$\{a\,J\,J_1\} = \{a\,J\,(J^\perp \oplus J_1)\} = \{a\,J\,\mathrm{Ker}(J)\} = \{0\},$$

and an element a in $J^\perp$ lies in $k(J^\perp)$ if and only if

$$\{a\,J^\perp J_1\} = \{a\,J^\perp (J \oplus J_1)\} = \{a\,J^\perp \mathrm{Ker}(J^\perp)\} = \{0\}.$$

This completes the proof. ∎

Before going on to discuss further properties of the central kernel, the following result is required.

LEMMA 3.9. *Let A be a JBW*-triple, let I be a weak*-closed ideal in A, and let J be a weak*-closed inner ideal in A, with J orthogonal to I. Then*,

$$J = (I \oplus J) \cap I^\perp, \quad \mathrm{Ker}(J) = I \oplus \mathrm{Ker}(I \oplus J).$$

Proof. By Lemma 3.2,

$$(I \oplus J) \cap I^\perp = (I \cap I^\perp) + (J \cap I^\perp) = \{0\} + J = J,$$

as required. Observe that

$$I \subseteq J^\perp \subseteq \mathrm{Ker}(J), \tag{3.3}$$

and that

$$\mathrm{Ker}(I + J) \subseteq \mathrm{Ker}(J). \tag{3.4}$$

By Lemma 3.1, $I \oplus J$ is a weak*-closed inner ideal in A and, therefore, using (3.3) and (3.4),

$$\begin{aligned} A &= (I \oplus J) \oplus \mathrm{Ker}(I \oplus J) = J \oplus (I \oplus \mathrm{Ker}(I \oplus J)) \\ &\subseteq J \oplus (\mathrm{Ker}(J) \oplus \mathrm{Ker}(J)) = J \oplus \mathrm{Ker}(J) = A. \end{aligned}$$

It follows that

$$\mathrm{Ker}(J) = I \oplus \mathrm{Ker}(I \oplus J),$$

and the proof is complete. ∎

It is now possible to describe the second main property of the central kernel.

THEOREM 3.10. *Let A be a JBW*-triple, and let J and K be elements of the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A. Then*

$$k(k(J) + K) = k(J) + k(K).$$

Proof. Let I be a weak*-closed ideal in A and let K be a weak*-closed inner ideal, which is orthogonal to I. Since I and $k(K)$ are orthogonal weak*-closed ideals in A, by Lemma 3.1, $I \oplus k(K)$ is also a weak*-closed ideal contained in the weak*-closed inner ideal $I \oplus K$. It follows that

$$I \oplus k(K) \subseteq k(I \oplus K). \tag{3.5}$$

On the other hand, let a be an element of $k(I \oplus K)$. Since $k(I \oplus K)$ is contained in $I \oplus K$, there exist elements b in I and c in K such that

$$a = b + c. \tag{3.6}$$

Since I is orthogonal to K, using Lemma 3.4, Theorem 3.7, and Lemma 3.9, it can be seen that

$$\begin{aligned} \{c\,K\,\mathrm{Ker}(K)\} &\subseteq \{a\,K\,\mathrm{Ker}(K)\} + \{b\,K\,\mathrm{Ker}(K)\} \\ &= \{a\,K\,\mathrm{Ker}(K)\} + \{0\} \\ &= \{a\,K\,(I \oplus \mathrm{Ker}(I \oplus K))\} \\ &= \{a\,K\,I\} + \{a\,K\,\mathrm{Ker}(I \oplus K)\} \\ &= \{0\} + \{a\,K\,\mathrm{Ker}(I \oplus K)\} \\ &\subseteq \{a\,(I \oplus K)\,\mathrm{Ker}(I \oplus K)\} = \{0\}. \end{aligned}$$

By Theorem 3.7 it follows that c lies in $k(K)$ and, from (3.6), a lies in $I \oplus k(K)$. Therefore, in this case

$$I \oplus k(K) = k(I \oplus K). \tag{3.7}$$

Now suppose that I is not necessarily orthogonal to K. In this case, however, I is orthogonal to the weak*-closed inner ideal $I^{\perp} \cap K$ in A. It follows from Lemma 3.1, (3.2), and (3.7) that

$$\begin{aligned} k(I + K) &= k(I \oplus (I^{\perp} \cap K)) = I \oplus k(I^{\perp} \cap K) \\ &= I \oplus (k(I^{\perp}) \cap k(K)) = I \oplus (I^{\perp} \cap k(K)) \\ &= I + k(K). \end{aligned}$$

Replacing I by $k(J)$ completes the proof of the theorem. ∎

Further properties of the relationships that exist between the central hull and central kernel are given in the following results.

THEOREM 3.11. *Let A be a JBW*-triple and let J be a weak*-closed inner ideal in A, having annihilator $J^{\perp}$, kernel* $\mathrm{Ker}(J)$, *central hull* $c(J)$, *and central kernel* $k(J)$. *Then*:

(i) $k(J) = c(\mathrm{Ker}(J))^{\perp}$;

(ii) $c(J) = c(\mathrm{Ker}(J^{\perp}))$.

Proof. Let J_0, J_1, and J_2 be the generalized Peirce spaces associated with the weak*-closed inner ideal J, and let $k(J)_0$ and $k(J)_2$ be the generalized Peirce spaces associated with the weak*-closed ideal $k(J)$. By [15, Corollary 6.7], J and $k(J)$ form a compatible pair of weak*-closed inner ideals. Their intersection table is given below.

$\cap$	J_2	J_1	J_0
$k(J)_2$	$k(J)$	$\{0\}$	$\{0\}$
$k(J)_0$	$J \cap k(J)^{\perp}$	$J_1 \cap k(J)^{\perp}$	$J^{\perp}$

It follows that J_1 and $J_1 \cap k(J)^{\perp}$ coincide, and, hence, that J_1 is contained in $k(J)^{\perp}$. Since $k(J)$ is contained in J, it follows that $J^{\perp}$, which is equal to J_0, is contained in $k(J)^{\perp}$. Therefore,

$$\mathrm{Ker}(J) = J_0 \oplus J_1 \subseteq k(J)^{\perp},$$

and, consequently,

$$c(\mathrm{Ker}(J)) \subseteq c(k(J)^{\perp}) = k(J)^{\perp}.$$

It follows that

$$k(J) \subseteq c(\mathrm{Ker}(J))^{\perp}. \tag{3.8}$$

Let a be an element of $c(\mathrm{Ker}(J))^{\perp}$. Since J is complemented, there exist elements b in J and c in $\mathrm{Ker}(J)$ such that

$$a = b + c.$$

Then, since a and c are orthogonal and from the definition of $\mathrm{Ker}(J)$,

$$\{a\,c\,b\} = 0, \ \ \{b\,c\,b\} = 0,$$

and it can be seen that

$$\{c\,c\,b\} = \{a\,c\,b\} - \{b\,c\,b\} = 0.$$

Therefore,

$$\{c\,c\,c\} = \{c\,c\,a\} - \{c\,c\,b\} = 0,$$

and, by the anisotropy of A, c is equal to zero. Hence, a is equal to b and therefore lies in J. It follows that

$$c(\mathrm{Ker}(J))^{\perp} \subseteq J,$$

and, from the definition of the central kernel,

$$c(\mathrm{Ker}(J))^{\perp} \subseteq k(J). \tag{3.9}$$

Then, (i) follows from (3.8) and (3.9).

To prove (ii), replace J by $J^{\perp}$ in (i) to obtain

$$k(J^{\perp}) = c(\mathrm{Ker}(J^{\perp}))^{\perp}.$$

Therefore, by Lemma 3.5,

$$c(\mathrm{Ker}(J^{\perp})) = k(J^{\perp})^{\perp} = c(J),$$

completing the proof of the result. ∎

This result allows some information about the central hull $c(J_1)$ of the Peirce-one space J_1 to be determined.

COROLLARY 3.12. *Let A be a JBW*-triple and let J be a weak*-closed inner ideal in A, having central hull $c(J)$, central kernel $k(J)$, and Peirce-one space J_1. Then,*

$$c(J_1) \subseteq c(J) \cap k(J)^{\perp}.$$

Proof. Observe that, by Theorem 3.11,

$$\begin{aligned} J_1 &= \mathrm{Ker}(J) \cap \mathrm{Ker}(J^{\perp}) \\ &\subseteq c(\mathrm{Ker}(J)) \cap c(\mathrm{Ker}(J^{\perp})) \\ &= k(J)^{\perp} \cap c(J), \end{aligned}$$

and the proof is complete. ∎

Before going on to prove the most significant results, the following lemma is required.

LEMMA 3.13. *Let A be a JBW*-triple, let J be a weak*-closed inner ideal in A, having generalized Peirce spaces J_0, J_1, and J_2. Then,*

$$\{J_1\, J_2\, J_1\} \subseteq \mathrm{Ker}(J).$$

Proof. From the definition of kernel, observe that

$$\{J_2\, J_1\, J_2\} = \{0\}.$$

Therefore, using (2.1),

$$\begin{aligned}\{J_2\, \{J_1\, J_2\, J_1\}\, J_2\} \subseteq \{\{J_2\, J_1\, J_2\}\, J_1\, J_2\} + \{J_2\, J_1\, \{J_2\, J_1\, J_2\}\} \\ + \{J_2\, J_1\, \{J_2\, J_1\, J_2\}\} = \{0\},\end{aligned}$$

and

$$\{J_1\, J_2\, J_1\} \subseteq \mathrm{Ker}(J),$$

as required. ∎

The final results are concerned with the annihilator $(J_1)^{\perp}$ of the Peirce-one space associated with the weak*-closed inner ideal J in the JBW*-triple A.

THEOREM 3.14. *Let A be a JBW*-triple, let J be a weak*-closed inner ideal in A, having Peirce-one space J_1 and central kernel $k(J)$. Then,*

$$k(J) = (J_1)^{\perp} \cap J.$$

Proof. Let K denote the weak*-closed inner ideal $(J_1)^{\perp} \cap J$. Using the intersection table in the proof of Theorem 3.11, observe that J_1 is contained in $k(J)^{\perp}$, from which it follows that

$$k(J) \subseteq (J_1)^{\perp} \cap J = K. \tag{3.10}$$

Let a_1 and c_1 be elements of J_1, let d and e be elements of K, and let b_2 be an element of J_2. Then, by Lemma 3.13,

$$\begin{aligned}\{e\, d\, \{a_1\, b_2\, c_1\}\} &\subseteq \{e\, d\, \mathrm{Ker}(J)\,\} \\ &= \{e\, d\, J_0\} + \{e\, d\, J_1\} \\ &\subseteq \{K\, K\, J_0\} + \{K\, K\, J_1\} = \{0\},\end{aligned}$$

since K is orthogonal to both J_0 and J_1. For the same reason, using (2.1) and (3.10),

$$\begin{aligned}\{a_1\, \{d\, e\, b_2\}\, c_1\} &= \{\{e\, d\, a_1\}\, b_2\, c_1\} + \{a_1\, b_2\, \{e\, d\, c_1\}\} \\ &\quad - \{e\, d\, \{a_1\, b_2\, c_1\}\} \\ &= \{0\, b_2\, c_1\} + \{a_1\, b_2\, 0\} - 0 = 0.\end{aligned}$$

Therefore,

$$\{J_1\, \{K\, K\, J_2\}\, J_1\} = \{0\}. \tag{3.11}$$

Now let a_2 and b_2 be elements of J_2, let d and e be elements of K, and let c_1 be an element of J_1. Then, since d is orthogonal to J_1, and, using [23, Theorem 4.5] and (2.1),

$$\begin{aligned}\{a_2\,\{b_2\,e\,d\}\,c_1\} &= \{\{e\,b_2\,a_2\}\,d\,c_1\} + \{a_2\,d\,\{e\,b_2\,c_1\}\} \\ &\quad - \{a_2\,d\,\{e\,b_2\,c_1\}\} \\ &= \{a_2\,d\,\{e\,b_2\,c_1\}\} \in \{a_2\,d\,\{J_2\,J_2\,J_1\}\} \\ &\subseteq \{a_2\,d\,J_1\} = \{0\}.\end{aligned}$$

Therefore,

$$\{J_2\,\{K\,K\,J_2\}\,J_1\} = \{0\}. \tag{3.12}$$

Since

$$\{K\,K\,J_2\} \subseteq \{J_2\,J_2\,J_2\} \subseteq J_2,$$

it follows that

$$\{J_0\,\{K\,K\,J_2\}\,J_1\} \subseteq \{J_0\,J_2\,J_1\} = \{0\}. \tag{3.13}$$

Therefore, adding (3.11)–(3.13),

$$\{A\,\{K\,K\,J_2\}\,J_1\} = \{0\},$$

from which it follows that

$$\{K\,K\,J_2\} \subseteq (J_1)^{\perp}. \tag{3.14}$$

Consequently,

$$\begin{aligned}\{K\,K\,A\} &= \{K\,K\,J_2\} + \{K\,K\,J_1\} + \{K\,K\,J_0\} \\ &\subseteq (J_1)^{\perp} \cap J_2 + \{K\,(J_1)^{\perp}\,J_1\} + \{K\,J_2\,J_0\} \\ &= (J_1)^{\perp} \cap J_2 = K.\end{aligned}$$

Hence, by [8, Proposition 3.1], the weak*-closed inner ideal K is, in fact, an ideal contained in J. It follows from the definition of $k(J)$ that K is contained in $k(J)$. Combining this fact with (3.10), it follows that K and $k(J)$ coincide. ∎

THEOREM 3.15. *Let A be a JBW*-triple, let J be a weak*-closed inner ideal in A having annihilator $J^{\perp}$ and Peirce-one space J_1, and let $k(J^{\perp})$ be the central kernel of $J^{\perp}$. Then*

$$k(J^{\perp}) = (J_1)^{\perp} \cap J^{\perp}.$$

Proof. The generalized Peirce decomposition of A corresponding to the weak*-closed inner ideal $J^{\perp}$ is given by

$$\begin{aligned} A &= (J^{\perp})_2 \oplus (J^{\perp})_1 \oplus (J^{\perp})_0 \\ &= J^{\perp} \oplus (\mathrm{Ker}(J^{\perp}) \cap \mathrm{Ker}(J^{\perp\perp})) \oplus J^{\perp\perp}. \end{aligned}$$

Since J is contained in $J^{\perp\perp}$ it follows that $\mathrm{Ker}(J^{\perp\perp})$ is contained in $\mathrm{Ker}(J)$ and, therefore,

$$(J^{\perp})_1 = \mathrm{Ker}(J^{\perp}) \cap \mathrm{Ker}(J^{\perp\perp}) \subseteq \mathrm{Ker}(J^{\perp}) \cap \mathrm{Ker}(J) = J_1.$$

It follows that

$$(J_1)^{\perp} \subseteq ((J^{\perp})_1)^{\perp},$$

and, by Theorem 3.14,

$$(J_1)^{\perp} \cap J^{\perp} \subseteq ((J^{\perp})_1)^{\perp} \cap J^{\perp} = k(J^{\perp}). \tag{3.15}$$

Furthermore, since

$$\mathrm{Ker}(J^{\perp}) \subseteq c(\mathrm{Ker}(J^{\perp})),$$

by Lemma 3.5, Theorem 3.11(i), and (3.15),

$$\begin{aligned} k(J^{\perp}) &= c(\mathrm{Ker}(J^{\perp}))^{\perp} \subseteq \mathrm{Ker}(J^{\perp})^{\perp} \\ &= (J_1 \oplus J)^{\perp} = (J_1)^{\perp} \cap J^{\perp} \\ &\subseteq k(J^{\perp}), \end{aligned}$$

and

$$k(J^{\perp}) = (J_1)^{\perp} \cap J^{\perp},$$

as required. ∎

The result above has the following corollary.

COROLLARY 3.16. *Let A be a JBW*-triple and let J be a weak*-closed inner ideal in A, having annihilator $J^{\perp}$. Then:*

(i) $k(J^{\perp}) = \mathrm{Ker}(J^{\perp})^{\perp}$;

(ii) $c(J) = \mathrm{Ker}(J^{\perp})^{\perp\perp}$.

Proof. As in the proof of Theorem 3.15,

$$k(J^{\perp}) = (J_1 \oplus J)^{\perp} = \mathrm{Ker}(J^{\perp})^{\perp},$$

which completes the proof of (i). Furthermore, by Lemma 3.5,

$$c(J) = k(J^{\perp})^{\perp} = \mathrm{Ker}(J^{\perp})^{\perp\perp},$$

as required. ∎

It should be noted that the preceding three results are remarkable because they demonstrate that the four weak*-closed inner ideals $(J_1)^\perp \cap J$, $(J_1)^\perp \cap J^\perp$, $\mathrm{Ker}(J^\perp)^\perp$ and $\mathrm{Ker}(J)^{\perp\perp}$ in A are in fact ideals in A. The final result of this section shows that, when J is a Peirce inner ideal, even more can be said.

THEOREM 3.17. *Let A be a JBW*-triple, let J be a Peirce weak*-closed inner ideal in A, having annihilator $J^\perp$ and Peirce-one space J_1, and let $k(J)$ and $k(J^\perp)$ be the central kernels of J and $J^\perp$, respectively. Then,*

$$(J_1)^\perp = k(J) \oplus_M k(J^\perp).$$

Proof. Let a be an element of $(J_1)^\perp$, and let

$$a = P_0(J)a + P_1(J)a + P_2(J)a = a_0 + a_1 + a_2,$$

be its generalized Peirce decomposition relative to J. Then, since a is orthogonal to J_1,

$$\{a_0\, a_1\, a_1\} + \{a_1\, a_1\, a_1\} + \{a_2\, a_1\, a_1\} = \{a\, a_1\, a_1\} = 0.$$

From the Peirce relations (1.2), it follows that $\{a_0\, a_1\, a_1\}$ lies in J_0, $\{a_1\, a_1\, a_1\}$ lies in J_1, and $\{a_2\, a_1\, a_1\}$ lies in J_2, and, therefore

$$\{a_0\, a_1\, a_1\} = \{a_1\, a_1\, a_1\} = \{a_2\, a_1\, a_1\} = 0.$$

Since A is anisotropic, it follows that a_1 is equal to zero, and, hence, that a lies in $J_0 \oplus J_2$. Therefore, the weak*-closed inner ideal $(J_1)^\perp$ is contained in $J \oplus J^\perp$. By [23, Lemma 4.2], J is a weak*-closed ideal in the weak*-closed subtriple $J \oplus J^\perp$ of A, with relative annihilator $J^\perp$, and the decomposition of $J \oplus J^\perp$ is, in fact, an M-decomposition. Therefore, by [15, Theorem 4.6],

$$(J_1)^\perp = ((J_1)^\perp \cap J) \oplus_M ((J_1)^\perp \cap J^\perp) = k(J) \oplus_M k(J^\perp),$$

by Theorems 3.14 and 3.15. ∎

In the special case in which J is Peirce, this result allows the central hull $c(J_1)$ of the Peirce-one space J_1 to be determined.

COROLLARY 3.18. *Let A be a JBW*-triple, let J be a Peirce weak*-closed inner ideal in A, having central hull $c(J)$, central kernel $k(J)$, and Peirce-one space J_1. Then*

$$c(J_1) = c(J) \cap k(J)^\perp.$$

Proof. Notice that, by Theorem 3.17, since $J_1 \subseteq c(J_1)$, it follows that

$$c(J_1)^\perp \subseteq k((J_1)^\perp) = k(J) \oplus_M k(J^\perp).$$

Therefore, taking annihilators and using Lemma 3.5,

$$c(J) \cap k(J)^\perp = (k(J) \oplus_M k(J^\perp))^\perp \subseteq c(J_1).$$

The result follows from Corollary 3.12. ∎

4. EXAMPLES AND APPLICATIONS

Let B be a W*-algebra, and let $\mathscr{P}(B)$ be the complete orthomodular lattice of self-adjoint idempotents in B. Let $Z(B)$ be the commutative W*-algebra which is the algebraic centre of B. Then $\mathscr{P}(Z(B))$ coincides with the complete Boolean lattice that is the orthomodular lattice centre $\mathscr{Z}\mathscr{P}(B)$ of $\mathscr{P}(B)$. Moreover, with respect to the Jordan triple product defined, for elements a, b, and c in B, by

$$\{a\, b\, c\} = \frac{1}{2}(ab^*c + cb^*a),$$

B is a JBW*-triple. For details, the reader is referred to [41–43].

For each element e in $\mathscr{P}(B)$, the *central support* $c(e)$ of e is defined by

$$c(e) = \bigwedge\{z \in \mathscr{Z}\mathscr{P}(B) : e \le z\}.$$

A pair (e, f) of elements of $\mathscr{P}(B)$ is said to be *centrally equivalent* if $c(e)$ and $c(f)$ coincide. The common central support is denoted by $c(e, f)$. When endowed with the product ordering, the set $\mathscr{C}\mathscr{P}(B)$ of centrally equivalent pairs of elements of $\mathscr{P}(B)$ forms a complete lattice in which the lattice supremum coincides with the supremum in the product lattice, but, in general, the lattice infimum does not. The results of [18] show that the mapping $(e, f) \mapsto eBf$ is an order isomorphism from $\mathscr{C}\mathscr{P}(B)$ onto $\mathscr{I}(B)$.

A JBW*-triple A is said to be *rectangular* if there exists a W*-algebra B and an element (p, q) of $\mathscr{C}\mathscr{P}(B)$ such that A is isomorphic to the JBW*-triple pBq. In what follows the rectangular JBW*-triples A and pBq will be identified. Let $\mathscr{C}\mathscr{P}(B)_{(p,q)}$ denote the principal order ideal in $\mathscr{C}\mathscr{P}(B)$ consisting of elements (e, f) such that

$$(e, f) \le (p, q).$$

Then, the mapping $(e, f) \mapsto eAf$ is an order isomorphism from $\mathscr{C}\mathscr{P}(B)_{(p,q)}$ onto the complete lattice $\mathscr{I}(B)$ of weak*-closed inner ideals in B. Therefore, there exists a corresponding order isomorphism from $\mathscr{C}\mathscr{P}(B)_{(p,q)}$ onto $\mathscr{S}(A)$.

The mapping $z \mapsto pz$ is an order isomorphism from the complete Boolean lattice $\mathscr{Z}\mathscr{P}(B)_{c(p,q)}$ onto $\mathscr{Z}\mathscr{P}(pBp)$ or, equivalently, $\mathscr{Z}(\mathscr{P}(B)_p)$. To simplify notation, for e in the principal order ideal $\mathscr{P}(B)_p$ of $\mathscr{P}(B)$, let

$$c^p(e) = \bigwedge\{zp\colon z \in \mathscr{Z}\mathscr{P}(B)_{c(p,q)}, e \le z\}. \tag{4.1}$$

It is clear that $c^p(e)$ coincides with $c(e)p$. The results of [24] show that the mapping μ, defined, for each element z of the complete Boolean lattice $\mathscr{Z}\mathscr{P}(B)_{c(p,q)}$ and each element a in A, by

$$\mu(z)(a) = za,$$

is an order isomorphism onto the complete Boolean lattice of M-projections on A. It follows that the mapping $z \mapsto zA$ is an order isomorphism from $\mathcal{ZP}(B)_{c(p,q)}$ onto the complete Boolean lattice $\mathcal{ZI}(A)$ of weak*-closed ideals in A.

For each element (e, f) in $\mathcal{CP}(B)_{(p,q)}$ and each element z in $\mathcal{ZP}(B)_{c(p,q)}$, write

$$e'^{p} = p - e, \quad f'^{q} = q - f, \quad z'^{c(p,q)} = c(p, q) - z.$$

For an element (e, f) in $\mathcal{CP}(B)_{(p,q)}$, let

$$(e, f)'^{(p,q)} = (c^q(f'^q)e'^p, c^p(e'^p)f'^q).$$

Then, the mapping $(e, f) \mapsto (e, f)'^{(p,q)}$ is order reversing, and if J is the weak*-closed inner ideal eAf in A, then the annihilator $J^{\perp}$ coincides with $c^q(e'^q)e'^p Ac^p(e'^p)f'^q$. It follows that the generalized Peirce decomposition of A corresponding to the weak*-closed inner ideal J is given by

$$J = J_0 \oplus J_1 \oplus J_2,$$

where

$$J_2 = eAf, \quad J_0 = c^q(f'^q)e'^p Ac^p(e'^p)f'^q,$$

and

$$J_1 = ec^q(f'^q)Ac(e, f)f'^q + c(e, f)e'^p Ac^p(e'^p)f.$$

Furthermore, every weak*-closed inner ideal J is a Peirce inner ideal.

It is now possible to investigate the properties of the central hull and central kernel in this example.

THEOREM 4.1. *Let B be a W^*-algebra, let (p, q) be an element of the complete lattice $\mathcal{CP}(B)$ of pairs of centrally equivalent projections in B, and let A be the rectangular JBW*-triple pBq. Let (e, f) be an element of the complete lattice $\mathcal{CP}(B)_{(p,q)}$, let J be the weak*-closed inner ideal eAf, and let J_0, J_1 and J_2 be the corresponding generalized Peirce spaces defined above. Then*:

(i) $c(J) = c(J^{\perp\perp}) = c(e, f)A$;

(ii) $k(J) = c(e'^p)'^{c(p,q)} c(f'^q)'^{c(p,q)} A$;

(iii) $c(J^{\perp}) = c(e'^p)c(f'^q)A$;

(iv) $k(J^{\perp}) = c(e, f)'^{c(p,q)} A$;

(v) $k(J^{\perp\perp}) = (c(e'^p)'^{c(p,q)} \vee c(f'^q)'^{c(p,q)})A$;

(vi) $(J_1)^{\perp} = c(e'^p)'^{c(p,q)} c(f'^q)'^{c(p,q)} A \oplus_M c(e, f)'^{c(p,q)} A$;

(vii) $c(J_1) = c(e, f)(c(e'^p) \vee c(f'^q))A$.

Proof. The proof of (i) is immediate from the definition of $c(J)$ and from Corollary 3.6. From above, there exists an element r of $\mathcal{ZP}(B)_{c(p,q)}$ such that $k(J)$ coincides with rA. Observe that, using [24, (2.2)],

$$\begin{aligned} r &= \bigvee\{z \in \mathcal{ZP}(B)_{c(p,q)} : zp \le e,\ zq \le f\} \\ &= \bigvee\{z \in \mathcal{ZP}(B)_{c(p,q)} : e'^p \le z'p,\ f'^q \le z'q\} \\ &= \left(\bigwedge\{z \in \mathcal{ZP}(B)_{c(p,q)} : e'^p \le zp,\ f'^q \le zq\}\right)'^{c(p,q)} \\ &= (c^p(e'^p) \vee c^q(f'^q))'^{c(p,q)} = (c^p(e'^p)'^{c(p,q)} \wedge c^q(f'^q)'^{c(p,q)}) \\ &= c^p(e'^p)'^{c(p,q)} c^q(f'^q)'^{c(p,q)}. \end{aligned}$$

It follows from (4.1) that

$$k(J) = c^p(e'^p)'^{c(p,q)} c^q(f'^q)'^{c(p,q)} A = c(e'^p)'^{c(p,q)} c(f'^q)'^{c(p,q)} A,$$

and the proof of (ii) is complete.

Since $J^\perp$ coincides with $c^q(f'^q)e'^p A c^p(e'^p) f'^q$, it follows from (i) that

$$c(J^\perp) = c^p(e'^p)c^q(f'^q)A = c(e'^p)c(f'^q)A.$$

This completes the proof of (iii), and (iv) follows from (i) and Lemma 3.5. The same lemma and (iii) show that (v) holds, and (vi) follows from Theorem 3.17 and (ii) and (iv). The proof of (vii) is immediate from Corollary 3.18. ∎

Observe that, by putting p and q both equal to the unit in B, Theorem 4.1 describes the situation for an arbitrary W*-algebra.

Now let α be a *-antiautomorphism of order 2 of the continuous W*-algebra B, and let A be the weak*-closed subspace $H(B,\alpha)$ of B of elements that are invariant under α. Then A is a subtriple of B and is, in fact, a JBW*-algebra, for the properties of which the reader is referred to [14, 31, 49, 50]. If 1 denotes the unit in B, then the mapping $T \mapsto T1$ is an isometric *-isomorphism from the centroid $\mathcal{Z}^b(A)$ of A onto the centre $Z(A)$ of the JBW*-algebra A. Furthermore, it was shown in [27] that the mapping $e \mapsto eA\alpha(e)$ is an order isomorphism from the complete lattice $\mathcal{P}(B,\alpha)$ of projections in B with α-invariant central supports onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A. The generalized Peirce decomposition of A corresponding to the weak*-closed inner ideal J, which coincides with $eA\alpha(e)$, is given by

$$J = J_0 \oplus J_1 \oplus J_2,$$

where

$$J_0 = e'A\alpha(e'),\ J_1 = eA\alpha(e') + e'A\alpha(e),\ J_2 = eA\alpha(e).$$

As before every weak*-closed inner ideal J is a Peirce inner ideal. When applied in this case the results of the previous section lead to the following theorem, the proof of which follows closely that of Theorem 4.1.

THEOREM 4.2. *Let B be a continuous W^*-algebra, let α be a *-anti-automorphism of B of order 2, and let A be the JBW^*-algebra $H(B, \alpha)$ of α-invariant elements of B. Let e be an element of the complete lattice $\mathcal{P}(B, \alpha)$ of projections in B having α-invariant central support, let J be the weak*-closed inner ideal $eA\alpha(e)$, and let J_0, J_1, and J_2 be the corresponding generalized Peirce spaces defined above. Then*:

(i) $c(J) = c(J^{\perp\perp}) = c(e)A$;

(ii) $k(J) = c(e')'A$;

(iii) $c(J^{\perp}) = c(e')A$;

(iv) $k(J^{\perp}) = c(e)'A$;

(v) $k(J^{\perp\perp}) = c(e')'A$;

(vi) $(J_1)^{\perp} = c(e')'A \oplus_M c(e)'A$;

(vii) $c(J_1) = c(e)c(e')A$.

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