Regularity for multi-phase variational problems

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Abstract

We prove $C^{1,\nu}$ regularity for local minimizers of the multi-phase energy:

$$w \mapsto \int_\Omega |Dw|^p + a(x)|Dw|^q + b(x)|Dw|^s \, dx,$$

under sharp assumptions relating the couples (p, q) and (p, s) to the Hölder exponents of the modulating coefficients $a(\cdot)$ and $b(\cdot)$, respectively.

1 Introduction and results

The aim of this paper is to analyze the regularity properties of non-autonomous variational integrals of the type

$$w \mapsto \int_\Omega F(x, Dw)\, dx \,, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain with $n \geq 2$, and emphasize a few new phenomena emerging when considering non-uniformly elliptic operators. Let us briefly review the situation. In the case of functionals satisfying standard polynomial growth and ellipticity of the type

$$F(x, Dw) \approx |Dw|^p \qquad \text{and} \qquad \partial_{zz} F(x, Dw) \approx |Dw|^{p-2} Id \,, \tag{1.2}$$

as for instance

$$w \mapsto \int_\Omega a(x)|Dw|^p \, dx \,, \qquad 0 < \nu \leq a(x) \leq L \,, \tag{1.3}$$

the regularity of minimizers is well-understood. In particular, assuming that the partial function $x \mapsto F(x, \cdot)$ is Hölder continuous with some exponent (for example, $a(\cdot)$ is locally Hölder continuous in the case of (1.3)), then it turns out that the gradient of minima is locally Hölder continuous. This is a well established theory, both in the scalar and in the vectorial case, for which we refer for instance to [30, 31, 34, 38]. The situation drastically changes when considering non-uniformly elliptic functionals. These are functionals so that the ellipticity ratio

$$\mathcal{R}(z, B) := \frac{\sup_{x\in B} \text{ highest eigenvalue of } \partial_{zz} F(x, z)}{\inf_{x\in B} \text{ lowest eigenvalue of } \partial_{zz} F(x, z)} \,,$$

where $B \subset \Omega$ is a ball, might become unbounded with $|z|$. This is the case, for instance, of the double phase functional given by

$$W^{1,H(\cdot)}(\Omega) \ni w \mapsto \int_\Omega |Dw|^p + a(x)|Dw|^q \, dx \,. \tag{1.4}$$

This functional has been introduced by Zhikov in the context of Homogenization and its integrand changes its growth - from p to q-rate - depending on the fact that x belongs to $\{a(\cdot) = 0\}$ or not (here is where the

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terminology double phase stems from). In the first case we have, following a terminology introduced in [11], the p-phase, in the other we have the (p,q)-phase. In the case of (1.4), the regularity of minimizers is regulated by a subtle interaction between the pointwise behaviour of the partial function $x \mapsto F(x,\cdot)$ and the growth assumption satisfied by $z \mapsto F(\cdot,z)$. For instance, as established in the work of Baroni, Colombo and Mingione [2, 3, 4, 11, 12], sufficient and necessary conditions for regularity of minimizers of the functional (1.4) are that

$$a(\cdot) \in C^{0,\alpha}(\Omega) \qquad \text{and} \qquad \frac{q}{p} \leq 1 + \frac{\alpha}{n}\,. \tag{1.5}$$

Specifically, if (1.5) holds, then minimizers of the functional (1.4) are locally C^{1,β_0}-regular, for some $\beta_0 \in (0,1)$, otherwise, they can be even discontinuous; see also [23, 24]. After these contributions, functionals with double phase type have become a topic of intense study, see for instance [7, 8, 9, 16, 17, 28, 29, 39, 40]. The condition in (1.5) plays a role also when considering more general functionals of the type in (1.1),under so called (p,q)-growth conditions, i.e.:

$$|Dw|^p \lesssim F(x,Dw) \lesssim |Dw|^q \qquad \text{and} \qquad |Dw|^{p-2}Id \lesssim \partial_{zz}F(x,Dw) \lesssim |Dw|^{q-2}Id\,.$$

For this we refer to [10, 21, 23]. Moreover, it intervenes in the validity of a corresponding Calderón-Zygmund theory [13, 15]. We refer to the papers of Marcellini [35, 36, 37] for more on general functionals with (p,q)-growth. The aim of this paper is to study a significant generalization of the functional (1.4), considering a functional that exhibit three phases. We shall indeed consider the following Multi-Phase variational energy

$$W^{1,H(\cdot)}(\Omega) \ni w \mapsto \mathcal{H}(w,\Omega) := \int_\Omega H(x,Dw)\,dx, \quad 1 < p<q\leq s\,, \tag{1.6}$$

with

$$H(x,z) := |z|^p + a(x)|z|^q + b(x)|z|^s, \tag{1.7}$$

and where the functions $a(\cdot)$ and $b(\cdot)$ satisfy the following assumptions

$$a \in C^{0,\alpha}(\Omega),\ \ a(\cdot) \geq 0,\ \ \alpha \in (0,1], \qquad b \in C^{0,\beta}(\Omega),\ \ b(\cdot) \geq 0,\ \ \beta \in (0,1]. \tag{1.8}$$

As not to trivialize the problem, we specifically focus on the case in which the strict inequality $(1.6)_2$ holds. The analysis of this functional then opens the way to that of functional exhibiting an arbitrary number of phases, and involves several subtle points. The main one can be described as follows. In the double phase case of the functional (1.4) the main game is to control the interaction between the potentially degenerate parte of the energy $a(x)|Dw|^q$ (here degenerate means that it can be $a(x) \equiv 0$) with the non-degenerate one $|Dw|^p$, that always provides a solid rate of ellipticity. This is done in [4, 11, 12] via a careful comparison scheme built in order to distinguish between the two phases. Here the situation changes and the matter becomes more delicate. Indeed, the problem is to control the interaction between the two possibly degenerate parts of the energy, that is $a(x)|Dw|^q$ and $b(x)|Dw|^s$. A new aspect in fact emerges here. We see that, in presence of a finer structure, conditions of the type in (1.5) can be in a sense relaxed. In fact, an immediate application of (1.5) would provide us with the conditions $a(\cdot), b(\cdot) \in C^{0,\alpha}(\Omega)$ with $q/p, s/p \leq 1 + \alpha/n$, by considering the global regularity of $x \mapsto F(x,\cdot)$. Instead, we see that the new condition coming into the play takes into account more precisely the way the presence of x affects the growth with respect to the gradient variable. Specifically, we shall assume that

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n} \quad \text{and} \quad \frac{s}{p} \leq 1 + \frac{\beta}{n}. \tag{1.9}$$

In other words, less regularity is needed on the coefficient affecting the q-growth, intermediate part of the energy density. The main result of the paper is the following (see the next section for more definitions and notation):

Theorem 1 **($C^{1,\nu}$-local regularity)** *Let u be a local minimizer of the functional* (1.6) *under assumptions* (1.8) *and* (1.9)*. Then there exists $\nu = \nu(\texttt{data}) \in (0,1)$ such that $u \in C^{1,\nu}_{\rm loc}(\Omega)$.*

We remark that the sharpness of both conditions in (1.9) can be obtained by the same counterexamples in [23, 24]. Moreover, as it is well-known from the regularity theory for the standard p-Laplacean case, the one in Theorem 1 is the maximal regularity obtainable for u. A worth singling-out intermediate result towards the proof of Theorem 1 is an intrinsic Morrey decay estimate, which reduces to a classical estimate in the case of the p-Laplacean and that extends to the multi-phase case the one proved in [4, 11, 12] for minima of functionals with a double phase.

Theorem 2 **(Intrinsic Morrey Decay)** *Let u be a local minimizer of the functional* (1.6) *under assumptions* (1.8) *and* (1.9)*. Then, for every $\vartheta \in (0,n)$, there exists a positive constant $c = c(\texttt{data}(\Omega_0), \vartheta)$ such that the decay estimate*

$$\int_{B_\varrho} H(x,Du)\,dx \le c\left(\frac{\varrho}{r}\right)^{n-\vartheta}\int_{B_r} H(x,Du)\,dx \tag{1.10}$$

holds whenever $B_\varrho \subset B_r \Subset \Omega_0 \Subset \Omega$ are concentric balls with $0 < \varrho \le r \le 1$.

Let us quickly describe the techniques we are employing to obtain the aforementioned theorems. The starting point is the recent proof of regularity of minimizers of double-phase variational problems appeared in [4], and based on a suitable use of harmonic type approximations lemmas (see also [12] for a first version). This is just a general blueprint we move from to treat the the real new difficulty here. Indeed, as we are dealing here with the presence of several phase transitions, and we have to carefully handle the regularity of solutions on the zero sets $\{x \in \Omega\colon a(x) = 0\}$ and $\{x \in \Omega\colon b(x) = 0\}$, that is, when the functional tends to loose part of its ellipticity properties and switch their kind of ellipticity. Therefore we have to handle the presence of two different transitions. We come up with a delicate scheme of alternatives and of nested exit time arguments, carefully controlling the interaction between the two phase transitions. It is then clear that the techniques introduced in this paper allow to prove regularity results for functionals with an arbitrary large numbers of phases, for instance,

$$w \mapsto \int_\Omega \left[|Dw|^p + \sum_{i=1}^m a_i(x)|Du|^{p_i}\right] dx$$

with

$$a_i(\cdot) \in C^{0,\alpha_i}(\Omega), \quad 1 < \frac{p_i}{p} \le 1 + \frac{\alpha_i}{n}, \quad 1 < p < p_1 \le \cdots \le p_m. \tag{1.11}$$

Finally, let us compare our result with the one obtained in [22, Theorem 1.1]. Both essentially give optimal regularity for the gradient of minimizers, but there are some evident differences. Even if [22, Theorem 1.1] treats a larger class of functionals than the one covered by our Theorem 1, we do not require any differentiability for the coefficients $a(\cdot)$ and $b(\cdot)$, cf. assumption (1.8). Moreover, we allow various rates of ellipticity, as prescribed in (1.11). As anticipated above, asking that $1 < \frac{q}{p} < \frac{s}{p} \le 1 + \frac{\min\{\alpha,\beta\}}{n}$ would result in a trivialization of the problem. With minor modification, see [4, 16, 33], our procedure also covers the regularity theory for minimizers of functionals with multi-phase structure, i.e. variational integrals of the type

$$W^{1,H(\cdot)}(\Omega) \ni w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega \left[f_p(Dw) + \sum_{i=1}^m a_i(x) f_{p_i}(Dw)\right] dx, \quad 1 < p < p_1 \le \cdots \le p_m,$$

with $f_p(z) \sim |z|^p$, $f_{p_i}(z) \sim |z|^{p_i}$ for all $z \in \mathbb{R}^n$, $a_i(\cdot) \in C^{0,\alpha_i}(\Omega)$ and $1 < \frac{p_i}{p} \le 1 + \frac{\alpha_i}{n}$ for any $i \in \{1, \cdots, m\}$.

2 Notation and preliminaries

In this section we establish some basic notation that we are going to use for the rest of the paper. As in the Introduction, Ω will denote an open, bounded subset of $\mathbb{R}^n$ with $n \geq 2$. As usual, we shall denote by c a general constant larger than one, which can vary from line to line. Relevant dependencies from certain parameters will be emphasized using brackets, i.e.: $c = c(n, p, q, s)$ means that c depends on n, p, q, s. We denote with $B_r(x_0) = \{x \in \mathbb{R}^n : \ |x - x_0| < r\}$ the n-dimensional open ball centered at x_0 and with radius $r > 0$; when non relevant or clear from the context, we will omit to indicate the centre as follows: $B_r = B_r(x_0)$. When not differently specified, in the same context, balls with different radius will share the same center. If $A \subset \mathbb{R}^n$ is any measurable subset with finite and positive Lebesgue measure $|A| > 0$ and $f : A \rightarrow \mathbb{R}^N$, $N \geq 1$ is a measurable map, we shall denote its integral average over A as

$$(f)_A := \fint_A f(x)\, dx = \frac{1}{|A|} \int_A f(x)\, dx.$$

When $A = B_r$, we shall write $(f)_r := (f)_{B_r}$. The integrand $H(\cdot)$ has already been defined in (1.7). With abuse of notation we shall denote $H(x, z)$ when $z \in \mathbb{R}^n$ and when $z \in \mathbb{R}$, that is when z is a scalar, so that we shall intend both $H : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$ and $H : \Omega \times \mathbb{R} \rightarrow [0, \infty)$. The modulating coefficients $a(\cdot)$ and $b(\cdot)$ will always satisfy (1.8). Here we recall that, if $f : \Omega \rightarrow \mathbb{R}$ is any γ-Hölder continuous map with $\gamma \in (0, 1)$ and $A \subset \Omega$, then its Hölder seminorm is defined as

$$[f]_{0,\gamma;A} := \sup_{x,y \in A,\ x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma}, \quad [f]_{0,\gamma} := [f]_{0,\gamma;\Omega}.$$

With "∂" we denote the partial derivative with respect to the z-variable. We are going to use several tools from the Orlicz space setting, therefore we start with the following preliminaries.

Definition 1 *A function $\varphi : [0, \infty) \rightarrow [0, \infty)$ is said to be a Young function if it satisfies the following conditions: $\varphi(0) = 0$ and there exists the derivative φ', which is right-continuous, non decreasing and satisfies*

$$\varphi'(0) = 0, \quad \varphi'(t) > 0 \quad \textit{for } t > 0, \quad \textit{and} \quad \lim_{t \to \infty} \varphi'(t) = \infty.$$

Remark 1 In order to extrapolate good regularity properties for minimizers of functionals with φ-growth, we need to assume something more. Precisely, from now on, in addition to the basic assumptions listed in Definition 1 we will also suppose that $\varphi \in C^1[0, \infty) \cap C^2(0, \infty)$ and that

$$i_\varphi \leq \frac{t\varphi''(t)}{\varphi'(t)} \leq s_\varphi \ \text{ unifornly in } t\,. \tag{2.1}$$

This is equivalent to the so-called Δ_2 condition, since $t \mapsto \varphi(t)$ is non decreasing, see [19], Section 2.

Definition 2 *Let φ be a Young function in the sense of Definition 1 and Remark 1. Given $\Omega \subset \mathbb{R}^n$, the Orlicz space $L^\varphi(\Omega)$ is defined as*

$$L^\varphi(\Omega) := \left\{ u : \Omega \rightarrow \mathbb{R} \text{ such that } \int_\Omega \varphi(|u|)\, dx < \infty \right\}$$

and, consequently,

$$W^{1,\varphi}(\Omega) := \left\{ u \in W^{1,1}(\Omega) \cap L^\varphi(\Omega) \text{ such that } Du \in L^\varphi(\Omega, \mathbb{R}^N) \right\}.$$

The definitions of the variants $W^{1,\varphi}_0(\Omega)$ and $W^{1,\varphi}_{\mathrm{loc}}(\Omega)$ come in an obvious way from the one of $W^{1,\varphi}(\Omega)$.

In connection to $H(\cdot)$, we also consider the following Orlicz-Musielak-Sobolev space

$$W^{1,H(\cdot)}(\Omega) := \left\{u \in W^{1,1}(\Omega) \colon \ H(\cdot, Du) \in L^1(\Omega)\right\}, \tag{2.2}$$

with local variant defined in an obvious way and $W^{1,H(\cdot)}_0(\Omega) = W^{1,H(\cdot)}(\Omega) \cap W^{1,p}_0(\Omega)$; we refer to [4, 28, 29] for more on such spaces. For later uses, we introduce also the auxiliary Young functions

$$\begin{cases} H_0(z) := |z|^p + a_0|z|^q + b_0|z|^s, \\ H_0^s(z) := |z|^p + b_0|z|^s, \\ H_0^q(z) := |z|^p + a_0|z|^q, \\ H_0^p(z) := |z|^p. \end{cases} \tag{2.3}$$

The values of the constants $a_0, b_0 \geq 0$ will vary according to the necessities, in particular they shall often assume the values $a_0 = a_i(B_r)$ and $b_0 = b_i(B_r)$, where

$$a_i(B_r) := \inf_{x \in B_r} a(x) \ \text{ and } \ b_i(B_r) := \inf_{x \in B_r} b(x). \tag{2.4}$$

In accordance to this terminology, we also mention the auxiliary Young functions

$$H^-_{B_r}(z) := |z|^p + a_i(B_r)|z|^q + b_i(B_r)|z|^s,$$
$$H^-(z) := |z|^p + \left[\inf_{x\in\Omega} a(x)\right]|z|^q + \left[\inf_{x\in\Omega} b(x)\right]|z|^s$$

In the following we will often use the vector field

$$V_t(z) := |z|^{(t-2)/2}z, \quad t \in \{p, q, s\}. \tag{2.5}$$

We recall from [19] important features of (2.5): there exists $c = c(n,t) > 0$ such that

$$|V_t(z_1) - V_t(z_2)|^2 \leq c\left(|z_1|^{t-2}z_1 - |z_2|^{t-2}z_2\right)\cdot(z_1 - z_2), \tag{2.6}$$

$$|V_t(z_1) - V_t(z_2)| \sim (|z_1| + |z_2|)^{\frac{t-2}{2}}|z_1 - z_2|, \tag{2.7}$$

where the constants implicit in (2.7) depend only on n, t and, for all $z \in \mathbb{R}^n$

$$|V_t(z)|^2 = |z|^t \quad . \tag{2.8}$$

We also introduce the following auxiliary functions

$$\begin{cases} \mathcal{V}_0(z_1,z_2)^2 := |V_p(z_1) - V_p(z_2)|^2 + a_0|V_q(z_1) - V_q(z_2)|^2 + b_0|V_s(z_1) - V_s(z_2)|^2, \\ \mathcal{V}_0^s(z_1,z_2)^2 := |V_p(z_1) - V_p(z_2)|^2 + b_0|V_s(z_1) - V_s(z_2)|^2, \\ \mathcal{V}_0^q(z_1,z_2)^2 := |V_p(z_1) - V_p(z_2)|^2 + a_0|V_q(z_1) - V_q(z_2)|^2, \\ \mathcal{V}_0^p(z_1,z_2)^2 := |V_p(z_1) - V_p(z_2)|^2. \end{cases} \tag{2.9}$$

Let us also recall some important tools in regularity. The first one is an iteration lemma from [25].

Lemma 1 *Let $h \colon [\varrho, R_0] \to \mathbb{R}$ be a non-negative bounded function and $0 < \theta < 1$, $0 \leq A$, $0 < \beta$. Assume that $h(r) \leq A(d-r)^{-\beta} + \theta h(d)$ for $\varrho \leq r < d \leq R_0$. Then $h(\varrho) \leq cA/(R_0 - \varrho)^{-\beta}$ holds, where $c = c(\theta, \beta) > 0$.*

Along the proof we shall make an intensive use of the regularity properties of φ-harmonic maps, so we recall definition and some reference estimates from [19].

Definition 3 *Let $U \Subset \Omega$ be an open set and $u_0 \in W^{1,\varphi}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be any function. With φ-harmonic map, we mean a map $h \in u_0 + W^{1,\varphi}_0(U, \mathbb{R}^N)$ solving the Dirichlet problem*

$$u_0 + W^{1,\varphi}_0(U, \mathbb{R}^N) \ni w \mapsto \min \int_U \varphi(|Dw|)\, dx.$$

The next proposition reports a Lipschitz type estimate for the gradient of φ-harmonic maps.

Proposition 1 *[19, Lemma 5.8] Let $\Omega \subset \mathbb{R}^n$ be open and $\varphi \in C^2(0,\infty) \cap C^1[0,\infty)$ be a Young function satisfying (2.1). If $h \in W^{1,\varphi}(\Omega, \mathbb{R}^N)$ is φ-harmonic on Ω in the sense of Definition 3, then for any ball B_r with $B_{2r} \Subset \Omega$ there holds*

$$\sup_{B_r} \varphi(|Dh|) \le c \fint_{B_{2r}} \varphi(|Dh|)\, dx,$$

where c depends only on $n, N, i_\varphi, s_\varphi$.

We conclude this section by giving the definition of a local minimizer of (1.6).

Definition 4 *A map $u \in W^{1,H(\cdot)}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the variational integral (1.6) if and only if $H(\cdot, Du) \in L^1_{\mathrm{loc}}(\Omega)$ and the minimality condition $\mathcal{H}(u, \mathrm{supp}(u-v)) \le \mathcal{H}(v, \mathrm{supp}(u-v))$ is satisfied whenever $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $\mathrm{supp}(u-v) \subset \Omega$.*

3 First regularity results

In this section we collect a few basic regularity results which can be proved with minor adjustments to the proofs contained in [4, 11, 12, 28, 39]. We start with the proof of Sobolev-Poincaré inequality.

Lemma 2 (Sobolev-Poincaré inequality) *Let $1 < p<q<s$ and $\alpha, \beta \in (0,1]$ verifying (1.8), (1.9). Then there exist a constant $c = c(n,p,q,s)$ and an exponent $d = d(n,p,q,s) \in (0,1)$ such that for any $w \in W^{1,H(\cdot)}(B_r)$ with $r \le 1$,*

$$\fint_{B_r} H\left(x, \frac{w-(w)_r}{r}\right) dx \le c\left(1 + [a]_{0,\alpha}\|Dw\|^{q-p}_{L^p(B_r)} + [b]_{0,\beta}\|Dw\|^{s-p}_{L^p(B_r)}\right)\left(\fint_{B_r} H(x, Dw)^d\, dx\right)^{\frac{1}{d}}. \tag{3.1}$$

Furthermore, the same is still true with $w-(w)_r$ replaced by w if we consider $w \in W^{1,H(\cdot)}_0(B_r)$.

Proof. We first look at the case

$$\sup_{x\in B_r} a(x) \le 4[a]_{0,\alpha} r^\alpha \quad \text{and} \quad \sup_{x \in B_r} b(x) \le 4[b]_{0,\beta} r^\beta, \tag{3.2}$$

and notice that, by virtue of (1.9), $q_* < p$ and $s_* < p$, where

$$t_* := \max\left\{\frac{nt}{n+t}, 1\right\}, \quad t \in \{p,q,s\}.$$

Then it follows from the classical Sobolev-Poincaré inequality, Holder inequality and (1.9) that

$$\fint_{B_r} a(x)\frac{|w-(w)_r|^q}{r^q}\, dx \le 4[a]_{0,\alpha} r^\alpha \fint_{B_r} \frac{|w-(w)_r|^q}{r^q}\, dx \le c[a]_{0,\alpha} r^\alpha \left(\fint_{B_r} |Dw|^{q_*} dx\right)^{\frac{q}{q_*}}$$

$$\leq c[a]_{0,\alpha}\left(\fint_{B_r}|Dw|^p\,dx\right)^{\frac{q-p}{p}}\left(\fint_{B_r}|Dw|^{pd_a}\,dx\right)^{\frac{1}{d_a}}$$

$$\leq c[a]_{0,\alpha}\|Dw\|^{q-p}_{L^p(B_r)}\left(\fint_{B_r}|Dw|^{pd_a}\,dx\right)^{\frac{1}{d_a}}, \tag{3.3}$$

with $c = c(n,q)$ and $d_a := \frac{q_*}{p} < 1$. In a similar fashion,

$$\fint_{B_r} b(x)\frac{|w-(w)_r|^s}{r^s}\,dx \leq 4[b]_{0,\beta}r^\beta \fint_{B_r}\frac{|w-(w)_r|^s}{r^s}\,dx \leq c[b]_{0,\beta}r^\beta\left(\fint_{B_r}|Dw|^{s_*}dx\right)^{\frac{s}{s_*}}$$

$$\leq c[b]_{0,\beta}\left(\fint_{B_r}|Dw|^p\,dx\right)^{\frac{s-p}{p}}\left(\fint_{B_r}|Dw|^{pd_b}\,dx\right)^{\frac{1}{d_b}}$$

$$\leq c[b]_{0,\beta}\|Dw\|^{s-p}_{L^p(B_r)}\left(\fint_{B_r}|Dw|^{pd_b}\,dx\right)^{\frac{1}{d_b}}, \tag{3.4}$$

where $c = c(n,s)$ and $d_b := \frac{s_*}{p} < 1$. In addition, it is clear that

$$\fint_{B_r}\frac{|w-(w)_r|^p}{r^p}\,dx \leq c\left(\fint_{B_r}|Dw|^{p_*}dx\right)^{\frac{p}{p_*}} = c\left(\fint_{B_r}|Dw|^{pd_0}\,dx\right)^{\frac{1}{d_0}},$$

for $c = c(n,p)$ and $d_0 := \frac{p_*}{p} < 1$. We remark from (1.6) that $p_* < q_* < s_*$. Combining estimates (3.3)-(3.4), we get

$$\fint_{B_r} H\left(x,\frac{w-(w)_r}{r}\right)dx \leq c\left(1+[a]_{0,\alpha}\|Dw\|^{q-p}_{L^p(B_r)}+[b]_{0,\beta}\|Dw\|^{s-p}_{L^p(B_r)}\right)\left(\fint_{B_r}|Dw|^{pd_b}dx\right)^{\frac{1}{d_b}}$$

$$\leq c\left(1+[a]_{0,\alpha}\|Dw\|^{q-p}_{L^p(B_r)}+[b]_{0,\beta}\|Dw\|^{s-p}_{L^p(B_r)}\right)\left(\fint_{B_r}H(x,Dw)^{d_b}\,dx\right)^{\frac{1}{d_b}}, \tag{3.5}$$

where $c = c(n,p,q,s)$. We now turn to the case

$$\sup_{x\in B_r} a(x) > 4[a]_{0,\alpha}r^\alpha \quad\text{and}\quad \sup_{x\in B_r} b(x) > 4[b]_{0,\beta}r^\beta, \tag{3.6}$$

and notice that

$$\inf_{x\in B_r} a(x) \geq \frac{1}{2}\sup_{x\in B_r} a(x) \quad\text{and}\quad \inf_{x\in B_r} b(x) \geq \frac{1}{2}\sup_{x\in B_r} b(x). \tag{3.7}$$

So, if we set $a_0 := a(x_0)$ and $b_0 := b(x_0)$ for $x_0 \in B_r$, we obtain

$$\begin{cases} a(x)t^q \leq 2\left[\inf_{x\in B_r} a(x)\right]t^q \leq 2a_0t^q \leq 4a(x)t^q \\ b(x)t^s \leq 2\left[\inf_{x\in B_r} b(x)\right]t^s \leq 2b_0t^s \leq 4b(x)t^s \end{cases} \quad\text{for all } x\in B_r,\ t\geq 0. \tag{3.8}$$

The content of the above display and the Sobolev Poincare inequality valid for the Young function H_0, see [18, Theorem 7], gives

$$\fint_{B_r} H\left(x,\frac{w-(w)_r}{r}\right)dx \leq 2\fint_{B_r} H_0\left(x,\frac{w-(w)_r}{r}\right)dx \leq c\left(\fint_{B_r} H_0(Dw)^{d_1}\,dx\right)^{\frac{1}{d_1}} \leq c\left(\fint_{B_r} H(x,Dw)^{d_1}\,dx\right)^{\frac{1}{d_1}}, \tag{3.9}$$

with $c = c(n,p,q,s)$ and $d_1 = d_1(n,p,q,s) < 1$. Let us examine the case

$$\sup_{x\in B_r} a(x) > 4[a]_{0,\alpha} r^{\alpha} \quad \text{and} \quad \sup_{x\in B_r} b(x) \le 4[b]_{0,\beta} r^{\beta} \tag{3.10}$$

and notice that, by $(3.10)_1$, $(3.7)_1$ and $(3.8)_1$ hold true. This, (3.4) and the Sobolev-Poincaré inequality valid for the Young function H_0^q render that

$$\begin{aligned} \fint_{B_r} H\left(x, \frac{w-(w)_r}{r}\right) dx \le & 2 \fint_{B_r} H_0^q\left(\frac{w-(w)_r}{r}\right) dx + c[b]_{0,\beta} \|Dw\|_{L^p(B_r)}^{s-p} \left(\fint_{B_r} |Dw|^{pd_b}\, dx\right)^{\frac{1}{d_b}} \\ \le & c\left(\fint_{B_r} H_0^q(Dw)^{d_q}\, dx\right)^{\frac{1}{d_q}} + c[b]_{0,\beta} \|Dw\|_{L^p(B_r)}^{s-p} \left(\fint_{B_r} |Dw|^{pd_b}\, dx\right)^{\frac{1}{d_b}} \\ \le & c\left(1 + [b]_{0,\beta} \|Dw\|_{L^p(B_r)}^{s-p}\right)\left(\fint_{B_r} H(x,Dw)^{d_2}\, dx\right)^{\frac{1}{d_2}}, \end{aligned} \tag{3.11}$$

with $c = c(n,p,q,s)$, $d_q = d_q(n,p,q) < 1$ and $d_2 := \max\{d_q, d_b\} < 1$. Finally, we look at the case

$$\sup_{x\in B_r} a(x) \le 4[a]_{0,\alpha} r^{\alpha} \quad \text{and} \quad \sup_{x\in B_r} b(x) > 4[b]_{0,\beta} r^{\beta}. \tag{3.12}$$

Inequality $(3.12)_2$ validates $(3.7)_2$ and $(3.8)_2$, therefore, using this time (3.3), the Sobolev Poincaré inequality holding for the Young function H_0^s we get

$$\begin{aligned} \fint_{B_r} H\left(x, \frac{w-(w)_r}{r}\right) dx \le & c[a]_{0,\alpha} \|Dw\|_{L^p(B_r(\Omega))}^{q-p} \left(\fint_{B_r} |Dw|^{pd_a}\, dx\right)^{\frac{1}{d_a}} + 2\fint_{B_r} H_0^s\left(\frac{w-(w)_r}{r}\right) dx \\ \le & c[a]_{0,\alpha} \|Dw\|_{L^p(B_r(\Omega))}^{q-p} \left(\fint_{B_r} H(x,Dw)^{d_a}\, dx\right)^{\frac{1}{d_a}} + c\left(\fint_{B_r} H_0^s(Dw)^{d_s}\, dx\right)^{\frac{1}{d_s}} \\ \le & c\left(1 + [a]_{0,\alpha} \|Dw\|_{L^p(B_r(\Omega))}^{q-p}\right)\left(\fint_{B_r} H(x,Dw)^{d_3}\, dx\right)^{\frac{1}{d_3}}, \end{aligned} \tag{3.13}$$

with $c = c(n,p,q,s)$, $d_s = d_s(n,p,s) < 1$ and $d_3 := \max\{d_a, d_s\} < 1$. Setting $d := \max\{d_b, d_1, d_2, d_3\} < 1$, from estimates (3.5), (3.9), (3.11) and (3.13) we obtain (3.1). □

Remark 2 An inequality of the type of (3.1) holds for general Sobolev maps $w \in W^{1,H(\cdot)}$ such that $w \equiv 0$ on a set A such that $|A| \ge \gamma |B_r|$. Precisely, we have that

$$\fint_{B_r} H\left(x, \frac{w}{r}\right) dx \le c\left(\fint_{B_r} H(x,Dw)^{d}\, dx\right)^{\frac{1}{d}}, \tag{3.14}$$

where $d < 1$ is the same as the one appearing in (3.1) and $c = c(\gamma, n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|Dw\|_{L^p(B_r)})$.

Next, we have a Caccioppoli-type inequality, which proof can be retrieved from those in [11, 28].

Lemma 3 (Caccioppoli Inequalities) *Let $u \in W^{1,H(\cdot)}_{\mathrm{loc}}(\Omega)$ be a local minimizer of (1.6), with $a(\cdot)$, $b(\cdot)$ and p, q, s satisfy (1.8) and (1.9) respectively. Then there exists a constant $c = c(n,p,q,s) > 0$ such that*

$$\fint_{B_\varrho} H(x,Du)\, dx \le c \fint_{B_r} H\left(x, \frac{u-(u)_r}{r-\varrho}\right) dx, \tag{3.15}$$

and for $\kappa \in \mathbb{R}$,

$$\int_{B_\varrho} H(x, D(u-\kappa)_\pm)\, dx \le c \int_{B_r} H\left(x, \frac{(u-\kappa)_\pm}{r-\varrho}\right) dx. \tag{3.16}$$

A direct consequence of (3.15) and (3.1) is the following inner local higher integrability result.

Lemma 4 (Gehring's Lemma) *There are* $c = c(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|Du\|_{L^p(B_r)}) > 0$ *and a positive integrability exponent* $\delta_g = \delta_g(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|Du\|_{L^p(B_r)})$ *such that if* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *is a local minimizer of the integral functional* (1.6) *under assumptions* (1.8)-(1.9)*, then*

$$H(\cdot, Du) \in L^{1+\delta_g}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \left(\fint_{B_{r/2}} H(x, Du)^{1+\delta_g}\, dx\right)^{\frac{1}{1+\delta_g}} \le c \fint_{B_r} H(x, Du)\, dx, \ \ \textit{for all} \ \ B_r \Subset \Omega. \tag{3.17}$$

From $(3.17)_1$ it follows that $u \in W^{1,p(1+\delta_g)}_{\mathrm{loc}}(\Omega)$, so $u \in W^{1,p(1+\delta_g)}(\Omega_0)$ for $\Omega_0 \Subset \Omega$. Moreover, by Hölder inequality, (3.17) is true if δ_g is replaced by any $\sigma \in (0, \delta_g)$. The next one is an up to the boundary higher integrability result for a solution of Dirichlet problems related to the multi-phase energy $H(\cdot)$. Clearly, when $a(\cdot) \equiv a_0 = \text{const}$ and $b(\cdot) \equiv b_0 = \text{const}$, it extends to the auxiliary Young functions H^p_0, H^q_0, H^s_0 and H_0. In this case, $[a]_{0,\alpha} = [a_0]_{0,\alpha} = 0$ and $[b]_{0,\beta} = [b_0]_{0,\beta} = 0$, so constants and exponents do not depend either on $[a]_{0,\alpha}$, $[b]_{0,\beta}$ nor on $\|Dv\|_{L^p(B_r)}$.

Lemma 5 (Higher integrability up to the boundary) *Let assumptions* (1.8)-(1.9) *be in force,* $B_r \Subset \Omega_0 \Subset \Omega$*, and* $v \in W^{1,H(\cdot)}_u(B_r)$ *be a solution to the Dirichlet problem*

$$u + W^{1,H(\cdot)}_0(B_r) \ni v \mapsto \min \int_{B_r} H(x, Dw)\, dx, \tag{3.18}$$

and $\delta_0 > 0$ *be such that* $u \in W^{1,H(\cdot)^{1+\delta_0}}(B_r)$*. Then there exists* $0 < \sigma_g < \delta_0$*, so that* $v \in W^{1,H(\cdot)^{1+\sigma_g}}(B_r)$ *and*

$$\fint_{B_r} H(x, Dv)^{1+\sigma}\, dx \le c \fint_{B_r} H(x, Du)^{1+\sigma}\, dx \ \ \textit{for all} \ \ \sigma \in (0, \sigma_g], \tag{3.19}$$

where $c = c(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^1(B_r)})$ *and* $\sigma_g = \sigma_g(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^1(B_r)})$.

Proof. With $x_0 \in B_r$, let us fix a ball $B_\varrho(x_0) \subset \mathbb{R}^n$. We start with the case in which it is $|B_\varrho(x_0) \setminus B_r| > \frac{|B_\varrho(x_0)|}{10}$. Let us fix $\varrho/2 < t < s < \varrho$ and take a cut-off function $\eta \in C^1_c(B_s(x_0))$ such that $\chi_{B_t(x_0)} \le \eta \le \chi_{B_s(x_0)}$ and $|D\eta| \le 2/(s-t)$. Since $v - u \in W^{1,H(\cdot)}_0(B_r)$ and $\eta|_{\partial B_s(x_0)} = 0$, the function $v - \eta(v-u)$ coincides with v on ∂B_r and on $\partial B_s(x_0)$ in the sense of traces and therefore, by the minimality of v and the features of η we obtain

$$\int_{B_s(x_0)\cap B_r} H(x, Dv)\, dx \le c \left\{ \int_{(B_s(x_0)\setminus B_t(x_0))\cap B_r} H(x, Dv)\, dx + \int_{B_s(x_0)\cap B_r} H(x, Du) + H\left(x, \frac{v-u}{r}\right) dx \right\},$$

with $c = c(n, p, q, s)$. By the classical hole-filling technique and Lemma 1, we can conclude that

$$\int_{B_{\varrho/2}(x_0)\cap B_r} H(x, Dv)\, dx \le c \int_{B_\varrho(x_0)\cap B_r} H(x, Du) + H\left(x, \frac{v-u}{r}\right) dx, \tag{3.20}$$

for $c = c(n, p, q, s)$. Notice that we can extend $v - u$ as zero outside B_r since $u - v \in W^{1,p}_0(B_r)$, so there are no discontinuities on $\partial(B_r \cap B_\varrho(x_0))$ and recall that $|B_\varrho(x_0)| \ge |B_\varrho(x_0) \setminus B_r| > \frac{|B_\varrho(x_0)|}{10}$. Poincaré's inequality (3.14) applies, thus getting

$$\fint_{B_\varrho(x_0)\cap B_r} H\left(x, \frac{v-u}{r}\right) dx \le c \left\{ \left(\fint_{B_\varrho(x_0)\cap B_r} H(x, Dv)^d\, dx \right)^{\frac{1}{d}} + \fint_{B_\varrho(x_0)\cap B_r} H(x, Du)\, dx \right\}, \tag{3.21}$$

with $c = c(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^1(B_r)})$. Here we dispensed c from the dependence of $\|Dv\|_{L^p(B_r)}$ by using the minimality of v, the fact that $v - u \in W_0^{1,H(\cdot)}(B_r)$ and observing that the constant appearing in (3.1) depends in an increasing fashion from the L^p-norm of the gradient. Merging (3.20) and (3.21) we obtain

$$\fint_{B_{\varrho/2}(x_0)\cap B_r} H(x, Dv)\, dx \leq c \left\{ \left(\fint_{B_\varrho(x_0)\cap B_r} H(x, Dv)^d\, dx \right)^{\frac{1}{d}} + \fint_{B_\varrho(x_0)\cap B_r} H(x, Du)\, dx \right\}.$$

We next consider the situation when it is $B_\varrho(x_0) \Subset B_r$, in which case the proof is analogous to the one for the interior case. As mentioned in Remark 2, we can assume that the exponent $d < 1$ from (3.1) and (3.14) is the same. The two cases can be combined via a standard covering argument. In fact, let us define

$$V(x) = \begin{cases} H(x, Dv(x))^d & \text{in } B_r \\ 0 & \text{in } \mathbb{R}^n \setminus B_r \end{cases} \quad \textit{and} \quad U(x) = \begin{cases} H(x, Du(x)) & \text{in } B_r \\ 0 & \text{in } \mathbb{R}^n \setminus B_r \end{cases},$$

we get

$$\fint_{B_{\varrho/2}(x_0)} V(x)^{\frac{1}{d}}\, dx \leq c \left\{ \left(\fint_{B_\varrho(x_0)} V(x)\, dx \right)^{\frac{1}{d}} + \fint_{B_\varrho(x_0)} U(x)\, dx \right\},$$

with $c = c(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^1(B_r)})$ and $0 < d < 1$. At this point the conclusion follows by a standard variant of Gehring's lemma, see [26, Theorem 3 and Proposition 1, Chapter 2]. □

Furthermore, u is locally bounded.

Lemma 6 *Let $u \in W^{1,H_M(\cdot)}_{\mathrm{loc}}(\Omega)$ be a local minimizer of (1.6) under assumptions (1.8) and (1.9). Then u is locally bounded in Ω and for any $\Omega_0 \Subset \Omega$ there holds that*

$$\|u\|_{L^\infty(\Omega_0)} \leq c(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^1(\Omega_0)}). \tag{3.22}$$

Proof. If $n < p(1 + \delta_g)$, where δ_g is the higher integrability exponent coming from Lemma 4, then there is nothing to prove. In fact, by Morrey's embedding theorem $u \in C^{0,\frac{p(1+\delta_g)-n}{p(1+\delta_g)}}(\Omega)$, therefore we can assume $n \geq p(1 + \delta_g)$. The boundedness result can be obtained as in [11, Section 10], as a consequence of (3.16) or by noticing that the generalized Young function in (1.7) under the assumptions (1.8) and (1.9) satisfies hypotheses (A0), (A1), (AInc) and (ADec) of [28, Theorem 1.3]. In fact, with the notation used in [28] (keep in mind also the terminology described in Section 2), it is easy to see that $H^+(\delta) \leq 1 \leq H^-(1)$ for $\delta := \min\left\{3^{-\frac{1}{p}}, (1 + \|a\|_{L^\infty(\Omega)})^{-\frac{1}{q-p}}, (1 + \|b\|_{L^\infty(\Omega)})^{-\frac{1}{s-p}}\right\} \in (0, 1)$. (A1) is true by choosing $\delta := \min\left\{3^{-\frac{1}{p}}, \left(2\,\mathrm{diam}(\Omega)^{\alpha-\frac{n(q-p)}{p}}[a]_{0,\alpha}\omega_n^{-\frac{1}{p}},\right)^{-\frac{1}{q-p}}, \left(2\,\mathrm{diam}(\Omega)^{\beta-\frac{n(s-p)}{p}}[b]_{0,\beta}\omega_n^{-\frac{1}{p}}\right)^{-\frac{1}{s-p}}\right\} \in (0, 1)$, where ω_n is the volume of the unit ball $B_1 \subset \mathbb{R}^n$. (AInc) clearly holds with $\gamma^- = p > 1$ and (ADec) is verified by $\gamma^+ = s \geq q > p > 1$. □

4 Different alternatives

For later uses, recall the quantities introduced in (2.4), which will play an important role along the proof. In fact, when dealing with those so called non uniformly elliptic problems, the question of the degeneracy of the coefficients is crucial. Precisely we will look at four different scenarios:

$$\begin{cases} \mathtt{deg}(B_r)\colon\ a_i(B_r) \le 4[a]_{0,\alpha} r^{\alpha-\gamma_a} & \text{and } \ b_i(B_r) \le 4[b]_{0,\beta} r^{\beta-\gamma_\beta} \\ \mathtt{deg}_\alpha(B_r)\colon\ a_i(B_r) \le 4[a]_{0,\alpha} r^{\alpha-\gamma_a} & \text{and } \ b_i(B_r) > 4[b]_{0,\beta} r^{\beta-\gamma_b} \\ \mathtt{deg}_\beta(B_r)\colon\ a_i(B_r) > 4[a]_{0,\alpha} r^{\alpha-\gamma_a} & \text{and } \ b_i(B_r) \le 4[b]_{0,\beta} r^{\beta-\gamma_b} \\ \mathtt{ndeg}(B_r)\colon\ a_i(B_r) > 4[a]_{0,\alpha} r^{\alpha-\gamma_a} & \text{and } \ b_i(B_r) > 4[b]_{0,\beta} r^{\beta-\gamma_b}\,, \end{cases}$$

where

$$\gamma_a := \begin{cases} 0 & \text{if } \ n \ge p(1+\delta_g) \\ \alpha - \frac{n(q-p)}{p} + \frac{n\delta_g(q-p)}{2p(1+\delta_g)} & \text{if } \ n < p(1+\delta_g) \end{cases} \tag{4.1}$$

and

$$\gamma_b := \begin{cases} 0 & \text{if } \ n \ge p(1+\delta_g) \\ \beta - \frac{n(s-p)}{p} + \frac{n\delta_g(s-p)}{2p(1+\delta_g)} & \text{if } \ n < p(1+\delta_g) \end{cases}, \tag{4.2}$$

where δ_g is the higher integrability exponent given by Lemma 4. The above four cases, suitably combined, will render the desired regularity. To shorten the notation, we shall summarize the dependencies from the characteristics of the integrand we are dealing with, as

$$\mathtt{data}(\Omega_0) := \begin{cases} \left(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta},\|H(\cdot,Du)\|_{L^1(\Omega_0)},\|H(\cdot,Du)\|_{L^{1+\delta_g}(\Omega_0)}\right) & \text{if } \ n \ge p(1+\delta_g) \\ \left(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta},[u]_{C^{0,\lambda_g}(\Omega_0)},\|H(\cdot,Du)\|_{L^1(\Omega_0)},\|H(\cdot,Du)\|_{L^{1+\delta_g}(\Omega_0)}\right) & \text{if } \ n < p(1+\delta_g) \end{cases},$$

$$\mathtt{data}_0(\Omega_0) := \begin{cases} \left(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta},\|H(\cdot,Du)\|_{L^1(\Omega_0)}\right) & \text{if } \ n \ge p(1+\delta_g) \\ \left(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta},[u]_{0,\lambda_g;\Omega_0}\right) & \text{if } \ n < p(1+\delta_g) \end{cases},$$

and

$$\mathtt{data} := (n,p,q,s,\alpha,\beta)\,.$$

Here, $\lambda_g := 1-\frac{n}{p(1+\delta_g)}$ is the Hölder continuity exponent coming from Sobolev-Morrey's embedding theorem when $n < p(1+\delta_g)$ and $\Omega_0 \Subset \Omega$ is any open set compactly contained in Ω. This will be helpful, since all the existing results we are going to use are of local nature. Exploiting the different phases (deg)-(ndeg) we obtain the various forms of Caccioppoli's inequality contained in Lemma 3. We collect them in the next Corollary. Moreover, the constants a_0 and b_0 appearing in the definition of the auxiliary Young functions H_0^p, H_0^q, H_0^s and H_0 will take the values $a_0 = a_i(B_r)$ and $b_0 = b_i(B_r)$.

Corollary 3 *Let $u \in W^{1,H(\cdot)}_{\mathrm{loc}}(\Omega)$ be a local minimizer of (1.6) under assumptions (1.8)-(1.9), and B_r, $r \in (0,1)$ be any ball such that $B_{2r} \Subset \Omega_0 \Subset \Omega$. Then the following is verified:*

$$\mathtt{deg}(B_r) \Rightarrow \fint_{B_r} H(x,Du)\,dx \le c \fint_{B_{2r}} H_0^p\left(\frac{u-(u)_{2r}}{r}\right) dx, \tag{4.3}$$

$$\mathtt{deg}_\alpha(B_r) \Rightarrow \fint_{B_r} H(x,Du)\,dx \le c \fint_{B_{2r}} H_0^s\left(\frac{u-(u)_{2r}}{r}\right) dx, \tag{4.4}$$

$$\mathtt{deg}_\beta(B_r) \Rightarrow \fint_{B_r} H(x,Du)\,dx \le c \fint_{B_{2r}} H_0^q\left(\frac{u-(u)_{2r}}{r}\right)dx, \tag{4.5}$$

$$\mathtt{ndeg}(B_r) \Rightarrow \fint_{B_r} H(x,Du)\,dx \le \bar{c} \fint_{B_{2r}} H_0\left(\frac{u-(u)_{2r}}{r}\right)dx, \tag{4.6}$$

for $c = c(\mathtt{data}_0(\Omega_0))$ *and* $\bar{c} = \bar{c}(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta})$.

Proof. First, notice that, by (1.9), $\gamma_a \ge 0$ and $\gamma_b \ge 0$. Moreover, if $n \ge p(1+\delta_g)$ we see that

$$\alpha - \gamma_a + p - q \ge \frac{n(q-p)}{p} - (q-p) \ge \delta_g(q-p) > 0, \tag{4.7}$$

$$\beta - \gamma_b + p - s \ge \frac{n(s-p)}{p} - (s-p) \ge \delta_g(s-p) > 0, \tag{4.8}$$

while, if $n < p(1+\delta_g)$,

$$\alpha - \gamma_a + (\lambda_g - 1)(q-p) = \frac{n\delta_g(q-p)}{2p(1+\delta_g)} > 0, \tag{4.9}$$

$$\beta - \gamma_b + (\lambda_g - 1)(s-p) = \frac{n\delta_g(s-p)}{2p(1+\delta_g)} > 0. \tag{4.10}$$

Assume $\mathtt{deg}(B_r)$. We observe that for any $x \in B_{2r}$, $a(x) \le 8[a]_{0,\alpha} r^{\alpha-\gamma_a}$ and $b(x) \le 8[b]_{0,\beta} r^{\beta-\gamma_b}$, since $\gamma_a, \gamma_b \ge 0$ and $r \in (0,1)$. If $n \ge p(1+\delta_g)$, from (3.15), Lemma 6, (4.7) and (4.8) we get,

$$\begin{aligned}
\fint_{B_r} H(x,Du)\,dx \le& c \fint_{B_{2r}} H\left(x, \frac{u-(u)_{2r}}{r}\right)dx \\
\le& c \fint_{B_{2r}} \left(1 + 8[a]_{0,\alpha} r^{\alpha-\gamma_a+p-q}\|u\|^{q-p}_{L^\infty(\Omega_0)} + 8[b]_{0,\beta} r^{\beta-\gamma_b+p-s}\|u\|^{s-p}_{L^\infty(\Omega_0)}\right)\left|\frac{u-(u)_{2r}}{r}\right|^p dx \\
\le& c \fint_{B_{2r}} H_0^p\left(\frac{u-(u)_{2r}}{r}\right)dx,
\end{aligned}$$

where $c = c(\mathtt{data}_0(\Omega_0))$. To determine the dependencies of c we also used (3.22). On the other hand, if $n < p(1+\delta_g)$ proceding as before but using Sobolev-Morrey's theorem and (4.9), (4.10) instead of (4.7), (4.8), we obtain

$$\begin{aligned}
\fint_{B_r} H(x,Du)\,dx \le& c \fint_{B_{2r}} H\left(x, \frac{u-(u)_{2r}}{r}\right)dx \\
\le& c \fint_{B_{2r}} \Big(1 + 8[a]_{0,\alpha} r^{\alpha-\gamma_a+(\lambda_g-1)(q-p)}[u]^{q-p}_{C^{0,\lambda_g}(\Omega_0)} \\
&+ 8[b]_{0,\beta} r^{\beta-\gamma_b+(\lambda_g-1)(s-p)}[u]^{s-p}_{C^{0,\lambda_g}(\Omega_0)}\Big)\left|\frac{u-(u)_{2r}}{r}\right|^p dx \\
\le& c \fint_{B_{2r}} H_0^p\left(\frac{u-(u)_{2r}}{r}\right)dx,
\end{aligned}$$

where $c = c(\mathtt{data}_0(\Omega_0))$. Now suppose $\mathtt{deg}_\alpha(B_r)$. If $n \ge p(1+\delta_g)$, we see from (3.15), (4.7), (4.8) and Lemma 6 that

$$\fint_{B_r} H(x,Du)\,dx \le c \fint_{B_{2r}} H\left(x, \frac{u-(u)_{2r}}{r}\right)dx$$

$$
\begin{aligned}
&\le c \fint_{B_{2r}} \left(1 + 8[a]_{0,\alpha} r^{\alpha-\gamma_a+p-q} \|u\|^{q-p}_{L^\infty(\Omega_0)}\right) \left|\frac{u-(u)_{2r}}{r}\right|^p \, dx \\
&\quad + c \fint_{B_{2r}} |b(x) - b_i(B_r)| \left|\frac{u-(u)_{2r}}{r}\right|^s + b_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^s \, dx \\
&\le c \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^p + [b]_{0,\beta} r^\beta \left|\frac{u-(u)_{2r}}{r}\right|^s + b_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^s \, dx \\
&\le c \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^p + b_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^s \, dx \le c \fint_{B_{2r}} H_0^s\left(\frac{u-(u)_{2r}}{r}\right) dx,
\end{aligned}
$$

since, being $r \in (0,1)$, $r^\beta \le r^{\beta-\gamma_b}$. Here, $c = c(\mathtt{data}_0(\Omega_0))$. If $n < p(1+\delta_g)$ we have, by exploiting (4.9) and (4.10),

$$
\begin{aligned}
\fint_{B_r} H(x, Du) \, dx &\le c \fint_{B_{2r}} H\left(x, \frac{u-(u)_{2r}}{r}\right) dx \\
&\le c \fint_{B_{2r}} \left(1 + 8[a]_{0,\alpha} r^{\alpha-\gamma_a+(\lambda_g-1)(q-p)} [u]^{q-p}_{C^{0,\lambda_g}(\Omega_0)}\right) \left|\frac{u-(u)_{2r}}{r}\right|^p \, dx \\
&\quad + c \fint_{B_{2r}} |b(x) - b_i(B_r)| \left|\frac{u-(u)_{2r}}{r}\right|^s + b_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^s \, dx \\
&\le c \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^p + [b]_{0,\beta} r^\beta \left|\frac{u-(u)_{2r}}{r}\right|^s + b_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^s \, dx \\
&\le c \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^p + b_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^s \, dx \le c \fint_{B_{2r}} H_0^s\left(\frac{u-(u)_{2r}}{r}\right) dx,
\end{aligned}
$$

with $c = c(\mathtt{data}_0(\Omega_0))$. If $\mathtt{deg}_\beta(B_r)$ is in force, then, as before, for $n \ge p(1+\delta_g)$, we have

$$
\begin{aligned}
\fint_{B_r} H(x, Du) \, dx &\le c \fint_{B_{2r}} H\left(x, \frac{u-(u)_{2r}}{r}\right) dx \\
&\le c \fint_{B_{2r}} \left(1 + 8[b]_{0,\beta} r^{\beta-\gamma_b+p-s} \|u\|^{s-p}_{L^\infty(\Omega_0)}\right) \left|\frac{u-(u)_{2r}}{r}\right|^p \, dx \\
&\quad + c \fint_{B_{2r}} |a(x) - a_i(B_r)| \left|\frac{u-(u)_{2r}}{r}\right|^q + a_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^q \, dx \\
&\le c \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^p + a_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^q \, dx \le c \fint_{B_{2r}} H_0^q\left(\frac{u-(u)_{2r}}{r}\right) dx,
\end{aligned}
$$

where $c = c(\mathtt{data}_0(\Omega_0))$. Moreover, if $n < p(1+\delta_g)$ we obtain

$$
\begin{aligned}
\fint_{B_r} H(x, Du) \, dx &\le c \fint_{B_{2r}} H\left(x, \frac{u-(u)_{2r}}{r}\right) dx \\
&\le c \fint_{B_{2r}} \left(1 + 8[b]_{0,\beta} r^{\beta-\gamma_b+(\lambda_g-1)(s-p)} [u]^{s-p}_{C^{0,\lambda_g}(\Omega_0)}\right) \left|\frac{u-(u)_{2r}}{r}\right|^p \, dx \\
&\quad + c \fint_{B_{2r}} |a(x) - a_i(B_r)| \left|\frac{u-(u)_{2r}}{r}\right|^q + a_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^q \, dx \\
&\le c \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^p + a_i(B_r) \left|\frac{u-(u)_{2r}}{r}\right|^q \, dx \le c \fint_{B_{2r}} H_0^q\left(\frac{u-(u)_r}{r}\right) dx,
\end{aligned}
$$

with $c = c(\mathtt{data}_0(\Omega_0))$. Finally, if $\mathtt{ndeg}(B_r)$ holds, then by (3.15), (1.8), the fact that either if $n \geq p(1+\delta_g)$ or if $n < p(1+\delta_g)$, $\alpha \geq \alpha - \gamma_a$ and $\beta \geq \beta - \gamma_b$, and the very definition of $\mathtt{ndeg}(B_r)$ we have

$$
\begin{aligned}
\fint_{B_r} H(x, Du)\, dx \leq & c \fint_{B_{2r}} H\left(\frac{u-(u)_{2r}}{r}\right) dx \\
\leq & c \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{2r}\right|^p + |a(x) - a_i(B_r)| \left|\frac{u-(u)_{2r}}{r}\right|^{(q-p)+p} \\
& + |b(x) - b_i(B_r)| \left|\frac{u-(u)_{2r}}{r}\right|^{(s-p)+p} + a_i(B_r)\left|\frac{u-(u)_{2r}}{r}\right|^q + b_i(B_r)\left|\frac{u-(u)_{2r}}{r}\right|^s dx \\
\leq & c \fint_{B_{2r}} \left|\frac{u-(u)_{2r}}{r}\right|^p + [a]_{0,\alpha} r^\alpha \left|\frac{u-(u)_{2r}}{r}\right|^q + [b]_{0,\beta} r^\beta \left|\frac{u-(u)_{2r}}{r}\right|^s dx \\
& + c \fint_{B_{2r}} H_0\left(\frac{u-(u)_{2r}}{r}\right) dx \leq \bar{c} \fint_{B_{2r}} H_0\left(\frac{u-(u)_{2r}}{r}\right) dx,
\end{aligned}
$$

with $\bar{c} = \bar{c}(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta})$. □

We conclude this section by recalling a quantitative Harmonic-approximation type result from [4]. We shall report it in the form that better fits our necessities.

Lemma 7 *Let $B_r \subset \mathbb{R}^n$ be a ball, $\varepsilon \in (0,1)$, $\tilde{H}$ be one of the Young functions defined in (2.3) and $v \in W^{1,\tilde{H}}(B_{2r})$ be a map satisfying the following estimates:*

$$
\fint_{B_{2r}} \tilde{H}(Dv)\, dx \leq \tilde{c}_1, \tag{4.11}
$$

and

$$
\fint_{B_r} \tilde{H}(Dv)^{1+\sigma_0}\, dx \leq \tilde{c}_2, \tag{4.12}
$$

where $\tilde{c}_1, \tilde{c}_2 \geq 1$ and $\sigma_0 > 0$ are fixed constants. Moreover, assume that

$$
\left| \fint_{B_r} \partial\tilde{H}(Dv) \cdot D\varphi\, dx \right| \leq \varepsilon \|D\varphi\|_{L^\infty(B_r)} \ \text{ for all } \varphi \in C_c^\infty(B_r), \tag{4.13}
$$

for some $\varepsilon \in (0,1)$. Then there exists a function $\tilde{h} \in v + W_0^{1,\tilde{H}}(B_r)$ such that the following conditions are satisfied:

$$
\fint_{B_r} \partial\tilde{H}(D\tilde{h}) \cdot D\varphi\, dx = 0 \ \text{ for all } \varphi \in C_c^\infty(B_r), \tag{4.14}
$$

$$
\fint_{B_r} \tilde{H}(D\tilde{h})^{1+\sigma_1}\, dx \leq c(n,p,q,s,\sigma_0)\tilde{c}_2, \tag{4.15}
$$

$$
\fint_{B_r} \widetilde{\mathcal{V}}(Dv, D\tilde{h})^2\, dx \leq c\varepsilon^m, \tag{4.16}
$$

where $\widetilde{\mathcal{V}}$ is the corresponding auxiliary function defined in (2.9), $c = c(n,p,q,s,\tilde{c}_1,\tilde{c}_2)$, $\sigma_1 = \sigma_1(n,p,q,s,\sigma_0) \in (0,\sigma_0)$, $m = m(n,p,q,s,\sigma_0) > 0$.

Proof. The cases of $\tilde{H} = H_0^p, H_0^q, H_0^s$ are treated in [4, Lemma 1], so we focus on $\tilde{H} = H_0$. The proof we provide is in some sense a simplified version of the original one since we do not need a powerful result such as [13, Theorem 5.1]. In fact we can recover some extra boundary integrability from Lemma 5. Define $h_0 \in W^{1,H_0}(B_r)$ to be the solution to the Dirichlet problem

$$v + W_0^{1,H_0}(B_r) \ni w \mapsto \min \int_{B_r} H_0(Dw)\, dx. \tag{4.17}$$

By minimality (4.14) is verified, since it is the Euler-Lagrange equation associated to the above variational problem. Moreover, it follows from (4.11) that

$$\fint_{B_r} H_0(Dh_0)\, dx \le \fint_{B_r} H_0(Dv)\, dx \le 2^n \tilde{c}_1. \tag{4.18}$$

Now, by the previous inequality, Lemma 5 with $a(\cdot) \equiv$ const and $b(\cdot) \equiv$ const, and (4.12), we obtain

$$\fint_{B_r} H_0(Dh_0)^{1+\sigma_g}\, dx \le c \fint_{B_r} H_0(Dv)^{1+\sigma_g}\, dx \le c\tilde{c}_2^{\frac{1+\sigma_g}{1+\sigma_0}} := \tilde{c}_3, \tag{4.19}$$

for some $0 < \sigma_g < \sigma_0$, which is (4.15) with $\sigma_1 = \sigma_g$. Here $\tilde{c}_3 = \tilde{c}_3(n, p, q, s, \sigma_g, \tilde{c}_1, \tilde{c}_2)$. Set $w = h_0 - v \in W_0^{1,H_0}(B_r)$ and let $\lambda \ge 1$ to be fixed later and consider $w_\lambda \in W_0^{1,\infty}(B_r)$, the Lipschitz truncation of w given by the main result in [1] and satisfying

$$\|Dw_\lambda\|_{L^\infty(B_r)} \le c(n)\lambda \quad \text{and} \quad \{w_\lambda \ne w\} \subset \{M(|Dw|) > \lambda\} \cup \textit{negligible set}. \tag{4.20}$$

Using such properties, the fact that $t \mapsto H_0(t)$ is increasing, Markov's inequality, (4.12), (4.19) and the maximal theorem we deduce that

$$\begin{aligned}
\frac{|\{w_\lambda \ne w\}|}{|B_r|} &\le \frac{|B_r \cap \{M(|Dw|) \ge \lambda\}|}{|B_r|} \le \frac{1}{H_0(\lambda)^{1+\sigma_g}} \fint_{B_r} H_0(M(|Dw|))^{1+\sigma_g}\, dx \\
&\le \frac{c}{H_0(\lambda)^{1+\sigma_g}} \fint_{B_r} H_0(Dw)^{1+\sigma_g}\, dx \le \frac{c}{H_0(\lambda)^{1+\sigma_g}} \fint_{B_r} H_0(Dh_0)^{1+\sigma_g} + H_0(Dv)^{1+\sigma_g}\, dx \\
&\le \frac{c}{H_0(\lambda)^{1+\sigma_g}} \left[\fint_{B_r} H_0(Dh_0)^{1+\sigma_g}\, dx + \left(\fint_{B_r} H_0(Dv)^{1+\sigma_0}\, dx \right)^{\frac{1+\sigma_g}{1+\sigma_0}} \right] \\
&\le \frac{c(\tilde{c}_3 + \tilde{c}_2^{\frac{1+\sigma_g}{1+\sigma_0}})}{H_0(\lambda)^{1+\sigma_g}} \le \frac{c(\tilde{c}_3 + \tilde{c}_2)}{H_0(\lambda)^{1+\sigma_g}},
\end{aligned} \tag{4.21}$$

where $c = c(n, p, q, s, \sigma_g, \sigma_0)$. Now we test (4.14) against w_λ, which is admissible by density, to get

$$\begin{aligned}
\text{(I)} &:= \fint_{B_r} (\partial H_0(Dh_0) - \partial H_0(Dv)) \cdot Dw_\lambda \chi_{\{w_\lambda = w\}}\, dx \\
&= -\fint_{B_r} \partial H_0(Dv) \cdot Dw_\lambda\, dx - \fint_{B_r} (\partial H_0(Dh_0) - \partial H_0(Dv)) \cdot Dw_\lambda \chi_{\{w \ne w_\lambda\}}\, dx =: \text{(II)} + \text{(III)}.
\end{aligned}$$

The properties of H_0 and (2.6) give

$$\text{(I)} \ge c \fint_{B_r} \mathcal{V}_0(Dv, Dh_0) \chi_{\{w_\lambda = w\}}\, dx, \tag{4.22}$$

where $c = c(n, p, q, s) > 0$. Moreover, by (4.13) and $(4.20)_1$ we see that

$$|(\mathrm{II})| \leq c\varepsilon\lambda, \tag{4.23}$$

with $c = c(n)$. Before estimating term (III), we recall a standard Young type inequality holding for H_0, see [4]: for all $\sigma \in (0, 1)$,

$$xy \leq \sigma^{1-s} H_0(x) + \sigma H_0^*(y), \tag{4.24}$$

where $H_0^*(y) = \sup_{x>0}\{yx - H_0(x)\}$ is the convex conjugate of H_0. Furthermore, there holds: $H_0^*\left(\frac{H_0(t)}{t}\right) \leq H_0(t)$, see [5] for more details. Now, using $(4.20)_1$, (4.11), (4.18), (4.24) and (4.21) we estimate, for a certain fixed $\sigma \in (0, 1)$,

$$\begin{aligned} |(\mathrm{III})| \leq & s\|Dw_\lambda\|_{L^\infty(B_r)} \fint_{B_r} \left(\frac{H_0(Dh_0)}{|Dh_0|} + \frac{H_0(Dv)}{|Dv|} \right) \chi_{\{w_\lambda \neq w\}}\, dx \\ \leq & \sigma \fint_{B_r} \left[H_0^*\left(\frac{H_0(Dh_0)}{|Dh_0|} \right) + H_0^*\left(\frac{H_0(Dv)}{|Dv|} \right) \right] dx + \frac{cH_0(\|Dw_\lambda\|_{L^\infty(B_r)})}{\sigma^{s-1}} \frac{|\{w_\lambda \neq w\}|}{|B_r|} \\ \leq & \sigma \fint_{B_r} H_0(Dh_0) + H_0(Dv)\, dx + \frac{c}{\sigma^{s-1} H_0(\lambda)^{\sigma_g}} \leq 2^{n+1}\sigma\tilde{c}_1 + \frac{c}{\sigma^{s-1}\lambda^{p\sigma_g}}, \end{aligned} \tag{4.25}$$

where we also used the fact that $H_0(\lambda) \geq \lambda^p$ since $\lambda \geq 1$. Here $c = c(n, p, q, s, \sigma_g)$. Collecting (4.22), (4.23) and (4.25) we obtain

$$\fint_{B_r} \mathcal{V}_0(Dv, Dh_0)^2 \chi_{\{w_\lambda = w\}}\, dx \leq c\left(\varepsilon\lambda + \sigma + \sigma^{1-s}\lambda^{-p\sigma_g} \right), \tag{4.26}$$

for $c = c(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, n, p, q, s, \sigma_g)$ and $\sigma \in (0, 1)$ to be fixed. For $\theta \in (0, 1)$, by Hölder's inequality and (4.26) we estimate

$$\left(\fint_{B_r} \mathcal{V}_0(Dv, Dh_0)^{2\theta} \chi_{\{w_\lambda = w\}}\, dx \right)^{\frac{1}{\theta}} \leq c\left(\varepsilon\lambda + \sigma + \sigma^{1-s}\lambda^{-p\sigma_g} \right).$$

Again, by Hölder's inequality, (4.21), (4.11) and (4.18) we have

$$\begin{aligned} \left(\fint_{B_r} \mathcal{V}_0(Dv, Dh_0)^{2\theta} \chi_{\{w_\lambda \neq w\}}\, dx \right)^{\frac{1}{\theta}} \leq & c\left(\frac{|\{w_\lambda \neq w\}|}{|B_r|} \right)^{\frac{1-\theta}{\theta}} \fint_{B_r} \mathcal{V}_0(Dv, D_0)^2\, dx \\ \leq & cH_0(\lambda)^{-\frac{(1+\sigma_g)(1-\theta)}{\theta}} \fint_{B_r} H_0(Dh_0) + H_0(Dv)\, dx \leq c\lambda^{-\frac{p(1-\theta)}{\theta}}, \end{aligned} \tag{4.27}$$

where $c = c(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, n, p, q, s, \sigma_g)$. Setting in (4.26) and (4.27) $\lambda = \varepsilon^{-1/2}$ and $\sigma = \varepsilon^{\frac{p\sigma_g}{4(s-1)}}$ we obtain that

$$\left(\fint_{B_r} \mathcal{V}_0(Dv, Dh_0)^{2\theta}\, dx \right)^{\frac{1}{\theta}} \leq c\varepsilon^{2m}, \tag{4.28}$$

with $c = c(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, n, p, q, s, \sigma_g)$ and $m := \frac{1}{2}\min\left\{ \frac{1}{2}, \frac{p\sigma_g}{4}, \frac{p\sigma_g}{4(s-1)}, \frac{p(1-\theta)}{2\theta} \right\}$. Notice that in the above estimates we still have a degree of freedom in θ. Applying Hölder's inequality with exponents $\frac{2(1+\sigma_g)}{1+2\sigma_g}$ and $2(1+\sigma_g)$ we obtain

$$\fint_{B_r} \mathcal{V}_0(Dv, Dh_0)^2\, dx \leq \left(\fint_{B_r} \mathcal{V}_0(Dv, Dh_0)^{\frac{2(1+\sigma_g)}{1+2\sigma_g}}\, dx \right)^{\frac{1+2\sigma_g}{2(1+\sigma_g)}} \left(\fint_{B_r} \mathcal{V}_0(Dv, Dh_0)^{2(1+\sigma_g)}\, dx \right)^{\frac{1}{1+\sigma_g}}$$

$$\leq c\varepsilon^m \left(\fint_{B_r} (H_0(Dh_0) + H_0(Dv))^{1+\sigma_g}\, dx \right)^{\frac{1}{1+\sigma_g}} \leq c\varepsilon^m,$$

with $c = c(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, n, p, q, s, \sigma_g)$. Here we used (4.12), (4.19), and (4.28) with $\theta = \frac{1+\sigma_g}{1+2\sigma_g} < 1$. Recalling (2.8), we can conclude from the previous estimate that

$$\fint_{B_r} \mathcal{V}_0(Dv, Dh_0)^2\, dx \leq c\varepsilon^m,$$

which is what we wanted. □

5 Morrey decay and Theorem 2

The proof of Theorem 2 goes in two moments: first, we prove that a suitable manipulation of a local minimizer u of (1.6) satisfies the assumptions of Lemma 7, then we exploit this to start an iteration which will eventually render the announced decay.

Step 1: Quantitative harmonic approximation. Define the quantities

$$E := E(u, B_{2r}) = \left(\fint_{B_{2r}} H(x, Du)\, dx \right)^{\frac{1}{p}} \quad \text{and} \quad v := \frac{u}{E},$$

where $u \in W^{1,H(\cdot)}_{\mathrm{loc}}(\Omega)$ is a local minimizer of (1.6) and $B_{2r} \Subset \Omega_0 \Subset \Omega$ is any ball of radius $r \leq \frac{1}{2}$. From now on, we will consider the following auxiliary Young functions

$$\begin{cases} H_0(z) := |z|^p + a_i(B_r)|z|^q + b_i(B_r)|z|^s, & \tilde{H}_0(z) := |z|^p + a_i(B_r)E^{q-p}|z|^q + b_i(B_r)E^{s-p}|z|^s, \\ H_0^s(z) := |z|^p + b_i(B_r)|z|^s, & \tilde{H}_0^s(z) := |z|^p + b_i(B_r)E^{s-p}|z|^s, \\ H_0^q(z) := |z|^p + a_i(B_r)|z|^q, & \tilde{H}_0^q(z) := |z|^p + a_i(B_r)E^{q-p}|z|^q, \\ H_0^p(z) := |z|^p, & \end{cases} \tag{5.1}$$

and

$$\begin{cases} \mathcal{V}_0(z_1, z_2)^2 := |V_p(z_1) - V_p(z_2)|^2 + a_i(B_r)|V_q(z_1) - V_q(z_2)|^2 + b_i(B_r)|V_s(z_1) - V_s(z_2)|^2, \\ \mathcal{V}_0^s(z_1, z_2)^2 := |V_p(z_1) - V_p(z_2)|^2 + b_i(B_r)|V_s(z_1) - V_s(z_2)|^2, \\ \mathcal{V}_0^q(z_1, z_2)^2 := |V_p(z_1) - V_p(z_2)|^2 + a_i(B_r)|V_q(z_1) - V_q(z_2)|^2, \\ \mathcal{V}_0^p(z_1, z_2)^2 := |V_p(z_1) - V_p(z_2)|^2, \end{cases}$$

where $a_i(\cdot)$ and $b_i(\cdot)$ are defined as in (2.4). Since u is a local minimizer of (1.6), a straightforward computation shows that v is a local minimizer of the functional

$$\tilde{\mathcal{H}}(w, \Omega) := \int_\Omega |Dw|^p + a(x)E^{q-p}|Dw|^q + b(x)E^{s-p}|Dw|^s\, dx.$$

Then, by scaling, it is easy to see that Lemma 4 holds true also for v with the same extra integrability exponent $\delta_g = \delta_g(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|Du\|_{L^p(B_r)})$ as u. For any open set $U \Subset \Omega$ it satisfies the Euler-Lagrange equation

$$0 = \int_U \left(p|Dv|^{p-2} + qa(x)E^{q-p}|Dv|^{q-2} + sb(x)E^{s-p}|Dv|^{s-2} \right) Dv \cdot D\varphi\, dx \quad \text{for all } \varphi \in C_c^\infty(U). \tag{5.2}$$

Moreover, if $\tilde{H}$ denotes H_0^p, $\tilde{H}_0^q$, $\tilde{H}_0^s$ or $\tilde{H}_0$, we see from the definition of v that

$$\fint_{B_{2r}} \tilde{H}(Dv)\,dx \le E^{-p} \fint_{B_{2r}} H(x, Du)\,dx \le 1, \tag{5.3}$$

which is (4.11) and, by (5.3) and Lemma 4 we obtain, for some $\tilde{\sigma}_g \in (0, \delta_g)$,

$$\fint_{B_r} \tilde{H}(Dv)^{1+\tilde{\sigma}_g}\,dx \le \fint_{B_r} H(x, Dv)^{1+\tilde{\sigma}_g}\,dx = \left(\fint_{B_{2r}} H(x, Du)\,dx\right)^{-(1+\tilde{\sigma}_g)} \left(\fint_{B_r} H(x, Du)^{1+\tilde{\sigma}_g}\,dx\right) \le c, \tag{5.4}$$

where $c = c(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|Du\|_{L^p(\Omega_0)})$ is the constant appearing in Lemma 4 and this verifies (4.12). So we see that conditions (4.11)-(4.12) of Lemma 7 are matched with $\sigma_0 = \tilde{\sigma}_g$ no matter what degeneracy (or non degeneracy) condition holds on B_{2r}. Clearly we have no problems of integrability, since $\tilde{\sigma}_g < \delta_g$, which is the exponent coming from Lemma 4. We now define

$$\sigma_a = \alpha - \gamma_a - \frac{n(q-p)}{p(1+\delta_g)} \quad \text{and} \quad \sigma_b = \beta - \gamma_b - \frac{n(s-p)}{p(1+\delta_g)}. \tag{5.5}$$

A simple computation shows that σ_a and σ_b are both positive numbers. We first assume $\mathtt{deg}(B_r)$. From (5.2) we deduce that

$$\left|\fint_{B_r} \partial H_0^p(Dv)\cdot D\varphi\,dx\right| \le qE^{q-p}\fint_{B_r} a(x)|Dv|^{q-1}|D\varphi|\,dx + sE^{s-p}\fint_{B_r} b(x)|Dv|^{s-1}|D\varphi|\,dx =: (\mathrm{I})_{\mathtt{deg}} + (\mathrm{II})_{\mathtt{deg}}.$$

From the very definition of condition $\mathtt{deg}$, Lemma 4, (5.3), Hölder's inequality and (5.5) we get

$$\begin{aligned}(\mathrm{I})_{\mathtt{deg}} \le & cE^{\frac{q-p}{q}} r^{\frac{\alpha-\gamma_a}{q}} \|D\varphi\|_{L^\infty(B_r)} \fint_{B_r} (E^{q-p}a(x))^{\frac{q-1}{q}} |Dv|^{q-1}\,dx \\ \le & c\|D\varphi\|_{L^\infty(B_r)} \left(\fint_{B_{2r}} H(x, Du)\,dx\right)^{\frac{q-p}{pq}} r^{\frac{\alpha-\gamma_a}{q}} \left(\fint_{B_r} E^{q-p-q}a(x)|Du|^q\,dx\right)^{\frac{q-1}{q}} \\ \le & c\|D\varphi\|_{L^\infty(B_r)} \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)}^{\frac{q-p}{pq}} r^{\frac{\alpha-\gamma_a}{q} - \frac{n(q-p)}{pq(1+\delta_g)}} \le c\|D\varphi\|_{L^\infty(B_r)} r^{\frac{\sigma_a}{q}}\end{aligned} \tag{5.6}$$

with $c_1 = c_1(n, p, q, [a]_{0,\alpha}, \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)})$. In a totally similar way we obtain

$$\begin{aligned}(\mathrm{II})_{\mathtt{deg}} \le & cE^{\frac{s-p}{s}} r^{\frac{\beta-\gamma_b}{s}} \|D\varphi\|_{L^\infty(B_r)} \fint_{B_r} (E^{s-p}b(x))^{\frac{s-1}{s}} |Dv|^{s-1}\,dx \\ \le & c\|D\varphi\|_{L^\infty(B_r)} \left(\fint_{B_{2r}} H(x, Du)\,dx\right)^{\frac{s-p}{ps}} r^{\frac{\beta-\gamma_b}{s}} \left(\fint_{B_r} E^{s-p-s}b(x)|Du|^s\,dx\right)^{\frac{s-1}{s}} \\ \le & c\|D\varphi\|_{L^\infty(B_r)} \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)}^{\frac{s-p}{ps}} r^{\frac{\beta-\gamma_b}{s} - \frac{n(s-p)}{ps(1+\delta_g)}} \le c_2\|D\varphi\|_{L^\infty(B_r)} r^{\frac{\sigma_b}{s}}\end{aligned} \tag{5.7}$$

where $c_2 = c_2(n, p, s, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)})$. Now we define $\tilde{\sigma}_p := \frac{1}{2}\min\{q^{-1}\sigma_a, s^{-1}\sigma_b\} > 0$ and fix a threshold radius $\tilde{R}_*^1$ such that $\max\{c_1, c_2\}(\tilde{R}_*^1)^{\tilde{\sigma}_p} \le \frac{1}{2}$ and assume that $0 < r \le \min\{\tilde{R}_*^1, 1\}$. In correspondence of such a choice, by (5.6) and (5.7) we can conclude that

$$\left|\fint_{B_r} \partial H_0^p(Dv)\cdot D\varphi\,dx\right| \le r^{\tilde{\sigma}_p}\|D\varphi\|_{L^\infty(B_r)}, \tag{5.8}$$

so the assumptions of Lemma 7 are matched and there exists a H_0^p-harmonic map $\tilde{h}_p$ satisfying in particular (4.16). It is clear that, if $h_p := E\tilde{h}_p$, then h_p is still H_0^p-harmonic, $h_p - u \in W_0^{1,H_0^p}(B_r)$ and, by (4.16),

$$\fint_{B_r} \mathcal{V}_0^p(Du, Dh_p)^2 \, dx \le cr^{m_p} \fint_{B_{2r}} H(x, Du) \, dx, \tag{5.9}$$

for $c = c(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)})$ and $m_p = m_p(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^1(\Omega_0)})$. Suppose now that $\mathrm{deg}_\alpha(B_r)$ holds. Then, by (5.2) we obtain

$$\begin{aligned}\left| \fint_{B_r} \partial \tilde{H}_0^s(Dv) \cdot D\varphi \, dx \right| \le & qE^{q-p} \fint_{B_r} a(x)|Dv|^{q-1}|D\varphi| \, dx + sE^{s-p} \fint_{B_r} (b(x) - b_i(B_r))|Dv|^{s-1}|D\varphi| \, dx \\ \le & qE^{q-p}\|D\varphi\|_{L^\infty(B_r)} \fint_{B_r} a(x)|Dv|^{q-1} \, dx + 2s[b]_{0,\beta} r^\beta E^{s-p}\|D\varphi\|_{L^\infty(B_r)} \fint_{B_r} |Dv|^{s-1} \, dx \\ =: & (\mathrm{I})_{\mathrm{deg}_\alpha} + (\mathrm{II})_{\mathrm{deg}_\alpha}.\end{aligned}$$

As in (5.6), we estimate

$$\begin{aligned}(\mathrm{I})_{\mathrm{deg}_\alpha} \le & c\|D\varphi\|_{L^\infty(B_r)} \left(\fint_{B_{2r}} H(x, Du) \, dx \right)^{\frac{q-p}{pq}} r^{\frac{\alpha-\gamma_a}{q}} \left(\fint_{B_r} E^{q-p-q} a(x)|Du|^q \, dx \right)^{\frac{q-1}{q}} \\ \le & c\|D\varphi\|_{L^\infty(B_r)} \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)}^{\frac{q-p}{pq}} r^{\frac{\alpha-\gamma_a}{q} - \frac{n(q-p)}{pq(1+\delta_g)}} \le c\|D\varphi\|_{L^\infty(B_r)} r^{\frac{\sigma_a}{q}}\end{aligned} \tag{5.10}$$

with $c_1 = c_1(n, p, q, [a]_{0,\alpha}, \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)})$, and

$$\begin{aligned}(\mathrm{II})_{\mathrm{deg}_\alpha} \le & c\|D\varphi\|_{L^\infty(B_r)} E^{\frac{s-p}{s}} r^{\frac{\beta}{s} + \frac{\gamma_b(s-1)}{s}} \left(\fint_{B_r} (E^{s-p} r^{\beta-\gamma_b})^{\frac{s-1}{s}} |Dv|^{s-1} \, dx \right) \\ \le & c\|D\varphi\|_{L^\infty(B_r)} r^{\frac{\gamma_b(s-1)}{s}} r^{\frac{1}{s}\left(\beta - \frac{n(s-p)}{p(1+\delta_g)}\right)} \left(\fint_{B_{2r}} E^{-p} b_i(B_r)|Du|^s \, dx \right)^{\frac{s-1}{s}} \le cr^{\frac{\gamma_b(s-1)}{s} + \frac{1}{s}\left(\beta - \frac{n(s-p)}{p(1+\delta_g)}\right)} \|D\varphi\|_{L^\infty(B_r)},\end{aligned} \tag{5.11}$$

where $c_2 = c_2(n, p, s, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)})$. Define $\tilde{\sigma}_s := \frac{1}{2} \min\left\{ \frac{\sigma_a}{q}, \frac{\gamma_b(s-1)}{s} + \frac{1}{s}\left(\beta - \frac{n(s-p)}{p(1+\delta_g)}\right) \right\} > 0$ and fix a threshold radius $\tilde{R}_*^2$ such that $\max\{c_1, c_2\}(\tilde{R}_*^2)^{\tilde{\sigma}_s} \le \frac{1}{2}$ and assume that $0 < r \le \min\{\tilde{R}_*^1, \tilde{R}_*^2, 1\}$. In correspondence of such a choice, by (5.10) and (5.11) we can conclude that

$$\left| \fint_{B_r} \partial \tilde{H}_0^s(Dv) \cdot D\varphi \, dx \right| \le r^{\tilde{\sigma}_s} \|D\varphi\|_{L^\infty(B_r)}, \tag{5.12}$$

so the assumptions of Lemma 7 are matched and there exists a $\tilde{H}_0^s$-harmonic map $\tilde{h}_s$ satisfying in particular (4.16). Clearly, if $h_s := E\tilde{h}_s$, then h_s is H_0^s-harmonic, $h_s - u \in W_0^{1,H_0^s}(B_r)$ and, by (4.16),

$$\fint_{B_r} \mathcal{V}_0^s(Du, Dh_s)^2 \, dx \le cr^{m_s} \fint_{B_{2r}} H(x, Du) \, dx, \tag{5.13}$$

where $c = c(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^{1+\delta_g}(\Omega_0)})$ and $m_s = m_s(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^1(\Omega_0)})$. This time assume $\mathrm{deg}_\beta(B_r)$ holds. Then, by (5.2) we obtain

$$\left| \fint_{B_r} \partial \tilde{H}_0^q(Dv) \cdot D\varphi \, dx \right| \le sE^{s-p} \fint_{B_r} b(x)|Dv|^{s-1}|D\varphi| \, dx + qE^{q-p} \fint_{B_r} (a(x) - a_i(B_r)) \, |Dv|^{q-1}|D\varphi| \, dx$$

$$=:(\mathrm{I})_{\mathtt{deg}_\beta}+(\mathrm{II})_{\mathtt{deg}_\beta}.$$

As in (5.7), we estimate

$$\begin{aligned}(\mathrm{I})_{\mathtt{deg}_\beta} \leq & c\|D\varphi\|_{L^\infty(B_r)}E^{\frac{s-p}{s}}r^{\frac{\beta-\gamma_b}{s}}\left(\fint_{B_r} E^{-p}b(x)|Du|^s\right)^{\frac{s-1}{s}}\\ \leq & c\|D\varphi\|_{L^\infty(B_r)}\|H(\cdot,Du)\|_{L^{1+\delta_g}(\Omega_0)}^{\frac{s-p}{sp}}r^{\frac{1}{s}\left(\beta-\gamma_b-\frac{n(s-p)}{p(1+\delta_g)}\right)} \leq cr^{\frac{\sigma_b}{s}}\|D\varphi\|_{L^\infty(B_r)},\end{aligned} \tag{5.14}$$

with $c_1=c_1(n,p,s,[b]_{0,\beta},\|H(\cdot,Du)\|_{L^{1+\delta_g}(\Omega_0)})$, and

$$\begin{aligned}(\mathrm{II})_{\mathtt{deg}_\beta} \leq & c\|D\varphi\|_{L^\infty(B_r)}E^{\frac{q-p}{q}}r^{\frac{\alpha}{q}+\frac{\gamma_a(q-1)}{q}}\left(\fint_{B_r}(E^{q-p}r^{\alpha-\gamma_a})^{\frac{q-1}{q}}|Dv|^{q-1}\,dx\right)\\ \leq & c\|D\varphi\|_{L^\infty(B_r)}r^{\frac{\gamma_a(q-1)}{q}}r^{\frac{1}{q}\left(\alpha-\frac{n(q-p)}{p(1+\delta_g)}\right)}\left(\fint_{B_{2r}} E^{-p}a_i(B_r)|Du|^q\,dx\right)^{\frac{q-1}{q}} \leq cr^{\frac{\gamma_a(q-1)}{q}+\frac{1}{q}\left(\alpha-\frac{n(q-p)}{p(1+\delta_g)}\right)}\|D\varphi\|_{L^\infty(B_r)},\end{aligned} \tag{5.15}$$

where $c_2=c_2(n,p,q,[a]_{0,\alpha},\|H(\cdot,Du)\|_{L^{1+\delta_g}(\Omega_0)})$. Let $\tilde{\sigma}_q:=\frac{1}{2}\min\left\{\frac{\sigma_b}{s},\frac{\gamma_a(q-1)}{q}+\frac{1}{q}\left(\alpha-\frac{n(q-p)}{p(1+\delta_g)}\right)\right\}>0$ and fix a threshold radius $\tilde{R}_*^3$ such that $\max\{c_1,c_2\}(\tilde{R}_*^3)^{\tilde{\sigma}_q}\leq\frac{1}{2}$ and assume that $0<r\leq\min\{\tilde{R}_*^1,\tilde{R}_*^2,\tilde{R}_*^3,1\}$. In correspondence of such a choice, by (5.10) and (5.11) we can conclude that

$$\left|\fint_{B_r}\partial\tilde{H}_0^q(Dv)\cdot D\varphi\,dx\right| \leq r^{\tilde{\sigma}_q}\|D\varphi\|_{L^\infty(B_r)}, \tag{5.16}$$

so the assumptions of Lemma 7 are satisfied and there exists a $\tilde{H}_0^q$-harmonic map $\tilde{h}_q$ satisfying in particular (4.16). Clearly, if $h_q:=E\tilde{h}_q$, then h_q is H_0^q-harmonic, $h_q-u\in W_0^{1,H_0^q}(B_r)$ and, by (4.16),

$$\fint_{B_r}\mathcal{V}_0^q(Du,Dh_q)^2\,dx\leq cr^{m_q}\fint_{B_{2r}}H(x,Du)\,dx, \tag{5.17}$$

where $c=c(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta})$ and $m_q=m_q(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta})$. Finally, suppose $\mathtt{ndeg}(B_r)$ holds. Then, by (5.2) we obtain

$$\begin{aligned}\left|\fint_{B_r}\partial\tilde{H}_0(Dv)\cdot D\varphi\,dx\right| \leq & sE^{s-p}\fint_{B_r}(b(x)-b_i(B_r))\,|Dv|^{s-1}|D\varphi|\,dx+qE^{q-p}\fint_{B_r}(a(x)-a_i(B_r))\,|Dv|^{q-1}|D\varphi|\,dx\\ =: & (\mathrm{I})_{\mathtt{ndeg}}+(\mathrm{II})_{\mathtt{ndeg}}.\end{aligned}$$

As in (5.11) we estimate

$$\begin{aligned}(\mathrm{I})_{\mathtt{ndeg}} \leq & c\|D\varphi\|_{L^\infty(B_r)}E^{\frac{s-p}{s}}r^{\frac{\beta}{s}+\frac{\gamma_b(s-1)}{s}}\left(\fint_{B_r}(E^{s-p}r^{\beta-\gamma_b})^{\frac{s-1}{s}}|Dv|^{s-1}\,dx\right)\\ \leq & c\|D\varphi\|_{L^\infty(B_r)}r^{\frac{\gamma_b(s-1)}{s}}r^{\frac{1}{s}\left(\beta-\frac{n(s-p)}{p(1+\delta_g)}\right)}\left(\fint_{B_{2r}} E^{-p}b_i(B_r)|Du|^s\,dx\right)^{\frac{s-1}{s}} \leq cr^{\frac{\gamma_b(s-1)}{s}+\frac{1}{s}\left(\beta-\frac{n(s-p)}{p(1+\delta_g)}\right)}\|D\varphi\|_{L^\infty(B_r)},\end{aligned} \tag{5.18}$$

with $c_1=c_1(n,p,s,[b]_{0,\beta},\beta,\|H(\cdot,Du)\|_{L^{1+\delta_g}(\Omega_0)})$, and, keeping (5.15) in mind,

$$(\mathrm{II})_{\mathtt{ndeg}} \leq c\|D\varphi\|_{L^\infty(B_r)}E^{\frac{q-p}{q}}r^{\frac{\alpha}{q}+\frac{\gamma_a(q-1)}{q}}\left(\fint_{B_r}(E^{q-p}r^{\alpha-\gamma_a})^{\frac{q-1}{q}}|Dv|^{q-1}\,dx\right)$$

$$\leq c\|D\varphi\|_{L^\infty(B_r)} r^{\frac{\gamma_a(q-1)}{q}} r^{\frac{1}{q}\left(\alpha-\frac{n(q-p)}{p(1+\delta_g)}\right)}\left(\fint_{B_{2r}} E^{-p}a_i(B_r)|Du|^q\,dx\right)^{\frac{q-1}{q}} \leq cr^{\frac{\gamma_a(q-1)}{q}+\frac{1}{q}\left(\alpha-\frac{n(q-p)}{p(1+\delta_g)}\right)}\|D\varphi\|_{L^\infty(B_r)}, \tag{5.19}$$

where $c_2 = c_2(n,p,q,[a]_{0,\alpha},\alpha,\|H(\cdot,Du)\|_{L^{1+\delta_g}(\Omega_0)})$. Let

$$\tilde{\sigma}_0 := \frac{1}{2}\min\left\{\frac{\gamma_a(q-1)}{q}+\frac{1}{q}\left(\alpha-\frac{n(q-p)}{p(1+\delta_g)}\right), \frac{\gamma_b(s-1)}{s}+\frac{1}{s}\left(\beta-\frac{n(s-p)}{p(1+\delta_g)}\right)\right\} > 0,$$

fix another threshold radius $\tilde{R}_*^4$ such that $\max\{c_1,c_2\}(\tilde{R}_*^4)^{\tilde{\sigma}_0} \leq \frac{1}{2}$ and assume that $0<r\leq \min\{\tilde{R}_*^1,\tilde{R}_*^2,\tilde{R}_*^3,\tilde{R}_*^4,1\}$. In correspondence of such a choice, by (5.18) and (5.19) we can conclude that

$$\left|\fint_{B_r}\partial\tilde{H}_0(Dv)\cdot D\varphi\,dx\right| \leq r^{\tilde{\sigma}_0}\|D\varphi\|_{L^\infty(B_r)}, \tag{5.20}$$

so the assumptions of Lemma 7 are satisfied and there exists a $\tilde{H}_0$-harmonic map $\tilde{h}_0$ satisfying in particular (4.16). Clearly, if $h_0 := E\tilde{h}_0$, then h_0 is H_0-harmonic, $h_0-u\in W_0^{1,H_0}(B_r)$ and, by (4.16),

$$\fint_{B_r}\mathcal{V}_0(Du,Dh_0)^2\,dx \leq cr^{m_0}\fint_{B_{2r}}H(x,Du)\,dx, \tag{5.21}$$

where $c=c(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta},\|H(\cdot,Du)\|_{L^{1+\delta_g}(\Omega_0)})$ and $m_0 = m_0(n,p,q,s,[a]_{0,\alpha},[b]_{0,\beta},\|H(\cdot,Du)\|_{L^1(\Omega_0)})$. Summarizing we got

$$\begin{aligned}
\mathtt{deg}(B_r) &\Rightarrow \fint_{B_r}\mathcal{V}_0^p(Du,Dh_p)^2\,dx \leq cr^{m_p}\fint_{B_{2r}}H(x,Du)\,dx\\
\mathtt{deg}_\alpha(B_r) &\Rightarrow \fint_{B_r}\mathcal{V}_0^s(Du,Dh_s)^2\,dx \leq cr^{m_s}\fint_{B_{2r}}H(x,Du)\,dx\\
\mathtt{deg}_\beta(B_r) &\Rightarrow \fint_{B_r}\mathcal{V}_0^q(Du,Dh_q)^2\,dx \leq cr^{m_q}\fint_{B_{2r}}H(x,Du)\,dx\\
\mathtt{ndeg}(B_r) &\Rightarrow \fint_{B_r}\mathcal{V}_0(Du,Dh_0)^2\,dx \leq cr^{m_0}\fint_{B_{2r}}H(x,Du)\,dx
\end{aligned}$$

where the above holds for $0<r\leq\tilde{R}_* := \min\{\tilde{R}_*^1,\tilde{R}_*^2,\tilde{R}_*^3,\tilde{R}_*^4,1\}$, and all the quantities involved are as described before. Finally, for the sake of clarity, we let $m:=\min\{m_p,m_q,m_s,m_0\}$. Now take a ball B_r with $0<r\leq\frac{1}{2}\tilde{R}_*$ such that $B_{2r}\Subset\Omega_0\Subset\Omega$. Fix $\tau_p\in\left(0,\frac{1}{8}\right)$ and assume $\mathtt{deg}(B_r)$ and $\mathtt{deg}(B_{\tau_p r})$. Notice that, by virtue of $\mathtt{deg}(B_{\tau_p r})$, there holds

$$a_i(B_{2\tau_p r}) \leq 8[a]_{0,\alpha}(\tau_p r)^{\alpha-\gamma_a} \text{ and } b_i(B_{2\tau_p r}) \leq 8[b]_{0,\beta}(\tau_p r)^{\beta-\gamma_b}. \tag{5.22}$$

We fix $\vartheta\in(0,n)$ and we estimate, by (4.3), (5.22), (3.1), Proposition 1 with $\varphi = H_0^p$, (2.8) and (5.9),

$$\begin{aligned}
\fint_{B_{2\tau_p r}}H(x,Du)\,dx \leq& c\fint_{B_{4\tau_p r}}H_0^p\left(\frac{u-(u)_{\tau_p r}}{4\tau_p r}\right)dx \leq c\fint_{B_{4\tau_p r}}H_0^p(Du)\,dx\\
\leq& c\left\{\fint_{B_{4\tau_p r}}\mathcal{V}_0^p(Du,Dh_p)^2\,dx+\fint_{B_{4\tau_p r}}|V_p(Dh_p)|^2\,dx\right\}\\
\leq& c\left\{\fint_{B_{4\tau_p r}}\mathcal{V}_0^p(Du,Dh_p)^2\,dx+|B_{4\tau_p r}|\sup_{B_{4\tau_p r}}H_0^p(Dh_p)\right\}
\end{aligned}$$

$$\leq c\left\{\int_{B_r} \mathcal{V}_0^p(Du, Dh_p)^2\,dx + \tau_p^n \int_{B_r} H_0^p(Dh_p)\,dx\right\}$$
$$\leq \tau_p^{n-\vartheta}\left(cr^m\tau^{\vartheta-n} + c\tau_p^{\vartheta}\right)\int_{B_{2r}} H(x, Du)\,dx, \tag{5.23}$$

where $c = c(\mathtt{data}(\Omega_0), \vartheta)$. For the ease of exposition we set $\varrho := 2r$ and adjusting the constants in (5.23) we get

$$\int_{B_{\tau_p\varrho}} H(x, Du)\,dx \leq \tau_p^{n-\vartheta}\left(c\varrho^m\tau_p^{\vartheta-n} + c\tau_p^{\vartheta}\right)\int_{B_\varrho} H(x, Du)\,dx.$$

Selecting τ_p in such a way that $c\tau_p^{\vartheta} \leq \frac{1}{2}$ and a threshold radius $R_*^1 \in (0, \tilde{R}_*]$ such that $cR^m\tau_p^{\vartheta-n} \leq \frac{1}{2}$, we can conclude that, for all $\varrho \in (0, R_*^1)$ and all $\vartheta \in (0, n)$,

$$\int_{B_{\tau_p\varrho}} H(x, Du)\,dx \leq \tau_p^{n-\vartheta}\int_{B_\varrho} H(x, Du)\,dx. \tag{5.24}$$

Now fix $\tau_s \in \left(0, \frac{1}{8}\right)$, assume $\mathtt{deg}_\alpha(B_r)$ and that $a_i(B_{\tau_s r}) \leq 4[a]_{0,\alpha}(\tau_s r)^{\alpha-\gamma_a}$, where $r < \frac{1}{2}R_*^1$. In this situation we have

$$a_i(B_{2\tau_s r}) \leq 8[a]_{0,\alpha}(\tau_s r)^{\alpha-\gamma_a} \ \text{ and } \ b_i(B_r) > 4[b]_{0,\beta}r^{\beta-\gamma_b}. \tag{5.25}$$

For $\vartheta \in (0, n)$, by (4.4), (5.25), (3.1), Proposition 1 with $\varphi = H_0^s$, (2.8) and (5.13) we obtain

$$\int_{B_{2\tau_s r}} H(x, Du)\,dx \leq c\int_{B_{4\tau_s r}} \left|\frac{u-(u)_{4\tau_s r}}{\tau_s r}\right|^p + b_i(B_{2\tau_s r})\left|\frac{u-(u)_{4\tau_s r}}{\tau_s r}\right|^s dx \pm c\int_{B_{4\tau_s r}} b_i(B_r)\left|\frac{u-(u)_{4\tau_s r}}{\tau_s r}\right|^s dx$$
$$\leq c\int_{B_{4\tau_s r}} H_0^s\left(\frac{u-(u)_{4\tau_s r}}{\tau_s r}\right)dx \leq c\int_{B_{4\tau_s r}} H_0^s(Du)\,dx$$
$$\leq c\left\{\int_{B_{4\tau_s r}} \mathcal{V}_0^s(Du, Dh_s)^2\,dx + \int_{B_{4\tau_s r}} H_0^s(Dh_s)\,dx\right\}$$
$$\leq c\left\{\int_{B_{4\tau_s r}} \mathcal{V}_0^s(Du, Dh_s)^2\,dx + |B_{4\tau_s r}|\sup_{B_{4\tau_s r}} H_0^s(Dh_s)\right\}$$
$$\leq c\left\{\int_{B_r} \mathcal{V}_0^s(Du, Dh_s)^2\,dx + \tau_s^n\int_{B_r} H_0^s(Dh_s)\,dx\right\}$$
$$\leq \tau_s^{n-\vartheta}\left(cr^m\tau_s^{\vartheta-n} + c\tau_s^{\vartheta}\right)\int_{B_{2r}} H(x, Du)\,dx, \tag{5.26}$$

where $c = c(\mathtt{data}(\Omega_0), \vartheta)$. Again, we name $\varrho := 2r$ thus getting

$$\int_{B_{\tau_s\varrho}} H(x, Du)\,dx \leq \tau_s^{n-\vartheta}\left(c\varrho^m\tau_s^{\vartheta-n} + c\tau_s^{\vartheta}\right)\int_{B_\varrho} H(x, Du)\,dx,$$

where, as before, $\vartheta \in (0, n)$ is arbitrary. Choose τ_s small enough so that $c\tau_s^{\vartheta} < \frac{1}{2}$ and a threshold R_*^2, $0 < R_*^2 \leq R_*^1$ such that $c(R_*^1)^m\tau_s^{\vartheta-n} \leq \frac{1}{2}$. Hence, for all $\varrho \in (0, R_*^2]$ and all $\vartheta \in (0, n)$ we get

$$\int_{B_{\tau_s\varrho}} H(x, Du)\,dx \leq \tau_s^{n-\vartheta}\int_{B_\varrho} H(x, Du)\,dx. \tag{5.27}$$

Consider $\tau_q \in \left(0, \frac{1}{8}\right)$, assume $\mathtt{deg}_\beta(B_r)$ and that $b_i(B_{\tau_q r}) \le 4[b]_{0,\beta}(\tau_q r)^{\beta-\gamma_b}$, where $r < \frac{1}{2}R_*^2$. Now,

$$a_i(B_r) > 4[a]_{0,\alpha}(2r)^{\alpha-\gamma_a} \quad\text{and}\quad b_i(B_{2\tau_q r}) \le 8[b]_{0,\beta}(\tau_q r)^{\beta-\gamma_b} \tag{5.28}$$

holds true. For $\vartheta \in (0, n)$, by (3.15), (5.28), (4.5), (3.1), Proposition 1 with $\varphi = H_0^q$ and (5.17) we obtain

$$\begin{aligned}
\int_{B_{2\tau_q r}} H(x, Du)\, dx \le & c \int_{B_{4\tau_q r}} \left|\frac{u-(u)_{4\tau_q r}}{\tau_q r}\right|^p + a_i(B_{2\tau_q r})\left|\frac{u-(u)_{4\tau_q r}}{\tau_q r}\right|^q dx \pm c\int_{B_{4\tau_q r}} a_i(B_r)\left|\frac{u-(u)_{4\tau_q r}}{\tau_q r}\right|^q dx \\
\le & c\int_{B_{4\tau_q r}} H_0^q\left(\frac{u-(u)_{4\tau_q r}}{\tau_q r}\right) dx \le c\int_{B_{4\tau_{s}t}} H_0^q(Du)\, dx \\
\le & c\left\{\int_{B_{4\tau_q r}} \mathcal{V}_0^q(Du, Dh_q)^2\, dx + \int_{B_{4\tau_q r}} H_0^q(Dh_q)\, dx\right\} \\
\le & c\left\{\int_{B_{4\tau_q r}} \mathcal{V}_0^q(Du, Dh_q)^2\, dx + |B_{4\tau_q r}| \sup_{B_{4\tau_q r}} H_0^q(Dh_q)\right\} \\
\le & c\left\{\int_{B_r} \mathcal{V}_0^q(Du, Dh_0)^2\, dx + \tau_q^n \int_{B_r} H_0^q(Dh_q)\, dx\right\} \\
\le & \tau_q^{n-\vartheta}\left(cr^m\tau_q^{\vartheta-n} + c\tau_q^{\vartheta}\right)\int_{B_{2r}} H(x, Du)\, dx,
\end{aligned} \tag{5.29}$$

where $c = c(\mathtt{data}(\Omega_0), \vartheta)$. Again, we set $\varrho = 2r$ thus obtaining

$$\int_{B_{\tau_q\varrho}} H(x, Du)\, dx \le \tau_q^{n-\vartheta}\left(c\varrho^m\tau_q^{\vartheta-n} + c\tau_q^{\vartheta}\right)\int_{B_\varrho} H(x, Du)\, dx,$$

where, as before, $\vartheta \in (0, n)$ is arbitrary. Take τ_q sufficiently small so that $c\tau_q^{\vartheta} < \frac{1}{2}$ and a threshold R_*^3, $0 < R_*^3 \le R_*^2$ such that $c(R_*^3)^m\tau_q^{\vartheta-n} \le \frac{1}{2}$. Hence, for all $\varrho \in (0, R_*^3]$ and all $\vartheta \in (0, n)$ we get

$$\int_{B_{\tau_q\varrho}} H(x, Du)\, dx \le \tau_q^{n-\vartheta}\int_{B_\varrho} H(x, Du)\, dx. \tag{5.30}$$

Finally, select $\tau_0 \in \left(0, \frac{1}{8}\right)$, assume $\mathtt{ndeg}(B_r)$, where $r \le \frac{1}{2}R_*^3$. In this scenario we observe that

$$a_i(B_r) > 4[a]_{0,\alpha}(r)^{\alpha-\gamma_a} \quad\text{and}\quad b_i(B_r) > 4[b]_{0,\beta}(r)^{\beta-\gamma_b}. \tag{5.31}$$

For $\vartheta \in (0, n)$, by (4.6), (5.31), (3.1), Proposition 1 with $\varphi = H_0$, (2.8) and (5.21) we obtain

$$\begin{aligned}
\int_{B_{2\tau_0 r}} H(x, Du)\, dx \le & c\int_{B_{4\tau_0 r}} \left|\frac{u-(u)_{4\tau_0 r}}{\tau_0 r}\right|^p + a_i(B_{2\tau_0 r})\left|\frac{u-(u)_{4\tau_0 r}}{\tau_0 r}\right|^q + b_i(B_{2\tau_0 r})\left|\frac{u-(u)_{4\tau_0 r}}{\tau_0 r}\right|^s dx \\
& \pm c\int_{B_{4\tau_0 r}} a_i(B_r)\left|\frac{u-(u)_{4\tau_0 r}}{\tau_0 r}\right|^q + b_i(B_r)\left|\frac{u-(u)_{4\tau_0 r}}{\tau_0 r}\right|^s dx \\
\le & c\int_{B_{4\tau_0 r}} H_0\left(\frac{u-(u)_{4\tau_0 r}}{4\tau_0 r}\right) dx \le c\int_{B_{4\tau_0 s}} H_0(Du)\, dx \\
\le & c\left\{\int_{B_{4\tau_0 r}} \mathcal{V}_0(Du, Dh_0)^2\, dx + \int_{B_{4\tau_0 r}} H_0^s(Dh_0)\, dx\right\}
\end{aligned}$$

$$\leq c\left\{\int_{B_{4\tau_0 r}} \mathcal{V}_0(Du, Dh_0)^2\,dx + |B_{4\tau_0 r}|\sup_{B_{4\tau_0 r}} H_0(Dh_0)\right\}$$
$$\leq c\left\{\int_{B_r} \mathcal{V}_0(Du, Dh_0)^2\,dx + \tau_0^n\int_{B_r} H_0^s(Dh_0)\,dx\right\}$$
$$\leq \tau_0^{n-\vartheta}\left(cr^m\tau_0^{\vartheta-n} + c\tau_0^{\vartheta}\right)\int_{B_{2r}} H(x, Du)\,dx, \tag{5.32}$$

where $c = c(\mathtt{data}(\Omega_0), \vartheta)$. Again, we set $\varrho = 2r$ thus obtaining

$$\int_{B_{\tau_0\varrho}} H(x, Du)\,dx \leq \tau_0^{n-\vartheta}\left(c\varrho^m\tau_0^{\vartheta-n} + \tau_0^{\vartheta}\right)\int_{B_\varrho} H(x, Du)\,dx,$$

where, as before, $\vartheta \in (0, n)$ is arbitrary. Take τ_0 sufficiently small so that $c\tau_0^{\vartheta} < \frac{1}{2}$ and a threshold R_*^4, $0 < R_*^4 \leq R_*^3$ such that $c(R_*^4)^m\tau_0^{\vartheta-n} \leq \frac{1}{2}$. Hence, for all $\varrho \in (0, R_*^4]$ and all $\vartheta \in (0, n)$ we get

$$\int_{B_{\tau_0\varrho}} H(x, Du)\,dx \leq \tau_0^{n-\vartheta}\int_{B_\varrho} H(x, Du)\,dx. \tag{5.33}$$

Step 2: double nested exit time and iteration

Now we are in position to develop the announced double nested exit time argument, which will connect estimates (5.24), (5.27), (5.30) and (5.33). Take $B_\varrho \Subset \Omega$ with $\varrho \in (0, R_*]$, where $R_* = \min_{i\in\{1,2,3,4\}}\{R_*^i\}$. For $\kappa \in \mathbb{N}\cup\{0\}$, we consider condition $\mathtt{deg}(B_{\tau_p^{\kappa+1}\varrho})$ and define the exit time index

$$t_p = \min\left\{\kappa \in \mathbb{N}\colon\ \mathtt{deg}(B_{\tau_p^{\kappa+1}\varrho}) \text{ fails}\right\}.$$

For any $\kappa \in \{1, \cdots, t_p\}$ we apply repeatedly (5.24) to obtain

$$\int_{B_{\tau_p^{\kappa}\varrho}} H(x, Du)\,dx \leq \tau_p^{\kappa(n-\vartheta)}\int_{B_\varrho} H(x, Du)\,dx. \tag{5.34}$$

The failure of $\mathtt{deg}(B_{\tau_p^{\kappa+1}\varrho})$ at $\kappa = t_p$, opens three different scenarios: either $\mathtt{deg}_\alpha(B_{\tau_p^{t_p+1}\varrho})$ or $\mathtt{deg}_\beta(B_{\tau_p^{t_p+1}\varrho})$ or directly $\mathtt{ndeg}(B_{\tau_p^{t_p+1}\varrho})$ is in force. Since the last condition is stable for shrinking balls, and the first two are described by similar procedures, we shall focus on the occurrence of $\mathtt{deg}_\alpha(B_{\tau_p^{t_p+1}\varrho})$. Let us introduce a second exit time index

$$t_s = \min\left\{\kappa \in \mathbb{N}\colon \mathtt{deg}_\alpha(B_{\tau_s^{\kappa+1}\tau_p^{t_p+1}\varrho}) \text{ fails}\right\}.$$

Iterating (5.27) with B_ϱ replaced by $B_{\tau_p^{t_p+1}\varrho}$, we obtain

$$\int_{B_{\tau_s^{\kappa}\tau_p^{t_p+1}\varrho}} H(x, Du)\,dx \leq \tau_s^{\kappa(n-\vartheta)}\int_{B_{\tau_p^{t_p+1}\varrho}} H(x, Du)\,dx. \tag{5.35}$$

If $\mathtt{deg}_\alpha(B_{\tau_s^{\kappa+1}\tau_p^{t_p+1}\varrho})$ fails at $\kappa = t_s$, the only chance we have is to look at $\mathtt{ndeg}(B_{\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho})$. Condition $\mathtt{ndeg}$ is stable, so we can iterate (5.33) for $\kappa \in \mathbb{N}$, thus getting

$$\int_{B_{\tau_0^{\kappa}\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho}} H(x, Du)\,dx \leq \tau_0^{\kappa(n-\vartheta)}\int_{B_{\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho}} H(x, Du)\,dx. \tag{5.36}$$

Now we only need to fillet estimates (5.33)-(5.36). For $0 < \varsigma < \varrho \le R_*$ we consider the following five cases.
Case (i): $\varrho > \varsigma \ge \tau_p^{t_p+1}\varrho$. Then there is $\bar\kappa \in \{0, \cdots, t_p\}$ such that $\tau_p^{\bar\kappa+1}\varrho \le \varsigma < \tau_p^{\bar\kappa}\varrho$. We obtain from (5.34) that,

$$\begin{aligned}\int_{B_\varsigma} H(x,Du)\,dx \le& \int_{B_{\tau_p^{\bar\kappa}\varrho}} H(x,Du)\,dx\\ \le& \tau_p^{\bar\kappa(n-\vartheta)}\int_{B_\varrho} H(x,Du)\,dx\\ \le& \tau_p^{(\bar\kappa+1)(n-\vartheta)}\tau_p^{\vartheta-n}\int_{B_\varrho} H(x,Du)\,dx \le c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx,\end{aligned} \tag{5.37}$$

where $c = c(\texttt{data}(\Omega_0), \vartheta)$.
Case (ii): $\tau_p^{t_p+1}\varrho > \varsigma \ge \tau_s\tau_p^{t_p+1}\varrho$. We see that, by (5.37),

$$\begin{aligned}\int_{B_\varsigma} H(x,Du)\,dx \le& \int_{B_{\tau_p^{t_p+1}\varrho}} H(x,Du)\,dx\\ \le& c\tau_p^{(t_p+1)(n-\vartheta)}\int_{B_\varrho} H(x,Du)\,dx\\ =& c(\tau_s\tau_p^{t_p+1})^{n-\vartheta}\tau_s^{\vartheta-n}\int_{B_\varrho} H(x,Du)\,dx \le c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx,\end{aligned} \tag{5.38}$$

with $c = c(\texttt{data}(\Omega_0), \vartheta)$.
Case (iii): $\tau_s\tau_p^{t_p+1}\varrho > \varsigma \ge \tau_s^{t_s+1}\tau_p^{t_p+1}\varrho$. So there is $\bar\kappa \in \{1, \cdots, t_s\}$ so that $\tau_s^{\bar\kappa}\tau_p^{t_p+1}\varrho > \varsigma \ge \tau_s^{\bar\kappa+1}\tau_p^{t_p+1}\varrho$. We have, by (5.35) and (5.37),

$$\begin{aligned}\int_{B_\varsigma} H(x,Du)\,dx \le& \int_{B_{\tau_s^{\bar\kappa}\tau_p^{t_p+1}\varrho}} H(x,Du)\,dx\\ \le& \tau_s^{\bar\kappa(n-\vartheta)}\int_{B_{\tau_p^{t_p+1}\varrho}} H(x,Du)\,dx\\ \le& \tau_s^{(\bar\kappa+1)(n-\vartheta)}\tau_s^{\vartheta-n}\tau_p^{(t_p+1)(n-\vartheta)}\int_{B_\varrho} H(x,Du)\,dx \le c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx,\end{aligned} \tag{5.39}$$

where $c = c(\texttt{data}(\Omega_0), \vartheta)$.
Case (iv): $\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho > \varsigma \ge \tau_s^{t_s+1}\tau_p^{t_p+1}\tau_0\varrho$. By (5.39) we obtain

$$\begin{aligned}\int_{B_\varsigma} H(x,Du)\,dx \le& \int_{B_{\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho}} H(x,Du)\,dx\\ \le& c(\tau_s^{t_s+1}\tau_p^{t_p+1})^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx\\ \le& c\tau_0^{\vartheta-n}(\tau_0\tau_s^{t_s+1}\tau_p^{t_p+1})^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx \le c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx,\end{aligned} \tag{5.40}$$

with $c = c(\texttt{data}(\Omega_0), \vartheta)$.
Case (v): $\tau_s^{t_s+1}\tau_p^{t_p+1}\tau_0\varrho > \varsigma > 0$. This condition renders a $\bar\kappa \in \mathbb{N}$ such that $\tau_0^{\bar\kappa+1}\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho \le \varsigma <$

$\tau_0^{\bar{\kappa}}\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho$. We then estimate, using (5.36) and (5.40),

$$
\begin{aligned}
\int_{B_\varsigma} H(x,Du)\,dx \le & \int_{B_{\tau_0^{\bar{\kappa}}\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho}} H(x,Du)\,dx \\
\le & \tau_0^{\bar{\kappa}(n-\vartheta)} \int_{B_{\tau_s^{t_s+1}\tau_p^{t_p+1}\varrho}} H(x,Du)\,dx \\
\le & c\tau_0^{\bar{\kappa}(n-\vartheta)}(\tau_s^{t_s+1}\tau_p^{t_p+1})^{(n-\vartheta)} \int_{B_\varrho} H(x,Du)\,dx \\
\le & \tau_0^{\vartheta-n}\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx = c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx,
\end{aligned} \tag{5.41}
$$

where $c = c(\texttt{data}(\Omega_0),\vartheta)$.
As mentioned before, the procedure is the same if, after $\texttt{deg}$ occurs $\texttt{deg}_\beta$ instead of $\texttt{deg}_\alpha$ and it is actually easier if, from $\texttt{deg}$ we jump directly to $\texttt{ndeg}$. All in all we can conclude that, for all $0<\varsigma<\varrho\le R_*$ and all $\vartheta\in(0,n)$ there holds

$$
\int_{B_\varsigma} H(x,Du)\,dx \le c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx, \tag{5.42}
$$

with $c = c(\texttt{data}(\Omega_0),\vartheta)$. Now, if $0<R_*\le\varsigma<\varrho\le 1$, we get

$$
\begin{aligned}
\int_{B_\varsigma} H(x,Du)\,dx \le & \left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\left(\frac{\varrho}{\varsigma}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx \\
\le & \left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\left(\frac{\varrho}{R_*}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx \le c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx,
\end{aligned} \tag{5.43}
$$

where $c = c(\texttt{data}(\Omega_0),\vartheta)$, by recalling the dependencies of R_*. Finally, if $0<\varsigma<R_*\le\varrho\le 1$, by (5.42) and (5.43) we have

$$
\begin{aligned}
\int_{B_\varsigma} H(x,Du)\,dx \le & c\left(\frac{\varsigma}{R_*}\right)^{n-\vartheta}\int_{B_{R_*}} H(x,Du)\,dx \\
\le & c\left(\frac{\varsigma}{R_*}\right)^{n-\vartheta}\left(\frac{R_*}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx = c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx,
\end{aligned} \tag{5.44}
$$

for $c = c(\texttt{data}(\Omega_0),\vartheta)$. Collecting estimates (5.42)-(5.44) we conclude that, for all $0<\varsigma<\varrho\le 1$ there holds

$$
\int_{B_\varsigma} H(x,Du)\,dx \le c\left(\frac{\varsigma}{\varrho}\right)^{n-\vartheta}\int_{B_\varrho} H(x,Du)\,dx, \tag{5.45}
$$

with $c = c(\texttt{data}(\Omega_0),\vartheta)$.

6 Gradient continuity

From (5.45) and a standard covering argument, we can conclude that for every open subset $\Omega_0\Subset\Omega$ and $\kappa>0$ there exists a constant $c = c(\texttt{data}(\Omega_0),\kappa)$ such that

$$
\fint_{B_r} H(x,Du)\,dx \le cr^{-\kappa} \tag{6.1}
$$

holds for every ball $B_r \Subset \Omega_0 \Subset \Omega$, $r \le 1$. Now, if h is any of the harmonic maps given by Lemma 7 and $\tilde{H}$ is one of the Young functions listed in (2.3) with $a_0 = a_i(B_r)$ or $b_0 = b_i(B_r)$, then, the theory in [33] applies rendering

$$\fint_{B_\varrho} \tilde{H}(Dh - (Dh)_{B_\varrho})\, dx \le c\left(\frac{\varrho}{r}\right)^{p\tilde{\nu}} \fint_{B_r} \tilde{H}(Dh)\, dx \le c\left(\frac{\varrho}{r}\right)^{p\tilde{\nu}} \fint_{B_r} H(x, Du)\, dx, \tag{6.2}$$

where c and $\tilde{\nu}$ depend at the most from n, p, q, s. Moreover, for $B_r \Subset \Omega_0$ with $0 < r \le R_*$, where R_* is the threshold radius introduced in the previous section, we obtain from Lemma 7 and (6.1) that

$$\fint_{B_r} \widetilde{\mathcal{V}}(Du, Dh)^2\, dx \le cr^m \fint_{B_{2r}} H(x, Du)\, dx \le cr^{m-\kappa} = cr^{\kappa_0},$$

by fixing $\kappa := \frac{m}{2}$, where $\widetilde{\mathcal{V}}$ is the corresponding auxiliary function defined in (2.9) and the constant c depends on $\mathtt{data}(\Omega_0)$. Arguing exactly as in [14, Proposition 3.3], we get

$$\fint_{B_r} \tilde{H}(Du - Dh)\, dx \le cr^{\kappa_1} \tag{6.3}$$

for some positive exponent $\kappa_1 = \kappa_1(n, p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|H(\cdot, Du)\|_{L^1(\Omega_0)})$. In this case, $c = c(\mathtt{data}(\Omega_0))$. Now, for $0 < \varrho < r \le R_*$, by (6.3), the minimality of h, (6.1) and (6.2) we see that

$$\begin{aligned}
\fint_{B_\varrho} |Du - (Du)_\varrho|^p\, dx \le & c\left\{\fint_{B_\varrho} |Dh - (Dh)_\varrho|^p\, dx + \fint_{B_\varrho} |Du - Dh|^p\, dx\right\} \\
\le & c\left\{\fint_{B_\varrho} \tilde{H}(Dh - (Dh)_\varrho)\, dx + \left(\frac{r}{\varrho}\right)^n \fint_{B_r} \tilde{H}(Du - Dh)\, dx\right\} \\
\le & c\left\{\left(\frac{\varrho}{r}\right)^{p\tilde{\nu}} \fint_{B_r} H(x, Du)\, dx + \left(\frac{r}{\varrho}\right)^n r^m \fint_{B_{2r}} H(x, Du)\, dx\right\} \\
\le & c\left(\varrho^{p\tilde{\nu}} r^{-p\tilde{\nu}-\kappa} + \varrho^{-n} r^{n+\kappa_1}\right),
\end{aligned} \tag{6.4}$$

with $c = c(\mathtt{data}(\Omega_0), \kappa)$. Now, first notice that there is no loss of generality in supposing $p\tilde{\nu} \le 1$. Setting $\varrho = r^{1+\frac{\kappa_1}{4n}}$ and $\kappa := \frac{\kappa_1 p\tilde{\nu}}{8n}$ in (6.4), we easily obtain

$$\fint_{B_\varrho} |Du - (Du)_\varrho|^p\, dx \le c\varrho^{\frac{\kappa_1 p\tilde{\nu}}{16n}}, \tag{6.5}$$

for all $\varrho \in (0, R_*)$, with $c = c(\mathtt{data}(\Omega_0))$. Now, by the integral characterization of Hölder continuity due to Campanato and Meyers we can conclude that $Du \in C^{0,\nu}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ for $\nu = \frac{\kappa_1 \tilde{\nu}}{16n}$. The full proof of Theorem 1 is still not complete, since ν depends on $\mathtt{data}(\Omega_0)$, while we announced that the Hölder continuity exponent of Du depends only on $\mathtt{data}$. So we will retain that, after a covering argument, $Du \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, therefore the non-uniform ellipticity of (1.6) becomes immaterial. Now, for $B_r \Subset \Omega_0 \Subset \Omega$, no matter what degeneracy condition holds, we compare u to $h \in W^{1,H0}(B_r)$ solution to the Dirichlet problem

$$u + W^{1,H(\cdot)}_0(B_r) \ni w \mapsto \min \int_{B_r} h_0(Dw)\, dx, \tag{6.6}$$

where $H_0(z) := |z|^p + a_i(B_r)|z|^q + b_i(B_r)|z|^s$. Notice that, for a functional like the one in (6.6), the Bounded Slope Condition holds, see [6], so there exists $c = c(n, p, q, s, \|Du\|_{L^\infty(B_r)})$ such that

$$\|Dh\|_{L^\infty(B_r)} \le c. \tag{6.7}$$

By strict convexity we obtain

$$\fint_{B_r} \mathcal{V}_0(Du, Dh)^2\, dx \le c \fint_{B_r} H_0(Du) - H_0(Dh)\, dx$$

$$= c\left\{\fint_{B_r} H_0(Du) - H(x, Du)\, dx + \fint_{B_r} H(x, Du) - H(x, Dh)\, dx + \fint_{B_r} H(x, Dh) - H_0(Dh)\, dx\right\} \le cr^{\gamma}, \quad (6.8)$$

with $\gamma := \min\{\alpha, \beta\}$ and $c = c(p, q, s, [a]_{0,\alpha}, [b]_{0,\beta}, \|a\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|Du\|_{L^\infty(\Omega_0)})$. We got this last estimate by using (1.8), the boundedness of $\|Du\|_{L^\infty_{\text{loc}}(\Omega)}$ and (6.7). Now we jump back to (6.4), thus getting

$$\fint_{B_\varrho} |Du - (Du)_\varrho|^p\, dx \le c\left\{\fint_{B_\varrho} H(x, Du - Dh)\, dx + \left(\frac{\varrho}{r}\right)^{p\tilde{\nu}} \fint_{B_r} H(x, Du)\, dx\right\} \le c\left(\varrho^{-n} r^{n+\gamma} + \varrho^{p\tilde{\nu}} r^{-p\tilde{\nu}}\right), \quad (6.9)$$

with $c = c(\mathtt{data}(\Omega_0), \|Du\|_{L^\infty(\Omega_0)})$. Equalizing in (6.9) as we did to get (6.4), we have

$$\fint_{B_\varrho} |Du - (Du)_\varrho|^p\, dx \le c\varrho^{\nu p},$$

with $\nu = \frac{\gamma\tilde{\nu}}{n + p\tilde{\nu}}$. This means, by the integral characterization of Hölder continuity due to Campanato and Mayers, that $Du \in C^{0,\nu}_{\text{loc}}(\Omega)$, and, recalling that $\tilde{\nu} = \tilde{\nu}(n, p, q, s)$, we see that now $\nu = \nu(\mathtt{data})$. This concludes the proof.

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