Kähler groups and Geometric Group Theory

Claudio Llosa Isenrich

Wolfson College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

Trinity 2017

To my parents

Acknowledgements

First, and above all, I want to express my deepest gratitude to my supervisor Martin Bridson for suggesting this challenging and rewarding topic to me, and for guiding me through it; for his endless help, encouragement and support; for the countless meetings and inspiring discussions; and for never losing faith in me, even at points when I had long lost it. I have no doubt that without him my DPhil studies would not have been nearly the same great and rewarding experience.

I am very grateful to Simon Donaldson for inspiring discussions and, in particular, for providing me with the key ideas for the proof of Theorem 9.3; to Mahan Mj for his hospitality during my visit to TIFR in Mumbai and for helpful discussions which, in particular, led me to simplify previous versions of some of the results in Chapter 4; to Alex Suciu for helpful discussions and the stimulus of asking me Question 3; and to Pierre Py for his hospitality during my visit to UNAM in Mexico City and for helpful discussions.

I am equally grateful to my fellow DPhil students and friends here in the Mathematical Institute for inspiring and helpful discussions, and for providing a great atmosphere to work in. In particular, I thank my office mate Alex Betts and my office mate and DPhil brother Giles Gardam for the many hours that we spent together in this room and all the mathematical and non-mathematical discussions; Simon Gritschacher, Tom Zeman, Matthias Wink and Antonio de Capua for the countless fun lunches; and my other two fellow DPhil brothers, whom I shared common time with here at Oxford, Rob Kropholler and Nici Heuer.

I gratefully acknowledge the financial support received from the EPSRC, the DAAD, and the Studienstiftung des deutschen Volkes e.V., and the support of the University of Oxford and Wolfson College who provided me with the perfect working and living environment. Through their generous support these institutions enabled me to conduct the research contained in this work.

It is also very important to me to thank my family and all of my friends outside the Oxford Maths department, who provided me with the best environment in Oxford and outside Oxford that I could have hoped for. In particular, my parents Pablo and Ursula and siblings Simon, Clara and Sofia for their endless support; Claudia, Corina, Luke, Maurits and Nina for the countless fun hours; Mika and all of the other guys in the football team; and my good friends Stefan and Randy from my undergraduate times in Munich.

To everyone else, who helped and supported me and shared time with me in the process of writing this thesis, whom I forgot to mention here – my gratitude of course extends to all of you.

Last, but definitely not least, I want to thank Lydia for her endless support, encouragement and motivation, for always standing by my side, and for living through this experience with me.

Abstract

In this thesis we study Kähler groups and their connections to Geometric Group Theory. This work presents substantial progress on three central questions in the field:

1. Which subgroups of direct products of surface groups are Kähler?
2. Which Kähler groups admit a classifying space with finite $(n-1)$-skeleton but no classifying space with finitely many n-cells?
3. Is it possible to give explicit finite presentations for any of the groups constructed in response to Question 2?

Question 1 was raised by Delzant and Gromov in [58].

Question 2 is intimately related to Question 1: the non-trivial examples of Kähler subgroups of direct products of surface groups never admit a classifying space with finite skeleton.

The only known source of non-trivial examples for Questions 1 and 2 are fundamental groups of fibres of holomorphic maps from a direct product of closed surfaces onto an elliptic curve; the first such construction is due to Dimca, Papadima and Suciu [62].

Question 3 was posed by Suciu in the context of these examples.

In this thesis we:

- provide the first constraints on Kähler subdirect products of surface groups (**Theorem 7.3.1**);
- develop new construction methods for Kähler groups from maps onto higher-dimensional complex tori (**Section 6.1**);
- apply these methods to obtain irreducible examples of Kähler subgroups of direct products of surface groups which arise from maps onto higher-dimensional tori and use them to show that our conditions in Theorem 7.3.1 are minimal (**Theorem A**);

- apply our construction methods to produce irreducible examples of Kähler groups that *(i)* have a classifying space with finite $(n-1)$-skeleton but no classifying space with finite n-skeleton *and (ii)* do not have a subgroup of finite index which embeds in a direct product of surface groups (**Theorem 8.3.1**);
- provide a new proof of Biswas, Mj and Pancholi's generalisation [24] of Dimca, Papadima and Suciu's construction to more general maps onto elliptic curves (**Theorem 4.3.2**) and introduce invariants that distinguish many of the groups obtained from this construction (**Theorem 4.6.2**); and
- construct explicit finite presentations for Dimca, Papadima and Suciu's groups thereby answering Question 3 (**Theorem 5.4.4**).

Statement of Originality

I declare that the work contained in this thesis is, to the best of my knowledge, original and my own work, unless indicated otherwise. I also declare that the work contained in this thesis has not been submitted towards any other degree at this institution or at any other institution.

Chapter 2 and Appendices A, B and C contain known results from the literature and, unless indicated otherwise, the results presented in these chapters are not my own work. The same is true for the results mentioned in the first part of Chapter 1.

Chapter 8 and Section 6.1.2 are based on my joint paper with my advisor Martin R. Bridson [32]. My main contributions to these results are the construction of these groups which is contained in Sections 8.1, 8.2 and 8.3 and the general results in Section 6.1.2. Furthermore, I made at least partial contributions to all sections of Chapter 8.

The contents of Chapters 3, 4, 5, 7 and 9 are my own original work, unless indicated otherwise. The same is true for Chapter 6 with the exception of Section 6.1.2 (as discussed above). Chapter 4 is based on my paper [98], Chapter 5 is based on my paper [99], and Chapter 6 is based on my paper [100].

Claudio Llosa Isenrich

Oxford, 4th December 2017

Contents

List of Figures

Chapter 1

Introduction

Misión, quizás en parte cumplida.

Javier Llosa García

In this thesis we study Kähler groups and their connections to Geometric Group Theory. A *Kähler group* is a group which can be realised as the fundamental group of a compact Kähler manifold. Kähler groups have been the focus of a field of active research for the last 70 years.

Being Kähler places strong constraints on a group. For instance, the group must have even first Betti number, be one-ended, be 1-formal, and it is not the fundamental group of a compact 3-manifold without boundary unless it is finite.

On the other hand, the class of Kähler groups is far from being trivial. It includes all finite groups, *surface groups* (fundamental groups of closed orientable surfaces), abelian groups of even rank; and direct products, as well as finite index subgroups of Kähler groups are again Kähler. In addition, there are examples of Kähler groups with exotic properties such as non-residually finite Kähler groups and non-coherent Kähler groups.

Despite being a field of active research for many decades, we are still remarkably far from understanding Kähler groups. In fact we do not even have a good guess as to what a classification of Kähler groups may look like.

The chances of answering the question of which groups are Kähler improve considerably when we restrict ourselves to specific classes of groups. It has already been mentioned that a Kähler groups which is the fundamental group of a compact 3-manifold without boundary must be finite, and there are other instances of this kind of result. For instance, Kähler groups which are 1-relator groups must be fundamental groups of closed orientable orbisurfaces and Kähler groups which have positive first l^2-Betti number are commensurable to surface groups.

Many of these results are based on a strong connection between Kähler groups and surface groups: any homomorphism from a Kähler group $G = \pi_1 M$ (M compact Kähler) onto a hyperbolic surface group is induced by a holomorphic map with connected fibres from M onto a closed Riemann surface. With their fundamental work on cuts in Kähler groups, Delzant and Gromov [58] initiated the more general study of the relation between Kähler groups and subgroups of direct products of surface groups. They provided criteria that imply that a Kähler group maps to a direct product of surface groups. Their work led them to ask the following question:

Question 1. *Which subgroups of direct products of surface groups are Kähler?*

Further impetus to their question has been given by the recent work of Py [107] and Delzant and Py [59], who showed that Kähler groups which act nicely on CAT(0) cube complexes are subgroups of direct products of surface groups.

Despite the significance of Delzant and Gromov's question and the fact that it has been around for more than 10 years now, our knowledge of Kähler subgroups of direct products of surface groups is very limited. There are only two known classes of examples. The first class consists of the finite index subgroups which are trivially Kähler. The second and much more interesting class is a class of subgroups arising as kernels of homomorphisms from a direct product of hyperbolic surface groups onto $\mathbb{Z}^2$. This class was constructed by Dimca, Papadima and Suciu by considering fibrations associated to 2-fold branched covers of elliptic curves [62].

Their work was motivated by the question of finding Kähler groups with exotic finiteness properties. A group G has *finiteness type* $\mathcal{F}_r$ if it has a classifying space $K(G,1)$ with finite r-skeleton. It has finiteness type $\mathcal{F}_\infty$ if it has a classifying space with finite r-skeleton for every r and finiteness type $\mathcal{F}$ if it has a finite classifying space. For every $r \geq 3$ they construct a subgroup of a direct product of surface groups of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$. This was a big breakthrough in the field, since it showed that Kähler groups can have exotic finiteness properties. We will refer to Dimca, Papadima and Suciu's examples as the *DPS groups*.

Given the close relation between Kähler groups and surface groups and the very good understanding of the finiteness properties of subgroups of direct products of surface groups following the work of Bridson, Howie, Miller and Short [29, 30], it comes as no surprise that the first examples of Kähler groups with exotic finiteness properties are subgroups of direct products of surface groups. In fact, it follows from the work of Bridson, Howie, Miller and Short that any non-trivial example of a Kähler subgroup of a direct product of surface groups must have exotic finiteness properties.

Thus, there is a close connection between Delzant and Gromov's question and the following question:

Question 2. *Which Kähler groups are of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$?*

In the context of the DPS groups, Suciu asked the question of

Question 3. *Is it possible to find explicit finite presentations for any of the groups constructed in response to Question 2?*

The significance of Question 3 lies in the fact that having explicit finite presentations opens up many possibilities for explicit interrogations of these groups.

With this thesis I contribute to an answer to each of these three questions. Concerning Question 1, I develop construction methods that lead to new classes of Kähler subgroups of direct products of surface groups (see Chapters 4 and 6); these examples have appeared in [98] and [100]. I also provide criteria which imply that a subgroup of a direct product of surface groups is not Kähler (see Chapter 7), as well as criteria which imply that a Kähler group is a subgroup of a direct product of surface groups (see Chapter 3). The focus in Chapters 4, 6 and 7 is on subgroups that arise as kernels from a direct product of surface groups onto an abelian group. These subgroups are called *coabelian*. They form an important class of subgroups – for three factors all finitely presented full subdirect products of surface groups are virtually coabelian while for more factors they are virtually conilpotent. We want to emphasise that the key constraints that we derive in Chapter 7 are very general: they apply to all Kähler subgroups of direct products of surface groups and also to Kähler groups which map to direct products of surface groups with finitely generated kernel.

We call a group *irreducible* if it has no finite index subgroup which splits as a direct product of two non-trivial groups. A subgroup of a direct product of groups is called *coabelian of even (odd) rank* if it arises as the kernel of a homomorphism onto an abelian group of even (odd) rank. The main consequence of our analysis of coabelian Kähler subgroups of direct products of surface groups can be summarised as follows:

Theorem A. *Let $G \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$ be a Kähler subgroup of a direct product of fundamental groups of closed Riemann surfaces S_{g_i} of genus $g_i \geq 2$, $1 \leq i \leq r$. Assume that G is of type $\mathcal{F}_m$ with $m \geq \frac{2r}{3}$ and has trivial centre.*

Then G has a finite index subgroup which is coabelian of even rank, and every finite index coabelian subgroup of G is coabelian of even rank.

Conversely, for any $r \geq 3$, $r - 1 \geq m \geq \frac{2r}{3}$ and $g_1, \cdots, g_r \geq 2$, there is a Kähler subgroup $K \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$ which is an irreducible full subdirect product of type $\mathcal{F}_m$ but not of type $\mathcal{F}_{m+1}$ (and has trivial centre).

The examples constructed in the context of Question 1 have exotic finiteness properties and thus contribute towards an answer of Question 2. Towards Question 2, we will also provide a completely different class of Kähler groups based on Kodaira surfaces (see Chapter 8). These groups do not have any finite index subgroups which are subgroups of direct products of surface groups. Their construction is contained in a joint paper with my advisor Martin R. Bridson [32]. Concerning Question 3, I construct explicit finite presentations for the DPS groups in Chapter 5. These results are published in [99].

1.1 Structure and Contents

In Chapter 2 we summarise relevant background material from the literature and explain the results discussed in the introduction in more detail. We start by recalling important notions concerning Group Theory required in this work in Section 2.1. In Section 2.2 we give a short general introduction to Kähler groups, expanding on some of the material in this chapter. In Section 2.3 we give a detailed overview of what is known about the deep connection between Kähler groups and surface groups and more generally Kähler groups and subgroups of direct products of surface groups. To highlight the significance of these results we discuss constraints on Kähler groups coming from Geometric Group Theory which are closely related to surface groups – these are contained in Section 2.4. In Section 2.5 we give an introduction to the finiteness properties of groups with a particular focus on the DPS groups.

In Chapter 3 we provide new criteria that imply that a Kähler group is a subgroup of a direct product of surface groups. These emphasise the importance of solving Delzant and Gromov's Question 1. In Section 3.1 we prove that every residually free Kähler group is a subdirect product of surface groups and a free abelian group (**Theorem 3.1.1**). In Section 3.2 we consider maps from Kähler groups onto torsion-free Schreier groups with non-vanishing first Betti number and show that any such map factors through a map onto a surface group (**Theorem 3.2.1**). This provides us with new criteria for a Kähler group to be a subgroup of a direct product of surface groups (**Theorem 3.2.3**).

In Chapter 4 we provide a new class of Kähler subgroups of direct products of r surface groups which are of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$ for every $r \geq 3$ (**Theorem**

4.3.2). All of our groups arise as fundamental group of fibres of holomorphic maps from direct products of closed hyperbolic Riemann surfaces onto elliptic curves which restrict to branched coverings on the factors. In fact, we show that the fundamental group of the fibre of any such map with at least three factors and connected fibres provides an example. From a group theoretic point of view, all of these examples are kernels of homomorphisms from a direct product of surface groups onto $\mathbb{Z}^2$. Our class is inspired by the DPS groups. This general construction is contained in Sections 4.1, 4.2 and 4.3. In Section 4.4 we give some very concrete examples which to us seem like the most natural generalisation of Bestvina–Brady groups in the setting of Kähler groups. In Sections 4.5 and 4.6 we introduce invariants that distinguish many of the groups obtained from our construction. In particular, they show that our class contains genuinely new examples (**Theorem 4.6.2**).

In Chapter 5 we construct explicit finite presentations for the DPS groups. In Sections 5.2 and 5.3 we apply a method by Bridson, Howie, Miller and Short [31] to obtain these explicit finite presentations (**Theorem 5.3.1**). In Section 5.4 we show how to simplify these presentations. This leads to presentations in which the relations naturally correspond to the standard relations in the direct product of surface groups (**Theorem 5.4.4**).

In Chapter 6 we provide a new method for constructing Kähler groups from maps onto higher-dimensional complex tori, and apply it to obtain a new class of examples of Kähler subgroups of direct products of surface groups; these arise as fundamental groups of the smooth generic fibres of these maps. Our construction method is explained in Section 6.1; the main result of this section is **Theorem 6.1.7**. We also consider two special cases of our construction which are of particular interest. The first one is a generalisation of Dimca, Papadima and Suciu's [62, Theorem C] to maps with fibrelong isolated singularities (see Definition 6.1.4 and **Theorem 6.1.5**). Theorem 6.1.5 and its proof are contained in a joint paper with Martin Bridson [32]. The second one is the special case of Theorem 6.1.7 when all singularities are isolated (**Theorem 6.1.3**). All of these results require the higher-dimensional tori to have certain symmetries. They are for instance satisfied when the torus is a k-fold direct product of an elliptic curve with itself. The construction methods developed in Section 6.1 allow us to construct new examples of irreducible Kähler subgroups of direct products of surface groups arising as kernels of homomorphisms from the direct product onto $\mathbb{Z}^{2k}$ (**Theorem 6.4.1**). These groups cover the existence part of Theorem A.

In Chapter 7 we give criteria which imply that a subgroup of a direct product of surface groups is not Kähler and, more generally, that a group which maps onto such a subgroup is not Kähler (**Theorem 7.3.1**). We use our results to provide constraints on coabelian subgroups of direct products of surface groups arising as kernels of a homomorphism onto a free abelian group. We we will show that in many cases in which the free abelian group has odd rank, the kernel of such a map is not Kähler. In particular, we will see that no subgroup arising as kernel of a homomorphism onto $\mathbb{Z}$ is Kähler (**Theorem 7.1.1**) and that for all k the kernel of a homomorphism from a product of r factors onto $\mathbb{Z}^{2k+1}$ cannot be Kähler if it is of finiteness type $\mathcal{F}_m$ with m at least $\frac{2r}{3}$. We use our results to prove that there are non-Kähler full subdirect products of surface groups which have even first Betti number (**Theorem 7.2.4**).

In Chapter 8 we construct classes of examples of Kähler groups with exotic finiteness properties of a different kind. They arise as fundamental groups of smooth generic fibres of holomorphic maps from a direct product of Kodaira fibrations onto an elliptic curve. We will provide two classes of examples. For the first and more interesting one, we modify Kodaira's construction of the first such fibrations with positive signature (see Section 8.2). In this way we obtain examples of irreducible Kähler groups of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$ ($r \geq 3$) which do not have any subgroup of finite index which is a subgroup of a direct product of surface groups (**Theorem 8.3.1**). For the second class, we consider Kodaira fibrations with signature zero and obtain examples of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$ which have finite index subgroups which are subgroups of direct products of surface groups (**Theorem 8.1.1**). However, these groups are in general not themselves subgroups of direct products of surface groups – we will give precise criteria for when they are such subgroups in Section 8.4. This Chapter is based on a joint paper with Martin R. Bridson [32].

In Chapter 9 we discuss a promising strategy for a proof of **Conjecture 6.1.2**, a conjecture we make in Chapter 6. This conjecture would provide a generalisation of Theorem 6.1.7 to holomorphic maps with isolated singularities and connected fibres onto general higher-dimensional complex tori, dropping any assumptions on symmetries of the tori. It is based on a putative induction argument which makes use of the local structure of a fibration with isolated singularities. A key ingredient in the induction is **Theorem 9.3.1** about lines in varieties. The proposed induction is explained in Section 9.4. The current gap in the argument stems from a lack of properness in a map that occurs in the induction step, as required in the current proof of Lemma 9.4.3. However, we believe that it should be possible to close this gap in future by means of a closer examination of the local topological structure of a fibration with isolated singularities.

There are three appendices. In Appendix A we discuss some classical results from homotopy theory that we make use of in Chapter 9. In Appendix B we give a brief discussion of the Lefschetz Hyperplane Theorem and its significance for the construction of Kähler groups. In Appendix C we introduce the basic notions of formality and explain some of its implications for Kähler groups.

Chapter 2

Background

In this chapter we want to provide an overview of the theory of Kähler groups with a particular focus on the relation between Kähler groups and subgroups of direct products of surface groups.

2.1 Group Theory

We begin by summarising some results from Group Theory that we will use in this work. In particular, we will give an introduction to residually free groups and their structure theory.

We start by recalling a few basic notions from Group Theory. Let G be a group, $H \leq G$ be a subgroup, and $gH = \{g \cdot h | h \in H\}$ be the left coset of g in G with respect to H. The *index* $[G : H] \in \mathbb{N} \cup \{\infty\}$ of H in G is the number of pairwise disjoint left cosets of H in G. We say that H is of *finite index* in G if $[G : H] < \infty$. The *centre* of G is the subset $Z(G) := \{g \in G \mid gh = hg \text{ for all } h \in G\}$.

For a group property $\mathcal{P}$ we call a group *virtually* $\mathcal{P}$ if the group has a finite index subgroup which has the property $\mathcal{P}$. For instance, we call a group virtually abelian if it has an abelian subgroup of finite index.

Two groups G and H are called *commensurable* if there are subgroups $G_1 \leq G$ and $H_1 \leq H$ of finite index such that $G_1 \cong H_1$. We say that G and H are *commensurable up to finite kernels* if there is a finite sequence $G = P_1, \dots, P_k = H$ of groups such that P_i is commensurable to P_{i+1} for $1 \leq i \leq k-1$. For residually finite groups commensurability up to finite kernels implies commensurability; this is false in general (see [51] for more details).

A *presentation of a group* G consists of a generating set $S = \{s_i\}_{i \in I}$ of G together with a set $R = \{r_j\}_{j \in J}$ of relations (words in the generators and their inverses) $r_j = s^{\pm 1}_{i_1(j)} \cdots s^{\pm 1}_{i_{k(j)}(j)}$, such that r_j is trivial in G and $G = F(S)/\langle\langle R \rangle\rangle$. Here $F(S)$ denotes the free group on the set S and $\langle\langle R\rangle\rangle$ denotes the normal closure of R in $F(S)$, that is, the smallest normal subgroup of $F(S)$ containing all words in R. We write $G = \langle S \mid R\rangle$.

We call a group G *finitely generated* if there exists a presentation of G with a finite generating set and we call G *finitely presented* if in addition, we can choose a finite set of relations.

An important connection between groups and geometry is provided by the *Cayley graph* $\mathrm{Cay}(G,S)$ of a group G with respect to a finite (or countable) generating set S. It is defined as the graph with vertex set $V = \{g \mid g \in G\}$ and edge set $E = \{(g, gs) \mid g \in G, s \in S\}$. If S is finite then every vertex of $\mathrm{Cay}(G,S)$ has *valency* $2|S|$, i.e., there are $2|S|$ edges incident to every vertex of $Cay(G,S)$. For every Cayley graph $\mathrm{Cay}(G,S)$ there is a canonical corresponding cell complex together with a geodesic metric with respect to which every edge has length one. In the following we do not distinguish between a Cayley graph and its realisation as cell complex and denote both by $\mathrm{Cay}(G,S)$.

For a topological space X we define its *set of ends* as the limit $\lim \pi_0(X \setminus K)$ over all compact subsets $K \subset X$. The cardinality of the set of ends is called the *number of ends* of X and is denoted by $e(X) \in \mathbb{N} \cup \{\infty\}$.

For a finitely generated group G, the number of ends of a Cayley graph $Cay(G,S)$ is independent of the choice of finite generating set S. Thus, we can define the number of ends of a finitely generated group G by $e(G) = e(Cay(G,S))$, where S is any finite generating set. The following theorem summarises some facts about the number of ends of a group. The proofs can be found in [111]:

Theorem 2.1.1. *Let G, H be finitely generated groups. Then the following hold:*

1. *if G and H are commensurable, then $e(G) = e(H)$;*
2. *G is finite if and only if $e(G) = 0$;*
3. *G is infinite and virtually cyclic if and only if $e(G) = 2$;*
4. *[Stallings [121]] G is an amalgamated product or an HNN extension over a finite group if and only if $e(G) \geq 2$.*

For a finite simplicial graph Γ, denote by $V(\Gamma)$ its vertex set and by $E(\Gamma)$ its edge set. We define the *Right Angled Artin group* (RAAG) A_Γ for Γ as the group with the finite presentation

$$A_\Gamma = \langle V(\Gamma) \mid [v,w] \text{ if } vw \in E(\Gamma)\rangle\,.$$

A group is called *virtually special* if it has a finite index subgroup which embeds in a RAAG. There are other equivalent definitions for a group to be virtually special, but for our purposes this definition is sufficient.

A *Coxeter group* C is a group that has a presentation

$$\langle g_1, \cdots, g_r \mid (g_i g_j)^{e_{ij}}, 1 \leq i, j \leq r \rangle$$

with $e_{ii} = 1$ and $e_{ij} \leq 2$ for $i \neq j$ with $e_{ij} \in (\mathbb{N} \cup \{\infty\})$. By convention, if $e_{ij} = \infty$ no relation is imposed.

For a group G we define its *k-th nilpotent quotient* by $G/\gamma_k(G)$, where $\gamma_k(G) = [\gamma_{k-1}(G), G]$ is the k-th term of the lower central series of G and $\gamma_1(G) = G$. We say that G is nilpotent of *nilpotency class* k if $\gamma_{k+1}(G)$ is trivial. We call G *residually nilpotent* if for every $g \in G$ there is $k \geq 1$ such that $\phi_k(g) \neq 1$ where $\phi_k : G \to G/\gamma_k(G)$ is the canonical projection. The maps ϕ_k have the universal property that any map $G \to H$ onto a nilpotent group H of nilpotency class at most $k-1$ factors through ϕ_k.

A group G is called solvable if its *derived series* $\mathcal{D}^{(k)}(G) = [\mathcal{D}^{(k-1)}(G), \mathcal{D}^{(k-1)}(G)]$, $\mathcal{D}^{(0)}(G) = G$, becomes trivial for sufficiently large k. Since $\mathcal{D}^{(k)}(G) \leq \gamma_{k+1}(G)$, every solvable group is nilpotent. The converse is false: an example of a solvable group which is not nilpotent is the fundamental group of the Klein bottle.

A subgroup $H \leq A_1 \times \cdots \times A_r$ of a direct product of groups A_i is called *subdirect* if the projection of H to each factor A_i is surjective. For $1 \leq i_1 < \cdots < i_k \leq r$, we will write $(A_{i_1} \times \cdots \times A_{i_k}) \cap H$ for the intersection of H with the subgroup corresponding to the image of the canonical inclusion $A_{i_1} \times \cdots \times A_{i_k} \hookrightarrow A_1 \times \cdots \times A_r$. The group H is called *full* if its intersection $H \cap A_i$ with every factor is non-trivial. For a group H we call the maximal integer k such that $\mathbb{Z}^k$ embeds in H the *abelian rank* of H. We call a group G *irreducible* if it does not have a finite index subgroup which splits as a direct product of two non-trivial groups.

We call a subdirect product $H \leq A_1 \times \cdots \times A_r$ *conilpotent* of class k if $\gamma_{k+1}(A_i) \leq H$ for $1 \leq i \leq r$. The group H is called coabelian if it is conilpotent of class one. In this case $H = \ker(A_1 \times \cdots \times A_r \to Q)$ is the kernel of the quotient homomorphism onto the abelian group $Q = (A_1 \times \cdots \times A_r)/H$. The same is not true for groups of strictly higher nilpotency class $k > 1$; they are group theoretic fibre products over nilpotent groups of class k. We call H *coabelian of even (odd)* rank if H is coabelian and the torsion-free rank of Q is even (odd).

An important class of groups that we will encounter in several instances in this work is the class of limit groups. A finitely generated group G is called a *limit group* (equivalently, *fully residually free*) if for every finite set $S \subset G$ there is a homomorphism $\phi : G \to F_2$ such that the restriction of ϕ to S is injective. More generally, a group G is *residually free* if for every $g \in G \smallsetminus \{1\}$ there is a homomorphism $\phi : G \to F_2$ such that $\phi(g) \neq 1$.

Limit groups can be seen as "approximately free groups": in many ways their behaviour closely resembles the behaviour of free groups. They come up naturally in Geometry, Group Theory and Logics, providing several different viewpoints and equivalent definitions. There has been an extensive study of limit groups in recent years [112, 86].

It is easy to see that direct products of residually free groups are residually free. In contrast, the product of two or more non-abelian limit groups is not a limit group and they behave in many ways like free groups. The following fundamental result describes the relation between residually free groups and limit groups.

Theorem 2.1.2 ([14], see also [31])**.** *A finitely generated group is residually free if and only if it is a subgroup of a direct product of finitely many limit groups.*

Two important classes of limit groups are finitely generated free groups and *surface groups* (fundamental groups of closed orientable surfaces). More generally the fundamental group of any closed hyperbolic surface (orientable or non-orientable) is a limit group [12] *except* $\Gamma_{-1} = \langle a, b, c \mid a^2b^2c^2 \rangle$, the fundamental group of the non-orientable closed surface with Euler characteristic -1, which is not residually free: in a free group, any triple of elements satisfying the equation $x^2y^2 = z^2$ must commute [102], so $[a, b]$ lies in the kernel of every homomorphism from Γ_{-1} to a free group.

The following Theorem summarises the most important properties of limit groups that we will frequently use.

Theorem 2.1.3. *Let Λ be a non-abelian limit group. Then the following hold:*

1. *Λ is torsion-free;*
2. *Λ is virtually special;*
3. *if $G \leq \Lambda$ is a finitely generated normal subgroup then G is either trivial or of finite index in Λ;*
4. *the centre of Λ is trivial.*

Proof. (2) is [132, Corollary 1.9]. (3) is [28, Theorem 3.1]. (1) and (4) are well-known and easy to prove. (1) is a direct consequence of the fact that any group homomorphism from a finite group to a free group is trivial. For (4) observe that for an element $g \in \Lambda$ and elements h and k in the centraliser of g their images under any homomorphism to a free group must lie in an infinite cyclic subgroup (see Theorem 2.1.4 below). Hence, the image of $[h,k]$ vanishes for any homomorphism to a free group implying that $[h,k]$ is trivial. □

A particularly strong result holds for the subgroup structure of surface groups.

Theorem 2.1.4. *Every subgroup $\Lambda' \leq \pi_1 S_g$ of a non-abelian surface group is either a surface group or a free group. The group Λ' is a surface group if and only if it is a finite index subgroup of $\pi_1 S_g$. In particular, the centraliser of any element in $\pi_1 S_g$ is infinite cyclic.*

Proof. See for instance [81, Theorem 1]. □

The work of Bridson, Howie, Miller and Short provides us with a very good understanding of the structure theory of residually free groups.

Theorem 2.1.5 ([31, Theorem C])**.** *Let G be residually free. Then there are non-abelian limit groups $\Gamma_1, \cdots, \Gamma_r$ such that $G/\mathrm{Z}(G)$ embeds as a full subdirect product of $\Gamma_1 \times \cdots \times \Gamma_r$. This embedding induces an embedding of G in $G_{ab} \times \Gamma_1 \times \cdots \times \Gamma_r$. Furthermore, if H is another residually free group with $\phi : H/\mathrm{Z}(H) \xrightarrow{\cong} G/\mathrm{Z}(G)$ and $\Lambda_1, \cdots, \Lambda_s$ are non-abelian limit groups such that $H/\mathrm{Z}(H)$ embeds as a full subdirect product of $\Lambda_1 \times \cdots \times \Lambda_s$, then $r = s$ and after reordering factors $\Gamma_i \cong \Lambda_i$ for $1 \leq i \leq r$. In particular, ϕ is induced by an isomorphism $\Lambda_1 \times \cdots \times \Lambda_r \to \Gamma_1 \times \cdots \times \Gamma_r$ which maps factors isomorphically to factors.*

Theorem 2.1.5 extends Theorem 2.1.2. We will make repeated use of this result and its consequences.

2.2 Kähler manifolds and fundamental groups

In this section we want to give a brief introduction to the theory of Kähler groups. We begin by summarising some basic notions in Algebraic Topology and Kähler manifolds.

2.2.1 Algebraic Topology

In handling Kähler groups, Algebraic Topology is an essential tool, and we want to recall some basic notions which we will use.

Let X, Y be topological spaces and $A \subset X$, $B \subset Y$ be subsets. Two continuous maps $f, g : (X, A) \to (Y, B)$ with $f(A) \subset B$, $g(A) \subset B$ are called *homotopic*, if there exists a continuous map $H : X \times [0,1] \to Y$ with $H(A \times [0,1]) \subset B$ and $H(\cdot, 0) = f$, $H(\cdot, 1) = g$. Write $f \simeq g$.

Let D^n be the closed n-dimensional unit disk in $\mathbb{R}^n$ and $S^{n-1} = \partial D^n$ its boundary. For a map $f : (D^n, S^{n-1}) \to (X, A)$ denote by $[f] = \{g \mid g \simeq f\}$ the homotopy class of f. The *n-th relative homotopy group* of the pair (X, A) is the set

$$\pi_n(X, A) = \left\{[f] \mid f : (D^n, S^{n-1}) \to (X, A) \text{ is continuous}\right\}.$$

The set $\pi_n(X, A)$ admits a natural group structure which is explained in [80, Chapter 4]. The most important case is $A = \{x_0\}$ for some $x_0 \in A$. In this case, provided that X is connected, we often just write $\pi_n(X)$ instead of $\pi_n(X, A)$, since a different choice of x_0 leads to an isomorphic group. We call $\pi_1(X)$ the *fundamental group* of X.

Other important notions from Algebraic Topology are homology $H_*(X, R)$ and cohomology $H^*(X, R)$ of a topological space with coefficients in a ring or $\pi_1 X$-module R. We don't want to define homology and cohomology here, but just refer to [80, Chapter 2 and 3].

A classifying space $K(G, 1)$ for a group G is a CW-complex X with $\pi_1(X) = G$ and $\pi_i(X) = \{0\}$ for $i \neq 1$. If $R = K$ is a field, then $H_*(X, K)$ and $H^*(X, K)$ are vector spaces and we define the *n-th Betti number* to be the dimension of $H^n(X, K)$. For a group G, we define its *group homology* (respectively cohomology) with coefficients in a $\mathbb{Z}G$-module R as $H_*(K(G, 1), R)$, respectively $H^*(K(G, 1), R)$, where $K(G, 1)$ is a classifying space.

On cohomology we can define the *cup product*

$$\cdot \cup \cdot : H^k(X, R) \times H^l(X, R) \to H^{k+l}(X, R),$$

which is bilinear and satisfies the equality $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$. Let K be a field. We call a subspace $V \subset H^1(X, K)$ *isotropic* if the restriction of the cup product to V vanishes, that is, $0 = \alpha \cup \beta$ for all $\alpha, \beta \in H^1(X, K)$.

2.2.2 Kähler manifolds

Next we want to introduce a few basic notions about Kähler manifolds. Recall that a *smooth real manifold* of dimension n is a topological space M together with a *smooth atlas* $\{(U_i, \phi_i)\}_{i \in I}$, with $U_i \subset M$ open, $M = \cup_{i \in I} U_i$, and $\phi_i : U_i \to \phi(U_i) \subset \mathbb{R}^n$ a homeomorphism, such that all *transition maps* $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ are smooth.

A smooth real n-manifold is called a *Riemannian manifold* with metric g, if $g : T_pM \times T_pM \to \mathbb{R}$ defines an inner product on every tangent space

$$T_pM \cong \{\dot{\gamma}(0) \mid \gamma : (-\epsilon, \epsilon) \to M \text{ smooth}, \epsilon > 0, \gamma(0) = p\} \cong \mathbb{R}^n,$$

varying smoothly with p.

A smooth real manifold of dimension $2n$ is called a *complex manifold* of dimension n, if in addition $\phi(U_i) \subset \mathbb{C}^n$ for all $i \in I$ and all transition maps are holomorphic. A complex manifold admits an *almost complex structure* J, that is, a map $J : TM \to TM$ whose restrictions $J_p : T_pM \to T_pM$, $p \in M$, are linear maps with $J_p^2 = -\mathbb{1}$. Note that the action of J_p on T_pM corresponds to multiplication by the complex number i in local coordinates.

A *Kähler manifold* (M, g, ω) is a complex Riemannian manifold (M, g) so that all tangent vectors $X, Y \in T_pM$ satisfy $g(JX, JY) = g(X, Y)$ and $\omega(X, Y) = g(JX, Y)$ is a closed non-degenerate 2-form on M. Usually we will just write M instead of (M, g, ω). Complex submanifolds of Kähler manifolds are Kähler, since the restrictions of g, J, ω are well-defined and inherit the required properties.

An important example of a Kähler manifold is the *n-dimensional complex projective space* $\mathbb{C}P^n = \mathbb{C}^{n+1} / \{(z_1, \cdots, z_{n+1}) \sim (\lambda z_1, \cdots, \lambda z_{n+1}), \lambda \in \mathbb{C}^*\}$. A complex submanifold of $\mathbb{C}P^n$ is called a *smooth projective variety.* It follows from Kodaira's embedding theorem [76, p.181] that many compact Kähler manifolds are smooth projective varieties. In the converse direction, Voisin proves in [129] and [130] that there exist compact Kähler manifolds which are not homotopy equivalent to any smooth projective variety.

For a complex manifold X, being a Kähler manifold imposes constraints on the de Rham and Dolbeault cohomology groups of X. For instance, all Betti numbers of odd degree with coefficients in $\mathbb{C}$ are even [76, p.117]. The constraints are a consequence of Hodge theory, a theory closely related to harmonic maps and forms on manifolds. This explains why harmonic maps are an important tool for many results about Kähler manifolds (and groups). For more background information on Kähler manifolds, [76] is a good reference.

2.2.3 Kähler groups

A *Kähler group* G is a group which can be realised as the fundamental group of a compact Kähler manifold. In particular, every Kähler group is finitely presented.

It has long been known that every finitely presented group can be realised as the fundamental group of a compact manifold of dimension at most four without boundary. This can be seen via the following classical construction. For a finitely presented group

$$\Gamma = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_k \rangle$$

consider the connected sum $M = \#_{i=1}^n S^3 \times S^1$ of n copies of $S^3 \times S^1$. Its fundamental group is a free group on n generators. Choose k pairwise non-intersecting simple loops $\gamma_1, \cdots, \gamma_k$ in M such that γ_i represents the conjugacy class of r_i in $\pi_1 M$. Using Dehn surgery we can replace a small tubular neighbourhood of γ_i (diffeomorphic to $S^1 \times D^3$) by $D^2 \times S^2$ for $i = 1, \cdots, k$. This yields a closed, smooth, orientable real 4-manifold $\widehat{M}$ with fundamental group Γ.

It is also known that every finitely presented group can be realised as fundamental group of an almost complex manifold of dimension four [91], of a symplectic manifold of dimension four [72], and of a complex manifold of complex dimension three [126]. In fact, for every finitely presented group, one can even find a 3-dimensional complex manifold which is also symplectic [72]. However, the symplectic and complex structure will in general not be compatible.

The whole story changes when for a finitely presented group G we try to find a compact Kähler manifold M with fundamental group G. This is due to strong constraints on compact Kähler manifolds coming from Hodge theory and, more generally, the theory of harmonic maps. Indeed, even one of the most direct consequences of Hodge theory, which is that the first Betti number $b_1(G) = b_1(M) = \dim(H_1(M, \mathbb{R}))$ of a Kähler group is even, provides strong constraints. For instance, it follows immediately that $\mathbb{Z}^{2k+1}$ is not Kähler for any k. On the other hand, there are non-trivial examples of Kähler groups, the simplest being the fundamental groups $\mathbb{Z}^{2k}$ of complex tori and the fundamental groups $\Gamma_g \cong \pi_1 S_g$ of closed orientable hyperbolic surfaces S_g of any genus g. Another class of examples are finite groups (Serre [113]; see Appendix B below for a proof).

Observe that finite covers of compact Kähler manifolds and direct products of compact Kähler manifolds naturally carry the structure of a compact Kähler manifold. Thus finite-index subgroups and direct products of Kähler groups are Kähler.

Combining all of these results already provides us with a variety of examples and non-examples of Kähler groups. In particular, we observe that free groups of any rank are not Kähler, since they have finite index subgroups with odd first Betti number. We shall mention at this point that while finite index subgroups of Kähler groups are Kähler, it is not true that finite extensions of Kähler groups are Kähler. For instance, the fundamental group of the Klein bottle is not Kähler, since it has first Betti number one, but it has the Kähler group $\mathbb{Z}^2$ as index two subgroup.

The question of which finitely presented groups are Kähler was first asked by Serre in the 1950s and has driven a field of very active research ever since. While a lot of very interesting results and constraints have been found, there is still no classification of Kähler groups. In fact, we do not even know what such a classification could look like. One of the key difficulties in understanding Kähler groups is the construction of new examples with interesting properties. Kähler groups with interesting group theoretic properties have been constructed. Among others, there are examples of non-residually finite Kähler groups (see Toledo [128], Catanese-Kollár [47]) and non-coherent Kähler groups (see Kapovich [84], Py [108]), showing that the class of Kähler groups is far from trivial. All of the known constructions employ very specialised techniques. As a result, the known classes of examples remain few and far between.

One aspect which makes the study of Kähler groups particularly interesting is that it lies at the meeting point of three fundamental fields in mathematics: Algebraic Geometry, Differential Geometry and Group Theory. As a consequence one can take different viewpoints on Kähler groups. In this work the focus will be on the connections between Kähler groups and Geometric Group Theory. Other approaches to Kähler groups include formality and the study of the de Rham fundamental group, which in particular provide results about Kähler groups which do not map onto surface groups (see Appendix C), and Non-Abelian Hodge theory (see [116] and also [3, Chapter 7]). For a general overview on Kähler groups see [3]. For a more recent survey with a focus on Geometric Group Theory see [38].

2.3 Kähler groups and their relation to surface groups

Some of the most classical results in the study of Kähler groups are on their relation to surface groups. Indeed, one of the first results revealing a connection between Kähler manifolds, surfaces, and algebra (in the form of differential forms) is the Theorem of

Castelnuovo-de Franchis [43, 50] which appeared in 1905, thereby long predating the study of Kähler groups. Their Theorem has since been generalised by Catanese [44].

Notation. We write S_g to denote the closed orientable surface of genus g.

Convention. Unless an explicit choice of complex structure on S_g has been made, we say that a map $f : X \to S_g$ is holomorphic if we can choose a complex structure on S_g such that f is holomorphic.

Theorem 2.3.1 ([44])**.** *Let M be a compact Kähler manifold and let $U \leq H^1(M, \mathbb{R})$ be a maximal isotropic subspace of dimension ≥ 2. Then there is a surjective holomorphic map $f : M \to S_g$, with $g \geq 2$, and a maximal isotropic subspace $V \leq H^1(S_g, \mathbb{R})$ such that $f^*V = U$.*

In fact, Siu [118] and Beauville [16] showed that there is a direct relation between the existence of a holomorphic map from a Kähler manifold M onto a closed Riemann surface S_g of genus at least two and the existence of a group homomorphism $\pi_1 G \to \pi_1 S_g$.

Theorem 2.3.2 ([118, 16])**.** *Let M be a compact Kähler manifold and let $G = \pi_1 M$. Then the following are equivalent:*

1. *there is an epimorphism $\phi : G \to \pi_1 S_g$, with $g \geq 2$;*
2. *there is $g' \geq g \geq 2$ and a surjective holomorphic map $f : M \to S_{g'}$ with connected fibres such that ϕ factors through $f_* : \pi_1 M \to \pi_1 S_{g'}$.*

A Kähler group is called *fibred* if any of the equivalent conditions in Theorem 2.3.2 holds. In fact there is a stronger version of Siu-Beauville's Theorem which first appears explicitly in Catanese's work [46], but was probably known much earlier (see discussion in [93]). Let S_g be a closed orientable surface of genus $g \geq 1$, let $D = \{p_1, \cdots, p_k\} \in S_g$ be a finite set of points together with multiplicities $\underline{m} = (m_1, \cdots, m_k)$ with $m_i \geq 1$. Choose loops $\gamma_1, \cdots, \gamma_r$ bounding small discs around $p_1, \cdots, p_r$. We define the *orbifold fundamental group* of S_g with respect to these multiplicities as the quotient

$$\pi_1^{orb} S_{g,\underline{m}} := \pi_1(S_g \setminus D) / \langle\langle \gamma_i^{m_i} \mid i = 1, \ldots, k \rangle\rangle .$$

There is a natural epimorphism $\pi_1^{orb} S_{g,\underline{m}} \to \pi_1 S_g$.

Throughout this thesis by a *closed orientable orbisurface* we will always mean a 2-orbifold with fundamental group $\pi_1^{orb} S_{g,\underline{m}}$ of this particular form; in particular, we do not consider any other 2-orbifolds.

Lemma 2.3.3 ([46, Lemma 4.2],[57, Theorem 2]). *Let M be a compact Kähler manifold. Assume that there is a surjective holomorphic map $f : M \to S_g$, $g \geq 1$, with connected fibres. Let $D = \{p_1, \cdots, p_k\} \subset S_g$ be the set of critical values of f, let m_i be the highest common factor of the multiplicities of the components of the divisor $f^{-1}(p_i)$ and let $\underline{m} = (m_1, \ldots, m_k)$.*

Then there is an induced homomorphism $\phi : \pi_1 M \to \pi_1^{orb} S_{g,\underline{m}}$ with finitely generated kernel such that $f_* : \pi_1 M \to \pi_1 S_g$ factors as $q_{\underline{m}} \circ \phi$, where $q_{\underline{m}}$ is the natural epimorphism $\pi_1^{orb} S_{g,\underline{m}} \to \pi_1 S_g$.

Conversely, if $\phi : G = \pi_1 M \to \pi_1^{orb} S_{g,\underline{m}}$ is an epimorphism with finitely generated kernel for some closed Riemann orbisurface $S_{g,\underline{m}}$ of genus $g \geq 2$, then ϕ is induced by a surjective holomorphic map $f : M \to S_{g,\underline{m}}$ with connected fibres.

Note that the orbifold part of the converse direction of Lemma 2.3.3 is contained in [57, Theorem 2]. Catanese provides other, equivalent, conditions for the fibering of Kähler groups in [46].

For a fixed Kähler manifold M and closed Riemann orbisurfaces S, S', we say that *two surjective holomorphic maps $f : M \to S$ and $f' : M \to S'$ are equivalent* if there is a biholomorphic map $h : S \to S'$ such that $f' = h \circ f$. The number of equivalence classes of surjective holomorphic maps with connected fibres onto closed Riemann orbisurfaces of genus at least two is finite. This result was stated explicitly and proved by Delzant [55, Theorem 2] and by Corlette and Simpson [49, Proposition 2.8] in 2008, but, as Delzant remarks, it was known well before then. In fact it was already implicit in Arapura's work [4], where the result was proved for surfaces of fixed genus. Indeed Arapura's work implies this result, since the rank of the abelianisation of a Kähler group gives an upper bound on the maximal genus of a surface group quotient. We want to state the following direct consequence of this result and Lemma 2.3.3.

Theorem 2.3.4. *Let G be a Kähler group. Then there is a finite number r of closed Riemann orbisurfaces Σ_i of genus $g_i \geq 2$ and epimorphisms $\phi_i : G \to \pi_1^{orb} \Sigma_i$ with finitely generated kernels, $1 \leq i \leq r$, such that any epimorphisms $G \to \pi_1 S_h$, with $h \geq 2$, factors through one of the ϕ_i.*

These results provide us with a very good understanding of the nature of maps from Kähler groups onto surface groups, in particular regarding the connection between algebra and geometry. They would however not be quite as significant if they would not be complemented by a variety of powerful criteria which provide us with maps from Kähler groups onto surface groups. This powerful combination lies at the heart of a lot of progress in restricting the class of fibred Kähler groups.

Most of the results of this form show that certain classes of Kähler groups must admit a map onto a surface group or, more generally, onto a subgroup of a direct product of surface groups. There have been two key approaches to providing criteria for the fibering of Kähler groups. The first one is to consider harmonic maps from a Kähler manifold (or its universal cover) to a suitable geometric space, for instance a tree, and then use these to generate a codimension one foliation of the Kähler manifold. Then one shows that this foliation comes from a holomorphic map onto a closed Riemann surface. The second approach originates in the work of Green and Lazarsfeld [74, 75] on character varieties. We start by discussing some of the consequences of the harmonic maps approach.

Two of the first explicit results of this form are Gromov's result from 1989 on Kähler groups with non-trivial first l^2-Betti number and Carlson and Toledo's results on maps from Kähler manifolds to hyperbolic manifolds (and more generally to locally symmetric spaces).

Theorem 2.3.5 (Gromov [77]). *A Kähler group has non-trivial first l^2-Betti number if and only if it is commensurable to a surface group $\pi_1 S_g$ with $g \geq 2$.*

Theorem 2.3.6 (Carlson, Toledo [42]). *Let $G = \pi_1 M$ be the fundamental group of a compact Kähler manifold M, let $\Gamma = \pi_1 N$ be the fundamental group of a compact hyperbolic manifold $N = \mathbb{H}^n_{\mathbb{R}}/\Gamma$ for $\Gamma \leq \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ a discrete cocompact lattice, $n \geq 2$, and let $\phi : G \to \Gamma$ be a homomorphism. Then ϕ factors through a homomorphism $\psi : G \to \pi_1 S_g$ ($g \geq 2$) such that either the image of ψ is infinite cyclic or ψ is induced by a surjective holomorphic map $f : M \to S_g$.*

Theorem 2.3.6 is based on the work on harmonic maps on Kähler manifolds by Siu [117], Sampson [110], and Eells-Sampson [66].

Based on Gromov's methods developed in [77], Arapura, Bressler and Ramachandran proved that Kähler groups must have zero or one end. By Stallings' Theorem 2.1.1(4) about ends of groups this can be stated in the following algebraic form:

Theorem 2.3.7 ([6]). *If a group G admits a decomposition as an amalgamated free product $G = A *_\Delta B$ or HNN-extension $G = A*_\Delta$ of groups A, B over a finite group Δ with $[A : \Delta] \geq 2$ and $[B : \Delta] \geq 2$ then G is not Kähler.*

Gromov and Schoen [78] showed that in fact the relation between Kähler groups, amalgamated free products and surface groups is even stronger

Theorem 2.3.8 ([78]). *Let X be a compact Kähler manifold and $G = \pi_1 X$. Assume that G splits as an amalgamated free product $G = G_1 *_\Delta G_2$ with $[G_1 : \Delta] \geq 2$ and $[G_2 : \Delta] \geq 3$.*

Then there is a representation $\rho : G \to \mathrm{PSL}(2, \mathbb{R})$ with discrete, cocompact image and a finite covering $X' \to X$ together with a holomorphic map $f : X' \to S_g$, $g \geq 2$, such that $\rho|_{\pi_1 X'} = f_*$.

An alternative, more recent, proof of Theorem 2.3.8 can be found in [105].

A different approach leading to fibred Kähler groups is based on the study of character varieties of Kähler groups, that is, the variety of homomorphisms $\mathrm{Hom}(G, \mathbb{C}^*)$ from a Kähler group G to the complex numbers $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. This approach goes back to the work of Green and Lazarsfeld [74, 75] and Beauville [15, 17]. Based on Beauville's work, Arapura [5, 4] gives a very explicit criterion in terms of the derived subgroup of a Kähler group.

Theorem 2.3.9 ([5, 4]). *If a Kähler group G is fibred then $H_1([G, G], \mathbb{R})$ is not finitely generated. Conversely, if $H_1([G, G], \mathbb{R})$ is not finitely generated then there is a finite index subgroup $G_0 \leq G$ and a surjection $G_0 \to \pi_1 S_g$, with $g \geq 2$.*

A partial generalisation of this result was subsequently proved by Napier and Ramachandran [104].

Theorem 2.3.10 ([104]). *Let $G = \pi_1 M$ be the fundamental group of a compact Kähler manifold M and let $\phi : G \to \mathbb{Z}$ be an epimorphism with $\ker\phi$ not finitely generated. Then there is a surjective holomorphic map $f : M \to S_g$, with $g \geq 2$, such that ϕ factors through $f_* : \pi_1 M \to \pi_1 S_g$.*

The connection between maps to surface groups and the derived subgroup, which was first observed by Beauville [15] and lead to Theorem 2.3.9, has proved to be of a deep nature and ultimately culminated in Delzant's alternative [56].

Theorem 2.3.11 ([56]). *Let G be a Kähler group. Then one of the following holds:*

1. *there is a finite index subgroup $G_0 \leq G$ which maps onto $\pi_1 S_g$, for some $g \geq 2$;*

2. *every solvable quotient of G is virtually nilpotent.*

A consequence of Theorem 2.3.11 is that every virtually solvable Kähler group is virtually nilpotent. Note that Theorem 2.3.11 was preceded by various partial results on polycyclic Kähler groups (see Arapura and Nori [7]) and solvable Kähler groups (see Campana [40], [41] and Brudnyi [37]). For a good overview of these results see [38]. We note that there are examples of nilpotent Kähler groups which are not abelian, for instance the $(2n+1)$-dimensional Heisenberg group which is Kähler for n at least two (see Sommese and Van de Ven [119], Campana [39]). It is not known if there is a Kähler group of nilpotency class larger than two.

These results provide us with an array of criteria which allow us to prove that a Kähler group is fibred. Considering that there are only finitely many equivalence classes of homomorphisms from Kähler groups onto surface groups, a natural next question to ask is whether there are cases in which we can in fact determine all such maps and (more strongly) if there are criteria which allow us to determine when a Kähler group is in fact a subgroup of a direct product of surface groups.

The study of finding maps from Kähler groups to direct products of surface groups was initiated in the fundamental work of Delzant and Gromov [58] on cuts in Kähler groups. In some sense their work can be seen as an attempt to generalise the work on the number of ends of a Kähler group to relative ends. Before stating their results we want to introduce some of the relevant notions.

For a proper geodesic metric space X and a subspace $X_0 \subset X$, we define *the space of relative ends*, denoted by $\mathrm{Ends}(X|X_0)$, to be the inverse limit of the subspaces $X_{-r} = \{x \in X \mid \mathrm{dist}(x, X_0) \geq r\}$, as $r \to \infty$. Note that this coincides with the space of ends of X if X_0 is compact, but can be different if X_0 is not compact – consider for instance $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{R}$.

If there is an action by a group H on X and X_0 is the orbit of a point in X, we define *the space of ends* of X with respect to H as $\mathrm{Ends}(X|H) = \mathrm{Ends}(X|X_0)$. If G is a finitely generated group, $X = \mathrm{Cay}(G, S)$ is its Cayley graph with respect to a finite generating set S, and $H \leq G$ is a subgroup, then we call $\mathrm{Ends}(G|H) = \mathrm{Ends}(Cay(G)|H)$ the *space of (relative) ends* of G with respect to H. We say that H *cuts* G (at infinity) if $|\mathrm{Ends}(G|H)| \geq 2$ and we say that H is a *branched cut* of G (at infinity) if $|\mathrm{Ends}(G|H)| \geq 3$.

Next we define stability at infinity. For this, let M be a complete connected Riemannian manifold, $x \in M$ any point and $B(x, R)$ a ball of radius R around x. Let E be an end with respect to $B(x, R)$, that is, a non-compact connected component of $M \setminus B(x, R)$. We define the *capacity of* E as $\mathrm{cap}(E) = \inf_{\phi \in \Phi} \int_M |\nabla \phi|^2$, where Φ is the set of smooth maps $\phi : X \to \mathbb{R}$ with $0 \le \phi \le 1$, $\phi|_{X \setminus E} = 0$, and $\phi|_E$ is equal to 1 on the complement of a compact subset of E.

Similarly, we define the capacity $c(x, R)$ of the complement $M \setminus B(x, R)$ of a ball of radius R around $x \in M$. We call M *stable at infinity* if $c(x, R) \to \infty$ as $R \to \infty$ and the convergence is uniform in x. An important implication of stability at infinity is the existence of certain harmonic maps which behave in a "good" way on the ends of M.

The definition of a *stable cut of a finitely presented group* G by a subgroup H is a bit different. One way to define it is to require that for a smooth manifold M with $G = \pi_1 M$, the quotient manifold $X = \widetilde{M}/H$ of the universal covering $\widetilde{M}$ of M satisfies a *strong isoperimetric inequality*, that is, an inequality of the form $\mathrm{vol}(\partial A) > k \cdot \mathrm{vol}(A)$ for some constant $k > 0$ and all compact subsets A of X with smooth boundary ∂A. Stability at infinity of X follows from this, but is not equivalent to it. For more details on stable branched cuts and their properties, see [58].

Let G be a group. We say that a subgroup $H \le G$ *induces a stable branched cut* of G if the group H is a stable branched cut of G. Then, following Delzant and Gromov, its *sbc-kernel* K is defined as the intersection of all its subgroups that induce stable branched cuts of G. We say, that K has *finite type* if there are finitely many subgroups of this sort so that K is the intersection of all of their conjugates. We call G of *sbc-type*, if $K = \{1\}$. This allows us to state an algebraic version of the main result of [58], the so-called sbc-Theorem.

Theorem 2.3.12 ([58, sbc-Theorem]). *Let G be a Kähler group with sbc-kernel K of finite type. Then there is a finite index subgroup $G_0 \le G$ and a subgroup F_0 of a direct product $F = \pi_1 S_{g_1} \times \cdots \times \pi S_{g_l}$, $g_i \ge 2$, such that F_0 is isomorphic to the quotient group $G_0/(K \cap G_0)$, where the isomorphism is induced by a short exact sequence*

$$1 \to K \cap G_0 \to G_0 \to F_0 \to 1.$$

The proof of Theorem 2.3.12 uses results and techniques related to the theory of harmonic maps to trees developed by Gromov and Schoen [78]: the existence of stable branched cuts implies that there are harmonic maps from the corresponding Kähler manifold to a tree and as a consequence there are finite index subgroups of G that map onto surface groups; the finite type hypothesis on the sbc-Kernel is then used to obtain the short exact sequence in Theorem 2.3.12.

Theorem 2.3.12 is particularly interesting in those cases where K is finite or trivial, as then G_0 and F_0 are commensurable up to finite kernels. In this context, we obtain the following Corollary [58]:

Corollary 2.3.13 ([58, sbc-Corollary])**.** *Let G be a torsion-free Kähler group which does not contain any non-trivial abelian normal subgroups. Then G has a subgroup $G_0 \leq G$ of finite index which is isomorphic to a subgroup of a direct product of surface groups if and only G is of finite sbc-type.*

Delzant and Gromov announced a generalisation of the sbc-Theorem [58] which does neither require the finiteness conditions on K, nor the stability condition on the cuts defining K, but details did not yet appear. However, significant progress in this direction has been made since. This will be discussed below.

One important class of groups to which Theorem 2.3.12 applies are hyperbolic groups which contain a quasi-convex subgroup. We call a cut coming from a quasi-convex subgroup a *convex cut*. These groups satisfy a stronger condition.

Theorem 2.3.14. *If a hyperbolic Kähler group admits a convex cut, then it is commensurable to a surface group.*

Another class of groups to which the methods of Delzant and Gromov are relevant is the class of Kähler groups acting on CAT(0) cube complexes. This idea was mentioned in [58], but was not explored there in much depth. It is only in the more recent work by Py [107] and Delzant and Py [59] that this question is studied for suitable actions.

More precisely, Py [107] studies Kähler groups that virtually map to Coxeter groups or virtually map to RAAGs.

Theorem 2.3.15 ([107, Theorem A,])**.** *Let M be a compact Kähler manifold and let $\Gamma = \pi_1 M$. Let W be a Coxeter group or a RAAG and assume that there is a homomorphism $\phi : \Gamma \to W$.*

Then there exists a finite covering $q : M_0 \to M$ and finitely many surjective holomorphic maps with connected fibres $p_i : M_0 \to \Sigma_i$ onto closed Riemann orbisurfaces Σ_i of genus $g_i \geq 2$, $1 \leq i \leq N$, such that the restriction of ϕ to $\Gamma_0 = \pi_1 M_0$ factorises through the map

$$\psi : \Gamma_0 \to (\Gamma_0)_{\mathrm{ab}} \times \pi_1^{\mathrm{orb}} \Sigma_1 \times \cdots \times \pi_1^{\mathrm{orb}} \Sigma_N.$$

The map ψ is induced by the p_i and the natural projection $\Gamma_0 \to (\Gamma_0)_{\mathrm{ab}}$ of Γ_0 onto its abelianisation.

In other words, there is a homomorphism $f : (\Gamma_0)_{\mathrm{ab}} \times \pi_1^{\mathrm{orb}}\Sigma_1 \times \cdots \times \pi_1^{\mathrm{orb}}\Sigma_N \to W$ *such that* $\phi \circ q_* = f \circ \psi$.

Corollary 2.3.16. *If a Kähler group* G *is a subgroup of a Coxeter group or a RAAG then there is a subgroup* $G_0 \leq G$ *of finite index which is isomorphic to a subgroup of the direct product of a free abelian group and finitely many surface groups.*

To prove Theorem 2.3.15, Py exploits the existence of walls in the Davis complex of a Coxeter group: he uses these to construct group actions of Kähler groups on trees and then deduces his results from the theory of Gromov and Schoen [78].

Delzant and Py consider the more general situation of Kähler groups acting on CAT(0) cube complexes [59]. Recall that we call an action of a group G by isometries on a metric space X *properly discontinuous* if X is locally compact and for each compact subset $K \subset X$ the set $\{g \in G \mid K \cap gK \neq \emptyset\}$ is finite. We call the action of G *cocompact* if the quotient X/G is compact. The action of G on X is called *geometric* if it is properly discontinuous and cocompact with finite point-stabilisers.

Delzant and Py proved very recently [59] that if a Kähler group acts nicely on a CAT(0) cube complex then this action virtually factors through a surface group. In the particular case of a geometric action they obtain a very strong constraint.

Theorem 2.3.17 ([59])**.** *Let* G *be a Kähler group. Assume that there is a CAT(0) cube complex* X *such that* G *acts on* X *by isometries. If the action of* G *on* X *is geometric then there is a finite index subgroup* $G_0 \leq G$ *which is isomorphic to a direct product* $\mathbb{Z}^k \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$ *with* $k, r \in \mathbb{N}$ *and* $g_i \geq 2$, $1 \leq i \leq r$.

Given the existence of maps from Kähler groups to direct products of surface groups for many classes of groups that have been at the centre of recent progress in Geometric Group Theory, such as RAAGs, a very natural question is the question of

Question 1. *Which subgroups of direct products of surface groups are Kähler?*

This question was first raised by Delzant and Gromov in [58]. However, to this point we do not know much about Kähler subgroups of direct products of surface groups and one of the central goals of this thesis is to extend our understanding of these groups.

2.4 Restrictions in low dimensions

We want to give some recent applications of the theory of fibred Kähler groups, in particular to Kähler groups G of low *cohomological dimension*

$$cd(G) := \sup\left\{i \mid \exists \text{ a } G\text{-module } M \text{ s.t. } H^i(G,M) \neq 0\right\}.$$

These results are based on advances in Geometric Group Theory in the era following Perelman's proof of Thurston's geometrisation conjecture, in particular Agol's proof of the virtual Haken conjecture [1] and the work of Wise [132], Haglund and Wise [79], and Przytycki and Wise [106] on special cube complexes. Their work provides us with a much better understanding of the structure of low-dimensional groups. For instance many of them are *large*, i.e. have finite index subgroups that map onto non-abelian free groups, which implies that they have finite index subgroups which map onto surface groups. See [21] for a recent survey of low-dimensional Kähler groups which covers similar results.

It is an open problem whether all Kähler groups of cohomological dimension at most three are commensurable to surface groups [24]. In particular, it is not known if there is a Kähler group of cohomological dimension three. Note that there are examples of Kähler groups of any cohomological dimension except 1 and 3. For even cohomological dimension, examples are provided by complex tori and products of Riemann surfaces, while for odd dimension we have the $(2n+1)$-dimensional Heisenberg group. Historically, the first examples of Kähler groups with arbitrary odd cohomological dimension ≥ 5 were cocompact lattices in $SU(n,1)$, which were constructed by Toledo [127]. There are no Kähler groups of cohomological dimension one, as groups of cohomological dimension one are free (see Stallings [121]).

A *3-manifold* group is a finitely presented group that arises as the fundamental group of a connected 3-manifold. We first want to address the question of which 3-manifold groups are Kähler groups. This question was posed by Donaldson and Goldman, in 1989, and the first non-trivial results were obtained by Reznikov [109] who came to the question independently, in 1993.

Dimca and Suciu [65] answered the question fully for fundamental groups of closed connected 3-manifolds in 2009 (quoting Perelman's solution to Thurston's Geometrisation conjecture). Their work initiated a sequence of recent results in this area.

Theorem 2.4.1 (Dimca, Suciu [65]). *Let G be a finitely presented group that is both, the fundamental group of a closed connected 3-manifold and a Kähler group. Then G is finite.*

The proof of Dimca and Suciu is obtained by comparing Kähler groups and 3-manifold groups with respect to their resonance varieties and with respect to Kazhdan's property (T). Alternative proofs of the same theorem have since been published by Biswas, Mj, and Seshadri [25], using results by Delzant and Gromov [58] on cuts in Kähler groups, and by Kotschick [94], using Poincaré Duality. The work of Biswas, Mj, and Seshadri gives a further result:

Theorem 2.4.2 ([25]). *Let Q be the fundamental group of a closed 3-manifold, let G be a Kähler group, and let N be a finitely generated group. Assume that G fits into a short exact sequence*

$$1 \to N \to G \to Q \to 1.$$

If Q is infinite then one of the following holds:

1. *Q is virtually a product $\mathbb{Z} \times \pi_1 S_g$;*
2. *Q is a finite index subgroup of the 3-dimensional Heisenberg group; or*
3. *Q is virtually cyclic.*

A *projective group* is a group that is the fundamental group of a compact projective manifold. A *quasi-Kähler group* G is a group that is the fundamental group of a manifold $X = \overline{X} \setminus D$, where $\overline{X}$ is a compact, connected Kähler manifold and D is a Divisor with normal crossings; G is called *quasi-projective* if $\overline{X}$ is projective. A 3-manifold M is called *prime* if whenever $M = M_1 \# M_2$ is the connected sum of two 3-manifolds M_1 and M_2 then $M = M_1$ or $M = M_2$. A *graph manifold* is a prime 3-manifold which can be cut along a finite number of embedded tori into disjoint pieces $M_1, \ldots, M_k$ such that every piece is a circle bundle $M_i \to B_i$ over a compact surface B_i (possibly with boundary).

After answering the question for Kähler groups, Dimca, Papadima, and Suciu raised, and partially answered, the question of which quasi-Kähler groups are 3-manifold groups [64]. The work of Dimca, Papadima and Suciu was generalised by Friedl and Suciu [69].

Theorem 2.4.3 ([69]). *Let G be a finitely presented group that is both, the fundamental group of a connected 3-manifold N with empty or toroidal boundary, and a quasi-Kähler group. Then all prime components of N are graph manifolds.*

Kotschick [95] gives a complete answer for arbitrary 3-manifolds (allowing boundary).

Theorem 2.4.4 ([95]). *Let G be an infinite group which is a Kähler group and the fundamental group of a 3-manifold. Then G is the fundamental group of a closed orientable surface.*

Biswas and Mj [23] completed the classification of quasi-projective 3-manifold groups

Theorem 2.4.5. *Let G be a quasi-projective 3-manifold group. Then one of the following holds:*

1. *G is the fundamental group of a closed Seifert-fibred manifold;*
2. *G is virtually free;*
3. *G is virtually $\mathbb{Z} \times F_r$ for some $r \geq 1$;*
4. *G is virtually a surface group.*

Another class of groups that can be studied using similar machinery is the class of 1-relator Kähler groups. A *one-relator group* is a finitely presented group G that admits a presentation of the form $\langle x_1, \cdots, x_l \mid r \rangle$ with one relator r. The question of which one-relator groups are Kähler was answered by Biswas and Mj in [22]:

Theorem 2.4.6 ([22, Theorem 1.1]). *A group G is an infinite one-relator Kähler group if and only if G is isomorphic to a group of the form*

$$\left\langle a_1, b_1, \cdots, a_g, b_g \mid \left(\prod_{i=1}^{g} [a_i, b_i] \right)^n \right\rangle.$$

Since the finite one-relator groups are precisely the finite cyclic groups and all finite groups are Kähler, this theorem provides a complete classification of one-relator Kähler groups. The proof of Biswas and Mj uses Delzant and Gromov's work on cuts in Kähler groups [58] and results about groups of cohomological dimension two. An alternative proof based on l^2-Betti numbers has since been given by Kotschick [93].

In the same work, Kotschick establishes two other constraints on Kähler groups using l^2-Betti numbers. For a finitely presented group G we define its *deficiency* as

$$\operatorname{def}(G) = \sup \{ n - k \mid G \cong \langle \mathcal{X} \mid \mathcal{R} \rangle, |\mathcal{X}| = n, |\mathcal{R}| = k \}.$$

Theorem 2.4.7 ([93]). *A Kähler group G has deficiency ≥ 2 if and only if it is the fundamental group of a closed Riemann orbisurface.*

Theorem 2.4.7 generalises Theorem 2.3.5 and work of Green and Lazarsfeld [75].

Theorem 2.4.8 ([93])**.** *A Kähler group G is a non-abelian limit group if and only if it is the fundamental group of a closed hyperbolic Riemann surface.*

We want to end this section with a very recent result by Friedl and Vidussi on rank gradients of Kähler groups [70]. Let G be a finitely generated group. Denote by $d(G)$ its minimal number of generators. Let $\{H_i\}_{i\in\mathbb{N}}$ be a descending sequence of finite index normal subgroups $H_{i+1} \trianglelefteq H_i \trianglelefteq G$. The *rank gradient* of the pair $(G, \{H_i\}_{i\in\mathbb{N}})$ is defined as

$$\mathrm{rg}(G, \{H_i\}) := \lim_{i\to\infty} \frac{d(H_i) - 1}{[G : H_i]}.$$

For a Kähler group G and a primitive (i.e. surjective) class $\phi \in H^1(G, \mathbb{Z}) = \mathrm{Hom}(G, \mathbb{Z})$, call the pair $(G, \{H_i\})$ a *Kähler pair* where $\{H_i\}$ is the sequence defined by $H_i = \ker\left(G \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}/i\mathbb{Z}\right) \trianglelefteq G$. The Kähler pair defined by ϕ is denoted by (G, ϕ).

Theorem 2.4.9 ([70])**.** *The rank gradient of a Kähler pair (G, ϕ) is zero if and only if the kernel of the homomorphism $\phi : G \to \mathbb{Z}$ is finitely generated.*

The proof of this result uses Napier and Ramachandran's Theorem 2.3.10.

2.5 Exotic finiteness properties

Question 1 is closely related to finiteness properties of groups. We say that a group G is of finiteness type $\mathcal{F}_r$ if it has a classifying $K(G, 1)$ with finite r-skeleton. In particular, G is of type $\mathcal{F}_1$ if and only if it is finitely generated and of type $\mathcal{F}_2$ if and only if it is finitely presented. It is clear that type $\mathcal{F}_{n+1}$ implies type $\mathcal{F}_n$. We say that G is of type $\mathcal{F}_\infty$ if G is of type $\mathcal{F}_n$ for every n and G is of type $\mathcal{F}$ if it has a finite classifying space.

Finiteness properties are well-behaved under passing to finite index subgroups and under finite extensions.

Lemma 2.5.1. *Let G and H be commensurable up to finite kernels. Then G is of type $\mathcal{F}_m$ if and only if H is of type $\mathcal{F}_m$.*

Proof. See for instance [71, Proposition 7.2.3]. □

While the same is not true for arbitrary subgroups and extensions, as we will see from the examples below, some of the implications remain true.

Lemma 2.5.2. *Let N, G and Q be groups that fit into a short exact sequence*

$$1 \to N \to G \to Q \to 1.$$

If N is of type $\mathcal{F}_{n-1}$ and G is of type $\mathcal{F}_n$ then Q is of type $\mathcal{F}_n$. Conversely, if N is of type $\mathcal{F}_n$ and Q is of type $\mathcal{F}_n$ then G is of type $\mathcal{F}_n$.

In particular, if N is of type $\mathcal{F}$ then G is of type $\mathcal{F}_n$ if and only if Q is of type $\mathcal{F}_n$.

Proof. See [71, Section 7.2] and also [20, Proposition 2.7]. □

2.5.1 Finiteness properties of residually free groups

The first example of a finitely presented group which is not of type $\mathcal{F}_3$ was constructed in 1963 by Stallings [120]. It arises as the kernel of a homomorphism $F_2 \times F_2 \times F_2 \to \mathbb{Z}$ which is surjective on factors. Stallings example was generalised by Bieri [20].

Theorem 2.5.3 ([120, 20])**.** *For $r \geq 1$ let $\phi_r : F_2 \times \cdots \times F_2 \to \mathbb{Z}$ be an epimorphism from a product of r free groups onto $\mathbb{Z}$ whose restriction to each factor is surjective. Then $\ker\phi_r$ is of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$.*

These examples are known as the Stallings–Bieri groups. They were subsequently generalised by Bestvina and Brady who produced large classes of groups with prescribed finiteness properties using combinatorial Morse theory [19]. Their groups arise as kernels of surjective maps from RAAGs to the integers and their finiteness properties are determined in terms of properties of the defining graph.

A class of groups whose finiteness properties are particularly well understood are subgroups of direct products of limit groups. It follows from the work of Bridson, Howie, Miller, and Short [29, 30, 31] that a subgroup of a direct product of limit groups is of type $\mathcal{F}_\infty$ if and only if it is virtually a direct product of limit groups. More precisely, they show:

Theorem 2.5.4 ([30, Theorem A])**.** *Let $n \geq 1$ and let $G \leq \Lambda_1 \times \cdots \times \Lambda_n$ be a subgroup of a direct product of limit groups Λ_i, $1 \leq i \leq n$. The group G is of type $\mathcal{F}_m$ for $m \geq n$ if and only if G has a finite index subgroup $H \leq G$ which is a direct product $H = \Lambda_1' \times \cdots \times \Lambda_n'$ of limit groups $\Lambda_i' \leq \Lambda_i$, $1 \leq i \leq n$.*

As a consequence of Theorem 2.5.4, subdirect products of finitely many surface groups which are of type $\mathcal{F}_\infty$ are Kähler, since finite index subgroups of Kähler groups are Kähler. This reduces Question 1 to Kähler subgroups of direct products of n surface groups which are not of type $\mathcal{F}_n$.

We will also need a result by Bridson and Miller about kernels of projections to surface or free group factors.

Theorem 2.5.5 ([34, Theorem 4.6]). *Let Λ be a non-abelian surface group, let A be any group, and let $G \leq \Lambda \times A$. Assume that G is finitely presented and that the intersection $\Lambda \cap G$ is non-trivial. Then $G \cap A$ is finitely generated.*

2.5.2 Finiteness properties of Kähler groups

In his book on Shafarevich maps and automorphic forms [89], Kollar raised the question if every *projective group* (fundamental group of a smooth compact projective manifold) is commensurable to the fundamental group of an aspherical quasi-projective variety. One way to give a negative answer to this question is to show that for some $r \geq 3$ there exists a projective group of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$, because aspherical quasi-projective manifolds are homotopy equivalent to finite CW-complexes and thus their fundamental groups are of type $\mathcal{F}$ [60].

Dimca, Papadima and Suciu showed that the Bestvina–Brady groups do not provide a negative answer to Kollar's question, since the only Bestvina–Brady groups which are Kähler are the free abelian groups of even rank which are of type $\mathcal{F}_\infty$ (see [61, Corollary 1.3]). However, Dimca, Papadima and Suciu [62] observed that one can imitate the Bestvina–Brady construction to obtain Kähler groups (indeed projective groups) with exotic finiteness properties, thus giving a negative answer to Kollar's question. Throughout this work we will refer to their groups as the *DPS groups*.

Let $r \geq 3$ and let $\Gamma_{g_i} \cong \pi_1 S_{g_i}$, $g_i \geq 2$, $1 \leq i \leq r$, be surface groups with presentation

$$\Gamma_{g_i} = \left\langle a_1^i, \cdots, a_{g_i}^i, b_1^i, \cdots, b_{g_i}^i \mid \left[a_1^i, b_1^i\right] \cdots \left[a_{g_i}^i, b_{g_i}^i\right] \right\rangle.$$

Consider the epimorphism $\phi_{g_1,\cdots,g_r} : \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \mathbb{Z}^2$ defined on factors by

$$\begin{array}{cccc} \phi_{g_i}: & \Gamma_{g_i} & \to & \mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle \\ & a_1^i, a_2^i & \mapsto & a \\ & b_1^i, b_2^i & \mapsto & b \\ & a_3^i, \cdots, a_{g_i}^i & \mapsto & 0 \\ & b_3^i, \cdots, b_{g_i}^i & \mapsto & 0. \end{array}$$

Although they did not describe them this way, the DPS groups are the groups $\ker\phi_{g_1,\cdots,g_r}$. We want to explain Dimca, Papadima and Suciu's geometric construction that underlies the map $\phi_{g_1,\cdots,g_r}$.

Let E be an elliptic curve, that is, a complex torus of dimension one, let $g \geq 2$, and let $B = \{b_1, \cdots, b_{2g-2}\} \subset E$ be a finite subset of even size. Choose a set of generators $\alpha, \beta, \gamma_1, \cdots, \gamma_{2g-2}$ of the first homology group $H_1(E \smallsetminus B, \mathbb{Z})$ such that γ_i is the boundary of a small disc centred at b_i, $i = 1, \cdots, 2g-2$, and α, β is a basis of $\pi_1 E$.

Then the map $H_1(E \smallsetminus B) \to \mathbb{Z}/2\mathbb{Z}$ defined by $\gamma_i \mapsto 1$ and $\alpha, \beta \mapsto 0$ induces a 2-fold normal covering of $\pi_1(E \smallsetminus B)$ which extends continuously to a 2-fold branched covering $f_g : S_g \to E$ from a topological surface S_g of genus $g \geq 2$ onto E. It is well-known that there is a unique complex structure on S_g such that the map f_g is holomorphic.

For $g_1, \cdots, g_r \geq 2$ as above, let f_{g_i} be the associated holomorphic branched covering maps. It is not hard to see that with a suitable choice of standard symplectic generating set of $\pi_1 S_{g_i}$ we can identify the induced maps $f_{g_i,*} : \Gamma_{g_i} \cong \pi_1 S_{g_i} \to \pi_1 E \cong \mathbb{Z}^2$ with the epimorphisms ϕ_{g_i} described above (see Section 5.1).

We use addition in the elliptic curve to define the holomorphic map $f = f_{g_1,\cdots,g_r} := \sum_{i=1}^r f_{g_i} : S_{g_1} \times \cdots \times S_{g_r} \to E$. The induced map f_* on fundamental groups is

$$\phi_{g_1,\cdots,g_r} : \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} \to \mathbb{Z}^2 = \pi_1 E.$$

Away from a finite subset $C \subset S_{g_1} \times \cdots \times S_{g_r}$ with $f(C) = B_1 \times \cdots \times B_r$, the map f is a proper submersion. Dimca, Papadima and Suciu show that f has connected fibres [62]. Hence, by the Ehresmann Fibration Theorem (see Appendix A) all of its *generic smooth fibres* $f^{-1}(p)$ over the open subset $p \in E \smallsetminus f(C)$ of regular values are homeomorphic. We denote by $H_{g_1,\cdots,g_r}$ the generic smooth fibre of $f_{g_1,\cdots,g_r}$. Note that $H_{g_1,\cdots,g_r}$ can be endowed with a Kähler structure, since it can be realised as complex submanifold of $S_{g_1} \times \cdots \times S_{g_r}$. Dimca, Papadima and Suciu proved that $H_{g_1,\cdots,g_r}$ has the following properties:

Theorem 2.5.6 ([62, Theorem A]). *For each $r \geq 3$ and $g_1, \cdots, g_r \geq 2$, the compact smooth generic fibre $H = H_{g_1,\cdots,g_r}$ of the surjective holomorphic map*

$$f = f_{g_1,\cdots,g_r} : S_{g_1} \times \cdots \times S_{g_r} \to E$$

is a connected smooth projective variety with the following properties:

1. *the homotopy groups $\pi_i H$ are trivial for $2 \leq i \leq r-2$ and $\pi_{r-1} H$ is non-trivial;*

2. *the universal cover $\widetilde{H}$ of H is a Stein manifold;*

3. *the fundamental group $\pi_1 H$ is a projective (and thus Kähler) group of finiteness type $\mathcal{F}_{r-1}$ but not of finiteness type $\mathcal{F}_r$;*

4. *the map f induces a short exact sequence*

$$1 \to \pi_1 H \to \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r} \xrightarrow{f_*} \pi_1 E = \mathbb{Z}^2 \to 1.$$

The construction of Dimca, Papadima and Suciu is the first construction of explicit examples of subgroups of direct products of surface groups which are Kähler and not type $\mathcal{F}_\infty$. Biswas, Pancholi and Mj use Lefschetz fibrations to give a more general construction of such examples as fundamental groups of fibres of holomorphic maps from a direct product of surface groups onto an elliptic curve. However, they do not show that the class of examples that can be obtained from their construction does not simply consist of disguised versions of the DPS groups. Before this work, these were the only known constructions of Kähler groups which are not of type $\mathcal{F}_\infty$.

Following the proof of existence of Kähler groups with exotic finiteness properties it seems natural to ask for the structure of this class of Kähler groups which leads us to the question of

Question 2. *Which Kähler groups are of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$?*

A challenging instance of this question would be to ask for examples which do not contain any subgroup of finite index which is isomorphic to a subdirect product of surface groups.

Another natural challenge in understanding Kähler groups with exotic finiteness properties is:

Question 3. *Is it possible to find explicit finite presentations for any of the groups constructed in response to Question 2?*

This question was posed by Suciu in the context of his examples with Dimca and Papadima. The recent results by Py [107] and Delzant and Py [59] have intensified the interest in understanding the Kähler subgroups of direct products of surface groups and we anticipate that finding explicit descriptions of such groups will be useful in this context.

In this thesis I will present substantial progress on Questions 1, 2 and 3. In particular, I will give new examples of Kähler subgroups of direct products of surface groups, provide new constraints on Kähler subgroups of direct products of surface groups, produce new examples of Kähler groups with exotic finiteness properties that are not commensurable to any subgroup of a direct product of surface groups, and construct explicit finite presentations for the DPS groups.

Chapter 3

Residually free groups and Schreier groups

In this chapter we are concerned with finding new criteria that imply a Kähler group is a subgroup of a direct product of surface groups. These results emphasise the importance of finding an answer to Delzant and Gromov's question of which subgroups of direct products of surface groups are Kähler.

In the first part of this chapter we will prove that a Kähler group is residually free if and only if it has a finite index subgroup which is a full subdirect product of finitely many surface groups and a free abelian group (see Theorem 3.1.1). This result is obtained by combining work of Py [107] and Wise [132] with the structure theory for residually free groups by Bridson, Howie, Miller and Short [31, 30, 29].

We then proceed to consider the relation between Kähler groups and *Schreier groups* (see Section 3.2). Following the terminology of de la Harpe and Kotschick [52], a Schreier group is a group G all of whose normal subgroups are either finite or of finite index in G. Interesting classes of Schreier groups are limit groups [28] and groups with non-trivial first l^2-Betti number (see [52] for other examples). One might view Schreier groups as a generalisation of non-abelian limit groups.

The significance of the Schreier property to the study of Kähler groups has first been recognised by Catanese [46]. Many of the ideas used in Section 3.2 are fairly standard. We will show that any epimorphism from a Kähler group to a Schreier group, which has no finite normal subgroups and virtually non-trivial first Betti number, factors through a map onto a surface group (see Theorem 3.2.1). It follows that any Kähler subdirect product of a direct product of Schreier groups of this form is a subdirect product of surface groups and a free abelian group (see Corollary 3.2.2). We will use these results to establish new constraints on Kähler groups.

3.1 Residually free Kähler groups

In this section we study Kähler groups that are residually free. The main result of this section is a classification of these groups.

Theorem 3.1.1. *Let G be a Kähler group. Then G is residually free if and only if there are integers $r, N \geq 0$ and $g_i \geq 2$, $1 \leq i \leq r$, such that G is a full subdirect product of $\mathbb{Z}^N \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$.*

As an easy consequence of Theorem 3.1.1 we obtain a new proof of Kotschick's Theorem 2.4.8. Another interesting consequence of Theorem 3.1.1 is a complete classification of Kähler subgroups of direct products of free groups, generalising work of Johnson and Rees [82] and of Dimca, Papadima and Suciu [64, 63].

Corollary 3.1.2. *A Kähler group is a subgroup of a direct product of free groups if and only if it is free abelian of even rank.*

Note that Bridson, Howie, Miller and Short [31] showed that there are examples of subgroups of direct products of free groups which are not virtually coabelian. These are not covered by the work of Dimca, Papadima and Suciu in [64, 63] who only considered coabelian subgroups of direct products of free groups.

The main constraint on Kähler groups that we want to use in this section is the following restated version of Py's Theorem 2.3.15.

Theorem 3.1.3. *Let G be a Kähler group. If G is virtually a subgroup of a Coxeter group or of a RAAG, then there are $r, N \in \mathbb{N}$ and $g_i \geq 2$, $1 \leq i \leq r$, such that G is virtually a subdirect product of $\mathbb{Z}^N \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$.*

Proof. Let X be a Kähler manifold with $G = \pi_1 X$. Since G is virtually a subgroup of a Coxeter group or a RAAG, Theorem 2.3.15 implies that there is a finite index subgroup $G_0 \leq G$ of G with corresponding finite-sheeted Kähler cover $X_0 \to X$ and holomorphic fibrations $p_i : X_0 \to \Sigma_i$, $1 \leq i \leq N$, onto closed hyperbolic orbisurfaces with connected fibres such that the p_i induce an injective map $G_0 \to (G_0)_{\mathrm{ab}} \times \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_N$. The induced maps $p_{i*} : \Gamma_0 \to \pi_1^{orb}\Sigma_i$, $1 \leq i \leq N$ are surjective. Passing to finite index covers $S_{g_i} \to \Sigma_i$ with $g_i \geq 2$, it follows that $G_1 := G_0 \cap ((G_0)_{\mathrm{ab}} \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_N}) \leq G_0 \leq G$ is a finite index subgroup of G that is subdirect in $(G_0)_{\mathrm{ab}} \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_N}$. □

The reason for restating Py's Theorem in this form is that *we shall later need that G is virtually a subdirect product.* While Py did not explicitly state his result in this form, he was certainly aware of this version of his result. Indeed it is mentioned in his recent paper with Delzant that G is subdirect [59].

Corollary 3.1.4. *Let G be a Kähler group. Then the following are equivalent:*

1. *G is virtually a subgroup of a Coxeter group;*
2. *G is virtually a subgroup of a RAAG;*
3. *G is virtually residually free;*
4. *G is virtually a full subdirect product of $\mathbb{Z}^N \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$, with $r, N \in \mathbb{N}$ and $g_i \geq 2$, $1 \leq i \leq r$.*

Proof. Theorem 3.1.3 implies that (1) ⇒ (4) and (2) ⇒ (4). RAAGs embed in right-angled Coxeter groups, so (2) ⇒ (1). Free abelian groups and fundamental groups of closed orientable surfaces are limit groups and hence so is their product. Thus (4) is stronger than (3). By Theorem 2.1.2, a group is residually free if and only if it is a subgroup of a direct product of finitely many limit groups. Since, by Theorem 2.1.3(2), limit groups are virtually special, every limit group virtually embeds in a RAAG. Direct products of RAAGs are RAAGs. Hence, every residually free group is virtually a subgroup of a RAAG and therefore (3) ⇒ (2). □

Since RAAGs are closed under taking direct products, in fact any Kähler subgroup of a direct product of virtually special groups is of the form of Corollary 3.1.4(4). Other interesting classes of virtually special groups are word-hyperbolic groups admitting a proper cocompact action on a CAT(0) cube complex [1] and fundamental groups of aspherical compact 3-manifolds which can be endowed with a metric of nonpositive curvature [106, Corollary 1.4]. Note that word-hyperbolic groups have no $\mathbb{Z}^2$-subgroups and therefore we retrieve the following result from [58] (Theorem 2.3.17 provides a different very recent proof).

Corollary 3.1.5. *If a Kähler group G is word-hyperbolic and admits a proper cocompact action on a CAT(0) cube complex, then it is virtually the fundamental group of a closed orientable hyperbolic surface.*

Proof. Since G is Kähler it is not $\mathbb{Z}$. By Agol [1] hyperbolic groups that admit a proper cocompact action on a CAT(0) cube complex are virtually special, so G is as in (4) of Corollary 3.1.4 and since G contains no $\mathbb{Z}^2$ it follows that $N = 0$ and $r = 1$. □

The proof of Theorem 3.1.1 requires a non-virtual version of Corollary 3.1.4. For this we will make use of

Lemma 3.1.6. *Let $G = \pi_1 X$ be a Kähler group and assume that there are $r, N \geq 0$ such that $G \leq \mathbb{Z}^N \times \pi_1 S_1 \times \cdots \times \pi_1 S_r$ is a full subdirect product where $S_1, \cdots, S_r$ are closed hyperbolic surfaces (possibly non-orientable) with Euler characteristic $\chi(S_i) \leq -2$, $1 \leq i \leq r$. Then $S_1, \cdots, S_r$ are closed orientable surfaces of genus at least 2.*

The main ingredient in the proof of Lemma 3.1.6 is a consequence of the theory of Kähler groups acting on $\mathbb{R}$-trees. One can think of an $\mathbb{R}$-tree as a generalisation of a tree. Since we do not make explicit use of $\mathbb{R}$-trees, we will not give a rigorous definition here. The connection between Kähler groups and actions on $\mathbb{R}$-trees was first investigated in [78]. Delzant [57, Theorem 6] summarised the main results, a proof of which was sketched in the last section of [78] (see also [90], [125]).

Theorem 3.1.7 (Delzant, [57, Theorem 6]). *Let G be a Kähler group and let T be an $\mathbb{R}$-tree which is not a line such that G acts on T minimally by isometries. Then there is a closed orientable orbisurface Σ and an epimorphism $\phi : \Gamma \to \pi_1^{orb}\Sigma$ such that $\pi_1^{orb}\Sigma$ acts on an $\mathbb{R}$-tree T' and there is a ϕ-equivariant map $T' \to T$.*

Proof of Lemma 3.1.6. Let $i \in \{1, \cdots, r\}$. Since $\chi(S_i) \leq -2$, it follows from [103, Theorem 1] that the surface group $\pi_1 S_i$ admits a minimal free action on an $\mathbb{R}$-tree T_i which is not a line. Consider the surjective projection $p_i : G \to \pi_1 S_i$. Theorem 3.1.7 implies that there is a closed orientable orbisurface Σ_i and a surjective map $\phi_i : G \to \pi_1^{orb}\Sigma_i$ such that $\pi_1 \Sigma_i^{orb}$ acts on an $\mathbb{R}$-tree T_i' and there is a ϕ_i-equivariant map $T_i' \to T_i$.

Since the action of $\pi_1 S_i$ on T_i is free, we obtain that p_i factors through ϕ_i, that is, there is an epimorphism $q_i : \pi_1^{orb}\Sigma_i \to \pi_1 S_i$ such that $p_i = q_i \circ \phi_i$. The kernel of q_i is the image of the kernel

$$K_i = G \cap \left(\mathbb{Z}^N \times \pi_1 S_1 \times \cdots \times \pi_1 S_{i-1} \times 1 \times \pi_1 S_{i+1} \times \cdots \times \pi_1 S_r\right) = \ker p_i$$

under the map ϕ_i.

The group G is finitely presented. Consequently, Theorem 2.5.5 implies that the group $K_i = \ker p_i$ is finitely generated. Hence, $\phi_i(K_i) = \ker q_i \trianglelefteq \pi_1^{orb}\Sigma_i$ is finitely generated. Finitely generated normal subgroups of fundamental groups of closed orientable orbisurfaces are either of finite index or trivial. Since q_i has infinite image it follows that $\ker q_i$ is trivial and $\pi_1\Sigma_i^{orb} \cong \pi_1 S_i$ and thus there is an isomorphism $\Sigma_i \cong S_i$. □

Note that the proofs in [107] are also based on actions on trees, but we could not see a way to extract Lemma 3.1.6 directly from there. An alternative proof of Lemma 3.1.6 which avoids the use of $\mathbb{R}$-trees can be obtained from the Schreier property of fundamental groups of closed hyperbolic orbisurfaces; it is obtained by applying the techniques used in Section 3.2 below.

We can now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Since free abelian groups and fundamental groups of closed orientable surfaces are limit groups the if direction follows from Theorem 2.1.2.

For the only if direction let H be residually free. Then, by Theorem 2.1.2, $H \leq \Lambda_0 \times \Lambda_1 \times \cdots \times \Lambda_r$ is a subgroup of a direct product of limit groups Λ_i, $0 \leq i \leq r$. The abelian limit groups are precisely the free abelian groups and all non-abelian limit groups have trivial centre. Since finitely generated subgroups of limit groups are limit groups, we may assume that Λ_0 is free abelian (possibly trivial), that $\Lambda_1, \cdots, \Lambda_r$ are non-abelian and that H is a full subdirect product (after passing to subgroups of the Λ_i and projecting away from factors which have trivial intersection with H).

Observe that $H \cap \Lambda_i$ is normal in H, since it is the kernel of the projection of H onto $\Lambda_0 \times \cdots \times \Lambda_{i-1} \times \Lambda_{i+1} \times \cdots \times \Lambda_r$. Since H projects onto Λ_i, it follows that $H \cap \Lambda_i$ is also normal in Λ_i. Hence, by Theorem 2.1.3(3), $H \cap \Lambda_i$ is either infinitely generated or of finite index in Λ_i for $1 \leq i \leq r$.

By Corollary 3.1.4, the group H has a finite index subgroup $\overline{H} \leq H$ which is isomorphic to a full subdirect product $\overline{G} \leq \mathbb{Z}^N \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_t}$ for $N, t \geq 0$ and $g_i \geq 2$, $1 \leq i \leq t$. By projecting $\overline{H}$ onto factors we obtain finite index subgroups $\overline{\Lambda}_i \leq \Lambda_i$, $1 \leq i \leq r$, such that $\overline{H} \leq \overline{\Lambda}_0 \times \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_r$ is a full subdirect product. In particular the intersections $\overline{H} \cap \overline{\Lambda}_i \trianglelefteq \overline{\Lambda}_i$ are normal and therefore either of finite index in $\overline{\Lambda}_i$ or infinitely generated for $1 \leq i \leq r$.

Since finite index subgroups of non-abelian limit groups are non-abelian, it follows that the centre of $\overline{H}$ is contained in $\overline{\Lambda}_0$ and the centre of $\overline{G}$ is contained in $\mathbb{Z}^N$. Therefore, $\overline{H}/\mathrm{Z}(\overline{H}) \leq \overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_r$ and $\overline{G}/\mathrm{Z}(\overline{G}) \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_t}$ are full subdirect products. Theorem 2.1.5 implies that $r = t$ and the isomorphism $\overline{H}/\mathrm{Z}(\overline{H}) \cong \overline{G}/\mathrm{Z}(\overline{G})$ is induced by an isomorphism $\overline{\Lambda}_1 \times \cdots \times \overline{\Lambda}_r \cong \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$. Furthermore, after reordering factors, this isomorphism is induced by isomorphisms $\overline{\Lambda}_i \cong \pi_1 S_{g_i}$, $1 \leq i \leq r$.

A torsion-free finite extension of a fundamental group of a closed hyperbolic surface is the fundamental group of a closed hyperbolic surface. Limit groups are torsion-free and therefore the finite extension Λ_i of $\overline{\Lambda}_i$ is the fundamental group of a closed hyperbolic surface for $1 \leq i \leq r$. The fundamental group of a closed hyperbolic surface is a limit group if and only if it has Euler characteristic ≤ -2 (see Section 2.1). Thus, we obtain that $H \leq \mathbb{Z}^M \times \pi_1 R_1 \times \cdots \times \pi_1 R_r$ is a full subdirect product where R_i is a closed hyperbolic surface of Euler characteristic ≤ -2 for $1 \leq i \leq r$. Lemma 3.1.6 implies that R_i is in fact a closed orientable hyperbolic surface. This completes the proof. □

Theorem 2.4.8 and Corollary 3.1.2 follow easily from Theorem 3.1.1.

Proof of Theorem 2.4.8. By Theorem 3.1.1, a Kähler limit group G is a full subdirect product of $\mathbb{Z}^N \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$ with $r, N \geq 0$ and $g_i \geq 2$. A limit group is either free abelian or every element has trivial centre. Hence, if G is not free abelian then G must be a subgroup of one of the $\pi_1 S_{g_i}$. Since G is full subdirect, it follows that $r = 1$, $N = 0$ and $G = \pi_1 S_{g_1}$. □

Proof of Corollary 3.1.2. Let G be a Kähler subgroup of a direct product of free groups. We may assume that G is a full subdirect product of a direct product of free groups and a free abelian group $\mathbb{Z}^K \times F_{l_1} \times \cdots \times F_{l_r}$ for some $K, r \geq 0$ and F_{l_i} free with $l_i \geq 2$, $1 \leq i \leq r$ (after projecting away from free factors with trivial intersection and passing to subgroups of the free factors which are again free). By Theorem 3.1.1, the group G is isomorphic to a full subdirect product of $\mathbb{Z}^N \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_t}$. Theorem 2.1.5 implies that $r = t$ and after reordering factors $F_{l_i} \cong \pi_1 S_{g_i}$. Since fundamental groups of closed hyperbolic surfaces are not free we obtain that in fact $r = t = 0$ and thus G is free abelian of even rank. □

3.2 Schreier groups

Our main result in this section is about maps from Kähler groups to subdirect products of Schreier groups.

Theorem 3.2.1. *Let X be a compact Kähler manifold, let $H = \pi_1 X$ be the corresponding Kähler group and let $G_1 \times \cdots \times G_r$ be a direct product of Schreier groups with $b_1(G_i) \neq 0$ such that G_i has no finite normal subgroups for $1 \leq i \leq r$.*

Then any homomorphism $\rho : H \to G_1 \times \cdots \times G_r$ with subdirect image factors through a homomorphism $\widehat{\rho} : H \to \mathbb{Z}^N \times \pi_1^{orb} \Sigma_1 \times \cdots \times \pi_1^{orb} \Sigma_k$ with full subdirect image for some closed orientable hyperbolic orbisurfaces Σ_i of genus $g_i \geq 2$, $1 \leq i \leq k$, with $k, N \geq 0$.

The projections $H \to \pi_1^{orb}\Sigma_i$ have finitely generated kernels and are induced by holomorphic fibrations $X \to \Sigma_i$.

Observe that the condition that H is subdirect in Theorem 3.2.1 is necessary: the free product of any torsion-free Kähler group G and $\mathbb{Z}$ is a Schreier group with non-trivial first Betti number which contains G as a Kähler subgroup.

Theorem 3.2.1 allows us to describe Kähler subdirect products of Schreier groups with non-trivial first Betti numbers and without finite normal subgroups.

Corollary 3.2.2. *A Kähler group $H \leq G_1 \times \cdots \times G_r$ is a subdirect product of Schreier groups G_i, with non-trivial first Betti number $b_1(G_i) \neq 0$ and without finite normal subgroups, $1 \leq i \leq r$, if and only if there are $k, N \geq 0$ and closed orientable hyperbolic orbisurfaces Σ_i of genus $g_i \geq 2$, $1 \leq i \leq k$, such that H is a full subdirect product of $\mathbb{Z}^N \times \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_k$.*

Proof. Consider the special case of Theorem 3.2.1 in which the map ρ is the inclusion map of $H \leq G_1 \times \cdots \times G_r$. It follows that H is a full subdirect product of a direct product $\mathbb{Z}^N \times \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_k$ where the Σ_i are closed orientable hyperbolic orbisurfaces of genus $g_i \geq 2$. This completes the proof of the only if direction.

For the converse note that $\mathbb{Z}$ and all fundamental groups of closed orientable hyperbolic orbisurfaces are Schreier groups with non-trivial first Betti number and without finite normal subgroups. ☐

The following equivalence between classes of Kähler groups summarises our results from Section 3.1 and 3.2.

Theorem 3.2.3. *Let G be a Kähler group. Then the following are equivalent:*

1. *G is virtually a subgroup of a Coxeter group;*
2. *G is virtually a subgroup of a RAAG;*
3. *G is virtually residually free;*
4. *G is virtually a subdirect product of a direct product of Schreier groups $G_1 \times \cdots \times G_r$ such that $b_1(G_i) \neq 0$ and G_i has no finite normal subgroups for $1 \leq i \leq r$;*
5. *G is virtually a full subdirect product of $\mathbb{Z}^N \times \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$, with $r, N \in \mathbb{N}$ and $g_i \geq 2$, $1 \leq i \leq r$.*

For the proof of Theorem 3.2.1 we want to restate Napier and Ramachandran's Theorem 2.3.10 in the following form.

Theorem 3.2.4. *Let X be a Kähler manifold, let $G = \pi_1 X$ be its fundamental group group and let $\phi : G \to \mathbb{Z}$ be an epimorphism whose kernel is not finitely generated. Then ϕ factors through an epimorphism $\psi : G \to \pi_1^{orb}\Sigma$ where Σ is the fundamental group of a closed orientable hyperbolic Riemann orbisurface of genus ≥ 2 and ψ has finitely generated kernel. The homomorphism ϕ is induced by a holomorphic fibration $X \to \Sigma$ for some complex structure on Σ.*

Proof. This result is mentioned in the proof of [70, Theorem 2.3]. It is a direct consequence of Lemma 2.3.3 and Theorem 2.3.10. □

Our proof of Theorems 3.2.1 follows by fairly standard methods from Theorem 3.2.4 and the following Lemma.

Lemma 3.2.5. *Let G, H_1, H_2, Q be infinite groups. Assume that H_2 is Schreier with no finite normal subgroups and that there is a commutative diagram*

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\phi"] & H_2 \arrow[d, "\nu"] \\ H_1 \arrow[r] & Q \end{tikzcd} \tag{3.1}$$

of epimorphisms such that $\ker\phi$ *is finitely generated. Then there is an epimorphism $\theta : H_1 \to H_2$ such that the diagram*

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d, "\phi"] & H_2 \arrow[d, "\nu"] \\ H_1 \arrow[r] \arrow[ur, "\theta"] & Q \end{tikzcd} \tag{3.2}$$

commutes.

Proof. By surjectivity of ψ the image $\psi(\ker(\phi))$ is a normal finitely generated subgroup of H_2. Since H_2 is Schreier with no finite normal subgroups it follows that either $\psi(\ker(\phi))$ is trivial or a finite index subgroup of H_2. If $\psi(\ker(\phi))$ is a finite index normal subgroup of H_2, then $\nu(\psi(\ker(\phi))) \trianglelefteq Q$ is a finite index normal subgroup of Q. This contradicts commutativity of the diagram (3.1). Thus, $\psi(\ker(\phi))$ is trivial and there is an induced homomorphism $\theta : H_1 \to H_2$ making the diagram (3.2) commutative. □

Proof of Theorem 3.2.1. Let $\overline{H} = \rho(H)$ be the image of the Kähler group H under ϕ. By assumption $\overline{H}$ is a subdirect product of the direct product of Schreier groups $G_1 \times \cdots \times G_r$ with $b_1(G_i) \neq 0$ such that G_i has no finite normal subgroups for $1 \leq i \leq r$.

Consider $1 \leq i \leq r$ such that G_i is non-abelian. Since $b_1(G_i) \neq 0$, there is an epimorphism $\psi_i : G_i \to \mathbb{Z}$. Its kernel is a non-trivial infinite index normal subgroup of G_i and therefore not finitely generated. The kernel of the surjective composition $\psi_i \circ p_i \circ \rho : H \to \mathbb{Z}$, where p_i is the surjective projection $p_i : \overline{H} \to G_i$ of $\overline{H}$ onto the ith factor, is also not finitely generated, since it maps onto the non-finitely generated group $\ker \psi_i$. Hence, Theorem 3.2.4 implies that there is a closed orientable hyperbolic Riemann orbisurface Σ_i^{orb} of genus $g_i \geq 2$ such that $\psi_i \circ p_i \circ \rho$ factors through an epimorphism $\phi_i : H \to \pi_1 \Sigma_i^{orb}$ with finitely generated kernel. The map ϕ_i is induced by a surjective holomorphic fibration $X \to \Sigma_i^{orb}$ with connected fibres. To simplify notation, we define $h_i := p_i \circ \rho$.

Denote by $q_i : \Sigma_i^{orb} \to \mathbb{Z}$ the epimorphism such that $\psi_i \circ h_i = q_i \circ \phi_i$. Then Lemma 3.2.5 implies that there is an induced map $f_i : \pi_1^{orb}\Sigma_i \to G_i$ such that the diagram

$$\begin{tikzcd}
H_1 \arrow[r, "h_i"] \arrow[d, "\phi_i"'] & G_i \arrow[d, "\psi_i"] \\
\pi_1^{orb}\Sigma_i \arrow[r, "q_i"'] \arrow[ur, "f_i"] & \mathbb{Z}
\end{tikzcd}$$

is commutative.

The only infinite abelian Schreier group without finite normal subgroups is $\mathbb{Z}$. Thus, after reordering factors such that $G_1, \cdots, G_l \cong \mathbb{Z}$ and $G_{l+1}, \cdots, G_r$ are non-abelian, it follows that $\rho : H \hookrightarrow G_1 \times \cdots \times G_r$ factors through a homomorphism $\widehat{\rho} : H \hookrightarrow \mathbb{Z}^l \times \pi_1^{orb}\Sigma_{l+1} \times \cdots \times \pi_1^{orb}\Sigma_r$ such that $\widehat{\rho}(H)$ is a full subdirect product. ☐

Observe that we can combine Theorem 3.2.1 and Theorem 2.3.15 to show:

Addendum 3.2.6. *Let X be a Kähler manifold, let $H = \pi_1 X$ be its fundamental group and let G_R be a RAAG. Let G_S be a direct product of Schreier groups, with virtually non-trivial first Betti numbers and without finite normal subgroups.*

Then for any homomorphism $\phi : H \to G_R \times G_S$ for which the projection of $\phi(H)$ to G_S is subdirect, there is a finite index subgroup $H_1 \leq H$ such that the restriction of ϕ to H_1 factors through a homomorphism $\widehat{\phi} : H_1 \to \mathbb{Z}^N \times \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ for some $N, r \geq 0$ and Σ_i closed orientable hyperbolic orbisurfaces of genus $g_i \geq 2$, $1 \leq i \leq r$.

Furthermore there is a finite-sheeted cover $X_1 \to X$ with fundamental group H_1, so that, after endowing the Σ_i with a suitable complex structure, the maps from H_1

onto the factors are induced by holomorphic fibrations with connected fibres, $X_1 \to \Sigma_i$, $1 \le i \le r$.

As a second application of Lemma 3.2.5 we can rephrase Lemma 2.3.3 and Theorem 2.3.4 in the following way.

Corollary 3.2.7. *Let X be a compact Kähler manifold and let $\widehat{G} = \pi_1 X$ be its fundamental group. Then there is $r \ge 0$ and closed hyperbolic Riemann orbisurfaces Σ_i of genus $g_i \ge 2$ together with surjective holomorphic maps $f_i : X \to \Sigma_i$ with connected fibres, $1 \le i \le r$, such that*

1. *the induced homomorphisms $f_{i,*} : \widehat{G} \to \pi_1^{orb}\Sigma_i$ are surjective with finitely generated kernel for $1 \le i \le r$;*
2. *the image of $\phi := (f_{1,*}, \cdots, f_{r,*}) : \widehat{G} \to \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ is full subdirect; and*
3. *every epimorphism $\psi : \widehat{G} \to \pi_1^{orb}\Sigma'$ onto a fundamental group of a closed orientable hyperbolic Riemann orbisurface Σ' of genus $h \ge 2$ factors through ϕ.*

Proof. By Theorem 2.3.4, there is $r \ge 0$, closed Riemann orbisurfaces Σ_i of genus $g_i \ge 2$, and epimorphisms $\phi_i : \widehat{G} \to \pi_1^{orb}\Sigma_i$ with finitely generated kernel, such that any epimorphism $\psi : \widehat{G} \to \pi_1 S_h$ with $h \ge 2$ factors through one of the ϕ_i. We may assume that the set of ϕ_i is minimal in the following sense: none of the homomorphisms ϕ_i factors through ϕ_j for $j \ne i$. We claim that $\phi := (\phi_1, \cdots, \phi_r)$ has the asserted properties.

By Lemma 2.3.3, each of the ϕ_i is induced by a surjective holomorphic map $f_i : X \to \Sigma_i$ with connected fibres. Thus (1) holds.

Let Σ' be a closed orientable hyperbolic Riemann orbisurface of genus $h \ge 2$ and let $\psi : \widehat{G} \to \pi_1^{orb}\Sigma'$ be an epimorphism. Then there is an epimorphism $\theta : \pi_1^{orb}\Sigma' \to \pi_1 S_h$ with $h \ge 2$. Thus, there is $1 \le i \le r$ and an epimorphism $\eta : \pi_1^{orb}\Sigma_i \to \pi_1 S_h$ such that the diagram

$$\begin{tikzcd} \widehat{G} \arrow[r, "\psi"] \arrow[d, "\phi_i"'] & \pi_1^{orb}\Sigma' \arrow[d, "\theta"] \\ \pi_1^{orb}\Sigma_i \arrow[r, "\eta"] & \pi_1 S_h \end{tikzcd}$$

commutes. Since the kernel of ϕ_i is finitely generated and $\pi_1^{orb}\Sigma'$ is Schreier without finite normal subgroups, we obtain from Lemma 3.2.5 that ψ factors through a homomorphism ϕ_i. In particular, ψ factors through ϕ and therefore (3) holds.

By definition, the image $\phi(\widehat{G})$ of ϕ is subdirect. It is full by the minimality assumption on the ϕ_i. This implies (2). $\square$

In our opinion, the viewpoint on the classical results about homomorphisms from Kähler groups to direct products of surface groups provided by Corollary 3.2.7 captures the essence of these maps: it shows that for every Kähler group $\widehat{G}$ there is a unique universal pair of a direct product $\pi_1^{orb}\Sigma_1 \times \cdots \times \times \pi_1^{orb}\Sigma_r$ of orbisurface fundamental groups together with a homomorphism $\phi : \widehat{G} \to \pi_1^{orb}\Sigma_1 \times \cdots \times \times \pi_1^{orb}\Sigma_r$ with full subdirect image. In Chapter 7 we will give constraints on the map ϕ (see in particular Theorems 7.4.2 and 7.3.1).

We want to conclude this section with some consequences of Theorem 3.2.1 and its proof.

An alternative proof of Theorem 3.1.1

Note that the only consequence of Py's work used in our proofs of Theorems 3.1.1 and 2.4.8, and Corollary 3.1.2 is Corollary 3.1.4(4). Non-abelian limit groups are torsion-free Schreier groups with non-trivial first Betti number and any finitely generated subgroup of a limit group is a limit group. Hence, this section yields different proofs of these results which do require neither the use of RAAGs, nor the fact that limit groups are virtually special.

Maps to free products of groups

Corollary 3.2.8. *Let H be a Kähler group, let G_1, G_2 be non-trivial groups and assume that $G_1 * G_2$ has virtually non-trivial first Betti number. Then for any epimorphism $\phi : H \to G_1 * G_2$ there is a finite index subgroup $H_0 \leq H$ and a closed orientable hyperbolic surface S_g such that the restriction of ϕ to H_0 splits through an epimorphism $\widehat{\phi} : H_0 \to \pi_1 S_g$.*

In particular for any group R the following holds: If a Kähler group H is a subgroup of the direct product $(G_1 * G_2) \times R$ such that the projection of H to $G_1 * G_2$ is surjective then $G_1 * G_2$ is virtually the fundamental group of a closed orientable hyperbolic surface.

Proof. By [11] every free product of two non-trivial groups is a Schreier group without normal finite subgroups. By assumption there is a finite index subgroup $G' \leq G_1 * G_2$ with non-trivial first Betti number. Thus, Theorem 3.2.1 and its proof yield that there is a finite index subgroup $H_0 \leq H$ and a closed surface S_g of genus g such that the restriction of ϕ to H_0 factors through an epimorphism $\widehat{\phi} : H_0 \to \pi_1 S_g$.

Choosing ϕ to be the inclusion map of a subgroup $H \leq (G_1 * G_2) \times R$ we obtain that $G_1 * G_2$ is virtually the fundamental group of a closed surface S_g of genus ≥ 2. $\square$

Corollary 3.2.8 generalises [82, Theorem 3]. It is known in the case when ϕ has finitely generated kernel (cf. [6, Theorem 4.2]), since free products other than $\mathbb{Z}_2 * \mathbb{Z}_2$ have non-vanishing first l^2-Betti number. While we would not be surprised if it were known, we could not find any treatment of the case when $\ker\phi$ is not finitely generated in the literature.

The condition that $G_1 * G_2$ has virtually non-trivial first Betti number is satisfied in many cases: It is sufficient that each of G_1 and G_2 has at least one finite quotient [94, Lemma 3.1]. Note that under these assumptions on G_1 and G_2, $G_1 * G_2$ is large unless G_1 and G_2 each only have the finite quotient $\mathbb{Z}_2$. Thus, we could also obtain a map to a surface group from this without using Theorem 3.2.4.

Torsion-free Schreier groups

As another consequence of Corollary 3.2.2 we obtain

Corollary 3.2.9. *Let G be Kähler. Then G is a torsion-free Schreier group with virtually non-trivial first Betti number if and only if $G \cong \pi_1 S_g$, with $g \geq 2$.*

Proof. Assume that G is a torsion-free non-abelian Schreier Kähler group with virtually nontrivial first Betti number. By [52, Lemma 3.2], finite index subgroups of Schreier groups are Schreier. Since the only free abelian Schreier group is $\mathbb{Z}$, which is not Kähler, it follows that G is not virtually abelian. Because G is torsion-free it has no finite normal subgroups and trivial centre. In particular, it follows from Corollary 3.2.2 that G is virtually the fundamental group of a closed orientable hyperbolic surface of genus at least two. Torsion-free groups which are virtually surface groups are surface groups. Hence, G is the fundamental group of a closed hyperbolic Riemann surface. □

Addendum 3.2.10. *Let G be Kähler. If G is a full subdirect product of non-abelian, torsion-free Schreier groups G_i with virtually non-trivial first Betti number then $G_i \cong \pi_1 S_{g_i}$, with $g_i \geq 2$, $1 \leq i \leq r$.*

The proof of this result is analogous to the proof of Corollary 3.2.9, since it follows from Corollary 3.2.2 that all factors have finite index subgroups which are fundamental groups of closed hyperbolic surfaces.

Paralimit groups

Another class of groups which are Schreier with no finite normal subgroups is the class of non-abelian *paralimit groups* [35, Theorem C and Section 8.6] (residually nilpotent groups with the same nilpotent quotients as a non-abelian limit group). For these groups we obtain:

Corollary 3.2.11. *The non-abelian paralimit Kähler groups are precisely the fundamental groups of closed hyperbolic Riemann surfaces.*

Proof. It is well-known that for a free group F_r on r generators all nilpotent quotients $F_r/\gamma_k(F_r)$ are torsion-free and that F_r is residually nilpotent. Let G be a limit group. Then for every $g \in G$ there is a homomorphism $\phi : G \to F_r$ onto a free group with $\phi(g) \neq 1$. As a consequence limit groups are also residually free with torsion-free nilpotent quotients $G/\gamma_k(G)$. Since paralimit groups are residually nilpotent, we obtain that paralimit groups are torsion-free. Since limit groups have non-trivial first Betti number, the same holds for paralimit groups. Corollary 3.2.9 implies that every non-abelian paralimit Kähler group is the fundamental group of a closed hyperbolic Riemann surface.

□

In the light of Corollary 3.2.11 and Theorem 3.1.1 it seems natural to ask if one can prove that a residually nilpotent Kähler group with the same nilpotent quotients as a residually free group must be a full subdirect products of finitely many surface groups and a free abelian group.

3-manifold groups and one-relator groups

The techniques presented in this chapter can also be used to improve results on Kähler groups and one-relator groups and on Kähler groups and 3-manifold groups. More precisely, techniques very similar to the ones used in this section can be combined with the work of Kotschick [93, 95], Biswas and Mj [22] and Biswas, Mj and Seshadri [25] to give constraints on groups Q fitting into a short exact sequence

$$1 \to N \to G \to Q \to 1$$

with N finitely generated, G Kähler and Q either the fundamental group of a compact 3-manifold (with or without boundary) or a coherent one-relator group.

In the case of fundamental groups of compact 3-manifolds we essentially obtain that the constraints for 3-manifolds without boundary given in [25] also hold for manifolds with boundary, with the difference that we need to allow Q to be commensurable to a surface group. For coherent one-relator groups we essentially obtain that Q must be commensurable to a surface group. We will get back to this in future work.

Chapter 4

Kähler groups from maps onto elliptic curves

In Section 2.5 we introduced Dimca, Papadima and Suciu's examples of Kähler groups of finiteness type $\mathcal{F}_{r-1}$ but not $\mathcal{F}_r$ $(r \geq 3)$ [62]. Recall that their examples arise as fundamental group of fibres of holomorphic maps from a product of Riemann surfaces onto an elliptic curve, where the restrictions to factors are branched double covers. Our main goals in this Chapter are to provide a new more general construction for Kähler groups with exotic finiteness properties from maps onto an elliptic curve and to introduce invariants which distinguish between many of the groups obtained from our construction. This class also provides new examples of subgroups of direct products of surface groups, thereby contributing towards an answer of Delzant and Gromov's Question 1.

We consider maps from products of closed Riemann surfaces onto elliptic curves which restrict to branched coverings on the factors. We show that if such a map induces a surjective map on fundamental groups, then its smooth generic fibre is connected (see Section 4.1) and its fundamental group is a Kähler group with exotic finiteness properties. The main result concerning our general construction is Theorem 4.3.2.

We then proceed to apply our construction to produce a new class of Kähler groups which to us seems like the most natural analogue of a specific subclass of the Bestvina–Brady groups [19] (see Section 4.4). To show that this new class of examples is not isomorphic to any of the DPS groups we introduce invariants for our examples in Sections 4.5 and 4.6. These invariants lead to a complete classification for the class of groups arising from *purely branched* coverings (see Definition 4.1.2). The main classification result is Theorem 4.6.2.

4.1 Connectedness of fibres

We give a precise criterion for the connectedness of the fibres of a map from a direct product of closed Riemann surfaces onto an elliptic curve defined by branched covers.

Theorem 4.1.1. *Let $r \geq 2$, let E be an elliptic curve and let $f_{g_i} : S_{g_i} \to E$ be holomorphic branched covers of E with $g_i \geq 2$.*

The map $f = \sum_{i=1}^{r} f_{g_i} : S_{g_1} \times \cdots \times S_{g_r} \to E$ has connected fibres if and only if it induces a surjective map on fundamental groups.

Proof. Every holomorphic map $h : X \to Y$ between compact complex manifolds X and Y with connected fibres induces a surjective map on fundamental groups, since it is a locally trivial fibration over the complement of a complex codimension one subvariety of Y. Hence, if f has connected fibres then it induces a surjective map on fundamental groups.

Assume now that f induces a surjective map on fundamental groups. If f does not have connected fibres then Stein factorisation [123] (also e.g. [18, Theorem 2.10]) yields a closed Riemann surface S and holomorphic maps $\alpha : S \to E$ and $\beta : S_{g_1} \times \cdots \times S_{g_r} \to S$ such that α is finite-to-one and β has connected fibres. Since holomorphic finite-to-one maps between closed Riemann surfaces are branched covering maps, it follows that α is a branched covering.

Choose a base point $(p_1, \cdots, p_r) \in S_{g_1} \times \cdots \times S_{g_r}$ and denote by β_i the restriction of β to the ith factor $\{(p_1, \cdots, p_{i-1})\} \times S_{g_i} \times \{(p_{i+1}, \cdots, p_r)\}$. Then there is $e_i \in E$ such that $\alpha \circ \beta_i = e_i + f_{g_i}$. Since β_i is holomorphic, it is non-trivial and finite-to-one, and hence a finite-sheeted holomorphic branched covering map. It now follows that $\beta_{i*}(\pi_1 S_{g_i}) \leq \pi_1 S$ is a finite index subgroup and therefore not cyclic for $i = 1, \cdots, r$.

It then follows from [24, Lemma 7.1] that S must itself be an elliptic curve. Since the argument is short we want to give it here: Choose $\gamma_1 \in \pi_1 S_{g_1}$ and $\gamma_2 \in \pi_1 S_{g_2}$ such that their images $\beta_1 \circ \gamma_1$ and $\beta_2 \circ \gamma_2$ do not lie in a common cyclic subgroup of $\pi_1 S$. Then $\beta_1 \circ \gamma_1$ and $\beta_2 \circ \gamma_2$ generate a $\mathbb{Z}^2$-subgroup of $\pi_1 S$ and the only closed Riemann surfaces with $\mathbb{Z}^2$-subgroups are elliptic curves.

Using Euler characteristic, it follows that any branched covering map between 2-dimensional tori is an unramified covering map. By assumption the map $f = \alpha \circ \beta$ induces a surjective map on fundamental groups. Hence, the map $\alpha : S \to E$ is an unramified holomorphic covering which is surjective on fundamental groups and therefore $S = E$ and α is biholomorphic. In particular, the map f has connected fibres. $\square$

We introduce a special class of branched covering maps of tori. Let X be a closed connected manifold and let Y be a torus of the same dimension k, let $D \subset Y$ be a complex analytic subvariety of codimension at least one or more generally a real analytic subvariety of codimension at least two. Let $f : X \to Y$ be a branched covering map with branching locus D, that is, $f^{-1}(D)$ is a nowhere dense set in X mapping onto D and the restriction $f : X \setminus f^{-1}(D) \to Y \setminus D$ is an unramified covering.

Assume that for a base point $z_0 \in Y \setminus D$ there are simple closed loops $\mu_1, \ldots, \mu_k : [0,1] \to Y \setminus D$ based at z_0 with the following properties:

- $\mu_i([0,1]) \cap \mu_j([0,1]) = \{z_0\}$ for $i \neq j$; and
- the loops $\mu_1, \ldots, \mu_k$ generate $\pi_1 Y$.

Definition 4.1.2. We call the map f *purely branched* if there exist loops $\mu_1, \ldots, \mu_k$ as above, which satisfy the condition

$$\langle\langle \mu_1, \cdots, \mu_k \rangle\rangle \leq f_*(\pi_1(X \setminus f_1^{-1}(D))) \leq \pi_1(Y \setminus D),$$

i.e. every lift to X of each μ_i is a loop.

Note that for Y a 2-torus, X a closed connected surface of genus $g \geq 1$ and $f : X \to Y$ a branched covering map with branching locus $D = \{p_1, \cdots, p_r\}$, the map f is purely branched if and only if there are simple closed loops $\mu_1, \mu_2 : [0,1] \to Y \setminus D$ which generate $\pi_1 Y$, intersect only in $\mu_1(0) = \mu_2(0)$ and have the property that every lift of μ_1 and μ_2 is a loop in X. We will revisit the notion of a purely branched covering in Section 6.2, where it will come up naturally.

4.2 A Kähler analogue of Bestvina–Brady groups

The DPS groups [62] can be described as kernels of maps from a direct product of surface groups $\Gamma_{g_i} \cong \pi_1 S_{g_i}$, $1 \leq i \leq r$, onto $\mathbb{Z}^2$.

$$\begin{array}{cccc} \phi_{g_1,\cdots,g_r} : & \Gamma_{g_1} \times \cdots \times \Gamma_{g_r} & \to & \mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle \\ & a_1^i, a_2^i & \mapsto & a \\ & b_1^i, b_2^i & \mapsto & b \\ & a_3^i, \cdots, a_{g_i}^i & \mapsto & 0 \\ & b_3^i, \cdots, b_{g_i}^i & \mapsto & 0, \end{array} \tag{4.1}$$

We will see that the branched coverings in Dimca, Papadima and Suciu's construction satisfy all the conditions of Theorem 4.3.2. This will provide us with a new proof of Theorem 2.5.6.

The case when $g_1 = \cdots = g_r = 2$ is a good surface group analogue of the Bestvina–Brady group corresponding to the direct product $F_2 \times \cdots \times F_2$ of r copies of the free group on two generators.

More generally, consider the direct product $\Lambda_{g_1} \times \cdots \times \Lambda_{g_r}$ of surface groups $\Lambda_{g_i} \cong \pi_1 S_{g_i}$, $g_i \geq 2$. Then, in analogy to the map $\phi_{2,\cdots,2}$, define the homomorphism

$$\begin{array}{rccc} \psi_{g_1,\cdots,g_r}: & \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} & \rightarrow & \mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle \\ & a_j^i & \mapsto & a \\ & b_j^i & \mapsto & b \end{array} \tag{4.2}$$

To prove that all groups arising as kernel of one of the $\psi_{g_1,\cdots,g_r}$ are not of type $\mathcal{F}_r$, we use a result by Bridson, Miller, Howie and Short [29, Theorem B]

Theorem 4.2.1. *Let $\Lambda_{g_1}, \cdots, \Lambda_{g_n}$ be surface groups with $g_i \geq 2$ and let $G \leq \Lambda_{g_1} \times \cdots \times \Lambda_{g_n}$ be a subgroup of their direct product. Assume that each intersection $L_i = G \cap \Lambda_{g_i}$ is non-trivial and arrange the factors in such a way that $L_1, \cdots, L_r$ are not finitely generated and $L_{r+1}, \cdots, L_n$ are finitely generated.*

If precisely $r \geq 1$ of the L_i are not finitely generated, then G is not of type FP_r.

For finitely presented groups, being of type FP_r is equivalent to being of type $\mathcal{F}_r$ [36, p. 197]. Since all the groups we consider will be finitely presented, we do not dwell on the meaning of the homological finiteness condition FP_r. As a consequence of Theorem 4.2.1 we can now prove

Theorem 4.2.2. *Let $\Lambda_{g_1}, \cdots, \Lambda_{g_r}$ be surface groups with $g_i \geq 2$ and let $r \geq 3$. Let $k \geq 1$ and let $\nu_i : \Lambda_{g_i} \rightarrow \mathbb{Z}^k$ be non-trivial homomorphisms. Then the kernel of the map*

$$\nu = \nu_1 + \cdots + \nu_r : \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} \rightarrow \mathbb{Z}^k$$

is not of type $\mathcal{F}_r$

Proof. Let $G = \ker(\nu) < \Lambda_{g_1} \times \cdots \times \Lambda_{g_r}$. Let $L_i = G \cap \Lambda_{g_i} := G \cap 1 \times \cdots \times \Lambda_{g_i} \times \cdots \times 1$ and note that $L_i = \ker(\nu_i)$. Then L_i is an infinite index normal subgroup of Λ_{g_i}. Infinite index normal subgroups of a surface group are infinitely generated free groups. Hence, none of the L_i are finitely generated and therefore G is not of type FP_r, by Theorem 4.2.1, and hence not of type $\mathcal{F}_r$. □

Corollary 4.2.3. *For all $r \geq 3$ and $g_1, \cdots, g_r \geq 2$, the groups $\ker(\phi_{g_1,\cdots,g_r})$ and the groups $\ker(\psi_{g_1,\cdots,g_r})$ are not of type $\mathcal{F}_r$.*

Remark 4.2.4. Under the additional condition that the maps $\nu_i : \Lambda_i \to \mathbb{Z}^k$ are surjective, all groups that arise in this way are group theoretic fibre products over $\mathbb{Z}^k$, and one can construct explicit finite presentations for them using the same methods as we use in Chapter 5 to construct finite presentations for the DPS groups.

The original proof that the DPS groups have arbitrary finiteness properties consists of an involved argument making use of characteristic varieties. Alternative proofs have been given since by Biswas, Mj and Pancholi [24] and by Suciu [124]. None of these proofs uses the work of Bridson, Howie, Miller and Short [29], but Dimca, Papadima and Suciu [62] mentioned without proof that the work of Bridson, Howie, Miller and Short could be used to obtain an alternative proof of the finiteness properties of their groups.

4.3 Constructing new classes of examples

Let E be a closed Riemann surface of positive genus and let X be a closed connected complex analytic manifold of dimension $r > 1$. An *irrational pencil* $h : X \to E$ is a surjective holomorphic map such that the smooth generic fibre H is connected.

Let M, N be complex manifolds and let $f : M \to N$ be a surjective holomorphic map. We say that the map f has *isolated singularities* if for every $y \in N$ and every $x \in f^{-1}(y)$ there is a neighbourhood U of x in $f^{-1}(y)$ such that $(U \cap f^{-1}(y)) \setminus \{x\}$ is smooth. It follows that a map f has isolated singularities if the set of singular points of f in $f^{-1}(y)$ is discrete. In the case of an irrational pencil $h : X \to E$ this is equivalent to the set of singular points of f being finite.

We will discuss the relevance of isolated singularities in the construction of new Kähler groups in more depth in Section 6.1 and Chapter 9. For a more detailed introduction to maps with isolated singularities and their local structure see Section 9.2.

The following result is due to Dimca, Papadima and Suciu. As it is stated this is an easy consequence of [62, Theorem C]. We will state the original version of their result in Section 6.1, where we will need it (see Theorem 6.1.1).

Theorem 4.3.1 ([62, Theorem C])**.** *Let $h : X \to E$ be an irrational pencil. Suppose that h has only isolated singularities. Then the following hold:*

1. *the inclusion $H \hookrightarrow X$ induces isomorphisms $\pi_i(H) \cong \pi_i(X)$ for $2 \leq i \leq r-1$;*

2. *the map h induces a short exact sequence $1 \to \pi_1 H \to \pi_1 X \to \pi_1 E \to 1$.*

A *Stein manifold* is a complex manifold that embeds biholomorphically as a closed submanifold in some affine complex space $\mathbb{C}^r$. Theorem 4.3.1 allows us to prove

Theorem 4.3.2. *Let $r \geq 3$ and let $g_1, \cdots, g_r \geq 2$. Let E be an elliptic curve and let $f_{g_i} : S_{g_i} \to E$ be a branched covering, for $i = 1, \cdots, r$. Assume that the map*

$$f = \sum_{i=1}^{r} f_{g_i} : S_{g_1} \times \cdots \times S_{g_r} \to E$$

induces an epimorphism on fundamental groups.

Then the generic fibre H of the map f is a connected $(r-1)$-dimensional smooth projective variety such that

1. *the homotopy groups $\pi_i H$ are trivial for $2 \leq i \leq r-2$ and $\pi_{r-1} H$ is nontrivial;*
2. *the universal cover $\widetilde{H}$ of H is a Stein manifold;*
3. *the fundamental group $\pi_1 H$ is a projective (and thus Kähler) group of finiteness type $\mathcal{F}_{r-1}$ but not of finiteness type $\mathcal{F}_r$;*
4. *the map f induces a short exact sequence*

$$1 \to \pi_1 H \to \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r} \to \pi_1 E \to 1$$

on fundamental groups.

Proof. It is well-known that there is a unique complex structure on S_{g_i} with respect to which the map f_{g_i} is holomorphic, since f_{g_i} is a finite-sheeted branched covering map. In particular, f_{g_i} has critical points the finite preimage $C_i = f_{g_i}^{-1}(D_i)$, where $D_i \subset E$ is the finite set of branching points of f_{g_i}.

This equips $X = S_{g_1} \times \cdots \times S_{g_r}$ with a projective structure with respect to which $f : X = S_{g_1} \times \cdots \times S_{g_r} \to E$ is a holomorphic submersion. The set of singular points of f is then the set of points $(x_1, \cdots, x_r) \in S_{g_1} \times \cdots \times S_{g_r}$ such that $0 = \mathrm{d}f(x_1, \cdots, x_r) = (\mathrm{d}f_{g_1}(x_1), \cdots, \mathrm{d}f_{g_r}(x_r))$. It follows that the set of singular points of f is the finite set $C_1 \times \cdots \times C_r$. In particular, f has isolated singularities.

By assumption all of the f_{g_i} are branched covering maps and f is surjective on fundamental groups. Thus, by Theorem 4.1.1, the map f has connected fibres.

It follows that f is an irrational pencil with isolated singularities. Hence, by Theorem 4.3.1, we obtain that f induces a short exact sequence

$$1 \to \pi_1 H \to \pi_1 X \xrightarrow{f_*} \pi_1 E \to 1$$

on fundamental groups proving assertion (5). Furthermore, we obtain that $\pi_i H \cong \pi_i X \cong 0$, for $2 \le i \le r-2$, where the last equality follows since X is a $K(\pi_1 X, 1)$. This implies the first part of assertion (1).

The group $\pi_1 H$ is projective, since the generic smooth fibre H of f is a complex submanifold of the compact projective manifold X.

Because $\pi_1 H = 0$ for $2 \le i \le r-2$, we obtain a $K(\pi_1 H, 1)$ from H by attaching cells of dimension $\ge r$. Since H is a compact complex manifold, it follows that H has a finite cell structure. Thus, the group $\ker(f_*) = \pi_i H$ is of finiteness type $\mathcal{F}_{r-1}$ and, by Theorem 4.2.2, it is not of type $\mathcal{F}_r$. This implies assertion (3) and the second part of assertion (1), since if $\pi_{r-1} H$ was trivial we could construct a $K(G,1)$ with finite r-skeleton.

Assertion (4) is an immediate consequence of the well-known result that two groups which are commensurable (up to finite kernels) have the same finiteness properties (see [20]).

Assertion (1) follows similarly as in the proof of [62, Theorem A]. Namely, the universal covering $\widetilde{X}$, $q : \widetilde{X} \to X$ of $X = S_{g_1} \times \cdots \times S_{g_r}$ is a contractible Stein manifold and the pair (X, H) is $(r-1)$-connected by Theorem 4.3.1. Hence, the preimage $q^{-1}(H)$ of H in $\widetilde{X}$ is a closed complex submanifold of the Stein manifold $\widetilde{X}$ which is biholomorphic to the universal covering $\widetilde{H}$ of H. Thus, $\widetilde{H}$ is Stein. □

Note that Theorem 4.3.2 is a generalisation of Theorem 2.5.6. Biswas, Mj and Pancholi [24] suggested a more general approach for arbitrary irrational Lefschetz pencils over surfaces of positive genus with singularities of Morse type. Although it is not explicit in [24], the class of examples constructed in this chapter can also be obtained as a consequence of their work. However, the techniques are quite different: Their approach is based on topological Lefschetz fibrations and these, by definition, have Morse type singularities; the result for non-Morse type singularities then follows by a deformation argument. They do not present any techniques to distinguish between different examples (cf. Theorem 4.6.2 in Section 4.6).

4.4 Constructing Bestvina–Brady type examples

We will now explain how one can realise the maps $\psi_{g_1,\cdots,g_r}$ in (4.2) geometrically: we will exhibit them as the induced maps on fundamental groups of maps satisfying the conditions of Theorem 4.3.2.

We start by observing that we can retrieve Theorem 2.5.6 from Theorem 4.3.2, since by construction (see Section 2.5) the groups in Theorem 2.5.6 satisfy all the assumptions of Theorem 4.3.2.

We will imitate the construction of these groups in order to produce holomorphic maps $f_{h_1}, \cdots, f_{h_r} : S_{h_i} \to E$ for all $h_1, \cdots, h_r \geq 2$ and $r \geq 3$ such that $f = \sum_{i=1}^{r} f_{h_i}$ realises the map $\psi_{h_1, \cdots, h_r}$ defined in (4.2). We will present two different constructions, each of which has advantages.

Construction 1: This is the more natural construction. It has the advantage that the maps f_{h_i} are normal branched coverings, but it comes at the cost that the singularities of f are not quadratic and therefore f is not a Morse function.

As above, let E be an elliptic curve and let $B = \{d_1, d_2\} \subset E$ be two arbitrary points. Let $\alpha, \beta, \gamma_1, \gamma_2$ be the same set of generators for the homology group $H_1(E \setminus B, \mathbb{Z})$ as in the paragraph preceding Definition 4.1.2, α, β are generators of $\pi_1 E$ intersecting positively with respect to the orientation induced by the complex structure on E and γ_1, γ_2 are the positively oriented boundary loops of small discs around b_1, respectively b_2. For $h \geq 2$, the surjective homomorphism $H_1(E \setminus B, \mathbb{Z}) \to \mathbb{Z}/h\mathbb{Z}$ defined by $\alpha, \beta \mapsto 0$, $\gamma_1 \mapsto 1$, $\gamma_2 \mapsto -1$, defines a h-fold normal branched covering $f_h : S_h \to E$ from a topological surface of genus h with branching locus B. We may assume that, after connecting the generators of $H_1(E \setminus B, \mathbb{Z})$ to a base point, the fundamental group of $\pi_1(E \setminus B)$ is

$$\pi_1 E \setminus B = \langle \alpha, \beta, \gamma_1, \gamma_2 \mid [\alpha, \beta] \gamma_1 \gamma_2 \rangle$$

It is well-known that there is a unique complex structure on S_h such that the map f_h is holomorphic. Denote by $C = \left\{c_1 = f_h^{-1}(d_1), c_2 = f_h^{-1}(d_2)\right\} \subset S_h$ the set of critical points of f_h.

In analogy to the DPS groups, we define the map f using the additive structure on E,

$$f = \sum_{i=1}^{r} f_{h_i} : S_{h_1} \times \cdots \times S_{h_r} \to E$$

for all $r \geq 3$ and $h_1, \cdots, h_r \geq 2$.

The maps f_{h_i} are branched coverings induced by the surjective composition of homomorphisms $\pi_1(E \setminus B) \to H_1(E \setminus B, \mathbb{Z}) \to \mathbb{Z}/h_i\mathbb{Z}$ defined by $\alpha, \beta \mapsto 0$, $\gamma_1 \mapsto 1$, $\gamma_2 \mapsto -1$. In particular, all of the f_{h_i} are purely branched, since α and β are elements of the kernel of this homomorphism, which is a normal subgroup of $\pi_1(E \setminus B)$. It follows that all of the f_{h_i} are surjective on fundamental groups.

Hence, Theorem 4.3.2 can be applied to f. This implies that the fundamental group $\pi_1 H$ of the generic fibre H of f is a projective (and thus Kähler) group of finiteness type $\mathcal{F}_{r-1}$, but not of finiteness type $\mathcal{F}_r$.

Since all of the f_{h_i} are purely branched, any lift of a generator $\alpha, \beta : [0,1] \to E \setminus B_i$ of $\pi_1 E$ to S_{h_i} is a loop. Choose a fundamental domain $F \subset S_{h_i}$ for the $\mathbb{Z}/h_i\mathbb{Z}$-action such that the images of α and β are contained in the image $f_{h_i}(U)$ of an open subset $U \subset F$ on which f_{h_i} restricts to a homeomorphism.

Denote the h_i lifts of α by $a_1^{(i)}, \cdots, a_{h_i}^{(i)}$ and the h_i lifts of β by $b_1^{(i)}, \cdots b_{h_i}^{(i)}$, where we choose lifts so that $a_j^{(i)}$ and $b_j^{(i)}$ are in the interior of the fundamental domain $\overline{j} \cdot F$ for $\overline{j} \in \mathbb{Z}/h_i\mathbb{Z}$. In particular, the loops $a_1^{(i)}, b_1^{(i)}, \cdots, a_{h_i}^{(i)}, b_{h_i}^{(i)}$ form a standard symplectic basis for the (symplectic) intersection form on $H_1(S_{h_i}, \mathbb{Z})$.

It is then well-known that we can find generators $\alpha_1^{(i)}, \beta_1^{(i)}, \cdots \alpha_{h_i}^{(i)}, \beta_{h_i}^{(i)}$ of $\pi_1 S_{h_i}$ such that the abelianisation is given by $\alpha_j^{(i)} \mapsto a_j^{(i)}$, $\beta_j^{(i)} \mapsto b_j^{(i)}$ and

$$\pi_1 S_{h_i} = \left\langle \alpha_1^{(i)}, \beta_1^{(i)}, \cdots \alpha_{h_i}^{(i)}, \beta_{h_i}^{(i)} \mid \left[\alpha_1^{(i)}, \beta_1^{(i)}\right] \cdots \left[\alpha_{h_i}^{(i)}, \beta_{h_i}^{(i)}\right] \right\rangle.$$

This is for instance an easy consequence of Theorem 4.5.5.

With respect to this presentation, the map on fundamental groups induced by f_{h_i} is given by

$$\begin{array}{rccc} f_{h_i *}: & \pi_1 S_{h_i} & \to & \pi_1 E \\ & \alpha_j^{(i)} & \mapsto & \alpha \\ & \beta_j^{(i)} & \mapsto & \beta \end{array}$$

For an illustration of the map f_h and the generators α_j, β_j, see Figure 4.1.

As a direct consequence, we obtain that f induces the map

$$\begin{array}{rccc} f_*: & \pi_1 S_{h_1} \times \cdots \times \pi_1 S_{h_r} & \to & \pi_1 E \\ & \alpha_j^{(i)} & \mapsto & \alpha \\ & \beta_j^{(i)} & \mapsto & \beta \end{array}$$

on fundamental groups. Thus, the induced map g_* on fundamental groups is indeed the map $\psi_{h_1, \cdots, h_r}$ in (4.2).

Construction 2: We will now give an alternative construction which realises $\psi_{h_1, \cdots, h_r}$ as the fundamental group of the generic fibre of a fibration over an elliptic curve with Morse type singularities only. This is at the expense of the maps f_{h_i} being regular branched coverings rather than normal branched coverings.

Let E be an elliptic curve, let $h \geq 2$, let $d_{1,1}, d_{1,2}, d_{2,1}, \cdots, d_{h-1,1}, d_{h-1,2}$ be $2(h-1)$ points in E and let $s_1, \cdots, s_{h-1} : [0,1] \to E$ be simple, pairwise non-intersecting paths

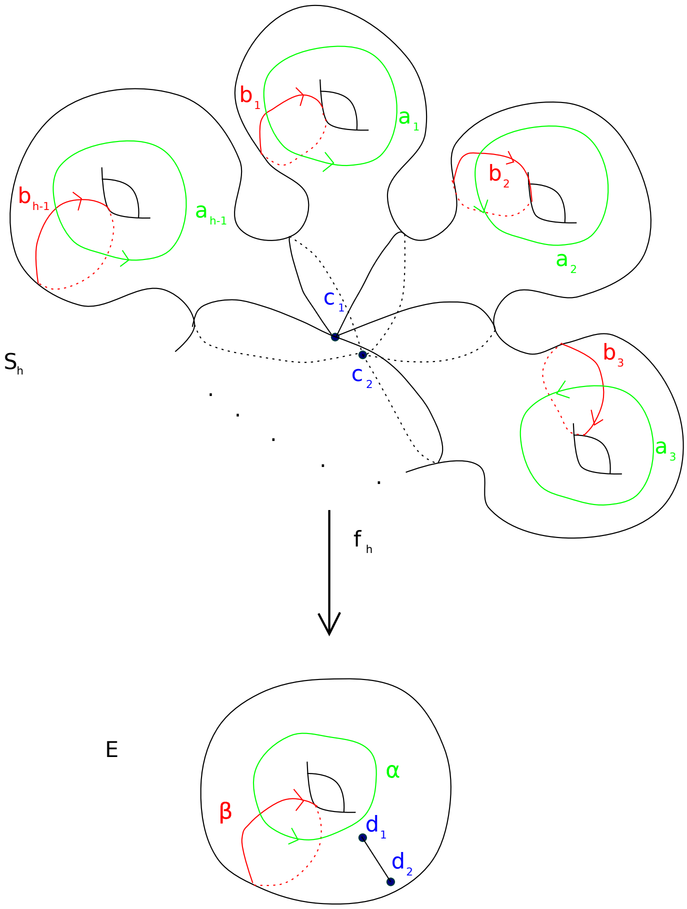

Figure 4.1: The h-fold branched normal covering f_h of E in Construction 1

with starting point $s_i(0) = d_{i,1}$ and endpoint $s_i(1) = d_{i,2}$ for $i = 1, \cdots, h-1$. Take g copies E_0, $E_1, \cdots, E_{h-1}$ of E, cut E_0 open along all of the paths s_i and cut E_i open along the path s_i for $1 \le i \le h-1$. This produces surfaces $F_0, \cdots, F_{h-1}$ with boundary.

Glue the boundary of F_i to the boundary component of F_0 corresponding to the cut produced by the path s_i where we identify opposite edges with respect to the identity homeomorphism $E_0 \to E_i$ for $i = 1, \cdots, h-1$. This yields a closed genus h surface R_h together with a $g-1$-fold branched covering map $f'_h : R_h \to E$ with critical points $c_{1,1}, c_{1,2}, c_{2,1}, \cdots, c_{h-1,1}, c_{h-1,2} \in R_h$, $f(c_{i,j}) = d_{i,j}$ of order two. Endow R_h with the unique complex structure that makes the map f'_h holomorphic.

It is clear that the map f'_h is purely branched and surjective on fundamental groups. In analogy to Construction 1 we find a standard generating sets α, β of $\pi_1 E$ and $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ of $\pi_1 S_g$ with respect to which the induced map on fundamental groups is given by $\alpha_i \mapsto \alpha$ and $\beta_i \mapsto \beta$. See Figure 4.2 for an illustration of the map f'_h.

For $h_1, \cdots, h_r \ge 2$, $r \ge 3$, the holomorphic map $f' = \sum_{i=1}^{r} f'_{h_i} : R_{h_1} \times \cdots \times R_{h_r}$ induces

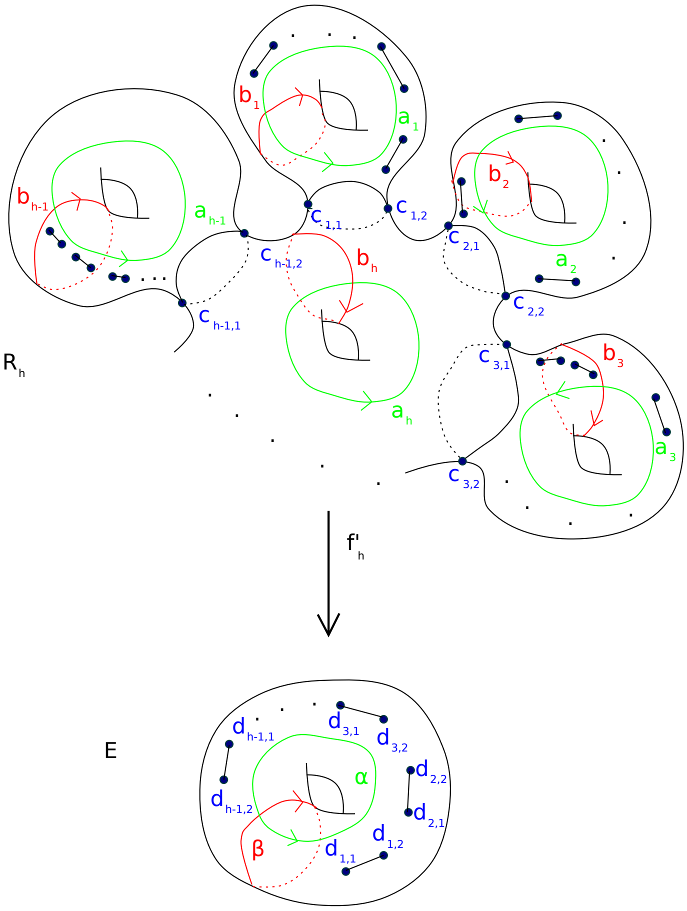

Figure 4.2: The h-fold branched covering f'_h of E with Morse type singularities in Construction 2

the map $\psi_{h_1,\cdots,h_r}$ on fundamental groups. The map f' has isolated singularities and connected fibres and in fact we can see, by considering local coordinates around the singular points, that all singularities of f' are of Morse type.

4.5 Reducing the isomorphism type of our examples to Linear Algebra

We will now show that our groups are not isomorphic to the DPS groups and thus provide genuinely new examples rather than being their examples in disguised form.

As before, let Λ_g be the fundamental group of a closed orientable surface of genus g. For $r, k \geq 1$, consider epimorphisms $\phi_{g_i} : \Lambda_{g_i} \to \mathbb{Z}^k$ and $\psi_{h_i} : \Lambda_{h_i} \to \mathbb{Z}^k$, where $g_i, h_i \geq 2$ and $1 \leq i \leq r$. Recall that $\ker(\phi_{g_i})$ and $\ker(\psi_{h_i})$ are infinitely generated free groups, since they are infinite index normal subgroups of surface groups.

Define maps $\phi_{g_1,\cdots,g_r} = \phi_{g_1} + \cdots + \phi_{g_r} : \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} \to \mathbb{Z}^k$ and $\psi_{h_1,\cdots,h_r} = \psi_{h_1} + \cdots + \psi_{h_r}$: $\Lambda_{h_1} \times \cdots \times \Lambda_{h_r} \to \mathbb{Z}^k$ and let

$$L_i := \Lambda_{g_i} \cap \ker(\phi_{g_1,\cdots,g_r}) = \ker(\phi_{g_i}),$$

$$K_i := \Lambda_{h_i} \cap \ker(\psi_{h_1,\cdots,h_r}) = \ker(\psi_{h_i}).$$

Then the following Lemma is a special case of Bridson, Howie, Miller and Short's Theorem 2.1.5.

Lemma 4.5.1. *Every isomorphism*

$$\theta : \ker(\phi_{g_1,\cdots,g_r}) \to \ker(\psi_{h_1,\cdots,h_r})$$

satisfies $\theta(L_i) = K_i$ *up to reordering of the factors. In particular,* θ *restricts to an isomorphism* $L_1 \times \cdots \times L_r \cong K_1 \times \cdots \times K_r$.

Furthermore, with the same reordering of factors, we have $\theta((\Lambda_{g_{i_1}} \times \cdots \Lambda_{g_{i_k}}) \cap \ker(\phi_{g_1,\cdots,g_r})) = (\Lambda_{h_{i_1}} \times \cdots \Lambda_{h_{i_k}}) \cap \ker(\psi_{h_1,\cdots,h_r})$ *for* $1 \le k \le r$ *and* $1 \le i_1 < \cdots < i_k \le r$.

Remark 4.5.2. It also follows from Theorem 2.1.5 that if we have an isomorphism between direct products of r surface groups $\Lambda_{g_1} \times \cdots \times \Lambda_{g_r}$ and $\Lambda_{h_1} \times \cdots \times \Lambda_{h_r}$, then after reordering of the factors it is induced by isomorphisms $\Lambda_{g_i} \cong \Lambda_{h_i}$ and, in particular, $g_i = h_i$ for $i = 1, \cdots, r$.

Theorem 4.5.3. *There is an isomorphism of the short exact sequences*

$$1 \to \ker(\phi_{g_1,\cdots,g_r}) \to \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} \xrightarrow{\phi_{g_1,\cdots,g_r}} \mathbb{Z}^k \to 1$$

and

$$1 \to \ker(\psi_{h_1,\cdots,h_r}) \to \Lambda_{h_1} \times \cdots \times \Lambda_{h_r} \xrightarrow{\psi_{h_1,\cdots,h_r}} \mathbb{Z}^k \to 1$$

if and only if (up to reordering factors) $g_1 = h_1, \cdots, g_r = h_r$ *and there are isomorphisms of the short exact sequences*

$$1 \to \ker(f_{g_i}) \to \Lambda_{g_i} \xrightarrow{\phi_{g_i}} \mathbb{Z}^k \to 1$$

and

$$1 \to \ker(g_{h_i}) \to \Lambda_{h_i} \xrightarrow{\psi_{h_i}} \mathbb{Z}^k \to 1$$

for all $i = 1, \cdots, r$ *such that the isomorphism* $A : \mathbb{Z}^k \to \mathbb{Z}^k$ *is independent of* i.

Proof. The if direction follows immediately by taking the Cartesian product of the isomorphisms $\theta_i : \Lambda_{g_i} \to \Lambda_{h_i}$ to be the isomorphism $\theta : \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} \to \Lambda_{h_1} \times \cdots \times \Lambda_{h_r}$ and the identity map to be the automorphism of $\mathbb{Z}^k$.

For the only if direction we use that, by Remark 4.5.2, the isomorphism $\Lambda_{g_1} \times \cdots \Lambda_{g_r} \to \Lambda_{h_1} \times \cdots \times \Lambda_{h_r}$ is realised by a direct product of isomorphisms $\theta_i : \Lambda_{g_i} \to \Lambda_{h_i}$, after possibly reordering factors. Restricting to factors then implies that the isomorphism θ_i and the identity on $\mathbb{Z}^k$ induce an isomorphism of short exact sequences for $i = 1, \cdots, r$. $\square$

The following theorem is well-known (see for instance [31, Section 7.5]):

Theorem 4.5.4. *The groups $H_1 = \ker(\phi_{g_1,\cdots,g_r})$ and $H_2 = \ker(\psi_{h_1,\cdots,h_s})$ are isomorphic if and only if there is an isomorphism of the short exact sequences*

$$1 \to \ker(\phi_{g_1,\cdots,g_r}) \to \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} \xrightarrow{\phi_{g_1,\cdots,g_r}} \mathbb{Z}^k \to 1$$

and

$$1 \to \ker(\psi_{h_1,\cdots,h_s}) \to \Lambda_{h_1} \times \cdots \times \Lambda_{h_s} \xrightarrow{\psi_{h_1,\cdots,h_s}} \mathbb{Z}^k \to 1.$$

Proof. Let $\theta : H_1 \to H_2$ be an abstract isomorphism of groups and let $L_i \le H_1$, $K_i \le H_2$ be as above. By Lemma 4.5.1, we may assume that after reordering factors $\theta(L_i) \le K_i$ and that $\theta(M_1) = M_2$ for $M_1 = H_1 \cap (1 \times \Lambda_{g_2} \times \cdots \times \Lambda_{g_r})$, $M_2 = H_2 \cap (1 \times \Lambda_{h_2} \times \cdots \times \Lambda_{h_s})$.

Since f_{g_i}, g_{h_j} are surjective for all $1 \le i \le r$, $1 \le j \le s$, we obtain that

$$\begin{aligned} H_1/M_1 &\cong \Lambda_{g_1}, \quad & H_1/L_1 &\cong \Lambda_{g_2} \times \cdots \times \Lambda_{g_r} \\ H_2/M_2 &\cong \Lambda_{h_1}, \quad & H_2/K_1 &\cong \Lambda_{h_2} \times \cdots \times \Lambda_{h_s}, \end{aligned}$$

where the isomorphisms are induced by the projection maps.

In particular, the map θ induces an isomorphism of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & K_1 \arrow[r] \arrow[d, "\theta"', "\cong"] & H_1/M_1 \times H_1/L_1 \cong \Lambda_{g_1} \times \cdots \times \Lambda_{g_r} \arrow[r] \arrow[d, "\theta"', "\cong"] & \mathbb{Z}^k \arrow[r] \arrow[d, "\overline{\theta}"', "\cong"] & 1 \\ 1 \arrow[r] & K_2 \arrow[r] & H_2/M_2 \times H_2/K_1 \cong \Lambda_{h_1} \times \cdots \times \Lambda_{h_s} \arrow[r] & \mathbb{Z}^k \arrow[r] & 1 \end{tikzcd}$$

proving the only if direction. The if direction is trivial. $\square$

It follows that in order to understand abstract isomorphisms of the kernels of maps of the form $\phi_{g_1,\cdots,g_r}$ and $\psi_{h_1,\cdots,h_r}$, it suffices to understand isomorphisms of short exact sequences of the form

$$1 \to N \to \Lambda_g \xrightarrow{\nu_g} \mathbb{Z}^k \to 1.$$

This reduces to Linear Algebra, as we now explain briefly. A detailed exposition of the subject can be found in [68] (see in particular Chapter 6). In the following, let S_g be a closed surface of genus g and let $\Lambda_g = \pi_1(S_g)$ be its fundamental group.

Let $MCG^{\pm}(S_g)$ be the *(extended) mapping class group* of S_g, that is, the group of homeomorphisms of S_g up to homotopy equivalences, where by the extended mapping class group we mean that we allow orientation reversing homeomorphisms. Let $Inn(\Lambda_g)$ be the group of *inner automorphisms* of Λ_g, that is, automorphisms of the form $a \mapsto b^{-1}ab$ for a fixed $b \in \Lambda_g$ and let $Out(\Lambda_g) = Aut(\Lambda_g)/Inn(\Lambda_g)$ be the group of *outer automorphisms* of Λ_g.

The map $\Lambda_g \to \mathbb{Z}^k$ factors through the abelianisation $H_1(\Lambda_g, \mathbb{Z})$ of Λ_g. Every automorphism τ of Λ_g induces an automorphism $\tau_* \in GL(2g, \mathbb{Z})$ of the abelianisation $H_1(\Lambda_g, \mathbb{Z})$. Since inner automorphisms act trivially on $H_1(\Lambda_g, \mathbb{Z})$, the induced homomorphism $Aut(\Lambda_g) \to GL(2g, \mathbb{Z})$ factors through $Out(\Lambda_g)$.

By the Dehn-Nielsen-Baer Theorem (cf. [68, Theorem 8.1]) the natural map $MCG^{\pm}(S_g) \to Out(\Lambda_g)$ is an isomorphism. In particular, we can realise any element of $Out(\Lambda_g)$ by a (up to homotopy) unique homeomorphism of S_g. It follows that for $\tau \in Aut(\Lambda_g)$, the map $\tau_* \in GL(2g, \mathbb{Z})$ is realised by some homeomorphism $\alpha \in MCG^{\pm}(S_g)$.

There is a natural symplectic form on $H_1(\Lambda_g, \mathbb{Z})$ induced by taking intersection numbers of representatives in S_g. Orientation preserving homeomorphism $\alpha \in MCG^{+}(S_g)$ of S_g preserve intersection numbers. Hence, the induced automorphism $\alpha_* \in H_1(\Lambda_g, \mathbb{Z})$ preserves the symplectic form which is equivalent to $A := \alpha_*$ of $Sp(2g, \mathbb{Z})$, where $Sp(2g, \mathbb{Z})$ is the group of *symplectic matrices* of dimension $2g$ with integer coefficients. It is defined by

$$Sp(2g, \mathbb{Z}) = \{A \in M_{2g}(\mathbb{Z}) \mid A^t J_{2g} A = J_{2g}\},$$

for J_{2g} the matrix representing the standard symplectic form given by the block diagonal matrix

$$J_{2g} = \begin{pmatrix} J & 0 & \cdots & & 0 \\ 0 & J & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & J \end{pmatrix}$$

with $J_2 = J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the standard symplectic form on $\mathbb{R}^2$.

For an orientation reversing homeomorphism we have that $A^tJA = -J$. We define the generalised symplectic group of dimension $2g$ with integer coefficients by

$$Sp^{\pm}(2g,\mathbb{Z}) = \{A \in M_{2g}(\mathbb{Z}) \mid A^tJA = J \text{ or } A^tJA = -J\}.$$

As a result, there is a natural homomorphism $\Psi : MCG^{\pm}(S_g) \to Sp^{\pm}(2g,\mathbb{Z})$.

Theorem 4.5.5 ([68, Theorem 6.4])**.** *The symplectic representation*

$$\Psi : MCG^{\pm}(S_g) \to Sp^{\pm}(2g,\mathbb{Z})$$

is surjective for $g \geq 1$*.*

Combining this with the isomorphism $MCG^{\pm}(S_g) \to Out(S_g)$, we obtain

Corollary 4.5.6. *For* $g \geq 1$*, the symplectic representation* Ψ *induces a surjective representation* $Out(S_g) \to Sp^{\pm}(2g,\mathbb{Z})$*.*

In particular, two short exact sequences

$$1 \to \ker(f) \to \Lambda_g \xrightarrow{\phi} \mathbb{Z}^k \to 1$$

and

$$1 \to \ker(h) \to \Lambda_g \xrightarrow{\psi} \mathbb{Z}^k \to 1$$

are isomorphic if and only if there exists $A \in Sp^{\pm}(2g,\mathbb{Z})$ *and* $A' \in GL(k,\mathbb{Z})$ *such that the following diagram commutes*

$$\begin{tikzcd} H_1(\Lambda_g,\mathbb{Z}) \arrow[r, "\phi_{ab}"] \arrow[d, "A"] & \mathbb{Z}^k \arrow[d, "A'"] \\ H_1(\Lambda_g,\mathbb{Z}) \arrow[r, "\psi_{ab}"] & \mathbb{Z}^k \end{tikzcd}$$

where ϕ_{ab} *and* ψ_{ab} *are the unique maps factoring* ϕ *and* ψ *through their abelianisation.*

Proof. The first part is a direct consequence of Theorem 4.5.5 and the isomorphism $MCG^{\pm}(S_g) \cong Out(\Lambda_g)$.

The only if in the second part follows directly from the fact that for any outer automorphism of Λ_g, the induced map on homology is in the generalised symplectic group. The if follows, since we can lift any $A \in Sp(2g,\mathbb{Z})$ to an element $\alpha \in Aut(\Lambda_g)$ inducing A by the first part. $\square$

4.6 Classification for purely branched maps

It follows from Section 4.5 that showing our groups are not isomorphic to the DPS groups amounts to comparing maps on abelianisations and therefore reduces to a question in Linear Algebra. In fact, it is possible to classify all maps on fundamental groups that arise from purely branched maps.

Theorem 4.6.1. *Let E be an elliptic curve and let $f_g : S_g \to E$, $g \geq 2$, be a purely branched covering map. Then the following are equivalent:*

1. *There is a standard symplectic generating set $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ of $\pi_1 S_g$ and α, β of $\pi_1 E$ with respect to which the induced map on fundamental groups is*

$$\begin{array}{rccc} f_{g*}: & \pi_1 S_g & \to & \pi_1 E \\ & \alpha_1, \cdots, \alpha_k & \mapsto & \alpha \\ & \beta_1, \cdots, \beta_k & \mapsto & \beta \\ & \alpha_{k+1}, \cdots, \beta_g & \mapsto & 0 \\ & \beta_{k+1}, \cdots, \beta_g & \mapsto & 0 \end{array}$$

2. *The map f_g is a k-fold purely branched covering map for $2 \leq k \leq g$.*

Furthermore, for $2 \leq k \leq g$, there exists a purely branched k-fold holomorphic covering map $f_g : S_g \to E$ with Morse type singularities.

As a consequence we can give a complete classification of all Kähler groups with arbitrary finiteness properties that arise from our construction in the case when all of the maps are purely branched.

Theorem 4.6.2. *Let E be an elliptic curve. Let $r, s \geq 3$, let $g_i, h_j \geq 2$, let S_{g_i} be a closed Riemann surface of genus $g_i \geq 2$ and let R_{h_i} be a closed Riemann surface of genus $h_i \geq 2$ for $1 \leq i \leq r$ and $1 \leq j \leq s$. Assume that there are purely branched k_i-fold holomorphic covering maps $p_i : S_{g_i} \to E$ and purely branched l_i-fold holomorphic covering maps $q_i : R_{h_i} \to E$. Define $p = \sum_{i=1}^{r} p_i : S_{g_1} \times \cdots \times S_{g_r} \to E$ and $q = \sum_{j=1}^{s} q_j : R_{h_1} \times \cdots \times R_{h_s} \to E$ and denote by H_p and H_q the smooth generic fibres of p, respectively q.*

Then the Kähler groups $\pi_1 H_p$ and $\pi_1 H_q$ are isomorphic if and only if $r = s$ and there is a permutation of the R_{h_i} and q_i such that $g_i = h_i$ and $k_i = l_i$ for $i = 1, \cdots, r$.

Proof. This is a direct consequence of Theorem 4.6.1, Theorem 4.5.4 and Theorem 4.3.2. ∎

Corollary 4.6.3. *The Kähler groups* $\ker(\phi_{g_1,\cdots,g_r})$ *obtained from* (4.1) *and the Kähler groups* $\ker(\psi_{h_1,\cdots,h_r})$ *obtained from* (4.2) *are isomorphic if and only if* $r = s$ *and* $g_1 = \cdots = g_r = h_1 = \cdots = h_r = 2$.

Proof. This is an immediate consequence of Theorem 4.6.2 and the fact that we constructed our groups as fundamental groups of the fibre of a sum of h_i-fold purely branched holomorphic maps in Section 4.4, while, as we also saw in Section 4.4, the DPS groups arise as fundamental groups of the fibre of a sum of 2-fold purely branched holomorphic maps. $\square$

The part of Theorem 4.6.1 that for $2 \leq k < l \leq g$ we obtain distinct maps on fundamental groups will follow from Section 4.5 and the following result in Linear Algebra.

Proposition 4.6.4. *For* $g \geq 2$ *and* $1 \leq k < l \leq g$ *there are no linear maps* $A \in Sp^{\pm}(2g,\mathbb{R})$ *and* $B \in Gl(2,\mathbb{Z}) = Sp^{\pm}(2,\mathbb{Z})$ *such that*

$$(\underbrace{I\cdots I}_{k\ times}\ \underbrace{0\cdots 0}_{g\text{-}k\ times}) \cdot A = B \cdot (\underbrace{I\cdots I}_{l\ times}\ \underbrace{0\cdots 0}_{g\text{-}l\ times}), \tag{4.3}$$

where $I = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *is the 2-dimensional identity matrix.*

Proof. The proof is by contradiction. Assume that there is $A \in Sp^{\pm}(2g,\mathbb{R})$ and $B \in Gl(2,\mathbb{Z}) = Sp^{\pm}(2,\mathbb{Z})$ satisfying Equation (4.3). Define $\gamma_1,\cdots,\gamma_{2g} \in \mathbb{R}^{2k}$ and $\alpha_1,\cdots,\alpha_{2g} \in \mathbb{R}^{2g-2k}$ by

$$A = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{2g} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{2g} \end{pmatrix}.$$

Then $A \in Sp^{\pm}(2g,\mathbb{R})$ implies that

$$\begin{aligned} \pm J_{2g} = A^t J A &= \begin{pmatrix} \gamma_1^t & \alpha_1^t \\ \vdots & \vdots \\ \gamma_{2g}^t & \alpha_{2g}^t \end{pmatrix} \cdot \begin{pmatrix} J_{2k} & 0 \\ 0 & J_{2g-2k} \end{pmatrix} \cdot \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{2g} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{2g} \end{pmatrix} \\ &= \begin{pmatrix} \gamma_1^t \\ \vdots \\ \gamma_{2g}^t \end{pmatrix} \cdot J_{2k} \cdot (\gamma_1 \cdots \gamma_{2g}) + \begin{pmatrix} \alpha_1^t \\ \vdots \\ \alpha_{2g}^t \end{pmatrix} \cdot J_{2g-2k} \cdot (\alpha_1 \cdots \alpha_{2g}) \\ &= \underbrace{[\gamma_i^t \cdot J_{2k} \cdot \gamma_j]_{i,j=1,\cdots,2g}}_{=:E} + \underbrace{[\alpha_i^t \cdot J_{2g-2k} \cdot \alpha_j]_{i,j=1,\cdots,2g}}_{=:F} \end{aligned}$$

The map E is of rank $\leq 2k$, since it splits through $\mathbb{R}^{2k}$ and the map F is of rank $\leq 2g - 2k$, since it splits through $\mathbb{R}^{2g-2k}$.

We will now prove that in fact E is of rank $\leq 2k-2$. Equation (4.3) implies that for

$$(\gamma_1 \cdots \gamma_{2g}) = \begin{pmatrix} A_{11} & \cdots & A_{1g} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kg} \end{pmatrix},$$

where $A_{ij} \in \mathbb{R}^{2\times 2}$ for $1 \leq i \leq k$ and $1 \leq j \leq g$, we have

$$(\underbrace{I \cdots I}_{\text{k times}}) \cdot \begin{pmatrix} A_{11} & \cdots & A_{1g} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kg} \end{pmatrix} = (\underbrace{B \cdots B}_{\text{l times}}\ \underbrace{0 \cdots 0}_{\text{g-l times}}).$$

It follows that

$$\sum_{i=1}^{k} A_{ij} = \begin{cases} B & \text{if } j \leq l \\ 0 & \text{if } j > l \end{cases}$$

and hence, that

$$(\gamma_1 \cdots \gamma_{2g}) = \begin{pmatrix} A_{11} & \cdots & A_{1l} & A_{1(l+1)} & \cdots & A_{1g} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{(k-1)1} & \cdots & A_{(k-1)l} & A_{(k-1)(l+1)} & \cdots & A_{(k-1)g} \\ B - \sum_{i=1}^{k-1} A_{i1} & \cdots & B - \sum_{i=1}^{k-1} A_{il} & -\sum_{i=1}^{k-1} A_{i(l+1)} & \cdots & -\sum_{i=1}^{k-1} A_{ig} \end{pmatrix} =: (MN)$$

with $M \in \mathbb{R}^{2k\times 2l}$ and $N \in \mathbb{R}^{2k\times 2(g-l)}$.

Then, we have

$$E = \begin{pmatrix} M^t J_{2k} M & M^t J_{2k} N \\ N^t J_{2k} M & N^t J_{2k} N \end{pmatrix}.$$

Define linear maps

$$M' = \left[\det B \cdot \left(B^{-1} A_{ij} - \frac{1}{\sqrt{k}-1} \sum_{l=1}^{k-1} B^{-1} A_{lj} + \frac{I_2}{\sqrt{k}} \right) \right]_{i=1,\cdots,k-1, j=1,\cdots,l} \in \mathbb{R}^{2(k-1)\times 2l},$$

$$N' = \left[\det \mathrm{B} \cdot \left(B^{-1} A_{im} - \frac{1}{\sqrt{k}-1} \sum_{r=1}^{k-1} B^{-1} A_{rm} \right) \right]_{i=1,\cdots,k-1, m=l+1,\cdots,g} \in \mathbb{R}^{2(k-1)\times 2(g-l)}.$$

Using that $B^t J_2 B = \det B \cdot J_2$, we obtain the following identities

$$M^t J_{2k} M = \frac{\det B}{k} \cdot \begin{pmatrix} J & \cdots & J \\ \vdots & \ddots & \vdots \\ J & \cdots & J \end{pmatrix} + M'^t J_{2k-2} M'$$

$$M^t J_{2k} N = M'^t J_{2k-2} N'$$

$$N^t J_{2k} M = N'^t J_{2k-2} M'$$

$$N^t J_{2k} N = N'^t J_{2k-2} N'$$

In particular, this implies that

$$E = \left[\gamma_i^t J_{2k} \gamma_j\right]_{i,j=1,\cdots,2g} = \begin{pmatrix} \frac{\det B}{k} J & \cdots & \frac{\det B}{k} J & \\ \vdots & \ddots & \vdots & 0 \\ \frac{\det B}{k} J & \cdots & \frac{\det B}{k} J & \\ & 0 & & 0 \end{pmatrix} + \underbrace{\begin{pmatrix} M'^t \\ N'^t \end{pmatrix} \cdot J_{2k-2} \cdot (M'N')}_{=:S}.$$

The linear map S splits through $\mathbb{R}^{2k-2}$, implying that $\operatorname{rank}(S) \leq 2k-2$. Furthermore, $\pm J_{2g} = E + F$ and $\det B = \pm 1$ imply that

$$F = \pm J_{2g} - E = \underbrace{\begin{pmatrix} (\pm 1 \pm \frac{1}{k})J & \pm\frac{1}{k}J & \cdots & \cdots & \pm\frac{1}{k}J & \\ \pm\frac{1}{k}J & \ddots & \ddots & & \vdots & \\ \vdots & \ddots & \ddots & \ddots & \vdots & 0 \\ \vdots & & \ddots & \ddots & \pm\frac{1}{k}J & \\ \pm\frac{1}{k}J & \cdots & \cdots & \pm\frac{1}{k}J & (\pm 1 \pm \frac{1}{k})J & \\ & & 0 & & & \pm J_{2(g-l)} \end{pmatrix}}_{=:R} - S. \tag{4.4}$$

Hence, F has rank $\geq 2g - (2k-2) = 2(g-k) + 2$, since by Lemma 4.6.5 below R is invertible for $l \neq k$. This contradicts $\operatorname{rank} F \leq 2(g-k)$, showing that there are no $A \in Sp^{\pm}(2g, \mathbb{R})$ and $B \in Gl(2, \mathbb{Z})$ satisfying (4.3). □

Lemma 4.6.5. *For $l \in \mathbb{Z}$, $k \in \mathbb{R}$, the linear map R defined in* (4.4) *is invertible if and only if $l \neq \pm k$.*

Proof. Clearly, it suffices to prove that for $k \neq l$, the matrix

$$R_{2l} = \begin{pmatrix} (\pm 1 \pm \frac{1}{k})J & \pm\frac{1}{k}J & \cdots & \cdots & \pm\frac{1}{k}J \\ \pm\frac{1}{k}J & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \pm\frac{1}{k}J \\ \pm\frac{1}{k}J & \cdots & \cdots & \pm\frac{1}{k}J & (\pm 1 \pm \frac{1}{k})J \end{pmatrix} \in \mathbb{R}^{2l \times 2l}$$

is invertible. We do row an column operations in order to compute the rank of R_{2l}.

Subtracting the last (double) row from all other rows yields

$$\begin{pmatrix} \pm J & 0 & \cdots & 0 & \mp J \\ 0 & \pm J & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \pm J & \mp J \\ \pm\frac{1}{k}J & \cdots & \cdots & \pm\frac{1}{k}J & (\pm 1 \pm \frac{1}{k})J \end{pmatrix}.$$

After subtracting multiples of the first $(l-1)$ (double) rows from the last row we obtain

$$\begin{pmatrix} \pm J & 0 & \cdots & 0 & \mp J \\ 0 & \pm J & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \pm J & \mp J \\ 0 & \cdots & \cdots & 0 & (\pm 1 \pm (\frac{1}{k} + \frac{l-1}{k}))J \end{pmatrix}.$$

Hence, R is invertible if and only if $\pm 1 \pm \frac{l}{k} \neq 0$. This is clearly the case for all choices of signs if and only if $l \neq \pm k$, completing the proof. □

For the other direction in Theorem 4.6.1 we will make use of

Lemma 4.6.6. *Let E be an elliptic curve and let $f : S_g \to E$ be a holomorphic k-fold purely branched covering map. Then there exist standard generating sets α_1, β_1, $\cdots$, α_g, β_g of $\pi_1 S_g$ and α, β of $\pi_1 E$ such that the induced map on fundamental groups is of the form described in Theorem 4.6.1(1).*

Proof. Let $B \subset E$ be the finite branching set of f. Since f is purely branched there are generators $\alpha, \beta : [0,1] \to E \smallsetminus B$ such that every lift of α and β with respect to the unramified covering $f : S_g \smallsetminus f^{-1}(B) \to E \smallsetminus B$ is a loop.

We may further assume that the only intersection point of α and β is the point $\alpha(0) = \beta(0)$ and that the intersection number $\iota(\alpha, \beta) = 1$ with respect to the orientation induced by the complex structure on E.

Since $f|_{S_g \smallsetminus f^{-1}(B)}$ is a k-fold unramified covering map, there are precisely k lifts $\alpha_1, \cdots, \alpha_k$ of α and $\beta_1, \cdots, \beta_k$ of β and we may choose them so that $\iota(\alpha_i, \beta_j) = \delta_{ij}$, $\iota(\alpha_i, \alpha_j) = 0$ and $\iota(\beta_i, \beta_j) = 0$.

Then the images of $\alpha_1, \beta_1, \cdots, \alpha_k, \beta_k$ in $H_1(S_g, \mathbb{Z})$ form part of a standard symplectic basis with respect to the symplectic intersection form on $H_1(S_g, \mathbb{Z})$. Extend by $\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g$ to a standard symplectic basis of $H_1(S_g, \mathbb{Z})$. We claim that with respect to this basis f_* takes the desired form.

We may assume that $\alpha_{k+1}, \beta_{k+1}, \cdots, \alpha_g, \beta_g$ are loops in $S_g \smallsetminus f^{-1}(B)$. Since α_j, $k+1, \leq j \leq g$ forms part of a symplectic basis we have that its intersection number with any of the α_i, β_i with $1 \leq i \leq k$ is zero. Since all of the lifts of α, β are given by $\alpha_1, \beta_1, \cdots, \alpha_k, \beta_k$ and the map f is holomorphic, thus orientation preserving, it follows that the intersection numbers $\iota(f \circ \alpha_j, \alpha)$ and $\iota(f \circ \alpha_j, \beta)$ satisfy

$$\iota(f \circ \alpha_j, \alpha) = \sum_{i=1}^{k} \iota(\alpha_j, \alpha_i) = 0,$$

$$\iota(f \circ \alpha_j, \beta) = \sum_{i=1}^{k} \iota(\alpha_j, \beta_i) = 0.$$

The nondegeneracy of the symplectic intersection form on $H_1(E, \mathbb{Z}) = \langle \alpha, \beta \mid [\alpha, \beta] \rangle = \pi_1 E$ then implies that $f \circ \alpha_j = 0$ in $\pi_1 E$. Similarly $f \circ \beta_j = 0$ in $\pi_1 E$ for $k+1 \leq j \leq g$.

Since by definition $f \circ \alpha_i = \alpha$ and $f \circ \beta_i = \beta$ for $1 \leq i \leq k$, it follows that with respect to the standard generating sets $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ of $\pi_1 S_g$ and α, β of $\pi_1 E$, the induced map on fundamental groups is indeed

$$\begin{array}{cccc} f_*: & \pi_1 S_g & \to & \pi_1 E \\ & \alpha_1, \cdots, \alpha_k & \mapsto & \alpha \\ & \beta_1, \cdots, \beta_k & \mapsto & \beta \\ & \alpha_{k+1}, \cdots, \beta_g & \mapsto & 0 \\ & \beta_{k+1}, \cdots, \beta_g & \mapsto & 0. \end{array}$$

□

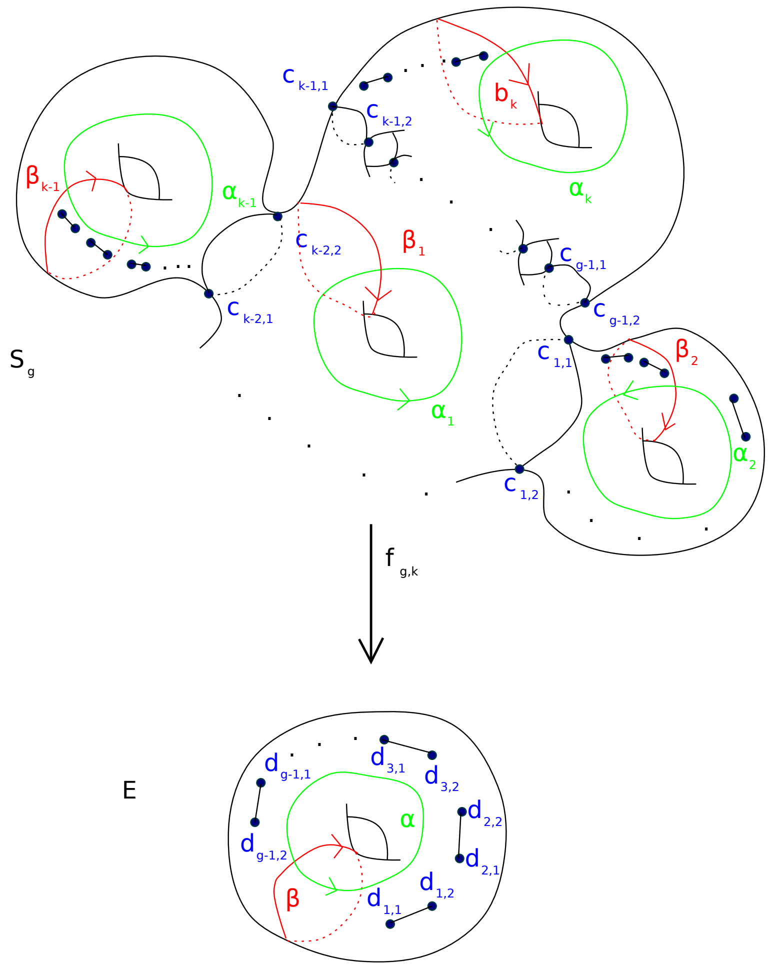

Figure 4.3: The k-fold purely branched covering map $f_{g,k}$ of E in Lemma 4.6.7

Finally, we prove the existence of a k-fold purely branched holomorphic covering map with Morse type singularities for every $g \geq 2$ and $2 \leq k \leq g$ by giving an explicit construction in analogy to Construction 2 in Section 4.4.

Proposition 4.6.7. *For every $g \geq 2$ and every $2 \leq k \leq g$ there is a k-fold purely branched holomorphic covering map $f_{g,k} : S_g \to E$ with Morse type singularities.*

Proof. Let E be an elliptic curve, let $g \geq 2$, let $1 \leq k \leq g$ and let $d_{1,1}, d_{1,2}, d_{2,1}, \cdots, d_{g-1,1}, d_{g-1,2}$ be $2(g-1)$ points in E. Let $s_1, \cdots, s_{g-1} : [0,1] \to E$ be simple, pairwise non-intersecting paths with starting point $s_i(0) = d_{i,1}$ and endpoint $s_i(1) = d_{i,2}$ for $i = 1, \cdots, g-1$. Take k copies E_0, $E_1, \cdots, E_{k-1}$ of E, cut E_0 open along all of the paths s_i, cut E_i open along the path s_i for $1 \leq i \leq k-2$ and cut E_{k-1} open along the paths $s_{k-1}, \cdots, s_{g-1}$. This produces surfaces $F_0, \cdots, F_{k-1}$ with boundary.

Gluing the surfaces $F_0, \cdots, F_{k-1}$ in the unique way given by identifying opposite edges in the corresponding boundary components F_0 and each of the F_i, we obtain a closed surface of genus g together with a continuous k-fold purely branched covering map $f_{g,k} : S_g \to E$. By choosing the unique complex structure on S_g that makes $f_{g,k}$ holomorphic we obtain the k-fold purely branched holomorphic covering map $f_{g,k} : S_g \to E$ pictured in Figure 4.3. Looking at this map in local coordinates it is immediate that all singularities are of Morse type. ☐

Proof of Theorem 4.6.1. (2) implies (1) by Lemma 4.6.6.

Next we prove that (1) implies (2): By Proposition 4.6.4 and Corollary 4.5.6 the integer k in (1) is an invariant of the map f_g up to isomorphisms and by Lemma 4.6.6 it must coincide with the degree of the branched covering map f_g.

The existence of a k-fold purely branched covering map for $2 \leq k \leq g$ now follows from Lemma 4.6.7. ☐

Remark 4.6.8. Note that Theorem 4.5.4 and Theorem 4.6.1 also allow us to distinguish Kähler groups arising from our construction for which not all maps are purely branched, provided that the purely branched maps do not coincide on fundamental groups up to reordering and choosing suitable standard generating sets.

It seems reasonable to us that there is a further generalisation of Proposition 4.6.4 to branched covering maps which are not purely branched. A suitable generalisation would allow us to classify all Kähler groups that can arise using our construction up to isomorphism. We are planning to address this question in future work. In particular, the purely branched maps should correspond precisely to the branched covering maps inducing maps of the form $(I \cdots I 0 \cdots 0)$ on homology which would allow us to remove the assumption that f_g is purely branched in Theorem 4.6.1.

Chapter 5

Constructing explicit finite presentations

Following the construction of the DPS groups (see Section 2.5), Suciu asked if it was possible to construct explicit presentations of such groups. In this chapter we will construct an explicit finite presentation for their examples, thus answering Suciu's question.

While the methods described in this section can be applied in general, we will focus on the case $g_1 = g_2 = \cdots = g_r = 2$ and denote the respective group by $K_r = \pi_1 H_{g_1,\cdots,g_r}$. We will show that for $r \geq 3$ there is an explicit finite presentation of K_r of the form $K_r \cong \left\langle \mathcal{X}^{(r)} \mid \mathcal{R}_1^{(r)} \cup \mathcal{R}_2^{(r)} \right\rangle$. The relations $\mathcal{R}_1^{(r)}$ correspond to the fact that elements of different factors in a direct product of groups commute and the relations $\mathcal{R}_2^{(r)}$ correspond to the surface group relations in the factors. We will give a similar presentation for K_3.

To obtain these presentations, we will first apply algorithms developed by Baumslag, Bridson, Miller and Short [13] and by Bridson, Howie, Miller and Short [31]. These lead to explicit presentations given in Theorem 5.3.1 (see Sections 5.2 and 5.3). We will then show by computations with Tietze transformations that these presentations can be simplified to the form of Theorem 5.4.4 (see Section 5.4).

Note that the techniques used here can be applied to give explicit finite presentations for many of the groups constructed in Chapter 4.

5.1 Notation

Recall from Section 2.5 that the DPS groups are obtained by considering 2-fold branched coverings $f_{g_i} : S_{g_i} \to E$ of an elliptic curve E with branching sets $B^{(i)} = \left\{ b_1^{(i)}, \cdots, b_{2g_i-2}^{(i)} \right\}$ of size $2g_i - 2$ as indicated in Figure 5.1.

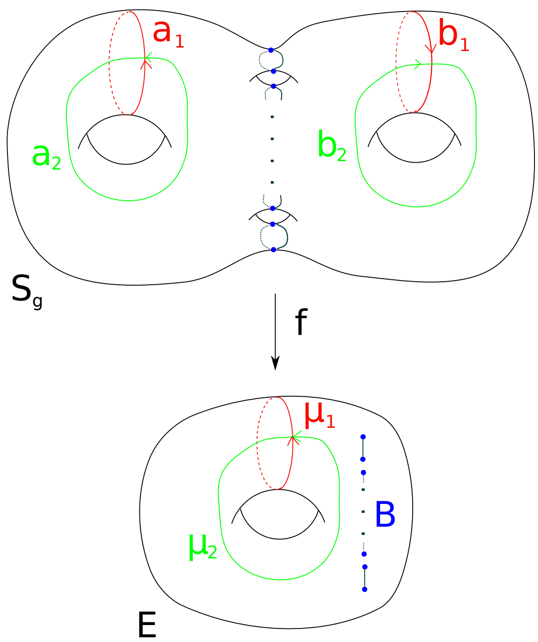

Figure 5.1: The 2-fold branched covering of E

We will now show how to construct an explicit finite presentation for their groups $\pi_1 H_{g_1,\cdots,g_r}$ for all $r \geq 3$. *To simplify our computations we will only consider the case where* $|B^{(i)}| = 2$ *and thus* $g_i = 2$ *for* $i = 1, \cdots, r$. A finite presentation for the general case can be constructed using the very same methods, but the ideas would be obscured by unnecessary complexity.

This means that in our situation the fundamental group of $S^{(i)} := S_{g_i}$ has a finite presentation

$$\pi_1 S^{(i)} = \left\langle a_1^{(i)}, a_2^{(i)}, b_1^{(i)}, b_2^{(i)} \;\middle|\; \left[a_1^{(i)}, a_2^{(i)}\right] \cdot \left[b_1^{(i)}, b_2^{(i)}\right] \right\rangle$$

where $a_j^{(i)}, b_j^{(i)}$ are as indicated in Figure 5.1 with an appropriate choice of base point. We see that with respect to the finite presentation

$$\pi_1 E = \langle \mu_1, \mu_2 \mid [\mu_1, \mu_2] \rangle$$

the induced map $f_{i,*} := f_{g_i,*}$ on fundamental groups is given by

$$\begin{aligned} f_{i,*} : \pi_1 S^{(i)} &\longrightarrow \pi_1 E \\ a_j^{(i)}, b_j^{(i)} &\longmapsto \mu_j, \;\; j = 1, 2. \end{aligned}$$

In particular it follows that the induced map $\phi_r := f_{r,*} = f_{g_1,\cdots,g_r,*}$ on fundamental groups is the map

$$\begin{aligned} \phi_r := f_{r,*} : G_r = \pi_1 S^{(1)} \times \cdots \times \pi_1 S^{(r)} &\longrightarrow \pi_1 E \\ a_j^{(i)}, b_j^{(i)} &\longmapsto \mu_j, \;\; i = 1, \cdots, r, \;\; j = 1, 2. \end{aligned} \tag{5.1}$$

and thus identical with the map $\psi_{g_1,\cdots,g_r}$ in (4.2). This is because we are in the situation where our new class of examples constructed in Chapter 4 coincides with the DPS groups. We will construct explicit finite presentations for the groups $K_r = \ker\phi_r$ (see Theorem 5.4.4)

Note that G_r has a presentation of the form

$$G_r = \left\langle a_i^{(k)}, b_i^{(k)}, i = 1,2, k = 1,\cdots,r \;\middle|\; \begin{array}{l} \left[a_1^{(k)}, a_2^{(k)}\right]\cdot\left[b_1^{(k)}, b_2^{(k)}\right], \\ \left[*^{(k)}, *^{(l)}\right], l \neq k, k, l = 1,\cdots,r \end{array} \right\rangle; \qquad (5.2)$$

Here $*^{(k)}$ runs over all elements of the form $a_i^{(k)}, b_i^{(k)}$, for $i = 1, 2$.

To simplify notation we will write $G = G_r$ and $K = K_r$ where this does not lead to confusion.

5.2 Some preliminary results

With the examples in hand we do now proceed to prove a few preliminary results which will allow us to focus on the actual construction of the finite presentations in Sections 5.3 and 5.4. Some of the results in this section will be fairly technical.

We will follow the definition of *words* given in [13, Section 1.3]: A *word* $w(A)$ is a function that assigns to an ordered alphabet A a word in the letters of $A \cup A^{-1}$ where A^{-1} denotes the set of formal inverses of elements of A. This allows us to change between alphabets where needed. For example if $A = \{a, b\}$, $w(A) = aba$ and $A' = \{a', b'\}$, then $w(A') = a'b'a'$.

For a word $w(A) = a_1 \cdots a_N$, with $a_1, \cdots, a_N \in A$, $N \in \mathbb{N}$, we will denote by $\overline{w}(A)$ the word $a_N \cdots a_1$ and denote by $w^{-1}(A)$ the word $w^{-1}(A) = a_N^{-1} \cdots a_1^{-1}$.

Following [13, Section 1.4] we want to derive from the short exact sequence

$$1 \to K \to G \xrightarrow{\phi} \pi_1 E \to 1$$

a finite presentation for G of the form

$$\langle \mathcal{X} \cup \mathcal{A} \mid S_1 \cup S_2 \cup S_3 \rangle,$$

where $\mathcal{X} = \left\{x_1 = a_1^{(1)}, x_2 = a_2^{(2)}\right\}$ is a lift of the generating set $\{\mu_1, \mu_2\}$ of $\pi_1 E = \langle \mu_1, \mu_2 \mid [\mu_1, \mu_2] \rangle$ under the homomorphism $\phi : G \to \pi_1 E$ and $\mathcal{A} = \{\alpha_1, \cdots, \alpha_n\}$ is a finite generating set of K. The relations are as follows:

- S_1 contains a relation $x_i^{\epsilon} \alpha_j x_i^{-\epsilon} \omega_{i,j,\epsilon}(A)$ for every $i = 1, 2$, $j = 1, \cdots, n$ and $\epsilon = \pm 1$, where $\omega_{i,j,\epsilon}(\mathcal{A}) \in K$ is a word in $\mathcal{A}$ which is equal to $x_i^{\epsilon} \alpha_j^{-1} x_i^{-\epsilon}$ in G;

- S_2 consists of one single relation of the form $[x_1, x_2]U(\mathcal{A})$ where $U(\mathcal{A})$ is an element of K which is equal to $[x_1, x_2]^{-1}$ in G;

- S_3 consists of a finite set of words in $\mathcal{A}$.

We start by deriving a finite generating set for $K = \ker\phi$. One could do this by following the proof of the asymmetric 0-1-2 Lemma (cf. [26, Lemma 1.3] and [27, Lemma 2.1]), but this would be no shorter than the more specific derivation followed here, which is more instructive.

Proposition 5.2.1. *For all $r \geq 2$, the group $K \leq \pi_1 S^{(1)} \times \cdots \times \pi_1 S^{(r)} = G$ is finitely generated with*

$$K = \langle \mathcal{K} \rangle,$$

where $\mathcal{K} = \left\{c_1^{(1)}, c_2^{(1)}, d, f_i^{(k)}, g_i^{(k)}, i = 1,2, k = 1,\cdots,r\right\}$ with the identifications $c_i^{(1)} = a_i^{(1)}(b_i^{(1)})^{-1}$, $d = \left[b_1^{(1)}, b_2^{(1)}\right]$, $f_i^{(k)} = a_i^{(1)}(a_i^{(k)})^{-1}$ and $g_i^{(k)} = b_i^{(1)}(b_i^{(k)})^{-1}$, $i = 1,2$, $k = 1,\cdots,r$.

To ease notation we introduce the ordered sets

$$X^{(k)} = \left\{a_1^{(k)}, a_2^{(k)}, b_1^{(k)}, b_2^{(k)}\right\},\ k = 2,\cdots,r,$$

$$Y^{(k)} = \left\{f_1^{(k)}, f_2^{(k)}, g_1^{(k)}, g_2^{(k)}\right\},\ k = 2,\cdots,r.$$

The proof of Proposition 5.2.1 will make use of

Lemma 5.2.2. *Let $\mathcal{K} = \left\{c_1^{(1)}, c_2^{(1)}, d, f_i^{(k)}, g_i^{(k)}, i = 1,2, k = 1,\cdots,r\right\}$ be as defined in Proposition 5.2.1. Let $w(X^{(1)} \cup \cdots \cup X^{(m)})$ be a word in $X^{(1)} \cup \cdots \cup X^{(m)}$ with $m \in \{1,\cdots,r-1\}$ and let $v(X^{(1)})$ be a word in $X^{(1)}$. Then the following hold:*

1. *In G we have the identity*

   $$v(X^{(1)}) \cdot w(X^{(1)} \cup \cdots \cup X^{(m)}) \cdot v^{-1}(X^{(1)}) = v(Y^{(k)}) w(X^{(1)} \cup \cdots \cup X^{(m)}) v^{-1}(Y^{(k)}).$$

   In particular if $w(X^{(1)} \cup \cdots \cup X^{(m)}) \in \langle \mathcal{K} \rangle$, then so are all its $\langle X^{(1)} \rangle$-conjugates.

2. *If $m = 1$, we can cyclically permute the letters of $w(X^{(1)})$ using conjugation by elements in $\langle \mathcal{K} \rangle$.*

3. *If $m = 1$, all commutators of letters in $X^{(1)}$ are contained in $\langle \mathcal{K} \rangle$.*

4. *If $m = 1$, we have $\phi(w(X^{(1)})) = 1$ if and only if the combined sum of the exponents of $a_i^{(1)}$ and $b_i^{(1)}$ is zero for both, $i = 1$ and $i = 2$.*

Proof. We obtain (1) using that $[X^{(k)}, X^{(l)}] = \{1\}$ for $1 \le k \ne l \le r$:

$$\begin{aligned}
(a_i^{(1)})^{\epsilon} \cdot w(X^{(1)} \cup \cdots \cup X^{(m)}) \cdot (a_i^{(1)})^{-\epsilon} =& ((a_i^{(1)})^{\epsilon}(a_i^{(k)})^{-\epsilon}) \cdot w(X^{(1)} \cup \cdots \cup X^{(m)}) \\
& \cdot ((a_i^{(k)})^{\epsilon}(a_i^{(1)})^{-\epsilon}) \\
=& (f_i^{(k)})^{\epsilon} w(X^{(1)} \cup \cdots \cup X^{(m)})(f_i^{(k)})^{-\epsilon}, \\
(b_i^{(1)})^{\epsilon} \cdot w(X^{(1)} \cup \cdots \cup X^{(m)}) \cdot (b_i^{(1)})^{-\epsilon} =& ((b_i^{(1)})^{\epsilon}(b_i^{(k)})^{-\epsilon}) \cdot w(X^{(1)} \cup \cdots \cup X^{(m)}) \\
& \cdot ((b_i^{(k)})^{\epsilon}(b_i^{(1)})^{-\epsilon}) \\
=& (g_i^{(k)})^{\epsilon} w(X^{(1)} \cup \cdots \cup X^{(m)})(g_i^{(k)})^{-\epsilon},
\end{aligned}$$

for all $i = 1, 2$, $\epsilon = \pm 1$, $k > m$.

We obtain (2) from (1). For instance $a_1^{(1)} w'(X^{(1)}) = f_1^{(k)} w'(X^{(1)}) a_1^{(1)} (f_1^{(k)})^{-1}$.

We obtain (3) from (1), (2) and the following identities in G

- $\left[b_1^{(1)}, b_1^{(2)}\right] = d$,
- $\left[a_1^{(1)}, a_2^{(1)}\right] = \left[b_1^{(1)}, b_2^{(1)}\right]^{-1}$ in $\pi_1 S^{(1)}$ and thus in G,
- $\left[a_i^{(1)}, b_i^{(1)}\right] = \left[f_i^{(k)}, (c_i^{(1)})^{-1}\right]$, for $i = 1, 2$ and $k = 2, \cdots, r$,
- $\left[a_i^{(1)}, b_j^{(1)}\right] = c_i^{(1)} \cdot d^{2j-3} \cdot g_j^{(k)} \cdot (c_1^{(1)})^{-1} \cdot (g_2^{(k)})^{-1}$, for $i, j = 1, 2$, $k = 2, \cdots, r$ and $i \ne j$.

With these commutators at hand we can use conjugation in $\langle \mathcal{K} \rangle$, cyclic permutation and inversion in order to obtain all other commutators.

We obtain (4) as an immediate consequence of the definition of the map ϕ given in (5.1). □

Proof of Proposition 5.2.1. It is immediate from the explicit form (5.1) of the map ϕ that all elements in $\mathcal{K}$ are indeed contained in K. Hence, we only need to prove that these elements actually generate K.

Let $g \in K$ be an arbitrary element. Since $[X^{(k)}, X^{(l)}] = \{1\}$ in G, there are words $w_1(X^{(1)})$, $\cdots$, $w_r(X^{(r)})$ such that

$$g = w_1(X^{(1)}) \cdot \cdots \cdot w_r(X^{(r)}).$$

Using $[X^{(1)}, X^{(l)}] = \{1\}$ for $l \ne 1$ we obtain that $w_k(X^{(k)}) = \overline{w_k}(X^{(1)}) \overline{w}_k^{-1}(Y^{(k)})$,

$k = 2, \cdots, r$, and consequently

$$\begin{aligned} g = & w_1(X^{(1)}) \cdot \ \cdots \ \cdot w_r(X^{(r)}) \\ = & w_1(X^{(1)}) \cdot w_2(X^{(2)}) \cdot \ \cdots \ \cdot w_{r-1}(X^{(r-1)}) \cdot \overline{w_r}(X^{(1)}) \cdot \overline{w_r}^{-1}(Y^{(r)}) \\ = & w_1(X^{(1)}) \cdot \overline{w_r}(X^{(1)}) \cdot w_2(X^{(2)}) \cdot \ \cdots \ \cdot w_{r-1}(X^{(r-1)}) \cdot \overline{w_r}^{-1}(Y^{(r)}) \\ = & \cdots \\ = & w_1(X^{(1)}) \cdot \overline{w_r}(X^{(1)}) \overline{w_{r-1}}(X^{(1)}) \cdot \ \cdots \ \cdot \overline{w_2}(X^{(1)}) \\ & \cdot \overline{w_2}^{-1}(Y^{(2)}) \cdot \ \cdots \ \cdot \overline{w}_{r-1}^{-1}(Y^{(r-1)}) \cdot \overline{w_r}^{-1}(Y^{(r)}) \end{aligned}$$

Since $g \in K$ and $\overline{w_2}^{-1}(Y^{(2)}) \cdot \ \cdots \ \cdot \overline{w_r}^{-1}(Y^{(r)}) \in \langle \mathcal{K} \rangle \leq K$, it suffices to prove that every element of K which is equal in G to a word $w(X^{(1)})$ is in $\langle \mathcal{K} \rangle$.

Due to Lemma 5.2.2(2),(3) we can use conjugation and commutators to obtain an equality

$$w(X^{(1)}) = u(\mathcal{K}) \cdot (a_1^{(1)})^{m_1} \cdot (b_1^{(1)})^{n_1} \cdot (a_2^{(1)})^{m_2} \cdot (b_2^{(1)})^{n_2} \cdot v(\mathcal{K})$$

in K for some $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ and words $u(\mathcal{K}), v(\mathcal{K}) \in \langle \mathcal{K} \rangle$.

By Lemma 5.2.2(4) we obtain that $n_1 = -m_1$ and $n_2 = -m_2$. Hence, another application of Lemma 5.2.2(2),(3) implies that

$$w(X^{(1)}) = u'(\mathcal{K}) \cdot (c_1^{(1)})^{m_1} \cdot (c_2^{(1)})^{m_2} \cdot v'(\mathcal{K}) \in \langle \mathcal{K} \rangle$$

in K for some words $u'(\mathcal{K}), v'(\mathcal{K}) \in \langle \mathcal{K} \rangle$. This completes the proof. □

We will use $\mathcal{K}$ as our generating set $\mathcal{A}$ and compute the elements of S_1 with respect to $\mathcal{K}$.

Lemma 5.2.3. *The following identities hold in G for all $i, j = 1, 2$, $k = 2, \cdots, r$ and $\epsilon = \pm 1$:*

$$\left[(a_i^{(1)})^\epsilon, f_j^{(k)}\right] = \left[(a_i^{(1)})^\epsilon, a_j^{(1)}\right] = \begin{cases} 1 & , \text{ if } i = j \\ d^{2i-3} & , \text{ if } i \neq j \text{ and } \epsilon = 1 \\ (f_i^{(k)})^{-1} d^{3-2i} f_i^{(k)} & , \text{ if } i \neq j \text{ and } \epsilon = -1. \end{cases} \tag{5.3}$$

$$\left[(a_i^{(1)})^\epsilon, g_j^{(k)}\right] = \left[(a_i^{(1)})^\epsilon, b_j^{(1)}\right] = \begin{cases} \left[(f_i^{(k)})^\epsilon, (c_i^{(1)})^{-1}\right] & , \text{ if } i = j \\ f_i^{(k)} (c_j^{(1)})^{-1} (f_i^{(k)})^{-1} d^{2i-3} c_j^{(1)} & , \text{ if } i \neq j \text{ and } \epsilon = 1 \\ (f_i^{(k)})^{-1} (c_j^{(1)})^{-1} d^{3-2i} f_i^{(k)} c_j^{(1)} & , \text{ if } i \neq j \text{ and } \epsilon = -1. \end{cases} \tag{5.4}$$

$$(a_i^{(1)})^\epsilon c_j^{(1)} (a_i^{(1)})^{-\epsilon} = (f_i^{(k)})^\epsilon c_j^{(1)} (f_i^{(k)})^{-\epsilon} \tag{5.5}$$

$$(a_i^{(1)})^\epsilon d (a_i^{(1)})^{-\epsilon} = (f_i^{(k)})^\epsilon d (f_i^{(k)})^{-\epsilon} \tag{5.6}$$

Proof. The vanishing $[X^{(l)}, X^{(k)}] = \{1\}$ for $k \neq l$ yields the equalities

$$\left[(a_i^{(1)})^\epsilon, f_j^{(k)}\right] = \left[(a_i^{(1)})^\epsilon, a_j^{(1)}\right],$$

$$\left[(a_i^{(1)})^\epsilon, g_j^{(k)}\right] = \left[(a_i^{(1)})^\epsilon, b_j^{(1)}\right],$$

as well as the equalities (5.5) and (5.6). In particular the commutators on the left of (5.3) and (5.4) are independent of k.

The equalities on the right of (5.3) and (5.4) are established similarly.

□

In the following we will denote by $V_{i,j,\epsilon}(\mathcal{K})$ the words in the alphabet $\mathcal{K}$ as defined on the right side of equation (5.3) and by $W_{i,j,\epsilon}(\mathcal{K})$ the words in the alphabet $\mathcal{K}$ as defined on the right side of equation (5.4), in both cases choosing $k = 2$.

With this notation we obtain

$$S_1 = \left\{ \begin{array}{l} \left[x_i^\epsilon, f_j^{(k)}\right] V_{i,j,\epsilon}^{-1},\ \left[x_i^\epsilon, g_j^{(k)}\right] W_{i,j,\epsilon}^{-1},\ x_i^\epsilon c_j x_i^{-\epsilon} (f_i^{(k)})^\epsilon c_j^{-1} (f_i^{(k)})^{-\epsilon}, \\ x_i^\epsilon d x_i^{-\epsilon} (f_i^{(k)})^\epsilon d^{-1} (f_i^{(k)})^{-\epsilon} \end{array} \right\}.$$

In fact we do not actually need all of the relations S_1, but we are able to express some of them in terms of the other ones

Lemma 5.2.4. *There is a canonical isomorphism*

$$\langle \mathcal{X}, \mathcal{K} \mid S_1' \rangle \cong \langle \mathcal{X}, \mathcal{K} \mid S_1 \rangle,$$

induced by the identity map on generators, with

$$S_1' = \left\{ \begin{array}{l} \left[x_i, f_j^{(k)}\right] V_{i,j,1}^{-1},\ \left[x_i, g_j^{(k)}\right] W_{i,j,1}^{-1},\ x_i^\epsilon c_j x_i^{-\epsilon} (f_i^{(k)})^\epsilon c_j^{-1} (f_i^{(k)})^{-\epsilon}, \\ x_i^\epsilon d x_i^{-\epsilon} (f_i^{(k)})^\epsilon d^{-1} (f_i^{(k)})^{-\epsilon} \end{array} \right\}.$$

Proof. Since the set of relations S_1' is a proper subset of S_1 it suffices to prove that all elements in $S_1 \setminus S_1'$ are products of conjugates of relations in S_1'. Indeed, we have the following equalities in $\langle \mathcal{X}, \mathcal{K} \mid S_1' \rangle$ using relations of the form (5.5) and (5.6) and the relations for $\epsilon = 1$:

$$\begin{aligned}
\left[x_i^{-1}, f_j^{(k)}\right] V_{i,j,-1}^{-1} &= x_i^{-1} f_j^{(k)} x_i (f_j^{(k)})^{-1} V_{i,j,-1}^{-1} \\
&= x_i^{-1} \left[x_i, f_j^{(k)}\right]^{-1} x_i V_{i,j,-1}^{-1} \\
&= \left\{ \begin{array}{ll} x_i^{-1} x_i & , \text{ if } i = j \\ x_i^{-1} d^{3-2i} x_i (f_i^{(k)})^{-1} d^{2i-3} f_i^{(k)} & , \text{ if } i \neq j \end{array} \right. \\
&= 1
\end{aligned}$$

and

$$
\begin{aligned}
\left[x_i^{-1}, g_j^{(k)}\right] W_{i,j,-1}^{-1} &= x_i^{-1} g_j^{(k)} x_i (g_j^{(k)})^{-1} W_{i,j,-1}^{-1} \\
&= x_i^{-1} \left[x_i, g_j^{(k)}\right]^{-1} x_i W_{i,j,-1}^{-1} \\
&= \begin{cases} x_i^{-1}\left[(c_i^{(1)})^{-1}, f_i^{(k)}\right] x_i \left[(c_i^{(1)})^{-1}, (f_i^{(k)})^{-1}\right] & \text{, if } i = j \\ x_i^{-1}(c_j^{(1)})^{-1} d^{3-2i} f_i^{(k)} c_j^{(1)} (f_i^{(k)})^{-1} x_i & \text{, if } i \neq j \\ \cdot (c_j^{(1)})^{-1} (f_i^{(k)})^{-1} d^{2i-3} c_j^{(1)} f_i^{(k)} & \end{cases} \\
&= \begin{cases} x_i^{-1}\left[(c_i^{(1)})^{-1}, f_i^{(k)}\right]\left[x_i (c_i^{(1)})^{-1} x_i^{-1}, x_i (f_i^{(k)})^{-1} x_i^{-1}\right] x_i & \text{, if } i = j \\ x_i^{-1}(c_j^{(1)})^{-1} d^{3-2i} f_i^{(k)} c_j^{(1)} (f_i^{(k)})^{-1} f_i^{(k)} (c_j^{(1)})^{-1} (f_i^{(k)})^{-1} \cdot & \text{, if } i \neq j \\ (f_i^{(k)})^{-1} f_i^{(k)} d^{2i-3} (f_i^{(k)})^{-1} f_i^{(k)} c_j^{(1)} (f_i^{(k)})^{-1} f_i^{(k)} x_i & \end{cases} \\
&= \begin{cases} x_i^{-1}\left[(c_i^{(1)})^{-1}, f_i^{(k)}\right]\left[f_i^{(k)} (c_i^{(1)})^{-1} (f_i^{(k)})^{-1}, (f_i^{(k)})^{-1}\right] x_i & \text{, if } i = j \\ 1 & \text{, if } i \neq j \end{cases} \\
&= 1.
\end{aligned}
$$

□

To obtain S_2 observe that using $[X^{(1)}, X^{(k)}] = \{1\}$ for $k \neq 1$ and the relation $\left[a_1^{(1)}, a_2^{(1)}\right] \cdot d$ in G, we obtain

$$[x_1, x_2] = d^{-1}.$$

Hence, the set of relations S_2 is given by

$$S_2 = \{[x_1, x_2]\, d\}.$$

To obtain the set S_3 we recall that G has a finite presentation of the form (5.2). Hence, it suffices to express all of the relations in the presentation (5.2) as words in $\mathcal{K}$ modulo relations of the form S_1 and S_2. For group elements g, h we write $g \sim h$ if g and h are in the same conjugacy class.

Lemma 5.2.5. *In the free group $F(\mathcal{X} \cup \mathcal{K})$ modulo the relations S_1 and S_2, and the identifications made in Proposition 5.2.1, we obtain the following equivalences of words:*

1. $\left[a_i^{(1)}, a_j^{(k)}\right] \sim 1$,
2. $\left[a_i^{(1)}, b_j^{(k)}\right] \sim 1$,
3. $\left[b_i^{(1)}, a_j^{(k)}\right] \sim 1$,

for all $i, j = 1, 2$ and $k = 2, \cdots, r$.

Proof.

(1) follows from $\left[x_i, f_j^{(k)}\right] V_{i,j,1}^{-1}$:

$$\left[a_i^{(1)}, a_j^{(k)}\right] = \left[x_i, (f_j^{(k)})^{-1} x_j\right] \sim [x_i, x_j] \left[f_j^{(k)}, x_i\right] = 1.$$

(2) follows from $\left[x_i, g_j^{(k)}\right] W_{i,j,1}^{-1}$:

$$\left[a_i^{(1)}, b_j^{(k)}\right] = \left[x_i, (g_j^{(k)})^{-1} (c_j^{(1)})^{-1} x_j\right] \sim \left[g_j^{(k)}, x_i\right] x_i (c_j^{(1)})^{-1} x_i^{-1} [x_i, x_j]\, c_j^{(1)} = 1.$$

(3) follows from $\left[x_i, f_j^{(k)}\right] V_{i,j,1}^{-1}$ and $x_j c_i^{(1)} x_j^{-1} f_j^{(k)} (c_i^{(1)})^{-1} (f_j^{(k)})^{-1}$:

$$\left[b_i^{(1)}, a_j^{(k)}\right] = \left[(c_i^{(1)})^{-1} x_i, (f_j^{(k)})^{-1} x_j\right] \sim x_j c_i^{(1)} x_j^{-1} f_j^{(k)} (c_i^{(1)})^{-1} (f_j^{(k)})^{-1} \left[f_j^{(k)}, x_i\right] [x_i, x_j] = 1.$$

□

Lemma 5.2.6. *In the free group $F(\mathcal{X} \cup \mathcal{K})$ modulo the relations S_1, S_2, the identifications made in Proposition 5.2.1, and the relations (1)-(3) from Lemma 5.2.5, we obtain the following equivalences of words:*

1. $\left[b_i^{(1)}, b_j^{(k)}\right] \sim \left[c_i^{(1)}, g_j^{(k)}\right] \left[(c_j^{(1)})^{-1} f_j^{(k)}, c_i^{(1)}\right]$,
2. $\left[a_i^{(k)}, a_j^{(l)}\right] = (f_i^{(k)})^{-1} (f_j^{(l)})^{-1} V_{i,j,1}^{-1} f_i^{(k)} f_j^{(l)} \sim \left[f_i^{(k)}, f_j^{(l)}\right] V_{i,j,1}^{-1}$,
3. $\left[a_i^{(k)}, b_j^{(l)}\right] \sim \left[f_i^{(k)}, g_j^{(l)}\right] W_{i,j,1}^{-1}$,
4. $\begin{array}{rcl} \left[b_i^{(k)}, b_j^{(l)}\right] & = & (c_i^{(1)} g_i^{(k)})^{-1} (g_j^{(l)} c_j^{(1)})^{-1} V_{i,j,1}^{-1} (c_i^{(1)} g_i^{(k)}) (c_j^{(1)} g_j^{(k)}) \\ & \sim & \left[(c_i^{(1)} g_i^{(k)}), (c_j^{(1)} g_j^{(k)})\right] V_{i,j,1}^{-1} \end{array}$,

for all $i, j = 1, 2$ and $k, l = 2, \cdots, r$.

Proof. (1) follows from Lemma 5.2.5(2) and the relation

$$x_j c_i^{(1)} x_j^{-1} f_j^{(k)} (c_i^{(1)})^{-1} (f_j^{(k)})^{-1} :$$

$$\begin{aligned} \left[b_i^{(1)}, b_j^{(k)}\right] &= b_i^{(1)} (a_i^{(1)})^{-1} b_j^{(k)} (b_j^{(1)})^{-1} b_j^{(1)} a_i^{(1)} (b_i^{(1)})^{-1} (b_j^{(1)})^{-1} b_j^{(1)} (b_j^{(k)})^{-1} \\ &\sim \left[c_i^{(1)}, g_j^{(k)}\right] b_j^{(1)} c_i^{(1)} (b_j^{(1)})^{-1} (c_i^{(1)})^{-1} \\ &= \left[c_i^{(1)}, g_j^{(k)}\right] b_j^{(1)} (a_j^{(1)})^{-1} a_j^{(1)} c_i^{(1)} (a_j^{(1)})^{-1} a_j^{(1)} (b_j^{(1)})^{-1} (c_i^{(1)})^{-1} \\ &= \left[c_i^{(1)}, g_j^{(k)}\right] \left[(c_j^{(1)})^{-1} f_j^{(k)}, c_i^{(1)}\right]. \end{aligned}$$

(2) follows from Lemma 5.2.5(1):

$$\begin{aligned}\left[a_i^{(k)}, a_j^{(l)}\right] &= a_i^{(k)}(a_i^{(1)})^{-1}a_j^{(l)}(a_j^{(1)})^{-1}\left[a_j^{(1)}, a_i^{(1)}\right]a_i^{(1)}(a_i^{(k)})^{-1}a_j^{(1)}(a_j^{(k)})^{-1}\\ &= (f_i^{(k)})^{-1}(f_j^{(l)})^{-1}V_{i,j,1}^{-1}f_i^{(k)}f_j^{(l)}\\ &\sim \left[f_i^{(k)}, f_j^{(l)}\right]V_{i,j,1}^{-1}.\end{aligned}$$

(3) follows from Lemma 5.2.5(2),(3) and (4) follows from Lemma 5.2.5(2) by similar calculations.

□

We introduce the words

$$S^{(k)}(\mathcal{K}) = (f_1^{(k)})^{-1}(f_2^{(k)})^{-1}df_1^{(k)}f_2^{(k)}$$

and

$$T^{(k)}(\mathcal{K}) = (c_1^{(1)}g_1^{(k)})^{-1}(c_2^{(1)}g_2^{(k)})^{-1}dc_1^{(k)}g_1^{(k)}c_2^{(1)}g_2^{(k)}$$

for $2 \le k \le r$. Notice that these words appear in the presentation in Theorem 5.4.4.

The only relation of G that we did not express, yet, is the relation

$$\left[a_1^{(1)}, a_2^{(2)}\right]\left[b_1^{(1)}, b_2^{(2)}\right].$$

Modulo S_1 and S_2 it satisfies

$$\begin{aligned}\left[a_1^{(1)}, a_2^{(1)}\right]\left[b_1^{(1)}, b_2^{(1)}\right] &= [x_1, x_2]\left[(c_1^{(1)})^{-1}x_1, (c_2^{(1)})^{-1}x_2\right]\\ &= d^{-1}(c_1^{(1)})^{-1}x_1(c_2^{(1)})^{-1}x_2x_1^{-1}c_1^{(1)}x_2^{-1}c_2^{(1)}\\ &= d^{-1}(c_1^{(1)})^{-1}f_1^{(k)}(c_2^{(1)})^{-1}(f_1^{(k)})^{-1}d^{-1}f_2^{(k)}c_1^{(1)}(f_2^{(k)})^{-1}c_2^{(1)}.\end{aligned}$$

We define the set of relations S_3 by

$$S_3 = \left\{\begin{array}{l}\left[f_i^{(k)}, f_j^{(l)}\right]V_{i,j,1}^{-1}, \left[f_i^{(k)}, g_j^{(l)}\right]W_{i,j,1}^{-1},\\ \left[c_i^{(1)}, g_j^{(k)}\right]W_{i,j,1}^{-1}c_i^{(1)}V_{i,j,1}^{-1}(c_i^{(1)})^{-1}, \left[g_i^{(k)}, c_jg_j^{(l)}\right]W_{j,i,1},\\ d^{-1}c_1^{-1}f_1^{(k)}c_2^{-1}(f_1^{(k)})^{-1}d^{-1}f_2^{(k)}c_1(f_2^{(k)})^{-1}c_2, S^{(k)}\cdot T^{(k)}\end{array}\right\}.$$

Theorem 5.2.7. *The group $G = \pi_1S^{(1)} \times \cdots \times \pi_1S^{(r)}$ is isomorphic to the group defined by the finite presentation $\langle \mathcal{X}, \mathcal{K} \mid \mathcal{R}\rangle =$*

$$\left\langle \begin{array}{l} x_1, x_2, c_1^{(1)}, c_2^{(1)}, d,\\ f_i^{(k)}, g_i^{(k)},\\ k = 2, \cdots, r,\\ i = 1, 2\end{array} \;\middle|\; \begin{array}{l}\left[x_i, f_j^{(k)}\right]V_{i,j,1}^{-1},\ \left[x_i, g_j^{(k)}\right]W_{i,j,1}^{-1},\ x_i^{\epsilon}(c_j^{(1)})x_i^{-\epsilon}(f_i^{(k)})^{\epsilon}(c_j^{(1)})^{-1}(f_i^{(k)})^{-\epsilon},\\ x_i^{\epsilon}dx_i^{-\epsilon}(f_i^{(k)})^{\epsilon}d^{-1}(f_i^{(k)})^{-\epsilon}, [x_1, x_2]\cdot d,\\ \left[c_i^{(1)}, g_j^{(k)}\right]\left[(c_j^{(1)})^{-1}f_j^{(k)}, c_i^{(1)}\right],\ \left[f_i^{(k)}, f_j^{(l)}\right]V_{i,j,1}^{-1},\ \left[f_i^{(k)}, g_j^{(l)}\right]W_{i,j,1}^{-1},\\ \left[c_i^{(1)}g_i^{(k)}, c_j^{(1)}g_j^{(l)}\right]V_{i,j,1}^{-1},\\ d^{-1}(c_1^{(1)})^{-1}f_1^{(k)}(c_2^{(1)})^{-1}(f_1^{(k)})^{-1}d^{-1}f_2^{(k)}c_1^{(1)}(f_2^{(k)})^{-1}c_2^{(1)},\ S^{(k)}\cdot T^{(k)},\\ i, j = 1, 2, \epsilon = \pm 1, k, l = 2, \cdots, r, l \ne k\end{array}\right\rangle.$$

Proof. By construction there is a canonical way of identifying G given by the presentation 5.2 with the presentation $\langle \mathcal{X}, \mathcal{K} \mid \mathcal{R} \rangle$. Namely, consider the map on generators $\mathcal{X} \cup \mathcal{K}$ defined by:

$$\begin{aligned} x_i &\mapsto a_i^{(1)} \\ c_i^{(1)} &\mapsto a_i^{(1)} (b_i^{(1)})^{-1} \\ d &\mapsto \left[b_1^{(1)}, b_2^{(1)} \right] \\ f_i^{(k)} &\mapsto a_i^{(1)} (a_i^{(k)})^{-1} \\ g_i^{(k)} &\mapsto b_i^{(1)} (b_i^{(k)})^{-1}. \end{aligned}$$

By construction of S_1', S_2 and S_3 the image of all relations in $\mathcal{R}$ vanishes in G. In particular this map extends to a well-defined group homomorphism $\psi : \langle \mathcal{X}, \mathcal{K} \mid \mathcal{R} \rangle \to G$. The map ψ is onto, because of the identities $a_i^{(1)} = \phi(x_i)$, $b_i^{(1)} = \phi(c_i^{(1)})^{-1} \phi(x_i)$, $a_i^{(k)} = \phi(f_i^{(k)})^{-1} \phi(x_i)$, $b_i^{(k)} = \phi(g_i^{(k)})^{-1} b_i^{(1)}$.

Hence, we only need to check that ϕ is injective. For this it suffices to construct a well-defined inverse homomorphism. It is obtained by considering the map on generators of G defined by

$$\begin{aligned} a_i^{(1)} &\mapsto x_i \\ b_i^{(1)} &\mapsto (c_i^{(1)})^{-1} x_i \\ a_i^{(k)} &\mapsto (f_i^{(k)})^{-1} x_i, \ k \geq 2 \\ b_i^{(k)} &\mapsto (g_i^{(k)})^{-1} (c_i^{(1)})^{-1} x_i, \ k \geq 2. \end{aligned}$$

Since we expressed all relations in the presentation 5.2 in terms of the generators of $\mathcal{R}$ under the identification given by ψ using only relations of the form S_1 and S_2, they vanish trivially under this map and thus there is an extension to a group homomorphism $\psi^{-1} : G \to \langle \mathcal{X}, \mathcal{K} \mid \mathcal{R} \rangle$ inverse to ψ.

For instance

$$\begin{aligned} \psi^{-1} \left(\left[a_1^{(1)}, a_2^{(1)} \right] \left[b_1^{(1)}, b_2^{(1)} \right] \right) &= [x_1, x_2] \left[(c_1^{(1)})^{-1} x_1, (c_2^{(1)})^{-1} x_2 \right] \\ &= d^{-1} (c_1^{(1)})^{-1} x_1 (c_2^{(1)})^{-1} x_2 x_1^{-1} c_1^{(1)} x_2^{-1} c_2^{(1)} \\ &= d^{-1} (c_1^{(1)})^{-1} f_1^{(k)} (c_2^{(1)})^{-1} (f_1^{(k)})^{-1} d^{-1} f_2^{(k)} c_1^{(1)} (f_2^{(k)})^{-1} c_2^{(1)}, \end{aligned}$$

which is indeed a relation in $\mathcal{R}$. Similarly we obtain that ψ^{-1} vanishes on all other relations by going through the proofs of Lemma 5.2.5 and 5.2.6.

□

We will now deduce a presentation for the group $\pi_1 S^{(r)} = \left\langle \left[a_1^{(r)}, a_2^{(r)} \right] \left[b_1^{(r)}, b_2^{(r)} \right] \right\rangle$ of the form of Remark 2.1(1) in [31] with respect to the epimorphism $\pi_1 S^{(r)} \to \pi_1 E$,

$a_i^{(r)}, b_i^{(r)} \mapsto \mu_i$ and the presentation $\langle \mu_1^{-1}, \mu_2^{-1} \mid [\mu_1^{-1}, \mu_2^{-1}] \rangle$ of $\pi_1 E$. That is, we derive a presentation of the form $\left\langle \{x_1, x_2\}, C \mid \widehat{R}, S \right\rangle$ such that $x_i \mapsto \mu_i^{-1}$, $c \mapsto 1$, $\widehat{R}$ consists of a relation of the form $[x_1, x_2]\, U(C)$ and S consists of a finite set of words in C^*. Here C^* is defined to be the set of conjugates of elements of C by words in the free group on X.

Proposition 5.2.8. *The finite presentation*

$$\left\langle x_1, x_2, c_1^{(r)}, c_2^{(r)}, \delta \mid [x_1, x_2]\, \delta, \delta^{-2} x_2 x_1 \left[(x_1 c_1^{(r)})^{-1}, (x_2 c_2^{(r)})^{-1} \right] x_2^{-1} x_1^{-1} \right\rangle$$

is a presentation for $\pi_1 S^{(r)}$ of the form described in the previous paragraph, with the isomorphism given by $x_i \mapsto (a_i^{(r)})^{-1}$, $c_1^{(r)} \mapsto a_i^{(r)} (b_i^{(r)})^{-1}$ and $\delta \mapsto \left[(a_2^{(r)})^{-1}, (a_1^{(r)})^{-1} \right]$.

Proof. Using Tietze transformations and the identifications $x_i = (a_i^{(r)})^{-1}$, $c_1^{(r)} = a_i^{(r)} \cdot (b_i^{(r)})^{-1}$, $\delta = \left[(a_2^{(r)})^{-1}, (a_1^{(r)})^{-1} \right]$, we obtain

$$\begin{aligned} \pi_1 S^{(r)} &= \left\langle a_1^{(r)}, a_2^{(r)}, b_1^{(r)}, b_2^{(r)} \mid \left[a_1^{(r)}, a_2^{(r)} \right] \left[b_1^{(r)}, b_2^{(r)} \right] \right\rangle \\ &= \left\langle x_1, x_2, c_1^{(r)}, c_2^{(r)}, \delta \mid [x_1, x_2]\, \delta, \left[x_1^{-1}, x_2^{-1} \right] \left[(x_1 c_1^{(r)})^{-1}, (x_2 c_2^{(r)})^{-1} \right] \right\rangle. \end{aligned}$$

Using Tietze transformations and the relation $[x_1, x_2]\, \delta$ we obtain

$$\begin{aligned} &\left[x_1^{-1}, x_2^{-1} \right] \left[(x_1 c_1^{(r)})^{-1}, (x_2 c_2^{(r)})^{-1} \right] \\ &= x_2^{-1} x_1^{-1} \delta^{-2} (x_2 x_1 (c_1^{(r)})^{-1} (x_2 x_1)^{-1}) (x_2 (c_2^{(r)})^{-1} x_2^{-1}) (x_1 c_1^{(r)} x_1^{-1}) (x_1 x_2 c_2^{(r)} (x_1 x_2)^{-1}) x_1 x_2 \\ &\sim \delta^{-2} \cdot (x_2 x_1 (c_1^{(r)})^{-1} (x_2 x_1)^{-1}) \cdot (x_2 (c_2^{(r)})^{-1} x_2^{-1}) \cdot (x_1 c_1^{(r)} x_1^{-1}) \cdot (x_1 x_2 c_2^{(r)} (x_1 x_2)^{-1}). \end{aligned}$$

This completes the proof. ∎

A subgroup $H \le \Gamma_1 \times \cdots \times \Gamma_r$ of a direct product is called *subdirect* if its projection to every factor is surjective.

Lemma 5.2.9. *The subgroup $K \le \pi_1 S^{(1)} \times \cdots \times \pi_1 S^{(r)}$ is subdirect; in fact its projection onto any $(r-1)$ factors is surjective.*

Proof. Since K is symmetric in the factors, it suffices to prove the second part of the assertion for the first $(r-1)$ factors. To see that the projection $K \to \pi_1 S^{(1)} \times \cdots \times \pi_1 S^{(r-1)}$ is surjective, observe that $f_i^{(r)} \mapsto a_i^{(1)}$ and $g_i^{(r)} \mapsto b_i^{(1)}$ under the projection map. Hence, we obtain that $(f_i^{(k)})^{-1} f_i^{(r)} \mapsto a_i^{(k)}$ and $(g_i^{(k)})^{-1} g_i^{(r)} \mapsto b_i^{(k)}$ under the projection map. ∎

5.3 Construction of a presentation

We will now follow the algorithms described in [31, Theorem 3.7] and [31, Theorem 2.2] in order to derive a finite presentation for K.

Theorem 5.3.1. *Let $r \geq 3$. Then the group defined by the finite presentation*

$$\left\langle \begin{array}{l} x_1, x_2, f_1^{(r)}, f_2^{(r)}, \\ A = \left\{ \begin{array}{l} c_1^{(1)}, c_2^{(1)}, d, \\ f_i^{(k)}, g_i^{(k)}, \\ k = 2, \cdots, r-1, \\ i = 1, 2 \end{array} \right\} \\ B = \left\{ c_1^{(r)}, c_2^{(r)}, \delta \right\} \end{array} \middle| \begin{array}{l} x_i^{\epsilon} c_j^{(1)} x_i^{-\epsilon} (f_i^{(k)})^{\epsilon} (c_j^{(1)})^{-1} (f_i^{(k)})^{-\epsilon},\ x_i^{\epsilon} d x_i^{-\epsilon} (f_i^{(k)})^{\epsilon} d^{-1} (f_i^{(k)})^{-\epsilon}, \\ [x_1, x_2] \cdot \delta \cdot d, \\ \left[c_i^{(1)}, g_j^{(k)}\right] \left[(c_j^{(1)})^{-1} f_j^{(k)}, c_i^{(1)}\right],\ \left[f_i^{(k)}, f_j^{(l)}\right] V_{i,j,1}^{-1}, \\ \left[f_i^{(k)}, g_j^{(l)}\right] W_{i,j,1}^{-1}, \left[c_i^{(1)} g_i^{(k)}, c_j^{(1)} g_j^{(l)}\right] V_{i,j,1}^{-1}, \\ d^{-1} (c_1^{(1)})^{-1} f_1^{(k)} (c_2^{(1)})^{-1} (f_1^{(k)})^{-1} d^{-1} f_2^{(k)} c_1^{(1)} (f_2^{(k)})^{-1} c_2^{(1)}, \\ S^{(k)} \cdot T^{(k)}, \delta^{-2} x_2 x_1 \left[(x_1 c_1^{(r)})^{-1}, (x_2 c_2^{(r)})^{-1}\right] x_2^{-1} x_1^{-1}, \\ [A, B], f_1^{(r)} = x_1, f_2^{(r)} = x_2, \\ i, j = 1, 2, \epsilon = \pm 1, k, l = 2, \cdots, r-1, l \neq k \end{array} \right\rangle$$

is isomorphic to the Kähler group K_r. Hence, it is Kähler of finiteness type $\mathcal{F}_{r-1}$, but not of finiteness type $\mathcal{F}_r$. Here $V_{i,j,\epsilon}(A)$, $W_{i,j,\epsilon}(A)$, $i, j = 1, 2, \epsilon = \pm 1$, $T^{(k)}$ and $S^{(k)}$ are the words in the free group on A defined above.

For $r = 3$ we define

$$\mathcal{X}^{(3)} = \{x_1, x_2, A, B\},$$

$$\mathcal{R}_1^{(3)} = \left\{ \begin{array}{l} x_i^{\epsilon} c_j^{(1)} x_i^{-\epsilon} (f_i^{(2)})^{\epsilon} (c_j^{(1)})^{-1} (f_i^{(2)})^{-\epsilon},\ x_i^{\epsilon} d x_i^{-\epsilon} (f_i^{(2)})^{\epsilon} d^{-1} (f_i^{(2)})^{-\epsilon}, [x_1, x_2] \cdot \delta \cdot d, \\ \left[c_i^{(1)}, g_j^{(2)}\right] \left[(c_j^{(1)})^{-1} f_j^{(2)}, c_i^{(1)}\right],\ \left[f_i^{(3)}, f_j^{(2)}\right] V_{i,j,1}^{-1},\ \left[f_i^{(3)}, g_j^{(2)}\right] W_{i,j,1}^{-1}, \\ [A, B], i, j = 1, 2,\ \epsilon = \pm 1 \end{array} \right\},$$

$$\mathcal{R}_2^{(3)} = \left\{ \begin{array}{l} d^{-1} (c_1^{(1)})^{-1} f_1^{(2)} (c_2^{(1)})^{-1} (f_1^{(2)})^{-1} d^{-1} f_2^{(2)} c_1^{(1)} (f_2^{(2)})^{-1} c_2^{(1)}, S^{(2)} \cdot T^{(2)}, \\ \delta^{-2} x_2 x_1 \left[(x_1 c_1^{(3)})^{-1}, (x_2 c_2^{(3)})^{-1}\right] x_2^{-1} x_1^{-1} \end{array} \right\}.$$

Proof of Theorem 5.3.1. By Proposition 5.2.1 the set

$$\mathcal{K} = \left\{ c_1^{(1)}, c_2^{(1)}, d, f_i^{(k)}, g_i^{(k)}, i = 1, 2, k = 1, \cdots, r \right\}$$

is a generating set of K where $c_i^{(1)} = a_i^{(1)} (b_i^{(1)})^{-1}$, $d = \left[b_1^{(1)}, b_2^{(1)}\right]$, $f_i^{(k)} = a_i^{(1)} (a_i^{(k)})^{-1}$ and $g_i^{(k)} = b_i^{(1)} (b_i^{(k)})^{-1}$. By Lemma 5.2.9 the projection $p_{ij}(\mathcal{K})$ to the group $\pi_1 S^{(i)} \times \pi_1 S^{(j)}$ is surjective, since $r \geq 3$ by assumption.

Hence, Theorem 3.7 in [31] provides us with an algorithm that will output a finite presentation of K. Since by Lemma 5.2.9 the projection $\Gamma_2 = q(K) = \pi_1 S^{(1)} \times \cdots \times \pi_1 S^{(r-1)}$ is surjective, Theorem 5.2.7 provides us with a finite presentation for Γ_2 given by

$$\left\langle \begin{array}{l} x_1, x_2, c_1^{(1)}, c_2^{(1)}, d, \\ f_i^{(k)}, g_i^{(k)}, \\ k = 2, \cdots, r-1, \\ i = 1, 2 \end{array} \;\middle|\; \begin{array}{l} \left[x_i, f_j^{(k)}\right] V_{i,j,1}^{-1},\ \left[x_i, g_j^{(k)}\right] W_{i,j,1}^{-1},\ x_i^{\epsilon} c_j^{(1)} x_i^{-\epsilon} (f_i^{(k)})^{\epsilon} (c_j^{(1)})^{-1} (f_i^{(k)})^{-\epsilon}, \\ x_i^{\epsilon} d x_i^{-\epsilon} (f_i^{(k)})^{\epsilon} d^{-1} (f_i^{(k)})^{-\epsilon}, [x_1, x_2] \cdot d, \\ \left[c_i^{(1)}, g_j^{(k)}\right]\left[(c_j^{(1)})^{-1} f_j^{(k)}, c_i^{(1)}\right],\ \left[f_i^{(k)}, f_j^{(l)}\right] V_{i,j,1}^{-1},\ \left[f_i^{(k)}, g_j^{(l)}\right] W_{i,j,1}^{-1}, \\ \left[c_i^{(1)} g_i^{(k)}, c_j^{(1)} g_j^{(l)}\right] V_{i,j,1}^{-1}, \\ d^{-1} (c_1^{(1)})^{-1} f_1^{(k)} (c_2^{(1)})^{-1} (f_1^{(k)})^{-1} d^{-1} f_2^{(k)} c_1^{(1)} (f_2^{(k)})^{-1} c_2^{(1)},\ S^{(k)} \cdot T^{(k)}, \\ i, j = 1, 2, \epsilon = \pm 1, k, l = 2, \cdots, r, l \neq k \end{array} \right\rangle. \tag{5.7}$$

For $\Gamma_1 = \pi_1 S^{(r)}$ we choose the finite presentation derived in Proposition 5.2.8

$$\Gamma_1 \cong \left\langle x_1, x_2, c_1^{(r)}, c_2^{(r)}, \delta \mid [x_1, x_2]\, \delta, \delta^{-2} x_2 x_1 \left[(x_1 c_1^{(r)})^{-1}, (x_2 c_2^{(r)})^{-1}\right] x_2^{-1} x_1^{-1} \right\rangle \tag{5.8}$$

and for $Q = \Gamma_1 / (\Gamma_1 \cap K) \cong \pi_1 E$ we choose the finite presentation

$$Q \cong \langle x_1, x_2 \mid [x_1, x_2] \rangle .$$

We further introduce the notation

$$\mathcal{X} = \{x_1, x_2\},$$

$$\mathcal{A} = \left\{c_1^{(1)}, c_2^{(1)}, d, f_i^{(k)}, g_i^{(k)}, k = 2, \cdots, r-1, i = 1, 2\right\},$$

$$\mathcal{B} = \left\{c_1^{(r)}, c_2^{(r)}, \delta\right\}.$$

Then the canonical projections $f_1 : \Gamma_1 \to Q$ and $f_2 : \Gamma_2 \to Q$ are given by the identity on $\mathcal{X}$ and by mapping $\mathcal{A}$ and $\mathcal{B}$ to 1. Note further that the presentation $\mathcal{Q} = \langle x_1, x_2 \mid [x_1, x_2] \rangle$ of Q satisfies $\pi_2 \mathcal{Q} = 1$.

By construction the group K in Section 5.1 is then isomorphic to the fibre product of the projections $f_1 : \Gamma_1 \to Q$ and $f_2 : \Gamma_2 \to Q$.

It follows that it suffices to apply the algorithm described in the proof of the Effective Asymmetric 1-2-3 Theorem [31, Theorem 2.2] by Bridson, Howie, Miller and Short in order to obtain a finite presentation of K. We will follow their notation as far as possible. For a more detailed explanation of the following steps we recommend the reader to have a look at the original source [31].

The first step of the algorithm considers the recursively enumerable class $\mathcal{C}(\mathcal{Q})$ of presentations of the form

$$\langle \mathcal{X} \cup A \cup B \mid S_1, S_2, S_3, S_4, S_5 \rangle,$$

with

- S_1 consists of a single relation of the form $[x_1, x_2]\, u(A)v(B^*)$, where $u(A)$ is a word in the free group on the letters of A and $v(B^*)$ is a word in the free group on the letters of B^*, the set of all formal conjugates of letters in B by elements in the free group on $\mathcal{X}$,

- S_2 consists of a relator $x^\epsilon a x^{-\epsilon}\omega_{a,x,\epsilon}$ for every $a \in A$, $x \in X$ and $\epsilon = \pm 1$ with $\omega_{a,x,\epsilon}(A)$ a word in the free group on A,

- $S_3 = \{aba^{-1}b^{-1} \mid a \in A, b \in B\}$,

- S_4 is a finite set of words in the free group on A,

- S_5 is a finite set of words in the free group on B.

Denote by H the group corresponding to this presentation, by N_A, respectively N_B, the normal closure of A, respectively B, in H and by $H_A = H/N_A$, respectively $H_B = H/N_B$, the corresponding quotients. Note in particular that there is a canonical isomorphism $Q \cong H/(N_A{\cdot}N_B)$ and hence there are canonical quotient maps $\pi_A : H_A \to Q$ and $\pi_B : H_B \to Q$.

The algorithm runs through presentations in the class $\mathcal{C}(\mathcal{Q})$ and stops when it finds a presentation such that there are isomorphisms $\phi_A : H_A \to \Gamma_1$ and $\phi_B : H_B \to \Gamma_2$ with the property that $f_1 \circ \phi_A = \pi_A$ and $f_2 \circ \phi_B = \pi_B$.

By construction of the presentations (5.7) and (5.8) and Lemma 5.2.4 it follows that the presentation

$$\left\langle \begin{array}{l} x_1, x_2, \\ A = \left\{ \begin{array}{l} c_1^{(1)}, c_2^{(1)}, d, \\ f_i^{(k)}, g_i^{(k)}, \\ k = 2, \cdots, r-1, \\ i = 1, 2 \end{array} \right\} \\ \mathcal{B} = \left\{ c_1^{(r)}, c_2^{(r)}, \delta \right\} \end{array} \middle| \begin{array}{l} \left[x_i^\epsilon, f_j^{(l)}\right] V_{i,j,\epsilon}^{-1},\ \left[x_i^\epsilon, g_j^{(l)}\right] W_{i,j,\epsilon}^{-1}, x_i^\epsilon c_j^{(1)} x_i^{-\epsilon} (f_i^{(k)})^\epsilon (c_j^{(1)})^{-1} (f_i^{(k)})^{-\epsilon}, \\ x_i^\epsilon d x_i^{-\epsilon} (f_i^{(k)})^\epsilon d^{-1} (f_i^{(k)})^{-\epsilon}, [x_1, x_2] \cdot \delta \cdot d, \\ \left[c_i^{(1)}, g_j^{(k)}\right]\left[(c_j^{(1)})^{-1} f_j^{(k)}, c_i^{(1)}\right],\ \left[f_i^{(k)}, f_j^{(l)}\right] V_{i,j,1}^{-1}, \\ \left[f_i^{(k)}, g_j^{(l)}\right] W_{i,j,1}^{-1},\ \left[c_i^{(1)} g_i^{(k)}, c_j^{(1)} g_j^{(l)}\right] V_{i,j,1}^{-1}, \\ d^{-1} (c_1^{(1)})^{-1} f_1^{(k)} (c_2^{(1)})^{-1} (f_1^{(k)})^{-1} d^{-1} f_2^{(k)} c_1^{(1)} (f_2^{(k)})^{-1} c_2^{(1)}, \\ S^{(m)} \cdot T^{(m)}, \delta^{-2} x_2 x_1 \left[(x_1 c_1^{(r)})^{-1}, (x_2 c_2^{(r)})^{-1}\right] x_2^{-1} x_1^{-1},\ [A, B] \\ i, j = 1, 2, \epsilon = \pm 1, k = 2, \cdots, r, l, m = 2, \cdots, r-1, l, m \neq k \end{array} \right\rangle \tag{5.9}$$

is of this form with f_1 and f_2 defined by the identity maps on $\mathcal{X}$, $\mathcal{A}$ and $\mathcal{B}$. Again we denote by H the group corresponding to this presentation.

The second part of the algorithm derives a finite generating set Z for the $\mathbb{Z}Q$-module $N_A \cap N_B$. The set Z is obtained from an arbitrary, but fixed, finite choice of identity sequences M that generate $\pi_2\mathcal{Q}$ as a $\mathbb{Z}Q$-module. In particular the elements of Z are in one-to-one correspondence with the elements of M. The details of the construction of the elements of M from the elements of Z can be found in the proof of [13, Theorem 1.2].

In our case we have $\pi_2 \mathcal{Q} = 1$. Hence, we can choose $M = \emptyset$ implying that $Z = \emptyset$ and in particular $N_A \cap N_B = 1$. But the algorithm tells us that $K \cong H/(N_A \cap N_B) = H$. Thus, the algorithm shows that the presentation (5.9) is indeed a finite presentation for K and by construction the isomorphism between K and H is induced by the map

$$\begin{aligned} x_i &\mapsto f_i^{(r)} \\ c_i^{(1)} &\mapsto c_i^{(1)} \\ d &\mapsto d \\ f_i^{(k)} &\mapsto f_i^{(k)} \\ g_i^{(k)} &\mapsto g_i^{(k)} \\ c_i^{(r)} &\mapsto (f_i^{(r)})^{-1} c_i^{(1)} g_i^{(r)} \\ \delta &\mapsto \left[f_2^{(r)}, f_1^{(r)} \right] d^{-1} \end{aligned}$$

on generating sets.

Applying Lemma 5.2.4 in order to reduce the relations of the form S_2 to a subset of S_2 and introducing the generators $f_i^{(r)} = x_i$ completes the proof.

□

5.4 Simplifying the presentation

We will now explain how one can simplify the presentation in Theorem 5.3.1 for $r \geq 4$ to obtain a presentation of the form $K_r \cong \left\langle \mathcal{X}^{(r)} \mid \mathcal{R}_1^{(r)} \cup \mathcal{R}_2^{(r)} \right\rangle$ with relations $\mathcal{R}^{(1)}$ corresponding to the fact that distinct factors of the direct product of surface groups commute and relations $\mathcal{R}^{(2)}$ corresponding to the surface group relation in the factors. For this we will make use of three auxiliary lemmas. We will give their proof at the end of this section.

Lemma 5.4.1. *Applying Tietze transformations to the presentation in Theorem 5.3.1, we can replace the set of relations*

$$\left\{ \begin{array}{l} \left[c_i^{(r)}, c_j^{(1)}\right], \left[c_i^{(r)}, f_j^{(k)}\right], \left[c_i^{(r)}, g_j^{(k)}\right], \\ \left[\delta, d\right], i = 1,2, k = 2, \cdots, r-1 \end{array} \right\} \subseteq [A, B]$$

by the set of relations

$$\left\{ \begin{array}{l} \left[c_i^{(1)}, g_j^{(r)}\right]\left[(c_j^{(1)})^{-1} f_j^{(r)}, c_i^{(1)}\right], \left[f_i^{(k)}, g_j^{(r)}\right] W_{i,j,1}^{-1}, \left[c_i^{(1)} g_i^{(r)}, c_j^{(1)} g_j^{(k)}\right] V_{i,j,1}^{-1}, \\ \left[\left[x_2, x_1\right], d\right] i, j = 1,2, k = 2, \cdots, r-1 \end{array} \right\}$$

under the identifications $x_i = f_i^{(r)}$, $g_i^{(r)} = (c_i^{(1)})^{-1} f_i^{(r)} c_i^{(r)}$ *and* $\delta = [x_2, x_1]\, d^{-1}$.

Denote by $\mathcal{M}$ the subset of the set of relations of the presentation for K in Theorem 5.3.1 defined by

$$\mathcal{M} = \left\{ \begin{array}{l} \left[d, c_i^{(r)}\right], \left[\delta, c_i^{(1)}\right], \left[\delta, f_i^{(k)}\right], \left[\delta, g_i^{(k)}\right], \\ i = 1, 2, k = 1, \cdots, r-1 \end{array} \right\},$$

and denote by $\mathcal{M}^C$ its complement in the set of all relations in the presentation for K given in Theorem 5.3.1.

Lemma 5.4.2. *For* $r \geq 4$, *all elements of* $\mathcal{M}$ *can be expressed as product of conjugates of relations in its complement* $\mathcal{M}^C$. *Therefore we can remove the set* $\mathcal{M}$ *from the set of relations of* K *using Tietze transformations.*

The third result we want to use is

Lemma 5.4.3. *In the presentation for* K *given in Theorem 5.3.1 we can replace the relation*

$$x_2^{-1} x_1^{-1} \delta^{-2} x_2 x_1 \left[(x_1 c_1^{(r)})^{-1}, (x_2 c_2^{(r)})^{-1}\right] = 1$$

by the relation

$$S^{(r)} T^{(r)} = 1$$

using Tietze transformations and the identifications $x_i = f_i^{(r)}$, $g_i^{(r)} = (c_i^{(1)})^{-1} f_i^{(r)} c_i^{(r)}$ *and* $\delta = [x_2, x_1]\, d^{-1}$.

These three lemmas allow us to obtain a simplified presentation for the groups K_r.

Theorem 5.4.4. *For each* $r \geq 3$, *the Kähler group* K_r *has an explicit finite presentation of the form*

$$K_r \cong \left\langle \mathcal{X}^{(r)} \middle| \mathcal{R}_1^{(r)}, \mathcal{R}_2^{(r)} \right\rangle,$$

where, for $r \geq 4$,

$$\mathcal{X}^{(r)} = \left\{ \begin{array}{l} c_i, d, f_i^{(k)}, g_i^{(k)}, \\ k = 2, \cdots, r, i = 1, 2 \end{array} \right\},$$

$$\mathcal{R}_1^{(r)} = \left\{ \begin{array}{l} (f_i^{(k)})^{\epsilon} c_j (f_i^{(k)})^{-\epsilon} (f_i^{(l)})^{\epsilon} (c_j)^{-1} (f_i^{(l)})^{-\epsilon}, \\ (f_i^{(k)})^{\epsilon} d (f_i^{(k)})^{-\epsilon} (f_i^{(l)})^{\epsilon} d^{-1} (f_i^{(l)})^{-\epsilon}, \left[\left[f_1^{(r)}, f_2^{(r)}\right], d\right], \\ \left[c_i, g_j^{(k)}\right]\left[(c_j)^{-1} f_j^{(k)}, c_i\right],\ \left[f_i^{(k)}, f_j^{(l)}\right] V_{i,j,1}^{-1},\ \left[f_i^{(k)}, g_j^{(l)}\right] W_{i,j,1}^{-1}, \\ \left[c_i g_i^{(k)}, c_j g_j^{(l)}\right] V_{i,j,1}^{-1}, i, j = 1, 2, \epsilon = \pm 1, k, l = 2, \cdots, r, l \neq k \end{array} \right\},$$

$$\mathcal{R}_2^{(r)} = \left\{ d^{-1}c_1^{-1}f_1^{(k)}c_2^{-1}(f_1^{(k)})^{-1}d^{-1}f_2^{(k)}c_1(f_2^{(k)})^{-1}c_2,\ S^{(k)}\cdot T^{(k)}, k = 2,\cdots,r \right\},$$

where $V_{i,j,1}(A)$, $W_{i,j,1}(A)$, $i,j = 1,2$, $S^{(k)}(A)$ and $T^{(k)}(A)$ are the words in the free group $F(A)$ in the generators $A = \left\{c_i, d, f_i^{(k)}, g_i^{(k)}, i = 1,2, k = 2,\cdots,r-1\right\}$ defined in Section 5.2.

And $\mathcal{X}^{(3)}$, $\mathcal{R}_1^{(3)}$ and $\mathcal{R}_2^{(3)}$ are as described in Theorem 5.3.1.

As a direct consequence of Theorem 5.4.4 we obtain

Corollary 5.4.5. *For $r \geq 3$, the first Betti number of the Kähler group K_r is $b_1(K_r) = 4r-2$.*

Proof. From the relations in the presentation in Theorem 5.4.4, the definition of $V_{i,j,1}$ and $W_{i,j,1}$ in Lemma 5.2.3, and the definition of $S^{(k)}$ and $T^{(k)}$ after Lemma 5.2.6, it is not hard to see that the only non-trivial relation in the canonical presentation for

$$H_1(K_r,\mathbb{Z}) = \pi_1 K_r / [\pi_1 K_r, \pi_1 K_r],$$

besides the commutator relations between the generators, is $d = 1$. Hence, $H_1(K_r,\mathbb{Z})$ is freely generated as an abelian group by c_i, $f_i^{(k)}$, $g_i^{(k)}$, where $i = 1,2$, $k = 2,\cdots,r$. It follows that $H_1(K_r,\mathbb{Z}) \cong \mathbb{Z}^{4r-2}$ and $b_1(K_r) = 4r-2$. □

Note that Corollary 5.4.5 will also follow as an immediate consequence of Theorem 7.1.5.

Proof of Theorem 5.4.4. Start with the presentation for K_r derived in Theorem 5.3.1 and use the identifications $x_i = f_i^{(r)}$, $g_i^{(r)} = (c_i^{(1)})^{-1}f_i^{(r)}c_i^{(r)}$ and $\delta = [x_2, x_1]\, d^{-1}$.

From Lemma 5.4.3 we obtain that using Tietze transformations we can replace the relation

$$\delta^{-2}x_2x_1\left[(x_1c_1^{(r)})^{-1}, (x_2c_2^{(r)})^{-1}\right]x_2^{-1}x_1^{-1}$$

by the relation

$$S^{(r)}T^{(r)}.$$

Lemma 5.4.2 implies that we can remove the relations

$$\left\{ \begin{array}{l} \left[d, c_i^{(1)}\right], \left[\delta, c_i^{(1)}\right], \left[\delta, f_i^{(k)}\right], \left[\delta, g_i^{(k)}\right], \\ i = 1,2, k = 1,\cdots,r-1 \end{array} \right\}$$

from our presentation.

Lemma 5.4.1 implies that we can replace the set of relations

$$\left\{ \begin{array}{l} \left[c_i^{(r)}, c_j^{(1)}\right], \left[c_i^{(r)}, f_j^{(k)}\right], \left[c_i^{(r)}, g_j^{(k)}\right], \\ \left[\delta, d\right], i = 1,2, k = 2,\cdots,r-1 \end{array} \right\} \subseteq [A, B]$$

by the set of relations

$$\left\{ \begin{array}{l} \left[c_i^{(1)}, g_j^{(r)}\right]\left[(c_j^{(1)})^{-1} f_j^{(r)}, c_i^{(1)}\right], \left[f_i^{(k)}, g_j^{(r)}\right] W_{i,j,1}^{-1}, \left[c_i^{(1)} g_i^{(r)}, c_j^{(1)} g_j^{(k)}\right] V_{i,j,1}^{-1}, \\ \left[\left[f_2^{(r)}, f_1^{(r)}\right], d\right] i,j = 1,2, k = 2, \cdots, r-1 \end{array} \right\}.$$

The identification $c_i^{(1)} = c_i$ thus shows that K_r is isomorphic to the group with the finite presentation

$$\left\langle \begin{array}{l} c_i, d, f_i^{(k)}, g_i^{(k)}, \\ k = 2, \cdots, r, i = 1,2 \end{array} \middle| \begin{array}{l} (f_i^{(k)})^{\epsilon} c_j (f_i^{(k)})^{-\epsilon} (f_i^{(l)})^{\epsilon} (c_j)^{-1} (f_i^{(l)})^{-\epsilon}, \\ (f_i^{(k)})^{\epsilon} d (f_i^{(k)})^{-\epsilon} (f_i^{(l)})^{\epsilon} d^{-1} (f_i^{(l)})^{-\epsilon}, \left[\left[f_1^{(r)}, f_2^{(r)}\right], d\right], \\ \left[c_i, g_j^{(k)}\right]\left[(c_j)^{-1} f_j^{(k)}, c_i\right],\ \left[f_i^{(k)}, f_j^{(l)}\right] V_{i,j,1}^{-1},\ \left[f_i^{(k)}, g_j^{(l)}\right] W_{i,j,1}^{-1}, \\ \left[c_i g_i^{(k)}, c_j g_j^{(l)}\right] V_{i,j,1}^{-1}, \\ d^{-1} c_1^{-1} f_1^{(k)} c_2^{-1} (f_1^{(k)})^{-1} d^{-1} f_2^{(k)} c_1 (f_2^{(k)})^{-1} c_2,\ S^{(k)} \cdot T^{(k)}, \\ i,j = 1,2, \epsilon = \pm 1, k,l = 2, \cdots, r, l \neq k \end{array} \right\rangle.$$

□

We will now prove the three lemmas:

Proof of Lemma 5.4.1. To simplify notation we enumerate the relations as follows:

1. $\left[c_i^{(1)}, g_j^{(r)}\right]\left[(c_j^{(1)})^{-1} f_j^{(r)}, c_i^{(1)}\right] = 1,$
2. $\left[f_i^{(k)}, g_j^{(r)}\right] W_{i,j,1}^{-1} = 1,$
3. $\left[c_i^{(1)} g_i^{(r)}, c_j^{(1)} g_j^{(l)}\right] V_{i,j,1}^{-1} = 1,$
4. $[[x_2, x_1], d] = 1.$

The following computation shows that relation (1) is equivalent to $\left[c_i^{(r)}, c_j^{(1)}\right] = 1$:

$$\begin{aligned} \left[c_i^{(1)}, g_j^{(r)}\right]\left[(c_j^{(1)})^{-1} f_j^{(r)}, c_i^{(1)}\right] =& c_i^{(1)} (c_j^{(1)})^{-1} f_j^{(r)} c_j^{(r)} (c_i^{(1)})^{-1} (c_j^{(r)})^{-1} (f_j^{(r)})^{-1} c_j^{(1)} \\ &\cdot \left[(c_j^{(1)})^{-1} f_j^{(r)}, c_i^{(1)}\right] \\ =& c_i^{(1)} (c_j^{(1)})^{-1} f_j^{(r)} (c_i^{(1)})^{-1} c_j^{(r)} (c_j^{(r)})^{-1} (f_j^{(r)})^{-1} c_j^{(1)} \\ &\cdot \left[(c_j^{(1)})^{-1} f_j^{(r)}, c_i^{(1)}\right] = 1. \end{aligned}$$

Now we show that relation (2) is equivalent to the relation $\left[c_j^{(r)}, f_i^{(k)}\right] = 1$:

$$\begin{aligned} \left[f_i^{(k)}, g_j^{(r)}\right] W_{i,j,1}^{-1} &= f_i^{(k)} (c_j^{(1)})^{-1} f_j^{(r)} c_j^{(r)} (f_i^{(k)})^{-1} (c_j^{(r)})^{-1} (f_j^{(r)})^{-1} c_j^{(1)} W_{i,j,1}^{-1} \\ &= f_i^{(k)} (c_j^{(1)})^{-1} f_j^{(r)} (f_i^{(k)})^{-1} c_j^{(r)} (c_j^{(r)})^{-1} (f_j^{(r)})^{-1} c_j^{(1)} W_{i,j,1}^{-1} \\ &= \left[f_i^{(k)}, (c_j^{(1)})^{-1} x_j\right] W_{i,j,1}^{-1} = 1. \end{aligned}$$

Next we show that relation (3) is equivalent to $\left[c_j^{(r)}, g_j^{(k)}\right]$ modulo all other relations:

$$\begin{aligned}\left[c_i^{(1)}g_i^{(r)}, c_j^{(1)}g_j^{(l)}\right]V_{i,j,1}^{-1} &= f_i^{(r)}c_i^{(r)}c_j^{(1)}g_j^{(l)}(c_i^{(r)})^{-1}(f_i^{(r)})^{-1}(g_j^{(l)})^{-1}(c_j^{(1)})^{-1}V_{i,j,1}^{-1}\\ &= f_i^{(r)}c_j^{(1)}g_j^{(l)}c_i^{(r)}(c_i^{(r)})^{-1}(f_i^{(r)})^{-1}(g_j^{(l)})^{-1}(c_j^{(1)})^{-1}V_{i,j,1}^{-1}\\ &= f_i^{(r)}c_j^{(1)}(f_i^{(r)})^{-1}W_{i,j,1}(c_j^{(1)})^{-1}V_{i,j,1}^{-1} = 1.\end{aligned}$$

Note that here the only relation from $[A, B]$ that we use is $\left[c_i^{(r)}, c_j^{(1)}g_j^{(l)}\right] = 1$ which modulo $\left[c_i^{(r)}, c_j^{(1)}\right] = 1$ is equivalent to $\left[c_i^{(r)}, g_j^{(l)}\right] = 1$. This means that modulo (1) the relation $\left[c_i^{(r)}, g_j^{(l)}\right] = 1$ is equivalent to the relation (3).

Equivalence of relation (4) and $[\delta, d]$ is immediate from $\delta = [x_2, x_1]d^{-1}$. □

Proof of Lemma 5.4.2. We need to show that the following relations follow from the relations in $\mathcal{M}^C$:

1. $\left[c_i^{(r)}, d\right] = 1$,
2. $\left[\delta, c_i^{(1)}\right] = 1$,
3. $\left[\delta, f_i^{(k)}\right] = 1$,
4. $\left[\delta, g_i^{(k)}\right] = 1$.

For (1), given $r \geq 4$ we can choose $2 \leq l \neq k \leq r-1$ with

$$\left[c_i^{(r)}, d\right] = \left[c_i^{(r)}, \left[c_1^{(1)}g_1^{(k)}, c_2^{(1)}g_2^{(l)}\right]\right] = 1.$$

For (2), using $d = \left[f_2^{(r)}, f_1^{(l)}\right]$ we obtain that it suffices to prove that

$$(c_i^{(1)})^{-1}f_2^{(r)}f_1^{(r)}(f_2^{(r)})^{-1}f_2^{(k)}(f_1^{(r)})^{-1}(f_2^{(k)})^{-1}c_i^{(1)} = f_2^{(r)}f_1^{(r)}(f_2^{(r)})^{-1}f_2^{(k)}(f_1^{(r)})^{-1}(f_2^{(k)})^{-1}.$$

This equality is a consequence of the relations $\left[f_i^{(k)}, f_j^{(l)}\right] = V_{i,j,\epsilon} = d^\epsilon$, $f_2^{(k)}d(f_2^{(k)})^{-1} = f_2^{(r)}d(f_2^{(r)})^{-1}$, and $f_2^{(k)}c_i^{(1)}(f_2^{(k)})^{-1} = f_2^{(r)}c_i^{(1)}(f_2^{(r)})^{-1}$:

$$\begin{aligned}&(c_i^{(1)})^{-1}f_2^{(r)}f_1^{(r)}(f_2^{(r)})^{-1}f_2^{(k)}(f_1^{(r)})^{-1}(f_2^{(k)})^{-1}c_i^{(1)}\\ =&f_2^{(r)}(f_2^{(l)})^{-1}(c_i^{(1)})^{-1}df_1^{(r)}f_2^{(l)}(f_2^{(r)})^{-1}f_2^{(k)}(f_2^{(l)})^{-1}(f_1^{(r)})^{-1}d^{-1}c_i^{(1)}f_2^{(l)}(f_2^{(r)})^{-1}\\ =&f_2^{(r)}(f_2^{(l)})^{-1}f_1^{(r)}(f_1^{(l)})^{-1}(c_i^{(1)})^{-1}df_1^{(l)}(f_2^{(r)})^{-1}f_2^{(k)}(f_1^{(l)})^{-1}d^{-1}c_i^{(1)}f_1^{(l)}(f_1^{(r)})^{-1}f_2^{(l)}(f_2^{(r)})^{-1}\\ =&f_2^{(r)}(f_2^{(l)})^{-1}f_1^{(r)}(f_1^{(l)})^{-1}(c_i^{(1)})^{-1}d(f_2^{(r)})^{-1}df_1^{(l)}(f_1^{(l)})^{-1}d^{-1}f_2^{(k)}d^{-1}c_i^{(1)}f_1^{(l)}(f_1^{(r)})^{-1}f_2^{(l)}(f_2^{(r)})^{-1}\\ =&f_2^{(r)}(f_2^{(l)})^{-1}f_1^{(r)}(f_1^{(l)})^{-1}(c_i^{(1)})^{-1}d(f_2^{(r)})^{-1}f_2^{(k)}d^{-1}c_i^{(1)}f_1^{(l)}(f_1^{(r)})^{-1}f_2^{(l)}(f_2^{(r)})^{-1}\\ =&f_2^{(r)}(f_2^{(l)})^{-1}f_1^{(r)}(f_2^{(r)})^{-1}f_2^{(k)}(f_1^{(r)})^{-1}f_2^{(l)}(f_2^{(r)})^{-1}\\ =&f_2^{(r)}f_1^{(r)}(f_2^{(r)})^{-1}f_2^{(k)}(f_1^{(r)})^{-1}(f_2^{(r)})^{-1}.\end{aligned}$$

where the last equality follows from the vanishing

$$\begin{aligned}
&(f_2^{(l)})^{-1} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(k)} (f_1^{(r)})^{-1} f_2^{(l)} f_1^{(r)} (f_2^{(k)})^{-1} f_2^{(r)} (f_1^{(r)})^{-1} \\
&\sim \left[f_2^{(l)}, (f_1^{(r)})^{-1}\right] (f_2^{(r)})^{-1} f_2^{(k)} \left[(f_1^{(r)})^{-1}, f_2^{(l)}\right] (f_2^{(k)})^{-1} f_2^{(r)} \\
&\sim (f_2^{(k)})^{-1} (f_2^{(r)})^{-1} d f_2^{(r)} f_2^{(k)} (f_2^{(r)})^{-1} (f_2^{(k)})^{-1} d^{-1} f_2^{(k)} f_2^{(r)} = 1.
\end{aligned}$$

Using $[[x_1, x_2], d]$ we show that relation (3) follows from $\mathcal{M}^C$. We prove the case $i = 1$, the case $i = 2$ is similar.

$$\begin{aligned}
&\left[[x_2, x_1] d^{-1}, f_1^{(k)}\right] \\
&= f_2^{(r)} f_1^{(r)} (f_2^{(r)})^{-1} (f_1^{(r)})^{-1} d^{-1} f_1^{(k)} d f_1^{(r)} f_2^{(r)} (f_1^{(r)})^{-1} (f_2^{(r)})^{-1} (f_1^{(k)})^{-1} \\
&\sim \left((f_2^{(r)})^{-1} (f_1^{(k)})^{-1} d^{-1} f_2^{(r)} f_1^{(k)}\right) (f_1^{(k)})^{-1} f_1^{(r)} (f_2^{(r)})^{-1} (f_1^{(r)})^{-1} f_1^{(k)} d f_1^{(r)} f_2^{(r)} (f_1^{(r)})^{-1} \\
&= f_1^{(r)} (f_1^{(k)})^{-1} (f_2^{(r)})^{-1} f_1^{(k)} (f_1^{(r)})^{-1} d f_1^{(r)} f_2^{(r)} (f_1^{(r)})^{-1} \\
&= f_1^{(r)} (f_2^{(r)})^{-1} (f_1^{(k)})^{-1} d^{-1} f_1^{(k)} (f_1^{(r)})^{-1} d f_1^{(r)} f_2^{(r)} (f_1^{(r)})^{-1} \\
&= f_1^{(r)} (f_2^{(r)})^{-1} (f_1^{(r)})^{-1} d^{-1} f_1^{(r)} (f_1^{(r)})^{-1} d f_1^{(r)} f_2^{(r)} (f_1^{(r)})^{-1} = 1.
\end{aligned}$$

We finish the proof by showing that relation (4) follows from the relations in $\mathcal{M}^C$. This is equivalent to proving that modulo $\mathcal{M}^C$ the equality

$$g_i^{(k)} f_2^{(r)} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} (f_2^{(l)})^{-1} (g_i^{(k)})^{-1} = f_2^{(r)} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} (f_2^{(l)})^{-1}$$

holds.

Using $\left[f_i^{(l)}, g_j^{(k)}\right] W_{i,j,1}^{-1} = 1$ we obtain that

$$\begin{aligned}
&g_i^{(k)} f_2^{(r)} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} (f_2^{(l)})^{-1} (g_i^{(k)})^{-1} \\
=&W_{2,i,1}^{-1} f_2^{(r)} W_{1,i,1}^{-1} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} W_{1,i,1} (f_2^{(l)})^{-1} W_{2,i,1}.
\end{aligned}$$

As in (3) we distinguish the cases $i = 1, 2$. Again we will only show how to prove the case $i = 1$, the case $i = 2$ being similar:

$$\begin{aligned}
&W_{2,1,1}^{-1} f_2^{(r)} W_{1,1,1}^{-1} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} W_{1,1,1} (f_2^{(l)})^{-1} W_{2,1,1} \\
=&W_{2,1,1}^{-1} f_2^{(r)} (c_1^{(1)})^{-1} f_1^{(r)} c_1^{(1)} (f_1^{(r)})^{-1} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} \\
&\cdot f_1^{(r)} (c_1^{(1)})^{-1} (f_1^{(r)})^{-1} c_1^{(1)} (f_2^{(l)})^{-1} W_{2,1,1} \\
=&W_{2,1,1}^{-1} f_2^{(r)} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} (f_2^{(l)})^{-1} W_{2,1,1} \\
=&(c_1^{(1)})^{-1} d f_2^{(r)} c_1^{(1)} (f_2^{(r)})^{-1} f_2^{(r)} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} (f_2^{(l)})^{-1} f_2^{(l)} (c_1^{(1)})^{-1} (f_2^{(l)})^{-1} d^{-1} c_1^{(1)} \\
=&f_2^{(r)} f_1^{(r)} (f_2^{(r)})^{-1} f_2^{(l)} (f_1^{(r)})^{-1} (f_2^{(l)})^{-1},
\end{aligned}$$

where in the last equality we use $\left[\delta, c_i^{(1)}\right] = 1$ and $[d, [x_1, x_2]] = 1$. Hence, relation (4) indeed follows from the relations $\mathcal{M}^C$, completing the proof. $\square$

Proof of Lemma 5.4.3. First observe that using $\delta^{-1} = [x_1, x_2]\, d$, $[[x_1, x_2], d]$, $df_1^{(r)} f_2^{(r)} = f_1^{(r)} (f_1^{(l)})^{-1} df_1^{(l)} f_2^{(r)}$ and $df_1^{(l)} f_2^{(r)} = f_2^{(r)} f_1^{(l)}$, we obtain

$$x_2^{-1} x_1^{-1} \delta^{-2} x_2 x_1 = \left\{ (f_1^{(r)})^{-1} (f_2^{(r)})^{-1} df_1^{(r)} f_2^{(r)} \right\} (f_2^{(r)})^{-1} (f_1^{(l)})^{-1} f_2^{(r)} f_1^{(l)}.$$

Using that $\left[c_1^{(1)} g_1^{(l)}, c_2^{(1)} g_2^{(r)}\right] d = 1$, we obtain

$$\begin{aligned} &\left[(c_1^{(1)} g_1^{(r)})^{-1}, (c_1^{(1)} g_2^{(r)})^{-1} \right] \\ =&(c_1^{(1)} g_1^{(r)})^{-1} (c_2^{(1)} g_2^{(r)})^{-1} (c_1^{(1)} g_1^{(l)}) (c_2^{(1)} g_2^{(r)}) (c_1^{(1)} g_1^{(l)})^{-1} (c_1^{(1)} g_1^{(r)}) \\ &\cdot \left\{ (c_1^{(1)} g_1^{(r)})^{-1} (c_2^{(1)} g_2^{(r)})^{-1} d (c_1^{(1)} g_1^{(r)}) (c_2^{(1)} g_2^{(r)}) \right\}. \end{aligned}$$

But this means that the equivalence of the two relations is equivalent to

$$\left[(f_2^{(r)})^{-1}, (f_1^{(l)})^{-1} \right] (c_1^{(1)} g_1^{(r)})^{-1} \left[(c_2^{(1)} g_2^{(r)})^{-1}, (c_1^{(1)} g_1^{(l)}) \right] (c_1^{(1)} g_1^{(r)}) = 1$$

modulo all other relations.

We will prove that this term does indeed vanish:

$$\begin{aligned} &\left[(f_2^{(r)})^{-1}, (f_1^{(l)})^{-1} \right] (c_1^{(1)} g_1^{(r)})^{-1} \left[(c_2^{(1)} g_2^{(r)})^{-1}, (c_1^{(1)} g_1^{(l)}) \right] (c_1^{(1)} g_1^{(r)}) \\ &= (f_2^{(r)})^{-1} (f_1^{(l)})^{-1} df_1^{(l)} f_2^{(r)} (c_1^{(1)} g_1^{(r)})^{-1} (c_2^{(1)} g_2^{(r)})^{-1} d^{-1} (c_2^{(1)} g_2^{(r)}) (c_1^{(1)} g_1^{(r)}) \\ &\overset{c_i^{(1)} g_i^{(r)} = f_i^{(r)} c_i^{(r)}}{=} (f_2^{(r)})^{-1} (f_1^{(l)})^{-1} df_1^{(l)} f_2^{(r)} (c_1^{(1)} g_1^{(r)})^{-1} (c_2^{(r)})^{-1} (f_2^{(r)})^{-1} d^{-1} f_2^{(r)} c_2^{(r)} (c_1^{(1)} g_1^{(r)}). \end{aligned}$$

Finally the relations $\left[c_j^{(r)}, d\right] = 1$, $\left[c_j^{(r)}, f_j^{(l)}\right] = 1$, $(f_i^{(r)})^{-1} d^{-1} f_i^{(r)} = (f_i^{(k)})^{-1} d^{-1} f_i^{(k)}$, $(f_1^{(l)})^{-1} (f_2^{(k)})^{-1} d^{-1} f_2^{(k)} f_1^{(l)} = (f_2^{(k)})^{-1} (f_1^{(l)})^{-1} d^{-1} f_1^{(l)} f_2^{(k)}$, and $c_1^{(1)} g_1^{(r)} = f_1^{(r)} c_1^{(r)}$ imply

$$\begin{aligned} &(f_2^{(r)})^{-1} (f_1^{(l)})^{-1} df_1^{(l)} f_2^{(r)} (c_1^{(1)} g_1^{(r)})^{-1} (c_2^{(r)})^{-1} (f_2^{(r)})^{-1} d^{-1} f_2^{(r)} c_2^{(r)} (c_1^{(1)} g_1^{(r)}) \\ &= (f_2^{(r)})^{-1} (f_1^{(l)})^{-1} df_1^{(l)} f_2^{(r)} (f_1^{(l)})^{-1} (f_2^{(r)})^{-1} d^{-1} f_2^{(r)} f_1^{(l)} \sim ddd^{-1} d^{-1} = 1. \end{aligned}$$

Hence, we can replace the relation $x_2^{-1} x_1^{-1} \delta^{-2} x_2 x_1 \left[(x_1 c_1^{(r)})^{-1}, (x_2 c_2^{(r)})^{-1} \right] = 1$ by the relation $S^{(r)} \cdot T^{(r)} = 1$ using Tietze transformations. ☐

Chapter 6

Kähler groups from maps onto higher-dimensional tori

This chapter consists of two parts. In the first part (Sections 6.1 and 6.2) we develop a new construction method for Kähler groups. The groups obtained from this method arise as fundamental groups of fibres of holomorphic maps onto higher-dimensional complex tori. In the second part (Sections 6.3 and 6.4) we address Delzant and Gromov's question by applying our construction method to provide Kähler subgroups of direct products of surface groups that are not commensurable with any of the previous examples. These subgroups arise as kernels of epimorphisms onto $\mathbb{Z}^{2k}$; they are irreducible.

We consider a holomorphic map $h : X \to Y$ from a compact Kähler manifold X onto a complex torus Y with connected smooth generic fibre H. The principal idea is that if h has well-behaved singularities then it induces a short exact sequence

$$1 \to \pi_1 H \to \pi_1 X \stackrel{h_*}{\to} \pi_1 Y \to 1$$

on fundamental groups. We conjecture that the right condition for h to induce such a short exact sequence is that the map h has isolated singularities (see Conjecture 6.1.2) or, more generally, that h has fibrelong isolated singularities (see Definition 6.1.4). This Conjecture is based on a proof strategy presented in Chapter 9, but there are some practical constraints originating in a lack of properness which mean that additional work is needed to prove it in full generality. Instead we look at the more specific setting when our torus admits a filtration by subtori and prove our conjecture in this situation.

The key result in our construction method is Theorem 6.1.7. It will be proved in Section 6.1. We present two special cases of Theorem 6.1.7 which are of particular interest (Theorem 6.1.3 and Theorem 6.1.5). Theorem 6.1.7 is complemented by a method for proving that the fibres of h are connected under suitable assumptions on the Kähler manifold X and the map h (see Theorem 6.2.1 in Section 6.2).

We expect that our methods can be applied to construct interesting new classes of Kähler groups. Indeed we provide two applications of our methods in this work: In this chapter we will use them to construct new classes of subgroups of direct products of surface groups; and in Chapter 8 we apply them to construct examples of Kähler groups with exotic finiteness properties which are not commensurable to any subgroup of a direct product of surface groups.

We will use the notation $E^{\times k} = \underbrace{E \times \cdots \times E}_{k \text{ times}}$ for the Cartesian product of k copies of an elliptic curve E. Our construction provides irreducible subgroups of direct products of surface groups arising as kernels of epimorphisms $\pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r} \to \pi_1 E^{\times k} \cong \mathbb{Z}^{2k}$ for $r \geq 3k$ and every $k \geq 1$. The idea is to consider branched coverings $\alpha_i : S_{\gamma_i} \to E$, compose these with linear maps $v_i : E \to E^{\times k}$ which embed E in the Cartesian product $E^{\times k}$ in the direction $v_i \in \mathbb{Z}^k$, and combine these maps using addition in $E^{\times k}$. By choosing distinct v_i for $i = 1, \cdots, r$, the smooth generic fibre of the resulting surjective holomorphic map $h : S_{\gamma_1} \times \cdots \times S_{\gamma_r} \to E^{\times k}$ will have fundamental group an irreducible subgroup of the direct product $\pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r}$ (Theorem 6.4.1). This construction is contained in Section 6.3 and the proof that these examples are irreducible is contained in Section 6.4. In Section 6.4 we determine the precise finiteness properties of our examples.

The coabelian subgroups of direct products of surface groups form an important subclass of the class of all subgroups of direct products of surface groups. Indeed, in the case of three factors any finitely presented full subdirect subgroup of $D = \pi_1 S_{\gamma_1} \times \pi_1 S_{\gamma_2} \times \pi_1 S_{\gamma_3}$ is virtually coabelian; with more factors any full subdirect subgroup is virtually conilpotent [31].

6.1 A new construction method

Let X and Y be complex manifolds and let $f : X \to Y$ be a surjective holomorphic map. Recall that a sufficient condition for the map f to have isolated singularities is that the set of singular points of f intersects every fibre of f in a discrete set.

Before we proceed we fix some notation: For a set M and subsets $A, B \subset M$ we will denote by $A \setminus B$ the set theoretic difference of A and B. If $M = T^n$ is an n-dimensional torus then we will denote by $A - B = \{a - b \mid a \in A, b \in B\}$ the group theoretic difference of A and B with respect to the additive group structure on T^n. We will be careful to distinguish $-$ from set theoretic $\smallsetminus$.

In this section we shall need Dimca, Papadima and Suciu's Theorem 4.3.1 in its original version.

Theorem 6.1.1 ([62, Theorem C]). *Let X be a compact complex manifold and let Y be a closed Riemann surface of genus at least one. Let $f : X \to Y$ be a surjective holomorphic map with isolated singularities and connected fibres. Let $\widehat{f} : \widehat{X} \to \widetilde{Y}$ be the pull-back of f under the universal cover $p : \widetilde{Y} \to Y$ and let H be the smooth generic fibre of $\widehat{f}$ (and therefore of f).*

Then the following hold:

1. $\pi_i(\widehat{X}, H) = 0$ *for* $i \leq \dim H$*;*
2. *if* $\dim H \geq 2$*, then* $1 \to \pi_1 H \to \pi_1 X \xrightarrow{f_*} \pi_1 Y \to 1$ *is exact.*

6.1.1 Conjecture

Having isolated singularities yields strong restrictions on the topology of the fibres near the singularities. We will only make indirect use of these restrictions here, by applying Theorem 6.1.1. For background on isolated singularities see Section 9.2.

Conjecture 6.1.2. *Let X be a compact connected complex manifold of dimension $n + k$ and let Y be a k-dimensional complex torus or a Riemann surface of positive genus. Let $h : X \to Y$ be a surjective holomorphic map with connected generic fibre.*

Let further $\widehat{h} : \widehat{X} \to \widetilde{Y}$ be the pull-back fibration of h under the universal cover $p : \widetilde{Y} \to Y$ and let H be the generic smooth fibre of $\widehat{h}$, or equivalently of h.

Suppose that h has only isolated singularities. Then the following hold:

1. $\pi_i(\widehat{X}, H) = 0$ *for all* $i \leq \dim H$*;*
2. *if, moreover,* $\dim H \geq 2$ *then the induced homomorphism $h_* : \pi_1 X \to \pi_1 Y$ is surjective with kernel isomorphic to $\pi_1 H$.*

Conjecture 6.1.2 is a generalisation of Theorem 6.1.1 to higher dimensions. It can be seen as a Lefschetz type result, since it says that in low dimensions the homotopy groups of the subvariety $H \subset \widehat{X}$ of codimension $n \geq 2$ coincide with the homotopy groups of $\widehat{X}$. The most classical Lefschetz type theorem is the Lefschetz Hyperplane

Theorem which is stated in Appendix B together with an application to illustrate its significance. For a detailed introduction to Lefschetz type theorems see [73].

In Chapter 9 we will provide strong evidence towards Conjecture 6.1.2 by presenting a proof strategy, but there is an annoying technical detail which we were not able to overcome yet. Here we will prove a special case of Conjecture 6.1.2.

Theorem 6.1.3. *Let X be a compact complex manifold of dimension $n+k$ and let Y be a complex torus of dimension k. Let $h : X \to Y$ be a surjective holomorphic map with connected smooth generic fibre H. Assume that there is a filtration*

$$\{0\} \subset Y^0 \subset Y^1 \subset \cdots \subset Y^{k-1} \subset Y^k = Y$$

of Y by complex subtori Y^l of dimension l such that the projections

$$h_l = \pi_l \circ h : X \to Y/Y^{k-l}$$

have isolated singularities, where $\pi_l : Y \to Y/Y^{k-l}$ is the holomorphic quotient homomorphism.

If $n = \dim H \geq 2$, then the map h induces a short exact sequence

$$1 \to \pi_1 H \to \pi_1 X \to \pi_1 Y = \mathbb{Z}^{2k} \to 1.$$

Furthermore, we obtain that $\pi_i(X, H) = 0$ for $2 \leq i \leq \dim H$.

In fact we will prove the more general Theorem 6.1.7 from which Theorem 6.1.3 follows immediately.

6.1.2 Fibrelong isolated singularities

Before stating and proving Theorem 6.1.7 we will first give a generalisation of Theorem 6.1.1 which relaxes the conditions on the singularities of h.

Definition 6.1.4. Let X, Y be compact complex manifolds. We say that a surjective map $h : X \to Y$ has *fibrelong isolated singularities* if it factors as

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "h"'] & Z \arrow[d, "f"] \\ & Y \end{tikzcd}$$

where Z is a compact complex manifold, g is a regular holomorphic fibration, and f is holomorphic with isolated singularities.

For holomorphic maps with connected fibrelong isolated singularities we obtain.

Theorem 6.1.5. *Let Y be a closed Riemann surface of positive genus and let X be a compact Kähler manifold. Let $h : X \to Y$ be a surjective holomorphic map with connected generic (smooth) fibre $\overline{H}$.*

If h has fibrelong isolated singularities, g and f are as in Definition 6.1.4, and f has connected fibres H of dimension $n \geq 2$, then the sequence

$$1 \to \pi_1 \overline{H} \to \pi_1 X \overset{h_*}{\to} \pi_1 Y \to 1$$

is exact.

Proof. By applying Theorem 6.1.1 to the map $f : Z \to Y$ we get a short exact sequence

$$1 \to \pi_1 H \to \pi_1 Z \to \pi_1 Y \to 1. \tag{6.1}$$

Let $p \in Y$ be a regular value such that $H = f^{-1}(p)$, let $j : H \hookrightarrow Z$ be the (holomorphic) inclusion map, let $F \subset X$ be the (smooth) fibre of $g : X \to Z$, and identify $\overline{H} = h^{-1}(p) = g^{-1}(H)$. The long exact sequence in homotopy for the fibration

$$\begin{tikzcd} F \arrow[r, hook] & \overline{H} \arrow[d] \\ & H \end{tikzcd}$$

begins

$$\cdots \to \pi_2 H \to \pi_1 F \to \pi_1 \overline{H} \to \pi_1 H \to 1 (= \pi_0 F) \to \cdots. \tag{6.2}$$

Let $\widehat{Z} \to Z$ be the regular covering with Galois group $\ker f_*$, let $\widehat{f} : \widehat{Z} \to \widetilde{Y}$ be a lift of f and, as in Theorem 6.1.1, identify H with a connected component of its preimage in $\widehat{Z}$.

In the light of Theorem 6.1.1(1), the long exact sequence in homotopy for the pair $(\widehat{Z}, H)$ implies that $\pi_i H \cong \pi_i \widehat{Z}$ for $i \leq \dim H - 1 = n - 1$ and that the natural map $\pi_n H \to \pi_n \widehat{Z}$ is surjective. In particular, $\pi_2 H \to \pi_2 \widehat{Z} \overset{\cong}{\to} \pi_2 Z$ is surjective for all $n \geq 2$; this map is denoted by η in the following diagram.

In this diagram, the first column comes from (6.2), the second column is part of the long exact sequence in homotopy for the fibration $g : X \to Z$, and the bottom row comes from (6.1). The naturality of the long exact sequence in homotopy assures us that the diagram is commutative. We must prove that the second row yields the short exact sequence in the statement of the theorem.

$$\begin{tikzcd}
 & \pi_2 H \arrow[r, "\eta"] \arrow[d] & \pi_2 Z \arrow[d] & & \\
 & \pi_1 F \arrow[r, "="] \arrow[d, "\kappa"'] & \pi_1 F \arrow[d, "\lambda"] & & \\
 & \pi_1 \overline{H} \arrow[r, "\iota"] \arrow[d, "\epsilon"'] & \pi_1 X \arrow[r, "h_*"] \arrow[d] & \pi_1 Y \arrow[r] \arrow[d] & 1 \arrow[d] \\
1 \arrow[r] & \pi_1 H \arrow[r, "\delta"] \arrow[d] & \pi_1 Z \arrow[r, "\alpha"] \arrow[d] & \pi_1 Y \arrow[r] & 1 \\
 & 1 \arrow[r] & 1 & &
\end{tikzcd}$$

We know that δ is injective and η is surjective, so a simple diagram chase (an easy case of the 5-Lemma) implies that the map ι is injective.

A further (more involved) diagram chase proves exactness at $\pi_1 X$, i.e., that $\mathrm{Im}(\iota) = \ker(h_*)$. $\square$

We will also need the following proposition. Note that the hypothesis on $\pi_2 Z \to \pi_1 F$ is automatically satisfied if $\pi_1 F$ does not contain a non-trivial normal abelian subgroup. This is the case, for example, if F is a direct product of hyperbolic surfaces.

Proposition 6.1.6. *Under the assumptions of Theorem 6.1.5, if the map $\pi_2 Z \to \pi_1 F$ associated to the fibration $g : X \to Z$ is trivial, then* (6.2) *reduces to a short exact sequence*

$$1 \to \pi_1 F \to \pi_1 \overline{H} \to \pi_1 H \to 1.$$

If, in addition, the fibre F is aspherical, then $\pi_i \overline{H} \cong \pi_i H \cong \pi_i X$ for $2 \le i \le n-1$.

Proof. The commutativity of the top square in the above diagram implies that $\pi_2 H \to \pi_1 F$ is trivial, so (6.2) reduces to the desired sequence.

If the fibre F is aspherical then naturality of long exact sequences of fibrations and Theorem 6.1.1(1) imply that we obtain commutative squares

$$\begin{tikzcd}
\pi_i \overline{H} \arrow[r] \arrow[d, "\cong"] & \pi_i X \arrow[d, "\cong"] \\
\pi_i H \arrow[r, "\cong"] & \pi_i Z
\end{tikzcd}$$

for $2 \le i \le n-1$. It follows that $\pi_i \overline{H} \cong \pi_i H \cong \pi_i X$ for $2 \le i \le n-1$. $\square$

The consequences of these two results which we will need in this Chapter are summarised in the following result.

6.1.3 Restrictions on $h : X \to Y$ for higher-dimensional tori

Let X be a compact complex manifold and let Y be a complex torus of dimension k. Let $h : X \to Y$ be a surjective holomorphic map. Assume that there is a filtration

$$\{0\} \subset Y^0 \subset Y^1 \subset \cdots \subset Y^{k-1} \subset Y^k = Y$$

of Y by complex subtori Y^l of dimension l, $0 \le l \le k$. Let $\pi_l : Y \to Y/Y^{k-l}$ be the canonical holomorphic projection.

Assume that the maps h and $h_l = \pi_l \circ h : X \to Y/Y^{k-l}$ have connected fibres and fibrelong isolated singularities. In particular, there are compact complex manifolds Z_l such that h_l factors as

$$\begin{tikzcd} X \arrow[r, "g_l"] \arrow[dr, "h_l"'] & Z_l \arrow[d, "f_l"] \\ & Y/Y^{k-l} \end{tikzcd} .$$

with g_l a regular holomorphic fibration and f_l surjective holomorphic with isolated singularities and connected fibres. Assume further that the smooth compact fibre F_l of g_l is connected and aspherical. We denote by $\overline{H}_l$ the connected smooth generic fibre of h_l and by H_l the connected smooth generic fibre of f_l.

For a generic point $x^0 = \left(x_1^0, \cdots, x_k^0\right) \in Y$ we claim that $x^{0,l} = x^{0,k} + Y^{k-l} \in Y/Y^{k-l}$ is a regular value of h_l for $0 \le l \le k$: For $1 \le l \le k$ there is a proper subvariety $V^l \subset Y/Y^{k-l}$ such that the set of critical values of h_l is contained in V^l; any choice of x^0 in the open dense subset $Y \smallsetminus \left(\cup_{l=1}^k \pi_l^{-1}(V^l)\right) \subset Y$ satisfies the assertion.

The smooth generic fibres $\overline{H}_l = h_l^{-1}(x^{0,l})$ of h_l form a nested sequence

$$\overline{H} = \overline{H}_k \subset \overline{H}_{k-1} \subset \cdots \subset \overline{H}_0 = X.$$

Consider the corestriction of h_l to the elliptic curve $x^{0,l} + Y^{k-l+1}/Y^{k-l} \subset Y/Y^{k-l}$. The map

$$h_l|_{\overline{H}_{l-1}} : h_l^{-1}\left(x^{0,l} + Y^{k-l+1}/Y^{k-l}\right) = h^{-1}\left(x^{0,k} + Y^{k-l+1}\right) = \overline{H}_{l-1} \to x^{0,l} + Y^{k-l+1}/Y^{k-l}$$

is holomorphic surjective with fibrelong isolated singularities and connected smooth generic fibre $\overline{H}_l = h_l^{-1}(x^{0,l} + Y^{k-l})$.

Assume that the induced map $\pi_2\overline{H}_{l-1} \to \pi_1 F_l$ is trivial for $1 \le l \le k$. Then the following result holds:

Theorem 6.1.7. *Assume that $h : X \to Y$ has all the properties described in Paragraph 6.1.3 and that $n := \min_{0\leq l\leq k-1} \dim H_l \geq 2$. Then the map h induces a short exact sequence*

$$1 \to \pi_1 \overline{H} \to \pi_1 X \overset{h_*}{\to} \pi_1 Y \cong \mathbb{Z}^{2k} \to 1$$

and $\pi_i(\overline{H}) \cong \pi_i(X)$ for $2 \leq i \leq n-1$.

Note that Theorem 6.1.3 is the special case of Theorem 6.1.7 with $Z_l = X$ and $g_l = id_X$ for $1 \leq l \leq k$.

Proof of Theorem 6.1.7. The proof uses an inductive argument reducing the statement to an iterated application of Theorem 6.1.5 and Proposition 6.1.6.

Since $\dim H_l \geq n \geq 2$, Theorem 6.1.5 and Proposition 6.1.6 imply the restriction $h_l|_{\overline{H}_{l-1}}$ induces a short exact sequence

$$1 \to \pi_1 \overline{H}_l \to \pi_1 \overline{H}_{l-1} \overset{h_{l*}}{\to} \pi_1 \left(x^{0,l} + Y^{k-l+1}/Y^{k-l}\right) = \mathbb{Z}^2 \to 1 \tag{6.3}$$

and that $\pi_i(\overline{H}_{l-1}) \cong \pi_i(\overline{H}_l)$ for $2 \leq i \leq \dim H_l - 1$, where $1 \leq l \leq k$. In particular, we obtain that $\pi_i(\overline{H}_{l-1}) \cong \pi_i(\overline{H}_l)$ for $2 \leq i \leq n-1$.

Hence, we are left to prove that the short exact sequences in (6.3) induce a short exact sequence

$$1 \to \pi_1 \overline{H} \to \pi_1 X \to \pi_1 Y = \mathbb{Z}^{2k} \to 1.$$

For this consider the commutative diagram of topological spaces

$$\begin{tikzcd}
\overline{H} \arrow[r, hook] & X = \overline{H}_0 = h_k^{-1}\left(x^{0,0} + V^k\right) \arrow[r, two heads, "h"] & x^{0,0} + V^k \\
\overline{H} \arrow[r, hook] \arrow[u, "="] & \overline{H}_1 = h^{-1}\left(x^{0,1} + V^{k-1}\right) \arrow[r, two heads, "h"] \arrow[u, hook] & x^{0,1} + V^{k-1} \arrow[u, hook] \\
\vdots \arrow[u, "="] & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] \\
\overline{H} \arrow[r, hook] \arrow[u, "="] & \overline{H}_{k-1} = h_k^{-1}\left(x^{0,k-1} + V^1\right) \arrow[r, two heads, "h"] \arrow[u, hook] & x^{0,k-1} + V^1 \arrow[u, hook] \\
\overline{H} \arrow[r, hook] \arrow[u, "="] & \overline{H} = \overline{H}_k = h_k^{-1}\left(x^{0,k} + V^0\right) \arrow[r, two heads, "h"] \arrow[u, hook] & x^{0,k} + V^0 \arrow[u, hook]
\end{tikzcd}$$

This induces a commutative diagram of fundamental groups

$$
\begin{tikzcd}
1 \arrow[r] & \pi_1\overline{H} \arrow[r, hook] & \pi_1 X \arrow[r, two heads, "h_*"] & \pi_1(x^{0,0}+V^k)=\mathbb{Z}^{2k} \arrow[r] & 1 \\
1 \arrow[r] & \pi_1\overline{H} \arrow[r, hook] \arrow[u, "="] & \pi_1\overline{H}_1 \arrow[r, two heads, "h_*"] \arrow[u, hook] & \pi_1(x^{0,1}+V^{k-1})=\mathbb{Z}^{2k-2} \arrow[r] \arrow[u, hook] & 1 \\
 & \vdots \arrow[u, "="] & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] & \\
1 \arrow[r] & \pi_1\overline{H} \arrow[r, hook] \arrow[u, "="] & \pi_1\overline{H}_{k-1} \arrow[r, two heads, "h_*"] \arrow[u, hook] & \pi_1(x^{0,k-1}+V^1)=\mathbb{Z}^2 \arrow[r] \arrow[u, hook] & 1 \\
1 \arrow[r] & \pi_1\overline{H} \arrow[r, hook] \arrow[u, "="] & \pi_1\overline{H} \arrow[r, two heads, "h_*"] \arrow[u, hook] & \pi_1(x^{0,k}+V^0)=1 \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}
\tag{6.4}
$$

where injectivity of the vertical maps in the middle column follows from (6.3). The last two rows in this diagram are short exact sequences: The last row is obviously exact and the penultimate row is exact by (6.3) for $l = k$.

We will now prove by induction (with l decreasing) that the l-th row from the bottom

$$1 \to \pi_1\overline{H} \to \pi_1\overline{H}_l \to \pi_1(x^{0,l}+V^{k-l}) \to 1$$

is a short exact sequence for $0 \le l \le k$.

Assume that the statement is true for l. We want to prove it for $l-1$. Exactness at $\pi_1\overline{H}$ follows from the sequence of injections $\pi_1\overline{H}_l \hookrightarrow \pi_1\overline{H}_{l-1}$.

For exactness at $\pi_1(x^{0,l-1}+Y^{k-l+1})$ observe that, by the Ehresmann Fibration Theorem, the fibration $\overline{H}_{l-1} \to x^{0,l-1}+Y^{k-l+1}$ restricts to a locally trivial fibration $\overline{H}^*_{l-1} \to (x^{0,l-1}+Y^{k-l+1})^*$ with connected fibre $\overline{H}$ over the complement $(x^{0,l-1}+Y^{k-l+1})^*$ of the subvariety of critical values of h in $x^{0,l-1}+Y^{k-l+1}$. Hence, the induced map $\pi_1\overline{H}^*_{l-1} \to \pi_1(x^{0,l-1}+Y^{k-l+1})^*$ on fundamental groups is surjective. Since the complements $\overline{H}_{l-1} \setminus \overline{H}^*_{l-1}$ and $(x^{0,l-1}+Y^{k-l+1}) \setminus (x^{0,l-1}+Y^{k-l+1})^*$ are contained in complex analytic subvarieties of real codimension at least two, the induced map $\pi_1\overline{H}_{l-1} \to \pi_1(x^{0,l-1}+Y^{k-l+1})$ is surjective.

For exactness at $\pi_1\overline{H}_{l-1}$ it is clear that $\pi_1\overline{H} \le \ker\left(\pi_1\overline{H}_{l-1} \to \pi_1(x^{0,l-1}+Y^{k-l+1})\right)$. Hence, the only point that is left to prove is that $\pi_1\overline{H}$ contains

$$\ker\left(\pi_1\overline{H}_{l-1} \to \pi_1(x^{0,l-1}+Y^{k-l+1}) = \mathbb{Z}^{2k-2(l-1)}\right).$$

Let $g \in \ker\left(\pi_1 \overline{H}_{l-1} \overset{h_*}{\to} \pi_1(x^{0,l-1} + Y^{k-l+1})\right)$. Then

$$g \in \ker\left(\pi_1 \overline{H}_{l-1} \overset{h_{l*}}{\to} \pi_1\left(x^{0,l} + Y^{k-l+1}/Y^{k-l}\right)\right),$$

since the map h_{l*} factors through $h_* : \pi_1 \overline{H}_{l-1} \to \pi_1(x^{0,l-1} + Y^{k-l+1})$.

By exactness of (6.3) for l, this implies that there is $h \in \pi_1 \overline{H}_l$ with $\iota_{l*}(h) = g$, where $\iota_l : \overline{H}_l \hookrightarrow \overline{H}_{l-1}$ is the inclusion map. It follows from commutativity of the diagram of groups (6.4) and injectivity of the vertical maps that $h \in \ker(\pi_1 \overline{H}_l \to \pi_1(x^{0,l} + Y^{k-l}))$. The induction assumption now implies that $h \in \mathrm{Im}(\pi_1 \overline{H} \to \pi_1 \overline{H}_l)$.

Hence, by Induction hypothesis, $g \in \pi_1 \overline{H}$ and therefore the map $h|_{\overline{H}_{l-1}}$ does indeed induce a short exact sequence

$$1 \to \pi_1 \overline{H} \to \pi_1 \overline{H}_{l-1} \to \pi_1 Y^{k-l+1} \to 1.$$

In particular, for $l = 0$ we then obtain that h induces a short exact sequence

$$1 \to \pi_1 \overline{H} \to \pi_1 X \to \pi_1 Y \to 1.$$

□

Note that our proof of Theorem 6.1.7 uses the fact that the maps h_l are proper: this is required to justify the application of the Ehresmann Fibration Theorem here, and again in the proof of Theorem 6.1.1, which we invoked in the proof of Theorem 6.1.5. A natural approach to Conjecture 6.1.2 fails at this point because a non-proper situation arises when pursuing a similar inductive technique. We will get back to this point in Chapter 9, where we present a promising strategy for a proof of our conjecture.

6.2 Connectedness of fibres

We will make use of the following result about the connectedness of fibres of maps onto complex tori.

Theorem 6.2.1. *Let X_1, X_2, and X_3 be connected compact manifolds, Y a torus and $y \in Y$. Assume that there are surjective maps $f_1 : X_1 \to Y$, $f_3 : X_3 \to Y$ and $f_2 : X_2 \to Y$, $g = f_2 + f_3 : X_2 \times X_3 \to Y$ with the following properties:*

1. *For any $u \in Y$ there is $x_3 \in f_3^{-1}(u)$ such that any path in Y starting at u lifts to a path in X_3 starting at x_3.*

2. *There is $w \in Y$, an open ball $B \subset Y$ with centre w and $x_2^0 \in f_2^{-1}(w)$ so that every loop in B based at w lifts to a loop in X_2 based at x_2^0.*

3. *There is $D_1 \subset Y$ such that $f_1 : X_1 \setminus f_1^{-1}(D_1) \to Y \setminus D_1$ is an unramified covering map and a basis $\mu_1, \cdots, \mu_k$ of standard generators of $\pi_1 Y$ satisfying assertion (4) such that its normal closure in $\pi_1(Y \setminus D_1)$ satisfies*

$$\langle\langle \mu_1, \cdots, \mu_k \rangle\rangle \leq f_{1*}(\pi_1(X_1 \setminus f_1^{-1}(D_1)).$$

Assume furthermore that the set $(X_2 \times X_3) \setminus g^{-1}(y - D_1)$ is path-connected for all $y \in Y$.

4. *Assume that there are $p_1, \cdots, p_l \in D_1$ and $g_1, \cdots, g_k \in \pi_1(Y \setminus D_1)$ such that $\pi_1(Y \setminus D_1) = \langle \mu_1, \cdots, \mu_k, b_1, \cdots, b_l \rangle$ and for any choice of open neighbourhoods U_i of p_i, $1 \leq i \leq l$ there are paths $\delta_1, \cdots, \delta_l : [0,1] \to Y \setminus D_1$ starting at a base point $z^0 \in Y \setminus D_1$ and ending at a point in U_i and loops $\nu_i : [0,1] \to Y \setminus D_1$ such that the concatenation $\beta_i = \delta_i \cdot \nu_i \cdot \delta_i^- 1$ is a representative of b_i with base point z^0.*

Let $h = f_1 + f_2 + f_3 : X_1 \times X_2 \times X_3 \to Y$ and $H_y = h^{-1}(y)$ be its fibre at y.

Then the projection map $\mathrm{pr} : H_y \to X_2 \times X_3$ is surjective, its restriction $\mathrm{pr} : H_y \setminus \mathrm{pr}^{-1}(g^{-1}(y - D_1)) \to (X_2 \times X_3) \setminus g^{-1}(y - D_1)$ is a covering map and the set $H_y \setminus \mathrm{pr}^{-1}(g^{-1}(y - D_1))$ is connected.

As an immediate Corollary we obtain

Corollary 6.2.2. *If, under the same assumptions, we further assume that $H_y = \overline{H_y \setminus \mathrm{pr}^{-1}(g^{-1}(y - D_1))}$, then h has connected fibres.*

Before proving Theorem 6.2.1, we want to give an intuition how the proof works: The basic idea is that the projection $\mathrm{pr} : H_y \to X_2 \times X_3$ behaves like a branched covering which is obtained purely by branching over a subset of $X_2 \times X_3$. The branching behaviour comes from the fact that the covering $f_1 : X_1 \setminus f_1^{-1}(D_1) \to Y \setminus D_1$ behaves like a branched covering over D_1.

This allows us to show connectedness of H_y by showing connectedness of the covering. After choosing a suitable point $(x_2^0, x_3^0) \in X_2 \times X_3$ and a point $x^0 = (x_1^0, x_2^0, x_3^0) \in \mathrm{pr}^{-1}(x_2^0, x_3^0)$, we need to prove that for any $x = (x_1, x_2^0, x_3^0) \in \mathrm{pr}^{-1}(x_2^0, x_3^0)$ there is a loop in $X_2 \times X_3$ whose lift to H_y connects x^0 to x. We obtain such a loop by first choosing a suitable path γ_1 in X_1. Since by properties (3) and (4) the covering f_1 comes purely from branching, we can choose this path to project onto a concatenation of loops of the form $\delta_i \cdot \nu_i \cdot \delta_i^{-1}$.

The path γ_1 is not contained in H_y though. To fix this problem we go forth and back along paths in X_3 to compensate for the δ_i contributions to γ_i and travel along small loops in X_2 to compensate for the ν_i contribution, yielding a loop in $X_2 \times X_3$. Properties (1) and (2) ensure that we can do this. We will now formalise this argument.

Proof of Theorem 6.2.1. We start by proving that the projection $\mathrm{pr} : H_y \to X_2 \times X_3$ is surjective and that the preimage of any point in $(X_2 \times X_3) \smallsetminus g^{-1}(y - D_1)$ has precisely m elements.

For a point $(x_2, x_3) \in X_2 \times X_3$ consider the intersection

$$\begin{aligned} H_y \cap \mathrm{pr}^{-1}(x_2, x_3) &= \{(x, x_2, x_3) \in X_1 \times X_2 \times X_3 \mid f_1(x) = y - g(x_2, x_3)\} \\ &= f_1^{-1}(y - g(x_2, x_3)) \times \{(x_2, x_3)\}. \end{aligned}$$

By surjectivity of f_1, this set is non-empty and thus pr is surjective. If, moreover, $(x_2, x_3) \in (X_2 \times X_3) \smallsetminus g^{-1}(y - D_1)$ then we obtain that $y - g(x_2, x_3) \in Y \smallsetminus D_1$ and thus by assumption (3) the intersection $H_y \cap \mathrm{pr}^{-1}(x_2, x_3)$ has precisely m elements.

In fact, the restriction of pr to $H_y \smallsetminus (\mathrm{pr}^{-1}(g^{-1}(y - D_1)))$ is an unramified covering:

Let $(x_2, x_3) \in (X_2 \times X_3) \smallsetminus g^{-1}(y - D_1)$ and let $U \subset Y \smallsetminus D_1$ be an open neighbourhood of $y - g(x_2, x_3)$ such that $f_1^{-1}(U)$ is the union of m pairwise disjoint open sets $V_1, \cdots, V_m$, with the property that $f_1|_{V_i} : V_i \to U$ is a homeomorphism for $i = 1, \cdots, m$. Such a U exists, since f_1 is an unramified covering on $X_1 \smallsetminus f_1^{-1}(D_1)$.

The preimage $\mathrm{pr}^{-1}(g^{-1}(y - U))$ consists of the disjoint union of the m open sets $H_y \cap (V_i \times g^{-1}(y - U))$, $i = 1, \cdots, m$. The restriction $\mathrm{pr} : H_y \cap (V_i \times g^{-1}(y - U)) \to g^{-1}(y - U)$ is continuous and bijective and has continuous inverse

$$(x_2, x_3) \mapsto \big((f|_{V_i})^{-1}(y - g(x_2, x_3)), x_2, x_3\big)$$

on the open set $g^{-1}(y - U)$. Thus, pr is indeed an unramified covering map over $(X_2 \times X_3) \smallsetminus g^{-1}(y - D_1)$ of covering degree equal to the covering degree of f_1.

We will now show how connectedness of $H_y \smallsetminus \mathrm{pr}^{-1}(g^{-1}(y - D_1)$ follows from conditions (1)-(4):

Let $z^0 \in Y \smallsetminus D_1$ be as in (4) and let $f_1^{-1}(z^0) = \{x_{1,1}^0, \cdots, x_{1,m}^0\}$. Let further w and x_2^0 be as in (2) and let $x_3^0 \in f_3^{-1}(y - z_0 - w)$ be as in (1).

Since by (3) the set $(X_2 \times X_3) \smallsetminus g^{-1}(y - D_1)$ is path-connected and we proved that pr is an m-sheeted covering over this set, it suffices to show that for $x^0 = (x_{1,1}^0, x_2^0, x_3^0)$ we can find paths

$$\alpha_1, \cdots, \alpha_m : [0, 1] \to H_y \smallsetminus \mathrm{pr}^{-1}(g^{-1}(y - D_1))$$

with $\alpha_i(0) = x^0$ and $\alpha_i(1) = (x^0_{1,j}, x^0_2, x^0_3)$, $i = 1, \cdots, m$.

For $p_1, \cdots, p_l$ as in (4), let $U_1, \cdots, U_l$ be open neighbourhoods such that for any point $v \in Y$ we have: if $w \in v - U_i$ then $v - U_i \subset B$, where w and B are as in (2).
Such U_i clearly always exist by choosing $\operatorname{diam}(U_i) \leq \frac{1}{2}\operatorname{diam}(B)$ with respect to the standard Euclidean metric on the torus $Y \cong \mathbb{R}^n/\mathbb{Z}^n$.

Since $f_1|_{X_1 \setminus f_1^{-1}(D_1)}$ is an m-sheeted covering, there exist coset representatives

$$s_1, \cdots, s_m : [0,1] \to Y \setminus D_1$$

for $\pi_1(Y \setminus D_1)/(f_{1*}\pi_1(X_1 \setminus f_1^{-1}(D_1))$ such that s_i lifts to a path in $X_1 \setminus f_1^{-1}(D_1)$ starting at $x^0_{1,1}$ and ending at $x^0_{1,i}$, $i = 1, \cdots, m$.

Since, by (3), $\langle\langle \mu_1, \cdots, \mu_k \rangle\rangle \leq f_{1*}\pi_1(X_1 \setminus f_1^{-1}(D_1))$ we may assume that the loops $s_1, \cdots, s_m$ represent elements of $\langle g_1, \cdots, g_l \rangle$ (see (4)).

Hence, by (4), each of the s_i is homotopic to a concatenation of loops of the form $\beta_j = \delta_j \cdot \nu_j \cdot (\delta_j)^{-1}$ and their inverses. Thus, without loss of generality we may assume that s_i is indeed a concatenation of such loops.

By (1) there is a lift $\epsilon_j : [0,1] \to X_3$ of the path $t \mapsto y - w - \delta_j(t)$ with $\epsilon_j(0) = x^0_3$, $j = 1, \cdots, l$.

Note that

$$w = y - f_3(\epsilon_j(1)) - \delta_j(1) = y - f_3(\epsilon_j(1)) - \nu_j(0) \in y - f_3(\epsilon_j(1)) - U_j.$$

Thus, the map

$$t \mapsto y - f_3(\epsilon_j(1)) - \nu_j(t) \text{ in } y - f_3(\epsilon_j(1)) - U_j \subset B$$

is a loop in B.

Hence, by (2), there is a loop $\lambda_j : [0,1] \to f_2^{-1}(B)$ with $\lambda_j(0) = \lambda_j(1) = x^0_2$ lifting the loop $t \mapsto y - f_3(\epsilon_j(1)) - \nu_j(t)$ to X_2.

By construction, the concatenation

$$t_j = (x^0_2, \epsilon_j) \cdot (\lambda_j, \epsilon_j(1)) \cdot (x^0_2, (\epsilon_j)^{-1})$$

is a loop in $(X_2 \times X_3) \setminus g^{-1}(y - D_1)$ such that $g \circ t_j + \beta_j \equiv y$.

Let $s_i = \beta^{\epsilon_1}_{j} \cdot \cdots \cdot \beta^{\epsilon_r}_{j_r}$ for $\epsilon_i \in \{\pm 1\}$ and $j_i \in \{1, \cdots, l\}$ and let $\widetilde{s}_i : [0,1] \to X_1 \setminus f_1^{-1}(D_1)$ be the unique lift of s_i with $\widetilde{s}_i(0) = x^0_{1,1}$. Then

$$\alpha_i = \left(\widetilde{s}_i, t^{\epsilon_1}_{j_1} \cdot \cdots \cdot t^{\epsilon_r}_{j_r}\right) : [0,1] \to H_y \setminus \operatorname{pr}^{-1}(g^{-1}(y - D_1))$$

defines a path in $H_y \setminus \operatorname{pr}^{-1}(g^{-1}(y - D_1))$ with $\alpha_i(0) = (x^0_{1,1}, x^0_2, x^0_3)$ and $\alpha_i(1) = (x^0_{1,i}, x^0_2, x^0_3)$. In particular, it follows that $H_y \setminus \operatorname{pr}^{-1}(g^{-1}(y - D_1))$ is connected.

$\square$

The following remark should make clear why the seemingly rather abstract conditions in the Theorem come up naturally:

Remark 6.2.3.

(a) It is well-known that condition (4) is satisfied for $E = \mathbb{C}/\Lambda$ an elliptic curve and $D_1 = \{p_1, \cdots, p_l\}$ a finite set of points. This follows by choosing the ν_i to be the boundary circles of small discs around p_i and the δ_i to be simple pairwise non-intersecting paths connecting z^0 to $\nu_i(0)$ inside a fundamental domain for the $\Lambda(\cong \mathbb{Z}^2)$-action on $\mathbb{C}$.

(b) Condition (2) is for instance satisfied if f_2 is an unramified covering on the complement of a closed proper subset $D_2 \subset Y$.

(c) Condition (1) is satisfied in many circumstances in which f_3 is surjective, for instance if f_3 satisfies the homotopy lifting property. It is also clearly satisfied if f_3 is of the form $q_1 + \cdots + q_n : X_{3,1} \times \cdots \times X_{3,n} \to Y = E = \mathbb{C}/\Lambda$ such that q_1 is a finite-sheeted branched covering.

(d) Path-connectedness of $(X_2 \times X_3) \smallsetminus g^{-1}(y - D_1)$ is for instance satisfied if f_2 and f_3 are holomorphic and surjective and D_1 is an analytic subvariety of Y of codimension ≥ 1 , since then $g^{-1}(y - D_1)$ is an analytic subvariety of codimension ≥ 1 in $X_2 \times X_3$.

Note that Condition (4) in Theorem 6.2.1 is satisfied if f_1 is purely branched (see Definition 4.1.2).

Remark 6.2.4. A change of a lift of the basepoint for the fundamental group of $X \smallsetminus f^{-1}(D)$ corresponds to conjugation by an element of $\pi_1(Y \smallsetminus D)$. Hence, an equivalent topological characterisation of the property that for a regular covering map $f : X \smallsetminus f^{-1}(D) \to Y \smallsetminus D$ the normal subgroup generated by elements $\mu_1, \cdots, \mu_k$ is in the image of f_* is that every lift of the μ_i to $X \smallsetminus f^{-1}(D)$ is a loop.

Addendum 6.2.5. *Conditions (1)-(4) of Theorem 6.2.1 are well-behaved under taking direct products. For instance if $f_1 : X_1 \to Y$ and $f_1' : X_1' \to Y'$ satisfy conditions (3) and (4) for sets $D_1 \subset Y$ and $D_1' \subset Y'$, then it is easy to see that also $(f_1, f_1') : X_1 \times X_1' \to Y \times Y'$ satisfies conditions (3) and (4) for the set $D = (Y \times D_1') \cup (D_1 \times Y)$, with the possible exception of connectedness of $((X_2 \times X_2') \times (X_3 \times X_3')) \smallsetminus g^{-1}((y, y') - D)$.*

In fact, we have the following stronger result in the setting of purely branched covering maps.

Lemma 6.2.6. *Let Y and Y' be tori and let $f : X \to Y$ and $f' : X' \to Y'$ be purely branched covering maps with branching loci D, respectively D'. Then the map $(f, f') : X \times X' \to Y \times Y'$ is purely branched with branching locus $(D \times Y') \cup (Y \times D')$.*

Proof. It is clear that the map is a branched covering with branching locus $(D \times X') \cup (X \times D')$. Let $\mu_1, \cdots, \mu_k$ be generators of $\pi_1 Y$ and $\mu'_1, \cdots, \mu'_{k'}$ be generators of $\pi_1 Y'$ such that $\langle\langle \mu_1, \cdots, \mu_k \rangle\rangle \le f_*(\pi_1(X \smallsetminus f^{-1}(D)))$ and $\left\langle\left\langle \mu'_1, \cdots, \mu'_{k'} \right\rangle\right\rangle \le f'_*(\pi_1(X' \smallsetminus f'^{-1}(D')))$. Then

$$
\begin{aligned}
\langle\langle \mu_1, \cdots, \mu_k, \mu'_1, \cdots, \mu'_{k'} \rangle\rangle &= \langle\langle \mu_1, \cdots, \mu_k \rangle\rangle \times \langle\langle \mu'_1, \cdots, \mu'_{k'} \rangle\rangle \\
&\le f_*(\pi_1(X \smallsetminus f^{-1}(D))) \times f'_*(\pi_1(X' \smallsetminus f'^{-1}(D'))) \\
&= (f, f')_* \left(\pi_1 \left((X \times X') \smallsetminus ((f^{-1}(D) \times X') \cup (X \times f^{-1}(D')))\right)\right).
\end{aligned}
$$

Hence, (f, f') is indeed purely branched. □

The proof that condition (4) is preserved under taking products is similar.

6.3 A class of higher dimensional examples

In this section we will construct a general class of examples of Kähler subgroups of direct products of surface groups arising as kernels of homomorphisms onto $\mathbb{Z}^{2k}$ for any $k \ge 1$. Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, let $r \ge 3$ and let

$$\alpha_i : S_{\gamma_i} \to E$$

be branched holomorphic coverings for $1 \le i \le r$.

Our groups will be the fundamental groups of the fibres of holomorphic surjective maps from the direct product $S_{\gamma_1} \times \cdots \times S_{\gamma_r}$ onto the k-fold direct product $E^{\times k}$ of E with itself. For vectors $w_1, \cdots w_n \in \mathbb{Z}^k$ we will use the notation $(w_1 \mid \cdots \mid w_n)$ to denote the $k \times n$-matrix with columns w_i. To construct these maps we make use of the following result

Lemma 6.3.1. *Let $v_1 = (v_{1,1}, \cdots, v_{k,1})^t, \cdots, v_r = (v_{1,r}, \cdots, v_{k,r})^t \in \mathbb{Z}^k$. Then the $\mathbb{C}$-linear map $B = (v_1 \mid v_2 \mid \cdots \mid v_r) \in \mathbb{Z}^{k \times r} \subset \mathbb{C}^{k \times r}$ descends to a holomorphic map*

$$\overline{B} : E^{\times r} \to E^{\times k}.$$

If, in addition, $r = k$ and $B \in \mathrm{GL}(\mathrm{k}, \mathbb{C}) \cap \mathbb{Z}^{\mathrm{k} \times \mathrm{k}}$ then $\overline{B}$ is a regular covering map. In particular, $\overline{B}$ is a biholomorphic automorphism of $E^{\times k}$ if $B \in \mathrm{GL}(\mathrm{k}, \mathbb{Z})$.

Proof. It suffices to prove that B preserves maps $\Lambda^{\times r}$ into $\Lambda^{\times k}$. For this let $\lambda_1, \lambda_2 \in \mathbb{C}$ be a $\mathbb{Z}$-basis for Λ and denote by $\lambda_{1,i}, \lambda_{2,i}$ the corresponding $\mathbb{Z}$-basis of the ith factor of $\Lambda^{\times r}$ and by $\lambda'_{1,j}, \lambda'_{2,j}$ the corresponding $\mathbb{Z}$-basis of the jth factor of $\Lambda^{\times k}$. Then we have

$$B\lambda_{i,j} = \sum_{l=1}^{k} v_{l,j} \lambda'_{i,l} \in \Lambda^{\times k} \text{ for } 1 \le i \le 2,\ 1 \le j \le r$$

It follows that B descends to a holomorphic map $\overline{B} : E^{\times r} \to E^{\times k}$. If $B \in \mathrm{GL}(\mathrm{k}, \mathbb{C}) \cap \mathbb{Z}^{\mathrm{k} \times \mathrm{k}}$ then $\overline{B}$ is a regular covering map, since B is a local homeomorphism. If, in addition, $B \in \mathrm{GL}(\mathrm{k}, \mathbb{Z})$, then it is immediate that $\overline{B}$ and $\overline{B^{-1}}$ are mutual inverses. ☐

We say that a set of vectors $\mathcal{C} = \{v_1, \cdots, v_r\} \subset \mathbb{Z}^k$ has property

(P1) if there is a partition $\mathcal{C} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ such that $\mathcal{E}_1$ is a $\mathbb{Z}$-basis for $\mathbb{Z}^k \subset \mathbb{C}^k$ and $\mathcal{E}_2$, $\mathcal{E}_3$ are both spanning sets for $\mathbb{C}^k$ as a $\mathbb{C}$-vector space.

(P1') if (P1) holds and $\mathcal{E}_1$ is the standard $\mathbb{Z}$-basis for $\mathbb{Z}^k$

(P2) if $\mathcal{C}$ has property (P1) and in addition any choice of k vectors in $\mathcal{C}$ is linearly independent.

By Lemma 6.3.1 for any set $\mathcal{C} = \{v_1, \cdots, v_r\} \subset \mathbb{Z}^k$ and $B = (v_1 \mid \cdots \mid v_r)$ we can define a holomorphic map

$$h = \overline{B} \circ (\alpha_1, \cdots, \alpha_r) = \sum_{i=1}^{r} v_i \cdot \alpha_i : S_{\gamma_1} \times \cdots \times S_{\gamma_r} \to E^{\times k}.$$

We will be interested in maps h for which the set $\mathcal{C}$ has properties (P1') and (P2). Note that after adjusting by a biholomorphic automorphism of $E^{\times k}$, say $A \in \mathrm{GL}(\mathrm{k}, \mathbb{Z})$, and after reordering the factors of $S_{\gamma_1} \times \cdots \times S_{\gamma_r}$, we may in fact assume that $\mathcal{E}_1 = \{v_1, \cdots, v_k\}$ and that $\{v_1, \cdots, v_k\}$ is the standard basis for $\mathbb{Z}^k$. In particular, we may assume that property (P1') holds if property (P1) holds.

The following result shows that such maps exist.

Proposition 6.3.2. *For all positive integers r, k there is a set $\mathcal{C} = \{v_1, \cdots, v_r\} \subset \mathbb{Z}^k$ with the property that for any integers $1 \le i_1 < i_2 < \cdots < i_k \le r$ the subset $\{v_{i_1}, \cdots, v_{i_k}\}$ is linearly independent. Moreover, if $r \ge 3k$ we may assume that $\mathcal{C}$ has properties (P1') and (P2).*

Proof. The proof is by induction on r. For $r = 1$ the statement is trivial. Assume that for a positive integer r we have a set $\mathcal{C} = \{v_1, \cdots, v_r\} \subset \mathbb{Z}^k$ of r vectors with the property that for any $1 \leq i_1 < i_2 < \cdots < i_k \leq r$ the subset $\{v_{i_1}, \cdots, v_{i_k}\} \subset \mathcal{C}$ is linearly independent.

Let I be the set of all $(k-1)$-tuples $\mathbf{i} = (i_1, \cdots, i_{k-1})$ of integers $1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq r$. For $\mathbf{i} \in I$ denote by $W_{\mathbf{i}} = \mathrm{span}_{\mathbb{C}} \{v_{i_1}, \cdots, v_{i_{k-1}}\}$ the $\mathbb{C}$-span of the linearly independent set $\{v_{i_1}, \cdots, v_{i_{k-1}}\}$. Then $W = \bigcup_{\mathbf{i} \in I} W_{\mathbf{i}}$ is a finite union of complex hyperplanes in $\mathbb{C}^k$ such that for any vector $v_{r+1} \in \mathbb{Z}^k \setminus W$ the set $\mathcal{C} \cup \{v_{r+1}\}$ has the desired property. The set $\mathbb{Z}^k \setminus W$ is nonempty, because $\mathbb{Z}^k \subset \mathbb{C}^k$ is Zariski-dense in $\mathbb{C}^k$.

If $r \geq 3k$ then choosing $\mathcal{E}_1 = \{v_1, \cdots, v_k\}$ to be the standard basis of $\mathbb{C}^k$, $\mathcal{E}_2 = \{v_{k+1}, \cdots, v_{2k}\}$ and $\mathcal{E}_3 = \{v_{2k+1}, \cdots, v_r\}$ ensures that properties (P1') and (P2) are satisfied. $\square$

Note that the proof of Proposition 6.3.2 shows that properties (P1') and (P2) are in some sense generic properties.

The main result of this section is:

Theorem 6.3.3. *Let $\mathcal{C} \subset \mathbb{Z}^k$ and $\overline{B}$ be as defined above. Assume that $\mathcal{C}$ satisfies properties (P1') and (P2), that $\mathcal{E}_1 = \{v_1, \cdots, v_k\}$ and that $\alpha_1, \cdots, \alpha_k$ are* purely *branched coverings. Then the smooth generic fibre $\overline{H}$ of h is connected and its fundamental group fits into a short exact sequence*

$$1 \to \pi_1 \overline{H} \to \pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r} \xrightarrow{h_*} \pi_1 E^{\times k} = \mathbb{Z}^{2k} \to 1.$$

Furthermore, $\pi_1 \overline{H}$ is a Kähler group of type $\mathcal{F}_{r-k}$ but not of type $\mathcal{F}_r$. In fact $\pi_j H = 0$ for $2 \leq j \leq r - k - 1$.

Denote by $\pi_l : E^{\times k} \to \{0\} \times E^{\times l}$ the canonical projection onto the last l factors and for a map h satisfying the conditions of Theorem 6.3.3 let $h_l = \pi_l \circ h$. Due to the assumptions on $\mathcal{C}$ the map h_l factors as $h_l = f_l \circ g_l$ for $1 \leq l \leq k$ with

$$f_l = (v_{k-l+1} \mid \cdots \mid v_r) \circ (\alpha_{k-l+1}, \cdots, \alpha_r) : S_{\gamma_{k-l+1}} \times \cdots \times S_{\gamma_r} \to Y^k / Y^{k-l} = \{0\} \times E^{\times l}$$

and

$$g_l : S_{\gamma_1} \times \cdots \times S_{\gamma_r} \to S_{\gamma_{k-l+1}} \times \cdots \times S_{\gamma_r}$$

the canonical projection with fibre $F_l := S_{\gamma_1} \times \cdots \times S_{\gamma_{k-l}}$, a product of closed hyperbolic surfaces (It follows from the fact that $v_1, \cdots, v_k \in \mathbb{Z}^k$ is a standard basis of $\mathbb{Z}^k$ that $h_l = f_l \circ g_l$ for $1 \leq l \leq k$).

Theorem 6.3.3 will be a consequence of Theorem 6.1.3 after checking that the maps h, h_l, g_l and f_l satisfy all necessary conditions.

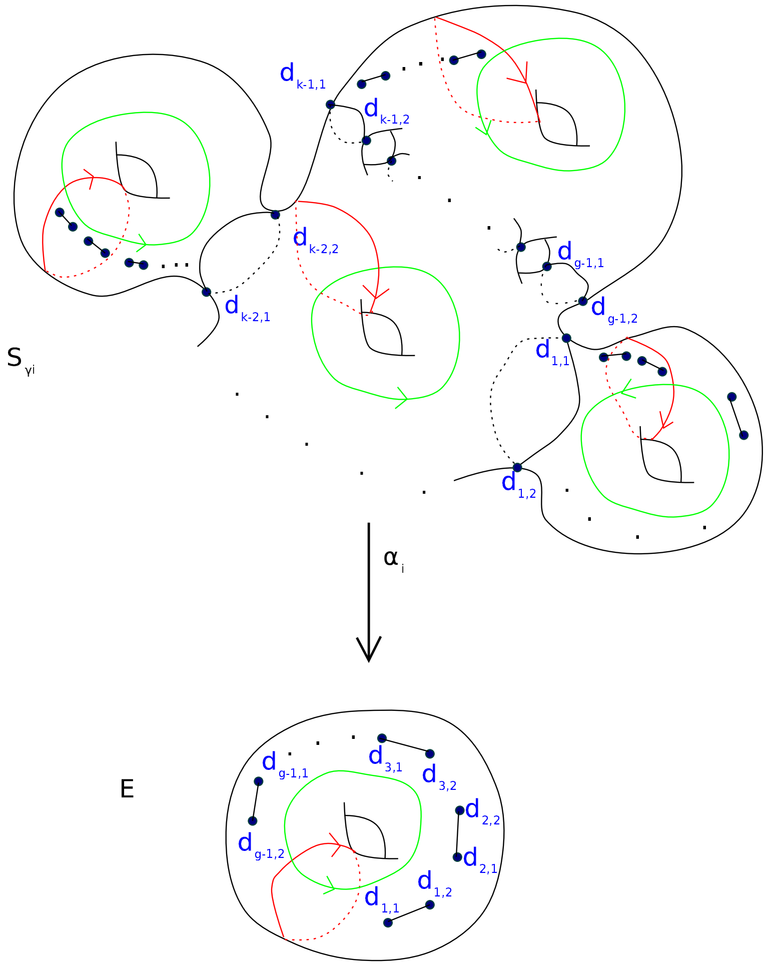

Figure 6.1: Example of a k-fold purely branched covering constructed in Chapter 4 with 2g-2 branching points $d_{1,1}, d_{1,2}, \ldots, d_{g-1,2}$

Proposition 6.3.4. *Under the assumptions in Theorem 6.3.3, the maps h, h_l, f_l, and g_l, $1 \leq l \leq k$, have connected fibres.*

Proof. We introduce the notation

$$A_1 = (v_1 \mid \cdots \mid v_k)\,,\ A_2 = (v_{k+1} \mid \cdots \mid v_{2k}) \text{ and } A_3 = (v_{2k+1} \mid \cdots \mid v_{3k})\,.$$

To simplify notation we will use the same notation for linear maps between $\mathbb{C}$-vector spaces and their induced maps on direct products of elliptic curves.

By assumption $A_1 = (v_1 \mid \cdots \mid v_k) = Id \in \mathrm{GL}(\mathrm{k}, \mathbb{Z})$. Since $\mathcal{C}$ has property (P2) we may further assume that $v_{k+1}, \cdots, v_{2k}$ and $v_{2k+1}, \cdots, v_{3k}$ are $\mathbb{C}$-bases of $\mathbb{C}^k$. We will prove connectedness of the fibres of h. Since $A_1 = Id$ it will be clear that connectedness of the fibres of f_l and h_l follow by the same argument. Connectedness of the fibres of g_l is trivial.

We want to apply Theorem 6.2.1 and Corollary 6.2.2 to the following maps and compact complex manifolds to show that the fibres of h are connected:

$$f_1 = A_1 \circ (\alpha_1, \cdots, \alpha_k) : X_1 = S_{\gamma_1} \times \cdots \times S_{\gamma_k} \to E^{\times k},$$

$$f_2 = A_2 \circ (\alpha_{k+1}, \cdots, \alpha_{2k}) : X_2 = S_{\gamma_{k+1}} \times \cdots \times S_{\gamma_{2k}} \to E^{\times k},$$

$$f_3 = (A_3 \mid v_{3k+1} \mid \cdots \mid v_r) \circ (\alpha_{2k+1}, \cdots, \alpha_r) : X_3 = S_{\gamma_{2k+1}} \times \cdots \times S_{\gamma_r} \to E^{\times k}.$$

Checking the conditions for Theorem 6.2.1: It is clear that f_1 and $g = f_2 + f_3$ are surjective holomorphic maps. We will now check that conditions (1) to (4) in Theorem 6.2.1 are satisfied for h.

Condition (1): Fix a point $p \in S_{\gamma_{3k+1}} \times \cdots \times S_{\gamma_r}$ and let

$$q := (v_{3k+1} \mid \cdots \mid v_r) \circ (\alpha_{3k+1}, \cdots, \alpha_r)(p).$$

Since $A_3 \in \mathrm{GL}(\mathrm{k}, \mathbb{C}) \cap \mathbb{Z}^{\mathrm{k} \times \mathrm{k}}$, there is an inverse $A_3^{-1} \in \mathrm{GL}(\mathrm{k}, \mathbb{Q})$ and a minimal integer $d_3 \in \mathbb{Z}$ with $T_3 = d_3 \cdot A_3^{-1} \in \mathbb{Z}^{k \times k}$. In particular, we have $T_3 \cdot A_3 = d_3 \cdot \mathrm{Id}$. By Lemma 6.3.1 the induced map $T_3 : E^{\times k} \to E^{\times k}$ is a regular covering. Consider the composition

$$\hat{f}_3 = T_3 \circ f_3(z_{2k+1}, \cdots, z_{3k}, p) = (d_3 \cdot \alpha_{2k+1}(z_{2k+1}), \cdots, d_3 \cdot \alpha_{3k}(z_{3k})) + q. \tag{6.5}$$

Note that $\hat{f}_3$ is surjective. It suffices to prove that for any $\underline{w} \in E^{\times k}$ and $(\underline{z}, p) \in f_3^{-1}(\underline{w})$ we can lift any path $\nu : [0,1] \to E^{\times k}$ with $\nu(0) = \underline{w}$ to a path $(\widetilde{\nu}, p) : [0,1] \to X_3$ with $\widetilde{\nu}(0) = \underline{z}$ under f_3. Since T_3 is a regular covering this is possible if and only if there is a lift of $T_3 \circ \nu$ to a path $(\widetilde{\nu}, p) : [0,1] \to X_3$ with $\widetilde{\nu}(0) = \underline{z}$. The latter follows immediately from equation (6.5) and the assumption that the α_i are branched covering maps.

Condition (2): By the same argument as above, there is a map $T_2 \in \mathrm{GL}(\mathrm{k}, \mathbb{Z})$ such that

$$T_2 \circ f_2 = (d_2 \cdot \alpha_{k+1}, \cdots, d_2 \cdot \alpha_{2k}) : \pi_1 S_{\gamma_{k+1}} \times \cdots \times \pi_1 S_{\gamma_{2k}} \to E^{\times k}$$

and since T_2 is a regular covering map Condition (2) holds if and only if it holds for the map $T_2 \circ f_2$. It is clear that Condition (2) holds for a regular value $\underline{w} \in E^{\times k}$ of $T_2 \circ f_2$ and a sufficiently small ball B around $\underline{w}$, since $T_2 \circ f_2$ restricts to an unramified finite-sheeted covering on the complement of a subset of complex codimension one in $E^{\times k}$.

Conditions (3) and (4): By assumption $A_1 = Id$ and $\alpha_1, \cdots, \alpha_k$ are purely branched coverings and therefore satisfy property (3) with the exception of the path-connectedness part, which we need to check separately. By Remark 6.2.3(b) the maps $\alpha_1, \cdots, \alpha_k$ satisfy condition (4).

Thus, an iterated application of Addendum 6.2.5 and Lemma 6.2.6 imply that $f_1 = (\alpha_1, \cdots, \alpha_k)$ satisfies all conditions in (3) and (4) with $D_1 \subset E^{\times k}$ the codimension one subvariety of critical values of f_1 except for connectedness of $(X_2 \times X_3) \setminus g^{-1}(y - D_1)$ for $y \in E^{\times k}$. The latter follows from Remark 6.2.3(d), since g is holomorphic and surjective and D_1 is a codimension one complex analytic subvariety of $E^{\times k}$.

Hence, properties (1)-(4) of Theorem 6.2.1 are indeed satisfied and it follows that $H_y \setminus (\mathrm{pr}^{-1}(g^{-1}(y - D_1)))$ is connected for all $y \in E^{\times k}$.

Applying Corollary 6.2.2: By Corollary 6.2.2 it suffices to show that $H_y = \overline{H_y \setminus (\mathrm{pr}^{-1}(g^{-1}(y - D_1)))}$ for $y \in E^{\times k}$ to obtain connectedness of H_y.

Recall that $A_1 = Id$. Let $(x_1, x_2, x_3) \in \mathrm{pr}^{-1}(g^{-1}(y - D_1))$. Since $g^{-1}(y - D_1)$ is an analytic variety of codimension one, its complement $(X_2 \times X_3) \setminus g^{-1}(y - D_1)$ is dense in $X_2 \times X_3$. Thus, there is a sequence $\{(x_{2,n}, x_{3,n})\}_{n \in \mathbb{N}} \subset (X_2 \times X_3) \setminus g^{-1}(y - D_1)$ converging to (x_2, x_3) as $n \to \infty$.

Since $\alpha_1, \cdots, \alpha_k$ are purely branched coverings and $y - g(x_2, x_3) \in D_1$, there is a neighbourhood U of $x_1 \in f_1^{-1}(y - g(x_2, x_3))$ in which f_1 takes the following form after an appropriate choice of coordinates:

$$(z_1, \cdots, z_k) \mapsto (z_1^{i_1}, \cdots, z_k^{i_k}) \text{ for some integers } i_1, \cdots, i_k \geq 1$$

and the set of critical values D_1 is

$$D_1 = f_1(U) \cap \left(\bigcup_{j : i_j \geq 2} \mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{k-j+1} \right).$$

We may further assume that $y - g(x_{2,n}, x_{3,n}) \in f_1(U)$ for all $n \in \mathbb{N}$.

It is now clear that we can choose a sequence $x_{1,n} \in f_1^{-1}(y - g(x_{2,n}, x_{3,n}))$ which converges to x_1 as $n \to \infty$.

Hence, the sequence $\{(x_{1,n}, x_{2,n}, x_{3,n})\}_{n \in \mathbb{N}} \subset H_y \setminus \mathrm{pr}^{-1}(g^{-1}(y - D_1))$ converges to (x_1, x_2, x_3) as $n \to \infty$ and in particular $\overline{H_y \setminus (\mathrm{pr}^{-1}(g^{-1}(y - D_1)))} = H_y$ for $y \in E^{\times k}$ is connected. □

Proposition 6.3.5. *Under the assumptions of Theorem 6.3.3 consider the filtration $Y^l = E^{\times l} \times \{0\}$ of $E^{\times k}$ where $\pi_l : E^{\times k} \to Y^k / Y^{k-l} = \{0\} \times E^{\times l}$ is the projection onto the last l coordinates.*

Then the map h satisfies the condition that $h_l = \pi_l \circ h$ has fibrelong isolated singularities for $0 \leq l \leq k$. More precisely, the factorisation $h_l = f_l \circ g_l$ satisfies that g_l is a regular fibration and f_l has isolated singularities. Furthermore, the dimension of the smooth generic fibre H_l of f_l is r-k for $1 \leq l \leq k$.

Proof. Recall that by definition of f_l and g_l we have $h_l = f_l \circ g_l$. The map g_l is clearly a regular fibration. To see that the map f_l has isolated singularities consider its differential

$$\mathrm{Df_l} = (\mathrm{D}\pi_\mathrm{l}(\mathrm{v_{k-l+1}}) \cdot \mathrm{d}\alpha_{\mathrm{k-l+1}}, \ldots, \mathrm{D}\pi_\mathrm{l}(\mathrm{v_r}) \cdot \mathrm{d}\alpha_\mathrm{r}) \,. \tag{6.6}$$

Note that by definition of π_l the vector $\mathrm{D}\pi_\mathrm{l}(\mathrm{v_i})$ is the vector in $\mathbb{Z}^l$ consisting of the last l entries of v_i. By property (P2) any k vectors in $\mathcal{C}$ form a linearly independent set. Furthermore we chose $\mathcal{C}$ such that $\mathcal{E}_1 = \{v_1, \ldots, v_k\}$ is the standard basis of $\mathbb{Z}^k$. This implies that the set

$$\mathcal{C}_l = \{\mathrm{D}\pi_\mathrm{l}(\mathrm{v_{k-l+1}}), \ldots, \mathrm{D}\pi_\mathrm{l}(\mathrm{v_r})\} \subset \mathbb{Z}^l$$

also has property (P2). In particular, any choice of l vectors in $\mathcal{C}_l$ forms a linearly independent set.

It follows from (6.6) that a point $(z_{k-l+1}, \cdots, z_r) \in S_{\gamma_{k-l+1}} \times \cdots \times S_{\gamma_r}$ is a critical point of f_l if and only if z_i is a critical point of α_i for at most $r-k+1$ of the z_i, where $k-l+1 \leq i \leq r$.

Thus, the set of critical points C_{f_l} of f_l is the union $C_{f_l} = \bigcup_{i \in I_l} B_{l,i}$ of a finite number of $(l-1)$-dimensional submanifolds $B_{l,i} \subset S_{\gamma_{k-l+1}} \times \cdots \times S_{\gamma_r}$ with the property that for every surface factor S_{γ_j} of $S_{\gamma_{k-l+1}} \times \cdots \times S_{\gamma_r}$ the projection of $B_{l,i}$ onto S_{γ_j} is either surjective or has finite image. Linear independence of any l vectors in $\mathcal{C}_l$ implies that the restriction of f_l to any of the $B_{l,i}$ is locally injective. Hence, the intersection $C_{f_l} \cap H_{l,y}$ is finite for any fibre $H_{l,y} = h_l^{-1}(y)$, $y \in \{0\} \times E^{\times l}$.

In particular, the map f_l has isolated singularities. It follows immediately that the smooth generic fibre H_l of f_l has dimension $r-(k-l)-l = r-k$. □

Proof of Theorem 6.3.3. The proof follows from Theorem 6.1.7, Proposition 6.3.4 and Proposition 6.3.5. Indeed, by Proposition 6.3.4 and Proposition 6.3.5 combined with the fact that F_l is a direct product of closed hyperbolic surfaces, all assumptions in Theorem 6.1.7 are satisfied with $n = r-k$. Thus, the map h induces a short exact sequence

$$1 \to \pi_1 \overline{H} \to \pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r} \xrightarrow{h_*} \pi_1 E^{\times k} \to 1$$

and isomorphisms $\pi_i \overline{H} \cong \pi_i(S_{\gamma_1} \times \cdots \times S_{\gamma_r}) \cong 0$ for $2 \leq i \leq r-k-1$.

Since $\overline{H}$ is the smooth generic fibre of a holomorphic map, it is a compact complex submanifold and thus a compact projective submanifold of the projective manifold $S_{\gamma_1} \times \cdots \times S_{\gamma_r}$. In particular, it is a compact Kähler manifold and $\pi_1 \overline{H}$ is a Kähler group. Furthermore, $\overline{H}$ can be endowed with the structure of a finite CW-complex.

It follows that we can construct a classifying space $K(\pi_1\overline{H},1)$ from the finite CW-complex $\overline{H}$ by attaching cells of dimension at least $r-k+1$. Hence, there is a $K(\pi_1\overline{H},1)$ with finitely many cells in dimension less than or equal to $r-k$. Thus, the group $\pi_1\overline{H}$ is of type $\mathcal{F}_{r-k}$. Since all α_i are finite-sheeted branched covers, the image of the induced map $v_i \cdot \alpha_{i*}$ in $\pi_1 E^{\times k} \cong \mathbb{Z}^{2k}$ is nontrivial for $1 \leq i \leq r$. By Theorem 4.2.2 the group $\pi_1\overline{H}$ is not of type $\mathcal{F}_r$. □

6.4 Finiteness properties and irreducibility

In this section we want to determine the precise finiteness properties of our examples and prove that they are irreducible.

Theorem 6.4.1. *Let $k \geq 0$ and $r \geq 3k$ be integers and let E be an elliptic curve. Let $\alpha_i : S_{\gamma_i} \to E$ be branched covers of E with $\gamma_i \geq 2$, $1 \leq i \leq r$.*

Then there is a surjective holomorphic map

$$h : S_{\gamma_1} \times \cdots \times S_{\gamma_r} \to E^{\times k}$$

with smooth generic fibre $\overline{H}$ such that the restriction of h to each factor S_{γ_i} factors through α_i; the map h induces a short exact sequence

$$1 \to \pi_1\overline{H} \to \pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r} \to \pi_1 E^{\times k} \cong \mathbb{Z}^{2k} \to 1;$$

and the group $\pi_1\overline{H}$ is Kähler of type $\mathcal{F}_{r-k}$ but not of type $\mathcal{F}_{r-k+1}$. Furthermore, $\pi_1\overline{H}$ is irreducible.

As a consequence of Theorem 6.4.1 and its proof we obtain.

Corollary 6.4.2. *For every $r \geq 3$, $\gamma_1, \cdots, \gamma_r \geq 2$ and $r-1 \geq m \geq \frac{2r}{3}$, there is a Kähler subgroup $G \leq \pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r}$ which is an irreducible full subdirect product of type $\mathcal{F}_m$ but not of type $\mathcal{F}_{m+1}$.*

Let $H \leq G = G_1 \times \cdots \times G_r$ be a subgroup of a direct product of groups $G_1, \cdots, G_r$. For every $1 \leq i_1 < \cdots < i_k \leq r$ denote by $p_{i_1,\cdots,i_k} : G \to G_{i_1} \times \cdots \times G_{i_k}$ the canonical projection. We say that the group H *virtually surjects onto k-tuples* if for every $1 \leq i_1 < \cdots < i_k \leq r$ the group $p_{i_1,\cdots,i_k}(H)$ has finite index in $G_{i_1} \times \cdots \times G_{i_k}$. We say that H is *surjective on k-tuples* if for every $1 \leq i_1 < \cdots < i_k \leq r$ we have equality $p_{i_1,\cdots,i_k}(H) = G_{i_1} \times \cdots \times G_{i_k}$. We say that H is *virtually surjective on pairs* (VSP) if H virtually surjects onto 2-tuples.

For subgroups of direct products of limit groups, a close relation between their finiteness properties and virtual surjection to k-tuples has been observed (see [31],[87], also [96]). In fact if a subgroup $H \leq G_1 \times \cdots \times G_r$ of a direct product of finitely presented groups is subdirect (i.e. surjects onto 1-tuples) then H is finitely generated; and if it is VSP then H is itself finitely presented [31, Theorem A]. The converse is not true in general; it is true though if $G_1, \cdots, G_r$ are (non-abelian) limit groups and H is full subdirect [31, Theorem D].

More generally it is conjectured [87] that, for $G_1, \cdots, G_r$ non-abelian limit groups and $H \leq G_1 \times \cdots \times \times G_r$ a full subdirect product, the following are equivalent:

1. H is of type $\mathcal{F}_k$;

2. H virtually surjects onto k-tuples.

Kochloukova proved that (1) implies (2) and gave conditions under which (2) implies (1).

Theorem 6.4.3 (Kochloukova [87, Theorem C])**.** *For $r \geq 1$ let $G_1, \ldots, G_r$ be non-abelian limit groups, let $H \leq G_1 \times \cdots \times G_r$ be a full subdirect product, and let $2 \leq k \leq r$. If H is of type $\mathcal{F}_k$ then H virtually surjects onto k-tuples. The converse is true if H is virtually coabelian.*

Note that in Kochloukova's original version of Theorem 6.4.3 the condition is that H has the homological finiteness type $\mathrm{FP_k}(\mathbb{Q})$. By [31, Corollary E] this is however equivalent to type $\mathcal{F}_k$ for subgroups of direct products of limit groups. In general we only have that $\mathcal{F}_k$ implies $\mathrm{FP_k}(\mathbb{Q})$ (see for instance [71, Section 8.2]).

We shall need the following auxiliary result which is a consequence of Theorem 6.4.3.

Lemma 6.4.4. *Let $G_1, \cdots, G_r$ be groups and Q be a finitely generated abelian group. Let $\phi : G_1 \times \cdots \times G_r \to Q$ be an epimorphism. Assume that the subgroup $H = \ker\phi \leq G_1 \times \cdots \times G_r$ virtually surjects onto m-tuples. Then the group $\phi(G_{i_1} \times \cdots \times G_{i_{r-m}}) \leq Q$ is a finite index subgroup of Q for every $1 \leq i_1 < \cdots < i_{r-m} \leq r$.*

Under the stronger assumption that H surjects onto m-tuples, the restriction of ϕ to $G_{i_1} \times \cdots \times G_{i_{r-m}}$ is surjective for all $1 \leq i_1 < \cdots < i_{r-m} \leq r$.

Proof. Assume that H virtually surjects onto m-tuples. Consider a product $G_{i_1} \times \cdots \times G_{i_{r-m}}$ of $r-m$ factors. We may assume that $i_j = j$.

Let $g \in Q$ be an arbitrary element. By surjectivity of ϕ there exist elements $h_1 \in G_1 \times \cdots \times G_{r-m}$ and $h_2 \in G_{r-m+1} \times \cdots \times G_r$ such that $g = \phi(h_1) \cdot \phi(h_2)$. Since H virtually surjects onto m-tuples there is $k \geq 1$ such that $h_2^k \in p_{r-m+1,\cdots,r}(H)$. Hence, there is $\overline{h}_1 \in G_1 \times \cdots \times G_{r-m}$ such that $\overline{h}_1 \cdot h_2^k \in H = \ker\phi$. In particular it follows that $\phi(h_2^k) = \phi((\overline{h}_1)^{-1})$. As a consequence we obtain that $g^k = \phi(h_1)^k \cdot \phi(h_2)^k = \phi(h_1)^k \cdot \phi((\overline{h}_1)^{-1}) \in \phi(G_1 \times \cdots \times G_{r-m})$.

We proved that the abelian group $Q/\phi(G_1 \times \cdots \times G_{r-m})$ has the property that each of its elements is torsion. This implies that $Q/\phi(G_1 \times \cdots \times G_{r-m})$ is finite and thus $\phi(G_1 \times \cdots \times G_{r-m})$ is a finite index subgroup of Q.

The second part follows immediately, since we can choose $k = 1$ in the proof if $\phi|_{G_1 \times \cdots \times G_{r-m}}$ is surjective. □

Corollary 6.4.5. *Let $\phi : \Lambda_1 \times \cdots \times \Lambda_r \to Q$ be an epimorphism, where $\Lambda_1, \cdots, \Lambda_r$ are non-abelian limit groups and Q is a finitely generated abelian group. If $\ker\phi$ is a full subdirect product of type $\mathcal{F}_m$ then the image $\phi(\Lambda_{i_1} \times \cdots \times \Lambda_{i_{r-m}}) \le Q$ is a finite index subgroup of Q for all $1 \le i_1 < \cdots < i_{r-m} \le r$.*

Proof. This is a direct consequence of Lemma 6.4.4 and Theorem 6.4.3. □

As another consequence of Theorem 6.4.3 we obtain a proof that our groups are irreducible.

Proposition 6.4.6. *If a subgroup $G \le \Lambda_1 \times \cdots \times \Lambda_r$ of a direct product of r limit groups Λ_i has type $\mathcal{F}_m$ with $m \ge \frac{r}{2}$ and it is virtually a product $H_1 \times H_2$, then at least one of H_1 and H_2 is of type $\mathcal{F}_\infty$ and there is $1 \le s \le r$ such that $H_1 \le \Lambda_1 \times \cdots \times \Lambda_s$ and $H_2 \le L_{s+1} \times \cdots \times \Lambda_r$.*

Proof. Project away from factors Λ_i which have abelian intersection with $H_1 \times H_2$. The image of $H_1 \times H_2$ under this projection is a direct product $\overline{H}_1 \times \overline{H}_2$ of at most r limit groups. By Lemma 2.5.2 the groups $\overline{H}_1$, $\overline{H}_2$ and $\overline{H}_1 \times \overline{H}_2$ have the same finiteness properties as H_1, H_2 and $H_1 \times H_2$. Thus, we may assume that $H_1 \times H_2$ intersects each of the Λ_i in a non-abelian group.

Let G be a subgroup of a direct product of r non-abelian limit groups of type $\mathcal{F}_m$ with $m \ge \frac{r}{2}$ and let $H_1 \times H_2 \le G$ be a finite index subgroup which is a direct product. Since non-abelian limit groups have trivial centre it follows that after reordering factors $H_1 \le \Lambda_1 \times \cdots \times \Lambda_s$ and $H_2 \le \Lambda_{s+1} \times \cdots \times \Lambda_r$ for some $1 \le s \le r$.

After possibly reducing the number of factors and replacing the limit groups by finitely generated subgroups, which are again limit groups, we may assume that $H_1 \times H_2$ is a full subdirect product of $\Lambda_1 \times \cdots \times \Lambda_r$ of type $\mathcal{F}_m$ (Note that neither decreasing r nor replacing the Λ_i by subgroups would affect the rest of the argument, thus we will not change notation here).

From Theorem 6.4.3 we obtain that $H_1 \times H_2$ virtually surjects onto m-tuples. Hence, $m \geq \frac{r}{2}$ implies that at least one of the following holds: H_1 is a finite index subgroup of $\Lambda_1 \times \cdots \times \Lambda_s$ or H_2 is a finite index subgroup of $\Lambda_{s+1} \times \cdots \times \Lambda_r$. Direct products of limit groups are of type $\mathcal{F}_\infty$ and finite index subgroups of groups of type $\mathcal{F}_\infty$ are of type $\mathcal{F}_\infty$. Thus, at least one of H_1 and H_2 is of type $\mathcal{F}_\infty$. □

We shall also need the following result by Kuckuck.

Proposition 6.4.7 ([96, Corollary 3.6])**.** *Let $G \leq \Lambda_1 \times \cdots \times \Lambda_r$ be a full subdirect product of a direct product of r non-abelian limit groups Λ_i, $1 \leq i \leq r$. If G virtually surjects onto m tuples for $m > \frac{r}{2}$ then G is virtually coabelian. In particular, G is virtually coabelian if G is of type $\mathcal{F}_m$.*

More precisely, we have that in either case there exist finite index subgroups $\Lambda_i' \leq \Lambda_i$, a free abelian group A and a homomorphism

$$\phi : \Lambda_1' \times \cdots \times \Lambda_r' \to A$$

such that $\ker\phi \leq G$ is a finite index subgroup.

We will require the following consequence of Theorem 6.4.3 and Proposition 6.4.7:

Corollary 6.4.8. *Let $r \geq 1$ and let $G \leq \Lambda_1 \times \cdots \times \Lambda_r$ be a full subdirect product of non-abelian limit groups Λ_i, $1 \leq i \leq r$. Assume that G is of type $\mathcal{F}_m$ with $m \geq 0$. For $k \geq 0$ with $m > \frac{k}{2}$ and $1 \leq i_1 < \cdots < i_k \leq r$ the projection $p_{i_1,\cdots,i_k}(G) \leq \Lambda_{i_1} \times \cdots \times \Lambda_{i_k}$ is of type $\mathcal{F}_m$.*

Proof. By Theorem 6.4.3 the group $G \leq \Lambda_1 \times \cdots \times \Lambda_r$ virtually projects onto m-tuples. Hence, the projection $Q := p_{i_1,\cdots,i_k}(G) \leq \Lambda_{i_1} \times \cdots \times \Lambda_{i_k}$ is full subdirect and virtually surjects onto m-tuples with $m > \frac{k}{2}$. By Proposition 6.4.7 the projection $Q := p_{i_1,\cdots,i_k}(G)$ is virtually coabelian. Hence, the subgroup $Q \leq \Lambda_{i_1} \times \cdots \times \Lambda_{i_k}$ is full subdirect, virtually coabelian, and virtually projects onto m-tuples. The converse direction of Theorem 6.4.3 then implies that Q is of type $\mathcal{F}_m$. □

As a consequence of the results in this section we can determine the precise finiteness properties of the groups arising from our construction in Theorem 6.3.3.

Theorem 6.4.9. *Under the assumptions of Theorem 6.3.3 and with the same notation, let $\phi = h_* : \pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r} \to \pi_1 E^{\times k}$ be the induced epimorphism on fundamental groups. Then $\ker\phi \cong \pi_1 \overline{H}$ is a Kähler group of type $\mathcal{F}_{r-k}$, but not of type $\mathcal{F}_{r-k+1}$, and $\ker\phi$ is irreducible.*

Proof. By Theorem 6.3.3 we know that $\ker\phi$ is of type $\mathcal{F}_{r-k}$. Hence, we only need to proof that $\ker\phi$ is not of type $\mathcal{F}_{r-k}$ and that $\ker\phi$ has no finite index subgroup which is a direct product of two non-trivial groups.

By definition, we obtain that $\phi = h_*$ is given by the surjective map

$$\phi(g_1,\cdots,g_r) = \sum_{i=1}^{r} v_i \cdot \alpha_i(g_i) \in (\pi_1 E)^{\times k} \cong (\mathbb{Z}^2)^k \cong \mathbb{Z}^{2k}$$

for $(g_1,\cdots,g_r) \in \pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r}$.

Since the maps α_i are finite sheeted branched coverings, the image $\alpha_{i,*}(\pi_1 S_{\gamma_i}) \leq \pi_1 E$ is a finite index subgroup for $1 \leq i \leq r$. The assumption that the v_i satisfy property (P2) implies that the image $\phi(\pi_1 S_{\gamma_{i_1}} \times \cdots \times \pi_1 S_{\gamma_{i_k}}) \leq \pi_1 E^{\times k}$ of any k factors is a finite index subgroup of $\pi_1 E^{\times k} \cong \mathbb{Z}^{2k}$, $1 \leq i_1 < \cdots < i_k \leq r$.

Since we have $r \geq 3k$ factors and any k factors map to a finite index subgroup of $\pi_1 E^{\times k}$ the kernel of

$$\phi_0 = \phi|_{\Lambda_1\times\cdots\times\Lambda_r} : \Lambda_1 \times \cdots \times \Lambda_r \to \pi_1 E^{\times k}.$$

is subdirect, after passing to finite index subgroups $\Lambda_i \leq \pi_1 S_{\gamma_i}$. Note that the image $\mathrm{im}\phi_0 \leq \pi_1 \mathrm{E}^{\times \mathrm{k}}$ is a finite index subgroup, thus isomorphic to $\mathbb{Z}^{2k}$, and that $\ker\phi_0 \leq \ker\phi$ is a finite index subgroup. The intersection $L_i = \Lambda_i \cap \ker\phi_0 \leq \Lambda_I$ is a non-trivial normal subgroup of infinite index in Λ_i, since $\phi(\Lambda_i) \cong \mathbb{Z}^2$. Thus, $\ker\phi_0$ is a full subdirect product of $\Lambda_1 \times \cdots \times \Lambda_r$.

Since the image of the restriction of ϕ_0 to any factor Λ_i is isomorphic to $\mathbb{Z}^2$, the image of the restriction of ϕ to any $k-1$ factors $\Lambda_{i_1} \times \cdots \times \Lambda_{i_{k-1}}$ $(1 \leq i_1 < \cdots < i_{k-1})$ is isomorphic to $\mathbb{Z}^{2(k-1)}$ (by the same argument as for k factors). In particular, $\phi(\Lambda_{i_1} \times \cdots \times \Lambda_{i_{k-1}})$ is not a finite index subgroup of the image $\mathrm{im}\phi_0 \cong \mathbb{Z}^{2\mathrm{k}}$. By Corollary 6.4.5, $\ker\phi_0$ and, therefore, its finite extension $\ker\phi \geq \ker\phi_0$ cannot be of type $\mathcal{F}_{r-k+1}$.

Assume that there is a finite index subgroup $H_1 \times H_2 \leq \ker\phi$ which is a product of two non-trivial groups H_1 and H_2. By Proposition 6.4.6 we may assume that (after reordering factors) $H_1 \leq \pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_s}$, $H_2 \leq \pi_1 S_{\gamma_{s+1}} \times \cdots \pi_1 S_{\gamma_r}$ and H_1 is of type $\mathcal{F}_\infty$, for some $1 \leq s \leq r$. It follows from Theorem 2.5.4 that H_1 is virtually a product of finitely generated subgroups $\Gamma_i \leq \pi_1 S_{\gamma_i}$, $1 \leq i \leq s$. Since $\ker\phi_0$ is subdirect in $\Lambda_1 \times \cdots \times \Lambda_r$ and $\ker\phi_0 \cap (H_1 \times H_2) \leq \ker\phi_0$ has finite index, the Γ_i must be finite index subgroups of the $\pi_1 S_{\gamma_i}$. This contradicts that the restriction of ϕ to any finite index subgroup of $\pi_1 S_{\gamma_i}$ has infinite image. It follows that $\ker\phi$ is irreducible. □

Addendum 6.4.10. *Note that the proof of Theorem 6.4.9 also shows that if we consider ϕ where the set $\mathcal{C}$, as defined in Theorem 6.3.3, does not have the generic property described in Proposition 6.3.2 then* $\ker\phi$ *must have finiteness type less than*

$\mathcal{F}_{r-k}$. *In fact it shows that the finiteness type of* $\ker\phi$ *is at most* $\mathcal{F}_{r-l}$ *where* $l-1$ *is the size of a maximal subset of* $\mathcal{C}$ *which does not form a basis of* $\mathbb{C}^k$.

Proof of Theorem 6.4.1. Theorem 6.4.1 is now a direct consequence of Theorem 6.3.3 and Theorem 6.4.9. The irreducibility of $\ker\phi$ can also be proved using elementary Linear Algebra and Theorem 2.1.5. □

Proof of Corollary 6.4.2. Let $k := r - m$. Then $r \geq 3k$, since $m \geq \frac{2r}{3}$. The only thing that does not follow immediately from Theorem 6.4.1 and its proof is that $\pi_1\overline{H}$ is a full subdirect product. Replacing the construction in Proposition 6.3.2 by a slightly more careful construction shows that for any $r \geq 3k$ there is a set $\mathcal{C} = \{v_1, \cdots, v_r\} \leq \mathbb{Z}^k$ with properties (P1') and (P2) such that $\{v_1, \cdots, v_k\}$ and $\{v_{k+1}, \cdots, v_{2k}\}$ are bases of $\mathbb{Z}^k$. Thus, the restrictions of h_* to $\pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_k}$ and to $\pi_1 S_{\gamma_{k+1}} \times \cdots \times \pi_1 S_{\gamma_{2k}}$ are both surjective (see proof of Theorem 6.4.9 for more details on h_*). It follows that $\pi_1\overline{H}$ is subdirect. It is full, because the image of the restriction of h_* to any factor is abelian. □

6.5 Potential generalisations

We finish with a brief discussion of some open questions arising from this chapter.

Potential generalisations of our examples

It seems reasonable to believe that the class of examples constructed here allows for further generalisations. In particular, we believe that one should be able to weaken the condition (P2) on the set $\{v_1, \cdots, v_r\} \subset \mathbb{Z}^k$ used in the construction of the map h in Section 6.3, as well as the condition that the branched covers $\alpha_1, \cdots, \alpha_k$ are purely branched. Indeed there is no obvious reason why these conditions should be minimal in any sense; they are required for purely technical reasons in the proof. It would be desirable to provide a unified approach wherein kernels that are direct products of smaller Kähler groups would also arise. One might be able to obtain such an approach by proving a suitable version of Conjecture 6.1.2. By Addendum 6.4.10 the finiteness properties of the kernel would vary in such a generalised approach.

In contrast, the condition $r \geq 3k$ in Theorem 6.4.1 is minimal, as the following example shows: For $0 \leq i \leq r-1$ we choose $v_{3i+1} = v_{3i+2} = v_{3i+3} = e_i$ to be the i-th vector of the standard basis $\{e_1, \cdots, e_k\}$ of $\mathbb{Z}^k$ and $\alpha_1, \cdots, \alpha_{3k}$ to be any choice of non-trivial finite-sheeted branched holomorphic coverings of E which are surjective on fundamental groups. Then it follows from Theorem 4.3.2 that the kernel of the associated map h is Kähler and fits into a short exact sequence induced by h as in Theorem 6.4.1. However, removing any coordinate factor from h will break the property that h induces a short exact sequence with kernel the smooth generic fibre of h. This is because the smooth generic fibre of any holomorphic map $S_{\gamma_1} \times S_{\gamma_2} \to E$ is a closed Riemann surface while by Theorem 4.2.2 the kernel of the map induced by $\alpha_i + \alpha_j : \pi_1 S_{\gamma_{j_1}} \times \pi_1 S_{\gamma_{j_2}} \to \pi_1 E$ for $3i + 1 \leq j_1 < j_2 \leq 3i + 3$ is not finitely presented.

Chapter 7

Restrictions on subdirect products of surface groups

In this Chapter we consider Delzant and Gromov's question from a different point of view. We will give criteria that imply that a subgroup of a direct product of surface groups is not Kähler. More generally we provide criteria on finitely presented subgroups G of direct products of surface groups which imply that no Kähler group can map onto G with finitely generated kernel.

We start by deriving general conditions that allow us to compute the first Betti number of the kernel of a homomorphism from a direct product of groups onto a free abelian group (see Theorem 7.1.5). This allows us to show that in many cases the kernel of a homomorphism from a direct product of surface groups onto a free abelian group of odd rank is not Kähler. In particular, we will see that the kernel of a non-trivial homomorphism from a direct product of surface groups onto $\mathbb{Z}$ is never Kähler (see Theorem 7.1.1).

In Section 7.2 we prove that every map from a Kähler group to a direct product of surface groups with finitely generated kernel and finitely presented image is induced by a holomorphic map (see Proposition 7.2.2).

We will then proceed to a more thorough analysis of homomorphisms from Kähler groups onto a finitely presented subgroup G of a direct products of surface groups in Section 7.3. The main result of this section is that if G is of type $\mathcal{F}_m$ then the image of the projection of G onto any $\leq \frac{3m}{2}$ factors is virtually coabelian of even rank (see Theorem 7.3.1).

In Section 7.4 we discuss some generalisations of results of the previous sections, as well as interesting consequences of this chapter.

7.1 Maps to free abelian groups

All of the non-trivial examples of Kähler subgroups of direct products of surface groups constructed so far are obtained as kernels of maps from a direct product of surface groups to a free abelian group. Hence, a natural special case of Delzant and Gromov's question is the following question.

Question 4. *Let $S_{g_1},\cdots,S_{g_r}$ be closed hyperbolic Riemann surfaces, let $k\in\mathbb{Z}$ and let $\phi:\pi_1S_{g_1}\times\cdots\times\pi_1S_{g_r}\to\mathbb{Z}^k$ be an epimorphism. When is $\ker(\phi)$ a Kähler group?*

Note that we may assume that $\ker(\phi)$ is subdirect in $\pi_1S_{g_1}\times\cdots\times\pi_1S_{g_r}$. If not then we can pass to finite index subgroups of the $\pi_1S_{g_i}$ such that $\ker(\phi)$ is subdirect. We will see in this section that for $k=1$ the answer to Question 4 is as follows:

Theorem 7.1.1. *Let $S_{g_1},\cdots,S_{g_r}$ be closed Riemann surfaces of genus $g_i\geq 2$ and let $\psi:\pi_1S_{g_1}\times\cdots\times\pi_1S_{g_r}\to\mathbb{Z}$ be any non-trivial homomorphism. Then $\ker(\psi)$ is not Kähler.*

For $k>1$ and odd we will show that under some additional assumptions on the restriction of ϕ to the factors the group $\ker(\phi)$ is not Kähler either (see Corollary 7.1.6 and Remark 7.1.7).

The proof of Theorem 7.1.1 is a consequence of a more general result about the first Betti numbers of subdirect products arising as kernels of maps to free abelian groups and the well-known

Lemma 7.1.2. *Let G be a Kähler group then the first Betti number $b_1(G_0)$ is even for every finite index subgroup $G_0\leq G$.*

Proof. Kähler groups have even first Betti number and every finite index subgroup of a Kähler group is itself Kähler and thus has even first Betti number. ☐

We want to mention a simple consequence of Lemma 7.1.2 which we shall need in Section 7.3.

Corollary 7.1.3. *Let $H\leq G:=\pi_1S_{g_1}\times\cdots\times\pi_1S_{g_r}$ be a finite index subgroup of a direct product of fundamental groups of closed Riemann surfaces S_{g_i} of genus $g_i\geq 0$, $1\leq i\leq r$, $r\geq 1$. Then the first Betti number $b_1(H)$ of H is even.*

Proof. The group G is Kähler. Hence, the first Betti number $b_1(H)$ of any finite index subgroup $H\leq G$ is even. ☐

We will also make use of the following easy and well-known fact

Lemma 7.1.4. *Let G and H be groups and let $\phi : G \to H$ be an injective homomorphism. Then the following are equivalent:*

1. *the induced map $\phi_{ab} : G_{ab} \to H_{ab}$ on abelianisations is injective;*
2. *$\phi([G,G]) = \phi(G) \cap [H,H]$.*

Our main technical result in this section is

Theorem 7.1.5. *Let $k \geq 1$, $r \geq 2$ be integers, let $G_1, \cdots, G_r$ be finitely generated groups and let $\psi : G = G_1 \times \cdots \times G_r \to \mathbb{Z}^k$ be an epimorphism. Assume further that (at least) one of the following two conditions is satisfied:*

1. *$k = 1$ and $\ker\psi$ is subdirect in $G_1 \times \cdots \times G_r$;*
2. *the restriction of ψ to G_i surjects onto $\mathbb{Z}^k$ for at least three different $i \in \{1, \cdots, r\}$.*

Then the map ψ induces a short exact sequence

$$1 \to (\ker\psi)_{ab} \to (G_1 \times \cdots \times G_r)_{ab} \to \mathbb{Z}^k \to 1 \tag{7.1}$$

on abelianisations and in particular the following equality of first Betti numbers holds:

$$b_1(G) = k + b_1(\ker\psi). \tag{7.2}$$

Proof. We will first give a proof under the assumption that Condition (2) is satisfied and will then explain how to modify our proof if Condition (1) is satisfied. Assume that Condition (2) holds and that (without loss of generality) the restriction of ψ to each of the first three factors is surjective.

It is clear that exactness of (7.1) implies the equality (7.2) of Betti numbers. Hence, we only need to prove that the sequence (7.1) is exact. Abelianisation is a right exact functor from the category of groups to the category of abelian groups. Hence, it suffices to prove that the inclusion $\iota : \ker\psi \to G$ induces an injection $\iota_{ab} : (\ker\psi)_{ab} \to G_{ab}$ of abelian groups.

Since the image of ψ is abelian, it follows that $[G,G] \leq \ker\psi$. We want to show that $[G,G] \leq [\ker\psi, \ker\psi]$. Since $[G,G] = [G_1,G_1] \times \cdots \times [G_r,G_r]$ it suffices to show that $[G_i,G_i] \leq [\ker\psi, \ker\psi]$ for $1 \leq i \leq r$.

We may assume that $i > 2$, since for $i = 1,2$ the same argument works after exchanging the roles of G_i and G_3. Fix $x, y \in G$. Since the restrictions $\psi|_{G_j} : G_j \to \mathbb{Z}^k$

are surjective for $j = 1, 2$ we can choose elements $g_1 \in G_1$ and $g_2 \in G_2$ with $\psi(g_1) = -\psi(x)$ and $\psi(g_2) = -\psi(y)$.

Then the elements $u := g_1^{-1} \cdot x \in G$ and $v := g_2^{-1} \cdot y \in G$ are in $\ker\psi$. Since $[G_i, G_j] = \{1\}$ for $i \neq j$ it follows that $[u, v] = [x, y]$. Thus, $[G_i, G_i] \leq [\ker\psi, \ker\psi]$ for $1 \leq i \leq r$. Consequently $[G, G] \leq [\ker\psi, \ker\psi]$ and therefore by Lemma 7.1.4 the map ι_{ab} is injective.

Now assume that Condition (1) holds. As before it suffices to prove that $[G_i, G_i] \leq [\ker\psi, \ker\psi]$ for $1 \leq i \leq r$. To simplify notation assume that $i = 1$. If we can prove that there is some element $\overline{g}_0 \in G_2 \times \cdots \times G_r$ such that for any $x \in G_1$ there is an integer $k \in \mathbb{Z}$ with $x \cdot \overline{g}_0^k \in \ker\psi$ then the same argument as before will show that $[G_1, G_1] \leq [\ker\psi, \ker\psi]$.

Observe that we have the following equality of sets

$$Q := \{\psi(g_1, 1) \mid (g_1, \overline{g}) \in \ker\psi \leq G_1 \times (G_2 \times \cdots \times G_r)\} = \{\psi(1, \overline{g}) \mid (g_1, \overline{g}) \in \ker\psi\} \leq \mathbb{Z}.$$

The set Q is a subgroup of $\mathbb{Z}$, since it is the image of the group $\ker\psi$ under the homomorphism $\psi \circ \iota_1 \circ \pi_1 : G \to \mathbb{Z}$ where $\pi_1 : G \to G_1$ is the canonical projection and $\iota_1 : G_1 \to G$ is the canonical inclusion. Let $\overline{g}_0 \in G_2 \times \cdots \times G_r$ be an element such that $\psi(1, \overline{g}_0) = l_0$ generates Q.

Since $\ker\psi$ is subdirect, for any $g_1, g_2 \in G_1$ there are elements $\overline{g}_1, \overline{g}_2 \in G_2 \times \cdots \times G_r$ such that $(g_1, \overline{g}_1), (g_2, \overline{g}_2) \in \ker\psi$ and therefore $\psi(g_1, 1) = k_1 \cdot l_0 . \psi(g_2, 1) = k_2 \cdot l_0 \in Q$. It follows that $(g_1, (\overline{g}_0)^{-k_1}), (g_2, (\overline{g}_0)^{-k_2}) \in \ker\psi$. Thus $[g_1, g_2] \in [\ker\psi, \ker\psi]$, completing the proof of the Theorem. □

As a direct consequence we obtain a constraint on Kähler groups

Corollary 7.1.6. *Let $r, k \geq 1$ be integers, let $G_1, \cdots, G_r$ be finitely generated groups and let $\psi : G_1 \times \cdots \times G_r \to \mathbb{Z}^k$ be an epimorphism satisfying one of the Conditions (1) or (2) in Theorem 7.1.5. If $b_1(G_1) + \cdots + b_1(G_r) - k$ is odd then $\ker\psi$ is not Kähler.*

Proof. By Theorem 7.1.5 the first Betti number of $\ker\psi$ is equal to $b_1(G_1) + \cdots + b_1(G_r) - k$ and therefore odd. Hence, $\ker\psi$ can not be Kähler by Lemma 7.1.2. □

This allows us to prove Theorem 7.1.1

Proof of Theorem 7.1.1. Observe that there are no factors $\pi_1 S_{g_i}$ which have trivial intersection with $\ker\psi$. Since $\ker\psi$ is Kähler it is finitely presented and thus any quotient of $\ker\psi$ is finitely generated. Normality of $\ker\psi$ in $\pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$ implies that the image $\Lambda_i \leq \pi_1 S_{g_i}$ of $\ker\psi$ under the projection to a factor $\pi_1 S_{g_i}$ is a normal finitely generated subgroup and therefore either trivial or of finite index. It can not be trivial, because $\ker\psi \cap \pi_1 S_{g_i}$ is non-trivial. Thus, the group Λ_i is a finite index subgroup of $\pi_1 S_{g_i}$.

Consider the restriction $\psi' := \psi|_{\Lambda_1\times\cdots\times\Lambda_r}$ of ψ to the finite index subgroup $\Lambda_1 \times \cdots \times \Lambda_r \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$. By definition of the Λ_i we have $\ker\psi \leq \ker\psi'$ and therefore $\ker\psi = \ker\psi'$. After replacing $\mathbb{Z}$ by its isomorphic subgroup $\mathrm{im}\psi'$, the map ψ' satisfies Condition (1) of Theorem 7.1.5.

Since the Λ_i are fundamental groups of closed Riemann surfaces, it is immediate that $b_1(\Lambda_i)$ is even for $1 \leq i \leq r$ and $b_1(\Lambda_1)+\cdots+b_1(\Lambda_r)-1$ is odd. Hence, by Corollary 7.1.6 the group $\ker\psi' = \ker\psi$ is not Kähler. $\square$

Remark 7.1.7. Corollary 7.1.6 provides large classes of examples of non-Kähler subgroups of direct products of surface groups with odd first Betti number. Indeed, choose any $r \geq 3$ and $\psi : \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r} \to \mathbb{Z}^{2k+1}$ such that at least three of the restrictions of ψ to factors are surjective. Then $b_1(\ker\psi)$ is odd, so $\ker\psi$ is not Kähler.

7.2 Holomorphic maps to products of surfaces

In this section we will generalise classical results about the existence of holomorphic maps from Kähler manifolds to surfaces, which we discussed in Section 2.3, to holomorphic maps from Kähler manifolds to products of surfaces (see Proposition 7.2.2). We apply this generalisation to prove the following constraint on Kähler groups which admit maps to direct products of surface groups.

Theorem 7.2.1. *Let $G = \pi_1 M$ be the fundamental group of a closed Kähler manifold M, let $H \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_s}$ be a finitely presented full subdirect product with $g_i \geq 2$, let $k \geq 0$, and let $1 \leq i_1 < \cdots < i_k \leq s$.*

Assume that there is a finite index subgroup $P \leq \pi_1 S_{g_{i_1}} \times \cdots \times \pi_1 S_{g_{i_k}}$ with $\overline{H} := p_{i_1,\cdots,i_k}(H) \leq P$ such that the induced map $\overline{H}_{ab} \to P_{ab}$ is injective, and an epimorphism $\phi : G \to H$ with finitely generated kernel $N = \ker\phi$.

Then the image of the induced injective map $\phi^* : H^1(\overline{H}, \mathbb{C}) \to H^1(G, \mathbb{C})$ is even-dimensional.

As a consequence of Theorem 7.2.1 we will produce examples of subdirect products of surface groups which have even first Betti number but are not Kähler (see Theorem 7.2.4 and Corollary 7.2.5). To prove Theorem 7.2.1 we will use the following auxiliary results.

Proposition 7.2.2. *Retaining in the assumptions of Theorem 7.2.1 and its notation, denote by $q : Y \to S_{g_{i_1}} \times \cdots \times S_{g_{i_k}}$ a finite sheeted cover with $q_*(\pi_1 Y) = P$.*

Then the composition $p_{i_1,\cdots,i_k} \circ \phi : G \to \overline{H} \leq P$ is induced by a holomorphic map $\widehat{f}_{i_1,\cdots,i_k} : M \to Y$ with respect to a suitable choice of complex structures on Y.

Lemma 7.2.3. *Let $G \cong \pi_1 M$ and $H \cong \pi_1 N$ be fundamental groups of closed Kähler manifolds M, N and let $\phi : G \to H$ be a homomorphism which can be realised by a holomorphic map $f : M \to N$. Then the induced map*

$$\phi^* : H^1(H, \mathbb{C}) \to H^1(G, \mathbb{C})$$

has even dimensional image.

Proof of Theorem 7.2.1. By Proposition 7.2.2 the homomorphism $\phi_{i_1,\cdots,i_k} := p_{i_1,\cdots,i_k} \circ \phi : G \to \overline{H} \leq P$ can be realised by a holomorphic map $\widehat{f}_{i_1,\cdots,i_k} : M \to Y$. To avoid confusion let $\psi := \widehat{f}_{i_1,\cdots,i_k,*} : G = \pi_1 M \to P = \pi_1 Y$ be the composition of $\phi_{i_1,\cdots,i_k}$ with the inclusion $\iota : \overline{H} \hookrightarrow P$.

By Lemma 7.2.3, the image of the homomorphism

$$\psi^* : H^1(P, \mathbb{C}) \to H^1(G, \mathbb{C})$$

is even-dimensional.

Since abelianisation is a right exact functor on groups the map $\phi_{i_1,\cdots,i_k}$ induces an epimorphism $\phi_{i_1,\cdots,i_k,*} : G_{ab} \cong H_1(G, \mathbb{Z}) \to \overline{H}_{ab} \cong H_1(\overline{H}, \mathbb{Z})$. By the Universal Coefficient Theorem (UCT) for fields of characteristic zero we obtain that the induced homomorphism

$$\phi^*_{i_1,\cdots,i_k} : H^1(\overline{H}, \mathbb{C}) \to H^1(G, \mathbb{C})$$

is injective.

By assumption the map $\iota_{ab} : \overline{H}_{ab} \to P_{ab}$ is injective and thus the UCT implies that the induced map

$$\iota^* : H^1(P, \mathbb{C}) \to H^1(\overline{H}, \mathbb{C})$$

is surjective.

Hence, the factorisation $\psi^* = \phi^*_{i_1,\cdots,i_k} \circ \iota^*$ of the induced map ψ^* on cohomology implies that

$$\mathrm{im}(\psi^* : H^1(P, C) \to H^1(G, \mathbb{C})) = \mathrm{im}(\phi^*_{i_1,\cdots,i_k} : H^1(\overline{H}, \mathbb{C}) \to H^1(G, \mathbb{C})).$$

This completes the proof. $\square$

Proof of Proposition 7.2.2. We start by showing that the homomorphism $\phi : G \to H$ is induced by a holomorphic map $f : M \to S_{g_1} \times \cdots \times S_{g_s}$. If $s = 1$ then $H = \pi_1 S_{g_1}$ and this is just Lemma 2.3.3. Assume that $s \geq 2$. Since H is finitely presented, by Theorem 2.5.5, the kernel

$$L_i := \ker p_i = (\pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_{i-1}} \times 1 \times \pi_1 S_{g_{i+1}} \times \cdots \times \pi_1 S_{g_s}) \cap H$$

of the projection $p_i : H \to \pi_1 S_{g_i}$ is finitely generated for $1 \leq i \leq s$.

Observe that the kernel of the map q_i defined by $q_i : G \xrightarrow{\phi} H \xrightarrow{p_i} \pi_1 S_{g_i}$ is an extension

$$1 \to N \to \ker q_i \to L_i \to 1$$

of a finitely generated group by a finitely generated group, so it is finitely generated.

Hence, by Lemma 2.3.3, the homomorphism p_i is induced by a holomorphic map $f_i : M \to S_{g_i}$ with respect to a suitable complex structure on S_{g_i}. It follows that

$$f := (f_1, \cdots, f_s) : M \to S_{g_1} \times \cdots \times S_{g_s}$$

is a holomorphic map inducing the composition $\iota \circ \phi$ on fundamental groups where $\iota : H \hookrightarrow \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_s}$ is the canonical inclusion.

For any $k \geq 0$ and $1 \leq i_1 < \cdots < i_k \leq s$, the projection $S_{g_1} \times \cdots \times S_{g_s} \to S_{g_{i_1}} \times \cdots S_{g_{i_k}}$ onto k factors is holomorphic and hence so is its composition $f_{i_1,\cdots,i_k} = (f_{i_1}, \cdots, f_{i_k}) : M \to S_{g_{i_1}} \times \cdots \times S_{g_{i_k}}$ with f. Thus, the homomorphism $p_{i_1,\cdots,i_k} \circ \phi : G \to \overline{H} = p_{i_1,\cdots,i_k}(H)$ is induced by the holomorphic map $f_{i_1,\cdots,i_k}$. In particular, $f_{i_1,\cdots,i_k,*}(G) = \overline{H}$.

The map q in the statement of the proposition induces a Kähler structure on the compact manifold Y with respect to which q is holomorphic.

Since $f_{i_1,\cdots,i_k,*}(G) = \overline{H} \leq q_*(\pi_1 Y)$ the map $f_{i_1,\cdots,i_k}$ lifts to a continuous map $\widehat{f}_{i_1,\cdots,i_k} : M \to Y$ such that the diagram

$$\begin{tikzcd} & Y \arrow[d, "q"] \\ M \arrow[ur, "\widehat{f}_{i_1,\cdots,i_k}"] \arrow[r, "f_{i_1,\cdots,i_k}"'] & S_{g_1} \times \cdots \times S_{g_r} \end{tikzcd}$$

is commutative.

By construction of the induced complex structure on Y, the map q is locally biholomorphic. Hence, $\widehat{f}_{i_1,\cdots,i_k}$ is locally a composition of holomorphic maps, thus holomorphic, and $\widehat{f}_{i_1,\cdots,i_k}$ induces the homomorphism ϕ on fundamental groups. $\square$

Proof of Lemma 7.2.3. The map f is holomorphic and therefore induces homomorphisms

$$f^* : H^k(N, \mathbb{C}) \to H^k(M, \mathbb{C})$$

of pure weight-k Hodge structures for all $k \in \mathbb{Z}$.

Since N is Kähler it follows that the odd-dimensional Betti numbers $b^{2k+1}(N)$ are even for all $k \in \mathbb{Z}$. Hence, the image $\mathrm{im}(f^* : H^{2k+1}(N, \mathbb{C}) \to H^{2k+1}(M, \mathbb{C}))$ is even-dimensional for all k.

Let $\phi := f_* : \pi_1 M \to \pi_1 N$ be the induced homomorphism on fundamental groups. The inclusion $M \hookrightarrow K(G,1)$ (respectively $N \hookrightarrow K(H,1)$) obtained by constructing a classifying $K(G,1)$ from M (respectively $K(H,1)$ from N) by attaching cells of dimension greater than two induces isomorphisms on fundamental groups and on first (co)homology. It is well-known (and easy to see) that, up to these isomorphisms on cohomology, we have

$$\phi^* = f^* : H^1(H, \mathbb{C}) \to H^1(G, \mathbb{C}).$$

Hence, the image of ϕ^* on first cohomology is even-dimensional. □

Theorem 7.1.5 allows us to construct interesting examples of subgroups of direct products of surface groups which are not Kähler. Indeed, we obtain that the direct product of any non-Kähler subdirect product of surface groups obtained from Theorem 7.1.1 and Remark 7.1.7 with an arbitrary finitely generated group is not Kähler. In particular, by taking the direct product of any two of the groups constructed in Theorem 7.1.1 and Remark 7.1.7, we obtain a subdirect product of closed orientable hyperbolic surface groups which has even first Betti number, but is not Kähler. This observation is summarised in the following Theorem.

Theorem 7.2.4. *For any l and any $r \geq 6$ there is an epimorphism*

$$\phi : \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r} \to \mathbb{Z}^l$$

with $g_i \geq 2$, $1 \leq i \leq r$, such that $\ker\phi$ *is full subdirect and not Kähler. Furthermore,* $b_1(\ker\phi) \equiv l \mod 2$.

More generally we obtain the following constraint:

Corollary 7.2.5. *For $s \geq 0$ let $G \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_s}$, $g_i \geq 2$, be a full subdirect product. Assume that there is $k \geq 0$ and $1 \leq i_1 < \cdots < i_k \leq r$, such that the image H of the projection $p_{i_1,\ldots,i_k} : G \to \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_k}$ is a finitely presented full subdirect product with odd first Betti number, the kernel of $p_{i_1,\ldots,i_k}$ is finitely generated, and the induced homomorphism $H_{ab} \to (\pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_k})_{ab}$ is injective. Then G is not Kähler.*

7.3 Restrictions from finiteness properties

In this section we give a strong constraint on Kähler groups which map into direct products of surface groups with finitely generated kernel and finitely presented image. This constraint in particular applies to Kähler subgroups of direct products of surface groups. As a consequence we obtain a proof of Theorem A.

Theorem 7.3.1. *Let $\widehat{G}$ be a Kähler group and let $G \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$, $g_i \geq 2$, be a full subgroup. Assume that G is of type $\mathcal{F}_m$ for $m \geq 2$ and that there is an epimorphism $\psi : \widehat{G} \to G$ with finitely generated kernel $N = \ker\psi$.*

Then, after reordering factors, there is $s \geq 0$ such that for any $k \leq \frac{3m}{2}$ and any $1 \leq i_1 < \cdots < i_k \leq s$ the projection $p_{i_1,\ldots,i_k}(G) \leq \pi_1 S_{g_{i_1}} \times \cdots \times \pi_1 S_{g_{i_k}}$ is virtually coabelian and all of its coabelian finite index subgroups are coabelian of even rank. Furthermore, $\mathrm{Z}(G) = G \cap (\pi_1 S_{g_{s+1}} \times \cdots \times \pi_1 S_{g_r}) \leq p_{s+1,\ldots,r}(G) \cong \mathbb{Z}^{r-s}$ is a finite index subgroup.

More precisely, there is $M \geq 0$, finite index subgroups $\pi_1 S_{h_{i_l}} \leq \pi_1 S_{g_{i_l}}$ and an epimorphism $\phi : \pi_1 S_{h_{i_1}} \times \cdots \times \pi_1 S_{h_{i_k}} \to \mathbb{Z}^M$ such that $\ker\phi \leq p_{i_1,\cdots,i_k}(G)$ is a finite index subgroup and for any choice of such finite index subgroups $\pi_1 S_{h_{i_l}}$ and homomorphism ϕ we have $b_1(\ker\phi) \equiv M \equiv 0$ mod 2.

It is tempting to combine Theorem 7.3.1 with the existence results for maps to products of surface groups due to Delzant and Gromov, as well as Py, and Delzant and Py, discussed in Section 2.3. Indeed our result offers the exciting possibilty of finding new constraints on Kähler groups admitting cuts and more generally Kähler groups which admit actions on CAT(0) cube complexes (see [58],[107],[59]).

For $G = \widehat{G} \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$ a full subgroup we can inductively construct $1 = s_1 < \cdots < s_N \leq r$ and a finite index subgroup $H_1 \times \cdots \times H_N \times \mathrm{Z}(G) \leq G$ such that $H_i \leq \pi_1 S_{g_{s_i}} \times \cdots \times \pi_1 S_{g_{s_{i+1}-1}}$ is irreducible with trivial centre. Note that Theorem 7.3.1 leads to a particularly nice result if the H_i have strong finiteness properties.

Corollary 7.3.2. *Assume that, with the notation of the previous paragraph, G is Kähler and H_i is of type $\mathcal{F}_{m_i}$ with $m_i \geq \frac{2(s_{i+1}-s_i)}{3}$ for $1 \leq i \leq N$. Then the subgroups $H_i \leq \pi_1 S_{g_{s_i}} \times \cdots \times \pi_1 S_{g_{s_{i+1}-1}}$ have finite index subgroups $H_{i,0} \leq H_i$ which are coabelian of even rank; more precisely, every coabelian finite index subgroup $H_{i,0} \leq H_i$ is coabelian of even rank and $b_1(H_{i,0}) \equiv 0$ mod 2. Moreover, $\mathrm{Z}(G) \cong \mathbb{Z}^{r-s_N}$ with $r - s_N$ even.*

Proof. It follows immediately by applying Theorem 7.3.1 to the projections $p_{s_i,\cdots,s_{i+1}-1}$ that for $1 \leq i \leq N$ there exist finite index subgroups $H_{i,0} \leq H_i$ which are coabelian of even rank and satisfy $b_1(H_{i,0}) \equiv 0$ mod 2. Since the group $H_{1,0} \times \cdots \times H_{N,0} \times \mathrm{Z}(G) \leq G$

has finite index it is Kähler. Hence, its first Betti number is even and, in particular, $r - s_N = \mathrm{rk}_{\mathbb{Z}}(\mathrm{Z}(G))$ is even. □

Note that if G in Corollary 7.3.2 is of type $\mathcal{F}_m$ then H_i is also of type $\mathcal{F}_m$ for $1 \le i \le k$. Hence, the conditions in Corollary 7.3.2 are satisfied whenever G itself has sufficiently strong finiteness properties. We can now prove Theorem A.

Proof of Theorem A. Since the Kähler group G in Theorem 6.4.1 is a full irreducible subgroup of a direct product of r surfaces and G is of type $\mathcal{F}_m$ with $m \ge \frac{2r}{3}$. The first part of Theorem 6.4.1 follows immediately from Corollary 7.3.2. The second part of Theorem 6.4.1 is a direct consequence of Corollary 6.4.2. □

A special case of this is the case $r = 3$.

Corollary 7.3.3. *Let G be Kähler. If $G \le \pi_1 S_{g_1} \times \pi_1 S_{g_2} \times \pi_1 S_{g_3}$ is a full subgroup, $g_i \ge 2$, then G is either virtually coabelian of even rank or G is virtually $\mathbb{Z}^2 \times \pi_1 S_h$ for some $h \ge 2$.*

Proof. This is immediate from Corollary 7.3.2. □

Theorem 7.3.1 is a consequence of Sections 6.4, 7.2 and the following result:

Proposition 7.3.4. *Let $r \ge 3$, $l \ge 1$, and let $\phi : G = \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r} \to A = \mathbb{Z}^l$ be an epimorphism with $g_i \ge 2$.*

If $H = \ker\phi$ is of type $\mathcal{F}_m$ with $m \ge \frac{2r}{3}$, then there is a finite index subgroup $P \le G$ such that the inclusion $H_0 \hookrightarrow P$ of $H_0 := P \cap H$ induces an injection $(H_0)_{ab} \to P_{ab}$ and $b_1(H_0) \equiv l \bmod 2$. Furthermore, there are finite index subgroups $\pi_1 S_{h_i} \le \pi_1 S_{g_i}$ such that $H_0, P \le \pi_1 S_{h_1} \times \cdots \times \pi_1 S_{h_r}$ are both full subdirect products.

Proof of Theorem 7.3.1. Let $G \le \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$ be a full subgroup of type $\mathcal{F}_m$ with $m \ge 2$.

Infinite index subgroups of surface groups are free and finite index subgroups are surface groups. Thus, after reordering factors, we may assume that there are integers $s, t \ge 0$ such that

- $p_i(G) = \pi_1 S_{h_i} \le \pi_1 S_{g_i}$ is a finite index surface subgroup for $1 \le i \le t$,
- $p_i(G) = F_{h_i} \le \pi_1 S_{g_i}$ is finitely generated free with $h_i \ge 2$ generators for $t+1 \le i \le s$, and
- $p_i(G) \cong \mathbb{Z} \le \pi_1 S_{g_i}$ is infinite cyclic for $s + 1 \le i \le r$.

Centralisers in subgroups of surface groups are infinite cyclic. It follows that $p_{s+1,\dots,r}(H)$ is free abelian with $\mathrm{Z}(G) = \ker(p_{1,\dots,s}|_G) \leq p_{s+1,\dots,r}(G) \cong \mathbb{Z}^{r-s}$ a finite index subgroup and therefore $N = r - s$.

Consider the case when $p_i(G) = F_{h_i}$ with $h_i \geq 2$ free. Since G is finitely presented and N is finitely generated it follows from Theorem 2.5.5 that the kernel of the composition $p_i \circ \psi$ is finitely generated (see Proof of Proposition 7.2.2). The group F_{h_i} is Schreier with $b_1(F_{h_i}) = h_i \neq 0$. Hence, by Theorem 3.2.4, there is a commutative diagram

$$\begin{tikzcd} \widehat{G} \arrow[r, "\text{f.g. kernel}"] \arrow[d, "\text{f.g. kernel}"'] & p_i(G) = F_{h_i} \arrow[d] \\ \pi_1^{orb} S_{\gamma_i} \arrow[r] & \mathbb{Z} \end{tikzcd},$$

with surjective maps onto the infinite group $\mathbb{Z}$ for $\pi_1^{orb} S_{\gamma_i}$ the fundamental group of a closed hyperbolic Riemann orbisurface. Since $p_i(G)$ and $\pi_1^{orb} S_{\gamma_i}$ are Schreier with no finite normal subgroups it follows from Lemma 3.2.5 that $\pi_1^{orb} S_{\gamma_i} \cong F_{h_i}$. This is impossible and therefore $t = s$.

By Lemma 2.5.2 the quotient $H = p_{1,\cdots,s}(G) = G/\mathrm{Z}(G) \leq \pi_1 S_{h_1} \times \cdots \times \pi_1 S_{h_s}$ is full subdirect of type $\mathcal{F}_m$. For $k \leq \frac{3m}{2}$ and $1 \leq i_1 < \cdots < i_k \leq s$ consider the projection $\overline{H} := p_{i_1,\cdots,i_k}(G) \leq \pi_1 S_{h_{i_1}} \times \cdots \times \pi_1 S_{h_{i_k}}$. By Corollary 6.4.8 the group $\overline{H}$ is full subdirect of type $\mathcal{F}_m$ with $m \geq \frac{2k}{3}$.

By Proposition 6.4.7 there are finite index subgroups $\pi_1 S_{h'_{i_j}} \leq \pi_1 S_{h_{i_j}}$, $M \geq 0$, $1 \leq j \leq k$, and an epimorphism $\phi : \pi_1 S_{h'_{i_1}} \times \cdots \times \pi_1 S_{h'_{i_k}} \to \mathbb{Z}^M$ such that $\ker\phi \leq \pi_1 S_{h'_{i_1}} \times \cdots \times \pi_1 S_{h'_{i_k}}$ is full subdirect, $\ker\phi \leq \overline{H}$ is a finite index subgroup, and $\ker\phi$ is of type $\mathcal{F}_m$ with $m \geq \frac{2k}{3}$. The remainder of the argument does not depend on the choice of finite index subgroups $\pi_1 S_{h'_{i_j}}$ and epimorphism ϕ. Thus, the consequences we derive below hold for all such choices.

Proposition 7.3.4 implies that there is a finite index subgroup $P \leq \pi_1 S_{h'_{i_1}} \times \cdots \times \pi_1 S_{h'_{i_k}}$ such that the inclusion $\overline{H}_1 \hookrightarrow P$ of $\overline{H}_1 := P \cap \overline{H}$ induces an injection $(\overline{H}_1)_{ab} \hookrightarrow P_{ab}$ and $b_1(\overline{H}_1) \equiv M \bmod 2$. Furthermore there are finite index subgroups $\pi_1 S_{h''_{i_j}} \leq \pi_1 S_{h'_{i_j}}$, $1 \leq j \leq k$, such that $P \leq \pi_1 S_{h''_{i_1}} \times \cdots \times \pi_1 S_{h''_{i_k}}$ and $\overline{H}_1 \leq \pi_1 S_{h''_{i_1}} \times \cdots \times \pi_1 S_{h''_{i_k}}$ are both full subdirect products.

Consider the finite index subgroup $G_1 := G \cap p^{-1}_{i_1,\dots,i_k}(\overline{H}_1) \leq G$. Then there are finite index subgroups $\pi_1 S_{h''_j} \leq \pi_1 S_{g_j}$, $1 \leq j \leq s$, such that the projection $p_{1,\dots,s}(G_1) \leq \pi_1 S_{h''_1} \times \cdots \times \pi_1 S_{h''_s}$ is full subdirect (the notation is deliberate, since for $j = i_l$, $1 \leq l \leq k$, the groups are indeed the groups $\pi_1 S_{h''_{i_l}}$ obtained above). By construction

$p_{i_1,\cdots,i_k}(G_1) = \overline{H}_1$ and G_1 is of type $\mathcal{F}_m$ for $m \geq 2$. In particular, G_1 is finitely presented.

It follows that $\widehat{G}_1 := \psi^{-1}(G_1) \leq \widehat{G}$ is a finite index Kähler subgroup. Consider the induced homorphism $p_{1,\cdots,s} \circ \psi : \widehat{G}_1 \to p_{1,\cdots,s}(G_1)$ onto the finitely presented full subdirect product $p_{1,\cdots,s}(G_1) \leq \pi_1 S_{h_1''} \times \cdots \times \pi_1 S_{h_s''}$. Its kernel is an extension

$$1 \to N \to \ker\left(p_{1,\cdots,s} \circ \psi|_{\widehat{G}_1}\right) \to \ker\left(p_{1,\cdots,s}|_{G_1}\right) \to 1.$$

$\ker\left(p_{1,\cdots,s}|_{G_1}\right) \leq \mathrm{Z}(G) \cong \mathbb{Z}^{r-s}$ is a finitely generated (abelian) group and N is finitely generated by assumption, so $\ker\left(p_{1,\cdots,s} \circ \psi|_{\widehat{G}_1}\right)$ is finitely generated. It follows from Theorem 7.2.1 that $M = b_1(\overline{H}_1) \equiv 0 \bmod 2$. Thus, there is $l \geq 0$ such that $M = 2l$. □

We will use Theorem 6.4.3 to prove Proposition 7.3.4.

Proof of Proposition 7.3.4. Assume that $H = \ker\phi$ is of type $\mathcal{F}_m$ and that $m \geq \frac{2r}{3}$. By Corollary 6.4.5 the group $\phi(\pi_1 S_{g_{i_1}} \times \cdots \times \pi_1 S_{g_{i_{r-m}}}) \leq A = \mathbb{Z}^l$ is a finite index subgroup for every $1 \leq i_1 < \cdots < i_{r-m} \leq r$.

Since $m \geq \frac{2r}{3}$ it follows that $r - m \leq \frac{r}{3}$. Hence, we can partition $\{1, \cdots, r\}$ into three subsets $B_1 = \{j_0 = 1, \cdots, j_1\}$, $B_2 = \{j_1 + 1, \cdots, j_2\}$ and $B_3 = \{j_2 + 1, \cdots, j_r = r\}$ of size $|B_i| \geq r - m$. Let $P_i = \pi_1 S_{g_{j_{i-1}+1}} \times \cdots \times \pi_1 S_{g_{j_i}}$ and let $A_i := \phi(P_i) \leq A = \mathbb{Z}^l$ be the corresponding finite index subgroups of A. The intersection $\overline{A} = A_1 \cap A_2 \cap A_3$ is itself a finite index subgroup of A and in particular $\overline{A} \cong \mathbb{Z}^l$.

Define finite index subgroups $P_{i,0} := \phi^{-1}(\overline{A}) \cap P_i \leq P_i$. Since $\phi(P_i) = A_i \geq \overline{A}$ we have $\phi(P_{i,0}) = \overline{A}$. Consider the restriction $\overline{\phi} : P_{1,0} \times P_{2,0} \times P_{3,0} \to \overline{A}$ to the finite index subgroup $P := P_{1,0} \times P_{2,0} \times P_{3,0} \leq G$.

After possibly passing to finite index subgroups $\pi_1 S_{h_i} \leq \pi_1 S_{g_i}$, we may assume that $P \leq \pi_1 S_{h_1} \times \cdots \times \pi_1 S_{h_r}$ is a full subdirect product. By construction $\overline{\phi}(P_{i,0}) = \overline{A}$ for $i = 1, 2, 3$. Thus, the projection of $H_0 := \ker\overline{\phi}$ onto the factors $P_{i,0}$ is surjective for $i = 1, 2, 3$ and in particular H_0 is itself a full subdirect product of $\pi_1 S_{h_1} \times \cdots \times \pi_1 S_{h_r}$. Since $P \leq G$ is a finite index subgroup the group $H_0 \leq \ker\phi$ is a finite index subgroup.

Since the restriction of the homomorphism $\overline{\phi} : P_{1,0} \times P_{2,0} \times P_{3,0} \to \overline{A}$ to every factor is surjective we can apply Theorem 7.1.5 to $\overline{\phi}$. It follows that the induced homomorphism $(\ker\overline{\phi})_{ab} = (H_0)_{ab} \to P_{ab}$ is injective and

$$b_1(H_0) = b_1(P) - l = b_1(P_{1,0}) + b_1(P_{2,0}) + b_1(P_{3,0}) - l.$$

Corollary 7.1.3 implies that $b_1(P_{i,0})$ is even for $i = 1, 2, 3$. Thus, we obtain

$$b_1(H_0) \equiv l \bmod 2.$$

□

7.4 Consequences and generalisations

In this section we discuss some consequences and generalisations of the results of this chapter. Most of the topics presented in this section offer tempting open questions and we are currently pursuing them.

7.4.1 Orbifold fundamental groups and the universal homomorphism

Most of the results in Section 7.2 and Section 7.3 also hold if we replace the surface groups $\pi_1 S_{g_i}$ by orbifold fundamental groups $\pi_1^{orb} S_{g_i}$ (as defined in Section 2.3). In particular, Theorem 7.2.1, Proposition 7.2.2, Proposition 7.3.4 and Theorem 7.3.1 hold in this more general setting. This is because Lemma 2.3.3 also applies to orbifold fundamental groups and the same holds for all of the required results about subgroups of direct products of surface groups (because we can always pass to finite index surface subgroups $\pi_1 S_{h_i} \leq \pi_1^{orb} S_{g_i}$ and this implies the analogous results for subdirect products of orbifold fundamental groups).

The range of potential applications of Theorem 7.3.1 becomes particularly clear if for a Kähler group $\widehat{G}$ we combine its orbifold version with the universal homomorphism $\phi : \widehat{G} \to \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ to a product of orbisurface fundamental groups from Corollary 3.2.7 and the following observation:

Lemma 7.4.1. *Let X be a compact Kähler manifold, let $\widehat{G} = \pi_1 X$ be its fundamental group and let $\phi : \widehat{G} \to \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ be the universal homomorphism to a product of orbisurface fundamental groups defined in Corollary 3.2.7.*

Then ϕ is induced by a holomorphic map $f : X \to \Sigma_1 \times \cdots \times \Sigma_r$ and the image $G := \phi(\widehat{G}) \leq \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ of ϕ is a finitely presented full subdirect product.

Proof. To simplify notation denote by $\Gamma_i := \pi_1^{orb}\Sigma_i$ the orbifold fundamental group of Σ_i for $1 \leq i \leq r$. The only part that is not immediate from Corollary 3.2.7 is that the image G of the restriction $\phi|_{\widehat{G}}$ is finitely presented.

To see this, recall that by Corollary 3.2.7 we have that the composition $p_i \circ \phi : \widehat{G} \to \pi_1^{orb}\Sigma_i$ of ϕ with the projection onto $\pi_1^{orb}\Sigma_i$ has finitely generated kernel. This implies that the kernel

$$N_i := \ker(p_i|_G) = G \cap (\Gamma_1 \times \cdots \times \Gamma_{i-1} \times 1 \times \Gamma_{i+1} \times \cdots \times \Gamma_r) \trianglelefteq G$$

of the surjective restriction $p_i|_G : G \to \Gamma_i$ is a finitely generated normal subgroup of G.

Finite presentability is trivial for $r = 1$, so assume that $r \geq 2$. Let $1 \leq i < j \leq r$. The image of the projection $p_{i,j}(G) \leq \Gamma_i \times \Gamma_j$ is a full subdirect product and $p_{i,j}(N_i) \trianglelefteq p_{i,j}(G)$ is a normal finitely generated subgroup. Since by definition $p_{i,j}(N_i) \leq 1 \times \Gamma_j$ it follows from subdirectness of $p_{i,j}(G)$ that in fact $p_{i,j}(N_i) \trianglelefteq 1 \times \Gamma_j$ is a normal finitely generated subgroup.

The group $\Gamma_j = \pi_1^{orb}\Sigma_j$ is Schreier without finite normal subgroups. Hence, $p_{i,j}(N_i)$ is either trivial or has finite index in Γ_j. The former is not possible, because G is full. It follows that $p_{i,j}(N_i) \trianglelefteq 1 \times G_j$ is a finite index subgroup. Thus, $p_{i,j}(G) \leq \Gamma_i \times \Gamma_j$ is a finite index subgroup. Since i and j were arbitrary we obtain that G has the VSP property. This implies that G is finitely presented. $\square$

Note that the only place in the proof of Theorem 7.3.1 where we used that the kernel N of the homomorphism ϕ is finitely generated was to obtain holomorphic maps inducing the epimorphisms $p_i \circ \phi|_{\widehat{G}_1} : \widehat{G}_1 \to \pi_1 S_{h_i'}$ obtained by restricting the projections $p_i \circ \phi : \widehat{G} \to \pi_1^{orb}\Sigma_i$ to finite index surface subgroups. Since the universal homomorphism ϕ is by definition induced by a holomorphic map, it is clearly possible to induce the restrictions $p_i \circ \phi|_{\widehat{G}_1}$ by holomorphic maps – the argument is the same as in the proof of Proposition 7.2.2. Thus, ϕ and its image satisfy all of the conclusions of Theorem 7.3.1. As a consequence we obtain a different version of Theorem 7.3.1.

Theorem 7.4.2. *For every Kähler group $\widehat{G}$ there are $r \geq 0$, closed orientable hyperbolic orbisurfaces Σ_i of genus $g_i \geq 2$ and a homomorphism $\phi : \widehat{G} \to \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ with the universal properties described in Corollary 3.2.7. Its image $\phi(\widehat{G}) \leq \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ is a finitely presented full subdirect product.*

If G is of type $\mathcal{F}_m$ for $m \geq \frac{2k}{3}$ then for every $1 \leq i_1 < \cdots < i_k \leq r$ the projection $p_{i_1,\ldots,i_k}(G) \leq \pi_1^{orb}\Sigma_{i_1} \times \cdots \times \pi_1^{orb}\Sigma_{i_k}$ has a finite index coabelian subgroup, and every finite index coabelian subgroup of $p_{i_1,\ldots,i_k}(G)$ is coabelian of even rank.

Lemma 7.4.1 and its proof raise the natural question if there is a geometric analogue of the VSP property in this setting. More precisely, it is natural to ask if the composition $f_{ij} : X \to \Sigma_i \times \Sigma_j$ of the holomorphic map $f : X \to \Sigma_1 \times \cdots \times \Sigma_r$ inducing the universal homomorphism and the holomorphic projection $\Sigma_1 \times \cdots \times \Sigma_r \to \Sigma_i \times \Sigma_j$ is surjective. The answer to this question is positive.

Proposition 7.4.3. *Let X be a compact Kähler manifold and let $\widehat{G} = \pi_1 X$. Let $f = (f_1, \cdots, f_r) : X \to \Sigma_1 \times \cdots \times \Sigma_r$ be a holomorphic realisation of the universal homomorphism $\phi : \widehat{G} \to \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ defined in Corollary 3.2.7.*

Then the holomorphic projection $f_{ij} = (f_i, f_j) : X \to \Sigma_i \times \Sigma_j$ is surjective for $1 \leq i < j \leq r$.

Proof. Since $\Sigma_i \times \Sigma_j$ is connected of complex dimension 2 and f_{ij} is holomorphic it suffices to show that the image of f_{ij} is 2-dimensional. After passing to finite covers $R_{\gamma_i} \to \Sigma_i$ by closed Riemann surfaces of genus $\gamma_i \geq 2$ and the induced finite-sheeted cover $X_0 \to X$ with $f_*(\pi_1 X_0) = (\pi_1 R_{\gamma_1} \times \cdots \times \pi_1 R_{\gamma_r}) \cap f_*(\pi_1 X)$, we may assume that the Σ_i are closed Riemann surfaces.

Let $g = (g_1, \ldots, g_r) : X_0 \to R_{\gamma_1} \times \cdots \times R_{\gamma_r}$ be the corresponding holomorphic map with full subdirect image $g_*(\pi_1 X_0) = f_*(\pi_1 X_0)$. For $1 \leq i < j \leq r$ consider the image $g_{ij}(X_0) \subset R_{\gamma_i} \times R_{\gamma_j}$ of the holomorphic map $g_{ij} = (g_i, g_j) : X_0 \to R_{\gamma_i} \times R_{\gamma_j}$. Assume for a contradiction that it is one-dimensional (equivalently, the image $f_{ij}(X) \subset \Sigma_i \times \Sigma_j$ of $f_{ij} = (f_i, f_j)$ is one-dimensional).

Then Stein factorisation provides us with a factorisation

$$\begin{tikzcd} X \arrow[r, "h_2"] \arrow[dr, "g_{ij}"'] & Y \arrow[d, "h_1"] \\ & g_{ij}(X_0) \end{tikzcd}$$

such that h_1 is holomorphic and finite-to-one, h_2 is holomorphic with connected fibre and Y is a complex analytic space of dimension one. It is well-known that smoothness and compactness of X and the fact that Y is one-dimensional imply that we may assume that Y is a closed Riemann surface. Since the argument is short, but hard to find in the literature, we want to sketch it for the readers convenience: Since X is smooth and compact it is a normal complex analytic space. By the universal property of the normalisation $Y_{nor} \to Y$ we may assume that h_2 factors through $h_2' : X \to Y_{nor}$ and it is easy to see that h_2' has connected fibres. Thus, we may assume that Y is a one-dimensional normal complex analytic space. Now use the fact that one-dimensional normal complex analytic spaces are smooth.

By projecting $g_{ij}(X_0)$ to factors we obtain factorisations

$$\begin{tikzcd} X_0 \arrow[r, "h_2"] \arrow[dr, "g_i"'] & Y \arrow[d, "q_i"] & X_0 \arrow[r, "h_2"] \arrow[dr, "g_j"'] & Y \arrow[d, "q_j"] \\ & R_{\gamma_i} & & R_{\gamma_j} \end{tikzcd},$$

in which all maps are surjective and holomorphic, and in particular q_i and q_j are finite-sheeted branched coverings.

Connectedness of the fibres of h_2 implies that the induced map $h_{2*} : \pi_1 X \to \pi_1 Y$ is surjective. Since the induced maps $g_{i,*} : \pi_1 X \to \pi_1 R_{\gamma_i}$ and $g_{j,*} : \pi_1 X \to \pi_1 R_{\gamma_j}$ are surjective it follows that $q_{i,*}$ and $q_{j,*}$ are surjective. Hence, the fact that the kernels of $g_{i,*}$ and $g_{j,*}$ are finitely generated implies that the kernels $\ker q_{i,*}$ and $q_{j,*}$ are finitely generated. The Schreier property of surface groups implies that $q_{i,*}$ and $q_{j,*}$ are isomorphisms.

It follows that g_{ij} factors as

$$\begin{tikzcd} X_0 \arrow[r, "h_2"] \arrow[dr, "g_{ij}"'] & Y \arrow[d, "(q_i, q_j)"] \\ & R_{\gamma_i} \times R_{\gamma_j} \end{tikzcd},$$

where the induced map $(q_{i,*}, q_{j,*}) : \pi_1 Y \to \pi_1 R_{\gamma_i} \times \pi_1 R_{\gamma_j}$ is injective. In particular the image of $(q_{i,*}, q_{j,*})$ does not contain $\mathbb{Z}^2$ as a subgroup, because the fundamental group of the closed hyperbolic surface $\pi_1 Y$ does not. In contrast Lemma 7.4.1 and its proof imply that the image $g_{ij,*}(\pi_1 X_0) \leq \pi_1 R_{\gamma_i} \times \pi_1 R_{\gamma_j}$ is a finite index subgroup and therefore contains $\mathbb{Z}^2$ as a subgroup. This contradicts the assumption. It follows that $g_{ij}(X_0)$ is 2-dimensional. $\square$

It is natural to ask if there is a generalisation of Proposition 7.4.3 to give surjective holomorphic maps onto products of k factors. The examples constructed in Theorem 4.3.2 show that this is certainly false for general $k \geq 3$ – for instance consider Theorem 4.3.2 with $r = 3$ and any choice of branched coverings satisfying all necessary conditions. More generally, we also note that all of the groups constructed in Theorem 4.3.2 are projective. Thus, we can use the Lefschetz Hyperplane Theorem (see Appendix B) to realise them as fundamental groups of compact projective surfaces. Hence, we can not even hope for holomorphic surjections onto k-tuples under the additional assumption that our groups are of type $\mathcal{F}_m$ and that $k \leq m$. Indeed, the correct way to phrase this question seems to be as follows:

Question. *Let X be a compact Kähler manifold with $\pi_i X = 0$ for $2 \leq i \leq m-1$ and let $\widehat{G} = \pi_1 X$. Let $\phi : \widehat{G} \to \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb}\Sigma_r$ be the universal homomorphism defined in Corollary 3.2.7 and let $f = (f_1, \ldots, f_r) : X \to \Sigma_1 \times \cdots \times \Sigma_r$ be a holomorphic map realising ϕ.*

If the image $\phi(\widehat{G}) \leq \pi_1^{orb}\Sigma_1 \times \cdots \times \pi_1^{orb} S_r$ has finiteness type $\mathcal{F}_m$ with $m \geq 2$, does this imply that for all $1 \leq i_1 < \cdots < i_k \leq r$ the corresponding holomorphic projection $(f_{i_1}, \cdots, f_{i_k}) : X \to \Sigma_{i_1} \times \cdots \times \Sigma_{i_k}$ onto k factors is surjective?

7.4.2 Delzant and Gromov's question in the coabelian case

Theorem 6.4.1 provides classes of examples of Kähler subgroups of direct products of surface groups arising as kernels of homomorphisms onto a abelian groups of arbitrary even rank. For any $k \geq 1$, Remark 7.1.7 provides large classes of non-Kähler subgroups of direct products of surface groups arising as kernel of a homomorphism onto a coabelian group of rank k. In particular, we obtain examples of such subgroups with even first Betti number which are not Kähler.

Following these results, the main question which remains open is if there are any Kähler subgroups of direct products of surface groups which arise as the kernel of a homomorphism onto a free abelian group of odd rank. This chapter shows that in many cases such groups can not be Kähler and we believe that they can never be Kähler. While we can not give a full proof of this result at this point, Theorem 7.3.1 and Corollary 7.3.2 make this result seem very plausible. Indeed we would not be surprised if a more careful analysis of the techniques and results developed here can be used to give a negative answer to this question; we are currently pursuing this. A particularly promising approach is to try and find projections to fewer factors for which the conclusions of Theorem 7.3.1 hold and to show that if the group is coabelian of odd rank then there is a projection with the same properties.

In fact, more strongly, we believe that for a coabelian group even the restrictions of the homomorphism to single factors should have even rank image in the free abelian group.

Reducing the number of surjections in Theorem 7.1.5

If in Theorem 7.1.5 we assume that the $G_i \cong \pi_1 S_{g_i}$ are fundamental groups of closed Riemann surfaces of genus $g_i \geq 2$ (or more generally non-abelian limit groups) and that $\ker\psi$ is finitely presented, then $\ker\psi$ virtually surjects onto pairs and we can apply Lemma 6.4.4 and methods similar to the ones used in the proof of Proposition 7.3.4 to show that it suffices to assume that the restriction of ψ to at least two factors (rather than three) is surjective.

Maps onto $\mathbb{Z}^2$ and $\mathbb{Z}^3$

Using combinatorial arguments and the VSP property we can also show that every finitely presented subgroup of a direct product of surface groups which arises as the kernel of an epimorphism from the product onto $\mathbb{Z}^2$ ($\mathbb{Z}^3$) has a finite index subgroup with even (odd) first Betti number. In particular, the kernel of an epimomorphism from a direct product of surface groups onto $\mathbb{Z}^3$ can not be Kähler, in analogy to Theorem 7.1.1. These combinatorial arguments do not generalise in any obvious way to $\mathbb{Z}^k$ with $k \geq 4$.

Potential generalisation of Theorem 7.3.1

The only obstruction to weakening the condition $k \leq \frac{3m}{2}$ on k in Theorem 7.3.1 comes from the same condition in Proposition 7.3.4. We want to explain why we expect that this condition can be reduced to $m > \frac{r}{2} + 1$.

Assume that the kernel $H = \ker\phi$ of an epimorphism $\phi : G = G_1 \times \cdots \times G_r \to \mathbb{Z}^l$ surjects onto m tuples for $m > \frac{r}{2} + 1$ if r is even or $m > \frac{r}{2}$ if r is odd. Then for any $i \in \{1, \cdots, r\}$ there is a partition of $\{1, \ldots, r\} \setminus \{i\}$ into two sets $A_i = \{a_{i,1}, \cdots, a_{i,k_i}\}$ and $B_i = \{b_{i,1}, \cdots, b_{i,n_i}\}$ with $k_i, n_i \geq m$. Thus, by Lemma 6.4.4, the restrictions $\phi|_{G_{a_{i,1}} \times \cdots \times G_{a_{i,k_i}}}$ and $\phi|_{G_{b_{i,1}} \times \cdots \times G_{b_{i,n_i}}}$ are surjective.

An argument very similar to the proof of Theorem 7.1.5 shows that ϕ induces a short exact sequence

$$1 \to H_{ab} \to G_{ab} \to \mathbb{Z}^l \to 1$$

and in particular $b_1(H) = b_1(G) - l$.

We hope to use this observation to improve the conditions on m in Proposition 7.3.4. Assume that we can show that for every group H of type $\mathcal{F}_m$ (m as above) which is the kernel of an epimorphism $\phi : \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r} \to \mathbb{Z}^l$, there is a subgroup $H_0 \leq H$ of finite index which is the kernel of the restriction $\phi : \pi_1 S_{h_1} \times \cdots \times \pi_1 S_{h_r} \to \mathbb{Z}^l$ to finite index subgroups $\pi_1 S_{h_1} \leq \pi_1 S_{g_1}$ and surjects onto m-tuples in $\pi_1 S_{h_1} \times \cdots \times \pi_1 S_{h_r}$. Then it follows from the previous paragraph that $b_1(H_0) \equiv l \bmod 2$. This means that under these assumptions we can reduce the condition in Proposition 7.3.4 to $m > \frac{r}{2} + 1$ if r is even and $m > \frac{r}{2}$ if r is odd.

Since by Theorem 6.4.3 any such group H of type $\mathcal{F}_m$ virtually surjects onto m-tuples it seems reasonable to expect that it thus indeed suffice to put these weaker conditions on m. However, it is not clear to us whether the approach described in the previous paragraph will lead to the desired result or if different techniques which avoid producing a situation where we have actual surjection to m-tuples will be needed. This is because so far we have not been able to establish the assumption required in the previous paragraph, since the most natural approaches seem to fail.

7.4.3 Construction of non-Kähler, non-coabelian subgroups

The general nature of Theorem 7.3.1 means that it has potential applications far beyond the realm of coabelian subgroups of direct products of surface groups. One particularly tempting instance of such a potential application is the question of finding irreducible examples of non-Kähler subgroups of direct products of surface groups which are not virtually coabelian.

Currently there is only one known construction of subgroups of direct products of limit groups which are not virtually coabelian. This construction is due to Bridson, Howie, Miller and Short and their examples are subgroups of direct products of free groups (see [31, Section 4]). We shall explain how the groups arising from their construction can be used to obtain examples of subgroups of direct products of surface groups which are not virtually coabelian and why we expect that some (and probably even all) of these subgroups are not Kähler.

The examples of Bridson, Howie, Miller and Short are finitely presented subdirect products $H_{r,2} \leq F_2^{(1)} \times \cdots \times F_2^{(r)} =: P_{2,r}$ of r 2-generated free groups $F_2^{(i)}$; they can readily be generalised to examples of finitely presented subdirect products $H_{r,n} \leq F_n^{(1)} \times \cdots \times F_n^{(r)}$ of n-generated free groups $F_n^{(i)}$ for all $n \geq 2$, $r \geq 3$. Finite presentability implies that the groups $H_{r,n}$ have the VSP property.

The key property of the groups $H_{r,n}$ is that they are conilpotent of nilpotency class $r-2$: the intersection $H_{r,n} \cap F_n^{(i)} = \gamma_{r-1}(F_n^{(i)})$ is the $(r-1)$-th term of the lower central series of $F_n^{(i)}$, $1 \leq i \leq r$. This property of the $H_{r,n}$ shows that there are no finite index subgroups $\Lambda_i \leq F_n^{(i)}$ such that $\gamma_k(\Lambda_i) \leq \gamma_{r-1}(F_n^{(i)}) = H_{r,n} \cap F_n^{(i)}$ for any $k \leq (r-2)$ — if there were, then this would mean that there is a homomorphism $\Lambda_i/\gamma_k(\Lambda_i) \to F_n^{(i)}/\gamma_{r-1}(F_n^{(i)})$ with finite index image of nilpotency class $k-1 < r-2$. In particular, the $H_{r,n}$ can not be virtually coabelian for $r \geq 4$.

Observe that the $H_{r,n}$ are irreducible, since if there was a finite index subgroup $H_1 \times H_2 \leq H_{r,2}$ with $H_1 \leq F_n^{(1)} \times \cdots \times F_n^{(s)}$ and $H_2 \leq F_n^{(s+1)} \times \cdots \times F_n^{(r)}$, then each factor H_i would be conilpotent of lower nilpotency class by [31, Theorem C(2)].

Let $\pi_1 S_n$ be the fundamental group of a surface of genus g and let $\alpha_1, \beta_1, \cdots, \alpha_n, \beta_n$ be a standard symplectic generating set for $\pi_1 S_n$. Let $q : \pi_1 S_n \to F_n = F(\{a_1, \dots, a_n\})$ be the epimorphism defined by $\alpha_i \mapsto a_i$, $\beta_i \mapsto 1$, $1 \leq i \leq n$. Take r copies $\pi_1 S_n^{(1)}, \dots, \pi_1 S_n^{(r)}$ of $\pi_1 S_n$ together with epimorphisms $q_i : \pi_1 S^{(i)} \to F_n^{(i)}$ of this form. Consider the surjective product homorphism

$$\psi = (q_1, \dots, q_r) : \pi_1 S_n^{(1)} \times \cdots \times \pi_1 S_n^{(r)} \to F_n^{(1)} \times \cdots \times F_n^{(r)}.$$

The preimage $G_{r,n} := \psi^{-1}(H_{r,n}) \leq \pi_1 S_n^{(1)} \times \cdots \times \pi_1 S_n^{(r)}$ is a full subdirect product and has the VSP property. Thus, the $G_{r,n}$ are finitely presented. By definition the intersections $N_i := G_{r,n} \cap \pi_1 S_n^{(i)}$ satisfy $\pi_1 S_n^{(i)}/N_i \cong F_n^{(i)}/\gamma_{r-1}(F_n^{(i)})$. In particular, it follows from the same argument as for $H_{r,n}$ that $G_{r,n}$ is conilpotent of nilpotency class precisely $r-2$ and has no finite index subgroup which is conilpotent of lower nilpotency class. Thus, the group $G_{r,n}$ is not virtually coabelian.

Proposition 6.4.7 and finite presentability of the $H_{r,n}$ imply that the projections $p_{i_1,i_2,i_3}(H_{r,n})$ are virtually coabelian for $1 \leq i_1 < i_2 < i_3 \leq r$. Assume that there are finite index subgroups $\Lambda_{i_k} \leq F_n^{(i_k)}$, $N \geq 0$ and an epimorphism $\phi : \Lambda_{i_1} \times \Lambda_{i_2} \times \Lambda_{i_3} \to \mathbb{Z}^{2N+1}$. Then the preimages $\widehat{\Lambda}_{i_k} = q_i^{-1}(\Lambda_{i_k}) \leq \pi_1 S_n^{(i_k)}$ are finite index subgroups. We obtain an induced epimorphism $\widehat{\phi} = \phi \circ (q_{i_1}, q_{i_2}, q_{i_3}) : \widehat{\Lambda}_{i_1} \times \widehat{\Lambda}_{i_2} \times \widehat{\Lambda}_{i_3} \to \mathbb{Z}^{2N+1}$. Since projection to factors commutes with ψ it follows that $\ker \widehat{\phi} \leq p_{i_1,i_2,i_3}(G_{r,n}) \leq \pi_1 S_n^{(i_1)} \times \pi_1 S_n^{(i_2)} \times \pi_1 S_n^{(i_3)}$ is a finite index subgroup of $p_{i_1,i_2,i_3}(G_{r,n})$ which is coabelian of odd rank. Thus, if such Λ_{i_k} and ϕ exist for any $1 \leq i_1 < i_2 < i_3 \leq r$, then by Theorem 7.3.1 the group $G_{r,n}$ is not Kähler.

Due to the general nature of the construction in [31] it would be remarkable if all finite index coabelian subgroups of the (finitely presented) projections $p_{i_1,i_2,i_3}(H_{r,n}) \leq F_n^{(i_1)} \times F_n^{(i_2)} \times F_n^{(i_3)}$ with $1 \leq i_1 < i_2 < i_3 \leq r$, $r \geq 3, n \geq 2$, were coabelian of even rank. Indeed it seems merely a question of actually finding finite index coabelian subgroups of odd rank. The explicit way in which the groups $H_{r,n}$ are defined in [31, Section 4] allows for a direct search for such finite index subgroups and we are currently pursuing this.

Chapter 8

Kähler groups and Kodaira fibrations

The finiteness properties of subgroups of direct products of surface groups are very well understood [31, 30], so it is natural that the first examples of Kähler groups with exotic finiteness properties should have been constructed as the kernels of maps from a product of hyperbolic surface groups to an abelian group, as explained in Section 2.5 and Chapter 4.

The main goal of this chapter is to construct two new classes of Kähler groups with exotic finiteness properties. Our construction of these groups presents a second application of the methods developed in Section 6.1.2. All of the groups constructed in this chapter arise as fundamental groups of generic fibres of holomorphic maps from a direct product of Kodaira fibrations onto an elliptic curve. A *Kodaira fibration* (also called a *regularly fibred surface*) is a compact complex surface X that admits a regular holomorphic map onto a smooth complex curve. Topologically, X is the total space of a smooth fibre bundle whose base and fibre are closed 2-manifolds (with restrictions on the holonomy).

The first and most interesting family arises from a detailed construction of complex surfaces of positive signature that is adapted from Kodaira's original construction of such surfaces [88] (see Sections 8.2 and 8.3, in particular Theorem 8.3.1). In fact, our surfaces are diffeomorphic to those of Kodaira but have a different complex structure. The required control over the finiteness properties of these examples comes from the results in Chapter 4. We will show that the groups are very different from all of the previous examples; they do not have any finite index subgroup which embeds in a direct product of surface groups (see Section 8.4).

Our second class of examples is obtained from Kodaira fibrations of signature zero (see Theorem 8.1.1). Here the constructions are substantially easier and do not take us far from subdirect products of surface groups. Indeed it is not difficult to see that all of the groups that arise in this setting have a subgroup of finite index that embeds in a direct product of surface groups — this problem is solved in Section 8.4.

8.1 Kodaira fibrations of signature zero

In this section we will prove the following result:

Theorem 8.1.1. *Fix $r \geq 3$ and for $i = 1, \dots, r$ let $S_{\gamma_i} \hookrightarrow X_i \xrightarrow{k_i} S_{g_i}$ be a topological surface-by-surface bundle such that X_i admits a complex structure and has signature zero. Assume that $\gamma_i, g_i \geq 2$. Let $X = X_1 \times \dots \times X_r$. Let E be an elliptic curve and let $\alpha_i : S_{g_i} \to E$ be branched coverings such that the map $\sum_{i=1}^r \alpha_i : S_{g_1} \times \dots \times S_{g_r} \to E$ is surjective on π_1.*

Then we can equip X_i and S_{g_i} with Kähler structures such that:

1. *the maps k_i and α_i are holomorphic;*
2. *the map $f := \sum_{i=1}^r \alpha_i \circ k_i : X \to E$ has connected smooth generic fibre $\overline{H} \xrightarrow{j} X$;*
3. *the sequence*
$$1 \to \pi_1 \overline{H} \xrightarrow{j_*} \pi_1 X \xrightarrow{f_*} \pi_1 E \to 1$$
is exact;
4. *the group $\pi_1 \overline{H}$ is Kähler and of type $\mathcal{F}_{r-1}$, but not $\mathcal{F}_r$;*
5. *$\pi_1 \overline{H}$ has a subgroup of finite index that embeds in a direct product of surface groups.*

Fibrations of the sort described in Theorem 8.1.1 have been discussed in the context of Beauville surfaces and, more generally, quotients of products of curves; see Catanese [45], also e.g. [10, Theorem 4.1], [53]. There are some similarities between that work and ours, in particular around the use of fibre products to construct fibrations with finite holonomy, but the overlap is limited.

8.1.1 The origin of the lack of finiteness

We want to summarise the results of Chapter 4 which will be needed to prove that the groups constructed in this chapter have exotic finiteness properties.

Let E be an elliptic curve and for $i = 1, \ldots, r$ let $h_i : S_{g_i} \to E$ be a branched cover, where each $g_i \geq 2$. Endow S_{g_i} with the complex structure that makes h_i holomorphic. Let $Z = S_{g_1} \times \cdots \times S_{g_r}$. Using the additive structure on E, we define a surjective map with isolated singularities and connected fibres

$$h = \sum_{i=1}^{r} h_i : Z \to E.$$

The following criterion summarises the parts of Theorem 4.3.2 which are relevant to this Chapter:

Theorem 8.1.2. *If $h_* : \pi_1 Z \to \pi_1 E$ is surjective, then the generic fibre H of h is connected and its fundamental group $\pi_1 H$ is a projective (hence Kähler) group that is of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$. Furthermore, the sequence*

$$1 \to \pi_1 H \to \pi_1 Z \xrightarrow{h_*} \pi_1 E \to 1$$

is exact.

8.1.2 Kodaira Fibrations

The following definition is equivalent to the more concise one that we gave at the beginning of this chapter.

Definition 8.1.3. A *Kodaira fibration* X is a Kähler surface (real dimension 4) that admits a regular holomorphic surjection $X \to S_g$. The fibre of $X \to S_g$ will be a closed surface, S_γ say. Thus, topologically, X is a S_γ-bundle over S_g. We require $g, \gamma \geq 2$.

These complex surfaces bear Kodaira's name because he [88] (and independently Atiyah [8]) constructed specific non-trivial examples in order to prove that the signature is not multiplicative in smooth fibre bundles. Kodaira fibrations should not be confused with Kodaira surfaces in the sense of [9, Sect. V.5], which are complex surfaces of Kodaira dimension zero that are never Kähler.

The nature of the holonomy in a Kodaira fibration is intimately related to the *signature* $S(X)$, which is the signature of the bilinear form

$$\cdot \cup \cdot : H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to H^4(X, \mathbb{R}) \cong \mathbb{R}$$

given by the cup product.

8.1.3 Signature zero: groups commensurable to subgroups of direct products of surface groups

We will make use of the following theorem of Kotschick [92] and a detail from his proof. Here, $\mathrm{Mod}(S_g)$ denotes the mapping class group of S_g.

Theorem 8.1.4. *Let X be a (topological) S_γ-bundle over S_g where $g, \gamma \geq 2$. Then the following are equivalent:*

1. *X can be equipped with a complex structure and $\sigma(X) = 0$;*
2. *the monodromy representation $\rho : \pi_1 S_g \to \mathrm{Out}(\pi_1 S_\gamma) = \mathrm{Mod}(S_\gamma)$ has finite image.*

The following is an immediate consequence of the proof of Theorem 8.1.4 in [92].

Addendum 8.1.5. *If either of the equivalent conditions in Theorem 8.1.4 holds, then for any complex structure on the base space S_g there is a Kähler structure on X with respect to which the projection $X \to S_g$ is holomorphic.*

We are now in a position to construct the examples promised in Theorem 8.1.1. Fix $r \geq 3$ and for $i = 1, \cdots, r$ let X_i be the underlying manifold of a Kodaira fibration with base S_{g_i} and fibre S_{γ_i}. Suppose that $\sigma(X_i) = 0$. Let $Z = S_{g_1} \times \cdots \times S_{g_r}$.

We fix an elliptic curve E and choose branched coverings $h_i : S_{g_i} \to E$ so that $h := \sum_i h_i$ induces a surjection $h_* : \pi_1 Z \to \pi_1 E$. We endow S_{g_i} with the complex structure that makes h_i holomorphic and use Addendum 8.1.5 to choose a complex structure on X_i that makes $p_i : X_i \to S_{g_i}$ holomorphic. Let $X = X_1 \times \cdots \times X_r$ and let $p : X \to Z$ be the map that restricts to p_i on X_i.

Theorem 8.1.6. *Let $p : X \to Z$ and $h : Z \to E$ be the maps defined above, let $f = h \circ p : X \to E$ and let $\overline{H}$ be the generic smooth fibre of f. Then $\pi_1 \overline{H}$ is a Kähler group of type $\mathcal{F}_{r-1}$ that is not of type $\mathcal{F}_r$ and there is a short exact sequence*

$$1 \to \pi_1 \overline{H} \to \pi_1 X \xrightarrow{f_*} \pi_1 E = \mathbb{Z}^2 \to 1.$$

Moreover, $\pi_1 \overline{H}$ has a subgroup of finite index that embeds in a direct product of surface groups.

Proof of Theorem 8.1.6. By construction, the map $f = p \circ h : X \to E$ satisfies the hypotheses of Theorem 6.1.5. Moreover, since Z is aspherical, $\pi_2 Z = 0$ and Proposition

6.1.6 applies. Thus, writing $\overline{H}$ for the generic smooth fibre of f and H for the generic smooth fibre of h, we have short exact sequences

$$1 \to \pi_1\overline{H} \to \pi_1 X \to \pi_1 E = \mathbb{Z}^2 \to 1$$

and

$$1 \to \pi_1 S_{\gamma_1} \times \cdots \times \pi_1 S_{\gamma_r} \to \pi_1\overline{H} \to \pi_1 H \to 1.$$

The product of the closed surfaces S_{γ_i} is a classifying space for the kernel in the second sequence, so Lemma 2.5.2 implies that $\pi_1\overline{H}$ is of type $\mathcal{F}_k$ if and only if $\pi_1 H$ is of type $\mathcal{F}_k$. Theorem 8.1.2 tells us that $\pi_1 H$ is of type $\mathcal{F}_{r-1}$ and not of type $\mathcal{F}_r$. Finally, the group $\pi_1\overline{H}$ is clearly Kähler, since it is the fundamental group of the compact Kähler manifold $\overline{H}$.

To see that $\pi_1\overline{H}$ is commensurable to a subgroup of a direct product of surface groups, note that the assumption $\sigma(X_i) = 0$ implies that the monodromy representation $\rho_i : \pi_1 S_{g_i} \to \mathrm{Out}(\pi_1 S_{\gamma_i})$ is finite, and hence $\pi_1 X_i$ contains the product of surface groups $\Gamma_i = \pi_1 S_{\gamma_i} \times \ker\rho_i$ as a subgroup of finite index. (Here we are using the fact that the centre of S_{γ_i} is trivial – cf. Corollary 8.IV.6.8 in [36]). The required subgroup of finite index in $\pi_1\overline{H}$ is its intersection with $\Gamma_1 \times \cdots \times \Gamma_r$. $\square$

In the light of Theorem 8.1.6, all that remains unproved in Theorem 8.1.1 is the assertion that in general $\pi_1\overline{H}$ is not itself a subgroup of a product of surface groups. We shall return to this point in the last section of this chapter.

8.2 New Kodaira Fibrations $X_{N,m}$

In 1967 Kodaira [88] constructed a family of complex surfaces $M_{N,m}$ that fibre over a complex curve but have positive signature. (See [8] for a very similar construction by Atiyah.) We shall produce a new family of Kähler surfaces $X_{N,m}$ that are Kodaira fibrations. We do so by adapting Kodaira's construction in a manner designed to allow appeals to Theorems 6.1.5 and 8.1.2. This is the main innovation in our construction of new families of Kähler groups.

Our surface $X_{N,m}$ is diffeomorphic to Kodaira's surface $M_{N-1,m}$ but it has a different complex structure. Because signature is a topological invariant, we can appeal to Kodaira's calculation of the signature

$$\sigma(X_{N,m}) = 8m^{4N} \cdot N \cdot m \cdot (m^2 - 1)/3. \tag{8.1}$$

The crucial point for us is that $\sigma(X_{N,m})$ is non-zero. It follows from Theorem 8.1.4 that the monodromy representation associated to the Kodaira fibration $X_{N,m} \to \Sigma$

has infinite image, from which it follows that the Kähler groups with exotic finiteness properties constructed in Theorem 8.3.6 are not commensurable to subgroups of direct products of surface groups, as we shall see in Section 8.4.

8.2.1 The construction of $X_{N,m}$

Kodaira's construction of his surfaces $M_{N,m}$ begins with a regular finite-sheeted covering of a higher genus curve $S \to R$. He then branches $R \times S$ along the union of two curves: one is the graph of the covering map and the other is the graph of the covering map twisted by a certain involution. We shall follow this template, but rather than beginning with a regular covering, we begin with a carefully crafted branched covering of an elliptic curve; this is a crucial feature, as it allows us to apply Theorems 6.1.5 and 8.1.2. Our covering is designed to admit an involution that allows us to follow the remainder of Kodaira's argument.

Let $\overline{E} = \mathbb{C}/\Lambda$ be an elliptic curve. Choose a finite set of (branching) points $B = \{b_1, \cdots, b_{2N}\} \subset \overline{E}$ and fix a basis $\overline{\mu}_1, \overline{\mu}_2$ of $\Lambda \cong \pi_1 \overline{E} \cong \mathbb{Z}^2$ represented by loops in $E \setminus B$. Let $p_{\overline{E}} : E \to \overline{E}$ be the double covering that the Galois correspondence associates to the homomorphism $\Lambda \to \mathbb{Z}_2$ that kills $\overline{\mu}_1$. Let μ_1 be the unique lift to E of $\overline{\mu}_1$ (it has two components) and let μ_2 be the unique lift of $2 \cdot \overline{\mu_2}$. Note that $\pi_1 E$ is generated by μ_2 and a component of μ_1.

E has a canonical complex structure making it an elliptic curve and the covering map is holomorphic with respect to this complex structure.

Let $\tau_E : E \to E$ be the generator of the Galois group; it is holomorphic and interchanges the components of $E \setminus \mu_1$.

Denote by $B^{(1)}$ and $B^{(2)}$ the preimages of B in the two distinct connected components of $E \setminus \mu_1$. The action of τ_E interchanges these sets.

Choose pairs of points in $\{b_{2k-1}, b_{2k}\} \subset B$, $k = 1, \cdots, N$, connect them by disjoint arcs $\gamma_1, \cdots, \gamma_N$ and lift these arcs to E. Denote by $\gamma_1^1, \cdots, \gamma_N^1$ the arcs joining points in $B^{(1)}$ and by $\gamma_1^2, \cdots, \gamma_N^2$ the arcs joining points in $B^{(2)}$.

Next we define a 3-fold branched covering of E as follows. Take three copies F_1, F_2 and F_3 of $E \setminus (B^{(1)} \cup B^{(2)})$ identified with $E \setminus (B^{(1)} \cup B^{(2)})$ via maps j_1, j_2 and j_3. We obtain surfaces G_1, G_2 and G_3 with boundary by cutting F_1 along all of the arcs $\gamma_1^2, \cdots, \gamma_N^2$, cutting F_2 along the arcs γ_k^i, $i = 1, 2$, $k = 1, \cdots, N$ and cutting F_3 along the arcs $\gamma_1^1, \cdots, \gamma_N^1$. Identify the two copies of the arc γ_k^1 in F_2 with the two copies of the arc γ_k^1 in F_3 and identify the two copies of the arc γ_k^2 in F_2 with the two copies of the arc γ_k^2 in F_1 in the unique way that makes the continuous

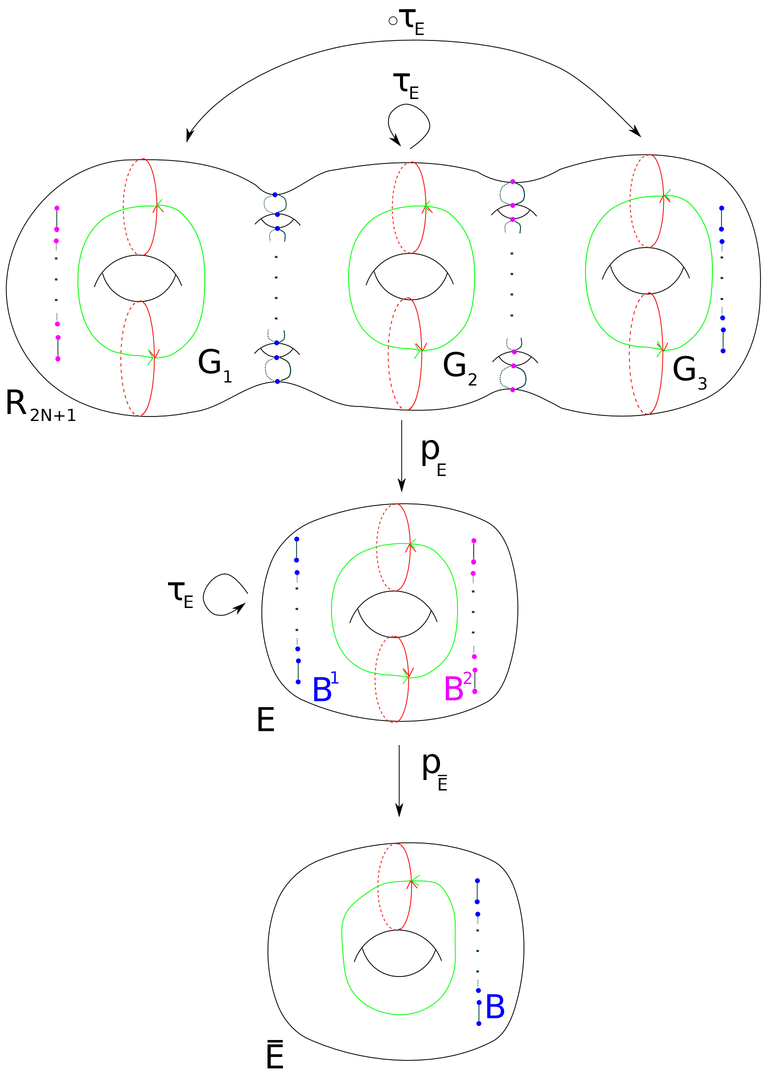

Figure 8.1: R_{2N+1} as branched covering of E together with the involution τ_E

map $p_E : G_1 \cup G_2 \cup G_3 \mapsto E \setminus (B^{(1)} \cup B^{(2)})$ induced by the identifications of F_i with $E \setminus (B^{(1)} \cup B^{(2)})$ a covering map. Figure 8.1 illustrates this covering map.

The map p_E clearly extends to a 3-fold branched covering map from the closed surface R_{2N+1} of genus $2N+1$, obtained by closing the cusps of $G_1 \cup G_2 \cup G_3$, to E. By slight abuse of notation we also denote this covering map by $p_E : R_{2N+1} \to E$. There is a unique complex structure on R_{2N+1} making the map p_E holomorphic.

The map τ_E induces a continuous involution $\tau_2 : G_2 \to G_2$ and a continuous involution $\tau_{1,3} : G_1 \sqcup G_3 \to G_1 \sqcup G_3$ without fixed points: these are defined by requiring the following diagrams to commute

$$\begin{tikzcd}
G_2 \arrow[r, "\tau_2"] \arrow[d, "j_2"'] & G_2 \arrow[d, "j_2"'] \\
E \smallsetminus (B^{(1)} \cup B^{(2)}) \arrow[r, "\tau_E"] & E \smallsetminus (B^{(1)} \cup B^{(2)}),
\end{tikzcd}$$

$$\begin{tikzcd}
G_1 \arrow[r, "\tau_{1,3}"] \arrow[d, "j_1"'] & G_3 \arrow[d, "j_3"'] \\
E \smallsetminus (B^{(1)} \cup B^{(2)}) \arrow[r, "\tau_E"] & E \smallsetminus (B^{(1)} \cup B^{(2)}),
\end{tikzcd}$$

$$\begin{tikzcd}
G_3 \arrow[r, "\tau_{1,3}"] \arrow[d, "j_3"'] & G_1 \arrow[d, "j_1"'] \\
E \smallsetminus (B^{(1)} \cup B^{(2)}) \arrow[r, "\tau_E"] & E \smallsetminus (B^{(1)} \cup B^{(2)}).
\end{tikzcd}$$

wherein j_i denotes the unique continuous extension of the original identification $j_i : F_i \to E \smallsetminus (B^{(1)} \cup B^{(2)})$.

The maps τ_2 and $\tau_{1,3}$ coincide on the identifications of $G_1 \sqcup G_3$ with G_2 and thus descend to a continuous involution $R_{2N+1} \smallsetminus p_E^{-1}(B^{(1)} \cup B^{(2)}) \to R_{2N+1} \smallsetminus p_E^{-1}(B^{(1)} \cup B^{(2)})$ which extends to a continuous involution $\tau_R : R_{2N+1} \to R_{2N+1}$.

Consider the commutative diagram

$$\begin{tikzcd}
R_{2N+1} \smallsetminus p_E^{-1}(B^{(1)} \cup B^{(2)}) \arrow[r, "\tau_R"] \arrow[d, "p_E"'] & R_{2N+1} \smallsetminus p_E^{-1}(B^{(1)} \cup B^{(2)}) \arrow[d, "p_E"'] \\
E \smallsetminus (B^{(1)} \cup B^{(2)}) \arrow[r, "\tau_E"] & E \smallsetminus (B^{(1)} \cup B^{(2)}).
\end{tikzcd}$$

As p_E is a holomorphic unramified covering onto $E \smallsetminus (B^{(1)} \cup B^{(2)})$ and τ_E is a holomorphic deck transformation mapping $E \smallsetminus (B^{(1)} \cup B^{(2)})$ onto itself, we can locally express τ_R as the composition of holomorphic maps $p_E^{-1} \circ \tau_E \circ p_E$ and therefore τ_R is itself holomorphic.

Since τ_R extends continuously to R_{2N+1}, it is holomorphic on R_{2N+1}, by Riemann's Theorem on removable singularities. By definition $\tau_R \circ \tau_R = \mathrm{Id}$. Thus $\tau_R : R_{2N+1} \to R_{2N+1}$ defines a holomorphic involution of R_{2N+1} without fixed points.

We have now manoeuvred ourselves into a situation whereby we can mimic Kodaira's construction. We replace the surface R in Kodaira 's construction [88, p.207-208] by R_{2N+1} and the involution τ in Kodaira's construction by the involution τ_R. The adaptation is straightforward, but we shall recall the argument below for the reader's convenience.

First though, we note that it is easy to check that for $m \geq 2$ we obtain a complex surface that is homeomorphic to the surface $M_{N-1,m}$ constructed by Kodaira, but in general our surface will have a different complex structure.

We denote this new complex surface $X_{N,m}$. Arguing as in the proof of [92, Proposition 1], we see that $X_{N,m}$ is Kähler.

8.2.2 Completing the Kodaira construction

Let $\alpha_1, \beta_1, \cdots, \alpha_{2N+1}, \beta_{2N+1}$ denote a standard set of generators of $\pi_1 R_{2N+1}$ satisfying the relation $[\alpha_1, \beta_1] \cdots [\alpha_{2N+1}, \beta_{2N+1}] = 1$, chosen so that the pairs α_1, β_1, α_2, β_2 and α_3, β_3 correspond to the preimages of μ_1 and μ_2 in G_1, G_2 and G_3 (with tails connecting these loops to a common base point).

For $m \in \mathbb{Z}$ consider the m^{2g}-fold covering $q_R : S \to R_{2N+1}$ corresponding to the homomorphism

$$\begin{aligned} \pi_1 R_{2N+1} &\to (\mathbb{Z}/m\mathbb{Z})^{2g} \\ \alpha_i &\mapsto (0, \cdots, 0, 1_{2i-1}, 0, 0, \cdots, 0) \\ \beta_i &\mapsto (0, \cdots, 0, 0, 1_{2i}, 0, \cdots, 0), \end{aligned} \tag{8.2}$$

where 1_i is the generator in the i-th factor. By multiplicativity of the Euler characteristic, we see that the genus of S is $2N \cdot m^{2g} + 1$.

To simplify notation we will from now on omit the index R in q_R and τ_R, as well as the index $2N+1$ in R_{2N+1}, and we denote the image $\tau(r)$ of a point $r \in R$ by r^*. Let $q^* = \tau \circ q : S \to R$, let $W = R \times S$ and let

$$\Gamma = \{(q(u), u) \mid u \in S\},$$

$$\Gamma^* = \{(q^*(u), u) \mid u \in S\}$$

be the graphs of the holomorphic maps q and q^*. Let $W'' = W \setminus (\Gamma \cup \Gamma^*)$. The complex surface $X_{N,m}$ is an m-fold branched covering of W branched along Γ and Γ^*. Its construction makes use of [88, p.209,Lemma]:

Lemma 8.2.1. *Fix a point $u_0 \in S$, identify R with $R \times u_0$ and let D be a small disk around $t_0 = q(u_0) \in R$. Denote by γ the positively oriented boundary circle of D. Then γ generates a cyclic subgroup $\langle\gamma\rangle$ of order m in $H_1(W'', \mathbb{Z})$ and*

$$H_1(W'', \mathbb{Z}) \cong H_1(R, \mathbb{Z}) \oplus H_1(S, \mathbb{Z}) \oplus \langle\gamma\rangle. \tag{8.3}$$

The proof of this lemma is purely topological and in particular makes no use of the complex structure on W''. From a topological point of view our manifolds and maps are equivalent to Kodaira's manifolds and maps, i.e. there is a homeomorphism that makes all of the obvious diagrams commute.

The composition of the isomorphism (8.3) with the surjective homomorphism $\pi_1 W'' \to H_1(W'', \mathbb{Z})$ induces an epimorphism $\pi_1 W'' \to \langle\gamma\rangle$. Consider the m-sheeted covering $X'' \to W''$ corresponding to the kernel of this map and equip X'' with the complex structure that makes the covering map holomorphic. The covering extends to an m-fold ramified covering on a closed complex surface $X_{N,m}$ with branching loci Γ and Γ^*.

The composition of the covering map $X_{N,m} \to W$ and the projection $W = R \times S \to S$ induces a regular holomorphic map $\psi : X_{N,m} \to S$ with complex fibre $R' = \psi^{-1}(u)$ a closed Riemann surface that is an m-sheeted branched covering of R with branching points $q(u)$ and $q^*(u)$ of order m. The complex structure of the fibres varies: each pair of fibres is homeomorphic but not (in general) biholomorphic.

8.3 Construction of Kähler groups

The main result of this section is:

Theorem 8.3.1. *For each $r \geq 3$ there exist Kodaira fibrations X_i, $i = 1, \ldots, r$, and a holomorphic map from $X = X_1 \times \cdots \times X_r$ onto an elliptic curve E, with generic fibre $\overline{H}$, such that the sequence*

$$1 \to \pi_1 \overline{H} \to \pi_1 X \to \pi_1 E \to 1$$

is exact and $\pi_1 \overline{H}$ is a Kähler group that is of type $\mathcal{F}_{r-1}$ but not $\mathcal{F}_r$.

Moreover, no subgroup of finite index in $\pi_1 \overline{H}$ embeds in a direct product of surface groups.

We fix an integer $m \geq 2$ and associate to each r-tuple of positive integers $\mathbf{N} = (N_1, \cdots, N_r)$ with $r \geq 3$ the product of the complex surfaces $X_{N_i,m}$ constructed in the previous section:

$$X(\mathbf{N}, m) = X_{N_1,m} \times \cdots \times X_{N_r,m}.$$

Each $X_{N_i,m}$ was constructed to have a holomorphic projection $\psi_i : X_{N_i,m} \to S_i$ with fibre R_i'.

By construction, each of the Riemann surfaces S_i comes with a holomorphic map $f_i = p_i \circ q_i$, where $p_i = p_{E,i} : R_{2N_i+1} \to E$ and $q_i = q_{R,i} : S_i \to R_{2N_i+1}$. We also need the homomorphism defined in (8.2), which we denote by θ_i.

We want to determine what $f_{i*}(\pi_1 S_i) \trianglelefteq \pi_1 E$ is. By definition $q_{i*}(\pi_1 S_i) = \ker(\theta_i)$, so $f_{i*}(\pi_1 S_i) = p_{i*}(\ker\theta_i)$. The map θ_i factors through the abelianisation $H_1(R_i, \mathbb{Z})$ of $\pi_1 R_i$, yielding $\overline{\theta}_i : H_1(R_i,\mathbb{Z}) \to (\mathbb{Z}/m\mathbb{Z})^{2g_i}$, which has the same image in $H_1(E,\mathbb{Z}) = \pi_1 E$ as $f_{i*}(\pi_1 S_i)$.

Now,

$$\ker\overline{\theta}_i = \langle m\cdot[\alpha_1], m\cdot[\alpha_1], m\cdot[\beta_1], \cdots, m\cdot[\alpha_{2N_i+1}], m\cdot[\beta_{2N_i+1}]\rangle \leq H_1(R_i,\mathbb{Z}).$$

and α_j, β_j were chosen such that for $1 \leq i \leq r$ we have

$$p_{i*}[\alpha_j] = \begin{cases} \mu_1 & \text{, if } j \in \{1,2,3\} \\ 0 & \text{, else} \end{cases} \quad \text{and } p_{i*}[\beta_j] = \begin{cases} \mu_2 & \text{, if } j \in \{1,2,3\} \\ 0 & \text{, else} \end{cases}.$$

(Here we have abused notation to the extent of writing μ_1 for the unique element of $\pi_1 E = H_1 E$ determined by either component of the preimage of $\overline{\mu}_1$ in E.) Thus,

$$f_{i*}(\pi_1 S_i) = \langle m\cdot\mu_1, m\cdot\mu_2\rangle \leq \pi_1 E. \tag{8.4}$$

There are three *loops* that are lifts $\mu_{1,i}^{(1)}, \mu_{1,i}^{(2)}, \mu_{1,i}^{(3)}$ of μ_1 with respect to p_i (regardless of the choice of basepoint $\mu_{1,i}^{(j)}(0) \in p_i^{-1}(\mu_1(0))$). The same holds for μ_2. And by choice of α_j, β_j for $j \in \{1,2,3\}$, we have $\left[\mu_1^{(j)}\right] = [\alpha_j] \in H_1(R_i,\mathbb{Z})$ after a permutation of indices.

Denote by $q_E : E' \to E$ the m^2-sheeted covering of E corresponding to the subgroups $f_{i*}(\pi_1 S_i)$. Endow E' with the unique complex structure making q_E holomorphic. By (8.4) the covering and the complex structure are independent of i.

Since $f_{i*}(\pi_1 S_i) = q_{E*}(\pi_1 E')$ there is an induced surjective map $f_i' : S_i \to E'$ making the diagram

$$\begin{tikzcd} S_i \arrow[r, "q_i"] \arrow[d, "f_i'"'] \arrow[dr, "f_i"] & R_i \arrow[d, "p_i"] \\ E' \arrow[r, "q_E"] & E \end{tikzcd} \tag{8.5}$$

commutative. The map f_i' is surjective and holomorphic, since f_i is surjective and holomorphic and q_E is a holomorphic covering map.

Lemma 8.3.2. *Let $B' = q_E^{-1}(B)$, $B_{S_i} = f_i^{-1}(B) = f_i'^{-1}(B')$. Let $\mu_1', \mu_2' : [0,1] \to E' \smallsetminus B'$ be loops that generate $\pi_1 E'$ and are such that $q_E \circ \mu_1' = \mu_1^m$, $q_E \circ \mu_2' = \mu_2^m$.*

Then the restriction $f_i' : S_i \smallsetminus B_{S_i} \to E' \smallsetminus B'$ is an unramified finite-sheeted covering map and all lifts of μ_1' and μ_2' with respect to f_i' are loops in $S_i \smallsetminus B_{S_i}$.

Proof. Since f_i and q_E are unramified coverings over $E \smallsetminus B$, it follows from the commutativity of diagram (8.5) that the restriction $f_i' : S_i \smallsetminus B_{S_i} \to E' \smallsetminus B'$ is an unramified finite-sheeted covering map.

For the second part of the statement it suffices to consider μ_1', since the proof of the statement for μ_2' is completely analogous. Let $y_0 = \mu_1'(0)$, let $x_0 \in f'^{-1}(y_0)$ and let $\nu_1 : [0,1] \to S_i \smallsetminus B_{S_i}$ be the unique lift of μ_1' with respect to f_i' with $\nu_1(0) = x_0$.

Since q_i is a covering map it suffices to prove that $q_i \circ \nu_1$ is a loop in R_i based at $z_0 = q_i(x_0)$ such that its unique lift based at x_0 with respect to q_i is a loop in S_i.

By the commutatitivity of diagram (8.5) and the definition of μ_1',

$$\mu_1^m = q_E \circ \mu_1' = q_E \circ f_i' \circ \nu_1 = p_i \circ q_i \circ \nu_1.$$

But the unique lift of μ_1^m starting at z_0 is given by $(\mu_{j_0}^1)^m$ where $j_0 \in \{1,2,3\}$ is uniquely determined by $\mu_{j_0}^{(1)}(0) = z_0$. Uniqueness of path-lifting gives

$$q_i \circ \nu_1 = (\mu_{j_0}^{(1)})^m.$$

Thus $(\mu_{j_0}^{(1)})^m \in \ker\theta_i = f_{i*}(\pi_1 S_i)$. Now, $\ker\theta_i$ is normal in $\pi_1 R_i$ and $q_i : S_i \to R_i$ is an unramified covering map, so all lifts of $(\mu_{j_0}^{(1)})^m$ to S_i are loops. In particular ν_1 is a loop in S_i. □

Lemma 8.3.2 implies

Corollary 8.3.3. *The holomorphic maps $f_i' : S_i \to E'$ are purely-branched covering maps for $1 \le i \le r$. In particular, the maps f_i' induce surjective maps on fundamental groups.*

Remark 8.3.4. The invariants which we introduced in Chapter 4 for the Kähler groups arising in Theorem 8.1.2 lead to a complete classification of these groups in the special case where all the coverings are purely-branched (see Theorem 4.6.2). Thus Corollary 8.3.3 ought to help in classifying the groups that arise from our construction.

Let

$$Z_{\mathbf{N},m} = S_1 \times \cdots \times S_r.$$

Using the additive structure on the elliptic curve E' we combine the maps $f_i' : S_i \to E'$ to define $h' : Z_{\mathbf{N},m} \to E'$ by

$$h : (x_1, \cdots, x_r) \mapsto \sum_{i=1}^{r} f_i'(x_i).$$

Lemma 8.3.5. *For all $m \geq 2$, all $r \geq 3$ and all $\mathbf{N} = (N_1, \ldots, N_r)$, the map $h : Z_{\mathbf{N},m} \to E'$ has isolated singularities and connected fibres.*

Proof. By construction, f_i' is non-singular on $S_i \setminus B_{S_i}$ and B_{S_i} is a finite set. Therefore, the set of singular points of h' is contained in the finite set

$$B_{S_1} \times \cdots \times B_{S_r}.$$

In particular, h' has isolated singularities.

Corollary 8.3.3 implies that the f_i' induce surjective maps on fundamental groups, so we can apply Theorem 8.1.2 to conclude that h' has indeed connected fibres. $\square$

Finally, we define $g : X_{\mathbf{N},m} \to Z_{\mathbf{N},m}$ to be the product of the fibrations $\psi_i : X_{N_i,m} \to S_i$ and we define

$$f = h' \circ g : X_{\mathbf{N},m} \to E'.$$

Note that g is a smooth fibration with fibre $F_{\mathbf{N},m} \coloneqq R_1' \times \cdots \times R_r'$.

With this notation established, we are now able to prove:

Theorem 8.3.6. *Let $f : X_{\mathbf{N},m} \to E'$ be as above, let $\overline{H}_{\mathbf{N},m} \subset X_{\mathbf{N},m}$ be the generic smooth fibre of f, and let $H_{\mathbf{N},m}$ be its image in $Z_{\mathbf{N},m}$. Then:*

1. *$\pi_1 \overline{H}_{\mathbf{N},m}$ is a Kähler group that is of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$;*

2. *there are short exact sequences*

$$1 \to \pi_1 F_{\mathbf{N},m} \to \pi_1 \overline{H}_{\mathbf{N},m} \xrightarrow{g_*} \pi_1 H_{\mathbf{N},m} \to 1$$

   and

$$1 \to \pi_1 \overline{H}_{\mathbf{N},m} \to \pi_1 X_{\mathbf{N},m} \xrightarrow{f_*} \mathbb{Z}^2 \to 1,$$

   such that the monodromy representations $\pi_1 H_{\mathbf{N},m} \to \mathrm{Out}(\pi_1 F_{\mathbf{N},m})$ and $\mathbb{Z}^2 \to \mathrm{Out}(\pi_1 F_{\mathbf{N},m})$ both have infinite image;

3. *No subgroup of finite index in $\pi_1 \overline{H}_{\mathbf{N},m}$ embeds in a direct product of surface groups (or of residually free groups);*

4. *$\pi_1 \overline{H}_{\mathbf{N},m}$ is irreducible.*

Proof. We have constructed $\overline{H}_{\mathbf{N},m}$ as the fundamental group of a Kähler manifold, so the first assertion in (1) is clear.

We argued above that all of the assumptions of Theorem 6.1.5 are satisfied, and this yields the second short exact sequence in (2). Moreover, $Z_{\mathbf{N},m} = S_1 \times \cdots \times S_r$ is aspherical, so Proposition 6.1.6 applies: this yields the first sequence.

$F_{\mathbf{N},m}$ is a finite classifying space for its fundamental group, so by applying Lemma 2.5.2 to the first short exact sequence in (2) we see that $\pi_1 \overline{H}_{\mathbf{N},m}$ is of type $\mathcal{F}_n$ if and only if $\pi_1 H_{\mathbf{N},m}$ is of type $\mathcal{F}_n$. Theorem 8.1.2 tells us that $\pi_1 H_{\mathbf{N},m}$ is of type $\mathcal{F}_{r-1}$ but not of type $\mathcal{F}_r$. Thus (1) is proved.

The holonomy representation of the fibration $\overline{H}_{\mathbf{N},m} \to H_{\mathbf{N},m}$ is the restriction

$$\nu = (\rho_1, \cdots, \rho_r)|_{\pi_1 H_{\mathbf{N},m}} : \pi_1 H_{\mathbf{N},m} \to \mathrm{Out}(\pi_1 R_1') \times \cdots \times \mathrm{Out}(\pi_1 R_r')$$

where ρ_i is the holonomy of $X_{N_i,m} \to S_i$. Since the branched covering maps f_i' are surjective on fundamental groups it follows from the short exact sequence induced by h' that the projection of $\nu(\pi_1 H)$ to $\mathrm{Out}(\pi_1 R_i')$ is $\rho_i(\pi_1 S_i)$. In particular, the map ν has infinite image in $\mathrm{Out}(\pi_1 F)$ as each of the ρ_i do. This proves (2).

Assertion (3) follows immediately from (2) and the group theoretic Proposition 8.4.1 below.

Assume that there is a finite index subgroup $\overline{G}_1 \times \overline{G}_2 \leq \pi_1 \overline{H}$ with $\overline{G}_1$ and $\overline{G}_2$ non-trivial; its projection $G_1 \times G_2 \leq \pi_1 H$ to $\pi_1 S_1 \times \cdots \times \pi_1 S_r$ is a finite index subgroup of $\pi_1 H$. By Proposition 6.4.6, $\pi_1 H$ is irreducible and thus either G_1 or G_2 is trivial, say G_1. Hence, $\overline{G}_1 \leq \pi_1 F_{\mathbf{N},m}$ and the finite index subgroup $(\overline{G}_1 \times \overline{G}_2) \cap \pi_1 F_{\mathbf{N},m} \leq \pi_1 F_{\mathbf{N},m}$ decomposes as a direct product $\overline{G}_1 \times \left(\overline{G}_2 \cap (\pi_1 R_{s+1}' \times \cdots \times \pi_1 R_r')\right)$ with $s \geq 1$. In particular, the projection of $\overline{G}_1 \times \overline{G}_2$ to $\pi_1 X_{N_1,m}$ yields a decomposition of a finite index subgroup into a direct product of surface groups; this is impossible by definition of the ρ_i and Proposition 8.4.1 below. Assertion (4) follows. □

Theorem 8.3.1 is now a direct consequence of Theorem 8.3.6.

Remark 8.3.7 (Explicit presentations)**.** The groups $\pi_1 \overline{H}_{\mathbf{N},m}$ constructed above are fibre products over $\mathbb{Z}^2$. Therefore, given finite presentations for the groups $\pi_1 X_{N_i,m}$, $1 \leq i \leq r$, we could apply the algorithm developed by Bridson, Howie, Miller and Short [31], which we used to obtain the finite presentations in Chapter 5, to also construct explicit finite presentations for these examples.

8.4 Commensurability to direct products

Each of the new Kähler groups $\Gamma := \pi_1\overline{H}$ constructed in Theorems 8.3.1 and 8.1.1 fits into a short exact sequence of finitely generated groups

$$1 \to \Delta \to \Gamma \to Q \to 1, \tag{8.6}$$

where $\Delta = \Sigma_1 \times \cdots \times \Sigma_r$ is a product of $r \geq 1$ closed surface groups $\Sigma_i := \pi_1 S_{g_i}$ of genus $g_i \geq 2$.

Such short exact sequences arise whenever one has a fibre bundle whose base B has fundamental group Q and whose fibre F is a product of surfaces: the short exact sequence is the beginning of the long exact sequence in homotopy, truncated using the observation that since Δ has no non-trivial normal abelian subgroups, the map $\pi_2 B \to \pi_1 F$ is trivial. For us, the fibration in question is $\overline{H} \to H$, and (8.6) is a special case of the sequence in Proposition 6.1.6. In the setting of Theorem 8.3.1, the holonomy representation $Q \to \mathrm{Out}(\Delta)$ has infinite image, and in the setting of Theorem 8.1.1 it has finite image.

In order to complete the proofs of the theorems stated in the introduction, we must determine (i) when groups such as Γ can be embedded in a product of surface groups, (ii) when they contain subgroups of finite index that admit such embeddings, and (iii) when they are commensurable with residually free groups. In this section we shall answer each of these questions.

Recall that a finitely generated group is residually free if and only if it is a subgroup of a direct product of finitely many limit groups. For a more detailed discussion of residually free groups and limit groups see Sections 2.1 and 2.5.

8.4.1 Infinite holonomy

Proposition 8.4.1. *If the holonomy representation $Q \to \mathrm{Out}(\Delta)$ associated to (8.6) has infinite image, then no subgroup of finite index in Γ is residually free, and therefore Γ is not commensurable with a subgroup of a direct product of surface groups.*

Proof. Any automorphism of $\Delta = \Sigma_1 \times \cdots \times \Sigma_r$ must leave the set of subgroups $\{\Sigma_1, \cdots, \Sigma_r\}$ invariant (cf. [33, Prop.4]). Thus $\mathrm{Aut}(\Delta)$ contains a subgroup of finite index that leaves each Σ_i invariant and $\mathcal{O} = \mathrm{Out}(\Sigma_1) \times \cdots \times \mathrm{Out}(\Sigma_r)$ has finite index in $\mathrm{Out}(\Delta)$.

Let $\rho : Q \to \mathrm{Out}(\Delta)$ be the holonomy representation, let $Q_0 = \rho^{-1}(\mathcal{O})$, and let $\rho_i : Q_0 \to \mathrm{Out}(\Sigma_i)$ be the obvious restriction. If the image of ρ is infinite, then the image of at least one of the ρ_i is infinite. Infinite subgroups of mapping class groups have to contain elements of infinite order (e.g. [83, Corollary 5.14]), so it follows that Γ contains a subgroup of the form $M = \Sigma_i \rtimes_\alpha \mathbb{Z}$, where α has infinite order in $\mathrm{Out}(\Sigma_i)$. If Γ_0 is any subgroup of finite index in Γ, then $M_0 = \Gamma_0 \cap M$ is again of the form $\Sigma \rtimes_\beta \mathbb{Z}$, where $\Sigma = \Gamma_0 \cap \Sigma_i$ is a hyperbolic surface group and $\beta \in \mathrm{Out}(\Sigma)$ (which is the restriction of α) has infinite order.

M_0 is the fundamental group of a closed aspherical 3-manifold that does not virtually split as a direct product, and therefore it cannot be residually free, by Theorem A of [30]. As any subgroup of a residually free group is residually free, it follows that Γ_0 is not residually free.

For the reader's convenience, we give a more direct proof that M_0 is not residually free. If it were, then by [14] it would be a subdirect product of limit groups $\Lambda_1 \times \cdots \times \Lambda_t$. Projecting away from factors that M_0 does not intersect, we may assume that $\Lambda_i \cap M_0 \neq 1$ for all i. As M_0 does not contain non-trivial normal abelian subgroups, it follows that the Λ_i are non-abelian. As limit groups are torsion-free and M_0 does not contain $\mathbb{Z}^3$, it follows that $t \leq 2$. Replacing each Λ_i by the coordinate projection $p_i(M_0)$, we may assume that $M_0 < \Lambda_1 \times \Lambda_2$ is a subdirect product (i.e. maps onto both Λ_1 and Λ_2). Then, for $i = 1, 2$, the intersection $M_0 \cap \Lambda_i$ is normal in $\Lambda_i = p_i(M_0)$. Non-abelian limit groups do not have non-trivial normal abelian subgroups, so $I_i = M_0 \cap \Lambda_i$ is non-abelian. But any non-cyclic subgroup of M_0 must intersect Σ, so $I_1 \cap \Sigma$ and $I_2 \cap \Sigma$ are infinite, disjoint, commuting, subgroups of Σ. This contradict the fact that Σ is hyperbolic. □

Corollary 8.4.2. *The group $\pi_1 \overline{H}$ constructed in Theorem 8.3.1 is not commensurable with a subgroup of a direct product of surface groups.*

8.4.2 Finite holonomy

When the holonomy $Q \to \mathrm{Out}(\Delta)$ is finite, it is easy to see that Γ is virtually a direct product.

Proposition 8.4.3. *In the setting of (8.6), if the holonomy representation $Q \to \mathrm{Out}(\Delta)$ is finite, then Γ has a subgroup of finite index that is residually free [respectively, is a subgroup of a direct product of surface groups] if and only if Q has such a subgroup of finite index.*

Proof. Let Q_1 be the kernel of $Q \to \mathrm{Out}(\Delta)$ and let $\Gamma_1 < \Gamma$ be the inverse image of Q_1. Then, as the centre of Δ is trivial, $\Gamma_1 \cong \Delta \times Q_1$.

Every subgroup of a residually free group is residually free, and the direct product of residually free groups is residually free. Thus the proposition follows from the fact that surface groups are residually free. □

Corollary 8.4.4. *Each of the groups $\pi_1\overline{H}$ constructed in Theorem 8.1.1 has a subgroup of finite index that embeds in a direct product of finitely many surface groups.*

Proof. Apply the proposition to each of the Kodaira fibrations X_i in Theorem 8.1.1 and intersect the resulting subgroup of finite index in $\pi_1 X_1 \times \cdots \times \pi_1 X_r$ with $\pi_1\overline{H}$. □

8.4.3 Residually-free Kähler groups

We begin with a non-trivial example of a Kodaira surface whose fundamental group is residually-free.

Example 8.4.5. Let G be any finite group and for $i = 1, 2$ let $q_i : \Sigma_i \to G$ be an epimorphism from a hyperbolic surface group $\Sigma_i = \pi_1 S_i$. Let $P < \Sigma_1 \times \Sigma_2$ be the fibre product, i.e. $P = \{(x, y) \mid q_1(x) = q_2(y)\}$. The projection onto the second factor $p_i : P \to \Sigma_2$ induces a short exact sequence

$$1 \to \Sigma_1' \to P \to \Sigma_2 \to 1$$

with $\Sigma_1' = \ker q_1 \trianglelefteq \Sigma_1$ a finite-index normal subgroup. The action of P by conjugation on Σ_1 defines a homomorphism $\Sigma_2 \to \mathrm{Out}(\Sigma_1')$ that factors through $q_2 : \Sigma_1 \to G = \Sigma_1/\Sigma_1'$.

Let $S_1' \to S_1$ be the regular covering of S_1 corresponding to $\Sigma_1' \trianglelefteq \Sigma_1$. Nielsen realisation [85] realises the action of Σ_2 on Σ_1' as a group of diffeomorphisms of S_1', and thus we obtain a smooth surface-by-surface bundle X with $\pi_1 X = P$, that has fibre S_1', base S_2 and holonomy representation q_2. Theorem 8.1.4 and Addendum 8.1.5 imply that X can be endowed with the structure of a Kodaira surface.

Our second example illustrates the fact that torsion-free Kähler groups that are virtually residually free need not be residually free.

Example 8.4.6. Let R_g be a closed orientable surface of genus g and imagine it as the connected sum of g handles placed in cyclic order around a sphere. We consider the automorphism that rotates this picture through $2\pi/g$. Algebraically, if we fix the usual presentation $\pi_1 R_g = \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \rangle$, this rotation (which has two fixed points) defines an automorphism ϕ that sends $\alpha_i \mapsto \alpha_{i+1}$, $\beta_i \mapsto \beta_{i+1}$ for

$1 \le i \le g-1$ and $\alpha_g \mapsto \alpha_1$, $\beta_g \mapsto \beta_1$. Thus $\langle\phi\rangle \le \mathrm{Aut}(\pi_1 R_g)$ is a cyclic subgroup of order g.

Let T_h be an arbitrary closed surfaces of genus $h \ge 2$ and let $\rho : \pi_1 T_h \to \langle\overline{\phi}\rangle \cong \mathbb{Z}/g\mathbb{Z} \le \mathrm{Out}(\pi_1 R_g)$ be the map defined by sending each element of a standard symplectic basis for $H_1(\pi_1 T_h, \mathbb{Z})$ to $\overline{\phi} := \phi \cdot \mathrm{Inn}(\pi_1 R_g)$. Consider a Kodaira fibration $R_g \hookrightarrow X' \to T_h$ with holonomy ρ. It follows from Proposition 8.4.9 that $\pi_1 X'$ is not residually free. And it follows from Theorem 8.4.10 that if the Kodaira surfaces in Theorem 8.1.6 are of this form then the Kähler group $\pi_1 \overline{H}$ is not residually free.

Lemma 8.4.7. *Let Σ be a hyperbolic surface group and let G be a group that contains Σ as a normal subgroup. The following conditions are equivalent:*

(i) the image of the map $G \to \mathrm{Aut}(\Sigma)$ given by conjugation is torsion-free and the image of $G \to \mathrm{Out}(\Sigma)$ is finite;

(ii) one can embed Σ as a normal subgroup of finite index in a surface group $\widehat{S}$ so that $G \to \mathrm{Aut}(\Sigma)$ factors through $\mathrm{Inn}(\widehat{\Sigma}) \to \mathrm{Aut}(\Sigma)$.

Proof. If (i) holds then the image A of $G \to \mathrm{Aut}(\Sigma)$ is torsion free and contains $\mathrm{Inn}(\Sigma) \cong \Sigma$ as a subgroup of finite index. A torsion-free finite extension of a surface group is a surface group, so we can define $\widehat{\Sigma} = A$. The converse follows immediately from the fact that centralisers of non-cyclic subgroups in hyperbolic surface groups are trivial. ☐

Lemma 8.4.7 has the following geometric interpretation, in which $\widehat{\Sigma}$ emerges as $\pi_1(\widetilde{S}/\Lambda)$.

Addendum 8.4.8. *With the hypotheses of Lemma 8.4.7, let S be a closed hyperbolic surface with $\Sigma = \pi_1 S$, let Λ be the image of $G \to \mathrm{Aut}(\Sigma)$ and let $\overline{\Lambda}$ be the image of $G \to \mathrm{Out}(\Sigma)$. Then conditions (i) and (ii) are equivalent to the geometric condition that the action $\overline{\Lambda} \to \mathrm{Homeo}(S)$ given by Nielsen realisation is free.*

Proof. Assume that condition (i) holds. Since $\overline{\Lambda}$ is finite, Kerckhoff's solution to the Nielsen realisation problem [85] enables us to realise Λ as a cocompact Fuchsian group: $\overline{\Lambda}$ can be realised as a group of isometries of a hyperbolic metric g on S and Λ is the discrete group of isometries of the universal cover $\widetilde{S} \cong \mathbb{H}^2$ consisting of all lifts of $\overline{\Lambda} \le \mathrm{Isom}(S, g)$. As a Fuchsian group, Λ is torsion-free if and only if its action on $\widetilde{S} \cong \mathbb{H}^2$ is free, and this is the case if and only if the action of $\overline{\Lambda} = \Lambda/\Sigma$ on Σ is free. ☐

As a consequence of Lemma 8.4.7 we obtain:

Proposition 8.4.9. *Consider a short exact sequence $1 \to F \to G \to Q \to 1$, where F is a direct product of finitely many hyperbolic surface groups Σ_i, each of which is normal in G. The following conditions are equivalent:*

- *(i) G can be embedded in a direct product of surface groups [resp. of non-abelian limit groups and Γ_{-1}];*
- *(ii) Q can be embedded in such a product and the image of each of the maps $G \to \mathrm{Aut}(\Sigma_i)$ is torsion-free and has finite image in $\mathrm{Out}(\Sigma_i)$.*

Proof. If (ii) holds then by Lemma 8.4.7 there are surface groups $\widehat{\Sigma}_i$ with $\Sigma_i \trianglelefteq \widehat{\Sigma}_i$ of finite index such that the map $G \to \mathrm{Aut}(\Sigma_i)$ given by conjugation factors through $G \to \mathrm{Inn}(\widehat{\Sigma}_i) \cong \widehat{\Sigma}_i$. We combine these maps with the composition of $G \to Q$ and the embedding of Q to obtain a map Φ from G to a product of surface groups. The kernel of the map $G \to Q$ is the product of the Σ_i, and each Σ_i embeds into the coordinate for $\widehat{\Sigma}_i$, so Φ is injective and (i) is proved.

We shall prove the converse in the surface group case; the other case is entirely similar. Thus we assume that G can be embedded in a direct product $\widehat{\Sigma}_1 \times \cdots \times \widehat{\Sigma}_m$ of surface groups. After projecting away from factors $\widehat{\Sigma}_i$ that have trivial intersection with G and replacing the $\widehat{\Sigma}_i$ with the coordinate projections of G, we may assume that $G \leq \Lambda_1 \times \cdots \times \Lambda_m$ is a full subdirect product, where each Λ_i is either a surface group, a non-abelian free group, or $\mathbb{Z}$. Note that $G \cap \Lambda_i$ is normal in Λ_i, since it is normal in G and G projects onto Λ_i.

By assumption $F = \Sigma_1 \times \cdots \times \Sigma_k$ for some k. We want to show that after reordering factors Σ_i is a finite index normal subgroup of Λ_i. Denote by $p_i : \Lambda_1 \times \cdots \times \Lambda_m \to \Lambda_i$ the projection onto the ith factor. Since F is normal in the subdirect product $G \leq \Lambda_1 \times \cdots \times \Lambda_m$ the projections $p_i(F) \trianglelefteq \Lambda_i$ are finitely-generated normal subgroups for $1 \leq i \leq m$. Since the Λ_i are surface groups or free groups, it follows, each $p_i(F)$ is either trivial or of finite index. (For the case of limit groups, see [28, Theorem 3.1].)

Since F has no centre, it intersects abelian factors trivially. Suppose Λ_i is non-abelian. We claim that if $p_i(F)$ is non-trivial, then $F \cap \Lambda_i$ is non-trivial. If this were not the case, then the normal subgroups F and $G \cap \Lambda_i$ would intersect trivially in G, and hence would commute. But this is impossible, because the centraliser in Λ_i of the finite-index subgroup $p_i(F)$ is trivial. Finally, since F does not contain any free abelian subgroups of rank greater than k, we know that F intersects at most k factors Λ_i.

After reordering factors we may thus assume that Λ_i is the only factor which intersects Σ_i nontrivially. It follows that the projection of F onto $\Lambda_1 \times \cdots \times \Lambda_k$ is injective and maps Σ_i to a finitely generated normal subgroup of Λ_i. In particular, Λ_i must be a surface group, and the action of G by conjugation on Σ_i factors through $\mathrm{Inn}(\Lambda_i) \to \mathrm{Aut}(\Sigma_i)$. □

Theorem 8.4.10. *Let the Kodaira surfaces $S_{\gamma_i} \hookrightarrow X_i \to S_{g_i}$ with zero signature be as in the statement of Theorem 8.1.1 and assume that each of the maps $\alpha_i : S_{g_i} \to E$ is surjective on π_1. Then the following conditions are equivalent:*

1. *the Kähler group $\pi_1\overline{H}$ can be embedded in a direct product of surface groups;*

2. *each $\pi_1 X_i$ can be embedded in a direct product of surface groups;*

3. *for each X_i, the image of the homomorphism $\pi_1 X_i \to \mathrm{Aut}(\pi_1 S_{\gamma_i})$ defined by conjugation is torsion-free.*

Proof. Proposition 8.4.9 establishes the equivalence of (2) and (3), and (1) is a trivial consequence of (2), so we concentrate on proving that (1) implies (2). Assume that $\pi_1\overline{H}$ is a subgroup of a direct product of surface groups.

The fibre of $X = X_1 \times \cdots \times X_r \to S_{g_1} \times \cdots \times S_{g_r}$ is $F = S_{\gamma_1} \times \cdots \times S_{\gamma_r}$, the restriction of the fibration gives $F \hookrightarrow \overline{H} \to H$. Each $\pi_1 S_{\gamma_i}$ is normal in both $\pi_1 X$ and $\pi_1\overline{H}$. By Proposition 8.4.9 (and our assumption on $\pi_1\overline{H}$), the image of each of the maps $\phi_i : \pi_1\overline{H} \to \mathrm{Aut}(\pi_1 S_{\gamma_i})$ given by conjugation is torsion-free, and the image in $\mathrm{Out}(\pi_1 S_{\gamma_i})$ is finite. The map ϕ_i factors through $\rho_i : \pi_1 X_i \to \mathrm{Aut}(S_{\gamma_i})$. Because $\pi_1 H \leq \pi_1 S_{g_1} \times \cdots \times \pi_1 S_{g_r}$ is subdirect, the image of ϕ_i coincides with the image of ρ_i. Therefore, the conditions of Proposition 8.4.9 hold for each of the fibrations $S_{\gamma_i} \hookrightarrow X_i \to S_{g_i}$. □

Chapter 9

Maps onto complex tori

In this chapter we are concerned with explaining a strategy for proving Conjecture 6.1.2. If successful, this would provide us with a construction method that may lead to many new examples of Kähler groups with interesting properties.

We consider a surjective holomorphic map $h : X \to Y$ with isolated singularities and connected smooth generic fibre H, where X is a connected compact complex manifold and Y is a complex torus. We want to show that h induces a short exact sequence $1 \to \pi_1 H \to \pi_1 X \overset{h_*}{\to} \pi_1 Y \to 1$.

For this the map h is lifted to a map $\widehat{h} : \widehat{X} \to \widetilde{Y} = \mathbb{C}^k$ to the universal cover $\widetilde{Y}$ of Y. We change coordinates on $\mathbb{C}^k$ so that any line in the coordinate directions intersects the critical locus D_h of h in a discrete set (see Section 9.3, in particular Theorem 9.3.1). The idea is that one should be able to obtain $\widehat{X}$ from H by attaching cells of dimension at least $\dim H$ due to the local topology of isolated singularities (see Section 9.2 and Section 9.4).

For the time being we are not able to prove Conjecture 6.1.2 in full generality, because we loose properness of the map $\widehat{h}$ in the induction process and properness is currently required to make our proof work (see Lemma 9.4.3 and discussion in Section 9.4). However, we saw in Chapter 6 that a version of the conjecture is true in a more specific situation in which we can guarantee properness due to additional symmetries of Y.

Conjecture 6.1.2 would generalise Theorem 6.1.3. The conjecture is inspired by Dimca, Papadima and Suciu's Theorem 6.1.1. In the context of our conjecture also see Shimada's result [115, Theorem 1.1] which is a related result about algebraic varieties. A successful proof of our conjecture would provide great potential for applications: for every compact Kähler manifold the Albanese map provides us with a natural holomorphic map to a complex torus and therefore there are plenty of potential sources for examples. It would also lead to an alternative proof of Theorem 6.3.3;

we would no longer need to consider a filtration of Y and the more general fibrelong isolated singularities.

9.1 Deducing the conjecture

We will first show how Conjecture 6.1.2 follows from the following Proposition:

Proposition 9.1.1. *Assume that the conclusions of Lemma 9.4.3 hold. Then we can find an increasing sequence $H \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots \subset \widehat{X}$ of open subsets such that $\widehat{X} = \cup_{n=1}^{\infty} X_n$, and $\pi_i(X_n, H) = 0$ for all $i \leq \dim H$. In particular $\pi_i(\widehat{X}, H) = 0$.*

Before explaining a strategy of proof for Proposition 9.1.1 which has the generality that would be required to obtain Conjecture 6.1.2, we will show how this conjecture can be derived from it.

Proof of Conjecture 6.1.2. The proof follows that of [62, Theorem C]. Proposition 9.1.1 proves the first part of Conjecture 6.1.2. It is only left to prove that $h_\# : \pi_1(X) \to \pi_1(Y)$ is surjective with kernel H.

Let C_h be the critical locus of h, $D_h = h(C_h)$ the discriminant locus, and $X^* = X \setminus h^{-1}(D_h)$, $Y^* = Y \setminus D_h$. Surjectivity and holomorphicity of h imply that $C_h \subsetneq X$ and $D_h \subsetneq Y$ are proper analytic subvarieties. Y^* is connected, since every proper subvariety of an analytic variety is of (complex) codimension ≥ 1. Furthermore $h^* : X^* \to Y^*$ is surjective without critical points. Hence, by the Ehresmann Fibration Theorem (see Appendix A) h^* defines a locally trivial fibration $H \hookrightarrow X^* \to Y^*$ and consequently a long exact sequence in homotopy

$$\cdots \to \pi_1(H) \to \pi_1(X^*) \to \pi_1(Y^*) \to \pi_0(H) \to \cdots.$$

Connectedness of H is equivalent to $\pi_0(H) = \{0\}$ and thus $h^*_\# : \pi_1(X^*) \to \pi_1(Y^*)$ is surjective. Since D_h has real codimension 2 in Y, the inclusion $\iota : Y^* \to Y$ induces a surjection $\iota_\# : \pi_1(Y^*) \to \pi_1(Y)$. By the same argument the inclusion $j : X^* \hookrightarrow X$ induces a surjection on fundamental groups. The diagram

$$\begin{tikzcd} X^* \arrow[r, "h^*"] \arrow[d, hook, "j"] & Y^* \arrow[d, hook, "\iota"] \\ X \arrow[r, "h"] & Y \end{tikzcd}$$

commutes and induces a diagram of fundamental groups. Thus, $h_\# : X \to Y$ is surjective and induces an isomorphism $\pi_1(X)/\ker(h_\#) \cong \pi_1(Y)$.

Next, we prove that the induced map $\widehat{p} : \widehat{X} \to X$ is the Galois covering corresponding to the subgroup $\widehat{p}_{\#}(\pi_1(\widehat{X})) = \ker(h_{\#}) \leq \pi_1(X)$. It is straightforward to show that $\widehat{p}$ is indeed a covering map.

We only need to check that $\widehat{p}_{\#}(\pi_1(\widehat{X})) = \ker(h_{\#})$. Commutativity of diagram 9.1 and $\pi_1(\widetilde{Y}) = \{0\}$ imply that $\widehat{p}_{\#}(\pi_1(\widehat{X})) \leq \ker(h_{\#})$.

$$\begin{tikzcd} H \arrow[r, hook, "\widehat{j}"] \arrow[dr, hook, "j"'] & \widehat{X} \arrow[r, "\widehat{h}"] \arrow[d, "\widehat{p}"] & \widetilde{Y} \arrow[d, "p"] \\ & X \arrow[r, "h"] & Y \end{tikzcd} \tag{9.1}$$

Let now $[\gamma] \in \ker(h_{\#})$ be represented by $\gamma : [0,1] \to X$. Then, as $\widehat{p}$ is a covering, γ lifts to a path $\widehat{\gamma} : [0,1] \to \widehat{X}$ with $\widehat{p}(\widehat{\gamma}(0)) = \widehat{p}(\widehat{\gamma}(1)) = \gamma(0)$. For the same reason $h \circ \gamma$ lifts to a loop $\widetilde{\gamma} : [0,1] \to \widetilde{Y}$ with respect to p.

Observe that for all $t \in [0,1]$

$$p((\widehat{h} \circ \widehat{\gamma})(t)) = (p \circ \widehat{h})(\widehat{\gamma}(t)) = (h \circ \widehat{p})(\widehat{\gamma}(t)) = h((\widehat{p} \circ \widehat{\gamma})(t)) = h(\gamma(t)) = p(\widetilde{\gamma}(t)).$$

Thus, the two paths $\widetilde{\gamma}$ and $\widehat{h} \circ \widehat{\gamma}$ are both lifts of γ under p and coincide if we choose $\widetilde{\gamma}$ to be the unique lift of γ with $\widetilde{\gamma}(0) = (\widehat{h} \circ \widehat{\gamma})(0)$. In particular, $\widehat{h} \circ \widehat{\gamma}(0) = \widetilde{\gamma}(0) = \widetilde{\gamma}(1) = \widehat{h} \circ \widehat{\gamma}(1)$ and consequently $\widehat{h} \circ \widehat{\gamma}$ is a loop.

Since the fibres of $\widehat{h}$ are connected, $\widehat{\gamma}(0)$ and $\widehat{\gamma}(1)$ lie in the same path-component of $\widehat{h}^{-1}(\widetilde{\gamma}(0))$. Thus, there is a loop $\widehat{\widehat{\gamma}} : [0,1] \to \widehat{X}$ with $\widehat{p} \circ \widehat{\widehat{\gamma}} \simeq \gamma$. This implies that $\gamma \in \operatorname{im}(\widehat{p}_{\#})$ and hence

$$\ker(h_{\#}) = \operatorname{im}(\widehat{p}_{\#}). \tag{9.2}$$

Assume now that $\dim H \geq 2$. Then Proposition 9.1.1 implies that $\pi_1(\widehat{X}, H) = \pi_2(\widehat{X}, H) = 0$. Thus, the long exact sequence of the pair $(\widehat{X}, H)$ in homotopy implies that $\pi_1 H \cong \pi_1 \widehat{X}$ where the isomorphism is induced by the inclusion map. Hence, diagram (9.1) induces the following commutative diagram on the level of fundamental groups:

$$\begin{tikzcd} \pi_1 H \arrow[r, "\widehat{j}_{\#}", "\cong"'] \arrow[dr, "j_{\#}"'] & \pi_1 \widehat{X} \arrow[r, "\widehat{h}_{\#}"] \arrow[d, hook, "\widehat{p}_{\#}"] & \pi_1 \widetilde{Y} \arrow[d, hook, "p_{\#}"] \\ & \pi_1 X \arrow[r, "h_{\#}"] & \pi_1 Y. \end{tikzcd}$$

This and equation (9.2) complete the proof of the Theorem, since

$$\ker(h_{\#}) = \operatorname{im}(\widehat{p}_{\#}) \cong \pi_1 H.$$

□

9.2 Isolated singularities

The proposed strategy of proof for Proposition 9.1.1 requires some knowledge of the theory of isolated singularities of fibrations. We want to give a brief introduction in this section. Our exposition is based on the contents and notation of [101].

Let M be a complex manifolds. Recall that a *(complex) analytic variety* is a subset X of M which can locally be described as zero set of finitely many holomorphic functions. We call an analytic variety X irreducible if whenever $X = X_1 \cup X_2$ for analytic subvarieties X_1 and X_2 then either $X_1 = X$ or $X_2 = X$. A point $x \in X$ is called *smooth* if X is a complex manifold in a neighbourhood of x, it is called *singular* if it is not smooth. Analytic varieties behave in many ways very similar to complex projective varieties; they admit a decomposition into *irreducible components*; the set of smooth points of an irreducible analytic variety forms an open dense subset and its complement is contained in a proper subvariety; and there is a well-defined notion of dimension: The *dimension* n of an analytic variety X at x is the maximal dimension of X as complex manifold at a non-singular point near x. We say that X has *pure dimension* n if X is n-dimensional at each of its points; irreducible analytic varieties have pure dimension.

The vanishing ideal I of a point $x \in X$ is the ideal of germs of holomorphic functions defined in a neighbourhood of x in M which vanish on X. Let $X^n \subset M^N$ be an analytic variety of pure dimension n and assume that M is N-dimensional. We call X *complete intersection at* x if the vanishing ideal I of X at x has a generating set $\{f_1, \dots, f_{N-n}\}$ with precisely $N - n$ elements. By considering the dimension of a nearby smooth point we see that every generating set of I has at least $N - n$ generators. Thus, the complete intersection condition means that we can reduce some generating set of I to $N - n$ elements. We call a point $x \in X$ an *isolated singularity* if there is an open neighbourhood U of x in X such that X is singular in x and smooth in every point of $U \setminus \{x\}$. An *isolated complete intersection singularity (icis)* x of X is a point which is both, a complete intersection point and an isolated singularity.

A setting where icis' come up naturally are surjective holomorphic maps between complex manifolds with well-behaved singularities. For $n, k \geq 0$, let X^{n+k} be an analytic variety with set of singular values X_{sing} (and set of regular values $X_{reg} = X \setminus X_{sing}$), let $U \subset \mathbb{C}^k$ be an open subset and let $f : X \to U$ be a holomorphic map. It is important to note that here and in all subsequent results of this section *we explicitly allow the case* $k = 0$; we will get back to this later. Let $C_f \subset X$ be the set of critical points of f and let $D_f = f(C_f)$ be its set of critical values. We call C_f

the *critical locus* of f and D_f the *discriminant locus* of f. We say that f has *isolated singularities* at $y \in U$ if the intersection $f^{-1}(y) \cap (C_f \cup X_{sing})$ with the fibre $f^{-1}(y)$ over y is discrete. We say that f has isolated singularities if f has isolated singularities at every $y \in U$. Similarly, we say that a holomorphic map $f : X^{n+k} \to N$ to a complex manifold N has *isolated singularities* at $y \in N$ if it has isolated singularities at y with respect to local coordinates on N, and f has isolated singularities if it has isolated singularities at every $y \in N$. An isolated singularity x of f is *icis* if x is an icis of the fibre $f^{-1}(f(x))$.

Assume that X is a smooth complex manifold and consider a surjective holomorphic map $f = (f_1, \dots, f_k) : X \to V \subset \mathbb{C}^k$ with V open. Assume that the set of singular points intersects each fibre $f^{-1}(y)$, $y \in V$, in a discrete set and let $x \in C_f \cap f^{-1}(y)$. Then the point x is an icis of the analytic variety $f^{-1}(y)$ with vanishing ideal defined by $f_1, \dots, f_k$.

Let $x \in X$ be an isolated singularity (not necessarily icis) of an analytic variety $X \subset \mathbb{C}^n$ and let $r : X \to [0, \infty)$ be the restriction of a real-analytic function $\widetilde{r}$ on U such that $r^{-1}(0) = \{x\}$. We say that r *defines the point x in X*. A standard choice for $\widetilde{r}$ is $\widetilde{r}(y) = |y - x|^2$, where $|\cdot|$ denotes the standard Hermitian inner product on $\mathbb{C}^n$.

Consider a surjective holomorphic map $f : X^{n+k} \to U \subset \mathbb{C}^k$ with an icis at $x \in X$, with U open. After a translation, we can assume that $f(x) = 0$. Furthermore, we can choose a real-analytic function $r : X \to [0, \infty)$ such that its restriction $r|_{f^{-1}(0)}$ defines x in $f^{-1}(0)$. We will use the notation

$$X_{r=\epsilon},\ X_{r\leq\epsilon},\ \cdots$$

for

$$X \cap \{x \mid r(x) = \epsilon\},\ X \cap \{x \mid r(x) \leq \epsilon\},\ \cdots$$

and $\epsilon > 0$. Then there is an open contractible neighbourhood S of 0 in $\mathbb{C}^k$ and $\epsilon > 0$ such that $f|_{X_{r=\epsilon}}$ is a submersion in all points of $f^{-1}(S)_{r=\epsilon}$. We define

$$\mathcal{X} := (f^{-1}(S))_{r<\epsilon},\ \overline{\mathcal{X}} := (f^{-1}(S))_{r\leq\epsilon}, \text{ and } \partial\overline{\mathcal{X}} := (f^{-1}(S))_{r=\epsilon}.$$

For $s \in S$ we denote by $X_s := \mathcal{X} \cap f^{-1}(s)$ and $\overline{X}_s := \overline{\mathcal{X}} \cap f^{-1}(s)$ the intersection of the fibre of f over s with $\mathcal{X}$, respectively $\overline{\mathcal{X}}$. Furthermore we introduce the notation $\mathcal{X}_A := \mathcal{X} \cap f^{-1}(A)$.

Definition 9.2.1. With the above notation call $f : \mathcal{X} \to S$ a *good representative* of f in x and $f : \overline{\mathcal{X}} \to S$ a *good proper representative* of f in x.

We summarise the most important properties of good representatives:

Theorem 9.2.2 ([101, Theorem 2.8]). *For a good proper representative $f : \overline{\mathcal{X}} \to S$ of $f : X^{n+k} \to \mathbb{C}^k$ in $x \in X$ the following hold:*

1. *f is proper and $f : \partial\overline{\mathcal{X}} \to S$ is a trivial smooth fibre bundle;*
2. *C_f is analytic in $\mathcal{X}$ and closed in $\overline{\mathcal{X}}$, and $f|_{C_f}$ is finite-to-one;*
3. *$\mathcal{X}_{sing}$ has dimension $\leq k$, $C_f \smallsetminus \mathcal{X}_{sing}$ is of pure dimension $k-1$, and D_f is analytic in S of the same dimension as C_f;*
4. *$f : (\overline{\mathcal{X}}_{S \smallsetminus D_f}, \partial\overline{\mathcal{X}}_{S \smallsetminus D_f}) \to S \smallsetminus D_f$ is a smooth fibre bundle pair with n-dimensional fibre with non-trivial boundary;*
5. *f has an icis at each of its singular points in $\overline{\mathcal{X}}_{reg} (= \overline{\mathcal{X}}$ if X is smooth).*

Definition 9.2.3. For $s \in S \smallsetminus D$, we call the smooth fibre X_s the *Milnor fibre* of the good representative $f : \mathcal{X} \to S$ and the fibration in (4) a *Milnor fibration.*

The following Lemma explains the local topology of fibrations with isolated singularities.

Lemma 9.2.4. *Assume that $X \smallsetminus \{x\}$ is non-singular. Then for any good proper representative $f : \overline{\mathcal{X}} \to S$ of f there exists an $\eta_0 > 0$ such that for every $\eta \in (0, \eta_0]$, $\overline{\mathcal{X}}_{|f| \leq \eta}$ is homeomorphic to the cone on its boundary $(\partial\overline{\mathcal{X}})_{|f|\leq\eta} \cup \overline{\mathcal{X}}_{|f|=\eta}$.*

Proof. See Lemma 2.10 in [101]. ☐

Note that the key conclusion of Lemma 9.2.4 does not lie in the fact that a neighbourhood of x is topologically a cone over its boundary – such neighbourhoods exist in every manifold; but in the fact that there is such a neighbourhood whose boundary has well-behaved intersection with the fibres of f. We will get back to this point later. The case when k is trivial in Lemma 9.2.4 can be used to describe the local topology of a fibre of a fibration with isolated singularities around a singular point.

Corollary 9.2.5. *Let M and N be complex manifolds and let $f : M \to N$ be a surjective holomorphic map with isolated singularities. Let $x \in C_f$ be a critical point of f, let $H_x = f^{-1}(f(x))$ be the fibre of f over $f(x)$ and let r define x in a neighbourhood $U \cap H_x \subset H_x$ of x with $U \subset M$ open.*

Then there is $\eta_0 > 0$ such that, for every $\eta \in (0, \eta_0]$, the set $r^{-1}(\eta) \cap H_x$ is contained in U, the trivial map $g : r^{-1}([0, \eta]) \to \{f(x)\}$ is a good proper representative in x for the trivial map $g : H_x \to \{f(x)\}$ and $r^{-1}([0, \infty]) \cap H_x$ is homeomorphic to the cone over $r^{-1}(\eta)$.

Proof. This is immediate from Lemma 9.2.4 and the fact that, as explained above, the point x is an icis of H_x – after choosing U small enough we may assume that $(U \cap H_x) \setminus \{x\}$ is non-singular. $\square$

While we will not make direct use of Corollary 9.2.5 it is useful to keep this local picture in mind. In particular, it provides us with the right picture for a good proper representative of a surjective holomorphic map $f : M \to U \subset \mathbb{C}$ with isolated singularities, defined on a complex manifold M, around one of its singularities. This is, because a good proper representative $f|_{\overline{\mathcal{X}}} : \overline{\mathcal{X}} \to U$, for an isolated singularity $x \in M$ of f, restricts to a good proper representative of the form of g in Corollary 9.2.5 on $f^{-1}(f(x))$. In particular, the intersection $f^{-1}(f(x)) \cap \overline{\mathcal{X}}_{r \leq \eta}$ is naturally homeomorphic to the cone over the Milnor fibre of $f|_{\overline{\mathcal{X}}}$ for small η. Figure 9.1 depicts a good proper representative for the holomorphic map $\mathbb{C}^2 \to \mathbb{C}$, $(w, z) \mapsto wz$ around $(0, 0)$, which illustrates this phenomenon.

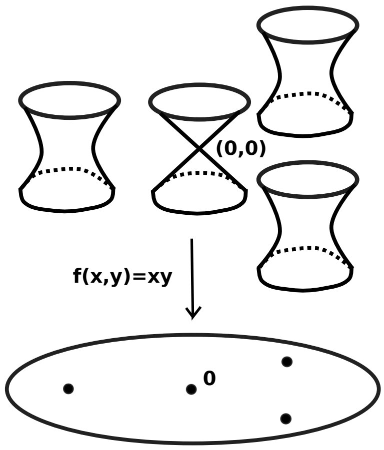

Figure 9.1: Good proper representative of the holomorphic map $f(w, z) = wz$.

We want to conclude this section by describing the topology of the Milnor fibre of a surjective holomorphic map.

Lemma 9.2.6. *Let $f : M^{n+k} \to N^k$ be a surjective holomorphic map between complex manifolds with fibres of complex dimension n and let $f|_{\overline{\mathcal{X}}} : \overline{\mathcal{X}} \to S$ be a good proper representative of an isolated singularity of f. Then every fibre of $f|_{\overline{\mathcal{X}}}$ is homotopy equivalent to a finite cell complex of real dimension $\leq n$.*

Furthermore, the Milnor fibre of $f|_{\overline{\mathcal{X}}}$ is $(n-1)$-connected and therefore homotopy equivalent to a finite bouquet of n-spheres.

Proof of Lemma 9.2.6. This Lemma follows immediately from Assertions (5.6) and (5.8) in [101, Section 5.B]. $\square$

Remark 9.2.7. The Milnor fibres of isolated singularities which are not complete intersection need not be (n-1)-connected [101, p.73]. However, this phenomenon only occurs in the case when M is not a manifold.

9.3 Lines in varieties

In this section we will prove the following result:

Theorem 9.3.1. *Let $D \subset \mathbb{C}^n$ be an analytic variety of codimension one and let $K \subset \mathbb{C}^n$ be a ball of finite radius around the origin. Then, there is a complex line $L \in \mathbb{C}P^{n-1}$ such that for all $x \in K$ the intersection $D \cap K \cap (L + x)$ is a finite set of points.*

The proof of Theorem 9.3.1 is independent of the rest of this chapter. We are extremely grateful to Simon Donaldson for providing us with the key ideas of this proof. Before we prove Theorem 9.3.1, we first need to introduce the variety of lines contained in a variety. Theorem 9.3.1 will play a crucial role in the induction step in the proposed strategy for proving Conjecture 6.1.2

For an analytic variety $D \subset \mathbb{C}^n$ of arbitrary codimension we define the *variety of lines $\mathcal{L}_D$ contained in D* by

$$\mathcal{L}_D = \left\{ (x, L) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid x \in D,\ (x + L) \cap D \text{ is not discrete } \right\} \subset D \times \mathbb{C}P^{n-1} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1},$$

where we view an element $L \in \mathbb{C}P^{n-1}$ as a line through the origin in $\mathbb{C}^n$. Note, if an affine line $x + L$ does not have discrete intersection with D then the intersection of D with $x + L$ is open in $x + L$.

For an open set $U \subset \mathbb{C}^n$ define by $\mathcal{L}_{U \cap D} = \mathcal{L}_D \cap (U \times \mathbb{C}P^{n-1})$ the set of affine lines contained in D through points in $D \cap U$.

The following Lemma shows that $\mathcal{L}_D$ is indeed a variety and therefore the name is justified:

Lemma 9.3.2. *The set $\mathcal{L}_D$ is an analytic subvariety of $D \times \mathbb{C}P^{n-1}$.*

This result seems to be classical and well-known, but since we could not find a proof of it in the literature we decided to provide one here.

Proof. Denote the complex coordinates on $\mathbb{C}^n$ by $z_1, \cdots, z_n$ and the homogeneous coordinates on $\mathbb{C}P^{n-1}$ by $[w_1 : \cdots : w_n]$. On $U_1 = \{w_1 \neq 0\} \subset \mathbb{C}P^{n-1}$ define affine coordinates by $(w_2, \cdots, w_n) \to [1 : w_2 : \cdots : w_n]$. Analogously, we can define affine coordinates on $U_i = \{w_i \neq 0\}$. Since U_i is open in $\mathbb{C}P^{n-1}$ and $\mathbb{C}P^{n-1} = \cup_{i=1}^n U_i$, it suffices to check that $\mathcal{L}_D \cap (D \times U_i)$ is an analytic variety for $1 \leq i \leq n$. We check that $\mathcal{L}_D \cap (D \times U_1)$ is analytic.

Let $z = (z_1, \cdots, z_n) \in D$ and let $V \subset \mathbb{C}^n$ be a neighbourhood of z such that there are holomorphic functions $f_1, \cdots, f_k : V \to \mathbb{C}$ with $D \cap V = \{f_1 = \cdots = f_k = 0\}$. We may assume that $z = 0$ and V is an open ball of radius $r > 0$ around the origin (with respect to the standard Hermitian metric on $\mathbb{C}^n$).

It suffices to show that $\mathcal{L}_D \cap (V \times U_1)$ is analytic in $\mathbb{C}^n \times \mathbb{C}P^{n-1}$. Consider an affine line $z + L = z + \lambda w$ with $z \in D \cap V$ and $w = (1, w_2, \cdots, w_n) \in U_i$. It is contained in $\mathcal{L}_D$ if and only if $f_1|_{(z+L)\cap V} = \cdots = f_k|_{(z+L)\cap V} = 0$.

Use the standard multiindex notation $I = (i_1, \cdots, i_n) \in \mathbb{Z}^n$, $|I| = i_1 + \cdots + i_n$, and $z^I = z_1^{i_1} \cdots z_n^{i_n}$. Multiindices are partially ordered by the relation $I \geq J$ if and only if $i_l \geq j_l$, $l = 1, \cdots, n$ ($<$, $>$, $\leq$ are defined analogously). Furthermore, for each $I \in \mathbb{Z}^n$, $0 \leq l \in \mathbb{Z}$, let $\alpha_{I,l} \in \mathbb{C}[z_1, \cdots, z_n, w_2, \cdots, w_n]$ be the unique polynomial such that

$$(z + \lambda w)^I = \sum_{l=0}^{\infty} \alpha_{I,l}(z_1, \cdots, z_n, w_2, \cdots, w_n) \cdot \lambda^l.$$

Note that $\alpha_{I,l} = 0$ for $l > |I|$.

We may assume that r was chosen small enough such that $f_1, \cdots, f_k$ can be written as power series in $z_1, \cdots, z_n$ which converges uniformly in $\{z = (z_1, \cdots, z_n) \mid |z| < r\}$. Thus, there are $c_I \in \mathbb{C}$, $I \geq 0$, such that

$$f_1(z_1, \cdots, z_n) = \sum_{I>0} c_I z^I$$

converges uniformly in $\{z \mid |z| < r\}$.

Using uniform convergence we obtain that for $|z + \lambda w| < r$

$$\begin{aligned} f_1(z + \lambda w) &= \sum_{I>0} c_I (z + \lambda w)^I \\ &= \sum_{I>0} \sum_{l=0}^{|I|} c_I \lambda^l \alpha_{I,l}(z, w) \\ &= \sum_{l=0}^{\infty} \left(\sum_{I>0} c_I \alpha_{I,l}(z, w) \right) \lambda^l. \end{aligned}$$

This shows that for $l \geq 0$

$$\beta_{1,l}(z, w) = \sum_{I>0} c_I \alpha_{I,l}(z, w)$$

converges uniformly in $\{z \mid |z + \lambda w| < r\}$. Independence of λ implies that $\beta_{1,l}(z, w)$ is holomorphic in $V \times U_1$. Furthermore for $(z, w) \in V \times U_1$ the following holds: $f_1(z+\lambda w) = 0$ for $|z + \lambda w| < r$ if and only if $\beta_{1,l}(z, w) \equiv 0$ for $l \geq 0$.

Similarly we obtain holomorphic functions $\beta_{i,l}(z,w) : V \times \mathbb{C}P^{n-1} \to \mathbb{C}$ such that for $(z,w) \in V \times U_1$ the following holds: $f_i(z+\lambda w) = 0$ for $|z+\lambda w| < r$ if and only if $\beta_{i,l}(z,w) \equiv 0$ for $l \geq 0$.

In particular, $(z,w) \in \mathcal{L}_D \cap (V \times U_1)$ if and only if $\beta_{i,l}(z,w) = 0$ for $i = 1, \cdots, k$ and $l \geq 0$. It follows that

$$\mathcal{L}_D \cap (V \times U_1) = \cap_{i=1,\cdots,k} \cap_{l=0}^{\infty} \{\beta_{i,l} = 0\} .$$

Thus $\mathcal{L}_D \cap (V \times U_1)$ is a countable intersection of analytic varieties and as such analytic by [48, p. 63, 5.7]. In particular, there are finitely many holomorphic functions $g_1, \cdots g_N : V \times U_1 \to \mathbb{C}$ such that $\mathcal{L}_D \cap (V \times U_1) = \{g_1 = \cdots = g_N = 0\}$. This completes the proof. □

In the proof of Theorem 9.3.1 we will also need the following Proposition, a proof of which can for instance be found in [48, p.41].

Proposition 9.3.3. *Let X, Y be complex manifolds and let $A \subset X$ be an analytic variety. Let $f : A \to Y$ be an analytic map and let A_{reg} be the set of smooth points of A. Let* $\dim f = \max\{\mathrm{rank}_z f \mid z \in A_{reg}\}$. *Then $f(A)$ is contained in a countable union of analytic subvarieties of dimension $\leq \dim f$ in Y.*

We are now ready to prove Theorem 9.3.1, following the proof outlined to us by Simon Donaldson.

Proof of Theorem 9.3.1. Let $\pi : \mathbb{C}^n \times \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ be the projection onto the second coordinate. It is analytic and therefore the restriction $p = \pi|_{\mathcal{L}_D}$ is analytic. Observe that a line $L \in \mathbb{C}P^{n-1}$ *does not* satisfy the condition in the Theorem if and only if $L \in p(\mathcal{L}_{D\cap K}) \subset \mathbb{C}P^{n-1}$.

It suffices to prove that $p(\mathcal{L}_{D\cap K})$ is a zero set in $\mathbb{C}P^{n-1}$. Assume that this is not the case. By Lemma 9.3.2, $\mathcal{L}_{D\cap K}$ is an analytic variety in K. Restrictions of analytic maps to analytic subvarieties are analytic. Hence, $p|_{\mathcal{L}_{D\cap K}} : (D \cap K) \times \mathbb{C}P^{n-1} \to \mathbb{C}P^{n-1}$ is analytic.

By Proposition 9.3.3, $p(\mathcal{L}_{D\cap K})$ is contained in a countable union of analytic subvarieties of $\mathbb{C}P^n$, each of which has dimension equal to the maximal rank of p on the set of smooth points of $\mathcal{L}_{D\cap K}$. Such a countable union is a zero set, unless there is a point $u = (x, L) \in \mathcal{L}_{D\cap K,reg}$ with $\mathrm{rank} d_u p = n-1$. Since we assumed that $p(\mathcal{L}_{D\cap K})$ is not a zero set, there is indeed a point $u \in \mathcal{L}_{D\cap K,reg}$ at which p has full rank. Therefore, there is an open subset $W \subset \mathcal{L}_{D\cap K,reg}$ such that $p|_W$ has full rank on W.

After choosing suitable complex coordinates $w = (w_1, w_2)$ on W, we may assume that $W = W_1 \times W_2$ with $\dim W_1 = n-1$, such that $\frac{\partial p}{\partial w_1}(u) : T_{w_1(u)} W_1 \times \{0\} \to T_{p(u)}\mathbb{C}P^{n-1}$ is invertible. Thus, the map

$$f : W_1 \times W_2 \to \mathbb{C}P^{n-1} \times W_2$$

$$(w_1, w_2) \mapsto (p(w_1, w_2), w_2)$$

has invertible Jacobian in u and, by the Inverse function Theorem, there exist open sets $u \in W' = W_1' \times W_2' \subset W$ and $U = p(W') \times W_2' \subset \mathbb{C}P^{n-1} \times W_2$ such that $f : W' \to U$ is invertible with analytic inverse $g : U \to W'$.

The restriction $\overline{g} = g|_{p(W') \times \{u_2\}}$ is analytic with $u = (u_1, u_2) \in W_1' \times W_2'$. Thus, the composition $f \circ \overline{g}$ is analytic and satisfies $(p \circ \overline{g}, u_2) = f \circ \overline{g} = \mathrm{id}_{p(W') \times \{u_2\}}$. Hence, the composition $h = \overline{g} \circ \iota$ defines an analytic right inverse of p on the open subset $p(W') \subset \mathbb{C}P^{n-1}$, with $\iota : p(W') \hookrightarrow p(W') \times \{u_2\} \subset p(W') \times W_2$ the natural inclusion. Let $\alpha : \tau = \{ (v, z) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n \mid z \in v\} \to \mathbb{C}P^{n-1}$ be the tautological line bundle. Consider its restriction $\beta = \alpha|_{\alpha^{-1}(p(W'))} : \tau|_{p(W')} = \alpha^{-1}(p(W')) \to p(W')$. We define an analytic map

$$F : \alpha^{-1}(p(W')) \to \mathbb{C}^n$$

$$(v, z) \mapsto (q \circ h)(v) + z,$$

where $q : \mathbb{C}^n \times \mathbb{C}P^{n-1} \to \mathbb{C}^n$ is the projection onto the first component.

By definition of h, the image of F is contained in D. After choosing affine coordinates $(u_2, \cdots, u_n)$ on $U_1 \subset \mathbb{C}P^{n-1}$ as in the proof of Lemma 9.3.2, we obtain a trivialisation of the tautological bundle β with respect to which the map F takes the form

$$((u_2, \cdots, u_n), \lambda) \mapsto (q \circ h)([1 : u_2 : \cdots : u_n]) + \lambda\,(1, u_2, \cdots, u_n)\,.$$

The Jacobian of F in these coordinates is

$$\mathrm{d}F((u_2, \cdots, u_n), \lambda) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ \lambda & 0 & \dots & 0 & u_2 \\ 0 & \lambda & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & u_{n-1} \\ 0 & 0 & 0 & \lambda & u_n \end{pmatrix} + \begin{pmatrix} \frac{\partial (q\circ h)_1}{\partial u_2} & \frac{\partial (q\circ h)_1}{\partial u_3} & \cdots & \frac{\partial (q\circ h)_1}{\partial u_{n-1}} & 0 \\ \frac{\partial (q\circ h)_2}{\partial u_2} & \ddots & & \vdots & 0 \\ \vdots & & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ \frac{\partial (q\circ h)_n}{\partial u_2} & \dots & \dots & \frac{\partial (q\circ h)_n}{\partial u_{n-1}} & 0 \end{pmatrix}.$$

This shows that for sufficiently large λ the rank of the Jacobian is n, irrespectively of the precise form of $q \circ h$, contradicting the assumption that the image of F is contained in an $(n-1)$-dimensional analytic subvariety $D \subset \mathbb{C}^n$. Therefore, the image of p is indeed a null set. This completes the proof. □

9.4 Strategy for proving the key result

We will now explain how we hope to combine results about isolated complete intersection singularities and ideas from the proof of Dimca, Papadima and Suciu's [62, Lemma 3.3] to derive Proposition 9.1.1. Unfortunately, the methods in their work which we want to use do not adapt directly to our situation, since they rely on the Ehresmann Fibration Theorem which only applies to proper maps. We will give a detailed version of their proof below to illustrate where the problem arises (see Lemma 9.4.3). In this section we will use the same notation as in Conjecture 6.1.2.

Lemma 9.4.1. *Let $R > 0$ and let $\widehat{h}$ be as as in Conjecture 6.1.2. Then there is a unitary linear coordinate transformation $A : \mathbb{C}^k \to \mathbb{C}^k$ such that the restriction*

$$h_k = A \circ \widehat{h} : h_k^{-1}(Z_{k,R}) \to Z_{k,R} = \Delta_{1,R} \times \cdots \times \Delta_{k,R}$$

satisfies that for all projections $\pi_{i,j} : \mathbb{C}^i \to \mathbb{C}^j$ onto the first j coordinates, $j \leq i$, the map

$$h_l = \pi_{k,l} \circ h_k : h_k^{-1}(Z_{k,R}) \to Z_{l,R}$$

has only isolated complete intersection singularities, where $Z_{l,R} := \Delta_{1,R} \times \cdots \times \Delta_{l,R}$ for $\Delta_{i,R} \subset \mathbb{C}$ the disc of radius $R > 0$ around 0 in the lth factor of $\mathbb{C}^k$.

Furthermore, we can choose the coordinates so that the discriminant locus D_l of h_l intersects $\{\eta\} \times \Delta_{l,R}$ only in isolated points for $\eta \in \Delta_{1,R} \times \cdots \times \Delta_{l-1,R}$.

Proof. Let $K^l_{\sqrt{l}R} \subset \mathbb{C}^l$ be the disc of radius $\sqrt{l}R$ around 0 and let C_l be the critical locus of h_l. We have $Z_{l,R} \subset K^l_{\sqrt{l}R}$.

Since $h_k^{-1}(Z_{k,R})$ is smooth we only need to prove that the h_l have isolated singularities (see Theorem 9.2.2(5)). The proof is by induction on l. We will prove: For a holomorphic map $\widehat{h}_l : U_l \to K^l_{\sqrt{l}R} \subset \mathbb{C}^l$ with isolated singularities there is a unitary (linear) change of complex coordinates A_l on $K^l_{\sqrt{l}R} \subset \mathbb{C}^l$, such that the restriction $h_l = A_l \circ \widehat{h}_l : h_l^{-1}(Z_{l,R}) \to Z_{l,R}$ is well-defined and satisfies the conclusions of the lemma.

Assume that the statement holds for $l-1$ and assume that we have a map $\widehat{h}_l$ with isolated singularities. The discriminant locus has codimension one in $\mathbb{C}^l$. By Theorem 9.3.1 there exists a complex line $L \subset \mathbb{C}^l$ through the origin, such that $(L+x) \cap K^l_{\sqrt{l}R} \cap D$ consists is a discrete finite set for every $x \in K^l_{\sqrt{l}R}$.

After a unitary (linear) change of coordinates $B_l : \mathbb{C}^l \to \mathbb{C}^l$, we may assume that $L = \{(0, \cdots, 0, x_l) \mid x_l \in \mathbb{C}\}$. In these new coordinates the discriminant locus D_l of $\widetilde{h}_l = B_l \circ \widehat{h}_l$ intersects $\{\eta\} \times \Delta_{l,R}$, $\eta \in K^{l-1}_{\sqrt{l-1}R}$, only in isolated points.

To apply the induction hypothesis, we need to prove that

$$\widehat{h}_{l-1} = \pi_{l,l-1} \circ \widetilde{h}_l : U_{l-1} = \widetilde{h}_l^{-1}(K^{l-1}_{\sqrt{l-1}R} \times \Delta_{l,R}) \to K^{l-1}_{\sqrt{l-1}R}$$

has only isolated singularities. Let $x \in \widehat{h}_{l-1}^{-1}(\eta) = \widetilde{h}_l^{-1}(\{\eta\} \times \Delta_{l,R})$ be a singular point in the fibre of $\widehat{h}_{l-1}$ over $\eta \in K^{l-1}_{\sqrt{l-1}R}$. In particular we have $x \in C_{l-1}$. It follows that $x \in C_l \supset C_{l-1}$ and $\widetilde{h}_l(x) = (\eta, t_0) \in (\{\eta\} \times \Delta_{l,R}) \cap D_l$. Since by assumption $(\{\eta\} \times \Delta_{l,R}) \cap D_l$ consists of isolated points, we can choose a neighbourhood S of t_0 in $\Delta_{l,R}$ so that $(\{\eta\} \times S) \cap D_l = \{(\eta, t_0)\}$. This implies that

$$x \in C_{l-1} \cap \widetilde{h}_l^{-1}(\{\eta\} \times S) \subset C_l \cap \widetilde{h}_l^{-1}(\{\eta\} \times S) \subset C_l \cap \widetilde{h}_l^{-1}(\eta, t_0).$$

Since $\widetilde{h}_l$ has only isolated singularities, the set $C_l \cap \widetilde{h}_l^{-1}(\eta, t_0)$ consists of isolated points. It follows that x is isolated in $\widetilde{h}_l^{-1}(\{\eta\} \times S)$ and thus in $\widehat{h}_{l-1}^{-1}(\eta)$. This implies that $\widehat{h}_{l-1}$ has isolated singularities.

We apply the induction hypothesis to $\widehat{h}_{l-1} : U_{l-1} \to K^{l-1}_{\sqrt{l-1}R}$, to obtain a unitary linear coordinate transformation A_{l-1} on $K^{l-1}_{\sqrt{l-1}R}$, with the property that $h_{l-1} = A_{l-1} \circ \widehat{h}_{l-1}$ satisfies the conclusions of the lemma. Let B_{l-1} be the unitary linear map defined by $B_{l-1} = A_{l-1} \times \mathrm{id}_{\mathbb{C}} : \mathbb{C}^l \to \mathbb{C}^l$, $(v_1, \dots, v_l) \mapsto (A(v_1, \dots, v_{l-1}), v_l)$.

Define $A_l = (A_{l-1} \times \mathrm{id}_{\mathbb{C}}) \circ B_l$. By definition of A_l and $\widehat{h}_{l-1}$ we have $h_{l-1} = \pi_{l,l-1} \circ h_l$. The identity $\pi_{l-1,i} \circ \pi_{l,l-1} = \pi_{l,i}$ and the equality of sets

$$h_{l-1}^{-1}(Z_{l-1,R}) = h_l^{-1}(\pi_{l,l-1}^{-1}(\Delta_{1,R} \times \cdots \times \Delta_{l-1,R})) = h_l^{-1}(\Delta_{1,R} \times \cdots \times \Delta_{l-1,R} \times \Delta_{l,R}) = h_l^{-1}(Z_{l,R})$$

imply the induction hypothesis for l.

For $l = 1$ the discrimant locus D_1 consists of isolated points, because it is of codimension one in $\mathbb{C}$. Thus, the arguments used in the induction step can be applied in this case. This completes the proof. □

Lemma 9.4.2. *With the same assumptions as in Lemma 9.4.1, there is a sequence of smooth fibres of $h_l : h_k^{-1}(Z_{k,R}) \to Z_{l,R}$ of the form*

$$H \cong H_k \subset H_{k-1} \subset \cdots \subset H_0 = \widehat{h}^{-1}(Z_{k,R}),$$

where $H_l \cong h_l^{-1}(x_1^0, \cdots, x_l^0)$ for some $x^0 = (x_1^0, \cdots, x_k^0) \in Z_{k,R}$.

Proof. $h_k^{-1}(Z_{k,R})$ is a smooth complex manifold, since it is an open subset of a smooth complex manifold. The proof is by induction on l.

By definition $h_1 : h_k^{-1}(Z_{k,R}) \to \Delta_{1,R}$ is a surjective holomorphic map and thus its generic fibre is smooth. Hence, we can choose $x_1^0 \in \Delta_{1,R}$ such that $H_1 = h_1^{-1}(x_1^0)$ is smooth.

Assume that $H_i = h_i^{-1}(x_1^0, \cdots, x_i^0)$ is smooth for $i \leq l$ and consider the surjective holomorphic map $h_{l+1} : h_k^{-1}(Z_{k,R}) \to \Delta_{1,R} \times \cdots \times \Delta_{l+1,R}$. Then

$$H_l = h_{l+1}^{-1}\left(\left\{(x_1^0, \cdots, x_l^0, x_{l+1}) | x_{l+1} \in \Delta_{l+1,R}\right\}\right)$$

is smooth and by Lemma 9.4.1 the set $\left(\left\{(x_1^0, \cdots, x_l^0)\right\} \times \Delta_{l+1,R}\right) \cap D_{l+1}$ consists of isolated points only. Choose $(x_1^0, \cdots, x_{l+1}^0) \in \left(\left\{ (x_1^0, \cdots, x_l^0)\right\} \times \Delta_{l+1,R}\right) \smallsetminus D_{l+1}$. Then $H_{l+1} = h_{l+1}^{-1}(x_1^0, \cdots, x_{l+1}^0)$ is smooth. □

The induction step in the proof of Proposition 9.1.1 currently relies on obtaining a generalised non-proper version of the following lemma (which in the version stated here is Lemma 3.3 in [62]). We do currently not know how to obtain such a result in general; we will discuss the obstacle we are facing and speculate about potential solutions after its proof.

Lemma 9.4.3. *Let $\Delta_R \subset \mathbb{C}$ be a disc of radius $R > 0$ and let $f : X \to \Delta_R$ be a smooth proper holomorphic map with only isolated singularities.*

Then X is obtained from the generic smooth fibre H of f, up to homotopy relative to H, by attaching finitely many cells of dimension n. In particular, $\pi_i(X, H) = 0$ for $i \leq r - 1$.

Proof. The proof we give is a detailed version of the proof of Lemma 3.3 in [62]; it is very similar to the induction step in the proof of 5.5 and 5.6 in [101]. We include this proof with the given level of detail, because it provides a very good intuition for the local structure of $\widehat{X}$ with respect to H and consequently for why we believe that this result should hold without assuming global properness.

Let C_f be the set of critical points and $D_f = f(C_f)$ be the discriminant locus of f. Since $f : X \to \Delta_R$ has only isolated singularities, the discriminant locus $D_f \subset \Delta_R$ is finite and since the fibres of f are compact and X is smooth, the set of singular points $C_f \subset X$ is also finite.

Let $D_f = \{s_1, \cdots, s_m\}$ and for every $\mu \in \{1, \cdots, m\}$ let $\left\{x_{\mu,1}, \cdots, x_{\mu,l_\mu}\right\} = f^{-1}(s_\mu) \cap C_f$. For each of the $x_{\mu,\alpha}$ there is a good proper representative $f : \overline{\mathcal{X}}_{\mu,\alpha} \to \overline{\Delta}_{\mu,\alpha}$ over a sufficiently small disc $\overline{\Delta}_{\mu,\alpha}$ around $x_{\mu,\alpha}$.

Let $\overline{\Delta}_\mu = \cap_{\alpha=1}^{l_\mu} \overline{\Delta}_{\mu,\alpha}$. Then the maps $f : \overline{\mathcal{X}}_{\mu,\alpha} \cap f^{-1}(\overline{\Delta_\mu}) \to \overline{\Delta}_\mu$ are good proper representatives. After possibly choosing smaller Δ_μ and $\epsilon_{\mu,\alpha} > 0$, we may assume that the conditions in Lemma A.0.8 and Lemma 9.2.4 hold for every μ and that the Δ_μ are pairwise disjoint. By slight abuse of notation we denote these new good proper representatives by $f : \overline{X}_{\mu,\alpha} \to \Delta_\mu$ and the radius of Δ_μ by η_μ.

Fix a regular value $t \in \Delta_R \setminus D_f$ and let $\gamma_1, \cdots, \gamma_m : [0,1] \to \Delta_R$ be embedded paths with $\gamma_\mu(0) = t$, $\gamma_\mu(1) = s_\mu$ such that the following conditions hold:

1. $\gamma_\mu([0,1]) \cap \gamma_\nu([0,1]) = \{t\}$ for $\mu \neq \nu$;
2. $\gamma_\mu([0,1]) \cap \Delta_\nu = \varnothing$ for $\mu \neq \nu$;
3. $\gamma_\mu([0,1]) \cap \Delta_\mu = \gamma_\mu([1-\delta_\mu, 1])$, for some $\delta_\mu > 0$.

Let $\Gamma = \cup_{\mu=1}^m \gamma_\mu([0,1])$, $A = \cup_{\mu=1}^m \overline{\Delta}_\mu \cup \Gamma$, $B = \Gamma \setminus \left(\cup_{\mu=1}^m \overline{\Delta}_\mu\right)$ and let $\overline{\mathcal{X}}_A = f^{-1}(A)$, $\overline{\mathcal{X}}_B = f^{-1}(B)$, and $\overline{\mathcal{X}}_{\{t\}} = f^{-1}(\{t\})$. Then A is a strong deformation retract of Δ_R and $\{t\}$ is a strong deformation retract of B.

Since the fibration $f : X \to \Delta_R$ is locally trivial over $\Delta_R \setminus D_f = \Delta_R \setminus \{s_1, \cdots, s_m\}$, we can apply the homotopy lifting property for fibre bundles (Proposition A.0.6) by the Ehresmann Fibration Theorem (Theorem A.0.5), and obtain that $\overline{\mathcal{X}}_A = f^{-1}(A)$ is a strong deformation retract of X. The same argument shows that $\overline{\mathcal{X}}_t$ is a strong deformation retract of $f^{-1}(t) = H$.

By applying Lemma A.0.8 to $\overline{\mathcal{X}}_A$, we obtain that $\overline{\mathcal{X}}_B \cup \left(\cup_{\mu,\alpha} \overline{\mathcal{X}}_{\mu,\alpha}\right)$ is a strong deformation retract of $\overline{\mathcal{X}}_A$.

Let $F_{\mu,\alpha} = \overline{\mathcal{X}}_B \cap \overline{\mathcal{X}}_{\mu,\alpha}$. Then, by Lemma 9.2.4, $\overline{\mathcal{X}}_{\mu,\alpha}$ is a cone over the space $Z_{\mu,\alpha} = \partial \overline{\mathcal{X}}_{\mu,\alpha} \cup \overline{\mathcal{X}}_{\mu,\alpha,|f-s_\mu|=\eta_\mu}$ which contains the cone $CF_{\mu,\alpha}$ over $F_{\mu,\alpha}$. In particular, we can apply Proposition A.0.7, yielding that the pairs $(\overline{\mathcal{X}}_{\mu,\alpha}, CF_{\mu,\alpha})$ are NDR-pairs.

The inclusion map $\sqcup_{\mu,\alpha} CF_{\mu,\alpha} \hookrightarrow \sqcup_{\mu,\alpha} \overline{\mathcal{X}}_{\mu,\alpha}$ is a homotopy equivalence, since both sets are disjoint unions of cones and hence deformation retract onto disjoint unions of points. Thus, we can apply Lemma A.0.2 to obtain that (E, E') is an NDR-pair, where $E = \overline{\mathcal{X}}_B \cup \left(\cup_{\mu,\alpha} \overline{\mathcal{X}}_B\right)$ and $E' = \overline{\mathcal{X}}_B \cup (\cup_{\mu,\alpha} CF_{\mu,\alpha})$. Lemma A.0.4 implies that the inclusion $E' \hookrightarrow E$ is a homotopy equivalence. Hence, E' is a strong deformation retract of E by Lemma A.0.3.

Since B is contractible, $f : \overline{\mathcal{X}}_B \to B$ is trivial. Thus, there is a map $r : \overline{\mathcal{X}}_B \to H$ such that $r|_H = \mathrm{id}_H$ and $(f, r) : \overline{\mathcal{X}}_B \to B \times H$ is a trivialisation. Let $E'' = H \cup (\cup_{\mu,\alpha} CF_{\mu,\alpha})$ be the space obtained from H by identifying H with $\sqcup_{\mu,\alpha} CF_{\mu,\alpha}$ via $r|_{\sqcup_{\mu,\alpha} F_{\mu,\alpha}}$. It is naturally homeomorphic to the space obtained from H by putting a cone over each $r(F_{\mu,\alpha})$. Then r induces a map $E' \to E''$ which is a homotopy equivalence relative to H.

Thus, we have constructed a finite sequence of homotopy equivalences relative to H which show that (X, H) is homotopy equivalent to (E'', H) relative to H. In particular, X is homotopy equivalent, relative to H, to a space E'' which is obtained from H by taking a finite number of cones over subspaces of the form $F_{\mu,\alpha}$ of H.

The good proper representatives $f : \overline{\mathcal{X}}_{\mu,\alpha} \to \overline{\Delta}_\mu$ define isolated complete intersection singularities with n-dimensional fibre. Hence, it follows from Lemma 9.2.6 that X is obtained from H up to homotopy equivalence relative to H by attaching finitely many cells of dimension n. $\square$

Note that we only make essential use of the properness of f in the proof of Lemma 9.4.3 at the point where we deformation retract the space X onto $\overline{\mathcal{X}}_B \cup \left({}_{\mu,\alpha}\overline{\mathcal{X}}_{\mu,\alpha}\right)$. The reason for rectracting is that this allows us to make direct use of the local topology of X around its finitely many singular points. While it is not clear to us how one could prove Lemma 9.4.3 without using strong deformation retracts it remains true that for a holomorphic map onto a disc with isolated singularities which is only locally proper (rather than globally) a neighbourhood of each singularity has a good proper representative of the form of $\mathcal{X}_{\mu,\alpha}$. In particular, it seems to us that topologically we will still be in a very similar situation and that finding a suitable version of Lemma 9.4.3 is a question of using the right tools.

One possible approach to overcome the difficulties that we currently face could be to replace the lines, used to reduce dimension in the induction argument of Lemma 9.4.3, by more general classes of complex submanifolds for which all maps are proper. This approach forms the base for the results of Chapter 6; in the situation considered there it is sufficient to use embedded subtori, meaning that we do not actually have to deviate from the general line of proof described in this chapter. Another approach is to avoid using the Ehresmann Fibration Theorem in its full strength – a suitable weaker version of it might suffice in order to globalise the local topological implications coming from isolated singularities in the proof of 9.4.3. However, we do currently not have any concrete ideas for a general proof of our conjecture which goes beyond these suggestions.

Under the assumption that X, H and f satisfy all conclusions of Lemma 9.4.3 we can now prove Proposition 9.1.1.

Proof of Proposition 9.1.1. The triple (X, Y, Z) of topological spaces defines a long exact sequence in relative homotopy

$$\cdots \to \pi_i(Y, Z) \to \pi_i(X, Z) \to \pi_i(X, Y) \to \pi_{i-1}(Y, Z) \to \cdots$$

As an immediate consequence we see that if, for some $i \in \mathbb{Z}$, $\pi_i(Y, Z) = \pi_i(X, Y) = \{0\}$ then also $\pi_i(X, Z) = 0$. Hence, if the pairs (X, Y) and (Y, Z) are k-connected for some $k \in \mathbb{Z}$, then (X, Z) is also k-connected.

Let now $R = M \in \mathbb{N}$. Assume that we are in the situation of Lemma 9.4.1 and denote by A_M the unitary linear coordinate transformation, let $h_{k,M} = A_M \circ \widehat{h}$, and let $X_M = h_{k,M}^{-1}(Z_{k,M})$. Fix a point $x^0 = (x_1^0, \cdots, x_k^0) \in Z_{k,M}$ for which the smoothness assumptions of Lemma 9.4.2 hold.

We want to prove that $H_{i-1,M}$ is obtained from $H_{i,M}$ up to homotopy equivalence relative to $H_{i-1,M}$ by attaching cells of dimension $n+k-i$ for $i = 1, \cdots, k$. This will imply that $\pi_j(H_{i-1,M}, H_{i,M}) = 0$ for $j \le n+k-i-1$ and the initial argument shows that $\pi_j(X_M, H) = 0$ for $j \le n-1$.

Since the A_{M_l} are linear and invertible, there is a sequence $\{M_l\}_{l\in\mathbb{N}}$ such that

$$A_{M_1}^{-1}(Z_{k,M_1}) \subset A_{M_2}^{-1}(Z_{k,M_2}) \subset \cdots \subset A_{M_l}^{-1}(Z_{k,M_l}) \subset A_{M_{l+1}}^{-1}(Z_{k,M_{l+1}}) \subset \cdots.$$

Thus, also

$$X_{M_l} = h_{k,M}^{-1}(Z_{k,M_l}) = \widehat{h}^{-1}(A_{M_l}^{-1}(Z_{k,M_l})) \subset \widehat{h}^{-1}(A_{M_{l+1}}^{-1}(Z_{k,M_{l+1}})) = X_{M_{l+1}}.$$

Note that $\cup_{l=1}^{\infty} X_{M_l} = \widehat{X}$ and that $\pi_j(X_{M_l}, H) = 0$ for $j \le n-1$ hold independently of the choice of smooth fibre H. We can therefore assume that $H = \widehat{h}^{-1}(p)$ for some $p \in A_{M_1}^{-1}(Z_{k,M_1})$. This yields inclusions of pairs

$$(X_{M_l}, H) \subset (X_{M_{l+1}}, H).$$

Thus, we can take the direct limit over all pairs (X_{M_l}, H). Since direct limits commute with taking homotopy groups, this implies that

$$\pi_j(\widehat{X}, H) = \varinjlim \pi_j(X_{M_l}, H) = \varinjlim \{0\} = \{0\}$$

for $j \le n-1$.

We finish the proof by showing that $H_{i-1,M}$ is obtained from $H_{i,M}$ by attaching a finite number of $(n+k-i)$-cells.

By Lemma 9.4.1 the discriminant locus $D_{i,M}$ of $h_{i,M}$ intersects $\{(x_1^0, \cdots, x_{i-1}^0)\} \times \Delta_{i,M}$ only in isolated points. This means that the restriction

$$h_{i,M} : h_{i,M}^{-1}\left(\left\{(x_1^0, \cdots, x_{i-1}^0)\right\} \times \Delta_{i,M}\right) \to \left\{(x_1^0, \cdots, x_{i-1}^0)\right\} \times \Delta_{i,M}$$

is a surjective holomorphic map onto a disc of radius M with isolated singularities. By Lemma 9.4.3, we obtain $h_{i,M}^{-1}(\{(x_1^0, \cdots, x_{i-1}^0)\} \times \Delta_{i,M})$ from the generic fibre $H_{i,M} = h_{i,M}^{-1}(x_1^0, \cdots, x_l^0)$ by attaching finitely many $(n+k-i)$-cells, up to homotopy equivalence relative to $H_{i,M}$. Since

$$H_{i-1,M} = h_{i-1,M}^{-1}\left(x_1^0, \cdots x_{i-1}^0\right) = h_{i,M}^{-1}\left(\left\{(x_1^0, \cdots, x_{i-1}^0)\right\} \times \Delta_{i,M}\right),$$

this completes the proof. □

Appendix A

Homotopy Theory

In this appendix we want to summarise the homotopy theory required in the proof of Lemma 9.4.3. Our main reference is Whitehead's book [131].

The first important ingredient in the proof of Lemma 9.4.3 is the theory of NDR-pairs which we introduce now. We call a topological space X *compactly generated* if X is Hausdorff and if every subset $A \subset X$ with the property that $A \cap C$ is closed for all $C \subset X$ compact is itself closed. It is easy to prove:

Lemma A.0.1. *Every metric space is compactly generated.*

Let X be compactly generated and $A \subset X$ be a subspace. We call (X, A) *an NDR-pair* if there are continuous maps $u : X \to [0,1]$ and $H : [0,1] \times X \to X$ with the following properties:

1. $A = u^{-1}(0)$;
2. $H(0,x) = x \ \forall \ x \in X$;
3. $H(t,x) = x \ \forall t \in [0,1], \ x \in A$;
4. $H(1,x) \in A \ \forall x \in X$ such that $u(x) < 1$.

We say that (u, H) *represents* (X, A) *as an NDR-pair*.

We want to discuss a simple example of an NDR-pair which serves as a model for a more involved example later on. Consider the cone $CX = [0,1] \times X / (\{1\} \times X)$ over a compactly generated space X and the subspace $\{0\} \times X$. The pair $(CX, \{0\} \times X)$ is an NDR-pair with maps defined by:

$$\begin{aligned} u : CX &\to [0,1] \\ [(t,x)] &\mapsto \begin{cases} 2t, & \text{if } t \in [0,1/2] \\ 1, & \text{if } t \in [1/2,1] \end{cases}, \end{aligned}$$

$$\begin{aligned} H : [0,1]\times CX &\to CX \\ (s,[(t,x)]) &\mapsto \begin{cases} [(0,x)], & \text{for } t\in[0,s/2] \\ [((2t-s)/(2-s),x)], & \text{if } t\in(s/2,1] \end{cases}, \end{aligned}$$

where $[(t,x)]$ represents the equivalence class of $(t,x)\in[0,1]\times X$ in CX.

The following Lemma is useful in constructing NDR-pairs:

Lemma A.0.2. *Let (X,A) be an NDR-pair, let B be compactly generated and let $h : A\to B$ be a continuous map. Then $(X\cup_h B, B)$ is an NDR-pair.*

We need NDR-pairs due to the following result:

Theorem A.0.3. *Let (X,A) be an NDR-pair and $i : A\hookrightarrow X$ be the inclusion. Then i is a homotopy equivalence if and only if A is a strong deformation retract of X.*

To apply this result we need the following Lemma:

Lemma A.0.4. *Let (X,A) be an NDR-pair and $h : A\to B$ a homotopy equivalence. Then $f : X\to X\cup_h B$ is a homotopy equivalence.*

The other ingredients that we use in the proof of Lemma 9.4.3 are the Ehresmann Fibration Theorem and the homotopy lifting property for fibre bundles. Both results are of large importance in Geometry and Topology. We will state the Ehresmann Fibration Theorem in the version given in [97, Section 3], since their version includes the case of manifolds with boundary. For the original version by Ehresmann see [67].

Theorem A.0.5 (Ehresmann Fibration Theorem)**.** *Let M, N be differentiable manifolds without boundary and $f : M\to N$ be a smooth proper submersion. Then $f : M\to N$ defines a locally trivial fibration.*

Let N be a differentiable manifold, M be a differentiable manifold with boundary ∂M and $f : M\to N$ be a smooth proper submersion. If $f|_{\partial M} : \partial M\to N$ is a smooth proper submersion then $f : (M,\partial M)\to N$ defines a pair of locally trivial fibrations.

We also need the following version of the homotopy lifting property for fibre bundles:

Proposition A.0.6 (Homotopy lifting property, see [122, 11.3])**.** *Let E and B be smooth manifolds, let $p : E\to B$ be a fibre bundle, let $H : [0,1]\times B\to B$ be a homotopy, and let $F : E\to E$ a lift of $H(0,\cdot)$ (i.e. $p\circ F = H|_{\{0\}\times B}\circ p$). Then H lifts to a fibre bundle homotopy $\widetilde{H} : [0,1]\times E\to E$. More precisely, $\widetilde{H}$ maps fibres homeomorphically to fibres and the diagram*

$$\begin{tikzcd} [0,1]\times E \arrow[r, "\widetilde{H}"] \arrow[d, "\mathrm{id}\times p"] & E \arrow[d, "p"] \\ [0,1]\times B \arrow[r, "H"] & B \end{tikzcd}$$

commutes.

Note that we can always lift the identity map on B to the identity map on E and therefore we can always lift homotopies of the identity on the base space and, in particular, strong deformation retractions of the base space. More general results of this form in the context of NDR-pairs can be found in [131, Section I.7].

While the next two results seem to be well-known, we could not find a source that contains a proof. We therefore decided to include them with their proof. We use the notation of Section 9.2. Let X be a smooth algebraic variety over $\mathbb{C}$, let $\overline{\Delta} \subset \mathbb{C}$ be a closed disc and let $f : X \to \overline{\Delta}$ a holomorphic map with isolated complete intersection singularities only. Let $x \in \overline{\Delta}$ be a point in the discriminant locus and let $f : \overline{\mathcal{X}} \to \overline{\Delta}_x$ be a good proper representative of f such that Δ_x is a disc of radius $\eta > 0$ in $\mathbb{C}$ with centre x. We will write f for the restriction of f to $\overline{\mathcal{X}}$. Let $y \in \partial\Delta_x$ be a point in the boundary and $F_y = f^{-1}(y) \subset \overline{\mathcal{X}}$ be the smooth fibre over y.

By Lemma 9.2.4, the set $\overline{\mathcal{X}}$ is a cone over $Z = \partial\overline{\mathcal{X}} \cup \overline{\mathcal{X}}_{|f|=\eta}$. It contains the cone CF_y over the fibre F_y.

Proposition A.0.7. *The pairs (Z, F_y) and $(\overline{\mathcal{X}}, CF_y)$ are NDR-pairs.*

Proof. We use the notation introduced in the previous two paragraphs. First we proof that (Z, F_y) is an NDR-pair. For this recall that, by Theorem 9.2.2(4),

$$f : \left(\overline{\mathcal{X}} \smallsetminus f^{-1}(x), \partial\overline{\mathcal{X}} \smallsetminus f^{-1}(x)\right) \to \overline{\Delta}_x \smallsetminus \{x\} \tag{A.1}$$

is a C^∞-fibre bundle pair.

Let $\overline{B_\delta}(y) \subset \overline{\Delta}_x$ be a closed ball of radius $\delta > 0$ around y with respect to the metric on $\overline{\Delta}_x$ induced by the standard metric on $\mathbb{C}$. By choosing δ sufficiently small we may assume that $x \notin \overline{B_\delta}(y)$.

There is a homotopy $H : [0,1] \times (\Delta_x \smallsetminus \{x\}) \to \overline{\Delta}_x \smallsetminus \{x\}$ that deformation retracts $B_{\delta/2}(y)$ onto $\{y\}$ and restricts to the identity outside $B_\delta(y)$ for all $t \in [0,1]$. We obtain it by defining $H(t,\cdot)|_{B_\delta(y)}$ to be the map that maps $B_\delta \smallsetminus \overline{B}_{\delta\frac{t}{2}}$ to $B_\delta \smallsetminus \{y\}$ by radial dilation by a factor of $\frac{2}{2-t}$ and maps $\overline{B}_{\delta\frac{t}{2}}$ to y.

There is also a continuous map $u : \Delta_x \to [0,1]$ with $0 \le u \le 1$, $u^{-1}(0) = \{y\}$ and $u(x) < 1$ if and only if $x \in B_{\delta/2}(y)$.

By Proposition A.0.6 and the fact that Equation (A.1) defines a fibre bundle pair, we obtain that H lifts to a homotopy of fibre bundles $\widetilde{H} : [0,1] \times (\overline{\mathcal{X}} \smallsetminus \{x\}) \to (\overline{\mathcal{X}} \smallsetminus \{x\})$ and in particular $\widetilde{H}([0,1] \times Z) \subset Z$. We observe that u lifts to $\widetilde{u} : \overline{\mathcal{X}} \to [0,1]$ with $u^{-1}(0) = F_y$. It follows that (Z, F_y) is an NDR-pair represented by $(\widetilde{u}, \widetilde{H})$.

Consider the pair of cones

$$(CZ = ([0,1] \times Z)/(\{1\} \times Z), CF_y = ([0,1] \times F_y)/(\{1\} \times F_y)) = (\overline{\mathcal{X}}, CF_y).$$

It is an NDR-pair represented by the pair of maps $(\overline{u}, \overline{H})$ defined by $\overline{u}(s,z) = (1 - s)\widetilde{u}(z)$ and $\overline{H}(t,(s,z)) = (t, H(s,z))$. □

Lemma A.0.8. *Let X be a smooth complex manifold, $\overline{\Delta}$ a closed disc around the origin and $f : X \to \overline{\Delta}$ a smooth proper holomorphic map with 0 as only critical value and with finite set of critical points $\{x_1, \cdots, x_l\}$. Let $f : \overline{\mathcal{X}}_\alpha \to \Delta$ be a good proper representative for x_α, $1 \leq \alpha \leq l$, and denote by $r_\alpha : X \to [0,\infty)$ the corresponding smooth map defining x_α in $f^{-1}(0)$. Assume that the following properties hold:*

1. *$\partial\overline{\mathcal{X}}_\alpha$ is defined by the same $\epsilon > 0$ for all α;*
2. *$\overline{\mathcal{X}}_\alpha \cap \overline{\mathcal{X}}_\beta = \varnothing$ for $\alpha \neq \beta$;*
3. *$dr_\alpha|_{f^{-1}(y)}(p)$ is a submersion for all $p \in \{r_\alpha = \epsilon\} \cap f^{-1}(y)$.*

Furthermore, let $s \in \partial\overline{\Delta}$ be a point in the boundary and let $f^{-1}(s)$ be the fibre of f over s.

Then $H \cup \left(\cup_{\alpha=1}^{m} \overline{\mathcal{X}}_\alpha\right)$ is a strong deformation retract of X.

Proof. Property (3) implies that the map $(f, r_\alpha) : X \to \overline{\Delta} \times [0,\infty)$ is a submersion in a neighbourhood of the set $\{r_\alpha = \epsilon\} \subset X$. In particular, there is $\delta > 0$ such that the restriction of (f, r_α) to $Y_\alpha = X \cap \{|r_\alpha - \epsilon| < \delta\}$ is a proper submersion. We may assume that we are using the same δ for all α and that δ is chosen such that the sets $\{r_\alpha \leq \epsilon + \delta\}$ are pairwise disjoint. Thus, the Ehresmann fibration Theorem implies that $(f, r_\alpha) : Y_\alpha \to \overline{\Delta} \times (\epsilon - \delta, \epsilon + \delta)$ is a locally trivial fibration. The map $f|_{X \smallsetminus (\cup_\alpha \{r_\alpha < \epsilon + \delta/2\})}$ is also a proper submersion. Thus, it defines a fibration by the Ehresmann Fibration Theorem. Since the base space is contractible for both maps, they are in fact globally trivial fibrations.

Let $H : [0,1] \times \overline{\Delta} \to \Delta$ be a strong deformation retraction of D onto $\{s\}$ along straight lines. By the homotopy lifting property, this induces a strong deformation retraction of $X \smallsetminus (\cup_\alpha \{r_\alpha < \epsilon + \delta/2\})$ onto $f^{-1}(s) \cap X \smallsetminus (\cup_\alpha \{r_\alpha < \epsilon + \delta/2\})$.

It is easy to see that for $\overline{\Delta}\times(\epsilon-\delta,\epsilon+\delta)$ there exists a strong deformation retraction onto

$$\left(\overline{\Delta}\times(\epsilon-\delta,\epsilon]\right)\cup\left(\{s\}\times[\epsilon,\epsilon+\delta)\right),$$

which on $\overline{\Delta}\times[\epsilon+\delta/2,\epsilon+\delta)$ is just the product of H and the identity on $[\epsilon+\delta/2,\epsilon+\delta)$.

By the Homotopy Lifting Property for locally trivial fibrations, this induces a strong deformation retraction H_α of Y_α to

$$\left(\overline{\mathcal{X}}_\alpha\cap\{r_\alpha>\epsilon-\delta\}\right)\cup\left(f^{-1}(s)\cap\{\epsilon\le r_\alpha<\epsilon+\delta\}\right).$$

Note that since all fibrations that appear are actually globally trivial, we can do the constructions of the strong deformation retractions explicitly and we can choose them such that H and H_α coincide on the intersections of their domains. Furthermore we can extend them by the identity to $\cup_\alpha\overline{\mathcal{X}}_\alpha$. Thus, they induce the desired strong deformation retraction of X to $H\cup\left(\cup_{\alpha=1}^{m}\overline{\mathcal{X}}_\alpha\right)$.

□

Appendix B

Lefschetz Hyperplane Theorem and finite groups

An important tool in the construction of Kähler groups is the Lefschetz Hyperplane Theorem [76].

Theorem B.0.1 (Lefschetz Hyperplane Theorem)**.** *Let $M \subset \mathbb{C}P^n$ be an m-dimensional smooth projective variety. Then for a generic hyperplane H the intersection $N = M \cap H$ is smooth and the inclusion $N \hookrightarrow M$ induces an isomorphism $\pi_i N \to \pi_i M$ for $0 \leq i \leq m-2$ and a surjection for $i = m-1$.*

The Lefschetz Hyperplane Theorem has been generalised in many different ways and a good introduction to the classical Theorem and its generalisations is [73]. One generalisation which is useful in the context of Kähler groups is:

Theorem B.0.2 ([3, Theorem 8.7])**.** *Let X be a smooth quasi-projective variety admitting a projective (possibly singular) compactification $\overline{X} \subset \mathbb{C}P^n$ such that $\overline{X} \setminus X$ has codimension at least three in $\overline{X}$. Then, there is a hyperplane $H \subset \mathbb{C}P^n$ such that $Y = X \cap H$ is a smooth projective variety and the inclusion $Y \hookrightarrow X$ induces an isomorphism on fundamental groups.*

As explained in Section 6.1.1, Conjecture 6.1.2 can also be seen as a Lefschetz type result.

A key strength of Lefschetz type theorems in constructing new examples of Kähler groups is that it allows us to construct them by first constructing a possibly singular projective variety with an interesting fundamental group and then intersecting it with a suitable hyperplane to obtain a smooth projective variety with the same fundamental group. We want to illustrate this by explaining Shafarevich's proof [114] of the following Theorem of Serre [113].

Theorem B.0.3 (Serre [113]). *Every finite group is Kähler.*

Proof of Theorem B.0.3. Since every finite group is a finite index subgroup of a symmetric group S_m for some $m \in \mathbb{N}$, it suffices to prove that S_m is Kähler for every $m \in \mathbb{N}$. The idea is to consider the permutation action of S_m on the Cartesian product $X = \mathbb{C}P^s \times \cdots \times \mathbb{C}P^s$ of m copies of $\mathbb{C}P^s$ for some $s \in \mathbb{N}$.

The action is free on an open subset $W \subset X$ with complement of codimension s in X. More precisely, W is the subset $\cap_{1\leq i<j\leq m}\{x_i \neq x_j\}$, where the $x_i = [x_{0i} : \cdots : x_{si}]$ denote homogeneous coordinates on the i-th copy of $\mathbb{C}P^s$. Furthermore there is an embedding $\iota : X \hookrightarrow \mathbb{C}P^M$ for some sufficiently large M induced by the so-called Segre embedding. For an appropriate choice of m and s, X will intersect a linear subspace of $\mathbb{C}P^M$ in a smooth compact subset $Y \subset W$ of dimension ≥ 2. An iterated application of the Lefschetz Hyperplane Theorem shows that Y is a simply connected compact smooth projective variety on which S_m acts freely.

Since S_m is finite and acts freely on Y, the projective variety Y/S_m is in fact a smooth compact projective variety with fundamental group $\pi_1(Y/S_m) = S_m$. This completes the proof. ☐

Other applications of the Lefschetz Hyperplane Theorem, or, more precisely, of a generalisation by Goresky and MacPherson [73], to Kähler groups can be found in [3, Chapter 8]. They include the first examples of non-abelian nilpotent Kähler groups by Sommese and Van de Ven [119] and Campana [39] and the first examples of Kähler groups that are not residually finite, by Toledo [128].

Appendix C

Formality of Kähler groups

The relevance of the concept of formality to Kähler groups stems from Deligne, Griffiths, Morgan and Sullivan's proof [54] that all Kähler manifolds are formal. This provides us with interesting constraints on Kähler groups. A more detailed discussion of formality and its implications can be found in [3, Chapter3], which also serves as our main reference for this appendix.

A *graded algebra* A is an algebra which is the direct sum $A = \oplus_{k=0}^{\infty} A_k$ of abelian groups such that $A_k \cdot A_l \subset A_{k+l}$ for all k and l. We call an element $x \in A$ *homogeneous of degree* $|x| = k$ if $x \in A_k$ for some k.

A *commutative graded algebra (CDGA)* is a graded algebra A which has the property of being *graded commutative*, that is,

$$x \wedge y = (-1)^{|x||y|} y \wedge x \text{ for all homogeneous elements } x, y \in A,$$

and there is a *boundary operator* $d : A \to A$ of degree 1, meaning that d satisfies

$$d^2 = 0 \text{ and } \mathrm{d}(x \wedge y) = \mathrm{d}x \wedge y + (-1)^{|x|} x \wedge \mathrm{d}y \text{ for all homogeneous elements } x, y \in A.$$

A *morphism* $\phi : A \to B$ of CDGAs is an algebra homomorphism $\phi : A \to B$ with $\phi(A_k) \subset B_k$ and $\phi \circ d = d \circ \phi$. One can define the homology $H_*(A)$ and cohomology $H^*(A)$ of a CDGA, and a morphism of CDGAs induces morphisms on homology and cohomology.

A morphism of CDGAs is called *quasi-isomorphism* if it induces an isomorphism on cohomology and two CDGAs A, B are called *weakly equivalent* if there is a sequence of quasi-isomorphisms of the form:

$$A \to C_1 \leftarrow C_2 \to \cdots \leftarrow B.$$

In the context of Kähler groups the most important examples of CDGAs are the de Rham algebra $\mathcal{E}^*(X)$ of smooth differential forms on a smooth manifold X with differential the exterior derivative and its real cohomology algebra $H^*(X) = H^*(X;\mathbb{R})$ with differential the zero differential.

A smooth manifold X is called formal if $\mathcal{E}^*(X)$ and $H^*(X)$ are weakly equivalent.

Theorem C.0.1 ([54])**.** *If X is a compact Kähler manifold then X is formal.*

We want to remark that there is an equivalent definition of formality using minimal models. It leads to a connection between the de Rham fundamental group, which is the $\mathbb{R}$-unipotent completion of the fundamental group of a smooth manifold and its real Malcev algebra. This connection yields constraints on the cup product structure on the first cohomology groups of Kähler groups. Rather than explaining the terminology used, we want to state a few results that should make the significance of formality clearer. The first one is about Massey triple products (see [3, Section 3.3]).

Let A be a CDGA and let $\alpha = [a], \beta = [b], \gamma = [c] \in H^*(A)$ be homogeneous elements with $\alpha \cup \beta = \beta \cup \gamma = 0$. Let $f, g \in A$ be such that $\mathrm{d}f = \alpha \cup \beta$ and $\mathrm{d}g = \beta \cup \gamma$. Then *the Massey triple product* of α, β and γ is defined by

$$\langle \alpha, \beta, \gamma \rangle = \left[f \cup c + (-1)^{|\alpha|-1} a \cup g \right] \in H^{|\alpha|+|\beta|+|\gamma|-1}(A) / \left(\alpha \cup H^{|\beta|+|\gamma|-1} + \gamma \cup H^{|\alpha|+|\beta|-1}(A) \right).$$

Proposition C.0.2. *Let X be a compact Kähler manifold. Then all Massey triple products on its real de Rham cohomology $H^*(X, \mathbb{R})$ are zero.*

Let $\Gamma = \pi_1 X$ be a Kähler group. Then all Massey triple products on its group cohomology $H^*(\Gamma, \mathbb{R})$ are zero.

One application of this is that the *Heisenberg group*

$$\mathcal{H}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Z}) \right\}$$

is not Kähler. Note, that historically the first proof of this is due to Serre in the 1960's (see [5]); Serre's proof does not involve the use of formality.

Another important area in which formality provides interesting constraints are *non-fibred* Kähler groups, that is, Kähler groups that admit no surjections onto surface groups (see [3, Section 3.5]). A *non-fibred Kähler manifold* is a compact Kähler manifold with non-fibred fundamental group.

Proposition C.0.3. *Let X be a non-fibred Kähler manifold. Then:*

$$\dim\left(\mathrm{Im}\cup : H^{(1,0)}(X)\wedge H^{(1,0)}(X)\to H^{(2,0)}(X)\right)\geq 2\dim H^{(1,0)}(X)-3;$$

$$\dim\left(\mathrm{Im}\cup : H^{(1,0)}(X)\otimes H^{(0,1)}(X)\to H^{(1,1)}(X)\right)\geq 2\dim H^{(1,0)}(X)-1.$$

From this we get lower bounds on the second Betti number of non-fibred Kähler groups and as consequence on the number of relations in a finite presentation:

Proposition C.0.4 ([2])**.** *Let $\Gamma = \pi_1 X$ be a non-fibred Kähler group, let*

$$\Gamma = \langle x_1, \cdots, x_n \mid r_1, \cdots, r_s\rangle$$

be a finite presentation, and define $q = b_1(\Gamma)/2$. The following inequalities hold:

1. *if $q = 0$, then $s \geq n$;*
2. *if $q = 1$, then $s \geq n - 1$;*
3. *if $q \leq 2$, then $s \geq n + 4q - 7$.*

Bibliography

[1] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Ian Agol, Daniel Groves and Jason Manning.

[2] J. Amorós. On the Malcev completion of Kähler groups. *Comment. Math. Helv.*, 71:192–212, 1996.

[3] J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo. *Fundamental groups of compact Kähler manifolds*, volume 44 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1996.

[4] D. Arapura. Higgs line bundles, Green-Lazarsfeld sets, and maps of Kähler manifolds to curves. *Bull. Amer. Math. Soc. (N.S.)*, 26(2):310–314, 1992.

[5] D. Arapura. Fundamental groups of smooth projective varieties. In *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, volume 28 of *Math. Sci. Res. Inst. Publ.*, pages 1–16. Cambridge Univ. Press, Cambridge, 1995.

[6] D. Arapura, P. Bressler, and M. Ramachandran. On the fundamental group of a compact kähler manifold. *Duke Math. Journal*, 68(3):477–488, 1992.

[7] D. Arapura and M. Nori. Solvable fundamental groups of algebraic varieties and Kähler manifolds. *Compositio Math.*, 116(2):173–188, 1999.

[8] M.F. Atiyah. The signature of fibre-bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 73–84. Univ. Tokyo Press, Tokyo, 1969.

[9] W.P. Barth, K. Hulek, C.A.M. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, second edition, 2004.

[10] I. Bauer, F. Catanese, F. Grunewald, and R. Pignatelli. Quotients of products of curves, new surfaces with $p_g = 0$ and their fundamental groups. *Amer. J. Math.*, 134(4):993–1049, 2012.

[11] B. Baumslag. Intersections of finitely generated subgroups in free products. *J. London Math. Soc.*, 41:673–679, 1966.

[12] G. Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962.

[13] G. Baumslag, M.R. Bridson, C.F. Miller III, and H. Short. Fibre products, non-positive curvature, and decision problems. *Comment. Math. Helv.*, 75(3):457–477, 2000.

[14] G. Baumslag, A. Myasnikov, and V. Remeslennikov. Algebraic geometry over groups. I. Algebraic sets and ideal theory. *J. Algebra*, 219(1):16–79, 1999.

[15] A. Beauville. Annulation du H^1 et systèmes paracanoniques sur les surfaces. *J. Reine Angew. Math.*, 388:149–157, 1988.

[16] A. Beauville. Appendix to: "Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations". *Inventiones Math.*, 104:263–289, 1991.

[17] A. Beauville. Annulation du H^1 pour les fibrés en droites plats. In *Complex algebraic varieties (Bayreuth, 1990)*, volume 1507 of *Lecture Notes in Math.*, pages 1–15. Springer, Berlin, 1992.

[18] S. R. Bell, J.-L. Brylinski, A. T. Huckleberry, R. Narasimhan, C. Okonek, G. Schumacher, A. Van de Ven, and S. Zucker. *Complex manifolds*. Springer-Verlag, Berlin, 1998. Corrected reprint of the 1990 translation [ıt Several complex variables. VI, Encyclopaedia, Math. Sci., 69, Springer, Berlin, 1990; MR1095088 (91i:32001)].

[19] M. Bestvina and N. Brady. Morse theory and finiteness properties of groups. *Invent. Math.*, 129(3):445–470, 1997.

[20] R. Bieri. *Homological dimension of discrete groups*. Queen Mary College Mathematical Notes. Queen Mary College, Department of Pure Mathematics, London, second edition, 1981.

[21] I. Biswas and M. Mj. Low dimensional projective groups. Preprint, 2012.

[22] I. Biswas and M. Mj. One-relator Kähler groups. *Geom. Topol.*, 16(4):2171–2186, 2012.

[23] I. Biswas and M. Mj. Quasiprojective three-manifold groups and complexification of three-manifolds. *Int. Math. Res. Not. IMRN*, (20):10041–10068, 2015.

[24] I. Biswas, M. Mj, and D. Pancholi. Homotopical Height. *Int. J. Math. Vol.*, 25(13), 2014.

[25] I. Biswas, M. Mj, and H. Seshardi. 3-manifold groups, Kähler groups and complex surfaces. *Commun. Cont. Math.*, 14(6, 1250038):24 pp., 2012.

[26] M.R. Bridson. Doubles, finiteness properties of groups, and quadratic isoperimetric inequalities. *J. Algebra*, 214(2):652–667, 1999.

[27] M.R. Bridson and F.J. Grunewald. Grothendieck's problems concerning profinite completions and representations of groups. *Ann. of Math. (2)*, 160(1):359–373, 2004.

[28] M.R. Bridson and J. Howie. Normalisers in limit groups. *Math. Ann.*, 337(2):385–394, 2007.

[29] M.R. Bridson, J. Howie, C.F. Miller III, and H. Short. The subgroups of direct products of surface groups. *Geometriae Dedicata*, 92:95–103, 2002.

[30] M.R. Bridson, J. Howie, C.F. Miller III, and H. Short. Subgroups of direct products of limit groups. *Ann. of Math.*, 170(2):1447–1467, 2009.

[31] M.R. Bridson, J. Howie, C.F. Miller III, and H. Short. On the finite presentation of subdirect products and the nature of residually free groups. *American Journal of Math.*, 135(4):891–933, 2013.

[32] M.R. Bridson and C. Llosa Isenrich. Kodaira fibrations, Kähler groups, and finiteness properties. *arXiv:1611.09382*, 2016.

[33] M.R. Bridson and C.F. Miller, III. Recognition of subgroups of direct products of hyperbolic groups. *Proc. Amer. Math. Soc.*, 132(1):59–65, 2004.

[34] M.R. Bridson and C.F. Miller III. Structure and finiteness properties of subdirect products of groups. *Proc. Lond. Math. Soc.*, 98(3):631–651, 2009.

[35] M.R. Bridson and A.W. Reid. Nilpotent completions of groups, Grothendieck pairs, and four problems of Baumslag. *Int. Math. Res. Not. IMRN*, (8):2111–2140, 2015.

[36] K.S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[37] A. Brudnyi. Solvable matrix representations of Kähler groups. *Differ. Geom. Appl.*, 19:167–191, 2003.

[38] M. Burger. Fundamental groups of Kähler manifolds and geometric group theory. *Astérisque*, (339):Exp. No. 1022, ix, 305–321, 2011. Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026.

[39] F. Campana. Remarques sur les groupes de Kähler nilpotents. *Ann. sci. de l' É.N.S., 4 série*, 28(3):307–316, 1995.

[40] F. Campana. Quotients résolubles des groupes de Kähler et ensembles de Green-Lazarsfeld. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(6):513–516, 1999.

[41] F. Campana. Ensembles de Green-Lazarsfeld et quotients résolubles des groupes Kähler. *J. Algebr. Geom.*, 10(4):599–622, 2001.

[42] J.A. Carlson and D. Toledo. Harmonic mappings of Kähler manifolds to locally symmetric spaces. *Inst. Hautes Études Sci. Publ. Math.*, (69):173–201, 1989.

[43] G. Castelnuovo. Sulle superficie aventi il genere aritmetico negativo. *Rend. Circ. Mat. Palermo*, 20:55–60, 1905.

[44] F. Catanese. Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations. *Invent. Math.*, 104(2):263–289, 1991.

[45] F. Catanese. Fibred surfaces, varieties isogenous to a product and related moduli spaces. *Amer. J. Math.*, 122(1):1–44, 2000.

[46] F. Catanese. Fibred Kähler and quasi-projective groups. *Adv. Geom.*, (suppl.):S13–S27, 2003. Special issue dedicated to Adriano Barlotti.

[47] F. Catanese and J. Kollár. *Trento Examples*, volume 1515. Springer Verlag, 1992.

[48] E.M. Chirka. *Complex analytic sets*, volume 46 of *Mathematics and its Applications (Soviet Series).* Kluwer Academic Publishers Group, Dordrecht, 1989. Translated from the Russian by R. A. M. Hoksbergen.

[49] K. Corlette and C. Simpson. On the classification of rank-two representations of quasiprojective fundamental groups. *Compos. Math.*, 144(5):1271–1331, 2008.

[50] M. de Franchis. Sulle superficie algebriche le quali contengono un fascio irrazionale di curve. *Rend. Circ. Mat. Palermo*, 20:49–54, 1905.

[51] P. de la Harpe. *Topics in geometric group theory.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[52] P. de la Harpe and D. Kotschick. Presentability by products for some classes of groups. *J. Topol. Anal.*, 9(1):27–49, 2017.

[53] T. Dedieu and F. Perroni. The fundamental group of a quotient of a product of curves. *J. Group Theory*, 15(3):439–453, 2012.

[54] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[55] T. Delzant. Trees, valuations and the Green-Lazarsfeld set. *Geom. Funct. Anal.*, 18(4):1236–1250, 2008.

[56] T. Delzant. L'invariant de Bieri-Neumann-Strebel des groupes fondamentaux des variétés kählériennes. *Math. Ann.*, 348(1):119–125, 2010.

[57] T. Delzant. A finiteness property on monodromies of holomorphic families. *arXiv:1402.4384v1*, 2014.

[58] T. Delzant and M. Gromov. Cuts in Kähler groups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 31–55. Birkhäuser, Basel, 2005.

[59] T. Delzant and P. Py. Cubulable Kähler groups. *arXiv:1609.08474*, 2016.

[60] A. Dimca. *Singularities and topology of hypersurfaces.* Universitext. Springer-Verlag, New York, 1992.

[61] A. Dimca, S. Papadima, and A.I. Suciu. Quasi-Kähler Bestvina–Brady groups. *J. Algebraic Geom.*, 17(1):185–197, 2008.

[62] A. Dimca, S Papadima, and A.I. Suciu. Non-finiteness properties of fundamental groups of smooth projective varieties. *J. Reine Angew. Math.*, 629:89–105, 2009.

[63] A. Dimca, S. Papadima, and A.I. Suciu. Topology and geometry of cohomology jump loci. *Duke Math. J.*, 148(3):405–457, 2009.

[64] A. Dimca, S. Papadima, and A.I. Suciu. Quasi-Kähler groups, 3-manifold groups, and formality. *Math. Z.*, 268(1-2):169–186, 2011.

[65] A. Dimca and A.I. Suciu. Which 3-manifold groups are Kähler groups? *J. Eur. Math. Soc. (JEMS)*, 11(3):521–528, 2009.

[66] J. Eells, Jr. and J.H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.

[67] C. Ehresmann. Sur les espaces fibrés différentiables. *C. R. Acad. Sci. Paris*, 224:1611–1612, 1947.

[68] B. Farb and D. Margalit. *A primer on Mapping Class groups*, volume 49 of *Princeton Math. Series*. Princeton University Press, Princeton, NJ, 2012.

[69] S. Friedl and A.I. Suciu. Kähler groups, quasi-projective groups and 3-manifold groups. *J. Lond. Math. Soc. (2)*, 89(1):151–168, 2014.

[70] S. Friedl and S. Vidussi. Rank gradients of infinite cyclic covers of Kähler manifolds. *J. Group Theory*, 19(5):941–957, 2016.

[71] R. Geoghegan. *Topological methods in group theory*, volume 243 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[72] R.E. Gompf. A new construction of symplectic manifolds. *Ann. of Math. (2)*, 142(3):527–595, 1995.

[73] M. Goresky and R. MacPherson. *Stratified Morse theory*, volume 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.

[74] M. Green and R. Lazarsfeld. Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville. *Invent. Math.*, 90(2):389–407, 1987.

[75] M. Green and R. Lazarsfeld. Higher obstructions to deforming cohomology groups of line bundles. *J. Amer. Math. Soc.*, 4(1):87–103, 1991.

[76] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[77] M. Gromov. Sur le groupe fondamental d'une variété kählérienne. *C. R. Acad. Sci. Paris Sér. I Math.*, 308(3):67–70, 1989.

[78] M. Gromov and R. Schoen. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Inst. Hautes Études Sci. Publ. Math.*, (76):165–246, 1992.

[79] F. Haglund and D.T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.

[80] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[81] W. Jaco. On certain subgroups of the fundamental group of a closed surface. *Proc. Cambridge Philos. Soc.*, 67:17–18, 1970.

[82] F.E.A. Johnson and E.G. Rees. On the fundamental group of a complex algebraic manifold. *Bull. London Math. Soc.*, 19(5):463–466, 1987.

[83] M. Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 2001.

[84] M. Kapovich. Noncoherence of arithmetic hyperbolic lattices. *Geom. Topol.*, 17(1):39–71, 2013.

[85] S.P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.

[86] O.G. Kharlampovich and A.G. Myasnikov. Irreducible affine varieties over a free group. II. Systems in triangular quasi-quadratic form and description of residually free groups. *J. Algebra*, 200(2):517–570, 1998.

[87] D.H. Kochloukova. On subdirect products of type FP_m of limit groups. *J. Group Theory*, 13(1):1–19, 2010.

[88] K. Kodaira. A certain type of irregular algebraic surfaces. *J. Analyse Math.*, 19:207–215, 1967.

[89] J. Kollár. *Shafarevich maps and automorphic forms.* M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1995.

[90] N. Korevaar and R. Schoen. Global existence theorems for harmonic maps to non locally compact spaces. *Comm. Anal. Geom.*, 5(2):333–387, 1997.

[91] D. Kotschick. All fundamental groups are almost complex. *Bull. London Math. Soc.*, 24(4):377–378, 1992.

[92] D. Kotschick. On regularly fibered complex surfaces. In *Geom. Topol. Monogr. 2*, pages 291–298. Geom. Topol. Publ., Coventry, 1999.

[93] D. Kotschick. The deficiencies of Kähler groups. *J. Topol.*, 5(3):639–650, 2012.

[94] D. Kotschick. Three-manifolds and Kähler groups. *Ann. Inst. Fourier (Grenoble)*, 62(3):1081–1090, 2012.

[95] D. Kotschick. Kählerian three-manifold groups. *Math. Res. Lett.*, 20(3):521–525, 2013.

[96] B. Kuckuck. Subdirect products of groups and the n-$(n+1)$-$(n+2)$ conjecture. *Q. J. Math.*, 65(4):1293–1318, 2014.

[97] K. Lamotke. The topology of complex projective varieties after S. Lefschetz. *Topology*, 20(1):15–51, 1981.

[98] C. Llosa Isenrich. Branched covers of elliptic curves and Kähler groups with exotic finiteness properties. *arXiv:1606.03142*, 2016.

[99] C. Llosa Isenrich. Finite presentations for Kähler groups with arbitrary finiteness properties. *J. Algebra*, 476:344–367, 2017.

[100] C. Llosa Isenrich. Kähler groups from maps onto higher dimensional tori. *arXiv:1701.01163*, 2017.

[101] E.J.N. Looijenga. *Isolated singular points on complete intersections*, volume 77 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1984.

[102] R.C. Lyndon. The equation $a^2b^2 = c^2$ in free groups. *Michigan Math. J*, 6:89–95, 1959.

[103] J. Morgan and P. Shalen. Free actions of surface groups on $\mathbb{R}$-trees. *Topology*, 30(2):143–154, 1991.

[104] T. Napier and M. Ramachandran. Hyperbolic Kähler manifolds and proper holomorphic mappings to Riemann surfaces. *Geom. Funct. Anal.*, 11(2):382–406, 2001.

[105] T. Napier and M. Ramachandran. Filtered ends, proper holomorphic mappings of Kähler manifolds to Riemann surfaces, and Kähler groups. *Geom. Funct. Anal.*, 17(5):1621–1654, 2008.

[106] P. Przytycki and D. Wise. Mixed 3-manifolds are virtually special. *arXiv:1205.6742v2*, 2012.

[107] P. Py. Coxeter groups and Kähler groups. *Math. Proc. Cambridge Philos. Soc.*, 155(3):557–566, 2013.

[108] P. Py. Some noncoherent, nonpositively curved Kähler groups. *Enseign. Math.*, 62(1-2):171–187, 2016.

[109] A. Reznikov. The structure of Kähler groups. I. Second cohomology. In *Motives, polyalgorithms and Hodge theory , Part II (Irvine, CA, 1998)*, pages 717–730. Int. Press Lect. Ser., 3, II, Int. Press Somerville, MA, 2002.

[110] J.H. Sampson. Applications of harmonic maps to Kähler geometry. In *Complex differential geometry and nonlinear differential equations (Brunswick, Maine, 1984)*, volume 49 of *Contemp. Math.*, pages 125–134. Amer. Math. Soc., Providence, RI, 1986.

[111] G.P. Scott and C.T.C. Wall. *Topological methods in group theory*, pages 137–203. in Homological Group Theory, London Math. Soc. Lecture Note Ser. **36**, Cambridge Univ. Press, Cambridge, 1979.

[112] Z. Sela. Diophantine Geometry over groups. I. Makanin-Razborov diagrams. *Publ. Math. Inst. Hautes Études Sci.*, 93:31–105, 2001.

[113] J.-P. Serre. Sur la topologie des variétés algébriques en caractéristique p. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[114] I.R. Shafarevich. *Basic algebraic geometry.* Springer-Verlag, Berlin-New York, study edition, 1977. Translated from the Russian by K. A. Hirsch, Revised printing of Grundlehren der mathematischen Wissenschaften, Vol. 213, 1974.

[115] I. Shimada. Fundamental groups of algebraic fiber spaces. *Comm. Math. Helv.*, 78(2):335–362, 2003.

[116] C.T. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, (75):5–95, 1992.

[117] Y.T. Siu. The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. *Ann. of Math. (2)*, 112(1):73–111, 1980.

[118] Y.T. Siu. Strong rigidity for Kähler manifolds and the construction of bounded holomorphic functions. In *Discrete groups in geometry and analysis (New Haven, Conn., 1984)*, volume 67 of *Progr. Math.*, pages 124–151. Birkhäuser Boston, Boston, MA, 1987.

[119] A.J. Sommese and A. Van de Ven. Homotopy groups of pullbacks of varieties. *Nagoya Math. J.*, 102:79–90, 1986.

[120] J.R. Stallings. A finitely presented group whose 3-dimensional integral homology is not finitely generated. *Amer. J. Math.*, 85:541–543, 1963.

[121] J.R. Stallings. On torsion-free groups with infinitely many ends. *Ann. of Math. (2)*, 88:312–334, 1968.

[122] N. Steenrod. *The Topology of Fibre Bundles.* Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.

[123] Karl Stein. Analytische Zerlegungen komplexer Räume. *Math. Ann.*, 132:63–93, 1956.

[124] A.I. Suciu. Characteristic varieties and Betti numbers of free abelian covers. *Int. Math. Res. Not. IMRN*, (4):1063–1124, 2014.

[125] X. Sun. Regularity of harmonic maps to trees. *Amer. J. Math.*, 125(4):737–771, 2003.

[126] C.H. Taubes. The existence of anti-self-dual conformal structures. *J. Differential Geom.*, 36(1):163–253, 1992.

[127] D. Toledo. Examples of fundamental groups of compact Kähler manifolds. *Bull. London Math. Soc.*, 22(4):339–343, 1990.

[128] D. Toledo. Projective varieties with non-residually finite fundamental group. *Inst. Hautes Études Sci. Publ. Math.*, (77):103–119, 1993.

[129] C. Voisin. On the homotopy types of compact Kähler and complex projective manifolds. *Invent. Math.*, 157(2):329–343, 2004.

[130] C. Voisin. On the homotopy types of Kähler manifolds and the birational Kodaira problem. *J. Differential Geom.*, 72(1):43–71, 2006.

[131] G.W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978.

[132] D. Wise. Research announcement: The structure of groups with a quasiconvex hierarchy. *Electron. Res. Announc. Math. Sci.*, 16:44–55, 2009.