The Yang-Mills Equations on Kähler Manifolds

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Abstract

Two special classes of solutions to the Yang-Mills equations are studied in this thesis; Hermitian-Einstein connections on holomorphic bundles over Kahler manifolds, and self-dual connections on bundles over Riemannian 4-manifolds.

We give a new proof of a theorem of Narasimhan and Seshadri, which characterizes those holomorphic bundles over an algebraic curve admitting projectively flat connections, and describe a conjecture of Hitchin and Kobayashi that would extend this to Hermitian-Einstein connections over any smooth projective variety. This conjecture is proved to be true for the simplest interesting case: bundles of rank 2 over $\mathbb{CP}^2$.

Moduli spaces of self-dual connections are studied from the point of view of differential topology. For bundles of Chern class -1 over a simply connected 4-manifold this moduli space can be compactified in a straightforward way and is, in a generic sense, an orientable manifold with quotient singularities. Applying the theory of cobordism to this moduli space we deduce that there are severe constraints on the matrices which can be realised by the intersection pairing on the second homology group of a smooth 4-manifold.

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INTRODUCTION.

The Yang-Mills equations are a system of partial differential equations for a connection on a bundle over a Riemannian or Lorentzian manifold. We shall be concerned with the Riemannian version, and we will suppose also that this base space is compact. Then, as in the Hodge Theory of harmonic forms, the solutions to the Yang-Mills equations can be viewed as the critical points of an "energy" functional (the Yang-Mills functional); and, like the Hodge Theory, the properties of these solutions are related to the geometrical and topological structure of the underlying manifold. In this thesis we study two of these relations.

The first problem that we discuss concerns the theory of connections on holomorphic bundles over a Kähler manifold. Kobayashi [3], [4] and independently Hitchin [2] defined the notion of a Hermitian-Einstein connection (this name was chosen because, for the torsion free connection on the tangent bundle, the condition reduces to the Einstein equation; from our point of view they are special examples of Yang-Mills connections), and they conjectured that a holomorphic bundle admits such a connection precisely when it is stable in the sense defined by Mumford and Takemoto.

When the base space is a Riemann surface the Hermitian-Einstein condition takes a very simple form (the connections are then projectively flat) and the truth of the conjecture follows from a comparatively old theorem of Narasimhan and Seshadri. The structure of connections on bundles over Riemann surfaces, and this relation with stability in algebraic geometry, were extensively studied by Atiyah and Bott in [1] and it is explained in that paper that the theorem of Narasimhan and Seshadri can best be understood as an infinite dimensional example of a general relation between complex, Riemannian and symplectic geometries under a group action. We will not use this infinite dimensional geometry very explicitly but we recall briefly the following facts ([5] page 158).

Suppose that G is a compact Lie group and that the complexification $G^{\mathbb{C}}$ acts on a smooth projective variety $A \subset \mathbb{CP}_n$ via some representation of $G^{\mathbb{C}}$ on $\mathbb{C}^{n+1}$. Then in algebraic geometry there is a criterion for selecting "stable" orbits of $G^{\mathbb{C}}$ in A, in such a way that the set of stable orbits is parametrised by a Hausdorff <u>moduli space</u>.

On the other hand we may give A a Kähler metric (or equivalently symplectic structure) invariant under the action of the compact group G. Then there is a way to define a positive G-invariant function f on A, the norm of the <u>moment mapping</u>:

$$A \xrightarrow{\mu} L(G)^* = \text{dual of the Lie algebra of } G.$$

With the property that the gradient flow of f on A lies within the $G^{\mathbb{C}}$ orbits.

Then by following the gradient flow one finds, for each $G^{\mathbb{C}}$ orbit o, a critical point v of f within the closure of o, and it turns out that the stability criterion is precisely that required to ensure that, if o is a stable orbit, v lies within o and is moreoever an absolute minimum of f on A.

Stability of algebraic vector bundles over a polarised variety was defined in algebraic geometry, again in order to get a good moduli space. As we shall see in Chapter 2 we may describe the theory of connections on holomorphic bundles over a Kähler manifold as a formal analogue of the situation described above, the analogue of the function f is essentially the Yang-Mills functional (in fact one has a choice of functionals to use, which we exploit in Chapter 1) and finding the corresponding critical points (Hermitian-Einstein connections) becomes a problem in the Calculus of Variations.

Whereas the Hodge Theory leads to linear partial differential equations, and the results apply equally well in all dimensions; the Yang-Mills equations are sensitive to the dimension of the base space and the "critical dimension", above which the direct method of the calculus of variations fails to apply, is (real) dimension 4. This is explained in the work of K.K. Uhlenbeck which we shall appeal to frequently throughout this thesis. There is a general survey of this work in [6].

First, in Chapter 1, we give a new proof of the theorem of Narasimhan and Seshadri for bundles over algebraic curves. In Chapter 2 we define Hermitian-Einstein connections, set down some of their basic properties and explain briefly why one should expect to modify the proof given in Chapter 1.

For Kähler surfaces the Hermitian-Einstein condition coincides with the notion of a (projective) anti self-dual connection, which is defined in purely Riemannian geometry. These connections share many of the properties of stable holomorphic bundles, in particular they are parametrised by a moduli space. In Chapter 3 we change course and carry out a detailed analysis of these moduli spaces under purely topological hypotheses on the four manifold. As we shall explain in the introduction to Chapter 3 the motive for this study is a theorem that we shall prove on the matrices realised by the intersection pairing:

$$H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}$$

on a smooth 4-manifold. This goes some way towards settling various long-standing problems in differential topology.

Of course in this more general case there is no complex structure present, but part of the interest of this subject is the close connection between the Riemannian and complex analytic points of view, generalising, in a way, the well known connections between the Hodge Theory on a Kähler manifold and sheaf cohomology. Thus in Chapter 3 the key point of the proof is the fact that a family of self-dual connections can only degenerate by acquiring " point singularities". The analogue in algebraic geometry is a family of sheaves whose generic member is locally free, but containing some sheaves with singularities along co-dimension 2 subvarieties.

We do not develop this correspondence explicitly here but in Chapter 4 we use the underlying ideas to establish a continuity property for families of Hermitian-Einstein connections, and deduce from this the truth of the conjecture of Hitchin and Kobayashi for bundles of rank two over $\mathbb{CP}^2$.

References

[1] Atiyah, M.F. and Bott, R. "The Yang-Mills equations over Riemann surfaces", to appear in Phil. Trans. Roy. Soc. London.

[2] Hitchin, N.J. "Non-linear problems in geometry", Proc. VI'th International Symposium, Division of Mathematics, The Taniguchi Foundation.

[3] Kobayashi, S. "First Chern Class and holomorphic tensor fields", Nagoya Math. J. 77 (1980) 5-11.

[4] Kobayashi, S. "Curvature and stability of vector bundles" Proc. Japan. Acad., 58, Ser. A (1982) 158-162.

[5] Mumford, D. and Fogarty, J. "Geometric Invariant Theory", 2nd edition (Springer-Verlag, 1982).

[6] Uhlenbeck, K.K. "Variational problems for gauge fields" Ann. Math. Studies 102 (Princeton U.P. 1982) 455-464.

CHAPTER ONE.

A NEW PROOF OF A THEOREM OF NARASIMHAN AND SESHADRI

§1. Introduction

In 1965 Narasimhan and Seshadri proved that the stable holomorphic vector bundles over a compact Riemann surface are precisely those arising from irreducible projective unitary representations of the fundamental group [5]. I shall give here a different, more direct, proof of this fact using the differential geometry of connections on holomorphic bundles. This complements, in a small way, the recent paper by Atiyah and Bott [1] in which the result of Narasimhan and Seshadri is used to calculate the cohomology of the moduli spaces of stable bundles, and which I take as a general reference for background and notation.

Let X be a compact Riemann surface with a Hermitian metric normalised to unit volume. If E is a vector bundle over X we write:

$$\mu(E) = \text{degree}(E) / \text{rank}(E)$$

(here the degree is obtained by evaluating $c_1(E)$ on the fundamental cycle). A holomorphic bundle $\mathcal{E}$ is defined to be

<u>indecomposable</u>: if it cannot be written as a proper direct sum,

<u>stable</u>: if for all proper holomorphic sub-bundes $\mathcal{F} < \mathcal{E}$, $\mu(\mathcal{F}) < \mu(\mathcal{E})$, or equivalently $\mu(\mathcal{E}/\mathcal{F}) > \mu(\mathcal{E})$.

Certainly a stable bundle is indecomposable. The theorem to be proved is:

THEOREM: <u>An indecomposable holomorphic bundle $\mathcal{E}$ over</u> X <u>is stable if and only if there is a unitary connection on $\mathcal{E}$ having constant central curvature</u> : $*F = -2\pi i.\mu(\mathcal{E})$. <u>Such a connection is unique up to isomorphism</u>.

<u>Note</u>: If $\deg(\mathcal{E}) = 0$ these connections are flat and so are given by unitary representations of the fundamental group. In the general case it is easy to prove the equivalence of this form of the result with the statement of Narasimhan and Seshadri ([1] Section 6).

§2. Definitions and notation, outline of proof.

If E is a C^∞ Hermitian vector bundle over X a unitary connection A on E gives an operator $d_A : \Omega^0(E) \to \Omega^1(E)$ which has a $(0,1)$ component $\bar\partial_A : \Omega^0(E) \to \Omega^{0,1}(E)$, and this defines a holomorphic structure $\mathcal{E}_A$ on E (see [1] Section 5 for a proof that there are sufficiently many local solutions of $\bar\partial_A s = 0$). Conversely if $\mathcal{E}$ is a holomorphic structure on E there is a unique way to define a unitary connection A such that $\mathcal{E} = \mathcal{E}_A$.

A connection on E induces a connection on all associated bundles, in particular on the bundle of endomorphism End E. The "gauge group" $\mathcal{G}$ of unitary automorphisms of E acts as a symmetry group on the affine space $\mathcal{A}$ of all unitary connections on E by

$$u(A) = A - d_A u\, u^{-1} \qquad u \in \mathcal{G},\ A \in \mathcal{A}.$$

The action extends to the complexification $\mathcal{G}^{\mathbb{C}}$ = group of general linear automorphisms of E:

$$g(A) = A - (\bar\partial_A g)g^{-1} + ((\bar\partial_A g)g^{-1})^* \qquad g \in \mathcal{G}^{\mathbb{C}},\ A \in \mathcal{A}$$

and connections define isomorphic holomorphic structures precisely when they lie in the same $\mathcal{G}^{\mathbb{C}}$ orbit. Thus the set of orbits parametrise all the holomorphic bundles of the same degree and rank as E (there are no further topological invariants of bundles over X). Given a holomorphic bundle $\mathcal{E}$, I write $o(\mathcal{E})$ for the corresponding orbit of connections on the appropriate C^∞ bundle.

Each connection A has a curvature $F(A) \in \Omega^2(\mathrm{End}\ E)$, and $F(A + a) = F(A) + d_A a + a \wedge a$.

The plan of the proof is this : we suppose inductively that the result has been proved for bundles of lower rank (the case of line bundles being an easy consequence of the Hodge Theory), then we choose a minimising sequence in $o(\mathcal{E})$ for a carefully constructed functional J of the curvature and extract a weakly convergent subsequence. Either the limiting connection is in $o(\mathcal{E})$ and we deduce the result by examining small variations within $o(\mathcal{E})$ or in another orbit $o(\mathcal{F})$ and we deduce that $\mathcal{E}$ is not stable. The main ingredient in this approach is a result of K. Uhlenbeck [6] on the weak compactness of the set of connections with L^2 bounded curvature.

In the intermediate stages of the argument we have to allow generalised connections of class L^2_1 (i.e. which differ from some fixed C^∞ connection by an element of the Hilbert space with norm $\|\alpha\|^2_{L^2_1} = \|\alpha\|^2 + \|\nabla\alpha\|^2$) with curvature in L^2 and gauge transformations in L^2_2. As explained in ([6] Section 1, [1] Section 14) the group actions and properties of curvature that we use extend without essential change (in particular $L^2_2 \hookrightarrow C^0$ so the topology of the bundle is preserved), and it is proved in ([1] Lemma 14.8) that each L^2_1 connection defines a holomorphic structure. For simplicity I shall work throughout with these more general objects with only occasional further comment.

<u>Definition of the functional J</u>

The "trace norm" is defined on $(n \times n)$ Hermitian matrices by:

$$\nu(M) = \operatorname{Tr}(M^*M)^{\frac{1}{2}} = \sum_{i=1}^{n} |\lambda_i|$$

where $\{\lambda_i\}$ are the eigenvalues of M, repeated according to multiplicity. We shall need to know that ν defines a norm and that if M is written in blocks: $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$ then $\nu(M) \geq |\operatorname{Tr} A| + |\operatorname{Tr} D|$. Both properties follow easily from the characterization:

$$\nu(M) = \max_{\{e_i\}} \sum_{i=1}^{n} |\langle Me_i, e_i\rangle|$$

where $\{e_i\}$ runs over all orthonormal frames for $\mathbb{C}^n$. (There is a complete account of such convex invariant functions in [1] Section 12).

Applying ν in each fibre we define, for any smooth self-adjoint section s in $\Omega^0(\operatorname{End} E)$:

$$N(s) = \left(\int_X \nu(s)^2\right)^{\frac{1}{2}}$$

Then N is a norm equivalent to the usual L^2 norm and so extends to the L^2 cross sections. For an L^2_1 connection A set:

$$J(A) = N\left(\frac{*F}{2\pi i} + \mu.1\right) \qquad \text{where} \quad \mu = \mu(E)\ .$$

Thus $J(A) = 0$ if and only if the connection is of the type required by the theorem. For bundles of rank 2 and degree 0, J is essentially the Yang-Mills functional $\|F\|_{L^2}$. For larger ranks the definition of J is chosen to make the inductive step (lemma 3) carry through easily, although the connections we find in the end are of course Yang-Mills connections. Although J is not smooth it does have the semi-continuity property : if $A_i \to A$ weakly in L_1^2, so $F(A_i) \to F(A)$ weakly in L^2, then $J(A) \leq \limsup J(A_i)$ (because for each $\varepsilon > o$ we can separate $\frac{*F(A)}{2\pi i}$ from the closed convex set $\{\alpha | N(\alpha + \mu 1) \leq J(A) - \varepsilon\}$ by a hyperplane).

§3. Proof of main lemma

This section contains the main part of the argument for which we need to extract the following proposition from ([6] Thm. 1.5).

PROPOSITION: Suppose that $A_i \in \mathcal{A}$ is a sequence of L_1^2 connections with $\|F(A_i)\|_{L^2}$ bounded. Then there is a subsequence $\{i'\} < \{i\}$ and L_2^2 gauge transformations $u_{i'}$ such that $u_{i'}(A_{i'})$ converges weakly in L_1^2.

From this I deduce:

LEMMA 1: Let $\mathcal{E}$ be a holomorphic bundle over X. Then either $\inf J\big|_{o(\mathcal{E})}$ is attained in $o(\mathcal{E})$ or there is a holomorphic bundle $\mathcal{F} \ncong \mathcal{E}$, of the same degree and rank as $\mathcal{E}$ and with $\inf J\big|_{o(\mathcal{F})} \leq \inf J\big|_{o(\mathcal{E})}$, $\mathrm{Hom}(\mathcal{E},\mathcal{F}) \neq 0$.

Proof

Pick a minimising sequence A_i for $J\big|_{o(\mathcal{E})}$. Since N is equivalent to the L^2 norm we have $\|F(A_i)\|_{L^2}$ bounded and can apply the proposition to deduce that, without loss of generality, $A_i \to B$ weakly in L_1^2 and $J(B) \leq \limsup J(A_i) = \inf J\big|_{o(\mathcal{E})}$.

Since B defines a holomorphic structure $\mathcal{E}_B$ the lemma will follow if we show $\mathrm{Hom}(\mathcal{E},\mathcal{E}_B) \neq 0$. (The two alternatives holding as $\mathcal{E} \cong \mathcal{E}_B$ or not). To see this define for any $A, A' \in \mathcal{A}$ a connection $d_{AA'}$ on the bundle $\mathrm{Hom}(E,E) = E^* \otimes E$ built from the connection A on the left hand factor and A' on the right, with a corresponding:

$$\bar\partial_{AA'} : \Omega^0(\mathrm{Hom}(E,E)) \to \Omega^{0,1}(\mathrm{Hom}(E,E)) .$$

Thus solutions of $\bar\partial_{AA'}\, s = 0$ correspond exactly to elements of $\mathrm{Hom}\,(\mathcal{E}_A, \mathcal{E}_{A'})$.

If in fact $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_B) = 0$ then $\bar\partial_{A_oB}$ has no kernel and since it is a first order elliptic operator:

$$\|\bar\partial_{A_oB} s\|_{L^2} \geq C\, \|s\|_{L^2_1} \qquad \text{for some } C, \text{ all } s.$$

By the Sobolev inequality $\|s\|_{L^2_1} \geq \mathrm{Const}\, \|s\|_{L^4}$ so :

$$\|\bar\partial_{A_oB} s\|_{L^2} \geq C_1\, \|s\|_{L^4} \quad \text{say.}$$

On the other hand $L^2_1 \hookrightarrow L^4$ is compact so $A_i \to B$ in L^4 norm and $\bar\partial_{A_oB} - \bar\partial_{A_oA_i}$ is the algebraic operator $s \to (B - A_i)_{0,1}.s$. Thus

$$\|(\bar\partial_{A_oB} - \bar\partial_{A_oA_i})\, s\|_{L^2} \leq C_2\, \|A_i - B\|_{L^4}\, \|s\|_{L^4} \quad \text{so for}$$

each i and all s : $\|\bar\partial_{A_oA_i} s\|_{L^2} \geq (C_1 - C_2\|A_i - B\|_{L^4})\|s\|_{L^4}$

Since $A_i \to B$ in L^4 norm this implies that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}_{A_i}) = 0$ for large enough i, contradicting $\mathcal{E}_{A_i} \cong \mathcal{E}$.

§4. Curvature and holomorphic extensions

We have to show that if $\mathcal{E}$ is stable the second alternative of lemma 1 does not occur. In general if $\alpha : \mathcal{E} \to \mathcal{F}$ is a (non-zero) holomorphic map of bundles over X there are proper extensions and a factorisation : (c.f. [5] Section 4).

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{P} & \to & \mathcal{E} & \to & \mathcal{Q} & \to & 0 \\
 & & & & \downarrow\alpha & & \downarrow\beta & & \\
0 & \leftarrow & \mathcal{N} & \leftarrow & \mathcal{F} & \leftarrow & \mathcal{M} & \leftarrow & 0
\end{array} \qquad *
$$

With rows exact, rank $\mathcal{Q}$ = rank $\mathcal{M}$, det $\beta \not\equiv 0$, deg $\mathcal{Q} \leq$ deg $\mathcal{M}$.

The next two lemmas give bounds on the curvature of bundles expressed as extensions- the first exactly as in ([1] Section 8), the second a little stronger, exploiting the special properties of J.

First some generalities : if we have any exact sequence of holomorphic bundles $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$ then any unitary connection A on $\mathcal{T}$ has the shape:

$$A = \begin{pmatrix} A_s & \beta \\ -\beta^* & A_u \end{pmatrix}$$ with A_s, A_u connections

on $\mathcal{S}$, $\mathcal{U}$ and β in $\Omega^{0,1}(\mathcal{U}^* \otimes \mathcal{S})$ because $\mathcal{S}$ is a holomorphic sub-bundle (β is a representative of the extension class in $H^1(\mathcal{U}^* \otimes \mathcal{S})$). In the corresponding curvature matrix:

$$F(A) = \begin{pmatrix} F(A_s) - \beta \wedge \beta^* & d\beta \\ -d\beta^* & F(A_u) - \beta^* \wedge \beta \end{pmatrix}$$

(Here $d : \Omega^1(\mathcal{U}^* \otimes S) \to \Omega^2(\mathcal{U}^* \otimes S)$ is built from A_u, A_s) the quadratic terms have a definite sign (this is the principle that "curvature decreases in holomorphic sub-bundles and increases in quotients"). For convenience normalise so that $*\mathrm{Tr}(\beta \wedge \beta^*) = -2\pi i|\beta|^2$.

Conversely, connections on $\mathcal{S}, \mathcal{U}$ and a representative β of the extension class define a unique connection on $\mathcal{T}$, and any non-zero multiple of β also gives a bundle isomorphic to $\mathcal{T}$ (although a different extension class).

LEMMA 2. <u>If</u> $\mathcal{F}$ <u>is a holomorphic bundle over</u> X <u>which can be expressed as an extension:</u> $0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{N} \to 0$ <u>and if</u> $\mu(\mathcal{M}) \geq \mu(\mathcal{F})$ <u>(so also</u> $\mu(\mathcal{F}) \geq \mu(\mathcal{N})$) <u>then for any unitary connection</u> A <u>on</u> $\mathcal{F}$:

$$J(A) \geq \deg\mathcal{M}\,(\mu(\mathcal{M}) - \mu(\mathcal{F})) + \deg\mathcal{N}(\mu(\mathcal{F}) - \mu(\mathcal{N})), = J_0 \text{ say,}$$

<u>with equality occurring only if the extension splits</u>.

<u>Proof</u>. Using the property of ν on block matrices mentioned in §3 and the notation of the discussion above we have:

$$\nu\left(\frac{*F(A)}{2\pi i} + \mu 1\right) \geq \left|\mathrm{Tr}\left(\frac{F(A_{\mathcal{M}}) - \beta\wedge\beta^*}{2\pi i} + \mu.1\right)\right| + \left|\mathrm{Tr}\left(\frac{F(A_{\mathcal{N}}) - \beta^*\wedge\beta}{2\pi i} + \mu.1\right)\right|$$

(Here $\mu = \mu(\mathcal{F})$)

$$\begin{aligned} \text{So}\quad J(A) &\geq \int_X \nu\left(\frac{*F}{2\pi i} + \mu 1\right) \\ &\geq \left|\int_X \left\{\mathrm{Tr}\left(\frac{*F(A_M)}{2\pi i} + \mu.1\right) - |\beta|^2\right\}\right| \\ &+ \left|\int_X \left\{\mathrm{Tr}\left(\frac{*F(A_N)}{2\pi i} + \mu.1\right) + |\beta|^2\right\}\right| \end{aligned}$$

But $\int_X \mathrm{Tr}(\frac{*F(A_M)}{2\pi i}) = \deg \mathcal{M} \leqslant \mu \int_X \mathrm{Tr} 1_{\mathcal{M}}$ by hypothesis,
so the first term on the right is $: \mathrm{rk}(\mathcal{M})(\mu(\mathcal{M}) - \mu(\mathcal{F})) + \|\beta\|^2$.
Similarly for the second term.

LEMMA 3. <u>Suppose that $\mathcal{E}$ is a stable holomorphic bundle and make the inductive hypothesis that the main theorem has been proved for bundles of lower rank. If $\mathcal{E}$ can be expressed as an extension</u> : $0 \to \mathcal{P} \to \mathcal{E} \to \mathcal{Q} \to 0$ (<u>so from the definition of stability</u> $\mu(\mathcal{P}) < \mu(\mathcal{E}) < \mu(\mathcal{Q})$) <u>then there is a connection</u> A <u>on</u> $\mathcal{E}$ <u>with</u>:

$$J(A) < \mathrm{rk}\,\mathcal{P}(\mu(\mathcal{E}) - \mu(\mathcal{P})) + \mathrm{rk}\,\mathcal{Q}(\mu(\mathcal{Q}) - \mu(\mathcal{E}))$$
$$= J_1 \quad \underline{\text{say}}.$$

<u>Proof</u>. Observe first that for a general extension:
$0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{U} \to 0$ the connections in $o(\mathcal{T})$ given by triples $(A_S, A_{\mathcal{U}}, t\beta)$ converge in C^∞ as $t \to o$ to a connection in $o(\mathcal{S} \oplus \mathcal{U})$. Now it is not hard to prove ([1] Section 7, [2]) that any holomorphic bundle $\mathcal{P}$ has a canonical "semi-stable filtration":

$$0 = \mathcal{P}_o < \mathcal{P}_1 < \ldots < \mathcal{P}_k = \mathcal{P}$$

with $\mathcal{P}_i/\mathcal{P}_{i-1}$ semi-stable and $\mu(\mathcal{P}_i/\mathcal{P}_{i-1})$ decreasing with i.

Each quotient $\mathcal{P}_i/\mathcal{P}_{i-1}$ has in turn a filtration [2] with associated quotients $\mathcal{C}_{ij}$ stable and $\mu(\mathcal{C}_{ij}) = \mu(\mathcal{P}_i/\mathcal{P}_{i-1}) < \mu(\mathcal{P}_1)$. If $\mathcal{P} < \mathcal{E}$ are the bundles in the statement of the lemma we must have $\mu(C_{ij}) < \mu(\mathcal{E})$ since $\mathcal{E}$ is stable. Also rank $\mathcal{C}_{ij}$ < rank $\mathcal{E}$ so we can suppose inductively that each $\mathcal{C}_{ij}$ has a connection with constant central curvature. Applying the observation to each of the steps in the filtrations we find connections $A^t_{\mathcal{P}} \in \mathcal{O}(\mathcal{P})$ $(t \neq o)$ converging as $t \to o$ to $A^o_{\mathcal{P}} \in \mathcal{O}(\bigoplus_{ij} C_{ij})$. And $*F(A^o_{\mathcal{P}}) = -2\pi i \Lambda_{\mathcal{P}}$ where $\Lambda_{\mathcal{P}}$ is the constant diagonal matrix with entries the $\mu(C_{ij}) < \mu(\mathcal{E})$.

Similarly there are $A^t_{\mathcal{Q}} \to A^o_{\mathcal{Q}}$, $*F(A^o_{\mathcal{Q}}) = -2\pi i \Lambda_{\mathcal{Q}}$ and the entries of $\Lambda_{\mathcal{Q}}$ are greater than $\mu(\mathcal{E})$.

For each t $A^t_{\mathcal{P}}$, $A^t_{\mathcal{Q}}$ give an operator d_t on the forms with values in the C^∞ bundle corresponding to $\mathcal{Q}^* \otimes \mathcal{P}$. For $t \neq o$ choose the harmonic representative β_t of the extension class corresponding to $\mathcal{E}$ i.e. $d_t\beta_t = 0$, scaled to $\|\beta_t\|_{L^2} = 1$.

Since $d_t \to d_o$ as $t \to o$ and the d_t are elliptic on $\Omega^{o,1}$ there is a uniform bound $\|\beta_t\|_{C^o} <$ Const (because for each t we have the usual elliptic bounds

$$\|\alpha\|_{L^2_{K+1}} \leq C_t(\|d_t\alpha\|_{L^2_K} + \|\alpha\|_{L^2_K})$$

and the C_t can be uniformly bounded since the d_t converge).

By our general discussion the triples $(A^t_{\mathcal{P}}, A^t_{\mathcal{Q}}, s\beta_t)$ $(s,t > o)$ give connections $A(s,t) \in \mathcal{O}(\mathcal{E})$ with curvature:

$$F(s,t) = \begin{pmatrix} F(A^t_{\mathcal{P}}) - s^2\beta_t\wedge\beta^*_t & o \\ o & F(A^t_{\mathcal{Q}}) - s^2\beta^*_t\wedge\beta_t \end{pmatrix}$$

It is clear that $J(A(s,t)) \to J_1$ as $s,t \to 0$. We have to check that for suitably chosen small s,t; $J(A(s,t)) < J_1$.

Since $\Lambda_{\mathcal{P}} - \mu.1_{\mathcal{P}}$ has all its eigenvalues negative, the same is true for nearby matrices, and for such matrices $\nu(\ ) = -T_r(\ )$. Using the unform bound on the β_t and $F(A^t_{\mathcal{P}}) \to \Lambda_{\mathbb{P}}$, together with the corresponding facts for the other component, we have for small s,t:

$$\nu\left(\frac{*F(s,t)}{2\pi i} + \mu.1\right) = J_1 - 2s^2|\beta_t|^2 + \varepsilon(t)$$

where $\varepsilon(t) \to o$ with t. So:

$$J(A(s,t))^2 = \int_X (J_1^2 - 2s^2|\beta_t|^2 + \varepsilon(t))^2$$

Choosing s so small that $s^4 \int_X |\beta_t|^4$ is much less than $s^2 \int_X |\beta_t|^2 = s^2$ (using the uniform bound again), and then t small enough for the terms in $\mathcal{E}$ to be negligible gives $J(A(s,t)) < J_1$.

§5. Proof of the theorem

First, the final clause of Lemma 2 shows that if $\mathcal{E}$ is an indecomposable bundle with a connection of the type required by the theorem (i.e. $J = o$) then $\mathcal{E}$ must be stable.

Conversely if $\mathcal{E}$ is stable and if the theorem has been proved for bundles of lower rank then $\inf J\big|_{o(\mathcal{E})}$ is attained in $o(\mathcal{E})$. For if not, lemma 1 constructs a bundle $\mathcal{F}$ with $\deg \mathcal{F} = \deg \mathcal{E}$, $\operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{E}$, $\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \neq o$ and $\inf J\big|_{o(\mathcal{E})} \geq \inf J\big|_{o(\mathcal{F})}$. In the corresponding diagram (*) we have

$$\mu(\mathcal{M}) \geq \mu(\mathcal{Q}) > \mu(\mathcal{E}) = \mu(\mathcal{F})$$

So we can apply lemma 2 to the bottom row of (*) to deduce

$$\inf J\big|_{o(\mathcal{F})} \geq J_o .$$

And lemma 3 to the top row to deduce:

$$\inf J\big|_{o(\mathcal{E})} < J_1 .$$

But: $\operatorname{rk} \mathcal{Q} = \operatorname{rk} \mathcal{M}$, $\operatorname{rk} \mathcal{P} = \operatorname{rk} \mathcal{N}$,

$$\deg \mathcal{Q} > \deg \mathcal{M} \qquad \deg \mathcal{P} < \deg \mathcal{N}$$

implies $J_1 \leq J_o$ and we obtain contradiction.

So suppose $\inf J\big|_{o(\mathcal{E})}$ is attained at $A \in o(\mathcal{E})$. The operator $d_A^* d_A$ acting on L_2^2 self-adjoint sections of End E has kernel the constant scalars (because any other element of the kernel would have eigenspaces that would decompose $\mathcal{E}$ holomorphically), and the projection of $*F/_{2\pi i}$ onto the scalars is $\mu(\mathcal{E})$. Thus by the Hodge Theory (in a version [4] adapted to the case where the coefficients need not be smooth) there is a self-adjoint section $h \in L_2^2$ such that:

$$d_A^* d_A h = 2\pi\mu - i{*F(A)}.$$

For small t, $1 + th = g_t \in \mathcal{G}^{\mathbb{C}}$; let $A_t = g_t(A) \in o(\mathcal{E})$ so we can compute the curvature:

$$F(A_t) = F(A) - \partial_A((\bar\partial_A g)g^{-1}) + \bar\partial_A(g^{-1}(\partial_A g))$$

$$- \bar\partial_A g\, g^{-2}\partial_A g - g^{-1}\partial_A g\bar\partial_A g g^{-1}$$

$$= F(A) - t(\partial_A\bar\partial_A - \bar\partial_A\partial_A)h + q(t,h) \quad \text{say,}$$

with $\|q(t,h)\|_{L^2} \le C(\|h\|_{L_2^2}) \cdot t^2$ for small t.

Since $d_A^* d_A = i{*}(\bar\partial_A\partial_A - \partial_A\bar\partial_A)$ we get:

$$N\left(\frac{*F(A_t)}{2\pi i} + \mu\right) = N\left(\frac{*F(A)}{2\pi i} + \mu\right) \cdot (1 - t) + O(t^2)$$

And if $J(A_t)$ is to be a minimum at $t = o$ we must have $\frac{*F(A)}{2\pi i} = -\mu$ as required.

Referring again to the paper by Uhlenbeck ([6] Corollary 1.4) we find that in some gauge the solution is smooth (i.e. there is $u \in \mathcal{G}$ with $u(A)$ smooth). Thus in each stable orbit $o(\mathcal{E})$ there is at least one connection of the required type.

Finally, to see that A is unique up to the action of $\mathcal{G}$ on $o(\mathcal{E})$, recall that any $\tilde{g} \in \mathcal{G}^{\mathbb{C}}$ can be factored $\tilde{g} = g.u$ with $g = g^*$, $u \in \mathcal{G}$ so if A, B are distinct solutions we can suppose $B = g(A)$ $g = g^*$. One computes the forumla:

$$F(A) = F(B) = \mu.1 \Longrightarrow \partial_A \bar{\partial}_A g^2 = - \{(\bar{\partial}_A g^2) g^{-1}\}\{(\bar{\partial}_A g^2) g^{-1}\}^*$$

Taking the trace $\tau = \mathrm{Tr}(g^2)$ of this gives : $\Delta\tau \le 0$ with equality if and only if $\bar{\partial}_A g^2 = o$. By the maximum principle the only possibility is $\Delta\tau = 0$ everywhere and $\bar{\partial}_A g^2 = \partial_A g^2 = 0$. Again since $\mathcal{E}$ is indecomposable we must have g a constant scalar and $A = B$. This completes the proof of the theorem.

References

1. M.F. Atiyah and R. Bott "The Yang-Mills equations over Riemann surfaces".
2. G. Harder and M.S. Narasimhan "On the cohomology groups of moduli spaces of vector bundles over curves" Math. Ann. 212 (1975), 215-248.
3. S. Kobayashi "First Chern class and holomorphic tensor fields" Nagoya Math. J. 77 (1980), 5-11.
4. M.S. Narasimhan and T.R. Ramadas: "Geometry of SU(2) gauge fields", Commun. Math. Phys. 67 (1979), 121-136.
5. M.S. Narasimhan and C.S. Seashadri "Stable and unitary vector bundles on a compact Riemann surface" Ann. of Maths. 82 (1965) 540-67.
6. K.K. Uhlenbeck "Connections with L^p bounds on curvature" Commun. Math. Phys. 3 (1981) 31-42.

CHAPTER TWO

Stable holomorphic bundles and Hermitian-Einstein connections

We suppose now that (X,ω) is a compact Kähler manifold of complex dimension n (usually ≥ 2) and carry over the notation of the previous chapter ; so that E is a $U(r)$ vector bundle over X and $\mathcal{A}$ is the space of connections on E. We will also denote by $\mathfrak{g}_E$ the bundle of Lie algebras corresponding to skew-adjoint elements of End E.

In the usual way we assign real numbers to the characteristic classes of E, for example:

$$c_2 = (c_2(E) \smile [\omega]^{n-2})[X]$$

$$c_1^2 = (c_1(E)^2 \smile [\omega]^{n-2})[X].$$

In the first section we relate these numbers to the <u>Yang-Mills functional</u> on $\mathcal{A}$:

$$\|F\|^2 = \int_X |F|^2 d\mu \qquad\qquad F = F(A).$$

§1. Decomposition of the curvature

First there is a natural orthogonal decomposition of the bundle $\mathfrak{g}_E$ into its' central and trace-free components, and so a corresponding splitting of the curvature of any connection:

$$F = F^+ + (\tfrac{1}{r})(\mathrm{Tr}F).1_E$$

$$\|F\|^2 = \|F^+\|^2 + \tfrac{1}{r}\|\mathrm{Tr}F\|^2.$$

Only the trace-free part F^+ has any real interest since the central component may be freely varied within the linear constraint given by the first Chern class, just as for line bundles.

On the other hand the complex structure of X gives a decomposition of the complexified 2-forms:

$$\Omega^2_{\mathbb{C}} = \Omega^{2,0} \oplus \Omega^{1,1} \oplus \Omega^{0,2}$$

and the $(1,1)$ forms decompose further into the components parallel and perpendicular to the Kähler form ω. Thus for $\alpha \in \Omega^2_{\mathbb{C}}$ we write:

$$\alpha = \alpha_{2,0} + \alpha_{0,2} + \alpha_{\perp} + (\tfrac{1}{n})\hat{\alpha}.\omega$$

where $\hat{\alpha} = \Lambda\alpha$ is a scalar.

There are two natural Hermitian forms on $\Omega^2_{\mathbb{C}}$; the norm $|\alpha|^2$ and the form ϕ defined by:

$$\phi(\alpha).\mathrm{vol} = \alpha \wedge \bar{\alpha} \wedge \omega^{n-2}$$

which are related to the decomposition above by the following equations (particular cases of the Hodge-Riemann bilinear relations [7], Chapter 5).

$$|\alpha|^2 = |\alpha_{2,0}|^2 + |\alpha_{0,2}|^2 + |\alpha_{\perp}|^2 + \tfrac{1}{n}|\hat{\alpha}|^2$$

$$\phi(\alpha) = (n-2)!\,(|\alpha_{0,2}|^2 + |\alpha_{2,0}|^2 + (\tfrac{n-1}{n})|\hat{\alpha}|^2 - |\alpha_{\perp}|^2).$$

(The decomposition is irreducible so one only need checks one term in each factor).

The norm used to compute the Yang-Mills functional comes from the tensor product of the Killing form $M \to -\mathrm{Tr}(M^2)$ on $\mathfrak{g}_E$ with the Hermitian norm on 2-forms. Using the form ϕ instead (on real 2-forms) we set:

$$\phi(F^+).\mathrm{vol} = -\mathrm{Tr}(F^+ \wedge F^+) \wedge \omega^{n-2}$$

$$= (\tfrac{1}{r}\,\mathrm{Tr}(F)^2 - \mathrm{Tr}(F \wedge F)) \wedge \omega^{n-2}$$

$-1/_{4\pi^2}\,\mathrm{Tr}(F \wedge F)$ is the Chern-Weil form corresponding to the characteristic class $c_1(E)^2 - 2c_2(E)$, and $-1/_{2\pi i}\,\mathrm{Tr}(F)$ to $c_1(E)$, hence on integrating we have:

$$\int_X \phi(F^+)d\mu = \frac{4\pi^2}{r}((r-1)c_1^2 - 2rc_2).$$

Whereas by the discussion above:

$$\int_X \phi(F^+)d\mu = (n-2)!\,(\|F^+_{0,2}\|^2 + \|F^+_{2,0}\|^2 + (\tfrac{n-1}{n})\|\hat{F}^+\|^2 - \|F^+_{\perp}\|^2).$$

This gives two lower bounds for the Yang-Mills functional depending upon the sign of the topological invariant $2rc_2 - (r-1)c_1^2$ (this is in fact an invariant of the projectivized bundle associated to E). When this number is positive we have:

PROPOSITION 1: <u>If</u> $2rc_2 - (r-1)c_1^2 \geq 0$ <u>then</u>:

$$\|F^+\|^2 \geq \frac{4\pi^2}{r(n-2)!}\,(2rc_2 - (r-1)c_1^2)$$

<u>with equality if and only if</u> $F^+_{0,2} = F^+_{2,0} = \hat{F}^+ = 0$. <u>Conversely if</u> E <u>admits a connection with these components of the curvature vanishing, then</u> $2rc_2 - (r-1)c_1^2 \geq 0$. (The calculation above was performed in more detail in [4]).

§ 2. Holomorphic bundles and the Hermitian-Einstein condition

In the previous chapter any connection A on E induced a holomorphic structure $\mathcal{E}_A$ on E via the operator $\bar\partial_A$. Over a manifold of larger dimension this remains true for those connections satisfying the integrability condition:

$$\bar\partial_A{}^2 = 0$$

that is, with $F_{0,2}$, and so also $F_{2,0}$, everywhere zero. This is plainly necessary for a connection to be compatible with a holomorphic structure, that is it is sufficient follows from the Newlander-Nirenberg theorem, as in [1] Prop. 3.1. We denote by $\mathcal{A}^{(1,1)}$ the subset of such integrable connections in $\mathcal{A}$; so by the calculation of § 1 for any A in $\mathcal{A}^{(1,1)}$:

$$\|F^+\|^2 = \frac{4\pi^2}{r(n-2)!}\,(2rc_2 - (r-1)c_1^2) + \|\hat{F}^+\|^2$$

or, replacing the central component :

$$\|F\|^2 = \frac{4\pi^2}{(n-2)!}\,(2c_2 - c_1^2) + \|\hat{F}\|^2 \qquad (1).$$

Thus within each topological type the Yang-Mills functional on $\mathcal{A}^{(1,1)}$ can be recovered from the single component $\hat{F}$.

DEFINITION 2: <u>A unitary connection on a holomorphic bundle $\mathcal{E}$ over X is a Hermitian-Einstein connection if the component $\hat{F} = \Lambda F$ of the curvature is the constant central endomorphism</u>

$$-\frac{2\pi i}{n!}\,\mu(\mathcal{E}).1_{\mathcal{E}}.$$

Here, as in Chapter 1,

$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\operatorname{rank}(\mathcal{E})} = \frac{1}{r}(c_1(\mathcal{E}) \smile \omega^{n-1})[X].$$

Clearly this definition reduces to that of Chapter 1 if we take X to be a Riemann surface. In higher dimensions we see by proposition 2 that there is a necessary topological condition: $2rc_2 - (r-1)c_1^2 \geq 0$ for a bundle to admit a Hermitian-Einstein connection.

§ 3. Geometry in the space of connections

We shall explain briefly why the same formal properties hold in the problem of finding Hermitian-Einstein connections as were present in the Riemann surface case.

If M is some (compact) manifold we may form the space $\mathcal{A}$ of connections on some unitary bundle over M, it is naturally a real affine space. Then any one of the three structures:

(1) Complex,

(2) Riemannian,

(3) Symplectic,

on M induces a corresponding structure on $\mathcal{A}$, invariant under the group of automorphisms $\mathcal{G}$. For example if (M^{2n}, ω) is symplectic then we define a skew pairing by:

$$(\alpha, \beta) = -\int_M \operatorname{Tr}(\alpha \wedge \beta) \wedge \omega^{n-1} \qquad (\alpha, \beta \in \Omega^1(\mathfrak{g}_E)).$$

(The Riemannian (Euclidean) structure comes likewise by integration, whereas the complex structure is defined pointwise).

We regard $\Omega^0(\mathfrak{g}_E)$ as the Lie algebra of $\mathcal{G}$, and $\Omega^{2n}(\mathfrak{g}_E)$ as its dual, then we have:

PROPOSITION 3. <u>The map</u> $\mathcal{A} \xrightarrow{m} \Omega^{2n}(\mathfrak{g}_E)$

$$m(A) = F \wedge \omega^{n-1}$$

<u>is a "moment map" for the action of</u> $\mathcal{G}$ <u>on</u> $\mathcal{A}$.

<u>Proof</u>: In general if a group G acts by symplectic diffeomorphisms on a symplectic manifold N, then a map:

$$\mu : N \to L(G)^*$$

is a moment map for the action if for each element v in the Lie algebra $L(G)$ the corresponding vector field: $v(n)$ on N is identified via the symplectic form with the derivative of the function:

$$(\mu(n), v) \qquad \text{on } N.$$

In our case, if $v \in \Omega^0(\mathfrak{g}_E)$ then the associated vector field (infinitesimal gauge transformation) on $\mathcal{A}$ is:

$$v(A) = -d_A v \in \Omega^1(\mathfrak{g}_E) .$$

Whereas the function $(m(A), v)$ is $\int_X \mathrm{Tr}(vF) \wedge \omega^{n-1}$.

Hence $(m(A + a), v) - (m(A), v)$

$$= \int_X \mathrm{Tr}(v[F(A + a) - F(A)] \wedge \omega^{n-1}$$

$$= \int_X \mathrm{Tr}(vd_A a) \wedge \omega^{n-1} + O(a)^2$$

Integrating by parts and using $d\omega = 0$, this is:

$$- \int_X \mathrm{Tr}(d_A v \wedge a) \wedge \omega^{n-1} + O(a)^2$$

$$= (v(A),a) + O(a^2) \quad \text{as required.}$$

When the base space X is a Kähler manifold the affine space $\mathcal{A}$ inherits a (flat) Kählerian structure as above, and the moment map is:

$$F \wedge \omega^{n-1} = (n-1)!\, \hat{F}.\mathrm{vol}.$$

So the moment map associated to the action of the gauge group is essentially the compoment of the curvature discussed above.

The subspace $\mathcal{A}^{(1,1)}$ is a complex subvariety of $\mathcal{A}$, cut out by the zeros of the smooth, complex differentiable map:

$$\mathcal{A} \to \Omega^{0,2}(\mathrm{End}\ E)$$

$$A \to F^{0,2}(A).$$

and restricted to this subspace the norm of the moment map squared is linearly related to the full Yang-Mills functional above. Then a Hermitian-Einstein connection $A \in \mathcal{A}^{(1,1)}$ on a holomorphic bundle $\mathcal{E}_A$ is an absolute minimum of this norm, when the moment map is the constant harmonic element of the centre of $L(\mathcal{G})$ prescribed by the degree of the bundle E.

PROPOSITION 4: (i) <u>Any Yang-Mills connection on a holomorphic bundle over</u> X <u>corresponds to a direct sum of Hermitian-Einstein connections</u>.

(ii) <u>The Yang-Mills flow preserves the subspace $\mathcal{A}^{(1,1)}$ and the holomorphic structure is invariant under the flow.</u>

(iii) <u>If an indecomposable holomorphic bundle $\mathcal{E}$ admits a Hermitian-Einstein connection then:</u>

$$H^{0}(\mathrm{End}\,\mathcal{E}) \cong \mathbb{C} .$$

<u>If also</u> $\deg(\mathcal{E}) \leq 0$ <u>then</u> $H^{0}(\mathcal{E}) = 0$.

<u>Proof</u>: Except perhaps for the vanishing of $H^{0}(\mathcal{E})$ these may all be proved in axiomatic fashion from the definition of a moment map and the interplay between the Euclidean, complex and symplectic structures on $\mathcal{A}$. However we give direct proofs:

We introduce the "twisted derivative" d_A^c on bundle valued forms , for example:

$$d_A^c = \Omega^{0}(\mathfrak{g}_E) \to \Omega^{1}(\mathfrak{g}_E)$$

$$d_A^c = i(\bar{\partial}_A - \partial_A) \qquad \text{(see [7] page 191-2).}$$

so that, just as for ordinary forms, we have the basic Kähler identity:

$$d_A^* = [\Lambda, d_A^c] \quad \text{on} \quad \Omega^{p}(\mathfrak{g}_E) \qquad \text{([7] (Cor. 4.9).}$$

For $A \in \mathcal{A}^{(1,1)}$ the curvature F is of type $(1,1)$ so that the Bianchi identity

$$d_A F = 0$$

implies $d_A^c F = 0$, hence
$$d_A^* F = -d_A^c \Lambda F = -d_A^c \hat{F} .$$

So, for (i), if $A \in \mathcal{A}^{(1,1)}$ is <u>Yang-Mills</u>, i.e. $d_A^* F = 0$ then $d_A^c \hat{F} = d_A \hat{F} = 0$ and the eigenspaces of $\hat{F}$ decompose the connection into Hermitian-Einstein factors. Similarly for (ii) the Yang-Mills flow:

$$\frac{\partial A}{\partial t} = - d_A^* F$$

becomes, on $\mathcal{A}^{(1,1)}$: $\frac{\partial A}{\partial t} = d_A^c \hat{F}$.

And, just as the tangent space to the $\mathcal{G}$ orbits is

$$\mathrm{Im}(d_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E))$$

the tangent space to the $\mathcal{G}^{\mathbb{C}}$-orbits is its complexification $\mathrm{Im}\, d_A \oplus \mathrm{Im}\, d_A^c$. The action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}$, complexification of the action of $\mathcal{G}$, is just as in Chapter 1, Section 2, and as there the orbits parametrise holomorphic structures, so the Yang-Mills flow preserves the $\mathcal{G}^{\mathbb{C}}$ orbits, i.e. the holomorphic structure is invariant under the flow. (We treat the flow formally here, without considering questions of existence and uniqueness).

To prove part (iii) we compute the three Laplacians:

$$\Delta_A = d_A^* d_A, \qquad \Delta_A' = \partial_A^* \partial_A, \qquad \Delta_A'' = \bar{\partial}_A^* \bar{\partial}_A$$

on $\Omega^0(\mathrm{End}\, E)$ or $\Omega^0(E)$.

Using the Kähler identities : $\partial_A^* = i[\Lambda, \bar{\partial}_A]$

$$\bar{\partial}_A^* = -i[\Lambda, \partial_A]$$

we get

$$\Delta_A = i\Lambda(\bar{\partial}_A - \partial_A)(\bar{\partial}_A + \partial_A) = \Delta_A' + \Delta_A'' = -i\Lambda d_A d_A^c$$

Whereas:

$$\Delta'_A - \Delta''_A = i\Lambda(\bar{\partial}_A\, \partial_A + \partial_A\, \bar{\partial}_A) = i\,\hat{F}\,.$$

So if A is Hermitian-Einstein then $\Delta_A = 2\Delta''_A$ on $\Omega^0(\text{End } E)$. Hence for any section s

$$\|\bar{\partial}_A s\|^2 = \langle \Delta''_A s, s\rangle = \tfrac{1}{2}\langle \Delta_A s, s\rangle = \tfrac{1}{2}\|d_A s\|^2\,.$$

So a holomorphic endomorphism is covariant constant and, if not a multiple of the identity, decomposes A.

Similarly on $\Omega^0(E)$ we have:

$$\|\bar{\partial}_A s\|^2 = \tfrac{1}{2}\|d_A s\|^2 - \frac{n!}{2\pi}\mu\,\|s\|^2\,.$$

So if $\mu < 0$ there are no holomorphic sections and if $\mu = 0$ any such section again decomposes the bundle.

(This is a special case of a vanishing theorem of Kobayashi and Wu [3]).

The following proposition extends the calculation alluded to in the last paragraph of Chapter 1.

PROPOSITION 5: _Let_ $A \in \mathcal{A}^{(1,1)}$ _and let_ g _be an element of the complexified gauge group_ $\mathcal{G}^{\mathbb{C}} \subset \Omega^0(\text{End } E)$. _Then if we write:_

$$h = g^*g \qquad B = g^{-1}(A) \qquad \tau = \text{Trace }(h) = |g|^2$$

we have:

$$\text{(i)}\quad d_A d_A^c h = 2gF_Bg^* - (hF_A + F_Ah) + 2(\partial_A h)h^{-1}(\bar{\partial}_A h)$$

$$\text{(ii)}\quad \Delta_A h = -i(2g\hat{F}_Bg^* - (h\hat{F}_A + \hat{F}_Ah)) - 2i(\partial_A h)h^{-1}(\bar{\partial}_A h)\,.$$

(iii) $\Delta\tau = -2i\ \mathrm{Tr}(g^*g\hat{F}_B - h\hat{F}_B) - 2i\Lambda\mathrm{Tr}(\partial_A h)h^{-1}(\bar{\partial}_A h)$.

Here (i) is a direct calculation, starting from

$$B = A + g^{-1}\bar{\partial}_A g - (g^{-1}\bar{\partial}_A g)^*$$

$$F_B = F_A + d_A(B - A) + (B - A) \wedge (B - A).$$

(ii) follows by applying $\Lambda : \Omega^{1,1}(\mathrm{End}\ E) \to \Omega^{0}(\mathrm{End}\ E)$, and (iii) by taking the trace in the bundle.

The force of each of these is that the quadratic term:

$$Q = i(\partial_A h)h^{-1}(\bar{\partial}_A h)$$

is positive (i.e. for any $v \in TX$ $Q(v \wedge \bar{v})$ is a positive endomorphism of the bundle) and in (iii):

$$\mathrm{Tr}\ \Lambda Q \geq 0$$

with equality if and only if $\partial_A h = \bar{\partial}_A h = 0$.

CORROLLARY 6: <u>If $\mathcal{E}$ is an indecomposable holomorphic bundle over a compact Kähler manifold, and if a Hermitian-Einsten connection on $\mathcal{E}$ exists, it is unique up to isomorphism.</u>

<u>Proof</u>: Same argument as the last paragraph of Chapter 1, using Proposition 5 (iii).

Thus most of the formal properties of the Yang-Mills functional on the space of connections over a Riemann surface hold in general for connections in holomorphic bundles over a Kähler manifold. We describe next the definition of a stable bundle in this case.

§ 4. Stable holomorphic bundles

To motivate the definition recall that in Chapter 1 we used the fact that for any holomorhic map $\alpha : \mathcal{E} \to \mathcal{F}$ between bundles over a Riemann surface there are bundles $\mathcal{Q}, \mathcal{M}$ of the same rank and a factorisation:

$$\mathcal{E} \xrightarrow{p} \mathcal{Q} \to \mathcal{M} \xrightarrow{i} \mathcal{F}$$

(p a projection, i an inclusion). It is easy to see that this will not be true for bundles $\mathcal{E}, \mathcal{F}$ over a complex manifold of larger dimension, but one can still make such a factorisation of the corresponding sheaves.

DEFINITION 7 ([6],[5]p.160): Let V be a smooth projective algebraic variety and H an ample line bundle over V. Then a vector bundle $\mathcal{E}$ over V is H semi-stable if, for all coherent subsheaves $\mathcal{S} \subset \mathcal{O}(\mathcal{E})$ we have

$$\mu(\mathcal{S}) \leq \mu(\mathcal{E})$$

if strict inequality holds then $\mathcal{E}$ is H-stable.

Notes: (i) Any such sheaf $\mathcal{S}$ is locally free except over a codimension two singular set. Thus the rank and degree and hence:

$$\mu(\mathcal{S}) = \frac{\text{degree } (\mathcal{S})}{\text{rank } (\mathcal{S})} = \frac{H^{n-1}.c_1(\mathcal{S})}{\text{rank } (\mathcal{S})}$$

can be defined, for example by restricting to the generic hyperplane section.

(ii) Only the polarisation (i.e. cohomology class $c_1(H)$) corresponding to H is used in the definition. Thus there is an obvious generalisation of this definition to bundles over any Kähler manifold, using the cohomology class determined by the metric.

The main property of stable bundles that we shall use is:

PROPOSITION 8 ([5] Corollary to Lemma 1.2.8.). Let $\alpha : \mathcal{E} \to \mathcal{F}$ be a non-zero holomorphic map between two semi-stable holomorphic bundles of the same rank and degree. Then if at least one of $\mathcal{E}, \mathcal{F}$ is stable α is an isomorphism.

This is proved by factorising α and applying the numerical criteria for stability to the resulting sub and quotient sheaves. We note finally that for bundles of rank two one need only consider sub-sheaves $\mathcal{S}$ of a very simple type:

PROPOSITION 9 ([5] Lemma 1.2.5 for the case when $V = \mathbb{P}_n$, the proof is general). Let $\mathcal{E}$ be a holomorphic bundle of rank 2 over V. Then $\mathcal{E}$ is stable (resp. semi-stable) if and only if for all line bundles $\mathcal{L}$ over V with a non-zero holomorphic map $\mathcal{L} \to \mathcal{E}$ we have:

$$\mu(\mathcal{L}) < \mu(\mathcal{E}) \qquad \text{(resp. } \mu(\mathcal{L}) \leq \mu(\mathcal{E})\text{)}.$$

§ 5. The conjecture of Hitchin and Kobayashi

It seems now very likely that the precise relationship that holds between Hermitian-Einstein connections and stable bundles over algebraic curves extends to varieties of all dimensions, perhaps in the form of:

CONJECTURE (Hitchin, Kobayashi: references cited in the Introduction). If X is a smooth complex projective variety

<u>with an ample line bundle</u> H <u>over</u> X <u>and a Kähler metric</u> ω <u>defining the same cohomology class as</u> $c_1(H)$, <u>then a holomorphic bundle</u> $\mathcal{E}$ <u>over</u> X <u>is</u> H <u>stable if and only if it admits a Hermitian-Einstein connection with respect to</u> ω.

Strong circumstantial evidence for the truth of the conjecture is given by the inequality of Proposition 1 above. A 2-plane bundle admitting a Hermitian-Einstein connection must have:

$$c_1^2 - 4c_2 \leq 0$$

and in algebraic geometry this same inequality has been shown to hold for, say, any stable bundle over $\mathbb{CP}^n$. ([5] p.234, Corollary to a Theorem of Barth).

Moreover Takemoto has proved ([6] Prop. 2.1.) that if $\mathcal{E}$ is a stable 2-plane bundle (over a minimal surface) with $c_1^2 - 4c_2 \geq 0$ then the stability condition is independent of the ample line bundle H chosen. This too is consistent with the conjecture, since any Hermitian-Einstein connection on a bundle with $c_1^2 - 4c_2 = 0$ must be projectively flat.

In Chapter 4 we will prove the conjecture for bundles of rank 2 over $\mathbb{CP}^2$. We observe next that, at least for 2-bundles, one half of the conjecture is easy to prove:

PROPOSITION 10. <u>Let</u> X,H,ω <u>be as in the conjecture above and</u> $\mathcal{E}$ <u>a holomorphic bundle of rank 2 over</u> X <u>admitting a Hermitian-Einstein connection with respect to</u> ω. <u>Then is semi-stable and, if indecomposable</u> H <u>stable.</u>

Proof: If $\mathcal{L} \to \mathcal{E}$ is a non zero map of a line bundle $\mathcal{L}$ to $\mathcal{E}$ then $\mathcal{E} \otimes \mathcal{L}^{-1}$ has a non-zero section, and also admits a Hermitian-Einstein connection. Then by Proposition 4(iii):

$$\mu(\mathcal{E} \otimes \mathcal{L}^{-1}) = \mu(\mathcal{E}) - \mu(\mathcal{L}) \geq 0$$

with strict inequality if $\mathcal{E} \otimes \mathcal{L}^{-1}$ is indecomposable. By Proposition 9 this completes the proof.

§6. General discussion of the conjecture for varieties of dimension at least two.

According to the analogy with the finite dimensional situation sketched in the introduction there should be a critical point of the Yang-Mills functional within the closure of any orbit $o(\mathcal{E}) \subset \mathcal{A}^{(1,1)}$ corresponding to a holomorphic bundle $\mathcal{E}$.

In Chapter 1 (Lemma 1) we essentially found this critical point by using the "weak compactness" theorem of Uhlenbeck and then showed, by considering extensions, that the stability condition was precisely that required to ensure that the critical point was contained in the orbit and was Hermitian-Einstein.

On the other hand, in the Riemann surface case discussed in Chapter One, if the bundle we chose was not stable, for example a 2-bundle given by an extension:

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O}_X \to 0 \qquad \mu(\mathcal{L}) \geq \mu(\mathcal{E}).$$

Then we would expect to find a critical point within the closure of $o(\mathcal{E})$ corresponding to the bundle:

$$\mathcal{L} \oplus \mathcal{O}_X$$

and again this agrees with the finite dimensional analogy.

When we go to manifolds of larger dimensions relatively few bundles may be expressed as extensions of line bundles: for example over $\mathbb{CP}^n$ all such extensions are trivial. Instead the typical phenomenon is for a 2-bundle to appear in a <u>Koszul complex</u> for the ideal sheaf J_Y of some subvariety Y of codimension 2; that is (up to twisting with a line bundle) as:

$$0 \to \mathcal{O}(\mathcal{L}) \to \mathcal{O}(\mathcal{E}) \to J_Y \to 0 \qquad \text{([6] Ch. I.5)}$$

(so $\mathcal{E}^* \cong \mathcal{E} \otimes (\det \mathcal{E})^{-1}$ has a section vanishing along Y). Of course this is the same phenomenon encountered in the definition of stability above.

Supposing then that $\mathcal{E}$ is not stable and appears in such a Koszul complex with $\mu(\mathcal{L}) \geq \mu(\mathcal{E})$. Then rough calculations, along the lines of Lemma 2 in Chapter 1, indicate that a minimising sequence A_i of connections on $\mathcal{E}$ converges to a connection on $\mathcal{L} \oplus \mathcal{O}_X$ except over the subvariety Y, and that the curvatures of the A_i are unbounded over Y; so that as currents of type (2,2):

$$c_2(A_i) \to T_Y \qquad \text{([3] p.386).}$$

Thus if we wish to associate a critical point with such an unstable bundle $\mathcal{E}$ it has to be some kind of formal connection, with a singularity along Y.

This is entirely consistent with the work of Uhlenbeck since in real dimensions ≥ 4 the weak compactness theorem does not hold true in the straightforward sense of Chapter 1.

Thus it is of interest to study examples of families of Yang-Mills connections whose curvature is unbounded, and in the next chapter we shall do this in some detail in the rather different setting of four-dimensional Riemannian geometry. To relate that chapter to this one we observe that if X is a Kähler surface we have the complex decomposition:

$$\Omega^2_{\mathbb{C}} = \Omega^{2,0} \oplus \Omega^{0,2} \oplus \mathbb{C}\omega \oplus \Omega^{1,1}_{\perp}$$

Whereas, regarding X simply as a Riemannian 4-manifold there is a decomposition of Ω^2 into <u>self-dual</u> and <u>anti self-dual</u> parts:

$$\Omega^2_{\mathbb{C}} = \Omega^2_+ \otimes \mathbb{C} \oplus \Omega^2_- \otimes \mathbb{C} \qquad \text{(defined in Chapter 3,I.3)}$$

and it is easy to check that:

$$\Omega^2_+ \otimes \mathbb{C} = \mathbb{C}\omega \oplus \Omega^{2,0} \oplus \Omega^{0,2}$$

$$\Omega^2_- \otimes \mathbb{C} = \Omega^{1,1}_{\perp} .$$

Thus, in the language of Chapter 3, a connection on a holomorphic bundle (with $c_1 = 0$) is anti self-dual if and only if it is Hermitian-Einstein, and conversely an anti self-dual connection defines a holomorphic bundle together with a Hermitian-Einstein connection.

<u>References</u>

[1] Atiyah, M.F., Hitchin, N.J. and Singer, I.M. "Self-duality in four dimensional Riemannian geometry" Proc. R. Soc. London A 362 (1978) 425-461.

[2] Griffiths, P. and Harris, J. "Principles of Algebraic Geometry" John Wiley, New York (1980).

[3] Kobayashi, S. and Wu, H.H. "On holomorphic sections of certain Hermitian Vector bundles", Math. Ann. 189 (1970) 1-4.

[4] Lübke, M. "Chern Classes of Hermitian-Einstein vector bundles"(to appear).

[5] Okonek, C., Schneider, M. and Spindler, M. "Vector bundles on complex projective space", Birkhauser, Boston (1980).

[6] Takemoto, F. "Stable vector bundles on algebraic surfaces II", Nagoya Math. J. 52 (1973) 173-195.

[7] Wells, R.O."Differential Analysis on complex manifolds" Springer Verlag New York (1980).

CHAPTER THREE.

An application of gauge theory to the topology of 4-manifolds

I.1. Introduction

The homotopy type of a compact, simply connected, oriented 4-manifold X is entirely determined by the intersection form:

$$Q : H^2(X;\mathbb{Z}) \to \mathbb{Z}$$
$$Q(\alpha) = (\alpha \smile \alpha)[X]$$

(dually, realised on homology by the self-intersection of cycles) a quadratic form on the free abelian group $H^2(X;\mathbb{Z})$, with determinant ± 1 by Poincaré duality. Many writers ([11],[12] for example) have discussed the problem of finding which forms may arise from 4-manifolds of various types.

Very recently M.H. Freedman has given a complete classification of simply connected topological 4-manifolds [8]. According to this classification such a manifold is specified, up to homeomorphism, by any form of determinant ± 1 and, if that form is of odd type (i.e. $Q(\alpha)$ odd for some α) the value of the Kirby-Siebenmann obstruction in $\mathbb{Z}/_2$ (this is an obstruction to smoothing the 5-manifold $X \times S^1$). We show here that there are severe constraints on the forms that may be realised by smooth 4-manifolds, in sharp contrast to the topological case. This is the theorem we shall prove, fulfilling the announcement in [7]:

THEOREM 1: Let X be a compact, smooth, simply connected oriented 4-manifold with the property that the associated form Q is positive definite. Then that form is equivalent, over the integers, to the standard diagonal form, so in some base: $Q(u_1,u_2,\ldots,u_r) = u_1^2 + u_2^2 \ldots + u_r^2$.

In the arithmetic of quadratic forms, there is a clear difference between the indefinite forms which are all explicitly known and are classified by three invariants: rank, signature and type, and the definite forms which occur in a large number of distinct isomorphism classes [16 Chapter 5]. Thus Theorem 1 implies that most of the manifolds constructed in [8] cannot be given a smooth structure.

The most obvious problem remaining in this area is to find for which values of m,n the manifold with quadratic form:

$$2mE_8 + n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

can be given a smooth structure. In particular if $m = 1$, then the form with $n = 3$ is realised by a K3 surface which is smooth; while for $n = 0$ the form cannot, by Theorem 1, be realised by a smooth simply-connected 4-manifold since $2E_8$ is definite. It is not yet clear whether the methods we use here can be extended to the intermediate cases.

On the other hand the fundamental group does not seem to play a very crucial role in the proof of Theorem 1, and one could hope to extend the result a long way in this direction. It will appear that the present proof works under the weaker, but rather strange, hypothesis that there be no non-trivial representations of $\pi_1(X)$ in $SU(2)$.

I.2. Method of proof

The bare structure of the proof of Theorem 1 is much simpler than the more technical material which makes up the bulk of this paper, so we give it now.

We will find associated to X a space $\mathcal{M}^{\sigma}(X)$ which can be compactified to an orientable 5-manifold with boundary X and a certain number of point singularities, one singularity for each pair $\pm\alpha$ of solutions of:

$$Q(\alpha) = 1 \qquad \alpha \in H^2(X;\mathbb{Z})\ .$$

Call this number $n(Q)$. A neighbourhood of one of these singular points in $\mathcal{M}^{\sigma}(X)$ will have the form of a cone on $\mathbb{CP}^2$, so if we remove these neighbourhoods we get an orientable manifold with boundary the disjoint union of X and $n(Q)$ copies of $\mathbb{CP}^2$. We do not know, at this stage of the argument, how the orientations of the copies of $\mathbb{CP}^2$ compare.

The only arithmetical fact about quadratic forms that we use is this observation:

LEMMA 2: <u>If</u> Q <u>is any positive definite quadratic form over</u> $\mathbb{Z}$ <u>then</u>:

$$n(Q) \le \mathrm{rank}(Q)$$

<u>with equality if any only if</u> Q <u>is equivalent to the standard form</u>.

<u>Proof</u>: By induction on $r = \mathrm{rank}(Q)$. If α is any solution of $Q(\alpha) = 1$ we may split:

$$\mathbb{Z}^r = \mathbb{Z}.\alpha \oplus \alpha^{\perp}$$

$$\beta \to (\beta.\alpha)\alpha \oplus (\alpha - (\beta.\alpha)\alpha)$$

and, because Q is definite: $n(Q) = n(Q\big|_{\alpha^{\perp}}) + 1$, $\operatorname{rk} Q = \operatorname{rk}(Q\big|_{\alpha^{\perp}}) + 1$.

On the other hand the <u>signature</u> of the manifold X is $\sigma(Q) = \operatorname{rank}(Q)$ (since Q is definite), and signature is an invariant of oriented cobordism so:

$$\operatorname{rank}(Q) \leq n(Q) \, . \, \sigma(\mathbb{CP}^2) = n(Q)$$

Thus, by our observation, we must have $\sigma(Q) = n(Q)$ and Q is the standard form.

I.3. <u>Connections and self-duality</u>

The space $\mathcal{M}^{\sigma} = \mathcal{M}^{\sigma}(X)$ will be defined by using the ideas and methods developed for the study of the "gauge theories" of Mathematical Physics. There are now several thorough expositions ([5],[13] for example) of these ideas so we will only say enough here to fix notation and the basic facts that we shall use.

If G is a compact Lie group, $\begin{smallmatrix} P \\ \downarrow \\ X \end{smallmatrix}$ a principal G bundle over the 4-manifold X one may define the space $\mathcal{A}$ of all <u>connections</u> on P. Any two connections A, B differ by an element:

$$A - B \in \Omega^1(\mathfrak{g}),$$

of the vector space of 1-forms with values in the bundle of Lie algebras associated to P, so $\mathcal{A}$ is an affine space. For any vector bundle V associated to P a connection A induces a differential operator:

$$d_A : \Omega^p(V) \to \Omega^{p+1}(V)$$

and if X has a Riemannian metric we get adjoints:

$$d_A^* : \Omega^{p+1}(V) \to \Omega^p(V)$$, and covariant derivatives

$$\nabla_A : \Omega^p(V) \to \Omega^p(V) \otimes \Omega^1.$$

The curvature $F(A)$ of any connection is an element of $\Omega^2(\mathfrak{g})$ and:

$$F(A + a) = F(A) + d_A a + \tfrac{1}{2}[a,a]. \qquad a \in \Omega^1(\mathfrak{g}).$$

The group $\mathcal{G}$ of automorphisms of P acts as a symmetry group on all this structure and connections are <u>equivalent</u> if they are in the same orbit of $\mathcal{G}$ on $\mathcal{A}$. Considering $\mathcal{G}$ as $\Gamma(P \times_{Ad} G)$ this action is given by:

$$g(A) = A - (d_A g)g^{-1}.$$

We will be concerned with the cases $G = SU(2)$, $U(1)$ and only very briefly with the second. The topological classification of such bundles is by:

$G = SU(2)$; P classified by $c_2(P) \in H^4(X;\mathbb{Z}) \cong \mathbb{Z}$

$G = U(1)$; P classified by $c_1(P) \in H^2(X;\mathbb{Z})$.

In the second case we summarise all we need in:

PROPOSITION 3. <u>If</u> P <u>is a</u> $U(1)$ <u>bundle over</u> X, A <u>any connection on</u> P, <u>then the image of</u> $c_1(P)$ <u>in</u> $H^2(X;\mathbb{R})$ <u>is represented in de Rham cohomology by the closed 2-form</u> $-1/_{2\pi i}\ F(A)$ ($\mathfrak{g}$ <u>is in this case a trivial bundle)</u>;

all 2-forms within this cohomology class occur as the curvature of some connection and, on the simply connected manifold X, any two connections with the same curvature are isomorphic. In particular, if X has a Riemannian metric, there is a unique connection on P, up to equivalence, with harmonic curvature; $\Delta F = 0$.

The corresponding characteristic class representation for an $SU(2)$ bundle is:

$$c_2(P) = 1/_{8\pi^2} \int_X Tr(F(A) \wedge F(A)) \in \mathbb{Z}$$

which also has a version for manifolds with boundary. If Y is a compact oriented 3-manifold, A a connection on an $SU(2)$ bundle over Y there is an invariant:

$$Tc_2(A) \in \mathbb{R}/_{\mathbb{Z}} \qquad \text{(See [5] Appendix 3)}$$

with the property that any time $Y = \partial Z$ and A extends to a connection $\tilde{A}$ over the oriented 4-manifold Z:

$$Tc_2(A) = 1/_{8\pi^2} \int_Z Tr(F(\tilde{A}) \wedge F(\tilde{A})) \quad \text{mod } \mathbb{Z} .$$

(The analogous invariant for $U(1)$ connections over circles is the holonomy around the circle).

Self-duality

On an oriented Riemannian 4-manifold the 2-forms decompose:

$$\Omega^2 = \Omega^2_+ \oplus \Omega^2_-$$

into the spaces of "self-dual" and "anti self-dual" forms, defined by the ± 1 eigenspaces of the operator $* : \Omega^2 \to \Omega^2$, thus by the property:

$$\alpha \in \Omega^2_{\pm} \ \text{ if } \ \alpha \wedge \alpha = \pm(\alpha \wedge *\alpha) = \pm|\alpha|^2.\text{vol}. \qquad (1).$$

There is a similar decomposition of bundle valued forms and so of the curvature $F(A)$ of a connection on P.

$$F(A) = F_+(A) + F_-(A) \in \Omega^2_+(\mathfrak{g}) \oplus \Omega^2_-(\mathfrak{g}).$$

The tool which we shall use to prove Theorem 1 is the notion of a <u>self-dual connection</u> [2], that is to say a connection with self-dual curvature, $F_-(A) = 0$. These were defined and studied for the following reason: for an $SU(2)$ connection A the "<u>Yang-Mills action</u>"

$$\|F(A)\|^2_{L^2} = \int_X |F(A)|^2 \, d\mu$$

$$= \int_X |F_+|^2 + |F_-|^2 \, d\mu$$

is bounded below by the absolute value of

$$-8\pi^2 c_2(P) = \int_X -\mathrm{Tr}(F \wedge F)$$

$$= \int_X |F_+|^2 - |F_-|^2 \, d\mu \quad \text{by (1) above,}$$

and for $c_2 \le 0$ there is equality if and only if A is self-dual. Conversely if the bundle P admits a self-dual connection then $c_2 \le 0$; and if $c_2 = 0$, so P is topologically trivial, any self-dual connection is <u>flat</u> i.e.

$F = 0$. In general a flat connection over a manifold is (up to isomorphism) equivalent to a representation of π_1 in the structure group, so for the simply connected manifold X of the theorem any flat connection is isomorphic to the standard product connection on $SU(2) \times X$, we call this connection θ.

This decomposition into self-dual and anti self-dual parts gives a differential operator acting on 1-forms:

$$d^- : \Omega^1 \to \Omega^2_-, \quad \text{the composite} \quad \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{\pi} \Omega^2_-$$

Similarly if we have any connection A there is:

$$d_A^- : \Omega^1(V) \to \Omega^2_-(V).$$

With Laplacians Δ_A on $\Omega^0(V)$, $\Omega^1(V)$, $\Omega^2_-(V)$ given respectively by $d_A^* d_A$, $d_A d_A^* + d_A^* d_A^-$, $d_A^- d_A^*$; and associated harmonic spaces H_A^0, H_A^1, H_A^2 (<u>if</u> A <u>is self-dual</u>, these are the harmonic spaces associated to the elliptic complex:

$$\Omega^0(V) \xrightarrow{d_A} \Omega^1(V) \xrightarrow{d_A^-} \Omega^2_-(V))$$

For the rest of this paper we suppose that X is given some Riemannian metric and that $\begin{smallmatrix} P \\ \downarrow \\ X \end{smallmatrix}$ is an $SU(2)$ bundle with $c_2(P) = -1$.

I.4. <u>Instantons, and conformal invariance</u>

All the self-dual connections on the bundle P ($c_2(P) = -1$) over the standard Riemannian S^4 are explicitly known, and this example provides the basic model for the general theory.

Notice first that the self-duality condition depends only upon the conformal class of the Riemannian metric. Moreover the action:

$$\int_U |F|^2 \, d\mu$$

contained in any open set U is likewise independent of conformal changes. This means on the one hand that the conformal group $SO(5,1)^+$ of S^4 acts on the set of self-dual connections on P; on the other hand that any self-dual connection on S^4 can be interpreted as a self-dual connection or "instanton" on $\mathbb{R}^4$, by the conformal equivalence $\mathbb{R}^4 \to S^4 \backslash \{pt\}$, and that the Chern class can be recovered from:

$$\int_{\mathbb{R}^4} |F|^2 \, d\mu. = -8\pi^2 c_2 . \tag{2}$$

A theorem of Uhlenbeck implies that the converse is also true. For later use we state a form of this result as:

PROPOSITION 4 ([20] Thm. 4.1). *If A is an $SU(2)$ connection (on the trivial bundle) over the punctured ball $B^4 \backslash \{0\}$, self-dual with respect to some smooth Riemannian metric on B^4 and with finite action:*

$$\int_{B^4 \backslash \{0\}} |F(A)|^2 \, d\mu. < \infty$$

then there is a bundle automorphism $g : B^4 \backslash \{0\} \to SU(2)$ such that $g(A)$ extends smoothly over B^4.

Hence we may recover a self-dual connection on P (up to isomorphism) from an instanton on $\mathbb{R}^4$ with total action $8\pi^2$.

According to the classification, established in [2] for example, the equivalence classes of self-dual connection on $\begin{smallmatrix} P \\ \downarrow \\ S^4 \end{smallmatrix}$ form a single orbit under the conformal group. There is a single SO(5) invariant class (coming from the natural connection on the fibration $\begin{smallmatrix} S^7 \\ \downarrow \\ S^4 \end{smallmatrix}$), hence the set of these equivalence classes is parametrised by a <u>moduli space</u> $\mathcal{M}(S^4) = SO(5,1)/_{SO(5)} = B^5$, the open 5-ball. Interpreted as instantons on $\mathbb{R}^4$ these connections can be specified by a "<u>centre</u>" in $\mathbb{R}^4$, about which they have SO(4) symmetry, and a <u>scale</u> which we can measure, for example, by the radius of the ball about the centre containing action $4\pi^2$ (i.e. one half of the total). Then the conformal group acts by translations on the centres and dilations $x \to \alpha . x$ on the radii. The instanton I_λ with centre 0 and radius λ can be given by the explicit connection matrix:

$$I_\lambda(x) = \mathrm{Im}\left(\frac{x d\bar{x}}{\lambda^2 + |x|^2}\right) \quad \{\text{Using quaternionic notation: } \mathbb{R}^4 \cong \mathbb{H},\ su(2) \cong \mathrm{Im}\,\mathbb{H}\}$$

with curvature: $F(I_\lambda) = \lambda^2 \dfrac{dx d\bar{x}}{(\lambda^2 + |x|)^2}$ (We shall not make any use of these equations).

Thus one sees that the compactification $\bar{B}^5 = \mathcal{M}(S^4) \cup S^4$ has an intrinsic interpretation in terms of the connections; a point x of S^4 represents the limit point of a sequence of connections whose curvature becomes concentrated in diminishing balls about x. The whole object of Sections II and III of this paper is to show that an analogous moduli space $\mathcal{M}(X)$ exists for the manifold X of the theorem, and that it may be compactified in the same way.

I.5. Taubes' Theorem on the existence of self-dual connections

According to the Hodge Theory the second real cohomology group of any compact Riemannian manifold is represented by the space of harmonic 2-forms. The $*$ operator commutes with the Laplacian Δ; so on a 4-manifold these harmonic forms decompose into self-dual and anti self-dual parts; $\mathcal{H}^2 = \mathcal{H}^2_+ \oplus \mathcal{H}^2_-$. By the defining property (1) the dimensions of these two spaces are just the numbers of positive and negative eigenvalues in a diagonalisation of the quadratic form:

$$Q_{\mathbb{R}}(\alpha) = \int_X \alpha \wedge \alpha \; : \quad \alpha \in H^2(X;\mathbb{R}).$$

(This is very close to some of the earliest applications of the theory of harmonic forms [9] Sec.52.2); so an equivalent form of the hypothesis in Theorem 1 that the intersection form is positive definite is the statement that the space $\mathcal{H}^2_-(X)$ of anti-self dual harmonic forms vanishes.

In the recent paper [18] C.H. Taubes constructs self-dual connections on the bundle P under precisely this hypothesis ([18] Theorem 1.1). They are constructed there by means of an "implicit function theorem" which we state here for later use:

<u>PROPOSITION 5</u> ([18] Theorem 2.2)

<u>Let</u> A_o <u>be a connection on</u> $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$. <u>There are constants</u> $C, \varepsilon_o > 0$, <u>depending only on the Riemannian structure of</u> X, <u>the</u> L^3<u>-norm</u> $\|F_-(A_o)\|_{L^3}$ <u>and the 1st eigenvalue</u> μ <u>of the</u>

operator $\Delta_{A_o} = d^-_{A_o} d^*_{A_o}$ on $\Omega^2_-(\mathfrak{g})$ such that if:

$$\delta(A_o) = \|F_-(A_o)\|_{L^2} + \|F(A_o)\|_{L^4} \|F_-(A_o)\|_{L^{4/3}} < \varepsilon_o$$

then there is a self-dual connection $A_o + d^*_{A_o} U = A_o + a = q(A_o)$ with $U \in \Omega^2_-(\mathfrak{g})$ and:

(i) $\|\nabla_{A_o} a\|_{L^2} \leq C.\delta(A_o)$

(ii) $\|a\|_{L^2} \leq C.\|F_-(A_o)\|_{L^{4/3}}$

The construction is $\mathcal{G}$-invariant and $q(A_o)$ varies smoothly with A_o.

Given any point of X and $\lambda > 0$, Taubes applies this to a connection A_o constructed to have most of its curvature in a ball of radius λ around that point . The hypothesis $\mathcal{H}^2_-(X) = 0$ means that the 1st eigenvalue of Δ on Ω^2_- is positive and ensures that $\mu(A_o)$ is bounded away from zero as $\lambda \to 0$.

A much easier observation, which we need to complete these preliminaries, is that for the 4-manifold X with positive definite form each $U(1)$ bundle has a unique self-dual connection, up to isomorphism. This follows directly from Proposition 3 and the fact that the harmonic curvature form is automatically self-dual.

II.1. Sobolev spaces, reducible connections and properties of the space $\mathcal{B}$.

We shall need to refer to Sobolev spaces L^p_k $(p \geq 1, k \in N)$ of sections of bundles associated to P; that is of sections locally represented by functions with their first k derivatives in L^p. A smooth connection A gives definite norms $\| \ \|_{L^p_k(A)}$ on these spaces for example

$$\|s\|^2_{L^2_1(A)} = \|\nabla_A s\|^2_{L^2} + \|s\|^2_{L^2} \cdot$$

$$= \int_X (|\nabla_A s|^2 + |s|^2)\, d\mu .$$

Moreover it is convenient in this Section II to widen the definition of $\mathcal{A}$ to allow A differing from a smooth connection by an element of $L^2_3(\Omega^1(\mathfrak{g}))$. This differentiability is high enough not to make any essential differences in the properties discussed in Sect. I (and also the norms $\| \ \|_{L^2_k(A)}$ are defined for $k \leq 3$) but allows us to work in Banach spaces.

We define the <u>moduli space</u> $\mathcal{M}(X)$ to be the set of equivalence classes of self-dual connections on P, that is to say a subset of the quotient space:

$$\mathcal{B} = \mathcal{A}/\mathcal{G}$$

and use the notation
$$p : \mathcal{A} \to \mathcal{B}$$
$$A \to [A].$$

The techniques that we use here to study the moduli space $\mathcal{M}$ are essentially those of [2], but we emphasise the role of the ambient space $\mathcal{B}$ rather more.

We distinguish first the <u>reducible</u> connections on P; they are characterised by any one of the properties:

(i) The holonomy group reduces to a subgroup of $SU(2)$. Since P is topologically non-trivial the only possibility is that this subgroup be a copy of $U(1)$.

(ii) The $\mathbb{C}^2$ vector bundle E associated to P decomposes as $E \cong L \oplus L^{-1}$ and the connection on P is induced from a $U(1)$ connection on the principal bundle corresponding to L.

(iii) $H^0_A = (\mathrm{Ker}\ \Delta_A \subset \Omega^0(\mathfrak{g})) \neq 0$, then for the same reason as (i) H^0_A is one dimensional, generated by an element (iu) say, where the ± 1 eigenspaces of u decompose E as in (ii).

(iv) There is a non-trival stabiliser $\Gamma_A \neq \{\pm 1\}$ of A in $\mathcal{G}$. Once we have chosen a generator iu, Γ_A is canonically isomorphic to $U(1)$ by $e^{i\theta} \to \exp(iu\theta)$.

From (ii) the possibilities for reducing P are classified, topologically, by line bundles L with:

$$c_1(L)^2 = -c_2(L \oplus L^{-1}) = -c_2(E) = 1$$

except that we must identify L with L^{-1} in this classification (corresponding to a choice of $\pm u$ in (iii)). Hence the number of such reductions is given in terms of cohomology by the number $n(Q) = \frac{1}{2}\ |\{\alpha \in H^2(X;\mathbb{Z}) : Q(\alpha) = 1\}|$ of Section I.2 and by Proposition 3 and the observation at the end of I.5 there is <u>a unique self-dual connection for each of these $n(Q)$ reductions</u>.

<u>THEOREM 6</u> (i) $\mathcal{B}$ <u>is a Hausdorff space and the open subset</u> $\mathcal{B}^*$ <u>of equivalence classes of irreducible connections forms a Banach manifold, with charts constructed from "slices"</u>:

$$T_{A,\varepsilon} = \{A + a \mid d_A^* a = 0, \|a\|_{L_3^2} < \varepsilon\}. \quad (\varepsilon \text{ sufficiently small}).$$

Moreoever $p : p^{-1}(\mathcal{B}^*) \to \mathcal{B}^*$ is a principle $\mathcal{G}/\pm 1$ bundle.

(ii) If A is reducible Γ_A acts on $T_{A,\varepsilon}$ and the map $T_{A,\varepsilon}/\Gamma_A \to \mathcal{B}$ is a homeomorphism to a neighbourhood of $[A]$ in $\mathcal{B}$, smooth off the fixed point set of Γ_A.

Proof

Results of this type are now standard ([2],[13]), so we shall be brief.

First, the groups $\mathcal{G}, \mathcal{G}/\pm 1$ are Lie groups in the L_4^2 topology and have Lie algebra $L_4^2(\Omega^0(\mathfrak{g}))$, moreover they act smoothly on $\mathcal{A}$. Second, we may always reduce the problem to a local one, in the following sense: if $A \in \mathcal{A}$, ε sufficiently small, then any $g \in \mathcal{G}$ for which there is an $A + a$ with $\|a\|_{L_3^2}$, $\|g(A + a) - A\|_{L_3^2}$ both less than ε may be factored as:

$$g = \gamma\tilde{g} \quad \text{with} \quad \gamma \in \Gamma_A, \qquad \|g - \mathbb{1}\|_{L_4^2} \leq \text{const } \varepsilon.$$

To see this, consider $\mathcal{G}$ as a subset of $L_4^2(\Omega^0(\mathfrak{g} \otimes \mathbb{C}))$ and write

$g = g_1 + g_2$, $g_1 \in H_A^0$, $g_2 \in H_A^{0\perp}$ ($\perp$ with respect to L^2 inner product). So : $d_A g_1 = 0$

$$\text{and} \quad \|d_A g_2\|_{L_k^2} \geq \text{const. } \|g_2\|_{L_{k+1}^2} \qquad k = 0,1,2,3.$$

Then : $b = g(A + a) - A = -d_A g\, g^{-1} + g\, a\, g^{-1}$

$$d_A g_2 = d_A g = bg - ga.$$

So, from $\|b\|_{L_3^2}$, $\|a\|_{L_3^2} < \varepsilon$ we deduce in turn that

$\|g_2\|_{L^2_k} <$ const.ε for $k = 1,2,3,4$; and so that the constant component g_1 differs from an element $\gamma \in \Gamma_A$ by $O(\varepsilon)$. Then write:

$$g = g_1 + g_2 = \gamma + (g_1 - \gamma + g_2)$$

$$= \gamma(1 + \gamma^{-1}\{g_1 - \gamma + g_2\}) = \gamma\tilde{g}.$$

By the same argument, if $A, B \in \mathcal{A}$ then for small enough ε any $g \in \mathcal{G}$ moving an element within ε of A to an element within ε of B can be factored into an automorphism sending A to B and an automorphism close to 1, so $\mathcal{B}$ is Haussdorf in the quotient topology.

Now the proof is by standard calculus:

(i) The set $\mathcal{B}^* \subset \mathcal{B}$ representing irreducible connections is open since the condition $H^0_A = 0$ is an open condition ("semi-continuity of cohomology"), and $\mathcal{G}/_{\pm 1}$ acts freely on $p^{-1}(\mathcal{B}^*)$. The smooth map

$$S : T_{A,\varepsilon} \times \mathcal{G}/_{\pm 1} \to \mathcal{A}$$

$$S(A + a, g) = g(A + a)$$

has derivative at $a = 0$, $g = 1$:

$$DS : \operatorname{Ker} d^*_A \times \Omega^0(\mathfrak{g}) \to \Omega^1(\mathfrak{g}).$$

$$(a,v) \to a + d_A v$$

which is an isomorphism since $\operatorname{Ker} d_A = H^0_A = 0$ and $\Omega^1(\mathfrak{g}) = \operatorname{Im} d_A \oplus \operatorname{Ker} d^*_A$. Hence S is locally a diffeomorphism and so, appealing to the discussion above, a diffeomorphism to its image. The cited properties of $p^{-1}(\mathcal{B}^*) \to \mathcal{B}^*$ follow easily from this.

(ii) This time DS is surjective but has Kernel H^0_A. Thus the restriction:

$$S : T_{A,\varepsilon} \times \exp (H^{0\perp}_A) \to \mathcal{A}$$

($H^{0\perp}_A$ the complement in the L^2 inner product) is a local diffeomorphism. Similarly, multiplication : $\Gamma_A \times \exp (H^0_A) \to \mathcal{G}$ has derivative 1 at the identity so there is a unique splitting of $g \in \mathcal{G}$ close to 1 as $g = \gamma\tilde{g} \quad \gamma \in \Gamma_A \quad \tilde{g} \in \exp (H^{0\perp}_A)$. So in this case a neighbourhood of $[A]$ in $\mathcal{B}$ is homeomorphic to $T_{A,\varepsilon}/\Gamma_A$, and $p : T_{A,\varepsilon} \to \mathcal{B}^*$ is clearly smooth off the fixed point set of Γ_A.

We see from Theorem 6 (ii) that in the decomposition, for reducible A, of:

$$\operatorname{Ker} d^*_A \subset \Omega^1(\mathfrak{g}) = \Omega^1(\langle iu \rangle) \oplus \Omega^1(L^2)$$

into real and complex spaces; the real component, on which Γ_A acts trivially, corresponds to the deformations of [A] within the space of equivalence classes of reducible connections and that there is a quotient singularity along this set in $\mathcal{B}$, given locally by the action of Γ_A on the complex component.

II.2. Local properties of the moduli space

If $A \in \mathcal{A}$ is self-dual the self-dual connections within the slice $T_{A,\epsilon}$ are given by the set $Z(\Phi)$ of zeros of the map:

$$T_{A,\varepsilon} = \{A + a \,|\, d_A^* a = 0, \|a\|_{L_3^2} < \varepsilon\} \xrightarrow{\Phi} L_2^2(\Omega_-^2(\mathfrak{g}))$$

$$\Phi(A + a) = F_-(A + a) = d_A^- a + \tfrac{1}{2}[a,a]_-$$

The essential fact that we use is that the derivative of Φ:

$$(D\Phi)_A : (\operatorname{Ker} d_A^* \subset L_3^2(\Omega^1(\mathfrak{g}))) \xrightarrow{d_A^-} L_2^2(\Omega_-^2(\mathfrak{g}))$$

is a Fredholm operator (recall that a bounded linear map is Fredholm if it has finite dimensional Kernel and cokernel and a closed range). This is an immediate consequence of the ellipticity of the differential operator: $d_A^* + d_A^-$ since it is well known that elliptic operators are Fredholm. Thus, first, it has an index:

$$\operatorname{index} (D\Phi)_A = \dim \operatorname{Ker} (D\Phi)_A - \dim \operatorname{coker}(D\Phi)_A$$

$$= \operatorname{index}(d_A^* + d_A^- : \Omega^1(\mathfrak{g}) \to \Omega_A^0(\mathfrak{g}) \oplus \Omega_-^2(\mathfrak{g})) + \dim H_A^0.$$

The index of $d_A^* + d_A^-$ is independent of $A \in \mathcal{A}$ and can be computed by the Atiyah-Singer index theorem in terms of the original data X, P to be [2]:

$$\operatorname{index} (d_A^* + d_A^-) = 8|c_2(P)| - \frac{3}{2}(\chi(X) - \tau(X))$$
$$= 5$$

(since by our assumptions on X ; $\chi = 2 + b_2, \tau = b_2$).

Thus the index of $(D\Phi)_A$ is 5 or 6 as A is irreducible or not, corresponding as we shall see to the fact that if A is reducible we have to divide $Z(\Phi)$ by Γ_A to get a true neighbourhood of $[A]$ in $\mathcal{B}$.

Second, it is a general property of non-linear maps of this type that we may reduce them, locally, to a linear part and a finite dimensional non-linear part. This was done by Kuranishi in [10] and by Smale in [16]; we shall use the following simple lemma, in which all maps are to be interpreted as germs about the origin.

Lemma 7: Let $E \xrightarrow{f} F$ $f(o) = 0$ be a smooth map between Banach spaces with the property that $L = (Df)_o$ is a Fredholm operator of index $k > 0$. If we choose complements:

$$E = \ker L \oplus E'$$
$$F = \operatorname{Im} L \oplus F'$$

(so that $\dim \operatorname{Ker} L = \dim F' + k$), there is a diffeomorphism ψ of E and a smooth map $\phi : E \to F'$ such that

$$f \circ \psi^{-1}(x) = Lx + \phi(x) \in \operatorname{Im} L \oplus F' = F.$$

Proof: By the open mapping theorem $L|_{E'} : E' \to \operatorname{Im} L$ has a bounded inverse $\tilde{L}$ which we can extend to F in the obvious way, so that $L\tilde{L} = 1 - \pi$; $\pi : F \to F'$ the projection.

Then $\psi : E \to E$

$$\psi(x') = x' + \tilde{L}(f(x') - Lx')$$

has derivative 1_E at the origin and is thus a local diffeomorphism with:

$$\begin{aligned} L\psi(x') &= Lx' + L\tilde{L}(f(x') - Lx') \\ &= Lx' + (1 - \pi)(f(x') - L(x')) \\ &= f(x') - \pi(f(x') - Lx') \end{aligned}$$

So if $x' = \psi^{-1}(x)$: $f(\psi^{-1}(x)) = Lx + \pi(f(\psi^{-1}(x)) - L\psi^{-1}x))$

$$= Lx + \phi(x)$$

as required.

Note also that, with the obvious definition (c.f. [16]), a point $y = y_1 + y_2 \in \operatorname{Im} L \oplus F' = \mathcal{F}$ is a <u>regular value</u> for f if and only if y_2 is a regular value for the finite dimensional map $\phi|_{L^{-1}(y_1)}$, and for any such regular value the pre-image $f^{-1}(y)$ is a smooth sub-manifold of E of dimension K.

Applying this first to the self-duality equations in $T_{A,\varepsilon}$, and choosing the harmonic space H_A^2 as the complement of $\operatorname{Im} d_A^-$, we have:

<u>PROPOSITION 8</u>: <u>For any self-dual</u> A, ε <u>sufficiently small there is a map</u> ϕ <u>from a neighbourhood of the origin in the harmonic space</u> H_A^1 <u>to the harmonic space</u> H_A^2 <u>such that: if</u> A <u>is irreducible a neighbourhood of [A] in</u> $\mathcal{M}$ <u>is carried by a diffeomorphism onto</u>

$$Z(\phi) = \phi^{-1}(0) \subset H_A^1 \quad (\text{and } \dim H_A^1 - \dim H_A^2 = 5)$$

<u>and if</u> A <u>is reducible so</u> H_A^1, H_A^2 <u>are complex spaces</u>, ϕ <u>is</u> Γ_A <u>equivariant and a neighbourhood of [A] is modelled on</u>

$$Z(\phi)/\Gamma_A \cong Z(\phi)/S^1 \quad (\underline{\text{and }\dim}_{\mathbb{R}} H_A^1 - \underline{\dim}_{\mathbb{R}} H_A^2 = 6).$$

This position follows straight from the lemma : the second part because the symmetry group Γ_A acts on the whole procedure.

Thus, just as H_A^0 detects the reducibility of a connection, the vanishing of the space H_A^2 signals whether or not 0 is a regular value for Φ, and so whether or not the moduli space has generic behaviour about A : either a smooth 5 manifold or the quotient of $H^1_A \cong \mathbb{C}^3$ by $\Gamma_A \cong S^1$. The 1st eigen-value μ of $\Delta_{A_o} = d^-_{A_o} d^*_{A_o}$ on $\Omega^2_-(\mathfrak{g})$ in Taubes implicit function theorem (Prop. 5) appears for essentially the same reason. Anticipating here Prop. 18 we can say that the sub-set $K \subset \mathcal{M}$ of classes where $H^2_A \neq 0$ is <u>compact</u>.

II.3. Perturbation of the moduli space

For the proof of the main theorem used in Section I.2 we have to establish the existence of a smooth 5-dimensional space with only finitely many, explicitly known, singularities. If the harmonic space H^2_A vanishes for all self-dual A then the moduli space $\mathcal{M}$ would have this property, but in general we know very little about these spaces so we argue that one may find a perturbation σ of the self duality equations such that the corresponding zero set $\mathcal{M}^\sigma$ has generic form.

This is the formal setting for the perturbation: the group $\mathcal{G}/_{\pm 1}$ acts on the Banach spaces:

$$L^2_3(\Omega^2_-(\mathfrak{g})) \hookrightarrow L^2_2(\Omega^2_-(\mathfrak{g}))$$

So we get a pair of bundles $\mathcal{E}^3 \hookrightarrow \mathcal{E}^2$ say, over the manifold $\mathcal{B}^*$, associated to the principle bundle $p^{-1}(\mathcal{B}^*)$; and the anti-self dual component of the curvature induces a canonical section Φ of $\mathcal{E}^2$. The allowable perturbations σ are to be smooth sections of $\mathcal{E}^3$: the "uniform norm" $\|\sigma(A)\|_{L^2_3(A)}$ in the fibre over $[A] \in \mathcal{B}^*$ is well defined as is the "covariant derivative" $(\nabla\sigma)_A$, a linear map:

$$(\nabla\sigma)_{[A]}: \operatorname{Ker} d^*_A \cong (T\mathcal{B}^*)_{[A]} \longrightarrow \mathcal{E}^3_A \cong L^2_3(\Omega^2_-(\mathfrak{g}))$$

(in effect the slices $T_{A,\varepsilon}$ define a connection on $\mathcal{E}^k$).
If we interpret σ as a section of $\mathcal{E}^2$, $\nabla(\Phi + \sigma)$ is Fredholm and of the same index as $\nabla\Phi$, Since $\nabla\sigma$ factors through L_3^2, and $L_3^2 \hookrightarrow L_2^2$ is compact.

All of this makes sense over the whole of $\mathcal{B}$ provided that σ is defined in a $T_{A,\epsilon}$ with the appropriate Γ_A symmetry. We give such σ the topology of uniform convergence of $\|\sigma\|_{L_3^2(A)}$ and convergence of $\nabla\sigma$ in operator norm, uniformly on compact sets.

It is convenient to treat the reducible connections first. Suppose that A is a reducible self-dual connection with $H_A^2 \cong \mathbb{C}^p$, $p > 0$ so that we have an S^1 equivariant map:

$$H_A^1 \cong \mathbb{C}^{3+p} \xrightarrow{\phi} H_A^2 \cong \mathbb{C}^p$$

with $(D\phi)_0 = 0$ by construction. Pick a complex linear epimorphism $\alpha : H_A^1 \to H_A^2$ and for sufficiently small $\eta > 0$ define $\tilde{\phi}$, also equivariant, by:

$$\tilde{\phi}(z) = \phi(z) + \eta.\beta\left(\frac{|z|}{\eta^{\frac{1}{2}}}\right) . \alpha(z) \in H_A^2, \qquad \beta \text{ a cut off}$$

function. Then in a small enough neighbourhood of 0, $\tilde{\phi}^{-1}(0)$ is a smooth 6 dimensional manifold and $\tilde{\phi}^{-1}(0)/_{S^1} \cong \mathbb{C}^3/_{S^1}$ is a cone on $\mathbb{C}\mathbb{P}^2$. In the notation of Lemma 7 we replace

$$\Phi : T_{A,\varepsilon} \to L_2^2(\Omega_-^2(\mathfrak{g}))$$

$$\Phi(\psi^{-1}x) = Lx + \phi(x)$$

by $\tilde{\Phi}$,

$\tilde{\Phi}(\psi^{-1}x) = Lx + \tilde{\phi}(x)$, equal to Φ outside a small neighbourhood of A.

Doing this for each of the $n(Q)$ reducible self-dual connections we can pass, without loss of generality, to the case when the compact set K is contained in $\mathcal{B}^*$ (since all our subsequent perturbations will be supported in $\mathcal{B}^*$). It is convenient then to prove:

<u>Lemma 9</u>. <u>If</u> $V \subset\subset U \subset \mathcal{B}^*$ <u>are open sets with</u> U <u>covered by finitely many slices</u> $T_{A,\epsilon}$ <u>then the set</u> G <u>of good perturbations</u> σ, <u>supported in</u> U <u>and such that the zero set</u> $\bar{V} \cap Z(\Phi + \sigma)$ <u>is cut out transversally (i.e.</u> $(\nabla(\Phi + \sigma))$ <u>is onto for all</u> $[A] \in \bar{V} \cap Z(\Phi + \sigma))$ <u>is open and dense</u>.

Given lemma 9 we can find three open sets $K \subset U_1 \subset\subset U_2 \subset\subset U_3$, each covered by finitely many slices $T_{A,\epsilon}$ (since K is compact). The set of good perturbations over U_1 supported in U_2 is open and dense, and by applying lemma 9 to $(U_2 \setminus \bar{U}_1) \subset U_3 \setminus \bar{U}_1$ and the fact that Φ is already good outside U_1, we can find the required small perturbations σ such that $\mathcal{M}^\sigma = Z(\Phi + \sigma)$ is a smooth 5-manifold with $n(Q)$ singular point, equal to $\mathcal{M}$ outside a compact set.

<u>Proof of lemma 9</u>

If σ is any perturbation, bounded in $\| \ \|_{L^2_3}$ and $0 \le R < \infty$, the set of $[A]$ in $\bar{V}$ such that $\|(\Phi + \sigma)(A)\|_{L^2_3(A)} \le R$ is compact (for in each of the slices $T_{A,\epsilon}$ covering U[1] an L^2_3

(1) <u>Footnote</u>. We ought, here, to suppose that the slices $T_{A,\varepsilon}$ are centred on smooth connections A – this is possible since the smooth connections are dense in $\mathcal{A}$.

bound on $\Phi + \sigma$ gives an L^2_3 bound on $(d^-_A + d^*_A)a$ and so an L^2_4 bound on a by ellipticity, then $L^2_4 \hookrightarrow L^2_3$ is compact).

That G is open, in the topology we defined, follows almost immediately from this and the fact that the set of surjective operators on a Banach space is open in operator norm.

Similarly, using this compactness, to prove any given σ is in $\bar{G}$ we may reduce to the case when U is contained in some $T_{A,\epsilon}$ and the local representation

$$f = \Phi + \sigma : E = \text{Ker } d^*_A \cap L^2_3 \to L^2_4 = F$$

can be decomposed in the manner of Lemma 7 over U (since a finite intersection of open dense sets is open and dense). We may then suppose that the finite dimensional complement F' lies in L^2_3 and so choose a regular value of f in F' arbitrarily small in L^2_3 norm, by applying the usual Sard lemma to $\phi : \text{Ker } L \to F'$. Then we extend this to get an element of G by using a bump function $\beta(\|a\|)$.

II.4. Orientability of $\mathcal{M}^\sigma$.

To prove that the 5-manifold $\mathcal{M}^\sigma \cap \mathcal{B}^*$ is orientable, for any good perturbation σ, we again use a general fact about Fredholm operators ([1],[3]). Suppose V,W are bundles over some (compact) manifold M and that T is a compact parameter space with bundles $\tilde{V}, \tilde{W}$ over $M \times T$ such that $\tilde{V}_t = \tilde{V}|_{M\times\{t\}} \cong V$, $\tilde{W}_t \cong W$. Then if we are given a family $\{L_t\}$ of Fredholm operators:

$$L_t : \Gamma(\tilde{V}_t) \to \Gamma(\tilde{W}_t)$$

varying continuously with $t \in T$, there is an element

$$\operatorname{ind}\{L_t\} \in KO(T)$$

which has the property that if $\dim \operatorname{Ker} L_t$ is constant for t in some closed subset $S \subset T$ - so that $\operatorname{Ker}\{L_t\}$, $\operatorname{coKer}\{L_t\}$ form vector bundles over S -

$$\operatorname{ind}\{L_t\}\Big|_S = \operatorname{Ker}\{L_t\} - \operatorname{CoKer}\{L_t\} \in KO(S). \qquad (*)$$

(The numerical index used previously is then the image of this KO index under the augmentation : $KO(T) \to \mathbb{Z}$).

The reducible connections are a nuisance here, so we choose a base point $x_o \in X$ and form the restricted gauge group $\mathcal{G}_\circ \subset \mathcal{G}$ of automorphisms equal to the identity on P_{x_o}. Then $\mathcal{G}_\circ$ acts freely on $\mathcal{A}$ so we get bundles:

$$\tilde{V} = \widetilde{(T^*X) \otimes \mathfrak{g}}\ , \qquad \tilde{W} = \widetilde{\mathfrak{g} \oplus \lambda^2(T^*X) \otimes \mathfrak{g}}$$

over the product $\mathcal{A}/_{\mathcal{G}_\circ} \times X = \hat{\mathcal{B}} \times X$ by glueing; and a family of operators:

$$L_{[A]} = d_A^* + d_A^- + (\nabla\sigma)_A : \Gamma(\tilde{V}_{[A]}) \to \Gamma(\tilde{W}_{[A]}).$$

Note first that the natural projection:

$$\hat{\mathcal{B}} \xrightarrow{\pi} \mathcal{B}$$

becomes a fibration when restricted to the dense open subset $\mathcal{B}^* \subset \mathcal{B}$ of irreducible connections, with fibre $SO(3)$; hence the tangent bundle to $\mathcal{M}^\sigma \cap \mathcal{B}^*$ will be orientable if and only if its lift to $\pi^{-1}(\mathcal{M}^\sigma \cap \mathcal{B}^*)$ is.

Second we recall that the characteristic class w_1 is defined on KO and since by definition:

$$\pi^*(T\mathcal{M}^\sigma) = \mathrm{Ind}\{L_{[A]}\} \text{ in } KO(\pi^{-1}(\mathcal{M}^\sigma \cap \mathcal{B}^*))$$

it suffices to prove that $\omega_1(\mathrm{Ind}\, L_{[A]}) = 0 \in H^1(\hat{\mathcal{B}};\mathbb{Z}/_2)$.

Thirdly observe that for all $t \in [0,1]$ the operator

$$d_A^* + d_A^- + t(\nabla\alpha)_A : L_3^2 \to L_2^2$$

is Fredholm, so that by forming the corresponding family of operators over a cylinder, we have:

$$\mathrm{Ind}\ \{L_{[A]}\} = \mathrm{Ind}\ \{d_A^* + d_A^-\} \in KO(\hat{\mathcal{B}}).$$

Now the next lemma, together with these preliminary remarks, proves that $\mathcal{M}^\sigma \cap \mathcal{B}^*$ is orientable.

LEMMA 10. Let M be a simply-connected 4-manifold, P_K an $SU(n)$ bundle over M with Chern class $c_2 = K$. Then if we form the space:

$$\hat{\mathcal{B}}(M,n,k)$$

of equivalence classes of connections on P_K, and the corresponding family of operators $d^* + d^-$ on Lie algebra valued forms we have:

$$\omega_1(\mathrm{Ind}(d^* + d^-)) = 0 \in H^1(\hat{\mathcal{B}}(M,n,k);\mathbb{Z}/_2).$$

Proof: The lemma is stated in more generality than we need since the proof will use the stable range $n \geq 3$.

The adjoint representation of $SU(3)$ on $\mathfrak{su}(3)$, when restricted to $SU(2)$ decomposes into the direct sum:

$$\mathfrak{su}(2) \oplus \mathbb{R} \oplus V$$

where V is a complex vector space. This means that if we regard

$\hat{\mathcal{B}}(M,2,k)$ as being included as a subset of reducible connections in $\hat{\mathcal{B}}(M,3,k)$, in the obvious way, the stable real vector bundle:

$$\text{ind}\{d^* + d^-\}(M,3,k)$$

when restricted to $\hat{\mathcal{B}}(M,2,k)$, differs from

$$\text{ind}\{d^* + d^-\}(M,2,k)$$

by the sum of $(\pm)$ a trivial real bundle and a complex stable bundle. Hence:

$$\omega_1(\text{ind}\{d^* + d^-\}(M,2,k))$$

is the restriction of $\omega_1(\text{ind}\{d^* + d^-\}(M,3,k)) \in H^1(\hat{\mathcal{B}}(M,3,k),\mathbb{Z}/_2)$ to $H^1(\hat{\mathcal{B}}(M,2,k);\mathbb{Z}/_2)$.

Then I claim that $\hat{\mathcal{B}}(M,n,k)$ is simply connected for $n \geq 3$; this will obviously complete the proof of the lemma. $\mathcal{G}_0 = \mathcal{G}_0(M,n,k)$ acts freely on the contractible affine space of connections so $\pi_1(\hat{\mathcal{B}}(M,n,k))$ is isomorphic to the set of connected components of $\mathcal{G}_0(M,n,k)$.

The 4-manifold M is simply connected so it is, homotopically, a wedge of 2-spheres with a 4-cell attached [12]. The bundle P_K is trivial over the 2-skeleton and, since $\pi_2(SU(n)) = 0$, any element of $\mathcal{G}_0$ can be deformed to be the identity on a neighbourhood of the 2-skeleton. Thus, collapsing the 2-skeleton, we may reduce to the case when $M = S^4$. Similarly P_K is trivial on the complement of a point, so we may reduce to the case when $k = 0$ and the bundle is trivial. Then the components of $\mathcal{G}_0(S^4,n,0)$ are the homotopy groups:

$$\pi_4(SU(n))$$

which vanish for $n \geq 3$.

Notes

(i) $\mathrm{ind}\{d^* + d^-\}$ is not, strictly, defined on the non-compact space $\hat{\mathcal{B}}$. However, this plainly does not affect the argument, which only involves compact subsets.

(ii) For the case at hand; when $k = 1$, $n = 2$, one may alternatively argue that the full quotient $\mathcal{B} = \hat{\mathcal{B}}/_{SO(3)}$ is simply connected.

(iii) The point to this proof is that we may regard the group $SU(2)$ as being the first member of either the family of symplectic groups $Sp(n)$ or of the special unitary groups $SU(n)$. For the manifold S^4, or more generally for any spin manifold, there is an essential loop in the spaces corresponding to the symplectic groups; and it is detected by the index of the Dirac operator [3]; whereas this loop does not persist in the other family, as we have seen above.

Section III

In this section we prove:

THEOREM 11. There is an open subset $\mathcal{M}_{\lambda_0}$ of the moduli space $\mathcal{M} = \mathcal{M}(X)$ of self-dual connections on P which is a smooth 5-manifold diffeomorphic to $X \times (0,\lambda_0)$ $(\lambda_0 > 0)$ and with the property that the complement $\mathcal{M} \setminus \mathcal{M}_{\lambda_0}$ is compact.

Once this has been done the proof of Theorem 1 will be complete - since the perturbed space $\mathcal{M}^\sigma$ of Section II is identical to $\mathcal{M}$ outside a compact set and so may be compactified to have boundary X by using the collar of Theorem 11.

The proof that we give may be regarded as an essay on the fundamental results of C.H. Taubes and K.K. Uhlenbeck [18],[20],[21]; and is divided into four parts:

III.1 Convergence of a sequence of connections over the 4-ball.

III.2 Convergence of a sequence of connections over X.

III.3 The definition of $\mathcal{M}_{\lambda_0}$ and of a covering map $\mathcal{M}_{\lambda_0} \xrightarrow{p} X \times (0,\lambda_0)$.

III.4 Proof that p is 1-sheeted, hence a diffeomorphism.

III.1.

PROPOSITION 12. There is a constant $C > 0$ such that if $\{m_i\}$ is a sequence of metrics on the 4-ball B^4, each sufficiently close to the Euclidean metric, and converging in $C^\infty(\bar{B}^4)$ to a limiting m_∞; and if $\{\tilde{A}_i\}$ is a sequence of connections (on the trivial bundle) over B^4 with $\tilde{A}_i$ self-dual with respect to metric m_i and satisfying:

$$\int_{B^4} |F(\tilde{A}_i)|^2 d\mu \leq C$$

<u>Then there is a subsequence</u> $\{i'\} \subset \{i\}$, <u>and</u> A_i <u>equivalent to</u> $\tilde{A}_i$ <u>such that</u> $A_{i'}$ <u>converge in</u> C^∞ <u>on the half sized ball</u>: $\frac{1}{2}B^4 = \{X \in \mathbb{R}^4 \mid |X| < \frac{1}{2}\}$ <u>to a limiting connection</u> A_∞; <u>self-dual with respect to the metric</u> m_∞.

This proposition follows from Theorem 1.3 of [21]. Identifying in the usual way a connection on the trivial bundle with a matrix of 1-forms, we may suppose by that result that the A_i are chosen to satisfy:

(i) $d^*A_i = 0$

(ii) $\|A_i\|_{L_1^p} \leq \text{const.} \|F(\tilde{A}_i)\|_{L^p}$ any $p \geq 2$.

Just as in Section II the constraint (i), together with the self-duality condition:

$$d^- A_i + \tfrac{1}{2}[A_i, A_i]_- = 0$$

gives an elliptic system of equations and, for C sufficiently small the standard convergence argument applies (by the a priori inequality Theorem 3.5 of [20] we can take $\|F(\tilde{A}_i)\|_{L^p}$, and so $\|A_i\|_{L_1^p}$, bounded for any given p).

III.2. The local result of III.1 above allows us to prove the following theorem for a sequence of connections defined over X:

THEOREM 13. <u>Let</u> $\tilde{A}_i \in \mathcal{A}$ <u>be a sequence of self-dual connections on the bundle</u> P <u>over</u> X. <u>Then there is a subsequence</u> $\{i'\}$ <u>such that either</u>:

(i) each $\tilde{A}_{i'}$ is equivalent to an $A_{i'} \in \mathcal{A}$, converging in C^∞ to a self-dual connection A_∞ on P; hence

$$[\tilde{A}_i] \to [A_\infty] \text{ in } \mathcal{M} \subset \mathcal{B}.$$

or

(ii) There is a point $x \in X$ and on the complement K of any geodesic ball about x bundle isomorphisms

$$\rho_{i'} : K \times su(2) \to P\Big|_K$$

such that $\rho_{i'}^*(A_{i'}) \to \Theta$ (the product connection) in $C^\infty(K)$.

There is considerable overlap in this theorem with the results of [15], one may compare also [14]. We prove a combinatorial lemma first.

LEMMA 14. Given $L, C > 0$ and a sequence of functions $f_i \geq 0$ on X with

$$\int_X f_i \, d\mu \leq L$$

one may find a subsequence $\{i'\} \subset \{i\}$, a finite set $\{x_1, x_2, \ldots, x_\ell\} \subset X$ and a countable collection $\{B_\alpha\}$ of small geodesic balls in X such that the half sized balls cover $X \setminus \{x_1, \ldots, x_\ell\}$ and for each α:

$$I(\alpha, i') = \int_{B_\alpha} f_{i'} \, d\mu$$

is eventually less than C.

Proof of Lemma 14. X has a countable base of neighbourhoods $\{B_q\}_{q \in N}$ made up of small geodesic balls, and for each q:

$$I(q,i) = \int_{B_q} f_i \, d\mu \le L \,. \qquad 3.32.$$

Thus we may find, for each fixed q, a subsequence i' such that $I(q,i')$ converges; and by a "diagonal argument" we can arrange that this happens for all q simultaneously: $I(q,i') \to I(q)$ say. This is just the standard fact that some subsequence of the f_i converge in the sense of distributions.

Suppose that $x_1,\ldots,x$ are points of X each of which lies in no ball B_q with $I(q) \le c/_2$ (i.e. the limiting distribution "contains δ-functions" at the points x_j). Then we may choose disjoint balls B_j containing x_j and having:

$$\int_{B_j} f_{i'} \, d\mu > c/_2 \qquad \text{for all large enough } i';$$

hence $L \ge \int_X f_{i'} \, d\mu \ge \sum_{j=1}^{\ell} \int_{B_j} f_{i'} \, d\mu \ge \ell . c/_2$.

So ℓ is at most $2L/_c$ and in particular there are only finitely many points of this type. Then we can select the required cover of $X \setminus \{x_1,x_2,\ldots,x_\ell\}$ from among the balls B_q.

<u>Proof of Theorem 13</u>.

Suppose that $B \subset X$ is any geodesic ball of radius r. We can find a geodesic coordinate system x^i in B such that:

$$ds^2 = (\delta_{ij} + O|x|^2) \, dx^i \, dx^j$$

Then if we expand this by a factor r^{-1} to give a map:

$$\chi_B : B^4 \to B$$
$$y^i \to ry^i = x^i .$$

The induced metric on B^4 is $r^2(\delta_{ij} + r^2 O|y|^2) dy^i dy^j$ which is conformally equivalent to a metric differing, with all its derivatives, by $O(r^2)$ from the Euclidean metric.

Thus we may apply Lemma 14 to the sequence of action densities:

$$f_i = |F(\tilde{A}_i)|^2 \qquad \int_X f_i \, d\mu = 8\pi^2 \text{ by I.3,}$$

with the constant C of Prop. 12 and choose the balls B_α so small that the hypotheses of Proposition 12 apply to the metrics on B^4 induced by these "conformal charts"

$$\chi_\alpha : B^4 \to B$$

Because of the <u>conformal invariance</u> of the Yang-Mills action we may apply Proposition 12 to deduce that, for each α, some sub-sequence of the $\chi_\alpha^*(A_{i'})$ converge after suitable bundle automorphism on $\frac{1}{2}B^4$ (all ensuing subsequences will be suppressed in the notation), and by a diagonal argument we may suppose this is true for all α simultaneously.

Thus in terms of local representations for the connections $\tilde{A}_{i'}$ over the cover $\frac{1}{2}B_\alpha$; we have connection matrices:

$$A_{i'}(\alpha) \to A_\infty(\alpha) \quad \text{in} \quad C^\infty(\tfrac{1}{2}B_\alpha)$$

with transition functions $g_{i'}(\alpha,\beta) : \frac{1}{2}B_\alpha \cap \frac{1}{2}B_\beta \to SU(2)$ satisfying the compatability condition:

$$(2) \quad A_{i'}(\alpha) = -dg_{i'}(\alpha,\beta)\, g_{i'}(\alpha,\beta)^{-1} + g_{i'}(\alpha,\beta) A_{i'}(\beta) g_{i'}(\alpha,\beta)^{-1}$$

The $g_{i'}(\alpha,\beta)$ are bounded since $SU(2)$ is compact, and the $A_{i'}(\alpha)$ converge in C^∞; so $dg_{i'}(\alpha,\beta)$ is uniformly bounded and we may thus find a subsequence converging uniformly.

Then equation (2) implies that this convergence is in C^∞. Once again we may arrange the sub-sequence so that this happens for all (α,β) simultaneously.

Then the data $(A_\infty(\alpha),\ g_\infty(\alpha,\beta))$ represents a self-dual connection on a bundle Q over $X \setminus \{x_1,\dots,x_\ell\}$; and if B_j is some small ball centred on x_j $(j = 1,\dots,\ell)$

$$(3) \qquad \int_{B_j\setminus\{x_j\}} |F(A_\infty)|^2 d\mu \leq \lim \int_{B_j\setminus\{x_j\}} |F(A_{i'})|^2 d\mu \leq 8\pi^2$$

The bundle $Q\Big|_{B_j\setminus\{x_j\}}$ is topologically trivial, so by Uhlenbeck's Removeability of Singularities Theorem (Proposition 4) the connection A_∞ and the bundle Q extend over all of X. Thus there is strict inequality in (3) above (by the definition of x_j in Lemma 14). On the other hand; all connections are self-dual so:

$$(4) \qquad \frac{1}{8\pi^2}\int_{B_j} |F(\tilde{A}_i)|^2 d\mu = \frac{1}{8\pi^2}\int_{B_j} \mathrm{Tr}(F \wedge F) = -Tc_2(\tilde{A}_i\Big|_{\partial B_j}) \bmod \mathbb{Z}\,.$$

Finally given any compact $K \subset X \setminus \{x_1,\dots,x_\ell\}$ we may construct bundle isomorphisms:

$$\rho_{i'} : Q\Big|_K \to P\Big|_K$$

such that $\rho^*_{i'}(\tilde{A}_{i'}) \to A_\infty$ in $C^\infty(K)$, by induction on the number of balls $\tfrac{1}{2}B_\alpha$ covering K just as in [21] Section 3. In particular:

$$Tc_2(A_\infty|_{\partial B_j}) = \lim_{i'} Tc_2(\tilde{A}_{i'}|_{\partial B_j})$$

so from (3),(4) $\ell = 0$ or 1 and we have <u>either</u>

(i) $\ell = 0$, $Q \cong P$; so we find the convergent sequence $A_{i'}$ in $\mathcal{A}$.

<u>or</u>

(ii) $\ell = 1$, Q is trivial and so by the discussion of I.3 $A_\infty \cong \theta$ and we find the required bundle maps over each K.

III.3. Theorem 13 shows that one needs to consider connections with curvature concentrated in small balls in order to compactify the moduli space $\mathcal{M}$ (just as for the case $X = S^4$ in I.4). Taubes constructs self-dual connections of this type depending upon a point in X and a scale $\lambda_0 > \lambda > 0$, thus he constructs a map:

$$X \times (0,\lambda_0) \xrightarrow{\Phi} \mathcal{M}$$

and one can prove directly [19] that this map is a diffeomorphism to its image. However we find it easier here to define a map p from an open subset of $\mathcal{M}$ representing these concentrated solutions to $X \times (0,\lambda_0)$ - thus approximately inverse to Taubes construction. The definition of any such map is bound to be rather arbitrary; the one we choose is slightly complicated but suits our later needs.

We could define the "radius" and position of a self-dual connection by taking the smallest ball that contains one half of the total action, just as for the instantons. However we will "smooth" this definition in the following way.

Let β be a smooth, even, bump function approximating, and dominated by, $\chi_{[-1,1]}$ and for any pair x, y of points in a Riemannian manifold and $s > o$ set:

$$\beta_s(x,y) = \beta\left(\frac{d(x,y)}{s}\right)$$

Then for the basic instanton action density:

$$f = |F(I_1)|^2 \qquad \text{on } \mathbb{R}^4$$

the function $R_I \; : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$

$$R_I(x,s) = \int_{\mathbb{R}^4} \beta_s(x,y) f(y) \, d\mu_y$$

has, as one can very easily see, the following properties:

(i) $\frac{\partial R_I}{\partial s} \neq 0$, so by the implicit function theorem and the fact that R_I is monotone there is a smooth function $s(x)$ on $\mathbb{R}^4$ such that $R_I(x,s(x)) = 4\pi^2$.

(ii) $s(x)$ (which is rotationally symmetric) has a unique non-degenerate minimum at 0 with minimum value K say.

Each of these is an open property so will be shared by functions sufficiently close to R_I.

DEFINITION 15. <u>For any connection</u> A <u>over a Riemannian define</u>:

(a) $R_A(x,s) = \int \beta_s(x,y) \, |F(A)|^2 \, d\mu_y$

(b) $\lambda(A) = K^{-1}\min\{s \mid \exists\, x, R_A(x,s) = 4\pi^2\}$.

Thus $\lambda(A)$ is essentially the radius of the smallest ball containing action $4\pi^2$, but this definition has been smoothed by β. Clearly, by Theorem 13, any sequence $[A_i] \in \mathcal{A}$ without convergent subsequences has $\lambda(A_i) \to 0$.

Suppose that A is a self-dual connection on P $\lambda(A) = \lambda$ and we pick a point $x \in X$ where the minimum in Def. 15(b) is attained. Then for some fixed small $r > 0$ we can take a "conformal chart" in the manner of Theorem 13, but this time scaling by a factor λ:

$$\chi_\lambda : (r/_\lambda)\, B^4 \subset \mathbb{R}^4 \to B(x,r) \subset X$$

Define $\hat{A} = \chi_\lambda^*(A)$ and let the induced metric on $(r/_\lambda)B^4$ be $\lambda^2\hat{m}$, so as in Theorem 13 $\hat{m}$ tends to the Euclidean metric as $\lambda \to 0$, uniformly with all its derivatives on bounded regions in $\mathbb{R}^4$.

THEOREM 16: (i) <u>On each bounded region of</u> $\mathbb{R}^4$ $\hat{A} \to I_1$ <u>uniformly in</u> C^∞, <u>for</u> $[A] \in \mathcal{M}$, <u>as</u> $\lambda(A) \to 0$.
(ii) <u>There is a function</u> $\delta(r)$, <u>tending to zero as</u> $r \to 0$ <u>such that the curvature densities satisfy a bound:</u>

$$|F(\hat{A}(y))| \le \text{const.} /_{|y|^{\{4 - \delta\}}} \qquad y \in \mathbb{R}^4.$$

(Note: We use the convention of writing const. for a general constant, depending at bottom only on the Riemannian metric of X and always independent of λ).

<u>Proof</u>: (i) We apply proposition 12 again. If A_i is a sequence of self-dual connections on P with $\lambda(A_i) \to 0$, then the corresponding $\hat{A}_i$ are self-dual in metric $\hat{m}_i$ and have:

$$\int_{\mathbb{R}^4} |F(\hat{A})|^2 d\mu \le 8\pi^2 .$$

The metrics $\hat{m}_i$ converge to the Euclidean metric on $\mathbb{R}^4$, thus we may run the argument of Theorem 13 again to deduce that there is a convergent subsequence with limit a self-dual connection on $\mathbb{R}^4$. This time we do not have curvature gathering over points because of the normalisation chosen: $\lambda(\hat{A}_i) = 1$. The convergence is uniform on bounded regions and again by the choice of normalisation the limit must be the instanton I_1 (by the classification discussed in Sect. I.4 and the property (ii) of R_I above).

Since the limit is the same for every possible subsequence the rate of convergence depends only upon λ.

(ii) This bound will become rather important in our development of III.4. To preserve continuity we give the proof in the Appendix.

COROLLARY 17: <u>There is a</u> $\lambda_o > 0$ <u>such that if</u> A <u>is a self-dual connection on</u> P <u>with</u> $\lambda(A) < \lambda_o$ <u>then the minimum in Definition 15(b) is attained at a unique point</u> $x(A)$ <u>in</u> X <u>and the map</u>:

$$p : \mathcal{M}_{\lambda_o} = \{[A] \in \mathcal{M} \mid \lambda(A) < \lambda_o\} \to X \times (0, \lambda_o)$$

$$p(A) = (x(A), \lambda(A))$$

<u>extends to a smooth map on some neighbourhood of</u> $\mathcal{M}_{\lambda_o}$ <u>in</u> $\mathcal{B}$. <u>If</u> $[A_t]$ <u>is any smooth path in</u> $\mathcal{B}$, $A_o \in \mathcal{M}_{\lambda_o}$ <u>there is a bound</u>:

$$\left| \frac{d}{dt} p(A_t) \right| \leq \text{const.} \quad \lambda(A_o) \left\| \frac{d}{dt} F(A_t) \right\|_{L^2} ,$$

<u>both derivatives being evaluated at</u> $t = 0$.

Proof: Any two minima must be separated by a distance of at most 2λ (since the ball of radius λ about each contains more than $\frac{1}{2}$ of the total action); since the question is also conformally invariant we may change scale and pass to $\hat{A}$.

The curvature densities $|F(\hat{A})|^2_{\hat{m}}$ converge uniformly on bounded regions of $\mathbb{R}^4$ to the instanton action density $f = |F(I_1)|^2$, as does the metric $\hat{m}$; so by the openness of the two properties of R_I noted before Def. 15 there will be a unique mimimum corresponding to $\hat{A}$ for sufficiently small λ, and the position of a non-degenerate minimum varies smoothly with parameters so the map p defined above is smooth.

Likewise for the second part; the statement is again scale invariant so we may pass to the corresponding path $\hat{A}_t$, $f_t = |F(\hat{A}_t)|^2$. Then:

$$\left.\frac{d\lambda(\hat{A})}{dt}\right|_{t=o} = \int_{\mathbb{R}^4} \beta_1(x(\hat{A}),y) \left.\dot{f}_t(y)\right|_{t=o} d\lambda_y$$

$$\leq \text{const.} \int_{\mathbb{R}^4} |\dot{f}_t(y)|\,d\mu_y \leq \text{const}\ \|F(\hat{A})\|_{L^2} \left\|\frac{d}{dt} F(\hat{A}_t)\right\|_{L^2}$$

And similarly:

$$\left.\frac{d}{dt}\,x(\hat{A})\right|_{t=o} = H^{-1}\left\{\int_{\mathbb{R}^4} \left[\text{grad}_y\, \beta_1(0,y)\right] \left.\dot{f}_t(y)\right|_{t=0} d\mu_y\right\}$$

where H is the Hessian corresponding to the minimum at $t = 0$.

To conserve notation we will allow ourselves to rechoose the positive number λ_o, throughout the rest of this section, to give the space $\mathcal{M}_{\lambda_0}$ the desired properties. In any case $\mathcal{M} \setminus \mathcal{M}_{\lambda_0}$ is compact by Theorem 13.

PROPOSITION 18: <u>For</u> $[A] \in \mathcal{M}_{\lambda_0}$:

(i) <u>Given</u> $r > 0 : [A] \to \theta$ <u>in</u> $C^\infty(X \setminus B(x(A),r))$ <u>as</u> $\lambda(A) \to 0$.

(ii) <u>The first eigenvalue</u> $\mu(A)$ <u>of</u> Δ_A <u>on</u> $\Omega^2_-(\mathfrak{g})$ <u>tends to the first eigenvalue</u> μ_1 <u>of</u> Δ <u>on</u> Ω^2_- <u>as</u> $\lambda(A) \to 0$. <u>Since</u> $\mu_1 > 0$, $\mathcal{M}_{\lambda_0}$ <u>is a smooth 5-manifold for</u> λ_0 <u>sufficiently small by the discussion of Section II</u>.

(iii) $\|F(A)\|_{L^4} \leq \text{const.}\ \lambda(A)^{-1}$.

<u>Proof</u>: (i) is just a re-statement of Theorem 13, alternative (ii), with uniformity in λ just as in Theorem 16.

(ii) If ω is the first (normalised) eigenfunction of Δ_A, so $\|d_A^*\omega\|^2_{L^2} = \mu(A)$, we compare ω with $\omega' = (1 - \beta_r)\omega$ where β_r is supported in $B(x(A),r)$, equal to 1 in the half-sized ball. Since $\|d_\theta^*\omega'\|^2_{L^2} \geq \mu_1\|\omega'\|^2_{L^2}$, and $A \to \theta$ on $\operatorname{supp}\omega'$ by (i) above, we deduce the result by taking r sufficiently small. This goes precisely as [18] Prop. 8.8 so we omit the details.

(iii) The L^4-norm on 2-forms transforms by a factor λ^{-1} under a scale change of λ, so it plainly suffices to show $\|F(\hat{A})\|_{L^4}$ bounded; but this follows from Theorem 16 (ii) once r, and so $\delta(r)$, is taken small enough for

$$1/|y|^{[4-\delta(r)]} \in L^4(\mathbb{R}^4 \setminus B^4)$$

THEOREM 19: <u>For sufficiently small</u> λ_0;

$$p : \mathcal{M}_{\lambda_0} \to X\times(0,\lambda_0)$$

<u>is a covering map</u>.

<u>Proof</u>: First note that Theorem 13 implies that p is a proper map, and it is easy to see that a proper local homeomorphism is a (finite) covering, so it suffices to prove that:

$$dp : (T\mathcal{M})_{[A]} \to (TX)_{x(A)} \times \mathbb{R}$$

is an isomorphism for $[A] \in \mathcal{M}$, $\lambda(A)$ sufficiently small. By the application of the index theorem described in II.2 we know each space is 5-dimensional so it suffices again to exhibit a linear map:

$$\alpha : (TX)_{x(A)} \times \mathbb{R} \to (T\mathcal{M})_{[A]}$$

such that $dp \circ \alpha$ is invertible, which will certainly be true if

$$|(dp \circ \alpha)\xi - \xi| < |\xi| \quad \text{for all} \quad \xi \in TX \times \mathbb{R} \qquad (1).$$

To define α, choose r so small that $\delta(r) < 1$ say, and take geodesic co-ordinates x^i in $B(x(A),r) \subset X$. Then given : $\xi = (v^i,\phi) \in TX \times \mathbb{R}$ let f_t be the flow on X given by the vector field, in local co-ordinates:

$$V = \beta_r\left(v^i \frac{\partial}{\partial x^i} + \phi\frac{\partial}{\partial \varrho}\right) .$$

Thus $f_t = 1$ outside $B(x(A),r)$. Lifting to the bundle f_t defines a flow on $\mathcal{B}$, set $[A_t] = q(\phi_t(A)) \in \mathcal{M}$ where q is Taubes' map as in Proposition 5, so A_t is defined for sufficiently small t (depending on λ).

Then put $\alpha(\xi) = \frac{d}{dt}[A_t]\Big|_{t=0} \in (T\mathcal{M})_{[A]}$

α is plainly linear since the whole process varies smoothly with ξ. To check property (1) : observe first that if m_t

is a path of Euclidean structures on $\mathbb{R}^4$ with m_0 the standard one, then for $\omega \in \Lambda^2\mathbb{R}^4$:

$$\frac{d}{dt}|\omega - *_{m_t}\omega| \le \text{const. } |\omega|\pi(\frac{dm_t}{dt})$$

(derivatives at $t = 0$), where π is the projection onto the "trace free" part. Hence on X:

$$\frac{d}{dt}|F_-(f_t(A)| \le \text{const. } |F_-(A)||\pi\mathcal{L}_V m|$$

(All derivatives $\frac{d}{dt}$ in this proof are to be understood as being evaluated at $t = 0$).

And one easily sees that, because x^i are geodesic and so V approximately conformal:

$$|\pi\mathcal{L}_V m| \le \text{const. } \varrho\,|\xi| \qquad o \le \rho \le r$$

where the left hand side is evaluated at $x \in X$ and $\varrho = d(x,x(A))$. Scaling down the bounds of Theorem 16 with $\delta = 1$ we have:

$$|F(A)| \le \text{const } \lambda^{-2}|\xi| \qquad o \le \rho \le \lambda$$

$$\le \text{const } \lambda\rho^{-3}|\xi| \qquad \lambda \le \rho \le r.$$

So $\frac{d}{dt}\|F_1(f_t(A))\|_{L^p} \le \text{const } \{\int_o^\lambda (\rho\lambda^{-2})^p \rho^3 d\rho + \int_\lambda^r (\lambda\rho^{-2})^p\rho^3 d\rho\}^{1/p} |\xi|$

which gives:

$$\frac{d}{dt}\|F_-(f_t(A))\|_{L^2}, \frac{d}{dt}\|F_-(f_t(A))\|_{L^{4/3}} \quad \text{each bounded by}$$

const $\lambda.|\xi|$. (We would improve the bound on the $L^{4/3}$ norm by taking δ smaller, but this is not necessary here).

So, using the bounds on $q(A_o) - A_o$ in Proposition 5; and recalling that $A_t = q(f_t(A))$:

$$\left\|\frac{d}{dt}(A_t - f_t(A))\right\|_{L_1^2(A)}\Big|_{t=o} \le \text{const.}\left\{\frac{d}{dt}\|F_1(f_t(a))\|_{L^2} + \right.$$

$$\left. + \lambda^{-1}\left\|\frac{d}{dt} \quad F_-(f_t())\right\|_{L^{4/3}}\right\} \le \text{const.}\ |\xi| \ .$$

Since by Prop. 18 $\|F(A)\|_{L^4} \le \text{const.}\lambda^{-1}$, and the eigenvalue $\mu(Z)$ is not small.

But this implies $\left\|\frac{d}{dt}\ (F(A_t) - F(f_t(A))\right\|_{L^2} \le \text{const.}\ |\xi|$.

So by Corollary 17 applied to the path $A_t - f_t(A)$:

$$\left|\frac{d}{dt}\ p(A_t) - \frac{d}{dt}\ p\ f_t(A)\right| \le \text{const.}\ \lambda|\xi|$$

i.e $\left|(dp \circ \alpha)\xi - \frac{d}{dt}\ p\ f_t(A)\right| \le \text{const.}\ \lambda|\xi|$

If the metric on X were flat in the ball of radius r about $x(A)$ then, by the construction, $\frac{d}{dt}\ p(f_t(A))$ would equal ξ exactly. In general one easily sees that the difference is bounded by $\text{const.}\lambda.|\xi|$ so:

$$|(dp \circ \alpha)\xi - \xi| \le \text{const.}\lambda.|\xi|.$$

And property (1) is satisfied once $\lambda = \lambda(A)$ is sufficiently small.

III.4. The proof that p is a diffeomorphism

To prove that $p\ ; \mathcal{M}_{\lambda_o} \to X\mathrm{x}(0,\lambda_o)$ is a diffeomorphism it suffices now to show that if we pick some point x_o in X then for λ sufficiently small any pair $[A],[B] \in \mathcal{M}_{\lambda}$ of self-dual connections with

$$\chi(A) = x(B) = x_o,$$

$$\lambda(A) = \lambda(B) = \lambda$$

can be joined by a short path in $\mathcal{M}$ (in a sense made precise below). The metric on X has always been at our choice and there is a small saving of labour here if we suppose it to be <u>flat</u> in some ball $B(x_o, r)$ - although this does not, of course, affect the truth of the result as one easily sees by deformation.

First we note a small lemma.

LEMMA 20. <u>There are constants</u> $K, C > 0$, <u>independent of</u> R, <u>such that if</u> A <u>is a connection defined on the trivial bundle over the annulus</u>

$$\{x \in \mathbb{R}^4 \mid R/2 < |x| < 2R\}$$

<u>with</u> $\|F(A)\|_{C^0} \le K/R^2$, <u>we may find a connection</u> $\tilde{A}$ <u>over the ball</u> $2RB^4$, <u>equal to</u> A <u>in</u> $R < |x| < 2R$, <u>and having</u>:

$$\|F(\tilde{A})\|_{C^0} \le C \cdot \|F(A)\|_{C^0} .$$

<u>Proof</u>: Under conformal re-scaling by factor R^{-1} the norms $\|F(A)\|_{C^0}$, $\|F(\tilde{A})\|_{C^0}$ each transform by a factor R^2, so it suffices to do this when $R = 1$.

Then once K is small enough we may find an "exponential gauge" in the manner of [20] Section 2, so that without loss of generality the connection matrix over the annulus satisfy:

$$\|A\|_{C^0} \le \text{const.} \|F(A)\|_{C^0} .$$

Then set $\tilde{A} = (1 - \beta)A$, whence

$$F(\tilde{A}) = d\tilde{A} + \tfrac{1}{2}[\tilde{A},\tilde{A}] = (1-\beta)F(A) - d\beta \wedge A + \tfrac{1}{2}(\beta^2 - 3\beta)[A,A]$$

so $\|F(\tilde{A})\|_{C^0} \le \text{const.}\|F(A)\|_{C^0}$ as required.

This lemma is used to compare the connection A on the bundle P over X with the instanton and with the flat connection. First we need some more notation: for the instanton $I_{y,\sigma}$ over $\mathbb{R}^4$ with centre y and scale σ write $I'_{y,\sigma}$ for the corresponding connection over S^4. Similarly the connection $\tilde{A}$ over the large ball $r\lambda^{-1}B^4 \subset \mathbb{R}^4$ has curvature bounded by:

$$|F(\hat{A})(y)| \le \text{const}/|y|^4 \qquad \text{(since in this case we may take}$$

$\delta = 0$ in Theorem 16). Thus we may conformally map this large ball to the complement in S^4 of a small cap C_λ over the north pole, of radius $O(\lambda)$; and the connection we get has uniformly bounded curvature, independent of λ. By lemma 20 we may extend this connection to a connection, A' say, defined over all of S^4, and maintain a uniform bound on the curvature of A'. Clearly A' has Chern class -1.

<u>Theorem 21</u>: (i) If A_1 be the restriction of the connection A to $X \setminus B(x_0,\sqrt{\lambda})$ then after a suitable bundle automorphism

$$\|A_1 - \theta\|_{L^p_1(\theta)} \le \text{const.}\ \lambda^{2/p} \qquad p > 1$$

(ii) Similarly, after a suitable bundle automorphism:

$$\|A' - I'_{0,1}\|_{L^p_1(I'_{0,1})} \le \text{const.}\ \lambda^{4/p}, \qquad p \ge 2.$$

<u>Proof</u>: (i) By Theorem 16 (ii), scaled down:

$$|F(A)_x| \le \text{const. } \lambda^2/\rho^4 \qquad \rho = d(x,x_0)$$

So the curvature is uniformly bounded, independent of λ, on the annulus $\frac{1}{2}\sqrt{\lambda} \le \rho \le 2\sqrt{\lambda}$ and, for λ sufficiently small, we may extend A_1 over all of X (by Lemma 20) to a connection on the trivial bundle, preserving a uniform bound on the curvature. Thus :

$$\begin{aligned} \|F_-(A_1)\|_{L^p} &\le \text{const.}[\text{vol.}B(x_1,\sqrt{\lambda})]^{1/p} \\ &\le \text{const.}\lambda^{2/p} . \end{aligned}$$

And, just as in Prop. 18, the eigenvalue $\mu(A_1)$ is not small so we may apply the implicit function theorem of Taubes (Prop. 5) to find a self-dual connection $A_1 + a = q(A_1)$, which is without loss of generality the product connection θ, with $\|a\|_{L_1^2(A)} \le \text{const. } \lambda$. Moroever by the form of the construction : $a = d^*_{A_1} u$ so that:

$$\text{(a)} \quad \begin{aligned} d^*_\theta a &= d^*_{A_1} a + \{u,a\} \\ &= \{F_-(A_1),u\} + \{a,a\} \end{aligned}$$

and

$$\text{(b)} \quad d^-_\theta a = \{a,a\} + F_-(A_1).$$

(We write $\{,\}$ for algebraic bilinear expressions whose particular form is not important here). By the ellipticity of $d^* + d^-$:

$$\|a\|_{L_1^p} \le \text{const.}\left[\|(d^* + d^-)a\|_{L^p} + \|a\|_{L^p}\right]$$

and combined with standard estimates of the quadratic terms in (a) (b) this gives:

$$\|a\|_{L_1^p} \leq \text{const.}\|F_-(A_1)\|_{L^p} \leq \text{const.}\ \lambda^{2/p}$$

once λ is sufficiently small. (The situation is not at all delicate here, compared with [18], because of the uniform bound on the curvature of A_1).

(ii) We apply the same argument to the connection A' over S^4 with:

$$\|F_-(A')\|_{L^p} \leq \text{const.}\ (\text{Vol.}C_\lambda)^{1/p}$$
$$\leq \text{const.}\ \lambda^{4/p}.$$

One obtains a lower bound on the eigenvalue $\mu(A')$ by using the Weitzenböch formula for connections over S^4 (18, Prop. 2.2., [2] P.145).

$$\Delta_A\omega = \tfrac{1}{2}\,\nabla_A^*\nabla_A\omega + \tfrac{2}{3}\,S.\omega + \{F_-,\omega\}\ ;\quad \omega \in \Omega_-^2(\mathfrak{g})$$

where s = Scalar curvature of $S^4 > 0$.

So this time we get $\|A' - I'_{y,\nu}\|_{L_1^p} \leq \text{const.}\lambda^{4/p}$. For some y,ν. Now in general, if $A, A + a$ are connections then:

$$F(A + a) = F(A) + d_A a + \tfrac{1}{2}[a,a]$$

implies $\|F(A + a) - F(A)\|_{L^2} \leq \text{const.}\{\|a\|_{L_1^2(A)} + \|a\|_{L^4}^2\}$

and in dimension 4 $L_1^2(A) \hookrightarrow L^4$ with embedding constant independent of A ([18] Lemma 5.2). Thus, in our case:

$$\|F(A') - F(I'_{y,\nu})\|_{L^2} \leq \text{const.}\ \lambda^2$$

from which one easily sees, by the normalisation $\lambda(\hat{A}) = 1$, $x(\hat{A}) = 0$, that $|y| + |\nu - 1| \leq \text{const.}\lambda^2$, so:

$$\|A' - I_{0,1}'\|_{L^p_1(I_1')} \leq \text{const.}\lambda^{4/p} \qquad p \geq 2.$$

The bounds (i), (ii) of Theorem 21 will also, of course, be satisfied by the putative connection B'; thus to complete the proof of Theorem 11 we have to use these to show that A and B can be joined, after suitable bundle automorphism, by a path in $\mathcal{A}$ which is short enough to project onto the moduli space. This is slightly easier to write down if we use a small modification of Proposition 5.

PROPOSITION 22: Let A be any self-dual connection on P, there are constants C, ϵ, depending only upon the first eigenvalue $\mu(A)$ and the metric on X, such that if $b \in \Omega^1(\mathfrak{g})$ and:

$$\delta(A,b) = \|\nabla_A b\|_{L^2} + \|b\|_{L^2}\|F(A)\|_{L^4} < \epsilon$$

we may construct a self-dual connection $A + b + q(b,A) = A + b + a$ with $\|\nabla_A a\|_{L^2} \leq C\{\|F_-(A+b)\|_{L^2} + \|F(A)\|_{L^4}\|F_-(A+b)\|_{L^{4/3}}\}$

$$\|a\|_{L^2} \leq C\|F_-(A+b)\|_{L^{4/3}}\,.$$

The construction is $\mathcal{G}$-invariant ($\mathcal{G}$ acting on A,a,b) and a varies smoothly with b.

This propositon follows by the method of proof of [18] Thm. 2.2., applied to the equation $F_-(A + b + d_A^* u) = 0$. In the equation corresponding to [18] (5.2) we lose the term in

$F_-(A_0)$, which means we do not need to involve $\|F_-\|_{L^3}$ in [18] (5.12.b).

LEMMA 23. Suppose that A, B are self-dual connections on P $x(A) = x(B) = x_0$, $\lambda(A) = \lambda(B) = \lambda$. Then if λ is sufficiently small we may suppose that, after a suitable bundle automorphism: $B = A + b$ with:

$$\|\nabla_A b\|_{L^2} \le \text{const.}\lambda^{1/2}$$

$$\|b\|_{L^2} \le \text{const.}\lambda^{3/2}$$

Proof of Theorem 11, given Lemma 23.

Suppose that, on the contrary, $p : \mathcal{M}_{\lambda_0} \to X \times (0, \lambda_0)$, which is a covering by Theorem 19, had at least two sheets. Then for arbitrarily small λ we could find A, B as in Lemma 23 representing different sheets. By Proposition 18 the eigenvalue $\dot{\mu}(A)$ is not small and $\|F(A)\|_{L^4} \le \text{const.}\lambda^{-1}$; thus we can apply Proposition 22 to the path $A + tb$ from A to B and find a path in $\mathcal{M}$, $[\beta(t)]$ say : $[\beta(0)] = [A]$

$[\beta(1)] = [B]$

and moreover $\|F(A + \beta(t)) - F(A)\|_{L^2} \le \text{const.}\ \lambda^{1/2}$.
From which one easily sees, as in Corollary 17, that:

$$|\lambda(A + \beta(t)) - \lambda(A)| + d(x(A + \beta(t)), x_0) \le \text{const.}\ \lambda^{3/2}$$

so, if λ is much less than λ_0, $\beta(t)$ stays in $\mathcal{M}_{\lambda_0}$ and $x(\beta(t))$ is close to x_0; hence $[A]$ $[B]$ are in the same local component of the covering and so equal.

On the other hand by Taubes construction there are self-dual connections on X with $\lambda(A)$ arbitrarily small, which one may see by estimating the curvature density of the solutions in [18], in the manner above.

Proof of Lemma 23

We proceed in three stages, corresponding to the three regions $X \setminus B(x_o, \sqrt{\lambda})$, $B_o(x_o, 2\sqrt{\lambda})$ and $W = B(x_o, 2\sqrt{\lambda}) \setminus B(x_o, \sqrt{\lambda})$

(a) Using theoreom 21(i) and the Sobolev embedding theorems in dimension 4 we may suppose that A is given over $X \setminus B(x_o, \sqrt{\lambda})$ by a connection matrix A_1 with

$$\|A_1\|_{L^2} \le \text{const.} \|A_1\|_{L_1^{4/3}} \le \text{const.}\ \lambda^{3/2}$$

and for $\varepsilon > 0$, $\|A_1\|_{C^0} \le C(\varepsilon)\|A_1\|_{L_1^{4+\epsilon}} \le C(\varepsilon)\ \lambda^{1/2-\varepsilon}$

, and $\|A_1\|_{L^2_1} \le$ const. λ

Similarly B is represented by some B_1, thus in this region $B = A + b_1$ (up to equivalence) with $\|b_1\|_{L_1^2} \le \text{const.}\lambda^{1/2}$

$$\|b_1\|_{L^2} \le \text{const.}\lambda^{3/2}.$$

(b) First we transform the bounds of Theorem 21(ii) on the connections A', B' over S^4 to the small ball $B(x_o, 2\sqrt{\lambda})$, by the obvious conformal transformation. Thus on S^4 we may suppose: $B' = A' + b'$;

$$\|b'\|_{L_1^p(A')} \le \text{const.}\lambda^{4/p} \quad p \ge 2.$$

So: $\|b'\|_{C^0} \le C(\varepsilon)\lambda^{1-\varepsilon} \le \text{const.}\lambda^{\frac{1}{2}}$ say.

$\|b'\|_{L^2} \le \text{Const.}\lambda^2.$

Taking account of conformal factors gives, over $B(x_0, 2\sqrt{\lambda})$,

$$B = A + b_2 \text{ say, with } \|b_2\|_{L^2} \le \text{const.}\lambda^2$$

$$\text{and } \|b_2\|_{C^0(W)} \le \text{const.}\lambda^{\frac{1}{2}}.$$

The norm on 1-forms $\|\nabla(\ )\|_{L^2}$ is independent of constant scale changes, but in going from S^4 to $B(x_0, 2\sqrt{\lambda})$ we have a varying conformal factor which introduces a lower order term: one easily calculates:

$$\|\nabla_{I_\lambda} b_2\|_{L^2} \le \|\nabla_{I_1}(\hat{A} - I_1)\|_{L^2}$$

$$\le \text{const.}\ \|b'\|_{L_1^2(I_1')} + \left[\int_{\mathbb{R}^4 \setminus B^4} \left(\frac{I_1 - \hat{A}}{|y|}\right)^2 d\mu_y\right]^{\frac{1}{2}}$$

$$\le \text{const.}\lambda^{\frac{1}{2}}, \text{ since } (\hat{A} - I_1)(y) \le \text{const}\ \frac{\|b'\|_{C^0}}{|y|^2}\ \left(y \in \mathbb{R}^4\right).$$

So also $\|b_2\|_{L_1^2(I_\lambda)} \le \text{const.}\lambda^{\frac{1}{2}}$, (Moreover it does not matter whether we use $L_1^2(I_\lambda)$ or $L_1^2(A)$ here).

(c) The region W corresponds under the conformal transformation of (b) to a small annulus of radius $O(\lambda^{\frac{1}{2}})$ about the north pole of S^4. Thus we may find connection matrics A_2, B_2 for A, B respectively over $B(x_0, 2\lambda^{\frac{1}{2}}) \setminus \{x_0\}$ such that in W:

$$\begin{aligned} &\|A_2\|_{L^2(W)} \le \text{const}.\lambda^{3/2} \\ \text{(I)}\qquad &\|A_2\|_{C^0(W)} \le \text{const}.\lambda^{\frac{1}{2}} \\ &\|\nabla A_2\|_{L^2(W)} \le \text{const}.\lambda^{\frac{1}{2}}. \end{aligned}$$

(Since we may find such a connection matrix for I_λ, corresponding to a stationary framing around the north pole of S^4, and then A is close to I_λ by (b) above).

On W we have transition functions:

$g,h : W \to SU(2)$.

$$\begin{aligned} &A_2 = -dgg^{-1} + gA_1g^{-1} \\ \text{(II)}\qquad &B_2 = -dhh^{-1} + hB_1h^{-1}. \end{aligned}$$

If it so happened that $g = h$ then we would be done, since the pair $A_2 - B_2$, $A_1 - B_1$ would represent a difference element $b \in \Omega^1(\mathfrak{g})$, $B = A + b$ after automorphism and satisfying the required bounds by parts (a),(b). So the final step is to check that we can find another set of data, $(B_1, \tilde{B}_2, g|_{W'})$ say, for B: where W' is the smaller annulus $B(x_0, \frac{3}{2}\lambda^{\frac{1}{2}}) \setminus B(x_0, \lambda^{\frac{1}{2}})$, with $\tilde{B}_2 = B_2$ outside of W, and keeping $A_2 - \tilde{B}_2$ small.

Since the connection θ is fixed by a constant rotation of the bundle we may suppose that at some point w of W $g = h = 1$ (note that this would not be true if we had an irreducible flat connection in place of θ). Then (i) and (ii), together with part (a) give:

$$\|dg\|_{C^0(W)} \leq \text{const.}\lambda^{\frac{1}{4}}$$

hence by integration: $\|g - 1\|_{C^0(W)} \leq \text{const.}\lambda^{\frac{3}{4}}$.

Thus if $u = gh^{-1} = \exp(s)$, $\|s\|_{C^0(W)} \leq \text{const.}\lambda^{\frac{3}{4}}$ and L^p_k bounds on u are equivalent to L^p_k bounds on s. Set then $V = \exp\,(\gamma_{\sqrt{\lambda}}\, s)$

$$\gamma_{\sqrt{\lambda}} = 0 \quad \text{on} \quad B(x_0, \sqrt{\lambda})$$

$$= 1 \quad \text{outside} \quad B(x_0, 3/_2\sqrt{\lambda})$$

so that $(B_1, \tilde{B}_2, g|_{W'})$ is another representation of B with:

$$\tilde{B}_2 = \tilde{\ } dvv^{-1} + vB_2v^{-1}.$$

Then one may check that $\|\tilde{B}_2\|_{L^2}$, $\|\tilde{B}_2\|_{L^2_1}$ are appropriately small. For example:

$$\|\nabla\tilde{B}_2\|_{L^2(W)} \leq \text{const.}\{\|\nabla B\|_{L^2(W)} + \|B_2\|_{L^2(W)} + \|\nabla\nabla v\|_{L^2(W)}\}$$

(since $\nabla v \to 0$ uniformly)

$$\leq \text{const.}(\lambda^{\frac{1}{2}} + \|(\nabla\nabla\gamma)_{\lambda^{\frac{1}{2}}} s\|_{L^2(W)} + \|\nabla\gamma_{\lambda^{\frac{1}{2}}}\nabla s\|_{L^2(W)}\}$$

$$\leq \text{const.}(\lambda^{\frac{1}{2}} + \lambda^{-1}\|s\|_{L^2(W)} + \lambda^{-\frac{1}{2}}\|\nabla s\|_{L^2(W)}\}$$

$$\leq \text{const.}(\lambda^{\frac{1}{2}} + \lambda^{-1}\lambda^{\frac{3}{4}}\,(\text{Vol}W)^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}\lambda^{3/2})$$

$$\leq \text{const.}\;(\lambda^{\frac{1}{2}} + \lambda^{\frac{3}{4}} + \lambda) \leqslant \text{const.}\lambda^{\frac{1}{2}}.$$

APPENDIX Curvature decay ; Proof of Theorem 16 (ii)

The removeability of singularities theorem (Prop. 4) implies, by conformal inversion in the unit sphere, that any self-dual connection I on $\mathbb{R}^4$ with finite action satisfies:

$$|F(I)(y)| \leq c(I)/_{|y|^4}, \quad y \in \mathbb{R}^4 \text{([20] Cor. 4.2)}$$

The result we need here is a small variant of this (since our connections are only defined over large balls $\lambda^{-1}rB^4 \subset \mathbb{R}^4$), however the method of proof of the Removeability of singularities theorem in [20] applies. We give an outline of this proof here for completeness, and because the proof becomes much simpler in our case (of self-dual connections), than in [20] which deals with the full Yang-Mills equations.

Suppose then that $[A] \in \mathcal{M}$ $\lambda(A) = \lambda$ and ignore the difference between the metric $\hat{m}$ on $\mathbb{R}^4$ and the Euclidean metric. Then define, for $R > 0$:

$$J(R) = 8\pi^2 - \int_{RB^4 \subset \mathbb{R}^4} |F(\hat{A})|^2 \, d\mu \geq 0$$

i.e. the amount of action of A outside the ball of radius λR about x. So:

(a) $$\frac{dJ}{dR} = - \int_{S(R)} |F(\hat{A})|^2 \, d\mu_S \text{, where}$$

$S(R)$ is the sphere of radius R about 0 in $\mathbb{R}^4$ and μ_S is the measure on $S(R)$.

In general, if we choose a distinguished $\mathbb{R}^3 \subset \mathbb{R}^4$ we get natural isomorphisms:

$$\Lambda^2\mathbb{R}^4 \cong \Lambda^2\mathbb{R}^3 \oplus \Lambda^1\mathbb{R}^3 \cong \mathbb{R}^3 \oplus \mathbb{R}^3$$

corresponding in "electro-magnetic" notation to

$$F \to (\underset{\sim}{E}, \underset{\sim}{B})$$

, and the self-duality condition becomes $\underline{E} = \underline{B}$. Applying this to the tangent spaces of the 3-space $S(R)$ gives, for self-dual $\hat{A}$:

(b) $|F(\hat{A})|^2 = 2|F(\hat{A}_R)|^2$

where we have written $\hat{A}_R$ for $\hat{A}|_{S(R)}$.

On the other hand, again because $\hat{A}$ is self-dual:

(c) $J = -Tc_2(\hat{A}_R)$ (as in I.3).

So both $J, \frac{dJ}{dR}$ depend only upon the connection $\hat{A}_R$ over $S(R)$. Now we know from Theorem 16 (i), that as $\lambda \to 0$ $|F(\hat{A})|^2 \to |F(I_1)|^2$ (uniformly on bounded regions) so, since $\|F(I_1)\|^2_{L^2(\mathbb{R}^4)} = 8\pi^2$, given $\varepsilon > 0$ we may find R_o such that, once λ is sufficiently small $J(R_o) < \varepsilon$.

Now the a priori bound [20] Theorem 3.5. for the curvature F of any self-dual connection says that there is a $k > o$ such that if, in a ball $2B$ say :

$$\int_{2B} |F|^2 < K \text{ then } \|F\|_{C^o(B)} \leqslant \frac{\text{const.}}{\text{radius}^2(B)} \|F\|_{L^2(2B)}.$$

Applying this to large balls outside R_oB^4 shows that, once we take $\mathcal{E}$ sufficiently small:

(d) $|F(\hat{A})y| \leq \text{const. } \varepsilon/_{|y|^2}$ $\qquad |y| \geq R_o$.

That is, if we rescale the sphere $S(R)$ to the unit sphere

$S(1) = S^3$ the induced connection A_R^* on S^3 has curvature uniformly bounded by const.ε. This means that the second order approximations in A_R^* :

$$Tc_2(A_R^*) = Tc_2(\hat{A}_R) = \int_{S^3} \mathrm{Tr}\{A_R^* \wedge (dA_R^* - [A_R^*, A_R^*])\}$$

$$= \int_{S^3} \mathrm{Tr}(A_R^* \wedge dA_R^*) + O(A_R^*)^3$$

and

$$\|F(A_R^*)\|^2_{L^2} = R\,\|F(\hat{A}_R)\|^2_{L^2}$$

$$= \int_{S^3} |d\hat{A}_R|^2 \, d\mu_S + O(A_R^*)^3$$

are valid with small error. For any $\omega \in \Omega^1(S^3)$:

$$\text{(d)} \quad \int_{S^3} \omega \wedge d\omega \leq 2 \int_{S^3} |d\omega|^2 d\mu_S$$

by the discussion of spherical harmonics in [13],[20].

$$\text{So} \quad Tc_2(\hat{A}_R) \leq 2R \int_{S(R)} |F(\hat{A}_R)|^2 \, d\mu_S + \text{small error}$$

$$\leq 4R \int_{S(R)} |F(A)|^2 \, d\mu_S + \text{small error}$$

by (b); hence by (d),(a): $J \leq -4R \dfrac{dJ}{dR}$ + small error.

Ignoring the "small error", which is accommodated by iteration of the argument just as in [20] Theorem 4.8, this differential inequality integrates to give $J(R) \leq$ const. R^{-4}, and hence the result by applying the a priori bound.

If we take account of the curvature of the metric $\hat{m}$ then the same argument applies with a small correction factor $\delta(r) \to 0$ as $r \to 0$.

References

[1] Atiyah, M.F. "K-Theory" (Benjamin, New York, 1967).

[2] Atiyah, M.F., Hitchin, N.J. and Singer, I.M. "Self-duality in four dimensional Riemannian geometry". Proc. R. Soc. London A 362 pp. 425-461 (1978).

[3] Atiyah, M.F. and Jones, J.D.S. "Topological aspects of Yang-Mills Theory" Commun. Math. Phys. 61 97 pp.97-118 (1978).

[4] Atiyah, M.F. and Singer, I.M. "The index of elliptic operators IV", Ann. of Maths. 93 pp. 119-138 (1971).

[5] Bourguignon, J.P. and Lawson, H. "Yang-Mills theory: its physical origins and differential geometric aspects" Ann. of Math. Studies 102 pp.395 - 421 (Princeton U.P. 1982).

[6] Chern, S.S. "Complex manifolds without potential theory" (Springer-Verlag New York 1979).

[7] Donaldson, S.K. "Self-dual connections and the topology of smooth 4-manifolds". To appear in the Bull. Amer. Math. Soc.

[8] Freedman, M.H. "The topology of four-dimensional manifolds". To appear in the Journal of Differential Geometry.

[9] Hodge, W.V.D. "The theory and applications of harmonic integrals" (Cambridge U.P. 1952).

[10] Kuranishi, M. "A new proof for the existence of locally complete families of complex structures", Proceedings of the conference on Complex Analysis, Minneapolis pp. 142-156 (Springer Verlag 1964).

[11] Mandelbaum, R. "Four dimensional topology; an introduction", Bull. Amer. Math. Soc. Vol. 2, No. 1. pp. 1-159 (1980).

[12] Milnor, J.W. "On simply connected 4-manifolds", Proc. of the International Symposium on Algebraic Topology, pp. 122-128 (University of Mexico, 1958).

[13] Parker, T. "Gauge theories on 4-dimensional Riemannian manifolds" Commun. Math. Phys. 85 pp. 563-602 (1982).

[14] Saks, J. and Uhlenbeck, K.K. "On the existence of minimal immersions of 2-spheres", Ann. of Math. 113 pp. 1-24 (1982).

[15] Sedlacek, S. "A direct method for minimising the Yang-Mills functional". To appear in Commun. Math. Phys.

[16] Serre, J-P. "A course in arithmetic" (Springer-Verlag 1973).

[17] Smale, S. "An infinite dimensional version of Sard's Theorem" Amer. Jour. Math. 87 pp.861-866 (1965).

[18] Taubes, C.H. "The existence of self-dual connections on non self-dual 4-manifolds". To appear in the Journal of Differential Geometry.

[19] Taubes, C.H. Private letter.

[20] Uhlenbeck, K.K. "Removeable singularities in Yang-Mills Fields" Commun. Math. Phys. 83 pp. 11-30 (1982).

[21] Uhlenbeck, K.K. "Connections with L^p bounds on curvature", Commun. Math. Phys. 83 pp. 31-42 (1982).

CHAPTER FOUR

Hermitian-Einstein connections over $\mathbb{CP}^2$

It seems likely that the direct variational method of Chapter 1 could be made to extend to prove the existence of Hermitian-Einstein connections, at least on bundles over algebraic surfaces. However, we shall revert, in this Chapter, to a method of proof similar to that used by Narasimhan and Seshadri, which will illustrate the connection between the Riemannian point of view of the last chapter and the complex analytic phenomena discussed at the end of Chapter 2. We prove:

THEOREM 1: An indecomposable holomorphic bundle of rank 2 over $\mathbb{CP}^2$ admits a Hermitian-Einstein connection (with respect to any Kähler metric on $\mathbb{CP}^2$) if and only if it is stable, and this connection is unique up to isomorphism.

Thus the conjecture of Hitchin and Kobayashi is true for these bundles. By Corollary 7 and Proposition 9 of Chapter 2 only the existence needs to be proved.

§1. Moduli spaces

The equivalence classes of stable holomorphic (hence algebraic) bundles over any smooth polarised variety are parametrised by a Hausdorff moduli space (this is not, of course, true for all holomorphic bundles), which may have complex analytic singularities. Many of these moduli of bundles over projective spaces are known explicitly ([4] Chapter 4).

Examples:

(i) The moduli space $M_{\mathbb{P}_2}(0,2)$ of stable 2-plane bundles over $\mathbb{CP}^2 = \mathbb{P}(V)$ with $c_1 = 0$, $c_2 = 2$ can be naturally identified with the space of non-singular conics in the dual space $\mathbb{P}(V^*)$.

(ii) The moduli space $M_{\mathbb{P}_2}(-1,2)$ (obvious notation) can be naturally identified with the set of unordered pairs of distinct points in $\mathbb{P}^2$.

(iii) The moduli space $M_{\mathbb{P}_3}(0,1)$ of 2-bundles over $\mathbb{P}_3 = \mathbb{P}(W)$ can be naturally identified with the set of non-singular skew forms on W (up to scalar multiplication).

Observe that in each of these three cases the moduli space can be compactified by adjoining points associated to codimension 2 subvarieties : in (i) we adjoin the set of singular conics i.e. line pairs in $\mathbb{P}^2_*$ or equivalently point pairs in $\mathbb{P}_2$, in (ii) we adjoin the points of $\mathbb{P}_2$ as the "diagonal" and in (iii) we adjoin the set of lines in $\mathbb{P}_3$, as the image of the Plücker embedding.

We will use the following general facts about the moduli spaces $M_{\mathbb{P}_2}(c_1,c_2)$ of rank 2 bundles over $\mathbb{CP}^2$. First note that, by tensoring with a line bundle, it suffices to consider the two cases $c_1 = 0$, $c_1 = -1$. Then we have ([4] : Lemma 1.2.7. and the remarks following, together with Sections 4.1 - 4.2):

PROPOSITION 2: (i) <u>There are stable 2-bundles over</u> $\mathbb{P}_2$ <u>with</u> $c_1 = 0$, $c_2 = k$ <u>for all</u> $k \geq 2$, <u>but not for any</u> $k < 2$. <u>The</u>

corresponding moduli spaces are smooth irreducible (hence connected) complex manifolds (Theorem of Barth).

(ii) There are stable 2-bundles over $\mathbb{P}_2$ with $c_1 = -1$, $c_2 = k$ for all $k \geq 1$ but not for $k < 1$. The moduli space for $c_2 = 1$ is a single point representing the twisted tangent bundle

$$T \otimes H^{-2} .$$

Again all the moduli spaces are irreducible (Theorem of Hulek).

We shall prove that for each of these moduli spaces the subset representing Hermitian-Einstein connections is open, non-empty, and closed. Since the moduli spaces are connected this will establish Theorem 1.

§ 2. Deformation of Hermitian-Einstein connections

For Kähler manifolds of any dimension we have:

PROPOSITION 3: If (X,ω) is a smooth projective algebraic variety with a Kähler metric and if $\mathcal{E}$ is a stable bundle over X admitting a Hermitian-Einstein connection with respect to ω then:

(i) There is a neighbourhood of the point in the moduli space corresponding to $\mathcal{E}$ such that all the holomorphic bundles associated to this neighbourhood admit Hermitian-Einstein connections.

(ii) <u>For any Kähler metric</u> ω' <u>sufficiently close to</u> ω, $\mathcal{E}$ <u>admits a Hermitian-Einstein connection with respect to</u> ω'.

<u>Proof</u>: We prove part (i), the second is similar.

Suppose that A is a Hermitian-Einstein connection on the C^∞ bundle E corresponding to $\mathcal{E} = \mathcal{E}_A$. Then by Proposition 5 (iii) of Chapter 2:

$$H^0(\operatorname{End}\mathcal{E}) = \operatorname{Ker}\left(\Delta_A : \Omega^0(\operatorname{End} E) \to \Omega^0(\operatorname{End} E)\right) \cong \mathbb{C}.$$

Observe now that it suffices to construct nearby connections with the trace-free part $\hat{F}^+$ of the curvature vanishing (for any such connection may be modified by a diagonal element of $\mathcal{G}^{\mathbb{C}}$ to a Hermitian-Einstein connection).

We have a smooth map:

$$\mathcal{G} \times \Omega^1(\mathfrak{g}_E) \xrightarrow{\Phi} \Omega^0(\mathfrak{g}_E)^+$$

$$\Phi(g,a) = \hat{F}^+(g(A+a))$$

whose derivative at $g = 1$, $a = 0$ is:

$$D\Phi(\theta,a) = (\Delta_A\theta)^+ + (d_A^{c*}a)^+$$

and since $\operatorname{Ker}\ \Delta_A \cong \mathbb{C}$ is perpendicular to the trace free part, the partial derivative with respect to θ is surjective and we may solve, by the implicit function theorem

$$\hat{F}(g(A+a))^+ = 0 \text{ for } g, \text{ given any sufficiently small } a$$

(that is we use the implicit function theorem in appropriate Sobolev spaces and then elliptic regularity gives smoothness, c.f. Chapter 3 Section 1, [1] Section 6).

Observe that this proof does not really use the existence or properties of the moduli space of stable bundles but only the topology in $\mathcal{A}$, and the result could be formulated in those terms.

§ 3. Construction of Hermitian-Einstein connections by the method of Taubes.

It was explained at the end of Chapter 2 that a Hermitian-Einstein connection on a holomorphic bundle over a Kähler surface is the same as an anti self-dual connection. The theorem of Taubes discussed in Chapter 3 does not apply directly to give anti self-dual connections over $\mathbb{CP}^2$, because the self-dual Kähler form is now an obstruction. Indeed, since we know that any indecomposable 2-bundle admitting a Hermitian-Einsten connection is stable (Prop. 11 of Chapter 2) we deduce from Proposition 2 that <u>for any Kähler metric on $\mathbb{CP}^2$ there are no anti self-dual</u> SU(2) <u>connections with</u> $c_2 = 1$.

However the proof of Taubes' theorem can easily be extended to overcome this obstruction, as we shall now describe. Suppose M is a compact, oriented Riemannian 4-manifold and we have some anti self-dual $SU(2)$ connection A over M with $c_2 = K$. Then we can try to manufacture an anti self-dual $SU(2)$ connection with $c_2 = K + 1$ by "glueing in" an anti-instanton, in the manner of Chapter 3. Then Taubes' proof (for the case when A is trivial) goes through to this situation and the relevant space of "obstructions" is now:

$H^2_A \subset \Omega^2_+(\mathfrak{g})$; the harmonic self-dual

2-forms with values in the bundle of Lie algebras. More generally if A is a $U(2)$ connection the same argument applies (for the whole discussion really concerns projective connections) and the relevant obstructions are now trace free harmonic self-dual 2-forms.

In a different direction suppose that M possesses an orientation preserving isometry σ with, for simplicity, $\sigma^2 = 1$. Then if we choose distinct points $x_1, x_2 = \sigma(x_1)$ we may construct, as in Taubes' paper, an "approximate solution" A_o with $c_2 = 2$, based on the points x_1, x_2 and depending upon a parameter λ. Moreover we may do it in such a way that the isometry σ lifts to the bundle and $\sigma^* A_o = A_o$. Then Taubes' implicit function theorem, for deforming A_o into an anti self-dual connection, goes through word for word under this σ action and the space of obstruction is now the $+1$ eigenspace of the action of σ on the harmonic self-dual 2-forms $\mathcal{H}^2_+(M)$. Thus:

PROPOSITION 4: Suppose that, in the situation above, σ acts as -1 on $\mathcal{H}^2_+(M)$. Then there is an anti self-dual $SU(2)$ connection over M with $c_2 = 2$.

Now we apply this to the case when $M = \mathbb{CP}^2$ with the Fubini-Study metric. First note that if A is an indecomposable Hermitian-Einstein connection the identification:

$$\Omega^2_+ \otimes \mathbb{C} \cong \Omega^{0,2} \oplus \Omega^{2,0} \oplus \mathbb{C}\omega$$

identifies the obstruction space H^2_A above with a subspace of the sheaf cohomology groups:

$$H^0(\mathrm{End}\,\mathcal{E}_A) \oplus H^2(\mathrm{End}\,\mathcal{E}_A)$$

corresponding to trace free endomorphisms. By Ch.2, Prop. 4 (iii); $H^0(\mathrm{End}\,\mathcal{E}_A) \cong \mathbb{C}$, and by Serre duality: $H^2(\mathrm{End}\,\mathcal{E}_A)^* \cong H^0(\mathrm{End}\,\mathcal{E}_A \otimes K_{\mathbb{P}_2}) = 0$; since the dual of $K_{\mathbb{P}_2}$ has holomorphic sections.

Thus these obstructions vanish in all cases for indecomposable Hermitian-Einstein connections over $\mathbb{CP}^2$ and so, in each of the families $c_1 = 0$, $c_1 = -1$, if we construct Hermitian-Einstein connections with $c_2 = K > 0$ then, by the discussion above, such connections will exist for all higher values of K.

The Fubini-Study connection on $\mathbb{TP}^2$ is Hermitian-Einstein so this gives representatives in all the moduli spaces with $c_1 = -1$ of bundles admitting Hermitian-Einstein connections. For the other family, with $c_1 = 0$, we use Proposition 4 above. If σ is any real structure on $\mathbb{CP}^2$, compatible with the metric, then σ gives an orientation preserving isometry of $\mathbb{CP}^2$ which reverses the complex structure; thus $\sigma^*(\omega) = -\omega$ and the hypotheses of Proposition 4 apply.

Hence in sum we have found <u>in each moduli space of stable 2-bundles over</u> $\mathbb{CP}^2$ <u>a holomorphic bundle admitting a Hermitian-Einstein connection.</u>

§ 4. A continuity argument.

Using the material developed in the previous chapter we have a good understanding of the "boundary" of the moduli space of anti self-dual connections, and shall now relate this to the moduli of stable bundles.

THEOREM 5: Let X be a smooth projective algebraic surface with a choice of polarisation. Suppose that a stable bundle $\mathcal{E}$ over X admits a Hermitian-Einstein connection with respect to some Kähler metric ω, consistent with this polarisation. Then:

(i) All stable bundles in the connected component of the moduli space defined by $\mathcal{E}$ admit Hermitian-Einstein connections.

(ii) For any other Kahler metric ω' consistent with the polarisation $\mathcal{E}$ admits a Hermitian-Einstein connection with respect to ω'.

Proof: (i) By Proposition 3 of this chapter it suffices to prove that the subset of the moduli space corresponding to Hermitian-Einstein connections is closed. Suppose that some holomorphic bundle $\mathcal{E}$ corresponds to a point in the closure of this set. Then on a fixed C^∞ bundle E we can find two sequences:

$$A_n, B_n \in \mathcal{A}^{(1,1)}$$

such that A_n is Hermitian-Einstein $A_n = g_n(B_n)$ for some $g_n \in \mathcal{G}^{\mathbb{C}}$ and the B_n converge in C^∞ to some (arbitrary) connection B_∞ corresponding to $\mathcal{E}$, i.e. $\mathcal{E} \cong \mathcal{E}_{B_\infty}$.

Now the A_i are (projective) anti self-dual connections, so by Theorem 13 of Chapter 3 (modified in the obvious way) we may suppose that, without loss of generality, the A_i converge except over a finite set of points $\{x_1, x_2, \ldots, x_\ell\} \subset X$, to an Hermitian-Einstein connection A_∞ over X. As in the previous chapter A_∞ may, a priori, be defined on a different bundle E', but the 2-forms:

$$-\frac{1}{2\pi i} \operatorname{Tr} F(A_i)$$

will all be the same harmonic form, so $c_1(E') = c_1(E)$. Then A_∞ defines a holomorphic structure $\mathcal{E}' = \mathcal{E}_{A_\infty}$ on E' and, since it admits a Hermitian-Einstein connection $\mathcal{E}'$ will be at least semi-stable by Proposition 11 of Chapter 2.

It suffices then to prove that $\operatorname{Hom}(\mathcal{E},\mathcal{E}') \neq 0$ (as in Section 2 of Chapter 1). For then since $\mathcal{E}$ is stable and $\mathcal{E}'$ is semi-stable, and the bundles have the same rank and degree, we will have $\mathcal{E} \cong \mathcal{E}'$ by Proposition 9 of Chapter 2 and $\mathcal{E}$ will admit a Hermitian-Einstein connection.

To prove that $\operatorname{Hom}(\mathcal{E},\mathcal{E}') \neq 0$, choose small balls B_j containing x_j $(j = 1, 2, \ldots, \quad)$ and such that both $\mathcal{E}$ and $\mathcal{E}'$ are holomorphically trivial in a neighbourhood of each B_j.

Then, as in Theorem 13 of Chapter 3, we may suppose that on the subest $X \setminus (UB_j)$:

$$\left.\begin{array}{l} B_n \to B \\ \\ A_n \to A \end{array}\right. \quad \text{in } C^\infty(X \setminus UB_j)$$

And we have endomorphisms g_n with : $\bar{\partial}_{A_n B_n} g_n = 0$ (in the notation of Chapter 1, Section 3). Moreover we may normalise the g_n by constant scale factors, so that, for example, $\|g_n\|_{L^4(X)} = 1$.

Thus since $\bar{\partial}_{A_n B_n} \to \bar{\partial}_{A_\infty B_\infty}$ we may, by the standard elliptic estimates, suppose that (perhaps after taking a subsequence) the g_n converge to some g_∞ on $X \setminus UB_j$ with $\bar{\partial}_{A_\infty B_\infty} g = 0$.

Then I claim that $g_\infty \neq 0$. To see this we use Proposition 6 (iii) of Chapter 2; that is, if $\tau_n = |g_n|^2$:

$$\Delta\tau_n \leq \mathrm{Tr}(g_n^* g_n \hat{F}_{B_n}) \leq \text{const. } \tau_n.$$

Since also $\tau_n \geq 0$ a standard argument ([3] Theorem 5.3.1. for example) gives:

$$\|\tau_n\|_{C^0(X)} \leq \text{const.}\|\tau_n\|_{L^2(X)} = \text{const}\|g_n\|^2_{L^4(X)} \leq \text{const.}$$

Hence, if the balls B_j were chosen small enough, we see that we cannot have both $\tau_n \to 0$ in $X \setminus \cup B_j$ and $\|\tau_n\|_{L^2} = 1$.

Thus there is a non-zero holomorphic map

$$\mathcal{E}\,|_{X\setminus \cup B_j} \to \mathcal{E}'\,|_{X\setminus \cup B_j}$$

and by the Theorem of Hartogs ([2] page 7) this extends over the balls B_j to give a non-zero element of $\mathrm{Hom}(\mathcal{E},\mathcal{E}')$.

The proof of part (ii) of Theorem 5 is almost identical; we can join the metrics ω, ω' by a linear path ω_t and the argument above shows that the set of t for which $\mathcal{E}$ admits a Hermitian-Einstein connection is open and closed. The essential point here is that the polarisation, and so the notion of stability does not vary with t.

Theorem 5 completes the proof of Theorem 1 at the beginning of this chapter. The same method can obviously be used more generally: for example it proves the conjecture of Hitchin and Kobayashi for 2-bundles with $c_1 = 0$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$, and can probably be extended to any rational surface. For a suitable Enriques surface we get again SU(2) Hermitian-Einstein connections with $c_2 \geq 2$ and so on its double cover, a K3 surface, with $c_2 \geq 4$. In both of these cases the same calculation that was used to prove the non-existence part of Proposition 2 shows that these values are the least possible, but I do not know that the relevant moduli spaces are connected. However it does not seem to be worthwhile to pursue these generalisations in detail, since one could hope that similar techniques give a more direct proof.

References

[1] Atiyah, M.F., Hitchin, N.J. and Singer, I.M. "Self-duality in four dimensional Riemannian geometry".

[2] Griffiths, P. and Harris, J. "Principles of Algebraic geometry".

[3] Morrey, C.B. "Multiple Integrals in the Calculus of Variations", Springer-Verlag, New York, 1966.

[4] Okonek, C., Scheider, M., and Spindler, M. "Vector bundles on complex projective space".